Particle Systems on the Positive Half-Line under Different Boundary Conditions



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Statement of Originality

I declare that this thesis contains no material which has been accepted or is currently being submitted for any other degree, diploma, certificate or other qualifications at the University of Oxford or elsewhere.

Chapter 3 is based on [40]. Chapter 4 contains unpublished work. Both chapters are based on joint work with my supervisor Ben Hambly and Andreas Søjmark from LSE. Chapter 5.1 contains material from [39] and is co-authored with Ben Hambly.

Abstract

We study interacting-particle systems on the positive half-line with different boundary conditions at zero. We focus on systems with a common noise and on boundaries that have a reflecting component, in particular instantaneous reflection, elastic reflection and sticky reflection. We also consider models driven by independent noises that have a stronger interaction with the boundary and experience feedback through elastic stopping times.

The use of these particle systems is motivated through applications in financial modelling. They can be employed to model large systems of entities with default risk such as large credit portfolios or a system of banks.

We show weak convergence of the particle systems with elastic and reflecting boundaries. We establish existence and uniqueness of solutions for the equation satisfied by the limiting empirical measure valued process for the surviving particles, which is a weak form for an SPDE with a noisy Robin boundary condition satisfied by the particle density. We show that this density process has good L^2-regularity properties in the interior of the domain but may exhibit singularities on the boundary at a dense set of times.

In a similar way we show that particle systems with a sticky boundary converge weakly to solutions of related limit SPDEs and that the limits have a density in the interior of the positive half-line.

We establish weak convergence of a particle system with feedback through elastic stopping times to the solution of a McKean-Vlasov equation that depends on the distribution of the elastic stopping time. In order to establish this weak convergence we prove uniqueness of solutions of the McKean-Vlasov equation given suitably defined solution concepts that respect the discontinuities that can appear in these systems. Moreover, we extend the existence result to two-phase systems that consist of a

population of particles on the positive half-line and a population on the negative half-line.

Contents

1 Introduction **1**

1.1 The Particle Systems 1
1.2 Motivating Applications 4
1.3 Elastic Boundaries 6
1.4 Sticky Boundaries 9
1.5 Feedback Models 10
1.6 Thesis Outline 13

2 Brownian Motion Reflected on Brownian Motion **14**

2.1 Reflected Brownian Motion 14
2.2 Heat Atoms and Heat Singularities 15

3 Particle Systems with Elastic Boundary **18**

3.1 Introduction 18
3.1.1 Summary of main results 20
3.1.2 Making sense of the elastic boundary at the origin 21
3.1.3 Literature on SPDEs with noisy boundary conditions 23
3.2 Convergence to an SPDE with elastic boundary 24
3.2.1 Functional convergence and well-posedness of the SPDE . . . 25
3.2.2 Reflecting and absorbing boundaries as limiting cases 27
3.3 Probabilistic Estimates for the Particle System 28
3.4 Convergence of the particle system 33
3.4.1 Evolution equation for the empirical measures 33
3.4.2 Convergence to the Limit SPDE 34
3.4.3 Regularity of the Limit Solution 38
3.5 Uniqueness of the SPDE 39
3.5.1 Weak Boundary Condition and Elastic Boundary Terms . . . 41
3.5.2 Proof of uniqueness for solutions to the SPDE 45

3.6 Existence of a density and its integrability . . . 48
3.6.1 Ruling out a Dirac mass at the origin . . . 54
3.7 Extension to nonlinear interactions in the drift . . . 56
3.8 Absorption and Reflection as Limiting Cases . . . 59
3.9 Application to Mortgage-Backed Securities . . . 64

4 Particle Systems with Sticky Boundary 67
4.1 Introduction . . . 67
4.2 Particle System Construction . . . 73
4.3 Probabilistic Estimates for Particles . . . 76
4.4 Convergence of the Particle System . . . 79
4.4.1 Finite Evolution Equation . . . 79
4.4.2 Tightness and Convergence . . . 80
4.5 The Uniqueness Proof in the Linear Case . . . 89
4.6 Regularity Results . . . 97
4.6.1 Existence and Regularity of a Density away from the Boundary 97
4.6.2 Regularity Results near the Boundary . . . 103
4.7 The Uniqueness Proof in the Nonlinear Case . . . 109

5 Feedback Models 115
5.1 One-Phase Model with Elastic Feedback . . . 117
5.1.1 Physical and Minimal Solutions . . . 118
5.1.2 PDE Formulation and Connection with the Kinetic Undercooled Stefan Problem . . . 122
5.1.3 Particle System Approximation . . . 123
5.1.4 Convergence to Absorbing and Reflecting Models . . . 130
5.2 Two-Phase Model with Elastic Feedback . . . 133
5.2.1 Existence of Solutions . . . 135
5.2.2 Particle Approximation . . . 137
5.3 One-Phase Model with Common Noise . . . 138
5.3.1 Existence of Solutions . . . 140
5.3.2 Provoking and Averting Blow-Ups . . . 141
5.4 One-Phase Absorbing Feedback Model with Upper Reflecting Boundary 142
5.4.1 Connected PDE . . . 144
5.4.2 Particle Approximation . . . 147

A Properties of the Elastic Heat Kernel 149

B Properties of the Sticky Heat Kernel **152**

C Heat Kernel Mollification in H^{-1} **155**

Bibliography **158**

List of Figures

4.1 Sticky Brownian Motion Path 68

5.1 A Realization and the Related Loss Process of the Elastic Feedback Model 117

Chapter 1

Introduction

1.1 The Particle Systems

The main objective of this thesis is to study the influence of different boundary conditions on a system of particles on the positive half-line. In particular we will focus on boundaries that have a reflection component. We discuss instantaneous reflection, elastic reflection and sticky reflection. Moreover we consider systems with a stronger boundary interaction through feedback. We begin by giving a brief overview of the models and discuss applications which motivate this work.

Our work builds on the recent literature on particle systems with absorbing boundary and a common noise [14, 37, 41]. These systems consist of N particles, $X^{i,N}, i = 1, \dots, N$, with dynamics

$$dX_t^{i,N} = \mu(t, X_t^{i,N}, \nu_t^N)dt + \sigma(t, X_t^{i,N})(\rho(t, \nu_t^N)dW_t^0 + \sqrt{1 - \rho(t, \nu_t^N)^2}dW_t^i) \quad (1.1.1)$$

with independent Brownian motions $W^0, W^1, \dots$. The term ν_t^N denotes the empirical measure given by

$$\nu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}} 1_{t<\tau^{i,N}} \quad (1.1.2)$$

with $\tau^{i,N}$ denoting the first hitting time of zero by $X^{i,N}$

$$\tau^{i,N} = \inf\{t \geqslant 0 : X_t^{i,N} \leqslant 0\}. \quad (1.1.3)$$

We will also refer to $\tau^{i,N}$ as the absorbing stopping time of the particle. We see that in a system with absorbing particles, a particle gets removed from the system the first time it hits the boundary at zero. Note that in these models the drift coefficient and the correlation coefficient are allowed to depend on the empirical measure ν_t^N

but, for technical reasons, not the volatility term σ. Moreover, dependence on x is only allowed in the drift and volatility component but not in the correlation. The empirical measure is the main object of interest when studying the particle system. Since the particle $X^{i,N}$ gets removed from the empirical measure at time $\tau^{i,N}$, the dynamics of $X^{i,N}$ do not affect the empirical measure ν_t^N for any $t > \tau^{i,N}$.

An analogous particle system with reflecting boundary consists of particles with dynamics given by

$$dX_t^{i,N} = \mu(t, X_t^{i,N}, \nu_t^N)dt + \sigma(t, X_t^{i,N})(\rho(t)dW_t^0 + \sqrt{1-\rho(t)^2}dW_t^i) + dL_t^{i,N}. \quad (1.1.4)$$

The term $L^{i,N}$ denotes the local time at zero of $X^{i,N}$ and ν_t^N the empirical measure given by

$$\nu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}. \quad (1.1.5)$$

Note that in (1.1.4) we need to restrict to a correlation coefficient that only depends on time but not the empirical measure.

The system with elastic reflection can be obtained by adding an elastic killing time for each particle to the system. This is the stopping time $\tau^{i,N}$ given by the first time the local time at zero of $X^{i,N}$ exceeds the value of an independent exponential random variable χ^i,

$$\tau^{i,N} = \inf\{t \geqslant 0 : L_t^{i,N} > \chi^i\}. \quad (1.1.6)$$

For the system with sticky reflection the particle dynamics of $X^{i,N}$ are given by the sticky diffusion process

$$dX_t^{i,N} = \frac{1}{2c}1_{X_t^{i,N}=0}dt + 1_{X_t^{i,N}>0}\left(\mu(X_t^{i,N}, \nu_t^N)dt + \sigma(X_t^{i,N})(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i)\right), \quad (1.1.7)$$

with parameter $c > 0$ controlling the strength of the stickiness at zero. Since our focus is on the influence of the boundary conditions we do not consider parameters that depend on time t in this setting to avoid further technical complications.

The final topic we study is the class of models with feedback through stopping times. In these models we have a system of N particles given by

$$\begin{cases} X_t^{i,N} = X_{0-}^i + B_t^i - \alpha\Lambda_t^N + L_t^{i,N} \\ \Lambda_t^N = \frac{1}{N}\sum_{i=1}^{N} 1_{\tau^{i,N}\leqslant t} \\ \tau^{i,N} = \inf\{t \geqslant 0 : L_t^{i,N} \geqslant \chi^i\}, \end{cases} \quad (1.1.8)$$

for $i = 1, \ldots, N$, with parameter $\alpha > 0$, independent Brownian motions $B^1, B^2, \ldots$, independent of the i.i.d. initial conditions X_{0-}^i and the independent exponential

random variables χ^i. The term $L^{i,N}$ originates from the Skorokhod map and ensures that the process $X^{i,N}$ stays nonnegative. The interaction with the boundary in this system is stronger than in the previous ones we discussed. There is a feedback to the other particles in the system if the elastic killing time at the boundary is reached for one of the particles. We will see that this is a singular form of interaction that can lead to discontinuities in the related limit process. When studying feedback models our main focus is on these singular interactions. This is the reason we consider the dynamics in (1.1.8) without additional drift or volatility terms.

In all the different cases of particle systems that we consider in this thesis, our first goal is to establish weak convergence of the empirical measure of the particle system and to characterize the limit as the unique solution to a related limit equation. Our second goal is to study regularity properties of the limit which can be done using the limit equation.

There are at least two different ways of viewing the empirical measure. We can define the empirical measure ν_t^N for each time t as done in (1.1.5) for the system with reflecting boundary. We can then consider the flow of measures $t \mapsto \nu_t^N$ as a process in a suitable space and establish convergence of this process as the number of particles N goes to infinity. This is the approach we will follow when studying the reflecting, elastic and sticky particle systems.

An alternative viewpoint is to define an empirical measure ν^N via

$$\nu^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^{i,N}_\cdot}. \tag{1.1.9}$$

In this case ν^N is a random measure on the space of paths in which the particles $X^{i,N}$ live. In this set up the goal is to show that the measure ν^N converges in a suitable sense to the law of a solution to a related limit equation. This is the approach we will pursue when considering the model with feedback. It relates to the notion of propagation of chaos for systems of exchangeable particles. Note that convergence of the empirical measure in (1.1.9) is a stronger notion than convergence of the flow of measures $t \mapsto \nu_t^N$ (see [55]).

As for the empirical measures, there are at least two different types of limit equations that we can find. One of the classical applications of particle methods is to approximate solutions to partial differential equations (PDEs). In that spirit limit equations for the limit of empirical measure processes (ν_t^N) can be viewed as weak formulations of PDEs. However, in the cases where a common noise is present, this

stochastic term will carry over to the limit and the limit equations will instead be weak formulations of stochastic partial differential equations (SPDEs).

An alternative approach is to formulate the limit as a solution for a McKean–Vlasov equation. These are equations where the law of the process appears in the equation. When studying feedback models we will use this approach. The related limit equation to our feedback model is a McKean–Vlasov equation with feedback through the distribution of the elastic stopping time at the boundary.

Before going into our analysis of the different particle systems in more detail, we will first discuss the financial motivation for these models.

1.2 Motivating Applications

Models involving particle systems with boundary interactions have been applied in a variety of different fields recently. The articles [24, 25] employed a particle system to find a probabilistic formulation of the leaky integrate-and-fire models [15, 16] used in neuroscience. In [20] a particle system with two reflecting boundaries is used to model the charging of a battery. Furthermore, [26] studied a probabilistic formulation of the supercooled Stefan problem that describes the evolution of the interface during the freezing of a supercooled liquid and [30] considered a particle system with moving elastic boundary to model an epidemic advancing through a susceptible population.

Our main motivation comes from financial applications, in particular the modeling of large baskets of entities with default risk and systemic risk in the financial sector. There is now a considerable literature developing particle models for these financial applications. The fallout from the Global Financial Crisis of 2008 showed the need for new models for credit portfolios, such as mortgage-backed securities. This motivated [14] to develop a structural credit model for a large basket of instruments with credit risk using a particle approach similar to (1.1.1)-(1.1.3). For a basket with a large number of instruments, the limit equation of the particle system can not only be used to analyse the behaviour of the system, but it can also be employed for pricing securities that are based on large credit portfolios such as mortgage-backed securities. In the case of very large portfolios it can be computationally more efficient to simulate one limit SPDE instead of a large number of stochastic differential equations for each individual particle. The articles [37, 41] built on this approach to extend the model of [14] to more general coefficients and [1] added the feature of early repayment of a mortgage to the model. There are alternative particle approaches to large credit portfolio modelling such as the reduced-form models by [32, 68].

We build on the models developed in [14, 37, 41] but change the behaviour at the boundary to reflect different default conditions. In these models, each particle represents the financial health associated with each mortgage, loan or other entity with default risk and the particle system as a whole represents the credit portfolio. The first time a particle hits the boundary, which we think of as the first time a payment is missed, corresponds to the default time in the models that have been considered in the literature so far. After default the particle is then removed from the system.

In the US, government sponsored enterprises (GSEs) such as Fannie Mae and Freddie Mac issue mortgage-backed securities for which the payments to investors are guaranteed by the issuing GSE. This means that, if a home buyer cannot make payments on interest or principal the GSE steps in and makes the payments to the investors. The corresponding mortgage is not removed from the security and the home buyer may resume payments at a later stage. This insurance is typically reflected in a higher price for the investor. These GSE-insured securities provide motivation for us to consider particle systems with a reflecting boundary instead of an absorbing boundary. In these reflecting models, a particle does not get removed from the system when hitting the boundary but stays within the positive half-line instead. This reflection at the boundary is used to model the GSE stepping in to make the payments to the investors.

If a home buyer misses payments on a mortgage it is typically due to the home buyer experiencing economic distress. This distress is often not instantaneously resolved. Thus, a model with instantaneous reflection at the boundary may not be the best model and models that take this observation into account may be better suited. The particle system with sticky boundary is an attempt to capture this aspect. When considering a model with sticky boundary, reflection at the boundary is not instantaneous but particles instead "stick to the boundary". Mathematically this is done through a time change that slows down the particle if it is at the boundary.

In a model based on a system with elastic boundary we relax the assumption of default at the first time a payment is missed but limit the number of payments that can be missed. The elastic killing time is the first time the local time at the boundary exceeds an exponential random variable. In a model using this boundary default only happens if several payments are missed, i.e. if the corresponding particle has accumulated too much local time at the boundary. This delay in the default allows for negotiations or debt restructuring.

Our study of models with feedback through stopping times is motivated by their use in modelling systemic risk in the banking sector. Each particle represents the financial health of a bank and the particle system models the banking sector as a whole. The balance sheets of banks are strongly connected through inter-bank lending and borrowing as well as over-the-counter trading activities between banks. These connections can lead to contagion effects in the system where financial difficulties for one bank can move other entities in the system into distress. Feedback models can capture these endogenous contagion effects. The killing time of a particle corresponding to a bank represents the time of default. If the killing time is reached and a bank defaults, this has a negative effect on the financial health of the other banks in the system and they are pushed closer towards the boundary through the feedback term. This move towards the boundary can trigger the default time of another bank and this process can continue until no more banks are moved into default and the system continues to evolve from there. The potential to create such default cascades is what makes feedback models suitable models for systemic risk. A series of papers have employed models with feedback through the first hitting time of the boundary to study endogenous contagion [38, 57, 58]. We change the model by considering feedback through elastic stopping times. The idea of using elastic killing times instead of first hitting times is again to allow for the potential for negotiations about additional funds or restructuring before the other banks are financially affected.

We will now discuss the different particle systems and the results that we establish in this thesis in more detail.

1.3 Elastic Boundaries

When analysing a particle system with elastic boundary we consider particles $X^{i,N}$ following the diffusion with reflection at the boundary given by (1.1.4). We then add the elastic killing time (1.1.6) to the definition of the empirical measure ν_t^N which we define as

$$\nu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}} 1_{t<\tau^{i,N}}. \tag{1.3.1}$$

While we will also discuss results for the case of a reflecting boundary, our analysis will focus on the closely related elastic particle system. The reason for this is that the proofs for the elastic system can easily be adjusted to the case of a reflecting

boundary. Moreover, we will see that they behave in a similar way when it comes to regularity near the boundary.

As discussed before the elastic killing time is given by the first time the local time at zero of the particle exceeds an exponential random variable. Thus, the boundary is reflecting until the particle has picked up too much local time at the boundary and then is absorbing. The elastic boundary can be seen as a mixture of reflecting and absorbing boundaries and the parameter κ of the exponential random variable governs the degree of mixing.

Our main contribution in the study of particle systems with elastic boundary and particles following the dynamics in (1.1.4) is a weak convergence result in a suitable space of the process of empirical measures (ν_t^N) given by (1.3.1) to the solution (ν_t) of an SPDE in weak formulation. This limit equation is given by

$$\begin{aligned}\langle \nu_t, \phi\rangle = \langle \nu_0, \phi\rangle + \int_0^t \langle \nu_s, \mu(s,\cdot,\nu_s)\phi'\rangle ds + \frac{1}{2}\int_0^t \langle \nu_s, \sigma^2(s,\cdot)\phi''\rangle ds \\ + \int_0^t \langle \nu_s, \rho(s)\sigma(s,\cdot)\phi'\rangle dW_s^0\end{aligned} \tag{1.3.2}$$

for all times $t \in [0,T]$ and all test functions $\phi \in \mathcal{C}_0^{E,\kappa}(\mathbb{R})$, with probability 1, where

$$\mathcal{C}_0^{E,\kappa}(\mathbb{R}) = \{\phi \in \mathcal{S} : \phi'(0) = \kappa\phi(0)\} \tag{1.3.3}$$

and $\mathcal{S}$ denotes the Schwartz space of rapidly decaying functions. In this equation the elastic boundary condition is encoded in the space of test functions against which we test the measures ν_t. Letting κ go to infinity in (1.3.3) yields the space of test functions for an absorbing boundary and taking κ to zero leads to the space for a reflecting boundary. We will proof that such convergence results also hold for solutions to the SPDE.

Our approach follows the techniques that have successfully been applied in the case of systems with absorbing boundary [14, 37, 41]. We mostly build on [37, 41]. We view the empirical-measure process ν^N as a càdlàg process taking values in the space of tempered distributions. After equipping this space with the M1-topology we can show tightness to establish the existence of limit points and prove that these satisfy the equation (1.3.2). The M1 topology is weaker than the more prominent J1 topology. It is a suitable choice for us because its modulus of continuity vanishes for monotone function. Thus, tightness of the process of mass lost at the boundary through killing can easily be inferred.

Denoting the density of the measure ν_t by V_t, the limit equation (1.3.2) for the measure-valued process ν can be used to formally derive an SPDE for the process of densities. The formal equation is given by

$$dV_t = \big(\partial_{xx}\big(\frac{\sigma^2(t,\cdot)}{2}V_t\big)(x) - \partial_x(\mu(t,\cdot,V_t)V_t)(x)\big)dt - \rho(t)\partial_x\left(\sigma(t,\cdot)V_t\right)(x)dW_t^0, \quad (1.3.4)$$

for $x > 0$, with boundary condition

$$\partial_x\big(\frac{\sigma^2(t,\cdot)}{2}V_t\big)(0) = \kappa\frac{\sigma^2(t,0)}{2}V_t(0) + \mu(t,0,V_t)V_t(0) + \rho(t)\sigma(t,0)V_t(0)\dot{W}_t^0. \quad (1.3.5)$$

Our main focus here is on the boundary condition. Given its dependence on $\dot{W}_t^0$ it is not clear how to make sense of the equation and it indicates irregular behaviour at the boundary. The absorbing case in contrast leads to the simple Dirichlet boundary condition $V_t(0) = 0$. While the process of densities also experiences irregular behaviour at the boundary in this case, it is easier to make sense of the boundary condition. Setting $\kappa = 0$ we obtain the boundary condition in the reflecting case. We can see that both, the reflecting and elastic cases, lead to a similar boundary condition involving the irregular term $\dot{W}_t^0$.

In order to show weak convergence of the entire sequence of empirical-measure processes (ν^N) it is then sufficient to prove uniqueness in a suitable class of processes that contains the limit points. We prove this uniqueness result in the case of a linear limit equation, i.e. in the case without a dependence on the measure in the drift term. This is done using a mollification with the elastic heat kernel and establishing energy estimates in H^{-1}, the dual of the first Sobolev space.

We furthermore prove that on a set of probability one we have existence of densities for the measures ν_t for all t and L^2-regularity strictly in the interior of the half-line.

The fact that we cannot establish L^2-regularity up to the boundary is the reason that we cannot show uniqueness in the case of a drift that depends on the measure. The methods we employ in the uniqueness proof rely on this regularity when used to show uniqueness for a nonlinear limit equation for systems with absorbing boundary [37, 41].

The formally derived SPDE for the density (1.3.4)-(1.3.5) shows that the regularity at the boundary is expected to be similar in both the elastic and reflecting case. When considering the simpler reflecting case we can relate the solution to the SPDE (1.3.2) to the law of a Brownian motion reflected on a fixed Brownian path. For these objects blow-ups of the density on a dense subset of times have been established in the literature [13]. These blow-ups suggest that the densities are not in L^2. In Chapter 2 we discuss Brownian motion reflected on Brownian motion in more detail.

1.4 Sticky Boundaries

We study a system of particles following the sticky SDE (1.1.7). The dynamics of a sticky particle are characterized through a standard diffusion in the interior of the domain and then a sticky component at the boundary that depends on the stickiness parameter c. The boundary component arises from the condition for a sticky boundary that relates the local time at the boundary and the time spent at the boundary. In the case of a particle on the positive half-line it is given by

$$\frac{1}{2c}\int_0^t 1_{\{X_s^{i,N}=0\}}ds = L_t^{i,N}. \tag{1.4.1}$$

The set of times t the particle spends at the boundary has positive Lebesgue measure. The process does not get reflected instantaneously and moves slower at the boundary. This is the reason it is referred to as sticky. The sticky boundary was first introduced by Feller [31]. Itô and McKean [43] showed that a sticky Brownian motion can be constructed by applying a time change to a reflected Brownian motion. Chitashiville [17] and later Engelbert and Peskir [28] found that sticky Brownian motion can be characterised as a weak solution to an SDE.

As mentioned in Section 1.2 our motivation to study sticky particle systems comes from the attempt to model mortgage-backed securities issued by US GSEs. The sticky boundary models the observation that home buyers that miss payments are likely to be in a state of economic distress that is not instantaneously resolved. Sticky boundaries have been used in the literature before to model economic conditions that may be difficult to escape. In [60] the author defines a Vasicek model with sticky boundary at zero for the evolution of interest rates. This model is based on the difficulty of central banks to exit zero-interest rate scenarios that have been observed over the last decade.

Systems of sticky particles without a common noise have been studied in the literature. In [34] Graham shows how a weak solution to a particle system driven by independent Brownian motions in a bounded subset of $\mathbb{R}^d$ with sticky boundary can be constructed. Building on this construction, Graham and Metivier [35] prove convergence of a solution to the particle system to the law of the solution to a McKean–Vlasov equation with sticky boundary.

We extend the known convergence results by adding a common noise in the case of a one-dimensional system on the positive half-line. We do this by showing convergence

of the empirical measure process ν with

$$\nu_t = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}$$

for particles $X^{i,N}$ following (1.1.7). We follow the same approach as described in Section 1.3 for the elastic system. We prove tightness in a suitable space and find that limit points satisfy the limit equation

$$\begin{aligned}\langle \nu_t, \phi\rangle = \langle \nu_0, \phi\rangle + \int_0^t \langle \tilde{\nu}_s, \mu(\cdot, \nu_s)\phi'\rangle ds + \frac{1}{2}\int_0^t \langle \tilde{\nu}_s, \sigma^2(\cdot)\phi''\rangle ds \\ + \int_0^t \rho\langle \tilde{\nu}_s, \sigma(\cdot)\phi'\rangle dW_s^0 + \frac{\phi'(0)}{2c}\int_0^t \nu_s(\{0\}) ds\end{aligned} \tag{1.4.2}$$

for $t \in [0,T]$ and $\phi \in \mathcal{S}$ where $\tilde{\nu}_t$ denotes the restriction of the measure ν_t to the interior of the positive half-line. The last term in (1.4.2) contains the point mass at zero of the measure ν_s. Its appearance is a consequence of the sticky boundary and shows that the point mass at the boundary in the distribution of the particles carries over to the equation that describes the limit of the particle system. We then proceed to show uniqueness of solutions to the limit equation in a class of processes that includes the limit points. Difficulties, similar to the ones in the elastic case, with regard to the regularity of the densities of the limit solution near the boundary require us to restrict the drift coefficients. We only allow drift coefficients for which the dependence on the measure vanishes near the boundary.

Note that an extension to such interactions that vanish at the boundary can also be achieved in the elastic case. The proof in that case follows the same steps as in the sticky case and in the proof for more general interactions in the absorbing case. Since the point masses at the boundary that appear when considering the sticky boundary pose some additional challenges, we decided to present this extension in the sticky case.

1.5 Feedback Models

Feedback models are used in finance to model systemic risk arising from contagion in the banking sector. The feedback through stopping times allows us to study the contagion effects that arise through the losses that banks suffer if another bank in the system defaults. These losses can lead to further defaults and can cause so-called default cascades of several banks getting moved into default. The focus in the existing literature has been on systems with feedback through first hitting times of zero. The

limits of such particle system as the number of particles grows to infinity can be described as the law of a solution to a McKean–Vlasov equation of the following form

$$\begin{cases} X_t = X_{0-} + B_t - \alpha \Lambda_t \\ \Lambda_t = \mathbb{P}\left(\tau \leqslant t\right) \\ \tau = \inf\{t \geqslant 0 : X_t \leqslant 0\}. \end{cases} \tag{1.5.1}$$

The article [38] showed that solutions to these McKean–Vlasov problems are discontinuous if the parameter $\alpha > 0$ is large enough. The resulting singular interactions are a cause for increased mathematical complexity when studying these models. At jump times there are in general multiple jump sizes that are consistent with the equations. This leads to nonuniqueness of solutions in general. In [24, 25], Delarue et al. introduced the concept of physical solutions to a related model coming from neuroscience. A physical solution is a solution with the smallest possible jump size. Physical solutions for models of the form (1.5.1) with absorbing stopping times have been studied extensively. In [38] the authors show existence of physical solutions as well as uniqueness up to the first jump time. A non-exhaustive list of further existence results include [6, 15, 16, 24, 38, 58]. Uniqueness in the case of weak feedback where solutions do not experience jumps is established in [53]. A first global uniqueness result for arbitrary feedback strengths under the assumption of non-oscillatory initial conditions can be found in [26]. The authors of [26] use that the McKean–Vlasov problem with feedback through absorbing stopping times can be viewed as a probabilistic representation of the supercooled Stefan problem, a free boundary value problem for the heat equation that models the evolution of the freezing front through a supercooled liquid. This result was later extended to allow for certain initial conditions with oscillations [56]. Recently, existence and uniqueness results for the supercooled Stefan problem in higher dimensions have been shown [36, 59].

In [21], the concept of minimal solutions to the feedback problem is studied. The authors use an approximating particle system to prove that minimal solutions are physical solutions.

Our focus is on extending these results to models with feedback through elastic stopping times. This means we consider the McKean–Vlasov problem

$$\begin{cases} X_t = X_{0-} + B_t - \alpha \Lambda_t + L_t \\ \Lambda_t = \mathbb{P}\left(\tau \leqslant t\right) \\ \tau = \inf\{t \geqslant 0 : L_t \geqslant \chi\} \end{cases} \tag{1.5.2}$$

with initial condition $X_{0-} \geqslant 0, \alpha > 0$, B a Brownian motion and χ an independent exponential random variable. The term L is the reflection term that arises from the

solution to the Skorokhod problem and insures that X stays nonnegative. Since X can have discontinuities L is not necessarily the local time at zero.

Our main contributions are existence and uniqueness results for physical and minimal solutions to the problem (1.5.2) under assumptions on the oscillations of the initial condition, similar to [26, 56]. We use the form of the feedback and the elastic stopping time to establish that any solution to the elastic feedback model is a solution to an adjusted feedback model with absorbing stopping times. This allows us to employ the results from e.g. [38] on existence and from [26, 56] and [21] on uniqueness.

Uniqueness of solutions allows us to prove convergence of the empirical measure of the corresponding particle system (1.1.8) to the McKean–Vlasov equation (1.5.2) following the approach in [21]. Similar to the model (1.5.1) being a probabilistic representation of the supercooled Stefan problem, we show that the elastic problem (1.5.2) is related to another free boundary problem for the heat equation, the supercooled Stefan problem with kinetic undercooling. This problem has been studied in [4] in the case without discontinuities. The authors furthermore show convergence of solutions of this problem to solutions to the supercooled Stefan problem. We provide a probabilistic extension of this convergence result by showing that solutions to (1.5.2) converge to solutions to (1.5.1) as the parameter κ of the exponential random variable χ converges to infinity. Moreover we obtain convergence to a reflecting Brownian motion when taking κ to zero.

We study extensions to the feedback model (1.5.2) that are inspired by extensions that have been made to the model with absorbing stopping times (1.5.1). The article [3] considers a system with two phases where in addition to the particles on the positive half-line there is a population of particles on the negative half-line. The particles in the two phases are coupled through feedback via their hitting times at zero. In [3] this is described as a model for systemic risk in two competing economies. Moreover, the corresponding free boundary problem is a model for the interface between a supercooled liquid and a superheated solid.

We employ a method similar to the one we used for the one-phase model to show existence of physical solutions to the two-phase McKean–Vlasov problem with elastic stopping times by referring to the existence result from [3]. Uniqueness of physical solutions is still an open problem for these models.

The articles [52] and [57] add a common noise to the particle system with absorbing stopping times. The corresponding limit equation turns into a conditional McKean–Vlasov equation in this case. The two articles use different methods to prove existence of solutions. We again find solutions to the elastic model with common noise by

relating it to an absorbing problem. The same method can be used to discuss the occurrence of discontinuities in the common noise model using the result from [52].

Our final result on feedback models is the existence of solutions to models with absorbing stopping times that have a reflecting upper boundary at one. In these models the feedback is related to the distance of the solution to its running supremum.

1.6 Thesis Outline

The structure of the thesis is as follows. We begin in Chapter 2 with a brief discussion of Brownian motion reflected on Brownian motion. Chapter 3 covers our study of the particle system with elastic boundary and summarizes related results for the case of a reflecting boundary. We continue with our analysis of systems with sticky boundary in Chapter 4. The majority of Chapter 5 is concerned with feedback models with elastic boundary on the positive half-line before discussing several extensions to this model. Some technical results on the different heat kernels we use, as well as on the mollification technique we employ, are gathered in the appendix.

Chapter 2

Brownian Motion Reflected on Brownian Motion

We will see in Section 3.6.1 that in its most simple case the limit SPDE of a particle system with reflecting boundary can be related to the study of a one-dimensional Brownian motion that is reflected off a fixed path of a Brownian motion. This can be viewed as a problem of a reflected Brownian motion in a time-dependent domain. Such problems have been studied in detail by Burdzy and co-authors in [10, 12, 11, 13]. In this chapter we discuss some of their results that are most relevant for the analysis of the limit SPDEs in our particle models. Thus, we restrict ourselves to the one-dimensional case and focus on the probabilistic approach to the problem instead of the purely analytic viewpoint in [11].

2.1 Reflected Brownian Motion

We are interested in a reflected Brownian motion in the time dependent domain $\mathcal{D} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : x > g(t)\}$ for a continuous function $g : \mathbb{R}_+ \to \mathbb{R}$. The reflected Brownian motion can be constructed via the Skorokhod lemma.

Lemma 2.1.1 (Skorokhod lemma)**.** *Let g be a continuous function from $\mathbb{R}_+$ to $\mathbb{R}$. Then, for every continuous function $y(t), t \geqslant 0$ with $y(0) \geqslant g(0)$ there exists a unique pair of functions $(x(t), l(t)), t \geqslant 0$ such that*

(i) $x(t) := y(t) + l(t) \geqslant g(t)$ *for all* $t \geqslant 0$

(ii) $l(t)$ *is a nondecreasing and continuous function with* $l(0) = 0$

(iii) $l(t)$ *only increases when* y *is at the boundary, i.e.*

$$l(t) = \int_0^t 1_{\{y(r)=g(r)\}} dl(r).$$

While the Skorokhod lemma is a deterministic result we can apply it pathwise to a Brownian motion W and a deterministic boundary g to construct a reflected Brownian motion.

Definition 2.1.2. Let g be a continuous function from $\mathbb{R}_+$ to $\mathbb{R}$ and W a Brownian motion with $W_0 \geqslant g(0)$. Then we call the process X defined via the Skorokhod lemma the reflected Brownian motion with boundary g and L the local time of X on g.

If we do not specify the boundary g we implicitly mean the constant boundary $g \equiv 0$. If the boundary function g has high enough regularity the analysis of the reflected Brownian motion with boundary g is relatively straight forward. Shifting the process by $-g$ we obtain a reflected Brownian motion with boundary zero and drift $-g$. If g is in H^1 the law of this process is absolutely continuous with respect to a reflecting Brownian motion with boundary zero. This allows us to establish absolute continuity of the law of the reflected Brownian motion with respect to the Lebesgue measure as well as an explicit formula for the density ([10, Theorem 3.9]).

2.2 Heat Atoms and Heat Singularities

While the analysis of reflected Brownian motion with boundary g gets simplified for g regular enough, an irregular boundary g can cause problems. Low regularity of g gives rise to situations in which the boundary moves very slowly and the domain of the reflected Brownian motion with boundary g contracts rapidly. Such sharp inwards movements of the boundary can cause singular behaviour of the Brownian motion at the boundary. One way to study this behaviour is by considering the related heat equation with reflecting boundary that is solved by the densities V_t of the reflected Brownian motion. If the boundary moves into the domain of the heat equation faster than the diffusion inside the domain can transport heat away, heat can get stuck at the boundary and create Dirac masses. We will see in Theorem 2.2.2 that this situation occurs at a boundary point $(t, g(t))$ for a boundary that is less regular than a Brownian motion in the sense that the map $s \mapsto g(t) - g(t-s)$ lies above paths of the Brownian motion with probability one on a small interval. The Dirac masses correspond to atoms in the distribution of the reflected Brownian motion with boundary g and are referred to as heat atoms. Another form of singular behaviour appears if the rapid contraction of the domain causes the density $V_t(x)$ of the reflected Brownian motion, which is a solution to the related heat equation, to

take very large values near the boundary. A heat singularity appears if

$$\limsup_{x \to g(t)} V_t(x) = \infty.$$

Both heat atoms and heat singularities are studied in detail in [12]. The authors show that existence of heat atoms and singularities are local properties of the boundary function g independent of the initial condition X_0. Moreover, surprisingly, any combination of existence or non existence of heat atoms and singularities at a boundary point $(t, g(t))$ is possible. The article [12] characterizes the existence of heat atoms using functions in the upper class of Brownian motion.

Definition 2.2.1. A function f belongs to the upper class of Brownian motion if

$$\mathbb{P}(\exists \varepsilon > 0 \quad \forall t \in (0, \varepsilon) : B_t < f(t)) = 1.$$

An example of such a function is $f(t) = t^{1/4}$. The result is then the following.

Theorem 2.2.2 ([12] Theorem 2.2)**.** *Suppose that $t > 0$. The point $(t, g(t))$ is a heat atom if and only if the function $s \mapsto g(t) - g(t-s)$ is an upper function of Brownian motion.*

We are interested in the case were the boundary g is a fixed path of a Brownian motion. Based on Theorem 2.2.2 the authors in [12] show that in this case there is no heat atom.

The study of heat singularities is more difficult as the existence of singularities is not monotone in the boundary. In [12] Burdzy and Nualart consider heat singularities in the case of g being the path of a Brownian motion. Their analysis is based on the slow points of Brownian motion (see [23]). These form a dense set of time points where the Brownian motion moves slower than $\frac{1}{2}\sqrt{t}$. Using points where the boundary contraction is sharper than $\frac{1}{2}\sqrt{t}$ they show the following result.

Theorem 2.2.3 ([12] Theorem 3.2)**.** *Let g be a Brownian motion. For almost all trajectories $g(t)$ every non-empty open interval of the real line contains a point s such that*

$$\limsup_{x \to g(s)} V_t(x) = \infty.$$

This means that for Brownian motion reflected on Brownian motion heat singularities are a dense set of time points. Moreover, the authors are not able to show that $\lim_{x \to g(s)} V_t(x) = \infty$, only that the statement holds for the limit superior. Therefore, we cannot rule out oscillations of the density near the boundary.

The heat singularities cause problems when trying to generalize the reflecting and elastic particle systems to allow for dependence on the empirical measure in the drift term. These generalizations lead to nonlinearities in the limit SPDE. In the uniqueness proofs for absorbing models in [37, 41] L^2-regularity of the density is used to deal with such nonlinearities. The absorbing boundary forces the density to be zero at the boundary. Therefore, establishing L^2-regularity is not a difficulty in that case. While a singularity does not rule out square integrability we conjecture that we do not have L^2-regularity here. Any boundary that causes a more rapid contraction than Brownian motion creates a heat atom. Thus, when considering a domain with a path of a Brownian motion as a boundary, we expect the singularity of the density at the boundary to be better than but very close to x^{-1}.

Chapter 3

Particle Systems with Elastic Boundary

3.1 Introduction

Consider a conditionally i.i.d. system of N reflected Itô diffusions $\{X^i\}_{1\leqslant i\leqslant N}$ living in the positive half-line $[0,\infty)$ with a given finite time horizon T. The initial values $\{X_0^i\}_{1\leqslant i\leqslant N}$ are chosen independently from a common distribution ν_0 on $(0,\infty)$, and each X^i has diffusive dynamics

$$\begin{aligned} X_t^i = X_0^i + \int_0^t \mu(s, X_s^i)ds + \int_0^t \sigma(s, X_s^i)\rho(s)dW_s^0 \\ + \int_0^t \sigma(s, X_s^i)(1-\rho(s)^2)^{\frac{1}{2}}dW_s^i + L_t^i, \quad i = 1, \dots, N, \end{aligned} \tag{3.1.1}$$

where L^i is the local time of X^i at the origin and $W^0, W^1, \dots, W^N$ are independent Brownian motions. The precise assumptions on the coefficients are left for Section 3.2.

We shall also refer to each X^i as the i'th particle. On top of the dynamics (3.1.1), we wish to consider the 'killing' of each particle after a suitable (random) amount of time has been spent at the boundary. To this end, we fix a parameter $\kappa > 0$ and define

$$\tau^i := \inf\{t > 0 : L_t^i > \chi^i\}, \quad i = 1, \dots, n, \tag{3.1.2}$$

for a family $(\chi)_{1\leqslant i\leqslant N}^i$ of i.i.d. exponential random variables with rate $\kappa > 0$. This captures the idea of each particle being killed elastically at the origin with parameter $\kappa > 0$, and we say τ^i is the elastic killing time of X^i. Note that the measure of dL_t^i, induced by the local time, is supported on the set $\{t : X_t^i = 0\}$, so we know that $X_{\tau^i}^i = 0$, meaning that each particle can only be killed when it is at the boundary.

The main object of study in this chapter is the family of empirical measures for the surviving particles, for every time t and any number of particles $N \geqslant 0$, given by

$$\nu_t^N(dx) := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}(dx) 1_{t<\tau^i}, \quad t \in [0,T], \quad N \geqslant 1. \tag{3.1.3}$$

In financial applications, τ^i could represent the *default time* of an entity whose *financial health* is modelled by X^i. We then say that an entity is in *distress* when X_t^i is zero. In practice, an entity could for example be an asset such as a credit instrument forming part of a larger credit default obligation or a defaultable loan in a mortgage backed security. It could also be a company or a financial institution. Either way, if enough time is spent in this distressed state, the inevitable will happen: the clock τ^i rings and the i'th entity is declared to be in default. Due to the structure of (3.1.2), where the underlying exponential random variables are not observed, we obtain the desirable properties that the default times are not predictable and given that default has not occurred up to a time t their laws conditional on the path only depend on the future increments of the local time.

In order to manage the risk of investments and regulate or monitor a financial system, it becomes important to understand, respectively, the pricing of credit derivatives on portfolios of a large number of defaultable entities or the computation of risk measures related to the overall health of a large number of financial institutions. Both cases will typically involve the computation of expected values $\mathbb{E}[F(\nu^N)]$ for some functional F of the paths $t \mapsto \nu_t^N$ up until a terminal time T. Viewing $\nu^N := (\nu_t^N)_{t\geqslant 0}$ as a measure-valued càdlàg stochastic process, it is then natural to look for a functional limit theorem as $N \to \infty$ and thereby seek to identify a good approximation $\mathbb{E}[F(\nu)]$, where the limit $\nu = (\nu_t)_{t\geqslant 0}$ satisfies an evolution equation driven only by the common factor W^0.

Closely related to our setting, [14] provides such a result motivated by the pricing of credit default swap indices, which involves computing the expected value of suitable functionals of the loss process $t \mapsto \mathcal{L}_t^N := 1 - \nu_t^N(0,\infty)$, giving the proportion of defaults. However, [14] does not work with elastic default times for reflected dynamics as we do here. Instead, they consider a constant coefficient version of the system (3.1.1) without reflection, where defaults are declared to happen immediately at the first time the financial health X^i hits zero. The paper [50] studies precise regularity properties of solutions to the resulting SPDE with absorbing boundary satisfied by ν (in the limit as $N \to \infty$). Moreover, [33, 62] proposed and analysed efficient numerical schemes for simulating this SPDE, based on finite difference schemes in combination with multi-level or multi-index Monte Carlo methods, which can yield substantial gains over the simulation of the finite system for reasonable sizes of N. Finally, [37,

41] considered the well-posedness of broader classes of such parabolic SPDEs with absorbing boundary.

In a class of sufficiently regular solutions, uniqueness of the SPDE with absorbing boundary in [14] can be deduced from the general Sobolev theory on Dirichlet problems for SPDEs developed in [47, 48]. As far as we are aware, no such theory exists for the type of SPDEs with elastic or reflecting boundaries that we consider in the present chapter. For details on the specific type, see (3.1.4)-(3.1.5) below with $\kappa = 0$ or $\kappa > 0$, respectively.

In [14], adapting the approach of [46, 49] for the whole space to a half-line with absorbing boundary, uniqueness of a priori measure-valued solutions is proved by means of uniform L^2 energy estimates for suitable mollifications of the solutions. This method only works because the authors can quantify the second moment of the mass near the boundary as having decay of at least order $\mathbb{E}[\int_0^T |\nu_t(0,\varepsilon)|^2 dt] = O(\varepsilon^{3+\gamma})$, as $\varepsilon \downarrow 0$, for some $\gamma > 0$. When attempting to generalize the model from [14] to allow for non-constant coefficients, it becomes unclear how to obtain such a strong order of decay. In [37], it was instead realised that, by working in the dual, H^{-1}, of the first Sobolev space, H^1, uniqueness in the case of an absorbing boundary can be obtained from energy estimates in H^{-1} as soon as we know that the first moment of the mass near the boundary satisfies $\mathbb{E}[\int_0^T |\nu_t(0,\varepsilon)| dt] = O(\varepsilon^{1+\gamma})$, as $\varepsilon \downarrow 0$, for some $\gamma > 0$, which is very easy to verify for the limit points of the empirical measures in the absorbing case.

The above first moment control, however, is much too strong a requirement for elastic and reflecting boundaries. The absolute best one could hope for is that $\mathbb{E}[\int_0^T |\nu_t(0,\varepsilon)| dt]$ is of order $O(\varepsilon)$ as $\varepsilon \downarrow 0$. Fortunately, by returning to the second moment, we are able to implement the H^{-1} approach from [37] by instead showing that limit points of (3.1.3) must satisfy $\mathbb{E}[\int_0^T |\nu_t(0,\varepsilon)|^2 dt] = O(\varepsilon^{1+\gamma})$, as $\varepsilon \downarrow 0$, for some $\gamma > 0$. This is enough for us to succeed in establishing uniqueness within a broad class of measure-valued solutions that includes the limit points of the empirical measures.

3.1.1 Summary of main results

By analogy with the connection between an elastically killed Brownian motion and the heat equation with Robin boundary, one heuristically expects that limit points of (3.1.3) should be given by a stochastic process $(V_t)_{t \geqslant 0}$, where each V_t is a random sub-

probability density on $[0,\infty)$ governed, in a suitable sense, by the evolution equation

$$dV_t = \big(\partial_{xx}\big(\frac{\sigma_t^2}{2}V_t\big) - \partial_x(\mu_t V_t)\big)dt - \rho_t\partial_x\left(\sigma_t V_t\right)dW_t^0, \tag{3.1.4}$$

in the interior, together with the noisy Robin type boundary condition

$$\partial_x\big(\frac{\sigma^2(t,\cdot)}{2}V_t\big)(0) = \kappa\frac{\sigma^2(t,0)}{2}V_t(0) + \mu(t,0)V_t(0) + \rho(t)\sigma(t,0)V_t(0)\dot{W}_t^0. \tag{3.1.5}$$

Note to simplify notation, we will often suppress the spatial dependence and simply write $\mu_t = \mu(t,x)$ and $\sigma_t = \sigma(t,x)$ as well as $\rho_t = \rho(t)$.

Our main contributions in this chapter are as follows. Firstly, we identify a suitable weak formulation of the SPDE (3.1.4)-(3.1.5), which is shown to uniquely characterize a functional limit theorem for the empirical measures (3.1.3) as $N \to \infty$ (Theorems 3.2.4 and 3.2.5). Secondly, we show that the unique solution ν of our weak formulation has a density V_t at all times, that this density is square-integrable for almost all times, and that it is square-integrable away from the boundary at all times (Theorem 3.2.6 and Proposition 3.2.7). Thirdly, we connect the study of absorbing, reflecting, and elastic boundaries within the class of linear parabolic SPDEs on a half-line analysed here. We do this by showing that well-posedness with a reflecting boundary is also captured by our arguments (Theorem 3.2.8) and that the cases of a reflecting or an absorbing boundary emerge as limiting cases when sending the parameter of elastic killing κ to 0 or $+\infty$, respectively (Theorem 3.2.9). Finally, towards the end of the chapter, we show that our arguments for identifying a limiting SPDE extend to drifts depending continuously on the empirical measures (Theorem 3.7.2), and we show that the resulting limiting solutions have McKean–Vlasov type probabilistic representations (Theorem 3.7.3). However, it does not seem straightforward for our H^{-1} approach to yield uniqueness in that setting.

3.1.2 Making sense of the elastic boundary at the origin

In this section, we present some heuristic arguments that motivate our notion of solution and highlight the reasons for working with a relaxed interpretation of the boundary condition (3.1.5).

If we had a smooth noise W^0 and (3.1.4)-(3.1.5) hold in the classical sense, we could integrate (3.1.4) with respect to x from zero to infinity. Applying the boundary condition (3.1.5) together with the fact that V_t and its derivatives vanish at ∞ we would obtain a loss of mass at the rate

$$\frac{d}{dt}\nu_t([0,\infty)) = -\kappa\frac{\sigma^2(t,0)}{2}V_t(0), \quad \text{for all} \quad t > 0. \tag{3.1.6}$$

This reveals a natural and slightly weaker way of imposing the elastic boundary condition without asking for spatial differentiability.

Nonetheless, we will have to drop the differentiability in time implied by (3.1.6), and, moreover, we will have to avoid evaluating the solution at the boundary, for reasons we discuss below. Thus, rather than taking (3.1.6) as a definition we derive a relaxed version for the case of a Brownian motion. Consider the density $G_\varepsilon^{E,\kappa}(x,y)$, at time $\varepsilon > 0$, of a Brownian motion on $[0,\infty)$ started at $y \geqslant 0$ and killed elastically at 0 with rate $\kappa > 0$ (see (A.3)). Then $y \mapsto G_\varepsilon^{E,\kappa}(0,y)$ approximates a Dirac mass at 0 from the right, tending to infinity at $y = 0$, while the function

$$\phi_\varepsilon^{E,\kappa}(x) := \int_0^\infty G_\varepsilon^{E,\kappa}(x,y) 1_{[0,\infty)}(y) dy = \int_0^\infty G_\varepsilon^{E,\kappa}(x,y) dy \tag{3.1.7}$$

defines a particular smooth approximation of $1_{[0,\infty)}$ which is close to 1 at $x = 0$. In Theorem 3.2.1, we introduce a succinct weak formulation of (3.1.4)-(3.1.5) that will hold for the limit points of our particle system. As we will see in Section 3.5.1, a key step in our uniqueness proof is that these weak solutions can be shown to satisfy

$$d\langle \nu_t, \phi_\varepsilon^{E,\kappa} \rangle = -\kappa \langle \nu_t, \frac{\sigma_t^2}{2} G_\varepsilon^{E,\kappa}(0,\cdot) \rangle dt + \langle \nu_t, \mu_t \bar{g}_\varepsilon^{E,\kappa} \rangle dt + \langle \nu_t, \rho_t \sigma_t \bar{g}_\varepsilon^{E,\kappa} \rangle dW_t^0 \tag{3.1.8}$$

for any $\varepsilon > 0$, for a suitable correction function $\bar{g}_\varepsilon^{E,\kappa} := g_\varepsilon^{E,\kappa}(0,\cdot)$ defined in (A.3). This may be viewed as a relaxed version of (3.1.6), where we have shifted attention from the boundary to arbitrarily small neighbourhoods of the boundary, at the cost of introducing two correction terms: an absolutely continuous term due to the drift μ and a local martingale term due to the driving noise W^0. As per the arguments in Section 3.5.1, we can take an expectation and send ε to zero, to obtain

$$\mathbb{E}[\nu_t([0,\infty))] = 1 - \lim_{\varepsilon \downarrow 0} \int_0^t \kappa \mathbb{E}\big[\langle \nu_s, \frac{\sigma_s^2}{2} G_\varepsilon^{E,\kappa}(0,\cdot) \rangle\big] ds. \tag{3.1.9}$$

Therefore, we may instructively think of (3.1.8) as enforcing (3.1.6) in a generalized sense in time and space, subject to taking expectations, when we assume nothing about the existence or regularity of a density or the noise being smooth.

While this is a much weaker formulation of the boundary condition (3.1.5) than (3.1.6), it will be enough for us to carry through our uniqueness proof, as we see in Section 3.5.2. Besides allowing for uniqueness, the relaxed formulation (3.1.8) is crucial for two reasons. Firstly, we need a sufficiently relaxed formulation so that it can be guaranteed to be satisfied by weak limit points of the particle system, for which it would seem intractable to ask for much more than the above. Secondly, even if we had more precise knowledge of the density for the limiting solutions, it would

not be possible to have (3.1.6) holding for all $t \in [0,T]$ with probability 1. As it turns out, for each realisation, the density can blow up as we approach the boundary for certain times (albeit a set of times of measure zero). We discuss this further in Section 3.2.1 together with our main results.

Intuitively, suppose dW_t^0 is badly behaved at some time t, pushing a non-negligible amount of mass towards the boundary in an infinitesimal amount of time while not pushing any mass the other way (with Brownian path for which $B_s - B_t < 0$ for s in some small right-neighbourhood of t). Since only a fraction of this mass leaves the domain, the rest must be instantaneously accommodated for near the boundary, which may then force $\limsup_{\varepsilon \downarrow 0} \langle \nu_t, G_\varepsilon^{E,\kappa}(0,\cdot)\rangle = +\infty$. In particular, this will also imply non-differentiably of the loss of mass at that time. As the paths $t \mapsto \nu_t([0,\infty))$ are decreasing, they are automatically differentiable at almost every time, but, for any given realisation, there can still be a dense set of times of measure zero where this fails for the aforementioned reasons.

3.1.3 Literature on SPDEs with noisy boundary conditions

Our focus in this chapter is on identifying the limit of the empirical measures (3.1.3) as the unique solution of a suitable weak formulation of the SPDE (3.1.4)-(3.1.5). Working with the relaxed condition (3.1.8), we will make no attempt at understanding (3.1.5) in a stronger sense. Nevertheless, it is worth emphasising that there are several results in the literature on SPDEs with irregular noise terms in the boundary condition.

Alos and Bonaccorsi [2] study SPDEs with a white noise Dirichlet boundary condition on the half-line, using the theory of fundamental solutions for linear stochastic evolution equations from [61] and techniques from Malliavin calculus. They prove existence of solutions in a weighted L^p space and determine the blow-up rate at the boundary. Da Prato and Zabczyk [22] study nonlinear stochastic evolution equations with white noise boundary conditions. They employ a stochastic version of the semigroup approach developed in [5] to show global existence and uniqueness of mild solutions. Moreover, DaPrato and Zabczyk establish that the solution in the Neumann case exhibits a higher regularity near the boundary. Maslowski [54] uses similar techniques based on the semigroup approach to analyse SPDEs on a bounded domain driven by space-dependent Gaussian noise and Robin-type boundary condition with bounded operators and independent noise at the boundary and shows existence and uniqueness of mild solutions. Sowers [67] studies stochastic reaction diffusion equations on an n-dimensional manifold with additive noise in the boundary

conditions showing existence and uniqueness of solutions. In work on stochastic dynamical boundary conditions, Chueshow and Schmalfus [18, 19] show well-posedness of a system of quasi-linear parabolic SPDEs on bounded domains with noisy boundary dynamics, and verify that solutions give rise to a random dynamical system. In more recent work, Shirikyan [66] studies 2d Navier–Stokes systems on a bounded domain driven by boundary noise with a piecewise independent structure.

The SPDE (3.1.4)-(3.1.5) differs from these problems in a number of ways. The equation involves a stochastic transport term where the derivative of the solution appears in front of the Brownian motion. The current literature is restricted to bounded operators in the multiplicative noise term. Moreover, the boundary noise is the same as the noise driving the equation in the interior while most of the literature is focused on the case of an independent noise at the boundary.

3.2 Convergence to an SPDE with elastic boundary

To have a unique strong solution of the finite particle system and establish the desired results on convergence and well-posedness of the limit, we impose the following assumptions on the initial condition and the coefficients of the system.

Assumption 3.2.1 (Initial Condition)**.** *The initial condition ν_0 is supported on $(0,\infty)$, and satisfies*

$$\nu_0(\lambda,\infty) = o(\exp(-\alpha\lambda)) \qquad \text{as } \lambda \to +\infty,$$

for every $\alpha > 0$. Furthermore, we assume that ν_0 has an L^2-density V_0.

Assumption 3.2.2 (Regularity in Space and Time)**.** *The coefficients μ, σ and ρ have the following regularity in the space and time variables*

(i) The maps $x \mapsto \mu(t,x)$ and $x \mapsto \sigma(t,x)$ are in $C^2([0,\infty))$ with

$$|\partial_x^n \mu(t,x)|, |\partial_x^n \sigma(t,x)| \leqslant C_{\sigma,\mu}$$

for all $t \in [0,T], x \in \mathbb{R}$ and $n = 0,1,2$, for constant $C_{\sigma,\mu} > 0$,

(ii) For all $t \in [0,T], x \in \mathbb{R}$, $\sigma(t,x) \geqslant C_{\sigma,\mu}^{-1} > 0$ and $0 \leqslant \rho(t) < 1$,

(iii) The map $t \mapsto \sigma(t,x)$ is in $C^1([0,T])$ and $\sup_{s\in[0,T]} \int_0^\infty |\partial_t \sigma(s,y)| dy < \infty$.

Assumption 3.2.1 is used in Section 3.3 to establish probabilistic estimates that are needed to show regularity results for limit points of our particle system. Moreover we employ it in Section 3.6 to prove regularity of the density of the limit process. Assumption 3.2.2 $(i) - (iii)$ are also used in the proofs of the probabilistic estimates in Section 3.3. We additionally rely on Assumption 3.2.2 (i) in the uniqueness proof in Section 4.5 and the proof of the density regularity in Section 3.6 to use several auxiliary results from [37].

Given Assumption 3.2.1 and Assumption 3.2.2, strong existence and uniqueness for the system (3.1.1) follows by classical results. Next, we define a class of processes capturing the fundamental regularity conditions satisfied by limit points of the empirical measure processes. Our uniqueness result will be for solutions to the limiting SPDE belonging to this class.

