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Quasi-hyperbolic semigroups ☆

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Abstract

We introduce the notion of quasi-hyperbolic operators and C_0-semigroups. Examples include the push-forward operator associated with a quasi-Anosov diffeomorphism or flow. A quasi-hyperbolic operator can be characterised by a simple spectral property or as the restriction of a hyperbolic operator to an invariant subspace. There is a corresponding spectral property for the generator of a C_0-semigroup, and it characterises quasi-hyperbolicity on Hilbert spaces but not on other Banach spaces. We exhibit some weaker properties which are implied by the spectral property.

Keywords: Semigroup; Operator; Lower bound; Expansive; Quasi-hyperbolic; Approximate point spectrum; Push-forward; Anosov

1. Introduction

The notion of hyperbolicity of operators and operator semigroups is central in differentiable dynamics and the theory of differential equations. For a general view on the subject one may consult [4] and [14].

Recall that a bounded linear operator T on a Banach space X is said to be *hyperbolic* if there is a splitting $X = X_s \oplus X_u$ where X_s and X_u are closed T-invariant subspaces of X, $T\restriction_{X_u}$ is

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invertible, and $\|(T\restriction_{X_s})^n\| \leqslant \frac{1}{2}$ and $\|(T\restriction_{X_u})^{-n}\| \leqslant \frac{1}{2}$ for some $n \in \mathbb{N}$. Then X_s and X_u are the *stable* and *unstable parts* of X for T. It is a standard result that T is hyperbolic if and only if the spectrum $\sigma(T)$ of T does not meet the unit circle $\mathbb{T}$. Hyperbolicity for a C_0-semigroup $(T(t))_{t\geqslant 0}$ on X can be defined in a similar way and it is equivalent to hyperbolicity of $T(1)$; see Section 3 below.

If A is the generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$ on X then for a variety of Banach spaces $\mathcal{F}$ of X-valued functions on $\mathbb{R}$ the existence and uniqueness of mild solutions to

$$\dot{x}(t) = Ax(t) + f(t) \quad (t \in \mathbb{R}) \tag{1.1}$$

in $\mathcal{F}$ for every $f \in \mathcal{F}$ is equivalent to $(T(t))_{t\geqslant 0}$ being hyperbolic, see for example [4, Chapters 4–5] and [26, Chapter 8]. Similar results concerning discrete versions of (1.1) can be found in [4, Chapters 4–5] and [32].

The interplay between solvability of (1.1) and hyperbolicity of the associated semigroups has motivated intensive studies of hyperbolic semigroups and exponentially dichotomic evolution families and their relationship to the properties of solutions to (1.1) and fine spectral structure of the associated differential operator $-\frac{d}{dt} + A$. Recent accounts of this area include, in particular, [4, Chapters 4–5], [2,3,19,22–24,37,38].

Moreover, hyperbolic operator semigroups arise very naturally in dynamical systems theory, in particular in the study of Anosov flows and diffeomorphisms. Let M be a compact smooth Riemannian manifold, and let TM be its tangent bundle. Denote by $C(TM)$ the real Banach space of continuous sections of the tangent bundle TM with the supremum norm. With any diffeomorphism $\varphi : M \mapsto M$ one can associate a linear operator E_φ on $C(TM)$, the so-called push-forward operator, defined by

$$(E_\varphi f)(\theta) = D\varphi\big(\varphi^{-1}\theta\big) f\big(\varphi^{-1}\theta\big) \quad (\theta \in TM),$$

where $D\varphi$ stands for the differential of φ. Recall that φ is said to be Anosov if the tangent bundle TM splits as a direct sum of $D\varphi$-invariant sub-bundles $TM = T_M^s \oplus T_M^u$ such that $D\varphi$ contracts T_M^s exponentially for positive times and contracts T_M^u exponentially for negative times. Anosov diffeomorphisms constitute one of the basic classes of mappings in differentiable dynamics possessing numerous specific features such as structural stability [14].

Consider the complexification of $C(TM)$ and the corresponding extension of E_φ to the complexification, for which we shall use the same notation. Mather's famous characterisation of Anosov diffeomorphisms obtained in [29, Theorem] says that φ is Anosov if and only if $1 \notin \sigma(E_\varphi)$. As noted in [29, Lemma] (see also [4, Lemma 6.28]), for aperiodic φ the spectrum of E_φ is invariant with respect to rotations. Thus, since Anosov diffeomorphisms are aperiodic, Mather's result can be restated as in [29, Theorem]: φ is Anosov if and only if E_φ is hyperbolic.

Answering a question of Hirsch concerning invariant hyperbolic submanifolds, Mañé introduced the notion of a quasi-Anosov diffeomorphism, that is a diffeomorphism $\varphi : M \mapsto M$ such that for every $\theta \in M$ and every non-zero x from the tangent space $T_\theta M$ of M the set $\{\|D\varphi^n(\theta)x\|\colon n \in \mathbb{Z}\}$ is unbounded. It is easy to show that every Anosov diffeomorphism is quasi-Anosov but the converse does not hold in general [11]. Mañé proved in [28, Theorem A] (see also [8, Theorem]) that φ is quasi-Anosov if and only if the approximate spectrum $\sigma_{\mathrm{ap}}(E_\varphi)$ of E_φ does not contain 1. Since quasi-Anosov diffeomorphisms are aperiodic, by the rotational symmetry of $\sigma_{\mathrm{ap}}(E_\varphi)$ for such φ (see the proof of [29, Lemma] and [4, Lemma 6.28]), we infer as above that φ is quasi-Anosov if and only if $\sigma_{\mathrm{ap}}(E_\varphi)$ is disjoint from the unit circle $\mathbb{T}$. Similar results hold in the context of smooth flows on Riemannian manifolds where the notion of a quasi-Anosov flow and the associated Mather (evolution) semigroup can be defined and studied similarly to the discrete setting. We refer to [5,6,17,41] for early developments of the operator approach to the study of dynamics of flows, and to the more recent book [4, Chapters 6, 7] for more general abstract results.

The Mather–Mañé spectral theory can be extended to the broader framework of Banach space valued cocycles over flows. Let Θ be a locally compact metric space and $\{\varphi^t\}_{t\in\mathbb{R}}$ be a continuous flow on Θ. If $\{\Phi^t\}_{t\geqslant 0}$ is a strongly continuous, exponentially bounded, $\mathcal{L}(X)$-valued cocycle over $\{\varphi^t\}_{t\in\mathbb{R}}$ then one can define the corresponding Mather (or evolution) C_0-semigroup $(E_\varphi^t)_{t\geqslant 0}$ on $C_0(\Theta, X)$ by

$$\left(E_\varphi^t f\right)(\theta) = \Phi^t\left(\varphi^{-t}\theta\right) f\left(\varphi^{-t}\theta\right) \quad (\theta \in \Theta). \tag{1.2}$$

Note that in general $(E_\varphi^t)_{t\geqslant 0}$ does not extend to a group. Hyperbolicity of $(E_\varphi^t)_{t\geqslant 0}$ is equivalent to invertibility of the generator of $(E_\varphi^t)_{t\geqslant 0}$ and is also equivalent to exponential dichotomy of $\{\Phi^t\}_{t\in\mathbb{R}}$—see [4, Chapters 6 and 7] for the details of these notions and results. In particular, by [4, Theorem 6.30] if $\{\varphi^t\}_{t\in\mathbb{R}}$ is aperiodic and G_φ is the generator of $(E_\varphi^t)_{t\geqslant 0}$ then

$$\sigma_{\mathrm{ap}}\left(E_\varphi^t\right) \setminus \{0\} = e^{t\sigma_{\mathrm{ap}}(G_\varphi)} \quad (t > 0). \tag{1.3}$$

An analogous spectral mapping theorem is true for Mather semigroups $(E^t)_{t\geqslant 0}$ defined on appropriate L_p-spaces (see [4, Theorem 6.37]). The spectral theory of Mather semigroups including the relation (1.3) turned out to be crucial for the study of dynamics of various partial differential equations such as Euler's equations—see [4, Chapter 8] and the more recent papers [24,39,40].

While hyperbolic Mather semigroups are well understood and the hyperbolic dynamics of Anosov diffeomorphisms and flows translate into hyperbolic dynamics of associated discrete or continuous operator semigroups on a Banach space of continuous sections, the situation with quasi-Anosov diffeomorphisms and flows is not so explicit at least from an operator-theoretic point of view. Thus we arrive at a need to generalise the notion of hyperbolicity for linear operators. In fact Eisenberg and Hedlund [9,15] introduced a relevant class of operators, without indicating any applications.

Let T be an *invertible* bounded linear operator on a Banach space X. According to [9] the operator T is *expansive* if for each $x \in X$ there exists $n \in \mathbb{Z}$ (depending on x) such that $\|T^n x\| \geqslant 2\|x\|$; and T is *uniformly expansive* if there exists $n \in \mathbb{N}$ (independent of x) such that $\max(\|T^n x\|, \|T^{-n} x\|) \geqslant 2\|x\|$ for all $x \in X$. Uniform expansiveness of T was characterised in [15, Theorem 1] (see also [9, Theorem 2]) by the condition

$$\sigma_{\mathrm{ap}}(T) \cap \mathbb{T} = \emptyset. \tag{1.4}$$

In the case of the push-forward operator $T = E_\varphi$, (1.4) clearly coincides with Mañé's spectral characterisation of quasi-Anosov diffeomorphisms φ.

We extend this notion to bounded linear operators T on X which are not necessarily invertible and we introduce a more suggestive terminology as follows.

Definition 1.1. We shall say that T is *quasi-hyperbolic* if there exists $n \in \mathbb{N}$ (independent of x) such that $\max(\|T^{2n} x\|, \|x\|) \geqslant 2\|T^n x\|$ for all $x \in X$.

