

Agenda-Manipulation in Ranking

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We study the susceptibility of committee governance (e.g. by boards of directors), modelled as the collective determination of a ranking of a set of alternatives, to manipulation of the order in which pairs of alternatives are voted on—*agenda-manipulation*. We exhibit an agenda strategy called *insertion sort* that allows a self-interested committee chair with no knowledge of how votes will be cast to do as well as if she had complete knowledge. Strategies with this “regret-freeness” property are characterized by their *efficiency*, and by their avoidance of two intuitive errors. What distinguishes regret-free strategies from each other is how they prioritize among alternatives; insertion sort prioritizes *lexicographically*.

Key words: Voting, Agenda, Committee, Ranking, Sorting, Regret, Amendment.

JEL Codes: C7, D71, D72, D8

1. INTRODUCTION

In many modern organizations, governance is entrusted not to a single individual, but to a committee: a group that reaches decisions by aggregating the individual votes of its members, usually via a majority or super-majority rule. Public firms and charities are run by boards of directors, for example.

To govern is to make decisions: to choose one alternative from among those that are available. These alternatives may be e.g. policies, candidates for a job, or investments, or they could be coarser things like *attributes* of policies, candidates or investments.

In practice, most governing committees meet too infrequently to be able to consider and solve each individual decision problem faced by their organization. Lacking the ability or time to foresee which decision problems (sets of available alternatives) will arise at any given future date, the committee cannot specify what alternative ought to be chosen. Committees therefore tend to set only high-level priorities, while delegating the details of day-to-day decision-making to executive officers. Such committee governance may be captured in stylized fashion by thinking of the committee as determining a *ranking* of all the potentially available alternatives, with

executives instructed to choose the highest-ranked alternative from among those that turn out to be available.

As observed by Condorcet (1785), the majority will of a committee can easily fail to be transitive: there may be a majority for alternative α over β , for β over γ , and for γ over α . Whenever there are such (Condorcet) cycles, the order in which questions are considered by the committee will affect what decision is reached. The chair of the committee can thus influence collective decisions via her control of the agenda.

We model agenda-setting by the chair as her determining the order in which pairs of alternatives are voted on. We assume that the committee is sovereign, so that a majority vote for α over β leads to α being ranked above β . Suppose that β had already been ranked above γ . Since rankings are by their nature transitive, it is then necessary also that α be ranked above γ . These are the rules: transitivity is imposed.

We abstract from strategic voting: we assume that if α garners a majority over β under one agenda-setting strategy of the chair, then the same is true under every agenda-setting strategy. This assumption buys tractability, but at a price: it is a strong idealization in many applied contexts, albeit empirically reasonable in some cases. We view this as the main limitation of our analysis.

The chair is uncertain how voters will vote on any given pair of alternatives, so that the only way to find out whether α would beat β or vice-versa is to offer a (binding) vote on this pair. In this uncertain world, how much influence can the chair exert via her control of the agenda? And in doing so, what sorts of agenda-setting strategies should we expect the chair to use?

Our first finding is that agenda-setting is surprisingly powerful. In particular, there exist *regret-free* agenda-setting strategies: no matter what the unknown voting behaviour of the committee, the ranking reached by such a strategy is not unambiguously worse (from the chair's perspective) than any other ranking that the chair could have reached had she perfectly known how voters would behave.

We prove this by exhibiting a regret-free strategy called *insertion sort*, which works as follows. Let's call the chair's k th-favourite alternative simply " k ". No votes involving k are offered until all of the alternatives that the chair considers strictly worse than k have been totally ranked. Once that has happened, insertion sort pits k against whichever of the worse alternatives is ranked highest; if k wins, then by transitivity, it is ranked above all of the worse alternatives. If not, then k is next pitted against the *second*-highest-ranked of the worse alternatives; if k wins, then it is ranked above all but the highest-ranked of the worse alternatives. If not, then k is pitted against the *third*-highest ranked of the worse alternatives; and so on.

Having established that regret-free agenda-setting is possible, we next ask what its qualitative properties are: that is, what do regret-free strategies have in common? We provide two characterizations of regret-free strategies. The first describes regret-free agenda-setting in terms of its outcomes: regret-free strategies are exactly those that (*ex post*) reach rankings that satisfy an *efficiency* property. The second is a qualitative description of behaviour: it identifies two intuitive types of *error* and shows that avoiding these is both necessary and sufficient for regret-freeness.

Finally, we ask what it is that distinguishes one regret-free strategy from another. The answer is *prioritization*: while one regret-free strategy focusses on getting a certain pair of alternatives ranked "right" (i.e. the way around that the chair prefers), another strategy may prioritize another pair. We show that insertion sort is characterized by *lexicographic* prioritization: it places the chair's favourite alternative as highly as possible, and subject to that, places her second-favourite alternative as well as possible, and so on.

1.1. *Related literature*

We contribute to the agenda-manipulation literature initiated by Farquharson (1969) and Black (1958), which asks how a committee's choice can be influenced by varying the order in which binary questions are voted on (the *agenda*). Two classes of agenda are emphasized: the *amendment procedure* used in Anglo-Saxon and Scandinavian legislatures, and the *successive procedure* widely used in continental Europe. Under complete information, for both sincere and strategic voting, Miller (1977) and Banks (1985) characterize which alternatives an agenda-setter can induce a committee to choose using (1) amendment agendas, (2) successive agendas, and (3) arbitrary agendas.^{1,2} Extensions include super- and sub-majority voting rules (Barberà and Gerber, 2017) and random agendas (Roessler, Shelegia and Strulovici, 2018).

Specifically, this article belongs to the literature on agenda-setting under incomplete information about voters' preferences. This literature long consisted of a single pioneering paper (Ordeshook and Palfrey, 1988), but has recently received interest from Kleiner and Moldovanu (2017), Gershkov, Moldovanu and Shi (2017, 2019, 2020), and Gershkov, Kleiner, Moldovanu and Shi (2020). We depart from these papers by considering a committee tasked not with choosing a single alternative, but rather with *ranking* all of the alternatives. We establish a link with the older literature by relating insertion sort to the amendment procedure (Section 6.4).

Also related is the social choice literature on *ranking methods*, meaning maps that assign to each (possibly cyclic) majority will a (transitive) ranking. "Impossibility" results such as Arrow's (1950, 1951, 1963) theorem assert that certain normatively appealing properties are inconsistent. The literature beginning with Zermelo (1929), Wei (1952), and Kendall (1955) studies ranking methods with at least *some* attractive normative properties.³ Our chair's problem can be formulated as a choice among ranking methods, but there is a feasibility constraint, and the objective reflects the chair's self-interest. This suggests that solutions to our chair's problem will bear little relation to the normative-motivated ranking methods in the literature; we confirm this in Supplementary Appendix F.

Both the voting and social choice literatures presume the existence of Condorcet cycles. Cycles are certainly *a priori* plausible.⁴ And they are empirically common, especially when there are few voters, as on a committee.⁵

Regret-based criteria for evaluating strategies appear in decision theory, including minimax regret (Savage, 1951) and "regret theory" (Bell, 1982; Loomes and Sugden, 1982; Fishburn, 1983). The online learning literature (Gordon, 1999; Zinkevich, 2003) studies (asymptotic) regret-freeness.

1. Part (1) under strategic voting due to Banks (1985); the rest are from Miller (1977). See Myerson (1991, Section 4.10) for a nice textbook treatment of (3) under strategic voting.

2. Related work by Apesteguia, Ballester and Masatlioglu (2014) and Horan (2021) axiomatizes the decision rules (which specify a choice for each choice set and preference profile) defined by various agendas under strategic voting.

3. For example, Copeland's (1951) method (Rubinstein, 1980), the Kemeny–Slater method (Kemeny, 1959; Slater, 1961; Young and Levenglick, 1978; Young, 1986, 1988) and the fair-bets method (Daniels, 1969; Moon and Pullman, 1970; Slutzki and Volij, 2005). See Charon and Hudry (2010) and González-Díaz, Hendrick and Lohmann (2014) for an overview.

4. For example, the probability of a cycle with five (seven) voters and five (six) alternatives is 39% (61%). See Gehrlein (1989) for a table of such probabilities.

5. The authoritative survey of Gehrlein and Lepelley (2011) includes data on 127 elections, of which 32 had small electorates (≤ 25 voters). A Condorcet cycle existed in 29% of all elections, and in 59% of small elections (see their Table 1.1).

1.2. Roadmap

We begin in Section 2 with two applications. We describe the environment and basic concepts in Section 3, and discuss interpretation and extensions in Section 4. We then (Section 5) establish that regret-free strategies exist by exhibiting one: insertion sort. In Section 6, we characterize regret-freeness in terms of efficiency (Theorem 2) and error-avoidance (Theorem 3) and show that both characterizations are tight (Propositions 1 and 2). In the same section, we show that regret-free strategies differ in their prioritization and characterize insertion sort in terms of its lexicographic prioritization (Theorem 4).

2. TWO APPLICATIONS

Before describing the abstract model, we fix ideas with two applications.

Hiring. *The alternatives \mathcal{X} are candidates for a job. Only some unknown subset of candidates would accept the job if offered it. The hiring committee decides the order in which offers should be made (a ranking): the job will be offered to the first candidate, then to the second if the first declined, and so on.*

Relabelling, we may instead think of the alternatives as investment projects of unknown viability. A firm's board (or a lower-level committee) ranks the projects, whereupon a manager evaluates the first project (e.g. by commissioning market research) and implements it if viable and otherwise evaluates the second project and implements that if viable, and so on.

Similarly, the committee could be a policy-making body, such as a ministerial cabinet or a parliamentary committee. Any given policy may turn out to be infeasible, for example due to a court ruling or political opposition. The committee therefore ranks the policies and tasks a bureaucrat with implementing the first policy if feasible, the second if not, and so on.

Party lists. *A political party's leadership committee must draw up a party list, meaning a ranking of the party's parliamentary candidates \mathcal{X} . The K top-ranked candidates will earn parliamentary seats, where K is the (uncertain) number of seats won by the party in a subsequent election.*

Electoral systems of this type, called party-list proportional representation, are used in most of the world's democracies. More precisely, we described the closed-list variant that gives voters no sway over party lists; this system is used in several dozen states.⁶ Other countries allow the electorate to influence party lists.⁷ In many of these, voters exert little influence on party lists in practice,⁸ making the closed-list system a reasonable idealization.

We may re-interpret this as a firm planning for downsizing. The firm will have to fire an uncertain number K of its workers \mathcal{X} . The board (or a lower-level committee) plans ahead by drawing up an order in which employees will (if necessary) be let go.

3. ENVIRONMENT

There is a finite set \mathcal{X} of alternatives. A ranking of the alternatives is a binary relation R , where xRy reads “ x is ranked above y according to R ”. Formally, a ranking is a binary relation that

6. For example, Argentina, Germany, Japan, South Africa, and Turkey.

7. For example, Brazil, Indonesia, Iraq, the Netherlands, and Ukraine.

8. In many countries, such as Indonesia and the Netherlands, the party list is only altered if a candidate receives a large number of personal votes. Furthermore, voting for an individual candidate is typically optional, and many voters do not: in Sweden, only about a quarter do (Oscarsson, 2019). Of course there are exceptions, e.g. voting for individual candidates is mandatory and important in Finnish parliamentary elections.

is transitive, irreflexive (no alternative is ranked above itself), and total (each distinct pair is ranked).⁹ To capture the notion of a “possibly incomplete ranking”, we use the term *proto-ranking* for relations which are transitive and irreflexive, but not necessarily total.

There is a committee and its *chair*. The committee comprises an odd number I of voters, $i \in \{1, \dots, I\}$. The committee meets to determine a ranking, and the chair sets the agenda.

3.1. Interaction

Initially, no alternatives are ranked. In each period, the chair offers a vote on an as-yet unranked pair of alternatives. The committee votes on this pair, and whichever alternative garners more votes wins. (The chair does not have a vote, though that can be allowed for: see Section 4.) The winning alternative is ranked above the losing one. In addition, transitivity is imposed: if alternative x is ranked above y and y above z , then x is considered ranked above z . The chair continues to offer votes until all alternatives are ranked.

More explicitly, the ranking decisions made by the end of a period t are captured by a proto-ranking R_t , where xR_ty if x has been ranked above y , and $x \not R_t y$ otherwise. In period $t+1$, the chair offers a vote on an R_t -unranked pair x, y of distinct alternatives. If x is the winner, then the new proto-ranking R_{t+1} is the transitive closure of $R_t \cup \{(x, y)\}$.¹⁰

A *history* is a sequence of pairs offered for votes and a winner of each vote. A *strategy* of the chair specifies what pair to offer after each non-terminal history. We give formal versions of these definitions in Appendix B.2.

3.2. The majority will

The chair need not keep track of individual votes: all that matters for each pair of alternatives is which one wins. This essential information is captured by the binary relation W such that xWy iff a majority of voters vote for x over y when the pair x, y is offered. We call W the *majority will*.

We consider all possible majority wills W , meaning all *total and asymmetric* relations. Clearly only such relations need be contemplated (for each pair of alternatives, one must win and the other lose). Conversely, all such relations must be considered because each of them is the majority will of *some* committee, as we show in Appendix B.1.

The *outcome* of a strategy under a majority will W is the ranking that results. If a history is visited by a strategy σ under some majority will, then we say that it belongs to the *path* of σ . We give formal definitions of outcomes and paths in Appendix B.2.

3.3. Reachable rankings

For a majority will W , we call a ranking *W-reachable* iff it is the outcome under W of some strategy of the chair. This captures *ex post* feasibility: were the chair to have perfect knowledge of W , she could achieve all and only *W-reachable* rankings by offering some sequence of pairwise votes.