Definition 3.2.3 (The class Λ)**.** We say that a distribution-valued càdlàg process $(\nu_t)_{t\in[0,T]}$ is of class Λ if ν takes values in the space of sub-probability measures $\mathbf{M}_{\leqslant 1}(\mathbb{R})$ and the following conditions are satisfied:

(i) (Support on positive half-line) For every $t \in [0,T]$, the sub-probability measure ν_t is supported on the positive half-line $[0,\infty)$,

(ii) (Exponential tails) For every $\alpha > 0$, we have

$$\mathbb{E}\left[\int_0^T \nu_t(\lambda,+\infty)dt\right] = o(e^{-\alpha\lambda}), \quad \text{as } \lambda \to \infty,$$

(iii) (Boundary decay) There exists $\gamma > 0$ such that

$$\mathbb{E}\left[\int_0^T (\nu_t(0,\varepsilon))^2 dt\right] = O(\varepsilon^{1+\gamma}), \quad \text{as } \varepsilon \to 0,$$

(iv) (Spatial concentration) There exist constants $C > 0$ and $\delta \in (0,1)$ such that

$$\mathbb{E}\left[\int_0^T \nu_t(a,b)dt\right] \leqslant C|b-a|^\delta, \quad \text{for all pairs } 0 < a < b.$$

3.2.1 Functional convergence and well-posedness of the SPDE

As discussed in the introduction, we are interested in a functional limit theorem for the empirical measures $\nu^N = (\nu_t^N)_{t\in[0,T]}$ seen as càdlàg stochastic processes. To this end, we proceed as in [37, 41, 51] and establish weak convergence on the Skorokhod space of $\mathcal{S}'$-valued càdlàg paths, denoted $D_{\mathcal{S}'} = D_{\mathcal{S}'}[0,T]$. As usual, $\mathcal{S}'$ is the space

of tempered distributions, forming the dual of $\mathcal{S}$, the space of Schwartz functions on $\mathbb{R}$. As per [51], we endow $D_{\mathcal{S}'}$ with Skorokhod's M1 topology and the corresponding Borel σ-field.

Theorem 3.2.4 (Functional limit theorem). *Let $\nu^N = (\nu_t^N)_{t\in[0,T]}$ be given by (3.1.3) with Assumptions 3.2.1-3.2.2 in place. Every subsequence of (ν^N, W^0) has a further subsequence converging in law on $(D_{\mathcal{S}'}, M1)\times(C_{\mathbb{R}}, \|\|_\infty)$. Moreover, for any limit point (ν, W^0), the marginal ν is in the class Λ and there is a filtration $\mathcal{F}^{\nu,W^0}$, for which the marginal W^0 is a Brownian motion and ν is adapted, so that the pair (ν, W^0) satisfies the SPDE*

$$\begin{aligned}\langle \nu_t, \phi\rangle = \langle \nu_0, \phi\rangle + \int_0^t \langle \nu_s, \mu(s,\cdot)\phi'\rangle ds + \frac{1}{2}\int_0^t \langle \nu_s, \sigma^2(s,\cdot)\phi''\rangle ds \\ + \int_0^t \langle \nu_s, \rho(s)\sigma(s,\cdot)\phi'\rangle dW_s^0\end{aligned} \tag{3.2.1}$$

for all times $t \in [0,T]$ and all test functions $\phi \in \mathcal{C}_0^{E,\kappa}(\mathbb{R})$, with probability 1, where

$$\mathcal{C}_0^{E,\kappa}(\mathbb{R}) = \{\phi \in \mathcal{S} : \phi'(0) = \kappa\phi(0)\}.$$

The particular choice of $\mathcal{C}_0^{E,\kappa}(\mathbb{R})$ is made to encode the elastic boundary condition in the space of test functions.

Our next result establishes uniqueness for measure-valued solutions to the SPDE (3.2.1) in the regularity class Λ. In particular, we thus obtain the full convergence in law of the empirical measure processes to the unique solution of (3.2.1) in this class.

Theorem 3.2.5 (Uniqueness). *Let (ν, W^0) and $(\tilde{\nu}, \tilde{W}^0)$ be solutions to the SPDE (3.2.1) such that ν and $\tilde{\nu}$ are in the class Λ, for Brownian motions W^0 and $\tilde{W}^0$. Then, we have*

(i) pathwise uniqueness if $W^0 = \tilde{W}^0$, that is,

$$\nu_t(S) = \tilde{\nu}_t(S) \quad \text{for every } t \in [0,T] \text{ and every Borel sets } S \subseteq \mathbb{R}. \tag{3.2.2}$$

(ii) uniqueness in law, that is, $\mathrm{Law}((\nu, W^0)) = \mathrm{Law}((\tilde{\nu}, \tilde{W}^0))$.

Remark. The uniqueness in law result of the second part of Theorem 3.2.5 follows from the pathwise uniqueness result in the first part using a Yamada-Watanabe type argument.

So far, our existence and uniqueness statements have been phrased solely in terms of properties of measure-valued solutions. Our third result concerns the extent to which we can guarantee L^2-regularity of the unique solution to the SPDE in the class Λ.

Theorem 3.2.6 (L^2-regularity)**.** *Let (ν, W^0) be the unique solution to the SPDE (3.2.1) in the class Λ. With probability 1, it holds for all $t \in [0,T]$ that the measure ν_t restricted to $(0,\infty)$ has a density V_t, which is in L^2 away from the origin. Furthermore, it holds for almost every $t \in [0,T]$ that V_t is the density of ν_t on all of $[0,\infty)$, and we have*

$$\mathbb{E}\Big[\int_0^T \|V_t\|_{L^2(0,\infty)} dt\Big] < \infty.$$

The above theorem highlights two interesting potential issues. Firstly, for a set of times of measure zero, there could fail to be a density of $\nu_t(\omega)$ on $[0,\infty)$. Secondly, even if we have a density for all times, it could fail to be in L^2 up to the boundary. The former turns out to not be an issue, as results of [12] on Brownian motion reflected off of a path of another Brownian motion will guarantee that there cannot be a point mass at zero. Moreover, based on the results of [13], we should expect to have $\limsup_{x \downarrow 0} V_t(x) = +\infty$ for a dense set of times. In line with this, we believe it is unlikely that L^2 integrability of the density can be guaranteed all the way up to the boundary for those times.

Proposition 3.2.7 (Density on all of $[0,\infty)$)**.** *With probability 1, there is no atom of ν_t at the origin, for any $t \in [0,T]$. Consequently, each V_t from Theorem 3.2.6 is in $L^1(0,\infty)$ and gives the density of ν_t on $[0,\infty)$ for all $t \in [0,T]$, with probability 1.*

While all our other results are proved in a self-contained manner, using mollification and energy estimates, we were not able to exploit this technique to rule out atoms at the origin, and hence a result of [12] is needed for the above proposition. We do not use this proposition for any of our other results, but we believe it is important to emphasise that there is indeed a density on all of $[0,\infty)$. In particular, we note that our proof of Theorem 3.2.4 deals explicitly with the a priori possibility of having atoms at zero.

3.2.2 Reflecting and absorbing boundaries as limiting cases

The elastic boundary condition in the SPDE (3.2.1) is encoded in the space of test functions. Changing the space of test functions, we in turn obtain an evolution

equation with a different boundary behaviour. Specifically, the space of test functions

$$\mathcal{C}_0^A(\mathbb{R}) \coloneqq \{\phi \in \mathcal{S}(\mathbb{R}) : \phi(0) = 0\}. \tag{3.2.3}$$

corresponds to an absorbing boundary, while the space of test functions

$$\mathcal{C}_0^R(\mathbb{R}) \coloneqq \{\phi \in \mathcal{S}(\mathbb{R}) : \phi'(0) = 0\} \tag{3.2.4}$$

captures a reflecting boundary. The well-posedness of (3.2.1) with a reflecting boundary can be inferred from our proofs in this chapter, and so we also get the following result.

Theorem 3.2.8. *The SPDE (3.2.1) with a reflecting boundary at zero, captured by the test function space $\mathcal{C}_0^R(\mathbb{R})$, has a unique solution in the class Λ, as in Theorem 3.2.5.*

Existence and uniqueness in the case of an absorbing boundary has been studied in [14, 50, 37, 41] with various additional features. Our final result shows that the elastic boundary gives rise to the absorbing and reflecting boundaries as limiting cases.

Theorem 3.2.9. *Let (ν^κ, W^0) denote the unique solution with ν^κ in the class Λ to the SPDE (3.2.1) with elastic boundary for a given $\kappa > 0$. Then we have that*

(i) as $\kappa \to \infty$, (ν^κ, W^0) converges in law to (ν^∞, W^0) which is a solution to the SPDE (3.2.1) with test function space $\mathcal{C}_0^A(\mathbb{R})$,

(ii) as $\kappa \to 0$, (ν^κ, W^0) converges in law to (ν^0, W^0) which is a solution to the SPDE (3.2.1) with test function space $\mathcal{C}_0^R(\mathbb{R})$.

3.3 Probabilistic Estimates for the Particle System

In this section we show a series of probabilistic estimates that will be used in Section 3.4.3 to prove the regularity of limit points. As a first step, we note that all particles in the system $\{(X_t^i 1_{t<\tau^i})_{t\in[0,T]}\}_{1\leqslant i\leqslant N}$ are identically distributed. As a result, we obtain for an arbitrary measurable set $S \subseteq \mathbb{R}$, $N \geqslant 1$ and $t \in [0, T]$

$$\mathbb{E}\left[\nu_t^N(S)\right] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left[1_{\{X_t^i \in S\}} 1_{t<\tau^i}\right] = \mathbb{P}(X_t^1 \in S, t < \tau^1). \tag{3.3.1}$$

We show how we can use a scale transformation and a change of measure to estimate (3.3.1) in terms of the distribution of a reflected Brownian motion $|W|$.

Lemma 3.3.1 (Scale transformation)**.** *Let X_t^1 be a particle from the particle system. Define the transformation function $\zeta : [0,T] \times \mathbb{R} \to \mathbb{R}$ as*

$$\zeta(t,x) \coloneqq \int_0^x \frac{dy}{\sigma(t,y)}.$$

Then the process $Z_t \coloneqq \zeta(t, X_t^1)$ is a semimartingale and its dynamics are given by $dZ_t = \tilde{\mu}_t dt + dB_t + dL_t(Z)$. The stochastic process B is a Brownian motion

$$B_t = \int_0^t \rho_s dW_s^0 + \int_0^t (1-\rho_s^2)^{\frac{1}{2}} dW_s^1,$$

L is the local time of Z at zero and $\tilde{\mu}$ a drift coefficient given by

$$\tilde{\mu}_t = (\frac{\mu}{\sigma} - \partial_x \sigma)(t, X_t^1) - \int_0^{X_t^1} \frac{\partial_t \sigma}{\sigma^2}(t,y)dy.$$

The coefficient $\tilde{\mu}_t$ is uniformly bounded in N and t.

Proof. We apply Itô's formula to obtain the dynamics of Z

$$\begin{aligned} dZ_t &= \partial_t \zeta(t, X_t^1)dt + \partial_x \zeta(t, X_t^1) dX_t^1 + \partial_{xx}\zeta(t, X_t^1) d[X^1]_t \\ &= \tilde{\mu}(t, X_t^1)dt + dB_t + \frac{1}{\sigma(t, X_t^1)} dL_t \end{aligned}$$

with $\tilde{\mu}$ as defined above. By [30] we can identify the last term as

$$\int_0^t \frac{1}{\sigma(s, X_s^1)} dL_s = L(Z_t)$$

establishing the equation for Z. The uniform boundedness for the drift parameter $\tilde{\mu}$ follows from the conditions on the coefficients in Assumption 3.2.2. □

Remark. The boundedness assumptions on σ give that, for all $t \in [0,T]$,

$$|\zeta(t,y) - \zeta(t,x)| \leqslant C_{\mu,\sigma}|y-x|, \qquad x, y \geqslant 0 \tag{3.3.2}$$

with the constant $C_{\mu,\sigma} > 0$ from Assumption 3.2.2. Moreover, note that the transformation $\zeta(t,\cdot)$ is invertible for all $t \in [0,T]$.

Remark. Note that the process Z_t obtained from the scale transformation is a reflected Brownian motion with drift. Thus, the process Z solves the Skorokhod problem for the process $dY_t = \tilde{\mu}dt + dB_t$ and, in particular, stays non-negative (see [63]).

After removing the volatility factor σ through the scale transformation, the next step is to remove the drift and obtain a reflected Brownian motion. This can be done through a change of measure and Girsanov's theorem.

Lemma 3.3.2 (Removing the drift)**.** *For all $\delta \in (0,1)$ there is a constant $c_\delta > 0$ so that*

$$\mathbb{P}(X_t^1 \in S, t < \tau^1) \leqslant c_\delta F_t(\zeta(t,S))^\delta, \qquad \text{for every measurable } S \subseteq \mathbb{R}.$$

The term F_t denotes the marginal law of a reflected Brownian motion at time t with initial distribution $\nu_0 \circ \zeta^{-1}(0,\cdot)$ where ν_0 is the initial distribution of X_0^1.

Proof. We know that Z is the solution to the Skorokhod problem for the process Y with $dZ_t = \tilde{\mu}_t dt + dB_t + dL_t(Z)$. As $\tilde{\mu}$ is bounded, we can apply Girsanov's Theorem with the change of measure

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t \tilde{\mu}_s dB_s - \frac{1}{2}\int_0^t (\tilde{\mu}_s)^2 ds\right) =: \Xi_t,$$

to see that there is a measure $\mathbb{Q}$ with a Brownian motion $B^{\mathbb{Q}}$ under which the dynamics of Z are those of a reflected Brownian motion

$$dZ_t = dB_t^{\mathbb{Q}} + dL_t(B^{\mathbb{Q}}).$$

Following the proof of [37, Lemma 4.2] we let $E \in \mathcal{F}_t$ and let q and p be Hölder exponents, i.e. $q^{-1} + p^{-1} = 1$. Then Hölder's inequality yields

$$\begin{aligned}\mathbb{P}(E) &= \mathbb{E}_{\mathbb{Q}}[\Xi_t^{-1} 1_E] \leqslant \mathbb{E}_{\mathbb{Q}}[\Xi_t^{-p}]^{\frac{1}{p}} \mathbb{Q}(E)^{\frac{1}{q}} \\ &= \mathbb{E}_{\mathbb{P}}[\Xi_t^{1-p}]^{\frac{1}{p}} \mathbb{Q}(E)^{\frac{1}{q}} = \exp(C_p \int_0^t \tilde{\mu}_s^2 ds) \mathbb{Q}(E)^{\frac{1}{q}} \leqslant C_q \mathbb{Q}(E)^{\frac{1}{q}}.\end{aligned}$$

From this estimate it easily follows that

$$\begin{aligned}\mathbb{P}(X_t^1 \in S, t < \tau^1) &\leqslant \mathbb{P}(X_t^1 \in S) \\ &\leqslant C_q \mathbb{Q}(Z_t \in \zeta(t,S))^{\frac{1}{q}} = C_q F_t(\zeta(t,S))^{\frac{1}{q}}\end{aligned}$$

and the assertion follows by setting $\delta = q^{-1}$. □

As an easy corollary we obtain good spatial concentration of the empirical measures.

Lemma 3.3.3. *There exist constants $c > 0$ and $\delta \in (0,1)$ such that property (iv) of Definition 3.2.3 is satisfied by $(\nu_t^N)_{t\in[0,T]}$ uniformly in $N \geqslant 1$.*

Proof. By Lemma 3.3.2 we have

$$\mathbb{E}\left[\nu_t^N(a,b)\right] \leqslant c_\delta F_t(\zeta(t,(a,b)))^\delta.$$

We note that $\zeta(t,(a,b)) \subseteq [\zeta(t,a),\zeta(t,b)]$ where $\zeta(t,(a,b)) = \{\zeta(t,x) : x \in (a,b)\}$. Using the reflecting heat kernel G_ε^R given in (A.2) we estimate

$$\begin{aligned}
F_t(\zeta(t,(a,b))) &\leqslant \int_0^\infty \int_{\zeta(t,a)}^{\zeta(t,b)} \frac{1}{\sqrt{2t\pi}} \Bigg[\exp\left(-\frac{(x-\zeta(0,x_0))^2}{2t}\right) \\
&\qquad + \exp\left(-\frac{(x+\zeta(0,x_0))^2}{2t}\right)\Bigg] dx\nu_0(dx_0) \\
&\leqslant 2(2\pi t)^{-1/2}(\zeta(t,b)-\zeta(t,a)) \\
&\leqslant 2C_{\mu,\sigma}(2\pi t)^{-1/2}(b-a)
\end{aligned}$$

where the last inequality follows from (3.3.2). Since the map $t \mapsto t^{-\delta/2}$ is integrable, the result then follows. □

Next we prove the boundary estimate that is a crucial part of the proof of uniqueness for the SPDE. First, we establish an estimate in the simple case where the particles follow a reflected Brownian motion with zero drift and $\sigma \equiv 1$.

Lemma 3.3.4. *Assume that we have two particles X and Y following a reflected Brownian motion with correlation ρ with $|\rho| < 1$ and initial values X_0 and Y_0, i.e. $X = |X_0 + W^1|$ and $Y = |Y_0 + W^2|$. Moreover, assume that X_0, Y_0 have the same distribution that we denote by π_0. Then we have the following estimate*

$$\mathbb{P}(0 < |X_0 + W_t^1| < \varepsilon, 0 < |Y_0 + W_t^2| < \varepsilon) \leqslant \frac{2}{\pi\sqrt{1-\rho^2 t}}\varepsilon^2. \tag{3.3.3}$$

Proof. We have, using the explicit form of the folded bivariate Gaussian density,

$$\begin{aligned}
&\mathbb{P}(0 < |X_0 + W_t^1| < \varepsilon, 0 < |Y_0 + W_t^2| < \varepsilon) \\
&= \int_0^\varepsilon \int_0^\varepsilon \int_0^\infty \int_0^\infty \frac{1}{2\pi\sqrt{1-\rho^2 t}} \times \\
&\Bigg[\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x-x_0)^2}{t} - 2\rho\frac{(x-x_0)(y-y_0)}{t} + \frac{(y-y_0)^2}{t}\right)\right) \\
&+ \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x+x_0)^2}{t} - 2\rho\frac{(x+x_0)(y+y_0)}{t} + \frac{(y+y_0)^2}{t}\right)\right) \\
&+ \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x+x_0)^2}{t} + 2\rho\frac{(x+x_0)(y-y_0)}{t} + \frac{(y-y_0)^2}{t}\right)\right) \\
&+ \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x-x_0)^2}{t} + 2\rho\frac{(x-x_0)(y+y_0)}{t} + \frac{(y+y_0)^2}{t}\right)\right)\Bigg] \pi_0(dx_0)\pi_0(dy_0)dxdy
\end{aligned}$$

Simple bounds on the Gaussian density give the result □

Proposition 3.3.5 (Boundary estimate). *For any $q > 1$ and $t \in [0,T]$ we have as $\varepsilon \to 0$ that for $i, j = 1, \dots, N$ with $i \neq j$*

$$\mathbb{P}(0 < X_t^i < \varepsilon, t < \tau^i, 0 < X_t^j < \varepsilon, t < \tau^j) = O(t^{-\frac{1}{q}} \varepsilon^{\frac{2}{q}}).$$

Proof. It is clear that we can estimate the probability from above by dropping the stopping time conditions

$$\mathbb{P}(0 < X_t^i < \varepsilon, t < \tau^i, 0 < X_t^j < \varepsilon, t < \tau^j) \leqslant \mathbb{P}(0 < X_t^i < \varepsilon, 0 < X_t^j < \varepsilon).$$

We apply the scale transformation ζ and then use the multidimensional version of Girsanov's theorem (see [44, §1.7.4]) that preserves the correlation structure between the Brownian motions. Note furthermore that $\zeta(t,0) = 0$ and by (3.3.2) we have $\zeta(t,\varepsilon) \leqslant C_{\mu,\sigma}\varepsilon$. As a result, we have

$$\mathbb{P}(0 < X_t^i < \varepsilon, 0 < X_t^j < \varepsilon) \leqslant \tilde{C}\mathbb{Q}(0 < |X_0^i + W_t^i| < C_{\mu,\sigma}\varepsilon, 0 < |X_0^j + W_t^j| < C_{\mu,\sigma}\varepsilon)^{1/q}$$

for $q > 1$ and a constant $\tilde{C} > 0$. Then, by Lemma 3.3.4 we get the desired estimate. □

Note that $t \mapsto t^{-\frac{1}{q}}$ is integrable because $q > 1$. A decay result can also be found for the mass escaping to infinity.

Proposition 3.3.6 (Tail estimate). *For every $\alpha > 0$, as $\lambda \to +\infty$*

$$\mathbb{E}\left[\nu_t^N(\lambda, +\infty)\right] = o(\exp(-\alpha\lambda)), \quad \textit{uniformly in } N \geqslant 1 \textit{ and } t \in [0,T].$$

Proof. We will again rely on the results and notation of Lemma 3.3.2. Setting $\pi_0 = \nu_0 \circ \zeta(0,\cdot)^{-1}$ we can estimate for $\lambda > 0$

$$\begin{aligned} F_t((\lambda,\infty)) &= \int_0^\infty \mathbb{P}(|W_t| > \lambda | |W_0| = x_0)\pi_0(dx_0) \\ &= \frac{1}{\sqrt{2\pi t}} \int_0^\infty \int_\lambda^\infty \left[e^{-\frac{(x-x_0)^2}{2t}} + e^{-\frac{(x+x_0)^2}{2t}} \right] dx \pi_0(dx_0) \\ &\leqslant \frac{2}{\sqrt{2\pi t}} \int_0^\infty \int_\lambda^\infty e^{-\frac{(x-x_0)^2}{2t}} dx \pi_0(dx_0) = 2\int_0^\infty \mathbb{P}(W_t > \lambda | W_0 = x_0)\pi_0(dx_0). \end{aligned}$$

Now we can argue as in [37] Prop.4.5 using Assumption 3.2.1 for the initial condition and applying Lemma 3.3.2. □

3.4 Convergence of the particle system

In this section we show that limit points of the particle system satisfy the SPDE (3.2.1). Furthermore, we verify that all limit points satisfy the regularity conditions of the class Λ and thus complete the proof of Theorem 3.2.4.

3.4.1 Evolution equation for the empirical measures

We begin by finding an evolution equation for the empirical-measure process ν^N of the particle system.

Proposition 3.4.1 (Evolution equation for ν^N)**.** *For all $N \geqslant 1$ the empirical measure process ν^N satisfies the evolution equation*

$$\begin{aligned}\langle \nu_t^N, \phi\rangle =& \langle \nu_0^N, \phi\rangle + \int_0^t \langle \nu_s^N, \mu(s,\cdot)\phi'\rangle ds + \frac{1}{2}\int_0^t \langle \nu_s^N, \sigma^2(s,\cdot)\phi''\rangle ds \\ &+ \int_0^t \langle \nu_s^N, \rho_s\sigma(s,\cdot)\phi'\rangle dW_s^0 + I_t^N(\phi) + J_t^N(\phi),\end{aligned} \tag{3.4.1}$$

with

$$I_t^N(\phi) = \frac{1}{N}\sum_{i=1}^N \int_0^t \langle \nu_s^N, (1-\rho_s^2)^{\frac{1}{2}}\sigma(s,\cdot)\phi'\rangle dW_s^i$$

and

$$J_t^N(\phi) = \phi(0)\left(\frac{1}{N\kappa}\sum_{i=1}^N L^i_{t\wedge\tau^i} - \frac{1}{N}\sum_{i=1}^N 1_{\tau^i\leqslant t}\right),$$

for all $\phi \in \mathcal{C}_0^{E,\kappa}(\mathbb{R})$, where $I_t^N(\phi) \to 0$ and $J_t^N(\phi) \to 0$ as $N \to \infty$.

Proof. Apply Itô's formula to $\phi(X^i_{t\wedge\tau^i})$ to get

$$\begin{aligned}\phi(X^i_{t\wedge\tau^i}) =& \phi(X_0^i) + \int_0^{t\wedge\tau^i} \phi'(X_s^i)\mu(s,X_s^i)ds + \int_0^{t\wedge\tau^i} \phi'(X_s^i)(1-\rho_s^2)^{\frac{1}{2}}\sigma(s,X_s^i)dW_s^i \\ &+ \int_0^{t\wedge\tau^i} \phi'(X_s^i)\rho_s\sigma(s,X_s^i)dW_s^0 + \frac{1}{2}\int_0^{t\wedge\tau^i}\phi''(X_s^i)\sigma^2(s,X_s^i)ds + \int_0^{t\wedge\tau^i}\phi'(X_s^i)dL_s^i.\end{aligned}$$

Recall that

$$\langle \nu_t^N, \phi\rangle = \frac{1}{N}\sum_{i=1}^N \phi(X_t^i)1_{t<\tau^i}.$$

Using $\phi(X_{t\wedge\tau^i}) = \phi(X_t)1_{t<\tau^i} + \phi(X_{\tau^i})1_{t\geqslant\tau^i}$, the elastic boundary condition of the test function ϕ and integrating against ν^N we obtain (3.4.1).

It remains to prove that the two terms $I^N(\phi)$ and $J^N(\phi)$ converge to zero as $N \to \infty$. We start by computing the quadratic variation of $I^N(\phi)$. By independence of the idiosyncratic Brownian motions W^i we obtain

$$[I^N_\cdot(\phi)]_t = \frac{1}{N^2} \sum_{i=1}^{N} \int_0^t (1-\rho_s^2)\sigma^2(s, X_s^i)(\phi'(X_s^i))^2 ds.$$

The boundedness of σ^2 and ϕ' yields

$$[I^N_\cdot(\phi)]_t = \|\phi'\|_\infty^2 \, O(N^{-1}), \quad \text{as } N \to \infty.$$

By Doob's martingale inequality it follows that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |I_t^N(\phi)|^2\right] \leqslant 4\mathbb{E}\left[I_T^N(\phi)^2\right] = 4\mathbb{E}\left[[I^N_\cdot(\phi)]_T\right].$$

The convergence $J_t^N(\phi) \to 0$ follows from [30] where the authors show that $J_t^N(\phi)$ is a martingale with respect to a suitable filtration. Similarly to the proof for $I_t^N(\phi)$ they establish that the quadratic variation of $J_t^N(\phi)$ vanishes as $N \to \infty$. The main ingredient in their argument is the fact that the continuity of the processes $X^{i,N}$ implies that with probability one the stopping times $\tau^{i,N}$ and $\tau^{j,N}$ with $i \neq j$ will be different. □

3.4.2 Convergence to the Limit SPDE

The next step is now to show that limit points of the empirical measure processes ν^N indeed solve our SPDE. First, we need to verify tightness and conclude that there exist limit points. To do this we follow the approach in [37], showing tightness in $D_{\mathcal{S}'}$ equipped with the M1 topology and then verifying that the limit points are sub-probability measure-valued processes.

Proposition 3.4.2. *For every $t \in [0,T]$ and $\eta > 0$, the empirical loss process, $\mathcal{L}^N$, obeys*

$$\lim_{\delta\to 0} \lim_{N\to\infty} \mathbb{P}\left(\mathcal{L}^N_{t+\delta} - \mathcal{L}^N_t \geqslant \eta\right) = 0, \quad \mathcal{L}^N_t := \frac{1}{N}\sum_{i=1}^{N} 1_{\tau^i \leqslant t}. \tag{3.4.2}$$

Proof. Using the triangle inequality we can estimate

$$\begin{aligned}\mathbb{P}\left(\mathcal{L}^N_{t+\delta} - \mathcal{L}^N_t \geqslant \eta\right) \leqslant & \mathbb{P}\left(\left|\mathcal{L}^N_{t+\delta} - \mathcal{L}^N_t - \frac{1}{N\kappa}\sum_{i=1}^{N}\left(L^i_{t+\delta\wedge\tau^i} - L^i_{t\wedge\tau^i}\right)\right| \geqslant \frac{\eta}{2}\right) \\ & + \mathbb{P}\left(\frac{1}{N\kappa}\sum_{i=1}^{N}\left(L^i_{t+\delta\wedge\tau^i} - L^i_{t\wedge\tau^i}\right) \geqslant \frac{\eta}{2}\right).\end{aligned}$$

Applying the convergence result for $J_t^N(\phi)$ as $N \to \infty$ in Proposition 3.4.1, which holds for all $\phi \in \mathcal{C}_0^{E,\kappa}(\mathbb{R})$, at the two time points t and $t+\delta$ we see that the first probability on the right-hand side vanishes as $N \to \infty$. Markov's inequality can be applied to the second term to obtain

$$\mathbb{P}\left(\frac{1}{N\kappa}\sum_{i=1}^{N}\left(L^i_{t+\delta\wedge\tau^i} - L^i_{t\wedge\tau^i}\right) \geqslant \frac{\eta}{2}\right) \leqslant \frac{2}{\eta\kappa}\mathbb{E}\left[L^i_{t+\delta} - L^i_t\right].$$

Since the particles X^i are continuous, the local time processes L^i are also continuous. Thus, we get convergence to 0 as $\delta \to 0$ uniform in $N \geqslant 1$. □

Proposition 3.4.3 (Tightness)**.** *The sequence of empirical-measure processes* (ν^N) *is tight on the space* $(D_{\mathcal{S}'}, M1)$ *and the sequence* (ν^N, W^0) *is tight on* $(D_{\mathcal{S}'}, M1) \times (C_{\mathbb{R}}, \|\cdot\|_\infty)$.

Proof. Using [51, Theorem 3.2] it suffices to show that, for arbitrary $\phi \in \mathcal{S}$, the process $\langle \nu^N, \phi\rangle$ is tight on $D_{\mathbb{R}}$ in the M1 topology. For all $t \in [0,T]$, we decompose $\langle \nu_t^N, \phi\rangle$ as

$$\langle \nu_t^N, \phi\rangle = \langle \hat{\nu}_t^N, \phi\rangle - \phi(0)\mathcal{L}_t^N, \quad \hat{\nu}_t^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{X^i_{t\wedge\tau^i}}, \tag{3.4.3}$$

with $\mathcal{L}_t^N$ as defined in (3.4.2). The term $\phi(0)\mathcal{L}_t^N$ is monotone and does not contribute to the M1-modulus of continuity. By [51, Propositions 4.1 and 4.2] and the arguments in the proof of [37, Proposition 5.1] it is then sufficient to establish that we have

$$\mathbb{E}\left[|\langle \hat{\nu}_t^N, \phi\rangle - \langle \hat{\nu}_s^N, \phi\rangle|^4\right] = O(|t-s|^2) \quad \text{as } |t-s| \to 0, \tag{3.4.4}$$

and that for any $\varepsilon > 0$

$$\lim_{\delta\to 0}\lim_{N\to\infty}\mathbb{P}\left(\sup_{t\leqslant\delta}|\langle \nu_t^N - \nu_0^N, \phi\rangle| + \sup_{t\in(T-\delta,T)}|\langle \nu_T^N - \nu_t^N, \phi\rangle| > \varepsilon\right) = 0. \tag{3.4.5}$$

For any $s,t \in [0,T]$ it holds that

$$\mathbb{E}\left[|\langle \hat{\nu}_t^N, \phi\rangle - \langle \hat{\nu}_s^N, \phi\rangle|^4\right] \leqslant \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|\phi(X^i_{t\wedge\tau^i}) - \phi(X^i_{s\wedge\tau^i})|^4\right] \leqslant \|\phi\|^4_{Lip}\,\mathbb{E}\left[|X^i_{t\wedge\tau^i} - X^i_{s\wedge\tau^i}|^4\right]$$

where $\|\phi\|_{Lip}$ denotes the Lipschitz constant of the function ϕ. It remains to deal with the expectation on the right-hand side. By using the inequality $(a+b+c)^4 \leqslant 27(a^4+b^4+c^4)$ we obtain

$$\begin{aligned}\mathbb{E}\left[|X^i_{t\wedge\tau^i} - X^i_{s\wedge\tau^i}|^4\right] \leqslant 27\Big(&\mathbb{E}\left[\left|\int_s^t 1_{u<\tau^i}\mu(u,X^i_u)du\right|^4\right] + \mathbb{E}\left[\left|\int_s^t 1_{u<\tau^i}\sigma(u,X^i_u)dB^i_u\right|^4\right]\\ &+ \mathbb{E}\left[\left|L^i_{t\wedge\tau^i} - L^i_{s\wedge\tau^i}\right|^4\right]\Big).\end{aligned}$$

Since the coefficient σ is bounded, the stochastic integral is a martingale and we can apply the Burkholder–Davis–Gundy inequality [64, Theorem IV.42.1] to find that

$$\mathbb{E}\left[\left|\int_s^t 1_{u<\tau^i}\sigma(u,X_u^i)dB_u^i\right|^4\right] = O(|t-s|^2), \quad \text{as } |t-s|\to 0.$$

The boundedness assumption in Assumption 3.2.2 (i) for the drift term μ yields

$$\mathbb{E}\left[\left|\int_s^t 1_{u<\tau^i}\mu(u,X_u^i)du\right|^4\right] = O(|t-s|^4), \quad \text{as } |t-s|\to 0.$$

The estimate $\mathbb{E}\left[\left|L^i_{t\wedge\tau^i}-L^i_{s\wedge\tau^i}\right|^4\right] = O(|t-s|^2)$, follows using the local time representation from the Skorokhod problem. Hence, we have $\mathbb{E}[|X_t^i - X_s^i|^4] = O(|t-s|^2)$ as $|t-s|\to 0$, and so we obtain (3.4.4).

To establish (3.4.5) it is enough to consider the small time interval $(0,\delta)$ as the result for the interval $(T-\delta,T)$ follows similarly. We can apply (3.4.3) to the left-hand side of (3.4.5) to get

$$\mathbb{P}\left(\sup_{t\leqslant\delta}|\langle\nu_t^N-\nu_0^N,\phi\rangle|>\varepsilon\right)\leqslant\mathbb{P}\left(\sup_{t\leqslant\delta}|\langle\hat{\nu}_t^N-\hat{\nu}_0^N,\phi\rangle|\geqslant\frac{\varepsilon}{2}\right)+\mathbb{P}\left(|\phi(0)|\mathcal{L}_\delta^N\geqslant\frac{\varepsilon}{2}\right).$$

We can then again use the estimates in the poof of [37, Proposition 5.1] combined with the arguments for (3.4.4) and obtain that the first term converges to 0 as $\delta\to 0$, uniformly in $N\geqslant 1$. Proposition 3.4.2 provides the same result for the second term and the tightness follows. □

At this stage we have only established that there exists a subsequence such that $(\nu^{N_k},W^0)\to(\nu^*,W^0)$ weakly in the space of $\mathcal{S}'$-valued càdlàg process equipped with M1 topology. However, we can deduce that the limiting process actually takes values in the space of sub-probability measures supported on $[0,\infty)$ by the same arguments as [37, Proposition 5.4].

Proposition 3.4.4. *Let (ν^*,W^0) realise the limiting law of (ν^{N_k},W^0). Then each ν_t^* is a sub-probability measure supported on $[0,\infty)$ for all $t\in[0,T]$, with probability 1.*

Next, we confirm that there is a suitable filtration for the limit points.

Lemma 3.4.5. *Let (ν^*,W^0) be the limit of a subsequence (ν^{N_k},W^0). Then there is a filtration $\mathcal{F}^{\nu^*,W^0}$ so that ν^* is adapted and W^0 is an $\mathcal{F}^{\nu^*,W^0}$-Brownian motion.*

Proof. On the sample space $\Omega = D_{\mathcal{S}'} \times C_{\mathbb{R}}$ we define the filtration

$$\mathcal{F}_t^{\nu^*,W^0} := \sigma\left(\{(\nu^*, W^0) \mapsto (\langle \nu_s^*, \phi\rangle, W_s^0) : s < t, \phi \in \mathcal{S}\}\right). \tag{3.4.6}$$

This is a sensible filtration because the projections $\pi_{t_1,\cdot,t_n}^{\phi^1,\cdot,\phi^n}$ given by

$$\pi_{t_1,\cdot,t_n}^{\phi^1,\cdot,\phi^n} : D_{\mathcal{S}'} \to \mathbb{R}^n, \pi_{t_1,\cdot,t_n}^{\phi^1,\cdot,\phi^n}(\xi) = (\xi_{t_1}(\phi_1), \dots, \xi_{t_n}(\phi_n)),$$

generate the Borel sets on $D_{\mathcal{S}'}$ as shown in [51]. It is clear that ν^* is adapted to $\mathcal{F}^{\nu^*,W^0}$. It remains to show that W^0 is a Brownian motion in this filtration. This follows with the same arguments as in [52, Section 4.1]. □

To show that limit points of the empirical-measure process ν^N solve the SPDE we will prove that solutions of the finite evolution equation of Proposition 3.4.1 converge to solutions of our SPDE. This will be done using the martingale approach of [37] and [41]. This approach easily allows for a higher level a generality that also covers an extension considered in Section 3.7. The martingale components are given by

$$\begin{aligned} M^\phi(\xi)(t) :=& \langle \xi_t, \phi\rangle - \langle \xi_0, \phi\rangle - \int_0^t \langle \xi_s, \mu(s,\cdot)\phi'\rangle ds \\ &- \frac{1}{2}\int_0^t \langle \xi_s, \sigma^2(s,\cdot)\phi''\rangle ds, \\ S^\phi(\xi)(t) :=& M^\phi(\xi)(t)^2 - \int_0^t \langle \xi_s, \sigma(s,\cdot)\rho(s)\phi'\rangle^2 ds, \\ C^\phi(\xi, w)(t) :=& M^\phi(\xi)(t)\cdot w(t) - \int_0^t \langle \xi_s, \sigma(s,\cdot)\rho(s)\phi'\rangle ds \end{aligned} \tag{3.4.7}$$

for $\xi \in D_{\mathcal{S}'}$, $w \in C_{\mathbb{R}}$ and $\phi \in \mathcal{C}_0^{E,\kappa}(\mathbb{R})$. Using the results from [37, Section 5] we obtain the weak convergences

$$\begin{aligned} M^\phi(\nu^{N_k})(t) &\to M^\phi(\nu^*)(t), \\ S^\phi(\nu^{N_k})(t) &\to S^\phi(\nu^*)(t), \\ C^\phi(\nu^{N_k}, W^0)(t) &\to C^\phi(\nu^*, W^0)(t) \qquad \text{in } \mathbb{R}. \end{aligned}$$

on a deterministic cocountable subset of $[0,T]$ for all $\phi \in \mathcal{C}_0^{E,\kappa}(\mathbb{R})$. Then, the convergence of the solutions follows as in [37, Proposition 5.11].

Proposition 3.4.6 (Evolution equation)**.** *Let (N_k) be a subsequence such that the weak convergence $(\nu^{N_k}, W^0) \to (\nu^*, W^0)$ holds. Then, for every $\phi \in \mathcal{C}_0^{E,\kappa}(\mathbb{R})$ the process $M^\phi(\nu^*), S^\phi(\nu^*)$ and $C^\phi(\nu^*, W)$ are martingales. As a consequence, ν^* and W^0 satisfy the limit SPDE from Theorem 3.2.4.*

3.4.3 Regularity of the Limit Solution

We now show that the solutions we constructed as limits from the particle system satisfy the regularity conditions of Definition 3.2.3 using the estimates from Section 3.3.

Proposition 3.4.7. *Let (ν^*, W^0) be a limit point of the sequence of empirical-measure processes. Then the process $(\nu_t^*)_{t\in[0,T]}$ is in the class Λ.*

Proof. In Proposition 3.4.4 we already showed that ν^* takes values in the space of sub-probability measures supported on $[0,\infty)$. It remains to prove the regularity conditions on the sub-probability measures.

Consider a finite open interval $I = (x, y) \subseteq \mathbb{R}$. Take $\delta > 0$ and any $\phi_\delta \in \mathcal{S}$ such that $\phi_\delta = 1$ on I, $\phi_\delta = 0$ on $(-\infty, x-\delta) \cup (y+\delta, \infty)$ and $\phi_\delta \in (0,1)$ otherwise. From $(\nu^{N_k}, W) \to (\nu^*, W)$ weakly we get that $\int_0^t \langle \nu_s^{N_k}, \phi_\delta\rangle ds \to \int_0^t \langle \nu_s^*, \phi_\delta\rangle \mathrm{ds}$ weakly by [74, Theorem 11.5.1]. We have

$$\mathbb{E}\left[\int_0^T \nu_t^*(I)dt\right] \leqslant \mathbb{E}\left[\int_0^T \langle \nu_t^*, \phi_\delta\rangle dt\right] = \lim_{k\to\infty} \mathbb{E}\left[\int_0^T \langle \nu_t^{N_k}, \phi_\delta\rangle dt\right].$$

Considering the bound in Lemma 3.3.3 and taking the limit $\delta \to 0$ yields (iv) in the definition of the class Λ.

Using the same approach we obtain that for $\varepsilon > 0$

$$\mathbb{E}\left[\nu_t^*(0,\varepsilon)^2\right] \leqslant \liminf_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^N \sum_{j\neq i, j=1}^N \mathbb{P}(0 < X_t^i < \varepsilon + \delta, 0 < X_t^j < \varepsilon + \delta).$$

By employing the estimate in Proposition 3.3.5 and taking δ to zero we obtain (iii) because $q > 1$ ensure the necessary integrability.

For the exponential-tails condition consider $I = (\lambda, \eta)$ for $\eta > 0$, $\lambda < \eta$. The approach for (iv) yields

$$\begin{aligned}\mathbb{E}\left[\int_0^T \nu_t^*(\lambda,\eta)dt\right] &\leqslant \liminf_{k\to\infty} \mathbb{E}\left[\int_0^T \nu_t^{N_k}(\lambda-\delta, \eta+\delta)dt\right] \\ &\leqslant \liminf_{k\to\infty} \mathbb{E}\left[\int_0^T \nu_t^{N_k}(\lambda-\delta, \infty)dt\right] = o(e^{-\alpha(\lambda-\delta)}),\end{aligned}$$

by using Proposition 3.3.6. Taking the limit $\delta \to 0$ and $\eta \to \infty$ yields (ii). □

Combining the results from Proposition 3.4.6 and Proposition 3.4.7 we can conclude our first main result Theorem 3.2.4.

3.5 Uniqueness of the SPDE

In this section we prove uniqueness of solutions to the SPDE in the class Λ. Before presenting the uniqueness proof we show how we can derive an alternative weak boundary formulation for the SPDE (3.2.1) that is helpful in dealing with boundary terms appearing in the uniqueness proof.

To overcome difficulties of working with solutions of low regularity, we regularize the weak solutions to our SPDE through convolution with a nice mollifier. The elastic heat kernel $G_\varepsilon^{E,\kappa}(x,y)$, defined in (A.3), is a natural candidate for this, since it is an element of the space of test functions $\mathcal{C}_0^{E,\kappa}(\mathbb{R})$ and since its explicit form makes it easy to perform manipulations. This leads us to consider the convolution operator $T_\varepsilon^{E,\kappa}$ given by $(T_\varepsilon^{E,\kappa}\zeta)(x) = \langle \zeta, G_\varepsilon^{E,\kappa}(x,\cdot)\rangle$. While we use a different operator, we take the same approach as in [37, 41] and work in the space H^{-1}, the dual of the first Sobolev space. This approach now rests on the following crucial inequality, connecting L^2 estimates for the regularized process with an H^{-1} estimate for the non-regularized process:

$$\|\zeta\|_{-1} \leqslant C \liminf_{\varepsilon\to 0} \left(\left\| \partial_x^{-1} T_\varepsilon^{E,\kappa}\zeta \right\|_{L^2(0,\infty)} + \left| \int_0^\infty T_\varepsilon^{E,\kappa}\zeta(y)dy \right| \right), \tag{3.5.1}$$

for any $\zeta \in H^{-1}$, where ∂_x^{-1} denotes the anti-derivative as defined in (C.3). The proof of this can be found in Lemma C.3 and further details are reserved for Appendix C.

Differently from [37, 41], here we work with H^{-1} as the dual of $H^1(0,\infty)$ rather than $H_0^1(0,\infty)$. In our proof of uniqueness, we will show that two different solutions to the SPDE, say ν and $\tilde{\nu}$, must satisfy $\mathbb{E}[\|\nu_t - \tilde{\nu}_t\|_{-1}] = 0$ for all $t \in [0,T]$. In doing so, the first term on the right-hand side of (3.5.1) is only concerned with the difference $\zeta = \nu_t - \tilde{\nu}_t$ restricted to $(0,\infty)$, while forcing the second term to zero ensures that potential point masses of ν_t and $\tilde{\nu}_t$ at the origin must be of the same value (recall the solutions are sub-probability measures, which a priori need not have the same mass at any given time). As discussed in Section 3.2.1, the unique solution turns out to have a density, but we are not making any such assumptions on our class of solutions at this point.

Before proceeding to show uniqueness, we first give two auxiliary results that will help us deal with the behaviour of our estimates near the boundary.

Lemma 3.5.1. *Let ν be a process in the class Λ and p_ε the Gaussian heat kernel defined in (A.1). Then we have*

$$\mathbb{E}\left[\int_0^T\int_0^\infty\left(\int_0^\infty p_\varepsilon(x+y)\nu_t(dy)\right)^2 dxdt\right]\to 0,\qquad \text{as } \varepsilon\to 0.$$

Proof. Arguing as in [37, Lemma 7.6] we have

$$\begin{aligned}|\langle\nu_t, p_\varepsilon(x+\cdot)\rangle| &\leqslant e^{-x^2/\varepsilon}\int_0^\infty p_\varepsilon(y)\nu_t(dy)\\ &\leqslant c_1 e^{-x^2/\varepsilon}\varepsilon^{-1/2}[\nu_t(0,\varepsilon^\eta)+\exp(-\varepsilon^{2\eta-1}/2)],\end{aligned}$$

for some $\eta\in(0,\frac{1}{2})$ and $c_1>0$. Squaring and integrating over $x>0$ gives

$$\int_0^\infty|\langle\nu_t, p_\varepsilon(x+\cdot)\rangle|^2dx\leqslant c_2\varepsilon^{-1/2}[\nu_t(0,\varepsilon^\eta)^2+\exp(-\varepsilon^{2\eta-1})],$$

for some $c_2>0$. Since ν is in the class Λ, the boundary decay yields a $\gamma>0$ so that

$$\mathbb{E}\left[\int_0^T\int_0^\infty|\langle\nu_t, p_\varepsilon(x+\cdot)\rangle|^2dx\right]=O(\varepsilon^{\eta(1+\gamma)-\frac{1}{2}})+O(\varepsilon^{-\frac{1}{2}}\exp(-\varepsilon^{2\eta-1})).$$

Therefore, we get the desired estimate by choosing η such that $(1+\gamma)^{-1}<2\eta<1$. □

Lemma 3.5.2. *Let ν be a process in the class Λ and let $g_\varepsilon^{E,\kappa}$ denote the correction term in the definition of the elastic heat kernel (A.3). Then we have that*

$$\mathbb{E}\left[\int_0^T\int_0^\infty\left(\int_0^\infty g_\varepsilon^{E,\kappa}(x,y)\nu_t(dy)\right)^2 dxdt\right]\to 0,\qquad \text{for } \varepsilon\to 0.$$

Proof. We start by looking at the inner integral. Using Lemma A.3 we get

$$\int_0^\infty g_\varepsilon^{E,\kappa}(x,y)\nu_t(dy)\leqslant\kappa e^{-\frac{x^2}{2\varepsilon}}\nu_t(0,\infty)\leqslant\kappa e^{-\frac{x^2}{2\varepsilon}}.$$

Now, considering the integral in x we have

$$\int_0^\infty\left(\int_0^\infty g_\varepsilon^{E,\kappa}(x,y)\nu_t(dy)\right)^2dx\leqslant\kappa^2\int_0^\infty e^{-\frac{x^2}{\varepsilon}}dx=\frac{1}{2}\kappa^2\sqrt{\pi}\varepsilon^{1/2}.$$

We then obtain

$$\mathbb{E}\left[\int_0^T\int_0^\infty\left(\int_0^\infty g_\varepsilon^{E,\kappa}(x,y)\nu_t(dy)\right)^2dxdt\right]=O(\varepsilon^{1/2})\tag{3.5.2}$$

and the result follows. □

3.5.1 Weak Boundary Condition and Elastic Boundary Terms

As discussed in Section 3.1.2, we need a particular weak formulation of the elastic boundary condition. This plays a critical role in dealing with the boundary terms appearing in (3.5.14) in the final uniqueness proof, presented in the next subsection. With $\phi_\varepsilon^{E,\kappa}$ as in (3.1.7), note that this gives a smooth function approximating 1 on $[0,\infty)$ and satisfying

$$\kappa\phi_\varepsilon^{E,\kappa}(0) = \partial_x\phi_\varepsilon^{E,\kappa}(0), \tag{3.5.3}$$

which it inherits from the elastic kernel $G_\varepsilon^{E,\kappa}$. We then obtain the following.

Proposition 3.5.3 (Weak boundary condition)**.** *Let (ν, W^0) be a solution to (3.2.1) with ν in the class Λ and let $\phi_\varepsilon^{E,\kappa}$ be as in (3.1.7). Then (3.1.8) holds for all $\varepsilon > 0$.*

Proof. By (3.5.3), we see that the convolution $\phi_\varepsilon^{E,\kappa}(x) = \int_0^\infty G_\varepsilon^{E,\kappa}(x,y)dy$ results in a valid test function in $\mathcal{C}_0^{E,\kappa}(\mathbb{R})$. Using Fubini's theorem to change the order of integration between $d\nu$ and dx, the assumption that ν is a solution to (3.2.1) gives

$$\begin{aligned}
\langle \nu_t, \phi_\varepsilon^{E,\kappa}\rangle - \langle \nu_0, \phi_\varepsilon^{E,\kappa}\rangle &= \int_0^\infty \langle \nu_t, G_\varepsilon^{E,\kappa}(x,\cdot)\rangle - \langle \nu_0, G_\varepsilon^{E,\kappa}(x,\cdot)\rangle dx \\
&= \int_0^\infty \Big(\int_0^t \langle \nu_s, \mu_s \partial_y G_\varepsilon^{E,\kappa}(x,\cdot)\rangle ds \\
&\qquad + \int_0^t \langle \nu_s, \frac{\sigma_s^2}{2}\partial_{yy} G_\varepsilon^{E,\kappa}(x,\cdot)\rangle ds + \int_0^t \rho_s \langle \nu_s, \sigma_s \partial_y G_\varepsilon^{E,\kappa}(x,\cdot)\rangle dW_s^0 \Big) dx.
\end{aligned}$$

Next, we employ Lemma A.2 to switch the derivatives from y to x. This yields

$$\begin{aligned}
&\langle \nu_t, \phi_\varepsilon^{E,\kappa}\rangle - \langle \nu_0, \phi_\varepsilon^{E,\kappa}\rangle \\
&= \int_0^\infty \Big(\int_0^t -\partial_x \langle \nu_s, \mu_s G_\varepsilon^E(x,\cdot)\rangle ds + \int_0^t \partial_{xx}\langle \nu_s, \frac{\sigma_s^2}{2} G_\varepsilon^E(x,\cdot)\rangle ds \\
&\quad - \int_0^t \rho_s \partial_x \langle \nu_s, \sigma_s G_\varepsilon^E(x,\cdot)\rangle dW_s^0 \Big) dx \\
&\quad + \int_0^\infty \left(2\int_0^t \partial_x \langle \nu_s, \mu_s p_\varepsilon(x+\cdot)\rangle ds + 2\int_0^t \rho_s \partial_x \langle \nu_s, \sigma_s p_\varepsilon(x+\cdot)\rangle dW_s^0 \right) dx \\
&\quad - \int_0^\infty \left(2\int_0^t \partial_x \langle \nu_s, \mu_s g_\varepsilon^{E,\kappa}(x,\cdot)\rangle ds - 2\int_0^t \rho_s \partial_x \langle \nu_s, \sigma_s g_\varepsilon^{E,\kappa}(x,\cdot)\rangle dW_s^0 \right) dx
\end{aligned}$$

with $g_\varepsilon^{E,\kappa}$ as defined in (A.3) and p_ε the Gaussian heat kernel given in (A.1). We can now interchange the order of integration again and compute the integrals in x to

obtain

$$\begin{aligned}\langle \nu_t, \phi_\varepsilon^{E,\kappa}\rangle - \langle \nu_0, \phi_\varepsilon^{E,\kappa}\rangle = &\int_0^t \langle \nu_s, \mu_s G_\varepsilon^{E,\kappa}(0,\cdot)\rangle ds - \int_0^t \partial_x \langle \nu_s, \frac{\sigma^2}{2} G_\varepsilon^{E,\kappa}(0,\cdot)\rangle ds \\ &+ \int_0^t \rho_s \langle \nu_s, \sigma_s G_\varepsilon^{E,\kappa}(0,\cdot)\rangle dW_s^0 \\ &- 2\int_0^t \langle \nu_s, \mu_s p_\varepsilon(\cdot)\rangle ds - 2\int_0^t \rho_s \langle \nu_s, \sigma_s p_\varepsilon(\cdot)\rangle dW_s^0 \\ &+ 2\int_0^t \langle \nu_s, \mu_s g_\varepsilon^{E,\kappa}(0,\cdot)\rangle ds + 2\int_0^t \rho_s \langle \nu_s, \sigma_s g_\varepsilon^{E,\kappa}(0,\cdot)\rangle dW_s^0.\end{aligned}$$

Using the specific form of the elastic heat kernel we have that

$$\begin{aligned}\int_0^t \langle \nu_s, \mu_s G_\varepsilon^{E,\kappa}(0,\cdot)\rangle ds =& 2\int_0^t \langle \nu_s, \mu_s p_\varepsilon(\cdot)\rangle ds - \int_0^t \langle \nu_s, g_\varepsilon^{E,\kappa}(0,\cdot)\rangle ds \\ \int_0^t \rho_s \langle \nu_s, \sigma_s G_\varepsilon^{E,\kappa}(0,\cdot)\rangle dW_s^0 =& 2\int_0^t \rho_s \langle \nu_s, \sigma_s p_\varepsilon(\cdot)\rangle dW_s^0 - \int_0^t \rho_s \langle \nu_s, \sigma_s g_\varepsilon^{E,\kappa}(0,\cdot)\rangle dW_s^0.\end{aligned}$$

By using these relations the equation above simplifies to

$$\begin{aligned}\langle \nu_t, \phi_\varepsilon^{E,\kappa}\rangle - \langle \nu_0, \phi_\varepsilon^{E,\kappa}\rangle = &-\int_0^t \partial_x \langle \nu_s, \frac{\sigma_s^2}{2} G_\varepsilon^{E,\kappa}(0,\cdot)\rangle ds \\ &+ \int_0^t \langle \nu_s, \mu_s g_\varepsilon^{E,\kappa}(0,\cdot)\rangle ds + \int_0^t \rho \langle \nu_s, \sigma_s g_\varepsilon^{E,\kappa}(0,\cdot)\rangle dW_s^0.\end{aligned}$$

Finally, we can then use the boundary condition of the elastic heat kernel to obtain the relaxed boundary condition (3.1.8). □

Before stating the next result on the elastic boundary term we need to introduce the following error terms $\mathcal{E}_{t,\varepsilon}^h$ that appear when taking a function $h_t(x) = h(t,x)$ outside of the integration against the measure ν_t

$$\mathcal{E}_{t,\varepsilon}^h := \langle \nu_t, h_t(\cdot) G_\varepsilon^{E,\kappa}(x,\cdot)\rangle - h_t(x) T_\varepsilon^{E,\kappa} \nu_t(x). \tag{3.5.4}$$

Lemma 3.5.4 (Elastic Boundary Term)**.** *Let ν and $\tilde{\nu}$ be two solutions to the SPDE (3.2.1) in the class Λ and denote their difference by Δ, i.e. $\Delta := \nu - \tilde{\nu}$. Then we have*

$$\begin{aligned}\mathbb{E}\Bigg[\int_0^t -\partial_x^{-1} T_\varepsilon^{E,\kappa} \Delta_s(0) \sigma_s^2(0) T_\varepsilon^{E,\kappa} \Delta_s(0) - \partial_x^{-1} T_\varepsilon^{E,\kappa} \Delta_s(0) \left(\mathcal{E}_{s,\varepsilon}^{\sigma^2} - \tilde{\mathcal{E}}_{s,\varepsilon}^{\sigma^2}\right)(0) ds\Bigg] \\ = -\frac{1}{\kappa}\mathbb{E}\left[(\partial_x^{-1} T_\varepsilon^{E,\kappa} \Delta_t(0))^2\right] + o(1)\end{aligned}$$

with $\mathcal{E}_{s,\varepsilon}^{\sigma^2}, \tilde{\mathcal{E}}_{s,\varepsilon}^{\sigma^2}$ error terms as defined in (3.5.4) for measures ν_s and $\tilde{\nu}_s$, respectively.

