Whose Mathematics, Which World?
Discrete and Continuous Quantity in the *Vestigia Trinitatis* and the Thought of Mary Hesse and Charles Sanders Peirce

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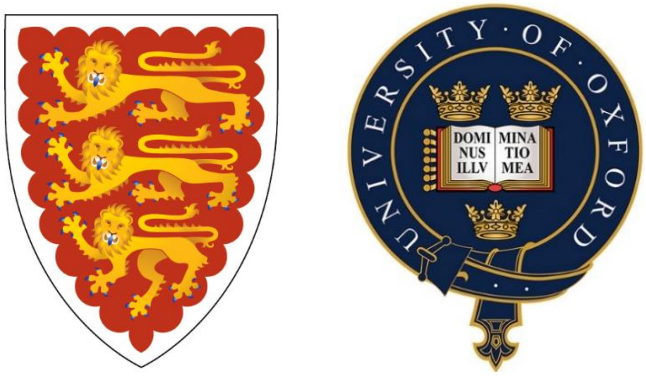

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Table of Contents

Title Page and Front Matter i

Short Abstract vii

Long Abstract ix

Acknowledgements xvi

Word Count xviii

Chapter One: Introduction to Discrete and Continuous Quantity and the *Vestigia Trinitatis* 1

Discrete Quantity and Continuous Quantity 4

Unity in Multiplicity as a Quantitative Aspect of the *Vestigia Trinitatis* 10

The Doctrine of the *Vestigia Trinitatis* Rests on Faith but is Developed by Reason 11

Is the Unity in Multiplicity of the World a Consequence of a Creating Trinity? 19

Is Quantity that Signifies the World Irreducibly Discrete and Irreducibly Continuous, Exhibiting Unity in Multiplicity and, if so, Can Such Quantity be Regarded as a *Vestigium Trinitatis*? 24

Mary Hesse and Charles Sanders Peirce as Dialog Partners 27

Chapter Two: Objections 31

First Objection: Mathematics and Formal Systems Need only Irreducibly Discrete Quantity to Signify the World 33

Reducing Continuous Quantity to Discrete Quantity in Mathematics 35

Logic and Set Theory Need Only Discrete Quantity to Signify the World 45

Second Objection: Mathematics and Formal Systems Need only Irreducibly Continuous Quantity to Signify the World 52

Only Irreducibly Continuous Quantity is Needed in Mathematics as Demonstrated by Traditional Algebra's Use of the Early Modern Number Concept..... 55

Since the Natural Numbers can be Derived from Continuous Magnitudes in Modern Formal Systems only Irreducibly Continuous Quantity is Needed in Mathematics 67

Because Mereological Wholes Can Signify Discrete Unities only Irreducibly Continuous Quantity is Needed in Formal Systems 75

Third Objection: Even if Quantity that Signifies the World is Irreducibly Discrete and Irreducibly Continuous, Such Quantity Still Cannot be Regarded as a *Vestigium Trinitatis*.....80

Should Only Human Persons be Regarded as *Vestigia Trinitatis*? 82

If Aspects of Non-Human Creation Qualify as *Vestigia Trinitatis*, Could Irreducibly Discrete and Irreducibly Continuous Quantity be Among Them? 90

Is Quantity a Metaphysical Principle which could be Regarded as a *Vestigium Trinitatis*? 92

Chapter Three: Do Mathematics and Formal Systems need both Irreducibly Discrete Quantity and Irreducibly Continuous Quantity to Signify the World? 103

Mary Hesse: Does Mathematics which Signifies the World in Scientific Theories Use both Irreducibly Discrete Quantity and Irreducibly Continuous Quantity?.....104

Atomicity and Continuity in the History of Physical Science..... 105

Discrete Phenomena Exhibit Continuous Aspects and Continuous Phenomena Exhibit Discrete Aspects..... 115

Hesse's Analogical Approach Diffuses the Logical Tension for Realists who Signify the World with both Irreducibly Discrete Quantity and Irreducibly Continuous Quantity..... 138

C.S. Peirce: Do Formal Systems Need both Discrete Quantity and Continuous Quantity to Signify the World through Persistent Metaphysical Concepts? 149

Peircean Conceptions of Discreteness and Continuity in Mathematics ... 152

Discrete and Continuous Aspects of Persistent Metaphysical Concepts in Peirce 160

The Peircean Notion of Discreteness and Continuity in Triadic Logic.... 179

The Relative Individual in Peirce-Schröder Logic..... 187

Implications of C.S. Peirce's Approach for Mereology and the Signification of Substance..... 197

Chapter Four: Discrete and Continuous Quantity and the Unity in Multiplicity of the World as a *Vestigium Trinitatis* 207

The Unity in Multiplicity of Quantity that Signifies the World and the *Vestigia Trinitatis*.....208

Measure, Number, and Weight as *Vestigia Trinitatis* and Quantity that Signifies the World 210

Is Unity in Multiplicity a General Mark of *Vestigia Trinitatis* and is it Exhibited by Irreducibly Discrete and Irreducibly Continuous Quantity that Signifies the World? 224

Key Aspects of the *Vestigia Trinitatis* like Substance, Persons, and Causal Metaphysical Principles Exhibit Unity in Multiplicity and should be Signified with Mutually Irreducible Discrete and Continuous Quantity.. 238

Reply to Objections 248

Reply to First Objection: Mathematics and Formal Systems Need More than just Discrete Quantity as an Irreducible Primitive in order to Signify the World 248

Reply to Second Objection: Mathematics and Formal Systems Need More than just Continuous Quantity as an Irreducible Primitive in order to Signify the World 258

Reply to Third Objection: Irreducibly Discrete and Irreducibly Continuous Quantity that Signifies the World can be Regarded as a *Vestigium Trinitatis*. 268

Chapter Five: Conclusion 277

Bibliography 283

Short Abstract

Implicit in the many formulations of the doctrine of the *vestigia Trinitatis*, the traces of the Trinity, is a theme of unity in multiplicity. Signifying unity and multiplicity in the world often involves quantity and numbers. Two notions of quantity have existed in mathematics through the millennia: discrete quantity underlying arithmetical approaches and continuous quantity underlying geometrical approaches. Does signifying the world with these seemingly incommensurate quantities exhibit unity in multiplicity, and if so, can such quantity thus be regarded as a *vestigium Trinitatis*? Some would say no. For example, some claim that continuous quantity is reducible to discrete quantity and others claim that discrete quantity is reducible to continuous quantity. Still others claim the mutual irreducibility of discrete and continuous quantity is irrelevant to any *vestigia Trinitatis*.

In this thesis I argue from the work of Mary Hesse and Charles Sanders Peirce that mutually irreducible discrete quantity and continuous quantity are both needed to signify the world. Hesse offers an analogical approach for mathematical semiotics which notes that "atomicity and continuity" form a pair of "apparently contradictory principles which continually reappear in the history of science." Peirce develops a philosophy of thirdness which advocates mutually irreducible discrete and continuous quantity for metaphysics, logic, and mathematics. I also show that such mutually irreducible quantity which signifies the world exhibits unity in multiplicity and can thus be regarded as a *vestigium Trinitatis*. I conclude that the choice of foundations in mathematics and formal systems is of the greatest importance for framing concepts of the world and its Creator.

Long Abstract

On a Nicene understanding of divine revelation, God the Holy Trinity is one substance and three persons. God is also the creator of the world, and since the time of Augustine theologians have asked whether there are traces of the Trinity, *vestigia Trinitatis*, in creation. Augustine developed a psychological doctrine of the *vestigia Trinitatis* focused on the memory, understanding, and will of human persons. But other theologians developed the doctrine of the *vestigia Trinitatis* to include aspects of non-human creation and metaphysical principles like unity in diversity as well. Throughout the doctrine's history the theme of unity in multiplicity seems implicit in its many formulations. Common to Augustine's search for unity in triadicity and Herman Bavinck's search for unity in diversity, the theme of unity in multiplicity underlies them. But the notions of unity and multiplicity relate to number and quantity, concepts dealt with by mathematics. Historically, there have been two central notions of quantity in mathematics, discrete quantity underlying arithmetical approaches and continuous quantity underlying geometrical approaches. Might the notions of unity and multiplicity in discrete quantity and continuous quantity have any relation to the theme of unity in multiplicity throughout the *vestigia Trinitatis*?

In this thesis I ask whether quantity that signifies the world is irreducibly discrete and irreducibly continuous, exhibiting unity in multiplicity and, if so, can quantity be regarded as a *vestigium Trinitatis*? There are a number of reasons to answer no to the central question I pose above. Some doubt whether unity in multiplicity is signified best by mutually irreducible discrete quantity and continuous quantity in mathematics and formal systems. And others ask whether unity in multiplicity should be considered as one of the *vestigia Trinitatis* at all? So before offering a positive case I consider three objections. The first two objections concern

mathematics and formal systems and the third concerns theology. The first objection asks, is continuous quantity reducible to discrete quantity, and thus is discrete quantity alone able to signify the world? Georg Cantor and Richard Dedekind argue that the continuum is reducible to discrete quantity. Bertrand Russell defends this position and develops a formal logic based on their notion of the arithmetical continuum which he uses to signify unity and multiplicity in the world, especially in metaphysical concepts such as identity-over-time.

The second objection asks the opposite question, is discrete quantity reducible to continuous quantity, and thus is continuous quantity alone able to signify the world? In the early modern period Simon Stevin argues that, "a continuous number corresponds to a continuous magnitude," thus reducing discrete quantity to continuous quantity.[1] This position influences others like René Descartes, John Wallis, and Isaac Newton. The conception has spread, often unrecognized, through the ambiguity inherent in the use of algebra and calculus, mathematical systems which profoundly signify the world. In addition, the basic relation in the formal system of mereology is the parthood relation patterned after the divisibility of the traditional continuum. Mereology is used by a wide range of thinkers to signify unity in multiplicity, or as they describe it parts and wholes. This use of mereology suggests that only continuous quantity is needed to signify the unity in multiplicity of entities and persistent metaphysical concepts like identity-over-time.

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[1] Simon Stevin, "L'Arithmetique," in *The Principal Works of Simon Stevin*, vol. 2, ed. Crone Ernst, trans. Dikshoorn C (Amsterdam: C. V. Swets & Zeitlinger, 1955), 501.

The final objection questions whether the signification of the unity in multiplicity of the world by discrete and continuous quantity is at all related to the *vestigia Trinitatis*. For those that hold an Augustinian view of the *vestigia Trinitatis*, the main focus of the doctrine should be on human persons and not on non-human creation or metaphysical principles like unity in multiplicity. For those that allow post-Augustinian development of the doctrine of the *vestigia Trinitatis*, none have explicitly identified unity in multiplicity or discrete and continuous quantity in their explanations of the doctrine. Therefore, it seems unlikely that the signification of unity in multiplicity by discrete and continuous quantity has anything to do with the *vestigia Trinitatis*.

On the other hand, the thinkers Mary Hesse and Charles Sanders Peirce offer compelling reasons why mutually irreducible discrete quantity and continuous quantity are both needed to signify the world. Hesse especially provides insight into why our best scientific theories need both discrete and continuous quantity to signify the world. A common response to the complementarity of quantum physics is to posit an instrumentalist view of science, such as that of Poincaré, in which mathematical quantities give no insight into the true nature of reality. Another response is that of Einstein who holds that fundamental reality is either discrete (particulate) or continuous (field) but only further research will reveal which of these two mutually exclusive positions is true.

Mary Hesse offers an analogical approach for mathematical semiotics that differs from both of these. Hesse notes that “atomicity and continuity” form a pair of “apparently

contradictory principles which continually reappear in the history of science."[2] By identifying the pervasiveness of this pair of discrete and continuous quantity in the physical sciences and by recognizing these as analogical signification of the entities under consideration, she offers an important way to signify unity in multiplicity. An entity considered mathematically by the physical sciences can be like a discrete billiard ball and like a continuous field at the same time without being exactly like either of them. By comparing Hesse's claims about quantitative signification to Aristotle's claims about quantitative signification it becomes clear that the nuanced position that Hesse offers accommodates the discoveries of modern physics, is realist, and displays why mutually irreducible discrete and continuous quantity are both needed to signify the world.

Charles Sanders Peirce offers further reasons why mutually irreducible discrete and continuous quantity are both needed to signify the world especially through persistent metaphysical concepts. Peirce develops a philosophy of thirdness which advocates mutually irreducible discrete and continuous quantity for metaphysics, for logic, and for mathematics. Peirce argues that both continuity and discreteness are needed to signify the persistent metaphysical concepts of identity-over-time and causality. Because Peirce's work substantially influences the formal systems of logic, set theory, and mereology, a study of his approach reveals how conceptions of discrete and continuous quantity affect foundational concepts in these systems and limit the manner in which entities can be signified. Together

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[2] Mary B. Hesse, *Forces and Fields. The Concept of Action at a Distance in the History of Physics.* (Nelson: London, 1961), 293.

these two thinkers show that one's choice of how to handle discrete quantity and continuous quantity in mathematics and formal systems has widespread effects on the conception of the world being signified, in particular whether that world reveals unity in multiplicity.

In the fourth chapter I make the case that unity in multiplicity is indeed an aspect of the *vestigia Trinitatis* and is exhibited by quantity that signifies the world. Augustine describes measure, number, and weight as a trinity. While number is linked to discrete quantity, measure is linked to both discrete and continuous quantity. Bonaventure develops this Augustinian theme into metaphysical causal principles. The most common aspect of all versions of the *vestigia Trinitatis* is unity in triadicity, and unity in triadicity implies unity in multiplicity. A mathematics or formal system which cannot signify unity in multiplicity, by default cannot signify unity in triadicity which is a species of unity in multiplicity. Another conception of the *vestigia Trinitatis* offered by Herman Bavinck, unity in diversity, also implies unity in multiplicity as a quantitative corollary. The traditional doctrines of the *vestigia Trinitatis* exhibit unity in multiplicity. And since irreducibly discrete and irreducibly continuous quantity that signifies the world also exhibits unity in multiplicity, this fact supports regarding such quantity as a *vestigium Trinitatis*. Finally, I argue that the notions of substance, person, and causality central to traditional conceptions of the *vestigia Trinitatis* are best signified by irreducible discrete and continuous quantity, and this implicitly supports including such quantity among the *vestigia Trinitatis*.

In the fourth chapter I also reply to the initial objections. My reply to the first two objections draws from a body of critics within philosophy of mathematics and philosophy of logic. One critic of the arithmeticization project was Paul du Bois-Reymond who wrote, "the resting, solid idea of space will never create the image of a precise, uniform line from a series

of dots, no matter how dense." [3] In reply to the second objection, I argue that one cannot derive discrete number from continuous magnitude alone as an additional element is needed: thought. It is exactly this component, thought, that mathematicians identify as ratio. The early modern number concepts based on continuity do not offer discrete quantity simply, they offer discrete quantity as a ratio of two continuous magnitudes. And this use of ratio smuggles in a discrete conception. And finally, I argue that the signification of unity in multiplicity by discrete and continuous quantity is indeed an aspect of the *vestigia Trinitatis*. First, though Augustine emphasized human persons as *vestigia Trinitatis*, he allowed that measure, number, and weight, reminiscent of discrete and continuous quantity, might also be *vestigia Trinitatis*. Second, the traditional conceptions of the *vestigia Trinitatis*, such as Augustine's psychological approach or Bonaventure's metaphysical principles rest on notions of identity-over-time or continuous causality. In order to signify these things mathematics and formal systems require irreducibly discrete and irreducibly continuous quantity as basic primitives.

In the conclusion, I note that this project has an apophatic and a positive aspect to it. Apophatically, on account of the analogical character of mathematical semiotics a gap exists between even the best accounts by mathematics or formal systems and a full account of the entities under consideration, including the mathematical *vestigia Trinitatis*. But positively, mathematics and formal systems that use mutually irreducible discrete and continuous

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[3] Paul du Bois-Reymond, *Die allgemeine Functionentheorie* (Tubingen: Verlag, 1882), 66, http://archive.org/details/dieallgemeinefu00boisgoog. Translated by Jan Bentz. See alternate translation given in Philip Ehrlich, *Real Numbers, Generalizations of the Reals, and Theories of Continua*, Synthese Library ; v. 242 (Dordrecht ; London: Kluwer Academic, 1994), x.

quantity can indeed signify unity in multiplicity and by doing so do positively illustrate the *vestigia Trinitatis*. I thus conclude that quantity that signifies the world is irreducibly discrete and irreducibly continuous, exhibiting unity in multiplicity, and therefore such quantity can be regarded as a *vestigium Trinitatis*. Furthermore, this conclusion has implications for choosing which sorts of mathematics and formal systems to use to signify the unity in multiplicity of the world as *vestigia Trinitatis*.

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Word Count

This thesis contains _86,765__ words.[4]

[4] This word count is calculated per the instructions outlined in the University of Oxford, Faculty of Theology and Religion, Handbook for Postgraduate Research Students. The count encompasses the thesis (all five chapters), including footnotes. Additionally, this thesis is formatted in accordance with the Faculty of Theology examination regulations as well as the University examination regulations applicable to my degree and entering year.

Chapter One: Introduction to Discrete and Continuous Quantity and the *Vestigia Trinitatis*

On a Nicene understanding of Christian revelation, God is the Holy Trinity, namely the Father, the Son, and the Holy Spirit. Whatever else can be stated about these Divine Persons, they are multiple distinct persons. Yet also on this understanding of God there is an absolute unity of substance. In other words, Christian theology marries principles of multiplicity and unity in its language of the Godhead. The Holy Trinity is simultaneously a unity of substance and a multiplicity of persons.

God is also the Creator of all things, and Christians since the time of St. Augustine have wondered if there is a Trinitarian imprint on creation. Might this unity in multiplicity in the Godhead leave *vestigia Trinitatis* (traces of the Trinity) exhibiting unity in multiplicity within creation? Science and philosophy seek to signify unity and multiplicity in the world. Could their signification of the unity and multiplicity of the world relate to the *vestigia Trinitatis*? Is the mathematics used to signify the sciences even able to signify what seems like a contradictory principle, namely simultaneous unity and multiplicity, or unity in multiplicity?

Mathematician, scientist, and philosopher Henri Poincaré offers a clue in his critique of a late 19th century development in mathematics called the arithmeticization of analysis. Of the arithmetical version of the continuum in which a line is composed of points Poincaré says, “[m]ultiplicity alone subsists, unity has disappeared—'the continuum is unity in

multiplicity…'" [5] What does Poincaré mean by this provocative statement that in the traditional continuum exists unity in multiplicity?

For most of its history, mathematics has held that two seemingly contradictory principles are needed in order to signify the world: discrete quantity and continuous quantity. As Aristotle held that there are two co-principles of a substance, namely form and matter, likewise traditional mathematics held that two co-principles of quantity are needed to signify all aspects of reality. Discrete quantity and continuous quantity were understood by nearly all mathematicians as two necessary yet incommensurable primitives from the time of Pythagoras until the 17th century and by many thereafter as well. Arithmetic studies discrete quantity and geometry studies continuous quantity. Because these two co-principles of quantity are incommensurable, they may be able to jointly signify a thing simultaneously without contradicting each other. When used together, they can signify simultaneous unity and multiplicity—or unity in multiplicity. As Poincaré says regarding the unity of a traditional continuum, "[b]etween the elements of the continuum exists an intimate connection making of it one whole, in which the point has no existence previous to the line, but the line does exist previous to the point." [6] But from another perspective the arithmetical continuum offers a different and incommensurable notion of a line altogether in which the existence of an infinite set of points, a collection of discrete quantity exists prior to or for some actually composes the line. Does seeing each of these perspectives

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[5] Henri Poincaré, *Science and Hypothesis*, trans. W. J. Greenstreet, with Joseph Larmor (London: Scott, 1905), 23, http://catalog.hathitrust.org/api/volumes/oclc/2156620.html.

[6] Poincaré, *Science and Hypothesis*, 23.

simultaneously as possible viewpoints reflect unity in multiplicity? If by use of the traditional continuum as well as a set of points one can signify "unity in multiplicity" then perhaps the irreducibly continuous line signifies the unity and a set of irreducibly discrete points residing on the line signify the multiplicity. Thus, together can this use of irreducibly continuous quantity and irreducibly discrete quantity signify unity in multiplicity? Moreover, if quantity as used to signify the world does reflect unity in multiplicity, might quantity itself be a *vestigium Trinitatis*?

There are objections to these ideas for multiple reasons. Many mathematicians today, following thinkers from the 19th century, believe that continuity is not basic in mathematics, but that continuity is established by reference to an infinite set of discrete points as noted above. This point-set conception of continuity holds that only discrete quantity is basic and what is called continuity can ultimately be reduced to discrete elements. On the other hand, in the last decade philosophers of mathematics have noted that there are multiple conceptions of continuity in mathematics. They claim that some ideas of continuity make certain notions in the sciences such as continuous (or gunky) time or continuous (or gunky) space harder to conceptualize. There is a revival of interest in the need for both discrete quantity and continuous quantity as two incommensurable and mutually irreducible primitives.

Definitions of discreteness and continuity have been highly contended over the past four hundred years. Thus, some clarifications are in order before proceeding.

Discrete Quantity and Continuous Quantity

Recent philosophers have distinguished between count nouns and mass nouns, and this distinction provides an intuitive understanding of discrete quantity versus continuous quantity. Mass nouns like 'water' describe something which is perceptually continuous, and count nouns like 'apple' describe something which is perceptually discrete.[7] Some refer to mass nouns as "stuff" and count nouns as "things". Scholars acknowledge a distinction between "things and stuff" though they disagree on whether this distinction is real or merely nominal.[8]

This distinction between things and stuff, or countable and non-countable entities, is well recognized and has engendered a sizable discussion in philosophy of language. Nonetheless, its significance for mathematics and logic is underappreciated in recent years. In the past decade, philosophers of mathematics have discussed the different notions of continuity that lie behind this discourse of "stuff" also referred to as "gunk". Geoffrey Hellman and Stuart Shapiro wrote *Varieties of Continua* in 2018 and then edited a volume

[7] "count NPs [noun phrases] use numerals, allow indefinite determiners, can be pluralized, and admit certain "individuating" quantifiers. Mass NPs do not do this. One very apparent difference between the two types of NP is that the count ones presuppose that there are separate, individual cars, boys, apples, books; the mass ones do not, but rather suggest some indeterminate amount of a given sort of matter…" Francis Jeffry Pelletier, Tibor Kiss, and Halima Husić, "Editorial Introduction: Background to the Count–Mass Distinction," in *Things and Stuff: The Semantics of the Count-Mass Distinction*, ed. Francis Jeffry Pelletier, Halima Husić, and Tibor Kiss (Cambridge: Cambridge University Press, 2021), 1, https://doi.org/10.1017/9781108937979.002.

[8] "Some authors call this distinction between individuative NPs [noun phrases] and non-individuative NPs a conceptual claim (that's how we think about whatever the NP describes); other authors think of it as an ontological claim (that's the way "reality" is organized). Either way seems to be a distinction between "things" and "stuff"…" Pelletier, Kiss, and Husić, "Editorial Introduction," 1.

History of Continua in 2021, which each highlight approaches to the continuum which cannot merely be modeled with discrete elements.[9] This renewed interest in continuity from highly esteemed philosophers of mathematics is suggestive. Perhaps the fact that many mathematicians *currently do not use* both discrete quantity and continuous quantity as mutually irreducible co-principles of mathematics does not mean that they *should not use* both as mutually irreducible co-principles of mathematics.

A key inspiration for Hellman and Shapiro's research is the lack of awareness among contemporary mathematicians of the Aristotelian continuum which for thousands of years was considered an irreducible primitive alongside discrete quantity in mathematics.[10] Hellman and Shapiro reconstruct with modern methods "a 'regions-based' one-dimensional

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[9] Geoffrey Hellman and Stewart Shapiro, *Varieties of Continua: From Regions to Points and Back* (Oxford, New York: Oxford University Press, 2018). Stewart Shapiro and Geoffrey Hellman, *The History of Continua: Philosophical and Mathematical Perspectives* (Oxford: University Press, Incorporated, 2021), https://doi.org/10.1093/oso/9780198809647.001.0001.

[10] "We were surprised to learn that many philosophers and even some mathematicians take the Dedekind–Cantor conception of continuity to be not only the right one, but the only one. Some were surprised to learn that there are any other notions. They may have heard something of the history, but many take it for granted that we now have the one correct account of continuity. The once long standing "intuitions" that support the other accounts are not to be taken seriously (and perhaps never should have been).

Our view is that there is no single, monolithic property of continuity. It is more of a cluster concept that can be sharpened, and developed rigorously, in mutually incompatible ways. Early on, we were led to the Aristotelian idea that continua are not composed of points, or indivisible parts. In terms of contemporary metaphysics, the theme is that continua are gunky: every part of a continuous substance has a proper part. This, of course, is not sanctioned in the contemporary Dedekind–Cantor account nor, arguably, in the intuitionistic accounts either (depending on what counts as a "point" in those contexts)." Shapiro and Hellman, *The History of Continua*, 1.

continuum, following the Aristotelian theme that continua are not composed of points." [11] One may quickly ascertain the idea of discrete things such as trees, people, or cars. But what makes a thing continuous is more difficult to understand. Since this conception of continuity is a "theme derived from Aristotle and maintained for two millennia," consider three important aspects of Aristotle's notion of continuity: potentially infinite divisibility, non-punctiform composition, and the fusion of boundaries.[12]

Aristotle wrote about discrete and continuous quantity explicitly. "'Quantity' means that which is divisible into constituent parts." [13] But the parts and hence quantity can be of two sorts: discrete or continuous. For Aristotle, discrete quantity is a multitude such as a flock of birds. One can divide a flock of birds into many individual birds or individual discrete 'parts'. A continuous quantity, like a line, on the other hand, is a "'magnitude' …which is potentially divisible into continuous parts." [14] Both kinds of quantity, discrete quantity and continuous quantity are divisible. But the divisibility of discrete quantity stops when one divides down to the units. In the example above the individual birds are the units and are indivisible as birds. Continuous quantity, on the other hand, is potentially infinitely

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[11] They continue: "Each part of our "gunky" space—each region—has a proper part. Although we show how "points" can be defined on the structure in terms of sets or sequences of regions, along lines similar to Alfred N. Whitehead's "extensive abstraction", there is no sense in which "points", so construed, are in, or are part of, regions. Indeed, "points", so construed, just are set-theoretic constructions on regions. We show that these "points" are isomorphic to the real numbers, and thus to the so-called "classical" point–set continuum of Dedekind and Cantor." Hellman and Shapiro, *Varieties of Continua*, 43.

[12] Shapiro and Hellman, *The History of Continua*, 1.

[13] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, Loeb Library (Cambridge, Mass.: Harvard University Press, 1935), 1020a. See Book V:13

[14] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1020a. See Book V:13

divisible, in an "unlimited potentiality". [15] A line can be divided up a potentially infinite number of times into smaller continuous line segments. This potentially infinite divisibility is the first aspect of Aristotelian continuity. While the infinite divisibility of continua is acknowledged today, most mathematicians refer to the 'actual' infinite divisibility of continua instead of Aristotle's 'potential' infinite divisibility.

The next important aspect of Aristotelian continuity is that "no continuum can be made up of indivisibles, as for instance a line out of points." [16] The conflict this creates between "the old orthodoxy (Aristotle) vs the new orthodoxy (Dedekind–Cantor)" is noted by Hellman and Shapiro.[17] "This [Aristotelian idea that continua are not composed of points, or indivisible parts] of course, is not sanctioned in the contemporary Dedekind–Cantor account nor, arguably, in the intuitionistic accounts either…" [18] Most schoolchildren learn the number line, which subtly implies that a line is composed of the points represented by the numbers. In geometry most learn that a circle (traditionally, a continuous curved line) is the set of all points equidistant from some center point. These ideas represent the Dedekind-Cantor view that a line is an actually infinite set of points. The Aristotelian view holds that the parts of a line are smaller lines or subregions. These

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[15] "The unlimited (ἄπειρον), then, exists only…as an unlimited potentiality of approximation by reduction of intervals. Unlimitedness is never actual…" Aristotle, *Physics*, trans. Philip H. Wicksteed and Francis Macdonald Cornford, with Harvard University Press, Loeb Classical Library 228 (Cambridge, MA: Harvard University Press, 1934), 206b. See Book III:6

[16] Aristotle, *Physics*, trans. Wicksteed and Cornford, 231a. See Book VI:1

[17] Hellman and Shapiro, *Varieties of Continua*, 1.

[18] Shapiro and Hellman, *The History of Continua*, 1.

subregions may have lengths measurable relative to a unit or a fixed length, but the subregions are not numbers nor points. Points arise when lines intersect as evident in the geometry of Euclid, who holds the same view of continuity as did Aristotle. But points do not compose a line, points for Aristotle and Euclid merely exist upon the line.

The third important aspect of Aristotelian continuity regards the boundaries of continuous objects which touch. Aristotle says this regarding contiguous objects:

> I call two things continuous when the limits of each, with which they touch and by which they are kept together, become one and the same, so that plainly the continuous is found in the things out of which a unity naturally arises in virtue of their contact.[19]

Here Aristotle wishes to distinguish between the waters of a river and rocks in a pile. When the waters of a river touch they become one stuff, one continuous water as perceived by unaided human perception. Therefore, the volumes of water represent continuous quantity. But each rock in a pile remains discrete from the others even though the rocks touch. Therefore, because the boundaries of the rocks do not fuse, the quantity of rocks is a discrete quantity. The fusion of boundaries is a subtle but important point regarding Aristotelian continuity which affects the way mathematics and formal systems signify continua.

Having summarized important aspects of continuous quantity, now consider discrete quantity and the distinction between plurality and multiplicity. While what makes a

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[19] Aristotle, *Metaphysics*, trans. W. D. Ross, Oxford scholarly editions online (Oxford: University Press, 2020), 1069a3-5. See Book XI:12.

quantity discrete may seem intuitive, the traditional Aristotelian concept of discrete quantity should be clarified. Discrete quantity requires an individual, indivisible unit. This unit (or unity) is not a quantity but the beginning, principle, or *arche* of quantity.[20] Thus, 'one', as in 'one man', is not a quantity in the traditional view. For Aristotle, "quantity means that which is divisible into constituent parts." [21] Because a man, as a man, is not divisible, one man is not a quantity and thus not a discrete quantity.

Traditionally mathematics did not consider 'one' itself to be a number. Two was considered the first number. One was considered the principle of number. A man, as one man, is not a quantity just as one is not a number but the principle of number. "Plurality means that which is potentially divisible into non-continuous parts; and 'magnitude' that which is potentially divisible into continuous parts." [22] Thus, plurality in this sense is only applicable to discrete things which are in multitudes. Multiplicity on the other hand is a more general term which can refer to either a plurality of things divisible into discrete parts or a continuous magnitude divisible into many continuous potential parts. In both cases, there are one thing and many things, but the unities and multiplicities are different in nature.

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[20] "The essence of 'one' is to be a kind of starting point [ἀρχῆ] of number; for the first measure is a starting point, because that by which first we gain knowledge of a thing is the first measure of each class of objects. "The one," then, is the starting-point of what is knowable in respect of each particular thing. But the unit is not the same in all classes, for in one it is the quarter-tone, and in another the vowel or consonant; gravity has another unit, and motion another. But in all cases the unit is indivisible, either quantitatively or formally." Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1016b18-25. See Book V:6:17-18.

[21] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1020a. See Book V:6:14.

[22] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1020a. See Book V:6:14.

Another contemporary philosopher of mathematics summarizes well the traditional understanding of discrete and continuous quantity and its importance for mathematics:

> From the time of Aristotle to the eighteenth century, one philosophy of mathematics dominated the field. Mathematics, it was said, is the "science of quantity". Quantity, one of Aristotle's basic categories of being, is divided into the discrete, studied by arithmetic, and the continuous, studied by geometry. That theory plainly gives an initially reasonable picture of at least elementary mathematics, with its emphasis on counting and measuring, and calculating with the resulting numbers.[23]

These distinctions between discrete quantity and continuous quantity for two millennia were considered central to the discipline of mathematics. That the traditional distinction between discrete and continuous quantity is no longer understood merely means that some translation work is needed in order to understand the terms of the discourse correctly. The explanations above are not meant to assert that one view of continuity and discreteness is correct and another is false. The explanations are merely meant to clarify the nature of the question which this thesis raises.