9. Definitions of some standard order-theoretic terms are collected in Appendix A.

10. Equivalently, $zR_{t+1}w$ iff either (1) zR_tw , or (2) both (a) $z=x$ or zR_tx and (b) $y=w$ or yR_tw . See Appendix C.1 (Observation 3, p. 19) for a proof of equivalence.

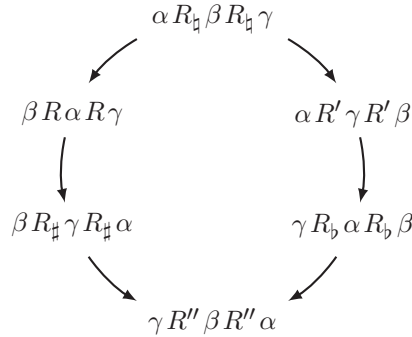
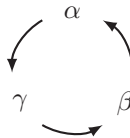


FIGURE 1

“More aligned with \succ than” for three alternatives $\mathcal{X} = \{\alpha, \beta, \gamma\}$, where $\alpha \succ \beta \succ \gamma$. In this (“Hasse”) diagram, there is a directed path from one ranking to another iff the former is more aligned with \succ .

Example 1. *There are three alternatives $\mathcal{X} = \{\alpha, \beta, \gamma\}$, and the majority will satisfies $\alpha W \gamma W \beta W \alpha$. Graphically:*



The ranking $\beta R \alpha R \gamma$ is W -reachable, achieved by offering β, α and α, γ . Similarly, the rankings $\alpha R' \gamma R' \beta$ and $\gamma R'' \beta R'' \alpha$ are W -reachable.

3.4. The chair's preferences

The chair has a strict preference over the alternatives \mathcal{X} , denoted \succ . We do not fully specify the chair's preferences over rankings. We assume only that she prefers a ranking over another whenever the former is *more aligned* with her own preference over alternatives.

Definition 1. *For rankings \succ, R and R' , we say that R is more aligned with \succ than R' iff for any pair $x, y \in \mathcal{X}$ of alternatives with $x \succ y$, if $x R' y$ then also $x R y$.*

In words, whenever a pair $x, y \in \mathcal{X}$ is ranked “right” by R' (viz. $x \succ y$ and $x R' y$), it is also ranked “right” by R (i.e. $x R y$).

Example 1 (continued). *Let the chair's preference be $\alpha \succ \beta \succ \gamma$. Figure 1 depicts how all rankings are ordered by “more aligned with \succ than”. Since R'' ranks no pairs “right”, every ranking is more aligned with \succ than R'' . The rankings R and R' are incomparable to each other, because R ranks β, γ “right” (while R' does not), whereas R' ranks α, β “right” (R does not).*

3.5. Regret-free strategies

Given a majority will W , we call a ranking W -unimprovable iff there is no other ranking that is both W -reachable and more aligned with \succ . Explicitly, R is W -unimprovable exactly if for any W -reachable ranking $R' \neq R$, some pair $x, y \in \mathcal{X}$ is ranked “right” by R and “wrong” by R' .

Example 1 (continued). *The ranking R is W -unimprovable, since neither of the other two W -reachable rankings is more aligned with \succ (R'' is less aligned, and R' is incomparable). R' is also W -unimprovable; R'' is not.*

The chair does not know the majority will. One would therefore expect her to face trade-offs: a strategy that does well against W may have a regrettable outcome under some W' . A *regret-free* strategy is one that has no such downside: its outcome under any majority will is unimprovable *ex post*.

Definition 2. *A strategy is regret-free iff for any majority will W , its outcome under W is W -unimprovable.*

Regret-freeness is a highly demanding optimality property. Our first result (Theorem 1 in Section 5) will be that, surprisingly, regret-free strategies exist.

4. DISCUSSION

Various aspects of the model deserve comment.

4.1. “More aligned than”

“More aligned than” captures *unambiguous* superiority of one ranking over another with respect to the chair’s preference \succ .¹¹ We illustrate by example:

Hiring (continued from Section 2). *Recall that the top-ranked candidate in X will be hired, where $X \subseteq \mathcal{X}$ is unknown. A ranking is more aligned with \succ exactly if it hires a weakly \succ -better candidate at every realization of the uncertainty X , as we show in Appendix B.4. Equivalently, a more aligned ranking is one that is preferred by every expected-utility preference consistent with \succ .*

This application demonstrates two general points. First, more aligned rankings are unambiguously better for the chair, given her preference \succ over the alternatives. Secondly, these comparisons are the *only* unambiguous ones: any further comparisons would have to be based on extraneous cardinal information, rather than on \succ alone.

Party lists (continued from Section 2). *Recall that the K highest-ranked candidates win parliamentary seats, where K is uncertain. If a ranking R is more aligned with \succ than R' , then the k th \succ -best candidate who wins a seat under R is weakly \succ -better than the k th \succ -best under R' . The converse is true provided there is some uncertainty about which candidates can take up seats.¹²*

The “more aligned than” criterion has its limitations. No extra credit is given for ranking *many* pairs of alternatives “right”, for example, nor for having pairs *near the top* ranked “right”.

11. It is an instance of *single-crossing dominance*, a general way of comparing rankings (or preferences); see Curello and Sinander (2022).

12. A party cannot alter its list after submitting it, but circumstances may render some of its candidates ineligible for parliamentary seats. For example, many countries disqualify candidates convicted of a serious crime, and all disqualify the deceased.

4.2. *Unimprovability as optimality*

Unimprovability is the strongest *ex post* optimality concept available without further assumptions about the chair's preference over rankings.¹³ It can therefore be thought of as optimality for a chair who is unable to make fine distinctions between rankings (which are complicated objects), or who is content to "satisfice".

Were we fully to specify the chair's preference over rankings, we could still break her problem under full information about W into two parts: first, reach the frontier (W -unimprovability), then choose among the frontier rankings.

Hiring (continued). *A ranking R is W -unimprovable exactly if any W -reachable ranking $R' \neq R$ hires a strictly \succ -worse candidate at some realization $X \subseteq \mathcal{X}$ of uncertainty.*

4.3. *The imposition of transitivity*

To understand why we assume that transitivity is imposed automatically, observe that the purpose of the interaction is to turn the will of the majority, which may contain (Condorcet) cycles, into a (by definition transitive) ranking. Some pairs will thus necessarily be ranked by fiat. We require that transitivity be imposed immediately after each vote because this is necessary and sufficient for *committee sovereignty*, the requirement that x be ranked above y if x beat y in a vote.¹⁴ Indeed, we contend that the protocol described in Section 3.1 is the only natural one, given that the interaction must end with a ranking: as shown in Supplementary Appendix H, any other protocol must violate either committee sovereignty or *democratic legitimacy*, the requirement that the chair offer enough votes to give the committee a fair say.

4.4. *Fixed voting*

By using the majority will, we implicitly assume fixed voting: xWy means that x garners a majority over y whenever x, y is offered, whatever the strategy of the chair. This is reasonable if voters are non-strategic: if they are unsophisticated, say, or vote "expressively" (to please their constituents, for example). Empirically, non-strategic voting appears to be the norm in many important institutions, such as the US Congress;¹⁵ whether the same is true of small groups of voters (such as committees) is unclear, however.

If voters were strategic, then they could potentially benefit from voting in a non-fixed way, by tailoring their behaviour to (their conjecture about) the chair's strategy. There are limits to their gains from strategizing, however: we show in Supplementary Appendix I that any insincerity by a voter risks producing a final ranking that is less aligned with her preference over the alternatives than the ranking that sincere voting would have delivered. By contrast, sincere voting carries no such risk: no strategizing can ever improve on its outcome in the (strong) "more aligned" sense.

13. It is a non-trivial optimality concept: we show in Supplementary Appendix G that not all W -reachable rankings are W -unimprovable exactly if W contains a (Condorcet) cycle, and that for a typical W , most W -reachable rankings are not W -unimprovable.

14. Sufficiency is obvious. For necessity, suppose the chair were allowed to offer x, z even though $xR_t yR_t z$; then committee sovereignty is violated whenever z beats x .

15. See Ladha (1994), Poole and Rosenthal (1997), and Wilkerson (1999), as well as the survey by Groseclose and Milyo (2010).

5. EXISTENCE OF REGRET-FREE STRATEGIES

In this section, we exhibit a regret-free strategy: *insertion sort*. We first (Section 5.1) introduce a notion of *efficiency* that implies regret-freeness, then (in Section 5.2) define insertion sort and show that it is efficient (Theorem 1).

5.1. Efficiency

A ranking R is called W -efficient iff every pair on which the chair and committee agree is ranked accordingly: if $x \succ y$ and $x W y$, then $x R y$.

Example 1 (continued). Recall the details from p. 6. Since $\alpha \succ \beta \succ \gamma$, W -efficiency requires precisely that α be ranked above γ . There are three such rankings: \succ itself, $\beta R \alpha R \gamma$ and $\alpha R' \gamma R' \beta$.

An *efficient strategy* is a strategy whose outcome under any majority will W is W -efficient. Efficiency matters because it is linked with regret-freeness:

Lemma 1. For any majority will W , every W -efficient ranking is W -unimprovable.

Corollary 1. Any efficient strategy is regret-free.

Proof of Lemma 1. Fix a majority will W , and let R be a W -efficient ranking. To establish that R is W -unimprovable, consider any W -reachable ranking $R' \neq R$; we must show that R' is not more aligned with \succ than R .

Since $R' \neq R$, there are alternatives $x, y \in \mathcal{X}$ such that $x R' y$ and $y R x$; furthermore, we may choose these to be R' -adjacent (i.e. $x R' z R' y$ holds for no $z \in \mathcal{X}$). Since R' is W -reachable, we must have $x W y$. (This follows from Observation 1 in Appendix B.3 (p. 17).) It must be that $y \succ x$, because otherwise the W -efficiency of R would require that $x R y$. Thus the pair x, y is ranked “right” by R ($y \succ x$ and $y R x$) and “wrong” by R' ($x R' y$), so that R' fails to be more aligned with \succ than R . \square

5.2. Insertion sort, a regret-free strategy

Insertion sort. Label the alternatives $\mathcal{X} \equiv \{1, \dots, n\}$ so that $1 \succ \dots \succ n$. We proceed in rounds indexed by $k \in \{n-1, \dots, 2, 1\}$, starting with $k = n-1$:

- By the k th round, those alternatives which the chair considers strictly worse than alternative k (i.e. the alternatives $\{k+1, \dots, n\}$) have already been totally ranked. Now “insert” alternative k into that ranking:
- First, pit alternative k against the highest-ranked worse alternative.
If k won, then $\{k, \dots, n\}$ are now totally ranked (with k on top).
- If k lost, pit k against the second-highest-ranked worse alternative.
If k won, then $\{k, \dots, n\}$ are now totally ranked (with k second).
- If k lost again, pit k against the third-highest-ranked worse alternative.
...
- Now $\{k, \dots, n\}$ are totally ranked. Decrease k by 1 and repeat.¹⁶

16. A detail: our definition of strategies requires them to specify behaviour after all histories, even those that cannot arise. (For example, a strategy that first offers x, y must still specify behaviour after histories where only y, z are ranked.) Thus formally, insertion sort defines not a strategy, but rather an equivalence class of strategies having the same path.

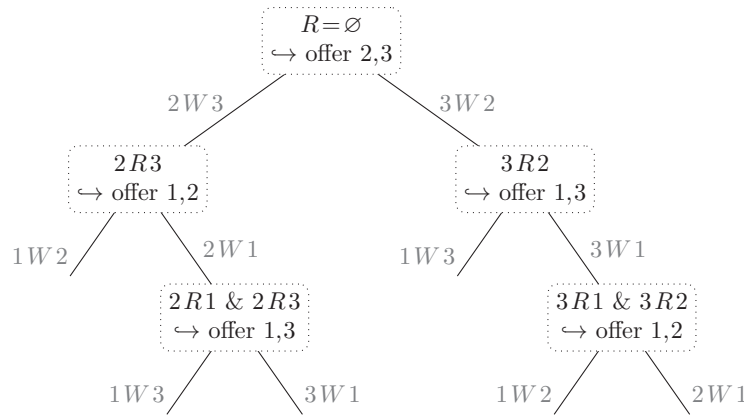


FIGURE 2

Insertion sort with three alternatives $\mathcal{X} = \{1, 2, 3\}$, where the chair's preference is $1 \succ 2 \succ 3$. R denotes the (evolving) proto-ranking. The path taken depends on the majority will W .

Insertion sort is illustrated in Figure 2 for the case of three alternatives.

Theorem 1. *Insertion sort is efficient, hence regret-free.*

Proof. Fix a majority will W , and let R be the outcome of insertion sort under W ; we must show that R is W -efficient. To that end, fix alternatives $x, y \in \mathcal{X}$ with $x \succ y$ and $x W y$; we shall prove that $x R y$.

Enumerate as $z_1 R \dots R z_K$ all of the alternatives that are strictly \succ -worse than x , and note that $z_k = y$ for some $k \leq K$. By definition of insertion sort, x will be pitted against z_1, z_2, \dots in turn until it wins a vote. If x loses against z_1, \dots, z_{k-1} , then it is next pitted against $z_k = y$, and wins since $x W y$ by hypothesis; thus $x R y$. If instead x wins against some z_ℓ with $\ell < k$, then $x R z_\ell R \dots R z_k = y$, so $x R y$ by transitivity of R . \square

Insertion sort is a standard sorting algorithm—see e.g. Knuth (1998, Section 5.2.1). In the sorting problem, there is a *transitive* and asymmetric relation W , and one seeks to reach a “sorted” ranking R , meaning that $x R y$ whenever $x W y$, by making (as few as possible) binary comparisons using W .