Proof. Combining Proposition 3.5.3 with the definition of the operator $T_\varepsilon^{E,\kappa}$, the anti-derivative and the error terms defined in (3.5.4) we obtain

$$\begin{aligned}\int_0^s \sigma_u^2(0) T_\varepsilon^{E,\kappa}\Delta_u(0)du = \frac{2}{\kappa}\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0) - \int_0^s (\mathcal{E}_{u,\varepsilon}^{\sigma^2} - \mathcal{E}_{u,\varepsilon}^{\tilde{\sigma}^2})(0)du \\ + \frac{2}{\kappa}\int_0^s \langle \Delta_u, \mu_u g_\varepsilon^{E,\kappa}(0,\cdot)\rangle du + \frac{2}{\kappa}\int_0^s \rho_u \langle \Delta_u, \sigma_u g_\varepsilon^{E,\kappa}(0,\cdot)\rangle dW_u^0.\end{aligned} \tag{3.5.5}$$

With this relation we obtain

$$\begin{aligned}-\mathbb{E}\left[\int_0^t \sigma_s(0)^2 \partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0) T_\varepsilon^{E,\kappa}\Delta_s(0)ds\right] \\ = -\mathbb{E}\left[\int_0^t \partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0) d\Big(\int_0^s \sigma_u^2(0)T_\varepsilon^{E,\kappa}\Delta_u(0)du\Big)\right] \\ = -\frac{1}{\kappa}\mathbb{E}\left[2\int_0^t \partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0) d(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0))\right] \\ + \mathbb{E}\left[\int_0^t \partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0)(\mathcal{E}_{s,\varepsilon}^{\sigma^2} - \tilde{\mathcal{E}}_{s,\varepsilon}^{\sigma^2})(0)ds\right] \\ - \frac{2}{\kappa}\mathbb{E}\left[\int_0^t \partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0)\langle \Delta_s, \mu_s g_\varepsilon^{E,\kappa}(0,\cdot)\rangle ds\right] \\ - \frac{2}{\kappa}\mathbb{E}\left[\int_0^t \partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0)\rho_s\langle \Delta_s, \sigma_s g_\varepsilon^{E,\kappa}(0,\cdot)\rangle dW_s^0\right].\end{aligned}$$

By combining the boundedness of σ with Lemma A.3 we get that the stochastic integral in the last line is a martingale and thus the term vanishes. For the first term we consider the fact that $\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_\cdot(0)$ is a continuous semimartingale. Hence, we have

$$(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t(0))^2 = 2\int_0^t \partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0) d(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0)) + [\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_\cdot(0)]_t.$$

Using equation (3.5.5) we obtain that

$$[\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_\cdot(0)]_t = \int_0^t \rho_s^2(\langle \Delta_s, \sigma_s g_\varepsilon^{E,\kappa}(0,\cdot)\rangle)^2 ds.$$

This yields

$$-\mathbb{E}\left[\int_0^t \sigma_s(0)^2 \partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0) T_\varepsilon^{E,\kappa}\Delta_s(0)ds\right] \tag{3.5.6}$$

$$= -\frac{1}{\kappa}\mathbb{E}\left[(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t(0))^2\right] + \mathbb{E}\left[\int_0^t \partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0)(\mathcal{E}_{s,\varepsilon}^{\sigma^2} - \tilde{\mathcal{E}}_{s,\varepsilon}^{\sigma^2})(0)ds\right] + J_1 + J_2 \tag{3.5.7}$$

with the two terms J_1 and J_2 defined as

$$J_1 := \frac{1}{\kappa}\mathbb{E}\left[\int_0^t \rho_s^2(\langle \Delta_s, \sigma_s g_\varepsilon^{E,\kappa}(0,\cdot)\rangle)^2 ds\right]$$
$$J_2 := -\frac{2}{\kappa}\mathbb{E}\left[\int_0^t \partial_x^{-1} T_\varepsilon^{E,\kappa}\Delta_s(0)\langle \Delta_s, \mu_s g_\varepsilon^{E,\kappa}(0,\cdot)\rangle ds\right].$$

From Lemma A.3 we have the estimate

$$g_\varepsilon^{E,\kappa}(0,y) \leqslant \kappa e^{-\frac{y^2}{2\varepsilon}}.$$

For a process ξ in the class Λ we get

$$\begin{aligned}\int_0^\infty g_\varepsilon^{E,\kappa}(0,y)\xi_s(dy) &\leqslant \kappa\int_0^{\varepsilon^{1/4}} e^{-\frac{y^2}{2\varepsilon}}\xi_s(dy) + \kappa\int_{\varepsilon^{1/4}}^\infty e^{-\frac{y^2}{2\varepsilon}}\xi_s(dy)\\ &\leqslant \kappa\xi_s(0,\varepsilon^{1/4}) + \kappa e^{-\frac{\varepsilon^{-1/2}}{2}}.\end{aligned} \tag{3.5.8}$$

The coefficients μ, σ, ρ are bounded by assumption and

$$|\partial_x^{-1} T_\varepsilon^{E,\kappa}\Delta_s(0)| \leqslant |\Delta_s(0,\infty)| \leqslant 2.$$

We can combine this with (3.5.8) to estimate

$$J_1 \leqslant C\left(\int_0^t \mathbb{E}\left[\nu_s(0,\varepsilon^{1/4})^2\right] ds + \int_0^t \mathbb{E}\left[\tilde{\nu}_s(0,\varepsilon^{1/4})^2\right] ds\right) + O(e^{-\varepsilon^{-1/2}}) \to 0, \tag{3.5.9}$$

as $\varepsilon \to 0$, because ν and $\tilde{\nu}$ are in the class Λ. For J_2 we similarly get

$$\begin{aligned}|J_2| &\leqslant C\mathbb{E}\left[\int_0^t \langle \nu_s, g_\varepsilon^{E,\kappa}(0,\cdot) + \langle \tilde{\nu}_s, g_\varepsilon^{E,\kappa}(0,\cdot)\rangle\rangle ds\right]\\ &\leqslant \int_0^t \mathbb{E}\left[\nu_s(0,\varepsilon^{1/4})\right] + \mathbb{E}\left[\tilde{\nu}_s(0,\varepsilon^{1/4})\right] ds + O(e^{-\varepsilon^{-1/2}/2}) \to 0, \quad \text{as } \varepsilon \to 0,\end{aligned} \tag{3.5.10}$$

again, by the properties of the class Λ. Combining (3.5.6), (3.5.9) and (3.5.10) we obtain

$$\begin{aligned}&-\mathbb{E}\left[\int_0^t \sigma_s(0)^2 \partial_x^{-1} T_\varepsilon^{E,\kappa}\Delta_s(0) T_\varepsilon^{E,\kappa}\Delta_s(0) ds\right]\\ &= -\frac{1}{\kappa}\mathbb{E}\left[(\partial_x^{-1} T_\varepsilon^{E,\kappa}\Delta_t(0))^2\right] + \mathbb{E}\left[\int_0^t \partial_x^{-1} T_\varepsilon^{E,\kappa}\Delta_s(0)(\mathcal{E}_{s,\varepsilon}^{\sigma^2} - \tilde{\mathcal{E}}_{s,\varepsilon}^{\sigma^2})(0) ds\right] + o(1)\end{aligned}$$

as $\varepsilon \to 0$, and so the result follows. □

3.5.2 Proof of uniqueness for solutions to the SPDE

We can now give the proof of Theorem 3.2.5. Lemma 3.5.4 from the previous section plays an important role in our arguments. It is used in (3.5.14) to deal with the boundary terms that arise when integrating by parts.

Proof of Theorem 3.2.5. Let (ν, W^0) be a solution to the SPDE (3.2.1) in the class Λ. Take the elastic heat kernel function $y \mapsto G_\varepsilon^{E,\kappa}(x,y)$ as a test function in the SPDE and apply the switching of derivatives from y to x using Lemma A.2 to obtain the following dynamics for the smoothed measure $T_\varepsilon^{E,\kappa}\nu_t$

$$\begin{aligned} dT_\varepsilon^{E,\kappa}\nu_t(x) = &- \partial_x\langle \nu_t, \mu_t G_\varepsilon^{E,\kappa}(x,\cdot)\rangle dt + \frac{1}{2}\partial_{xx}\langle \nu_t, \sigma_t^2 G_\varepsilon^{E,\kappa}(x,\cdot)\rangle dt \\ &- \rho_t \partial_x\langle \nu_t, \sigma_t G_\varepsilon^{E,\kappa}(x,\cdot)\rangle dW_t^0 \\ &+ 2\partial_x\langle \nu_t, \mu_t p_\varepsilon(x+\cdot)\rangle dt + 2\rho_t\partial_x\langle \nu_t, \sigma_t p_\varepsilon(x+\cdot)\rangle dW_t^0 \\ &- 2\partial_x\langle \nu_t, \mu_t g_\varepsilon^{E,\kappa}(x,\cdot)\rangle dt - 2\rho_t\partial_x\langle \nu_t, \sigma_t g_\varepsilon^{E,\kappa}(x,\cdot)\rangle dW_t^0. \end{aligned}$$

Next, we integrate to introduce the anti-derivative defined in (C.3)

$$\begin{aligned} d\partial_x^{-1}T_\varepsilon^{E,\kappa}\nu_t(x) = &- \langle \nu_t, \mu_t G_\varepsilon^{E,\kappa}(x,\cdot)\rangle dt + \frac{1}{2}\partial_x\langle \nu_t, \sigma_t^2 G_\varepsilon^{E,\kappa}(x,\cdot)\rangle dt \\ &- \rho_t \langle \nu_t, \sigma_t G_\varepsilon^{E,\kappa}(x,\cdot)\rangle dW_t^0 \\ &+ 2\langle \nu_t, \mu_t p_\varepsilon(x+\cdot)\rangle dt + 2\rho_t\langle \nu_t, \sigma_t p_\varepsilon(x+\cdot)\rangle dW_t^0 \\ &- 2\langle \nu_t, \mu_t g_\varepsilon^{E,\kappa}(x,\cdot)\rangle dt - 2\rho_t\langle \nu_t, \sigma_t g_\varepsilon^{E,\kappa}(x,\cdot)\rangle dW_t^0. \end{aligned}$$

We can take the coefficients μ, σ and σ^2 outside of the integration against the measure ν_t by introducing error terms $\mathcal{E}_{t,\varepsilon}^h$ as defined in 3.5.4 with h representing μ, σ or σ^2. The equation then becomes

$$\begin{aligned} d\partial_x^{-1}T_\varepsilon^{E,\kappa}\nu_t = &- (\mu_t T_\varepsilon^{E,\kappa}\nu_t + \mathcal{E}_{t,\varepsilon}^\mu)dt + \frac{1}{2}\partial_x(\sigma_t^2 T_\varepsilon^{E,\kappa}\nu_t + \mathcal{E}_{t,\varepsilon}^{\sigma^2})dt \\ &- \rho_t(\sigma_t T_\varepsilon^{E,\kappa}\nu_t + \mathcal{E}_{t,\varepsilon}^\sigma)dW_t^0 \\ &+ 2\langle \nu_t, \mu_t p_\varepsilon(x+\cdot)\rangle dt + 2\rho_t\langle \nu_t, \sigma_t p_\varepsilon(x+\cdot)\rangle dW_t^0 \\ &- 2\langle \nu_t, \mu_t g_\varepsilon^{E,\kappa}(x+\cdot)\rangle dt - 2\rho_t\langle \nu_t, \sigma_t g_\varepsilon^{E,\kappa}(x+\cdot)\rangle dW_t^0. \end{aligned}$$

To simplify the notation, we denote by $o_{sq}(1)$ any family of functions $\{(f_{t,\varepsilon})_{t\in[0,T]}\}_{\varepsilon>0}$ such that

$$\mathbb{E}\int_0^T \|f_{t,\varepsilon}\|_{L^2(0,\infty)}^2\, dt \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.5.11}$$

By using the limit behaviour from Lemma 3.5.1 and Lemma 3.5.2, as well as the results for $\mathcal{E}^h_{t,\varepsilon}$ from [37, Lemma 8.1] , which hold also in the elastic case because of the regularity of the class Λ, we have

$$\begin{aligned} d\partial_x^{-1}T_\varepsilon^{E,\kappa}\nu_t =& -\mu_t T_\varepsilon^{E,\kappa}\nu_t dt + \frac{1}{2}\partial_x(\sigma_t^2 T_\varepsilon^{E,\kappa}\nu_t + \mathcal{E}^{\sigma^2}_{t,\varepsilon})dt \\ & - \rho_t\sigma_t T_\varepsilon^{E,\kappa}\nu_t dW_t^0 + o_{sq}(1)dt + o_{sq}(1)dW_t^0. \end{aligned} \tag{3.5.12}$$

Let $(\tilde{\nu}, W^0)$ be another solution to the linear SPDE with the same Brownian motion W^0 and define the difference of the two solutions as $\Delta := \nu - \tilde{\nu}$. The dynamics of this are

$$\begin{aligned} d\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t =& -\mu_t T_\varepsilon^{E,\kappa}\Delta_t dt + \frac{1}{2}\partial_x(\sigma_t^2 T_\varepsilon^{E,\kappa}\Delta_t + \mathcal{E}^{\sigma^2}_{t,\varepsilon} - \tilde{\mathcal{E}}^{\sigma^2}_{t,\varepsilon})dt \\ & - \rho_t\sigma_t T_\varepsilon^{E,\kappa}\Delta_t dW_t^0 + o_{sq}(1)dt + o_{sq}(1)dW_t^0. \end{aligned}$$

Next we use Itô's formula to find an equation for the square of the anti-derivative

$$\begin{aligned} d(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t)^2 =& -2\mu_t(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t)T_\varepsilon^{E,\kappa}\Delta_t dt \\ & + (\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t)\partial_x(\sigma_t^2 T_\varepsilon^{E,\kappa}\Delta_t + \mathcal{E}^{\sigma^2}_{t,\varepsilon} - \tilde{\mathcal{E}}^{\sigma^2}_{t,\varepsilon})dt \\ & + \sigma_t^2\rho_t^2(T_\varepsilon^{E,\kappa}\Delta_t)^2 dt \\ & - 2(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t)\sigma_t\rho_t T_\varepsilon^{E,\kappa}\Delta_t dW_t^0 \\ & + \rho_t\sigma_t T_\varepsilon^{E,\kappa}\Delta_t o_{sq}(1)dt \\ & + (\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t)o_{sq}(1)dt \\ & + (\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t)o_{sq}(1)dW_t + o_{sq}(1)^2 dt. \end{aligned}$$

We are interested in estimates for $\mathbb{E}[\|\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t\|_2^2]$ to use the H^{-1}-estimate (3.5.1). We obtain these by integrating over $x > 0$ and taking expectation. We deal with the terms on the right-hand side individually. Combining the subexponential-tails property of ν_t and $\tilde{\nu}_t$ with [37, Lemma 8.3 and Lemma 8.7], which are easily adapted to the elastic heat kernel, the stochastic integral terms are martingales and thus vanish when we take expectation. For the first term we use the generalized Young's inequality with free parameter $\eta > 0$ to obtain

$$\begin{aligned} &\mathbb{E}\left[-2\int_0^t\int_0^\infty \mu_s(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s)T_\varepsilon^{E,\kappa}\Delta_s dx ds\right] \\ &\qquad \leqslant c_\eta\int_0^t \mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s\right\|_2^2\right] ds + \eta\mathbb{E}\left[\int_0^t\int_0^\infty (\mu_s)^2|T_\varepsilon^{E,\kappa}\Delta_s|^2 dx ds\right]. \end{aligned} \tag{3.5.13}$$

We employ integration by parts followed by the generalized Young's inequality for the second term. Lemma 3.5.4 from Section 3.5.1 allows us to handle the boundary term that appears when integrating by parts. Thus, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\int_0^\infty(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s)\partial_x(\sigma_s^2T_\varepsilon^{E,\kappa}\Delta_s+\mathcal{E}_{s,\varepsilon}^{\sigma^2}-\tilde{\mathcal{E}}_{s,\varepsilon}^{\sigma^2})dxds\right]\\
&\leqslant-\frac{1}{\kappa}\mathbb{E}\left[(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0))^2\right]-\mathbb{E}\left[\int_0^t\int_0^\infty\sigma_s^2|T_\varepsilon^{E,\kappa}\Delta_s|^2dxds\right]\\
&\quad+\eta\mathbb{E}\left[\int_0^t\int_0^\infty|T_\varepsilon^{E,\kappa}\Delta_s|^2dxds\right]+o(1).
\end{aligned}\tag{3.5.14}$$

We leave the third term as it is. For the remaining terms we proceed to use the generalized Young's inequality as in (3.5.13). Putting everything back together we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t\right\|_2^2\right]+\frac{1}{\kappa}\mathbb{E}\left[(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t(0))^2\right]\leqslant c_\eta\mathbb{E}\left[\int_0^t\left\|\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta\right\|_2^2ds\right]\\
&\quad-\mathbb{E}\left[\int_0^t\int_0^\infty(\sigma_s^2-\sigma_s^2\rho_s^2(1+\eta)-\mu_s^2\eta-\eta)|T_\varepsilon^{E,\kappa}\Delta_s|^2dxds\right]+o(1).
\end{aligned}$$

Since $\rho_s^2<1$ we can choose the free parameter η small enough such that

$$\sigma_s^2-\sigma_s^2\rho_s^2(1+\eta)-\mu_s^2\eta-\eta\geqslant c_0\tag{3.5.15}$$

for all x and s and for some constant $c_0>0$. This yields

$$\begin{aligned}
&\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta\right\|_2^2\right]+\frac{1}{\kappa}\mathbb{E}\left[(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t(0))^2\right]\\
&\quad\leqslant c_\eta\int_0^t\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s\right\|_2^2\right]ds-c_0\mathbb{E}\left[\int_0^t\left\|T_\varepsilon^{E,\kappa}\Delta_s\right\|_2^2ds\right]+o(1).
\end{aligned}\tag{3.5.16}$$

We then get the estimate

$$\begin{aligned}
&\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta\right\|_2^2\right]+\frac{1}{\kappa}\mathbb{E}\left[(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t(0))^2\right]\\
&\quad\leqslant c_\eta\int_0^t\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s\right\|_2^2\right]ds+c_\eta\int_0^t\frac{1}{\kappa}\mathbb{E}\left[(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_s(0))^2\right]ds+o(1).
\end{aligned}$$

By Gronwall's lemma it follows that

$$\lim_{\varepsilon\to0}\left(\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t\right\|_2^2\right]+\frac{1}{\kappa}\mathbb{E}\left[(\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t(0))^2\right]\right)=0.\tag{3.5.17}$$

Now, recall the H^{-1}-estimate (3.5.1). We can find another constant C_1 such that

$$\|\Delta_t\|_{-1}\leqslant C_1\liminf_{\varepsilon\to0}\left(\left\|\partial_x^{-1}T_\varepsilon^{E,\kappa}\Delta_t\right\|_{L^2(0,\infty)}+\frac{1}{\sqrt{\kappa}}\left|\int_0^\infty T_\varepsilon^{E,\kappa}\Delta_t(y)dy\right|\right).\tag{3.5.18}$$

Using $(a+b)^2 \leqslant 2(a^2+b^2)$ and Fatou's lemma, we obtain from (3.5.17) that

$$\mathbb{E}\left[\liminf_{\varepsilon\to 0}\left(\left\|\partial_x^{-1}T_\varepsilon^E\Delta_t\right\|_{L^2(0,\infty)}+\frac{1}{\sqrt{\kappa}}\left|\int_0^\infty T_\varepsilon^E\Delta_t(y)dy\right|\right)^2\right]=0$$

and so, as a consequence of (3.5.18), $\mathbb{E}\left[\|\Delta_t\|_{-1}\right]=0$. In turn, we can deduce the equality $\nu_t=\tilde{\nu}_t$ for all $t\in[0,T]$, with probability 1, which concludes the proof. □

3.6 Existence of a density and its integrability

In this section we establish the existence and regularity results for the densities of the sub-probability measures $\nu_t, t\in[0,T]$, for a solution to the SPDE (3.2.1). We use an approach based on energy estimates in L^2, similar to the arguments of Section 3.5, to establish existence of a density of ν_t restricted $(0,\infty)$. Unfortunately, these arguments to do not allow us to rule out the presence of an atom at the boundary. However, as we see in Section 3.6.1, this can be achieved by a result of [12] for reflected Brownian motion.

Lemma 3.6.1. *Let (ν, W^0) be the unique solution to the SPDE (3.2.1) with ν in the class Λ. Then we have*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t\left\|T_\varepsilon^{E,\kappa}\nu_s\right\|_2^2 ds\right]<\infty \qquad \text{for all} \quad t\in[0,T].$$

Proof. The proof follows by performing the same estimates as in the proof of Theorem 3.2.5 but for ν instead of the difference of two solutions. Instead of disregarding the negative term on the right-hand side involving c_0 as in (3.5.16), we move it over to the left-hand side and suitably adjust the application of Gronwall's lemma. □

Using this result we deduce that there is an L^2-density for a.e. $(\omega,t)\in\Omega\times[0,T]$.

Proposition 3.6.2. *Let (ν, W^0) be the unique solution to the SPDE (3.2.1) with ν in the class Λ. Then for almost every $(\omega,t)\in\Omega\times[0,T]$ the measure $\nu_t(\omega)$ has a density $V_t(\omega)$ on $[0,\infty)$ and we have the integrated estimate*

$$\mathbb{E}\left[\int_0^t\|V_s\|_2^2\,ds\right]<\infty, \qquad \text{for all} \quad t\in[0,T]. \tag{3.6.1}$$

Proof. Fatou's lemma applied to the result of Lemma 3.6.1 yields a bounded sequence $(T_{\varepsilon_n}^{E,\kappa}\nu_t)_n$ in L^2 for almost every $(\omega,t)\in\Omega\times[0,T]$. Combining the weak compactness of bounded sequences in a Hilbert space with the weak convergence of the smoothed measure, along with Schwartz functions $\mathcal{S}$ being dense in L^2, gives the result. □

Fixing ω (in an almost sure set), the above proposition only guarantees that $V_t(\omega)$ exists in $L^2(0,\infty)$ for almost every $t \in [0,T]$. Restricting each measure ν_t to $(0,\infty)$, we obtain instead the existence of a density for all $t \in [0,T]$, with probability 1, but we only show L^2 integrability away from the origin, which of course leaves the possibility of a Dirac mass at zero. Before verifying this, we first show several auxiliary results.

Take sets $U \Subset W \Subset (0,\infty)$, where $\Subset$ denotes compact containment, and let ψ be a smooth cut-off function such that $\psi = 1$ on U, $\psi \in (0,1)$ on $W \setminus U$ and $\psi = 0$ otherwise. We have

$$|\partial_x \psi| + |\partial_{xx}\psi| \leqslant C 1_{W\setminus U}$$

where the constant C only depends on U and W. Moreover, we define the error term

$$\bar{\mathcal{E}}^h_{t,\varepsilon}(x) = \partial_x \nu_t(h_t G^{E,\kappa}_\varepsilon(x,\cdot)) - h_t(x)\partial_x T^{E,\kappa}_\varepsilon \nu_t(x) + \partial_x h_t(x)\mathcal{H}_{t,\varepsilon}(x), \tag{3.6.2}$$

where

$$\mathcal{H}_{t,\varepsilon}(x) = \nu_t((x-y)\partial_x G^{E,\kappa}_\varepsilon(x,\cdot)), \tag{3.6.3}$$

for a function h representing μ, σ^2 or σ.

Lemma 3.6.3. *Let (ν, W^0) be the unique solution to the SPDE (3.2.1) with ν in the class Λ. Then we have the convergence*

$$\mathbb{E}\left[\int_0^t \int_0^\infty \psi^2(x)\left(\int_0^\infty \partial_x p_\varepsilon(x+y)\nu_s(dy)\right)^2 dx ds\right] \to 0, \quad \text{as } \varepsilon \to 0$$

with the cut-off function ψ specified above.

Proof. Splitting the derivative of the Gaussian heat kernel into two parts we get

$$\begin{aligned}
&\int_0^\infty \psi^2(x)\left(\int_0^\infty \frac{1}{\sqrt{2\pi\varepsilon}}\frac{x+y}{\varepsilon} e^{-\frac{(x+y)^2}{2\varepsilon}}\nu_s(dy)\right)^2 dx \\
&\leqslant 2\int_0^\infty \psi^2(x) e^{-x^2/\varepsilon}\left(\int_0^\infty \frac{1}{\sqrt{2\pi\varepsilon}}\frac{y}{\varepsilon} e^{-y^2/2\varepsilon}\nu_s(dy)\right)^2 dx \\
&+ 2\int_0^\infty \psi^2(x) e^{-x^2/\varepsilon}\frac{x^2}{\varepsilon^2}\left(\int_0^\infty \frac{1}{\sqrt{2\pi\varepsilon}} e^{-y^2/2\varepsilon}\nu_s(dy)\right)^2 dx.
\end{aligned}$$

For the first term on the right-hand side we have

$$\begin{aligned}
&\int_0^\infty \psi^2(x) e^{-x^2/\varepsilon}\frac{1}{2\pi\varepsilon^3}\left(\int_0^\infty y e^{-y^2/2\varepsilon}\nu_s(dy)\right)^2 dx \\
&\leqslant \int_0^\infty \psi^2(x) e^{-x^2/\varepsilon}\frac{1}{2\pi\varepsilon^2} dx = O(\varepsilon^{-3/2} e^{-w^2/2\varepsilon})
\end{aligned}$$

where $w = \inf W$. Then we obtain

$$\mathbb{E}\left[\int_0^t \int_0^\infty \psi^2(x) e^{-x^2/\varepsilon} \left(\int_0^\infty \frac{1}{\sqrt{2\pi\varepsilon}} \frac{y}{\varepsilon} e^{-y^2/2\varepsilon} \nu_s(dy)\right)^2 dxds\right] = O(\varepsilon^{-3/2} e^{-w^2/2\varepsilon}).$$

We can argue similarly for the second term and hence obtain the desired convergence. □

A similar estimate holds for the elastic correction term $g_\varepsilon^{E,\kappa}$.

Lemma 3.6.4. *Let (ν, W^0) be the unique solution to the SPDE (3.2.1) with ν in the class Λ. Then we have the convergence*

$$\mathbb{E}\left[\int_0^t \int_0^\infty \psi^2(x) \left(\int_0^\infty \partial_x g_\varepsilon(x,y) \nu_s(dy)\right)^2 dxds\right] \to 0, \quad \text{as } \varepsilon \to 0$$

with the cut-off function ψ specified above.

Proof. The derivative of the elastic correction term $g_\varepsilon^{E,\kappa}$ is given by

$$\begin{aligned}
\partial_x g_\varepsilon(x,y) =& \partial_x \left(\kappa e^{\kappa(x+y)} e^{\frac{\kappa^2\varepsilon}{2}} \left(1 - \operatorname{Erf}\left(\frac{x+y+\kappa\varepsilon}{\sqrt{2\varepsilon}}\right)\right)\right) \\
=& 2\kappa^2 e^{\kappa(x+y)} e^{\frac{\kappa^2\varepsilon}{2}} \frac{1}{\sqrt{2\pi\varepsilon}} \int_0^\infty \exp\left(-\frac{(z+x+y+\kappa\varepsilon)^2}{2\varepsilon}\right) dz \\
& - 2\kappa e^{\kappa(x+y)} e^{\frac{\kappa^2\varepsilon}{2}} \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{(y+x+\kappa\varepsilon)^2}{2\varepsilon}\right).
\end{aligned}$$

We can then estimate

$$2\kappa^2 e^{\kappa(x+y)} e^{\frac{\kappa^2\varepsilon}{2}} \frac{1}{\sqrt{2\pi\varepsilon}} \int_0^\infty \exp\left(-\frac{(z+x+y+\kappa\varepsilon)^2}{2\varepsilon}\right) dz \leqslant \kappa^2 e^{-\frac{(x+y)^2}{2\varepsilon}}$$

by Lemma A.3. For the other term we get

$$2\kappa e^{\kappa(x+y)} e^{\frac{\kappa^2\varepsilon}{2}} \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{(y+x+\kappa\varepsilon)^2}{2\varepsilon}\right) \leqslant \frac{2\kappa}{\sqrt{2\pi\varepsilon}} e^{-\frac{(x+y)^2}{2\varepsilon}}.$$

Thus, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_0^\infty \psi^2(x) \left(\int_0^\infty \partial_x g_\varepsilon(x,y) \nu_s(dy)\right)^2 dxds\right] \\
&\leqslant \mathbb{E}\left[\int_0^t \int_0^\infty \psi^2(x) \kappa^4 \left(\int_0^\infty e^{-\frac{(x+y)^2}{2\varepsilon}} \nu_s(dy)\right)^2 dxds\right] \\
&\quad + \mathbb{E}\left[\int_0^t \int_0^\infty \psi^2(x) 4\kappa^2 \left(\int_0^\infty \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{(x+y)^2}{2\varepsilon}} \nu_s(dy)\right)^2 dxds\right] \\
&\leqslant \mathbb{E}\left[\int_0^t \int_0^\infty \psi^2(x) \kappa^4 e^{-\frac{x^2}{\varepsilon}} dxds\right] + \mathbb{E}\left[\int_0^t \int_0^\infty \psi^2(x) \frac{e^{-x^2/\varepsilon}}{2\pi\varepsilon} dxds\right] = O(\varepsilon^{-1/2} e^{-w^2/2\varepsilon})
\end{aligned}$$

with $w = \inf W$. □

The necessary controls on the error term $\bar{\mathcal{E}}^h_{t,\varepsilon}$ are proved in the following lemmas.

Lemma 3.6.5. *Let $\bar{\mathcal{E}}^h_{s,\varepsilon}$ be the error term defined in (3.6.2) and ψ the cut-off function specified above. Then we have the convergence*

$$\mathbb{E}\left[\int_0^t \left\|\psi\bar{\mathcal{E}}^h_{s,\varepsilon}\right\|_2^2 ds\right] \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. We can combine the estimates in [37] Lemma 8.2 with the definition of the elastic heat kernel to get

$$\begin{aligned}
|\bar{\mathcal{E}}^h_{s,\varepsilon}| &\leqslant C\left\|\partial_{xx}h\right\|_\infty \int_0^\infty |x-y|^2 |\partial_x G^{E,\kappa}_\varepsilon| \nu_s(dy) \\
&\leqslant C\left\|\partial_{xx}h\right\|_\infty \int_0^\infty |x-y|^3 \varepsilon^{-1} p_\varepsilon(x-y)\nu_s(dy) \\
&\quad + C\left\|\partial_{xx}h\right\|_\infty \int_0^\infty |x-y|^2 |\partial_x p_\varepsilon(x+y)|\nu_s(dy) \\
&\quad + C\left\|\partial_{xx}h\right\|_\infty \int_0^\infty |x-y|^2 |\partial_x g^{E,\kappa}_\varepsilon(x,y)|\nu_s(dy).
\end{aligned}$$

The result follows for the first term by the results from the absorbing case in [37] Lemma 8.2. Similar to the calculations in Lemma 3.6.3 and Lemma 3.6.4 we can show that

$$\mathbb{E}\left[\int_0^t \int_0^\infty \psi^2(x)\left(\int_0^\infty |x-y|^2 |\partial_x g_\varepsilon(x,y)^{E,\kappa}|\nu_s(dy)\right)^2 dxds\right] = O(\varepsilon^{-1/2} e^{-w^2/2\varepsilon})$$

and

$$\mathbb{E}\left[\int_0^t \int_0^\infty \psi^2(x)\left(\int_0^\infty |x-y|^2 |\partial_x p_\varepsilon(x+y)|\nu_s(dy)\right)^2 dxds\right] = O(\varepsilon^{-5/2} e^{-w^2/2\varepsilon}).$$

□

To simplify the notation, we denote by $o^\psi_{sq}(1)$ any family of functions $\{(f_{t,\varepsilon})_{t\in[0,T]}\}_{\varepsilon>0}$ such that

$$\mathbb{E}\left[\int_0^T \|\psi f_{t,\varepsilon}\|_2^2 dt\right] \to 0, \qquad \text{as } \varepsilon \to 0.$$

The final auxiliary result is a control on the other error term $\mathcal{H}_{t,\varepsilon}$.

Lemma 3.6.6. *Let $\mathcal{H}_{t,\varepsilon}$ be the error term defined in (3.6.3). Then we have the estimate*

$$|\mathcal{H}_{t,\varepsilon}| \leqslant c|T^{E,\kappa}_{2\varepsilon}\nu_t| + o^\psi_{sq}(1)$$

for $\varepsilon > 0$ small enough.

Proof. We can use the definition of the elastic heat kernel $G_\varepsilon^{E,\kappa}$ to estimate

$$|\mathcal{H}_{t,\varepsilon}| \leqslant |\nu_t((x-y)\partial_x p_\varepsilon(x-y))| + |\nu_t((x-y)\partial_x p_\varepsilon(x+y))| + |\nu_t((x-y)\partial_x g_\varepsilon^{E,\kappa}(x,y))|.$$

For the first term we get

$$|\nu_t((x-y)\partial_x p_\varepsilon(x-y))| \leqslant c\nu_t(p_{2\varepsilon}(x-y))$$

by [37] Lemma 8.2. Moreover, we have the estimate

$$g_\varepsilon^{E,\kappa}(x,y) \leqslant \kappa e^{-\frac{(x+y)^2}{2\varepsilon}} \leqslant p_\varepsilon(x+y)$$

for ε small enough. This yields

$$|\nu_t((x-y)\partial_x p_\varepsilon(x-y))| \leqslant C T_{2\varepsilon}^{E,\kappa}\nu_t.$$

The estimates for the other two terms follows as in Lemma 3.6.3 and Lemma 3.6.4. □

Now we have all the tools to prove the existence of a density in the interior.

Proposition 3.6.7. *Let (ν, W^0) be the unique solution to the SPDE (3.2.1) with ν in the class Λ. The measure ν_t restricted to $(0,\infty)$ has a density V_t for all $t \in [0,T]$ with probability 1. The density is square integrable on (δ,∞) for every $\delta > 0$.*

Proof. We proceed similarly to the uniqueness proof in Section 3.5. We use the test function $y \mapsto G_\varepsilon^{E,\kappa}(x,y)$ in the SPDE (3.2.1), switch derivatives from y to x and take the coefficients outside the integration against the measure. This creates the error terms defined in (3.6.2) and (3.6.3). Applying Itô's formula we can then derive the dynamics

$$\begin{aligned}
d(T_\varepsilon\nu_t)^2 =& -2T_\varepsilon^{E,\kappa}\nu_t\left(\mu_t\partial_x T_\varepsilon^{E,\kappa}\nu_t - \partial_x\mu_t\mathcal{H}_{t,\varepsilon} + \bar{\mathcal{E}}_{t,\varepsilon}^{\mu}\right)dt \\
&+ T_\varepsilon^{E,\kappa}\nu_t\partial_x\left(\sigma_t^2\partial_x T_\varepsilon^{E,\kappa}\nu_t - \partial_x\sigma_t^2\mathcal{H}_{t,\varepsilon} + \bar{\mathcal{E}}_{t,\varepsilon}^{\sigma^2}\right)dt \\
&- 2T_\varepsilon^{E,\kappa}\nu_t\rho_t\left(\sigma_t\partial_x T_\varepsilon^{E,\kappa}\nu_t - \partial_x\sigma_t\mathcal{H}_{t,\varepsilon} + \bar{\mathcal{E}}_{t,\varepsilon}^{\sigma}\right)dW_t^0 \\
&+ 4T_\varepsilon^{E,\kappa}\partial_x\nu_t(\mu_t p_\varepsilon(x+\cdot))dt + 4T_\varepsilon^{E,\kappa}\nu_t\partial_x\nu_t(\rho_t\sigma_t p_\varepsilon(x+\cdot))dW_t^0 \\
&- 4T_\varepsilon^{E,\kappa}\nu_t\partial_x\nu_t(\mu_t g_\varepsilon^{E,\kappa}(x,\cdot))dt - 4T_\varepsilon^{E,\kappa}\nu_t\partial_x\nu_t(\rho_t\sigma_t g_\varepsilon^{E,\kappa}(x,\cdot))dW_t^0 \\
&+ \left(\rho_t\left(\sigma_t\partial_x T_\varepsilon^{E,\kappa}\nu_t - \partial_x\sigma_t\mathcal{H}_{t,\varepsilon} + \bar{\mathcal{E}}_{t,\varepsilon}^{\sigma}\right) + 2\partial_x\nu_t(\rho_t\sigma_t p_\varepsilon(x+\cdot)) - 2\partial_x\nu_t(\rho_t\sigma_t g_\varepsilon(x,\cdot))\right)^2 dt.
\end{aligned}$$

Next, we multiply the equation by ψ^2 and integrate over x. We then estimate all the terms on the right-hand side individually using the results of Lemma 3.6.3,

Lemma 3.6.4, Lemma 3.6.5 and Lemma 3.6.6 together with the generalized Young's inequality. For the first term we have

$$
\begin{aligned}
&-2\int_0^t\int_0^\infty \psi^2 T_\varepsilon^{E,\kappa}\nu_s\left(\mu_s\partial_x T_\varepsilon^{E,\kappa}\nu_s-\partial_x\mu_s\mathcal{H}_{s,\varepsilon}+\bar{\mathcal{E}}^{\mu}_{s,\varepsilon}\right)dxds\\
&\leqslant c_\eta\int_0^t\left\|\psi T_\varepsilon^{E,\kappa}\nu_s\right\|_2^2 ds+c_\eta\int_0^t\left\|\psi T_{2\varepsilon}^{E,\kappa}\nu_s\right\|_2^2 ds+\eta\int_0^t\int_0^\infty \mu_s^2\psi^2|\partial_x T_\varepsilon^{E,\kappa}\nu_s|^2dxds+o(1).
\end{aligned}
$$

For the term in the second line we apply integration by parts and obtain

$$
\begin{aligned}
&\int_0^t\int_0^\infty \psi^2 T_\varepsilon^{E,\kappa}\nu_s\partial_x\left(\sigma^2\partial_x T_\varepsilon^{E,\kappa}\nu_s-\partial_x\sigma_s^2\mathcal{H}_{s,\varepsilon}+\bar{\mathcal{E}}^{\sigma^2}_{s,\varepsilon}\right)dxds\\
&=-\int_0^t\int_0^\infty \psi^2\sigma^2|\partial_x T_\varepsilon^{E,\kappa}\nu_s|^2dxds-2\int_0^t\int_0^\infty \psi\psi' T_\varepsilon^{E,\kappa}\nu_s\partial_x T_\varepsilon^{E,\kappa}\nu_s\sigma_s^2dxds\\
&\quad+\int_0^t\int_0^\infty \psi^2\partial_x T_\varepsilon^{E,\kappa}\nu_s\partial_x\sigma_s^2\mathcal{H}_{s,\varepsilon}dxds+2\int_0^t\int_0^\infty \psi\psi' T_\varepsilon^{E,\kappa}\nu_s\partial_x\sigma_s^2\mathcal{H}_{s,\varepsilon}dxds\\
&\quad-\int_0^t\int_0^\infty \psi^2\partial_x T_\varepsilon^{E,\kappa}\nu_s\bar{\mathcal{E}}^{\sigma^2}_{s,\varepsilon}-2\int_0^t\int_0^\infty \psi\psi' T_\varepsilon^{E,\kappa}\nu_s\bar{\mathcal{E}}^{\sigma^2}_{s,\varepsilon}dxds\\
&\leqslant-\int_0^t\int_0^\infty \psi^2\sigma^2|\partial_x T_\varepsilon^{E,\kappa}\nu_s|^2dxds+\eta\int_0^t\int_0^\infty \psi^2|\partial_x T_\varepsilon^{E,\kappa}\nu_s|^2dxds\\
&\quad+c_\eta\int_0^t\int_0^\infty |T_\varepsilon^{E,\kappa}\nu_s|^2dxds+c_\eta\int_0^t\int_0^\infty |T_{2\varepsilon}^{E,\kappa}\nu_s|^2dxds+o(1).
\end{aligned}
$$

For now, we leave the terms involving stochastic integrals unchanged. The remaining terms in the fourth and fifth line can be estimated via

$$
\begin{aligned}
&4\int_0^t\int_0^\infty \psi^2 T_\varepsilon^{E,\kappa}\nu_s\partial_x\nu_s(\mu_s p_\varepsilon(x+\cdot))dxds+4\int_0^t\int_0^\infty \psi^2 T_\varepsilon^{E,\kappa}\nu_s\partial_x\nu_s(\mu_s g_\varepsilon^{E,\kappa}(x,\cdot))dxds\\
&\leqslant c_\eta\int_0^t\left\|\psi T_\varepsilon^{E,\kappa}\nu_s\right\|_2^2 ds+o(1).
\end{aligned}
$$

We can write the last term as

$$
\begin{aligned}
&\int_0^t\int_0^\infty \psi^2\Big(\rho_s\left(\sigma_s\partial_x T_\varepsilon^{E,\kappa}\nu_s-\partial_x\sigma_s\mathcal{H}_{s,\varepsilon}+\bar{\mathcal{E}}^{\sigma}_{s,\varepsilon}\right)+o^{\psi}_{sq}(1)\Big)^2dxds\\
&=\int_0^t\int_0^\infty \psi^2\rho_s^2\left(\sigma_s\partial_x T_\varepsilon^{E,\kappa}\nu_s-\partial_x\sigma_s\mathcal{H}_{s,\varepsilon}+\bar{\mathcal{E}}^{\sigma}_{s,\varepsilon}\right)^2dxds\\
&\quad+\int_0^t\int_0^\infty \psi^2\rho_s\left(\sigma_s\partial_x T_\varepsilon^{E,\kappa}\nu_s-\partial_x\sigma_s\mathcal{H}_{s,\varepsilon}+\bar{\mathcal{E}}^{\sigma}_{s,\varepsilon}\right)o^{\psi}_{sq}(1)dxds+o(1)\\
&\leqslant(1+\eta)\int_0^t\int_0^\infty \psi^2\rho_s^2\sigma_s^2|\partial_x T_\varepsilon^{E,\kappa}\nu_s|^2dxds\\
&\quad-(1+\eta)\int_0^t\int_0^\infty \psi^2\rho_s^2\sigma_s\partial_x T_\varepsilon^{E,\kappa}\nu_s\left(\partial_x\sigma_s\mathcal{H}_{s,\varepsilon}-\bar{\mathcal{E}}^{\sigma}_{s,\varepsilon}\right)dxds\\
&\quad+(1+\eta)\int_0^t\int_0^\infty \psi^2\rho^2\left(\partial_x\sigma\mathcal{H}_{s,\varepsilon}-\bar{\mathcal{E}}^{\sigma}_{s,\varepsilon}\right)^2dxds+o(1).
\end{aligned}
$$

We obtain

$$\int_0^t \int_0^\infty \psi^2 \Big(\rho_s \left(\sigma_s \partial_x T_\varepsilon^{E,\kappa} \nu_s - \partial_x \sigma_s \mathcal{H}_{s,\varepsilon} + \bar{\mathcal{E}}^\sigma_{s,\varepsilon} \right) + o^\psi_{sq}(1) \Big)^2 dx ds$$
$$\leqslant (1 + 2\eta + \eta^2) \int_0^t \int_0^\infty \psi^2 \rho_s^2 \sigma_s^2 |\partial_x T_\varepsilon^{E,\kappa} \nu_s|^2 dx ds + c_\eta \int_0^t \left\| \psi T_{2\varepsilon}^{E,\kappa} \nu_s \right\|_2^2 ds + o(1).$$

Putting everything back together and choosing $\eta > 0$ small enough such that

$$\sigma_s^2 - \rho_s^2 (1 + 2\eta + \eta^2) \sigma_s^2 - \eta \mu_s^2 - \eta \geqslant c_0$$

for some $c_0 > 0$ yields

$$\begin{aligned}
\left\| \psi T_\varepsilon^{E,\kappa} \nu_t \right\|_2^2 \leqslant & \left\| T_\varepsilon^{E,\kappa} \nu_0 \right\|_2^2 + c_\eta \int_0^t \left\| T_\varepsilon^{E,\kappa} \nu_s \right\|_2^2 ds + c_\eta \int_0^t \left\| T_{2\varepsilon}^{E,\kappa} \nu_s \right\|_2^2 ds + o(1) \\
& - 2 \int_0^t \int_0^\infty \psi^2 T_\varepsilon^{E,\kappa} \nu_s \rho_s \left(\sigma_s \partial_x T_\varepsilon^{E,\kappa} \nu_s - \partial_x \sigma_t \mathcal{H}_{t,\varepsilon} + \bar{\mathcal{E}}^\sigma_{t,\varepsilon} \right) dx dW_s^0 \\
& + 4 \int_0^t \int_0^\infty \psi^2 T_\varepsilon^{E,\kappa} \nu_s \partial_x \nu_s (\rho_s \sigma_s p_\varepsilon(x + \cdot)) dx dW_s^0 \\
& - 4 \int_0^t \int_0^\infty \psi^2 T_\varepsilon^{E,\kappa} \nu_s \partial_x \nu_s (\rho_s \sigma_s g_\varepsilon^{E,\kappa}(x, \cdot)) dx dW_s^0.
\end{aligned}$$

Next, we take supremum over t and then expectation. Using [37, Lemma 8.5] to estimate the stochastic integrals, which can easily be adapted to this case, we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{s \in [0,t]} \left\| \psi T_\varepsilon^{E,\kappa} \nu_s \right\|_2^2 \right] \leqslant & c' \mathbb{E}\left[\left\| T_\varepsilon^{E,\kappa} \nu_0 \right\|_2^2 \right] + c' \mathbb{E}\left[\int_0^t \left\| T_\varepsilon^{E,\kappa} \nu_s \right\|_2^2 ds \right] \\
& + c' \mathbb{E}\left[\int_0^t \left\| T_{2\varepsilon}^{E,\kappa} \nu_s \right\|_2^2 ds \right] + o(1)
\end{aligned}$$

with a constant $c' > 0$. Taking the limit $\varepsilon \to 0$ and using Lemma 3.6.1, Assumption 3.2.1 and Lemma C.1 we obtain

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\sup_{s \in [0,t]} \left\| T_\varepsilon^{E,\kappa} \nu_s \right\|_{L^2(U)}^2 \right] \leqslant \lim_{\varepsilon \to 0} \mathbb{E}\left[\sup_{s \in [0,t]} \left\| \psi T_\varepsilon^{E,\kappa} \nu_s \right\|_2^2 \right] < \infty. \tag{3.6.4}$$

Thus, we have the existence of an L^2-density in the set U using the same arguments as in the proof of Proposition 3.6.2 and since U was arbitrary the result follows. □

3.6.1 Ruling out a Dirac mass at the origin

As a special case of Theorem 3.7.3 in the next section, we can characterize the unique solution (ν, W^0) to the SPDE (3.2.1) as the conditional law

$$\nu_t(dx) = \mathbb{P}\left(X_t \in dx,\, t < \tau \middle| \mathcal{F}_t^{W^0} \right) \tag{3.6.5}$$

of a given reflected diffusion

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)\rho(t)W_t^0 + \sigma(t, X_t)(1-\rho(t)^2)^{\frac{1}{2}}dW_t^1 + dL_t, \tag{3.6.6}$$

where W^1 is a Brownian motion independent of W^0 and τ is the elastic killing time defined by a standard exponential random variable independent of X. This point of view makes it easy to deduce from [12] that there cannot be an atom at the origin.

Proof of Proposition 3.2.7. Express the unique solution ν as (3.6.5). We can then apply the scale transformation ζ defined in Lemma 3.3.1 to (3.6.6) in order to obtain

$$\nu_t(A) \leqslant \mathbb{P}\left(Z_0 + \rho W_t^0 + \sqrt{1-\rho^2}W_t^1 + \int_0^t \tilde{\mu}_s ds + L_t \in \zeta(t, A)\middle|\mathcal{F}_t^{W^0}\right),$$

for any Borel set A in $[0, \infty)$. By Lemma 3.3.1, we have that

$$Z_0 + \rho W_t^0 + \sqrt{1-\rho^2}W_t^1 + \int_0^t \tilde{\mu}_s ds \geqslant Z_0 + \rho W_t^0 + \sqrt{1-\rho^2}W_t^1 - Ct \tag{3.6.7}$$

for all $t \in [0, T]$ almost surely with a constant $C > 0$. Note that the processes in (3.6.7) only differ by the drift term. Moreover the drift $-Ct$ is always decreasing by more than the drift $\int_0^t \tilde{\mu}_s ds$. Thus we have

$$\begin{aligned}
&\left\{Z_0 + \rho W_t^0 + \sqrt{1-\rho^2}W_t^1 + \int_0^t \tilde{\mu}_s ds + L_t = 0\right\} \\
&= \left\{Z_0 + \rho W_t^0 + \sqrt{1-\rho^2}W_t^1 + \int_0^t \tilde{\mu}_s ds = \inf_{u \leqslant t}\left\{Z_0 + \rho W_u^0 + \sqrt{1-\rho^2}W_u^1 + \int_0^u \tilde{\mu}_s ds\right\}\right\} \\
&\subseteq \left\{Z_0 + \rho W_t^0 + \sqrt{1-\rho^2}W_t^1 - Ct = \inf_{u \leqslant t}\left\{Z_0 + \rho W_u^0 + \sqrt{1-\rho^2}W_u^1 - uC\right\}\right\} \\
&= \left\{Z_0 + \rho W_t^0 + \sqrt{1-\rho^2}W_t^1 - Ct + \tilde{L}_t = 0\right\}
\end{aligned}$$

where $\tilde{L}$ is the local time of the process in the last line. As a result we obtain

$$0 \leqslant \nu_t(\{0\}) \leqslant \mathbb{P}\left(Z_0 + \rho W_t^0 + \sqrt{1-\rho^2}W_t^1 - Ct + \tilde{L}_t = 0\middle|\mathcal{F}_t^{W^0}\right),$$

for all $t \in [0, T]$ almost surely. With $f(t) := Ct$, we obviously have that $f(t)/\sqrt{t}$ is non-decreasing, so f is not in the upper class of Brownian motion by [45, p.144]. Thus, [12, Theorem 2.2 and Theorem 2.5] gives that the probability on the right-hand side is zero for all $t \in [0, T]$ almost surely, and hence ν_t cannot have an atom at zero for any t. Combining this with Proposition 3.6.7 we can conclude that we have a density everywhere on the positive half-line. □

While the previous theorem establishes existence of a density for the entire positive half-line, we do not get the square integrability all the way up to the boundary, beyond a set of times of full measure as guaranteed by Theorem 3.2.6. It is unclear to us how to extend this result to include all $t \in [0, T]$ and we actually conjecture it to be wrong for all $t \in [0, T]$. To see where the problem lies we consider the simpler case of a reflecting boundary and set the drift μ to zero and the volatility σ to one. The solution to the SPDE then takes the form

$$\nu_t(dx) = \mathbb{P}\left(\rho W_t^0 + \sqrt{1-\rho^2} W_t^1 + L_t \in dx \,|\, \mathcal{F}_t^{W^0}\right).$$

We can interpret this conditional distribution as the distribution of the scaled Brownian motion $\sqrt{1-\rho^2}W^1$ reflected on a given fixed Brownian path $-\rho W^0$. This is a particular instance of a Brownian motion reflected in a time-dependent domain, which, together with the related heat equations, have been studied in a series of papers [10, 12, 11, 13]. In [12] the authors show that, at the boundary of such domains, both singularities of the density as well as atoms are possible. While they establish that there are no atoms when the boundary is a path of a Brownian motion, in [13] it is shown that in this case the density exhibits blow-up at the boundary on a dense subset of times t. These effects arise when the boundary sharply moves into the domain and the diffusion cannot transport the heat away fast enough. The authors show that, for any boundary that locally moves in sharper than Brownian motion (i.e., is in the upper class of Brownian motion), an atom occurs. This leads us to suspect that the blow-up is too strong to allow for square integrability of the density at the origin.

3.7 Extension to nonlinear interactions in the drift

In this section we discuss some of the results we can obtain and some of the difficulties associated to a natural extension of the particle system which allows for interaction through a dependence on the empirical measure ν_t^N in the drift coefficient μ. The particle system then becomes

$$\begin{aligned} X_t^{i,N} = X_0^i &+ \int_0^t \mu(s, X_s^{i,N}, \nu_s^N) ds + \int_0^t \sigma(s, X_s^{i,N}) \rho(s) dW_s^0 \\ &+ \int_0^t \sigma(s, X_s^{i,N})(1-\rho(s)^2)^{\frac{1}{2}} dW_s^i + L_t^{i,N}, \end{aligned} \tag{3.7.1}$$

together with the elastic stopping times $\tau^{i,N}$. In this case we need additional assumptions on the regularity of μ with respect to the measure variable.

Assumption 3.7.1 (Regularity in the Measure Variable)**.** *The drift coefficient μ is Lipschitz continuous in the measure variable ν on the space of sub-probability measures $\mathbf{M}_{\leqslant 1}(\mathbb{R})$ with respect to the bounded Lipschitz distance d_0 which is given by*

$$d_0(\nu, \tilde{\nu}) \coloneqq \sup\{|\langle \psi, \nu - \tilde{\nu}\rangle| : \|\psi\|_{Lip} \leqslant 1, \|\psi\|_\infty \leqslant 1\}, \tag{3.7.2}$$

i.e., there exists a fixed constant $c > 0$ such that $|\mu(t, x, \nu) - \mu(t, x, \tilde{\nu})| \leqslant c d_0(\nu, \tilde{\nu})$.

Through the empirical measure in the drift coefficient the elastic stopping times $\tau^{i,N}$ become part of the equations for the particles. Thus, well-posedness of the particle system needs additional arguments compared to the system (3.1.1). The existence of such a particle system can be established by adjusting the arguments in [30] to this case. In [30] the authors construct a particle system on the positive half-line with elastic boundary that moves upwards as more particles get killed. They consider the case where the particles are driven by independent noise terms and the interaction in the drift term is through the loss term that represents the number of particles that have been killed at the boundary. When applying their approach to our system we need to add a common noise in the filtrations defined in [30]. Another difference is that in our case the interactions are through the empirical measure not the loss function. An important step in the construction in [30] is, for a given particle i, to consider the system without particle i given that $t < \tau^{i,N}$. In our case such a system will, in contrast to the system in [30], not be independent of the particle i because of the empirical-measure interaction. However, given the condition $t < \tau^{i,N}$ we can consider this system with the empirical measure

$$\nu_t^{N,-i}(dx) = \frac{1}{N}\left(\sum_{j=1, j\neq i}^{N} \delta_{X_t^{j,N}}(dx)\mathbb{1}_{t<\tau^j} + \delta_{X_t^{i,N}}(dx)\right).$$

Thus, the system without particle i depends on W^i and X_0^i but not χ^i. Considering this fact, the construction in [30] carries through in our case. Using Assumptions 3.2.1, 3.2.2 and 3.7.1 a standard argument relying on the Lipschitz continuity and Gronwall's lemma can be used to establish uniqueness.

Following the same arguments as in Section 3.4 we obtain the existence result in this case. The methods we used in the proofs in Section 3.4 can easily be generalised to allow for interaction in the drift. In particular the martingale approach can be extended in a straight forward way to μ depending also on ξ in (3.4.7).

Theorem 3.7.2. *Let (ν^N, W^0) be a sequence with ν^N being empirical-measure processes corresponding to the particle system (3.7.1) that satisfy Assumptions 3.2.1,*

3.2.2 and 3.7.1. Then (ν^N, W^0) possesses converging subsequences in $(D_{\mathcal{S}'}, M1) \times (C_{\mathbb{R}}, \|\|_\infty)$. Moreover, for any limit point (ν, W^0) the process ν is in the class Λ and (ν, W^0) satisfies the SPDE

$$\begin{aligned}\langle \nu_t, \phi \rangle = \langle \nu_0, \phi \rangle + \int_0^t \langle \nu_s, \mu(s, \cdot, \nu_s)\phi' \rangle ds + \frac{1}{2}\int_0^t \langle \nu_s, \sigma^2(s, \cdot)\phi'' \rangle ds \\ + \int_0^t \langle \nu_s, \rho(s)\sigma(s, \cdot)\phi' \rangle dW_s^0\end{aligned} \tag{3.7.3}$$

for all times $t \in [0, T]$ and all test functions $\phi \in \mathcal{C}_0^{E,\kappa}(\mathbb{R})$, where

$$\mathcal{C}_0^{E,\kappa}(\mathbb{R}) = \{\phi \in \mathcal{S} : \phi'(0) = \kappa\phi(0)\}.$$

Note that in this case the limit equation becomes a nonlinear SPDE with a non-linearity in the drift term μ. A convenient way of obtaining regularity for solutions to this nonlinear SPDE is to realise that there is a probabilistic representation.

Theorem 3.7.3. *Let (ν, W^0) be a solution to the SPDE (3.7.3) with ν in the class Λ. Then, for all $t \in [0, T]$ we have the following representation of ν_t*

$$\nu_t = \mathbb{P}\left(X_t \in \cdot, t < \tau | \, \mathcal{F}_t^{\nu, W^0}\right) \tag{3.7.4}$$

where X is a reflecting particle with dynamics given by

$$dX_t = \mu(t, X_t, \nu_t)dt + \sigma(t, X_t)\rho_t W_t^0 + \sigma(t, X_t)(1 - \rho_t^2)^{\frac{1}{2}} dW_t^1 + dL_t,$$

with W^0, W^1 independent Brownian motions and X_0 being distributed according to ν_0. The stopping time τ is the elastic stopping time associated with X and $(\mathcal{F}_t^{\nu,W^0})$ the filtration generated by ν and W^0.