Unity in Multiplicity as a Quantitative Aspect of the *Vestigia Trinitatis*

What is the doctrine of the *vestigia Trinitatis*, and how is it variously conceived through the millennia? Theologians since the time of St. Augustine to the present have advocated the doctrine of the *vestigia Trinitatis*, and they have conceptualized the doctrine differently. But each conception is a response to the same question: Given that God is a Trinity, does creation bear any marks that reflect the Trinitarian nature of its Creator? The

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[23] James Franklin, *An Aristotelian Realist Philosophy of Mathematics: Mathematics as the Science of Quantity and Structure* (London: Palgrave Macmillan UK, 2014), 32.

most influential contributors to the doctrine of the *vestigia Trinitatis* conclude the Trinitarian nature of God from Scripture, not from nature nor from reason alone. Hence for these theologians, the doctrine of the *vestigia Trinitatis* is not an attempt to prove the Trinity apart from revelation. On the other hand, this doctrine is intended to strengthen the faith of believers by showing that the Trinity is not against reason but comports with reason when reason is rightly understood. The doctrine of the *vestigia Trinitatis* has changed through the years, but in most iterations of the doctrine the concept of unity in multiplicity arises.

The Doctrine of the *Vestigia Trinitatis* Rests on Faith but is Developed by Reason

The foundation for the doctrine of the Trinity can be found in various passages of Holy Scripture. While it is beyond the scope of this thesis to provide a comprehensive defense or explanation of the doctrine of the Trinity, nonetheless for the purposes of illustration consider a small sample of Scriptural passages that support the doctrine.[24] The unity of God is stated prominently in Deuteronomy 6:4, "Hear, O Israel: The Lord our God, the Lord is one." Regarding the singularity of God Isaiah 45:5 reads, "I am the Lord, and there is no other; apart from me there is no God." About the unity and multiplicity of the persons of the Trinity, Christ says, "I and the Father are one." [25] Christ accepts worship

[24] A good account of pro-Nicene Trinitarian theology and Scriptural interpretations at the time of Augustine can be found in chapter two of Lewis Ayres, *Augustine and the Trinity*, with Cambridge University Press, Cambridge Core. (Cambridge: University Press, 2010), 42–71.

[25] John 10:30 (New International Version). All Biblical citations are from the NIV unless otherwise noted.

from the disciples, an idolatrous practice were Jesus not God, and Christ accepts direct address referring to him as God as when doubting Thomas exclaims, "My Lord and my God!" [26]

Christ also refers to the three persons of the Trinity when charging the disciples to go out to all nations "baptizing them in the name of the Father and of the Son and of the Holy Spirit." [27] In the gospel of John, Christ refers to the Holy Spirit as proceeding from the Father, "When the Advocate comes, whom I will send to you from the Father—the Spirit of truth who goes out from the Father—he will testify about me." [28] And elsewhere in the New Testament the Holy Spirit is explicitly identified as God. "The Spirit searches all things, even the deep things of God. For who knows a person's thoughts except their own spirit within them? In the same way no one knows the thoughts of God except the Spirit of God." [29] Thus, though the unity of God is evident in Scripture, the three divine persons of the Trinity are also indicated in Scripture.

Augustine's interest in the *vestigia Trinitatis* is founded in his faith that the Scriptures, such as those referenced above, are true. Augustine seeks to "guard against the sophistries of those who disdain to begin with faith, and are deceived by a crude and

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[26] Matthew 28:17 (NIV) and John 20:28 (NIV)

[27] Matthew 28:19 (NIV)

[28] John 15:26 (NIV)

[29] 2 Corinthians 2:10-11 (NIV)

perverse love of reason." [30] That the persons of the Holy Trinity are "a divine unity of one and the same substance in an indivisible equality" is concluded by all those "Catholic expounders of the divine Scriptures Old and New" and is therefore a received doctrine of the church.[31] Augustine describes his efforts to establish the *vestigia Trinitatis*:

> First, however, we must demonstrate, according to the authority of the Holy Scriptures, whether the faith be so. Then…we may perhaps…enable [those talkative arguers] to find something which they are not able to doubt…[so that] they may be led to lay the fault to their own minds, rather than to the truth itself or to our reasonings…[32]

In other words, in his book *On the Holy Trinity* Augustine first wishes to demonstrate that Scripture teaches the doctrine of the Holy Trinity. Augustine shows this in Books I through VII. Next, Augustine wishes to demonstrate that the doctrine of the Trinity is not against reason but it is reasonable. The troublesome arguers should attribute "the fault to their own minds, rather than to the truth itself or to our reasonings." Augustine wishes to show that the doctrine of the Trinity is not unreasonable when reason is rightly understood. In order to show that the doctrine of the Holy Trinity is reasonable he wishes to show that there are

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[30] Augustine of Hippo, *On the Holy Trinity*, ed. Philip Schaff, trans. Arthur W. Haddan, A Select Library of the Nicene and Post-Nicene Church Fathers (Grand Rapids, Michigan: Wm. B. Eerdmans Publishing Company, 1978), 1.1.1.

[31] "All those Catholic expounders of the divine Scriptures, both Old and New, whom I have been able to read, who have written before me concerning the Trinity, Who is God, have purposed to teach, according to the Scriptures, this doctrine, that the Father, and the Son, and the Holy Spirit intimate a divine unity of one and the same substance in an indivisible equality; and therefore that they are not three Gods, but one God." Augustine of Hippo, *On the Holy Trinity*, 1.4.7.

[32] Augustine of Hippo, *On the Holy Trinity*, 1.2.4.

traces of the Trinity, *vestigia Trinitatis*, among created things. He does this in Books VIII-XV.

Through the millennia, many authors writing on the doctrine of the *vestigia Trinitatis* have restated Augustine's principle of beginning with faith. They often point out that the Holy Trinity is a divine mystery first revealed by faith. The traces of the Trinity once discovered are an encouragement to this faith, but they do not constitute a proof by reason alone which could then discard the faith. Thomas Aquinas offers a characteristic statement of this kind of thought where the *vestigia Trinitatis* are like one kind of argument which "does not prove a point conclusively but shows that its acceptance fits in with the observed effects…[it is in this manner that] reasons avail to prove the Trinity; as, when assumed to be true, such reasons confirm it." [33] If the doctrine of the Trinity is "assumed to be true" what effects would flow from the truth of this doctrine? If those effects (i.e. the *vestigia Trinitatis*) are observed, then that observation confirms the truth of the doctrine of the Trinity, but it does not prove the truth of the doctrine conclusively.

Other thinkers who discuss the *vestigia Trinitatis* describe similar approaches. Richard Baxter says, "It is one thing to *prove* the Sacred Trinity of Persons, by such reason (or to undertake *fully to open* the mystery) and it is another thing to prove that the Doctrine

[33] Thomas Aquinas, *Summa Theologica*, trans. Fathers of the English Dominican Province (London: T. Baker, 1920), 1.32.1ad2. First section of translation taken from James A. Weisheipl, "The Commentary of St. Thomas on the De Caelo of Aristotle," in *Thomas Aquinas: Contemporary Philosophical Perspectives*, ed. Brian Davies (Oxford University Press, 2002), 49.

is neither *incredible* nor *unlikely* to be true."[34] Herman Bavinck echoes the sentiment, "Scripture alone is the final ground for the doctrine of the Trinity… Nevertheless, the arguments advanced to shed light on the dogma of the Trinity are not devoid of all value… For Scripture itself points to this truth by saying that all created beings will show these imprints and human beings will exhibit the image of the triune God."[35] These thinkers acknowledge that it is right and an encouragement to faith to look for the effects of the Trinity upon creation, even though noting these effects, these *vestigia Trinitatis*, do not prove the Trinity from reason alone.

This style of argumentation for the *vestigia Trinitatis* tradition can be considered an abductive argument. It may also be considered a kind of natural theological argument of the sort described by Alister McGrath in his *Scientific Theology* project. "Natural theology is to be understood as the enterprise of seeing nature as creation, which both presupposes and

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[34] To the objection, "*Will you pretend to prove the Trinity by natural reason?*" Baxter replies, "It is one thing to *prove* the Sacred Trinity of Persons, by such reason (or to undertake *fully to open* the mystery) and it is another thing to prove that the Doctrine is neither *incredible* nor *unlikely* to be true; and that it implieth *no contradiction* or *discordancy*, but rather seemeth *very congruous* both to the frame of nature, and of certain moral verities. This only is my task against the Infidel." Richard Baxter, *The Reasons of the Christian Religion* (London: R. White, 1667), 375.

[35] "Now, over against all those who want to base the doctrine of the Trinity on rational grounds, we must undoubtedly maintain that we owe our knowledge of this doctrine solely to God's special revelation. Scripture alone is the final ground for the doctrine of the Trinity. Reason can at most somewhat clarify this doctrine a posteriori. Nevertheless, the arguments advanced to shed light on the dogma of the Trinity are not devoid of all value. In the first place, Scripture itself gives us the freedom to use them when it says that the entire creation and especially humankind is a work of the triune God. Certainly, all God's works ad extra are undivided and common to all three persons. Prominent in these works, therefore, is the oneness of God rather than the distinction of the persons. In this unity, however, the diversity cannot be lacking. For Scripture itself points to this truth by saying that all created beings will show these imprints and human beings will exhibit the image of the triune God." Herman Bavinck, *Reformed Dogmatics. Volume 2, God and Creation [Electronic Resource]*, with John Bolt and John Vriend (Grand Rapids, Michigan: Baker Academic, 2015), 255–56.

reinforces fundamental Christian theological affirmations." [36] It is neither deductive nor inductive in tenor but abductive.[37] It seeks to show that Christian doctrine is consonant with reason when reason is rightly understood.

Generations before Augustine, Tertullian, the first to coin the word Trinity, had compared the Trinity to a spring, a river, and a stream; a root, a tree, and its fruit; the sun, a ray, and the ray's apex. Tertullian described these as "analogies" of the Trinity.[38] While there is a similarity between Augustine's argument regarding the traces of the Trinity and Tertullian's explanations using analogies of the Trinity, these are slightly different approaches. For Augustine, an analogy of the Trinity such as Tertullian's spring, river, and stream would not have counted as a *vestigium Trinitatis*. These analogies do not have all of the requirements distinguishing a trace of the Trinity, a subtle but important point that will

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[36] Alister E. McGrath, *The Science of God: An Introduction to Scientific Theology* (Grand Rapids, Mich.: William B. Eerdmans Pub., 2004), 113.

[37] Ravi Scott Jain, "Science and Theology as Abductive Responses to Reality," *Science and Christian Belief* 35, no. 2 (October 2023): 158–65.

[38] "For God sent forth the Word, as the Paraclete also declares, just as the root puts forth the tree, and the fountain the river, and the sun the ray…I should not hesitate, indeed, to call the tree the son or offspring of the root, and the river of the fountain, and the ray of the sun; because every original source is a parent, and everything which issues from the origin is an offspring. Much more is (this true of) the Word of God, who has actually received as His own peculiar designation the name of Son. But still the tree is not severed from the root, nor the river from the fountain, nor the ray from the sun; nor, indeed, is the Word separated from God. Following, therefore, the form of these analogies, I confess that I call God and His Word — the Father and His Son — two. For the root and the tree are distinctly two things, but correlatively joined; the fountain and the river are also two forms, but indivisible; so likewise the sun and the ray are two forms, but coherent ones. Everything which proceeds from something else must needs be second to that from which it proceeds, without being on that account separated. Where, however, there is a second, there must be two; and where there is a third, there must be three. Now the Spirit indeed is third from God and the Son; just as the fruit of the tree is third from the root, or as the stream out of the river is third from the fountain, or as the apex of the ray is third from the sun." Tertullian, "Against Praxeas," in *Latin Christianity: Its Founder, Tertullian*, vol. 3, ed. Alexander Roberts, James Donaldson, and A Cleveland Cox, trans. Peter Holmes, Nicene and Post-Nicene Fathers (Buffalo, NY: Christian Literature Co., 1885), bk. 8.

be discussed further in Chapter Two. On the other hand, Augustine does say that all things have a "likeness of God" no matter how remote that likeness is to "the supreme good". [39] What is the relationship between analogy, likeness, and trace? To understand the differences one must understand Augustine's notion of a sign.

Augustine's use of the word *vestigium* involves his notion of sign, since a trace (or footprint, Latin *vestigium*) is a species of sign. [40] Bonaventure indicates the same saying, "vestiges…are divinely given signs." [41] As in the case of a footprint, a sign is ontologically connected by causality to the thing it signifies. A sign carries an imprint of a sort. Bonaventure says, "every effect is a sign of its cause". [42] An imprint or trace of the Trinity is thus somewhat different from an analogy of the Trinity. An analogy of the Trinity, like Tertullian's spring, river, and stream analogy, is a help to conceptualization, an epistemological category. A trace of the Trinity is a feature of creation that is connected to God inasmuch as an effect is to a cause, thus invoking an ontological category. Charles Sanders Peirce, situated within the Augustinian semiotic tradition, says likewise, "a sign

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[39] "for what is there that has not a likeness of God, in proportion to its kind and measure, seeing that God made all things very good, and for no other reason except that He Himself is supremely good? In so far, therefore, as anything that is, is good, in so far plainly it has still some likeness [*similitudinem*] of the supreme good, at however great a distance…" Augustine of Hippo, *On the Holy Trinity*, 11.5.

[40] "[A] sign is a thing which, over and above the impression it makes on the senses, causes something else to come into the mind as a consequence of itself: as when we see a footprint [*vestigium*], we conclude that an animal whose footprint this is has passed by…" Augustine of Hippo, *On Christian Doctrine*, ed. Philip Schaff, trans. J. F. Shaw, A Select Library of the Nicene and Post-Nicene Church Fathers (Grand Rapids, Michigan: Wm. B. Eerdmans Publishing Company, 1978), 1.1.

[41] Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, trans. Zachary Hayes, Works of St. Bonaventure ; II (St. Bonaventure, New York: Franciscan Institute Publications, 2002), 2.11.

[42] Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, 2.12.

must have some real connection with the thing it signifies…"[43] Peirce also contends that signs are connected to their causes.

Throughout the literature thinkers do refer to Augustine's *vestigia*, "memory, understanding, and will" as a psychological analogy of the Trinity even though Augustine did not use this term. Augustine uses the word, likeness (Latin, *similitudinem*) for the relationship many things bear to God as noted above.[44] And Augustine even referred to man as a likeness of the Trinity as well as an image of the Trinity. If a man walking on a beach did not leave just a footprint in the sand but laid his whole body down, one could say that the resulting impression is a likeness of the man as well as an impression. It is both his imprint (a trace or *vestigium*) and a likeness (*similitudinem*). But it is not an analogy. Some signs are both a likeness and an impression.

But likenesses are different from analogies. A picture of a woman is a likeness of her not an analogy. The picture of the woman is ontologically connected to the woman by the woman being the cause of the picture (even if mediated through an artist's brush or a photographer's camera). While mistaking the notion of a likeness for an analogy is an understandable confusion, it is worth disentangling these issues at the start. Augustine is the first one to develop this category of vestiges or traces of the Trinity which can include likenesses but is not identical with the category of likenesses.

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[43] Charles S. Peirce, *The Writings of Charles S. Peirce, a Chronological Edition [Electronic Resource]*., with Max H. Fisch, Edward C. Moore, and Nathan Houser, Past Masters (InteLex Corporation : Online) (Charlottesville, Va.: InteLex Corporation, 2008), 3.66-68.

[44] Augustine of Hippo, *On the Holy Trinity*, 11.5.

Thus, scholars should avoid this common confusion between an argument for the Trinity by its traces and an argument for the Trinity by analogy. The traces of the Trinity argument claims that the traces are an effect of a creating Trinity—a sign of the cause. An argument from analogy says one thing is like another thing because of a similarity. Nonetheless, discovering such traces as sought in the *vestigia Trinitatis* tradition is a difficult and subtle task not least because, as Augustine notes, "the Trinity works indivisibly in everything that God works…" [45] So what are the traces of the Trinity?

Is the Unity in Multiplicity of the World a Consequence of a Creating Trinity?

As the initiator of the doctrine, St. Augustine's characterization of the *vestigia Trinitatis* dominates the tradition. Augustine primarily conceives the *vestigia Trinitatis* as dealing with the souls of humans. In particular, Augustine believes that the human mind's aspects of memory, understanding, and will together are traces of the Trinity. In general, Augustine seeks for a thing which is one substance (i.e., the mind) that also might be considered three substances (i.e., memory, understanding, and will). Triadicity, or unity in triadicity, marks Augustine's search for the *vestigia Trinitatis*.

Because of this focus on the triadicity of things that are one substance, Augustine's psychological Trinity has a strong likeness to the Trinity as well and is also easily

[45] Augustine of Hippo, *On the Holy Trinity*, 1.5.8. The Latin is, "*inseparabiliter operari Trinitatem in omni re quam Deus operator*…" Augustine of Hippo, *De Trinitate*, vol. 42, ed. J. P. Migne, Patrologia Latina (France, 1864), 1.5.8.

considered an analogy of the Trinity by other thinkers. But this thesis will not primarily explore likenesses or analogies of the Trinity such as Tertullian's spring, river, and stream, but instead it will focus on the traces of the Trinity even as some of the traces will also be likenesses. As effects are signs of their causes, so the *vestigia Trinitatis* point to some aspect of the Trinity. But just as footprints are not full likenesses of the man who made them, so the traces of the Trinity need not be likenesses.

What trace of the Trinity in particular will this thesis consider? This thesis will primarily consider whether unity in multiplicity is a latent and implicit theme within the *vestigia Trinitatis* tradition. For example, three things imply a multiplicity of things. Thus, unity in triadicity seems necessarily to imply some conception of unity in multiplicity. If unity in multiplicity is categorically impossible or against reason, then so too is unity in triadicity.

More recently Herman Bavinck has articulated the *vestigia Trinitatis* as unity in diversity.[46] Unity in multiplicity also seems implicit in unity in diversity. A tree is one unified organism with a trunk and diverse leaves. This is not only a statement of unity in diversity, but it is also a statement of unity in multiplicity. Counting the number of leaves will show that they are multiple. Multiplicity is the quantitative counterpart to qualitative diversity. But is Bavinck's conception of the *vestigia Trinitatis* a good one? Even if unity in

[46] Bavinck, *Reformed Dogmatics. Volume 2, God and Creation [Electronic Resource]*, 255–56. See also James Perman Eglinton, *Trinity and Organism: Towards a New Reading of Herman Bavinck's Organic Motif*, T & T Clark Studies in Systematic Theology (London: T & T Clark, 2012), 86.

multiplicity is implied in unity in diversity, perhaps neither of these should be considered as *vestigia Trinitatis*.

The phrase "creating Trinity" can be found in St. Bonaventure's exploration of the doctrine of the *vestigia Trinitatis*. Bonaventure says, "the creature is an effect of the creating Trinity by virtue of a three-fold causality…These, as vestiges of the Creator, are found in all creatures..." [47] Bonaventure includes metaphysical principles such as causality in the doctrine of the *vestigia Trinitatis*. Unity in multiplicity plays a role in many metaphysical topics such as causality. A key question of this thesis is how unity in multiplicity can be signified. If something like "three-fold causality" requires a way of signifying a causality which is both one and many, then the sign-system used to signify causality must be able to signify unity in multiplicity. In other words, if the sign-system (i.e. formal logic) does not allow one to signify the unity in multiplicity of causality, and unity in multiplicity in causality is an aspect of the *vestigia Trinitatis*, then the sign-system precludes conversations about the *vestigia Trinitatis*. If the dominant ways of discussing contemporary metaphysics, i.e. formal logic or mereology, preclude conversations about the *vestigia Trinitatis*, then maybe a different sign-system should be employed when discussing contemporary metaphysics which does not preclude conversations about the *vestigia Trinitatis*.

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[47] Bonaventure, *Breviloquium*, trans. Dominic Monti, Works of St. Bonaventure 9 (St. Bonaventure, New York: Franciscan Institute Publications, 2005), 2.2. See also Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, 1.11.

If unity in multiplicity is not an effect of the creating Trinity, then how to signify the unity in multiplicity is of little or no theological importance. But if all the various conceptualizations of the *vestigia Trinitatis* rest on some implicit or explicit notion of unity in multiplicity, then a sign-system which blocks the signification of unity in multiplicity also blocks the signification of the *vestigia Trinitatis*. Since mathematics and formal systems dominate the discourse of the sciences and analytic philosophy, this would mean that one cannot even discuss the possibility that the world has a Trinitarian imprint in science and metaphysics. If contemplating the *vestigia Trinitatis* is blocked, then human reason is deprived of an encouragement to faith in the Holy Trinity.

The question of how human reason regarding number relates to the Trinity is not an obvious one. Tertullian inquired how number signifies the persons of the Trinity. Tertullian explores the relationship between unity in multiplicity in a tree and a root and says they "are distinctly two things, but correlatively joined". And "everything which proceeds from something else must needs be second to that from which it proceeds…Now the Spirit indeed is third from God and the Son; just as the fruit of the tree is third from the root." [48] The root and the tree are two discrete things but also "correlatively joined" or continuous. In this passage Tertullian considers the way unity and number applies to the Godhead, especially as God is one substance and three persons.

[48] See also footnote 38. Tertullian, "Against Praxeas," bk. 8.

Augustine explores a similar theme in *De Trinitate*. He notes that we must understand the Creator, "by the things that are made [and thus] we must needs understand the Trinity of whom there appear traces in the creature". [49] In so doing Augustine sanctions a positive case for theological understanding of the Trinity by its traces. But as Augustine continues he warns, "here in corporeal things, one thing alone is not as much as three together, and two are something more than one; but in that highest Trinity one is as much as the three together, nor are two anything more than one." [50] Augustine uses notions of unity and multiplicity, one, two, and three in his language of the Godhead like Tertullian, but he distinguishes the corporeal understanding of three from the way the term three refers to God. To say that the divine persons are three is a meaningful statement, but it does not mean exactly the same thing as saying that a corporeal quantity is three. As the Athanasian Creed states, "the Father is Lord, the Son Lord, and the Holy Ghost Lord. And yet not three Lords, but one Lord." [51] Thus, to explore the notion of unity in multiplicity as a *vestigium Trinitatis* by no means suggests that unity in multiplicity means exactly the same thing in reference to creation and in reference to the Holy Trinity.

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[49] Augustine of Hippo, *On the Holy Trinity*, 6.10.12.

[50] Augustine of Hippo, *On the Holy Trinity*, 6.10.12.

[51] Philip Schaff, tran., "Athanasian Creed," in *Creeds of Christendom, Volume 2: The Creeds of the Greek and Latin Churches* (New York: Harper & Brothers Franklin Square, 1877), 67, http://archive.org/details/creedsofchristen0002unse_g3q6.

Having considered the nature of discrete and continuous quantity as well as the doctrine of the *vestigia Trinitatis*, the central linked questions of this thesis can now be posed.

Is Quantity that Signifies the World Irreducibly Discrete and Irreducibly Continuous, Exhibiting Unity in Multiplicity and, if so, Can Such Quantity be Regarded as a *Vestigium Trinitatis*?

The 'orthodox' position in contemporary mathematics holds that only discrete quantity is a truly basic irreducible primitive and continua can be modeled by infinite sets of discrete elements, e.g., an infinite point-set to model a continuous line. Things which are modeled as continuous unities are then always reducible to an infinite set of discrete unities. But if continuous unities in reality actually possess different properties than a multitude of discrete unities (e.g. the fusion of boundaries), then these properties are not adequately represented by treating these unities merely as a multitude of discrete quantity. On the opposite side, if discrete quantity is treated merely as a model built upon the truly basic notion of irreducible continuous quantity, then any properties unique to discrete unities (e.g. their indivisibility) are lost in a model which only treats continuity as basic.

Many metaphysical positions throughout history have emphasized the interplay of the one and the many in reality. At times a thing may seem to be a unity and at times the same thing may seem to be a multiplicity. A tree is one thing but has many branches, leaves, and roots. Some may contend that the tree is only a multiplicity, a multitude of atoms. And some may contend that the tree is only a single organism. But if the tree is considered simultaneously as a unity and a multiplicity, or a unity in multiplicity, then how can that be signified in mathematics or formal systems? A tree is a discrete organism but it

has continuously connected parts. In one sense the tree is discrete, in another sense the tree is continuous with potentially many parts. Are discrete quantity and continuous quantity both needed by mathematics and formal systems in order to signify the simultaneous unity and multiplicity of the tree?

To suggest that the tree is merely a multitude of atoms is often associated with forms of physicalist reductionism. To suggest that the tree has no parts is sometimes associated with idealist monism.[52] But to suggest that a tree is simultaneously one and many, a unity in multiplicity, has less clear philosophical implications. If unity in multiplicity is an aspect of the *vestigia Trinitatis*, then perhaps seeing this unity in multiplicity can be an encouragement to Christian faith. But if the most prominent systems of mathematics and formal logic do not permit the tree to be signified as a unity in multiplicity, then this form of apologetic argument would be cut off.

It is for this reason that this thesis raises its central linked questions. Is quantity that signifies the world irreducibly discrete and irreducibly continuous, exhibiting unity in multiplicity and, if so, can quantity be regarded as a *vestigium Trinitatis*? A positive answer to this question would not prove the Holy Trinity, nor would a negative answer disprove it. To conclude that quantity that signifies the world is irreducibly discrete and irreducibly continuous and thus exhibits unity in multiplicity would interest some philosophers. Even so, this conclusion may imply little to those who do not already believe in the Trinity. But

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[52] For example, Bertrand Russell charged F.H. Bradley with a form of Hegelian idealist monism and responded by developing his logical atomism. Bertrand Russell, *My Philosophical Development* (London: G. Allen & Unwin, 1959), 54–64.

for those Trinitarian Christians that work with mathematics and formal systems, a link between discrete and continuous quantity and the Trinity would comport with their seeing the world "as creation," the project of natural theology as described above by McGrath. Moreover, if the world indeed does have a Trinitarian imprint, then describing the world the way it actually is provides greater insights to the sciences and philosophies which seek to understand the world.

This argument also has important limits. This thesis considers only "quantity that signifies the world" or applied quantity. Some who consider themselves pure mathematicians may like G.H. Hardy contend they have "never done anything 'useful,'" and that "[n]o discovery of [theirs] has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world". [53] To these pure mathematicians who are concerned only with ideal quantities, this thesis has little to say. It may be the case that ideal quantity *that does not signify the world* is also a *vestigium Trinitatis*. But this thesis does not make that argument. On the other hand, if the work of some pure mathematician is unexpectedly employed in order to signify the world, then this thesis may still be of relevance. In the second half of the third chapter, formal systems like set theory are considered, and these systems though initiated by pure mathematicians were eventually used to signify the world. Because cases like this instance of crossover between

[53] G. H. Hardy, *A Mathematician's Apology*, Canto Classics (Cambridge: Cambridge University Press, 2012), 150.

pure mathematics and applied mathematics are more abundant than G.H. Hardy's boast implies, a cautious pure mathematician may still find this thesis of interest.

Mary Hesse and Charles Sanders Peirce as Dialog Partners

Why consider the thought of Mary Hesse and Charles Sanders Peirce in this thesis? Mary Hesse (1924-2016) was a Cambridge professor of philosophy of science whose primary interest was the physical sciences in which mathematics is heavily used. In her book *Forces and Fields* she suggests that "atomicity and continuity" form a pair of "apparently contradictory principles which continually reappear in the history of science."[54] Examining Hesse's work permits a deeper consideration of discreteness and continuity as signified by mathematics in the physical sciences. In addition, Hesse as a practicing Anglican participated in interfaith dialog and engaged others even on the doctrine of the Trinity. Thus, Hesse's thought provides a good reference to ascertain whether the signification of discrete and continuous quantity by mathematics in the physical sciences may point to the Trinity.

C.S. Peirce (1839-1914) was an American thinker influential in the early stages of the development of mathematical logic. Some have credited Peirce both with the discovery of logical quantifiers and the discovery of the logic of relations, topics of utmost importance

[54] Hesse, *Forces and Fields*, 293.

in philosophical logic.[55] Others praise him as the father of contemporary semiotics (of relevance to signification) and the originator of abductive reasoning.[56] Peirce had a high view of both continuity and discreteness and believed that each played a role in

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[55] Hilary Putnam credited Peirce as the 'effective discover' of logical quantifiers and Alfred Tarski credited him with discovering the logic of relations. Hilary Putnam comments on the widespread influence of Peirce and Schröder. "The success they experienced was impressive. While, to my knowledge, no one except Frege ever published a single paper in Frege's notation, many famous logicians adopted Peirce-Schröder notation, and famous results and systems were published in it. Löwenheim stated and proved the Löwenheim theorem…in Peircian notation. In fact, there is no reference in Löwenheim's paper to any logic other than Peirce's. To cite another example, Zermelo presented his axioms for set theory in Peirce-Schröder notation, and not, as one might have expected, in Russell-Whitehead notation…Since I have mentioned Peano, I should remark that he was not only well acquainted with Peirce-Schröder logic, but he had actually corresponded with Peirce…I looked at Whitehead's *Universal Algebra*. This is a work squarely in the tradition to which Boole, Schröder, and Peirce belonged, the tradition that treated general algebra and logic as virtually one subject. And here, before Whitehead worked with Russell, there is no mention of Frege, but there is a citation of "suggestive papers" by Peirce's students O. H. Mitchell and Christine Ladd-Franklin. The topic, of course, is the quantifier. In sum, Whitehead certainly came to his knowledge of quantification through 'Peirce and his students.'". Hilary Putnam, "Peirce the Logician," *Historia Mathematica* 9, no. 3 (1982): 297–98, 3, https://doi.org/10.1016/0315-0860(82)90123-9.

Regarding his role in the logic of relations Tarski says, "The title of creator of the theory of relations was reserved for C. S. Peirce. In several papers published between 1870 and 1882, he introduced and made precise all the fundamental concepts of the theory of relations and formulated and established its fundamental laws. Thus Peirce laid the foundation for the theory of relations as a deductive discipline; moreover he initiated the discussion of more profound problems in this domain… It is true that A. N. Whitehead and B. Russell, in *Principia Mathematica*, included the theory of relations in the whole of logic, made this theory a central part of their logical system, and introduced many new and important concepts connected with the concept of relation…[Nonetheless,] *Principia Mathematica* contributed but slightly to the intrinsic development of the theory of relations as an independent deductive discipline." Alfred Tarski, "On the Calculus of Relations," *The Journal of Symbolic Logic* (New York, USA) 6, no. 3 (1941): 73–89, https://doi.org/10.2307/2268577.

[56] Regarding abduction Fisch says, "Peirce was such a student of the *Organon*, and he detected what he thought were "two blunders (7.248) in Book II of the Prior Analytics. When these blunders are corrected, Aristotle's theory of induction (*epagoge*) in Chapter XXIII, of analogy (*paradeigma*) in XXIV, and of abduction (*apagoge*) in XXV, becomes a coherent and intelligible body of doctrine, and the otherwise immense superiority of the Epicurean logic in respect of synthetic inference is somewhat reduced." Regarding semiotics Deely says, "In modern times, after Descartes (1596-1650), the hard-won Latin notion of sign in general disappears, to be replaced by the notion of ideas as self-representing objects, until the Latin notion is taken up again in the writings of Charles Sanders Peirce (1839-1914) under the banner of 'semiotics'" Max H. Fisch, "Peirce's Arisbe: The Greek Influence in His Later Philosophy," *Transactions of the Charles S. Peirce Society* (Buffalo, N. Y) 7, no. 4 (1971): 202. John N. Deely, *Four Ages of Understanding: The First Postmodern Survey of Philosophy from Ancient Times to the Turn of the Twenty-First Century*, Toronto Studies in Semiotics (Toronto, Ontario: University of Toronto Press, 2001), 740.

mathematics, formal systems, and metaphysical concepts. Moreover, while he was raised as a Unitarian, he converted to Trinitarianism early in his adulthood after philosophical reflection. Peirce sometimes called his philosophy thirdness and at times his philosophy seemed to point him to the Trinity as a sort of abductive argument (similarly to how the *vestigia Trinitatis* argument functions).[57] Thus, Peirce provides an excellent dialog partner to examine whether any aspect of discrete and continuous quantity in mathematics, formal systems, and metaphysics may point to the Trinity.