6. PROPERTIES OF REGRET-FREE STRATEGIES

Insertion sort is regret-free, but it is not the only such strategy. Which of its characteristics are essential for regret-freeness, and which are extraneous?

We give two answers in this section. Our first characterization, Theorem 2 (Section 6.1), is in terms of outcomes: the regret-free strategies are exactly the efficient ones, i.e. those reaching a W -efficient ranking under every majority will W . We show this characterization to be tight (Proposition 1).

The second characterization, Theorem 3 (Section 6.2), is in terms of behaviour: regret-freeness requires precisely that two intuitive errors be avoided. We show in addition that the advice offered by Theorem 3 can be operationalized myopically: avoiding errors ensures that a non-error pair will always be available to be offered next (Proposition 2).

What distinguishes one regret-free strategy from another is prioritization: *which* pairs of alternatives are ranked “right” when not all of them can be. We prove that *lexicographic* prioritization characterizes insertion sort (Theorem 4, Section 6.3): the chair’s favourite alternative is ranked as highly as possible, her second-favourite alternative is ranked as highly as possible subject to that, and so on. One consequence (Section 6.4) is that insertion sort is outcome-equivalent to recursively applying the much-studied *amendment procedure*.

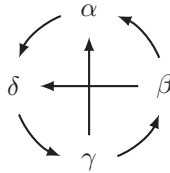
6.1. Characterization of regret-free strategies: outcomes

Efficiency is not only sufficient for regret-freeness, but also necessary:

Theorem 2. *A strategy is regret-free iff it is efficient.*

The proof, in Appendix C, establishes Theorem 2 jointly with Theorem 3 below (Section 6.2). The “if” direction was already established in Corollary 1 (p. 9). The following gives a feel for the “only if” direction:

Example 2. Consider alternatives $\mathcal{X} = \{\alpha, \beta, \gamma, \delta\}$ with $\alpha \succ \beta \succ \gamma \succ \delta$ and the following majority will W :¹⁷



There is exactly one W -reachable ranking that ranks α above β , namely $\alpha R \delta R \gamma R \beta$.¹⁸ Since no other W -reachable ranking ranks α above β , R is W -unimprovable. But R fails to be W -efficient, for it ranks δ above β despite the fact that $\beta W \delta$. (Thus the converse of Lemma 1 (p. 9) is false.)

Let σ be a strategy that (1) first offers β, γ , (2) then, in case γ won, offers γ, δ , and (3) next, in case γ and then δ won, offers α, δ . This strategy has outcome R under W , so fails to be efficient. To see that it is not regret-free, consider a majority will W' that differs from W only in that $\delta W' \alpha$. The outcome of σ under W' is $\delta R' \gamma R' \beta R' \alpha$.¹⁹ This ranking fails to be W' -unimprovable since $\gamma R'' \beta R'' \delta R'' \alpha$ is W' -reachable and is more aligned with \succ by inspection. Thus σ fails to be regret-free.

The broader insight underlying the “only if” part of Theorem 2 is that reaching a non- W -efficient outcome necessarily involves a sacrifice. (In the example, σ forgoes the opportunity to rank β above δ .) For some majority wills, such as W , the sacrifice pays off. (It allows α to be ranked above β , something that no W -reachable W -efficient ranking achieves.) But for other majority wills, such as W' , no reward materializes, yielding a non- W' -unimprovable outcome.

The characterization in Theorem 2 is tight, in the following sense:

17. Symbolically, $\alpha W \delta W \gamma W \beta W \alpha$, $\gamma W \alpha$ and $\beta W \delta$.

18. This can be verified directly. Alternatively, since there is exactly one W -path from α to β (namely $(\alpha, \delta, \gamma, \beta)$), it follows by Observation 1 in Appendix B.3 (p. 17).

19. First $\delta R' \gamma R' \beta$ and $\delta R' \alpha$ are determined. Then α, β and α, γ are offered (in some order that does not matter), and α loses in both votes.

Proposition 1. *For any majority will W and any W -reachable W -efficient ranking R , some regret-free strategy has outcome R under W .*

Thus no statement sharper than Theorem 2 can be made about the outcomes under W of regret-free strategies: every W -reachable W -efficient ranking is admissible. We give the proof of Proposition 1 in Supplementary Appendix E. A non-trivial argument is required because a given W -reachable W -efficient ranking can be reached in many ways, not all of which form part of a regret-free strategy.²⁰

6.2. Characterization of regret-free strategies: behaviour

To understand the behavioural content of regret-freeness, we begin with a simple intuition for why insertion sort is regret-free in the case of three alternatives (refer to Figure 2 on p. 10). Suppose first that the initial vote on 2,3 went well ($2W3$). Offering 1,2 next is a good idea because it affords an opportunity of a favourable imposition of transitivity: if the chair gets lucky ($1W2$), then she obtains $1R3$ “for free” from $1R2$ and $2R3$. (Even though it may be that $3W1$.) Offering 1,3 would miss this opportunity.

Suppose instead that the initial vote on 2,3 went badly ($3W2$). Offering 1,2 next would risk an unfavourable imposition of transitivity: were the vote to go against her ($2W1$), then 3 would be ranked above 1 since $3R2$ and $2R1$. (Even though it may be that $1W3$.)

To summarize, an intuition for why insertion sort is regret-free is that it never (1) misses an opportunity for a favourable imposition of transitivity nor (2) risks an unfavourable imposition of transitivity. Our second characterization formalizes this intuition.

Definition 3. *Let R be a non-total proto-ranking, and let $x \succ y$ be unranked alternatives (i.e. $x, y \in \mathcal{X}$ such that $x \not R y \not R x$).*

1. *Offering x, y for a vote misses an opportunity (at R) iff there is an alternative $z \in \mathcal{X}$ such that $x \succ z \succ y$ and $y \not R z \not R x$.*
2. *Offering x, y for a vote takes a risk (at R) iff there is an alternative $z \in \mathcal{X}$ such that either*

- (a) $z \succ y$, $x R z$ and $y \not R z$, or
- (b) $x \succ z$, $z R y$ and $z \not R x$.

The “missed opportunity” in (1) is that $x R y$ (the hoped-for outcome when offering x, y for a vote) could potentially have been obtained “for free” by offering votes on x, z and z, y , via a “favourable imposition of transitivity”. The “risk” in (2)(a) is that if the vote on x, y were to go badly (so that $y R x$), then $y R z$ would follow—an “unfavourable imposition of transitivity”. Similarly for (2)(b).

We say that a strategy *never misses an opportunity (never takes a risk)* iff it does not miss an opportunity (take a risk) on the path.

Theorem 3. *A strategy is regret-free iff it never misses an opportunity or takes a risk.*

20. To see why, return to Example 1 (pp. 6 and 9). The W -efficient ranking $\alpha R' \gamma R' \beta$ may be reached by offering α, γ and then γ, β . But a strategy that does this cannot be regret-free, because it has outcome R' also under the majority will $\alpha W' \beta W' \gamma W' \alpha$, and R' is not a W' -unimprovable ranking (since the more aligned ranking \succ is W' -reachable).

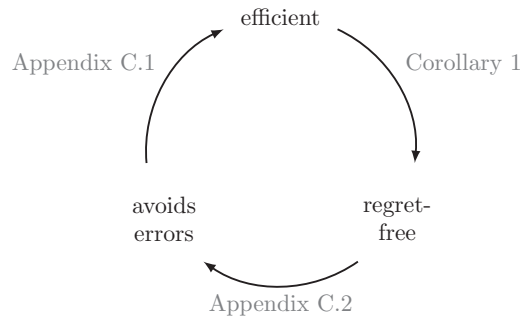


FIGURE 3

Proof of Theorems 2 and 3.

We give a joint proof of Theorems 2 and 3 in Appendix C. The argument is illustrated in Figure 3.

The “only if” part of Theorem 3 asserts that opportunity-missing and risk-taking are errors, in the sense that committing one of them will lead to a non- W -unimprovable ranking under some W . This is intuitive but requires some work to show. To appreciate why, suppose that σ offers a pair $x \succ y$ under W , and that doing so either misses an opportunity or takes a risk. It is then easy to find another majority will W' such that the outcome R of σ under W' ranks y above x , whereas some other W' -reachable ranking R' ranks $xR'y$. But that is not enough: if R' is to be more aligned with \succ than R , then *every* pair z, w ranked “right” by R ($z \succ w$ and zRw) must also be ranked “right” by R' ($zR'w$). We construct W' and R' with these properties in Appendix C.2.

The “if” part of Theorem 3 asserts that these are the *only* errors. Thus to achieve regret-freeness, it suffices to avoid missing opportunities and taking risks, separately after each history. The proof in Appendix C.1 begins with an arbitrary non-efficient strategy σ , meaning one whose outcome R under some majority will W ranks some pair $x, y \in \mathcal{X}$ as yRx even though $x \succ y$ and xWy . The pair x, y cannot have been voted on (else x would have prevailed), so must have been ranked by an imposition of transitivity. We show that avoiding the two errors suffices to preclude such “unfavourable” impositions of transitivity; thus σ must have committed the one or the other.

Theorem 3 tells us that non-error pairs are attractive, but it does not promise that they exist. In particular, there could conceivably be histories along which the chair has committed no errors but is now forced to do so because no unranked pairs remain that can be offered without missing an opportunity or taking a risk. The following rules out this scenario:

Proposition 2. *After any history along which the chair has not missed an opportunity or taken a risk, if some pairs remain unranked, then there is a pair which can be offered without missing an opportunity or taking a risk.*

The proof is in Supplementary Appendix E. Proposition 2 shows that the characterization in Theorem 3 of regret-free behaviour is tight: for any W and sequence of pairs that is error-free under W , some regret-free strategy offers this sequence under W .²¹

Theorem 3, augmented by Proposition 2, allows us to give the following simple “myopic” advice to the chair. After each history, inspect the current proto-ranking to identify an unranked pair of alternatives that would not miss an opportunity or take a risk; that such a pair exists is

21. In particular, a strategy can be constructed which offers this sequence under W and commits no errors under other majority wills. Such a strategy is regret-free by Theorem 3.

guaranteed by Proposition 2. Offer any such pair for a vote. By Theorem 3, the outcome will be W -unimprovable whatever the majority will W .

6.3. Priorities, and a characterization of insertion sort

We now show that regret-free strategies differ (only) in how they *prioritize*, and that insertion sort is characterized by its *lexicographic* prioritization.

Given a majority will W , call a pair $x \succ y$ an *agreement pair* iff xWy , and a *disagreement pair* otherwise. All regret-free strategies rank the agreement pairs “right” (i.e. xRy) by Theorem 2. A disagreement pair is ranked “wrong” (as yRx) if voted on, and ranked “right” if instead ranked by transitivity, since regret-free strategies permit only *favourable* impositions of transitivity by Theorem 3.

What distinguishes regret-free strategies is thus precisely which disagreement pairs are voted on and which are held back. Held-back pairs have more opportunities to be ranked “right” by transitivity.

Example 1 (continued). Recall the details from p. 6. The disagreement pairs are α, β and β, γ . Insertion sort prioritizes the former: its outcome is $\alpha R' \gamma R' \beta$. Other regret-free strategies prioritize β, γ instead, having outcome $\beta R \alpha R \gamma$.²²

To understand the priorities of insertion sort, label the alternatives $\mathcal{X} \equiv \{1, \dots, n\}$ so that $1 \succ \dots \succ n$. Insertion sort leaves alternative 1 for last, by totally ranking the alternatives $\{2, \dots, n\}$ before offering any votes involving 1. This intuitively maximizes opportunities for favourable impositions of transitivity involving 1, and so optimizes the placement of this alternative.

Apart from alternative 1, alternative 2 is left for last by insertion sort, suggesting that this alternative is placed optimally, subject to optimal placement of alternative 1. This suggests that insertion sort prioritizes lexicographically, by placing alternative 1 as well as possible, and subject to that placing alternative 2 as well as possible, and so on. We shall formalize this intuition.

A simple cardinal measure of how well a strategy σ places a given alternative $x \in \mathcal{X}$ is *how many* \succ -worse alternatives y are ranked below x , i.e.

$$N_x^\sigma(W) := |\{y \in \mathcal{X} : x \succ y \text{ and } xR^\sigma(W)y\}|,$$

where $R^\sigma(W)$ denotes the outcome of σ under majority will W . We call a strategy σ *better for* x than another strategy σ' iff N_x^σ first-order stochastically dominates $N_x^{\sigma'}$, i.e.

$$|\{W : N_x^\sigma(W) \geq k\}| \geq |\{W : N_x^{\sigma'}(W) \geq k\}| \quad \text{for each } k \in \{1, \dots, n-1\}.$$

If a strategy σ belongs to a set Σ of strategies and is better for x than any other $\sigma' \in \Sigma$, then we say that σ is *best for alternative* x *among* Σ .

Call a strategy *lexicographic* iff among all strategies, it is best for the chair's favourite alternative; among such strategies, it is best for her second-favourite; among *such* strategies, it is best for her third-favourite; etc. Say that two strategies are *outcome-equivalent* iff they have the same outcome under W , for every majority will W .

22. Some regret-free strategy has outcome R under W by Proposition 1 (p. 12). An example is “reverse insertion sort”: in its k th round, the chair's top $k-1$ alternatives have already been ranked, and now her k th-favourite is pitted in turn against the lowest-ranked of these \succ -better alternatives, against the second-lowest ranked, and so on until it loses a vote.

Theorem 4. *A strategy is outcome-equivalent to insertion sort iff it is lexicographic.*

The proof is in Appendix D.