Proof. The proof is based on a decoupling argument and the linear uniqueness result. Let ν be a weak solution to the nonlinear SPDE which exists by Theorem 3.7.2. We then define a stochastic process X as the solution to the SDE

$$dX_t = \mu_t^\nu dt + \sigma(t, X_t)(\rho_t dW_t^0 + (1 - \rho_t^2)^{1/2} dW_t^1) + dL_t \tag{3.7.5}$$

where the drift term μ^ν is given by

$$\mu_t^\nu := \mu(t, X_t, \nu_t)$$

and W^0, W^1 are independent Brownian motions. The associated elastic stopping time is $\tau^{lin} := \inf\{t > 0 : L_t > \chi\}$ for an independent exponential random variable

$\chi \sim \text{Exp}(\kappa)$. Define a process (ν_t^{lin}) taking values in the space of sub-probability measures as

$$\langle \nu_t^{lin}, \phi \rangle := \mathbb{E}\left[\phi(X_t) 1_{t<\tau^{lin}} | \mathcal{F}_t^{\nu, W^0}\right] \tag{3.7.6}$$

for $\phi \in \mathcal{S}$. Take a test function $\phi \in \mathcal{C}_0^{E,\kappa}(\mathbb{R})$, apply Itô's formula, rearrange and take conditional expectation to find that ν_t^{lin} solves the linear SPDE

$$\begin{aligned} d\langle \nu_t^{lin}, \phi \rangle =& \langle \nu_t^{lin}, \mu_t^\nu \phi' \rangle dt + \frac{1}{2} \langle \nu_t^{lin}, \sigma_t^2 \phi'' \rangle dt \\ &+ \langle \nu_t^{lin}, \rho_t \sigma_t \phi' \rangle dW_t^0 \end{aligned} \tag{3.7.7}$$

for all $\phi \in \mathcal{C}_0^{E,\kappa}(\mathbb{R})$. By the same arguments as for the particle system in Section 3.3, one easily checks that the solution is in the class Λ. Since ν is a solution to the nonlinear SPDE, it also solves the SPDE (3.7.7). By Therorem 3.2.5 solutions to (3.7.7) are unique in the class Λ, so we have $\nu_t = \nu_t^{lin}$. This yields the desired result. □

Based on this representation the same regularity results as in Section 3.6 can be shown. The question of uniqueness of solutions to the nonlinear SPDE is more subtle. In the case of an absorbing boundary, uniform L^2 regularity is used to deal with the nonlinearity by employing an argument involving a sequence of stopping times (see [37, 41]). As discussed in Section 3.6 we do not expect such regularity to hold in the elastic case. This makes the extension to the nonlinear case seem significantly more difficult than in the absorbing case and an entirely different approach may be needed.

3.8 Absorption and Reflection as Limiting Cases

Intuitively, an elastic boundary condition acts as a mixture of an absorbing boundary and a reflecting boundary – with the positive parameter κ controlling the balance of the two. This can be seen from the elastic condition

$$\phi'(0) = \kappa \phi(0), \tag{3.8.1}$$

if we take the limits $\kappa \to 0$ and $\kappa \to \infty$. When taking κ to ∞ we obtain the absorbing boundary condition $\phi(0) = 0$ and taking κ to 0 yields the reflecting boundary condition $\phi'(0) = 0$. This shows that the absorbing and reflecting cases can be obtained as limits of the elastic case. Moreover, we observe the same when analysing the elastic stopping times

$$\tau = \inf\{t > 0 : L_t > \chi^\kappa\}, \quad \chi^\kappa \sim \text{Exp}(\kappa).$$

In the limit $\kappa \to \infty$ we get $\chi^\kappa \to 0$ and the stopping time τ becomes the first time the process X hits the boundary at 0, *i.e.* the absorbing stopping time. For $\kappa \to 0$ we have the convergence $\chi^\kappa \to \infty$ and the elastic killing time τ^i converges to ∞. As a result the limit process becomes purely reflecting. The goal of this section is to show that we also have this convergence at the level of the measure-valued processes as we let κ go to 0 or ∞. We will write ν^κ for the solution to the elastic SPDE to emphasis the dependence on the elastic-killing parameter κ.

SPDEs of the type (3.2.1) with an absorbing boundary are studied in [37] and in [41]. Existence and uniqueness of solutions is proved in these articles. The absorbing nature of the boundary is defined through the space of test functions

$$\mathcal{C}_0^A(\mathbb{R}) \coloneqq \{\phi \in \mathcal{S}(\mathbb{R}) : \phi(0) = 0\}. \tag{3.8.2}$$

By using the space of test functions

$$\mathcal{C}_0^R(\mathbb{R}) \coloneqq \{\phi \in \mathcal{S}(\mathbb{R}) : \phi'(0) = 0\} \tag{3.8.3}$$

in the formulation of the SPDE (3.2.1) we have instead a reflecting boundary. With a few adjustments to the proof for the elastic case, we can also prove Theorem 3.2.8.

Proof of Theorem 3.2.8. The existence and necessary regularity of the weak solution follows through a particle approximation as done in Section 3.3 and Section 3.4. Note that the required estimates for the reflecting system were established alongside the estimates for the elastic system in Section 3.3.

The kernel smoothing method with the reflecting heat kernel G_ε^R can be employed to prove uniqueness. In the reflecting case the weak formulation of the boundary simplifies because the process takes values in the space of probability measures. Denoting by T_ε^R the convolution operator with G_ε^R and by ν^0 a solution to the reflecting SPDE we see that for the anti-derivative at the boundary, an application of Fubini's theorem and the fact that G_ε^R integrates to 1 gives $\partial_x^{-1} T_\varepsilon^R \nu_t(0) = -\nu_t[0, \infty)$. The rest of the proof now follows with the same arguments as in the elastic case. □

Now that we have existence and uniqueness for the three boundary cases, we can prove Theorem 3.2.9 on the weak convergence of the solution to the elastic equation to the absorbing and reflecting counter parts in the following way

- verify tightness of (ν^κ) to establish existence of limit points
- show that limit points solve the absorbing or reflecting SPDEs, respectively,

- deduce weak convergence from uniqueness of solutions.

The first step concerns tightness. As in the case of the particle approximations we consider tightness on the space of càdlàg processes with values in the space of tempered distributions.

Lemma 3.8.1. *Let (ν^κ, W^0) be a sequence of solutions with ν^κ in the class Λ to the SPDE (3.2.1) with elastic boundary condition for parameter $\kappa > 0$. Then, the sequence (ν^κ, W^0) is tight on $(D_{\mathcal{S}'}, M1) \times (C_{\mathbb{R}}, \|\cdot\|_\infty)$ for both cases, $\kappa \to \infty$ and $\kappa \to 0$.*

Proof. We follow the same approach as in Proposition 3.4.3. By the probabilistic representation from Theorem 3.7.3 we know that for $\phi \in \mathcal{S}$ we have

$$\langle \nu_t^\kappa, \phi \rangle = \mathbb{E}\left[\phi(X_t) 1_{t<\tau^\kappa} | \mathcal{F}_t^{\nu^\kappa, W^0}\right]. \tag{3.8.4}$$

We can then again apply the decomposition of [51, Proposition 4.2] and obtain that the first condition we need to show is

$$\mathbb{E}\left[|\langle \hat{\nu}_t^\kappa, \phi \rangle - \langle \hat{\nu}_s^\kappa, \phi \rangle|^4\right] = O(|t-s|^2), \qquad \text{as } |t-s| \to 0 \tag{3.8.5}$$

for

$$\langle \hat{\nu}_t^\kappa, \phi \rangle = \mathbb{E}\left[\phi(X_{t\wedge\tau^\kappa}) | \mathcal{F}_t^{\nu^\kappa, W^0}\right].$$

The definition of X is independent of κ and we can conclude condition (3.8.5) with the same methods as in Proposition 3.4.3. To conclude tightness using the approach of Proposition 3.4.3 we need to verify that for all $t \in [0,T]$ and $\eta > 0$

$$\lim_{\delta\to 0} \limsup_{\kappa\to 0} \mathbb{P}(\mathcal{L}_{t+\delta}^\kappa - \mathcal{L}_t^\kappa \geqslant \eta) = 0 \tag{3.8.6}$$

where

$$\mathcal{L}_t^\kappa = \mathbb{P}\left(\tau^\kappa \leqslant t | \mathcal{F}_t^{\nu^\kappa, W^0}\right)$$

to obtain tightness in the case $\kappa \to 0$ and the same condition with $\limsup_{\kappa\to\infty}$ for the other case. Markov's inequality yields

$$\mathbb{P}(\mathcal{L}_{t+\delta}^\kappa - \mathcal{L}_t^\kappa \geqslant \eta) \leqslant \eta^{-1} \mathbb{P}(t < \tau^\kappa \leqslant t+\delta)$$

We consider a process $\tilde{Y}$ given by

$$\tilde{Y}_t^\kappa = X_0^\kappa + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

where $dW_s = \rho_s dW_s^0 + \sqrt{1-\rho_s^2} dW_s^1$. The reflecting particle X is then given by

$$X_t = \tilde{Y}_t + L_t$$

using the Skorokhod problem. Moreover, we set

$$Y_t^\kappa = \tilde{Y}_t^\kappa + \chi^\kappa.$$

We can apply this together with the definition of the stopping time τ^κ to obtain

$$\mathbb{P}\left(\tau^\kappa \leqslant t\right) = \mathbb{P}\left(L_t \geqslant \chi^\kappa\right) = \mathbb{P}\left(\inf_{s\in[0,t]} \tilde{Y}_s \leqslant -\chi^\kappa\right) = \mathbb{P}\left(\inf_{s\in[0,t]} Y_s^\kappa \leqslant 0\right).$$

Taking $\varepsilon > 0$ we have

$$\begin{aligned}\mathbb{P}\left(t < \tau^\kappa \leqslant t+\delta\right) =& \mathbb{P}\left(t < \tau^\kappa \leqslant t+\delta, Y_t^\kappa \geqslant \varepsilon\right) + \mathbb{P}\left(t < \tau^\kappa \leqslant t+\delta, Y_t^\kappa \in (0,\varepsilon)\right) \\ &+ \mathbb{P}\left(t < \tau^\kappa \leqslant t+\delta, Y_t^\kappa \leqslant 0\right).\end{aligned}$$

The last term is zero because $Y_t^\kappa \leqslant 0$ implies $\tau^\kappa \leqslant t$. This gives the estimate

$$\mathbb{P}\left(t < \tau^\kappa \leqslant t+\delta\right) \leqslant \mathbb{P}\left(t < \tau^\kappa \leqslant t+\delta, Y_t^\kappa \geqslant \varepsilon\right) + \mathbb{P}\left(Y_t^\kappa \in (0,\varepsilon)\right) \tag{3.8.7}$$

To deal with the second term on the right-hand side we apply the scale transformation and change of measure discussed in Section 3.3 to transform Y^κ into a Brownian motion. The only dependence on κ is then left in the initial condition. However, note that χ^κ converges to zero or infinity almost surely if we take the limit $\kappa \to \infty$ or $\kappa \to 0$, respectively. Since the function ζ is continuous in the x variable and the measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ we also get $\mathbb{Q}$-almost sure convergences to zero or infinity for $\zeta(t, \chi^\kappa)$. Using the properties of Brownian motion we then have

$$\limsup_{\kappa\to\infty} \mathbb{P}(Y_t^\kappa \in (0,\varepsilon)) \leqslant C_q \mathbb{Q}\left(B_t \in (0, C_{\mu,\sigma}\varepsilon)\right)^{1/q} = o(1), \quad \text{as } \varepsilon \to 0,$$

for $q > 1$, a constant $C_q > 0$ depending on q and B a Brownian motion under $\mathbb{Q}$. In the case $\kappa \to 0$ we get

$$\limsup_{\kappa\to 0} \mathbb{P}(Y_t^\kappa \in (0,\varepsilon)) = 0.$$

We follow the structure in the proof of [37, Proposition 4.7] to estimate the other term in (3.8.7). We get

$$\begin{aligned}\mathbb{P}\left(t < \tau^\kappa \leqslant t+\delta, Y_t^\kappa \geqslant \varepsilon\right) &\leqslant \mathbb{P}\left(\inf_{s\in[t,t+\delta]} Y_s^\kappa \leqslant 0, Y_t^\kappa \geqslant \varepsilon\right) \\ &\leqslant \mathbb{P}\left(\inf_{s\in[t,t+\delta]} (Y_s^\kappa - Y_t^\kappa) \leqslant -\varepsilon\right).\end{aligned}$$

Using the scale transformation ζ from Section 3.3 we define $U_s^\kappa := \zeta(t+s, Y_{t+s}^\kappa) - \zeta(t+s, Y_t^\kappa)$. The dynamics of U^κ are given by

$$dU_s^\kappa = u_s^\kappa ds + dW_s$$

with uniformly bounded drift u_s^κ. This means we can find a constant $c_1 > 0$ such that

$$\begin{aligned}\mathbb{P}\left(\inf_{s\in[t,t+\delta]}(Y_s^\kappa - Y_t^\kappa) \leqslant -\varepsilon\right) &= \mathbb{P}\left(\inf_{s\in[0,\delta]} U_s^\kappa \leqslant -\kappa\right) \\ &\leqslant \mathbb{P}\left(\inf_{s\in[0,\delta]} W_s \leqslant -c_1(\varepsilon-\delta)\right) = \Phi(-c_1\delta^{-1/2}(\varepsilon-\delta))\end{aligned}$$

where Φ is the normal c.d.f. We need that $\varepsilon - \delta > 0$ and the convergences

$$\delta^{-1/2}(\varepsilon(\delta) - \delta) \to \infty, \quad \varepsilon(\delta) \to 0.$$

Choosing $\varepsilon(\delta) = \delta^{1/2}\log(1/\delta)$ gives the desired convergences and we get (3.8.6). This completes the tightness proof. □

To show weak convergence, it is now sufficient to prove that the limit points solve the respective SPDE for the reflecting and absorbing cases. To do this, we again rely on the martingale approach described in Section 3.4.2. Consider the maps $M^\phi(\nu^\kappa), S^\phi(\nu^\kappa)$ and $C^\phi(\nu^\kappa)$ defined in (3.4.7). We need to show that, as we take the limit $\kappa \to 0/\infty$, we obtain the corresponding quantities for the reflecting and absorbing cases, respectively.

Proposition 3.8.2. *Fix $t \leqslant T$ and define $\Psi_h : \{\xi \in D_{\mathcal{S}'} : \xi_s \in \mathbf{M}_{\leqslant 1}(\mathbb{R})\} \times \mathcal{S} \to \mathbb{R}$ by*

$$\Psi_h(\xi, \phi) := \int_0^t \langle \xi_s, h(s,\cdot)\phi\rangle ds \tag{3.8.8}$$

with h denoting a placeholder for μ, σ^2 or $\rho\sigma$. If $\nu^\kappa \to \nu^*$ weakly on $(D_{\mathcal{S}'}, M1)$ and $\phi^\kappa \to \phi^*$ in $\mathcal{S}$, then we also have the convergence $\Psi_h^\kappa \to \Psi_h^*$ weakly on $\mathbb{R}$.

Proof. We show the result for the case $\kappa \to 0$. The other case follows using the same line of argument. We fix a bounded Lipschitz function $f \in Lip(\mathbb{R})$. By Lemma 3.8.1 and the resulting relative compactness we can find a weakly convergent subsequence, again denoted by $(\nu^\kappa, \phi^\kappa)$. By Skorokhod representation we can assume almost sure convergence and use the triangle inequality and the Lipschitzness of f to obtain

$$\begin{aligned}|\mathbb{E}[f(\Psi_h^\kappa)] - \mathbb{E}[f(\Psi_h^*)]| &\leqslant C\Bigg(\mathbb{E}\left[\left|\int_0^t \langle \nu_s^\kappa - \nu_s^*, h(s,\cdot)\phi^*\rangle ds\right|\right] \\ &\quad + \mathbb{E}\left[\left|\int_0^t \langle \nu_s^\kappa, h(s,\cdot)\phi^* - h(s,\cdot)\phi^\kappa\rangle ds\right|\right]\Bigg) \\ &=: C\left(\mathbb{E}[I_1^\kappa] + \mathbb{E}[I_2^\kappa]\right)\end{aligned}$$

for some $C > 0$. We can treat the first term as in [41, Proposition 4.7]. For the second term involving I_2^κ, using that h is bounded and ν_s^κ is a sub-probability measure gives

$$I_2^\kappa \leqslant c \left\| \phi^* - \phi^\kappa \right\|_\infty \to 0 \quad \text{as } \kappa \to 0.$$

This completes the proof for convergence to the reflecting case. The same arguments show the result for the case $\kappa \to \infty$. □

Proof of Theorem 3.2.9. Combining the martingale approach we used in Section 3.4.2 with the results of Lemma 3.8.1 and Proposition 3.8.2 it follows that ν^κ converges to a solution to the reflecting or absorbing SPDE when taking the limit $\kappa \to 0$ and $\kappa \to \infty$, respectively. The necessary regularity results follow in the same way as in Section 3.4.3. We can then conclude weak convergence from the uniqueness result of Theorem 3.2.8 for the reflecting case and [41, Theorem 2.6] for the absorbing case. □

3.9 Application to Mortgage-Backed Securities

As discussed in the introduction, one motivation to consider a reflecting boundary comes from mortgage-backed securities guaranteed by government- sponsored enterprises (GSE) such as Fannie Mae and Freddie Mac. Holders of mortgage-backed securities pay a so-called *guaranty fee* for this insurance by a GSE. One important question then becomes how a fair guaranty fee can be calculated. This fee should reflect how much money the GSE expects to inject into the system if the mortgage owner is unable to pay. In our modeling approach, this is related to the local time process L as it is the term that needs to be added to keep the process in our model within the positive half-line.

Under simplifying assumptions we can derive a formula for the guaranty fee. Recall that we can express the local time at 0 as follows

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{[0,\varepsilon]}(X_s) d[X]_s.$$

If we assume constant coefficients

$$L_t = \sigma^2 \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{[0,\varepsilon]}(X_s) ds.$$

We first want to compute the conditional expectation of this local time given the common noise

$$\mathbb{E}\left[L_t | \mathcal{F}_t^{W^0} \right] = \sigma^2 \mathbb{E}\left[\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{[0,\varepsilon]}(X_s) ds \middle| \mathcal{F}_t^{W^0} \right].$$

Assuming we can interchange the order of integration and limit we obtain

$$\mathbb{E}\left[L_t|\mathcal{F}_t^{W^0}\right] = \int_0^t \lim_{\varepsilon\to 0} \frac{\nu_s(0,\varepsilon)}{\varepsilon} ds$$

where we used the probabilistic representation of the measure ν_s in the case of a reflecting boundary. We then have

$$\begin{aligned}\mathbb{E}\left[L_t|\mathcal{F}_t^{W^0}\right] =& \sigma^2 \int_0^t \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^\varepsilon V_s(x) dx ds \\ =& \sigma^2 \int_0^t V_s(0) ds\end{aligned}$$

assuming continuity of the density up to the boundary. Taking expectations yields

$$\mathbb{E}\left[L_t\right] = \sigma^2 \int_0^t \mathbb{E}\left[V_s(0)\right] ds.$$

The regularity assumptions we have to impose on the measures ν_s to obtain this expression for the expected local time relate to the regularity issues we discussed in Section 3.6.1.

For a fair value of the guaranty fee c, the present value of the fee payments has to be equal to the amount of money put into the system to keep it solvent. We assume that the payment dates for the fee are given by $t_1, \ldots, t_m = T$ and the risk-free interest rate is denoted by r. The present value of the fees is then given by

$$PV(\text{fees}) = \mathbb{E}\left[\sum_{i=1}^m e^{-\int_0^{t_i} r_u du} c\right]$$

where we assume that we take expectation under the pricing measure. The payment from the GSE is a continuous stream of payments. Its present value is

$$PV(\text{GSE Payment}) = \mathbb{E}\left[\int_0^T e^{-\int_0^s r_u du} dL_s\right].$$

Setting the two present values equal we find the fair rate c

$$c = \frac{\mathbb{E}\left[\int_0^T e^{-\int_0^s r_u du} dL_s\right]}{\mathbb{E}\left[\sum_{i=1}^m e^{-\int_0^{t_i} r_u du}\right]}.$$

Under the simplifying assumption of a constant interest rate we can use the expression for the expected local time to write the fee in terms of the density at zero

$$c = \frac{\int_0^T e^{-rs}\sigma^2 \mathbb{E}\left[V_s(0)\right] ds}{\sum_{i=1}^m e^{-rt_i}}. \tag{3.9.1}$$

The difficulty in computing this fee numerically mainly comes form the term $\mathbb{E}[V_s(0)]$. Assuming we can find a method to solve the equation for the density given in (3.1.4)-(3.1.5) for a fixed path of a Brownian motion and $\kappa = 0$, we can use a Monte-Carlo approach to approximate the expectation. The steps are then the following

1. Choose a number M of Monte-Carlo paths and a grid $t_0 < t_1, \ldots < t_m$ to discretize the time variable
2. Simulate M paths of the Brownian motion $W^{0,k}$, $k = 1, \ldots, M$
3. find the solution V^k to the equation (3.1.4)-(3.1.5) with path $W^{0,k}$ for $k = 1, \ldots, M$
4. calculate
$$\mathbb{E}\left[V_{t_j}(0)\right] \approx \frac{1}{M} \sum_{k=1}^{M} V_{t_j}^k(0), \qquad \text{for } j = 0, \ldots, m$$
5. use a numerical integration method to find the numerator in 3.9.1.

Chapter 4

Particle Systems with Sticky Boundary

4.1 Introduction

We study a particle system with common noise on the positive half-line. When hitting the boundary at zero the particles experience a sticky reflection. We prove weak convergence of the empirical-measure process of this particle system to a process that is characterized as the unique solution to an SPDE. Related particle models with absorbing boundaries have been studied in the past to model large credit portfolios such as mortgage-backed securities [1, 14, 37] and as models of systemic risk in the banking sector [41]. In Chapter 3 the assumption of immediate default after the first hitting time at zero is relaxed by considering models with elastic boundaries and the results therein also apply to a boundary with instantaneous reflection. This work completes the study of the most common boundary conditions by examining sticky reflection at the boundary. The financial motivation for a sticky boundary comes from mortgage-backed securities issued by US government-sponsored enterprises (GSE) Fannie Mae and Freddie Mac. In our model, each particle represents the financial health of a mortgage and the particle system a mortgage-backed security. The first hitting time at zero can be seen as the first time payments on the mortgage are missed. In the case of mortgage-backed securities issued by Fannie May or Freddie Mac, the GSE insures payments for these securities if mortgage payments are missed. Thus, in case of missed payments the mortgages do not get removed from the security. In our particle system this is modeled by replacing the assumption of an absorbing boundary by a reflecting one. This means that if a particle hits the boundary, i.e. if a payment is missed, the particle does not vanish from the system but gets reflected at the boundary.

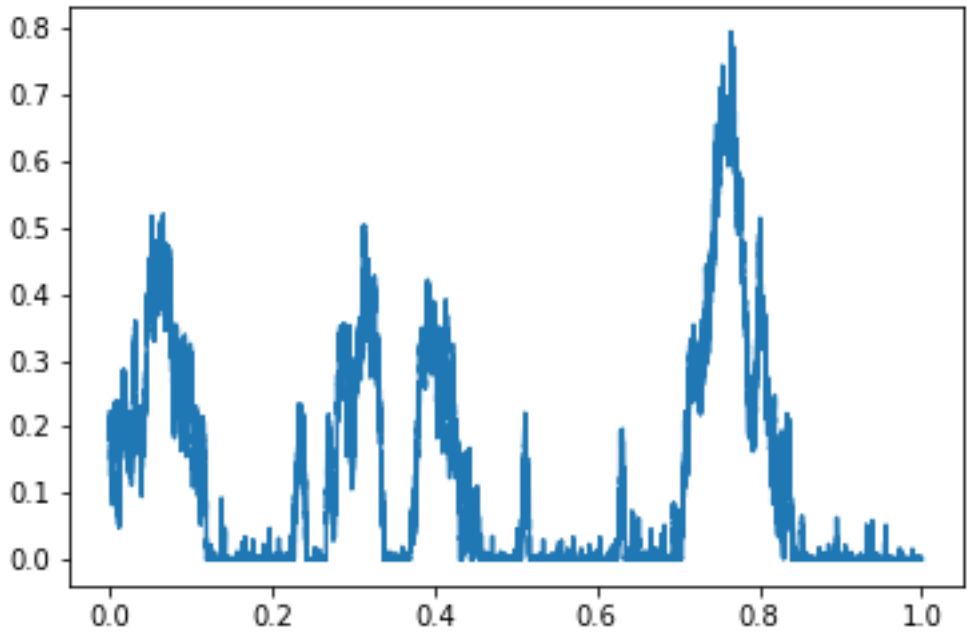


Figure 4.1: Sticky Brownian Motion Path

It is likely that a home buyer who missed payments is likely to stay in economic distress for a longer time. Thus, a sticky reflection may be a better model than the instantaneous reflection in the case of a Neumann boundary.

Sticky Brownian motion with stickiness parameter $c > 0$ is characterized through the domain of its generator which consists of bounded $C^2[0, \infty)$-functions ϕ satisfying the boundary condition

$$\phi'(0) = c\phi''(0).$$

The condition was introduced by Feller [31] in his search for the most general boundary conditions for diffusion processes. Compared to a process with instantaneous reflection the sticky particle gets slowed down when it hits the boundary. For a sticky process X the set of time spent at the boundary $A := \{t \in [0, T] : X_t = 0\}$ has positive Lebesgue measure but the process does not spend positive time at the boundary at any visit. Thus, the set A is a random generalized Cantor set. Itô and McKean [43] studied the sample paths of sticky processes and characterized them as time-changed reflecting paths. Figure 4.1 shows an example of a path of a sticky Brownian motion.

Chitashiville [17] and Engelbert and Peskir [28] showed that sticky Brownian motion is the unique weak solution to a sticky SDE as well as the fact that there does not exist a strong solution to the SDE. This fact makes the analysis of sticky processes more complicated as we have to account for additional randomness in the SDE. Building on the work for reflecting particles in [69], Graham [34] showed existence of a particle system with sticky boundary in bounded domains in $\mathbb{R}^d$ where the particles are driven by independent Brownian motions. Furthermore, Graham and Metivier [35] proved convergence of this particle system to the solution to a McKean–Vlasov equation. We add to this list of results in the one-dimensional case of the positive half-line by adding a common noise to the particle system.

We show weak convergence by following the general framework used in [37, 41]. After establishing probabilistic estimates on the particles we prove tightness in $D_{\mathcal{S}'}$, the space of càdlàg functions taking values in the space of tempered distributions equipped with the M1 topology and show that limit points are solutions to an SPDE. Then, we use energy estimates in H^{-1}, the dual space of the Sobolev space H^1, to obtain uniqueness of solutions to the SPDE in a class of processes with sufficient regularity and conclude weak convergence. We employ a mollification with the sticky heat kernel to find these estimates. While the overall structure of the weak convergence proof is similar to the cases of absorbing or elastic boundaries, the individual steps need new ideas. One source of difficulty is the fact that the distributions of sticky diffusion processes possess a point mass at the boundary. We regularly employ the decomposition of the marginal distributions into the point mass at zero and a part in the interior of the positive half-line. An important property that is used in the uniqueness proof is the boundary decay of the interior part of the SPDE solution. To show the necessary boundary decay of the finite particle limit we employ estimates involving two particles as has been done in [14]. We then rely on results of Warren [73] on the joining of two sticky Brownian motions to find the required regularity.

Model

We fix a finite time horizon $T > 0$ and consider a system of N particles $X^{i,N}, i = 1, \dots, N$ following dynamics given by

$$dX_t^{i,N} = \frac{1}{2c} 1_{X_t^{i,N}=0} dt + 1_{X_t^{i,N}>0} \left(\mu(X_t^{i,N}, \nu_t^N) dt + \sigma(X_t^{i,N})(\rho dW_t^0 + \sqrt{1-\rho^2} dW_t^i) \right), \tag{4.1.1}$$

where $W^0, W^1, \ldots, W^N$ are independent one-dimensional Brownian motions and $t \in [0,T]$. The dynamics in (4.1.1) display the typical form of a sticky SDE. The first term in (4.3.4) is a result of the sticky boundary. The boundary causes a change of speed of the particle at zero. The coefficient c controls the speed at which the particle leaves the boundary again. The change of speed at zero is then connected to c and the speed at the boundary. This is related to the boundary condition of the generator of a sticky process involving a relation between the first and second derivative of a function at the boundary.

When considering absorbing boundaries ([37, 41]) or elastic boundaries (Chapter 3), the cited works consider particles where μ and σ are allowed to depend on time t. When studying sticky processes here we do not consider time-dependent μ and σ. The main reason for this is the fact that this simplifies the probabilistic estimates we establish in Section 4.3. Therein we can use estimates involving the sticky heat kernel and do not have to consider time-dependent stickiness parameters. The extension to time-dependent coefficients and the study of sticky processes with time-dependent stickiness parameter is an interesting direction for future research.

We define the empirical-measure process ν^N of the particle system by

$$\nu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}.$$

We impose the following assumptions on our model

Assumption 4.1.1 (Initial Condition)**.** *The initial condition ν_0 is supported on $(0,\infty)$, and satisfies*

$$\nu_0(\lambda,\infty) = o(\exp(-\alpha\lambda)) \quad \text{as } \lambda \to \infty$$

for every $\alpha > 0$. Moreover, we assume that ν_0 has an L^2-density V_0.

Assumption 4.1.2 (Regularity in Space)**.** *The coefficients μ, σ and ρ have the following regularity*

(i) μ and σ are in $C^2([0,\infty))$ and there exists a constant $C > 0$ such that

$$|\partial_x^n \mu(x,\nu)|, |\partial_x^n \sigma(x)| \leqslant C$$

for all $x \in \mathbb{R}, \nu \in \mathcal{P}([0,\infty))$ and $n = 0,1,2$,

(ii) for all $x \in \mathbb{R}$

$$\sigma(x) \geqslant C^{-1} > 0, \quad 0 < \rho < 1$$

Assumption 4.1.3 (Regularity in the Measure Variable)**.** *The drift coefficient μ is Lipschitz continuous in the measure variable ν with respect to the bounded Lipschitz distance on the space of probability measures away from the boundary and does not depend on ν near the boundary, i.e. there exist constants $\lambda, c > 0$ such that for all $x \geqslant \lambda$*

$$|\mu(x,\nu) - \mu(x,\bar{\nu})| \leqslant c d_0(\nu,\bar{\nu})$$

with d_0 given by

$$d_0(\nu,\bar{\nu}) \coloneqq \sup\{|\langle \nu - \bar{\nu}, \psi\rangle| : \|\psi\|_{Lip} \leqslant 1, \|\psi\|_\infty \leqslant 1|\}$$

and for all $x \in [0,\lambda)$

$$|\mu(x,\nu) - \mu(x,\bar{\nu})| = 0.$$

This means we allow the drift coefficient to depend on the empirical measure of our particle system as long as the particle is away from the boundary, while dependence on x is allowed everywhere. An example of a form of such a drift coefficient is given by

$$\mu(x,\nu) = c(x) + b(x)\Phi(\nu)$$

where c is a bounded C^2-function with bounded derivatives, Φ is a map that is Lipschitz continuous with respect to the bounded Lipschitz distance d_0 on the space of measures and b is a bounded C^2-function with bounded derivatives such that $b(x) = 0$ for $x \in [0,\lambda)$.
We will see that for technical reasons, our uniqueness proof requires that we have no dependence on the measure near the boundary. Similar to Chapter 3 the problem is a lack of L^2-regularity of the densities of the measures of the limit process restricted to the interior. When analysing the regularity of limit points of the sequence of empirical-measure processes we will discuss this in more detail.

As the number of particles N grows to ∞ we will see that the empirical measure process ν^N converges to the solution of the limit SPDE

$$\begin{aligned}\langle \nu_t, \phi\rangle = \langle \nu_0, \phi\rangle &+ \int_0^t \langle \nu_s, \mu(\cdot,\nu_s)\phi'\rangle ds + \frac{1}{2}\int_0^t \langle \nu_s, \sigma^2(\cdot)\phi''\rangle ds \\ &+ \int_0^t \rho\langle \nu_s, \sigma(\cdot)\phi'\rangle dW_s^0 - \int_0^t \rho\sigma(0)\phi'(0)\nu_s(\{0\})dW_s^0 \\ &+ \int_0^t \left[\left(\frac{1}{2c} - \mu(0,\nu_s)\right)\phi'(0) - \frac{1}{2}\sigma^2(0)\phi''(0)\right]\nu_s(\{0\})ds\end{aligned} \tag{4.1.2}$$

for $t \in [0,T]$ and $\phi \in \mathcal{S}$, the Schwartz space of rapidly decaying smooth functions. We will often work with an alternative, more compact formulation of the SPDE. We

denote by $\tilde{\nu}_t$ the restriction of ν_t to the interior of the positive half-line, i.e. we set $\tilde{\nu}_t := \nu_t - \nu_t(\{0\})\delta_0$, where δ_0 denotes the Dirac measure at zero. The limit SPDE can then be expressed as

$$\begin{aligned}\langle \nu_t, \phi \rangle = \langle \nu_0, \phi \rangle + \int_0^t \langle \tilde{\nu}_s, \mu(\cdot, \nu_s)\phi' \rangle ds + \frac{1}{2}\int_0^t \langle \tilde{\nu}_s, \sigma^2(\cdot)\phi'' \rangle ds \\ + \int_0^t \rho \langle \tilde{\nu}_s, \sigma(\cdot)\phi' \rangle dW_s^0 + \frac{\phi'(0)}{2c}\int_0^t \nu_s(\{0\})ds\end{aligned} \tag{4.1.3}$$

for $t \in [0, T]$ and $\phi \in \mathcal{S}$.

Main Results

We first define a class of stochastic processes $\mathbb{D}$. We will show that limit points of the empirical measure process are in this class and our uniqueness result will be limited to this class. However, it is sufficient to conclude weak convergence.

Definition 4.1.4. We define $\mathbb{D}$ to be a class of càdlàg processes $(\nu_t)_{t\in[0,T]}$ taking values in $\mathcal{P}(\mathbb{R})$, the space of probability measures on $\mathbb{R}$, and that satisfy

(i) for all $t \in [0, T]$, ν_t is supported on $[0, \infty)$,

(ii) for every $a > 0$, as $\lambda \to \infty$

$$\mathbb{E}\left[\int_0^T \nu_t(\lambda, \infty)dt\right] = o(\exp(-a\lambda)),$$

(iii) there exists a constant $\gamma > 0$ such that

$$\mathbb{E}\left[\int_0^T (\tilde{\nu}_t(0, \varepsilon))^2 dt\right] = O(\varepsilon^{1+\gamma})$$

as $\varepsilon \to 0$,

(iv) there exist constants $C > 0, \delta > 0$ such that

$$\mathbb{E}\left[\int_0^T \nu_t(a, b)dt\right] \leqslant C(b - a)^\delta$$

for $0 < a < b$.

Our first main result is the existence of limit points of the sequence of empirical-measure processes and the fact that these limit points are solutions to the limit equation (4.1.2) in the class $\mathbb{D}$. We consider our processes to be elements in $D_{\mathcal{S}'}$, the space of càdlàg paths with values in the space of tempered distributions $\mathcal{S}'$. Following

the approach in [37, 41] we equip this space with the M1-topology. Moreover we denote by $(C_{\mathbb{R}}, \|\cdot\|_\infty)$ the space of continuous functions with the topology of uniform convergence. We can then formulate the following theorem.

Theorem 4.1.5. *Let (ν^N, W^0) be a sequence with ν^N denoting the empirical-measure process corresponding to the particle system (4.1.1) and assume that Assumptions 4.1.1-4.1.3 are satisfied. Then (ν^N, W^0) possesses weakly converging subsequences in $(D_{\mathcal{S}'}, M1) \times (C_{\mathbb{R}}, \|\cdot\|_\infty)$. Moreover, the limit points are solutions to the SPDE (4.1.2) and elements of the class $\mathbb{D}$.*

The second main result is a uniqueness statement for the limit SPDE (4.1.2). This yields well-posedness of the equation (4.1.2) in the class $\mathbb{D}$. In addition, it establishes weak convergence of the empirical-measure process of the particle system because the uniqueness implies that there is only one limit point.

Theorem 4.1.6. *Assume that (ν, W^0) and $(\bar{\nu}, \bar{W}^0)$ are solutions to the SPDE (4.1.2) and that ν and $\bar{\nu}$ are in the class $\mathbb{D}$. Then we have*

(i) pathwise uniqueness if $W^0 = \bar{W}^0$, i.e.

$$\nu_t(S) = \bar{\nu}_t(S) \quad \text{for every } t \in [0,T] \text{ and every Borel set } S \subseteq \mathbb{R},$$

(ii) uniqueness in law, i.e. $\mathcal{L}(\nu, W^0) = \mathcal{L}(\bar{\nu}, \bar{W}^0)$.

Outline

In Section 4.2 we show how the particle system (4.1.1) can be constructed. Section 4.3 contains a series of probabilistic estimates for the particles $X^{i,N}$ and Section 4.4 the proof of Theorem 4.1.5. Uniqueness in the case without a nonlinear interaction term in the drift is proved in Section 4.5. In Section 4.6 we show several regularity results including the existence of a density and an L^2-regularity result that is then used in Section 4.7 to establish uniqueness in the nonlinear case and conclude weak convergence.

4.2 Particle System Construction

It is not obvious how to construct the particle system with sticky boundary. We will follow the approach by Graham presented in [34]. The classical techniques of time-change to obtain existence and uniqueness as used in [28] do not work to construct

a solution to the particle system. The reason for this is that if we use time-change on each particle separately their time-scales will not coincide any more and we lose temporal coherence of the system. If we use a global time-change, it will perturb the equation of a particle inside the domain far too much as soon as another particle is on the boundary.

We want to construct a system of N particles X^i given by

$$\begin{cases} dX_t^i = \mu(X_t^i, \nu_t^N)d\langle M^i\rangle_t + \sigma(X_t^i)dM_t^i + dL_t(X_t^i) \\ 2cL_t(X_t^i) = \int_0^t 1_{X_s^i=0}ds. \end{cases} \tag{4.2.1}$$

with $dM_t^i = 1_{X_t^i>0}(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i)$.

Proposition 4.2.1. *Given the Assumption 4.1.1-4.1.3. There exists a weak solution to the particle system with dynamics (4.2.1).*

Proof. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a right-continuous and complete filtration $(\mathcal{F}_t)$ and $N+1$ independent one-dimensional Brownian motions $W^0, W^1, \dots, W^N$. For $\varepsilon > 0$ we are going to define recursively an increasing sequence of stopping times T_n, a sequence of subsets J_n of $\{1, \dots, N\}$ and the processes X^i, L^i, M^i between T_n and T_{n+1}. To particle X^i we associate the clock

$$C_t^i = 2cL_t(X^i).$$

Set $T_0 = 0, J_0 = \emptyset$ and suppose we have defined $T_0, \dots, T_N, J_0, \dots, J_N$ and X^i, L^i, M^i up to T_n. Then, starting at T_n we freeze X^j, L^j, M^j for $j \in J_n$ and let the other particles X^i for $i \notin J_n$ move following the equation with $c = 0$, starting at $X_{T_n^i}$ with

$$M_{T_n+s}^i - M_{T_n}^i = \rho(W_{T_n+s}^0 - W_{T_n}^0) + \sqrt{1-\rho^2}(W_{T_n+s}^i - W_{T_n}^i).$$

The existence of these processes follows by the results for the reflecting case in [69]. We then set

$$\varepsilon_t^i = C_t^i - \int_0^t 1_{X_s^i=0}ds$$

and

$$T_{n+1} = \inf\{t \geqslant T_n, \exists i \notin J_n, \varepsilon_t^i \geqslant \varepsilon, \text{ or } \exists j \in J_n, \varepsilon_t^i \leqslant 0\},$$

$$J_{n+1} = \{i \notin J_n, \varepsilon_t^i = \varepsilon\} \cup \{j \in J_n, \varepsilon_t^j \neq 0\}.$$

Note that $0 \leqslant \varepsilon_t^i \leqslant \varepsilon, \varepsilon^i$ is a process of bounded variation with $|\varepsilon_t^i|_t \leqslant 2t + \varepsilon$ and that for $j \in J_n, X_{T_n}^j = 0$. Whenever a particle X^i has not spent enough time on the

boundary, that is whenever ε_t^i reaches ε, we stop it and wait before freeing it again (and ε_t^i will have fallen to 0). We have $T_{n+2N} > T_n + \varepsilon$ so that $\lim T_n = +\infty$. Thus, we have defined a process on $\mathbb{R}_+$. The constructed system solves (4.2.1) with the second equation replaced by

$$2cL_t(X^{i,\varepsilon}) = \int_0^t 1_{X_s^{i,\varepsilon}=0} ds + \varepsilon_t^i.$$

Thus, the particle dynamics are

$$\begin{cases} dX_t^{i,\varepsilon} = \mu(X_t^{i,\varepsilon}, \nu_t^{N,\varepsilon}) 1_{X_t^{i,\varepsilon}>0} dt + \sigma(X_t^{i,\varepsilon}) 1_{X_t^{i,\varepsilon}>0} (\sqrt{1-\rho^2} dW_t^i + \rho dW_t^0) + dL_t(X^{i,\varepsilon}) \\ 2cL_t(X_t^i) = \int_0^t 1_{X_s^{i,\varepsilon}=0} ds + \varepsilon_t^i \end{cases}$$

for $i = 1, \dots, N$ with empirical measure

$$\nu_t^{N,\varepsilon} = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,\varepsilon}}.$$

Define for $f \in C_b^2([0,\infty)), x \in [0,\infty)$ and $p \in \mathcal{P}([0,\infty))$ the differential operator

$$Lf(x,p) = \frac{1}{2}\sigma^2(x)\partial_x^2 f(x) + \mu(x,p)\partial_x f(x).$$

Then the law of our constructed system will solve the N martingale problems

$$M_t^{f,i,\varepsilon} = f(X_t^{i,\varepsilon}) - f(X_0^i) - \int_0^t Lf(X_s^{i,\varepsilon}, \nu_t^{N,\varepsilon})(ds - 2cdL_s(X^{i,\varepsilon}) + d\varepsilon_s^i) - \partial_x f(0) L_t(X^{i,\varepsilon}) \tag{4.2.2}$$

is a martingale with

$$d\langle M_t^{f,i}, M_t^{f,j}\rangle = \rho^2 \partial_x f(X_t^i)\partial_x f(X_t^j)\sigma(X_t^i)\sigma(X_t^j)dt, \quad \text{if } i \neq j \tag{4.2.3}$$

and

$$ds - 2cdL_s(X^{i,\varepsilon}) + d\varepsilon_s^i \geqslant 0.$$

We can estimate

$$\begin{aligned} \mathbb{E}|X_t^{i,\varepsilon} - X_s^{i,\varepsilon}|^4 &\leqslant C\left(\mathbb{E}|\int_s^t \mu(X_u^{i,\varepsilon}, \nu_u^{N,\varepsilon})du|^4 + \mathbb{E}|\int_s^t \sigma(X_u^{i,\varepsilon})dB_u^i|^4 + |t-s|^4 + |\varepsilon_t^i - \varepsilon_s^i|^4\right) \\ &\leqslant O(|t-s|^2) \quad \text{as } |t-s| \to 0 \end{aligned}$$

by using Burkholder-Davis Gundy inequality [64, Thm. IV.42.1] to obtain

$$\mathbb{E}|\int_s^t \sigma(X_s^{i,\varepsilon})dB_u^i|^4 = O(|t-s|^2), \quad \text{as } |t-s| \to 0$$

and the local time regularity. Using [42] we obtain tightness for $(X^{i,\varepsilon}, L^{i,\varepsilon})$ for $0 < \varepsilon < 1$. We can then use the continuity of the coefficients, $0 \leqslant \varepsilon_t^i \leqslant \varepsilon, |\varepsilon_t^i| \leqslant \varepsilon + 2t$ and [34] Proposition I.17 to take the limit $\varepsilon \to 0$. A limit point of this sequence solves the martingale problem (4.2.2) with $\varepsilon^i = 0$ and $ds \geqslant 2cdL_s(X^i)$. The law solving this martingale problem is a weak solution to the particle system (4.2.1).

□

4.3 Probabilistic Estimates for Particles

In this section we show a series of probabilistic estimates that will be used in Section 4.4.2 to prove the regularity of our SPDE solution. As a first step, we note that all particles in the system $(X^{i,N})_{1 \leqslant i \leqslant N}$ are identically distributed. As a result, we obtain for an arbitrary measurable set $S \subseteq \mathbb{R}, N \geqslant 1$ and $t \in [0, T]$

$$\mathbb{E}\left[\nu_t^N(S)\right] = \frac{1}{N}\mathbb{E}\left[\sum_{i=1}^{N} 1_{X_t^{i,N} \in S}\right] = \mathbb{P}(X_t^{i,N} \in S). \tag{4.3.1}$$

We show how we can use a scale transformation and a change of measure to estimate (4.3.1) in terms of the distribution of a sticky Brownian motion.

Lemma 4.3.1 (Scale transformation)**.** *Let $X^{1,N}$ be a sticky particle with dynamics given in (4.1.1). Define the scale transform $\zeta(x) = \int_0^x \frac{dy}{\sigma(y)}$. Then the process $Z_t := \zeta(X_t^{1,N})$ is given by*

$$dZ_t = \frac{1}{2c\sigma(0)} 1_{Z_t = 0} dt + 1_{Z_t > 0}\left(\tilde{\mu}(X_t^{1,N}, \nu_t^N)dt + dB_t^1\right) \tag{4.3.2}$$

with Brownian motion B^1 given by

$$B_t^1 = \rho W_t^0 + \sqrt{1-\rho^2} W_t^1$$

and uniformly bounded drift

$$\tilde{\mu}(X_t^{1,N}, \nu_t^N) = \left(\frac{\mu}{\sigma} - \partial_x \sigma\right)(X_t^{1,N}, \nu_t^N).$$

Proof. Equation (4.3.2) follows from an application of Itô's lemma. The coefficient $\tilde{\mu}$ is uniformly bounded by Assumption 4.1.2 on μ and σ. □

The next step is to remove the drift and obtain a sticky Brownian motion.

Lemma 4.3.2. *For every $\delta \in (0,1)$ there exists a constant $c_\delta > 0$ such that*

$$\mathbb{P}(X_t^{1,N} \in S) \leqslant c_\delta F_t^{c\sigma(0)}(\zeta(S))^\delta, \quad \text{for every measurable } S \subseteq \mathbb{R},$$

where $F_t^{c\sigma(0)}$ is the marginal law of a sticky Brownian motion with stickiness parameter $c\sigma(0)$ at time t.

Proof. We consider the process Z defined in (4.3.2) and a change of measure via the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t = \exp\left(-\int_0^t \tilde{\mu}(X_s^{1,N}, \nu_s^N)dB_s^1 - \int_0^t \frac{\tilde{\mu}^2}{2}(X_s^{1,N})ds\right). \tag{4.3.3}$$

By Girsanov's Theorem the process $B_t^{\mathbb{Q}} = B_t^1 + \int_0^t \tilde{\mu}(X_s^{1,N}, \nu_s^N)ds$ is a Brownian motion under the measure $\mathbb{Q}$. We have

$$dZ_t = \frac{1}{2c\sigma(0)} 1_{Z_t=0} dt + 1_{Z_t>0} dB_t^{\mathbb{Q}}. \tag{4.3.4}$$

This means that Z is a sticky Brownian motion under the measure $\mathbb{Q}$ with stickiness parameter $c\sigma(0)$. Following the arguments in [37, Lemma 4.2] completes the proof. □

We use this result to find a series of probabilistic estimates.

Proposition 4.3.3. *There exist constants $c > 0$ and $\delta \in (0,1)$ such that*

$$\mathbb{E}\left[\int_0^T \nu_t^N(a,b)dt\right] \leqslant c(b-a)^\delta$$

for all $0 < a < b$ and all $N \geqslant 1$.

Proof. By Lemma 4.3.2 we have

$$\mathbb{E}\left[\nu_t^N(a,b)\right] \leqslant c_\delta F_t^{c\sigma(0)}(\zeta((a,b)))^\delta.$$

for some $\delta \in (0,1)$. Set $\tilde{c} := c\sigma(0)$. Note that $\zeta((a,b)) \subseteq (\zeta(a), \zeta(b))$. We can apply Lemma B.3 to estimate the sticky heat kernel $G_t^{S,\tilde{c}}$, defined in (B.3), via

$$G_t^{S,\tilde{c}}(x,y) \leqslant 3p_t(x-y)$$

where p_ε denotes the Gaussian heat kernel defined in (B.1). Using this together with the estimate

$$|\zeta(b) - \zeta(a)| \leqslant C|b-a|,$$

for a constant $C > 0$, for the scale transformation ζ we get

$$F_t^{\tilde{c}}(\zeta((a,b))) \leqslant \mathbb{P}(\zeta(a) < B_t^{\text{sticky}} < \zeta(b)) \leqslant 3\int_{\zeta(a)}^{\zeta(b)} p_t(x-y)dy \leqslant 3Ct^{-1/2}(b-a).$$

This yields the estimate

$$\int_0^t \mathbb{E}\left[\nu_t^N(a,b)\right] dt \leqslant c(b-a)^\delta, \quad \delta \in (0,1)$$

because the map $t \mapsto t^{1/2}$ is integrable. □

Proposition 4.3.4. *For any $q > 1$ and $t \in [0,T]$ we have that for $i,j = 1,\dots,N$ with $i \neq j$*

$$\mathbb{P}(0 < X_t^i < \varepsilon, 0 < X_t^j < \varepsilon) = t^{-\frac{1}{(1+\rho)q}} O(\varepsilon^{\frac{2}{q(1+\rho)}}).$$

Proof. Following the approach in Lemma 4.3.2, we can apply the scale transformation and a change of measure to obtain

$$\mathbb{P}(0 < X_t^i < \varepsilon, 0 < X_t^j < \varepsilon) \leqslant \tilde{C}\mathbb{Q}(0 < Y_t^i < C\varepsilon, 0 < Y_t^j < C, \varepsilon)^{\frac{1}{q}} \tag{4.3.5}$$

for $q > 1$ and a constant $\tilde{C} > 0$, where Y^i, Y^j are sticky Brownian motions. We can then follow the arguments in [73] and use Lemma 5 therein as our correlation is strictly bounded away from 1, $\rho < 1$. We then have that for two sticky Brownian motions Y^1 and Y^2 and a bounded path functional Φ the hypercontractivity result from [71, Lemma 6.5] holds and we get

$$\mathbb{E}\left[\Phi(Y^1)\Phi(Y^2)\right] \leqslant \mathbb{E}\left[\Phi(Y)\right]^{2/(1+\rho)}$$

where Y is another sticky Brownian motion.
The estimate

$$\mathbb{E}\left[1_{0<Y_t<\varepsilon}\right] = t^{-1/2}O(\varepsilon)$$

for the sticky Brownian motion (see [73, Proof of Theorem 6]) together with the hypercontractivity yield the desired estimate

$$\mathbb{E}\left[1_{0<Y_t^1<\varepsilon}1_{0<Y_t^1<\varepsilon}\right] = t^{-\frac{1}{1+\rho}}O(\varepsilon^{\frac{2}{1+\rho}}). \tag{4.3.6}$$

Combining (4.3.5) and (4.3.6) yields the result.

□

Proposition 4.3.5 (Tail Estimate)**.** *For every $\alpha > 0$ as $\lambda \to \infty$*

$$\mathbb{E}\left[\int_0^T \nu_t^N(\lambda, +\infty)dt\right] = o(\exp(-\alpha\lambda)), \quad \textit{uniformly in } N \geqslant 1 \textit{ and } t \in [0,T].$$

Proof. Let B_t^{sticky} be a sticky Brownian motion and set $\pi_0 = \nu_0 \circ \zeta^{-1}$. For any $\lambda > 0$ we can estimate

$$\begin{aligned} F_t^c((\lambda,\infty)) &= \int_0^\infty \mathbb{P}\left(B_t^{\text{sticky}} > \lambda \middle| B_0^{\text{sticky}} = x_0\right) \pi_0(dx_0) \\ &= \int_0^\infty \int_\lambda^\infty G_\varepsilon^{S,c}(x,x_0)dx\pi_0(dx_0) \\ &\leqslant 3\int_0^\infty \mathbb{P}\left(W_t > \lambda \middle| W_0 = x_0\right) \pi_0(dx_0). \end{aligned}$$

for a Brownian motion W. In the last line we used Lemma B.3 to bound the sticky heat kernel by the Gaussian heat kernel. The result then follows by [37, Proposition 4.5] and Lemma 4.3.2. □

4.4 Convergence of the Particle System

In this section we show that the sequence of empirical-measure processes $(\nu^N)_{N\geqslant 1}$ possesses limit points and that these limit points are solutions to the limit SPDE (4.1.2). Moreover, we show that limit points satisfy the regularity conditions of the class $\mathbb{D}$. We do this by establishing that the empirical-measure processes satisfy a finite evolution equation and that the sequence $(\nu^N)_{N\geqslant 1}$ is tight in the M1-topology. We then show that solutions to the finite evolution equation converge to solutions to the limit SPDE (4.1.2). The probabilistic estimates we derived in the previous section can then be used to obtain the regularity conditions of the class $\mathbb{D}$.

4.4.1 Finite Evolution Equation

The first step is to derive an evolution equation for the empirical-measure process ν^N that describes our particle system.

Proposition 4.4.1. *For all $N \geqslant 1$ the empirical measure process satisfies the evolution equation*

$$\begin{aligned} d\langle \nu_t^N, \phi\rangle =& \langle \nu_t^N, \mu(\cdot,\nu_t^N)\phi'\rangle dt + \frac{1}{2}\langle \nu_t^N, \sigma^2(\cdot)\phi''\rangle dt + \rho\langle \nu_t^N, \sigma(\cdot)\phi'\rangle dW_t^0 \\ &+ \left(\left[\frac{1}{2c} - \mu(0,\nu_t^N)\right]\phi'(0) - \frac{1}{2}\sigma^2(0)\phi'(0)\right)\nu_t^N(\{0\})dt \\ &- \rho\phi'(0)\sigma(0)\nu_t^N(\{0\})dW_t^0 + I_t^N(\phi) \end{aligned}$$

with

$$I_t^N(\phi) = \sqrt{1-\rho^2}\frac{1}{N}\sum_{i=1}^N \int_0^t \phi'(X_t^i)\sigma(X_t^i)1_{X_t^i>0}dW_t^i. \tag{4.4.1}$$

for all $\phi \in \mathcal{S}$ *where* $I_t^N(\phi) \to 0$ *as* $N \to \infty$.

Proof. Take a test function $\phi \in \mathcal{S}$ and apply Itô's formula to obtain

$$\begin{aligned} d\phi(X_t^{i,N}) =& \mu(X_t^{i,N}, \nu_t^N)\phi'(X_t^{i,N})dt + \frac{1}{2}\sigma^2(X_t^{i,N})\phi''(X_t^{i,N})dt \\ &+ \sqrt{1-\rho^2}\phi'(X_t^{i,N})\sigma(X_t^{i,N})1_{X_t^{i,N}>0}dW_t^i + \rho\phi'(X_t^{i,N})\sigma(X_t^{i,N})dW_t^0 \\ &+ \left(\left[\frac{1}{2c} - \mu(0,\nu_t^N)\right]\phi'(0) - \frac{1}{2}\sigma^2(0)\phi''(0)\right)1_{X_t^{i,N}=0}dt \\ &- \rho\phi'(0)\sigma(0)1_{X_t^{i,N}=0}dW_t^0. \end{aligned}$$

Summing over all particles and dividing by N then yields the desired evolution equation. By independence of the Brownian motions W^i we obtain

$$[I_\cdot^N(\phi)]_t = \frac{1}{N^2}(1-\rho^2)\sum_{i=1}^N \int_0^t 1_{X_t^{i,N}>0}\sigma^2(X_s^{i,N})(\phi'(X_s^{i,N}))^2 ds.$$

Boundedness of σ^2 and ϕ' then yields

$$[I_\cdot^N(\phi)]_t = \|\phi'\|_\infty^2 O(N^{-1}), \quad \text{as } N \to \infty,$$

for all $t \in [0,T]$. By Doob's martingale inequality we have the estimate

$$\mathbb{E}\left[\sup_{t\in[0,T]}|I_t^N(\phi)|^2\right] \leqslant 4\mathbb{E}\left[I_T^N(\phi)^2\right] = 4\mathbb{E}\left[\left[I_\cdot^N(\phi)\right]_T\right]$$

and we obtain the desired convergence to zero. □

4.4.2 Tightness and Convergence

The first step in establishing convergence of the particle system and proving existence of solutions to the SPDE (4.1.2) is to show tightness for the empirical measure process ν^N. We follow the approach in [37, 41] and prove tightness in $D_{\mathcal{S}'}$, the space of càdlàg functions taking values in the space of tempered distributions $\mathcal{S}'$ equipped with M1 topology. This space is discussed in detail in [50].

Proposition 4.4.2. *Let* (ν^N) *be a sequence of empirical-measure processes corresponding to the sticky particle systems (4.1.1). Then* (ν^N) *is tight on* $(D_{\mathcal{S}'}, M1)$. *Moreover, considering* (ν^N) *and the common noise* W^0 *in (4.1.1) the sequence* (ν^N, W^0) *is tight on the product space* $(D_{\mathcal{S}'}, M1) \times (C_{\mathbb{R}}, \|\cdot\|_\infty)$.

Proof. Following the arguments in [37, Proposition 5.1] it is enough to show that for any $s, t \in [0,T]$, any $N \geqslant 1$ and $\phi \in \mathcal{S}$ we have

$$\mathbb{E}\left[\left|\langle \nu_t^N, \phi\rangle - \langle \nu_s^N, \phi\rangle\right|^4\right] = O(|t-s|^2), \quad \text{as } |t-s| \to 0 \tag{4.4.2}$$

to prove tightness in the M1-topology. We can estimate

$$\begin{aligned}
&\mathbb{E}\left[\left|\langle \nu_t^N, \phi\rangle - \langle \nu_s^N, \phi\rangle\right|^4\right] \leqslant \|\phi\|_{Lip}^4 \, \mathbb{E}\left[\left|X_t^i - X_s^i\right|^4\right] \\
&\leqslant C\,\|\phi\|_{Lip}^4 \left(\mathbb{E}\left[\left|\frac{1}{2c}\int_s^t 1_{X_u^i=0} du\right|^4\right] + \mathbb{E}\left[\left|\int_s^t 1_{X_u^i>0}\left(\mu(X_u^i,\nu_t^N)du + \sigma(X_u^i)dB_u^i\right)\right|^4\right]\right) \\
&\leqslant C\left(|t-s|^4 + \mathbb{E}\left[\left|\int_s^t 1_{X_t^{i,N}>0}\mu(X_u^i,\nu_t^N)du\right|^4\right] + \mathbb{E}\left[\left|\int_s^t 1_{X_u^{i,N}>0}\sigma(X_u^i)dB_u^i\right|^4\right]\right) \\
&= O(|t-s|^4) + \mathbb{E}\left[\left|\int_s^t 1_{X_u^{i,N}>0}\sigma(X_u^i)dB_u^i\right|^4\right]
\end{aligned}$$

where we used that μ is bounded by Assumption 4.1.2. The Burkholder-Davis-Gundy inequality [64, Theorem IV.42.1] yields

$$\mathbb{E}\left[\left|\int_s^t 1_{X_u^{i,N}>0}\sigma(X_u^i)dB_u^i\right|^4\right] = O(|t-s|^2), \quad \text{as } |t-s| \to 0$$

which yields (4.4.2) and the desired tightness.