If T is quasi-hyperbolic, then there exist $a > 0$ and $c > 0$ such that $\max(\|T^{2n}x\|, \|x\|) \geqslant ce^{an}\|T^n x\|$ for all $x \in X$ and all $n \in \mathbb{N}$. Moreover, for each $x \in X$, the sequence $(\|T^n x\|)_{n\in\mathbb{N}}$ either diverges exponentially or it decays exponentially.

It is clear that any hyperbolic operator is quasi-hyperbolic. Moreover, an invertible operator is uniformly expansive if and only if it is quasi-hyperbolic. The push-forward operator associated with a quasi-Anosov diffeomorphism is a quasi-hyperbolic operator. Quasi-hyperbolic operators appear also in the theory of hyperbolic partial differential equations (see Example 2.3). Many weighted shifts are quasi-hyperbolic (see Example 2.6).

While quasi-hyperbolic operators have arisen naturally in applications, no systematic treatment of quasi-hyperbolicity has been made so far. The aim of the paper is to fill the gap in the literature and to pursue the study of quasi-hyperbolicity mainly in the context of C_0-semigroups on Banach spaces. Our approach is based on a deep extension theorem of Read and Müller for operators, a new notion of a *lower Fourier multiplier* and the techniques of evolution semigroups. The paper can be considered as an extension of [21,20,18] where hyperbolic C_0-semigroups on Banach spaces were characterised in terms of resolvents of their generators.

The paper is organised as follows. Section 2 is devoted to the study of quasi-hyperbolicity for discrete semigroups, i.e. iterates of a single operator. Quasi-hyperbolic operators are identified there as restrictions of hyperbolic operators to subspaces. This is a striking analogue of a part of [28, Theorem A] identifying quasi-Anosov diffeomorphisms with restrictions of diffeomorphisms to invariant submanifolds with hyperbolic structure. In Section 3 we characterise quasi-hyperbolic C_0-semigroups in terms of their generators and lower Fourier multipliers. In particular we prove in Corollary 3.10 that quasi-hyperbolicity of a C_0-semigroup $(T(t))_{t\geqslant 0}$ on a Hilbert space X with generator A is equivalent to the lower bound

$$\big\|(A - is)x\big\| \geqslant c\|x\| \quad \big(s \in \mathbb{R}, x \in \mathcal{D}(A)\big), \tag{1.5}$$

for some $c > 0$, where $\mathcal{D}(A)$ is the domain of A. However the simple condition (1.5) fails to characterise hyperbolicity of semigroups on general Banach spaces as our Example 3.3 shows, and the lower Fourier multiplier property is difficult to verify in Banach spaces. Thus in Section 4 we study what kind of asymptotic properties of $(T(t))_{t\geqslant 0}$ can be deduced from (1.5) if X is merely a Banach space. We show that (1.5) ensures that non-zero complete trajectories of $(T(t))_{t\geqslant 0}$ grow faster than polynomially in a pointwise sense (Theorem 4.1) and they grow exponentially in an integral norm (Theorem 4.5). It remains an open question whether they grow exponentially in a pointwise sense.

2. Quasi-hyperbolic operators

In order to give a spectral characterisation of quasi-hyperbolic operators, we recall an extension theorem. The appropriate construction was first carried out by Read in the context of Banach algebras [34]. Later Read adapted it to operators [35], while Müller independently exhibited a neat way to deduce the operator case from the algebra case (see [31, Theorem 9.22]).

Theorem 2.1 *(Read). Let T be a bounded linear operator on X. There is a Banach space Y and a bounded operator S on Y such that X is isometrically embedded in Y, $S\restriction_X = T$, $\|S\| = \|T\|$ and $\sigma(S) = \sigma_{\mathrm{ap}}(T)$.*

In the following result the equivalence of (i) and (iii) for invertible T was proved by Hedlund [15]. Read's Theorem allows us to introduce the equivalent condition (iv) which makes the properties of quasi-hyperbolic operators rather transparent. It also provides an alternative proof of the non-trivial part of Hedlund's result.

Theorem 2.2. *Let T be a bounded linear operator on X. The following are equivalent*:

(i) *T is quasi-hyperbolic.*
(ii) *There exists $c > 0$ such that $\|(T - \lambda)x\| \geqslant c\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{T}$.*
(iii) $\sigma_{\mathrm{ap}}(T) \cap \mathbb{T} = \emptyset$.
(iv) *There is a Banach space Y and a hyperbolic operator S on Y such that X is isometrically embedded in Y and $S{\restriction}_X = T$.*

In condition (iv) *one may arrange that $\|S\| = \|T\|$ and $\sigma(S) = \sigma_{\mathrm{ap}}(T)$.*

Proof. (i) ⇒ (ii). Suppose that $\max(\|T^{2n}x\|, \|x\|) \geqslant 2\|T^n x\|$ for all $x \in X$, and (ii) is false. We aim to obtain a contradiction.

There are sequences $(x_k)_{k\geqslant 1}$ in X and $(\lambda_k)_{k\geqslant 1}$ in $\mathbb{T}$ such that $\|x_k\| = 1$ for all k and $\|Tx_k - \lambda_k x_k\| \to 0$. Passing to a subsequence, we may assume that $\lambda_k \to \lambda \in \mathbb{T}$. Then $\|T^{2n}x_k - \lambda^{2n}x_k\| \to 0$ and $\|T^n x_k - \lambda^n x_k\| \to 0$, so $\|T^{2n}x_k\| \to 1$ and $\|T^n x_k\| \to 1$. But

$$\max\left(\left\|T^{2n}x_k\right\|, \|x_k\|\right) \geqslant 2\left\|T^n x_k\right\|.$$

Letting $k \to \infty$ gives a contradiction.

It is immediate from Read's Theorem that (iii) implies (iv), while (ii) ⇒ (iii) is trivial and (ii) ⇒ (i) is elementary. □

Example 2.3. The following hyperbolic equation with periodically moving boundary has been studied in [7]. Let

$$\Omega := \left\{(x,t)\colon\ 0 < x < 1 + \frac{\sin(\pi t)}{2\pi}\right\} \subseteq \mathbb{R}^2.$$

The boundary value problem

$$u_{tt} - u_{xx} = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

is well posed for data $u(\cdot,0), u_t(\cdot,0)$ in the energy space $\mathcal{H} = H_0^1(0,1) \times L_2(0,1)$, so that the monodromy operator

$$U(2,0) : \mathcal{H} \mapsto \mathcal{H},$$
$$U(2,0)\left[u(\cdot,0), u_t(\cdot,0)\right] = \left[u(\cdot,2), u_t(\cdot,2)\right],$$

is well defined. Moreover, $U(2,0)$ is invertible in $\mathcal{H}$, and by [7, p. 304 and Theorem 3.4]

$$\sigma\big(U(2,0)\big) = \left\{\lambda \in \mathbb{C}\colon\ \frac{1}{\sqrt{3}} \leqslant |\lambda| \leqslant \sqrt{3}\right\}, \qquad \sigma_{\mathrm{ap}}\big(U(2,0)\big) \cap \mathbb{T} = \emptyset. \tag{2.3}$$

By Theorem 2.2, $U(2,0)$ is quasi-hyperbolic and it follows that there exist $a>0$ and $c>0$ such that for every non-zero solution u of (2.1) one has

$$\text{either} \quad \|u(t)\|_{\mathcal{H}} \geqslant ce^{at} \quad \text{for all } t>0 \quad \text{or} \quad \|u(t)\|_{\mathcal{H}} \geqslant ce^{a|t|} \quad \text{for all } t<0.$$

Thus the energy of the system governed by (2.1), (2.2) grows exponentially in either positive or negative time for any choice of non-zero initial values. In fact, as noted in [7, p. 304],

$$\lim_{t\to\pm\infty} \frac{\log \|u(t)\|_{\mathcal{H}}}{|t|} = \frac{\ln 3}{8}.$$

General statements describing fine structure of the spectrum of monodromy operators for (2.1), (2.2) on a large class of domains Ω can be found in [7, Theorem 3.4, Theorem 2.4] and in [27, Theorem 1].

Other examples of quasi-hyperbolic operators on Hilbert space that are not hyperbolic were given in [9, Examples 4, 5] (see Example 2.6). We shall show that those examples are particular cases of a large class of quasi-hyperbolic operators.

Example 2.4. Let $w:\mathbb{Z}\mapsto\mathbb{R}$ be a (strictly) positive bounded sequence. Consider the weighted right shift operator S_w^r on $l_p(\mathbb{Z})$ $(1\leqslant p<\infty)$, defined by

$$S_w^r\big((x_n)_{n\in\mathbb{Z}}\big) = \big(w(n)x_{n-1}\big)_{n\in\mathbb{Z}}. \tag{2.4}$$

Let

$$u_n = \begin{cases} w(0)w(1)\cdots w(n-1) & (n>0),\\ 1 & (n=0),\\ (w(n)\cdots w(-1))^{-1} & (n<0). \end{cases} \tag{2.5}$$

Define

$$i^+\big(S_w^r\big) = \lim_{n\to\infty}\Big(\inf_{k>0} u_{n+k}/u_k\Big)^{1/n}, \qquad i^-\big(S_w^r\big) = \lim_{n\to\infty}\Big(\inf_{k<0} u_k/u_{k-n}\Big)^{1/n},$$
$$r^+\big(S_w^r\big) = \lim_{n\to\infty}\Big(\sup_{k>0} u_{n+k}/u_k\Big)^{1/n}, \qquad r^-\big(S_w^r\big) = \lim_{n\to\infty}\Big(\sup_{k<0} u_k/u_{k-n}\Big)^{1/n}.$$

Then the spectral radius of S_w^r is

$$r\big(S_w^r\big) = \max\big(r^-\big(S_w^r\big), r^+\big(S_w^r\big)\big),$$

and the inner spectral radius is

$$i\big(S_w^r\big) := \min\big\{|\lambda|:\ \lambda\in\sigma\big(S_w^r\big)\big\} = \min\big(i^+\big(S_w^r\big), i^-\big(S_w^r\big)\big).$$

If S_w^r is invertible then $i(S_w^r) = r((S_w^r)^{-1})^{-1}$.