6.4. Insertion sort and the amendment procedure

The following procedure figures prominently in the literature (see Section 1.1):

Amendment procedure. *Label the alternatives $\mathcal{X} \equiv \{1, \dots, n\}$ so that $1 \succ \dots \succ n$. First, pit $n-1$ against n . Then pit $n-2$ against the winner. Then pit $n-3$ against the previous round's winner. And so on. Call the winner of the final round the final winner.*

The amendment procedure is designed to choose a single alternative. In particular, the final winner is ranked top, since any other alternative either lost a vote to it, or lost a vote to an alternative that lost to it, etc. The natural way to obtain a ranking is to apply amendment recursively:

Recursive amendment. *Label the alternatives $\mathcal{X} \equiv \{1, \dots, n\}$ so that $1 \succ \dots \succ n$. First, run the amendment procedure on \mathcal{X} , and write y_1 for the final winner. Next, run the amendment procedure on $\mathcal{X} \setminus \{y_1\}$, writing y_2 for the final winner.²³ Then run the amendment procedure on $\mathcal{X} \setminus \{y_1, y_2\}$ to obtain a final winner y_3 . And so on. The resulting ranking is $y_1 R \dots R y_{n-1} R y_n$, where y_n denotes the unique element of $\mathcal{X} \setminus \{y_1, \dots, y_{n-1}\}$.²⁴*

Recursive amendment is distinct from insertion sort.²⁵ Nonetheless:

Proposition 3. *Recursive amendment is outcome-equivalent to insertion sort.*

This follows from one of the lemmata in our proof of Theorem 4:

Proof. Recall from Appendix D.1 (p. 23) the set Σ_{n-2} of strategies. Clearly recursive amendment belongs to Σ_{n-2} . Thus by Lemma 5 (p. 24), recursive amendment is outcome-equivalent to insertion sort. \square

In the sorting context, recursive amendment is called *selection sort*—see e.g. Knuth (1998, Section 5.2.3).

7. CONCLUDING REMARKS

We close by considering some alternative interpretations of our model and of our results, as well as a few extensions.

7.1. Broader interpretations

The model admits interpretations in which W arises from something other than majority voting by a committee. It could be the (expressed) preference of a single individual, for example: in this

23. If the amendment procedure demands a vote on a pair x, y that is already ranked as (wlog) $x R y$, treat this as offering x, y and obtaining outcome $x W y$.

24. Like insertion sort, recursive amendment describes only a path, so formally defines an equivalence class of strategies; refer to footnote 16 on p. 9.

25. For example, if $\mathcal{X} = \{1, 2, 3, 4\}$ with $1 \succ 2 \succ 3 \succ 4$, $1 W 4 W 3$ and $4 W 2 W 3$, then insertion sort offers 3, 4, then 2, 4, then 2, 3, then 1, 4, whereas recursive amendment offers 3, 4, then 2, 4, then 1, 4, then 2, 3. The two strategies *do* coincide when there are three alternatives.

case, cycles reflect inconsistencies in his judgment, and a crafty advisor (“the chair”) exploits these by asking him to make pairwise comparisons in a well-chosen order.

In practice, the chair may be more constrained in her choice of agenda-setting strategy. In that case, our analysis provides an upper bound on how well she can do, and gives a qualitative indication of what incremental adjustments to agenda-setting might most benefit her.

Finally, there need not literally be a committee chair: more generally, our results speak to how institutional or procedural design can influence collective decisions, in the direction of a(ny) given priority \succ .

7.2. Extensions

Varying the committee’s rules merely alters W , and so our analysis continues to apply. For example, we can accommodate a super-majority voting rule, an even electorate $I \in \mathbb{N}$, and abstentions. To determine W , we need only specify which alternative wins in case of an indecisive vote, e.g. by assuming that there is a status quo ranking that prevails in such cases.

A more substantial variation is to permit the chair (sometimes) to decide how two alternatives are to be ranked following an indecisive vote on them. This happens if the chair has a vote, for example. We extend all of our results to allow for this in Supplementary Appendix J.

Our definition of a strategy rules out conditioning on who voted how in the past, since a history records only which alternative in each pair wins (rather than the full vote tally).²⁶ This is merely to avoid uninteresting complications: we show in Supplementary Appendix K that our results hold for “extended strategies” that can condition on past votes cast.

Appendices

A. STANDARD DEFINITIONS

This appendix collects the definitions of order-theoretic concepts used in this article. Let \mathcal{A} be a non-empty set, and Q a binary relation on it. Recall that Q is formally a subset of $\mathcal{A} \times \mathcal{A}$, and that “ aQb ” is shorthand for “ $(a, b) \in Q$ ”. For $a, b \in \mathcal{A}$, we write $a \not Q b$ iff it is not the case that aQb . For $a, b \in \mathcal{A}$ such that aQb , we denote by $[a, b]_Q$ the *Q-order interval*

$$[a, b]_Q := \{a, b\} \cup \{c \in \mathcal{A} : aQcQb\}.$$

Distinct $a, b \in \mathcal{A}$ with aQb are *Q-adjacent* iff $[a, b]_Q = \{a, b\}$.

Q is *reflexive* (irreflexive) iff aQa ($a \not Q a$) for every $a \in \mathcal{A}$, *total* iff aQb or bQa for any distinct $a, b \in \mathcal{A}$, *complete* iff it is reflexive and total, *asymmetric* iff aQb implies $b \not Q a$ for $a, b \in \mathcal{A}$, and *transitive* iff $aQbQc$ implies aQc for $a, b, c \in \mathcal{A}$.

The *transitive closure* of Q is the smallest (in the sense of set inclusion) transitive relation that contains Q .²⁷ The *strict part* of Q is the binary relation $\text{str}Q$ such that $a \text{str}Q b$ iff aQb and $b \not Q a$. For two binary relations Q and Q' on \mathcal{A} , Q' is an *extension* of Q iff both $Q \subseteq Q'$ and $\text{str}Q \subseteq \text{str}Q'$.

B. ADDITIONAL MATERIAL FOR SECTION 3

This appendix complements the exposition of the environment in Section 3. We show that all and only total and asymmetric relations can be majority wills (Section B.1), formally define histories, strategies, outcomes and paths (Section B.2), and provide characterizations of W -reachability (Section B.3) and of “more aligned with than” (Section B.4).

26. Of course, this is only a restriction if the chair can observe individual votes.

27. Every relation possesses a transitive closure because the maximal relation $\mathcal{A} \times \mathcal{A}$ is transitive and the intersection of transitive relations is transitive.

B.1. Which binary relations are majority wills? (Section 3.2)

In this appendix, we prove that all and only total and asymmetric relations are legitimate majority wills. A *voting behaviour* V_i specifies for each pair $x, y \in \mathcal{X}$ whether voter i will vote for x ($x V_i y$) or for y ($y V_i x$). Formally:

Definition 4. A voting behaviour is a total and asymmetric relation. A voting profile is a collection $(V_i)_{i=1}^I$ of voting behaviours.

The majority will of a voting profile $(V_i)_{i=1}^I$ is the relation W such that $x W y$ iff $x V_i y$ for a majority of voters $i \in \{1, \dots, I\}$. The following shows that all (and only) total and asymmetric relations are the majority will of some voting profile $(V_i)_{i=1}^I$, even if we insist that each voter's behaviour V_i be transitive (meaning that it can be rationalized as sincere):

Fact 1. For a binary relation W on \mathcal{X} , the following are equivalent:

1. W is total and asymmetric.
2. For every $I \in \mathbb{N}$ odd, W is the majority will of some profile $(V_i)_{i=1}^I$ of voting behaviours.
3. For some $I \in \mathbb{N}$ odd, W is the majority will of some profile $(V_i)_{i=1}^I$ of transitive voting behaviours.

Proof. It is immediate that 2 and 3 (separately) imply 1. To see that 1 implies 2, simply observe that a total and asymmetric relation W is the majority will of the voting profile $(V_i)_{i=1}^I = (W)_{i=1}^I$ for any $I \in \mathbb{N}$. The fact that 1 implies 3 follows from McGarvey's (1953) theorem. \parallel

B.2. Formal definitions (Sections 3.1 and 3.2)

In this appendix, we formally define histories, strategies, outcomes, and paths. A history is a sequence $((x_t, y_t))_{t=1}^T$, where $\{x_t, y_t\}$ is the pair offered in period t , and x_t is the winner (i.e. $x_t W y_t$):

Definition 5. There is exactly one history of length 0 (the “empty history”). A history of length $T \in \mathbb{N}$ is a sequence $((x_t, y_t))_{t=1}^T$ of ordered pairs of alternatives satisfying $x_t \neq y_t$ and $x_t R_{t-1} y_t R_{t-1} x_t$ for each $t \in \{1, \dots, T\}$, where $R_0 := \emptyset$ and $R_t := \text{tr}(R_{t-1} \cup \{(x_t, y_t)\})$ for each $t \in \{1, \dots, T\}$. (Here, “tr” denotes the transitive closure.) The history is terminal iff R_T is total.

Let \mathcal{H}_T be the set of all non-terminal histories of length $T \geq 0$ and write $\mathcal{H} := \bigcup_{T=0}^{\infty} \mathcal{H}_T$ for all non-terminal histories. A strategy offers, after each non-terminal history, a pair of alternatives that are unranked at that history:

Definition 6. A strategy of the chair is a map $\sigma: \mathcal{H} \rightarrow 2^{\mathcal{X}}$ such that for each non-terminal history $h \in \mathcal{H}$, we have $\sigma(h) = \{x, y\}$ for some alternatives $x, y \in \mathcal{X}$ such that $(h, (x, y))$ and $(h, (y, x))$ are histories.

A strategy σ and a majority will W generate a terminal history $((x_t, y_t))_{t=1}^T$ as follows: for each $t \in \{1, \dots, T\}$, (x_t, y_t) is given by

$$\{x_t, y_t\} := \sigma\left(\left((x_s, y_s)\right)_{s=1}^{t-1}\right) \quad \text{and} \quad x_t W y_t.$$

This history is associated with a sequence of proto-rankings $(R_t)_{t=0}^T$, as outlined in Definition 5 above.²⁸

Definition 7. The outcome of a strategy σ under a majority will W is the ranking R_T associated with the terminal history they generate.

A strategy and a majority will also generate a sequence of non-terminal histories—namely, all truncations of their generated terminal history. If a non-terminal history is generated by σ and some majority will, we say that it belongs to the *path* of σ .

B.3. A characterization of W -reachability (Section 3.3)

This appendix contains a characterization of reachability used in our proofs.

28. Namely, $R_0 = \emptyset$ and $R_t = \text{tr}(R_{t-1} \cup \{(x_t, y_t)\}) = \text{tr}\left(\bigcup_{s=1}^t \{(x_s, y_s)\}\right)$ for $t \geq 1$.

Observation 1. Let W be a majority will, and R a ranking. The following are equivalent:

1. R is W -reachable.
2. For any R -adjacent $x, y \in \mathcal{X}$ with xRy , we have xWy .

Condition 2 admits a graph-theoretic interpretation. Think of W as a directed graph with vertices \mathcal{X} and a directed edge from x to y iff xWy (as in Example 1 on p. 6), and think of a ranking R as a sequence of alternatives: the highest-ranked, the second-highest-ranked, and so on.²⁹ Condition 2 requires precisely that R be a directed path in W .

Proof. 2 implies 1: Let R satisfy 2. Then R is the outcome under W of any strategy that offers a vote on each R -adjacent pair of alternatives.

1 implies 2: Let R be W -reachable, and let $x, y \in \mathcal{X}$ be R -adjacent with xRy . By W -reachability, there is a strategy σ whose outcome under W is R . Along its induced history, it is determined that xRy . Since x, y are R -adjacent, this cannot be via an imposition of transitivity. So it must occur in a vote on $\{x, y\}$, in which x wins—thus xWy . \parallel

B.4. A characterization of “more aligned with than” (Section 3.4)

In this appendix, we provide a characterization of “more aligned with than”, and use it to prove the claims made in Section 3.4 (p. 6) about the applications.

Observation 2. For rankings \succ, R and R' , the following are equivalent:³⁰

1. R is more aligned with \succ than R' .
2. For every non-empty set $X \subseteq \mathcal{X}$, the R -highest alternative in X is identical to or \succ -better than the R' -highest alternative in X .

Proof. 1 implies 2: We prove the contra-positive. Suppose that R, R' do not satisfy 2, so that there is a non-empty $X \subseteq \mathcal{X}$ whose R -highest element x is (strictly) \succ -worse than its R' -highest element x' . Then R' ranks x, x' “right” ($x' \succ x$ and $x' R' x$) whereas R ranks them “wrong” (xRx'), so R is not more aligned with \succ than R' .

2 implies 1: We prove the contra-positive. Suppose that R is not more aligned with \succ than R' , so that there are alternatives $x, x' \in \mathcal{X}$ with $x' \succ x$, $x' R' x$ and xRx' . Then the R -highest alternative in $X := \{x, x'\}$ is (strictly) \succ -worse than the R' -highest. \parallel

Hiring (continued). We claimed that a more aligned ranking is precisely one that hires a weakly \succ -better candidate at every realization of uncertainty. This follows immediately from Observation 2.

Party lists (continued). Enrich the model so that only a random subset $X \subseteq \mathcal{X}$ of candidates is available.³¹ We claim that R is more aligned with \succ than R' iff for every realization (K, X) and every $k \leq K$, the k th \succ -best candidate hired by R is weakly \succ -better than the k th \succ -best hired by R' .