□

Tightness implies the existence of a limit point, but this might not be unique. Moreover, we have only shown that any limit point is an element in $D_{\mathcal{S}'}$. We also need to show that these points actually take values in the space of probability measures.

Proposition 4.4.3. *Let (ν^*, W) realise a limiting law. Then, ν_t^* is a probability measure supported on $[0,\infty)$ for every $t \in [0,T]$, with probability 1.*

Proof. We adapt the proof for the case of an absorbing boundary from [37] where it is established that a limit point takes values in the space of sub-probability measures. After following their argument it remains to show that $\nu_t^*(\mathbb{R}) = 1$, i.e. ν_t^* is actually a probability measure.
Take a subsequence such that $(\nu^{N_k}, W) \to (\nu^*, W)$ weakly. Fix $\phi \in \mathcal{S}$, then $\langle \nu^{N_k}, \phi\rangle \to \langle \nu^*, \phi\rangle$ weakly in $(D_{\mathbb{R}}, M1)$ and a further implication is the weak convergence

$$\sup_{t\in[0,T]} |\langle \nu_t^{N_k}, \phi\rangle| \to \sup_{t\in[0,T]} |\langle \nu_t^*, \phi\rangle|, \qquad \text{on } \mathbb{R}.$$

If we then use the characterization of weak convergence by convergence for open sets given in the portmanteau theorem [7, Theorem 2.1], we obtain

$$\mathbb{P}(\sup_{t\in[0,T]}|\langle\nu_t^*,\phi\rangle|>\|\phi\|_\infty)\leqslant\liminf_{k\to\infty}\mathbb{P}(\sup_{t\in[0,T]}|\langle\nu_t^{N_k},\phi\rangle|>\|\phi\|_\infty)=0$$

because ν^{N_k} is a probability measure. Similar arguments yield that $\langle\nu_t^*,\phi\rangle$ is non-negative for non-negative ϕ and $\langle\nu_t^*,\phi\rangle=0$ for ϕ supported on $(-\infty,0)$. This means, that ν_t^* is a positive linear functional on $\mathcal{S}$ and extends to a positive linear functional on C_0, the space of continuous and compactly supported functions on $\mathbb{R}$ with uniform topology, denoted by ξ_t. By the Riesz representation theorem [65, Theorem 2.14], for every t, there exists a regular Borel measure ζ_t such that

$$\langle\xi_t,\phi\rangle=\int_{\mathbb{R}}\phi(x)\zeta_t(dx),\qquad\text{for every }\phi\in C_0.$$

Associating ζ and ν^* yields that the process takes values in the space of sub-probability measures. It remains to show that $\nu_t^*(\mathbb{R})=1$ for every t.
For $\lambda\in\mathbb{N}$ define $\phi_\lambda\in\mathcal{S}$ to be any function satisfying $\phi_\lambda=1$ on $[-\lambda,\lambda]$, $\phi_\lambda=0$ on $(-\infty,-2\lambda)\cup(2\lambda,\infty)$ and $\phi_\lambda\in(0,1)$ otherwise. Then we have

$$\begin{aligned}\mathbb{P}(\inf_{t\in[0,T]}\nu_t^*(\mathbb{R})<1)&\leqslant\mathbb{P}(\inf_{t\in[0,T]}\langle\nu_t^*,\phi_\lambda\rangle<1)\\&\leqslant\liminf_{k\to\infty}\mathbb{P}(\inf_{t\in[0,T]}\langle\nu^{N_k},\phi_\lambda\rangle<1)\to0\qquad\text{as }\lambda\to\infty,\end{aligned}$$

where we used that ν^{N_k} is a probability measure for all k. This means that $\nu_t^*(\mathbb{R})=1$ for all $t\in[0,T]$ with probability 1. Thus, we obtain that ν^* takes values in the space of probability measures. □

The next step is to show that a limit point is a solution to the SPDE (4.1.2). We do this by following the martingale approach that has been used in [37, 41]. We define the following processes

$$\begin{aligned}M^\phi(\nu^N)(t)\coloneqq&\langle\nu_t^N,\phi\rangle-\langle\nu_0^N,\phi\rangle-\int_0^t\langle\nu_s^N,\mu(\cdot,\nu_s^N)\phi'\rangle ds-\frac{1}{2}\int_0^t\langle\nu_s^N,\sigma^2(\cdot)\phi''\rangle ds\\&-\int_0^t\left[\left(\frac{1}{2c}-\mu(0,\nu_s^N)\right)\phi'(0)-\frac{\sigma^2(0)}{2}\phi''(0)\right]\nu_s^N(\{0\})ds\qquad(4.4.3)\\S^\phi(\nu^N)(t)\coloneqq&M^\phi(\nu^N)(t)^2-\int_0^t(\langle\nu_s^N,\sigma(\cdot)\rho\phi'\rangle-\nu_s^N(\{0\})\sigma(0)\rho\phi'(0))^2ds\\C^\phi(\nu^N,W^0)(t)\coloneqq&M^\phi(\nu^N)(t)\cdot W^0(t)-\int_0^t(\langle\nu_s^N,\sigma(\cdot)\rho\phi'\rangle-\nu_s^N(\{0\})\sigma(0)\rho\phi'(0))ds.\end{aligned}$$

Moreover, for the limit points ν^*, we set

$$\begin{aligned} M_*^\phi(\nu^*)(t) \coloneqq &\langle \nu_t^*, \phi\rangle - \langle \nu_0^*, \phi\rangle - \int_0^t \langle \nu_s^*, \mu(\cdot, \nu_s^*)\phi'\rangle ds - \frac{1}{2}\int_0^t \langle \nu_s^*, \sigma^2(\cdot)\rangle ds \\ &- \frac{\phi'(0)}{2c} Z_t^* + \frac{\sigma^2(0)\phi''(0)}{2} Z_t^* + \int_0^t \mu(0, \nu_s^*) dZ_s^* \end{aligned} \tag{4.4.4}$$

$$\begin{aligned} S_*^\phi(\nu^*)(t) \coloneqq &M_*^\phi(\nu^*)(t)^2 - \int_0^t \langle \nu_s^*, \sigma(\cdot)\rho\phi'(\cdot)\rangle^2 ds + 2\rho\sigma(0)\phi'(0)\int_0^t \langle \nu_s^*, \rho\sigma(\cdot)\phi'\rangle dZ_s^* \\ &- \sigma^2(0)\phi'(0)^2\rho^2 Y_t^* \\ C_*^\phi(\nu^*, W^0)(t) \coloneqq &M_*^\phi(\nu^*)(t)\cdot W_t^0 - \left(\int_0^t \langle \nu_s^*, \rho\sigma(\cdot)\phi'\rangle ds - \rho\sigma(0(\phi'(0)Z_t^*\right) \end{aligned}$$

for two continuous and increasing processes Z^* and Y^*. We first establish a series of auxiliary weak convergence results and we define

$$Z_t^N \coloneqq \int_0^t \nu_s^N(\{0\}) ds.$$

Lemma 4.4.4. *Let ν^N be a weakly convergent sequence in M1 with limit ν^*. Then, there exists an increasing continuous process Z^* such that the following weak convergences in the M1 topology hold*

(i) $Z^N \to Z^*$

(ii) $\int_0^t \langle \nu_s^N, \rho\sigma(\cdot)\phi\rangle dZ_s^N \to \int_0^t \langle \nu_s^*, \rho\sigma(\cdot)\phi\rangle dZ_s^*$,

(iii) $\int_0^t \mu(0, \nu_s^N) dZ_s^N \to \int_0^t \mu(0, \nu_s^*) dZ_s^*$

as $N \to \infty$, for $\phi \in \mathcal{S}$.

Proof. (i) Note that the process Z^N is continuous and increasing, hence we get convergence in the $M1$ topology to a process Z^*. Convergence in the M1-topology implies pointwise convergence on a cocountable subset I of $[0, T]$. For $s, t \in I$ we get by bounded convergence

$$\mathbb{E}\left[(Z_t^* - Z_s^*)^2\right] = \lim_{N\to\infty} \mathbb{E}\left[(Z_t^N - Z_s^N)^2\right] \leqslant |t - s|^2$$

and Kolmogorov's continuity criterion together with the càdlàg property of Z^* yield that Z^* is a continuous process.

(ii) Fix a bounded $f \in \mathrm{Lip}(\mathbb{R})$. By the triangle inequality and the definition of Z^N we have

$$\begin{aligned}
&\left|\mathbb{E}\left[f\left(\int_0^t \langle \nu_s^N, \rho\sigma(\cdot)\phi\rangle dZ_s^N\right)\right] - \mathbb{E}\left[f\left(\int_0^t \langle \nu_s^*, \rho\sigma(\cdot)\phi\rangle dZ_s^*\right)\right]\right| \\
&\leqslant \|f\|_{Lip}\left(\mathbb{E}\left[\left|\int_0^t \langle \nu_s^N, \sigma(\cdot)\rho\phi'\rangle \nu_s^N(\{0\}))ds - \int_0^t \langle \nu_s^*, \sigma(\cdot)\rho\phi'\rangle \nu_s^N(\{0\})ds\right|\right]\right. \\
&\qquad \left. + \mathbb{E}\left[\left|\int_0^t \langle \nu_s^*, \sigma(\cdot)\rho\phi'\rangle dZ_s^N - \int_0^t \langle \nu_s^*, \sigma(\cdot)\rho\phi'\rangle dZ_s^*\right|\right]\right).
\end{aligned}$$

Using the Skorokhod representation theorem we can assume almost sure convergence of $\nu^N \to \nu^*$ and $Z^N \to Z^*$. Thus, the proof reduces to estimating

$$\begin{aligned}
&\left|\int_0^t \langle \nu_s^N, \sigma(\cdot)\rho\phi'\rangle \nu_s^N(\{0\}))ds - \int_0^t \langle \nu_s^*, \sigma(\cdot)\rho\phi'\rangle \nu_s^N(\{0\})ds\right| \\
&\quad + \left|\int_0^t \langle \nu_s^*, \sigma(\cdot)\rho\phi'\rangle dZ_s^N - \int_0^t \langle \nu_s^*, \sigma(\cdot)\rho\phi'\rangle dZ_s^*\right|.
\end{aligned} \tag{4.4.5}$$

In order to estimate the first term of (4.4.5) we let ψ^ε be a mollifier and set $\sigma^\varepsilon(x) \coloneqq \sigma(x) * \psi^\varepsilon \in C^\infty$. We estimate

$$\begin{aligned}
&\left|\int_0^t \langle \nu_s^N, \sigma(\cdot)\rho\phi'\rangle \nu_s^N(\{0\}))ds - \int_0^t \langle \nu_s^*, \sigma(\cdot)\rho\phi'\rangle \nu_s^N(\{0\})ds\right| \\
&\leqslant \left|\int_0^t \langle \nu_s^N, \sigma^\varepsilon(\cdot)\rho\phi'\rangle \nu_s^N(\{0\})ds - \int_0^t \langle \nu_s^*, \sigma^\varepsilon(\cdot)\rho\phi'\rangle \nu_s^N(\{0\}))ds\right| \\
&\quad + 2\int_0^t \sup_{x\in\mathbb{R}} |\phi(x)||\sigma^\varepsilon(x) - \sigma(x)|ds.
\end{aligned} \tag{4.4.6}$$

We have

$$\begin{aligned}
|\langle \nu_s^N, \sigma^\varepsilon(\cdot)\rho\phi'\rangle \nu_s^N(\{0\})) - \langle \nu_s^*, \sigma^\varepsilon(\cdot)\rho\phi'\rangle \nu_s^N(\{0\})| &\leqslant |\langle \nu_s^N, \sigma^\varepsilon(\cdot)\phi'\rangle - \langle \nu_s^*, \sigma^\varepsilon(\cdot)\phi'\rangle| \\
&\to 0, \quad \text{as } N \to \infty,
\end{aligned} \tag{4.4.7}$$

for almost all s in a set of full measure, where we used that ν^N is a probability measure and that $\rho < 1$. Thus the first term on the right-hand side of (4.4.6) vanishes by bounded convergence. The second term on the right-hand side of (4.4.6) vanishes by the same arguments as in the proof of [37, Corollary 5.7]. It remains to study the second term of (4.4.5). Note that continuity of the process Z^* together with the equation (4.4.4) for M_*^ϕ yield that $\langle \nu^*, \phi\rangle$ is a continuous

process for any $\phi \in \mathcal{S}$. We again mollify σ through convolution with ψ^ε. The triangle inequality together with similar arguments to (4.4.6) and (4.4.7) means we are left with

$$\left| \int_0^t \langle \nu_s^*, \sigma^\varepsilon(\cdot)\rho\phi' \rangle dZ_s^N - \int_0^t \langle \nu_s^*, \sigma^\varepsilon(\cdot)\rho\phi' \rangle dZ_s^* \right|. \tag{4.4.8}$$

Since the functions Z^N and Z^* are increasing càdlàg functions on $[0,t]$ and are bounded above by t, they are distribution functions of finite measures λ^N and λ^* on $[0,t]$. Pointwise convergence of the sequence of functions (Z^N) to Z^* on a set of full measure implies weak convergence of the sequence of measures (λ^N) to λ^* [8, Proposition 8.1.8]. This implies convergence of the term in (4.4.8) to 0. We can then conclude the weak convergence

$$\int_0^t \langle \nu_s^N, \sigma(\cdot)\rho\phi' \rangle \nu_s^N(\{0\}) ds \to \int_0^t \langle \nu_s^*, \sigma(\cdot)\rho\phi' \rangle dZ.$$

(iii) By Assumption 4.1.3 the drift coefficient μ does not depend on the measure variable if $x = 0$. Thus, the convergence statement becomes

$$\mu^*(0) Z_t^N \to \mu^*(0) Z_t^*$$

as $N \to \infty$ weakly with respect to the M1 topology, where $\mu^*(0)$ denotes the value of the drift coefficient for $x = 0$. The result then follows by (i).

□

Lemma 4.4.5. *For every fixed $\phi \in \mathcal{S}$ there exists a deterministic cocountable subset of $[0,T]$ such that the weak convergences*

$$\begin{aligned} M^\phi(\nu^{N_k})(t) &\to M_*^\phi(\nu^*)(t), \quad S^\phi(\nu^{N_k})(t) \to S_*^\phi(\nu^*)(t) \\ C^\phi(\nu^{N_k}, W^0)(t) &\to C_*^\phi(\nu^*, W^0) \quad \textit{in } \mathbb{R}. \end{aligned}$$

Proof. Since $\langle \nu^{N_k}, \phi \rangle$ is a weakly converging subsequence in the space $(D_{\mathbb{R}}([0,T]), M1)$ with limit point $\langle \nu^*, \phi \rangle$, for any $\phi \in \mathcal{S}$, we obtain that $\langle \nu_0^{N_k}, \phi \rangle \to \langle \nu_0^*, \phi \rangle$ weakly and $\langle \nu_t^{N_k}, \phi \rangle \to \langle \nu_t^*, \phi \rangle$ weakly for all t in a cocountable subset of $[0,T]$. The weak convergence for the integrals in the first line of $M^\phi(\nu^N)$, as defined in (4.4.3), follows from [37, Corollary 5.7]. Using Lemma 4.4.4 we obtain the existence of a weak limit point Z^* for Z^N.

Next, we show weak convergence for the term $S^\phi(\nu^N)$. The convergence of $M^\phi(\nu^N)(t)^2$ follows by boundedness. Then, we need to examine the integral term more closely

$$\int_0^t \left(\langle \nu_s^N, \sigma(\cdot)\rho\phi'\rangle - \nu_s^N(\{0\})\sigma(0)\rho\phi'(0)\right)^2 ds = \int_0^t \langle \nu_s^N, \sigma(\cdot)\rho\phi'\rangle^2 ds + \sigma^2(0)\rho^2\phi'(0)^2 \int_0^t \nu_s^N(\{0\})^2 ds - 2\rho\sigma(0)\phi'(0) \int_0^t \langle \nu_s^N, \sigma(\cdot)\rho)\phi'\rangle \nu_s^N(\{0\}) ds$$

For the first term on the right-hand side we have the weak convergence

$$\int_0^t \langle \nu_s^N, \sigma(\cdot)\rho\phi'\rangle^2 ds \to \int_0^t \langle \nu_s^*, \sigma(\cdot)\rho\phi'\rangle^2 ds.$$

The process in the second line is increasing and càdlàg. Thus we obtain the existence of an increasing càdlàg process Y^* such

$$\int_0^t \nu_s^N(\{0\})^2 ds \to Y_t^*$$

weakly in M1. The convergence for the final term follows using Lemma 4.4.4. □

Proposition 4.4.6. *Suppose we have the weak convergence* $(\nu^{N_k}, W^0) \to (\nu^*, W^0)$. *Then, for every* $\phi \in \mathcal{S}$ *the processes* $M_*^\phi(\nu^*), S_*^\phi(\nu^*)$ *and* $C_*^\phi(\nu^*, W^0)$ *are martingales.*

Proof. We proceed as in the proof of [37, Proposition 5.11]. Fix $\phi \in \mathcal{S}$ and take $\mathbb{T}$ to be a cocountable subset of $[0, T]$. To show that $M_*^\phi(\nu^*)$ is a martingale it is sufficient to show that for any arbitrary $k \geqslant 1, s, t \in \mathbb{T}, s_1, \ldots, s_k \in [0, s] \cap \mathbb{T}$ and $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{R}$ continuous and bounded

$$F(\nu^*) := \left(M_*^\phi(\nu^*)(t) - M_*^\phi(\nu^*)(s)\right) \prod_{i=1}^k f_i(M_*^\phi(\nu^*)(s_i))$$

satisfies $\mathbb{E}\left[F(\nu^*)\right] = 0$. By weak convergence of $M^\phi(\nu^N)$ to $M_*^\phi(\nu^*)$ and boundedness and continuity of the f_i's we get

$$\mathbb{E}\left[F(\nu^*)\right] = \lim_{N\to\infty} \mathbb{E}\left[F(\nu^N)\right]$$

By the finite evolution equation $M^\phi(\nu^N)$ is a martingale and therefore $\mathbb{E}F(\nu^N) = 0$ and as a result $M_*^\phi(\nu^*)$ is a martingale.
For $S_*^\phi(\nu^*)$ we define

$$G^\phi(\nu^*) := \left(S_*^\phi(\nu^*)(t) - S_*^\phi(\nu^*)(s)\right) \prod_{i=1}^k f_i(S_*^\phi(\nu^*)(s_i)).$$

By Itô's formula

$$S^{\phi}(\nu^N)(t) = S^{\phi}(\nu^N)(0) + \text{martingale term} + 2[I^N(\phi)]_t$$

with $I^N(\phi)$ as defined in (4.4.1), and by boundedness of the f_i

$$\mathbb{E}\left[G(\nu^N)\right] = O(N^{-1})$$

which yields $\mathbb{E}\left[G(\nu^*)\right] = 0$ and as a result $S^{\phi}_*(\nu^*)$ is a martingale. Similarly we obtain the result for C^{ϕ}. □

Thus, we have that $M^{\phi}_*(\nu^*)(t)$ is a martingale with quadratic variation

$$\begin{aligned}[M^{\phi}_*(\nu^*)]_t = &\int_0^t \langle \nu^*_s, \sigma(\cdot)\rho\phi'\rangle^2 ds + \sigma^2(0)\rho^2\phi'(0)^2 Y^*_t \\ &- 2\rho\sigma(0)\phi'(0)\int_0^t \langle \nu^*_s, \sigma(\cdot)\rho\phi'\rangle dZ^*_s.\end{aligned} \tag{4.4.9}$$

Proposition 4.4.7. *Suppose we have the weak convergence $(\nu^{N_k}, W^0) \to (\nu^*, W^0)$. Then the pair (ν^*, W^0) satisfy the limit SPDE (4.1.2).*

Proof. Let $\phi_n \in \mathcal{S}$ such that $\phi_n, \phi_n' \to 0$ as $n \to \infty$ and $\phi_n''(x) \to 1_{\{0\}}$ as $n \to \infty$. We use ϕ_n as a test function in the limit equation. Using bounded convergence we obtain

$$\lim_{n\to\infty}\langle \nu^*_t, \phi_n\rangle = 0, \quad \lim_{n\to\infty}\langle \nu^*_0, \phi_n\rangle = 0, \quad \lim_{n\to\infty}\int_0^t \langle \nu^*_s, \mu(\cdot)\phi_n'\rangle ds = 0, \quad \lim_{n\to\infty}\int_0^t \langle \tilde{\nu}^*_s, \sigma^2(\cdot)\phi_n''\rangle ds = 0.$$

We can apply bounded convergence to the expression in (4.4.9) to obtain $[M^{\phi_n}_*(\nu^*)] \to 0$. Thus, since $(M^{\phi_n}_*(\nu^*))$ is a sequence of martingales whose quadratic variation converges to zero, we can conclude that $M^{\phi_n}_*(\nu^*) \to 0$ a.s. potentially after taking a subsequence. Using these convergence result in the expression for $M^{\phi}_*(\nu^*)$, given in (4.4.4), we obtain that

$$Z^*_t = \int_0^t \nu^*_s(\{0\})ds.$$

Plugging this expression into (4.4.4), $M^{\phi}_*(\nu^*)$ becomes

$$\begin{aligned}M^{\phi}_*(\nu^*)(t) =&\langle \nu^*_t, \phi\rangle - \langle \nu^*_0, \phi\rangle - \int_0^t \langle \nu^*_s, \mu(\cdot)\phi'\rangle ds - \frac{1}{2}\int_0^t \langle \nu^*_s, \sigma^2(\cdot)\phi''\rangle ds \\ &- \frac{1}{2c}\phi'(0)\int_0^t \nu^*_s(\{0\})ds + \phi'(0)\int_0^t \mu(0)\nu^*_s(\{0\})ds - \frac{\sigma^2(0)}{2}\phi''(0)\int_0^t \nu^*_s(\{0\})ds.\end{aligned}$$

Using Itô's formula we get

$$\begin{aligned}\langle \nu^*_t, \phi\rangle^2 =&\langle \nu^*_0, \phi\rangle^2 + 2\int_0^t \langle \nu^*_s, \phi\rangle\langle \tilde{\nu}^*_s, \mu(\cdot)\phi'\rangle ds + \int_0^t \langle \nu^*_s, \phi\rangle\langle \tilde{\nu}^*_s, \sigma^2(\cdot)\phi''\rangle ds \\ &+ \frac{1}{c}\phi'(0)\int_0^t \langle \nu^*_s, \phi\rangle\nu^*_s(\{0\})ds + \int_0^t \langle \nu^*_s, \phi\rangle dM^{\phi}_*(\nu)(s) + [M^{\phi}_*(\nu)]_t.\end{aligned}$$

Take expectation to obtain

$$\mathbb{E}\left[\langle \nu_t^*, \phi\rangle^2\right] = \mathbb{E}\left[\langle \nu_0^*, \phi\rangle^2\right] + 2\mathbb{E}\left[\int_0^t \langle \nu_s^*, \phi\rangle \langle \tilde{\nu}_s^*, \mu(\cdot)\phi'\rangle ds\right] + \mathbb{E}\left[\int_0^t \langle \nu_s^*, \phi\rangle \langle \tilde{\nu}_s^*, \sigma^2(\cdot)\phi''\rangle ds\right] + \frac{1}{c}\phi'(0)\mathbb{E}\left[\int_0^t \langle \nu_s^*, \phi\rangle \nu_s^*(\{0\}) ds\right] + \mathbb{E}\left[M_*^\phi(\nu^*)\right]_t.$$

Pick $\phi_n \in \mathcal{S}$ such that $\phi_n \to 0$ and $\phi_n' \to 1_{\{0\}}$. Using bounded convergence we obtain $\mathbb{E}\left[M_*^\phi(\nu^*)\right]_t = 0$ which implies $[M_*^\phi(\nu^*)]_t = 0$ a.s. We thus have

$$\rho^2\sigma^2(0)\int_0^t \nu_s^*(\{0\})^2 ds - 2\rho^2\sigma^2(0)\int_0^t \nu_s^*(\{0\})^2 ds + \sigma^2(0)\rho^2 Y_t^* = 0.$$

This implies

$$Y_t^* = \int_0^t \nu_s^*(\{0\})^2 ds$$

and

$$[M_*^\phi(\nu^*)]_t = \int_0^t \langle \tilde{\nu}_s^*, \rho\sigma(\cdot)\phi'\rangle^2 ds.$$

Combining this with the expression for $C_*^\phi(\nu^*)$ we see that

$$M_*^\phi(\nu^*)(t) = \int_0^t \langle \tilde{\nu}_s^*, \rho\sigma(\cdot)\phi'\rangle dW_s^0.$$

□

Proposition 4.4.8. *If (ν^*, W) realises a limiting law, then ν^* satisfies the regularity assumptions of the class $\mathbb{D}$.*

Proof. We have already established that ν^* takes values in the space of probability measures, but it remains to show that ν^* also satisfies the other regularity properties of processes in the class $\mathbb{D}$.

Consider a finite open interval $I = (x, y) \subseteq \mathbb{R}$. Take $\delta > 0$ and any $\phi_\delta \in \mathcal{S}$ such that $\phi_\delta = 1$ on I, $\phi_\delta = 0$ on $(-\infty, x-\delta) \cup (y+\delta, \infty)$ and $\phi_\delta \in (0,1)$ otherwise. From $(\nu_k^N, W) \to (\nu^*, W)$ weakly for some subsequence N_k we get that $\int_0^t \langle \nu_s^{N_k}, \phi_\delta\rangle ds \to \int_0^t \langle \nu_s^*, \phi_\delta\rangle ds$ weakly by [75, Theorem 11.5.1] . We have

$$\mathbb{E}\left[\int_0^T \nu_t^*(I) dt\right] \leqslant \mathbb{E}\left[\int_0^T \langle \nu_t^*, \phi_\delta\rangle dt\right] = \lim_{k\to\infty} \mathbb{E}\left[\int_0^T \langle \nu_t^{N_k}, \phi_\delta\rangle dt\right].$$

The spatial concentration property (iv) then follows by the estimate for ν^{N_k} in Proposition 4.3.3 and taking the limit $\delta \to 0$.

Using the same approach we obtain that for $\varepsilon > 0$

$$\mathbb{E}\left[\tilde{\nu}_t^*(0,\varepsilon)^2\right] \leqslant \liminf_{N\to\infty} \frac{1}{N^2}\sum_{i=1}^N \sum_{j\neq i, j=1}^N \mathbb{P}(0 < X_t^i < \varepsilon + \delta, 0 < X_t^j < \varepsilon + \delta).$$

We employ the estimate in Proposition 4.3.4 and take δ to zero. We then obtain (iii) in Definition 4.1.4 because we have $\rho < 1$ and we can choose $q > 1$ arbitrarily close to 1 to find the required $\gamma > 0$.
For the exponential tails condition we take $I = (\lambda, \eta)$ for $\eta > 0, \lambda < \eta$. Then, the above approach yields

$$\mathbb{E}\left[\int_0^T \nu_t^*(\lambda,\eta)dt\right] \leqslant \liminf_{k\to\infty} \mathbb{E}\left[\int_0^T \nu_t^{N_k}(\lambda-\delta,\eta+\delta)dt\right] = o(e^{-\alpha(\lambda-\delta)})$$

by Proposition 4.3.5. Taking the limit $\delta \to 0$ and $\eta \to \infty$ yields (ii). □

4.5 The Uniqueness Proof in the Linear Case

In this section we employ the kernel smoothing method to show uniqueness in the case where the limit SPDE (4.1.2) is linear, i.e. there is no nonlinearity in the drift coefficient. After establishing regularity results in the next section we will then show in Section 4.7 how the uniqueness proof can be extended to allow for nonlinearities that satisfy Assumption 4.1.3. The kernel smoothing method is described in [14] in the case of L^2-estimates and was then adjusted to work in the space H^{-1} instead in [37, 41], where H^{-1} denotes the dual space of the Sobolev space H^1. We do this through mollification with the sticky heat kernel $G_\varepsilon^{S,c}$, that is defined in (B.3). We define

$$T_\varepsilon^{S,c}\nu_t(x) = \int_0^\infty G_\varepsilon^{S,c}(x,y)\nu_t(dy)$$

for a process ν. In the case of a sticky boundary it is enough to show

$$\liminf_{\varepsilon\to 0} \left\|\partial_x^{-1} T_\varepsilon^{S,c}\Delta_t\right\|_2 = 0$$

with Δ denoting the difference of two arbitrary solutions in the class $\mathbb{D}$ and ∂_x^{-1} denoting an anti-derivative given by

$$\partial_x^{-1} f(x) = -\int_x^\infty f(y)dy \tag{4.5.1}$$

for an integrable function $f : \mathbb{R} \to \mathbb{R}$. Note that we have that $\partial_x \partial_x^{-1} f = f$.

As discussed in Chapter 3 this only establishes uniqueness in the interior while allowing for the potential of different point masses at zero. However, in the case of a sticky boundary we know that the solution takes values in the space of probability measures and therefore uniqueness in the interior together with the total mass of one determines that point masses at zero have to be equal. Another important fact is,

that for all $t \in [0,T]$ the measure ν_t can be decomposed via $\nu_t = \tilde{\nu}_t + \nu_t(\{0\})\delta_0$ with $\tilde{\nu}_t := \nu_t - \nu_t(\{0\})$. Thus we obtain a decomposition into the point mass at zero, $\nu_t(\{0\})\delta_0$, and a part away from zero, $\tilde{\nu}_t$. Before discussing the uniqueness proof we need several auxiliary results that will be used repeatedly in the proof

Lemma 4.5.1. *Let ν be a process in the class $\mathbb{D}$ and p_ε the Gaussian heat kernel defined in (B.1). Then we have*

$$\int_0^t \int_0^\infty \left(\int_0^\infty p_\varepsilon(x+y) d\tilde{\nu}_s(dy) \right)^2 dxds \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. Using the properties of the class of process $\mathbb{D}$ the result can be proved by following the arguments in [37, Lemma 7.6]. □

Lemma 4.5.2. *Let ν be a process in the class $\mathbb{D}$ and let $g_\varepsilon^{S,c}$ be the sticky correction term defined in (B.4). Then we have*

$$\int_0^t \int_0^\infty \left(\int_0^\infty g_\varepsilon^{S,c}(x,y) d\tilde{\nu}_s(dy) \right)^2 dxds \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. The proof works in the same way as in the elastic case discussed in Chapter 3. □

An advantage in the case of an absorbing boundary is that the Dirichlet heat kernel vanishes at zero and thus when performing integration by parts boundary terms are zero. In the sticky case we need to take care of the boundary terms. A helpful relation between the smoothed measure and its anti-derivative at zero is given in the following lemma.

Lemma 4.5.3. *Let M be a signed measure on $[0,\infty)$ with $M([0,\infty)) = 0$. Then, for all $\varepsilon > 0$ we have*

$$\partial_x^{-1} T_\varepsilon^{S,c} M(0) = c T_\varepsilon^{S,c} M(0).$$

Proof. Using the definition of the anti-derivative and changing the order of integration we obtain

$$\begin{aligned} \partial_x^{-1} T_\varepsilon^{S,c} M(0) &= -\int_0^\infty \int_0^\infty G_\varepsilon^{S,c}(x,y) M(dy) dx \\ &= -\int_0^\infty \int_0^\infty G_\varepsilon^{S,c}(x,y) dx M(dy) \end{aligned} \tag{4.5.2}$$

Note that the inner integral on the right-hand side of (4.5.2) is the probability mass in the interior of the positive-half line for a sticky Brownian motion at time ε.

Thus, adding the point mass at zero of this sticky Brownian motion we obtain the mass 1. Thus, we have

$$\partial_x^{-1} T_\varepsilon^{S,c} M(0) = -M([0,\infty)) + c\int_0^\infty G_\varepsilon^{S,c}(0,y)M(dy) = cT_\varepsilon^{S,c}M(0).$$

where we used that $M([0,\infty)) = 0$. □

Note that for two solutions to the SPDE in the class $\mathbb{D}$ their difference takes values in the space of signed measures and satisfies the conditions of Lemma 4.5.3.

Lemma 4.5.4. *Let (ν, W^0) be a solution to the SPDE (4.1.2) with ν in the class $\mathbb{D}$, $G_\varepsilon^{S,c}$ the sticky heat kernel and $g_\varepsilon^{S,c}$ the sticky correction term, defined in (B.3), with $0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 < 1$. Then, the following convergence results hold*

(i) $\int_0^\infty G_\varepsilon^{S,c}(x,0)dx \to 0 \quad$ *as* $\varepsilon \to 0$,

(ii) $\int_0^\infty (G_\varepsilon^{S,c}(x,0))^2 dx = \int_0^\infty (g_\varepsilon^{S,c}(x,0))^2 dx \to 0 \quad$ *as* $\varepsilon \to 0$,

(iii) $T_\varepsilon^{S,c}\tilde{\nu}_t(0) \to 0 \quad$ *as* $\varepsilon \to 0$

Proof. *i*) The form of the sticky heat kernel gives

$$G_\varepsilon^{S,c}(x,0) = G_\varepsilon^a(x,0) + g_\varepsilon^{S,c}(x,0) = g_\varepsilon^{S,c}(x,0)$$

using the boundary condition of the Dirichlet heat kernel G_ε^a, defined in (B.2). Considering the term $g_\varepsilon^{S,c}$ it follows from the properties of the error function that $g_\varepsilon^{S,c}(x,0) \to 0$ as $\varepsilon \to 0$ for all $x > 0$. We can estimate $g_\varepsilon^{S,c}$ in the following way

$$\begin{aligned}
g_\varepsilon^{S,c}(x,0) &= \frac{1}{c} e^{\frac{x}{c}+\frac{\varepsilon}{2c^2}} \left[1 - \mathrm{Erf}\left(\frac{x}{\sqrt{2\varepsilon}} + \frac{\sqrt{\varepsilon}}{\sqrt{2}c}\right)\right] \\
&= \frac{2}{c} e^{\frac{x}{c}+\frac{\varepsilon}{2c^2}} \frac{1}{\sqrt{2\pi\varepsilon}} \int_x^\infty \exp\left(-\left[\frac{z}{\sqrt{2\varepsilon}} + \frac{\sqrt{\varepsilon}}{\sqrt{2}c}\right]^2\right) dz \\
&\leqslant \frac{2}{c} e^{\frac{x}{c}+\frac{\varepsilon}{2c^2}} \frac{1}{\sqrt{2\pi\varepsilon}} \int_x^\infty e^{-z^2/(2\varepsilon)} e^{-\varepsilon/(2c^2)} e^{-z/c} dz \\
&\leqslant \frac{2}{c} \frac{1}{\sqrt{2\pi\varepsilon}} \int_x^\infty e^{-z^2/(2\varepsilon)} dz \\
&\leqslant \frac{2}{c}\left(1 - \Phi(\frac{x}{\sqrt{\varepsilon}})\right) \\
&\leqslant \frac{2}{c}\Phi(-\frac{x}{\sqrt{\varepsilon}}) \leqslant \frac{2}{c}\Phi(-x).
\end{aligned}$$

Therefore, the assertion follows by dominated convergence.
ii) follows by the same arguments as *i*).
iii) Note that

$$T_\varepsilon^{S,c}\tilde{\nu}_t(0) = \int_0^\infty G_\varepsilon^{S,c}(0,y)\tilde{\nu}_t(dy) = \int_0^\infty g_\varepsilon^{S,c}(0,y)\tilde{\nu}_t(dy).$$

The measure $\tilde{\nu}_t$ does not have a point mass at zero by construction. Therefore, $g_\varepsilon^{S,c}(0,y)$ converges to 0 $\tilde{\nu}_t$-a.s. as $\varepsilon \to 0$ and the result follows by the same arguments as in *i*). □

Given a bounded function $h(x,\nu_t)$ we define

$$\mathcal{E}_{t,\varepsilon}^h(x) = \langle \tilde{\nu}_t, h(\cdot,\nu_t)G_\varepsilon^{S,c}(x,\cdot)\rangle - h(x,\nu_t)T_\varepsilon^{S,c}\tilde{\nu}_t(x). \tag{4.5.3}$$

We need the following results for this error term.

Lemma 4.5.5.

$$\int_0^T \left\|\mathcal{E}_{t,\varepsilon}^h\right\|^2_{L^2(0,\infty)} dt \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. Follows by the same arguments as in the proof for the absorbing case (see [37, Lemma 8.1.]) because $\tilde{\nu}_t$ has exponential tails since ν is a process in the class $\mathbb{D}$. □

Now we have all the tools needed to prove the following linear version of Theorem 4.1.6

Theorem 4.5.6. *Assume that (ν, W^0) and $(\bar{\nu}, \bar{W}^0)$ are solutions to the SPDE*

$$\begin{aligned}\langle \nu_t, \phi\rangle = \langle \nu_0, \phi\rangle + \int_0^t \langle \tilde{\nu}_s, \mu(\cdot)\phi'\rangle ds + \frac{1}{2}\int_0^t \langle \tilde{\nu}_s, \sigma^2(\cdot)\phi''\rangle ds \\ + \int_0^t \rho\langle \tilde{\nu}_s, \sigma(\cdot)\phi'\rangle dW_s^0 + \frac{\phi'(0)}{2c}\int_0^t \nu_s(\{0\})ds\end{aligned} \tag{4.5.4}$$

for $t \in [0,T]$ and $\phi \in \mathcal{S}$ and that ν and $\bar{\nu}$ are in the class $\mathbb{D}$. Then, we have

(i) pathwise uniqueness if $W^0 = \bar{W}^0$, i.e.

$$\nu_t(S) = \bar{\nu}_t(S) \quad \text{for every } t \in [0,T] \text{ and every Borel set } S \subseteq \mathbb{R},$$

(ii) uniqueness in law, i.e. $\mathcal{L}(\nu, W^0) = \mathcal{L}(\bar{\nu}, \bar{W}^0)$.

Proof. Let (ν, W^0) be a solution to the SPDE (4.1.2) with ν in the class $\mathbb{D}$. We plug the sticky heat kernel $y \mapsto G_\varepsilon^{S,c}(x,y)$ as a test function in the equation, use Lemma B.2 to switch the derivative from y to x and the boundary condition of the sticky heat kernel. Moreover, we integrate over x to introduce the anti-derivative ∂_x^{-1}. This yields

$$\begin{aligned} d\partial_x^{-1}T_\varepsilon^{S,c}\nu_t =& -\langle\tilde{\nu}_t, \mu_t(\cdot)G_\varepsilon^{S,c}(x,\cdot)\rangle dt + \frac{1}{2}\partial_x\langle\tilde{\nu}_t, \sigma^2(\cdot)G_\varepsilon^{S,c}(x,\cdot)\rangle dt \\ &+ \rho\langle\tilde{\nu}_t, \sigma(\cdot)G_\varepsilon^{S,c}(x,\cdot)\rangle dW_t^0 \\ &- 2\langle\tilde{\nu}_t, \mu_t(\cdot)p_\varepsilon(x+\cdot)\rangle dt + 2\langle\tilde{\nu}_t, \mu_t(\cdot)g_\varepsilon^{S,c}(x,\cdot)\rangle dt \\ &- 2\langle\tilde{\nu}_t, \sigma(\cdot)p_\varepsilon(x+\cdot)\rangle dW_t^0 + 2\langle\tilde{\nu}_t, \sigma(\cdot)g_\varepsilon^{S,c}(x,\cdot)\rangle dW_t^0 \\ &+ \frac{1}{2}\partial_x G_\varepsilon^{S,c}(x,0)\nu_t(\{0\})dt. \end{aligned}$$

To simplify notation we write $o_{sq}(1)$ for any family of $L^2(0,\infty)$-processes $\{(f_{t,\varepsilon})_{t\in[0,T]}\}_{\varepsilon>0)}$ such that

$$\mathbb{E}\left[\int_0^T \|f_{t,\varepsilon}\|_2^2\, dt\right] \to 0 \quad \text{as } \varepsilon \to 0.$$

We can move the coefficients outside of the integration against the measure. This creates error terms as defined in (4.5.3). We can use Lemma 4.5.1, Lemma 4.5.2 and Lemma 4.5.5 and the decay class $o_{sq}(1)$ to obtain

$$\begin{aligned} d\partial_x^{-1}T_\varepsilon^{S,c}\nu_t =& -\mu_t(x)T_\varepsilon^{S,c}\tilde{\nu}_t dt + \frac{1}{2}\partial_x\left(\sigma^2(x)T_\varepsilon^{S,c}\tilde{\nu}_t + \mathcal{E}_{t,\varepsilon}^{\sigma^2}\right)dt \\ &+ \rho\sigma(x)T_\varepsilon^{S,c}\tilde{\nu}_t dW_t^0 + o_{sq}(1)dt + o_{sq}(1)dW_t^0 \\ &+ \frac{1}{2}\partial_x G_\varepsilon^{S,c}(x,0)\nu_t(\{0\})dt. \end{aligned}$$

The next step is to apply Itô's formula which yields the dynamics of the squared anti-derivative

$$\begin{aligned} d(\partial_x^{-1}T_\varepsilon^{S,c}\nu_t)^2 =& -2\mu_t(x)(\partial_x^{-1}T_\varepsilon^{S,c}\nu_t)(T_\varepsilon^{S,c}\tilde{\nu}_t)dt \\ &+ (\partial_x^{-1}T_\varepsilon^{S,c}\nu_t)\partial_x\left(\sigma^2(x)(T_\varepsilon^{S,c}\tilde{\nu}_t) + \mathcal{E}_{t,\varepsilon}^{\sigma^2}\right)dt \\ &+ 2\rho\sigma(x)(\partial_x^{-1}T_\varepsilon^{S,c}\nu_t)(T_\varepsilon^{S,c}\tilde{\nu}_t)dW_t^0 + (\partial_x^{-1}T_\varepsilon^{S,c}\nu_t)o_{sq}(1)dt \\ &+ (\partial_x^{-1}T_\varepsilon^{S,c}\nu_t)o_{sq}(1)dW_t^0 + \rho^2\sigma^2(x)(T_\varepsilon^{S,c}\tilde{\nu}_t)^2dt + o_{sq}^2(1)dt \\ &+ \rho\sigma(x)(T_\varepsilon^{S,c}\tilde{\nu}_t)o_{sq}(1)dt + (\partial_x^{-1}T_\varepsilon^{S,c}\nu_t)\partial_x G_\varepsilon^{S,c}(x,0)\nu_t(\{0\})dt. \end{aligned}$$

Let $(\bar{\nu}, W^0)$ be another solution to the SPDE (4.1.2) with the same Brownian motion and with $\bar{\nu}$ in the class $\mathbb{D}$. Set their difference to be $\Delta := \nu - \bar{\nu}$. We then easily obtain the dynamics for $\partial_x^{-1}T_\varepsilon^{Sc}\Delta$. We are interested in estimates for the squared

L^2-norm of $\partial_x^{-1}T_\varepsilon^{S,c}\Delta_t$. In order to obtain these, we integrate over $x > 0$ and then we take expectations. The process Δ has sub-exponential tails as the difference of two elements of the class $\mathbb{D}$. Combining this tail-decay property with Lemma 8.3 and Lemma 8.7 from [37], which can easily be adapted to the case of mollification with the sticky heat kernel, we obtain that the stochastic integral terms on the right-hand side are martingales. This means they vanish when we take expectation. The equation for the expected squared L^2-norm then turns into

$$\begin{aligned}
\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_t\right\|_2^2\right] =& -2\mathbb{E}\left[\int_0^t\int_0^\infty \mu_t(x)(\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s)(T_\varepsilon^{S,c}\tilde{\Delta}_s)dxds\right] \\
&+\mathbb{E}\left[\int_0^t\int_0^\infty (\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s)(\partial_x\left(\sigma^2(x)(T_\varepsilon^{S,c}\Delta_s)+\mathcal{E}_{s,\varepsilon}^{\sigma^2}-\bar{\mathcal{E}}_{s,\varepsilon}^{\sigma^2}\right)dxds\right] \\
&+\mathbb{E}\left[\int_0^t\int_0^\infty (\partial_x^{-1}T_\varepsilon^{S,c}\tilde{\Delta}_s)o_{sq}(1)dxds\right] \\
&+\mathbb{E}\left[\int_0^t\int_0^\infty \rho^2\sigma^2(x)|T_\varepsilon^{S,c}\tilde{\Delta}_s|^2dxds\right] \\
&+\mathbb{E}\left[\int_0^t\int_0^\infty \rho\sigma(x)(T_\varepsilon^{S,c}\tilde{\Delta}_s)o_{sq}(1)dxds\right] \\
&+\mathbb{E}\left[\int_0^t\int_0^\infty (\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s)\partial_x G_\varepsilon^{S,c}(x,0)\Delta_s\{0\}dxds\right]+o(1).
\end{aligned} \tag{4.5.5}$$

We treat each term individually. For the first term on the right-hand side of (4.5.5) we use the generalized Young's inequality with free parameter $\eta > 0$

$$\begin{aligned}
&-2\mathbb{E}\left[\int_0^t\int_0^\infty \mu_t(x)(\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s)(T_\varepsilon^{S,c}\tilde{\Delta}_s)dxds\right] \\
&\leqslant c_\eta\mathbb{E}\left[\int_0^t\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\right\|_2^2ds\right]+\eta\mathbb{E}\left[\int_0^t\int_0^\infty(\mu_t(x))^2|T_\varepsilon^{S,c}\tilde{\Delta}_s|^2dxds\right].
\end{aligned}$$

For the term in the second line of (4.5.5) we apply integration by parts and Lemma 4.5.3 for the boundary terms

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\int_0^\infty(\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s)(x)\partial_x\left(\sigma^2(x)(T_\varepsilon^{S,c}\tilde{\Delta}_s)(x)+\mathcal{E}_{s,\varepsilon}^{\sigma^2}-\bar{\mathcal{E}}_{s,\varepsilon}^{\sigma^2}\right)dxds\right] \\
&=-c\sigma^2(0)\mathbb{E}\left[\int_0^t T_\varepsilon^{S,c}\Delta_s(0)T_\varepsilon^{S,c}\tilde{\Delta}_s(0)ds\right]-c\mathbb{E}\left[\int_0^t(T_\varepsilon^{S,c}\Delta_s)(0)(\mathcal{E}_{s,\varepsilon}^{\sigma^2}(0)-\bar{\mathcal{E}}_{s,\varepsilon}^{\sigma^2}(0))ds\right] \\
&-\mathbb{E}\left[\int_0^t\int_0^\infty\sigma^2(x)T_\varepsilon^{S,c}\Delta_sT_\varepsilon^{S,c}\tilde{\Delta}_sdxds\right]-\mathbb{E}\left[\int_0^t\int_0^\infty(T_\varepsilon^{S,c}\Delta_s)(x)(\mathcal{E}_{s,\varepsilon}^{\sigma^2}(x)-\bar{\mathcal{E}}_{s,\varepsilon}^{\sigma^2}(x))dxds\right].
\end{aligned} \tag{4.5.6}$$

For the first term on the right-hand side of (4.5.6) we have

$$-c\sigma^2(0)\mathbb{E}\left[\int_0^t T_\varepsilon^{S,c}\Delta_s(0)T_\varepsilon^{S,c}\tilde{\Delta}_s(0)ds\right] = -c\sigma^2(0)\Big(\mathbb{E}\left[\int_0^t |T_\varepsilon^{S,c}\tilde{\Delta}_s(0)|^2 ds\right] + \mathbb{E}\left[\int_0^t g_\varepsilon^{S,c}(0,0)\tilde{\Delta}_s(\{0\})T_\varepsilon^{S,c}\tilde{\Delta}_s(0)ds\right]\Big). \tag{4.5.7}$$

Using Lemma 4.5.4 we get

$$-c\sigma^2(0)\mathbb{E}\left[\int_0^t T_\varepsilon^{S,c}\Delta_s(0)T_\varepsilon^{S,c}\tilde{\Delta}_s(0)ds\right] = o(1), \quad \text{as } \varepsilon \to 0.$$

The second term on the right-hand side of (4.5.7) vanishes by Lemma 4.5.4. The term

$$-c\mathbb{E}\left[\int_0^t T_\varepsilon^{S,c}\Delta_s(0)(\mathcal{E}_{s,\varepsilon}^{\sigma^2}(0) - \bar{\mathcal{E}}_{s,\varepsilon}^{\sigma^2}(0))ds\right]$$

vanishes by considering again splitting Δ_s into the point mass at zero and the remaining part and using Lemma 4.5.4 and Lemma 4.5.5. Moreover we have

$$-\mathbb{E}\left[\int_0^t\int_0^\infty \sigma^2(x)T_\varepsilon^{S,c}\Delta_s T_\varepsilon^{S,c}\tilde{\Delta}_s dxds\right] = -\mathbb{E}\left[\int_0^t\int_0^\infty \sigma^2(x)|T_\varepsilon^{S,c}\tilde{\Delta}_s|^2 dxds\right] - \mathbb{E}\left[\int_0^t\int_0^\infty \sigma^2(x)g_\varepsilon^{S,c}(x,0)\Delta_s\{0\})T_\varepsilon^{S,c}\tilde{\Delta}_s(x)dxds\right].$$

For the last term we use the generalized Young's inequality and Lemma 4.5.4 to get

$$\begin{aligned}
&-\mathbb{E}\left[\int_0^t\int_0^\infty \sigma^2(x)g_\varepsilon^{S,c}(x,0)\Delta_s\{0\})T_\varepsilon^{S,c}\tilde{\Delta}_s(x)dxds\right] \\
&\leqslant c_\eta\mathbb{E}\left[\int_0^t\int_0^\infty (\sigma^2(x))^2 g_\varepsilon^{S,c}(x,0)^2\Delta_s(\{0\})^2 dxds\right] + \eta\mathbb{E}\left[\int_0^t\int_0^\infty |T_\varepsilon^{S,c}\tilde{\Delta}_s|^2 dxds\right] \\
&= \eta\mathbb{E}\left[\int_0^t\int_0^\infty |T_\varepsilon^{S,c}\tilde{\Delta}_s|^2 dxds\right] + o(1).
\end{aligned}$$

We can handle the last term in (4.5.6) by again splitting the measure into two components. We then use Lemma 4.5.4 and Lemma 4.5.5 this yields

$$-\mathbb{E}\left[\int_0^t\int_0^\infty T_\varepsilon^{S,c}\Delta_s(x)(\mathcal{E}_{s,\varepsilon}^{\sigma^2}(x) - \bar{\mathcal{E}}_{s,\varepsilon}^{\sigma^2}(x))dxds\right] \leqslant \eta\mathbb{E}\left[\int_0^t\int_0^\infty |T_\varepsilon^{S,c}\tilde{\Delta}_s(x)|^2 dxds\right] + o(1).$$

We deal with the term in the third and fifth line of (4.5.5) by simply applying the generalized Young's inequality with parameter $\eta > 0$. The term in the fourth line

remains unchanged. For the last term in (4.5.5), integration by parts yields

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\int_0^\infty \partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\partial_x G_\varepsilon^{S,c}(x,0)\Delta_s(\{0\})dxds\right]\\
&= -\mathbb{E}\left[\int_0^t \partial_x^{-1}T_\varepsilon^{S,c}\Delta_s(0)g_\varepsilon^{S,c}(0,0)\Delta_s(\{0\}dxds\right]\\
&\quad - \mathbb{E}\left[\int_0^t\int_0^\infty T_\varepsilon^{S,c}\Delta_s g_\varepsilon^{S,c}(x,0)\Delta_s(0)dxds\right]
\end{aligned}$$

where we used that $G_\varepsilon^{S,c}(x,0) = g_\varepsilon^{S,c}(x,0)$. As before we apply Lemma 4.5.3 to deal with the boundary term

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\int_0^\infty \partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\partial_x G_\varepsilon^{S,c}(x,0)\Delta_s(\{0\})dxds\right]\\
&= -c\mathbb{E}\left[\int_0^t T_\varepsilon^{S,c}\Delta_s(0)g_\varepsilon^{S,c}(0,0)\Delta_s(\{0\})ds\right]\\
&\quad - \mathbb{E}\left[\int_0^t\int_0^\infty T_\varepsilon^{S,c}\Delta_s g_\varepsilon^{S,c}(x,0)\Delta_s(0)dxds\right].
\end{aligned}$$

The next step is now again to split the smoothed measure in the two terms on the right-hand side into the part involving the point mass at zero and the part on the positive half line $\tilde{\Delta}$. This together with Lemma 4.5.4 yields

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\int_0^\infty \partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\partial_x G_\varepsilon^{S,c}(x,0)\Delta_s(\{0\})dxds\right]\\
&= -c\mathbb{E}\left[\int_0^t (g_\varepsilon^{S,c}(0,0))^2\Delta_s(\{0\})^2 ds\right]\\
&\quad - \mathbb{E}\left[\int_0^t\int_0^\infty T_\varepsilon^{S,c}\tilde{\Delta}_s g_\varepsilon^{S,c}(x,0)\Delta_s^{lin}(\{0\})dxds\right] + o(1).
\end{aligned}$$

We apply the generalized Young's inequality to the second term and again Lemma 4.5.4

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\int_0^\infty \partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\partial_x G_\varepsilon^{S,c}(x,0)\Delta_s(\{0\})dxds\right]\\
&\leqslant \eta\mathbb{E}\left[\int_0^t\int_0^\infty |T_\varepsilon^{S,c}\tilde{\Delta}_s|^2 dxds\right] + o(1) - c\mathbb{E}\left[\int_0^t T_\varepsilon^{S,c}\Delta_s(0)g_\varepsilon^{S,c}(0,0)\Delta_s(\{0\})ds\right].
\end{aligned}$$

Putting everything back together yields

$$\begin{aligned}
\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_t\right\|_2^2\right] &\leqslant c_\eta\int_0^t \mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\right\|_2^2\right]ds\\
&\quad - \mathbb{E}\left[\int_0^t\int_0^\infty \left(\sigma^2(x)(1-(1+\eta)\rho^2) - \eta(\mu_t)^2(x) - 3\eta\right)|T_\varepsilon^{S,c}\tilde{\Delta}_s(x)|^2 dxds\right]\\
&\quad - c(g_\varepsilon^{S,c}(0,0))^2\int_0^t \mathbb{E}\left[\Delta_s(\{0\})^2\right]ds + o(1).
\end{aligned} \tag{4.5.8}$$

If we choose $\eta > 0$ small enough we can find a constant $c_0 > 0$ such that

$$\sigma^2(x)(1-(1+\eta)\rho^2) - \eta(\mu)^2(x) - 4\eta > c_0 \tag{4.5.9}$$

for all $x \geqslant 0$ because $\rho < 1$ and σ and μ are bounded. We can drop the second term on the right-hand side of (4.5.8) and are left with

$$\begin{aligned}\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_t\right\|_2^2\right] &\leqslant c_\eta \int_0^t \mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\right\|_2^2\right] ds - c\mathbb{E}\left[\int_0^t (g_\varepsilon^{S,c}(0,0)\Delta_s(\{0\}))^2 ds\right] + o(1)\\ &\leqslant c_\eta \int_0^t \mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\right\|_2^2\right] ds + o(1)\end{aligned}$$

and by Gronwall's lemma we obtain

$$\lim_{\varepsilon\to 0}\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_t\right\|_2^2 = 0.$$

The uniqueness then follows by the fact that, for all $t \in [0,T]$, Δ_t is the difference of two probability measures and by the H^{-1}-estimates from [41, 37] and Chapter 3. □

4.6 Regularity Results

In this section we first show existence of a density and L^2-regularity for almost every $t \in [0,T]$ as well as L^2-regularity away from the boundary for all $t \in [0,T]$. We then discuss several further regularity results that can be obtained, particularly under the assumption of constant coefficients.

4.6.1 Existence and Regularity of a Density away from the Boundary

In this section we modify the techniques used in Section 3.6 to prove existence of a density V_t of the measure measure $\tilde{\nu}_t$, for all $t \in [0,T]$, for a solution $(\nu_t)_{t\in[0,T]}$, to the limit SPDE (4.1.2) in the class $\mathbb{D}$.