The following result was proved by Ridge [36, Theorem 3] for the shifts defined on $l_2(\mathbb{Z})$. Its extension to $l_p(\mathbb{Z})$-spaces with $p \in [1, +\infty)$ presents no difficulties (see [30] and [25, Section I.6]).

Proposition 2.5 *(Ridge). Let S_w^r be the shift operator defined in* (2.4). *Then the following hold.*

(a) $\sigma(S_w^r) = \{\lambda \in \mathbb{C}\colon\, i(S_w^r) \leqslant |\lambda| \leqslant r(S_w^r)\}$.
(b) *If $r^-(S_w^r) < i^+(S_w^r)$ then*

$$\sigma_{\mathrm{ap}}(S_w) = \{\lambda \in \mathbb{C}\colon\, i^-(S_w^r) \leqslant |\lambda| \leqslant r^-(S_w^r)\} \cup \{\lambda \in \mathbb{C}\colon\, i^+(S_w^r) \leqslant |\lambda| \leqslant r^+(S_w^r)\}.$$

Otherwise,

$$\sigma_{\mathrm{ap}}(S_w^r) = \sigma(S_w^r).$$

It follows from Proposition 2.5 and Theorem 2.2 that S_w^r is quasi-hyperbolic, but not hyperbolic, if and only if $r^-(S_w^r) < 1 < i^+(S_w^r)$. In particular, if

$$\lim_{n\to\pm\infty} w(n) = w_\pm \quad \text{and} \quad w_+ > w_-,$$

then

$$\sigma_{\mathrm{ap}}(S_w^r) = \{\lambda \in \mathbb{C}\colon\, |\lambda| = w_- \text{ or } |\lambda| = w_+\},$$

and S_w^r is quasi-hyperbolic, but not hyperbolic, if and only if $w_- < 1 < w_+$.

There is a corresponding result for weighted left shifts S_w^l on $\ell_p(\mathbb{Z})$. Define the invertible isometry $J : l_p(\mathbb{Z}) \mapsto l_p(\mathbb{Z})$ by $J((x_n)_{n\in\mathbb{Z}}) = ((x_{-n})_{n\in\mathbb{Z}})$, and let $\tilde{w}(n) = w(-n)$. Then

$$S_w^l((x_n)_{n\in\mathbb{Z}}) = (w(n)x_{n+1})_{n\in\mathbb{Z}} = (J^{-1}S_{\tilde{w}}^r J)((x_n)_{n\in\mathbb{Z}}).$$

Thus the approximate point spectrum of S_w^l coincides with that of $S_{\tilde{w}}^r$ which allows a description by means of Proposition 2.5.

Examples 2.6. 1. A particular case of the weighted shift operator S_w^r with

$$w(n) = \begin{cases} 2\sqrt{2} & (n \geqslant 0), \\ \frac{1}{2\sqrt{2}} & (n < 0), \end{cases}$$

was considered in [9, Example 4].

2. We show here that [9, Example 5] can be put into the framework of weighted shifts discussed above. Let A be the annulus $\{z \in \mathbb{C}\colon\, \frac{1}{2} < |z| < 2\}$ and X be the Hilbert space of all holomorphic functions belonging to the space $L_2(A)$, where A is equipped with planar Lebesgue measure. Let $M : X \to X$ be defined by $(Mf)(z) = zf(z)$. It was shown in [13] that

$$\sigma(M) := \left\{\lambda \in \mathbb{C}\colon\, \frac{1}{2} \leqslant |\lambda| \leqslant 2\right\}, \qquad \sigma_{\mathrm{ap}}(M) := \left\{\lambda \in \mathbb{C}\colon\, |\lambda| = \frac{1}{2} \text{ or } |\lambda| = 2\right\}. \tag{2.6}$$

Thus, M is quasi-hyperbolic, by Theorem 2.2 or by a direct argument given in [9].

If $e_n := z^n/\|z^n\|$ $(n \in \mathbb{Z})$, then $\{e_n\colon n \in \mathbb{Z}\}$ is an orthonormal basis in $L_2(A)$, and moreover

$$Me_n = w_n e_{n+1}, \quad \text{where } w_n := \|z^{n+1}\| \|z^n\|^{-1} \ (n \in \mathbb{Z}).$$

Thus M can be identified with the right shift S_w^r on $l_2(\mathbb{Z})$. By an easy calculation

$$\lim_{n\to\infty} w_n = 2, \qquad \lim_{n\to-\infty} w_n = \frac{1}{2},$$

so we get (2.6) by Ridge's results (Proposition 2.5).

In the spirit of Theorem 2.2(iii) and with a view towards Section 4, given an operator T on X, we would like to characterise the property that

(HE) X can be continuously embedded in a Banach space Y such that $T = S\restriction_X$ for some quasi-hyperbolic operator S on Y.

It follows from Theorem 2.2 that we can assume that S is hyperbolic, and (HE) can be reformulated as follows:

(HE)$'$ There is a norm $\|\cdot\|'$ on X and there are constants $c > 0$ and C such that, for each $x \in X$,
- (a) $\|x\|' \leqslant C\|x\|$,
- (b) $\|Tx\|' \leqslant C\|x\|'$,
- (c) $\|(T-\lambda)x\|' \geqslant c\|x\|'$ for all $\lambda \in \mathbb{T}$.

If Y and S exist, then we can take $\|\cdot\|'$ to be the norm of Y. Conversely, if $\|\cdot\|'$ exists, then we can take Y to be the completion of $(X, \|\cdot\|')$. We aim now to limit the search to a narrow class of norms.

A necessary condition for the existence of such a pair (Y, S) is the following property of uniform exponential growth in a bilateral sense:

(UE) There exists $\alpha > 1$ such that, for each $x \in X \setminus \{0\}$ there exists $c_x > 0$ such that either
- (a) $\|T^n x\| \geqslant c_x \alpha^n$ for all $n \in \mathbb{N}$, or
- (b) $\|y\| \geqslant c_x \alpha^n$ whenever $n \in \mathbb{N}$, $y \in X$ and $T^n y = x$.

This implies each of the following two conditions concerning firstly a notion similar to that of an expansive operator, and secondly the point spectrum:

(EX) If $x \in \bigcap_{n\in\mathbb{N}} \operatorname{Ran} T^n$, then there exists $n \in \mathbb{N}$ such that either $\|T^n x\| \geqslant 2\|x\|$ or $\|y\| \geqslant 2\|x\|$ for each $y \in X$ with $T^n y = x$ (so, T is expansive if T is invertible);

(PS) $\overline{\sigma_{\mathrm{p}}(T)} \cap \mathbb{T} = \emptyset$, where $\sigma_{\mathrm{p}}(T)$ is the point spectrum of T.

There are simple examples of invertible operators on Hilbert space satisfying (EX) but not (PS) (a diagonal operator on $\ell_2(\mathbb{N})$ as in [9, Example 2]), and vice versa (an unweighted shift on $\ell_2(\mathbb{Z})$), and also an example of an invertible operator on Hilbert space satisfying both (EX) and (PS) but not (UE) (the shift S_w^r on $\ell_2(\mathbb{Z})$ where $w(n) \geqslant 1$ for all $n \in \mathbb{Z}$, $\prod_{n=0}^{\infty} w(n) = \infty$ and $\lim_{|n|\to\infty} w(n) = 1$).

Let us assume (for simplicity) that T is surjective. For $\alpha > 1$ and $x \in X$, define

$$q_{\alpha+}(x) = \inf_{n\in\mathbb{N}} \frac{\|T^n x\|}{\alpha^n}, \qquad q_{\alpha-}(x) = \inf\left\{ \frac{\|y\|}{\alpha^n}:\ y \in X,\ n \in \mathbb{N},\ T^n y = x \right\}.$$

Note that (UE) is equivalent to

$$\max\bigl(q_{\alpha+}(x), q_{\alpha-}(x)\bigr) > 0 \quad \bigl(x \in X \setminus \{0\}\bigr).$$

However, the functionals $q_{\alpha+}$ and $q_{\alpha-}$ may not satisfy the triangle inequality, so it is not clear whether (UE) implies (HE)$'$. Instead we define

$$\begin{aligned} p_{\alpha+}(x) &= \inf\left\{ \sum_{r=1}^{n} q_{\alpha+}(x_r):\ n \in \mathbb{N},\ x_r \in X,\ x = \sum_{r=1}^{n} x_r \right\}, \\ p_{\alpha-}(x) &= \inf\left\{ \sum_{r=1}^{n} q_{\alpha-}(x_r):\ n \in \mathbb{N},\ x_r \in X,\ x = \sum_{r=1}^{n} x_r \right\}, \\ p_\alpha(x) &= \max\bigl(p_{\alpha+}(x), p_{\alpha-}(x)\bigr). \end{aligned}$$

It is routine to verify that $p_{\alpha+}$, $p_{\alpha-}$ and p_α are seminorms on X, and the following inequalities hold

$$p_{\alpha+}(x) \leqslant \|x\|, \qquad p_{\alpha-}(x) \leqslant \|x\|,$$

$$p_{\alpha+}(Tx) \leqslant \|T\| p_{\alpha+}(x), \qquad p_{\alpha-}(Tx) \leqslant \|T\| p_{\alpha-}(x), \tag{2.7}$$

$$q_{\alpha+}(Tx) \geqslant \alpha q_{\alpha+}(x), \qquad p_{\alpha-}(x) \geqslant \alpha p_{\alpha-}(Tx). \tag{2.8}$$

The assumption that T is surjective ensures that

$$p_{\alpha+}(Tx) \geqslant \alpha p_{\alpha+}(x). \tag{2.9}$$

Theorem 2.7. *Let T be a bounded surjection on X. The following are equivalent*:

(i) *There exists $\alpha > 1$ such that p_α is a norm on X.*
(ii) *There exist a Banach space Y and a quasi-hyperbolic operator S on Y such that X is continuously embedded in Y and $T = S\restriction_X$.*
(iii) *There exist a Banach space Z and a hyperbolic operator U on Z such that X is continuously embedded in Z and $T = U\restriction_X$.*

Proof. It is immediate from Theorem 2.2 that (ii) implies (iii).