To prove the “only if” part, let R be more aligned with \succ than R' , and fix an $X \subseteq \mathcal{X}$ and a K . Assume without loss of generality that $K \leq |X|$. Label the candidates $\{x_1, \dots, x_K\}$ hired by R under X so that $x_1 R \dots R x_K$, and similarly write $x'_1 R' \dots R' x'_K$ for those hired by R' . We must show that $x_k \geq x'_k$ for every $k \in \{1, \dots, K\}$. We proceed by strong induction on k . The base case $k=1$ is immediate from Observation 2.

For the induction step, suppose for some $k \in \{2, \dots, K\}$ that $x_\ell \geq x'_\ell$ for every $\ell < k$. Define $Y := X \setminus \{x_1, \dots, x_{k-1}, x'_1, \dots, x'_{k-1}\}$. We have $x_\ell \geq x'_\ell > x'_k$ for every $\ell < k$ by the induction hypothesis. It follows that $x'_k \in Y$, and hence that x'_k is the R' -highest alternative in Y . If $x_k \in Y$, then x_k is the R -highest alternative in Y , whence $x_k \geq x'_k$ by Observation 2. If instead $x_k \notin Y$, then $x_k = x'_\ell > x'_k$ for some $\ell < k$.

For the “if” part, we prove the contra-positive. Suppose that R is not more aligned with \succ than R' . Then by Observation 2, there is a subset $X' \subseteq \mathcal{X}$ such that R hires a strictly \succ -worse candidate than R' at the realization $(K, X) = (1, X')$ of uncertainty.

29. Formally, identify R with the sequence $(x_k)_{k=1}^{|\mathcal{X}|}$ such that $x_1 R x_2 R \dots R x_{|\mathcal{X}|}$.

30. This is an instance of the Milgrom–Shannon (1994) comparative statics theorem: viewing (\mathcal{X}, \succ) as an ordered set of actions and R, R' as (strict) preferences, 1 says that R single-crossing dominates R' , and 2 says that R chooses higher actions than R' does.

31. The grand set $X = \mathcal{X}$ can occur with high probability, if desired.

C. PROOF OF THEOREMS 2 AND 3 (SECTION 6, PP. 11 AND 12)

We prove Theorems 2 and 3 jointly, in the manner depicted in Figure 3 (p. 13). We already showed that efficiency implies regret-freeness (Corollary 1, p. 9). We shall establish the other two parts in Sections C.1 and C.2.

C.1. *Proof that error-avoiding strategies are efficient*

The proof relies on two intermediate results, Lemma 2 and Corollary 2 below. We first require an abstract fact about the transitive closure operation:

Observation 3. Consider a proto-ranking R and unranked alternatives $x, y \in \mathcal{X}$ (i.e. $x \not R y \not R x$). Let R' be the transitive closure of $R \cup \{(z, w)\}$, and suppose that $y R' x$. Then (a) either $y R z$ or $y = z$, and (b) either $w R x$ or $w = x$.

Proof. Since R' is the transitive closure of $R \cup \{(z, w)\}$ and $y R' x$, there must be a sequence $(z_k)_{k=1}^K$ of alternatives with $z_1 = y$, $z_K = x$ and

$$(z_k, z_{k+1}) \in R \cup \{(z, w)\} \quad \text{for every } k < K.$$

Since $y R x$ and R is transitive, we must have $(z_k, z_{k+1}) = (z, w)$ for some $k < K$. The result follows since R is transitive. \square

Definition 8. Let R be a proto-ranking. An ordered pair of alternatives $(x, y) \in \mathcal{X}$ is a missed opportunity in R iff $y R x$ and there is an alternative $z \in \mathcal{X}$ such that $x \succ z \succ y$ and $y R z \not R x$.

Lemma 2. Consider a proto-ranking R that contains no missed opportunities. Let $x \succ y$ be alternatives with $y R x$. Suppose that offering $\{z, w\}$ (where $z R w \not R z$) does not take a risk at R , and that doing this leads to a proto-ranking R' such that $y R' x$. Then $\{z, w\} = \{x, y\}$.

Proof. Let R, x, y, z, w and R' satisfy the hypothesis of the lemma, and assume wlog that $z \succ w$. We must show that $z = x$ and $w = y$.

Claim. $w R' z$.

Proof of the claim. Suppose toward a contradiction that $w \not R' z$. We will show that R contains a missed opportunity.

Since R' is induced from R by offering $\{z, w\}$, and $w R' z$, it must be that R' is the transitive closure of $R \cup \{(z, w)\}$. Since $y R x$ and $y R' x$, it follows by Observation 3 that (a) either $y R z$ or $y = z$, and (b) either $w R x$ or $w = x$. Now consider two cases.

Case 1: $z \succ x$ or $z = x$. We will show that $z \succ x \succ y$, $y R z$, and $y R x \not R z$, so that (z, y) is a missed opportunity in R . Both $x \succ y$ and $y R x$ hold by hypothesis. For $x R z$, suppose to the contrary that $x \not R z$; then since $w R x$ or $w = x$ by property (b), we have $w R z$ by transitivity of R , a contradiction.

To obtain $y R z$, observe that $z \succ y$ since by hypothesis $z \succ x$ or $z = x$, and we know that $x \succ y$ and that \succ is transitive. Thus, $z \neq y$, whence $y R z$ follows by property (a). To see that $z \succ x$, simply note that $z = x$ is impossible because $y R z$ and $y \not R x$.

Case 2: $x \succ z$. We will show that $x \succ z \succ w$, $w R x$, and $w R z \not R x$, so that (x, w) is a missed opportunity in R . Both $x \succ z \succ w$ and $w R z$ hold by hypothesis. For $z R x$, suppose to the contrary that $z \not R x$; then since $y R z$ or $y = z$ by property (a), we have $y R x$ by transitivity of R , a contradiction.

To obtain $w R x$, observe that $x \succ w$ since $x \succ z \succ w$ and \succ is transitive. Thus $w \neq x$, whence $w R x$ follows by property (b). \square

In light of the claim, R' must be the transitive closure of $R \cup \{(w, z)\}$. Since $y R x$ and $y R' x$, applying Observation 3 yields that (a) either $y R w$ or $y = w$, and (b) either $z R x$ or $z = x$.

We claim that

$$z \neq x \succ w \tag{C.1}$$

is impossible. Suppose toward a contradiction that (C.1) holds; we will show that offering $\{z, w\}$ takes a risk at R , i.e. that $x \succ w$, $z R x$ and $w \not R x$. We have $x \succ w$ by (C.1), and $z R x$ by (C.1) and property (a). To see that $w \not R x$, suppose to the contrary that $w R x$; then since $y R w$ or $y = w$ by property (a), it follows by transitivity of R that $y R x$, a contradiction.

Now suppose toward a contradiction that $\{z, w\} \neq \{x, y\}$. We claim that

$$z \succ y \neq w. \tag{C.2}$$

If $z=x$, then $z \succ y$ is immediate, and $y \neq w$ follows since $\{z, w\} \neq \{x, y\}$ by hypothesis. Suppose instead that $z \neq x$. Since (C.1) cannot hold, it must be that either $w \succ x$ or $w = x$. Since $x \succ y$, it follows by transitivity of \succ that $w \succ y$, so that $y \neq w$. Furthermore, since $z \succ w$, transitivity of \succ yields $z \succ y$. So (C.2) holds.

It remains to derive a contradiction from $\{z, w\} \neq \{x, y\}$, using the fact that (C.2) must hold. We shall show that $z \succ y$, yRw and $y \not R z$, so that offering $\{z, w\}$ takes a risk at R . We obtain $z \succ y$ from (C.2), and yRw from (C.2) and property (a). And it must be that $y \not R z$ because yRz together with property (b) and transitivity of R imply the contradiction yRx . \parallel

Corollary 2. *Suppose that R contains no missed opportunities, and that offering $\{z, w\}$ (where $z \not R wRz$) does not miss an opportunity or take a risk at R . Then, the proto-ranking R' induced by offering $\{z, w\}$ contains no missed opportunities.*

Proof. Let R , z , w , and R' be as in the hypothesis of the lemma and suppose toward a contradiction that there is a missed opportunity (x, y) in R' .

We claim that yRx and $yR'x$, so that Lemma 2 is applicable. It must be that yRx , for otherwise (x, y) would be a missed opportunity in R . That $yR'x$ is immediate from the fact that (x, y) is a missed opportunity in R' .

It follows by Lemma 2 that $\{z, w\} = \{x, y\}$. But since (x, y) is a missed opportunity in R' , there is an alternative $z' \in \mathcal{X}$ such that $x \succ z' \succ y$ and $y \not R z'$, and thus $y \not R z'$ since $R \subseteq R'$. Thus offering $\{z, w\} = \{x, y\}$ misses an opportunity at R —a contradiction. \parallel

Armed with Lemma 2 and Corollary 2, we are ready to prove that error-avoiding strategies are efficient.

Proposition. *A strategy that never misses an opportunity or takes a risk is efficient.*

Proof. Take a strategy σ that is not efficient, and suppose that it never misses an opportunity or takes a risk; we shall derive a contradiction. Since σ is not efficient, there exists a majority will W such that the outcome R of σ under W fails to be W -efficient, which is to say that $x \succ y$, xWy and yRx for some alternatives $x, y \in \mathcal{X}$.

Write $\emptyset = R_0 \subseteq R_1 \subseteq \dots \subseteq R_{T'} = R$ for the sequence of proto-rankings associated with the terminal history generated by σ and W . Let $T \leq T'$ be the first period in which x, y are ranked ($y \not R_{T-1} xR_{T-1} y$ and $yR_T x$). Since xWy and $yR_T x$, it cannot be that $\{x, y\}$ is voted on in period T .

Because $R_0 = \emptyset$ contains no missed opportunities and σ never misses an opportunity or takes a risk, Corollary 2 promises that R_{T-1} contains no missed opportunities. Thus by Lemma 2, σ must offer $\{x, y\}$ in period T —a contradiction. \parallel

C.2. Proof that regret-free strategies avoid errors

The proof relies on two lemmata. For the first, recall from Appendix A the notation $[x, y]_R$ for R -order intervals.

Lemma 3. *Given a pair of alternatives $x, y \in \mathcal{X}$, let R' be a ranking such that $xR'y$, and let W be a majority will that agrees with R' on every pair $\{z, w\} \not\subseteq [x, y]_{R'}$. Then the outcome under W of any strategy agrees with R' on every pair $\{z, w\} \not\subseteq [x, y]_{R'}$.*

Proof. Let x, y, R' , and W satisfy the hypothesis, and let R be the outcome under W of some strategy of the chair.

Claim. *If $\{z, w\} \not\subseteq [x, y]_{R'}$ satisfy (a) $zR'w$ and (b) either $z \not R x$ or $y \not R w$, then zRw .*

Proof of the claim. Assume that $z \not R x$; we omit the similar argument for the case $y \not R w$. Suppose toward a contradiction that wRz . Label the alternatives $[w, z]_R \equiv \{x_1, \dots, x_K\}$ so that

$$w = x_1 R \dots R x_K = z.$$

Since R is W -reachable, we have $x_1 W \dots W x_K$ by Observation 1 (Appendix B.3, p. 18). Suppose that $yR'x_k$ for every $k < K$. Then, $\{x_k, x_{k+1}\} \not\subseteq [x, y]_{R'}$ for every $k < K$. Since R' agrees with W on pairs $\{z', w'\} \not\subseteq [x, y]_{R'}$, it follows that $x_k R' x_{k+1}$ for every $k < K$, whence $wR'z$ by transitivity of R' , contradicting (a).

Suppose instead that $x_k R'y$ for some $k < K$, and let k' be the smallest such k . It must be that $xR'w$, since otherwise $zR'w$ and the transitivity of R' would produce the contradiction $zR'x$. Thus $xR'w$. Then $w \notin [x, y]_{R'}$, as $\{z, w\} \not\subseteq [x, y]_{R'}$, $zR'w$, and $z \not R x$. Hence $yR'w = x_1$, so that $k' > 1$. By definition of k' , we have $x_{k'} R'yR'x_{k'-1}$. On the one hand, the transitivity of R' demands that $x_{k'} R'x_{k'-1}$. On the other hand, since $\{x_{k'-1}, x_{k'}\} \not\subseteq [x, y]_{R'}$ (because $yR'x_{k'-1}$) and $x_{k'-1} W x_{k'}$, we must have $x_{k'-1} R'x_{k'}$ —a contradiction. \square

Now fix a pair $\{z, w\} \not\subseteq [x, y]_{R'}$ such that $zR'w$; we must show that zRw . If either $xR'z$ or $wR'y$, then zRw follows from the claim.

Suppose instead that $zR'x$ and $yR'w$. Observe that $\{z, x\} \not\subseteq [x, y]_{R'}$, (a) $zR'x$ and (b) $y \not R'x$. We may therefore apply the claim to $\{z, x\}$ to obtain zRx . Similarly applying the claim to $\{x, w\}$ yields xRw , whence zRw follows by transitivity of R . \parallel

Lemma 4. *Let R be a proto-ranking, and let $x, y, z \in \mathcal{X}$ be such that*

$$\{x, y, z\}^2 \cap R \subseteq \{(x, z)\}.$$

Then there exists a ranking $R' \supseteq R \cup \{(x, z), (z, y)\}$ such that $[x, y]_{R'} = [x, z]_R \cup \{y\}$.