Since this thesis is broad-ranging and controversial, before an affirmative response to the question is considered, the seriousness of the objections should be examined. The objections are many and substantial.

[57] Andrew Robinson and Christopher Southgate also explore links between Peirce's philosophy of Thirdness, the Trinity, and the doctrine of the *vestigia Trinitatis*, commending this exploration as a fruitful "research program". Andrew John Nottage Robinson and Christopher Southgate, "Semiotics as a Metaphysical Framework for Christian Theology," *Zygon* 45, no. 3 (September 2010): 690.

Chapter Two: Objections

This thesis raises the question, is quantity that signifies the world irreducibly discrete and irreducibly continuous, exhibiting unity in multiplicity and, if so, can such quantity be regarded as a *vestigium Trinitatis*? Consider three reasons why the answer to this question is no, quantity that signifies the world is not irreducibly discrete and irreducibly continuous; it does not exhibit unity in multiplicity; and therefore such quantity should not be considered as a *vestigium Trinitatis*. The fist two objections are from the domains of mathematics and philosophical logic, the last objection is from the domain of theology.

The first objection contends that mathematics and formal systems need only discrete quantity as an irreducible primitive and therefore quantity is not also of a kind that is irreducibly continuous. This objection is compelling because most mathematicians today derive continuous quantity from discrete quantity, and thus treat only discrete quantity as an irreducible primitive. In the 19th century, Georg Cantor and Richard Dedekind posited a mathematical continuum composed of discrete points, and their usage has become the standard one especially through set theory. Seemingly continuous quantities in the world can be signified by using the real numbers as constructed upon discrete number. Because of set theory's central role in both mathematics and formal logic, discrete quantity has also become basic in formal systems like the predicate logic used by analytic philosophers. Thus, discrete quantity as used in formal systems can also represent the world through philosophical ideas which contemporary metaphysicians wish to signify.

The second objection is the reverse of the first. Mathematics and formal systems need only continuous quantity as an irreducible primitive and therefore quantity is not also

irreducibly discrete. Many contend that discrete quantity such as the natural numbers are constructible from continuous quantity alone. How so? Consider the marks on a ruler. In any apparent continuum such as a length, marks can be made which measure off discrete numbers. Advocates of this position range from Simon Stevin and René Descartes in the 17th century to Cesare Burali-Forti in the early 20th century. Thus, in mathematical contexts discrete numbers can be derived from continuous quantity.

In addition, mereology is a wide-ranging formal system used by analytic philosophers and mathematicians to discuss wholes and their parts. Mereology can signify aspects of continuity more easily than set theory because of its parthood relation which it holds as basic which is inspired by traditional continuity. In traditional mereology wholes are composed of divisible parts which have properties similar to a continuum such as non-unique decompositions. Thus, wholes and discrete quantity can be constructed from continuous quantity, and therefore only continuous quantity is needed as an irreducible primitive in mathematics and formal systems.

The third objection is that even if quantity that signifies the world is irreducibly discrete and irreducibly continuous such quantity is unrelated to the *vestigia Trinitatis*. When St. Augustine first employed the phrase *vestigia Trinitatis* he used it primarily of human persons. He acknowledged only the slightest possibility of the *vestigia Trinitatis* outside of human persons. Thus, the idea that any aspect of the world beyond human persons is a *vestigium Trinitatis* does not comport with Augustine's notion. Moreover, even if non-human aspects of creation are potential *vestigia Trinitatis* as advocated by Bonaventure, Aquinas and other theologians, irreducibly discrete and irreducibly

continuous quantity are not highlighted by them. Thus, the mutual irreducibility of quantity seems irrelevant to any plausible *vestigia Trinitatis* suggested in the tradition.

First Objection: Mathematics and Formal Systems Need only Irreducibly Discrete Quantity to Signify the World

The most forceful objection to the thesis that quantity that signifies the world is irreducibly discrete and irreducibly continuous is the claim that continuous quantity can be reduced to discrete quantity. While a black line drawn on a piece of white paper may look continuous, when examined under a microscope at sufficient resolution the seemingly continuous mark appears full of skips and jumps. The apparent continuity is an illusion which can be ignored. If this is true of all seemingly continuous entities, then in reality only discrete quantity is needed to signify the unity in multiplicity of these seemingly continuous entities and only discrete quantity is needed as basic in mathematics and formal systems. Mathematics has a new notion of continuity, Cantor-Dedekind continuity, constructed upon discrete number which accommodates all the previous mathematics based on the outdated notion of Aristotelian continuity. Does mathematics really need continuous quantity as a basic irreducible primitive? Instead cannot continuous quantity just be derived showing that continuous quantity is reducible to discrete quantity? Thus, it seems that a mathematics which treats only discrete quantity as a basic irreducible primitive can signify everything in the world that previously was signified by continuous quantity.

There are two main reasons to conclude that discrete quantity is the only irreducible primitive needed by mathematics and formal systems to signify the world. The first reason comes from mathematics, the second reason comes additionally from philosophical logic and formal systems in general. First, the methods of Georg Cantor and Richard Dedekind

provide examples of the construction of continuous quantity from discrete quantity in mathematics. The vast majority of mathematicians today, following Cantor and Dedekind, conceive a continuous line as a set of an infinite number of discrete points. This conception contradicts the Aristotelian claim which held for millennia that a continuum cannot be made out of indivisibles such as points. Thus, because mathematicians can reduce continuous quantity to discrete quantity, it seems only discrete quantity is needed as a basic irreducible primitive in mathematics.

Second, philosophical logic provides another reason why only discrete quantity is needed in mathematics and formal systems. The mathematical approach of Cantor and Dedekind also leads to the exclusive use of discrete quantity as an irreducible primitive in set theory. What is the link between mathematics, logic, and set theory? Set theory is foundational for much of mathematics and sets are also widely used in philosophical logic. In mathematics set theory can represent continuity by discrete quantity. But outside of mathematics is set theory able to represent all aspects of the world, including those philosophical concepts which are seemingly continuous? Set theory is designed to represent the unity of a diverse array of things. The multiplicity of things is made one by the set. Because of this unifying power of set theory which uses only discrete quantity, only discrete quantity is needed as basic in mathematics and formal systems.

A particular example of how seemingly continuous things can be reduced to discrete things through set theory illustrates the point. What seemingly continuous things need to be represented in philosophical logic through set theory? Identity-over-time and substance are each conceptions requiring notions of continuity. But the seeming continuity of identity-over-time, can be represented by a set composed of a time-sequence of snapshots of an

entity over many discrete instants. While in this approach the continuity of identity is an illusion, it can be signified by the discrete quantities used in sets. In addition to identity-over-time another notion that seems to have continuous aspects is a substance. Russell's use of sets can also represent a substance. Since set theory can represent these two important and seemingly continuous notions of identity-over-time and substance, the other seemingly continuous conceptions of philosophical logic can likely also be represented by set theory. Since set theory only uses discrete quantity to signify these seemingly continuous concepts, only discrete quantity is needed in formal systems. Thus, because only discrete quantity is needed in mathematics and formal systems to signify seemingly continuous things in the world, there is no need to treat continuous quantity as irreducible.

Reducing Continuous Quantity to Discrete Quantity in Mathematics

Can discrete quantity alone adequately signify the world? For more than two thousand years the key thinkers in the Western mathematical tradition employed two co-principles of quantity: discrete quantity and continuous quantity. Both were equally basic and neither could be fully reduced to the other. But in the nineteenth century mathematicians concluded that only discrete quantity was required for mathematics. "Gauss and Cauchy, Bolzano and Dirichlet, Weierstrass, Heine, Cantor, Dedekind, Peano, and others had laboured throughout the century to put the theory of real numbers on a strictly arithmetical (and ultimately set theoretic) foundation, and to exclude geometric intuition

from the foundations of the calculus." [58] These thinkers showed that continuous quantity can be reduced to an infinite set of discrete numbers, and a linear continuum can be reduced to an infinite set of discrete points. By doing this they demonstrated that only discrete quantity is needed as basic in mathematics. Cantor and Dedekind were the leading advocates of treating the linear continuum as an infinite set of discrete points, thereby developing the "arithmetic continuum". [59] Consider their approach.

What precisely does it mean to reduce continuous quantity to discrete quantity in mathematics? Consider two aspects: 1) a continuous magnitude must be composable from discrete quantity and 2) boundary values of regions should be the same sort of things as the interior parts of regions. Cantor and Dedekind both conceive of a continuum as a set of points reflecting the first aspect. And Dedekind's work especially displays the implications of the point-set approach for boundary values which addresses the second aspect of reducing continuous quantity to discrete quantity. Consider Cantor's ideas first.

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[58] William Bragg Ewald Jr. and William Bragg Ewald, *From Kant to Hilbert Volume 2* (Oxford, United Kingdom: Oxford University Press, 2007), 956, http://ebookcentral.proquest.com/lib/oxford/detail.action?docID=422427.

[59] "[T]he modem Arithmetico-set-theoretical conception of a real number emerged during the latter half of the 19th century when a number of mathematicians including Cantor (1872) and Dedekind (1872) introduced theories of real numbers that were designed to be independent of the former notion, and intended to make possible an analysis which banished the latter. The newly constructed ordered field of real numbers was dubbed the arithmetic continuum because it was held that this number system is completely adequate for the analytic representation of all types of continuous phenomena. In accordance with this view, the geometric linear continuum was assumed to be isomorphic with the arithmetic continuum, the axioms of geometry being so selected to insure this would be the case. In honor of Cantor and Dedekind, who first proposed the thesis, the presumed correspondence between the two structures has come to be called the Cantor-Dedekind axiom." Ehrlich, *Real Numbers, Generalizations of the Reals, and Theories of Continua*, vii–viii.

Cantor's Axiom asserts that the Continuum can be Represented by Discrete Number

Cantor's continuum utilizes "point-sets," which mean sets of discrete indivisible points. The points are designated by the real numbers. Not all sets of points qualify as continua though. The set of points designated along a number line by the numbers {3, 3.14, π, 4}, is clearly not a continuum as it is merely a finite set of four points. An infinite point-set of integers, e.g. {…-2, -1, 0, 1, 2, 3…}, is also discontinuous. Sets like the integers or the rational numbers are only countably infinite, meaning that their elements can be put into a one-to-one matching relation with the natural numbers. "[A]ll infinite point-sets P have either the power [cardinal-number] of the first number-class (I) or the power [cardinal-number] of the second number-class (II)". [60]

The cardinal number of the first number-class is what is today known as $\aleph_0$ (aleph-null), and the cardinal number of the second number-class is greater than $\aleph_0$, what would be known today as $\aleph_1$ (aleph-one) or greater. Sets with cardinality $\aleph_0$, aleph-null, are countably infinite, which means their elements can be put into a one-to-one correspondence with the natural numbers. Sets with cardinality $\aleph_1$ or greater are uncountably infinite, meaning elements of these sets cannot be put into a one-to-one correspondence with the natural numbers. The real numbers are uncountably infinite and thus have a cardinality above $\aleph_0$

[60] Georg Cantor, "Cantor's Grundlagen," in From Kant to Hilbert Volume 2, ed. William Bragg Ewald (Oxford, United Kingdom: Oxford University Press, 2007), 905, http://ebookcentral.proquest.com/lib/oxford/detail.action?docID=422427.

(perhaps the cardinality of $\aleph_1$, if not greater than $\aleph_1$). According to Cantor, for a point-set to qualify as a continuum it must have the cardinality of $\aleph_1$ or greater.

While developing his notion of cardinality, Cantor posits the continuum hypothesis, a notoriously evasive theory that has eluded direct proof by the best mathematicians.[61] A subtle issue in the continuum hypothesis that is seldom addressed is that the name itself, "continuum hypothesis," contains an assumption which is not proved. The term "continuum" in the phrase "continuum hypothesis" draws from an unproven axiom. Cantor explains the axiom,

> … to every numerical magnitude [*Zahlengröße*] there corresponds a definite point of the line, whose coordinate is equal to that numerical magnitude.
>
> I call this proposition an *axiom*, since it is in the nature of this statement that it cannot be proven.[62]

Cantor says that the identification of the points on a line with numbers or "numerical magnitudes" cannot be proven. That the point-set of a line corresponds to the set of real numbers is axiomatic for Cantor and does not require an argument with a definite and certain conclusion.

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[61] Cantor, "Cantor's Grundlagen," 904–5. Formally, the continuum hypothesis has been proven independent of the axioms of Zermelo-Frankel set theory through the work of Kurt Gödel and Paul Cohen.

[62] "...zu jeder Zahlengrösse ein bestimmter Punkt der Geraden gehört, dessen Coordinate gleich ist jener Zahlengrösse, und zwar in dem Sinne gleich, wie solches in diesem [Section] erklart wird.

Ich nenne diesen Satz ein Axiom, weil es in seiner Natur liegt, nicht allgemein beweisbar zu sein." Georg Cantor, "Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen," Mathematische annalen 5, no. 1 (1872): 128, https://doi.org/10.1007/BF01446327. Translation provided in Akihiro Kanamori, "Cantor and Continuity," in The History of Continua: Philosophical and Mathematical Perspectives, ed. Stewart Shapiro and Geoffrey Hellman (Oxford University Press, 2020), 226–27, https://doi.org/10.1093/oso/9780198809647.003.0010.

Thus, Cantor's continuum is composed of an actual uncountable infinity of points and these points according to Cantor can be put into a one-to-one correspondence with the real numbers. Cantor also believes, though he does not prove, his axiom that "to every numerical magnitude there corresponds a definite point of the line." [63] This means that for Cantor the cardinality of the continuum is the same as the cardinality of the real numbers. And therefore to hypothesize on the cardinality of the real numbers is the same as to hypothesize on the cardinality of the continuum. When considering discussions by mathematicians about the continuum hypothesis, one observes that the material of their study is the real numbers not mathematical objects traditionally associated with continuity like lines, areas, or durations. The continuous nature of the real numbers is assumed by Cantor's axiom.

In summary, Cantor, considered the father of set theory, holds axiomatically that the continuum is composed of indivisibles, like a line composed of points. Cantor and the tradition he established maintain that continuous quantity can be reduced to discrete quantity, and therefore only discrete quantity is needed as a basic irreducible primitive in mathematics.

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[63] Note, that this axiom is not the continuum hypothesis. The continuum hypothesis is a different concept altogether. See also footnote 61. In laymen's terms the continuum hypothesis holds that there is no intermediate cardinality between that of the natural numbers and that of the real numbers (which are assumed to be continuous).

Dedekind's Axiom and Boundary Values as Number-Points

Next consider the character of boundary values as exemplified in the work of Richard Dedekind. Dedekind's notion of a continuum shares many properties with Cantor's notion, especially the property that the continuum is composed of points. But the emphasis of Dedekind's definition is different:

> I find the essence of continuity… in the following principle: "If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions…
>
> We can then say that every rational number 'a' produces one cut or, strictly speaking, two cuts, which, however, we shall not look upon as essentially different"[64]

In this passage, Dedekind's interest in the "straight line" regards its "points" and their relations. "[T]he essence of continuity" requires a division, which later comes to be known as a Dedekind cut, a topic often taught in contemporary advanced calculus or analysis. When Dedekind says, "there exists one and only one point which produces this division of all points into two classes," the point he means is the cut and it defines a real number. At first, Dedekind refers to the cut (the division) as a point, "only one point which produces this division". But immediately after Dedekind associates the cut with a rational number

[64] Richard Dedekind, "Continuity and Irrational Numbers," in *From Kant to Hilbert Volume 2*, trans. William Bragg Ewald (Oxford, United Kingdom: Oxford University Press, 2007), 771–72, http://ebookcentral.proquest.com/lib/oxford/detail.action?docID=422427.

"every rational number 'a' produces one cut". Notice that, as with Cantor, Dedekind too maintains the correspondence of a point to a number.

Dedekind says there are two ways to get this boundary point which makes the cut, "which…we shall not look upon as essentially different." What does he mean by this? For Dedekind the cut itself is defined by the two classes. If a Dedekind cut designates a rational number, then the point itself at which the cut occurs must belong to either the left class or the right class of numbers. Here the word class is interchangeable with the word set.

To understand the concept of a Dedekind cut, consider dividing up the real numbers on the interval $(-\infty, \infty)$ by the x-value of 1. In today's notation an open bracket excludes an endpoint and a closed bracket includes it. Therefore cutting the line at $x=1$ would lead to two intervals. The first interval according to Dedekind could be $(-\infty, 1)$ meaning all the numbers less than the number one. And then the second interval would be $[1, \infty)$ meaning all the numbers equal to one or greater than it. On the other hand, the closed bracket (indicating the adjacent number is included in the interval) could belong to the first interval yielding $(-\infty, 1]$ and $(1, \infty)$. In other words, the point $x=1$ cannot simultaneously be contained by both intervals under the Dedekind understanding of points and real numbers. This is why Dedekind says, "strictly speaking, two cuts, which, however, we shall not look upon as essentially different." The number 1, a rational number, cannot simultaneously belong to both classes. Because of this ambiguity of where the point itself belongs, there are two ways to create the classes for Dedekind defining a rational number.

There is a binary character to the options here. If the point is the member of one class (or set), then it cannot be a member of the other class. It is excluded from one class or the other. In this case, the point at $x=1$ is a boundary value of these two intervals, and including

the point in one interval means that it must be excluded from the other interval. This is because the line is nothing more than points, and the endpoint of a line is the same kind of part of the line (a point) as the parts of the rest of the line (which is composed of points). This is the most simple example of the exclusionary character of boundary values in the Cantor-Dedekind concept of continuity.[65] But it also applies in a similar manner to other boundaries such as any straight or curved perimeter line bounding an area or any surface bounding a volume.

In the pre-Dedekind traditional understanding of a continuum, a boundary value is a different sort of thing than the continuous region itself. For example, the boundary of a volume is a surface which is dimensionally different from the subregions of the volume which are volumes. The boundary of a line is a point, which is dimensionally different from the subregions, or parts, of the line which are line segments. But for Dedekind, the boundary values are exclusionary because they are merely point-parts of the line in the same way the rest of the line is composed of point-parts. Moreover, if a line is composed of points and an area is composed of lines and a volume is composed of areas, then all of these, whether volumes, areas, or lines, are actually just composed of sets of points arranged in various dimensions.

[65] Dedekind also has a second kind of a Dedekind cut which defines an irrational number, but his explanation adds nothing more to the present argument.

Like Cantor, Dedekind too wishes to find for continuity a "true origin in the elements of arithmetic and thus…to secure a real definition of the essence of continuity." [66] Along with Cantor, Dedekind also claims the basis for his "essence of continuity" is axiomatic:

> … I am unable to adduce any proof of its correctness, nor has anyone the power. The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity, by which we think continuity into the line.[67]

Dedekind likens the cuts defining the real numbers and the cuts defining a system of points on a line. Dedekind asserts that the essence of continuity is in the similar property between the cuts as the real numbers and the cuts as points of a line. He does not seek nor offer a "proof", because like Cantor he claims it is an "axiom". Thus Dedekind describes a point-based continuity which is similar to Cantor's point-set approach. Dedekind maintains that continuous quantity can be derived from discrete quantity alone. Therefore through Dedekind's definition of the real numbers and through his axiom Dedekind reduces continuous quantity to discrete quantity and asserts that only discrete quantity is truly basic in mathematics.

Later commentators have called the identification of the points on a line with the real numbers the Cantor-Dedekind axiom.[68] That the two mathematicians agree with each other upon the axiomatic nature of this claim is evident in the preface to Dedekind's essay.

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[66] Dedekind, "Continuity and Irrational Numbers," 767.

[67] Dedekind, "Continuity and Irrational Numbers," 771–72.

[68] It was called the Cantor-Dedekind axiom as early as 1898. Emmylou Haffner and Dirk Schlimm, "Dedekind on Continuity," in The History of Continua: Philosophical and Mathematical Perspectives, ed. Stewart Shapiro and Geoffrey Hellman (Oxford University Press, 2020), 267, https://doi.org/10.1093/oso/9780198809647.003.0011.

Dedekind, having just read Cantor's 1872 paper which introduces Cantor's version of the axiom, says, "the axiom given in Section II of [Cantor's] paper, aside from the form of presentation, agrees with what I designate in Section III as the essence of continuity."[69]

Dedekind and Cantor are likely drawing on earlier intuitions of a number line.[70] Cantor is certainly aware of atomistic alternatives to the traditional approach associated with Aristotle and Euclid.[71] And Cantor and Dedekind make explicit their reduction of continuous quantity to discrete quantity and defend it in a way that establishes this reduction as the new norm for mathematics as a discipline. This new norm has largely persisted into the 21st century with the majority of mathematicians today holding only discrete quantity as basic.

Together these efforts from Cantor and Dedekind show two things. First, they each display the reduction of the continuum to a set of discrete points. Second, the work of Dedekind especially illustrates how a boundary value like a point is exclusionary, belonging only to one set or the other but not both simultaneously. In these two aspects

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[69] Dedekind, "Continuity and Irrational Numbers," 767.

[70] The prominence of these ideas in Europe increased greatly in the 16th century. The first handbook of geometry in the English language in 1551 also departs from Euclid. "Nowe of a great number of these prickes, is made a Lyne, as you may perceiue by this form ensuying. ……..

Where as I have set a number of prickes, so if you with your pen will set in more other prickes between euerye two of these, then will it be a lyne, as here you may see _____ and this lyne, is called of Geometricians, Lengthe without breadth." Robert Recorde, *Pathway to Knowledge* (London: R. Wolfe, 1551), 587.

[71] Cantor is aware not only of Aristotle's views but also of the views of the atomists "Epicurus and Lucretius construct [a continuum] out of their atoms considered as finite things." See Section 10. Cantor, "Cantor's Grundlagen," 903.

they demonstrate how modern mathematics can reduce continuous quantity to discrete quantity. This reduction shows that discrete quantity alone is needed to signify the world through mathematics.

Logic and Set Theory Need Only Discrete Quantity to Signify the World

Philosophers often use logic and set theory to signify the world. Though an expansive review of logic for philosophers is beyond the scope of this argument, consider an example to illustrate how discrete quantity is sufficient in philosophical logic to signify the world. The way identity-over-time is represented in mathematical logic and set theory has implications for argumentation in all of philosophy. In the 18th century Hume explains that there is no true connectedness or continuity in what is called identity-over-time. Then in the 20th century Bertrand Russell is able to represent this Humean insight through logic and set theory. Since identity-over-time does not need continuous quantity, neither do the popularly conceived notions of substance or persons which depend on identity-over-time.

Continuity Not Needed in the Conception of Identity-over-Time

David Hume's conception of identity influences later notions of identity in the British philosophical tradition. Moreover, Hume's conception of identity appears to flow from and resemble his better known critique of causality. Going against centuries of tradition, Hume argues that what is popularly conceived of as causality is really just a habit of the mind, "the understanding never observes any real connexion among objects, and that even the union of cause and effect, when strictly examin'd, resolves itself into a customary

association of ideas." [72] Hume dispenses with causality because there is no observed "real connexion among objects". Continuity or "real connexion" is not observed, it is only inferred. Because one cannot prove continuity based on one's sense observations, it is not real.

Hume repeats this notion more than once and describes why objects cannot be in a relationship of continuity. "I immediately perceive, that they are *contiguous* in time and place, and that the object we call cause *precedes* the other we call effect." [73] When two discrete objects hit each other, like one billiard ball striking another, they will be in contact or "contiguous" in "place". Though the spatial contiguity of the situation is obvious, his more controversial claim is that they are "contiguous" in "time". The state of affairs before the balls strike is different from the state of affairs after striking. And the state of affairs beforehand precedes the state of affairs afterwards. Hume suggests there is no continuity between the state of affairs beforehand and the state of affairs afterwards. They are merely contiguous and successive states of affairs. Aristotle claims that contiguity is necessary for continuity but not sufficient for continuity.[74] Hume demonstrates that all one has sensible access to is contiguity and succession.

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[72] David Hume, *David Hume: A Treatise of Human Nature*, Second edition., with L. A. Selby-Bigge and P. H. Nidditch, Clarendon Edition of the Works of David Hume (Oxford: University Press, 2014), 1.4.6.17.

[73] Hume, *David Hume*, 1.3.14.1.

[74] "The continuous is a subdivision of the contiguous: things are called continuous when the touching limits of each become one and the same and are, as the word implies, contained in each other: continuity is impossible if these extremities are two." Aristotle, "Physics," in *The Complete Works of Aristotle: The Revised Oxford Translation*, vol. 1, trans. Jonathan Barnes, Bollingen Series 71 (Princeton ; Guildford: Princeton University Press, 1984), 227a.

Hume's demonstration that temporal contiguity does not entail that there is a real connectedness between the cause and the effect has many implications, some of which he enumerates. A key concept Hume examines is identity-over-time. "We now proceed to explain the nature of *personal identity*, which has become so great a question in philosophy, especially of late years in *England*…The identity, which we ascribe to the mind of man, is only a fictitious one, and of a like kind with that which we ascribe to vegetables and animal bodies."[75] Identity is only a multiplicity of perceptions, there is no true unity among them. The unity is only an assumption of the mind. "the identity, which we attribute to the human mind, however perfect we may imagine it to be, is not able to run the several different perceptions into one, and make them lose their characters of distinction and difference, which are essential to them."[76] The truth about identity is multiplicity: "several different perceptions". It is the mind that runs them "into one". But the unity does not exist in the thing, only in the mind.

Hume explains in further detail, when "we suppose the whole train of perceptions to be united by identity, a question naturally arises concerning this relation of identity; whether it be something that really binds our several perceptions together, or only associates their ideas in the imagination…identity is nothing really belonging to these different perceptions, and uniting them together; but is merely a quality, which we attribute to them, because of the union of their ideas in the imagination, when we reflect upon

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[75] Hume, *David Hume*, 1.4.6.16.

[76] Hume, *David Hume*, 1.4.6.17.

them." [77] Not only is identity something attributed to the human mind, but it is "in the imagination".

> The thing itself which some claim to have identity is not really self-connected:
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> > These are the uniting principles in the ideal world, and without them every distinct object is separable by the mind, and may be separately consider'd, and appears not to have any more connexion with any other object, than if disjoin'd by the greatest difference and remoteness. 'Tis, therefore, on some of these three relations of resemblance, contiguity and causation, that identity depends.[78]

The "uniting principles" of the world of ideas are based on the "relations of resemblance, contiguity and causation". Continuity is not needed as a principle of unity nor is it able to be observed. Instead contiguity is enough to lead to what the mind puts together as unity. Moreover, though Hume references causality as a relation on which identity depends, he has already dismissed it as merely contiguity and succession. Since discrete quantity can exhibit contiguity and succession, Hume shows that only discrete quantity is needed as basic to undergird a modern conception of identity.

Continuity not Needed for the Signification of Identity-over-Time, Substance, or Persons in Set Theory

Bertrand Russell's conception of identity is very much like Hume's conception. But Russell describes explicitly the mechanism by which the mind brings together the "several

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[77] Hume, *David Hume*, 1.4.6.17.

[78] Hume, *David Hume*, 1.4.6.17.

different perceptions" into one "in the imagination." The mind brings together the different perceptions through classes or sets. Russell claims, "for the sake of argument, one might identify Socrates with the series of his experiences. He would be really a series of classes [or sets], because one has many experiences simultaneously." [79] What these experiences have in common is contiguity and succession. They are discrete experiences. What they are not is connected or continuous in any temporal or causal way. Instead using Cantor's idea that a set unifies many discrete things into one, Russell makes the Humean "several different perceptions" into a "series of classes". Each class represents the set of all of Socrates' experiences at one instant in time. Then take all of those sets as a sequence of discrete sets and one obtains as the sum total the identity of Socrates.

The previous example of Socrates is offered by Russell at the end of an argument about the identity of Piccadilly Train Station in London. Russell claims, "Therefore [Socrates] comes to be very like Piccadilly." His entire argument is this:

> "Piccadilly," on the face of it, is the name for a certain portion of the earth's surface, and I suppose, if you wanted to define it, you would have to define it as a series of classes of material entities, namely those which, at varying times, occupy that portion of the earth's surface. So that you would find that the logical status of Piccadilly is bound up with the logical status of series and classes, and if you are going to hold Piccadilly as real, you must hold that series of classes are real, and whatever sort of metaphysical status you assign to them, you must assign to it. As you know, I believe that series and classes are of the nature of logical fictions: therefore that thesis, if it can be maintained, will dissolve Piccadilly into a fiction.

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[79] Bertrand Russell, *The Philosophy of Logical Atomism* (Minneapolis: Dept. of Philosophy, University of Minnesota, 1918), 9.

> Exactly similar remarks will apply to other instances: Rumania, Twelfth Night, and Socrates.[80]

Piccadilly, Socrates, Romania, and Twelfth Night are best understood as "a series of classes of material entities…which, at varying times, occupy that portion of the earth's surface." And because these entities are best understood as a series of classes they can be represented in set theory and mathematical logic. The experiences are discrete members of sets and the sets are discrete parts of a series. By doing this, the key philosophical concept, identity-over-time, can be represented through only the discrete quantities of set theory. Therefore, formal systems only need discrete quantity as a basic irreducible primitive to signify the key philosophical concept identity-over-time.

Since both the notions of substance and person depend upon identity-over-time, these two concepts are not considered of separate importance in Russell's philosophy. The passage above about identity implicitly contains his views on substance and persons which he elsewhere makes more explicit:

> [T]he notion of 'substance', at any rate in any sense involving permanence, must be shut out from our thoughts if we are to achieve a philosophy in any way adequate either to modern physics or to modern psychology.[81]

A persisting substance, one with identity-over-time, is not necessary in Russell's thought. Because traditionally substance had been a general concept that undergirded so many other

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[80] Russell, *The Philosophy of Logical Atomism*, 9.

[81] Bertrand Russell, *An Outline of Philosophy*, Routledge Classics (Abingdon, Oxon, England ; Routledge, 2009), 268–69.

important notions, Russell's rejection of substance leads to a rejection of other key concepts:

> And in psychology, equally, the 'ego' has disappeared as an ultimate conception, and the unity of a personality has become a peculiar causal nexus among a series of events…And it must be understood that the same reasons which lead to the rejection of substance lead also to the rejection of 'things' and 'persons' as ultimately valid concepts.[82]

Russell's rejection of 'the ego', 'things', and 'persons' all flow from the "same reasons". From his explanation in the paragraphs above, these reasons resemble Hume's rejection of identity. It cannot be conceptualized because there is no real notion of continuity. "Unity" is "a peculiar causal nexus among a series of events." Russell elaborates elsewhere, "I think the universe is all spots and jumps, without unity, without continuity…" [83] Thus, because things, substances, and persons are not ultimate conceptions, they do not need to be represented as greater unities. They can be represented well as merely a series of discrete sets in time composed of discrete events. Therefore only discrete quantity is needed to represent the philosophical concepts of things, substance, and persons. Given the centrality of identity-over-time, substance, things, and persons to philosophical discourse, this

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[82] Russell, *An Outline of Philosophy*, 269.

[83] "Academic philosophers, ever since the time of Parmenides, have believed that the world is a unity. This view has been taken over from them by clergymen and journalists, and its acceptance has been considered the touchstone of wisdom. The most fundamental of my intellectual beliefs is that this is rubbish. I think the universe is all spots and jumps, without unity, without continuity, without coherence or orderliness or any of the other properties that governesses love. Indeed, there is little but prejudice and habit to be said for the view that there is a world at all." Bertrand Russell, *The Scientific Outlook*, second edition (London: George Allen & Unwin, 1949), 98.

example illustrates why discrete quantity alone suffices to signify the world in logic and set theory.