Suppose that (iii) holds. Each $x \in X$ can be written uniquely as $x = z_{x+} + z_{x-}$ where z_{x+} and z_{x-} are respectively in the unstable and stable parts of Z for U. Hence there are constants C, $c > 0$ and $\alpha > 1$, independent of x, such that $\|z_{x+}\|_Z + \|z_{x-}\|_Z \leqslant C\|x\|_X$ and

$\|U^n z_{x+}\|_Z \geqslant c\alpha^n \|z_{x+}\|_Z$ and $\|z\|_Z \geqslant c\alpha^n \|z_{x-}\|_Z$ if $n \geqslant 0$, $z \in Z$ and $U^n z = z_{x-}$. Moreover, $z_{(T^n x)+} = U^n z_{x+}$. Hence

$$\|T^n x\|_X \geqslant \frac{1}{C}\|U^n z_{x+}\|_Z \geqslant \frac{c\alpha^n}{C}\|z_{x+}\|_Z.$$

It follows that $q_+(x) \geqslant \frac{c}{C}\|z_{x+}\|_Z$. If $y \in X$ and $T^n y = x$, then $U^n z_{y-} = z_{x-}$ and

$$\|y\|_X \geqslant \frac{1}{C}\|z_{y-}\|_Z \geqslant \frac{c\alpha^n}{C}\|z_{x-}\|_Z.$$

It follows that $q_-(x) \geqslant \frac{c}{C}\|z_{x-}\|_Z$.

If $x = \sum_{r=1}^n x_r$, then

$$\sum_{r=1}^n q_+(x_r) \geqslant \frac{c}{C}\sum_{r=1}^n \|z_{x_r+}\|_Z \geqslant \frac{c}{C}\|z_{x+}\|_Z.$$

Hence $p_+(x) \geqslant \frac{c}{C}\|z_{x+}\|_Z$. Similarly, $p_-(x) \geqslant \frac{c}{C}\|z_{x-}\|_X$. If $x \neq 0$, then either $z_{x+} \neq 0$ or $z_{x-} \neq 0$, and it follows that $p_+(x) \neq 0$ or $p_-(x) \neq 0$. This establishes (i).

Now suppose that (i) holds. Let Y be the completion of (X, p_α). It follows from (2.7) that T extends to a bounded linear operator S on Y.

Let $x \in X$. For $n \geqslant 0$, it follows from (2.9) and (2.8) that

$$\|S^{2n}x\|_Y = p_\alpha(T^{2n}x) \geqslant p_{\alpha+}(T^{2n}x) \geqslant \alpha^n p_{\alpha+}(T^n x),$$
$$\|x\|_Y = p_\alpha(x) \geqslant p_{\alpha-}(x) \geqslant \alpha^n p_{\alpha-}(T^n x).$$

Thus,

$$\max(\|S^{2n}x\|_Y, \|x\|_Y) \geqslant \alpha^n p_\alpha(T^n x) = \alpha^n \|S^n x\|_Y.$$

It now follows by density of X in Y that S is quasi-hyperbolic on Y. □

3. Quasi-hyperbolic semigroups

Now we consider the corresponding notions for a C_0-semigroup $\mathcal{T} := (T(t))_{t\geqslant 0}$ on a Banach space X. It is standard terminology that $\mathcal{T}$ is *hyperbolic* if there is a splitting $X = X_s \oplus X_u$ where X_s and X_u are closed $\mathcal{T}$-invariant subspaces of X, $T(t)\restriction_{X_u}$ is invertible for some (or all) $t > 0$, and $\|T(t)\restriction_{X_s}\| < 1$ and $\|(T(t)\restriction_{X_u})^{-1}\| < 1$ for some $t > 0$.

Definition 3.1. We say that $\mathcal{T} = (T(t))_{t\geqslant 0}$ is *quasi-hyperbolic* if there exists $t > 0$ (independent of x) such that $\max(\|T(2t)x\|, \|x\|) \geqslant 2\|T(t)x\|$ for all $x \in X$.

It is easy to see that $\mathcal{T}$ is quasi-hyperbolic or hyperbolic if and only if $T(1)$ is quasi-hyperbolic or hyperbolic, respectively. If $\mathcal{T}$ is quasi-hyperbolic, then, by Theorem 2.2, $T(1)$ can be extended to a hyperbolic operator $S(1)$ on a larger space Y. It can be seen from the construction in [35], or more easily from that in the proof of [31, Theorem 9.22], that each operator $T(t)$ also extends to an operator $S(t)$ on Y with $\|S(t)\| = \|T(t)\|$ and $S(t_1)S(t_2) = S(t_1 + t_2)$. Replacing

Y if necessary by the maximal subspace on which S is strongly continuous, we can arrange that $(S(t))_{t\geqslant 0}$ is a C_0-semigroup on Y, and then it is hyperbolic. Hence quasi-hyperbolic and hyperbolic C_0-semigroups can be characterised in terms of spectral properties of $T(1)$ as in Section 2. It would be more useful to have results involving spectral properties of the generator A of $\mathcal{T}$, but the failure of the spectral mapping theorem for C_0-semigroups means that this is not straightforward.

The natural analogue of Theorem 2.2(ii) for A is the lower bound (1.5). One simple general result is the following.

Proposition 3.2. *Let A be the generator of a quasi-hyperbolic C_0-semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X. Then* (1.5) *holds.*

Proof. Let t be such that $\max(\|T(2t)x\|, \|x\|) \geqslant 2\|T(t)x\|$ for all $x \in X$. For $x \in \mathcal{D}(A)$ and $s \in \mathbb{R}$,

$$\int_0^t e^{-is\tau} T(\tau)(A - is)x \, d\tau = e^{-ist} T(t)x - x. \tag{3.1}$$

Hence

$$tM_t \big\|(A - is)x\big\| \geqslant \big|\big\|T(t)x\big\| - \|x\|\big|,$$

where $M_t = \sup_{|\tau|\leqslant t} \|T(\tau)\|$. Replacing x by $T(t)x$ in (3.1),

$$tM_{2t} \big\|(A - is)x\big\| \geqslant \big|\big\|T(2t)x\big\| - \big\|T(t)x\big\|\big|.$$

Let $c = (2tM_{2t})^{-1}$. Then

$$\begin{aligned}
\big\|(A - is)x\big\| &\geqslant 2c \max\big(\big|\big\|T(t)x\big\| - \|x\|\big|, \big|\big\|T(2t)x\big\| - \big\|T(t)x\big\|\big|\big) \\
&\geqslant 2c\big\|T(t)x\big\| \\
&\geqslant 2c\big(\|x\| - \big|\big\|T(t)x\big\| - \|x\|\big|\big) \\
&\geqslant 2c\|x\| - \big\|(A - is)x\big\|. \qquad \square
\end{aligned}$$

The converse of Proposition 3.2 does not hold, even for a C_0-group of positive operators on a reflexive Banach lattice. This is shown by the following example of a C_0-group with bad spectral properties, adapted from an example originally due to Wolff [43] (see [12, pp. 102–103]).

Example 3.3. Let X be the reflexive Banach space $L_p(\mathbb{R}, e^{2x}\, dx) \cap L_q(\mathbb{R}, w(x)\, dx)$, where $1 < p < 2 < q < \infty$, and

$$w(x) = \begin{cases} e^{ax} & (x \leqslant 0), \\ 1 & (x > 0), \end{cases}$$

for some $a > 2q/p$. Here,

$$\|f\|_X = \left(\int_{\mathbb{R}} |f(x)|^p e^{2x}\, dx\right)^{1/p} + \left(\int_{\mathbb{R}} |f(x)|^q w(x)\, dx\right)^{1/q}.$$

Let $(T(t)f)(s) = f(s+t)$ $(s, t \in \mathbb{R})$. Then $(T(t))_{t\in\mathbb{R}}$ is a C_0-group on X, $\|T(t)\| = 1$ $(t \geqslant 0)$, $\sigma(A) \cap i\mathbb{R} = \emptyset$ and $\sup_{s\in\mathbb{R}} \|(is - A)^{-1}\| < \infty$ [12]. Hence, (LB) holds. However, for $t > 0$, there exists $f \in X$ such that $\|f\|_X = 1$ and $\|T(-t)f\|_X < 2\|f\|_X$ (because $\|T(t)\| > 1/2$). Moreover $\|T(t)f\|_X < 2\|f\|_X$ (because $\|T(t)\| < 2$). So $T(t)$ is not quasi-hyperbolic.

Note that if $Y = L_p(\mathbb{R}, e^{2x}\, dx)$ then X is continuously and densely embedded in Y, and the C_0-group $(S(t))_{t\in\mathbb{R}}$ of shifts on Y is hyperbolic with trivial unstable part because $\|S(t)\| = e^{-2t/p}$ $(t \in \mathbb{R})$. Moreover, $S(t)\restriction_X = T(t)$. Thus the continuous-parameter analogue of (HE) holds in this example.