To interpret the conclusion, observe that the properties of R' are equivalent to the following: (a) $xR'z$ and $[x, z]_{R'} = [x, z]_R$, and (b) $zR'y$ and $[z, y]_{R'} = \{z, y\}$. In words, Lemma 4 runs as follows. Suppose that a proto-ranking R ranks x above z , and has nothing else to say about $\{x, y, z\}$.³² Call the (possibly empty) set of alternatives ranked below x and above z (i.e. $[x, z]_R \setminus \{x, z\}$) the “in-between set”. The lemma asserts that R may be extended to a ranking R' that (1) adds nothing to the in-between set ($[x, z]_{R'} = [x, z]_R$) and that (2) ranks y immediately below z ($zR'y$ and $[z, y]_{R'} = \{z, y\}$).

The proof of Lemma 4 relies on the following general extension principle. Recall from Appendix A the definition of “extension”.

Extension lemma. *Let R be a proto-ranking, and let $A \subseteq \mathcal{X}$ be such that $[x, y]_R \subseteq A$ for all $x, y \in A$ with xRy . Then the binary relation $R \cup A^2$ admits a complete and transitive extension.*

Proof of the extension lemma. Let R and A satisfy the hypothesis; we seek a complete and transitive extension of the relation $Q := R \cup A^2$. By Suzumura’s extension theorem,³³ it suffices to show that for any finite sequence of alternatives $(w_k)_{k=1}^K$ such that $w_1 Q \cdots Q w_K$, we have either $w_1 Q w_K$ or $w_1 Q w_K Q w_1$. There are two cases.

Case 1: $w_k R w_{k+1}$ for every $k < K$. Then $w_1 R w_K$ since R is transitive (being a proto-ranking), so $w_1 Q w_K$.

Case 2: $\{w_k, w_{k+1}\} \subseteq A$ for some $k < K$. Let k' (k'') be the smallest (largest) such $k < K$, so that $w_1 R \cdots R w_{k'}$ if $k' > 1$ and $w_{k''+1} R \cdots R w_K$ if $k'' < K - 1$. Assume toward a contradiction that $w_K Q w_1$ and $w_1 Q w_K$. Since $\{w_1, w_K\} \not\subseteq A$ (as otherwise $w_1 Q w_K Q w_1$), it must be that $w_K R w_1$ and either $k' > 1$ or $k'' < K - 1$. By transitivity of R , $w_1 R w_{k'}$ if $k' > 1$, and $w_{k''+1} R w_K$ if $k'' < K - 1$; in either case, $\{w_K, w_1\} \subseteq [w_{k''+1}, w_{k'}]_R$. Note that $[w_{k''+1}, w_{k'}]_R \subseteq A$ since $w_{k''+1}, w_{k'} \in A$ by construction. Therefore $\{w_K, w_1\} \subseteq A$, which implies that $w_K Q w_1 Q w_K$ —a contradiction. \parallel

Proof of Lemma 4. Let a proto-ranking R and alternatives $x, y, z \in \mathcal{X}$ satisfy the hypothesis. Define $A := [x, z]_R \cup \{y\}$.

Claim. *For any $x', y' \in A$ such that $x'Ry'$, we have $[x', y']_R \subseteq A$.*

Proof of the claim. Fix alternatives $x', y' \in A$ with $x'Ry'$. By definition of A , it suffices to show that $\{x', y'\} \not\cong y$. So suppose toward a contradiction that $x' = y$; the other case is similar. We have $y' \neq y$ since $x'Ry'$ and R is irreflexive (being a proto-ranking). Since $y' \in A$, it follows that $y' \in [x, z]_R$. By $x'Ry'$ and the transitivity of R , we obtain $y = x'Rz$. But yRz by hypothesis—a contradiction. \square

By the claim, the extension lemma is applicable, so there exists a complete and transitive extension Q of the binary relation $R \cup A^2$. Since $z' Q w' Q z'$ for any $z', w' \in A$, we have in particular that $w Q y Q w$ for any $w \in [x, z]_R$. We may therefore obtain the desired ranking R' by appropriately breaking indifferences in Q . \parallel

With Lemmata 3 and 4 in hand, we are ready to prove that regret-free strategies avoid errors.

Proposition. *A regret-free strategy never misses an opportunity or takes a risk.*

Proof. We shall prove the contra-positive. To that end, fix a strategy σ and a majority will W such that σ misses an opportunity or takes a risk under W . We shall construct a majority will W' such that the outcome R of σ under W' fails to be W' -unimprovable. In particular, we shall find a W' -reachable ranking $R' \neq R$ that is more aligned with \succ than R .

Let T be the first period in which σ either misses an opportunity or takes a risk under W . Write R_{T-1} for the associated end-of-period- $(T-1)$ proto-ranking, and let $\{x, y\}$ be the pair offered in period T .

32. For simplicity, neglect the case $\{x, y, z\}^2 \cap R = \emptyset$.

33. See e.g. Bossert and Suzumura (2010, p. 45).

We shall consider three cases, based on the behaviour of σ under W in period T . By hypothesis, $\{x, y\}$ either misses an opportunity or takes a risk at R_{T-1} . If $\{x, y\}$ misses an opportunity, there is an alternative $z \in \mathcal{X}$ such that $x \succ z \succ y$ and $y R_{T-1} z R_{T-1} x$. It cannot be that $x R_{T-1} z R_{T-1} y$, as this would imply that $x R_{T-1} y$, contradicting the fact that $\{x, y\}$ is offered in period T . Thus one of the following must hold:

- (a) $x R_{T-1} z R_{T-1} y$,
- (b) $x R_{T-1} z R_{T-1} y$, or
- (c) $x R_{T-1} z R_{T-1} y$.

If instead $\{x, y\}$ takes a risk, then there is a $z \in \mathcal{X}$ such that either

- (d) $z \succ y$, $x R_{T-1} z$, and $y R_{T-1} z$, or
- (e) $x \succ z$, $z R_{T-1} y$, and $z R_{T-1} x$.

This yields three cases, as follows. Case 1 is (a). Case 2 encompasses both (b) and (d) under the (slightly more general) hypothesis that “there exists a $z \in \mathcal{X}$ such that $z \succ y$, $z R_{T-1} y R_{T-1} z$, and $x R_{T-1} z$ ”. Finally, case 3 encompasses (c) and (e) under the hypothesis that “there exists a $z \in \mathcal{X}$ such that $x \succ z$, $z R_{T-1} x R_{T-1} z$ and $z R_{T-1} y$ ”. Since cases 2 and 3 are analogous, we omit the proof for the latter.

Case 1: $\exists z \in \mathcal{X}$ such that $x \succ z \succ y$ and $\{x, z, y\}^2 \cap R_{T-1} = \emptyset$. By Lemma 4, there is a ranking R' such that

$$R' \supseteq R_{T-1} \cup \{(x, z), (z, y)\} \quad \text{and} \quad [x, y]_{R'} = \{x, y, z\}.$$

Define W' to equal R' , except that $y W' x$. Clearly W' is a majority will (total and asymmetric), and R' is W' -reachable by Observation 1 (Appendix B.3, p. 18) since x, y are not R' -adjacent. Denote by R the outcome of σ under W' . It remains to show that $R \neq R'$, and that R' is more aligned with \succ than R .

For the former, since $x R' y$, it suffices to show that $y R x$. To this end, observe that $R_{T-1} \subseteq W'$. Thus the history of length $T-1$ generated by σ and W' is the same as that generated by σ and W , which means in particular that $\{x, y\}$ is offered in period T . Since $y W' x$, it follows that $y R x$, as desired.

To show that R' is more aligned with \succ than R , observe that W' agrees with R' on every pair $\{w, w'\} \not\subseteq \{x, y, z\} = [x, y]_{R'}$. It follows by Lemma 3 that R and R' agree on every pair $\{w, w'\} \not\subseteq \{x, y, z\}$. Since $x \succ z \succ y$ and $x R' z R' y$, it follows that R' is more aligned with \succ than R .

Case 2: $\exists z \in \mathcal{X}$ such that $z \succ y$, $z R_{T-1} y R_{T-1} z$ and $x R_{T-1} z$. We shall begin with an auxiliary ranking R'' , then use it to construct our majority will W' and ranking R' . By Lemma 4, there is a ranking

$$R'' \supseteq R_{T-1} \cup \{(x, z), (z, y)\}$$

such that

$$[x, y]_{R''} = [x, z]_{R_{T-1}} \cup \{y\}. \quad (\text{C.3})$$

Define

$$X := \{w \in [x, z]_{R_{T-1}} \setminus \{x\} : w \succ y\},$$

and let W' be such that

- (i) $w W' y$ for every $w \in X$,
- (ii) $y W' w$ for every $w \in [x, z]_{R_{T-1}} \setminus X$, and
- (iii) W' agrees with R'' on every other pair.

Denote by R the outcome of σ under the majority will W' .

Observe that (i) $y R_{T-1} w$ for every $w \in X$ (since otherwise $y R_{T-1} w R_{T-1} z$, contradicting the case-2 hypothesis), (ii) $w R_{T-1} y$ for every $w \in [x, z]_{R_{T-1}} \setminus X$ (otherwise either $x = w R_{T-1} y$ or $x R_{T-1} w R_{T-1} y$, whence $x R_{T-1} y$), and (iii) $R_{T-1} \subseteq R''$. Thus $R_{T-1} \subseteq W'$ by definition of the latter. It follows that the history of length $T-1$ generated by σ and W' is the same as that generated by σ and W , which means in particular that $\{x, y\}$ is offered in period T . Since $y W' x$, we thus have $y R x$.

Since $X \subseteq [x, z]_{R_{T-1}} \subseteq [x, y]_{R''}$ (by definition of X and (C.3)), W' agrees with R'' on every pair $\{w, w'\} \not\subseteq [x, y]_{R''}$. It follows by Lemma 3 that R agrees with R'' on every pair $\{w, w'\} \not\subseteq [x, y]_{R''}$. This, together with (C.3), $R_{T-1} \subseteq R$ and $y R x$, implies that y, x are R -adjacent, whence

$$[y, z]_R = \{y\} \cup [x, z]_R = \{y\} \cup [x, z]_{R_{T-1}} = [x, y]_{R''}. \quad (\text{C.4})$$

It follows that $X \cup \{x\} \subseteq [y, z]_R$.

Define $X' := X \cup \{x\}$, and label its elements $X' \equiv \{a_1, \dots, a_K\}$ so that $a_1 R \dots R a_K$. Similarly, label $[y, z]_R \setminus X' \equiv \{b_1, \dots, b_L\}$ so that $b_1 R \dots R b_L$.³⁴ Let R' be the ranking that

34. $[y, z]_R \setminus X'$ is non-empty since y belongs to it.

- (I) agrees with R on any pair $\{w, w'\} \not\subseteq [y, z]_R$, and
- (II) ranks the elements of $[y, z]_R$ as $a_1 R' \dots R' a_K R' b_1 R' \dots R' b_L$.

We have now constructed a majority will W' and a ranking R' . Recall that R is the outcome of σ under W' . It remains to show that

- (dist) R' is distinct from R ,
- (align) R' is more aligned with \succ than R , and
- (feas) R' is W' -reachable.

For (dist), observe that since $x \in X'$ and $y \in [y, z]_R \setminus X'$, we have $x = a_k R' b_\ell = y$ for some k and ℓ .³⁵ Since $y R x$, it follows that $R' \neq R$.

For (align), fix a pair $w, w' \in X'$ with $w R w'$ and $w' R' w$; we must show that $w' \succ w$. By definition of R' , it must be that $w' \in X' = X \cup \{x\}$ and that $w \in [y, z]_R \setminus X'$. Thus $w' \succ y$ (by $x \succ y$ and the definition of X) and either $y = w$ or $y \succ w$, whence $w' \succ w$ by transitivity of \succ .

It remains to establish (feas). To this end (recalling Observation 1 in Appendix B.3, p. 18), fix an R' -adjacent pair $w, w' \in X'$ with $w' R' w$; we must show that $w' W' w$. There are two cases.

Sub-case (2)(a): $\{w, w'\} \not\subseteq [y, z]_R$. Then $w' R w$ by part (I) of the definition of R' . Since R agrees with R'' on any pair $\{z', z''\} \not\subseteq [y, z]_R$, it follows that $w' R'' w$. It therefore suffices to show that W' agrees with R'' on $\{w, w'\}$. Recalling the definition (i)–(iii) of W' ,

- If $\{w, w'\} \not\preceq y$, then W' agrees with R'' on $\{w, w'\}$ by (iii).
- If $w' = y \in [y, z]_R$, then $w \notin [y, z]_R \supseteq [x, z]_{R_{T-1}} \supseteq X$ by (C.4), so neither (i) nor (ii) applies to the pair $\{w, y\} = \{w, w'\}$. Thus by (iii), W' agrees with R'' on $\{w, w'\}$.
- If $w = y \in [y, z]_R$, then $w' \notin [y, z]_R \supseteq [x, z]_{R_{T-1}} \supseteq X$ by (C.4), so neither (i) nor (ii) applies to the pair $\{y, w'\} = \{w, w'\}$. Thus by (iii), W' agrees with R'' on $\{w, w'\}$.

Sub-case (2)(b): $\{w, w'\} \subseteq [y, z]_R$. Recall the definition (i)–(iii) of W' . Recall also part (II) of the definition of R' . Observe that $y = b_1$ since y is R -highest in $[y, z]_R$, and $y \notin X' = X \cup \{x\}$. Furthermore, $x = a_1$ since y, x are R -adjacent (recall (C.4)) and $x \in X$. Finally, remark that $K \geq 2$ since $z \in X' = X \cup \{x\}$ and $z \neq x$.

Suppose first that $w' = y = b_1$. Then since w', w are R' -adjacent with $w' R' w$, we have $w = b_2 \notin X' \supseteq X$. Thus, part (i) does not apply to the pair $\{w, y\} = \{w, w'\}$. So by (C.4), part (ii) applies, yielding $w' = y W' w$.