Second Objection: Mathematics and Formal Systems Need only Irreducibly Continuous Quantity to Signify the World

Another objection to the thesis that quantity which signifies the world is irreducibly discrete and irreducibly continuous is the opposite claim to the one just considered: mathematics and formal systems need only continuous quantity as a basic irreducible primitive in order to signify the world. Some contend that discrete quantity can be reduced to, or derived from, continuous quantity. The natural numbers can arise from a continuous magnitude, as when a length is subdivided into equal partitions. Using the first partition as a measure, a person can count off equal partitions. The marks on a ruler exemplify this idea: centimeter marks on a meter stick can be counted up to one hundred. But each centimeter can be subdivided further into millimeters and so on. Thus, though the length itself is continuous, discrete quantity as a number of centimeters or a number of millimeters can be derived from this continuous length. A variety of thinkers have espoused this position through the centuries.

Three reasons support the position that only continuous quantity is needed as a basic irreducible primitive in mathematics and formal systems. First, only continuous quantity was needed as an irreducible primitive for the early modern number concept as developed in the 16th and 17th centuries at the advent of modern mathematical methods such as algebra and calculus. And the roots of the early modern number concept date back to Euclid's ancient work. Second, recent mathematicians using formal systems have shown that discrete quantity can be reduced to continuous quantity, and thus only continuous quantity

is needed as a basic irreducible primitive in mathematics and formal systems. Third, mereology as a formal system treats the fundamental property of continuous quantity (divisibility or parthood) as basic but not the fundamental property of discrete quantity (indivisibility or membership). Because of this, mereology as a formal system can represent discrete wholes while treating only the fundamental property of continuous quantity as basic. Therefore only continuous quantity is needed as a basic irreducible primitive in formal systems.

The first reason only irreducible continuous quantity is needed in mathematics is because the early modern number concept, which emerged in the late 16th and early 17th centuries, dispenses with discrete quantity as basic and instead derives discrete quantity from continuous magnitudes. Thinkers associated with the foundation of early modern algebra like Simon Stevin, Francis Vieta, and René Descartes together helped establish the modern number concept which was influential over the next three centuries and signifies the world in countless ways.

This early modern reduction of discrete quantity to continuous quantity draws from an insight in Euclid's Book V related to the given example of the centimeter marks on a ruler. While Euclid ultimately retained both notions of discrete quantity and continuous quantity as mutually irreducible primitives, his work provides a method by which discrete quantity may be derived from continuous quantity. Thus, because even traditional approaches to mathematics from the ancient and early modern period demonstrate that discrete quantity can be reduced to continuous quantity, therefore only continuous quantity is needed as a basic irreducible primitive in mathematics.

The second reason why only irreducible continuous quantity is needed as basic is because after the 19th century advent of mathematical logic various thinkers have reduced discrete quantity to continuous quantity. For example, consider Burali-Forti, a mathematician who in 1899 claimed that continuous magnitudes are fundamental and discrete whole numbers can be derived from ratios of these continuous magnitudes. More recently John Bigelow and Robert Pargetter have developed a three-tiered approach to quantities that builds discrete numbers and other real numbers from magnitudes. Both approaches to quantity above, that of Burali-Forti and that of Bigelow and Pargetter, acknowledge their indebtedness to previous notions of quantity. Since these mathematicians using formal systems have shown that discrete quantity can be reduced to continuous quantity, only continuous quantity is needed as a basic irreducible primitive in mathematics and formal systems.

The third reason why only irreducible continuous quantity is needed as basic is because discrete wholes can be signified through the formal system of mereology which does not require discrete quantity as basic. In set theory, the foundational relation is membership which assigns discrete elements to discrete sets. But the foundational relation in mereology is parthood, which describes parts as parts of some whole. But parts are not necessarily discrete. Parts can overlap. Parts can also be contiguous and have overlapping but non-exclusionary boundaries, a feature of Aristotelian continuity. In other words, unlike set theory, mereology does not assume that a continuous whole, like a line, consists of indivisibles, such as points. In fact, traditional mereology (classical extensional mereology) which does not use singletons does not have a basic way across all its implementations to signify an indivisible such as a point. In mereology this lack of entities that have no parts, the lack of indivisibles, means that mereology treats divisibility as more basic than

indivisibility. Because mereology as a formal system treats the property of continuous quantity as basic (divisibility or parthood) but not the property of discrete quantity (indivisibility or membership), formal systems need only continuous quantity as basic in order to signify the world.

Moreover, mereological wholes and mereological fusions can represent real wholes in the world better than set theory, because mereology can show the interconnected nature of wholes in time and space. Using set theory a philosopher signifies identity-over-time as a sequence of sets, each of which represents a time-slice of an object's existence. With mereology, an entity with identity-over-time is a whole and the entity has temporal parts. These temporal parts are continuous intervals intrinsically connected to each other and not discrete, and this better signifies identity-over-time.

Because mereological wholes lack a foundation in discrete singleton entities, mereology effectively treats the parts of wholes as continuous entities. Thus, in mereology discrete wholes are constructed from continuous parts. Because mereology as a formal system can represent discrete wholes while treating only continuous quantity as basic, only continuous quantity is needed in formal systems.

Only Irreducibly Continuous Quantity is Needed in Mathematics as Demonstrated by Traditional Algebra's Use of the Early Modern Number Concept

A quote from Isaac Newton illustrates the early modern number concept. "By Number we understand not so much a Multitude of Unities, as the abstracted Ratio of any

Quantity, to another Quantity of the same kind, which we take for Unity." [84] Here, Newton clarifies that what he means by number is actually "ratio". And while this ratio could be between two discrete quantities, it need not be. He references Euclid's *Elements* books V, VII, and X to establish his point. Books V and VII of Euclid advance a theory of proportions and ratios for discrete number and continuous magnitude separately. Book X addresses incommensurable quantities which Newton describes as "surd numbers". [85] In this work, Newton is at the end of a long line of changes to the concept of number. The modern number concept which holds that number is ratio was initiated in the late 16th century by Francis Vieta and Simon Steven and later advanced by René Descartes. While it drew from Euclidean insights, ultimately it was a departure from the Euclidean model.

Aristotle, Euclid, and a Universal Object of Mathematics Anticipate the Early Modern Number Concept of Number as Ratio

During the 16th and 17th centuries, some mathematicians sought to abolish discrete quantity and reestablish the concept of number upon continuous magnitude for a variety of reasons. By founding numbers upon ratios they were able to do this. Their approach hinged on whether there existed an 'object of the general analytic'. For reference, imagine an

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[84] Isaac Newton, *Universal Arithmetick: Or, A Treatise of Arithmetical Composition and Resolution*, with Mr (Samuel) Cunn, Joseph Raphson, and Theaker Wilder (London: Printed for W. Johnston, 1769), 2, http://archive.org/details/universalarithm00wildgoog.

[85] Newton, *Universal Arithmetick*, 2.

object of the general analytic to be a variable like 'x'. Because ratios can represent a ratio among discrete things (12 apples:2 trees is 12:2 or 6:1) or a ratio of two continuous things (24 ft:4 ft is 24:4 or 6:1), the notion of ratio was common to both discrete and continuous quantity.

Aristotle was aware of the pursuit of a universal mathematics (a general science covering both arithmetic and geometry) which had been advocated by Eudoxus. He referred to it in Metaphysics E, "geometry and astronomy deal with a particular kind of entity, whereas universal mathematics appeals to all kinds alike." [86] Here Aristotle means that geometry and astronomy study particular kinds of continuous quantity but universal mathematics studies both discrete quantity and continuous quantity of no particular kind. According to the ancient divisions, geometry and astronomy studied continuous magnitude while arithmetic and music studied discrete number. Aristotle in the quote above is not saying that geometry studies continuous quantity and arithmetic studies discrete quantity, he is saying something different. He is saying that geometry studies continuous quantity, astronomy studies continuous quantity in motion, and (since everybody knows that arithmetic studies discrete quantity) there is another subject, the universal mathematics, which studies both discrete quantity and continuous quantity alike. Whether universal mathematical objects actually exist and what they are is a separate question. If they were to exist, these universal mathematical objects would be found in the theory of ratios and proportions.

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[86] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1026a27-28. See Book VI:1:11

One of the topics in the theory of proportions is the alternation of proportionals. The idea that proportionals alternate (if A:B=C:D then A:C=B:D) was proved twice in Euclid's *Elements,* first for continuous magnitudes then for discrete numbers.[87] Euclid proves it twice because there is no general object of universal mathematics, an object representing both discrete quantity and continuous quantity, for proof. He can separately prove the alternation of proportional continuous magnitudes and the alternation of proportional discrete numbers. But he cannot prove both theorems at the same time, because he does not have a general object which refers to both the continuous magnitude and the discrete number at the same time.

An important passage of Aristotle recognizes the parallelism between ratios of numbers and ratios of magnitudes and references the question whether there is a general object of mathematics. Aristotle begins with a reference to the alternation of proportionals, that if A:B=C:D then A:C=B:D,

> [T]he law that *proportionals alternate* might be supposed to apply to numbers *qua* numbers, and similarly to lines, solids and periods of time; as indeed it used to be demonstrated of these subjects separately.[88]

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[87] Euclid first proves it for continuous magnitudes in Bk V.12 "If any number of magnitudes are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents." Euclid then proves it separately for discrete numbers in Bk VII.9, "If a number is a part of a number, and another is the same part of another, then alternately, whatever part or parts the first is of the third, the same part, or the same parts, the second is of the fourth."

[88] Aristotle, "Posterior Analytics," in *Posterior Analytics Topica*, trans. Hugh Tredennick, with Aristotle, E. S. Forster, and Harvard University Press, Loeb Classical Library (Cambridge, Mass: Harvard University Press, 2014), 74a18-27. See book I.5

In geometry and in arithmetic the alternation of proportionals had been proved using separate methods. But was this the only or best option?

> It could, of course, have been proved of them all by a single demonstration, but since there was no single term to denote the common quality of numbers, lengths, time and solids, and they differ in species from one another, they were treated separately; [89]

The "single term to denote the common quality of numbers, lengths, time, and solids" is the object of universal mathematics, what will later be known as the object of the general analytic. Whether in proportions or ratios, discrete numbers and continuous magnitudes follow parallel logic. But in the Euclidean approach they could not be signified simultaneously.

Aristotle continues, "but now the law [of the alternation of proportionals] is proved universally; for the property did not belong to them qua lines or qua numbers, but qua possessing this special quality which they are assumed to possess universally." In this summary Aristotle is referring to the Eudoxian program which does advocate an object of the general analytic.[90] What Aristotle describes is a putative object which has just these properties. For the object of universal mathematics the proportions are not thought of as signifying properties of existing things or relations between existing things. Instead the object of universal mathematics is just that abstract notion which possesses these properties. In as much as an existing thing possesses these properties, then it would be an

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[89] Aristotle, "Posterior Analytics," 74a18-27. See book I.5

[90] Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, 1st edition, trans. Eva Brann and J. Winfree Smith, with Vieta (M.I.T. Press, 1968), 159.

instantiation of these properties of proportionality. Euclid begins with the things and abstracts to the proportions. Eudoxus on the other hand begins with proportions as the starting point. The objects are just those things which exist as an instantiation of these proportions. The general abstract concept is the object of universal mathematics.

Whether properties and relations precede things or whether things precede properties and relations is a major question Aristotle asks in Metaphysics M.[91] Since the proportions are more abstract than the things, it is not always clear how the proportions apply to existing things. In some cases the proportions may not apply at all. If they do apply, they only represent an aspect of a thing—an objectified thing—not a thing in its entirety. Moreover, if a proportion which could be an accurate relation between two continuous magnitudes is used to describe a relation between two discrete quantities, then the proportion may or may not be applicable. For example, π diameters truly measure the circumference of a circle, but can π people fit on a motorcycle?

This Eudoxian notion of quantity as an abstract conception is parallel to the early modern number concept. Both Eudoxus and later Proclus recognize the possibility of the science of the general analytic. Scholar Jacob Klein recalls Proclus' notion of "the 'one science' which gathers all mathematical knowledge into one". [92] It has,

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[91] For example consider this passage which he uses to discuss his philosophy of geometry. "For if attributes such as 'moving' or 'white,' do not exist apart from their substances, 'white' will be prior in formula to 'white man', but not in substantiality; for it cannot exist in separation, but always exists conjointly with the concrete whole—by which I mean 'white man'." Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1077b5-10. See Book XIII:2:17-18

[92] Klein, *Greek Mathematical Thought and the Origin of Algebra*, 159.

> ‘common theorems’ which can, it is true, be studied ‘in numbers and magnitudes and motions’ but which are identical neither with arithmetical nor with geometric nor with astronomical theorems. To this [one] science belong, above all, the theorems of the theory of ratios and proportions.[93]

Here Proclus seeks one science which can be studied in number or magnitudes but transcends them both. Klein continues showing that while the ancients entertained the existence of a general science, its most influential thinkers still rejected the idea that this science had an object:

> For ancient science, the existence of a ‘general object’ is by no means a simple consequence of the existence of a ‘general theory.’ Thus Aristotle says: ‘the general propositions in mathematics are not about separate things which exist outside of and alongside the [geometric] magnitudes and numbers, but are just about these; not, however, insofar as they are such as to have a magnitude or to be divisible [into discrete units]…There cannot be a specific mathematical object which is ‘neither a [determinate] number [of monads] nor [indivisible, geometric] points, nor an [arbitrarily divisible geometric] magnitude, nor a [determinate period of] time.[94]

Klein first notes that a theory does not imply the existence of a single object for the Greeks. For example, a theory can study the relations among relations of objects, which suggests there is no single object for the study but many objects. Klein points to Aristotle’s insistence that mathematics abstracts from concrete instances of geometric “magnitudes and numbers” claiming that these are the real objects. Klein further asserts that this rejection of a general object of the analytic was not unique to Aristotle and his followers but was shared by the other major mathematicians of the period such as Euclid, Apollonius, and

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[93] Klein, *Greek Mathematical Thought and the Origin of Algebra*, 159.

[94] Klein, *Greek Mathematical Thought and the Origin of Algebra*, 161–62. Brackets are original to Klein to clarify Aristotle’s source text.

Diophantus.[95] The ancient mathematicians and philosophers recognized the possibility of building a universal mathematics upon a theory of ratios and proportions, though most of them avoided it.

But even recognizing the possibility of building a theory on an object of the universal mathematics, like the variable *x*, is notable. The ancient mathematicians recognized that an object of universal mathematics could serve as a replacement for discrete number. And since ratios can be that object of universal mathematics and ratios can be obtained between continuous magnitudes, discrete number is not necessary to build the object of universal mathematics. Nonetheless, ratios can yield the whole numbers, for example whenever one magnitude is twice another, thrice another, and so on. Even the ancient mathematicians recognized that mathematically, if not metaphysically, discrete number could be founded upon continuous quantity. Therefore, only continuous quantity is needed as a basic irreducible primitive in mathematics in order to signify the world.

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[95] "And the Platonic tradition likewise, as it makes itself heard one last time in Proclus, assigns no special mathematical object to the 'general study.' This may, to be sure, be seen as an attempt to understand the possibility of any 'ratio,' 'proportion,' and 'harmony' on the basis of a 'common property' of a primordial kind, but as soon as this attempt is realized the realm of the properly 'mathematical' has been far superseded. This same fundamental conception of the objects of mathematics certainly obtains in Diophantus, namely insofar as in his problems and solutions he admits as we have seen only determinate numbers of monads." Klein, *Greek Mathematical Thought and the Origin of Algebra*, 162.

Vieta, Stevin, and the Modern Concept of Number as Ratio

Francis Vieta was a 16th century mathematician who had a marked influence on the development of algebra and the modern number concept. Like the ancients Vieta saw that there are parallel notions of ratio in geometry and arithmetic. He acknowledged the ancients' focus on determinate magnitudes and determinate numbers, *arithmos*. "[A]lgebra, as Theon, Apollonius, Pappus and the other ancient analysts passed it down, is generally geometrical and always exhibits the magnitudes which are sought either as an object [i.e. a visible construction] or in number, or else it will be an irrational or absurd problem [which here simply means impossible]." [96] In the end, the answer given was always determinate either in discrete units, *arithmos*, or as a magnitude in a construction. In other words, the ancients like Pappus did use algebraic variables but they strictly separated variables that indicated discrete quantity and variables that indicated continuous quantity. Modern mathematical notation does this when treating *x*, *y*, and *z* as varying over the real numbers and treating *l*, *m*, and *n* as varying over the integers. It is analogous to computer programmers who distinguish between variables which are integers and variables which are floating point numbers.

But Vieta departs from the approach of the ancients and instead identifies the variable precisely with the object of the general analytic. Klein comments on "the generalization which the *eidos* concept undergoes at Vieta's hands, a generalization through

[96] Vieta as cited by Klein, *Greek Mathematical Thought and the Origin of Algebra*, 157.

which the species becomes the objects of a 'general' mathematical discipline which can be identified neither with geometry nor with arithmetic." [97] Klein elaborates later, "[t]he modern concept of 'number', as it underlies symbolic calculi, is itself, as is that which it intends, symbolic in nature—it is identical with Vieta's concept of species." [98] What is it symbolic of? It is symbolic of a ratio.

Klein explains,

> this symbolic concept of 'number'…lies at the origin of that direct route which leads, *via* the "*characteristica universalis*" of Leibniz, straight to modern theories of 'logistic' (i.e., that branch of symbolic logic dealing with the foundations of mathematics).[99]

Klein claims that there is a link between Leibniz' concept of number, the "symbolic concept" of number, and the branches of "symbolic logic dealing with the foundations of mathematics." What is the common thread? In none of these cases are numbers understood primarily as a "definite number of a definite number of things." [100] Instead, numbers are understood as just those abstract entities which satisfy certain logical properties common to both magnitudes and multitudes such as the alternation of proportionals. "The condition for this whole development is the transformation of the ancient concept of *arithmos* [number] and its transfer into a new conceptual dimension." [101] Vieta advocates a conceptual notion

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[97] Klein, *Greek Mathematical Thought and the Origin of Algebra*, 172.

[98] Klein, *Greek Mathematical Thought and the Origin of Algebra*, 176.

[99] Klein, *Greek Mathematical Thought and the Origin of Algebra*, 185.

[100] Klein, *Greek Mathematical Thought and the Origin of Algebra*, 46.

[101] Klein, *Greek Mathematical Thought and the Origin of Algebra*, 185.

of number, not a notion of number that abstracts from particular instances in the world. Vieta begins with this conceptual notion, does mathematics with this notion, and as a consequence this conceptual notion of number only relates to the world secondarily. Answers to the problems of arithmetic no longer, of necessity, yield determinate discrete numbers. They may happen to yield determinate discrete numbers, but they may happen not to yield determinate discrete numbers as well. The answers to the mathematical problems now yield concepts or to be more precise, ratios.

Katherine Neal explains the significance of Vieta's developments,

> François Viète (1540-1603) introduced an improved form of symbolism: unknowns were distinguished from given magnitudes by being labelled with capital vowels while the given magnitudes were denoted by capital consonants. This powerful new symbolism was used to denote both unknown magnitudes and numbers; it indicated that numbers and magnitudes were, in a sense, interchangeable. This association of numbers with magnitudes encouraged the notion that numbers could also be treated as though they were continuous in the Aristotelian sense. Thus, the traditional idea of discrete number *versus* continuous magnitude was challenged in the early modern period in several ways.[102]

Vieta through his development of algebra shows that there is no need to have both discrete numbers and continuous magnitudes as they are "interchangeable". Instead, all that is needed are continuous magnitudes in order to construct the ratios which the variables actually represent.

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[102] Katherine Neal, *From Discrete to Continuous: The Broadening of Number Concepts in Early Modern England*, Australasian Studies in History and Philosophy of Science, 16 (Dordrecht: Kluwer, 2002), 1–2.

Another early modern mathematician who contributed to the modern number concept was Simon Stevin. Stevin contends that there is no such thing as discrete number. All number is continuous. Stevin explains,

> Number is some thing in magnitude as humidity is in water, it penetrates like this into every part of its magnitude; and just as a continuous water corresponds to a continuous humidity, so a continuous number corresponds to a continuous magnitude[103]

Stevin dismisses any notion that number is discrete and instead emphasizes the continuous nature of number. Of Stevin, Neal says,

> Simon Stevin (1548-1620) insisted in his 1585 *Arithmétique* that the traditional Greek notion of numbers was wrong: he believed that numbers were continuous rather than discrete. Stevin also developed a system for indefinite decimal expansions of number that implicitly contained the idea of a numerical continuum.[104]

While Vieta retains the idea that the variables of algebra represent the object of the general analytic which could be derived from discrete numbers or continuous magnitudes, Stevin holds that there is no such concept as discrete number. All numbers are continuous, and Stevin can appeal both to a decimal expansion like $3.33\overline{3}$ or to Vieta's object of the general analytic for support. Stevin's explanation that number is continuous shows that a separate notion of discrete quantity is not needed for mathematics.

The modern number concept was strongly shaped by Vieta and Stevin as is shown in Jacob Klein's *Greek Mathematics and the Origins of Algebra*. The ideas of these two

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[103] Stevin, "L'Arithmetique," 501.

[104] Neal, *From Discrete to Continuous*, 1–2.

thinkers influenced the ensuing generations strongly. Mathematicians like René Descartes, John Wallis, and Isaac Newton depended on the modern number concept for much of their work.[105] This is exemplified by Newton's quote, "By Number we understand not so much a Multitude of Unities, as the abstracted Ratio of any Quantity, to another Quantity of the same kind, which we take for Unity." [106] The modern number concept needs only continuous quantity and led to the rise of algebra and calculus which signify the world. This shows that discrete quantity is not needed by mathematics in order to signify the world.

Since the Natural Numbers can be Derived from Continuous Magnitudes in Modern Formal Systems only Irreducibly Continuous Quantity is Needed in Mathematics

While the dominant approach to developing the real numbers in contemporary formal systems is the arithmetic approach of Cantor and Dedekind, other philosophers of mathematics offer an alternative approach. In his formal system the influential 19th century thinker, Cesare Burali-Forti, developed the real numbers from continuous magnitude in a way consonant with Francis Vieta's and Simon Stevin's approach. And in recent decades, philosophers of mathematics John Bigelow and Robert Pargetter following a clue from Gottlob Frege have redeveloped a three-tiered relational theory of number which they believe is the traditional one used by Newton, Euler, and Frege, but which they have

[105] See Neal, *From Discrete to Continuous*.

[106] Newton, *Universal Arithmetick*, 2.

adapted for use in contemporary formal systems.[107] The collective effect of this diverse array of mathematical logicians developing concepts of number based on continuous quantity shows that discrete quantity is not needed in formal systems.

Burali-Forti Creates a Formal System with only Continuous Quantity as Basic

The role of formal systems in mathematics greatly increased beginning in 1872 at the time when Cantor and Dedekind published their arithmetical conceptions of the continuum and Cantor initiated set theory. And while the arithmeticization project had great influence, not all mathematicians were convinced that continuous magnitudes should be based on discrete number:

> At the turn of the last century, many important mathematicians (Stolz, Helmholtz, Burali Forti, Hölder, Hilbert, Huntington, to mention a few), elaborated various axiomatic systems for what they called a theory of magnitude. These various 'theories of magnitude' formed a little 'genre' in the mathematical literature of the time…The mere fact that these various scientists tried to formalise and precisely define the old notion of quantity shows that, at the time, magnitude was not universally regarded as a mere outgrowth of arithmetical concepts.[108]

Cesare Burali-Forti was an Italian mathematician who had a lasting influence. One of his hallmarks was the lack of a role for discrete numbers as basic in his system. Burali-Forti explains,

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[107] J. Bigelow and R. Pargetter, "Quantities," *Philosophical Studies* (Heidelberg) 54, no. 3 (1988): 299, https://doi.org/10.1007/BF00646273.

[108] Sébastien Gandon, "Which Arithmeticisation for Which Logicism? Russell on Relations and Quantities in The Principles of Mathematics," *History and Philosophy of Logic* 29, no. 1 (2008): 5–7.

> Chapter I of this book contains the properties of the magnitudes which do not depend on the idea of number (integer or fraction or irrational). The idea of a whole number is logically derived from the usual and concrete idea of magnitude. Once the sum of two integers [is] defined, we found that the whole numbers constitute a homogenous class of magnitudes; and once checked which are the primitive propositions about magnitudes which are also true for the integers, we already get an important group of propositions about the whole numbers, by replacing the word magnitude by the word integer in the propositions of Chap. I.
>
> An analogous procedure is followed in chapters III and IV, devoted to the rationals and the irrationals.[109]

Not only does Bural-Forti note that he will not base continuous quantity in discrete quantity, but he additionally claims that he will do the reverse. Burali-Forti does not need a separate idea of discrete quantity, because like Simon Stevin he just understands whole numbers as some particular points in the sea of magnitude. Moreover, once whole numbers are constructed this way, he has the essential logic to construct both the rational numbers and the real numbers.

While Burali-Forti is going against the tide with his construction, he identifies that his approach is much simpler than the alternatives:

> A first conclusion of the method we just exposed is the swiftness with which we can teach, at school, the formal properties of the algebraic operations, by including the elements magnitudes and numbers which are usually examined separately. But another much more important conclusion is reached, which consists in making it possible to obtain the general idea of number in a concrete shape by deriving it from

[109] C. Burali-Forti, "Les Propriétés Formales Des Opérations Logiques," trans. Sébastien Gandon, *Rivista Di Matematica* 6 (1899): 141–77. English translation here and following as given in Gandon, "Which Arithmeticisation for Which Logicism?," 5–7.

> the concrete usual idea of magnitude, which is essential for the metrical part of geometry as well.[110]

In this passage Burali-Forti shows the degree to which he agrees with Stevin. In a formal system he can now "obtain the general idea of number in a concrete shape by deriving it from the concrete usual idea of magnitude…" While in one sense this conception of number had dominated mathematical discourse for the previous three hundred years, in another sense this was a new development. Burali-Forti could now express through mathematical logic the modern number concept which had never before been formalized this way. Burali-Forti provides a formal system which shows it is only necessary to treat continuous quantity as basic in order to construct discrete quantity upon it.

John Bigelow and Robert Pargetter Develop Number from Continuous Magnitude

While the arithmetical construction of the real numbers following Cantor and Dedekind has dominated mathematics in the past century, the work of John Bigelow and Robert Pargetter has revived the early interest in deriving numbers from continuous magnitudes. Some of the interest is on account of their anti-Platonism. "Bigelow (1988) is a development of a physicalist conception, akin to Burali Forti's (but Burali-Forti is not referred to)." [111]

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[110] Burali-Forti, "Les Propriétés Formales Des Opérations Logiques."

[111] Sébastien Gandon, "On Quantity and Number in Principia Mathematica: A Plea for an Ontological Interpretation of the Application Constraint," in *The Palgrave Centenary Companion to Principia Mathematica*, History of Analytic Philosophy (London: Palgrave Macmillan UK, n.d.), 433, https://doi.org/10.1057/9781137344632_15.

Bigelow and Pargetter offer a three-tiered construction of numbers which depends on the ratios of magnitudes of continuous properties of individual things in relation. "Individuals stand… in a variety of relationships to one another." [112] Whether one object is twice as massive as another, three times as dense, or five times as fast, these all represent different relations. From this they claim,

> for an individual to have a particular determinate property is just for it to stand in a particular range of relationships to other individuals. For instance, to have such-and-such a mass is to stand in a certain set of relationships to other massive objects. For an object to be a particular shade of blue, is to stand in a set of relationships to other blue objects. And so on.[113]

Thus from one point of view, mass as a property of an object could also be understood as mass as a relation of one object in proportion with another.

From here the notion of continuous quantities arises in the thought of Bigelow and Pargetter that "follows a path which is closely analogous to the orthodox treatment of mass-terms." [114] In short, the theory needs to be able to account for ordering relations of more or less among mass terms like more or less water in a glass or more or less of an amount of matter. While, "the relational theory handles differences rather well," Bigelow and Pargetter ask whether, "it adequately explains what individuals have in common? Things

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[112] Bigelow and Pargetter, "Quantities," 296.

[113] Bigelow and Pargetter, "Quantities," 296.

[114] Bigelow and Pargetter, "Quantities," 297.

which have, for instance, different determinate masses do, nevertheless, have something in common." [115] They continue,

> We could simply add a property into the theory, a property which is shared by all the things which stand in mass-relations to one another. Call this property simply mass. When things stand in mass-relations to one another, this entails that they share the property of mass. And when two things share the property of mass, this entails that there is some mass-relation which holds between them.[116]

The origin of continuous quantity in their approach are the relations that properties of things have to each other, such as one length to another length. These could be spatial and temporal relations, or they could be mass relations as discussed above.[117]

But Bigelow and Pargetter believe that the account of quantity offered above is incomplete. What can be said from above is that a Bulldog is twice as massive as a Terrier or that a Great Dane is two-and-a-half times as massive as a Terrier. But a question remains. How can these relations be ordered to show that the mass of the Bulldog is closer to the mass of the Terrier? To do that Bigelow and Pargetter argue that another tier should be added: relations among relations. "The relational theory is, nonetheless, still incomplete. What has not yet been explained, is the way that some mass-relations are in some sense 'closer' to one another than to others." [118] And so they explain a three tiered system which

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[115] Bigelow and Pargetter, "Quantities," 297.

[116] Bigelow and Pargetter, "Quantities," 297.

[117] Although, it is a live question whether these other relations like mass-relations can only be measured by spatio-temporal means. See Eugene Hecht, "There Is No Really Good Definition of Mass," *The Physics Teacher* 44, no. 1 (January 2006): 40–45, https://doi.org/10.1119/1.2150758.

[118] Bigelow and Pargetter, "Quantities," 299.

consists of "(1) individuals; (2) determinate relationships between individuals; (3) relations of proportion between those determinate relationships." [119]

"Double the volume" and "double the mass of" both include the notion of two. It is in this way, through proportions among relations that all of the real numbers can be introduced in a manner similar to what was described previously regarding the early modern number concept. Thus, by developing this three-tiered approach to the origin of quantity as 1) individuals, 2) determinate proportions of properties, and 3) proportions of relations, they provide a compelling case for those who believe that quantity is part of the real world. Bigelow and Pargetter's account holds that notions of quantity are abstractions from the world as did nearly all mathematical accounts before the 19th century. Individuals possess determinate "mass-relations", which represent continuous magnitudes not discrete

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[119] They explain the theory in more detail. "Indeed, using level (3) proportions, we can generate classifications which cut across the equivalence classes—that is, which cut across the classes of mass-relations, volume-relations, velocity-relations, and so forth. For instance, two level (2) mass-relations will stand in some specific proportion; and two level (2) volume-relations may stand in exactly the same proportion.

This explains how it can be that when an object is "twice the mass of", and also "twice the volume of" another, these two relations are at the same time both the same, and yet also different. These relations are different, in that one is a mass-relation, standing in proportions to other mass-relations, while the other is not a mass-relation, and stands in no proportions to mass-relations. And yet there is also something these two different relations have in common, as is reflected in the way we describe them: "twice the mass of" -- "twice the volume of". This common factor, too, is explained by level (3) proportions. There is a proportion between "twice the mass of" and "the same mass as"; and this very same proportion holds also between "twice the volume of" and "the same volume as". And that is why the same word "twice" occurs in the descriptions of both." Bigelow and Pargetter, "Quantities," 299–300.

quantities.[120] Bigelow and Pargetter's third tier, proportions, represent dimensionless magnitudes, most akin to the idea of numbers as ratios.[121] Bigelow and Pargetter also show a method of developing whole numbers upon continuous magnitude. Thus, in addition to all the previous mathematicians, contemporary mathematicians too show that only continuous quantity is needed as a basic irreducible primitive in mathematics and formal systems in order to signify the world.

Bigelow and Pargetter provide insight into contemporary mathematical practice. But their thought also offers a more precise explanation of how thinkers in the 16th and 17th centuries could subsume discrete number under the broader category of continuous magnitude and develop the "modern number concept". The pervasiveness of these recognitions, suggests that there is a stable core to the generalized modern number concept, the object of the general analytic. Together these thinkers, Burali-Forti, Bigelow, and Pargetter, demonstrate that since the natural numbers can be derived from continuous

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[120] "Individuals either exist, or they don't: they do not come in degrees. Likewise, properties and relations are either instantiated by given things, or they aren't: they cannot be partly instantiated or barely instantiated, or almost fully instantiated, and so on.