We shall now characterise quasi-hyperbolicity of C_0-semigroups in terms of Fourier multiplier properties of linear pencils induced by semigroup generators. In particular this will establish that the converse of Proposition 3.2 holds on Hilbert spaces (Corollary 3.10).

Let A be a densely defined, closed linear operator on a Banach space X. By $S(\mathbb{R}, X)$ denote the space of X-valued Schwartz functions. For $\varphi \in \mathcal{S}(\mathbb{R})$ and $x \in \mathcal{D}(A)$, write $(\varphi \otimes x)(t) = \varphi(t)x$ $(t \in \mathbb{R})$. Let $S_A(\mathbb{R}, X) := \operatorname{lin}\{\varphi \otimes x\colon x \in \mathcal{D}(A),\ \varphi \in \mathcal{S}(\mathbb{R})\}$ and $\mathcal{F}$ be the Fourier transform on $S(\mathbb{R}, X)$. Note that $S_A(\mathbb{R}, X)$ is dense in $L_p(\mathbb{R}, X)$ for every $p \in [1, +\infty)$.

Definition 3.4. The linear operator

$$M_{A-i\cdot} : S_A(\mathbb{R}, X) \mapsto S(\mathbb{R}, X),$$
$$(M_{A-i\cdot} f)(s) = (A - is)f(s) \quad (s \in \mathbb{R}),$$

is said to be *a lower $L_p(\mathbb{R}, X)$-Fourier multiplier* if there exists $c > 0$ such that

$$\left\|\mathcal{F}^{-1} M_{A-i\cdot} \mathcal{F} f\right\|_{L_p} \geqslant c\|f\|_{L_p} \tag{3.2}$$

for every $f \in S_A(\mathbb{R}, X)$. Note that

$$\left(\mathcal{F}^{-1} M_{A-i\cdot} \mathcal{F} f\right)(s) = Af(s) - f'(s). \tag{3.3}$$

Remark 3.5. If $i\mathbb{R} \subset \rho(A)$ and $R(\lambda, A) = (\lambda - A)^{-1}$ denotes the resolvent of A then (3.2) is equivalent to the fact that the operator $M_{R(i\cdot,A)}$, defined on $S_A(\mathbb{R}, X)$ by

$$(M_{R(i\cdot,A)} f)(s) = R(is, A)f(s) \quad (s \in \mathbb{R}),$$

is an $L_p(\mathbb{R}, X)$-Fourier multiplier, i.e. that $R(i\cdot, A)$ is bounded from $\mathbb{R}$ to $\mathcal{L}(X)$ and $\mathcal{F}^{-1} M_{R(i\cdot,A)} \mathcal{F}$ extends to a bounded linear operator on $L_p(\mathbb{R}, X)$. To see this, it is sufficient to show that (3.2) implies the boundedness of $R(i\cdot, A)$—then the claim is apparent.

Let $x \in \mathcal{D}(A)$ and $s \in \mathbb{R}$. Choose $\varphi \in \mathcal{S}(\mathbb{R})$ with $\|\varphi\|_{L_p} = 1$, and let $\varphi_n(t) = e^{-ist} n^{-1/p} \varphi(t/n)$ ($n \in \mathbb{N}$). Then (3.3) and (3.2) imply that

$$\left\| -\varphi_n' \otimes x + \varphi_n \otimes Ax \right\|_{L_p} = \left\| \mathcal{F}^{-1}(A - i\cdot)\mathcal{F}(\varphi_n \otimes x) \right\|_{L_p} \geqslant c\|x\|.$$

By passing to the limit in the above inequality one gets

$$\left\| (A - is)x \right\| \geqslant c\|x\| \quad \big(x \in \mathcal{D}(A),\ s \in \mathbb{R}\big).$$

Thus, $R(i\cdot, A) \in L_\infty(\mathbb{R}, \mathcal{L}(X))$ and (3.2) implies the claim.

It is well known that $M_{R(i\cdot,A)}$ is an $L_p(\mathbb{R}, X)$-Fourier multiplier if and only if $(T(t))_{t\geqslant 0}$ is hyperbolic, see [21, Theorem 2.7].

Our approach to the study of lower Fourier multipliers will be based on evolution semigroups (see [4]). The same technique was employed for the study of Fourier multiplier properties of resolvents in [20] (see also [21]). Recall that given a strongly continuous evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ on a Banach space X, the associated evolution semigroup $(E(t))_{t\geqslant 0}$ on $L_p(\mathbb{R}, X)$ $(1 \leqslant p < \infty)$ is defined by

$$\big(E(t)f\big)(s) = U(s, s-t)f(s-t) \quad \big(f \in L_p(\mathbb{R}, X),\ t \geqslant 0,\ s \in \mathbb{R}\big). \tag{3.4}$$

A special case occurs when $(U(t,s))_{t\geqslant s\geqslant 0}$ is induced by a C_0-semigroup $(T(t))_{t\geqslant 0}$ on X with generator A (see [4, Section 2.2]). Then (3.4) takes the form

$$\big(E(t)f\big)(s) = T(t)f(s-t) \quad \big(f \in L_p(\mathbb{R}, X),\ t \geqslant 0,\ s \in \mathbb{R}\big).$$

In this case, $(E(t))_{t\geqslant 0}$ is the product of two commuting semigroups on $L_p(\mathbb{R}, X)$: the multiplication semigroup $(M(t))_{t\geqslant 0}$, $(M(t)f)(s) = T(t)f(s)$, and the shift semigroup $(S(t))_{t\geqslant 0}$, $(S(t)f)(s) = f(s-t)$, with the generators D, $Df = -\frac{df}{dt}$, and M_A, $M_A f = Af(s)$, respectively defined on their *maximal* domains. Thus, by well-known criteria, the generator G of $(E(t))_{t\geqslant 0}$ can be identified as the closure of the sum $D + M_A$ defined on $I := \mathcal{D}(D) \cap \mathcal{D}(M_A)$. In other words, I is a core for G.

In the next lemma we diagonalise G on an appropriate domain contained in I.

Lemma 3.6. *Let G be the generator of the evolution semigroup $(E(t))_{t\geqslant 0}$ on $L_p(\mathbb{R}, X)$, associated with an evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ on X, where $1 \leqslant p < \infty$. Then*

$$G = \overline{\mathcal{F}^{-1} M_{A-i\cdot} \mathcal{F}\restriction_{\mathcal{S}_A}}. \tag{3.5}$$

Proof. Observe that $\mathcal{S}_A(\mathbb{R}; X)$ is dense in $\mathcal{D}(G)$ and is $(E(t))_{t\geqslant 0}$-invariant. Thus, $\mathcal{S}_A(\mathbb{R}, X)$ is a core for G, i.e. $\overline{G\restriction_{\mathcal{S}_A}} = G$. On the other hand, by (3.3) for every $f \in \mathcal{S}_A(\mathbb{R}, X)$,

$$Gf = \mathcal{F}^{-1} M_{A-i\cdot} \mathcal{F} f. \qquad \square \tag{3.6}$$

We shall now discuss the relation between the spectral properties of G, $E(1)$ and certain weighted shift operators. Recall that

$$M_\xi^{-1} G M_\xi = -i\xi + G, \qquad M_\xi^{-1} E(1) M_\xi = e^{-i\xi} E(1) \quad (\xi \in \mathbb{R}), \tag{3.7}$$

where $(M_\xi f)(s) := e^{i\xi s} f(s)$. Hence the spectrum and the approximate point spectrum of G are invariant under vertical translations, and the spectrum and the approximate point spectrum of $E(1)$ are rotationally invariant (see [4, p. 65]). Moreover, the following are equivalent:

(G1) $\sigma_{\mathrm{ap}}(G) \cap i\mathbb{R} = \emptyset$,
(G2) G is bounded from below, i.e.

$$\|Gf\|_{L_p} \geqslant c\|f\|_{L_p} \quad \big(f \in \mathcal{D}(G)\big), \tag{3.8}$$

for some $c > 0$,
(G3) G satisfies (1.5), i.e.

$$\big\|(G - is)f\big\|_{L_p} \geqslant c\|f\|_{L_p} \quad \big(f \in \mathcal{D}(G),\ s \in \mathbb{R}\big),$$

for some $c > 0$.

For a bounded sequence of bounded linear operators $(W_n)_{n\in\mathbb{Z}}$ on X, denote by $S^r_{\{W_n\}}$ the associated right shift operator on $\ell_p(\mathbb{Z}, X)$ defined by

$$S^r_{\{W_n\}}\big((x_n)_{n\in\mathbb{Z}}\big) = (W_n x_{n-1})_{n\in\mathbb{Z}} \quad \big((x_n)_{n\in\mathbb{Z}} \in l_p(\mathbb{Z}, X)\big).$$

As in (3.7),

$$M_\xi^{-1} S^r_{\{W_n\}} M_\xi = e^{-i\xi} S^r_{\{W_n\}} \quad (\xi \in \mathbb{R}),$$

where $M_\xi((x_n)_{n\in\mathbb{Z}}) := (e^{in\xi} x_n)_{n\in\mathbb{Z}}$. Hence, the spectrum and the approximate point spectrum of $S^r_{\{W_n\}}$ are rotationally invariant.

When the weight $\{W_n\}$ does not depend on n and $W_n = W$ for all $n \in \mathbb{Z}$, the corresponding shift operator $S^r_{\{W_n\}}$ will be denoted by S^r_W. We can relate the approximate point spectrum of S^r_W to the approximate point spectrum of its weight W as in the following lemma. By rotation-invariance and a scaling argument, this lemma shows that $\sigma_{\mathrm{ap}}(S^r_W)$ is determined by $\sigma_{\mathrm{ap}}(W)$.