Suppose instead that $w = y = b_1$. Then since w', w are R' -adjacent with $w' R' w$, we have $w' = a_K \in X' = X \cup \{x\}$. Since $x = a_1$ and $K \geq 2$, we have $w' \neq x$. Thus $w' \in X$, so that (i) applies to the pair $\{y, w'\} = \{w, w'\}$, yielding $w' W' y = w$.

Finally, suppose that $\{w, w'\} \not\preceq y$. Then W' and R' agree on the pair $\{w, w'\}$ by (iii), so it suffices to show that $w' R'' w$. Since $b_1 = y \notin \{w, w'\}$, we have either $\{w, w'\} \subseteq X'$ or $\{w, w'\} \subseteq [y, z]_R \setminus X'$. Thus R and R' agree on $\{w, w'\}$ by part (II) of the definition of R' , so that $w' R w$.

Now label $[w', w]_R \equiv \{z_1, \dots, z_J\}$ so that $z_1 R \dots R z_J$. Since R is W' -reachable, we have $z_1 W' \dots W' z_J$ by Observation 1 (Appendix B.3, p. 18). By the hypotheses $w' \in [y, z]_R$ and $w' \neq y$, we must have $y R w'$ and thus $y \notin [w', w]_R$. This together with the fact that $z_j W' z_{j+1}$ for each $j < J$ implies, via part (iii) of the definition of W' , that $z_j R'' z_{j+1}$ for each $j < J$. It follows by transitivity of R'' that $w' = z_1 R'' z_J = w$, as desired. \parallel

D. PROOF OF THEOREM 4 (SECTION 6.3, P. 15)

In this appendix, we first prove Theorem 4 using two lemmata (Section D.1), and then prove these lemmata (Sections D.2–D.4). Throughout, we label the alternatives $X \equiv \{1, \dots, n\}$ so that $1 \succ \dots \succ n$.

D.1. Proof using lemmata

Definition 9. Let Σ_0 be the set of all strategies. For every integer $k \in \{1, \dots, n-2\}$, let Σ_k be the set of all strategies σ with the following property: for any majority will W and alternative $j \leq k$, labelling the alternatives $\{j+1, \dots, n\} \equiv \{x_{j+1}, \dots, x_n\}$ as

$$x_{j+1} R^\sigma(W) \dots R^\sigma(W) x_n,$$

the first vote involving j that σ offers under W is on $\{j, x_{j+1}\}$; if j loses, then a second vote involving j is offered, namely on $\{j, x_{j+2}\}$; if j loses again, then a third vote involving j is offered, namely on $\{j, x_{j+3}\}$; and so on.

35. In fact, $k = \ell = 1$ since y is R -highest in $[y, z]_R$ and (recall) y, x are R -adjacent.

The definition of Σ_{n-2} describes a natural generalization of insertion sort: for each alternative j , given how the \succ -worse alternatives $\{x_{j+1}, \dots, x_n\}$ are ultimately ranked, the same votes involving j are offered, in the same order, though not necessarily in adjacent periods.³⁶ Each Σ_k for $k < n-2$ is defined by the same property restricted to those alternatives j that are \succ -better than or equal to k , so that $\Sigma_0 \supseteq \Sigma_1 \supseteq \dots \supseteq \Sigma_{n-2}$.

Lemma 5. *A strategy is outcome-equivalent to insertion sort iff it belongs to Σ_{n-2} .*

Lemma 6. *Given $k \in \{1, \dots, n-2\}$, a strategy $\sigma \in \Sigma_{k-1}$ is best for k among Σ_{k-1} iff it belongs to Σ_k .*

Proof of Theorem 4. Let $B_0 = \Sigma_0$ be the set of all strategies, and for each $k \in \{1, \dots, n-2\}$, let B_k be the set of strategies in B_{k-1} that are best for k among B_{k-1} .³⁷ A lexicographic strategy is precisely one that lives in B_{n-2} .

By Lemma 5, a strategy is outcome-equivalent to insertion sort iff it lives in Σ_{n-2} ; so what must be shown is that $\Sigma_{n-2} = B_{n-2}$. We shall prove the stronger claim that $\Sigma_k = B_k$ for each $k \in \{0, \dots, n-2\}$ by (weak) induction on k . The base case $k=0$ holds by definition of Σ_0 and B_0 . For the induction step, suppose that $\Sigma_{k-1} = B_{k-1}$; then $\Sigma_k = B_k$ by Lemma 6. \parallel

The remainder of this appendix is devoted to proving Lemmata 5 and 6. We begin in Section D.2 with two preliminary results, then prove Lemma 6 in Section D.3 and Lemma 5 in Section D.4.

D.2. Preliminary results

The following lemma is used in the proof of Lemma 5.

Lemma 7. *Given a $k \in \{1, \dots, n-2\}$, consider a strategy $\sigma \in \Sigma_k$, a majority will W , an alternative $j \leq k$ and some $m \in \{1, \dots, n-j\}$. Suppose that under W , σ offers at least m votes involving j , and that j loses the first $m-1$ of these. Let ℓ be the m th alternative pitted against j , and let R be the proto-ranking associated with the history after which the vote on $\{j, \ell\}$ occurs. Then, $\ell R i$ for any $i \neq \ell$ such that $i > j$ and j did not lose against i prior to the vote on $\{j, \ell\}$.*

Proof. Suppose toward a contradiction that $\ell \not R i$ for some $i \neq \ell$ with $i > j$ such that j did not lose to i prior to the vote on $\{j, \ell\}$. Then there exists a ranking W' such that $R \subseteq W'$ and $i W' \ell$. (Note that a ranking is precisely a transitive majority will.) Being a ranking, W' is the only W' -reachable ranking (by Observation 1 in Appendix B.3, p. 18), so that $R^\sigma(W') = W'$, and in particular $i R^\sigma(W') \ell$.

Let T be the period in which σ offers $\{j, \ell\}$ (i.e. the m th vote involving j) under W . Since $R \subseteq W'$, the history of length $T-1$ generated by σ and W' is the same as that generated by σ and W . So in particular, under W' , σ offers $\{j, \ell\}$ in period T , σ does not offer $\{j, i\}$ in an earlier period, and j does not win a vote in any earlier period. Since $\sigma \in \Sigma_k$, it follows that $\ell R^\sigma(W') i$, a contradiction. \parallel

The proof of Lemma 6 relies on the following.

Lemma 8. *Given $k \in \{2, \dots, n-2\}$, if $N_k^\sigma(W) > 0$ for some $\sigma \in \Sigma_{k-1}$ and majority will W , then prior to winning its first vote, k is pitted only against \succ -worse alternatives.*

Proof. We prove the contra-positive. Let $k \in \{2, \dots, n-2\}$, $\sigma \in \Sigma_{k-1}$ and a majority will W be such that σ pits k against some $j < k$ under W and k wins no vote before the one against j ; we must show that $N_k^\sigma(W) = 0$. Let R be the proto-ranking associated with the history after which the vote on $\{j, k\}$ occurs. It suffices to show that $\ell R k$ for all $\ell > k$.

Claim. *j wins no vote prior to the one against k .*

Proof of the claim. Suppose toward a contradiction that the first alternative against which j wins is $\ell \neq k$. Then $j R \ell$. Since $\sigma \in \Sigma_{k-1}$, the vote on $\{j, \ell\}$ is the m th involving j , for some $m < n-j$. It follows by Lemma 7 (above) that $\ell R k$, which together with $j R \ell$ and the transitivity of R yields $j R k$. On the other hand, since $\{j, k\}$ is offered after a history with proto-ranking R , we must have $j \not R k$. Contradiction! \square

36. Note a subtlety in the definition: although we label $\{x_{j+1}, \dots, x_n\}$ according to the outcome $R^\sigma(W)$ of σ under W , a strategy in Σ_k need not (as insertion sort would) have totally ranked $\{j+1, \dots, n\}$ before offering votes involving j .

37. Let $B_k := \emptyset$ if B_{k-1} is empty. (It is in fact non-empty, but we have not proved it yet.)

Fix an $\ell > k$; we shall show that $\ell R k$. Since $\sigma \in \Sigma_{k-1}$, the vote on $\{j, k\}$ is the m th involving j , for some $m < n - j$. It must be that σ offers $\{j, \ell\}$ prior to $\{j, k\}$, since otherwise Lemma 7 would yield $k R \ell$, contradicting the hypothesis that k wins no votes prior to the one against j . Since j wins no vote prior to the one against k and $m < n - j$, Lemma 7 yields $\ell R k$, as desired. \parallel

D.3. Proof of Lemma 6 (Section D.1, p. 24)

Let \mathcal{W} denote the set of all majority wills (total and asymmetric relations) on \mathcal{X} . We shall use the probabilistic notation $\Pr(E|F) := |E \cap F|/|F|$ for the fraction of majority wills in $F \subseteq \mathcal{W}$ that belong to $E \cap F \subseteq F$, and similarly $\Pr(E) := \Pr(E|\mathcal{W})$. This corresponds the thought experiment in which the majority will W is drawn uniformly at random from \mathcal{W} .

We must establish that membership of Σ_k is necessary and sufficient for being best for k among Σ_{k-1} . We prove sufficiency and necessity in turn, making use of Lemma 4 from the previous section.

Proof of sufficiency. Fix k , a strategy $\sigma \in \Sigma_{k-1}$ and an $m \in \{1, \dots, n - k\}$. We shall derive an upper bound for $\Pr(N_k^\sigma \geq m)$ then show that it is attained if $\sigma \in \Sigma_k$.

For each $\ell \in \{1, \dots, n - k\}$, let $F_\ell \subseteq \mathcal{W}$ be the set of majority wills under which σ offers at least ℓ votes involving alternative k , with alternative k losing the first $\ell - 1$ of these and winning the ℓ th.

Claim. $\Pr(F_\ell) \leq 1/2^\ell$ for each $\ell \in \{1, \dots, n - k\}$, with equality if $\sigma \in \Sigma_k$.

Proof of the claim. A majority will W lies in F_ℓ iff under σ and W ,

- alternative k loses its first vote (probability $1/2$),
- a second vote involving alternative k occurs (probability ≤ 1 , with equality if $\sigma \in \Sigma_k$) and k loses (probability $1/2$),
- ...
- an $(\ell - 1)$ th vote involving alternative k occurs (probability ≤ 1 , with equality if $\sigma \in \Sigma_k$) and k again loses (probability $1/2$), and
- an ℓ th vote involving alternative k occurs (probability ≤ 1 , with equality if $\sigma \in \Sigma_k$) and k wins (probability $1/2$).

Thus,

$$\Pr(F_\ell) \leq \frac{1}{2} \times \left(1 \times \frac{1}{2}\right)^{\ell-1} = \frac{1}{2^\ell}, \quad \text{with equality if } \sigma \in \Sigma_k,$$

which completes the proof. \square

By Lemma 4 (Section D.2, p. 24), if $N_k^\sigma(W) > 0$, then prior to winning its first vote, k was only pitted against $>$ -worse alternatives. There are $n - k$ of these: $\{k + 1, \dots, n\}$. Thus if $N_k^\sigma(W) \geq m$ holds, then alternative k cannot have lost strictly more than $n - m - k$ votes and must have won at least one—so in particular, $W \in F_1 \cup \dots \cup F_{n-k-m+1}$. Thus,

$$\begin{aligned} \Pr(N_1^\sigma \geq m) &= \sum_{\ell=1}^{n-k-m+1} \Pr(F_\ell) \Pr(N_1^\sigma \geq m | F_\ell) \\ &\leq \sum_{\ell=1}^{n-k-m+1} \Pr(F_\ell) \tag{\#} \\ &\leq \sum_{\ell=1}^{n-k-m+1} \frac{1}{2^\ell}, \tag{b} \end{aligned}$$

where (b) holds by the claim.

Now suppose that $\sigma \in \Sigma_k$; we shall show that (#) and (b) hold with equality, so that σ attains the bound. For (b), this follows from the claim. For (#), fix an $\ell \in \{1, \dots, n - k - m + 1\}$ and a $W \in F_\ell$; we must show that $N_k^\sigma(W) \geq m$. Label $\{k + 1, \dots, n\} \equiv \{x_{k+1}, \dots, x_n\}$ so that

$$x_{k+1} R^\sigma(W) \dots R^\sigma(W) x_n.$$

Since $W \in F_\ell$, we have by definition of Σ_k that $k R^\sigma(W) x_{\ell+1}$. Thus, $k R^\sigma(W) x_{\ell'}$ for each $\ell' \in \{\ell + 1, \dots, n\}$, so that $N_k^\sigma(W) \geq n - \ell \geq m$. \parallel

Proof of necessity. Take a strategy σ in $\Sigma_{k-1} \setminus \Sigma_k$. Since σ belongs to Σ_{k-1} , it satisfies the inequalities (#) and (b) in the sufficiency argument. Suppose that one or the other holds strictly for some $m \in \{1, \dots, n - k\}$, so that σ fails to attain the

bound in the sufficiency proof. Since any $\sigma' \in \Sigma_k$ attains the bound for every m by the (just-proved) sufficiency part, it follows that σ is not best for k among Σ_{k-1} . It therefore suffices to find an $m \in \{1, \dots, n-k\}$ such that either (\sharp) or (b) holds strictly.