Yet some things in the world seem not to be quite like that. Individuals can be counted: but a substance, like plutonium for instance, cannot be counted, but only measured. We can say "how much", but not "how many". Individuals are reflected in language by so-called count nouns like "door" and "key"; substances correspond to mass terms like "bread" and "water"." Bigelow and Pargetter, "Quantities," 287.

[121] "We have argued that, given the three levels -- individuals ; determinates; and proportions -- we can account for the complex patterns of sameness and difference, which characterize quantities. The level (3) relations generate a rich network of second-order properties and relations among objects. And not only does it generate all these second-order properties and relations, but it also explains the pattern of entailment relations which hold among them." Bigelow and Pargetter, "Quantities," 302.

magnitudes in modern formal systems only continuous quantity is needed as a basic irreducible primitive in order to signify the world.

Because Mereological Wholes Can Signify Discrete Unities only Irreducibly Continuous Quantity is Needed in Formal Systems

As mentioned above, various "axiomatic systems for… a theory of magnitude… formed a little 'genre'" in the late 19th century. But because these theories had different axiomatic bases, no single formal system emerged as the sole basis for discourse regarding continuous quantities. But in the 20th century an axiomatic system called mereology emerged that, like Burali-Forti's system, was not founded on the membership operator of Peano arithmetic. For the *Principia* of Russell and Whitehead, Peano-membership is a basic operation allowing for the inclusion of discrete entities in sets. For mereology, the parthood relation is the basic relation which allows continuity to be easily formalized but not discrete quantity. In this mereological formal system in which the relation needed for continuous quantity (parthood) is more basic than the relation needed for discrete quantity (membership), wholes can still be robustly discussed in philosophical logic. That philosophical concepts regarding wholes can be robustly discussed in a formal system which treats continuous quantity as more basic than discrete quantity shows that only continuous quantity is needed as a basic irreducible primitive in formal systems.

While it had forerunners, mereology as a formal system was initially founded by Stanisław Leśniewski in the publication of his 1916 work, *Foundation of the General Theory of Collections*. The word mereology, first used by Leśniewski in 1927, comes from the Greek word μερος (*meros*) for part, thus meaning literally the study of parts. In his first

text he sought to reestablish the concept of a whole in analogy with traditional Aristotelian continuity. He describes his conception of the "whole object" in analogy with a line AB consisting of subsegments AC, CD, DE, EG, GF, and FB (where points C, D, E, G, and F lie on a line between points A and B in that order). Leśniewski says, "we use the expression '*m*' to denote every segment AC, CD, DE, and DB, which is a part of segment AB." [122] A commentator explains, "[t]hen the class of (all the) m just is the whole object AB, an object that we would now call the mereological sum of the objects m." [123] Importantly, Leśniewski understands the concept of whole in terms of the traditional notion of continuity. In traditional continuity, as for example expounded in Aristotle, a line is not made up of points but of line segments. The explanation offered by a commentator for the above begins with this conception of a whole, "Consider a line segment AB, and suppose it is composed of subsegments…" [124] In addition, Leśniewski reinterprets in Cantor's work

[122] Stanisław Leśniewski et al., *Collected Works*, Nijhoff International Philosophy Series (Dordrecht: Kluwer Academic Publishers, 1992), 1:135.

[123] The commentator continues, "As is clear from Lesniewski's explanations of his terminology, he considered this to be the only sensible way to understand the notions of 'class of m' (the sum of all the m), 'set of m' (a sum of some but not necessarily all the m), and 'element of m' and 'subset of m' (a proper or improper part of the class of the m). The monograph declares this to give the true understanding of 'set' (*Menge*) in Cantor, and it is in effect a bid to take over the terminology of the then exploding discipline of set theory and turn it in Lesniewski's direction." Peter Simons, "Leśniewski and Mereology," in *The Lvov-Warsaw School. Past and Present*, ed. Ángel Garrido and Urszula Wybraniec-Skardowska (Cham: Springer International Publishing, 2018), 214–15, https://doi.org/10.1007/978-3-319-65430-0_26.

[124] Simons, "Leśniewski and Mereology," 213.

the words "element" and "subset" to be instead "proper part" and "improper part". [125] Thus, Leśniewski establishes the key distinction between the traditional understanding of set theory and its main 20th century rival among formal systems, mereology.

Leśniewski's very foundation of mereology shows that the parthood relation is conceived as a relation which formalizes traditional continuity since its prime example describes a line as a whole composed of line segments not points. It departs from "official" set theory over the concept of discrete "members" which are indivisible instead of "parts" that have no single decomposition and are indefinitely divisible. As one author notes, "With ordinary set theory, we are forced to assume that everything is ultimately composed of atoms, represented by singleton sets." [126] Mereology, on the other hand, is agnostic about whether its fundamental entities are atomic or not.

Regarding atomicity, mereology is underdetermined. "By themselves, the axioms of CEM [Classical Extensional Mereology] do not specify whether atoms exist or not: among the models they describe, there are some in which everything is made up of atoms (in particular, this includes all models that contain finitely many elements), some in which

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[125] "That his understanding is clearly not that of Cantor and other set theorists is shown by the fact that there are four objects m [the line subsegments of AB] but many more (including AB and all its parts) that are elements of m, and that a single object such as AB can be the class of many distinct groups of objects n, even such as have none in common with the m, all of which runs counter to Cantor, and standard set theory." Simons, "Leśniewski and Mereology," 213.

[126] Lucas Champollion and Manfred Krifka, "Mereology," in *The Cambridge Handbook of Formal Semantics*, ed. Maria Aloni and Paul Dekker, Cambridge Handbooks in Language and Linguistics (Cambridge: Cambridge University Press, 2016), 377, https://doi.org/10.1017/CBO9781139236157.014.

there are no atoms at all, and intermediate cases." [127] Because of this ability to signify an atomless universe of discourse, mereology can formalize traditional notions of continuity and geometry well. "[P]oint-free geometries and topology – more generally continua…[provide] examples in which uses of mereology are made explicit and play a major role." [128] One example is, "Tarski's recovery of three dimensional geometry based on Euclid." [129] Another is the recent work of Hellman and Shapiro which "presents regions-based axiom systems able to recover n-dimensional Euclidean geometry along with theories of classical continua, R^n." In these, "[m]ereology plays a crucial role throughout: It enters to specify that every region has another region as proper part, so that atoms are ruled out from the start." [130] So while mereology allows the signification of atomic conceptions through additional axioms, its basic form allows the signification of atomless conceptions.

Moreover, with additional primitives and syntax mereology can provide a foundation for arithmetic and mathematics more generally. As Hellman describes it, "it is only in combination with other primitives that mereology functions in accounts of mathematics…"[131] Thus, mereology, which only treats continuous quantity as basic, with additional structure can signify both discrete and continuous quantity. Many philosophers

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[127] Champollion and Krifka, "Mereology," 378.

[128] Geoffrey Hellman, "Mereology in the Philosophy of Mathematics," in *Handbook of Mereology*, ed. Stamatios Gerogiorgakis et al., Analytica (München: Philosophia, 2017), 415.

[129] Hellman, "Mereology in the Philosophy of Mathematics," 415.

[130] Hellman, "Mereology in the Philosophy of Mathematics," 416–17.

[131] Hellman, "Mereology in the Philosophy of Mathematics," 214.

use mereology to signify the world, for example Ted Sider's four-dimensionalist approach to identity-over-time.[132] Therefore only irreducible continuous quantity is needed to signify the world because discrete wholes can be signified by continuous quantity.

This concludes the second objection to the thesis that discrete and continuous quantity should both be treated as mutually irreducible primitives in order to best signify the world. In fact, treating continuity as basic has a long-standing pedigree in mathematics. Since the time of Francis Vieta and Simon Stevin mathematicians have recognized that discrete quantity could be understood merely as a part of continuous quantity in algebra. In addition, even though the mathematical foundations movement of the late 19th century favored the arithmeticization project, a number of mathematicians, including Burali-Forti, recognized that discrete whole numbers were just a subset of continuous magnitude. And finally, the contemporary formal system of mereology can signify many seemingly discrete philosophical concepts like wholeness using a system that treats continuity as more basic than discreteness. Together, all of these facts show that the unity in multiplicity of the world can be well signified by treating only continuous quantity as a basic irreducible primitive.

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[132] "Following Leonard and Goodman's [mereology as described in the] "Calculus of Individuals", four dimensionalists tend to speak of the parts of an object simpliciter, rather than the parts it has at this time or that….Thus, a four dimensionalist will say that my current temporal part is, atemporally, sitting, 69 inches tall, and wearing a hat; and a four dimensionalist will say that this temporal part is, atemporally, part of the larger space-time worm that is me." Theodore Sider, "Four-Dimensionalism," *The Philosophical Review* (Boston) 106, no. 2 (1997): 199, https://doi.org/10.2307/2998357. Also, "Four-dimensionalism is often supported by appeal to its utility in solving various traditional puzzles about identity over time." Sider, "Four-Dimensionalism," 213.

Third Objection: Even if Quantity that Signifies the World is Irreducibly Discrete and Irreducibly Continuous, Such Quantity Still Cannot be Regarded as a *Vestigium Trinitatis*

Another objection of a more theological sort could also be raised. Suppose one dismisses the two objections above and agrees that quantity that signifies the world is irreducibly discrete and irreducibly continuous. Instead the question arises, why would such quantity that signifies the world have any relation to the traces of the Trinity, the *vestigia Trinitatis*? First, the *vestigia Trinitatis* as developed by Augustine pertain almost exclusively to human persons. Therefore, the signification of the non-human world by quantity seems irrelevant. Second, though Aquinas suggests that other aspects of non-human creation reveal *vestigia Trinitatis*, mutually irreducible discrete and continuous quantity is not among the instances he cites. And third, Bonaventure believes that triadic metaphysical principles are *vestigia Trinitatis*, which is a position similar to that of Richard Baxter, but these also lack a link to mutually irreducible discrete and continuous quantity. Thus, quantity that signifies the world, even if it is irreducibly discrete and irreducibly continuous, does not appear to have the kind of traditional theological precedents one would expect for a *vestigium Trinitatis*.

First, Augustine's use of the phrase *vestigia Trinitatis* most often pertains to humans. When Augustine considers possible *vestigia Trinitatis* he develops criteria to evaluate them. He examines many entities as potential *vestigia Trinitatis*, but Augustine's criteria are quite strict. Augustine struggles to find possible entities besides humans that could satisfy his stringent criteria for the *vestigia Trinitatis*. The majority of the books in *De Trinitate* regarding the *vestigia Trinitatis*, books VIII-XV, focus almost exclusively on

humans and their capacities. Therefore, it would seem that the mutual irreducibility of discrete and continuous quantity would not be relevant to the *vestigia Trinitatis*.

Second, even if one considers the broader tradition of the *vestigia Trinitatis* beyond Augustine which allows for the existence of non-human *vestigia Trinitatis* in the world, nonetheless mutually irreducible discrete quantity and continuous quantity are not among them. Aquinas allows that non-human creation reveals *vestigia Trinitatis*. But he does not explicitly associate the *vestigia Trinitatis* with irreducibly discrete and irreducibly continuous quantity.

Third, Bonaventure claims that "the creature is an effect of the creating Trinity," and Bonaventure examines the effects as *vestigia Trinitatis*.[133] But when doing so, Bonaventure primarily considers metaphysical principles as traces of the Trinity such as a tri-fold causality he identifies as efficient cause, exemplary cause, and final cause. In a similar move an early modern thinker Richard Baxter highlights power, wisdom, and goodness as *vestigia Trinitatis* to parallel his tri-fold causality: efficient cause, constituent cause, and final cause. But neither Bonaventure's nor Baxter's tri-fold metaphysical principles explicitly reference discrete or continuous quantity. Thus, even if quantity that signifies the world is irreducibly discrete and irreducibly continuous, such quantity never seems to have been regarded as a *vestigium Trinitatis*.

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[133] Bonaventure, *Breviloquium*, 2.2. See also Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, 1.11.

Should Only Human Persons be Regarded as *Vestigia Trinitatis*?

Augustine's inquiry into the *vestigia Trinitatis* in his treatise *De Trinitate* begins a theological discourse that peaks in the medieval period. Not only did Augustine initiate the *vestigia Trinitatis* tradition of inquiry, but he is also one of the most restrictive thinkers in its application. While Augustine considers many things in the world as potential *vestigia Trinitatis*, he focuses on the souls of human persons as the most perfect *vestigia Trinitatis*.

At the end of Book VI of *De Trinitate*, Augustine introduces the notion of the *vestigia Trinitatis*, "When therefore we regard the Creator, who is understood by the things that are made we must needs understand the Trinity of whom there appear traces in the creature, as is fitting."[134] This sentence is Augustine's first reference in *De Trinitate* to traces of the Trinity.[135] For scriptural support he alludes to Romans 1:20, which acknowledges that the Creator is "understood by the things that are made…" Neither the scriptural passage nor Augustine claim that the Creator can be understood in all aspects apart from revelation. In particular, the Trinity is a mystery, and discovering any traces of the Trinity in creation is a difficult task. But the doctrine of the Trinity comports with reason, and this comportment can be seen when examining creation. In *De Trinitate*, the first seven books are about the doctrine of the Trinity as taught by scripture. The last eight books explore potential traces

[134] Augustine of Hippo, *On the Holy Trinity*, 6.10.12.

[135] His mention of the Creator being "understood by the things that are made" is an allusion to Romans 1:20

of the Trinity in creation to show how the doctrine revealed in scripture comports with reason.

The first *vestigium Trinitatis* Augustine introduces is love at the end of book VIII, where he describes how love itself has a triadic character:

> Now love is of someone who loves, and something is loved with love. So then there are three: the lover, the beloved, and the love. What else is love, therefore, except a kind of life which binds or seeks to bind some two together, namely, the lover and the beloved? [136]

In this passage Augustine does not use the phrase *vestigia Trinitatis* explicitly, but he invokes the language of threeness. In book XV he recapitulates his argument and notes book VIII is about "love, which in the Holy Scriptures is called God, by which, first of all, those who have understanding begin also, however feebly, to discern the Trinity, to wit, one that loves, and that which is loved, and love." [137] This example, as well as all the other examples Augustine explores as potential *vestigia Trinitatis,* is a triad. Augustine never seems to consider a generic unity in multiplicity but only triads as potential *vestigia Trinitatis*.

The triadic character of love is the first example of the *vestigia Trinitatis* for Augustine. The triadicity of the lover, the beloved, and the love, is the beginning of his

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[136] This section is often titled, though not by Augustine, "There are Three Things in Love, as It Were a Trace of the Trinity". Augustine of Hippo, *Augustine: On the Trinity Books 8-15*, ed. Gareth B. Matthews, trans. Stephen McKenna, Cambridge Texts in the History of Philosophy (Cambridge: Cambridge University Press, 2002), 8.10, https://doi.org/10.1017/CBO9781139164658.

[137] Augustine of Hippo, *On the Holy Trinity*, 15.3.5.

argument, from which he will "weave the rest of [the] discourse." [138] The love Augustine notes is always between persons. He does not dwell on carnal love but instead asks, "What does the soul love in a friend except the soul?" [139] This passage provides the first evidence that Augustine locates the *vestigia Trinitatis* primarily in the inner man of human persons.

Second, in book IX Augustine says, "in these three, when the mind knows itself and loves itself, there remains a trinity: mind, love, knowledge" [140] But this trinity of mind, love, and knowledge he does not dwell on. Instead, Augustine develops this tentative trinity in book IX into a more confident trinity in book X. "[T]hese three, memory, understanding, will, are not three lives, but one life; nor three minds, but one mind…neither are they three substances, but one substance… hence these three are one…" [141] Interestingly, in these books Augustine does not use the phrase traces of the Trinity but instead uses the phrase *imago Trinitatis* or image of the Trinity, indicating the image of God in man. Not until

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[138] Augustine of Hippo, *On the Trinity*, 8.10.14.

[139] This section is often titled, though not by Augustine, "There are Three Things in Love, as It Were a Trace of the Trinity". Augustine of Hippo, *On the Trinity*, 8.10.

[140] Augustine of Hippo, *On the Holy Trinity*, 9.5.

[141] "Since, then, these three, memory, understanding, will, are not three lives, but one life; nor three minds, but one mind; it follows certainly that neither are they three substances, but one substance. Since memory, which is called life, and mind, and substance, is so called in respect to itself; but it is called memory, relatively to something. And I should say the same also of understanding and of will, since they are called understanding and will relatively to something; but each in respect to itself is life, and mind, and essence. And hence these three are one, in that they are one life, one mind, one essence; and whatever else they are severally called in respect to themselves, they are called also together, not plurally, but in the singular number. But they are three, in that wherein they are mutually referred to each other; and if they were not equal, and this not only each to each, but also each to all, they certainly could not mutually contain each other; for not only is each contained by each, but also all by each…And, therefore, while all are mutually comprehended by each, and as wholes, each as a whole is equal to each as a whole, and each as a whole at the same time to all as wholes; and these three are one, one life, one mind, one essence." Augustine of Hippo, *On the Holy Trinity*, 10.11.

book X does Augustine argue explicitly that memory, understanding, and will are an image of the Trinity, and even then he does not use the exact phrase.

Though neither the phrase '*imago Trinitatis*' nor '*vestigia Trinitatis*' is used often in books IX and X, when Augustine reviews his argument in book XV, he notes that both books IX and X discuss the image of the Trinity:

> In the ninth, the argument advances as far as to the image of God, viz. man in respect to his mind; and in this we found a kind of trinity, i.e. the mind, and the knowledge whereby the mind knows itself, and the love whereby it loves both itself and its knowledge of itself; and these three are shown to be mutually equal, and of one essence. In the tenth, the same subject is more carefully and subtly handled, and is brought to this point, that we found in the mind a still more manifest trinity of the mind, viz. in memory, and understanding, and will.[142]

Two qualifications that Augustine notes above for a true *vestigia Trinitatis* include: mutual equality and unity of essence. Moreover, all the potential *vestigia Trinitatis* Augustine lists are triads. But again, in these new conceptions of the *vestigia Trinitatis*, each conception concerns the inner man. In book IX the conception of the *vestigia Trinitatis* consists in the mind, its knowledge, and its self-love. In book X the *vestigia Trinitatis* consists in memory, understanding, and will. But in both cases these conceptions of the *vestigia Trinitatis* are about the inner man.

In book XI of *De Trinitate* Augustine discusses the traces of the Trinity discernible in the outer man. "Let us seek, then, so far as we can, in that which perishes, some image of the Trinity, if not so express, yet perhaps more easy to be discerned. For that outer man also

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[142] Augustine of Hippo, *On the Holy Trinity*, 15.3.

is not called man to no purpose, but because there is in it some likeness of the inner man."[143] While Augustine believes that all five of the senses reveal this trinitarian structure, he uses sight as his paradigm.[144] Augustine uses sight to display the triadic character of all sensation: "First, the object itself which we see…next, vision or the act of seeing…in the third place…the attention of the mind."[145] So, for Augustine the triad of sight consists of 1) the object 2) the vision and 3) the attention of the mind. In as much as every sense impression depends upon an object, the sense itself, and the attention of the mind, then all senses have the same triadic structure as does sight. One could easily extend Augustine's example from sight to hearing as consisting of 1) the sound-emitting object 2) the hearing and 3) the attention of the mind.

While Augustine has identified a triad in vision, is a triad enough to be a *vestigium Trinitatis*? Augustine requires more criteria. Just as the fatherhood relation is different from the sonship relation, so Augustine believes *vestigia Trinitatis* will have relational

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[143] Augustine of Hippo, *On the Holy Trinity*, 10.1.

[144] The beginning of the quote reads, "The outer man, then, endued as he is with the bodily sense, is conversant with bodies. And this bodily sense, as is easily observed, is fivefold; seeing, hearing, smelling, tasting, touching. But it is both a good deal of trouble, and is not necessary, that we should inquire of all these five senses about that which we seek. For that which one of them declares to us, holds also good in the rest." Augustine of Hippo, *On the Holy Trinity*, 10.1.

[145] Augustine of Hippo, *On the Holy Trinity*, 10.2.

differences.[146] "These three, then, the body that is seen, and vision itself, and the attention of mind which joins both together, are manifestly distinguishable…" [147] The body, the vision, and the attention are indeed related to each other differently. But Augustine also expects *vestigia Trinitatis* to display unity of substance, essence, or nature which they do not possess, "…[they are distinguishable] not only on account of the properties of each, but also on account of the difference of their natures." [148]

Augustine summarizes his evaluation of the sense of sight as a vestige of the Trinity:

> A kind of image of the Trinity is pointed out, even in the outer man; first of all, in those things which are perceived from without, viz. in the bodily object that is seen, and in the form that is impressed by it upon the sight of the seer, and in the purpose of the will that combines the two; although these three are neither mutually equal, nor of one substance.[149]

Thus, vision does not completely satisfy Augustine's criteria for a *vestigia Trinitatis* because the three are not "mutually equal, nor of one substance", but he nevertheless calls vision a kind of image of the Trinity. This statement explicitly shows two of Augustine's

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[146] "Wherefore love and knowledge are not contained in the mind as in a subject, but these also exist substantially, as the mind itself does; because, even if they are mutually predicated relatively, yet they exist each severally in their own substance. Nor are they so mutually predicated relatively as color and the colored subject are; so that color is in the colored subject, but has not any proper substance in itself, since colored body is a substance, but color is in a substance; but as two friends are also two men, which are substances, while they are said to be men not relatively, but friends relatively." Unity of substance is not enough for Augustine, he is also interested in the relational differences of a potential *vestigia Trinitatis*. Augustine of Hippo, *On the Holy Trinity*, 9.4.

[147] Augustine of Hippo, *On the Holy Trinity*, 10.1.

[148] Augustine of Hippo, *On the Holy Trinity*, 10.1.

[149] Introduction to book 11 in Augustine of Hippo, *On the Holy Trinity*, 11.

criteria for a *vestigia Trinitatis*, mutual equality and unity of substance. His four criteria noted above are then: triadicity, relational difference, mutual equality, and unity of substance.[150]

Augustine does not leave the argument here and abandon his search for *vestigia Trinitatis* discovered from the outer man. He uses this hesitating attempt as a stepping stone and forges ahead. Augustine links vision to measure, number, and weight in Book XI, but there is no indication that he intends anything technical about discrete or continuous quantity by this reference.[151] Augustine instead claims the more perfect trinity of sight is taken in by the outer man but consists in its effect on the inner man:

> Next, a kind of trinity, in three somewhats of one substance, is observed to exist in the mind itself, as it were introduced there from those things that are perceived from without; viz. the image of the bodily object which is in the memory, and the impression formed therefrom when the mind's eye of the thinker is turned to it, and the purpose of the will combining both. And this latter trinity is also said to pertain to the outer man, in that it is introduced into the mind from bodily objects, which are perceived from without.[152]

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[150] "If the human mind is a trinity, we can perceive in it the triune God as in a mirror or in an image. Augustine makes this the program of the second half of *De Trinitate*. To conduct this project, Augustine has to show that the four principles of trinitarian ontology: unity of substance, threefoldness, radical equality, and relational difference apply, not only to God, but also to the human mind. This is the meaning of Augustine's theory of the human mind as laid out in books 9–15 of De Trinitate. According to Augustine, the key to an understanding of the trinitarian character of the human mind is the notion of its self-relatedness." Johannes Brachtendorf, "Augustine on Self-Knowledge and Human Subjectivity," in *Self-Knowledge: A History*, ed. Ursula Renz (Oxford University Press, 2017), 102–3, https://doi.org/10.1093/acprof:oso/9780190226411.003.0007.

[151] Augustine of Hippo, *On the Holy Trinity*, 11.11.

[152] Introduction to book 11 in Augustine of Hippo, *On the Holy Trinity*, 11.

This latter trinity imitates Augustine's trinity of the inner man from Book X consisting of memory, understanding, and will. The memory stores the image from the senses, the understanding holds an impression from the mind's eye, and the will purposes to combine them. Because Augustine has already identified memory, understanding, and will as a trinity, he can now affirm that an inner sense of vision is a true *vestigia Trinitatis*.

These four books of *De Trinitate* form the backbone of Augustine's argument describing what the *vestigia Trinitatis* are. He then deploys this analysis in books XII-XIV.[153] Augustine almost exclusively considers human persons as *vestigia Trinitatis*. And even when he examines the "outer man" of human persons, he concludes that the "outer man" does not exhibit *vestigia Trinitatis* as clearly as the inner man. Because most possibilities for *vestigia Trinitatis* outside of humans do not possess triadicity, relational difference, unity of substance, and radical equality, they do not qualify for him as *vestigia Trinitatis* in any way. Moreover, it is especially the faculties of the human soul that can satisfy the requirements of unity of substance and radical equality that comprise true *vestigia Trinitatis*.

Because the only things that qualify as *vestigia Trinitatis* for Augustine are human persons, the treatment of discrete or continuous quantity as mutually irreducible primitives in mathematics seems to have little relevance to the *vestigia Trinitatis*.

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[153] Book XIV provides a good example of how he draws from the trinity of memory, understanding, and will in his later analysis. "This trinity, then, of the mind is not therefore the image of God, because the mind remembers itself, and understands and loves itself; but because it can also remember, understand, and love Him by whom it was made." Augustine of Hippo, *On the Holy Trinity*, 14.12.

If Aspects of Non-Human Creation Qualify as *Vestigia Trinitatis*, Could Irreducibly Discrete and Irreducibly Continuous Quantity be Among Them?

The *vestigia Trinitatis* tradition of inquiry begins with Augustine but peaks in the medieval period persisting into the Renaissance. Augustine restricts the concept of the *vestigia Trinitatis* almost exclusively to human persons, but his doctrine is extended by medieval thinkers to stretch into new domains. Aquinas distinguishes between the image of the Trinity which is restricted to humans and the traces of the Trinity which applies more broadly. Following this path, Renaissance thinkers search for triads throughout creation. But while the post-Augustinian *vestigia Trinitatis* display great variety, the idea that mutually irreducible discrete and continuous quantity is a trace of the Trinity never seems to appear.

Augustine uses the phrases image of God and trace of God almost exclusively for man. Aquinas, on the other hand seems to extend the notion of vestige or trace more broadly, "in all creatures there is found the trace of the Trinity." [154] Aquinas explains, "[w]hile in all creatures there is some kind of likeness to God, in the rational creature alone we find a likeness of "image" as we have explained above (Articles 1 and 2); whereas in other creatures we find a likeness by way of a 'trace.'" [155] As Aquinas expands his point he

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[154] "But in all creatures there is found the trace of the Trinity, inasmuch as in every creature are found some things which are necessarily reduced to the divine Persons as to their cause." Aquinas, *Summa Theologica*, 1.45.7.

[155] Aquinas, *Summa Theologica*, 1.93.3.

also offers a succinct summary of the Augustinian doctrine of the *vestigia Trinitatis* which Aquinas describes as an image of the Trinity:

> Likewise as the uncreated Trinity is distinguished by the procession of the Word from the Speaker, and of Love from both of these, as we have seen (I:28:3) so we may say that in rational creatures wherein we find a procession of the word in the intellect, and a procession of the love in the will, there exists an image of the uncreated Trinity, by a certain representation of the species. In other creatures, however, we do not find the principle of the word, and the word and love; but we do see in them a certain trace of the existence of these in the Cause that produced them.[156]

In both cases the creature points to the Creator. But Aquinas limits the image of the Trinity to the image of God in man. For other creatures he notes not an image but a trace of the "existence of these [word and love] in the Cause that produced them." Creatures show that their cause was Trinitarian by word and love though the non-human creatures do not themselves possess these. Thus, Aquinas claims that creatures do show traces of the "uncreated Trinity", but only humans image the Trinity.

Aquinas explains that traces of God are signs which relate effect and cause:

> An "image" represents something by likeness in species, as we have said; while a "trace" represents something by way of an effect, which represents the cause in such a way as not to attain to the likeness of species. For imprints which are left by the movements of animals are called "traces"; so also ashes are a trace of fire, and desolation of the land a trace of a hostile army.[157]

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[156] Aquinas, *Summa Theologica*, 1.93.3.

[157] Aquinas, *Summa Theologica*, 1.93.3.

The explanation that Aquinas provides here, tracing effects to causes, is a typical explanation of natural signs such as that given by Augustine.[158] These explanations connect Aquinas' conception of the *vestigia Trinitatis* with his more general apologetic strategy, notably the five ways. Aquinas recognizes that tracing effects to causes, as done in the *vestigia Trinitatis* tradition, is a distinct sort of reasoning.[159] As mentioned in chapter one, this sort of reasoning is best understood in today's categories as abductive. The end result of Aquinas' arguments on the matter is to restrict the image of the Trinity to human persons but to expand the *vestigia Trinitatis* more broadly as an apologetic strategy applying to many things in creation. Even though Aquinas expands the *vestigia Trinitatis* to include aspects of non-human creation, irreducibly discrete quantity and irreducibly continuous quantity are not among them.

Is Quantity a Metaphysical Principle which could be Regarded as a *Vestigium Trinitatis*?

Along with Aquinas, Bonaventure conceives the *vestigia Trinitatis* as applying to non-human creation in addition to human persons:

> Now that we have presented a summary review of the Trinity of God, we need to say a few things about the creation of the world. Concisely put, we should maintain

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[158] This resembles Augustine's teaching on signs. "[A] sign is a thing which, over and above the impression it makes on the senses, causes something else to come into the mind as a consequence of itself: as when we see a footprint, we conclude that an animal whose footprint this is has passed by; and when we see smoke, we know that there is fire beneath; and when we hear the voice of a living man, we think of the feeling in his mind; and when the trumpet sounds, soldiers know that they are to advance or retreat, or do whatever else the state of the battle requires." Augustine of Hippo, *On Christian Doctrine*, 1.1.

[159] Aquinas, *Summa Theologica*, 1.32.1ad2. Weisheipl, "The Commentary of St. Thomas on the De Caelo of Aristotle," 49.

> the following belief concerning this: namely, that the entire world machine [*machina mundi*] was brought into existence in time and from nothing by one First Principle, unique and supreme, whose power, though immeasurable, has arranged all things in measure, number, and weight.[160]

Bonaventure speaks of creation after discussing the Trinity and expands Augustine's notion of the *vestigia Trinitatis*. Augustine briefly mentions measure, number, and weight, but Bonaventure emphasizes this triad in the myriad structures of creation:

> [W]hen we say in measure, number, and weight, we indicate that the creature is an effect of the creating Trinity by virtue of a three-fold causality: efficient, through which there is in the creature unity, mode, and measure; exemplary, from which the creature derives truth, form, and number; and final, from which it is endowed with goodness, order, and weight. These, as vestiges of the Creator, are found in all creatures, whether corporeal, spiritual, or composites of both.[161]

Bonaventure claims that "the creature is an effect of the creating Trinity" and "these, as vestiges of the Creator, are found in all creatures." Bonaventure, like Aquinas, thus expands Augustine's program. No longer are the *vestigia Trinitatis* only found in man made in the image of God, an image of the Trinity, but now the vestiges of the Trinity are to be "found in all creatures". By this Bonaventure appears to mean all of creation, not just living creatures.

Augustine's use of the word *vestigium* includes the notion of sign, since a trace (or footprint) is a species of sign.[162] Bonaventure points out this theme explicitly in a section of

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[160] Bonaventure, *Breviloquium*, 2.1.

[161] Bonaventure, *Breviloquium*, 2.2. See also Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, 1.11.

[162] Augustine of Hippo, *On Christian Doctrine*, 1.1.

his *Itinerarium* discussing "On Seeing God through his Vestiges in the Sensible World". "They are vestiges, images, and spectacles proposed to us for the contuition of [so that we could contemplate] God. They are divinely given signs." [163] Not only does he identify vestiges as signs of God, but he also explains why this is so in a manner similar to Aquinas:

> For the created beings of this sensible world signify the invisible things of God partly because God is the origin, exemplar, and goal of all creation, and every effect is a sign of its cause; every copy is a sign of its exemplar; and the road is a sign of the goal to which it leads. They function this way partly by reason of natural representation, partly by reason of prophetic prefiguration, partly by reason of angelic operation, and partly by reason of additional institution.[164]

In this quote, Bonaventure claims that the created world is a sign of God as "the origin, exemplar, and goal of all creation." Bonaventure's philosophy of signification is still Augustinian bringing together "natural" signs and artificial signs (by "reason of additional institution") under the same concept.[165]

But Bonaventure's distinction between images and vestiges, like Aquinas, seems to depart from Augustine ever so slightly. For Augustine the clearest and perhaps only true *vestigia Trinitatis* is the image of God in man, especially in the inner man. But for Bonaventure, there are traces of the creating Trinity in all creatures:

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[163] Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, 2.11.