Lemma 3.7. *Let $W \in \mathcal{L}(X)$ and $p \in [1, \infty)$. Then*

(a) $\sigma(W) \cap \mathbb{T} = \emptyset \Leftrightarrow \sigma(S^r_W) \cap \mathbb{T} = \emptyset$;
(b) $\sigma_{\mathrm{ap}}(W) \cap \mathbb{T} = \emptyset \Leftrightarrow \sigma_{\mathrm{ap}}(S^r_W) \cap \mathbb{T} = \emptyset$.

Proof. The statement (a) and the implication $\Leftarrow$ in (b) are proved in [4, Lemma 2.37]. Thus, it remains to show the implication $\Rightarrow$ in (b).

Assume that

$$\sigma_{\mathrm{ap}}(W) \cap \mathbb{T} = \emptyset,$$

i.e. W is quasi-hyperbolic. Let V be a hyperbolic extension of W to a Banach space $Y \supset X$ provided by Theorem 2.2. Consider the shift operator S_V^r on $l_p(\mathbb{Z}, X)$ associated with the operator V. Since $\sigma(V) \cap \mathbb{T} = \emptyset$, we have

$$\sigma\left(S_V^r\right) \cap \mathbb{T} = \emptyset$$

by the property (a). Moreover,

$$S_V^r x = S_W^r x \quad \left(x \in l_p(\mathbb{Z}, X)\right).$$

Thus, S_V^r is a hyperbolic extension of S_W^r to $l_p(\mathbb{Z}, Y) \supset l_p(\mathbb{Z}, X)$, so

$$\sigma_{\mathrm{ap}}\left(S_W^r\right) \cap \mathbb{T} = \emptyset$$

by Theorem 2.2. □

Now we return to our consideration of an evolution family $(U(t,s))_{t \geqslant s \geqslant 0}$, and we recall the relations between the approximate spectra of the functional operator $E(1)$, the difference operator $S^r_{\{U(n,n-1)\}}$ and the differential operator G.

Lemma 3.8. *Let $(E(t))_{t \geqslant 0}$ be an evolution semigroup on $L_p(\mathbb{R}, X)$ associated with an evolution family $(U(t,s))_{t \geqslant s \geqslant 0}$, and let G be the generator of $(E(t))_{t \geqslant 0}$. Then the following properties are equivalent.*

(i) $\sigma_{\mathrm{ap}}(E(1)) \cap \mathbb{T} = \emptyset$,
(ii) $\sigma_{\mathrm{ap}}(S^r_{\{U(n,n-1)\}}) \cap \mathbb{T} = \emptyset$,
(iii) $\sigma_{\mathrm{ap}}(G) \cap i\mathbb{R} = \emptyset$.

The equivalence (i) ⇔ (iii) follows from the spectral mapping theorem for approximate point spectrum satisfied by evolution semigroups (see, for example, [4, Lemma 2.41, Theorem 3.13, p. 83] or [33, Proposition 2.2]). For the proof of the equivalence (ii) ⇔ (iii) see [42, Theorem 2], [22, Theorem 1.2 and p. 1004] or [23, Theorem 1.4]. (In [22] and [23] the equivalence was proved in a more general context of closed ranges.)

The following theorem and its corollary provide a characterisation of quasi-hyperbolic semigroups on X in terms of lower Fourier multipliers. Their proofs are based on a combination of spectral properties of shifts given in Lemmas 3.7 and 3.8. As one may expect, such a characterisation becomes especially transparent when X is a Hilbert space.

Theorem 3.9. *Let $(T(t))_{t \geqslant 0}$ be a C_0-semigroup on a Banach space X with generator A. Then $(T(t))_{t \geqslant 0}$ is quasi-hyperbolic if and only if $M_{A-i\cdot}$ is a lower $L_p(\mathbb{R}, X)$-Fourier multiplier for all/some $p \in [1, +\infty)$.*

Proof. Let $p \in [1, \infty)$, $(E(t))_{t \geqslant 0}$ be the evolution semigroup on $L_p(\mathbb{R}, X)$ associated with the semigroup $(T(t))_{t \geqslant 0}$, G be the generator of $(E(t))_{t \geqslant 0}$, and $S^r_{T(1)}$ be the (right) shift on $l_p(\mathbb{Z}, X)$ associated with the operator $T(1)$.

From Lemma 3.6 one easily gets that $M_{A-i\cdot}$ is a lower $L_p(\mathbb{R}, X)$-Fourier multiplier if and only if (3.8) holds. Now (3.8) is equivalent to (G1), and Lemma 3.8 shows that (G1) is equivalent to

$$\sigma_{\mathrm{ap}}\big(S^r_{T(1)}\big) \cap \mathbb{T} = \emptyset. \tag{3.9}$$

In turn, by Lemma 3.7, (3.9) is equivalent to

$$\sigma_{\mathrm{ap}}\big(T(1)\big) \cap \mathbb{T} = \emptyset,$$

i.e. $T(1)$ and then $(T(t))_{t\geqslant 0}$ are quasi-hyperbolic. □

If X is a Hilbert space, then the Fourier transform is a unitary operator on $L_2(\mathbb{R}, X)$, and the following corollary of Theorem 3.9 is almost immediate.

Corollary 3.10. *Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on a Hilbert space X with generator A. Then $(T(t))_{t\geqslant 0}$ is quasi-hyperbolic if and only if* (1.5) *holds.*

A description of $\sigma_{\mathrm{ap}}(T(t))$ $(t > 0)$ can also be given in terms of lower $L_p(\mathbb{T})$-Fourier multipliers, in a similar way to [4, Section 2.2.1]. By rescaling it suffices to characterise when $1 \in \sigma_{\mathrm{ap}}(T(2\pi))$. Define

$$P(\mathbb{T}, X) := \left\{ \sum_{k=-n}^{n} \xi^k x_k \colon \xi \in \mathbb{T},\ x_k \in X,\ n \in \mathbb{N} \right\},$$
$$P_A(\mathbb{T}, X) := \left\{ \sum_{k=-n}^{n} \xi^k x_k \colon \xi \in \mathbb{T},\ x_k \in \mathcal{D}(A),\ n \in \mathbb{N} \right\}.$$

Definition 3.11. The linear operator

$$M_{A-i\cdot} : P_A(\mathbb{T}, X) \mapsto P(\mathbb{T}, X),$$
$$M_{A-i\cdot}\left(\sum_{k=-n}^{n} \xi^k x_k \right) := \sum_{k=-n}^{n} \xi^k (A - ik) x_k,$$

is said to be a *lower $L_p(\mathbb{T}, X)$-Fourier multiplier*, where $p \in [1, \infty)$, if there exists $c > 0$ such that

$$\left\| \sum_{k=-n}^{n} \xi^k (A - ik) x_k \right\|_{L_p(\mathbb{T}, X)} \geqslant c \left\| \sum_{k=-n}^{n} \xi^k x_k \right\|_{L_p(\mathbb{T}, X)} \tag{3.10}$$

for all $n \in \mathbb{N}$, $x_k \in \mathcal{D}(A)$.

The following counterpart of Theorem 3.9 holds.

Theorem 3.12. *Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on a Banach space X with generator A. Then $1 \notin \sigma_{\mathrm{ap}}(T(2\pi))$ if and only if $M_{A-i\cdot}$ is a lower $L_p(\mathbb{T}, X)$-Fourier multiplier for all/some $p \in [1, +\infty)$.*

The proof follows the same lines as the proof of Theorem 3.9, so we give only a sketch.

Given a C_0-semigroup $(T(t))_{t\geqslant 0}$ on X define an evolution semigroup $(E_{\mathrm{per}}(t))_{t\geqslant 0}$ on $L_p([0, 2\pi], X)$ by the formula

$$\big(E_{\mathrm{per}}(t)f\big)(s) = T(t)f\big([s-t](\operatorname{mod} 2\pi)\big) \quad \big(s \in [0, 2\pi]\big). \tag{3.11}$$

Let G_{per} be the generator of $(E_{\mathrm{per}}(t))_{t\geqslant 0}$. Note that as in the case of evolution semigroups on $L_p(\mathbb{R}, X)$ one has

$$1 \in \sigma_{\mathrm{ap}}\big(E_{\mathrm{per}}(2\pi)\big) \quad \Longleftrightarrow \quad 0 \in \sigma_{\mathrm{ap}}(G_{\mathrm{per}})$$

by the spectral mapping theorem for the approximate point spectrum of evolution semigroups [4, Lemma 2.29]. Since

$$\begin{aligned} &\big(E_{\mathrm{per}}(2\pi)f\big)(t) = T(2\pi)f(t) \quad \big(t \in [0, 2\pi]\big), \\ &1 \in \sigma_{\mathrm{ap}}\big(E_{\mathrm{per}}(2\pi)\big) \quad \Longleftrightarrow \quad 1 \in \sigma_{\mathrm{ap}}\big(T(2\pi)\big). \end{aligned}$$

(Unlike the case of multipliers on $L_p(\mathbb{R}, X)$ above, there is no need to invoke shift operators on $\ell_p(\mathbb{Z}, X)$ here.) Finally the property of $M_{A-i\cdot}$ being a lower $L_p(\mathbb{T}, X)$-Fourier multiplier is equivalent to the boundedness from below of G_{per} since

$$\big(\mathcal{F}^{-1}G_{\mathrm{per}}\mathcal{F}f\big)(\xi) = \sum_{k=-n}^{n} (A - ik)x_k\xi^k \quad (\xi \in \mathbb{T}),$$

for $f \in P_A(\mathbb{T}, X)$, $f(\xi) = \sum_{k=-n}^{n} x_k\xi^k$.