Lemma 4.6.1. *Let (ν, W^0) be the solution to the limit SPDE (4.1.2) in the class $\mathbb{D}$. Then for almost every $(\omega,t) \in \Omega\times[0,T]$ the measure $\tilde{\nu}_t(\omega)$ has a density $V_t(\omega)$ on $(0,\infty)$ and we have the integrated estimate*

$$\mathbb{E}\left[\int_0^t \|V_s\|_2^2\, ds\right] < \infty, \qquad \text{for all } t \in [0,T]. \tag{4.6.1}$$

Proof. Adjusting the uniqueness proof of Section 4.5 in the same way as described in Lemma 3.6.1 we obtain the estimate

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t \|T_\varepsilon\tilde{\nu}_s\|_2^2\,ds\right] < \infty, \qquad \text{for all } t\in[0,T]. \tag{4.6.2}$$

The result then follows by the arguments in the proof of Proposition 3.6.2. □

We follow the approach outlined in Section 3.6 to show existence of a density for $\tilde{\nu}_t$ and L^2-regularity away from the boundary. However, in the case of a sticky boundary, it turns out to be more convenient to work with the Gaussian heat kernel p_ε as a mollifier in the proof. Hence, we need an estimate similar to (4.6.2) for mollification with the Gaussian heat kernel.

Take sets $B \Subset A \Subset (0,\infty)$, where $\Subset$ denotes compact containment. We can easily obtain the following estimate from (4.6.2)

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t \left\|T_\varepsilon^{S,c}\nu_s\right\|_{L^2(A)}^2 ds\right]\\
&\leqslant \lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t \left\|T_\varepsilon^{S,c}\nu_s\right\|_2^2 ds\right]\\
&= \lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t\int_0^\infty \left(T_\varepsilon^{S,c}\tilde{\nu}_s + g_\varepsilon^{S,c}(x,0)\nu_s(\{0\})\right)^2 dxds\right]\\
&\leqslant 2\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t \left\|T_\varepsilon^{S,c}\tilde{\nu}_s\right\|_2^2 ds\right] ds + 2\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t\int_0^\infty g_\varepsilon^{S,c}(x,0)^2\nu_s(\{0\})^2 dxds\right]\\
&= 2\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t \left\|T_\varepsilon^{S,c}\tilde{\nu}_s\right\|_2^2 ds\right] ds < \infty
\end{aligned} \tag{4.6.3}$$

where we used that the integral involving $g_\varepsilon^{S,c}(x,0)^2$ vanishes as $\varepsilon\to 0$ by Lemma 4.5.4.

We denote by $\bar{T}_\varepsilon$ the operator of convolution with the Gaussian heat kernel p_ε and by T_ε^a convolution with the absorbing heat kernel G_ε^a.

Lemma 4.6.2. *Let $A \Subset (0,\infty)$ and (ν, W^0) a solution to the limit SPDE (4.1.2) with ν in the class $\mathbb{D}$. Then we have*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t \left\|\bar{T}_\varepsilon\nu_s\right\|_{L^2(A)}^2 ds\right] < \infty, \qquad \textit{for all } t\in[0,T].$$

Proof. The definition of the sticky heat kernel in (B.3) shows that it can be decomposed into the absorbing heat kernel G_ε^a and the term $g_\varepsilon^{S,c}$. We can then expand the

square under the integrals and obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^t \left\|T_\varepsilon^{S,c}\nu_s\right\|_{L^2(A)}^2 ds\right] =&\mathbb{E}\left[\int_0^t\int_A \left(T_\varepsilon^{S,c}\nu_s\right)^2 dxds\right]\\ =&\mathbb{E}\left[\int_0^t\int_A \left(T_\varepsilon^{a}\nu_s\right)^2 dxds\right]+\mathbb{E}\left[\int_0^t\int_A\left(\int_0^\infty g_\varepsilon^{S,c}(x,y)\nu_s(dy)\right)^2 dxds\right]\\ &+2\mathbb{E}\left[\int_0^t\int_A T_\varepsilon^a\nu_s\int_0^\infty g_\varepsilon^{S,c}(x,y)\nu_s(dy)\right]dxds.\end{aligned}$$

We obtain the convergence

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t\int_A\left(\int_0^\infty g_\varepsilon^{S,c}(x,y)\nu_s(dy)\right)^2 dxds\right]\to 0$$

following the arguments in Lemma 4.5.2 and noting that the infimum of the set A is strictly bigger than zero and thus $g_\varepsilon^{S,c}(x,0)$ converges to zero for all $x\in A$. Using standard estimates on the Dirichlet heat kernel and following the argument in the proof of Lemma 3.6.4 we then get

$$\begin{aligned}&\mathbb{E}\left[\int_0^t\int_A T_\varepsilon^a\nu_s\int_0^\infty g_\varepsilon^{S,c}(x,y)\nu_s(dy)dxds\right]\\ &\leqslant (2\pi\varepsilon)^{-1/2}\,\mathbb{E}\left[\int_0^t\int_A\int_0^\infty g_\varepsilon^{S,c}(x,y)\nu_s(dy)dxds\right]\to 0,\quad \text{as } \varepsilon\to 0.\end{aligned}\tag{4.6.4}$$

We can further estimate

$$\begin{aligned}&\mathbb{E}\left[\int_0^t\int_A\left(T_\varepsilon^a\nu_s\right)^2 dxds\right]\\ &=\mathbb{E}\left[\int_0^t\int_A\left(\int_0^\infty p_\varepsilon(x-y)\nu_s(dy)\right)^2 dxds\right]+\mathbb{E}\left[\int_0^t\int_A\left(\int_0^\infty p_\varepsilon(x+y)\nu_s(dy)\right)^2 dxds\right]\\ &\quad-2\mathbb{E}\left[\int_0^t\int_A\int_0^\infty p_\varepsilon(x-y)\nu_s(dy)\int_0^\infty p_\varepsilon(x+y)\nu_s(dy)dxds\right].\end{aligned}$$

The second term on the right-hand side vanishes by the arguments as in the proof of Lemma 4.5.1 and the observation that the infimum of the set A is strictly positive. The third term converges to zero using the same approach as in (4.6.4) and following the arguments in the proof of Lemma 3.6.3. Thus we obtain

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t\left\|T_\varepsilon^{S,c}\nu_s\right\|_{L^2(A)}^2 ds\right]=\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t\int_A\left(\int_0^\infty p_\varepsilon(x+y)\nu_s(dy)\right)^2 dxds\right]$$

and therefore the result follows by (4.6.3).

□

Let ψ be a smooth cut-off function such that $\psi = 1$ on B, $\psi \in (0,1)$ on $A \setminus B$ and $\psi = 0$ otherwise. We have

$$|\partial_x \psi| + |\partial_{xx} \psi| \leqslant C 1_{A \setminus B}$$

where the constant C only depends on A and B.

Moreover, we define the error term

$$\bar{\mathcal{E}}^h_{t,\varepsilon}(x) = \partial_x \langle \nu_t, h_t p_\varepsilon(x - \cdot) \rangle - h_t(x) \partial_x \bar{T}_\varepsilon \nu_t(x) + \partial_x h_t(x) \mathcal{H}_{t,\varepsilon}(x), \tag{4.6.5}$$

where

$$\mathcal{H}_{t,\varepsilon}(x) = \langle \nu_t, (x - \cdot) \partial_x p_\varepsilon(x - \cdot) \rangle, \tag{4.6.6}$$

for a function h representing μ, σ^2 or σ.

Lemma 4.6.3. *Let $\bar{\mathcal{E}}^h_{s,\varepsilon}$ be the error term defined in (4.6.5) and ψ the cut-off function specified above. Then we have the convergence*

$$\mathbb{E}\left[\int_0^t \left\| \psi \bar{\mathcal{E}}^h_{s,\varepsilon} \right\|_2^2 ds \right] \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. Follows by the same arguments as Lemma 3.6.5. □

Lemma 4.6.4. *Let $\mathcal{H}_{t,\varepsilon}$ be the error term defined in (4.6.6). Then there is a constant $c_{\mathcal{H}}$ such that*

$$|\mathcal{H}_{t,\varepsilon}| \leqslant c_{\mathcal{H}} |\bar{T}_{2\varepsilon} \nu_t|$$

Proof. Follows by the same arguments as Lemma 3.6.6. □

Theorem 4.6.5. *Let (ν, W^0) be a solution to the limit SPDE with (4.1.2) with ν in the class $\mathbb{D}$. Then with probability 1, for all $t \in [0,T]$ $\tilde{\nu}_t$ has a density V_t on $(0,\infty)$ and V_t is in $L^2(\lambda, \infty)$ for all $\lambda > 0$.*

Proof. We use the Gaussian heat kernel $p_\varepsilon(x-y)$ as a test function, switch derivatives from y to x and apply Itô's formula to obtain the dynamics for $(\bar{T}_\varepsilon \nu_t)^2$. Next, we multiply by ψ^2 and integrate over $x > 0$. We then obtain

$$\begin{aligned}
\left\| \psi \bar{T}_\varepsilon \nu_t \right\|_2^2 = \left\| \psi \bar{T}_\varepsilon \nu_0 \right\|_2^2 &- 2 \int_0^t \int_0^\infty \psi^2 \bar{T}_\varepsilon \nu_s \left(\mu_s \partial_x \bar{T}_\varepsilon \tilde{\nu}_s - \partial_x \mu_s \mathcal{H}_{s,\varepsilon} + \bar{\mathcal{E}}^\mu_{s,\varepsilon} \right) dx ds \\
&+ \int_0^t \int_0^\infty \psi^2 \bar{T}_\varepsilon \nu_s \partial_x \left(\sigma^2(x) \partial_x \bar{T}_\varepsilon \tilde{\nu}_s - \partial_x \sigma^2(x) \mathcal{H}_{s,\varepsilon} + \bar{\mathcal{E}}^{\sigma^2}_{s,\varepsilon} \right) dx ds \\
&- \int_0^t \int_0^\infty \psi^2 \bar{T}_\varepsilon \nu_s \rho \left(\sigma(x) \partial_x \bar{T}_\varepsilon \tilde{\nu}_s - \partial_x \sigma(x) \mathcal{H}_{s,\varepsilon} + \bar{\mathcal{E}}^{\sigma}_{s,\varepsilon} \right) dx dW^0_s \\
&+ \int_0^t \int_0^\infty \psi^2 \bar{T}_\varepsilon \nu_s \nu_s(\{0\}) \partial_x p_\varepsilon(x) \\
&+ \int_0^t \int_0^\infty \psi^2 \rho^2 \left(\sigma(x) \partial_x \bar{T}_\varepsilon \tilde{\nu}_s - \partial_x \sigma(x) \mathcal{H}_{s,\varepsilon} + \bar{\mathcal{E}}^{\sigma}_{s,\varepsilon} \right)^2 dx ds.
\end{aligned}$$

Using the generalized Young's inequality and the estimates from Lemma 4.6.3 and Lemma 4.6.4 we can estimate the first term by

$$
\begin{aligned}
&-2\int_0^t\int_0^\infty \psi^2\bar{T}_\varepsilon\nu_s\left(\mu_s\partial_x\bar{T}_\varepsilon\tilde{\nu}_s-\partial_x\mu_s\mathcal{H}_{s,\varepsilon}+\bar{\mathcal{E}}^{\mu}_{s,\varepsilon}\right)dxds\\
&\leqslant c_\eta\int_0^t\left\|\psi\bar{T}_\varepsilon\nu_s\right\|_2^2ds+\eta\int_0^t\int_0^\infty\psi^2\mu_s^2|\partial_x\bar{T}_\varepsilon\tilde{\nu}_s|^2dxds+c_\eta\int_0^t\left\|\psi\bar{T}_{2\varepsilon}\nu_s\right\|_2^2ds+o(1).
\end{aligned}
$$

For the second term we apply integration by parts to obtain

$$
\begin{aligned}
&\int_0^t\int_0^\infty\psi^2\bar{T}_\varepsilon\nu_s\partial_x\left(\sigma^2(x)\partial_x\bar{T}_\varepsilon\tilde{\nu}_s-\partial_x\sigma^2(x)\mathcal{H}_{s,\varepsilon}+\bar{\mathcal{E}}^{\sigma^2}_{s,\varepsilon}\right)dxds\\
&=-\int_0^t\int_0^\infty\psi^2\sigma^2(x)\partial_x\bar{T}_\varepsilon\nu_s\partial_x\bar{T}_\varepsilon\tilde{\nu}_sdxds-\int_0^t\int_0^\infty2\psi\psi'\sigma^2(x)\bar{T}_\varepsilon\nu_s\partial_x\bar{T}_\varepsilon\tilde{\nu}_sdxds\\
&-\int_0^t\int_0^\infty\psi^2\partial_x\bar{T}_\varepsilon\nu_s\partial_x\sigma^2(x)\mathcal{H}_{s,\varepsilon}dxds-\int_0^t\int_0^\infty2\psi\psi'\sigma^2(x)\bar{T}_\varepsilon\tilde{\nu}_s\partial_x\sigma^2(x)\mathcal{H}_{s,\varepsilon}dxds\\
&-\int_0^t\int_0^\infty\psi^2\partial_x\bar{T}_\varepsilon\nu_s\bar{\mathcal{E}}^{\sigma^2}_{s,\varepsilon}dxds-\int_0^t\int_0^\infty2\psi\psi'\sigma^2(x)\bar{T}_\varepsilon\tilde{\nu}_s\bar{\mathcal{E}}^{\sigma^2}_{s,\varepsilon}dxds\\
&\leqslant-\int_0^t\int_0^\infty\psi^2\sigma^2(x)|\partial_x\bar{T}_\varepsilon\tilde{\nu}_s|^2dxds+\eta\int_0^t\int_0^\infty\psi^2|\partial_x\bar{T}_\varepsilon\tilde{\nu}_s|^2dxds\\
&\quad+c_\eta\int_0^t\int_A|\bar{T}_\varepsilon\nu_s|^2dxds+c_\eta\int_0^t\int_A|\bar{T}_{2\varepsilon}\nu_s|^2dxds+o(1).
\end{aligned}
$$

We leave the term involving the stochastic integral unchanged for now. For the fourth term, the generalized Young's inequality together with the estimate

$$
\int_0^\infty\psi^2(x)\left(\partial_{xx}p_\varepsilon(x)\right)^2dx\to0,\quad\text{as }\varepsilon\to0 \tag{4.6.7}
$$

yields

$$
\int_0^t\int_0^\infty\psi^2\bar{T}_\varepsilon\nu_s\nu_s(\{0\})\partial_{xx}p_\varepsilon(x)dxds\leqslant c_\eta\int_0^t\int_0^\infty\left\|\psi\bar{T}_\varepsilon\nu_s\right\|_2^2ds+o(1).
$$

The convergence (4.6.7) follows from the fact that ψ is a cut-off function and that we only integrate over the set A with the infimum of A being strictly positive. The decay of the heat kernel derivative in the interior of the positive half-line then yields the convergence to zero.

For the final term we get

$$\begin{aligned}
&\int_0^t \int_0^\infty \psi^2 \rho^2 \left(\sigma(x)\partial_x \bar{T}_\varepsilon \tilde{\nu}_s - \partial_x \sigma(x) \mathcal{H}_{s,\varepsilon} + \bar{\mathcal{E}}^\sigma_{s,\varepsilon}\right)^2 dxds \\
&= (1+\eta) \int_0^t \int_0^\infty \psi^2 \rho^2 \sigma^2(x) |\partial_x \bar{T}_\varepsilon \tilde{\nu}_s|^2 dxds \\
&\quad + (1+c_\eta) \int_0^t \int_0^\infty \psi^2(x) \left(\bar{\mathcal{E}}^\sigma_{s,\varepsilon}\right)^2 dxds + (1+c_\eta) \int_0^t \int_0^\infty \psi^2 \left(\mathcal{H}_{s,\varepsilon}\right)^2 dxds \\
&\leqslant (1+\eta) \int_0^t \int_0^\infty \psi^2 \rho^2 \sigma^2(x) |\partial_x \bar{T}_\varepsilon \tilde{\nu}_s|^2 dxds + c_\eta \int_0^t \left\|\psi \bar{T}_\varepsilon \nu_s\right\|_2^2 dxds + o(1).
\end{aligned}$$

Putting everything back together and choosing $\eta > 0$ small enough such that

$$\sigma_s^2 - \rho_s^2(1 + 2\eta + \eta^2)\sigma_s^2 - \eta \mu_s^2 - \eta \geqslant c_0$$

for some $c_0 > 0$ yields

$$\begin{aligned}
\left\|\psi \bar{T}_\varepsilon \nu_t\right\|_2^2 \leqslant \left\|\bar{T}_\varepsilon \nu_0\right\|_{L^2(W)}^2 + c_\eta \int_0^t \left\|\bar{T}_\varepsilon \nu_s\right\|_{L^2(W)}^2 ds + \int_0^t \left\|\bar{T}_{2\varepsilon} \nu_s\right\|_{L^2(A)}^2 ds + o(1) \\
- \int_0^t \int_0^\infty \psi^2 \bar{T}_\varepsilon \nu_s \rho \left(\sigma(x)\partial_x \bar{T}_\varepsilon \tilde{\nu}_s - \partial_x \sigma(x) \mathcal{H}_{s,\varepsilon} + \bar{\mathcal{E}}^\sigma_{s,\varepsilon}\right) dx dW_s^0.
\end{aligned}$$

Next, we take supremum over t and then expectation. Using [37, Lemma 8.5] to estimate the stochastic integrals, which can easily be adapted to this case, we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]} \left\|\psi T_\varepsilon^{E,\kappa} \nu_s\right\|_2^2\right] \leqslant & c' \mathbb{E}\left[\left\|T_\varepsilon^{E,\kappa} \nu_0\right\|_{L^2(A)}^2\right] + c' \mathbb{E}\left[\int_0^t \left\|T_\varepsilon^{E,\kappa} \nu_s\right\|_{L^2(A)}^2 ds\right] \\
& + c' \mathbb{E}\left[\int_0^t \left\|T_{2\varepsilon}^{E,\kappa} \nu_s\right\|_{L^2(A)}^2 ds\right] + o(1)
\end{aligned}$$

with a constant $c' > 0$. Taking the limit $\varepsilon \to 0$ and applying Lemma 4.6.2 we obtain

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{s\in[0,t]} \left\|\psi \bar{T}_\varepsilon \nu_t\right\|_2^2\right] \leqslant \lim_{\varepsilon\to 0} \mathbb{E}\left[\left\|\psi \bar{T}_\varepsilon \nu_0\right\|_2^2\right] + 2 \lim_{\varepsilon\to 0} c_\eta \mathbb{E}\left[\int_0^t \int_A \left(\bar{T}_\varepsilon \nu_s\right)^2 dxds\right] < \infty.$$

Using the definition of this test function ψ this yields

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{s\in[0,t]} \left\|\bar{T}_\varepsilon \nu_t\right\|_{L^2(B)}^2\right] < \infty \tag{4.6.8}$$

and we obtain that ν_t has a density on $(0,\infty)$ or alternatively $\tilde{\nu}_t$ for all $t \in [0,T]$ almost surely. Moreover, this density is in $L^2(\lambda,\infty)$ for all $\lambda > 0$.

□

We can also establish the following uniform L^2-regularity.

Lemma 4.6.6. *Let (ν, W^0) be a solution to the limit SPDE (4.1.2) with ν in the class $\mathbb{D}$. Then, with probability one we have that*

$$\sup_{s\in[0,T]} \sup_{\varepsilon>0} \left\| T_\varepsilon^{S,c} \nu_s \right\|^2_{L^2(\lambda,\infty)} < \infty.$$

Proof. Follows from (4.6.8) by applying the arguments from [37, Proposition 7.3]. □

4.6.2 Regularity Results near the Boundary

In the previous subsection we established existence of a density and L^2 regularity away from the boundary for solutions to the limit SPDE (4.1.2) in the class $\mathbb{D}$. We now restrict to the case where the coefficients μ and σ are constant. In this case we show further regularity results for the measure ν_t near the boundary at zero. The two important components of the proof are a probabilistic representation of solutions and the following result by Warren [72] on the conditional distribution of a sticky Brownian motion given the driving Brownian noise.

Proposition 4.6.7 (Warren 1997)**.** *Suppose that X is a sticky Brownian motion with $X_0 = 0$ and that W is the driving Wiener process, also starting from zero. Letting $L_t = \sup_{s\leqslant t}(-W_s)$ we have*

$$\mathbb{P}(X_t \leqslant x | \mathcal{F}_t^W) = \exp\left(-\frac{1}{c}(W_t + L_t - x)\right)$$

a.s. for $x \in [0, W_t + L_t]$, where $\mathcal{F}^W$ denotes the filtration generated by the Brownian motion W.

The next theorem gives the probabilistic representation and indicates how Proposition 4.6.7 can be helpful.

Theorem 4.6.8. *Let (ν, W^0) be the unique solution to the SPDE (4.1.2) with ν in the class $\mathbb{D}$. Then,*

$$\langle \nu_t, \phi \rangle = \mathbb{E}\left[\phi(X_t) | \mathcal{F}_t^{W^0}\right] \tag{4.6.9}$$

where

$$dX_t = 1_{X_t>0}\left(\mu dt + \sigma(\sqrt{1-\rho^2} dW_t^1 + \rho dW_t^0)\right) + \frac{1}{2c} 1_{X_t=0} dt. \tag{4.6.10}$$

Proof. Using the approximation via a particle system we discussed in Section 4.4 and applying Birkhoff's ergodic theorem we obtain the desired representation for $\langle \nu_t, \phi \rangle$. □

Using this we can prove our regularity result.

Theorem 4.6.9. *Let (ν, W^0) be a the solution to the limit SPDE (4.1.2) with constant coefficients in the class $\mathbb{D}$. Then we have the estimate $\tilde{\nu}_t(0, \varepsilon) \leqslant C_t \varepsilon$ as $\varepsilon \to 0$ for a constant $C_t > 0$.*

Proof. Let (ν, W^0) be the unique solution to the SPDE (4.1.2). Then, by Theorem 4.6.8, ν has the probabilistic representation (4.6.9)-(4.6.10).

Note that X is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and W^0 and W^1 are $(\mathcal{F}_t)$-Brownian motions. However, $\mathcal{F}$ is larger than the filtration generated by the two Brownian motions.

We set $Z_t = \zeta(X_t)$ with ζ being the scale transformation used in Lemma 4.3.1 and define a measure $\mathbb{Q}$ through the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(-\tilde{\mu} W_t^{0,1} - \frac{1}{2}\tilde{\mu}^2 t\right) := \Xi_t$$

where

$$W_t^{0,1} = \rho W_t^0 + \sqrt{1-\rho^2} W_t^1.$$

Using Girsanov's theorem we then obtain that the process $W^{\mathbb{Q}}$ given by

$$W_t^{\mathbb{Q}} := W_t^{0,1} + \tilde{\mu} t$$

is a Brownian motion under the measure $\mathbb{Q}$ started at 0 and the process Z is a sticky Brownian motion driven by $W^{\mathbb{Q}}$ under $\mathbb{Q}$. Moreover we define the local time process $L_t^{\mathbb{Q}} := -\inf_{s \leqslant t} W_s^{\mathbb{Q}}$.

Define the stopping time τ by $\tau := \inf\{t > 0 : X_t \leqslant 0\}$. We then have

$$\begin{aligned} \mathbb{P}\left(0 < X_t < \varepsilon \middle| \mathcal{F}_t^{W^0}\right) =& \mathbb{P}\left(0 < Z_t < \zeta(\varepsilon) \middle| \mathcal{F}_t^{W^0}\right) \\ =& \mathbb{P}\left(0 < Z_t < \zeta(\varepsilon), t < \tau \middle| \mathcal{F}_t^{W^0}\right) \\ &+ \mathbb{P}\left(0 < Z_t < \zeta(\varepsilon), \tau \leqslant t \middle| \mathcal{F}_t^{W^0}.\right) \end{aligned} \tag{4.6.11}$$

Note that on the set $\{t < \tau\}$ the process Z is a Brownian motion with drift $\tilde{\mu}$. Thus, the first term on the right-hand side is the conditional probability given $\mathcal{F}^{W^0}$

of the process Z killed when it hits the boundary at zero. By [37, Corollary 7.4] the measure

$$\mathbb{P}\left(Z_t \in \cdot, t < \tau \middle| \mathcal{F}_t^{W_0}\right) = \mathbb{P}\left(Z_0 + \sqrt{1-\rho^2}W_t^1 + \rho W_t^0 + \tilde{\mu}t \in \cdot, t < \tau \middle| \mathcal{F}_t^{W^0}\right)$$

has an L^2 density p_t^{abs} for all $t \in [0,T]$ almost surely. We can use this to estimate

$$\mathbb{P}\left(0 < Z_t < \zeta(\varepsilon), t < \tau \middle| \mathcal{F}_t^{W^0}\right) = \int_0^{\zeta(\varepsilon)} p_t^{\text{abs}}(x)dx \leqslant \int_0^{C\varepsilon} p_t^{\text{abs}}(x)dx = C\int_0^{\varepsilon} p_t^{\text{abs}}(Cx)dx.$$

Furthermore we have the estimate

$$\mathbb{P}\left(0 < Z_t < \zeta(\varepsilon), t < \tau \middle| \mathcal{F}_t^{W^0}\right) \leqslant C_t\varepsilon \tag{4.6.12}$$

almost surely for all $t \in [0,T]$.

For the second term on the right-hand side of (4.6.11) we calculate

$$\begin{aligned}\mathbb{P}\left(0 < Z_t < \zeta(\varepsilon), \tau \leqslant t \middle| \mathcal{F}_t^{W^0}\right) &= \mathbb{E}^{\mathbb{P}}\left[1_{0<Z_t<\zeta(\varepsilon)} 1_{\zeta(\varepsilon)<Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}} 1_{\tau\leqslant t} \middle| \mathcal{F}_t^{W^0}\right] \\ &\quad + \mathbb{E}^{\mathbb{P}}\left[1_{0<Z_t<\zeta(\varepsilon)} 1_{Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}\leqslant\zeta(\varepsilon)} 1_{\tau\leqslant t} \middle| \mathcal{F}_t^{W^0}\right].\end{aligned}$$

For the first conditional expectation on the right-hand side we apply the tower property to obtain

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[1_{0<Z_t<\zeta(\varepsilon)} 1_{\zeta(\varepsilon)<Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}} 1_{\tau\leqslant t} \middle| \mathcal{F}_t^{W^0}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{P}\left(0 < Z_t < \zeta(\varepsilon) \middle| \mathcal{F}_t^{W^0,W^1} \vee F_\tau\right) 1_{\zeta(\varepsilon)<Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}} 1_{\tau\leqslant t} \middle| \mathcal{F}_t^{W^0}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\Xi_t\Xi_t^{-1} 1_{0<Z_t<\zeta(\varepsilon)} \middle| \mathcal{F}_t^{W^0,W^1} \vee F_\tau\right] 1_{\zeta(\varepsilon)<Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}} 1_{\tau\leqslant t} \middle| \mathcal{F}_t^{W^0}\right].\end{aligned} \tag{4.6.13}$$

Since Ξ_t^{-1} is measurable with respect to $\mathcal{F}_t^{W^0,W^1}$ we can take it outside of the inner conditional expectation and then use the Bayes' rule to perform a change of measure for the inner conditional expectation. This yields

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[1_{0<Z_t<\zeta(\varepsilon)} 1_{\zeta(\varepsilon)<Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}} 1_{\tau\leqslant t} \middle| \mathcal{F}_t^{W^0}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{Q}\left(0 < Z_t < \zeta(\varepsilon) \middle| \mathcal{F}_t^{W^0,W^1} \vee F_\tau\right) 1_{\zeta(\varepsilon)<Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}} 1_{\tau\leqslant t} \middle| \mathcal{F}_t^{W^0}\right].\end{aligned}$$

Under $\mathbb{Q}$ the process Z is a sticky Brownian motion started at Z_0. Sticky Brownian motion is a strong Markov process by [60, Proposition 2.2] thus we can define

$$\tilde{Z}_t := Z_{t+\tau}.$$

Then, given $\mathcal{F}_\tau$ the process $\tilde{Z}$ is a sticky Brownian motion with $\tilde{Z}_0 = Z_\tau = 0$ driven by

$$\tilde{W}_t := W_{t+\tau}^{\mathbb{Q}} - W_\tau^{\mathbb{Q}}$$

which is a standard Brownian motion with $\tilde{W}_0 = 0$ given $\mathcal{F}_\tau$. Note that for the stopping time τ we have

$$\begin{aligned}\tau = \inf\{t > 0 : X_t \leqslant 0\} = \inf\{t > 0 : Z_t \leqslant 0\} &= \inf\{t > 0 : Z_0 + W_t^{\mathbb{Q}} \leqslant 0\} \\ &= \inf\{t > 0 : W_t^{\mathbb{Q}} \leqslant -Z_0\}.\end{aligned}$$

This yields $W_\tau^{\mathbb{Q}} = -Z_0$. We can then use Proposition 4.6.7 to calculate

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[\mathbb{Q}\left(0 < Z_t < \zeta(\varepsilon)\middle|\mathcal{F}_t^{W^0,W^1} \vee F_\tau\right) 1_{\zeta(\varepsilon)<Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}} 1_{\tau\leqslant t}\middle|\mathcal{F}_t^{W^0}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{Q}\left(0 < \tilde{Z}_{t-\tau} < \zeta(\varepsilon)\middle|\mathcal{F}_t^{W^0,W^1} \vee F_\tau\right) 1_{\zeta(\varepsilon)<Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}} 1_{\tau\leqslant t}\middle|\mathcal{F}_t^{W^0}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\frac{1}{c}\int_0^{\zeta(\varepsilon)} \exp\left(-\frac{1}{c}\left(\tilde{W}_{t-\tau} + \tilde{L}_{t-\tau} - x\right)\right) dx 1_{\zeta(\varepsilon)<Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}} 1_{\tau\leqslant t}\middle|\mathcal{F}_t^{W^0}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\frac{1}{c}\int_0^{\zeta(\varepsilon)} \exp\left(-\frac{1}{c}\left(Z_0 + W_t^{\mathbb{Q}} + L_t^{\mathbb{Q}} - x\right)\right) dx 1_{\zeta(\varepsilon)<Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}} 1_{\tau\leqslant t}\middle|\mathcal{F}_t^{W^0}\right] \\ &\leqslant \frac{1}{c}\zeta(\varepsilon) \leqslant C\varepsilon.\end{aligned}$$

For the second term we obtain in a similar way

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[\mathbb{Q}\left(0 < Z_t < \zeta(\varepsilon)\middle|\mathcal{F}_t^{W^0,W^1} \vee \mathcal{F}_\tau\right) 1_{Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}\leqslant\zeta(\varepsilon)} 1_{\tau\leqslant t}\middle|\mathcal{F}_t^{W^0}\right] \\ &\leqslant \mathbb{E}^{\mathbb{P}}\left[\left(1 - \exp\left(-\frac{1}{c}\left(Z_0 + W_t^{\mathbb{Q}} + L_t^{\mathbb{Q}}\right)\right)\right) 1_{Z_0+W_t^{\mathbb{Q}}+L_t^{\mathbb{Q}}\leqslant\zeta(\varepsilon)}\middle|\mathcal{F}_t^{W^0}\right] \\ &= \int_0^{\zeta(\varepsilon)} \left(1 - e^{-\frac{1}{c}x}\right) p_t^{\text{refl}}(x)dx \\ &\leqslant \int_0^{C\varepsilon} \left(1 - e^{\frac{1}{c}x}\right) p_t^{\text{refl}}(x)dx \\ &= C\int_0^{\varepsilon} \left(1 - e^{-\frac{C}{c}x}\right) p_t^{\text{refl}}(Cx)dx\end{aligned}$$

where p_t^{refl} is the density of the measure

$$\mathbb{P}\left(Z_0 + W_t^{\mathbb{Q}} + L_t^{\mathbb{Q}} \in \cdot\middle|\mathcal{F}_t^{W^0}\right) = \mathbb{P}\left(Z_0 + \sqrt{1-\rho^2}W_t^1 + \rho W_t^0 + \tilde{\mu}t + L_t \in \cdot\middle|\mathcal{F}_t^{W^0}\right)$$

which exists for all $t \in [0, T]$ almost surely by adjusting the arguments from Chapter 3 to the case of a reflecting boundary.

We then have the estimate

$$\mathbb{P}\left(0 < X_t < \varepsilon\middle|\mathcal{F}_t^{\nu,W^0}\right) \leqslant \int_0^{\varepsilon} C\left(1 + p_t^{\text{abs}}(x) + \left(1 - e^{-\frac{C}{c}x}\right) p_t^{\text{refl}}(x)\right) dx$$

with

$$\left(1 - e^{-Cx/c}\right) = O(x), \quad \text{as } x \to 0$$

and $p_t^{\text{refl}} \in L^1(0, \varepsilon)$. Thus we obtain the bound

$$\mathbb{P}\left(0 < X_t < \varepsilon \middle| \mathcal{F}_t^{\nu, W^0}\right) \leqslant C_t \varepsilon$$

for some constant $C_t > 0$. □

Remark. The reason we need to restrict to constant coefficients in this section is that we exploit the fact that for constant coefficients Ξ_t and Ξ_t^{-1} are measurable functions of the Brownian motions W^0 and W^1. Since X and Z are not adapted to $\mathcal{F}^{W^0, W^1}$ we lose this measurably when considering coefficients that depend on X. In (4.6.13) we take advantage of this measurability.

A similar argument can be employed to find an expression for the point mass at zero.

Theorem 4.6.10. *Let (ν, W^0) be the unique solution to the SPDE (4.1.2) with ν in the class $\mathbb{D}$ with constant coefficients μ and σ. Then the point mass at zero is given by*

$$\nu_t(\{0\}) = \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\frac{1}{c}\left(\frac{X_0}{\sigma} + \rho W_t^0 + \sqrt{1-\rho^2} W_t^1 + \tilde{\mu} t + L_t\right)\right) 1_{\tau \leqslant t} \middle| \mathcal{F}_t^{W^0}\right]$$

where τ denotes the first hitting time at zero of the process X defined in (4.6.10).

Proof. We can follow the same ideas as in the proof of Theorem 4.6.9 setting $Z_t = \zeta(X_t)$ with ζ the scale transformation from Lemma 4.3.1

$$\begin{aligned}
\mathbb{P}\left(X_t = 0 | \mathcal{F}_t^{W^0}\right) =& \mathbb{P}\left(Z_t = 0 | \mathcal{F}_t^{W^0}\right) \\
=& \mathbb{P}\left(Z_t = 0, t < \tau | \mathcal{F}_t^{W^0}\right) \\
&+ \mathbb{P}\left(Z_t = 0, \tau \leqslant t | \mathcal{F}_t^{W^0}\right) \\
=& \mathbb{P}\left(Z_t = 0, \tau \leqslant t | \mathcal{F}_t^{W^0}\right).
\end{aligned}$$

We can then calculate

$$\begin{aligned}
\mathbb{P}\left(Z_t = 0, \tau \leqslant t | \mathcal{F}_t^{W^0}\right) =& \mathbb{E}^{\mathbb{P}}\left[1_{Z_t = 0} 1_{\tau \leqslant t} \middle| \mathcal{F}_t^{W^0}\right] \\
=& \mathbb{E}^{\mathbb{P}}\left[\mathbb{P}\left(Z_t = 0 \middle| \mathcal{F}_t^{W^0, W^1} \vee \mathcal{F}_\tau\right) 1_{\tau \leqslant t} \middle| \mathcal{F}_t^{W^0}\right].
\end{aligned}$$

Proceeding as is the proof of Theorem 4.6.9 we then obtain

$$\nu_t(\{0\}) = \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\frac{1}{c}(Z_0 + \rho W_t^0 + \sqrt{1-\rho^2}W_t^1 + \tilde{\mu}t + L_t)\right) 1_{\tau \leqslant t} \middle| \mathcal{F}_t^{W^0}\right]$$

□

The above discussion shows some of the limitations of the results about the behaviour of the density near the boundary that we can establish. The fact that we cannot show continuity of the density is one of the main problems. However, we can show weaker versions of known properties in the case without a common noise. One of the interesting properties of a sticky Brownian motion B^{sticky} is a relation between the point mass at zero and the right limit of the density at zero, i.e

$$\mathbb{P}\left(B_t^{\text{sticky}} = 0\right) = cG_t^{S,c}(x_0, 0+).$$

The following lemma presents a weak version of this result.

Proposition 4.6.11. *Let (ν, W^0) be a solution to the SPDE (4.1.2) with ν in the class $\mathbb{D}$, and let ϕ_n be a sequence of test functions in $\mathcal{S}$ such that $\phi_n \to 0$ and $\phi_n' \to 1_{\{0\}}$ as $n \to \infty$. Then, for all $t \in [0,T]$ we have the convergence*

$$\int_0^t \nu_s(\{0\})ds = -\lim_{n\to\infty} c\int_0^t \langle \tilde{\nu}_s, \sigma^2(\cdot)\phi_n''\rangle ds.$$

Proof. Plug the test function ϕ_n into the limit SPDE (4.1.2) to obtain

$$\begin{aligned}\langle \nu_t, \phi_n\rangle = \langle \nu_0, \phi_n\rangle &+ \int_0^t \langle \tilde{\nu}_s, \mu(\cdot, \nu_s)\phi_n'\rangle ds + \frac{1}{2}\int_0^t \langle \tilde{\nu}_s, \sigma^2(\cdot)\phi_n''\rangle ds \\ &+ \int_0^t \langle \tilde{\nu}_s, \rho\sigma(\cdot)\phi_n'\rangle dW_s^0 + \frac{\phi_n'(0)}{2c}\int_0^t \nu_s(\{0\})ds.\end{aligned}$$

Taking the limit $n \to \infty$, using bounded convergence and the convergence to zero of the quadratic variation of the stochastic integral we obtain the desired equation

$$\int_0^t \nu_s(\{0\})ds = -\lim_{n\to\infty} c\int_0^t \langle \tilde{\nu}_s, \sigma^2(\cdot)\phi_n''\rangle ds. \tag{4.6.14}$$

□

Note that the definition of the sequence ϕ_n implies that $-\phi_n''$ is an approximation of the Dirac delta at zero. Thus, if the density of $\tilde{\nu}_s$ were continuous up to the boundary, the integrand on the right-hand side of (4.6.14) would converge to a multiple of the right-limit of the density at zero.

Our results are not sufficient to establish L^2-regularity of the density V_t of $\tilde{\nu}_t$ near the boundary. It is possible that the common noise causes the density to strongly oscillate near the boundary at zero. These oscillations could cause the density to not be L^2-integrable near the boundary. This is similar to some of the issues that we discussed in the reflecting case in Chapter 2 where the results on heat singularities can only be stated in terms of the limes-superior of the density near the boundary because of potential oscillations.

The exact regularity near the boundary is a potential topic for further research that could also help to complete the uniqueness proof in the case where the dependence on the measure in the drift term μ is present all the way to the boundary.

4.7 The Uniqueness Proof in the Nonlinear Case

In this section we prove Theorem 4.1.6 and establish uniqueness in the case of nonlinearities that vanish near the boundary. Our proof of this result relies on the uniform L^2-regularity of the smoothed solutions $T_\varepsilon^{S,c}\nu_t$ away from the boundary that we proved in Section 4.6. The proof closely follows the argument is [41].

We let (ν, W^0) and $(\bar{\nu}, W^0)$ be two weak solutions to the nonlinear SPDE (4.1.2) with the same Brownian motion and with ν and $\bar{\nu}$ in the class $\mathbb{D}$ and set their difference to be $\Delta := \nu - \bar{\nu}$. Moreover, we define the difference of the nonlinearities $\delta_t^\mu := \mu(x, \nu_t) - \mu(x, \bar{\nu}_t)$. The same approach as in the proof in the linear case in Section 4.5 can be used to find the dynamics of $\partial_x^{-1} T_\varepsilon^{S,c}\Delta_t$,

$$\begin{aligned} d\partial_x^{-1} T_\varepsilon^{S,c}\Delta_t = &- \left(\mu(x, \tilde{\nu}_t) T_\varepsilon^{S,c}\tilde{\Delta}_t + \delta_t^\mu T_\varepsilon^{S,c}\tilde{\nu}_t\right) dt \\ &+ \frac{1}{2}\partial_x \left(\sigma^2(x) T_\varepsilon^{S,c}\tilde{\Delta}_t + \mathcal{E}_{t,\varepsilon}^{\sigma^2} - \bar{\mathcal{E}}_{t,\varepsilon}^{\sigma^2}\right) dt \\ &+ \rho\sigma(x) T_\varepsilon^{S,c}\tilde{\Delta}_t dW_t^0 + o_{sq}(1)dt + o_{sq}(1)dW_t^0 \\ &+ \frac{1}{2}\partial_x G_\varepsilon^{S,c}(x, 0)\Delta_t(\{0\})dt. \end{aligned}$$

The difference compared to the linear case is the term involving δ_t^μ that appears from the nonlinearity of the equation. We can apply Itô's formula to get

$$
\begin{aligned}
d(\partial_x^{-1}T_\varepsilon^{S,c}\Delta_t)^2 =& -2\partial_x^{-1}T_\varepsilon^{S,c}\Delta_t\left(\mu(x,\bar\nu_t)T_\varepsilon^{S,c}\tilde\Delta_t+\delta_t^\mu T_\varepsilon^{S,c}\tilde\nu_t\right)dt\\
&+\partial_x^{-1}T_\varepsilon^{S,c}\Delta_t\partial_x\left(\sigma^2(x)T_\varepsilon^{S,c}\tilde\Delta_t+\mathcal{E}_{t,\varepsilon}^{\sigma^2}-\bar{\mathcal{E}}_{t,\varepsilon}^{\sigma^2}\right)dt\\
&+2\rho\sigma^2(x)\partial_x^{-1}T_\varepsilon^{S,c}\Delta_tT_\varepsilon^{S,c}\tilde\Delta_tdW_t^0+\partial_x^{-1}T_\varepsilon^{S,c}\Delta_to_{sq}(1)dt\\
&+\partial_x^{-1}T_\varepsilon^{S,c}\Delta_to_{sq}(1)dW_t^0+\rho^2\sigma^2(x)(T_\varepsilon^{S,c}\tilde\Delta_t)^2dt\\
&+o_{sq}^2(1)dt+\rho\sigma(x)T_\varepsilon^{S,c}\tilde\Delta_to_{sq}(1)dt\\
&+\partial_x^{-1}T_\varepsilon^{S,c}\Delta_t\partial_xG_\varepsilon^{S,c}(x,0)\Delta_t(\{0\})dt.
\end{aligned}
$$

We proceed as in the linear case, i.e. we integrate over $x\geqslant 0$ to obtain the squared L^2-norm of $\partial_x^{-1}T_\varepsilon^{S,c}\Delta_t$ and take expectation. We treat the terms on the right-hand side in the same way as in Section 4.5 and for now, we leave the term involving δ_t^μ. This yields the estimate

$$
\begin{aligned}
\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_t\right\|_2^2\right]\leqslant& c_\eta\int_0^t\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\right\|_2^2\right]ds\\
&-2\mathbb{E}\left[\int_0^t\int_0^\infty\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\delta_s^\mu T_\varepsilon^{S,c}\tilde\nu_sdxds\right]\\
&-\mathbb{E}\left[\int_0^t\int_0^\infty(\sigma^2(x)(1-(1+\eta)\rho^2)-\eta\mu^2(x,\bar\nu_s)-3\eta)|T_\varepsilon^{S,c}\tilde\Delta_s(x)|^2dxds\right]\\
&-c(g_\varepsilon^{S,c}(0,0))^2\mathbb{E}\left[\int_0^t(\Delta_s(\{0\}))^2ds\right]+o(1).
\end{aligned}
\tag{4.7.1}
$$

When estimating the term involving δ_t^μ we first note that, by Assumption 4.1.3, δ_t^μ vanishes for $x<\lambda$ for some $\lambda>0$. Thus we have

$$
\left|\mathbb{E}\left[\int_0^t\int_0^\infty\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\delta_s^\mu T_\varepsilon^{S,c}\tilde\nu_sdxds\right]\right|\leqslant\ \mathbb{E}\left[\int_0^t\left|\int_\lambda^\infty\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\delta_s^\mu T_\varepsilon^{S,c}\tilde\nu_sdx\right|ds\right]
$$

Next, we apply Cauchy-Schwarz' inequality and use the Lipschitz continuity of μ in the measure variable. This yields

$$
\left|\mathbb{E}\left[\int_0^t\int_0^\infty\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\delta_s^\mu T_\varepsilon^{S,c}\tilde\nu_sdxds\right]\right|\leqslant c\mathbb{E}\left[\int_0^t\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\right\|_2\left\|T_\varepsilon^{S,c}\tilde\nu_s\right\|_{L^2(\lambda,\infty)}d_0(\nu_s,\bar\nu_s)ds\right]
$$

We define a sequence of stopping times (t_n) by

$$
t_n:=\inf\left\{t\geqslant 0:\sup_{s\leqslant t}\sup_{\varepsilon>0}\left\|T_\varepsilon^{S,c}\tilde\nu_s\right\|_{L^2(\lambda,\infty)}>n\right\}\wedge T.
$$

The uniform L^2-regularity from Lemma 4.6.6 yields that $t_n \to T$ a.s. as $n \to \infty$. Moreover, we can again choose $\eta > 0$ such that we can find a constant $c_0 > 0$ with

$$(\sigma^2(x)(1-(1+\eta)\rho^2) - \eta\mu^2(x,\bar{\nu}_s) - 3\eta) \geqslant c_0.$$

Our estimate (4.7.1) then becomes

$$\begin{aligned}
&\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_{t\wedge t_n}\right\|_2^2\right] + c_0\mathbb{E}\left[\int_0^{t\wedge t_n}\left\|T_\varepsilon^{S,c}\tilde{\Delta}_s\right\|_2^2 ds\right] \\
&\leqslant c_\eta\int_0^t \mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_{s\wedge t_n}\right\|_2^2 ds\right] \\
&\quad + c_\eta n\mathbb{E}\left[\int_0^{t\wedge t_n}\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\right\|_2 d_0(\nu_s,\bar{\nu}_s)ds\right] \\
&\quad - c(g_\varepsilon^{S,c}(0,0))^2\mathbb{E}\left[\int_0^{t\wedge t_n}(\Delta_s(\{0\}))^2 ds\right] + o(1).
\end{aligned}$$

Applying the integrating factor $\exp(-c_\eta t)$ yields

$$\begin{aligned}
\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_{t\wedge t_n}\right\|_2^2\right] \leqslant& c_n\mathbb{E}\left[\int_0^{t\wedge t_n}\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\right\|_2 d_0(\nu_s,\bar{\nu}_s)ds\right] \\
&- c_n(g_\varepsilon^{S,c}(0,0))^2\mathbb{E}\left[\int_0^{t\wedge t_n}(\Delta_s(\{0\}))^2 ds\right] \\
&- c_0\mathbb{E}\left[\int_0^{t\wedge t_n}\left\|T_\varepsilon^{S,c}\tilde{\Delta}_s\right\|_2^2 ds\right] + o(1).
\end{aligned}$$

The following Lemma is an adaptation of the result in the absorbing case [41, Lemma 5.2] and helps to find an upper bound for the bounded Lipschitz distance $d_0(\nu_s, \tilde{\nu}_s)$.

Lemma 4.7.1. *There exists $c > 0$ so that for all $\delta \in (0,1)$ and all $\lambda > 1$ and almost all all $s \leqslant T$*

$$d_0(\nu_s,\bar{\nu}_s) \leqslant c\sqrt{\lambda+\delta^{-1}}\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\right\|_2 + c\delta^{\frac{1}{2}}\left\|\frac{d\tilde{\Delta}_s}{dx}\right\|_2 + c|\Delta_s(\{0\})| + x_s^\lambda + z_s^\varepsilon + o(1)$$

as $\varepsilon \to 0$ where $(x_t^\lambda)_{t\leqslant T}$ is a process which satisfies that for all $a > 0$ there exists a constant c_a only depending on a such that $\mathbb{E}\left[\int_0^T |x_s^\lambda|^2 ds\right] \leqslant c_a e^{-a\lambda}$, and $(z_s^\varepsilon)_{s\leqslant T}$ is a process such that $\mathbb{E}\left[\int_0^T |z_s^\varepsilon|^2 ds\right] = o(1)$ as $\varepsilon \downarrow 0$.

Remark. We can establish the bound in Lemma 4.7.1 only for almost all $s \leqslant T$ because we have L^2-regularity of the density only for almost all $s \leqslant T$ by Lemma 4.6.1.

Proof. Fix $\psi \in \mathcal{C}_{d_0} \coloneqq \{f : \|f\|_{Lip} \leqslant 1, \|f\|_\infty \leqslant 1\}$. Furthermore, fix $\lambda > 1$ and $\delta > 0$. We then take a cutoff function $\chi \in C_c^\infty(\mathbb{R})$ equal to 1 on $[\delta, \lambda - 1]$ and supported in $(\delta/2, \lambda)$ such that

$$|\chi| \leqslant 1, \quad |\partial\chi| \leqslant C_\chi/\delta \text{ on } [\delta/2, \delta] \quad \text{and} \quad |\partial\chi| \leqslant C_\chi \text{ on } [\lambda - 1, \lambda].$$

The function $\chi\psi$ is then an element of the Sobolev space of functions with zero trace and one derivative in L^∞, $\mathcal{W}_0^{1,\infty}(\delta/2, \lambda)$, and we have the bound

$$\begin{aligned}\|\partial(\chi\psi)\|_2^2 &\leqslant \int_0^\delta |\partial\chi|^2|\psi|^2 dx + \int_{\lambda-1}^\lambda |\partial\chi|^2|\psi|^2 dx + \int_0^\lambda |\chi|^2|\partial\psi|^2 dx \\ &\leqslant C_\chi^2(\delta^{-1} + 1)\|\psi\|_{L^\infty(0,\lambda)}^2 + \lambda.\end{aligned}$$

We can observe that

$$|\langle \Delta_s, \psi\rangle| \leqslant |\langle \Delta_s, \chi\psi\rangle| + |\langle \Delta_s, (\chi - 1)\psi 1_{[0,\delta]}\rangle| + \langle \nu_s + \bar{\nu}_s, |\psi| 1_{[\lambda-1,\infty)}\rangle.$$

We focus on the second term on the right-hand side and estimate

$$\begin{aligned}|\langle \Delta_s, (\chi - 1)\psi 1_{[0,\delta]}\rangle| &= |\int_0^\infty (\chi - 1)\psi 1_{[0,\delta]} \Delta_s(dx)| \\ &\leqslant |\int_0^\infty (\chi - 1) 1_{[0,\delta]} \psi \frac{d\tilde{\Delta}_s}{dx} dx| + \|\psi\|_\infty |\Delta_s(\{0\})| \\ &\leqslant C\delta^{1/2} \left\| \frac{d\tilde{\Delta}_s}{dx} \right\|_2 + C|\Delta(\{0\})|.\end{aligned}$$

Thus, we obtain

$$|\langle \Delta_s, \psi\rangle| \leqslant |\langle \Delta_s, \chi\psi\rangle| + C\delta^{1/2} \left\| \frac{d\tilde{\Delta}_s}{dx} \right\|_2 + C|\Delta(\{0\})| + C\langle \nu_s + \bar{\nu}_s, \mathbb{1}_{[\lambda-1,\infty)}\rangle.$$

Using integration by parts and the Cauchy-Schwarz inequality yields

$$\begin{aligned}|\langle \Delta_s, \chi\psi\rangle| &\leqslant |(T_\varepsilon^{S,c}\Delta_s, \chi\psi)| + |(T_\varepsilon^{S,c}\Delta_s, \chi\psi) - \langle \Delta_s, \chi\psi\rangle| \\ &\leqslant \left\|\partial_x^{-1} T_\varepsilon^{S,c}\Delta_s\right\|_2 \|\partial_x(\chi\psi)\|_2 + |(T_\varepsilon^{S,c}\Delta_s, \chi\psi) - \langle \Delta_s, \chi\psi\rangle|\end{aligned}$$

where $(\cdot, \cdot)$ denotes the L^2-inner product. By the Arzela-Ascoli theorem applied to $\{\chi f : f \in \mathcal{C}_{d_0}\}$ we can find a finite family of functions $\{\varphi_i \in \mathrm{Lip}(\mathbb{R}) : i = 1, \ldots, k(\delta, \lambda)\}$ with support in $[\delta/2, \lambda]$ so that for any $f \in \mathcal{C}_{d_0}$,

$$\sup_{x\in\mathbb{R}} |\varphi_i(x) - \chi f(x)| \leqslant \delta^{1/2}\lambda^{-1/2} \quad \text{for some } i = 1, \ldots, k(\delta, \lambda).$$

Thus, we can find a φ_i such that

$$\begin{aligned}|(T_\varepsilon^{S,c}\Delta_s,\chi\psi)-\langle\Delta_s,\chi\psi\rangle| =&|(T_\varepsilon^{S,c}\Delta_s,\chi\psi-\varphi_i+\varphi_i)-\langle\Delta_s,\chi\psi-\varphi_i+\varphi_i\rangle|\\ \leqslant&|(T_\varepsilon^{S,c}\Delta_s,\varphi_i)-\langle\Delta_s,\varphi_i\rangle|+|(T_\varepsilon^{S,c}\Delta_s,\chi\psi-\varphi_i)|\\ &+|\langle\Delta_s,\chi\psi-\varphi_i\rangle|.\end{aligned}$$

Using the Cauchy-Schwarz inequality we get

$$|\langle\Delta_s,\chi\psi-\varphi_i\rangle|\leqslant\left\|\frac{d\tilde{\Delta}_s}{dx}\right\|_2\|\varphi_i-\chi\psi\|_2\leqslant\sqrt{\delta}\left\|\frac{d\tilde{\Delta}_s}{dx}\right\|_2$$

and

$$|(T_\varepsilon^{S,c}\Delta_s,\chi\psi-\varphi_i)|\leqslant\sqrt{\delta}\left\|\frac{d\tilde{\Delta}_s}{dx}\right\|_2+o(1).$$

Note that $(T_\varepsilon^{S,c}\Delta_s,\varphi_i)\to\langle\Delta_s,\varphi_i\rangle$ as $\varepsilon\to0$; so defining

$$z_s^\varepsilon=z_s^\varepsilon(\delta,\lambda)\coloneqq\sup\{|(T_\varepsilon^{S,c}\Delta_s,\varphi_i)-\langle\Delta_s,\varphi_i\rangle||:i=1,\dots,k(\delta,\lambda)\}$$

we have $\mathbb{E}\left[\int_0^T|z_s^\varepsilon|^2\right]ds\to0$ as $\varepsilon\to0$ by bounded convergence. Consequently we have

$$|\langle\Delta_s,\chi\psi-\varphi_i\rangle|\leqslant\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\right\|_2\|\partial_x(\chi\psi)\|_2+2\sqrt{\delta}\left\|\frac{d\tilde{\Delta}_s}{dx}\right\|_2+z_s^\varepsilon+o(1).$$

Combining everything and taking the supremum over all $\psi\in\mathcal{C}_{d_0}$ yields

$$d_0(\nu_s,\bar{\nu}_s)\leqslant c\sqrt{1+\delta^{-1}}\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_s\right\|_2+c\delta^{1/2}\left\|\frac{d\tilde{\Delta}_s}{dx}\right\|_2+c|\Delta_s(\{0\})|+x_s^\lambda+z_s^\varepsilon+o(1)$$

where $x_s^\lambda\coloneqq\langle\nu_s+\bar{\nu}_s,\mathbb{1}_{[\lambda-1,\infty)}\rangle$. Since ν and $\bar{\nu}$ are processes in the class $\mathbb{D}$ the exponential-tail property ensures the required decay of x_s^λ. □

Using the estimate for the distance d_0 and applying the generalized Young's inequality with free parameter $\eta>0$ we obtain

$$\begin{aligned}\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_{t\wedge t_n}\right\|_2^2\right]\leqslant&\delta c_n'\mathbb{E}\left[\int_0^{t\wedge t_n}\left\|\frac{d\tilde{\Delta}_s}{dx}\right\|_2^2ds\right]-c_0\mathbb{E}\left[\int_0^{t\wedge t_n}\left\|T_\varepsilon^{S,c}\tilde{\Delta}_s\right\|_2^2ds\right]\\ &+c_n'\sqrt{\lambda+\delta^{-1}}\int_0^t\mathbb{E}\left[\left\|\partial_x^{-1}T_\varepsilon^{S,c}\Delta_{s\wedge t_n}\right\|_2^2ds\right]\\ &-c_n'\mathbb{E}\left[\int_0^t(g_\varepsilon^{S,c}(0,0)^2-\eta)|\Delta_s(\{0\})|^2ds\right]+c_a'e^{-a\lambda}\\ &+o(1)\end{aligned}$$

as $\varepsilon \to 0$. We have $\left\| T_\varepsilon^{S,c} \tilde{\Delta}_s \right\|_2 \to \left\| \frac{d\tilde{\Delta}_s}{dx} \right\|_2$ for almost every s and by the monotone convergence theorem

$$\mathbb{E}\left[\int_0^{t\wedge t_n} \left\| T_\varepsilon^{S,c} \Delta_s \right\|_2^2 ds \right] \to \mathbb{E}\left[\int_0^{t\wedge t_n} \left\| \frac{d\tilde{\Delta}_s}{dx} \right\|_2^2 ds \right] \quad \text{as } \varepsilon \downarrow 0.$$

Take $\delta \coloneqq c_0/2c_n'$, the sum of the first two terms on the right-hand side is eventually non-positive for small $\varepsilon > 0$. Moreover, we fix an $0 < \varepsilon_0 < 1$ and choose $\eta > 0$ such that

$$\eta < g_\varepsilon^{S,c}(0,0)^2 \quad \text{for all } \varepsilon \leqslant \varepsilon_0.$$

Thus the first term in the third line will also eventually be non-positive for small $\varepsilon > 0$. We can apply the integrating factor $\exp(-c_n'\sqrt{1+\delta^{-1}})$ to the anti-derivative term and deduce that

$$\mathbb{E}\left[\left\| \partial_x^{-1} T_\varepsilon^{S,c} \Delta_{t\wedge t_n} \right\|_2^2 \right] \leqslant c_a' c_n' e^{c_n'\sqrt{\lambda+\delta^{-1}}T - a\lambda} + o(1)$$

as $\varepsilon \downarrow 0$. Taking the limit $\lambda \to \infty$ we get

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\left\| \partial_x^{-1} T_\varepsilon^{S,c} \Delta_{t\wedge t_n} \right\|_2^2 \right] = 0.$$

Then, we take the limit $n \to \infty$ and note that $t_n \uparrow T$. Using that ν_t and $\bar{\nu}_t$ are probability measures yields the desired pathwise uniqueness for the nonlinear SPDE.