Since $\sigma \in \Sigma_{k-1} \setminus \Sigma_k$, there is a majority will W such that, labelling the alternatives $\{k+1, \dots, n\} \equiv \{x_{k+1}, \dots, x_n\}$ so that

$$x_{k+1} R^\sigma(W) \dots R^\sigma(W) x_n,$$

one of the following holds under W :

- (a) σ pits k against some $j \neq x_{k+1}$ prior to pitting it against x_{k+1} .
- (b) For some $\ell \in \{k+1, \dots, n-k\}$, σ offers at least ℓ votes involving k , the first $\ell-1$ of which are against x_{k+1}, \dots, x_ℓ and are all lost by k , and the ℓ th of which is against some $j \neq x_{\ell+1}$.
- (c) For some $\ell \in \{k+1, \dots, n-k\}$, σ offers exactly $\ell-1$ votes involving k , against x_{k+1}, \dots, x_ℓ , each of which is lost by k .

Case (a). We shall exhibit a majority will $W' \in F_1$ such that $N_k^\sigma(W') < n-k$, so that (\sharp) holds strictly for $m=n-k$. This is trivial if $N_k^\sigma(W)=0$ (let $W' := W$), so suppose that $N_k^\sigma(W) > 0$. Let $j \neq x_{k+1}$ be the first alternative pitted against k by σ under W , and let R be the proto-ranking associated with the history after which this occurs. Clearly $j R k R x_{k+1}$. Since $\sigma \in \Sigma_{k-1}$ and $N_k^\sigma(W) > 0$, Lemma 4 (Section D.2, p. 24) implies that $j > k$. So $x_{k+1} R^\sigma(W) j$ (since $j \in \{k+1, \dots, n\} = \{x_{k+1}, \dots, x_n\}$ and $j \neq x_{k+1}$), and thus $j R x_{k+1}$.

It follows that there is a ranking W' such that $R \subseteq W'$ and $x_{k+1} W' k W' j$.³⁸ (Note that a ranking is precisely a transitive majority will.) Let T be the period in which σ offers $\{j, k\}$ under W . Since $R \subseteq W'$, the history of length $T-1$ generated by σ and W' is the same as that generated by σ and W , and thus $\{j, k\}$ is the first pair involving k that σ offers under W' . Thus $W' \in F_1$ since $k W' j$. Being a ranking, W' is the only W' -reachable ranking (by Observation 1 in Appendix B.3, p. 18), so $R^\sigma(W') = W'$. Thus $N_k^\sigma(W') < n-k$ since $x_{k+1} W' k$.

Case (b). If $j > k$, then the case-(a) argument yields a $W' \in F_\ell$ such that $N_k^\sigma(W') < n-k-\ell+1$, so that (\sharp) holds strictly for $m=n-k-\ell+1$. Suppose instead that $j < k$. Let W' be the majority will that agrees with W on every pair, except that $k W' j$. Clearly $W' \in F_\ell$. By (the contra-positive of) Lemma 4, we have $N_k^\sigma(W') = 0$. Thus $\Pr(N_k^\sigma \geq 1 | F_\ell) < 1$, so that (\sharp) holds strictly for $m=1$.

Case (c). Let E be the set of majority wills under which σ offers at least $\ell-1$ votes involving k , the first $\ell-1$ of which k loses. Then $\Pr(E) > 0$, and the probability that σ offers at least ℓ votes conditional on E is strictly less than 1. The argument for the claim in the sufficiency proof therefore yields $\Pr(F_\ell) < 1/2^\ell$, so that (b) holds strictly for e.g. $m=1$. ||

D.4. Proof of Lemma 5 (Section D.1, p. 24)

We must show that membership of Σ_{n-2} is necessary and sufficient for outcome-equivalence to insertion sort. For the sufficiency part, we shall make use of Lemma 7 in Section D.3.

Proof of sufficiency. Let $\sigma \in \Sigma_{n-2}$, fix a majority will W , and let $R (R')$ be the outcome of σ (of insertion sort) under W . We will show for each $k \in \{n-2, \dots, 1\}$ that R agrees with R' on $\{k, \dots, n\}$, using induction on k . For the base case $k=n-2$, let $\{j, \ell\}$ be the first pair offered by σ , where $j < \ell$; it suffices to show that $j=n-1$. Suppose to the contrary; then by Lemma 7 (Section D.3, p. 24), prior to the first vote, ℓ is already ranked above every $i > j$ such that $i \neq \ell$, which is absurd. The induction step is immediate from the fact that $\sigma \in \Sigma_{n-2}$. ||

The proof of necessity refers to the argument for Lemma 6 in Section D.3 above.

Proof of necessity. Let $\sigma \notin \Sigma_{n-2}$; we must show that it is not outcome-equivalent to insertion sort. Note that there is a (unique) $k \in \{1, \dots, n-2\}$ such that $\sigma \in \Sigma_{k-1} \setminus \Sigma_k$. Recall the bound in the proof of the sufficiency part of Lemma 6 (Section D.3 above). Since insertion sort belongs to Σ_k , it attains the bound for every $m \in \{1, \dots, n-k\}$ by the Lemma 6 sufficiency argument, and thus all of its outcome-equivalents do. By the Lemma 6 necessity argument, σ fails to attain the bound for some $m \in \{1, \dots, n-k\}$, so is not among the outcome-equivalents of insertion sort. ||

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38. To see why, observe that the transitive closure R' of $R \cup \{(x_{k+1}, k)\}$ is a proto-ranking since $k R x_{k+1}$. Since $j \neq x_{k+1}$ and $j R x_{k+1}$, we have $j R k$ by Observation 3 (Appendix C.1, p. 19), and thus the transitive closure R'' of $R' \cup \{(k, j)\}$ is also a proto-ranking. Let W' be any ranking that contains R'' .

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Supplementary Data

Supplementary data are available at *Review of Economic Studies* online.

REFERENCES

- APESTEGUIA, J., BALLESTER, M. A. and MASATLIOGLU, Y. (2014), "A Foundation for Strategic Agenda Voting", *Games and Economic Behavior*, **87**, 91–99.
- ARROW, K. J. (1950), "A Difficulty in the Concept of Social Welfare", *Journal of Political Economy*, **58**, 328–346.
- ARROW, K. J. (1951), *Social Choice and Individual Values* (New York, NY: Wiley).
- ARROW, K. J. (1963), *Social Choice and Individual Values*, 2nd edn (New Haven, CT: Yale University Press).
- BANKS, J. S. (1985), "Sophisticated Voting Outcomes and Agenda Control", *Social Choice and Welfare*, **1**, 295–306.
- BARBERÀ, S. and GERBER, A. (2017), "Sequential Voting and Agenda Manipulation", *Theoretical Economics*, **12**, 211–247.
- BELL, D. E. (1982), "Regret in Decision Making under Uncertainty", *Operations Research*, **30**, 961–981.
- BLACK, J. D. (1958). *The Theory of Committees and Elections* (Cambridge: Cambridge University Press).
- BOSSERT, W. and SUZUMURA, K. (2010). *Consistency, Choice, and Rationality* (Cambridge, MA: Harvard University Press).
- CHARON, I. and HUDRY, O. (2010), "An Updated Survey on the Linear Ordering Problem for Weighted or Unweighted Tournaments", *Annals of Operations Research*, **175**, 107–158.
- THE MARQUIS DE CONDORCET (1785), *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix* (Paris: L'imprimerie royale).
- COPELAND, A. (1951), "A Reasonable Social Welfare Function" (Notes from University of Michigan Seminar on Applications of Mathematics to the Social Sciences).
- CURELLO, G. and SINANDER, L. (2022), "The Preference Lattice" (Working Paper).
- DANIELS, H. E. (1969), "Round-Robin Tournament Scores", *Biometrika*, **56**, 295–299.
- FARQUHARSON, R. (1969), *Theory of Voting* (New Haven, CT: Yale University Press).
- FISHBURN, P. C. (1983), Transitive Measurable Utility. *Journal of Economic Theory*, **31**, 293–317.
- GEHRLEIN, W. V. (1989), "The Probability of Intransitivity of Pairwise Comparisons in Individual Preference", *Mathematical Social Sciences*, **17**, 67–75.
- GEHRLEIN, W. V. and LEPELLEY, D. (2011), *Voting Paradoxes and Group Coherence: The Condorcet Efficiency of Voting Rules*. Studies in Choice and Welfare (Berlin: Springer).
- GERSHKOV, A., KLEINER, A., MOLDOVANU, B. and SHI, X. (2020), "The Art of Compromising: Voting with Interdependent Values and the Flag of the Weimar Republic" (Working Paper).
- GERSHKOV, A., MOLDOVANU, B. and SHI, X. (2017), "Optimal Voting Rules", *Review of Economic Studies*, **84**, 688–717.
- GERSHKOV, A., MOLDOVANU, B. and SHI, X. (2019), "Voting on Multiple Issues: What To Put on the Ballot?", *Theoretical Economics*, **14**, 555–596.
- GERSHKOV, A., MOLDOVANU, B. and SHI, X. (2020), "Monotonic Norms and Orthogonal Issues in Multidimensional Voting", *Journal of Economic Theory*, **189**.
- GONZÁLEZ-DÍAZ, J., HENDRICK, R. and LOHMANN, E. (2014), "Paired Comparisons Analysis: An Axiomatic Approach to Ranking Methods", *Social Choice and Welfare*, **42**, 139–169.
- GORDON, G. J. (1999), "Regret Bounds for Prediction Problems", in *Proceedings of the Twelfth Annual Conference on Computational Learning Theory* (New York, NY: Association for Computing Machinery) 29–40.
- GROSECLOSE, T. and MILYO, J. (2010), "Sincere versus Sophisticated Voting in Congress: Theory and Evidence", *Journal of Politics*, **72**, 60–73.
- HORAN, S. (2021), "Agendas in Legislative Decision-Making", *Theoretical Economics*, **16**, 235–274.
- KEMENY, J. G. (1959), "Mathematics Without Numbers", *Dædalus*, **88**, 577–591.
- KENDALL, M. G. (1955), "Further Contributions to the Theory of Paired Comparisons", *Biometrics*, **11**, 43–62.
- KLEINER, A. and MOLDOVANU, B. (2017), "Content-based Agendas and Qualified Majorities in Sequential Voting", *American Economic Review*, **107**, 1477–1506.
- KNUTH, D. E. (1998), *The Art of Computer Programming, Vol. 3: Sorting and Searching*, 2nd edn. (Upple Saddle River, NJ: Addison–Wesley).
- LADHA, K. K. (1994), "Coalitions in Congressional Voting", *Public Choice*, **78**, 43–63.
- LOOMES, G. and SUGDEN, R. (1982), "Regret Theory: An Alternative Theory of Rational Choice under Uncertainty", *The Economic Journal*, **92**, 805–824.

- MCGARVEY, D. C. (1953), "A Theorem on the Construction of Voting Paradoxes", *Econometrica*, **21**, 608–610.
- MILGROM, P. and SHANNON, C. (1994), "Monotone Comparative Statics", *Econometrica*, **62**, 157–180.
- MILLER, N. R. (1977), "Graph-Theoretical Approaches to the Theory of Voting", *American Journal of Political Science*, **21**, 769–803.
- MOON, J. W. and PULLMAN, N. J. (1970), "On Generalized Tournament Matrices", *SIAM Review*, **12**, 384–399.
- MYERSON, R. B. (1991), *Game Theory* (Cambridge, MA: Harvard University Press).
- ORDESHOOK, P. C. and PALFREY, T. R. (1988), "Agendas, Strategic Voting, and Signaling with Incomplete Information", *American Journal of Political Science*, **32**, 441–466.
- OSCARSSON, H. (2019), "Rekordlåg Personröstning", in Andersson, U., Rönnerstrand, B., Öhberg, P. and A. Bergström (eds) *Storm och Stiltje* (Gothenburg: SOM-institutet) 107–116.
- POOLE, K. T. and ROSENTHAL, H. (1997), *Congress: A Political-Economic History of Roll Call Voting* (New York, NY: Oxford University Press).
- ROESSLER, C., SHELEGIA, S. and STRULOVICI, B. (2018), "Collective Commitment", *Journal of Political Economy*, **126**, 347–380.
- RUBINSTEIN, A. (1980), "Ranking the Participants in a Tournament", *SIAM Journal on Applied Mathematics*, **38**, 108–111.
- SAVAGE, L. J. (1951), "The Theory of Statistical Decision", *Journal of the American Statistical Association*, **46**, 55–67.
- SLATER, P. (1961), "Inconsistencies in a Schedule of Paired Comparisons", *Biometrika*, **48**, 303–312.
- SLUTZKI, G. and VOLIJ, O. (2005), "Ranking Participants in Generalized Tournaments", *International Journal of Game Theory*, **33**, 255–270.
- WEI, T.-H. (1952), *Algebraic Foundations of Ranking Theory* (Doctoral Thesis, University of Cambridge).
- WILKERSON, J. D. (1999), "'Killer' Amendments in Congress", *American Political Science Review*, **93**, 535–552.
- YOUNG, H. P. (1986), "Optimal Ranking and Choice from Pairwise Comparisons", in Grofman, B. and Owen, G. (eds) *Information Pooling and Group Decision Making* (Greenwich, CT: JAI Press) 113–122.
- YOUNG, H. P. (1988), "Condorcet's Theory of Voting", *American Political Science Review*, **82**, 1231–1244.
- YOUNG, H. P. and LEVENGLICK, A. (1978), "A Consistent Extension of Condorcet's Election Principle", *SIAM Journal on Applied Mathematics*, **35**, 285–300.
- ZERMELO, E. (1929), "Die Berechnung der Turnier-Ergebnisse als ein Maximumproblem der Wahrscheinlichkeitsrechnung", *Mathematische Zeitschrift*, **29**, 436–460.
- ZINKEVICH, M. (2003), "Online Convex Programming and Generalized Infinitesimal Gradient Ascent", in *Proceedings of the Twentieth International Conference on Machine Learning* (Washington, DC: Association for the Advancement of Artificial Intelligence) 928–935.