[164] Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, 2.12.

[165] Augustine united the distinct conceptions of natural signs, like a dog's footprints, with the conception of symbols made by intelligent artifice like names and words under one broadened category of signs. "[T]he fusion between a theory of signs and a theory of words (albeit vaguely foreseen by Stoics) is definitely sanctioned only by Augustine who is the first to explicitly propose a 'general semiotics', that is, a general science of *signa*, the sign being the genus of which both words and natural symptoms are species." Umberto Eco, "Latratus Canis," *Tijdschrift Voor Filosofie* 47, no. 1 (1985): 5. See also Deely, *Four Ages of Understanding*, 214–18.

> The two preceding steps, which have led us to God by means of the vestiges through which God shines forth in every creature, have brought us to a point where we can enter again into ourselves, that is, into the mind itself in which the divine image shines forth.[166]

While divine "vestiges" shine "forth in every creature," as for Augustine so also for Bonaventure man alone bears the image of the Trinity. Bonaventure distinguishes the *vestigia Trinitatis* from the image of the Trinity:

> Therefore, in this third stage, as we leave the outer court and enter into ourselves, we should try to see God through a mirror, as it were, in the holy place—namely in the area in front of the tabernacle. It is here that the light of truth shines like the light of a candelabrum upon the face of our mind in which the image of the most blessed Trinity shines in splendor.[167]

In the passages above, Bonaventure clearly separates the vestiges from the image. After seeing God through his vestiges the contemplator then proceeds to seek God through "the image of the most blessed Trinity."

Like Aquinas Bonaventure warrants the search for the *vestigia Trinitatis* in all things. And both Bonaventure and Aquinas link the centrality of traces in the finding of causes from effects. But differently than Aquinas Bonaventure emphasizes God's triadic causal role in creation even offering a three-fold causality of efficient cause, exemplary cause, and final cause. Thus, Bonaventure's program is very much like Augustine's program though it can be seen to differ slightly from both Augustine's approach and Aquinas' approach.

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[166] Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, 3.1.

[167] Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, 3.1.

Renaissance and early modern thinkers expand the carefully nuanced arguments of Aquinas and Bonaventure and embrace the notion of the *vestigia Trinitatis* more expansively. John Donne's interest was characteristic of many others of his time:

> And therefore it is a lovely and religious thing, to finde out *Vestigia Trinitatis*, Impressions of the Trinity, in as many things as we can; and it is a reverent obedience to embrace the wisdom of our Church, in renewing the Trinity to our contemplation…Let us therefore, with *S. Bernard*, consider *Trinitatem Creatricem*, and *Trinitatem Creatam*, a Creating, and a Created Trinity; a Trinity, which the Trinity in Heaven, Father, Son, and Holy Ghost, hath created in our soules, Reason, Memory, and Will.[168]

While in typical Augustinian fashion Donne emphasizes the memory, understanding, and will as vestiges of the Trinity in man, he also commends looking for the *vestigia Trinitatis* "in as many things as we can".

For Augustine the four requirements for an entity to qualify as a vestige of the Trinity keeps the bar high. But Renaissance thinkers generally look only for two of the four marks, triadicity and relational difference. The marks of unity of substance and radical equality are disregarded. The question then becomes, can any triple qualify as a *vestigium Trinitatis*? The search for triads is further encouraged by the rising influence of neo-Platonism as represented in figures such as Pico de Mirandola and Marsilio Ficino.[169] In this vein, William Metham of York goes so far as to say that even heaven, earth, and hell display "the

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[168] From Donne's sermon on Trinity Sunday, 1620. John Donne, *The Oxford Edition of the Sermons of John Donne*, with Peter McCullough et al. (Oxford: University Press, 2013), 10.

[169] Dennis R. Klinck, "Vestigia Trinitatis in Man and His Works in the English Renaissance," *Journal of the History of Ideas* (Philadelphia, Pa., etc) 42, no. 1 (1981): 15, https://doi.org/10.2307/2709415.

Trinitie". [170] More common themes are those of the sun, its light, and its heat; or a source, spring, and river.[171]

While imagining every conceivable triad to reflect the Trinity seems absurd, a weaker form of this principle is defended by more than one thinker. The main argument is that because all things have some slight resemblance to their creator, everything reflects the Trinity in some way. Some things, like man reflect him very strongly—even as an image. While others reflect him only dimly, perhaps merely in the sense that they exist.[172]

Richard Fowns explicitly mentions the imperfect nature of the resemblance of creation to God through the *vestigia Trinitatis*:

> The glorious and incomprehensible TRINITY in vnitie, hath so created all things in the spacious Theatre of this round world, that although nothing can perfectly comprehend or expresse his cause; yet in euery thing there is a certaine image and resemblance of the great Creator" (1); this principle of elements "differing, but not divided: sundry, and yet the same" occurs in "euery thing, which Arte, or Nature doth bring forth.[173]

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[170] Klinck, "Vestigia Trinitatis in Man and His Works in the English Renaissance," 14.

[171] Klinck, "Vestigia Trinitatis in Man and His Works in the English Renaissance," 16.

[172] While this sentiment reflects the general notion of the transcendental properties of being, the origins of this idea can also be found in Augustine's *On the Trinity*. In referencing a trinity from the outer man which does not have unity of substance he at first dismisses it. "And therefore that trinity is not an image of God since it is produced in the mind itself through the bodily sense, from the lowest, that is, the corporeal creature, than which the mind is higher." But then Augustine reconsiders and admits that even though it is not a perfect trinity, it has some resemblance. "Yet neither is it altogether dissimilar [from a true trinity]: for what is there that has not a likeness of God, in proportion to its kind and measure, seeing that God made all things very good, and for no other reason except that He Himself is supremely good? In so far, therefore, as anything that is, is good, in so far plainly it has still some likeness of the supreme good, at however great a distance…" Augustine of Hippo, *On the Holy Trinity*, 11.5.

[173] Richard Fowns, in his Trisagion or the Three Holy Offices of Jesus Christ (1618) as cited by Klinck, "Vestigia Trinitatis in Man and His Works in the English Renaissance," 16.

Any triad whose members are somehow related reflects the Trinity in so far as the members are "differing but not divided: sundry and yet the same." All things resemble the highest good, God, because the existence of any created thing is good. Thus, all triads resemble the Trinity in the same way that all things resemble God.

Some Renaissance thinkers in a manner similar to Bonaventure link the *vestigia Trinitatis* to metaphysical principles. Bonaventure describes the *vestigia Trinitatis* using metaphysical terms like "causality", "form", and "final cause" as seen above. These associations develop further in the Renaissance and Reformation era as especially seen in the puritan thinker Richard Baxter. Baxter's approach to the *vestigia Trinitatis* provides insight into how this tradition draws from while expanding the Augustinian conception of it and even interacted with the rising mechanical philosophy.

Richard Baxter, who developed a Trinitarian metaphysics, wrestled with medieval ideas of the *vestigia Trinitatis* such as those of Aquinas and Bonaventure as well as those of other Renaissance and Reformation thinkers such as Donne's.[174] In Baxter's *Method of Christian Theology* he asks, "Is there a trace of the Trinity in created things and in the image [of God] in humanity?" He answers his question affirmatively:

> [Yes,] So almost all the Fathers thought, especially Augustine [in] *De Trinitate* books 10 & 11 & 14, especially book 10, chapter 11, & almost all the Scholastics [including Lombard and Aquinas]…yet the Image of the Trinity is only in the mind;

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[174] David S. Sytsma, *Richard Baxter and the Mechanical Philosophers*, Oxford Studies in Historical Theology (New York: Oxford University Press, 2017), 123.

> but in the same way a trace [*vestigium*] is in the lower faculties in other Creatures…[175]

In the above Baxter diminishes the differences between Augustine and the latter thinkers by suggesting that Augustine believes that there are *vestigia Trinitatis* "in the lower faculties of Creatures." But otherwise Baxter here is reiterating a view so commonplace in medieval thought that he says, "there is no need…to cite words or pages." [176]

In the early pages of *Method* Baxter mentions that Aristotle's fourfold causality is not completely accurate. Instead, reminiscent of Bonaventure, three causes suffice: efficient cause, constitutive cause (subsuming both material and formal cause), and final cause.[177] In the next paragraph he uses phrases written in all capitalized letters describing how the causes are effective through the "VIRTUE OF POWER, THE LIGHT OF WISDOM, and LOVE or GOODNESS." He reiterates this triadic theme a couple of pages later.[178] Almost the same theme appears in Aquinas who says, the pagan philosophers nevertheless "knew some of the essential attributes appropriated to the persons, as power to the Father, wisdom

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[175] Richard Baxter, *Methodus theologiae Christianæ* (London: White and Snowden, 1681), 80, http://archive.org/details/bim_early-english-books-1641-1700_methodus-theologiae-chr_baxter-richard_1681. translation provided by Sytsma, *Richard Baxter and the Mechanical Philosophers*, 122.

[176] Baxter, *Methodus theologiae Christianæ*, 80. translation provided by Sytsma, *Richard Baxter and the Mechanical Philosophers*, 122.

[177] Baxter, *Methodus theologiae Christianæ*, 1.4.

[178] Baxter, *Methodus theologiae Christianæ*, 1.4-6.

to the Son, goodness to the Holy Ghost…" [179] Baxter considers these three principles: power, wisdom, and goodness, as a trinity of essentialities spread throughout reality.[180]

Baxter does not believe that this trinity of principles is proof of the three persons of the Holy Trinity. But he does believe that given that the doctrine of the Trinity is revealed by scripture as true, these threefold principles discoverable in nature of power, wisdom, and goodness are consonant with the truth of revelation through reason:

> That the Trinity of essentialities is the same as the Trinity of persons I have never affirmed in words, nor in my mind, and nor do I think it should be affirmed. I assert only this: while this Trinity of essentialities or primalities shows its footsteps [*vestigia*] through the whole nature of things, if it is not the Trinity of persons yet it renders it easily credible to mortals. For no reason can be given, when one Trinity in the unity of essence is most known to God from the light of nature, why another should remain in any way incredible.[181]

In the above passage, Baxter describes the *vestigia Trinitatis* in terms of metaphysical principles. He does not wish to suggest that the principles are the same as the Trinity of

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[179] Power, wisdom, and goodness as attributes appropriated to the divine persons are commented on by Aquinas in his discussion of "Whether the Trinity of Divine Persons Can be Known by Natural Reason." "The philosophers did not know the mystery of the trinity of the divine persons by its proper attributes, such as paternity, filiation, and procession, according to the Apostle's words, 'We speak the wisdom of God which none of the princes of the world'—i.e. the philosophers—'knew'. Nevertheless, they knew some of the essential attributes appropriated to the persons, as power to the Father, wisdom to the Son, goodness to the Holy Ghost; as will later on appear." Aquinas, *Summa Theologica*, 1.32.1ad1.

[180] These appropriations also appear in Richard of St. Victor's *De Trinitate*. The formulation of the *vestigia Trinitatis* as power, wisdom, and love is attributed to Abelard. "…the doctrine of Abelard which represented the three persons as the power, wisdom, and love of the Divine Being 'became, on the whole, current in the middle ages.' George Park Fisher, "The Theology of Richard Baxter," *Bibliotecha Sacra* 9, no. 33 (January 1852): 157. Baxter also mentions Tomasso Campanella as an influence who identified "three 'primalities' or principles of being: power, knowledge, and love." Sytsma, *Richard Baxter and the Mechanical Philosophers*, 134.

[181] Baxter, *Methodus theologiae Christianæ*, 1.121. Translation given by Sytsma, *Richard Baxter and the Mechanical Philosophers*, 126.

persons, nonetheless they are signs of the Trinity and indicate the reasonableness of the doctrine of the Trinity though not its provability from reason alone.

Like Bonaventure, Baxter affirms triadic metaphysical principles as *vestigia Trinitatis*. Touching on the mechanical philosophy, he even uses these metaphysical principles such as constitutive cause to illustrate how multiple things, like Boyle's corpuscles, can act as one.[182] But even though in his conversation about constitutive cause Baxter mentions both material and formal cause, he notes neither discreteness nor continuity.[183]

Baxter and Bonaventure then provide further examples of thinkers who while affirming the importance of the *vestigia Trinitatis* in non-human creation, especially the *vestigia* as metaphysical principles, do not make any mention of irreducibly discrete or irreducibly continuous quantity.

In summary, while the Augustinian and post-Augustinian conceptions of the *vestigia Trinitatis* range widely, none provide a reason to think that irreducibly discrete and irreducibly continuous quantity that signifies the world is a *vestigium Trinitatis*. First, Augustine claims only human persons count as *vestigia Trinitatis*. Second, though Aquinas

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[182] "Although Baxter to a certain degree accepted Boyle's corpuscularism in the realm of passive natures, when he came to active and living beings, he adopted a concept of form at odds with corpuscular explanation. Active natures have three basic inadequate concepts or principles: substantiality (*Substantialitas*), disposition of substance (*Substantiae peculiaris dispositio*), and power or formal power (*Virtus, seu Vis Formalis*). He refers to "active form" as a synonym for formal power, and this form can be considered according to its power (in the first concept), immanent act (in the second concept), and action ad extra (in the third concept)." Sytsma, *Richard Baxter and the Mechanical Philosophers*, 145.

[183] Baxter, *Methodus theologiae Christianæ*, 1.5.

expands the *vestigia Trinitatis* to include aspects of non-human creation, mutually irreducible discrete and continuous quantity is not among them. And third, Bonaventure believes that triadic metaphysical principles are *vestigia Trinitatis*, which is a position similar to that of Richard Baxter. But neither does the conception of the *vestigia Trinitatis* of Bonaventure and Baxter support the idea that irreducibly discrete and irreducibly continuous quantity are *vestigia Trinitatis*. This metaphysically rich account of quantity is neither a human person, a triad, nor a metaphysical principle. Thus, there is no precedent among these leading thinkers to regard irreducibly discrete and irreducibly continuous quantity that signifies the world as a *vestigium Trinitatis*.

This chapter has offered three major objections to the thesis. First, some systems successfully reduce continuous quantity to discrete quantity, thereby showing that quantity that signifies the world is not irreducibly continuous. Second, other systems reduce discrete quantity to continuous quantity, thereby showing that quantity that signifies the world is not irreducibly discrete. And finally, the mutual irreducibility of discrete and continuous quantity is shown to be irrelevant to any possible conception of the *vestigia Trinitatis*. Together these three reasons suggest that quantity that signifies the world is not both irreducibly discrete and irreducibly continuous and even if it were it should not be considered a *vestigium Trinitatis*.

Chapter Three: Do Mathematics and Formal Systems need both Irreducibly Discrete Quantity and Irreducibly Continuous Quantity to Signify the World?

This thesis has raised the question: is quantity that signifies the world irreducibly discrete and irreducibly continuous, exhibiting unity in multiplicity and, if so, can such quantity be regarded as a *vestigium Trinitatis*? The previous chapter has considered important objections to this thesis. Two initial objections arise from mathematics and logic. First, most contemporary mathematicians treat only discrete quantity as a basic irreducible primitive and construct continuous quantity from it. Since these mathematicians can derive continuous quantity from discrete quantity there is no need to treat continuous quantity as a basic irreducible primitive to signify anything. The second objection is the opposite: discrete quantity can be constructed from continuous quantity. Thus, discrete quantity need not be treated as a basic irreducible primitive to signify anything. Besides these two objections arising from within mathematics and logic, others may object to the thesis simply on theological grounds. Of theologians that have written on the *vestigia Trinitatis* none have explicitly mentioned the significance of discrete or continuous quantity. Therefore quantity and its mutual irreducibility seems unlikely to have any relevance to the *vestigia Trinitatis*. These objections are substantial, and for many they will be persuasive.

But on the contrary side, perhaps quantity that signifies the world is irreducibly discrete and irreducibly continuous, and this is of significance for the doctrine of the *vestigia Trinitatis*. The first part of the question will be addressed in this chapter. Is quantity that signifies the world irreducibly discrete and irreducibly continuous? Two

distinct but complementary inquiries address this question. The first inquiry arises from philosopher Mary Hesse's analysis of the alternation between "atomicity and continuity" in the history of physical science. Her work has implications for whether natural science needs a mathematics with irreducibly discrete and irreducibly continuous quantity to signify the world. The second inquiry focuses on the thought of mathematician, logician, and philosopher Charles Sanders Peirce. Peirce contends that irreducibly discrete and irreducibly continuous quantity have important roles in mathematics and formal systems when signifying the world in both science and metaphysics. Hesse's thought illustrates the place of discrete and continuous quantity in the work of applied mathematicians and physical scientists who mathematicise the world. Peirce offers insights more relevant for pure mathematicians, philosophers, and logicians who are often uninterested in the applications of mathematics but whose systems of quantity nevertheless do get applied to the world.

Mary Hesse: Does Mathematics which Signifies the World in Scientific Theories Use both Irreducibly Discrete Quantity and Irreducibly Continuous Quantity?

The work of Mary Hesse (1924-2016) traces the development of discrete and continuous models in physical science from the ancient period until the mid-20th century. Her analysis offers three key topics for consideration. First, she considers the pervasiveness of discrete and continuous quantity in signifying the world. "[A]tomicity and continuity" are "one of those pairs of apparently contradictory principles which continually reappear in

the history of science." [184] Second, she notes that irreducibly discrete and irreducibly continuous quantity both seem to signify the world in scientific theories. She explores whether "in all apparent continuous action there are some hidden discontinuities" and whether "in all apparent actions at a distance there are hidden chains of continuous causes."[185] The hiddenness she describes leads to the question of whether apparently continuous quantities in the world also exhibit irreducibly discrete aspects and whether apparently discrete quantities in the world also exhibit irreducibly continuous aspects. Third, Hesse offers an interpretive framework to ease the logical tension for realists who simultaneously use irreducibly discrete and irreducibly continuous quantity. She does this by considering the relationship between "mathematical models" and reality and exploring whether a univocal, instrumental, or analogical approach is best.[186]

Atomicity and Continuity in the History of Physical Science

In her book *Forces and Fields* Hesse explores the role of continuous and discrete notions in the history of physical science. She especially considers the concept of force and compares it to the concept of field, both the concept of field from the contemporary period as well as premodern anticipations of the concept. Hesse begins her analysis with the

[184] Hesse, *Forces and Fields*, 293.

[185] Hesse, *Forces and Fields*, 294.

[186] Hesse, *Forces and Fields*, 24.

ancient period and explains what she describes as "primitive analog[ies]" which nonetheless had some technical information.[187]

Hesse especially explores the emergence of the modern concept of field as it developed in the 19th century under Faraday and Maxwell and recognizes that even at its inception, there was a duality. A tension exists between theories of discrete bodies operating through forces at a distance (as implied by Newton's principle of universal gravitation) and theories describing entities that are infinitely extended through space (as cryptically posited by Leibniz).[188] This tension reflects the tension between irreducibly discrete quantity and irreducibly continuous quantity that signifies the world.

An influential but lesser known figure, Roger Boscovich, has a role in the development of these coupled themes. In the 18th century Boscovich "derives his metaphysical principles [e.g. the law of continuity] from Leibniz, but integrates them with

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[187] For example, Stoics offered explanations "in terms of tension, pressure, and wave-motion in the continuous and pervasive pneuma [spirit]." Following other scholars Hesse interprets the Greek notion of pneuma as a precursor to the modern concept of field and considers it among the first suggestions towards "the fundamental contact theories of modern physics." Hesse, *Forces and Fields*, 76.

[188] In Leibniz' *Monadology* number 61 he says something about the continuity of all things which for modern ears may be best heard by musing on the nature of gravitational waves. "For all is a plenum (and thus all matter is connected together) and in the plenum every motion has an effect upon distant bodies in proportion to their distance, so that each body not only is affected by those which are in contact with it and in some way feels the effect of everything that happens to them, but also is mediately affected by bodies adjoining those with which it itself is in immediate contact. Wherefore it follows that this inter-communication of things extends to any distance, however great. And consequently every body feels the effect of all that takes place in the universe, so that he who sees all might read in each what is happening everywhere, and even what has happened or shall happen, observing in the present that which is far off as well in time as in place: σύμπνοια πάντα [sympnoia panta], as Hippocrates said. But a soul can read in itself only that which is there represented distinctly; it cannot all at once unroll everything that is enfolded in it, for its complexity is infinite." Gottfried Wilhelm Leibniz, *The Monadology and Other Philosophical Writings*, trans. Robert Latta (Oxford: Clarendon Press, 1898), 251, http://archive.org/details/monadologyandot01lattgoog.

Newtonian physics." [189] A duality originates in physics through Boscovich's reconciliation of Leibniz and Newton. Hesse explains,

> [Boscovich] suggests that matter consists of identical points of no extent having no essential properties except inertia and the capacity of exerting forces on one another of magnitudes depending on their mutual distances. These point centres of force are spatial versions of Leibniz's monads, and Boscovich claims that Leibniz, as well as Newton, requires every particle to be connected with every other at a distance.[190]

The innovation described above in Boscovich's thought concentrates matter at a dimensionless point with forces (instead of the mass itself) as the extended part. Newton recognized that masses, like the earth and the moon, could be treated as though their mass was concentrated at a center, and Newton could make calculations upon this assumption. But Boscovich extends this assumption to its theoretical limit and makes it an ontological principle. Hesse summarizes Boscovich, "the primary elements of matter must be simple points. The force exerted between two point particles is a continuous function of the distance between them…" [191] The duality of irreducibly discrete points and irreducibly continuous lines starts to become evident in contemporary physics in the thought of Boscovich.

Michael Faraday, often credited for discovering the idea of field, develops this Boscovichian duality. Faraday's view, "which he ascribes to Boscovich, [is] that an atom is

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[189] Hesse, *Forces and Fields*, 164.

[190] Hesse, *Forces and Fields*, 164.

[191] Hesse, *Forces and Fields*, 164.

a *point* with 'an atmosphere of force grouped around it'." [192] This "atmosphere of force" is developed by Faraday and Maxwell into the concept of the continuous field. Both discrete points and continuous "atmospheres of force" exist side by side, mutually dependent on each other for intelligibility. This is the picture of charge that most elementary students of physics have today. A discrete point charge (like an electron) with continuous field lines radiating out from it is an image met today in introductory physics textbooks. Irreducibly discrete quantity and irreducibly continuous quantity are invoked side-by-side. What is less clear though, both to the student and to the theorists, is whether there is some causal order of priority between the two, i.e. between the point charge and the field radiating from it. Is the discrete central point the cause of the emanating field? Or is the convergence of field the cause of the discrete point? This leads to deep questions in physics regarding the priority of particles or fields.

Albert Einstein's account of the competition between particle theories and field theories and its relevance for his revolutionary work accentuates aspects of Mary Hesse's explanation and contextualizes her concerns. Consider an excerpt from an address by Einstein on "Maxwell's Influence on the Development of the Conception of Physical Reality":

> …[B]efore Maxwell people thought of physical reality — in so far as it represented events in nature — as material points, whose changes consist only in motions which are subject to total differential equations. After Maxwell they thought of physical reality as represented by continuous fields, not mechanically explicable, which are

[192] Hesse, *Forces and Fields*, 200.

> subject to partial differential equations. This change in the conception of reality is the most profound and the most fruitful that physics has experienced since Newton; but it must also be granted that the complete realisation of the programme implied in this idea has not by any means been carried out yet.[193]

Einstein here notes that instead of "material points" now reality is represented by "continuous fields". Einstein characterizes the change as an opposition between the two conceptions, a contest for priority. The fundamental conception of reality in early modern science consisted of discrete entities (whether discrete extended 'atoms' or merely point masses) with properties in continuous space and time. And after Maxwell it consisted in exclusively continuous entities (fields) and continuous space and time. Einstein believes that the continuous field program will "surely be brought back to the attempt to realise the programme which may suitably be designated as Maxwellian: a description of physical reality in terms of fields which satisfy partial differential equations in a way that is free from [discrete] singularities." [194]

But contrary to what Einstein expected, there were further alternations between conceptualizing reality as discrete particles and conceptualizing it as continuous fields. Quantum theory led to quantum field theory, a field theory which includes virtual particle exchange. Initially, discrete formulations must reconcile with continuous formulations, but

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[193]Albert Einstein, "Maxwell's Influence on the Development of the Conception of Physical Reality," in *A Dynamical Theory of the Electromagnetic Field*, ed. Thomas F. Torrance (Eugene, OR: Wipf & Stock, 1996), 29–32.
[194]Einstein, "Maxwell's Influence on the Development of the Conception of Physical Reality."

ultimately it appears that any continuous formulation must also incorporate notions of the discrete, even if only virtually.

Over the course of Einstein's life the discovery of electron and neutron diffraction patterns only added to the logical tension of the new physics because these discoveries underlined the logical contradiction of wave-particle duality. That discrete particles of matter also have a continuous wave nature was predicted by De Broglie in 1924 and confirmed for electrons in the Davisson-Germer experiment of 1927. The same property was confirmed for neutrons in 1945 by Ernest O. Wollan. These entities, similar to the photon Einstein posited decades prior to interpret the photoelectric effect, now had other qualities logically incompatible with mere billiard ball particles. In the above account often referred to as complementarity, a term introduced by Niels Bohr, from the evidence of experiments reality can neither be said to be exclusively discrete nor exclusively continuous.[195] A highly idealized experiment done in 2025 with atomic precision at the Massachusetts Institute of Technology confirmed Bohr's account of complementarity

[195] "Bohr has brought to my attention [that] the uncertainty in our observation does not arise exclusively from the occurrence of discontinuities, but is tied directly to the demand that we ascribe equal validity to the quite different experiments which show up in the [particulate] theory on one hand, and in the wave theory on the other hand." Werner Heisenberg as cited by Jim Baggott, *The Quantum Story: A History in 40 Moments*, 1st edition (Oxford: Oxford University Press, 2011), 102. Taken from Werner Heisenberg, *Zeitschrift für Physik*, **43**, 1927, pp.172-198. See Wheeler and Zurek, *Quantum Theory and Measurement*, Princeton University Press, 1983 pp. 83-84.

confirming that an object's wave nature and particle nature cannot be observed at the same time.[196]

This all seemed like a very new dilemma arising in 20th century physics. Mary Hesse summarizes the problem for quantum physics like this,

> Irreducible atomicity has been contemplated before in the history of physics, but mathematical, as opposed to qualitative, theories have been developed almost exclusively in terms of field theory, and although the notion of atomicity can be introduced into these by development of the analogy of discrete 'natural frequencies', it may be that this is not enough, and the whole notion of the continuous field and its accompanying mathematical apparatus may have to be abandoned.[197]

In this quote one can see that Hesse recognizes that the quantities being considered in quantum theory are irreducibly discrete. But what exactly does "[i]rreducible atomicity" mean in terms of a mathematical theory? Hesse mentions the "analogy of natural frequencies". Here she is almost certainly thinking of Schrödinger's interpretation of the wave equation solution to the energy levels of the electron in hydrogen. They are described by $E_n = \frac{m_e e^4}{8\varepsilon_0^2 h^2 n^2}$ where, for the single electron in hydrogen, all of the variables are given except for E and n, where n is a counting number.[198] Bohr's quantization of the energy levels of an electron in the hydrogen atom (his use of a discrete 'n') comes about as a result

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[196] Vitaly Fedoseev et al., "Coherent and Incoherent Light Scattering by Single-Atom Wave Packets," *Physical Review Letters* 135, no. 4 (July 2025): 043601, https://doi.org/10.1103/zwhd-1k2t.

[197] Hesse, *Forces and Fields*, 260.

[198] The meanings of the variables are: E_n is the energy level; m_e is the mass of the electron; *e* is the elementary charge of the electron; ε_0 is the permittivity of free space; h is Planck's constant; and n is the quantum number.

of signifying empirical experimental results. Schrödinger's quantization, on the other hand, offers an additional theoretical reason. Schrödinger suggests the discrete 'n' arises because the energy levels are a result of natural frequencies of a standing matter wave.

How so? The Schrödinger equation is naturally derived from a few principles. One is the De Broglie relation that supposes that matter, such as an electron, has wave properties. The De Broglie relation, $\lambda = \frac{h}{p}$, correlates the momentum (p) of a particle and its theoretical wavelength (λ) through Planck's constant (h). The Schrödinger equation can be derived by taking the three-dimensional wave equation $\nabla^2\psi = -k^2\psi$, using De Broglie's principle as an assumption, and inserting known quantities. By doing this, Schrödinger was able to get the quantization principle which Bohr discovered for the electron in the hydrogen atom. But there is one big difference: the integers are not arbitrarily introduced. Instead, they reflect discrete frequencies of a standing wave.

Schrödinger explains,

> In this paper I wish to consider, first, the simple case of the hydrogen atom (non-relativistic and unperturbed), and show that the customary quantum conditions can be replaced by another postulate, in which the notion of "whole numbers", merely as such, is not introduced. Rather when integralness does appear, it arises in the same natural way as it does in the case of the node-numbers of a vibrating string. The new conception is capable of generalisation, and strikes, I believe, very deeply at the true nature of the quantum rules.[199]

[199] Erwin Schrödinger, *Collected Papers on Wave Mechanics* (London: Blackie and Sons, 1928), 1.

While both Bohr's approach and Schrödinger's approach use the counting numbers, they get to the counting numbers in different ways. In a wave, the possible frequencies of the wave are assumed to be continuous. Thus, in Schrödinger's approach the discreteness of a particular frequency emerges from a more fundamental continuity of all possible frequencies. On the other hand, for Bohr the integers are just an empirical observation which have no deeper explanation.

One of the most ancient observations in mathematics is that only certain string lengths (out of all possible continuous lengths) produce mellifluous tones when played together. Pythagoras observed that these lengths are in integral numerical proportions and this fact is said to have inspired him to develop his four disciplines of mathematics: arithmetic, geometry, astronomy, and music. While the strings of a lute are continuous, dividing the string up in ratios of 1:2, 1:3, 1:4, etc. make beautiful tones which please the ear. The recognition that irreducibly discrete number and irreducibly continuous magnitudes are both required and interplay with each other was fundamental to the division of the four subjects of mathematics through the millennia. That this interplay between arithmetical and geometrical formulations continues on today as represented in the work of Bohr and Schrödinger is noteworthy. Hence rises Mary Hesse's question, whether it is enough that "the notion of atomicity can be introduced…by the analogy of discrete 'natural frequencies'" Or whether "the whole notion of the continuous field and its accompanying mathematical apparatus may have to be abandoned." The tension between irreducibly discrete quantity and irreducibly continuous quantity signifying the world fits a longstanding pattern.

Thus, one of Mary Hesse's key contributions is to demonstrate that this tension between discrete quantity and continuous quantity is not unique to 20th century physics. By tracing the alternation of discrete conceptions of fundamental quantities and continuous conceptions of fundamental quantities through history, she situates modern complementarity within a much longer narrative. Hesse shows that issues of "atomicity and continuity" are not a dichotomy peculiar to modern quantum physics, but a feature present in physics since the time of the "primitive analog[ies]" such as the Stoic *pneuma*. Irreducibly discrete quantity and irreducibly continuous quantity have existed side by side with each other in the signification of the world for millennia and our current struggles with them are not unique. "[A]tomicity and continuity" are "one of those pairs of apparently contradictory principles which continually reappear in the history of science." [200] Thus, Hesse's work leads to the conclusion that instead of being an aberration in the history of physics, as Einstein thought, a pervasive feature of quantity that signifies the world is that it is irreducibly discrete and irreducibly continuous. And thus, it is plausible that this tension between discrete quantity that signifies the world and continuous quantity that signifies the world at some level will always exist.

[200] Hesse, *Forces and Fields*, 293.