Chapter 5

Feedback Models

In this chapter we study models with a stronger interaction with the boundary through feedback via stopping times. We consider particles on the positive half-line and associate either the first hitting time or an elastic stopping time at the boundary with each particle. When a stopping time is reached, all the other particles in the system jump closer to the boundary. This move towards the boundary can trigger further stopping times and lead to cascades of particles getting absorbed at the boundary. In the large particle limit these systems are connected to McKean–Vlasov problems with feedback through the distribution of hitting times. We will see that these dependencies create singular interactions that can lead to jumps and cause complexity in the mathematical analysis such as non-uniqueness of solutions in general.

Such McKean–Vlasov equations with positive feedback through hitting times have recently been used in the modelling of a range of phenomena, from the self-excitation of neurons [24, 25], through the modelling of default and systemic risk in banking networks, [3, 38, 52, 53, 57, 58], to a probabilistic representation for the supercooled Stefan problem [26].

The focus in the literature has been on models with feedback through absorbing stopping times, i.e. first hitting times at zero. Our focus in this chapter is mainly on extending these models by allowing for elastic stopping times, i.e. the first time the local time at the boundary exceeds an independent exponential random variable. These problems are interesting from a mathematical point of view but can also be applied in several areas. For example, they can be used to model default cascades and systemic risk in the banking sector when default does not happen immediately. The elastic boundary can represent factors that can influence default such as negotiations over additional funds.

In Section 5.1 we study a one-phase model with elastic feedback. This means that we consider particles on the positive half-line and the related McKean–Vlasov problem

with elastic feedback. We show existence and uniqueness for two solution concepts, physical and minimal solutions, using the results from [26] and [21], respectively. We prove convergence of the particle system to the law of the McKean–Vlasov problem following the approach in [21] and establish that the unique minimal solution coincides with the unique physical solution. Moreover, we find a connection to the Stefan problem with kinetic undercooling discussed in [4] and provide probabilistic extensions to the convergence results in [4].

In Section 5.2 we extend our analysis to an elastic two-phase model. We consider both particles on the positive and negative half-line that are connected via feedback though the elastic stopping times at zero. A version of this system with absorbing stopping time has been studied in [3] as a model for systemic risk in two competing regions. We use techniques similar to the ones we established in Section 5.1 to prove existence of solutions to the two-phase McKean–Vlasov problem and find a connected PDE problem. Additionally, we show convergence of the related particle system to the McKean–Vlasov problem.

A further extension to the elastic feedback model is discussed in Section 5.3. Therein, we consider a model with common noise. This means we add a common Brownian motion in the particle system. In our systemic risk model, this common noise represents the evolution of the market risk factor. The related limit equation is then a conditional McKean–Vlasov equation with feedback through the conditional distribution of the stopping time conditioned on the common noise. Such equations have been studied in the case of absorbing stopping times in [52] and [57] where existence of solutions is shown as well as conditions for the existence of blow-ups. We use the techniques we established in Section 5.1 and Section 5.2 to show existence of solutions. Moreover, we employ the results from [52] to find blow-up conditions. Uniqueness is still an open problem for the common noise equation, even in the absorbing case.

Finally, in Section 5.4 we study a model with absorbing feedback at the origin. We adjust the model studied in the literature by adding an upper reflecting boundary at 1. We show that the feedback in this case can be studied by considering the first time a process on the whole-space deviates from its running supremum by more than 1. Moreover, we show existence of solutions via particle approximation and find the connected PDE problem. Uniqueness of solutions in this model is an open problem and may be an interesting direction for future research.

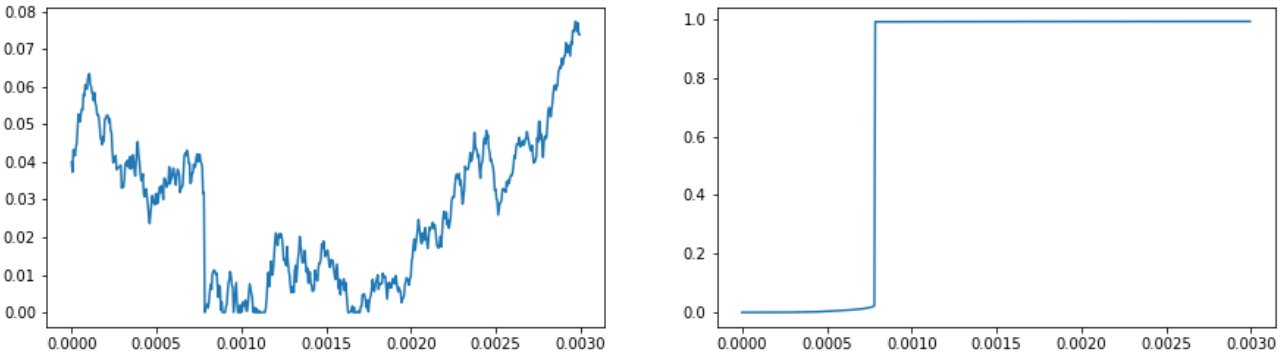


Figure 5.1: A Realization and the Related Loss Process of the Elastic Feedback Model

5.1 One-Phase Model with Elastic Feedback

In this section we study a one-phase model, i.e. a system of particles on the positive half-line. One-phase models with feedback through absorbing stopping times have been an active area of research in recent years [6, 21, 38, 26].

Our aim in this section is to consider a variant in which the feedback is provided through elastic stopping times. The McKean–Vlasov equation we consider is

$$\begin{cases} X_t = X_{0-} + B_t - \alpha\Lambda_t + L_t, \quad t \geqslant 0 \\ \Lambda_t = \mathbb{P}(\tau \leqslant t), \quad t \geqslant 0, \\ \tau = \inf\{t \geqslant 0 : L_t \geqslant \chi\}, \end{cases} \tag{5.1.1}$$

where χ is an exponential random variable with parameter $\kappa > 0$, X_{0-} is a random variable with law ν_{0-}, $\alpha > 0$ a constant, B a standard Brownian motion, independent of χ and X_{0-}, and L the reflection term we need to add to obtain a non-negative process. In the case of a continuous process, L is the local time but as we will see the process X may possess discontinuities and thus L is the positivity preserving function in the corresponding Skorokhod problem. Feedback models with absorbing stopping times can be obtained by setting $\chi = 0$. Figure 5.1 shows a realized path and the related loss function of the elastic model.

We will see that the elastic feedback model is closely connected to the supercooled Stefan problem with kinetic undercooling discussed in [4].

The discontinuities that can occur as a result of the feedback lead to non-uniqueness. To resolve this problem, in the absorbing case, [24] introduced the concept of a physical solution to such a McKean–Vlasov equation. Existence of such solutions has been established for several different feedback models [6, 15, 16, 24, 38, 58], while

uniqueness remained an open problem until Delarue et al. [26] were able to use the connection to the supercooled Stefan problem to prove uniqueness for a class of initial conditions with restricted oscillations. In [21], the authors analyzed minimal solutions as an alternative natural solution concept. They were able to prove the existence of minimal solutions and show that minimal solutions are physical solutions.

In Section 5.1.1 we will establish existence and uniqueness of physical and minimal solutions to the elastic feedback problem by exploiting a simple connection to a corresponding absorbing model. In Section 5.1.2 we show the connection to the Stefan problem with kinetic undercooling. We show weak convergence of a particle system to the McKean–Vlasov problem (5.1.1) in Section 5.1.3. Finally, in Section 5.1.4, we prove convergence of (5.1.1) to a reflecting Brownian motion and an absorbing feedback model when letting κ go to zero or infinity, respectively. This is a probabilistic extension of the result in [4] where convergence to the absorbing feedback model is shown under the assumption of continuous solutions.

5.1.1 Physical and Minimal Solutions

A solution to the problem (5.1.1) is a deterministic càdlàg function $t \mapsto \Lambda_t$, called the loss function, that starts at 0, is increasing and satisfies the problem (5.1.1). We will define a physical and a minimal solution for this problem. We will then show how to exploit the results for absorbing equations to prove that (5.1.1) has unique physical and minimal solutions and that these coincide.

The process X is a reflected process that is the result of an application of the Skorokhod map $\Theta : D([0,\infty)) \to D([0,\infty))$ that maps a real-valued càdlàg function to the corresponding non-negative reflected process (see [75, Chapter 13.5]). We see that the process X can be constructed by taking a process Y defined as

$$Y_t = X_{0-} + B_t - \alpha\Lambda_t \tag{5.1.2}$$

and applying the Skorokhod map Θ, i.e. $X = \Theta(Y)$. The term L that is needed to keep X nonnegative is given by

$$L_t = (-\inf_{0\leqslant s\leqslant t} Y_s) \vee 0. \tag{5.1.3}$$

We can use this to find the following expression for the elastic stopping time

$$\tau = \inf\{t \geqslant 0 : L_t \geqslant \chi\} = \inf\{t \geqslant 0 : Y_t \leqslant -\chi\}$$

and the loss function Λ_t becomes

$$\Lambda_t = \mathbb{P}(\tau \leqslant t) = \mathbb{P}(\inf_{0 \leqslant s \leqslant t} Y_s \leqslant -\chi).$$

A first result that is important in the elastic case uses a simple argument, similar to that proving [38, Theorem 1.1] , to show that if the feedback parameter α is large enough, any solution to (5.1.1) cannot be continuous for all times $t \geqslant 0$.

Lemma 5.1.1. *Let $m_{0-} \coloneqq \int_0^\infty x\nu_{0-}(dx)$. If $\alpha > 2(m_{0-} + \frac{1}{\kappa})$, then any solution Λ to (5.1.1) cannot be continuous for all times $t \geqslant 0$.*

Proof. Assume $\alpha > 2(m_{0-} + \frac{1}{\kappa})$ and Λ is a continuous solution. We have that

$$0 \leqslant X_{t\wedge\tau} = X_{0-} + B_{t\wedge\tau} - \alpha\Lambda_{t\wedge\tau} + L_{t\wedge\tau}.$$

After taking expectations and rearranging

$$m_{0-} \geqslant \alpha\mathbb{E}[\Lambda_{t\wedge\tau}] - \mathbb{E}[L_{t\wedge\tau}].$$

Taking the limit as $t \to \infty$ and noting that $\Lambda_\infty = 1$, as $\inf_{t\geqslant 0} B_t = -\infty$, yields

$$m_{0-} \geqslant \alpha \int_0^\infty \Lambda_s d\Lambda_s - \mathbb{E}[\chi] = \frac{1}{2}\alpha(\Lambda_\infty^2 - \Lambda_0^2) - \frac{1}{\kappa} = \frac{1}{2}\alpha - \frac{1}{\kappa}.$$

Thus we get $2(m_{0-} + \frac{1}{\kappa}) \geqslant \alpha$, a contradiction. □

The fact that, as in the absorbing case, solutions in general are not continuous means we need to define a suitable solution concept. The two most natural ones to consider are that of a physical solution and a minimal solution.

Definition 5.1.2 (Minimal Solution)**.** A minimal solution $\underline{\Lambda}$ is a solution to (5.1.1) such that for any other solution Λ we have

$$\underline{\Lambda}_t \leqslant \Lambda_t, \quad t \geqslant 0.$$

In the case of absorbing feedback models these solutions have been introduced and studied in detail in [21].

Earlier the idea of a physical solution was introduced by Delarue et al. [24]. A physical solution, in the absorbing case, is defined as a solution Λ such that the condition

$$\Delta\Lambda_t = \inf\{x \geqslant 0 : \mathbb{P}(t < \tau^a, 0 < X_{0-} + B_t - \alpha\Lambda_{t-} < \alpha x) < x\} \tag{5.1.4}$$

is satisfied, where $\tau^a = \inf\{t \geqslant 0 : X_t \leqslant 0\}$ denotes the absorbing stopping time.

Recalling the definitions of the elastic stopping time in (5.1.1) and the map L from (5.1.3) an equivalent condition for a solution Λ to the elastic feedback model can be defined.

Definition 5.1.3 (Physical Solution)**.** A physical solution Λ to (5.1.1) is a solution such that the condition

$$\Delta\Lambda_t = \inf\{x \geqslant 0 : \mathbb{P}(t < \tau, 0 < X_{0-} + \chi + B_t - \alpha\Lambda_{t-} < \alpha x) < x\} \tag{5.1.5}$$

holds for all $t \geqslant 0$.

We will establish the existence and uniqueness of a physical solution for (5.1.1). The key idea is that we can relate the elastic feedback model to an absorbing feedback model for which these are known results by [26].

Lemma 5.1.4. *Let Λ be an increasing càdlàg function with $\Lambda_0 = 0$. Then Λ is a solution to the McKean–Vlasov problem (5.1.1) if and only if it is a solution to the absorbing feedback model*

$$\begin{cases} Y_t = Y_{0-} + B_t - \alpha\Lambda^a_t, \quad t \geqslant 0 \\ \Lambda^a_t = \mathbb{P}(\tau^a \leqslant t), \quad t \geqslant 0, \\ \tau^a = \inf\{t \geqslant 0 : Y_t \leqslant 0\} \end{cases} \tag{5.1.6}$$

with initial condition $Y_{0-} \coloneqq X_{0-} + \chi$.

Proof. Let Λ be a solution to the problem (5.1.1). We can rewrite this using the definition of the stopping time τ and the Skorokhod map Θ in the following way

$$\begin{aligned} \Lambda_t = \mathbb{P}(\tau \leqslant t) = \mathbb{P}(L_t \geqslant \chi) &= \mathbb{P}(\inf_{s \leqslant t}(X_{0-} + B_s - \alpha\Lambda_s) \leqslant -\chi) \\ &= \mathbb{P}(\inf_{s \leqslant t}(\chi + X_{0-} + B_s - \alpha\Lambda_s) \leqslant 0) \\ &= \Lambda^a_t. \end{aligned}$$

Therefore, a solution to the elastic feedback model (5.1.1) is also a solution to the absorbing model (5.1.6). Conversely, any solution to (5.1.6) solves the elastic model. □

Theorem 5.1.5. *Let X_{0-} be a random variable with bounded density f such that f changes monotonicity at most finitely often on any compact interval. Then there exists a unique physical solution to the elastic feedback model (5.1.1).*

Proof. By Lemma 5.1.4 we know that an increasing càdlàg process Λ solves the McKean–Vlasov problem (5.1.1) if and only if it solves the problem (5.1.6). Additionally, considering the definition of a physical solution for the absorbing model (5.1.4) and for the elastic model (5.1.5), we see that the same holds for physical solutions. Hence a map Λ is a physical solution to the elastic feedback model if and

only if it is a physical solution to the absorbing feedback model (5.1.6). The random variable Y_{0-} with the law of $X_{0-} + \chi$, has a density g which inherits the properties from f, i.e. g is bounded and changes monotonicity at most finitely often on compact intervals. Moreover, since χ is an exponential random variable we obtain from the convolution formula that $g(0) = 0$. We can apply the results from [26] for absorbing feedback models and conclude that the elastic feedback model (5.1.1) has a unique physical solution. □

Similarly, Lemma 5.1.4 can be combined with the ideas from [21] about minimal solutions.

Theorem 5.1.6. *The elastic feedback model has a unique minimal solution. Moreover, if the conditions of Theorem 5.1.5 hold, then the unique minimal solution and the unique physical solution coincide.*

Proof. Using the reformulation in terms of an absorbing feedback problem from Lemma 5.1.4, by [21] there exists a unique minimal solution and the minimal solution is a physical solution. If we additionally have that the density of X_{0-} is bounded and changes monotonicity at most finitely often on compact intervals, Theorem 5.1.5 applies and the physical solution is unique. □

Building on the results of [26] for initial conditions with densities that change monotonicity only finitely often on any compact interval, [56] extended the uniqueness result to allow for oscillatory initial conditions. This extension is inspired by space discretized particle approximations such as [27] where the initial condition can be viewed as a discretized versions of white noise. We can adjust [56, Theorem 1.1] to extend that uniqueness result to our problem.

Theorem 5.1.7. *Let X_{0-} be an initial condition and χ an independent exponential random variable such that $X_{0-} + \chi$ has a density f such that*

(i)
$$f \leqslant 1, \quad \int_0^\infty x f(x) dx < \infty,$$

(ii) there exists an $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$ and for all $\mu \in [0, 1]$
$$\int_\mu^{\mu+1} f(\lambda x) dx \leqslant 1 - g(\lambda(\mu + 1))$$

for a non-decreasing function $g : (0, \infty) \to (0, \infty)$.

Then (5.1.1) *started from* X_{0-} *and with elastic stopping time depending on* χ *has a unique physical solution.*

Remark. The results in this section and in Section 5.1.3 only rely on the random variable χ being nonnegative, L^1 and such that the density of $X_{0-} + \chi$ satisfies the conditions of [26]. Thus we could extend our McKean–Vlasov equation to this more general formulation. However, since we are mainly interested in elastic boundary behaviour, we focus on the case of χ having an exponential distribution.

5.1.2 PDE Formulation and Connection with the Kinetic Undercooled Stefan Problem

In [4] the authors study a free boundary problem called the supercooled Stefan problem with kinetic undercooling as an approximation to the supercooled Stefan problem. They work in the regime without jumps and use a probabilistic representation of solutions based on a Feynmac-Kac formula to derive their results. In this section we show, under the assumption of continuous solutions, that the elastic feedback model is a probabilistic formulation of the Stefan problem with kinetic undercooling. Thus, the results in Section 5.1.1 on existence and uniqueness, as well as the approximation results discussed in Section 5.1.4, are a probabilistic generalisation of the results in [4]. We define a measure ν_t by its actions on test functions $\phi \in C^2$ with $\kappa\phi(0) = \phi'(0)$ as $\langle \nu_t, \phi \rangle \coloneqq \mathbb{E}[\phi(X_t)1_{t<\tau}]$. To obtain the dynamics we calculate $\phi(X_{t\wedge\tau})$ by applying Itô's formula

$$d\phi(X_{t\wedge\tau}) = \phi'(X_{t\wedge\tau})dB_t - \alpha\phi'(X_{t\wedge\tau})d\Lambda_t + \phi'(X_{t\wedge\tau})dL_t + \frac{1}{2}\phi''(X_{t\wedge\tau})dt.$$

As $X_\tau = 0$, we have

$$\phi(X_{t\wedge\tau}) = \phi(X_t)1_{t<\tau} + \phi(0)1_{\tau\leqslant t}.$$

Using this fact and taking expectations we have the measure dynamics

$$d\langle \nu_t, \phi \rangle = \frac{1}{2}\langle \nu_t, \phi'' \rangle dt - \alpha\langle \nu_t, \phi' \rangle d\Lambda_t + \kappa\phi(0)\mathbb{E}[L_{t\wedge\tau}] - \phi(0)\mathbb{P}(\tau \leqslant t).$$

We focus on the last two terms and compute

$$\begin{aligned}\kappa\mathbb{E}[L_{t\wedge\tau}] - \mathbb{P}(\tau \leqslant t) &= \kappa\mathbb{E}[L_t 1_{t<\tau}] + \kappa\mathbb{E}[\chi 1_{t\geqslant\tau}] - \mathbb{P}(\tau \leqslant t) \\ &= \kappa\mathbb{E}[L_t e^{-\kappa L_t}] - \kappa\mathbb{E}[L_t e^{-\kappa L_t}] + \mathbb{P}(\tau \leqslant t) - \mathbb{P}(\tau \leqslant t) = 0.\end{aligned} \tag{5.1.7}$$

To see this we first calculate the first expectation on the right-hand side by using the definition of the stopping time τ and conditioning on L. We then get

$$\mathbb{E}\left[L_t 1_{\tau<t}\right] = \mathbb{E}\left[L_t 1_{\chi>L_t}\right] = \mathbb{E}\left[L_t \mathbb{E}\left[1_{\chi>L_t}|L_t\right]\right] = \mathbb{E}\left[L_t e^{-\kappa L_t}\right].$$

We can employ the same approach to obtain an expression for the other expectation in (5.1.7)

$$\begin{aligned}
\mathbb{E}\left[\chi 1_{t\geqslant\tau}\right] = \mathbb{E}\left[\chi 1_{L_t\geqslant\chi}\right] &= \mathbb{E}\left[\mathbb{E}\left[\chi 1_{L_t\geqslant\chi}|L\right]\right] \\
&= \mathbb{E}\left[\int_0^{L_t} \kappa x e^{-\kappa x} dx\right] \\
&= \mathbb{E}\left[\left[-xe^{-\kappa x}\right]_0^{L_t} - \frac{1}{\kappa}\int_0^{L_t} \kappa e^{-\kappa x} dx\right] \\
&= \mathbb{E}\left[-L_t e^{-\kappa L_t}\right] + \frac{1}{\kappa}\mathbb{P}\left(\chi \leqslant L_t\right).
\end{aligned}$$

Thus we obtain the weak form of the nonlinear PDE

$$\begin{cases} \langle \nu_t, \phi\rangle = \langle \nu_0, \phi\rangle + \frac{1}{2}\int_0^t \langle \nu_s, \phi''\rangle ds - \alpha \int_0^t \langle \nu_s, \phi'\rangle d\Lambda_s, \\ \Lambda_t = 1 - \int_0^\infty \nu_t(dx). \end{cases}$$

By removing the continuous drift using Girsanov's theorem and applying the Radon-Nikodym Theorem it can be shown that the measure ν_t possesses a density $V(t,x)$. We can use integration by parts to derive the PDE governing the evolution of the density

$$\begin{cases} \partial_t V(t,x) = \frac{1}{2}\partial_{xx} V(t,x) + \alpha\dot{\Lambda}_t \partial_x V(t,x), \\ \frac{1}{2}\partial_x V(t,0) = \left(\frac{\kappa}{2} - \alpha\dot{\Lambda}_t\right) V(t,0). \end{cases}$$

Now we reparametrize and set $\beta = \frac{2}{\alpha}$, $\mathcal{L} = \frac{\alpha}{2}\Lambda$, and for an $\varepsilon > 0$ set $\kappa = \beta/\varepsilon$. Finally, writing $u(t,x) = V(t, x - \mathcal{L}_t)$, we have that u solves the free boundary problem

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_{xx} u, \quad \text{on } \Gamma := \{(t,x) = [0,\infty)^2 : x \geqslant \mathcal{L}_t\}, \\ u(t, \mathcal{L}_t) = \varepsilon\dot{\mathcal{L}}_t, \quad t \geqslant 0, \\ (\beta - 2u(t,\mathcal{L}_t))\dot{\mathcal{L}}_t = \partial_x u(t,\mathcal{L})_t, \quad t \geqslant 0, \quad \mathcal{L}_0 = 0. \end{cases}$$

This is the Stefan problem with kinetic undercooling as discussed in [4].

5.1.3 Particle System Approximation

We consider a system of particles with elastic feedback given by

$$\begin{cases} X_t^{i,N} = X_{0-}^i + B_t^i - \alpha\Lambda_t^N + L_t^{i,N}, \quad t \geqslant 0 \\ \Lambda_t^N = \frac{1}{N}\sum_{i=1}^N 1_{\tau_{i,N}\leqslant t}, \quad t \geqslant 0, \\ \tau_{i,N} = \inf\{t \geqslant 0 : L_t^{i,N} \geqslant \chi^i\}, \end{cases} \tag{5.1.8}$$

where the parameters are as discussed for (5.1.1) and the random variables χ^i are i.i.d. We can use the ideas from Section 5.1.1 and formulate this as a system of particles with absorbing boundary and initial conditions $X^i_{0-} + \chi^i$. This way, we see that the theory for absorbing particle systems also holds for the elastic ones. As for our McKean–Vlasov problem, minimal and physical solutions can be defined for this particle system. In [21] existence of a minimal solution is shown. Moreover, the authors establish that the physical solution is equal to the minimal solution.

The goal of this section is to establish weak convergence, on the Skorokhod space with the M1-topology [75], of the system (5.1.8) to a solution to the McKean–Vlasov equation (5.1.1). After verifying tightness in the M1-topology we show that physical solutions to the particle system converge to physical solutions of (5.1.1). We can then refer to the uniqueness results in Theorem 5.1.5 to deduce weak convergence. The main idea is again to use the structure of the processes $X^{i,N}_t$ in terms of the Skorokhod map Θ, that is

$$X^{i,N}_t = \Theta(Y^{i,N}_t) = \Theta(X^i_{0-} + B^i_t - \alpha\Lambda^N_t)$$

and exploit the fact that the Skorokhod map is continuous on the Skorokhod space with the M1-topology (see [75] Theorem 13.5.1).

We closely follow the approach in [21]. We consider the loss processes Λ^N to be elements of the space M defined by

$$M \coloneqq \{l : \bar{\mathbb{R}} \to [0,1] | l \text{ càdlàg and increasing, } l_{0-} = 0, l_\infty = 1\}. \tag{5.1.9}$$

Writing $\mathcal{P}(E)$ for the space of probability measures on a Polish space E, the space M can be identified with distribution functions of measures in $\mathcal{P}([0,\infty])$. If we equip it with the topology of weak convergence it becomes a compact Polish space, because the topology is metrizable (for example using the Lévy metric). Define the spaces $\bar{E}$ and $\hat{E}$ as

$$\bar{E} \coloneqq C([0,\infty)) \times M, \quad \hat{E} \coloneqq \bar{E} \times \mathbb{R}.$$

Theorem 5.1.8. *Endowed with the product topology induced by compact convergence on $C([0,\infty))$, and the Lévy-metric on M, the space $\bar{E}$ and then also $\hat{E}$ are Polish. For $w \in C([0,\infty))$ and $l \in M$, define*

$$\hat{w}_t \coloneqq \begin{cases} w_0 & t \in [-1,0) \\ w_t & t \in [0,\infty) \end{cases} \qquad \check{l}_t \coloneqq \begin{cases} 0 & t \in [-1,0) \\ l_t & t \in [0,\infty). \end{cases}$$

For any $\alpha \in \mathbb{R}$ the embedding $\iota_\alpha : \hat{E} \to (D([-1,\infty)) \times \mathbb{R})$ defined via

$$\iota_\alpha(w, l, z) = (\hat{w} - \alpha \check{l}, z)$$

is continuous.

Proof. The space $\hat{E}$ is Polish as the product space of Polish spaces. The fact that the map $(w, l) \mapsto (\hat{w} - \alpha \check{l})$ is a continuous map from $\bar{E}$ to $D([-1,\infty))$ then follows as in the proof of [21, Theorem 4.2]. As a consequence ι_α is a continuous map. □

Theorem 5.1.9. *Let $(X^N)_{N\in\mathbb{N}}$ be a sequence of stochastic processes with paths in $D([0,\infty))$. Suppose that X^N is of the form*

$$X^N = \Theta(Y^N) = \Theta(Z^N - \alpha_N \Lambda^n)$$

with α_N a real number, Z^N continuous and $\Lambda^N \in M$. Suppose that $(Z^N)_{N\in\mathbb{N}}$ is tight on $C([0,T])$ for each $T > 0$ and $(\alpha_N)_{N\in\mathbb{N}}$ is tight on $\mathbb{R}$. Furthermore, let $(\chi^N)_{N\in\mathbb{N}}$ be a tight sequence of random variables on $\mathbb{R}$. Following the notation from Theorem 5.1.8 we denote by $\hat{Y}^N$ the extension of Y^N to $D([-1,\infty))$ and set $\hat{X}^N = \Theta(\hat{Y}^N)$. Then the random variables $((Z^N, \Lambda^N, \chi^N))_{N\in\mathbb{N}}$ are tight on $\hat{E}$, the random variables $((\hat{Y}^N, \chi^N))_{N\in\mathbb{N}}$ are tight on $D([-1,\infty)) \times \mathbb{R}$ and the random variables $(\hat{X}^N)_N$ are tight on $D([-1,\infty))$.

Proof. By [21, Theorem 4.4] the random variables $((Z^N, \Lambda^N))_{N\in\mathbb{N}}$ are tight on $\bar{E}$ and the random variables $(Y^N)_{N\in\mathbb{N}}$ are tight on $D([-1,\infty)$. Since $(\chi^N)_{N\in\mathbb{N}}$ are tight on $\mathbb{R}$ by assumption it follows that $((Z^N, \Lambda^N, \chi^N))_{N\in\mathbb{N}}$ are tight on $\hat{E}$ and $((\hat{Y}^N, \chi^N))_{N\in\mathbb{N}}$ are tight on $D([-1,\infty)) \times \mathbb{R}$. By continuity of the Skorokhod map it then follows that the random variables $(\hat{X}^N)_{N\in\mathbb{N}}$ are tight on $D([-1,\infty))$. □

In the rest of this section we will no longer distinguish between processes on $D([0,\infty))$ and their extensions to $D([-1,\infty))$. Theorem 5.1.9 yields tightness of the processes we consider, but we need tightness for the corresponding empirical measures. By the following result of [70], if we have exchangeability, then this tightness follows from the tightness of a sequence of particles.

Proposition 5.1.10 ([70], Proposition 2.2.)**.** *Let E be a Polish space and let $X^N := (X^{1,N}, \dots, X^{N,N})$ be N-exchangeable on E^N for every $N \in \mathbb{N}$, then the $\mathcal{P}(E)$-valued random variables $(\mu_N)_{N\in\mathbb{N}}$ given by*

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^{i,N}}$$

are tight if and only if $(X^{1,N})_{N\in\mathbb{N}}$ is tight on E.

Corollary 5.1.11. *Suppose that X^N has dynamics*

$$X_t^{i,N} = \Theta(Y^{i,N}) = \Theta(X_{0-}^{i,N} + B_t^i - \alpha\Lambda_t^N)$$

where $\alpha > 0$, $(X_{0-}^N)_{N\in\mathbb{N}}$ and $(\chi^{i,N})_{N\in N}$ are N-exchangeable random vectors, $(B_i)_{i\in\mathbb{N}}$ independent Brownian motions, and $\Lambda^N \in M$. If $(X_{0-}^{1,N})_{N\in\mathbb{N}}$ and $(\chi^{1,N})_{N\in\mathbb{N}}$ are tight on $\mathbb{R}$, then the empirical measures

$$\mu_N^\Theta = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}} \quad \mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{(Y^{i,N},\chi^{i,N})}, \quad \text{and} \quad \zeta_N = \frac{1}{N}\sum_{i=1}^N \delta_{(X_{0-}^{i,N}+B^i,\Lambda^N,\chi^{i,N})} \tag{5.1.10}$$

are tight on $\mathcal{P}(D([-1,\infty)))$, $\mathcal{P}(D([-1,\infty))\times\mathbb{R})$, and $\mathcal{P}(\hat{E})$ respectively.

Proof. Since $(X_{0-}^{1,N})_{N\in\mathbb{N}}$ is tight on $\mathbb{R}$ it easily follows that $(X_{0-}^{1,N} + B^1)_{N\in\mathbb{N}}$ is tight on $C[0,T]$ for all $T > 0$. Thus, the conditions of Theorem 5.1.9 are satisfied and we can combine this with Proposition 5.1.10 to finish the proof. □

We will write $f(\mu)$ for the pushforward of a measure μ by a map f.

Corollary 5.1.12. *Let $(\mu_N^\Theta)_{N\in\mathbb{N}}$ and $(\mu_N)_{N\in\mathbb{N}}$ be as in the Corollary 5.1.11. Then there are $\mathcal{P}(\hat{E})$-valued random variables ζ, ζ_N such that, at least for a subsequence, $law(\mu_N) = law(\iota_\alpha(\zeta_N)), \zeta_N \to \zeta$ and $\iota_\alpha(\zeta_N) \to \iota_\alpha(\zeta)$ a.s. It holds that $law(\mu_N^\Theta) = law(\Theta(\mu_N))$ and $\Theta(\mu_N) \to \Theta(\iota_\alpha(\zeta))$ a.s. Moreover,*

$$law(\zeta_N) = law\left(\frac{1}{N}\sum_{i=1}^N \delta_{(X_{0-}^{i,N}+B^i,\Lambda^N,\chi^{i,N})}\right).$$

Proof. The result follows by combining Corollary 5.1.11, the Skorokhod representation theorem and the continuity of the maps involved. □

We consider the setting of Corollary 5.1.11. The three empirical measures given in (5.1.10) have the following relations

$$\text{law}(\mu_N) = \text{law}(\iota_\alpha(\zeta_N)); \quad \text{law}(\mu_N^\Theta) = \text{law}(\Theta(\mu_N)).$$

Definition 5.1.13. For $t \in \mathbb{R}$, $x \in D([-1,\infty))$ and $z \in \mathbb{R}$ define the following path functionals

$$\tau_0(x,z) := \inf\{s \geqslant 0 : x_s \leqslant -z\} \quad \text{and} \quad \lambda_t(x,z) := 1_{\tau_0(x,z)\leqslant t}$$

Lemma 5.1.14. *Assume that* $(x^n, z^n) \to (x, z)$ *in* $(D([-1,\infty))\times\mathbb{R})$ *with* $(x^n, z^n), (x, z) \in \iota_\alpha(\hat{E})$ *and* x *satisfies the crossing property*

$$\inf_{0\leqslant s\leqslant h}(x_{\tau(x,z)+s} - x_{\tau(x,z)}) < 0, \quad h > 0.$$

Then we have that

$$\lim_{n\to\infty} \lambda_t(x^n, z^n) = \lambda_t(x, z)$$

for all $t \in [0,\infty)$ *in the complement of a countable set.*

Proof. Define the map

$$\Xi : (D([-1,\infty)) \times \mathbb{R}) \to D([-1,\infty)); (x, z) \mapsto x + z.$$

This map is continuous with respect to the $M1$-topology and the function $\Xi(x, z)$ satisfies the crossing property. We have

$$\lambda_t(x, z) = \lambda_t(\Xi(x, z), 0).$$

The convergence then follows as in the proof of [21, Lemma 5.4]. □

Lemma 5.1.15. *Assume that* $(\zeta^n)_{n\in\mathbb{N}}$ *is a convergent sequence of probability measures on* $\hat{E}$ *with limit* ζ. *Define* $\mu^n := \iota_\alpha(\zeta^n)$ *and* $\mu := \iota_\alpha(\zeta)$. *If* μ*-a.e path satisfies the crossing property*

$$\mu(\{(x, z) \in D([-1,\infty)) \times \mathbb{R} : \inf_{0\leqslant s\leqslant h}(x_{\tau_0(x,z)+s} - x_{\tau_0(x,z)}) = 0\}) = 0, \quad h > 0, \quad (5.1.11)$$

then $\lim_{n\to\infty}\langle\mu^n, \lambda\rangle = \langle\mu, \lambda\rangle$ *in* M.

Proof. The map ι_α is continuous. As a result, the convergence $\mu^n \to \mu$ in $\mathcal{P}(D[-1,\infty)\times\mathbb{R})$ follows. We set $l^n_t := \langle\mu^n, \lambda_t\rangle$ and note that for all $n \in \mathbb{N}$ the map l^n is in M. As mentioned at the beginning of the section, the space M is compact. Thus, we have the existence of a limit point $l \in M$. We need to prove that $l = \langle\mu, \lambda\rangle$. This can be done following the arguments in the proof of [21] Lemma 5.5 and replacing their path functionals by the paths functional $\tau_0(x, z)$ and $\lambda_t(x, z)$ considered here. □

The following two lemmas are important tools in the proof of convergence to the McKean–Vlasov problem. Their proofs follow from the argument used in [21], replacing the path functionals with the functionals $\tau_0(x, z)$ and $\lambda_t(x, z)$, and extending the measures involved to the spaces $\hat{E}$ and $D([-1,\infty)) \times \mathbb{R}$.

Lemma 5.1.16. *For almost every realization (ω, z), if $law((W, \Lambda, \chi)) = \zeta(\omega, z)$, then $W - W_0$ is a Brownian motion with respect to the filtration generated by (W, Λ). In particular, $W - W_0$ is independent of W_0.*

Lemma 5.1.17. *Suppose that $law(\mu_N) \to law(\mu)$ for some random variable μ in $\mathcal{P}(D[-1, \infty) \times \mathbb{R})$ with μ_N and μ of the same form as in Lemma 5.1.15. Then μ satisfies property (5.1.11) almost surely.*

Proposition 5.1.18. *For $N \in \mathbb{N}$, let (X^N, Λ^N) be a solution to the particle system*

$$\begin{cases} X_t^{i,N} = \Theta(Y^{i,N})_t = \Theta(X_{0-}^{i,N} + B_t^i - \alpha\Lambda_t^N) = X_{0-}^{i,N} + B_t^i - \alpha\Lambda_t^N + L_t^{i,N} \\ \Lambda_t^n = \frac{1}{N}\sum_{i=1}^N 1_{\tau_{i,N} \leqslant t}, \\ \tau_{i,N} = \inf\{t \geqslant 0 : L_t^{i,N} \geqslant \chi^{i,N}\} \end{cases}$$

for $\chi^{i,N}, i = 1, \dots, n$, independent, exponentially distributed random variables with parameter $\kappa > 0$. Let μ^N and μ_N^Θ denote the corresponding empirical measures defined in (5.1.10). Suppose that for some random measure μ^Θ and some measure $\nu_{0-} \in \mathcal{P}(\mathbb{R}_+)$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \delta_{X_{0-}^{i,N}} = \nu_{0-}$$

and $law(\mu_N^\Theta) \to law(\mu^\Theta)$ along some subsequence. Then μ^Θ corresponds almost surely to the law of a solution to the McKean–Vlasov problem (5.1.1) with $law(X_{0-}) = \nu_{0-}$.

Proof. We can assume that the empirical measures are given as $\mu_N = \iota_\alpha(\zeta_N)$, $\mu_N^\Theta = \Theta(\mu_N)$ with ζ_N as in (5.1.10). The same can be assumed for the limit points $\mu = \iota_\alpha(\zeta)$ and $\mu^\Theta = \Theta(\mu)$ by continuity. The map $t \mapsto \mathbb{E}[\langle \mu, \lambda_t \rangle]$ is increasing. Hence, it has at most countably many points of discontinuity, and the same holds for the map $t \mapsto \mathbb{E}[\int l_t d\zeta(\omega, l, z)]$. We denote the set of discontinuities of these maps by J and fix a t that is a continuity point of all these maps, i.e. $t \notin J$. Then $\langle \mu, \lambda_{t-} \rangle = \langle \mu, \lambda_t \rangle$ almost surely. From Lemma 5.1.17 the crossing property holds and we can use Lemma 5.1.15 to deduce the almost sure convergence $\lim_{N\to\infty} \langle \mu_N, \lambda_t \rangle = \langle \mu, \lambda_t \rangle$. The goal is to show that the measure μ is the law of the process (Y, χ) of the McKean–Vlasov problem.

Step 1: We show that for almost every realization (ω, z), if $\text{law}(W, \Lambda, \chi) = \zeta(\omega, l, z)$, then $\Lambda = \langle \mu(\omega, z), \lambda \rangle$ almost surely. We obtain the estimate

$$\mathbb{E}\left[\int_{\bar{E}} \Big| |l_t - \langle \mu, \lambda_t \rangle| - |l_t - \langle \mu_N, \lambda_t \rangle| \Big| d\zeta_N(\omega, l, z)\right] \leqslant \mathbb{E}\left[|\langle \mu, \lambda_t \rangle - \langle \mu_N, \lambda_t \rangle|\right]$$

and see that the right-hand side vanishes as $N \to \infty$ using the DCT. Since $t \notin J$ the map $l \mapsto l_t$ is continuous on M and it follows that for ζ-almost every l we have

$$\mathbb{E}\left[\int_{\hat{E}} |l_t - \langle \mu, \lambda_t \rangle| d\zeta(\omega, l, z)\right] \leqslant \lim_{N\to\infty} \mathbb{E}\left[\int_{\hat{E}} |l_t - \langle \mu_N, \lambda_t \rangle| d\zeta_N(\omega, l, z)\right] = 0$$

because $\langle \mu_N, \lambda \rangle = l$ for ζ_N-almost every $l \in M$, almost surely. Let t range through a countable dense subset of $[0, \infty) \setminus J$ and use right-continuity to conclude.

Step 2: We show that for a.e. realization (ω, z) if $\text{law}(W, \Lambda, \chi) = \zeta(\omega, z)$, then $\text{law}(W_0) = \nu_{0-}$. Let the evaluation map $\hat{\pi}_0$ defined on $\hat{E}$ be given by $(\omega, l, z) \to \omega_0$. Since $\hat{\pi}_0$ is a continuous map, the continuous mapping theorem yields that for the pushforward of ζ by $\hat{\pi}_0$ we obtain

$$\hat{\pi}_0(\zeta) = \lim_{N\to\infty} \hat{\pi}_0(\zeta_N) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X_{0-}^{i,N}} = \nu_{0-}$$

Step 3: We have $\text{law}(X_{0-}) = \nu_{0-}$ and $\iota_\alpha(\zeta) = \mu$. Step 1 yields that if $\text{law}(W, \Lambda, \chi) = \zeta(\omega, z)$ the equation

$$\text{law}((W_0, W - W_0)) = \text{law}((X_{0-}, Y - X_{0-} + \alpha\langle \mu, \lambda \rangle))$$

holds. Using Lemma 5.1.16 we obtain that $Y - X_{0-} + \langle \mu, \lambda \rangle$ is a Brownian motion.

If we now combine the results from Step 2, Step 3 and apply the pushforward with the Skorokhod map we obtain the theorem by the continuous mapping theorem. □

Theorem 5.1.19. *Suppose that X_{0-} has density f that is bounded and changes monotonicity finitely often on any compact interval. Let $(X^{i,N}, \Lambda^N)$ be a physical solution to the particle system (5.1.8) and let μ^Θ be the law of the unique physical solution to the McKean–Vlasov problem (5.1.1). Then we have the convergence*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X^{i,N}} = \mu^\Theta \tag{5.1.12}$$

weakly on $\mathcal{P}(D[-1, \infty))$.

Proof. Corollary 5.1.11 yields tightness of the empirical-measure process of the particle system (5.1.8). Thus, we know that limit points exist. Moreover, Proposition 5.1.18 shows that limit points are laws of solutions to the McKean–Vlasov problem. If we now restrict to physical solutions to the particle system, we can use the result from [24] that physical solutions converge to physical solutions and combine it with Lemma 5.1.4 and the corresponding result for the particle systems. The conditions on the initial density imply uniqueness of physical solutions using Theorem 5.1.5 and we can conclude weak convergence. □

5.1.4 Convergence to Absorbing and Reflecting Models

The elastic boundary is a mixture of absorbing and reflecting boundaries. Thus we can expect to obtain the absorbing and reflecting models as limit cases of the elastic model. In this section we show that as $\kappa \to \infty$ the solution to the elastic model converges to the solution of the absorbing feedback model. On the other hand, if we consider the limit as $\kappa \to 0$, the elastic model converges to a reflecting Brownian motion.

Our first goal is to show convergence for the loss process Λ^κ as $\kappa \to 0$ and $\kappa \to \infty$.

Proposition 5.1.20. *Let Λ^κ be a physical solution to the McKean–Vlasov problem (5.1.1) with parameter $\kappa > 0$. Then we have the two limits*

$$\lim_{\kappa \to 0} \Lambda^\kappa = 0, \quad \text{in } M1, \quad \text{and} \quad \lim_{\kappa \to \infty} \Lambda^\kappa = \Lambda^\infty, \quad \text{in } M1. \tag{5.1.13}$$

Moreover, Λ^∞ is a physical solution to the absorbing feedback model.

Proof. Let $\kappa_1, \kappa_2 > 0$ be such that $\kappa_1 < \kappa_2$ and consider the two related McKean–Vlasov problems, one with χ^{κ_1} and one with χ^{κ_2}. Following [21] we define the operator $\Gamma_\kappa : M \to M$ by

$$\begin{cases} X_t^{\kappa,l} = X_{0-} + \chi^\kappa + B_t - \alpha l_t, \quad t \geqslant 0 \\ \Gamma_\kappa[l]_t = \mathbb{P}(\tau^{\kappa,l} \leqslant t), \quad t \geqslant 0, \\ \tau^{\kappa,l} = \inf\{t \geqslant 0, X_t^{\kappa,l} \leqslant 0\}. \end{cases} \tag{5.1.14}$$

A solution to the absorbing feedback model is a fixed point of the operator Γ_κ and in [21] the authors show that the minimal solution and therefore also the unique physical solution can be obtained by $\lim_{n\to\infty} \Gamma_\kappa^{(n)}[0]$. Moreover, by Lemma 5.1.5 the solution to this absorbing model is a solution to (5.1.1). Using the operators $\Gamma_{\kappa_i}, i = 1, 2$ we have

$$\Gamma_{\kappa_1}[0]_t = \mathbb{P}(\inf_{s \leqslant t} X_{0-} + \chi^{\kappa_1} + B_s \leqslant 0) \leqslant \mathbb{P}(\inf_{s \leqslant t} X_{0-} + \chi^{\kappa_2} + B_s \leqslant 0) = \Gamma_{\kappa_2}[0]_t,$$

for all $t \geqslant 0$. Assume that

$$\Gamma_{\kappa_1}^{(n)}[0]_t \leqslant \Gamma_{\kappa_2}^{(n)}[0]_t, \quad \text{for all } t \geqslant 0, \tag{5.1.15}$$

for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \Gamma_{\kappa_1}^{(n+1)}[0]_t = \Gamma_{\kappa_1}[\Gamma_{\kappa_1}^{(n)}[0]]_t &= \mathbb{P}(\inf_{s \leqslant t} X_{0-} + \chi^{\kappa_1} + B_s - \Gamma_{\kappa_1}^{(n)}[0]_s \leqslant 0) \\ &\leqslant \mathbb{P}(\inf_{s \leqslant t} X_{0-} + \chi^{\kappa_2} + B_s - \Gamma_{\kappa_2}^{(n)}[0]_s \leqslant 0) \\ &= \Gamma_{\kappa_2}[\Gamma_{\kappa_2}^{(n)}[0]]_t = \Gamma_{\kappa_2}^{(n+1)}[0]_t \end{aligned}$$

for all $t \geqslant 0$ and by induction the relation (5.1.15) is true for all $n \in \mathbb{N}$. Therefore, we have that, for all $t \geqslant 0$, Λ_t^κ is increasing in κ. Moreover it is bounded between zero and one. This means that limit points exist in both cases and we can define

$$\Lambda_t^0 = \lim_{\kappa \to 0} \Lambda_t^\kappa, \quad \text{for all } t \geqslant 0, \quad \text{and} \quad \Lambda_t^\infty = \lim_{\kappa \to \infty} \Lambda_t^\kappa, \quad \text{for all } t \geqslant 0.$$

In the first case we can use the reverse Fatou lemma. This yields

$$\Lambda_t^0 = \lim_{\kappa \to 0} \Lambda_t^\kappa \leqslant \mathbb{P}(\lim_{\kappa \to 0} X_{0-} + \chi^\kappa + B_t - \alpha \Lambda_t^\kappa \leqslant 0) = 0,$$

for all $t \geqslant 0$. Thus, we obtain $\Lambda^0 = 0$. By [75, Corollary 12.5.1] convergence in the M1-topology follows.

For the second case we have

$$\begin{aligned} \Lambda_t^\infty = \lim_{\kappa \to \infty} \Lambda_t^\kappa &= \lim_{\kappa \to \infty} \mathbb{P}(X_{0-} + \chi^\kappa + B_t - \alpha \Lambda_t^\kappa \leqslant 0) \\ &= \mathbb{P}(X_{0-} + B_t - \alpha \Lambda_t^\infty \leqslant 0) \end{aligned}$$

where we applied the Portmanteau Theorem in the last step which is possible because the law of the random variable under consideration has no atoms. Hence Λ^∞ is a solution to the absorbing feedback model with initial condition X_{0-}. Another application of [75, Corollary 12.5.1] yields the desired convergence in M1.

To show that Λ^∞ is a physical solution we let Λ^a be a solution to the absorbing feedback model. Then we have $\Lambda^\kappa \leqslant \Lambda^a$ for all $\kappa > 0$ and by taking limits we obtain $\Lambda^\infty \leqslant \Lambda^a$. Thus Λ^∞ is the minimal solution and therefore by [21] a physical solution. □

Proposition 5.1.21. *Let τ^κ be the elastic stopping time in the feedback model (5.1.1) with parameter $\kappa > 0$ and denote by τ^∞ the absorbing stopping time in the absorbing feedback model. Then we have the following almost sure convergences in the $M1$-topology*

$$1_{t<\tau^\kappa} \to 1, \quad \text{as } \kappa \to 0; \quad 1_{t<\tau^\kappa} \to 1_{t<\tau^\infty}, \quad \text{as } \kappa \to \infty. \tag{5.1.16}$$

Proof. As in Section 5.1.3 we can see that the crossing property is satisfied. We can then use Lemma 5.1.14 to infer convergence for all $t \in [0, \infty)$ in the complement of a countable set. Since the functions we consider in this proposition are all monotone, the convergence in M1 follows from [75, Corollary 12.5.1] . □

Equipped with these results we can turn to our goal and show convergence for $1_{t<\tau^\kappa} X^\kappa$.

Theorem 5.1.22. *Let X^κ be the elastic feedback process corresponding to a physical solution to the McKean–Vlasov problem (5.1.1) with parameter $\kappa > 0$ and τ^κ the associated elastic stopping time. Then*

(i) as $\kappa \to \infty$, $1_{t<\tau^\kappa} X^\kappa$ converges almost surely in the M1-topology to $1_{t<\tau^\infty} X^\infty$ with X^∞ given by

$$X^\infty = X_{0-} + B - \alpha\Lambda^\infty$$

with Λ^∞ being a physical solution to the absorbing feedback model.

(ii) as $\kappa \to 0$, $1_{t<\tau^\kappa} X^\kappa$ converges almost surely in the M1-topology and uniformly to X^0 given by

$$X^0 = \Theta(X_{0-} + B).$$

The law of X^0 is a reflecting Brownian motion started at X_{0-}.

Proof. (i) From Proposition 5.1.20 we know that Λ^κ converges to Λ^∞ in $M1$ and that Λ^∞ is a physical solution. Since the Brownian motion B has continuous paths a.s., the convergence

$$X_{0-} + B - \alpha\Lambda^\kappa \to X^\infty, \quad \text{as } \kappa \to \infty$$

in M1 follows from [75, Theorem 12.7.3]. By continuity of the Skorokhod map Θ with respect to the M1-topology we can deduce the convergence

$$X^\kappa \to \Theta(X^\infty) \quad \text{as } \kappa \to \infty$$

in M1. Moreover, by Proposition 5.1.21 we have the M1-convergence $1_{t<\tau^\kappa} \to 1_{t<\tau^\infty}$. Note that both processes $\Theta(X^\infty)$ and $1_{t<\tau^\infty}$ only have downward jumps. Thus, we can use [75, Theorem 13.3.2] to obtain the desired convergence

$$1_{t<\tau^\kappa} X^\kappa \to 1_{t<\tau^\infty}\Theta(X^\infty), \quad \text{as } \kappa \to \infty,$$

in M1. Since τ^∞ is the first hitting time of zero by X^∞ we have the equality

$$1_{t<\tau^\infty}\Theta(X^\infty) = 1_{t<\tau^\infty} X^\infty,$$

which finishes the proof.
(ii) Using the results from Proposition 5.1.20 and Proposition 5.1.21 and following the same arguments as in (i) yields the convergence

$$1_{t<\tau^\kappa} X^\kappa \to X^0, \quad \text{as } \kappa \to 0$$

in M1. Since X^0 has continuous paths almost surely we obtain the uniform convergence almost surely by [75, Theorem 12.4.1]. The characterization of X^0 as a reflecting Brownian motion then follows by applying Tanaka's formula to $X_{0-} + B$ and using [63, Chapter VI Corollary 2.2]. □

5.2 Two-Phase Model with Elastic Feedback

A natural extension to the feedback models discussed in Section 5.1 is to add a second phase to the model by considering a process on the positive half-line and a process on the negative half-line simultaneously and connect them via the feedback term. A two-phase system with feedback through absorbing stopping times has been studied in [3]. The authors motivate this system as a model for systemic risk in two competing regions with the particles X^i representing the asset values of companies in the $\mathcal{X}$-region and $-Y^i$ the asset values of companies in a competing $\mathcal{Y}$-region. Then, a default of a company in the $\mathcal{X}$-region causes a flight to quality where investors move capital from companies in the $\mathcal{X}$-region to companies in the $\mathcal{Y}$-region. This leads to a downward move in capital for the $\mathcal{X}$-companies and an upward jump in capital for the $\mathcal{Y}$-companies. The downward move has the potential to move further companies into default which can cause default cascades as well as large inflows of capital into the $\mathcal{Y}$-region. Defaults of $\mathcal{Y}$-companies have the same flight-to-quality effect on the $\mathcal{X}$-region. In [3], the authors discuss the connection of the McKean–Vlasov problem for the two-phase model to a two-phase Stefan problem. This PDE describes the evolution of an interface between a supercooled liquid and a superheated solid. Moreover, they establish existence of a solution to the McKean–Vlasov problem with feedback through absorbing stopping times by proving convergence of an approximating particle system. The main difference compared to the one-phase model is that, through the interaction of the two-phases, a solution Λ to the two-phase model is in general not a monotone function. Thus, tightness in the M1-topology of the approximating terms Λ^N is not immediate. Instead, [3] establishes tightness directly by studying the M1-oscillations and tightness criteria from [74, Theorem 12.13.3]. We consider a version of the two-phase model with feedback through elastic stopping times. Thus, we study the following McKean–Vlasov problem for a process $X \geqslant 0$ and a process $Y \leqslant 0$, coupled via the feedback Λ ,

$$\begin{cases} X_t = X_{0-} + B_t - \Lambda_t + L_t(X) \\ Y_t = Y_{0-} + W_t - \Lambda_t - L_t(Y) \\ \Lambda_t = \alpha \mathbb{P}\left(\sigma \leqslant t\right) - \beta \mathbb{P}\left(\tau \leqslant t\right) \\ \sigma = \inf\{t \geqslant 0 : L_t(X) \geqslant \chi^X\} \\ \tau = \inf\{t \geqslant 0 : L_t(Y) \geqslant \chi^Y\}. \end{cases} \tag{5.2.1}$$

The two initial conditions are given by independent random variable $X_{0-} \geqslant 0, Y_{0-} \leqslant 0$. B and W are two independent Brownian motions, independent of X_{0-} and Y_{0-}.

Moreover, χ^X and χ^Y are independent exponential random variables with parameters κ^X and κ^Y, respectively. We ensure that the processes X and Y stay within the positive or negative half-line, respectively, by adding the reflection terms at zero $L(X), L(Y)$ given by

$$L_t(X) = (-\inf_{0\leqslant s\leqslant t} \tilde{X}_s) \vee 0 \qquad L_t(Y) = (\sup_{0\leqslant s\leqslant t} \tilde{Y}_s) \vee 0 \tag{5.2.2}$$

with the whole-space processes $\tilde{X}, \tilde{Y}$ defined by

$$\begin{aligned} \tilde{X}_t &= X_{0-} + B_t - \Lambda_t \\ \tilde{Y}_t &= Y_{0-} + W_t - \Lambda_t \end{aligned} \tag{5.2.3}$$

A solution to the McKean–Vlasov problem (5.2.1) is given by a deterministic càdlàg function $\Lambda : [0, \infty) \to \mathbb{R}$, started at 0, that satisfies the system of equations. Following the arguments in [3], similar to the one-phase case, a first result we can establish is that, given a large enough difference in the feedback parameters, the system must experience a blow-up.

Lemma 5.2.1. *If the condition*

$$\alpha\mathbb{E}\left[X_{0-}\right] - \beta\mathbb{E}[Y_{0-}] < \frac{(\alpha-\beta)^2}{2} - \frac{\beta}{\kappa^Y} - \frac{\alpha}{\kappa^X}$$

is satisfied, then any solution Λ to (5.2.1) cannot be continuous for all times $t \geqslant 0$.