Remark 3.13. Theorems 3.9 and 3.12 remain valid if the L_p-spaces are replaced by $C_0(\mathbb{R}, X)$ and $C(\mathbb{T}, X)$ respectively. The definitions of lower Fourier multipliers on these spaces, and the proofs, remain valid, mutatis mutandis.

In the following we describe a class of natural examples of quasi-hyperbolic C_0-semigroups.

Example 3.14. Let $w : \mathbb{R} \mapsto \mathbb{R}$ be a (strictly) positive continuous function such that

$$\frac{w(s)}{w(s-t)} \leqslant ce^{\omega t} \quad (s \in \mathbb{R},\ t \geqslant 0),$$

for some $c, \omega \in \mathbb{R}$. Define a weighted right shift C_0-semigroup $(T(t))_{t\geqslant 0}$ on $L_p(\mathbb{R})$ $(1 \leqslant p < \infty)$, by the formula

$$\big(T(t)f\big)(s) = \frac{w(s)}{w(s-t)} f(s-t).$$

Note that $(T(t))_{t\geqslant 0}$ is an evolution semigroup on $L_p(\mathbb{R})$ associated with the strongly continuous evolution family $(U_w(t,s))_{t\geqslant s}\subset\mathcal{L}(\mathbb{C})$, given by $U_w(t,s)=w(t)/w(s)$ $(t\geqslant s)$. Then by Lemma 3.8

$$\sigma_{\mathrm{ap}}\big(T(1)\big)\cap\mathbb{T}=\emptyset \quad\Longleftrightarrow\quad \sigma_{\mathrm{ap}}\big(S^r_{w^*}\big)\cap\mathbb{T}=\emptyset,$$

where $S^r_{w^*}$ is the right shift operator on $\ell_p(\mathbb{Z})$ $(1\leqslant p<\infty)$ associated with the weight $w^*(n):=w(n)/w(n-1)$, i.e.,

$$S^r_{w^*}\big((x_n)_{n\in\mathbb{Z}}\big)=\left(\frac{w(n)}{w(n-1)}x_{n-1}\right)_{n\in\mathbb{Z}}.$$

Hence quasi-hyperbolicity of $T(1)$ is equivalent to the quasi-hyperbolicity of the operator $S^r_{w^*}$, and that is described completely by Proposition 2.5. Thus the class of weighted shift semigroups provides a variety of examples of quasi-hyperbolic semigroups. For instance, for w such that

$$\lim_{n\to\pm\infty}\frac{w(n)}{w(n-1)}=w_\pm \quad\text{and}\quad w_+>w_-, \tag{3.12}$$

we have

$$\sigma_{\mathrm{ap}}\big(T(1)\big)=\sigma_{\mathrm{ap}}\big(S^r_w\big)=\big\{\lambda\in\mathbb{C}\colon\ |\lambda|=w_-\ \text{or}\ |\lambda|=w_+\big\}.$$

Thus $(T(t))_{t\geqslant 0}$ is quasi-hyperbolic, but not hyperbolic, if and only if $w_-<1<w_+$. Recall also that under the assumptions (3.12),

$$\sigma\big(T(1)\big)=\big\{\lambda\in\mathbb{C}\colon\ w_-\leqslant|\lambda|\leqslant w_+\big\}.$$

4. Weaker properties

Since the lower bounds (1.5) for $A-is$ do not imply quasi-hyperbolicity of $T(t)$ in general, it is of interest to find weaker properties of $T(t)$ which are implied by lower bounds. The spectral mapping theorem for the point spectrum of a semigroup [10, Theorem IV.3.7] shows that (1.5) implies that $T(t)$ satisfies (PS), and we shall see in Corollary 4.2 that it also implies (EX). A reasonable question is whether (1.5) implies the analogue of (HE) that there is a Banach space Y in which X is continuously embedded and a (quasi-)hyperbolic C_0-semigroup $(S(t))_{t\geqslant 0}$ on Y with $T(t)=S(t)\restriction_X$. A positive answer to that question would imply the analogue of (UE) so that, for each non-zero x, either $\|T(t)x\|$ increases at an exponential rate in t or $\inf\{\|y\|\colon y\in X,\ T(t)y=x\}$ increases at an exponential rate in t. We are unable to show that is the case, but we shall give two slightly weaker results in Corollary 4.2 and Theorem 4.5.

In this section, $\mathcal{F}$ denotes the Fourier transform on $L_1(\mathbb{R})$.

Theorem 4.1. *Let $\mathcal{T}=(T(t))_{t\geqslant 0}$ be a C_0-semigroup with generator A, and assume that the lower bound* (1.5) *holds for some $c>0$. For each $n\geqslant 1$ and $a>0$, there exists α_n (depending on n, a and $\mathcal{T}$) such that*

$$\frac{\|T(t_2)x\|}{t_2^n} + \frac{\|y\|}{t_1^n} \geqslant \frac{c^n}{n!}\left\| \int_0^a (\mathcal{F}f)(\tau)T(\tau)x\,dt \right\| - \alpha_n\|x\|\left(\frac{1}{t_1} + \frac{1}{t_2}\right) \tag{4.1}$$

whenever $t_1, t_2 > a$, $x, y \in X$, $T(t_1)y = x$, $f \in L_1(\mathbb{R})$, $\|f\|_{L_1} = 1$ *and* $\operatorname{supp}(\mathcal{F}f) \subseteq [0, a]$.

Proof. Let $n \geqslant 1$ and $a > 1$ be fixed, and let t_1, t_2, x and y be as in the statement of the theorem. Let

$$g(\tau) = \begin{cases} (1 + \frac{\tau}{t_1})^n & (-t_1 \leqslant \tau < 0), \\ (1 - \frac{\tau}{t_2})^n & (0 \leqslant \tau \leqslant t_2). \end{cases}$$

Let $s \in \mathbb{R}$ and

$$y_r = \int_{-t_1}^{t_2} g^{(r)}(\tau)e^{-is\tau}T(\tau + t_1)y\,d\tau \quad (r = 0, 1, \ldots, n).$$

Then $y_r \in \mathcal{D}(A)$ and

$$(A - is)y_r = y_{r+1} + \big(g^{(r)}(0-) - g^{(r)}(0+)\big)x \quad (r = 0, 1, \ldots, n-1),$$
$$(A - is)y_n = g^{(n)}(0+)e^{-ist_2}T(t_2)x - g^{(n)}(0-)e^{ist_1}y + \big(g^{(n)}(0-) - g^{(n)}(0+)\big)x.$$

By the assumption (1.5),

$$\|y_{r+1}\| \geqslant c\|y_r\| - \|x\|\big|g^{(r)}(0-) - g^{(r)}(0+)\big| \quad (r = 0, 1, \ldots, n-1),$$
$$\big\|g^{(n)}(0+)e^{-ist_2}T(t_2)x - g^{(n)}(0-)e^{ist_1}y\big\| \geqslant c\|y_n\| - \|x\|\big|g^{(n)}(0-) - g^{(n)}(0+)\big|.$$

Noting that $g^{(r)}(0-) = \frac{n!}{(n-r)!t_1^r}$ and $g^{(r)}(0+) = \frac{(-1)^r n!}{(n-r)!t_2^r}$, and in particular $g(0-) = g(0+)$, we obtain

$$\begin{aligned} \frac{\|T(t_2)x\|}{t_2^n} + \frac{\|y\|}{t_1^n} &\geqslant \frac{c}{n!}\|y_n\| - \|x\|\left(\frac{1}{t_1^n} + \frac{1}{t_2^n}\right) \\ &\geqslant \frac{c^2}{n!}\|y_{n-1}\| - \|x\|\left(\frac{n}{t_1^{n-1}} + \frac{n}{t_2^{n-1}} + \frac{1}{t_1^n} + \frac{1}{t_2^n}\right) \\ &\vdots \\ &\geqslant \frac{c^n}{n!}\|y_0\| - \|x\|\sum_{r=1}^n \frac{n!}{(n-r)!}\left(\frac{1}{t_1^r} + \frac{1}{t_2^r}\right). \end{aligned}$$

Now let $f \in L_1(\mathbb{R})$ with $\|f\|_{L_1} = 1$ and $\operatorname{supp}(\mathcal{F}f) \subseteq [0, a]$. Then

$$\frac{c^n}{n!}\left\|\int_0^a g(\tau)(\mathcal{F}f)(\tau)T(\tau)x\,dt\right\| = \frac{c^n}{n!}\left\|\int_{\mathbb{R}} f(s)\int_{-t_1}^{t_2} g(\tau)e^{-is\tau}T(\tau+t_1)y\,d\tau\,ds\right\|$$
$$\leqslant \frac{\|T(t_2)x\|}{t_2^n} + \frac{\|y\|}{t_1^n} + \|x\|\sum_{r=1}^n \frac{n!}{(n-r)!}\left(\frac{1}{t_1^r}+\frac{1}{t_2^r}\right).$$

However, for $\tau > 0$,

$$g(\tau) = 1 + \sum_{r=1}^n \frac{(-1)^r n!}{r!(n-r)!t_2^r}\tau^r.$$

Hence

$$\frac{c^n}{n!}\left\|\int_0^a (\mathcal{F}f)(\tau)T(\tau)x\,dt\right\|$$
$$\leqslant \frac{c^n}{n!}\left\|\int_0^a g(\tau)(\mathcal{F}f)(\tau)T(\tau)x\,dt\right\| + \sum_{r=1}^n \frac{n!}{r!(n-r)!}\left\|\int_0^a \tau^r(\mathcal{F}f)(\tau)T(\tau)x\,d\tau\right\|$$
$$\leqslant \frac{\|T(t_2)x\|}{t_2^n} + \frac{\|y\|}{t_1^n} + \|x\|\sum_{r=1}^n \frac{n!}{(n-r)!}\left(1+\frac{a^{r+1}M_a}{r!}\right)\left(\frac{1}{t_1^r}+\frac{1}{t_2^r}\right)$$
$$\leqslant \frac{\|T(t_2)x\|}{t_2^n} + \frac{\|y\|}{t_1^n} + \alpha_n\|x\|\left(\frac{1}{t_1}+\frac{1}{t_2}\right)$$

where $M_a = \sup\{\|T(t)\|\colon\ 0 \leqslant t \leqslant a\}$ and α_n is suitably chosen. □