Discrete Phenomena Exhibit Continuous Aspects and Continuous Phenomena Exhibit Discrete Aspects

A question not addressed in the above account is whether quantity that signifies the world is alternately discrete and alternately continuous in competing theories or is it simultaneously discrete and simultaneously continuous in every theory? Hesse and Einstein note that fundamental models vie with each other for dominance. So perhaps, it is either discrete quantity that signifies the world (i.e. particle theory) at one time, or it is continuous quantity that signifies the world (i.e. a field theory) at another time. One might suppose that discrete quantity and continuous quantity do not signify the world simultaneously. On the other hand, Hesse summarizes her research in a provocative way. "In all apparent continuous action there are some hidden discontinuities" and "in all apparent actions at a distance there are hidden chains of continuous causes." [201] This perspective suggests that even if a theory has fundamental quantities that are continuous, there will be discrete aspects to the theory. And if a theory has fundamental quantities that are discrete, there will be continuous aspects to the theory. If this is the case, then is quantity that signifies the world always both discrete and continuous?

There are two different ways to consider this question. The first is to understand what Mary Hesse herself means by making the claim that there will be "hidden discontinuities" in "continuous action" and "hidden chains of continuous causes" in

[201] Hesse, *Forces and Fields*, 294.

"apparent actions at a distance". The other way is to extend Hesse's idea and ask what in the mathematics might harbor these hidden discontinuities or hidden continuities. The answer to the first question will lead naturally to the second question.

Mary Hesse considers action at a distance models which need posit only particles, and continuous action models which invoke continuous fields. Hesse explains how these models are connected to reality,

> Action-at-a-distance theories [or particle theories] are…not necessarily lacking in the content which makes good models. So they may, like continuous-action theories, become candidates for reality in so far as it is accepted that theories are in intention factual descriptions and not merely heuristic devices.[202]

Both "action-at-a-distance theories" and "continuous-action theories" have content which makes good models. Hesse holds that these theories are connected to and properly signify reality. Hesse thinks both a discrete model of reality and a continuous model of reality simultaneously provide true insight. But this is a logical dilemma. Both theories imply that the entities they describe are "good candidates for reality", and yet they are logically incompatible.

From here Hesse applies one of her hallmark critiques of 20th century epistemology. She critiques both the verificationist criterion of the logical positivists as well as the falsifiability criterion of Karl Popper:

> In their most general form, statements asserting ultimate action at a distance or continuous action are unconfirmable and unfalsifiable. The thesis that 'Every action between bodies is made up of spatially indivisible intervals' cannot be confirmed,

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[202] Hesse, *Forces and Fields*, 292.

> because, given two bodies apparently acting across empty space, there may either be practically unobservable activity going on within it, or the meaning of continuous action may be changed in such a way that the intervals become divisible, yielding parts in which the newly defined action is still found. On the other hand the thesis [of action-at-a-distance] cannot be falsified because every apparently continuous action may turn out to be composed of smaller indivisible parts.[203]

The central idea of this statement above is that the theory of discrete action-at-a-distance between bodies is "unconfirmable and unfalsifiable". Any action-at-a-distance theory between discrete particles is always subject to slight Quinean modifications making it unfalsifiable. Hesse claims that action-at-distance theories, which conceive fundamental reality as made of discrete points, contain metaphysical assertions and are not merely conclusions from the evidence. For these discrete mathematical models, discrete mathematical idealizations are needed to signify these quantities.

> But it is equally conceivable that the opposite is the case:
>
> It follows that the antithesis that every action between bodies takes place by a continuous action through the intervening space' can be neither falsified nor confirmed.[204]

The central idea of this statement is that the theory of continuous action "can be neither falsified nor confirmed". Any continuous action theory is always subject to slight Quinean modifications. Continuous action theories are metaphysical assertions and not merely conclusions from the evidence. For these models continuous mathematical idealizations are best suited to signify these continua.

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[203] Hesse, *Forces and Fields*, 293–94.
[204] Hesse, *Forces and Fields*, 293–94.

While it appears that because reality cannot be simultaneously discrete and continuous in the same way at the same time, either one case or the other case should hold. Falsifying the action-at-a-distance model should vindicate the contact action model or vice versa as Einstein seems to expect. But what occurs in the history of physics is that both of these fundamental models have endured. Both action-at-a-distance models and continuous action models can be slightly modified in order to accommodate changing observations. Neither model is simply verifiable or falsifiable.

By rejecting the absoluteness of both the falsifiability principle and the verification principle, Hesse shows her preference for the Quinean notion of the underdetermination of theory by evidence. The Quinean objection to falsifiability asserts that even if a hypothesis is falsified by new evidence, a small modification can usually be made to accommodate the offending evidence.

The Quinean objection to verifiability reasserts the central problem of induction. No finite number of observations of white swans can ensure that all swans are white. A black swan may yet be discovered. It is impossible to verify a hypothesis absolutely simply by adducing more and more instances which confirm it.

On the other hand, not only are these theories underdetermined and unverifiable, but they are also logically inconsistent with each other. The logical tension with holding both theories simultaneously has bothered scientists from Einstein to Feynman. From these three points, Hesse begins her main proposal. This debate regarding incommensurable opposites through the ages, such as discreteness and continuity, is not an exceptional case in physics. In fact, these kinds of dialectical tensions have consistently occurred and have often been unresolved throughout the history of science:

> Action at a distance and contact action form one of those pairs of apparently contradictory principles which continually reappear in the history of science: atomicity and continuity, mechanism and vitalism, determinism and freedom, causation and teleology, body and mind.[205]

Her charge is that "atomicity and continuity" are only one sort of a number of things which form pairs of "contradictory principles which reoccur in the history of science". "[A]tomicity and continuity" are, to use a phrase reintroduced by Iain McGilchrist, one pair in the coincidence of opposites, *coincidentia oppositorum*.[206] Among other pairs are "mechanism and vitalism, determinism and freedom, causation and teleology, body and mind."

This observation leads Hesse to the conclusion that there are metaphysical questions that need to be addressed by those working on fundamental notions in natural science which are not incidental to the science itself:

> There is clearly a sense in which the dispute can be said to be a metaphysical one, for it shows all the signs of progressive withdrawal from tentative empirical positions as soon as they are falsified, which characterise the metaphysical assertion. Matter acts only by contact, so, faced with matter attracting at a distance with no apparent material medium, subtle matter of dubious status and properties has to be postulated. All attempts to describe these precisely are refuted. So the medium is described in terms of stresses and tensions, in such a way that energy is the only material property which is located in it, and this is said to show that action is after all continuous.[207]

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[205] Hesse, *Forces and Fields*, 293.

[206] McGilchrist says, "that at the origin of everything, there lies a coincidence of opposites that is profoundly generative, indeed necessary for creation, and gives rise to all that we know; and that this coincidence of opposites is by no means contrary to reason." Iain McGilchrist, *The Matter with Things: Our Brains, Our Delusions, and the Unmaking of the World* (London: Perspectiva Press, 2021), 841.

[207] Hesse, *Forces and Fields*, 293.

In the quote above, Hesse examines whether a scientist who is committed to the notion of continuous action, can make modifications to the theory in order to allow the newly discovered facts to still justify it. Likewise, can a scientist who wants to justify the action at a distance of discrete points make similar modifications? Hesse concludes that these scientists can and do modify their theories. Whether one thinks that reality is basically discrete or whether one thinks that reality is basically continuous, these are metaphysical claims according to Hesse. While questions of discrete and continuous quantity seem like merely positive claims about the facts, Hesse notes that this is not merely a battle between positivism and metaphysics in general.[208] Debates about discreteness and continuity are actually a metaphysical debate between two metaphysical positions.

According to Hesse, claims that ultimate reality is discrete or that ultimate reality is continuous are metaphysical claims which while in tension have both generated important research in mathematical physics. If Hesse's analysis is correct, one cannot dispense with either discrete quantity or continuous quantity in order to signify the world. Both are needed if mathematical signs are connected to reality, and reality should not be understood

[208] Ockham, Newton and Bohr speak "in the form 'Do not postulate unobservable intermediate entities'". But Hesse dismisses the notion that action-at-a-distance theorists are merely motivated by positivist concerns. There is more to the story than that. "Continuous action therefore appears to be more powerful as a predictive model, and to make more claims upon the facts, and this would seem to explain why action at a distance has usually been associated with positivist views of theory, where it is desired to assert no more than is already justified by experiment. But further consideration of historical examples suggests that this is too simple a view of the matter." Hesse, *Forces and Fields*, 291–92.

as merely discrete nor merely continuous.[209] For Hesse, reality is *simultaneously* both discrete in one way and continuous in another. Therefore, in order for the mathematics used by natural science to signify the world both discrete and continuous quantity are needed as mutually irreducible primitives.

Now if one allows for Hesse's own 'metaphysical claim' that there will always be hidden continuities in discrete phenomena and hidden discontinuities in continuous phenomena, how might this appear in the mathematics of the theory? Where might one look in the formulas for hidden discontinuities or hidden continuities? And are there resources from philosophy of mathematics that can help here? Exploring Hesse's claim requires delving more deeply into some equations. Once those specific examples are in view, they can be examined through the philosophy of quantity of Bigelow and Pargetter met in the previous chapter. Perhaps Bigelow and Pargetter offer a way to interpret the mathematical semiotics so as to make sense of Mary Hesse's claim that scientific theories will always have both discrete and continuous aspects to them.

What equations provide good examples of Hesse's point and how might equations themselves reflect simultaneous discreteness and continuity? Towards the end of their many-year writing project of *Principia Mathematica* A.N. Whitehead wrote to Bertrand

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[209] The fuzziness Anthony O'Hear posits in the objects of science may be due to just this feature of mathematical semiotics. "The world may be less regular than a universal science supposes and the objects in it more fuzzy than its laws envisage." O'Hear goes on to cite Hesse approvingly and concur that in spite of this fuzziness there is real scientific progress. Anthony O'Hear, *The Element of Fire: Science, Art and the Human World* (Oxford, England: Taylor & Francis Group, 2013), 81, http://ebookcentral.proquest.com/lib/oxford/detail.action?docID=1713396.

Russell. “The modern arithmeticization of mathematics is an entire mistake”. [210] Arithmeticization creates a “limitation of proof” which “leaves the whole theory of applied mathematics (measurement etc.) unproved.” [211] Whitehead’s surprising words here offer a clue. Measurement is key.

How are quantities in physical science measured? What kinds of units are involved? And how do these units reflect discreteness or continuity? The answers to these questions are best grasped by considering a number of examples.

The question of whether a force or a field is discrete or continuous is related to how it is measured. So, how does one measure force? Weights are the oldest measure of force. But measuring force with weights is not practical in every context, and weight must also be disambiguated from mass. An early alternative to weights as a measure of force was the use of springs. If a repeatable practice to measure force which correlates with weight can be found, then the more general principle of force can be established. In addition, mass can then be disambiguated from weight, a step required to formalize Newton’s laws of motion. As an example of this process, consider Hooke’s Law, named after Robert Hooke, which offers a look at an important early measure of force.

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[210] Alfred North Whitehead, “On Quantity—the Whole Theory of Applied Mathematics Is Unproved.,” September 14, 1909, Russell Letter no. 0246, RA1, document number 057422, p. 1, Bertrand Russell Archives, McMaster University Libraries, https://bracers.mcmaster.ca/81610.

[211] Whitehead, “On Quantity—the Whole Theory of Applied Mathematics Is Unproved.,” 1.

Consider the role of discrete and continuous quantity in the establishment of Hooke's Law. Hooke's Law is today given by the equation F = kx, where 'F' is a force exerted by a spring, 'k' is the spring constant, and 'x' is the displacement from the spring's resting point. At first examination discrete quantity does not seem to appear in the above equation. F, k, and x are all real numbers, and real numbers are held to be at least apparently continuous by both the Cantor-Dedekind tradition and the Burali-Forti tradition. But to understand how there are hidden discrete aspects of the above equation one must consider how an experimenter would establish Hooke's equation.

An experimenter could establish Hooke's equation in the following manner. Consider a hanging spring and two weights in a proportion known by a balance scale. The weight of the first, for simplicity, is 1 lb and is considered the unit, and the weight of the second is determined by a balance scale as 2.5 lbs. When these two weights are each hung upon a spring in separate trials, they distend the spring different distances. The first weight distends the spring 4 cm, the second weight distends the spring 10 cm. This might lead the experimenter to a hypothesis. The hypothesis is that the linear distension of the spring from its resting point is proportional to the weight hung upon it.[212] In the first case, the force (F_1) is 1 lb and the distance of distension (x_1) is 4 cm. In the second case, the force (F_2) is 2.5 lbs and the distance of distension (x_2) is 10 cm.

[212] Some limitations apply, e.g. that the weights are neither too heavy so as to break the spring, nor too light so as to not budge the spring.

The starting abstract equation given traditionally as a proportionality would look like this, $\frac{F_1}{F_2} = \frac{x_1}{x_2}$ where F_1 is the first weight, F_2 is the second weight, x_1 is the first distension, and x_2 is the second distension. Filling in the values for the first experiment yields $\frac{1\ lb}{2.5\ lb} = \frac{4\ cm}{10\ cm}$, which is a true equation. This bodes well for the experimenter, but it does not establish a principle or a law. Also, notice that the dimensions above will not matter, because they will cancel out with each other because they are in a ratio.

Next, by filling in the values for the first instance of the mass on a spring as the hypothesis for a proportionality, the equation $\frac{1\ lb}{F_n} = \frac{4\ cm}{x_n}$ is obtained. F_n stands for the force from the nth weight to be hung on the spring and x_n stands for the nth distance that the spring distends when measured with the nth weight. If this relation holds true for more instances than just a second weight noted above, then one can use this proportionality or equation as a general rule to describe the spring.

A few other weights should be tried and they are. A weight of 3 lbs is hung which distends the spring 12cm. And then a weight of 1.5 lbs is hung which distends the weight 6 cm. These additional weights confirm the hypothesis that the distension of the spring, 'x', is proportional to the weight, 'F', hung on the spring. Solving for 'F_n' in this hypothetical equation yields $F_n=(0.25)x_n$. If one drops the subscripts, the equation is now $F=(0.25)x$. Entering the data for the other trials in either the initial hypothetical equation or in the solved hypothetical equation yields true results.

$\frac{1\ lb}{3\ lb} = \frac{4\ cm}{12\ cm}$ or $3=(.25)12$ and again,

$\frac{1\ lb}{1.5\ lb} = \frac{4\ cm}{6\ cm}$ or $1.5=(.25)6$

This data empirically confirms that the equation F=(0.25)x holds for this spring. Doing a similar experiment for a range of other springs and confirming that this relationship holds for them as well would establish Hooke's Law over those springs (with different 'k' values) as well.

How are discrete and continuous quantity used in the signification of the world through this equation? By looking at the first hypothetical equation $\frac{F_1}{F_n} = \frac{x_1}{x_n}$ some observations about quantity emerge. First, what is being considered consists of a relationship between at least two different discrete instances. Instance '1' of a weight on a spring and instance 'n' of a weight on a spring. Note the subscripts. While these two instances may be related, they are discrete instances. The numerals 1 and 2 (or 1 and n) signify discrete quantity. There is not a 1.276^{th} and a 3.14^{th} trial. There are 1, 2, 3, 4, etc. discrete trials. Thus discrete quantity is inherent in the establishment of this equation because these are discrete instances tried.

Second, there are two different properties of the 'weight on a spring' system being considered. First the weight or 'F' is considered, and second the distension or 'x' is considered. The weight and the distension are two distinct (and therefore discrete from each other) aspects of the phenomenon. If one objects that the equation F=kx does not have any subscripts, that is only because the subscripts on the F and the x have been dropped for convenience. To establish the proportionality (or the equation) originally the subscripts would have been needed or implied.

Nonetheless, there is a further nuance. When applying this equation to one particular spring a 'k' value is needed. That 'k' value is obtained by an experiment where

as described above k= $\frac{F_1}{x_1}$. In other words, for this above experiment, k=$\frac{1\ lb}{4\ cm}$. And the application of this 'k' value to any other instance implies at least one other discrete instance of a 'weight on a spring' system. What the constant 'k' implies therefore is that another $\frac{F_n}{x_n}$ (or another discrete instance) will be equal to the ratio of $\frac{F_1}{x_1}$. All these aspects reflect how discrete quantity is 'hidden' in an equation that seems to contain only continuous quantities.

Are there other ways that this early measure of force displays continuous quantity? This is of special interest because some mathematicians like Cantor and Dedekind might contend that the real numbers given in 'F', 'k', and 'x' are reducible to discrete quantity and not truly continuous. The distance that a spring is distended at the time of Hooke would have been considered a continuous magnitude. Even thinkers like Dedekind, Frege, and Cantor did not say that time and space were discrete. They only suggested that the numbers in mathematics should not appeal to time and space intuitions (because time and space seem continuous). Since 1) this equation is offered as a way to measure force, and 2) spatial distance is the correlated variable, then 3) force would also have been considered a continuous magnitude.

But the question of how to measure a continuous distance also raises an important point. What is 2.5 cm? The unit centimeter is not just any magnitude length but a distinct designated fixed magnitude. 2.5 cm is itself a ratio between two distinct (or discrete from each other) length segments, 2.5cm:1cm. Each segment is continuous and assumed infinitely divisible in itself. But they are also discrete lengths from all other lengths. Thus a claim that a distance is 2.5 cm, at least traditionally understood, already represents a ratio

between two continuous magnitudes. On the other hand, once the two length magnitudes in the Hooke's Law equations are put into a ratio, the units are irrelevant because they will cancel out. 2.5cm/10cm is the same as 25mm/100mm or as 0.98in/3.92in. On the other hand a claim that the 'k' value is 0.25 lb/cm does involve units that do not cancel out and therefore involves ratios of continuous magnitudes of different sorts.

That this relation F=kx holds for a number of different 'weight on spring' systems has also raised questions about discreteness and continuity for philosophers. Is there a metaphysical continuity between these instances? If so, in what way? This is perhaps the deeper question that leads to considerations of whether a force is a manifestation of a field or a field is an abstraction from instances of forces. These questions will be considered later.

The example of Hooke's Law provides helpful context to understand Bigelow and Pargetter's description of how quantity signifies the world as referenced in chapter two. Bigelow and Pargetter explained a three tiered system which consists of "(1) individuals; (2) determinate relationships between individuals; (3) relations of proportion between those determinate relationships." [213] The notion of quantity as ratio which they develop depends upon both irreducibly discrete and irreducibly continuous quantity. The individuals are the two 'weight on spring' systems (discrete and individual as indicated by the subscripts). There are "determinate relationships between the two individuals", namely the property of

[213] Bigelow and Pargetter, "Quantities," 299–300.

weight between the two weights ($\frac{F_1}{F_2}$) and a determinate relationship between the property of distance between the two distances ($\frac{x_1}{x_2}$). And finally the equation itself, $\frac{F_1}{F_2} = \frac{x_1}{x_2}$ is a "relation of proportion between those determinate relationships". In contrast to the interpretation in the last chapter that the early modern number concept as ratio depends only on continuous magnitude, Bigelow and Pargetter's proposal can be interpreted differently. The conclusion of philosopher of mathematics James Franklin is that Bigelow's philosophy of mathematics has "an extensively developed Aristotelian view of quantity, both discrete and continuous (though preferring to call it physicalist, Platonist and Pythagorean)." [214]

An important nuance emerges then when tracing equations back to what they signify in the world and how they do so. Equations often signify both discrete and continuous aspects of things, but in different ways. In general, an equation like F = kx is able to be applied over many discrete individual phenomena and signifies an insight gleaned from many discrete instances. But the variables the equation employs may themselves signify continuous entities as in force and distance. This observation is reinforced by the canonical form of the equation given as, $\vec{F} = -k\vec{x}$. In this form the arrows above the letters (or elsewhere a boldface font) indicate that the quantities are

[214] Franklin, *An Aristotelian Realist Philosophy of Mathematics*, 127.

vectors possessing both magnitude and direction. The term magnitude is the traditional technical way to indicate a continuous quantity.

Consider now an equation relating the force, 'F', exerted between two electric charges, 'q_1' and 'q_2', some distance, 'r', away from each other. Coulomb's Law, ($\vec{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{r}_{12}$), describes two particles conceived as point charges interacting with each other and exhibiting a coordinated motion. This coordinated motion is described in Newton's Third Law. "To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts." [215] Newton restates the law in terms of mutuality, "the mutual actions" of the bodies. Thus, every force law as an instance in Newton's system signifies particles bound together in a coordinated motion. For the situation above, the force of the first charge on the second $\vec{F}_{12}$, is equal and opposite to the force of the second charge on the first $\vec{F}_{21}$, or $\vec{F}_{12} = -\vec{F}_{21}$ as described by Newton's Third Law.

The notion of discrete individuals drawn together in a coordinated motion is explored by Aristotle in more than one way. He claims that a bundle of individual sticks "is made continuous by its string, and pieces of wood by glue…'Continuous' means that whose

[215] Isaac Newton, *Mathematical Principles of Natural Philosophy*, vol. 32, Great Books of the Western World (Chicago; London: Encyclopaedia Britannica, 1990), bk. 1.

motion is essentially one, and cannot be otherwise". [216] The sticks themselves are not naturally continuous with each other. Though those things "which are naturally continuous are one in a truer sense than those which are artificially continuous". Nonetheless, since these sticks exhibit one unified motion when moved about, Aristotle conceives them as continuous in at least this aspect. Likewise discrete charged particles that exhibit one unified motion can also be conceived as 'continuous' in one sense even as they can be conceived as discrete in another sense.

In Metaphysics Δ Aristotle also says, "things are one…analogically…which have the same relation as something else to some third object." [217] This could also be translated as things are one 'by proportion' since the Greek word for proportion is *analogos* (αναλογος). Aristotle indicates above and others reiterate that an analogy or proportion is a relation of relations.[218] Therefore, a unity of proportion signifies entities which have the same kind of relation to each other as some other entities have to each other. Having the same kind of

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[216] Aristotle describes this regarding a bundle of sticks which are made one artificially. "[S]ome [things] are said to be so by virtue of their continuity, e.g., a faggot is made continuous by its string, and pieces of wood by glue; and a continuous line, even if it is bent, is said to be one, just like each of the limbs; e.g., the leg or arm. And of these things themselves those which are naturally continuous are one in a truer sense than those which are artificially continuous "Continuous " means that whose motion is essentially one, and cannot be otherwise; and motion is one when it is indivisible, i.e. indivisible in time." Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1015b. See Book V:6

[217] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1016b31-35. See Book V:6:21-22

[218] "As already stated, metaphor is the transfer of a name from one object to another on the basis of analogical similarity, i.e. of likeness through 'equality of ratios' (ἡ γὰρ ἀναλογία ἰσότης ἐστὶ λόγων, καὶ ἐν τέτταρσιν ἐλαχίστοις). Analogy is a similarity of relationship, a correspondence of proportion – or, as both Alasdair MacIntyre and Martin Heidegger term it, a 'relation of relations'." Fran O'Rourke, *Aristotelian Interpretations* (Co. Kildare, Ireland: Irish Academic Press, 2016), 107.

relation is what underlies the force laws which show coordinated motion. This coordinated motion Aristotle calls continuous ‘in one sense’. In one way the ‘bundle of sticks’ is continuous. In another way the bundle of sticks is a discrete quantity. There are degrees of continuity. A thing which is continuous ‘by nature’ is more continuous than that which is continuous ‘by art’. Should the proportions or equations which indicate coordinated motion lead the scientist to understand the phenomena as a unity like a bundle of sticks “made continuous by its string”, or should the scientist examine the sticks themselves as the truly existent entities? Considering objects as individuals in force pairs is like looking at individual sticks, while considering the electric field as whole is like examining the motion of the unified bundle.

This ambiguity is more clearly manifest in certain problems in electrostatics, for example the calculation of the electric force upon a charged particle inside a charged conducting spherical shell. It is a tenet of electrostatics that the charge upon a conductor will equally distribute itself upon the surface of the conductor creating a symmetric charge distribution on a spherical shell. Once the charge is symmetrically distributed, there is a potential ambiguity in textbook presentations regarding whether the charge is one continuous distribution upon the spherical shell or whether it is a large but finite number of individual charge carriers distributed throughout the shell. Most textbooks imply that while it is in actuality the second phenomena (a large number of discrete charge carriers, e.g. charged iron atoms), it can be treated like the first (a continuous spherical shell of charge).

For most applications as the number of individuals approaches infinity there is no measurable difference between the discrete charge distribution of individually charged iron molecules and the continuous charge distribution. The first approach (the continuous

approach) allows for the elegant application of the laws of Gauss to show that the force inside the sphere upon a charged particle would be nothing, while the second approach calculates the same answer according to Coulomb's force law described above. Gauss' Law for the electric flux is given by $\int_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{inside}}{\varepsilon_0}$. Gauss' formulation offers a computational equivalence between electric charge (q_{inside}) and electric field ($\boldsymbol{E}$).

If this was the entire story, then most physicists would conclude simply that the existence of a multitude of discrete charged iron atoms is the reality and a continuous charge distribution is a useful fiction for calculation. But the significance of the Gaussian formulation increased over the course of the 19th century as it became clear that moving charge creates changes in the region of space around it. In 1819 Hans Christian Oersted discovered that moving charge (electric current) deflects a compass needle and thus concluded that it generates a magnetic field. This led to the formulation of Ampere's Law, $\int_C \mathbf{B} \cdot d\mathbf{S} = \mu_0 I$. A similar law for the creation of an electric field from a moving magnet was discovered by Faraday, $\int_C \mathbf{E} \cdot d\mathbf{S} = \frac{-d[\boldsymbol{B}\cdot\boldsymbol{A}]}{dt}$. James Clerk Maxwell noted the parallelism and added a term to Ampere's Law resulting in the Ampere-Maxwell Equation, $\int_S \mathbf{B} \cdot d\mathbf{S} = \mu_0(I + \varepsilon_0 \frac{d[\boldsymbol{E}\cdot\boldsymbol{A}]}{dt})$. [219]

By the use of Maxwell's equations (which includes Gauss' formulation of the electric field equation), Maxwell was able to show that light itself is an alternation of electric and

[219] There are many different equivalent formulations of these equations. I have chosen the integral formulations and simplified the flux in the above equations for the sake of explanation.

magnetic fields. Given that light is 'something', this gave great credence to the notion that the electric field and magnetic fields are also 'something'. Thus, the ambiguity inherent in Gauss' formulation, $\int_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{inside}}{\varepsilon_0}$, increased. Are the discrete charges (q) the fundamental thing, or is the continuous electric field (E) the fundamental thing?

Deploying the framework developed above, consider the mathematical semiotics associated with the two competing formulations: Coulomb's Law, $\mathbf{F} = k\frac{q_1 q_2}{r^2}\hat{\boldsymbol{r}}$, which signifies force at a distance and Gauss' Law in electricity, $\int_S \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0}$, which signifies contact force when another charge is introduced ($\mathbf{F}$=q$\mathbf{E}$). First, imagine the two equations written out as proportions. Writing out Coulomb's Law as a proportionality without the constant 'k' yields, $\frac{F_a}{F_b} = \frac{r_b^2 q_{1a} q_{2a}}{r_a^2 q_{1b} q_{2b}}$. Writing out Gauss' Law without the constant 'ε_0' yields, , $\frac{\int_S \boldsymbol{E}_1 \cdot d\boldsymbol{A}_1}{\int_S \boldsymbol{E}_2 \cdot d\boldsymbol{A}_2} = \frac{q_{inside\,1}}{q_{inside\,2}}$. What are the similarities and differences between these equations and how do they relate to discrete and continuous quantity?

First, consider the similarities between the two equations. In both cases the proportionalities reveal discrete measurements by the subscripts. In the initial unitized equations with the constants only certain units (i.e. SI units) give true results, and those units are hidden within the 'k' or 'ε_0'. Also in both cases there are spatial variables like distance (r) and area (A) understood, at least by their first advocates, as continuous. Thus, both discrete quantity and continuous quantity are implicit in these two formulations representing force at a distance or contact force.

Second, consider the differences. A typical application of Gauss' Law examines concentric spheres of charge; the largest sphere contains all of the other spheres and they

share a common center. The term q_{inside} refers to the charge (q) inside a Gaussian sphere. This means that the charge (q) may be a continuous distribution as determined by the underlying field.[220] The continuous electric field within the largest sphere can be understood to be one continuous entity with different subregions each having different values. On the other hand, in Coulomb's Law the charges (q_1 and q_2) are always understood to be discrete point charges in coordinated motion due to Newton's 3rd Law.

This formulation of Gauss' Law allows an ambiguity. Which is fundamental charge or field? Is the fundamental reality a discrete particle which has a property (charge) which interacts with other particles with the same property? Or is the fundamental reality a continuous field, which when concentrated in space resembles a particle? [221] In the 19th

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[220] The charge in the Gaussian sphere may be continuous as given by the equation $Q = \int dq = \int \rho dV$, where ρ signifies a uniform charge density $\rho = \frac{Q}{V} = \frac{dq}{dV}$, or the charge may be a discrete point charge, or it may be an array of discrete point charges. This formulation of Gauss' Law allows for the underlying conception of the charge to be either discrete or continuous.

[221] Because a similar idea can be associated with the gravitational force equation, matter itself can express this ambiguity. Those having finished a secondary school physics course may recognize the equation known as Newton's Universal Principle of Gravitation, $F = G\frac{m_1 m_2}{r^2}$. Fewer readers will have ever encountered the partial differential formulation, $\boldsymbol{\nabla} \cdot \mathbf{F} = -4\pi G\rho$ where density (ρ) arises from the divergence of the gravitational field, **F**. An integral formulation of the same equation which parallels Gauss' Law is described by historian of physics Max Jammer in his book *Concepts of Mass*:

> A more abstract and less elementary definition of mass in terms of gravitation was suggested in 1904 at the International Congress of Philosophy at Geneva by Rene de Saussure. Let the vector **F** denote the gravitational field intensity and d**S** the element of a closed surface in the interior of which particles with the total mass M are enclosed. Then by Gauss's theorem.
>
> $$\iint \boldsymbol{F}\, d\boldsymbol{S} = -4\pi GM$$

century after Maxwell but before the discovery of the photoelectric effect, there did not seem to be any discrete particles (like charged iron atoms) inherent in a wave of light. Could that mean that the electric field is fundamentally continuous?

But in modern physics the story is turned. Light, presumed after Maxwell to be an electromagnetic wave with a definite velocity, was determined in the 20th century to have particulate properties. Albert Einstein published an article on the photoelectric effect which detailed how light possessed these particle qualities. Nonetheless, Einstein was not sure what this meant for fundamental physics.

While Einstein considers the continuous formulations (e.g., Maxwell's Laws like $\int_S \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0}$) superior to the discrete formulations (e.g. Coulomb's Law, , $\mathbf{F} = k\frac{q_1 q_2}{r^2}\hat{\boldsymbol{r}}$, he

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where G is the constant of gravitation. If the latter is again taken as unity, Gauss's theorem can be expressed by the statement that the mass enclosed in a given surface is equal to the gravitational flux through the surface divided by 4π.

Max Jammer, *Concepts of Mass: In Classical and Modern Physics* (Cambridge, Mass: Harvard University Press, 1961), 124.

It is easy enough to reorder the above equation to get an explicit definition for mass, $\frac{\iint \boldsymbol{F}\, d\boldsymbol{S}}{-4\pi G} = M$. The differential form of this equation is what is given above, $\boldsymbol{\nabla} \cdot \mathbf{F} = -4\pi G\rho$. In this case, ρ stands for density, which easily yields mass after multiplying by volume. The difference between the integral and differential forms of the equations above mainly regards dimensional convenience. In both of the above equations mass (or density leading to mass) is defined in terms of fields. The second equation ($\boldsymbol{\nabla} \cdot \mathbf{F}$) is a partial differential equation. In the latter field formulation, the $\boldsymbol{\nabla} \cdot \mathbf{F}$ which can be written as $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$ uses partial differentials to analyze a continuous gravitational field.