Proof. Assume that Λ_t is continuous for all $t \geqslant 0$. We consider the stopped processes

$$\begin{aligned} X_{t\wedge\sigma} &= X_{0-} + B_{t\wedge\sigma} - \Lambda_{t\wedge\tau} + L_{t\wedge\tau}(X) \\ Y_{t\wedge\tau} &= Y_{0-} + W_{t\wedge\tau} - \Lambda_{t\wedge\tau} - L_{t\wedge\tau}(Y). \end{aligned}$$

Taking expectations and rearranging yields

$$\begin{aligned} \mathbb{E}\left[X_{0-}\right] &\geqslant \mathbb{E}\left[\Lambda_{t\wedge\sigma}\right] - \mathbb{E}\left[L_{t\wedge\sigma}(X)\right] \\ \mathbb{E}\left[Y_{0-}\right] &\leqslant \mathbb{E}\left[\Lambda_{t\wedge\tau}\right] + \mathbb{E}\left[L_{t\wedge\tau}(Y)\right]. \end{aligned}$$

Next, we take limits $t \to \infty$. Since we assumed that Λ is continuous, the reflection terms $L(X)$ and $L(Y)$ are also continuous. Thus, the definition of the stopping times τ and σ gives

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}\left[\Lambda_{t\wedge\sigma}\right] - \mathbb{E}\left[L_{t\wedge\sigma}(X)\right] &= \mathbb{E}\left[\Lambda_\sigma\right] - \mathbb{E}\left[\chi^X\right] = \int_0^\infty \Lambda_s \mathbb{P}\left(\sigma \leqslant s\right) - \frac{1}{\kappa^X} \\ \lim_{t\to\infty} \mathbb{E}\left[\Lambda_{t\wedge\tau}\right] + \mathbb{E}\left[L_{t\wedge\tau}(Y)\right] &= \mathbb{E}\left[\Lambda_\tau\right] + \mathbb{E}\left[\chi^Y\right] = \int_0^\infty \Lambda_s \mathbb{P}\left(\tau \leqslant s\right) + \frac{1}{\kappa^Y}. \end{aligned}$$

Combining these two equations we obtain

$$\begin{aligned}\alpha\mathbb{E}[X_{0-}] - \beta\mathbb{E}[Y_{0-}] &\geqslant \int_0^\infty \Lambda_s d\Lambda_s - \frac{\beta}{\kappa^Y} - \frac{\alpha}{\kappa^X} \\ &= \frac{\Lambda_\infty^2}{2} - \frac{\beta}{\kappa^Y} - \frac{\alpha}{\kappa^X} \\ &= \frac{(\alpha-\beta)^2}{2} - \frac{\beta}{\kappa^Y} - \frac{\alpha}{\kappa^X}.\end{aligned}$$

This provides a contradiction to the assumption of continuity and completes the proof. □

Note that, when we set the parameter β to zero, the blow-up condition in Lemma 5.2.1 simplifies to the condition for the one-phase model in Lemma 5.1.1.

5.2.1 Existence of Solutions

Similar to the one-phase model we define the notion of a physical solution, the solution with the smallest permitted jump size. [3] defines a physical solution for the two-phase model with feedback through absorbing stopping times. We combine the idea behind their definition with the definition of a physical solution for the elastic one-phase model in Section 5.1.

Definition 5.2.2. A physical solution Λ to (5.2.1) is a solution such that for all $t \geqslant 0$ with $\Delta\Lambda_t \neq 0$ the condition

$$|\Delta\Lambda_t| = \begin{cases} \inf\left\{x \geqslant 0 : \mathbb{P}\left(t < \sigma, X_{0-} + \chi^X + B_t - \Lambda_{t-} \in [0, \alpha x]\right) < x\right\}, \text{if } \Delta\Lambda_t > 0 \\ \inf\left\{y \geqslant 0 : \mathbb{P}\left(t < \tau, Y_{0-} - \chi^Y + W_t - \Lambda_{t-} \in [-\beta y, 0]\right) < y\right\}, \text{if } \Delta\Lambda_t < 0 \end{cases} \tag{5.2.4}$$

is satisfied.

We can apply a similar method as in the case of the one-phase model to establish the existence of solutions. We show that the elastic problem is related to a modified absorbing problem using the definition of the stopping times and the feedback term Λ. We can then infer existence by invoking the results for the absorbing model from [3].

Lemma 5.2.3. *A càdlàg function $\Lambda : [0, \infty) \to \mathbb{R}$ is a physical solution to the two-phase McKean–Vlasov problem (5.2.1) if and only if it is a physical solution to the*

following McKean–Vlasov problem

$$\begin{cases} X_t^a = X_{0-} + \chi^X + B_t - \Lambda_t^a \\ Y_t^a = Y_{0-} - \chi^Y + W_t - \Lambda_t^a \\ \Lambda_t^a = \alpha \mathbb{P}\left(\sigma^a \leqslant t\right) - \beta \mathbb{P}\left(\tau^a \leqslant t\right) \\ \sigma^a = \inf\{t \geqslant 0 : X_t^a \leqslant 0\} \\ \tau^a = \inf\{t \geqslant) : Y_t^a \geqslant 0\}. \end{cases} \tag{5.2.5}$$

Proof. Let Λ be a physical solution to (5.2.1). Using the definitions for the reflection terms and the stopping times we can rewrite Λ in the following way

$$\begin{aligned} \Lambda_t &= \alpha P(\sigma \leqslant t) - \beta \mathbb{P}\left(\tau \leqslant t\right) \\ &= \alpha \mathbb{P}\left(\inf_{0 \leqslant s \leqslant t} \tilde{X}_s \leqslant -\chi^X\right) - \beta \mathbb{P}\left(\sup_{0 \leqslant s \leqslant t} \tilde{Y}_s \geqslant \chi^Y\right) \\ &= \alpha \mathbb{P}\left(\inf_{0 \leqslant s \leqslant t} (\chi^X + \tilde{X}_s) \leqslant 0\right) - \beta \mathbb{P}\left(\sup_{0 \leqslant s \leqslant t} (\tilde{Y}_s - \chi^Y) \geqslant 0\right) \\ &= \alpha \mathbb{P}\left(\inf_{0 \leqslant s \leqslant t} X_s^a \leqslant 0\right) - \beta \mathbb{P}\left(\sup_{0 \leqslant s \leqslant t} Y_s^a \geqslant 0\right) = \Lambda_t^a. \end{aligned}$$

This shows that Λ is also a solution to the modified system (5.2.5). Considering Definition 5.2.2 and [3, Definition 1.1], we see it satisfies the conditions of a physical solution for (5.2.5). The other direction follows in the same way. □

We can now prove an existence result.

Theorem 5.2.4. *Assume that the parameters in the McKean–Vlasov problem* (5.2.1) *satisfy* $\alpha, \beta > 0$, $\mathbb{E}[X_{0-}] < \infty, \mathbb{E}[Y_{0-}] > -\infty$. *Moreover assume that the initial conditions have no atoms at zero, i.e.* $\mathbb{P}\left(X_{0-} = 0\right) = \mathbb{P}\left(Y_{0-} = 0\right) = 0$, *then there exists a physical solution to* (5.2.1).

Proof. We can apply Lemma 5.2.3 to see that any physical solution to the modified problem with absorbing feedback (5.2.5) is a physical solution to (5.2.1) in the sense of Definition (5.2.2). [3] then yields the existence of a physical solution to (5.2.5). □

As discuss in [3], the integrability assumption for the initial condition is equivalent to the most general assumption known that allows for existence in the one-phase models. Moreover, the authors argue that the assumption of no atoms at zero is a natural condition as it would not be possible to decide in a canonical way to which of the two phases the mass at zero belongs.

5.2.2 Particle Approximation

The particle system correspoding to the two-phase McKean–Vlasov problem is given by

$$\begin{cases} X_t^{i,N} = X_{0-}^{i,N} + B_t^i - \Lambda_t^N + L_t(X^{i,N}) \\ Y_t^{i,N} = Y_{0-}^{i,N} + W_t^i - \Lambda_t^N - L_t(Y^{i,N}) \\ \Lambda_t = \frac{\alpha}{N}\sum_{i=1}^N 1_{\sigma^{N,i}\leqslant t} - \frac{\beta}{N}\sum_{i=1}^N 1_{\tau^{i,N}\leqslant t} \\ \sigma^{i,N} = \inf\{t \geqslant 0 : L_t(X^{i,N}) \geqslant \chi^{X^i}\} \\ \tau^{i,N} = \inf\{t \geqslant 0 : L_t(Y^{i,N}) \geqslant \chi^{Y^i}\} \end{cases}$$

where $X_{0-}^{i,N} \geqslant 0$ and $Y_{0-}^{i,N} \leqslant 0, i = 1, \dots, N$ are independent i.i.d. sequences of random variables, that are also independent of the Brownian motions B^i and W^i and the exponential random variables χ^{X^i} and χ^{Y^i}. Moreover, L_t denotes the reflection terms at 0. We can again write $X^{i,N}$ and $Y^{i,N}$ as images of processes $\tilde{X}^{i,N}, \tilde{Y}^{i,N}$ under the Skorkhod reflections Θ^+ and Θ^-, that keep the process on the positive or negative half-line respectively. The proof of convergence to the McKean–Vlasov problem then works by combining the ideas of Section 5.1.3 and [3]. As in Section 5.1.3 we show weak convergence of the system involving the processes on the whole space and then use continuity of the Skorokhod map with respect to the $M1$-topology to conclude the convergence for the elastic system we are interested in. The individual steps in the proof can then be achieved using the approach from [3]. As mentioned at the start of the section, the main complication of the two-phase model is the fact that the process Λ^N is not monotone as in the one-phase model and as a result the tightness proof becomes more challenging. The $M1$-oscillilation, u, of a function $x \in D[S,T]$ is given by

$$u_{[S,T]}(x,\delta) \coloneqq v_{[S,T]}(x,S,\delta) \vee v_{[S,T]}(x,T,\delta) \vee \sup_{t\in[S,T]} w_{[S,T]}(x,t,\delta)$$

with

$$v_{[S,T]}(x,t,\delta) \coloneqq \sup_{S\vee(t-\delta)\leqslant t_1\leqslant t_2\leqslant T\wedge(t+\delta)} |h(t_2) - h(t_1)|$$

and

$$w_{[S,T]}(x,h,y) = \sup_{S\vee(t-\delta)\leqslant t_1<t_2<t_3\leqslant T\wedge(t+\delta)} \inf_{\theta\in[0,1]} |\theta h(t_1) + (1-\theta)h(t_3) - h(t_2)|.$$

The tightness criterion [74, Theorem 12.12.3] is then the following, where we denote by $\tilde{D}[S,T]$ the subspace of $D([S,T])$ of paths that are left-continuous at S, and using

our usual extension to start our process from -1 instead of 0 we always work with such functions.

Lemma 5.2.5. *Let $(\mathbb{P}_N)_{N\geqslant 1}$ be a sequence of probability measures on the space $(\tilde{D}[S,T], M1)$. Then the sequence is tight if and only if*

(i) for all $\varepsilon > 0$ there exists a constant $C < \infty$ such that

$$\mathbb{P}_N(\{x \in \tilde{D}[S,T] : \|x\|_\infty > C\}) < \varepsilon, \quad \text{for all } N \geqslant 1$$

(ii) for all $\varepsilon > 0$ and all $r > 0$ there exists some $\delta > 0$ such that

$$\mathbb{P}\left(x \in \tilde{D}[S,T] : u_{[S,T]}(x,\delta) \geqslant r\right) < \varepsilon, \quad \text{for all } N \geqslant 1.$$

The crucial result from [3] is a bound on the oscillations $w_{[0,T]}$ given in the following lemma.

Lemma 5.2.6 ([3] Lemma 3.2)**.** *For any $N \geqslant 1, t \in [0,T], \delta > 0$ and $r > 0$,*

$$\mathbb{P}(w_{[0,T]}(\Lambda^N, t, \delta) \geqslant r) \leqslant \frac{4(\alpha+\beta)}{r}\Phi(\frac{-r}{2\sqrt{2\delta}}).$$

where Φ denotes the cdf of the standard normal distribution.

Using this result it is straightforward to show tightness of the empirical measure process by establishing tightness of $X^{1,N}$ and $Y^{1,N}$ respectively and applying Proposition 5.1.10. Proceeding as in in the proof of [3, Theorem 4.1.] then yields the desired convergence result.

5.3 One-Phase Model with Common Noise

In this section we consider a particle system on the positive half-line with elastic feedback similar to the system (5.1.8), with an added common noise. Thus, the system we are interested in is given by

$$\begin{cases} X_t^{i,N} = X_{0-}^i + (1-\rho)^{1/2}B_t^i + \rho B_t^0 - \alpha\Lambda_t^N + L_t^{i,N}, \quad t \geqslant 0 \\ \Lambda_t^N = \frac{1}{N}\sum_{i=1}^N 1_{\tau_{i,N}\leqslant t}, \quad t \geqslant 0 \\ \tau_{i,N} = \inf\{t \geqslant 0, L_t^{i,N} \geqslant \chi^i\} \end{cases} \tag{5.3.1}$$

for $i = 1, \ldots, N$ where $B^0, \ldots, B^N$ are independent Brownian motions and $X_{0-}^1, \ldots, X_{0-}^i$ independent and identically-distributed random variables that are also independent

of the Brownian motions. Moreover, χ^i are independent exponential random variables with parameter $\kappa > 0$, independent of the initial conditions and the Brownian motions. The term $L_t^{i,N}$ denotes the reflection term coming from the Skorokhod problem that ensures that $X^{i,N}$ stays nonnegative.

As a result of the common noise we expect that the corresponding limit equation is not a McKean–Vlasov problem but a conditional McKean–Vlasov problem, i.e. the equation contains the conditional distribution given the common noise. It takes the form

$$\begin{cases} X_t = X_{0-} + (1-\rho^2)^{1/2} B_t^1 + \rho B_t^0 - \alpha \Lambda_t + L_t \\ \Lambda_t = \mathbb{P}\left(\tau \leqslant t \middle| B^0\right) \\ \tau = \inf\{t \geqslant 0 : L_t \geqslant \chi\} \end{cases} \tag{5.3.2}$$

where B^1 and B^0 are independent Brownian motion that are independent of the initial condition X_{0-} and the exponential random variable χ with parameter $\kappa > 0$. The interesting questions for the conditional McKean–Vlasov problem (5.3.2) are the same as in the case without a common noise, i.e. we are interested in existence and uniqueness of solutions, existence of blow-ups under different initial conditions and convergence of solutions to the particle system (5.3.1) to solutions to (5.3.2). The conditional distribution in (5.3.1) significantly increases the difficulty in addressing these questions compared to the case without a common noise discussed in Section 5.1. [52] and [57] provide results for some of these questions in the case of an absorbing feedback model with common noise. In [57] the authors employ a generalized Schauder fixed point argument to construct a solution to the conditional McKean–Vlasov. [52] uses a particle approximation to establish existence of solutions to a relaxed version of the conditional McKean–Vlasov problem that relaxes some of the measurability conditions implied in the original problem. Moreover, the authors show that solutions to the conditional McKean–Vlasov problem can be characterized as solutions to a SPDE. They also address the question of whether there exist conditions on the initial condition that ensure that blow-ups will happen or will never happen. While the conditions in [52] are not sharp, they show that, in contrast to the case without common noise, there are initial conditions that lead to a probability that is positive but strictly less than one for the occurrence of a blow-up. Intuitively this is possible because the common noise can transport enough mass to or away from the boundary to provoke or prevent a blow-up, respectively. Uniqueness results for feedback models with common noise remain an open problem.

In this section we discuss results for the elastic feedback model with common noise. We use the same ideas as in Section 5.1 when considering the case without

a common noise and exploit the close connection of the elastic model to a modified absorbing feedback model.

5.3.1 Existence of Solutions

We recall that a particle X solving the conditional McKean–Vlasov problem (5.3.2) is constructed by applying the Skorokhod map Θ that maps a real-valued càdlàg function to the corresponding non-negative reflected process. Thus, if we define a process Y given by

$$Y_t = X_{0-} + (1-\rho^2)^{1/2} B_t^1 + \rho B_t^0 - \alpha \Lambda_t, \tag{5.3.3}$$

the process X can be constructed by applying the Skorokhod map Θ to Y, i.e. $X = \Theta(Y)$. The reflection term L is defined by

$$L_t = (-\inf_{0 \leqslant s \leqslant t} Y_s) \vee 0. \tag{5.3.4}$$

Equipped with this we can establish a connection between solutions to the elastic feedback problem and a modified absorbing problem, similar to Lemma 5.1.4.

Lemma 5.3.1. *Let Λ be an increasing càdlàg process starting from 0. Then Λ is a solution to the conditional McKean–Vlasov problem (5.3.2) if an only if it is a solution to the following conditional McKean–Vlasov problem with absorbing stopping time.*

$$\begin{cases} Y_t = Y_{0-} + (1-\rho^2)^{1/2} B_t^1 + \rho B_t^0 - \alpha \Lambda_t^a \\ \Lambda_t^a = \mathbb{P}\left(\tau^a \leqslant t \middle| B^0\right) \\ \tau^a = \inf\{t \geqslant 0 : Y_t \leqslant 0\} \end{cases} \tag{5.3.5}$$

with $Y_{0-} = X_{0-} + \chi$.

Proof. The proof closely follows the arguments in the case without common noise. Let Λ be a solutions to (5.3.2). Using the definitions of the elastic stopping time τ and the reflection term L we obtain

$$\begin{aligned} \Lambda_t = \mathbb{P}\left(\tau \leqslant t \middle| B^0\right) &= \mathbb{P}\left(L_t \geqslant \chi \middle| B^0\right) \\ &= \mathbb{P}\left(\inf_{0 \leqslant s \leqslant t} Y_s \leqslant 0 \middle| B^0\right) \\ &= \Lambda_t^a. \end{aligned}$$

This completes the proof. □

5.3.2 Provoking and Averting Blow-Ups

We saw in Section 5.1.1 that in the case without common noise the solution to the McKean–Vlasov problem with elastic feedback has to experience a blow-up if

$$\mathbb{E}\left[X_{0-}\right] < \frac{\alpha}{2} - \frac{1}{\kappa}. \tag{5.3.6}$$

Thus, having a blow-up depends on the initial condition, the strength of the feedback and the parameter of the exponential random variable χ. In particular, for a given initial condition and κ, the solution will have a blow-up if the feedback parameter is large enough. Moreover, for a given feedback, the solution is continuous for all $t \geqslant 0$ if the initial condition is supported far enough away from zero.

The picture is more complicated when adding the common noise. The occurrence of a blow-up also depends on the path of the common noise in this case. The realized path of the common noise can transport mass towards the boundary provoking a blow-up and also transport it away from the boundary to temporally avert a blow-up until the diffusion and the boundary disperse enough mass to insure that no blow-ups occur. We can exploit the connection of solutions to the elastic feedback model and a modified absorbing feedback model, that we established in Lemma 5.3.1, and the result from [52] to obtain the following results.

Proposition 5.3.2. *Let the law of the initial condition X_{0-} be supported on the interval (x_*, ∞) for some $x_* > 0$. Then, for a solution Λ to the conditional McKean–Vlasov problem (5.3.2) it holds that*

$$\mathbb{P}\left(\Lambda \text{ experiences blow up}\right) < 1$$

for all $\alpha > 0$.

Proof. Follows from Lemma 5.3.1 and [52, Proposition 2.9]. □

Proposition 5.3.2 shows the ability of the common noise to avert blow-ups. Considering the system without a common noise, equation (5.3.6) shows that, for a given initial condition X_{0-} with finite expectation and a given $\kappa > 0$, the system experiences a blow-up if the feedback parameter $\alpha > 0$ is chosen to be sufficiently large. When the common noise is added, and the distribution of the initial condition is supported away from the boundary, the probability of not experiencing a blow-up is positive, for all feedback parameters $\alpha > 0$.

Proposition 5.3.3. *Let the law of the initial distribution X_{0-} be supported on the interval (x_*, ∞) for some $x_* > 0$. Moreover assume that $\mathbb{E}[X_{0-}] - x_* < \frac{1}{2}\alpha - \frac{1}{\kappa}$. Then the conditional McKean–Vlasov problem (5.3.2) with feedback parameter $\alpha > 0$ satisfies*

$$0 < \mathbb{P}\left(\Lambda \text{ experiences blow up}\right) < 1.$$

Proof. By Proposition 5.3.2 the blow-up probability is strictly less than 1. Combining Proposition 5.3.1 and [52, Proposition 2.10] yields that the probability of a blow-up is strictly positive. □

The most interesting conclusion of Proposition 5.3.3 is that if the feedback parameter α is large enough and such that $\frac{\alpha}{2} - \frac{1}{\kappa} > 0$, the conditional McKean–Vlasov problem (5.3.2) with a fixed starting point $X_{0-} = x_* > 0$ has a positive probability of experiencing a blow-up for any fixed starting point $x_* > 0$. This behaviour is again in contrast to the case without a common noise. If a common noise is not present blow-ups do not occur if the feedback parameter is fixed and the process is started from a fixed point $x_0 > 0$ far enough away from the boundary. In that case the heat equation cannot transport sufficient mass fast enough to the boundary to create a blow-up. However, when adding a common noise we can find a set of realizations of the common noise that has positive probability and consists of paths that move sufficient mass to the boundary to create a blow-up.

Proposition 5.3.2 and Proposition 5.3.3 show how the common noise impacts the occurrence of blow-ups in the conditional McKean–Vlasov problem (5.3.2). A solution to (5.3.2) is a stochastic processes and the event of experiencing a blow-up is random. It depends on the interplay of the initial condition, the parameter κ of the exponential random variable, the feedback parameter α and the realized path of the common noise.

5.4 One-Phase Absorbing Feedback Model with Upper Reflecting Boundary

The final feedback model we consider is a variation of the feedback model with absorbing stopping times studies in [6, 21, 26, 38, 53]. We change the model by adding a reflecting upper boundary at $x = 1$ to avoid particles moving too far away from the feedback boundary. We characterise the feedback process in this case in terms of the distribution of a stopping time for a process on the whole space. Moreover, we derive the PDE connected to this feedback model and prove existence of solutions through a

particle approximation similar to Section 5.1.3 and [21]. Due to the reflection term in the feedback the results of [26] cannot be applied immediately to obtain uniqueness results and uniqueness of solutions remains an open problem.

The McKean–Vlasov problem we are interested in is given by

$$\begin{cases} X_t = X_{0-} + B_t - \alpha\Lambda_t - L_t^1 \\ \Lambda_t = \mathbb{P}(\tau \leqslant t) \\ \tau = \inf\{t \geqslant 0 : X_t \leqslant 0\} \end{cases} \tag{5.4.1}$$

where $X_{0-} \geqslant 0$ is the initial condition, B a Brownian motion, independent of X_{0-} and L^1 denotes the reflection term at 1 given by

$$L_t^1 = \left(\sup_{0 \leqslant s \leqslant t} (X_{0-} + B_s - \alpha\Lambda_s) - 1 \right) \vee 0.$$

Using this expression we can then rewrite the stopping time τ in the following way

$$\begin{aligned} \tau &= \inf\{t \geqslant 0 : X_t \leqslant 0\} \\ &= \inf\{t \geqslant 0 : X_{0-} + B_t - \alpha\Lambda_t \leqslant L_t^1\} \\ &= \left\{ \inf t \geqslant 0 : X_{0-} + B_t - \alpha\Lambda_t \leqslant \left(\sup_{0 \leqslant s \leqslant t} (X_{0-} + B_s - \alpha\Lambda_s) - 1 \right) \vee 0 \right\}. \end{aligned}$$

Defining the whole-space process Y via

$$Y_t = X_{0-} + B_t - \alpha\Lambda_t \tag{5.4.2}$$

the stopping time τ becomes

$$\tau = \inf \left\{ t \geqslant 0 : Y_t \leqslant \left(\sup_{0 \leqslant s \leqslant t} (Y_s) - 1 \right) \vee 0 \right\}. \tag{5.4.3}$$

We see that the stopping time τ is the first time the process Y hits zero, given that it has not hit one before, i.e. if there has been no reflection, or if it has hit one before zero, τ is the first time the difference between the running supremum of Y and the process Y exceeds one. If we denote the first hitting times at zero and one of the process Y by τ^0 and τ^1 respectively we can express the feedback Λ as

$$\begin{aligned} \Lambda_t &= \mathbb{P}(\tau \leqslant t) = \mathbb{P}\left(\tau \leqslant t, \tau^0 < \tau^1\right) + \mathbb{P}\left(\tau \leqslant t, \tau^1 < \tau^0\right) \\ &= \mathbb{P}\left(\inf_{0 \leqslant s \leqslant t} (Y_s) \leqslant 0, \tau^0 < \tau^1 \right) + \mathbb{P}\left(\sup_{0 \leqslant s \leqslant t} \left(\sup_{0 \leqslant u \leqslant s} (Y_u) - Y_s \right) \geqslant 1, \tau^1 < \tau^0 \right). \end{aligned}$$

This decomposition of the stopping time explains the behaviour of the feedback term. Moreover, it shows that the feedback can be expressed through a process Y without

a reflection. Thus, if we are able to construct the process Y we can obtain X using the Skorokhod reflection map at 1, $\Theta^1 : D([0,\infty)) \to D([0,\infty))$ because X is given by

$$X = \Theta^1(Y).$$

As in the previous feedback models we analysed a first result shows that jumps have to occur if the feedback parameter $\alpha > 0$ is large enough.

Lemma 5.4.1. *The map $t \mapsto \Lambda_t$ cannot be continuous for all $t \geqslant 0$ if $2\mathbb{E}[X_{0-}] < \alpha$*

Proof. We assume that $\alpha > 2\mathbb{E}[X_{0-}]$ and that Λ is a continuous solution. We have

$$0 \leqslant X_{0-} + B_{t\wedge\tau} - \alpha\Lambda_{t\wedge\tau} - L^1_{t\wedge\tau}.$$

Taking expectation and rearranging we get

$$\mathbb{E}[X_{0-}] \geqslant \alpha\mathbb{E}[\Lambda_{t\wedge\tau}] + \mathbb{E}[L_{t\wedge\tau}] \geqslant \alpha\mathbb{E}[\Lambda_{t\wedge\tau}].$$

We then take the limit $t \to \infty$ and obtain

$$\mathbb{E}[X_{0-}] \geqslant \alpha\mathbb{E}\left[\Lambda_\tau\right] = \alpha\int_0^\infty \Lambda_s d\Lambda_s = \frac{\alpha}{2}.$$

This is a contradiction and we can conclude that Λ cannot be continuous for all $t \geqslant 0$. □

In particular, as $\mathbb{E}X_{0-} \leqslant 1$ we obtain the crude jump condition $\alpha > 2$.

5.4.1 Connected PDE

In this section we derive a nonlinear and nonlocal PDE that is connected to the McKean–Vlasov problem (5.4.1).

Let $\phi \in C_b^2$ such that $\phi(0) = 0$ and $\phi'(1) = 0$ and define a measure ν_t via $\langle \nu_t, \phi\rangle = \mathbb{E}\left[\phi(X_t)1_{t<\tau}\right]$. By Itô's formula we have

$$\begin{aligned}\phi(X_{t\wedge\tau}) = \phi(X_0) &+ \int_0^t \phi'(X_s)1_{s<\tau}dB_s - \alpha\int_0^t \phi'(X_s)1_{s<\tau}d\Lambda_s - \int_0^t \phi'(X_s)1_{s<\tau}dL^1_s \\ &+ \frac{1}{2}\int_0^t \phi''(X_s)1_{s<\tau}ds.\end{aligned}$$

Note that $\phi(X_\tau) = \phi(0) = 0$ and L^1 is only supported on the set $\{t \geqslant 0 : X_t = 1\}$. The condition $\phi'(1) = 0$ then causes the term involving the local time to vanish. Taking expectation yields

$$\langle \nu_t, \phi\rangle = \langle \nu_0, \phi\rangle + \frac{1}{2}\int_0^t \langle \nu_s, \phi''\rangle ds - \alpha\int_0^t \langle \nu_s, \phi'\rangle d\Lambda_s$$

and we obtain an equation for the measure-valued process ν

$$\begin{cases} \langle \nu_t, \phi \rangle = \langle \nu_0, \phi \rangle + \frac{1}{2} \int_0^t \langle \nu_s, \phi'' \rangle ds - \alpha \int_0^t \langle \nu_s, \phi' \rangle d\Lambda_s \\ \Lambda_t = 1 - \int_0^1 \nu_t(dx) \end{cases}$$

for all $\phi \in C_b^2$ such that $\phi(0) = 0 = \phi'(1)$. Assuming that ν_t has a density V_t we can use integration by parts and calculate

$$\langle \nu_t, \phi' \rangle = \int_0^t V_t \phi' dx = [V_t \phi]_0^1 - \int_0^1 \partial_x V_t \phi dx = V_t(1)\phi(1) - \int_0^1 \partial_x V_t \phi dx$$

and

$$\begin{aligned} \langle \nu_t, \phi'' \rangle = \int_0^1 V_t \phi'' dx &= [V_t \phi']_0^1 - \int_0^1 \partial_x V_t \phi' dx \\ &= [V_t \phi']_0^1 - [\partial_x V_t \phi]_0^1 + \int_0^1 \partial_{xx} V_t \phi dx \\ &= -V_t(0)\phi'(0) - \partial_x V_t(1)\phi(1) + \int_0^t \partial_{xx} V_t \phi dx. \end{aligned}$$

Plugging this into the measure equation and differentiating with respect to t yields

$$\begin{aligned} \int_0^1 \partial_t V_t \phi dx = \frac{1}{2} \int_0^1 \partial_{xx} V_t \phi dx - \frac{1}{2} V_t(0)\phi'(0) - \frac{1}{2} \partial_x V_t(1)\phi(1) \\ + \alpha \Lambda_t' \int_0^1 \partial_x V_t \phi dx - \alpha \Lambda_t' V_t(1)\phi(1) \end{aligned}$$

After rearranging we have

$$\int_0^1 \phi \left[\partial_t V_t = \frac{1}{2} \partial_{xx} V_t - \alpha \Lambda_t' \partial_x V_t \right] dx = \phi'(0) V_t(0) - \phi(1) \left[\alpha \Lambda_t' V_t(1) + \frac{1}{2} \partial_x V_t(1) \right]$$

and we obtain the three equations

$$\begin{cases} \partial_t V_t - \frac{1}{2} \partial_{xx} V_t - \alpha \Lambda_t' \partial_x V_t, \quad 0 < x < 1 \\ V_t(0) = 0 \\ \frac{1}{2} \partial_x V_t(1) + \alpha \Lambda_t' V_t(1) = 0. \end{cases}$$

Moreover we can use the second equation in the measure-valued PDE to compute

$$\begin{aligned}\Lambda_t' = -\int_0^1 \partial_t V_t dx &= -\int_0^1 \frac{1}{2}\partial_{xx} V_t + \alpha\Lambda_t' \partial_x V_t dx \\ &= -\frac{1}{2}\int_0^1 \partial_{xx} V_t dx - \alpha\Lambda_t' \int_0^1 \partial_x V_t dx \\ &= -\frac{1}{2}[\partial_x V_t(1) - \partial_x V_t(0_] - \alpha\Lambda_t'[V_t(1) - V_t(0)] \\ &= \frac{1}{2}\partial_x V_t(0) + \alpha\Lambda_t' V_t(0) - \left[\frac{1}{2}\partial_x V_t(1) + \alpha\Lambda_t' V_t(1)\right] \\ &= \frac{1}{2}\partial_x V_t(0).\end{aligned}$$

Thus, we can complete our PDE problem

$$\begin{cases} \partial_t V_t = \frac{1}{2}\partial_{xx} V_t - \alpha\Lambda_t' \partial_x V_t, \quad 0 < x < 1 \\ \Lambda_t' = \frac{1}{2}\partial_x V_t(0) \\ \frac{1}{2}\partial_x V_t(1) + \alpha\Lambda_t' V_t(1) = 0 \\ V_t(0) = 0. \end{cases}$$

We examine the reflecting boundary condition more closely

$$\frac{1}{2}\partial_x V_t(1) + \alpha\Lambda_t' V_t(1) = 0 = \frac{1}{2}\partial_x V_t(1) + \alpha\frac{1}{2}\partial_x V_t(0) V_t(1) = 0$$

which simplifies to

$$\partial_x V_t(1) + \alpha\partial_x V_t(0) V_t(1) = 0$$

and note that

$$\alpha\partial_x V_t(0) V_t(1) \geqslant 0.$$

Thus, we must have $\partial_x V_t(1) \leqslant 0$. If the system experiences a jump we have $\Lambda_t' = \infty$ and as a result $V_t(1) = 0$.

A typical transformation of the problem involves setting $u(t,x) \coloneqq V_t(x - \alpha\Lambda_t)$. In the setting without an upper reflecting boundary this leads to the supercooled Stefan's problem. In the case we consider here we obtain

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\partial_{xx} u(t,x), \qquad \alpha\Lambda_t < x < \alpha\Lambda_t + 1, \quad t > 0 \\ \frac{1}{2}\partial_x u(t, \alpha\Lambda_t) = \Lambda_t' \qquad t > 0 \\ \frac{1}{2}\partial_x u(t, 1 + \alpha\Lambda_t) + \alpha\Lambda_t' u(t, 1 + \alpha\Lambda_t) = 0 \qquad t > 0 \\ u(t, \alpha\Lambda_t) = 0 \qquad t > 0. \end{cases} \tag{5.4.4}$$

5.4.2 Particle Approximation

The proof of convergence of the approximating particle system to the solution to the McKean–Vlasov equation works in a very similar way to the proof in the one-phase model we discussed in Section 5.1.3 and follows the approach of [21].

We consider the particle system

$$\begin{cases} X_t^{i,N} = X_0^{i,N} + B_t^i - \alpha\Lambda_t^N - L_t^{1,i,N} \\ \Lambda_t^N = \frac{1}{N}\sum_{i=1}^N 1_{\tau^{i,N}\leqslant t} \\ \tau^{i,N} = \inf\{t \geqslant 0 : X_t^{i,N} \leqslant 0\} \end{cases}$$

with the local time at one of particle $X^{i,N}$ given by

$$L_t^{1,i,N} = \sup_{0\leqslant s\leqslant t}(X_0^{i,N} + B_s^i - \alpha\Lambda_s^{i,N} - 1) \vee 0.$$

Similar to the McKean–Vlasov equation we can consider particles $Y^{i,N}$ without reflection given by

$$Y_t^{i,N} = X_0^{i,N} + B_t^i - \alpha\Lambda_t^N \tag{5.4.5}$$

and we get that $X^{i,N} = \Theta^1(Y^{i,N})$, where Θ^1 is the Skorokhod reflection map.

We define the empirical measures ν^N and ν_Θ^N by

$$\nu^N := \frac{1}{N}\sum_{i=1}^N \delta_{Y^{i,N}}, \qquad \nu_\Theta^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}. \tag{5.4.6}$$

The main difference between the model in Section 5.1.3 and the one here, is the different form of the feedback. Here, the feedback depends on the difference between the process and its running supremum. Thus, we need a different path functional and the main part of the argument is to show continuity with respect to the $M1$-topology of this functional. The proof of tightness and convergence to the law of a solution to the McKean–Vlasov equation then follows as in Section 5.1.3.

Definition 5.4.2. For $t \in \mathbb{R}, x \in D([-1,\infty))$, define the following path functionals

$$\tau_0(x) := \inf\{s \geqslant 0 : x_s - (\sup_{0\leqslant u\leqslant s}(x_u) - 1) \vee 0) \leqslant 0\}, \quad \lambda_t(x) := 1_{[\tau_0(x)\leqslant t]}$$

Lemma 5.4.3. *Let $x^n, x \in \iota_\alpha(\bar{E})$ and suppose that $x^n \to x$ in $D([-1,\infty))$. Then, if $t \geqslant 0$ is any point of continuity of x, it holds that*

$$\lim_{n\to\infty}\inf_{0\leqslant s\leqslant t}\left(x_s^n - (\sup_{0\leqslant u\leqslant s}(x_u^n) - 1) \vee 0)\right) = \inf_{0\leqslant s\leqslant t}\left(x_s - (\sup_{0\leqslant u\leqslant s}(x_u) - 1) \vee 0)\right)$$

Proof. We only need to show that $x \mapsto x_s - (\sup_{0\leqslant u\leqslant s}(x_u) - 1) \vee 0)$ is continuous in the M1-topology for $x \in \iota_\alpha(\bar{E})$. The rest of the proof then follows as in Section 5.1.3. Note that any $x \in \iota_\alpha(\bar{E})$ only has downward jumps. Moreover, the running supremum is a nondecreasing process. So x and $-(\sup_{0\leqslant u\leqslant s}(x_u) - 1) \vee 0)$ cannot have jumps in opposite directions. Thus, the sum is continuous with respect to the $M1$ topology and the proof follows. □

Once we have establish the continuity of the path functional, the rest of the convergence proof follows in the same way as in Section 5.1.3 and we obtain the result.

Proposition 5.4.4. *For $N \geqslant 1$, let (X^N, Λ^N) be a solution to the particle system*

$$\begin{cases} X_t^{i,N} = \Theta^1(Y_t^{i,N}) = \Theta^1(X_{0-}^{i,N} + B_t^i - \alpha\Lambda_t^N) = X_{0-}^{i,N} + B_t^i - \alpha\Lambda_t^N + L_t^{1,i,N} \\ \Lambda_t^N = \frac{1}{N}\sum_{i=1}^N 1_{\tau_{i,N}\leqslant t} \\ \tau_{i,N} = \inf\{t \geqslant 0 : Y_t^{i,N} - (\sup_{0\leqslant s\leqslant t}(Y_s^{i,N} - 1) \vee 0) \leqslant 0\} \end{cases}$$

and let ν^N and $(\nu_N^{\Theta^1})_{N\geqslant 1}$ be the corresponding empirical measures defined in (5.4.6). Suppose that for some random measure ν^{Θ^1} and some measure $\nu_{0-} \in \mathcal{P}(\mathbb{R}_+)$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \delta_{X_{0-}^{i,N}} = \nu_{0-}$$

and $Law(\nu_N^{\Theta^1}) \to Law(\nu^{\Theta^1})$ along some subsequence. Then ν^{Θ^1} corresponds almost surely to the law of a solution to the McKean–Vlasov (5.4.1) problem with $Law(X_{0-}) = \nu_{0-}$.

Appendix A

Properties of the Elastic Heat Kernel

We begin by introducing several heat kernels, corresponding to a standard Brownian motion either on the whole real line or on the positive half-line with either reflection or elastic killing imposed at the origin.

Recall first that the Gaussian heat kernel on $\mathbb{R}$ with variance ε, which we denote by p_ε, is given by

$$p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}. \tag{A.1}$$

Starting from this, the reflecting (or Neumann) heat kernel G_ε^R on $[0,\infty)$ can then be defined as

$$G_\varepsilon^R(x,y) := p_\varepsilon(x-y) + p_\varepsilon(x+y). \tag{A.2}$$

Finally, the elastic (or Robin) heat kernel $G_\varepsilon^{E,\kappa}$ is given by [see [9] Appendix 1 Section 10]

$$\begin{aligned} G_\varepsilon^{E,\kappa}(x,y) &= \frac{1}{\sqrt{2\pi\varepsilon}} \left[e^{-(x-y)^2/2\varepsilon} + e^{-(x+y)^2/2\varepsilon} \right] \\ &\quad - \kappa \exp\left(\kappa(x+y) + \frac{\kappa^2\varepsilon}{2} \right) \left(1 - \operatorname{Erf}\left(\frac{x+y+\kappa\varepsilon}{\sqrt{2\varepsilon}} \right) \right) \\ &= G_\varepsilon^R(x,y) - g_\varepsilon^{E,\kappa}(x,y) \end{aligned} \tag{A.3}$$

where Erf is the error function

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz.$$

Lemma A.1. *The two maps $x \mapsto G_\varepsilon^{E,\kappa}(x,y)$, for $y \geqslant 0$, and $y \mapsto G_\varepsilon^{E,\kappa}(x,y)$, for $x \geqslant 0$, satisfy the elastic boundary condition and are elements of the test function space $\mathcal{C}_0^{E,\kappa}(\mathbb{R})$ for all $\varepsilon > 0$.*

Proof. Consider the map $x \mapsto G_\varepsilon^E(x,y)$. We have that

$$\kappa G_\varepsilon^E(0,y) = 2\kappa p_\varepsilon(y) - \kappa^2 \exp\left(\kappa y + \frac{\kappa^2\varepsilon}{2}\right)\left(1 - \mathrm{Erf}\left(\frac{y+\kappa\varepsilon}{\sqrt{2\varepsilon}}\right)\right)$$

and for the derivative at zero we use linearity to obtain

$$\partial_x G_\varepsilon^E(0,y) = \partial_x G_\varepsilon^r(0,y) - \partial_x g_\varepsilon(0,y).$$

Then we get from the two terms that

$$\partial_x G_\varepsilon^r(0,y) = \frac{1}{\sqrt{2\pi\varepsilon}}\left[\frac{y}{\varepsilon}e^{-y^2/2\varepsilon} - \frac{y}{\varepsilon}e^{-y^2/2\varepsilon}\right] = 0$$

and by product rule for the second term

$$\begin{aligned}\partial_x g_\varepsilon(0,y) &= \kappa^2 \exp\left(\kappa y + \frac{\kappa^2\varepsilon}{2}\right)\left(1 - \mathrm{Erf}\left(\frac{y+\kappa\varepsilon}{\sqrt{2\varepsilon}}\right)\right) \\ &\quad - \frac{2\kappa}{\sqrt{2\pi\varepsilon}} \exp\left(\kappa y + \frac{\kappa^2\varepsilon}{2} - \frac{(y+\kappa\varepsilon)}{2\varepsilon}\right) \\ &= \kappa^2 \exp\left(\kappa y + \frac{\kappa^2\varepsilon}{2}\right)\left(1 - \mathrm{Erf}\left(\frac{y+\kappa\varepsilon}{\sqrt{2\varepsilon}}\right)\right) - 2\kappa p_\varepsilon(y).\end{aligned}$$

Putting these together yields the desired relation

$$\kappa G_\varepsilon^E(0,y) = \partial_x G_\varepsilon^E(0,y).$$

The other case follows in a similar way. □

We need a relation between derivatives in x and y of $G_\varepsilon^{E,\kappa}$ to be able to switch derivatives from one variable to the other. This is the content of the following lemma.

Lemma A.2. *The derivatives of the elastic heat kernel $G_\varepsilon^{E,\kappa}$ satisfy*

(i) $\partial_y G_\varepsilon^{E,\kappa}(x,y) = -\partial_x G_\varepsilon^{E,\kappa}(x,y) + 2\partial_x p_\varepsilon(x+y) - 2\partial_x g_\varepsilon^{E,\kappa}(x,y)$

(ii) $\partial_{yy} G_\varepsilon^{E,\kappa}(x,y) = \partial_{xx} G_\varepsilon^{E,\kappa}(x,y)$.

Proof. (i) We have

$$\begin{aligned}\partial_y G_\varepsilon^{E,\kappa}(x,y) &= \partial_y G_\varepsilon^R(x,y) - \partial_y g_\varepsilon^{E,\kappa}(x,y) \\ \partial_x G_\varepsilon^{E,\kappa}(x,y) &= \partial_x G_\varepsilon^R(x,y) - \partial_x g_\varepsilon^{E,\kappa}(x,y).\end{aligned}$$

Note that by symmetry of $g_\varepsilon^{E,\kappa}$ in x and y we have $\partial_y g_\varepsilon^{E,\kappa}(x,y) = \partial_x g_\varepsilon^{E,\kappa}(x,y)$. Recall that the reflecting heat kernel is given by

$$G_\varepsilon^R(x,y) = p_\varepsilon(x-y) + p_\varepsilon(x+y).$$

Thus, we get the relation

$$\partial_y G_\varepsilon^R(x,y) = -\partial_x G_\varepsilon^R(x,y) + 2\partial_x p_\varepsilon(x+y).$$

Considering the two equations for the elastic heat kernel we have

$$\partial_y G_\varepsilon^{E,\kappa}(x,y) = -\partial_x G_\varepsilon^{E,\kappa}(x,y) + 2\partial_x p_\varepsilon(x+y) - 2\partial_x g_\varepsilon^{E,\kappa}(x,y).$$

(ii) It is easy to see that

$$\partial_{yy} G_\varepsilon^R(x,y) = \partial_{xx} G_\varepsilon^R(x,y).$$

The same relation holds for $g_\varepsilon^{E,\kappa}$ by symmetry in x and y which yields the desired result.

□

We also need a bound on the elastic correction term $g_\varepsilon^{E,\kappa}$.

Lemma A.3. *For all $x, y \geqslant 0$ and all $\varepsilon > 0$ we have*

$$g_\varepsilon^{E,\kappa}(x,y) \leqslant \kappa e^{-\frac{(x+y)^2}{2\varepsilon}}.$$

Proof. Using the definition of the error function and an change of variable we can estimate

$$\begin{aligned}
g_\varepsilon^{E,\kappa}(x,y) &= \kappa e^{\kappa(x+y)+\frac{\kappa^2\varepsilon}{2}} \left(1 - \operatorname{Erf}\left(\frac{x+y+\kappa\varepsilon}{\sqrt{2\varepsilon}}\right)\right) \\
&= 2\kappa e^{\kappa(x+y)+\frac{\kappa^2\varepsilon}{2}} \frac{1}{\sqrt{2\pi\varepsilon}} \int_0^\infty e^{-\frac{(z+x+y+\kappa\varepsilon)^2}{2\varepsilon}} dz \\
&= 2\kappa e^{\kappa(x+y)+\frac{\kappa^2\varepsilon}{2}} \frac{1}{\sqrt{2\pi\varepsilon}} \int_0^\infty e^{-\frac{(z+x+y)^2}{2\varepsilon}} e^{-\frac{\kappa^2\varepsilon}{2}} e^{-\kappa(z+x+y)} dz \\
&\leqslant \kappa \frac{2}{\sqrt{2\pi\varepsilon}} \int_0^\infty e^{-\frac{z^2}{2\varepsilon}} e^{-\frac{(x+y)^2}{2\varepsilon}} dz \\
&\leqslant \kappa e^{-\frac{(x+y)^2}{2\varepsilon}}.
\end{aligned}$$

□

Appendix B

Properties of the Sticky Heat Kernel

We introduce the sticky heat kernel and prove several useful properties.

The Gaussian heat kernel with variance ε is given by

$$p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}. \tag{B.1}$$

Using this, we can define the absorbing (or Dirichlet) heat kernel G^a_ε on $[0,\infty)$ as

$$G^a_\varepsilon(x,y) \coloneqq p_\varepsilon(x-y) - p_\varepsilon(x+y). \tag{B.2}$$

The sticky heat kernel $G^{S,c}_\varepsilon$ with parameter $c > 0$ is then given by

$$\begin{aligned} G^{S,c}_\varepsilon(x,y) &= \frac{1}{\sqrt{2\pi\varepsilon}} \left[e^{-(x-y)^2/(2\varepsilon)} - e^{-(x+y)^2/(2\varepsilon)} \right] + \frac{1}{c} e^{\frac{x+y}{c} + \frac{\varepsilon}{2c^2}} \left[1 - \operatorname{Erf}\left(\frac{x+y}{\sqrt{2\varepsilon}} + \frac{\sqrt{\varepsilon}}{\sqrt{2}c} \right) \right] \\ &= G^a_\varepsilon(x,y) + \frac{1}{c} e^{\frac{x+y}{c} + \frac{\varepsilon}{2c^2}} \left[1 - \operatorname{Erf}\left(\frac{x+y}{\sqrt{2\varepsilon}} + \frac{\sqrt{\varepsilon}}{\sqrt{2}c} \right) \right] \end{aligned} \tag{B.3}$$

where Erf denotes the error function

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz.$$

Moreover, we denote by $g^{S,c}_\varepsilon$ the sticky correction term

$$g^{S,c}_\varepsilon(x,y) = \frac{1}{c} e^{\frac{x+y}{c} + \frac{\varepsilon}{2c^2}} \left[1 - \operatorname{Erf}\left(\frac{x+y}{\sqrt{2\varepsilon}} + \frac{\sqrt{\varepsilon}}{\sqrt{2}c} \right) \right]. \tag{B.4}$$

Lemma B.1. *The map $y \mapsto G^{S,c}_\varepsilon(x,y)$ for $x \geqslant 0$ satisfies the sticky boundary condition*

$$\partial_y G^{S,c}_\varepsilon(x,0) = c \partial_{yy} G^{S,c}_\varepsilon(x,0).$$

Lemma B.2. *The following relations for the derivatives of the sticky heat kernel $G_\varepsilon^{S,c}$ in x and y hold*

(i) $\partial_y G_\varepsilon^{S,c}(x,y) = -\partial_x G_\varepsilon^{S,c}(x,y) - 2\partial_x p_\varepsilon(x+y) + 2\partial_x g_\varepsilon^{S,c}(x,y)$

(ii) $\partial_{yy} G_\varepsilon^{S,c}(x,y) = \partial_{xx} G_\varepsilon^{S,c}(x,y)$

Proof.

(i) Note that the sticky heat kernel can be written as

$$G_\varepsilon^{S,c}(x,y) = G_\varepsilon^{a}(x,y) + g_\varepsilon^{S,c}(x,y)$$

and the function $g_\varepsilon^{S,c}$ is symmetric in x and y.
For the absorbing heat kernel G_ε^a we have the relation

$$\partial_y G_\varepsilon^{a}(x,y) = -\partial_x G_\varepsilon^{a}(x,y) - 2\partial_x p_\varepsilon(x+y).$$

Moreover, symmetry in x and y yields $\partial_y g_\varepsilon^{S,c}(x,y) = \partial_x g_\varepsilon^{S,c}(x,y)$. We can combine these to obtain

$$\begin{aligned}\partial_y G_\varepsilon^{S,c}(x,y) &= \partial_y G_\varepsilon^{a}(x,y) + \partial_y g_\varepsilon^{S,c}(x,y)\\ &= -\partial_x G_\varepsilon^{a}(x,y) + \partial_x g_\varepsilon^{S,c}(x,y) - 2\partial_x p_\varepsilon(x+y)\\ &= -\partial_x G_\varepsilon^{S,c}(x,y) - 2\partial_x p_\varepsilon(x+y) + 2\partial_x g_\varepsilon^{S,c}(x,y).\end{aligned}$$

(ii) Note that $\partial_{yy} G_\varepsilon^{A}(x,y) = \partial_{xx} G_\varepsilon^{A}(x,y)$ and the same property holds for $g_\varepsilon^{S,c}(x,y)$ by symmetry. Thus, we get

$$\begin{aligned}\partial_{yy} G_\varepsilon^{S,c}(x,y) = \partial_{yy} G_\varepsilon^{a}(x,y) + \partial_{yy} g_\varepsilon^{S,c}(x,y) &= \partial_{xx} G_\varepsilon^{a}(x,y) + \partial_{xx} g_\varepsilon^{S,c}(x,y)\\ &= \partial_{xx} G_\varepsilon^{S,c}(x,y).\end{aligned}$$

□

We need an estimate for the sticky heat kernel $G_\varepsilon^{S,c}$ in terms of the Gaussian heat kernel p_ε.

Lemma B.3. *For all $\varepsilon > 0$ and all $x, y > 0$ we have the estimate*

$$G_\varepsilon^{S,c}(x,y) \leqslant 3p_\varepsilon(x-y)$$

Proof. We can decompose the sticky heat kernel in the following way

$$G_\varepsilon^{S,c}(x,y) = G_\varepsilon^a(x,y) + g_\varepsilon^{S,c}(x,y) \leqslant p_\varepsilon(x-y) + g_\varepsilon^{S,c}(x,y).$$

Considering the definition of the elastic heat kernel $G_\varepsilon^{E,\kappa}$ (see [9, Appendix 1, Sect. 10])

$$G_\varepsilon^{E,\kappa}(x,y) = p_\varepsilon(x-y) + p_\varepsilon(x+y) - g_\varepsilon^{E,\kappa}(x,y)$$

where the elastic correction term $g_\varepsilon^{E,\kappa}$ satisfies $g_\varepsilon^{S,c}(x,y,) = g_\varepsilon^{E,\frac{1}{c}}(x,y)$ for $c > 0$. Since the elastic heat kernel is nonnegative we get the estimate

$$g_\varepsilon^{S,c}(x,y) \leqslant p_\varepsilon(x-y) + p_\varepsilon(x+y) \leqslant 2p_\varepsilon(x-y)$$

for any $x, y > 0$ which completes the proof.

□

Appendix C

Heat Kernel Mollification in H^{-1}

Let ζ be a finite signed measure and let p_ε denote the Gaussian heat kernel. The convolution $\zeta * p_\varepsilon$ given by

$$\zeta * p_\varepsilon(x) = \int_{\mathbb{R}} p_\varepsilon(x-y)\zeta(dy) \tag{C.1}$$

is a function in $C^\infty(\mathbb{R})$ and the sequence $(\zeta * p_\varepsilon)$ converges weakly to ζ as $\varepsilon \to 0$, i.e. for every bounded and continuous function $\phi : \mathbb{R} \to \mathbb{R}$ it holds that

$$\int_{\mathbb{R}} \phi(x)(\zeta * p_\varepsilon)(x)dx \to \int_{\mathbb{R}} \phi(x)\zeta(dx), \quad \text{as } \varepsilon \to 0.$$

This smooth approximation is the basis of the kernel smoothing method but instead of the Gaussian heat kernel we use the elastic heat kernel $G_\varepsilon^{E,\kappa}$. We define the smoothed measure $T_\varepsilon^{E,\kappa}\zeta$ as

$$T_\varepsilon^{E,\kappa}\zeta(x) = \langle \zeta, G_\varepsilon^{E,\kappa}(x,\cdot)\rangle = \int_0^\infty G_\varepsilon^{E,\kappa}(x,y)\zeta(dy). \tag{C.2}$$

The reason for this choice is that it is an element of the test function space $\mathcal{C}_0^{E,\kappa}(\mathbb{R})$. This means we can use it in the weak formulation of the SPDE. Furthermore, it is known in an explicit form which enables explicit computations such as the switching of derivatives in Lemma A.2. The weak convergence still holds for $T_\varepsilon^{E,\kappa}\zeta$ in the case we are interested in. Furthermore, we have the following property.

Proposition C.1 (Contraction)**.** *Let $f \in L^2(0,\infty)$. Then $\left\|T_\varepsilon^{E,\kappa} f\right\|_2 \leqslant \|f\|_2$ for all $\varepsilon > 0$.*

Proof. Note that $G_\varepsilon^{E,\kappa} \geqslant 0$ and apply Cauchy-Schwarz inequality

$$\begin{aligned}|T_\varepsilon^{E,\kappa} f(x)|^2 &= |\int_0^\infty G_\varepsilon^{E,\kappa}(x,y) f(y) dy|^2 \\ &= |\int_0^\infty (G_\varepsilon^{E,\kappa}(x,y))^{\frac{1}{2}} (G_\varepsilon^{E,\kappa}(x,y))^{\frac{1}{2}} f(y) dy|^2 \\ &\leqslant \int_0^\infty G_\varepsilon^{E,\kappa}(x,y) dy \cdot \int_0^\infty G_\varepsilon^{E,\kappa}(x,y) f(y)^2 dy.\end{aligned}$$

Integrating over $x \geqslant 0$ yields the result. □

The space in which we perform the energy estimates is H^{-1}, the dual space of H^1. The Sobolev space H^1 is the space of L^2-functions with weak derivative in L^2 equipped with the norm

$$\|f\|_{H^1(0,\infty)} \coloneqq \left(\|f\|_{L^2(0,\infty)}^2 + \|\partial_x f\|_{L^2(0,\infty)}^2 \right)^{1/2}.$$

The space H^{-1} is its dual space given by the linear functionals on H^1 with norm

$$\|\zeta\|_{-1} \coloneqq \sup_{\|\phi\|_{H^1(0,\infty)}=1} |\zeta(\phi)|.$$

The following proposition is adapted from [37]. It justifies why we work on H^{-1}.

Proposition C.2. *Let ζ be a finite signed measure. Then $\zeta \in H^{-1}$.*

Another tool we need for the energy estimates is the notion of an anti-derivative ∂_x^{-1} that we define as

$$\partial_x^{-1} f(x) = -\int_x^\infty f(y) dy \tag{C.3}$$

for an integrable function $f : \mathbb{R} \to \mathbb{R}$. Note that we have that $\partial_x \partial_x^{-1} f = f$. The reason we introduce the anti-derivative is the following estimate for the H^{-1}-norm.

Lemma C.3. *Let $\zeta \in H^{-1}$. Let furthermore the convolution operator $T_\varepsilon^{E,\kappa}$ and the anti-derivative ∂_x^{-1} be defined as above. Then there exists a constant $C > 0$ such that*

$$\|\zeta\|_{-1} \leqslant C \liminf_{\varepsilon \to 0} \left(\left\| \partial_x^{-1} T_\varepsilon^{E,\kappa} \zeta \right\|_{L^2(0,\infty)} + \left| \int_0^\infty T_\varepsilon^{E,\kappa} \zeta(y) dy \right| \right) \tag{C.4}$$

Proof. Take a function $\phi \in H^1(0,\infty)$. By integration by parts we have

$$\begin{aligned}\langle \zeta, T_\varepsilon^{E,\kappa} \phi \rangle = (T_\varepsilon^{E,\kappa} \zeta, \phi)_{L^2(0,\infty)} &= \int_0^\infty \partial_x \partial_x^{-1} T_\varepsilon^{E,\kappa} \zeta(x) \phi(x) dx \\ &= \left[\partial_x^{-1} T_\varepsilon^{E,\kappa} \zeta(x) \phi(x) \right]_0^\infty - \int_0^\infty \partial_x^{-1} T_\varepsilon^{E,\kappa} \zeta(x) \partial_x \phi(x) dx \\ &= -\partial_x^{-1} T_\varepsilon^{E,\kappa} \zeta(0) \phi(0) - \int_0^\infty \partial_x^{-1} T_\varepsilon^{E,\kappa} \zeta(x) \partial_x \phi(x) dx\end{aligned}$$

Note that $\phi(0) \leqslant \|\phi\|_\infty$ and by Morrey's inequality [29] $\|\phi\|_\infty \leqslant C\,\|\phi\|_{H^1}$ for a universal constant $C > 0$. Therefore, we obtain

$$|\zeta(\phi)| \leqslant \liminf_{\varepsilon \to 0} \left(C |\partial_x^{-1} T_\varepsilon^{E,\kappa} \zeta(0)\, \|\phi\|_{H^1}| + \left\| \partial_x^{-1} T_\varepsilon^{E,\kappa} \zeta \right\|_2 \|\phi\|_{H^1} \right).$$

Then take the supremum over all ϕ with $\|\phi\|_{H^1} = 1$. This yields the result. □

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