Corollary 4.2. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X, and assume that* (1.5) *holds. Let $x \in \bigcap_{t>0}\operatorname{Ran}(T(t))$ with $x \neq 0$. Then at least one of the following holds*:

(i) *For each $n \in \mathbb{N}$, $\|T(t)x\|/t^n \to \infty$ as $t \to \infty$;*
(ii) *For each $n \in \mathbb{N}$,*

$$\frac{\inf\{\|y\|\colon\ T(t)y = x\}}{t^n} \to \infty \quad \text{as } t \to \infty.$$

Proof. Choose $f \in L_1(\mathbb{R})$ with $\|f\|_{L_1} = 1$, $\operatorname{supp}(\mathcal{F}f) \subseteq [0,a]$ for some $a > 0$, and $\int_0^a (\mathcal{F}f)(t)T(t)x\,dt \neq 0$. This is possible by strong continuity of $(T(t))_{t\geqslant 0}$.

If neither (i) nor (ii) held, then one could choose values of $n \in \mathbb{N}$, $t_1 > a$ and $t_2 > a$ such that (4.1) failed. □

Corollary 4.3. *Let A be the generator of a C_0-group $(T(t))_{t\in\mathbb{R}}$ on a Banach space X, and assume that* (1.5) *holds. Then $(T(t))_{t\geqslant 0}$ is expansive.*

Remark 4.4. We can introduce a norm $\|\cdot\|_{T,a}$ on X defined by

$$\|x\|_{T,a} = \sup\left\{ \left\| \int_0^a (\mathcal{F}f)(\tau)T(\tau)x\,d\tau \right\| : \ f \in L_1(\mathbb{R}),\ \|f\|_1 = 1,\ \operatorname{supp}(\mathcal{F}f) \subseteq [0,a] \right\}.$$

The different norms for different values of a are equivalent to each other. Now (4.1) can be written as

$$\frac{\|T(t_2)x\|}{t_2^n} + \frac{\|y\|}{t_1^n} \geqslant \frac{c^n}{n!}\|x\|_{T,a} - \alpha_n\|x\|\left(\frac{1}{t_1} + \frac{1}{t_2}\right).$$

This suggests that the norm $\|\cdot\|_{T,a}$ is a possible candidate to show that $T(t)$ satisfies (HE)$'$, but it would be necessary to obtain an inequality of this type with this norm on the left-hand side.

Our second partial result indicates exponential growth in an integral sense. A very similar result is given in [16, Theorem 2.5]. However the proof there seems to be incomplete (in general, boundedness of the Carleman transform of u near $z_0 \in i\mathbb{R}$ does not imply that z_0 is not in the Carleman spectrum of u). So we give a complete proof here.

Recall that a function $g : \mathbb{R} \to X$ is a *complete trajectory* of $\mathcal{T}$ if $g(t+s) = T(t)g(s)$ ($t \geqslant 0, s \in \mathbb{R}$). Any complete trajectory is continuous on $\mathbb{R}$. When $\mathcal{T}$ is a C_0-group, the complete trajectories are of the form $g(t) = T(t)x$ ($t \in \mathbb{R}$) for some $x \in X$.

Theorem 4.5. *Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on a Banach space X with generator A, and assume that* (1.5) *holds. If $g : \mathbb{R} \mapsto X$ is a non-zero complete trajectory of $(T(t))_{t\geqslant 0}$ then there exists $\varepsilon > 0$ such that $\|g(\cdot)\| \notin L_1(\mathbb{R}, e^{-\varepsilon|t|}\,dt)$.*

Proof. Let $c > 0$ be as in (1.5). Let $\lambda = a + is \in \mathbb{C}$, where $|a| \leqslant c/2$. Then

$$\left\|(A-\lambda)x\right\| \geqslant \left\|(A-is)x\right\| - |a|\|x\| \geqslant \frac{c}{2}\|x\| \quad \left(x \in \mathcal{D}(A)\right). \tag{4.2}$$

Let g be a complete trajectory and suppose that $\|g(\cdot)\| \in L_1(\mathbb{R}, e^{-\varepsilon|t|}\,dt)$ for every $\varepsilon > 0$. Let $x = g(0)$. The Carleman transform $\hat{g}$ of g is defined for $\operatorname{Re}\lambda \neq 0$ by:

$$\hat{g}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} g(t)\,dt & (\operatorname{Re}\lambda > 0), \\ -\int_0^\infty e^{\lambda t} g(-t)\,dt & (\operatorname{Re}\lambda < 0). \end{cases}$$

Then $\hat{g}$ is holomorphic, and bounded on $\{\lambda\colon\ |\operatorname{Re}\lambda| > \varepsilon\}$ for each $\varepsilon > 0$. In particular, $\hat{g}$ is bounded on $\{\lambda\colon\ |\operatorname{Re}\lambda| \geqslant c/2\}$.

Consider λ with $\operatorname{Re}\lambda > 0$, and let $k > 0$. Then $\int_0^k e^{-\lambda t} g(t)\,dt \in \mathcal{D}(A)$ and

$$(A-\lambda)\left(\int_0^k e^{-\lambda t} g(t)\,dt\right) = e^{-\lambda k} g(k) - x.$$

Since $(A-\lambda)$ is closed, letting $k \to \infty$ (through a suitable sequence) shows that

$$\hat{g}(\lambda) \in D(A) \quad \text{and} \quad (A-\lambda)\hat{g}(\lambda) = -x. \tag{4.3}$$

A variation of this argument shows that (4.3) also holds for $\operatorname{Re}\lambda < 0$. It follows from (4.2) that $\|\hat{g}(\lambda)\| \leqslant 2\|x\|/c$ for $0 < |\operatorname{Re}\lambda| < c/2$. Thus $\hat{g}$ is bounded on $\mathbb{C} \setminus i\mathbb{R}$.

Now suppose that $0 < |\operatorname{Re}\lambda| \leqslant c/2$ and $0 < |\operatorname{Re}\mu| \leqslant c/2$. Then

$$\begin{aligned}\|\hat{g}(\lambda) - \hat{g}(\mu)\| &\leqslant \frac{2}{c}\|(A-\lambda)(\hat{g}(\lambda) - \hat{g}(\mu))\| \\ &= \frac{2}{c}\|-x - (-x + (\mu-\lambda)\hat{g}(\mu))\| \leqslant \frac{4}{c^2}|\lambda - \mu|\|x\|.\end{aligned}$$

Now by Cauchy's criterion, $\hat{g}$ extends continuously to $i\mathbb{R}$, and then $\hat{g}$ is a bounded entire function. Since $\hat{g}(a) \to 0$ as $a \to \infty$, $\hat{g}$ is identically zero. Then $g = 0$ by the uniqueness theorem for Laplace transforms. □

Finally we consider the situation where (1.5) holds but no trajectory grows exponentially in forward time. Recall that the *growth bound* $\omega(\mathcal{T})$ of $(T(t))_{t\geqslant 0}$ is the unique quantity in $\mathbb{R} \cup \{-\infty\}$ such that the spectral radius of $T(t)$ is $\exp(t\omega(\mathcal{T}))$ for some/all $t > 0$. If $\omega(\mathcal{T}) < 0$, then (1.5) holds, and $\mathcal{T}$ is exponentially stable and therefore trivially hyperbolic. If $\omega(T) = 0$, then no trajectory grows exponentially in forward time, and we now show that (1.5) implies that all non-zero trajectories grow exponentially in reverse time, with the same exponent. Example 3.3 is an illustration of this situation.

Theorem 4.6. *Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup with generator A and with growth bound* 0*, and assume that the lower bound* (1.5) *holds. Then*

(i) *$(T(t))_{t\geqslant 0}$ is not quasi-hyperbolic,*
(ii) *A is invertible,*
(iii) *there exist constants $\varepsilon > 0$ and $\kappa > 0$ such that*

$$\|y\| \geqslant \kappa e^{\varepsilon t}\|A^{-1}x\|$$

whenever $t > 0$, $x, y \in X$ and $T(t)y = x$.

Proof. Since the growth bound is 0, $\mathbb{T}$ contains a point in the boundary of $\sigma(T(1))$ and hence in $\sigma_{\mathrm{ap}}(T(1))$, so $\mathcal{T}$ is not quasi-hyperbolic. Moreover, $R(\lambda, A)$ exists whenever $\operatorname{Re}\lambda > 0$ and $\|R(\lambda, A)\|$ is bounded for $\operatorname{Re}\lambda \geqslant a$ for any $a > 0$. It follows from (4.2) that $\|R(\lambda, A)\|$ is bounded for $0 < \operatorname{Re}\lambda < a$ for some $a > 0$, and then it follows from the Neumann series for the resolvent that $R(\lambda, A)$ exists and is uniformly bounded for $\operatorname{Re}\lambda > -\varepsilon$ for some $\varepsilon > 0$. By a theorem of Weis and Wrobel [1, Theorem 5.1.9], there exists a constant C such that $\|T(t)A^{-1}\| \leqslant Ce^{-\varepsilon t}$ $(t > 0)$. Hence $\|y\| \geqslant C^{-1}e^{\varepsilon t}\|A^{-1}x\|$ if $T(t)y = x$. □

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