In the former equation, $F = G\frac{m_1 m_2}{r^2}$ the discrete point masses, m_1 and m_2, give rise to a discrete gravitational force, F (or force pair). The forces are a result of a property of the discrete particles, namely their two individual masses. In the latter the mass (or density) is a result of the continuous field which is either a self-existent continuous entity or a property of the space and time. Either way the field is held to be continuous.

admits, as seen above, that neither discrete formulations nor continuous formulations have been able to fully encapsulate all the phenomena.[222]

What remains unclear from the above analysis is whether the two individual measurements of electric field needed to create a proportion in Gauss' Law are two discrete measurements of one unified individual (continuous) field or whether they are two measurements of two discrete individual (continuous) fields. Might the fields be properties of particles? [223] Or might the fields just be subsegments of one grand overarching field? This is the equation level root cause for the ambiguity that Mary Hesse expresses about hidden discreteness within continuous fundamental phenomena and hidden continuities within discrete fundamental phenomena. Tracing these themes further into quantum physics is beyond the scope of this thesis. But it should be noted that contemporary quantum physics is essentially a systematic attempt to deal with the ambiguity of this wave-particle duality, as "each particle has associated with it a wavefunction $\psi(x,t)$." [224] This suggests

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[222] Einstein, "Maxwell's Influence on the Development of the Conception of Physical Reality."

[223] Hesse describes a point from Wheeler and Feynman "The fields which act on a given particle arise only from other particles." Hesse, *Forces and Fields*, 282.

[224] "We found that entities such as the electron are particles in the classical sense in that when detected they seem to carry all their energy, momentum, charge, etc. in localized form; and at the same time they are not particlelike in that assuming they move along definite trajectories leads to conflict with experiment. It appears that each particle has associated with it a wavefunction $\psi(x,t)$, such that $\| \psi(x,t) \|$ gives the probability of finding it at a point x at time t. This is called *wave-particle duality*." Ramamurti Shankar, *Principles of Quantum Mechanics*, 2nd ed. (New York: Springer Science Business Media, 1994), 113.

that in quantum theory both discrete quantity and continuous quantity will still be needed.[225]

For the reasons given above Hesse believes the decision whether to prioritize the discrete or continuous picture of reality is a metaphysical one. Should reality be conceived as 1) particles with emanating fields as properties or 2) an overarching grand field with subsegments exhibiting occasional singularities and particle-like concentrations. Bohr discovered discreteness empirically by using equations with the discrete counting numbers. Schrödinger explained similar phenomena with different equations but argued that those same counting numbers resulted from a constraint imposed by continuous physical phenomena. Discrete and continuous aspects of the phenomena are exhibited in all cases. But whether there is an absolute priority to either discrete quantity or continuous quantity is not clear merely from the phenomena. Thus, Mary Hesse's claim that there will always be

[225] That these themes could be traced further and bear fruit for this project is likely. Physicist John Bell while defending Bohmian mechanics says, "While the founding fathers agonized over the question 'particle' or 'wave' de Broglie in 1925 proposed the obvious answer 'particle' and 'wave'." Moreover, this current project can be seen in light of Mark Harris' remarks on quantum fundamentalism, "the belief that quantum physics captures what is most fundamental in our physical world". Harris says, "Christian theologians…have found that QM is a fantastically rich source of metaphor…even for that deep enigma of Christian thought: the nature of God's Trinitarian being. Characteristic features of QM—wave/particle duality, indeterminacy, and entanglement—crop up repeatedly…these features offer stunning analogies for theological mysteries…" Acknowledging that wave/particle duality may present an analogy for the mystery of the Trinity may not be far from contending that discrete and continuous quantity reveal traces of the Trinity. J. S. Bell, "Six Possible Worlds of Quantum Mechanics," in *Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy*, 2nd ed. (Cambridge: Cambridge University Press, 2004), 191–92, https://doi.org/10.1017/CBO9780511815676.022. Mark Harris, "Quantum Fundamentalism and Theological Liberty," *Zygon: Journal of Religion and Science* 59, no. 3 (October 2024), https://doi.org/10.16995/zygon.17702.

hidden continuities and hidden discontinuities supports the point that quantity that signifies the world is irreducibly discrete and irreducibly continuous.

Hesse's Analogical Approach Diffuses the Logical Tension for Realists who Signify the World with both Irreducibly Discrete Quantity and Irreducibly Continuous Quantity

Along with contemporary physicists, Hesse notes the logical tension that arises for the orthodox interpretation of quantum physics as explored above. The most common approach to the mathematical semiotics used to resolve this tension among physicists is instrumentalism. In instrumentalism the relationship between mathematical theories and reality is equivocal. A passage of Henri Poincaré is often quoted to express an instrumentalist point of view regarding the relationship between mathematical formulas and the entities they represent:

> The object of mathematical theories is not to reveal to us the real nature of things; that would be an unreasonable claim. Their only object is to co-ordinate the physical laws with which physical experiment makes us acquainted, the enunciation of which, without the aid of mathematics, we should be unable to effect. Whether the ether exists or not matters little—let us leave that to the metaphysicians; what is essential for us is, that everything happens as if it existed, and that this hypothesis is found to be suitable for the explanation of phenomena.[226]

Whether "the real nature" of light is either particulate or a wave is not a concern of mathematical physics according to instrumentalism. The mathematical theories and their quantities offer no fundamental insight into the nature of the entities under consideration

[226] Poincaré, *Science and Hypothesis*, 235–36.

because the mathematical signs are not connected to reality.[227] Because of this break between the mathematical signs and the reality of the entities under consideration, instrumentalism does not claim to signify the world. It makes no claims about the actual entities in the world.

But other thinkers seem fundamentally dissatisfied with this approach to mathematical semiotics. In a journal article arguing for the statement, "[t]here are no particles, there are only fields" the author defended his realist claim. Realism "is an important topic, because too many physicists, disheartened by the apparent conundrums of standard quantum physics, are giving up on science's endeavor to describe the real world."[228] Einstein shared a similar sentiment, "In a particular area of theoretical physics the continuous field appeared side by side with the material point as the representative of physical reality. This dualism has to this day not disappeared, disturbing as it must be to any systematic mind."[229] In this statement Einstein expresses the nub of the problem:

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[227] A more recent view of instrumentalism associated with a "naturalized" philosophy of science is offered by Larry Laudan "epistemic justification…is simply a species of the genus instrumental rationality." Larry Laudan, "Aim-Less Epistemology?," *Studies in History and Philosophy of Science. Part A* (Oxford) 21, no. 2 (1990): 318, https://doi.org/10.1016/0039-3681(90)90029-8.

[228] This metaphysical conundrum still exists for contemporary physicists. A series of comments followed a 2013 article by physicist Art Hobson in which he claimed against the standard interpretation of quantum physics, "there are no particles, there are only fields." When Hobson rebutted one critic he described his motivation for writing the article with these words, "I thank R. J. Sciamanda for the nice comments, but must disagree with his views about realism. It is an important topic, because too many physicists, disheartened by the apparent conundrums of standard quantum physics, are giving up on science's endeavor to describe the real world." For Hobson, reality can be continuous as a field or it can be discrete as a particle, but it cannot be both at the same time. And he suggests that many physicists who have tried to hold to the standard position have been led by it to abandon realism. Robert J. Sciamanda and Art Hobson, "There Are No Particles, and There Are No Fields," *American Journal of Physics* 81, no. 9 (2013): 645, https://doi.org/10.1119/1.4812316.

[229] Einstein, "Maxwell's Influence on the Development of the Conception of Physical Reality."

discrete points and continuous fields can represent different aspects of one thing. But as they are incommensurable, they cannot both be univocal representations of the same thing without disturbing one's sense of an orderly system.

While it exists for all, the logical tension would seem to be accentuated by those who hold that discrete quantity and continuous quantity are mutually irreducible. One strategy to soften the logical tension in the doctrine of complementarity is to suggest that continuous quantity can be reduced to discrete quantity. By doing this, seemingly continuous phenomena like waves may eventually be discerned as simply another version of discrete particles. The reverse could also be done.

The logical tension would seem sharper for those realists who signify the world simultaneously with irreducibly discrete and irreducibly continuous quantity. But even though this tension will still exist, Hesse offers philosophical resources to diffuse it. Hesse raises ontological questions about whether entities must themselves be exclusively discrete or exclusively continuous or whether these are models whose relation to reality is nonetheless more than instrumental.

Hesse's conception of scientific models and analogies is pertinent to the notion of discrete and continuous quantities discussed in the previous sections:

> When we take a collection of billiard balls in random motion as a model for a gas, we are not asserting that billiard balls are in all respects like gas particles, for billiard balls are red or white, and hard and shiny, and we are not intending to suggest that gas molecules have these properties. We are in fact saying that gas

> molecules are analogous to billiard balls, and the relation of analogy means that there are some properties of billiard balls which are not found in molecules.[230]

Just as gas molecules have some properties in common with billiard balls, so gas molecules and billiard balls have some properties in common with idealized discrete units. They do not share all properties, but they share some properties.

Hesse notes that many fields "as diverse as cosmology and nuclear physics, brain physiology and Freudian psychology" use models. She thus concludes, "[i]t therefore becomes appropriate to speak of 'mathematical models' alongside these more traditional types." [231] While the phrase mathematical modelling has become popular today, Hesse is one of the first to mention the idea. What is notable is that she recognizes it among even the most simple notions of mathematics, like the discrete billiard balls in the kinetic theory of gases.

Hesse is also aware of the strict separation between discrete and continuous quantity that exists in the classical tradition of mathematics. Analyzing the same passage of Aristotle regarding the alternation of proportionals encountered in chapter two, Hesse concurs with Aristotle. Hesse says, "the properties of proportionals in lines and in numbers are

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[230] She continues, "Let us call those properties we know belong to billiard balls and not to molecules the negative analogy of the model. Motion and impact, on the other hand, are just the properties of billiard balls that we do want to ascribe to molecules in our model, and these we can call the positive analogy. Now the important thing about this kind of model-thinking in science is that there will generally be some properties of the model about which we do not yet know whether they are positive or negative analogies; these are the interesting properties, be-cause, as I shall argue, they allow us to make new predictions. Let us call this third set of properties the neutral analogy." Mary B. Hesse, *Models and Analogies in Science*, 2. print, Philosophy (Notre Dame, Ind: Univ. of Notre Dame Press, 1970), 8.

[231] Hesse, *Forces and Fields*, 24.

analogous".[232] The same property cannot be proven for magnitudes and for numbers simultaneously because they are fundamentally different sorts of quantity. Nonetheless, a strong analogy holds between them. This is of considerable interest because Hesse notes in an article on "Aristotle's Logic of Analogy" that discrete quantity and continuous quantity are not identical, and later she commends the use of "mathematical models". By joining these two observations together she creates the possibility of perceiving discrete unity and continuous unity as two different sorts of mathematical models.

While neither of these mathematical models, idealized discrete quantity or idealized continuous quantity, would be univocal with things in the world, when used as signs of entities in the world they can each provide insight into the nature of the entities under consideration. If Hesse's work sanctions thinking about these two sorts of quantity as mathematical models, or as analogies to entities in reality, she may be drawing this idea from Aristotle. In Metaphysics M Aristotle says,

> For man, *qua* man, is one indivisible thing; and the arithmetician assumes man to be one indivisible thing, and then considers whether there is any attribute of man *qua* indivisible. And the geometrician considers man neither *qua* man nor *qua* indivisible, but *qua* something solid. For clearly the attributes which would have belonged to "man" even if man were somehow not indivisible can belong to man irrespectively of his humanity or indivisibility. Hence for this reason the geometricians are right in what they maintain, and treat of what really exists; i.e., the objects of geometry really exist. For things can exist in two ways, either in complete reality or as matter. [233]

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[232] Hesse, *Models and Analogies in Science*, 137.

[233] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1078a23-28. See Book XIII:3:9

Aristotle says above, that "things can exist in two ways". A man "as man" is discrete. Not every aspect of his being is discrete. But in the context of being one irreducible indivisible person, man when counted "as man" is discrete. But man "as something solid" is continuous. This is not to say that man is in every way continuous. As an author of books of anatomy Aristotle knows that there are spaces inside men's lungs filled with air. But when man is considered "as something solid" he is analogous to an idealized 3-dimensional continuum. Thus, from Aristotle's point of view man is simultaneously "really" discrete and "really" continuous. The continuity of a man is real, and so is the discreteness of a man. But the man is a discrete unity and continuous unity with respect to different properties.[234] A man is like a continuum but only in certain ways. A man is like a discrete unity, but only in regards to certain aspects.

So how does this analogical approach to quantity diffuse the logical tension for realists who use irreducibly discrete quantity and irreducibly continuous quantity to signify reality? The analogical approach to quantity mitigates the logical tension in two ways. First, it establishes that quantities are always about particular measures, e.g. man "as man" or man "as something solid". Quantities which signify the world have a referent and are not merely abstract numbers. Numbers which refer to the world are always numbers 'of

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[234] "Substance, no doubt, is receptive of contrary qualifications, but not in such way that a man at the same time is both sick and healthy, a thing black and white simultaneously" Aristotle, "Categories," in *The categories: On interpretation*, trans. Harold P. Cooke, Loeb Library (Cambridge, Mass: Harvard University Press, 1938), 5b.

something'. And because one can see man in more than one way, either as discrete or as continuous, neither sign will exhaustively signify the totality of what man is.

And second, the analogical approach mitigates expectations of how encompassing a system of quantity can be in signifying the world. Because quantitative statements attach to only one aspect of an entity, e.g. man 'as something solid', it should surprise no one when statements regarding man 'as man' are irreconcilable with statements about man 'as something solid'. Statements about man 'as man' will follow the logic of discrete quantity as far as the analogy holds. Statements about man 'as something solid' will follow the logic of continuous quantity as far as the analogy holds. But a single man in his complete existence is neither exclusively discrete nor exclusively continuous, and thus both discrete and continuous quantity are needed to signify him fully.

This double approach to quantity is expressed by many thinkers in the mathematical philosophy of the ancient and medieval period. Thomas Aquinas restates Aristotle's point of view more than one thousand years later:

> I call that matter intelligible which is considered in the nature of a continuum, and sensible, that which is physical matter….I call matter designated if it is considered together with the determination of its dimensions…In this connection, it must be noted that designated matter is the principle of individuation, from which every intellect abstracts…The intellect of the mathematician…abstracts entirely from sensible matter…Hence, it is clear that abstraction, which is common to all intellects, makes a form universal.[235]

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[235] Thomas Aquinas, *The Disputed Questions on Truth*, trans. Robert W. Mulligan, Library of Living Catholic Thought (Chicago: Henry Regnery, 1952), 2.6.

In this passage, Aquinas makes explicit a link between matter and continuity, which is also a theme in Aristotle.[236] Intelligible matter "is considered in the nature of a continuum". He also describes how designated (de-sign-ated) matter has a "principle of individuation" resembling an Aristotelian comment that form "delimits" matter. In the above Aquinas echoes the basic Aristotelian notion that associates matter with continuity and form with discreteness. Thus, for Aquinas matter-form compounds simultaneously have a continuous aspect in the material principle and a discrete aspect in the formal principle.[237] All embodied things are matter-form compounds, and thus all embodied things have this simultaneous discreteness and continuity.[238]

Whether one agrees with the doctrine of form and matter is not the issue. The above shows that across more than a millennium an influential philosophy of mathematics existed which held that quantity which signifies the world is both discrete and continuous. If one replaces the term 'matter' with the phrase, 'a principle of continuity' and replaces the term form with 'a principle of discreteness', the above analysis still holds. In either case, the

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[236] Aristotle, *Physics*, trans. Wicksteed and Cornford, 209b. See Book IV:2

[237] "In all material substances, for Aristotle, there is a dialectic between form and matter…Without matter embodied forms could not exist because their being consists in organizing a foundation into the actuality and unity it only anticipates potentially by itself. But without form, matter would be a disorganized heap of forces without unity or meaning in itself." Robert Sokolowski, "Matter, Elements and Substance in Aristotle," *Journal of the History of Philosophy* (Berkeley, Calif) 8, no. 3 (1970): 284, https://doi.org/10.1353/hph.2008.1672.

[238] "How can this [matter] do the job of distinguishing without formal differentiation? Precisely because it is the very essence of matter, as quantitatively extended, to be spread out in space, parts outside of parts—one part of a material body can never be in the identical same place as another (by the Pauli exclusion principle in physics.)" W. Norris Clarke, *The One and the Many: A Contemporary Thomistic Metaphysics* (Notre Dame, Ind.: University of Notre Dame Press, 2001), 97.

recognition of a simultaneously discrete aspect and continuous aspect to things in the world had not previously caused a logical tension. It has always been understood that one kind of quantity does not signify the totality of an entity, but one kind of quantity only signifies one aspect of an entity. Because of this, there was no logical contradiction.

Logical contradictions arise when one grounds all aspects of a thing on some one property as in early modern atomism. Thinkers in the 17th century who advocated the doctrine of primary qualities, e.g. extension and motion, attempted such a project. If numbers are thought to univocally signify the 'point masses' then all aspects of a thing must reduce down to these discrete numbers. It is this overidentification of one model of quantity, e.g. discrete number, with the entirety of the entity that leads to logical tensions. If the entities under consideration are understood to be more than mere point masses, then there is no expectation that all aspects of a thing will be signified univocally by one model of quantity.

Consider a contemporary example of how the assumption of univocal interpretation with an arithmetical continuum in mathematics can lead to difficulties for scientists. Philosopher of physics Frank Arntzenius admits that "It is standardly assumed that space and time consist of extensionless points…[and] that all matter in the universe has point-sized parts. We are not often explicitly reminded of these very basic assumptions. But they are there …" Arntzenius considers the "alternative…that space and time and matter are 'pointless', or 'gunky'." But he qualifies this suggestion, "Let me emphasize how radical this idea is."

But why is the punctiform conception of space, time, and matter so commonly assumed and why is the gunky conception now considered radical? Philosophers of mathematics Hellman and Shapiro reply to Arntzenius,

> Of course, the thesis that space and time, or space–time, is punctiform is now the dominant orthodoxy, to the extent that the view is sometimes called "classical" (perhaps due to its reliance on classical logic). This can be attributed, of course, to the overwhelming success of the Dedekind–Cantor accounts of continuity and the real numbers. And *that* is due, in large part, to the power of actual infinity—to the "paradise" that Cantor bequeathed us.
>
> But there is no evidence, so far as we know, that Aristotle, Euclid, and the medieval and early modern mathematicians and scientists, found gunky space and gunky time to be a "dizzying prospect", nor was this a "radical" idea. Indeed, for most of our history, it was the received view.[239]

Helman and Shapiro describe above how mathematical conceptions can constrain physical theory or at least bias physical theory towards punctiform or gunky conceptions. Whether points or regions are used in a mathematical account of continuity does have implications for the conceptions of physical theory itself. That the Dedekind-Cantor account of continuity has become the default mathematical account, means it has also become the default account of the composition of space-time. The way the world is signified is not incidental to scientific theorizing of the world. On the contrary, it is profoundly influential. Thus, the way discrete quantity and continuous quantity are conceived of as either reducible to each other or as mutually irreducible in mathematics strongly influences the way scientists perceive the world.

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[239] Hellman and Shapiro, *Varieties of Continua*, 174–75.

It is not a sufficient argument for mathematicians to say that because continuous quantity can be reduced to discrete quantity, therefore continuous quantity is not needed as a basic irreducible primitive. If important intrinsic features of either discrete or continuous quantity are lost in the reduction of one sort of quantity to the other, then the reduction is lossy. If a scientist then uses that lossy concept of continuous quantity to signify a perceptually continuous quantity in the world, important features of the phenomena may never manifest in the mind of the researcher. Thus, if the signification of reality by discrete quantity and the signification of reality by continuous quantity have meaningful differences, then both should be retained in the signification of reality.

The search for a single theory of everything is a powerful driver in research and experimentation. But it may be that, as Hesse has suggested, there will always be an inevitable alternation between at least two incommensurable theories, one based in "atomicity" and the other based in "continuity". If Hesse is right, then it may be most profitable not to discard competing incommensurable theories, but for researchers to use competing theories constructively to discern new aspects of one's own theory.

The analogical approach to quantitative signification therefore diffuses the tension for realists who use both irreducibly discrete and irreducibly continuous quantity to signify the world, because from the beginning all aspects of a thing are presumed not to fit into a single system. Aristotle and Aquinas are each considered realists *par excellence* and they both hold that quantity that signifies the world is irreducibly discrete and irreducibly continuous. In neither case does it lead them to either abandon realism nor to a logical contradiction. Hesse brings the logic of analogy to bear in the context of modern physics. As Aristotle describes man "as man" or "as something solid", so Hesse recognizes that one

can conceive of a gas ‘as discrete billiard balls’ or ‘as a continuous fluid,’ and in neither case is the analogy complete nor should one presume to find a complete analogy. “[W]e are not asserting that billiard balls are in all respects like gas particles, for billiard balls are red or white, and hard and shiny, and we are not intending to suggest that gas molecules have these properties.”

It is precisely this recognition, that quantity that signifies the world is irreducibly discrete and irreducibly continuous, which produces the conviction that theories in both models of quantity should be simultaneously pursued and informed by each other. This conception is no longer conceived by the realist as a sharp logical tension which must be reconciled but as a generative difference. Theories in both models of quantity offer true insights and yet neither could ever be fully complete without the other.

C.S. Peirce: Do Formal Systems Need both Discrete Quantity and Continuous Quantity to Signify the World through Persistent Metaphysical Concepts?

Having argued that quantity that signifies the world in scientific theories is both irreducibly discrete and irreducibly continuous why should one now consider formal systems? Mathematics is a vast discipline, and many mathematicians consider the traditional algebraic and calculus symbols discussed above almost irrelevant to their conception of mathematics. Today much mathematics is conducted in formal systems, often Zermelo-Fraenkel set theory with the axiom of choice (abbreviated as ZFC), which uses the

Cantor-Dedekind arithmetical notion of reducible continuity as seen in chapter two. Set theory and formal systems like it are used in various ways to signify the world, both in the mathematical sciences like physics but also in philosophy.

Mathematical logic and philosophical logic grew highly intertwined beginning in the late 19th century. Some pure mathematicians such as G.H. Hardy may dislike this entanglement; if "no discovery" of pure mathematics will make "the least difference to the amenity of the world" then why should mathematicians concern themselves with signifying the world at all? [240] Whether pure mathematicians wish it or not, mathematics has in fact had a strong influence on the way the world is signified through set theory and other formal systems.

In the early 20th century Bertrand Russell and Alfred North Whitehead attempted to create a new foundation for mathematics based on set theory and mathematical logic. While their formalism for mathematics had inherent problems and was displaced by the set theory of Ernst Zermelo and Abraham Fraenkel, many philosophers took up the general project which Russell and Whitehead had popularized. The new formal logic along with set theory became a common language among analytic philosophers. The scientific topics discussed in the previous section are not of central interest to pure mathematicians. On the other hand, the foundations of analytic philosophy are of considerably more interest to pure

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[240] "I have never done anything 'useful.' No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world." Hardy, *A Mathematician's Apology*, 150.

mathematicians because mathematicians and analytic philosophers use a similar language for proof. Thus, in order to address fully the question of whether mathematics uses both irreducibly discrete quantity and irreducibly continuous quantity to signify the world, formal systems should be discussed. Since an important way that formal systems signify the world is through metaphysics, discussing the role of discrete and continuous quantity in metaphysics provides an excellent lens to understand whether formal systems that signify the world need both irreducibly discrete and irreducibly continuous quantity.

The work of C.S. Peirce (1839-1914) provides a helpful vantage point from which to examine how the foundations of mathematics intersect with metaphysics and formal systems. While C.S. Peirce and Bertrand Russell disagreed on many aspects of metaphysics and mathematical logic, they were both eminent in the field. Russell even said of Peirce, "It is, however, beyond doubt that [Peirce] was one of the most original minds of the later nineteenth century, and certainly the greatest American thinker ever." [241]

Peirce takes great interest in continuity and discreteness and their role in mathematics, metaphysics, and formal systems. To understand why requires an examination of each of these three fields through his eyes. Unlike the other leading mathematicians of his day, Peirce doubts that the Cantor-Dedekind continuum presents a true continuum. In metaphysics Peirce's thought on identity-over-time, substance, persons,

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[241] Bertrand Russell, *Wisdom of the West: A Historical Survey of Western Philosophy in Its Social and Political Setting* (New York, NY: Doubleday, 1959), 276.

and causality emphasize continuity as well as discreteness. In formal systems, his invention of triadic logic, his use of the relative individual, and his contributions to the mereological tradition are all influenced by his philosophy which references both discreteness and continuity. Thus, examining Peirce's thought may prove fruitful for considering whether quantity that signifies the world is irreducibly discrete and irreducibly continuous.

Peircean Conceptions of Discreteness and Continuity in Mathematics

Peirce, unlike Cantor, Dedekind, and the many mathematicians who followed them, seems to hold a more traditional view of discrete quantity and continuous quantity seeing them as mutually irreducible. Two key aspects of his mathematical work shed light on the issue. First, Peirce explicitly addresses discrete and continuous quantity in mathematics. Second, parallel to Aristotle's explanation of the fusion of boundaries in adjacent continuous magnitudes, Peirce discusses the generative nature of boundary values in continuous quantities.

First, Peirce offers many comments explicitly on the theme of discrete and continuous quantity in mathematics. For example, he says "A system of simple quantity is either continuous, discrete, or mixed." [242] But what does he mean by continuous or discrete?

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[242] Charles S. Peirce, "On the Logic of Number," in *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks, Past Masters (Charlottesville, Va: InteLex Corporation, 1994), 3.256.

> A continuous system is one in which every quantity greater than another is also greater than some intermediate quantity greater than that other. A discrete system is one in which every quantity greater than another is next greater than some quantity (that is, greater than without being greater than something greater than).[243]

The key phrase to hear in the quote above is "next greater than" another which regards Peirce's language of successors. For the natural numbers 4 is the successor of 3. It is the "next greater than" quantity. In neither the traditional continuum nor in the Cantor-Dedekind continuum does continuous quantity have a "next greater than some quantity". Having unique successors is a key distinction of discrete quantity.

Aristotle mentions the importance of "next-in-succession" and this is a core idea of the Peano axioms for arithmetic.[244] For traditional discrete quantity each quantity has a

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[243] Peirce continues, "A mixed system is one in which some quantities greater than others are next greater than some quantities, while some are continuously greater than some quantities." Peirce, "On the Logic of Number," 3.256. Also consider Peirce's opinion of the paper, "This is not the occasion to explore at all deeply just what it is that Peirce achieves in his "On the Logic of Number" of 1881 (3.252-3.288). He apparently liked this paper, for c. 1905 he commented upon it as follows (4.331): "As for Dedekind, his little book Was sind und was sollen die Zahlen? is most ingenious and excellent. But it proves no difficult theorem that I had not proved or published years before, and my paper had been sent to him." A detailed comparison of this paper with Dedekind's book would be of interest, but it is doubtful that the result would be to Peirce's advantage." R. M. Martin, "On Individuality and Quantification in Peirce's Published Logic Papers, 1867-1885," *Transactions of the Charles S. Peirce Society* 12, no. 3 (1976): 241.

[244] "The terms 'continuous,' ' contiguous,' and 'next-in-succession' have been defined above as follows: things are 'continuous' if (while they are themselves distinct in the sense of occupying different places) their limits are one, 'contiguous' if their limits are together, "next-in-succession' if they have nothing of the same nature as themselves between them. If these definitions are accepted, it follows that no continuum can be made up of indivisibles, as for instance a line out of points, granting that the line is continuous and the point indivisible. For two points cannot have identical limits, since in an indivisible there can be no distinction of a limit from some part other than the limit; and (for the same reason) neither can the limits be together, for a thing that has no parts has no limit, since a limit must be dis- tinct from what it limits. Yet the points would have to be either continuous or contiguous if they were to make a continuum." Aristotle, *Physics*, trans. Wicksteed and Cornford, 231a. See Book VI:1 Also "Closer scrutiny reveals that in [Peano, 1891] he formulates the axioms as conditions on a set N, its subsets, and a successor function." Ivor Grattan-Guinness, *Landmark Writings in Western Mathematics 1640-1940* (Chantilly, Netherlands: Elsevier Science & Technology, 2005), 622, http://ebookcentral.proquest.com/lib/oxford/detail.action?docID=269620.

unique successor, but for continuous quantity (either traditional or contemporary) each quantity does not have a unique successor. In other words, it is not clear what is next greater than 3.14. Is 3.15 the number next greater than 3.14? Or is 3.141 next greater? Or is the next greater some other number altogether such as π? There is no clear successor.

Peirce states his view of number, "Numbers express nothing whatsoever except order, discrete order. . . Number cannot express continuity" [245] Number in the traditional number concept before Simon Stevin was understood as discrete quantity. Extended magnitudes like lines, areas, and volumes are continuous quantities. While Peirce's emphasis on order differs from the older tradition's emphasis on multitude, both Peirce and the traditional view stress the fundamental distinction between discrete quantity and continuous quantity.

Furthermore, "Peirce thinks that the collection of real numbers is a discrete one…[and] Cantor's real numbers do not form a continuum" [246] Thus Peirce altogether opposes the point-set conception of continuity encountered in the previous chapter, which claims that continuous magnitudes (like geometric lines) are nothing more than infinite sets of points.

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[245] Charles S. Peirce, *The New Elements of Mathematics* (Hague: Mouton Publishers, 1976), 3.93.

[246] Jérôme Havenel, "Peirce's Clarifications of Continuity," *Transactions of the Charles S. Peirce Society* (Bloomington, IN) 44, no. 1 (2008): 110–11.

While discreteness is important, continuity is an even more major theme for Peirce. He describes continuity as "the master-key which … unlocks the arcana of philosophy." [247] Indeed, he sometimes referred to his philosophy as synechism, from the Greek word *syneches* for continuous. So, understanding Peirce's notion of the continuous has important metaphysical implications for him beyond mere mathematical interest.

Peirce from the beginning acknowledges that collections (his word for sets) are always gatherings of discrete things and it is for this reason that they cannot be built into a continuum. "A collection is a whole whose being consists in the independent being of its members; a line, on the contrary, has a being from which the being of its points is derived and in which they, as possibilities, are involved." [248] The reality of points can be derived from the reality of a continuum.[249] They are possibilities upon it. But the reality of points alone cannot construct the reality of the continuum. The above explicit statements about the distinction between discrete and continuous quantity suffice to show that Peirce sees these two sorts of quantity as mutually irreducible in mathematics.

The second topic which sheds light on Peirce's more traditional Aristotelian notion of the continuum regards how points as possibilities upon a line suggest that the continuum

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[247] Charles S. Peirce, "Fallibilism, Continuity, and Evolution," in *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks, Past Masters (Charlottesville, Va: InteLex Corporation, 1994), 1.163.

[248] Peirce, *The New Elements of Mathematics*, 2.531.

[249] This anticipates the move that A.N. Whitehead will make decades later with his process of "extensive abstraction". Whitehead in his writings after *Principia Mathematica* held that points can arise from intervals and not merely the other way around.

is generative.[250] One way that Peirce's continuum cleaves to the traditional continuum is that it allows for points to be fused, united, or welded together. In Aristotle when two line segments are brought together they become fused at the overlapping endpoint. Peirce's points can also unite:

> [A] continuum is a collection of so vast a multitude that in the whole universe of possibility there is not room for them to retain their distinct identities; but they become welded into one another. Thus the continuum is all that is possible, in whatever dimension it be Continuous'. [251]

The kind of fusions or weldings that Peirce describes above are more general in nature. He does not describe them as occurring only at boundary points as did Aristotle.[252] Nonetheless, the mere possibility of fusion or weldings of points distinguishes Peirce's notion of continuity from point-set notions of continuity. In point-set continuity when an

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[250] While Peirce's views on the continuum evolve through a number of phases, he generally upheld an Aristotelian version of the continuum over and against the Cantor-Dedekind version. He did have a Cantorian period from 1884-1892. Havenel, "Peirce's Clarifications of Continuity," 91.

[251] Charles S. Peirce, *Philosophy of Mathematics: Selected Writings*, with Matthew E. Moore (Bloomington, Ind.: Indiana University Press, 2010), 174.

[252] "Of quantities that are discrete we may here instance number and speech, of quantities that are continuous line, superficies, and solid, to which time and place may be added. Consider the parts of a number. You find there is no common limit at which they may join or unite. For example, two fives will make ten. These, however, are wholly distinct, there is no common limit whatever at which these two fives coalesce. And the same with the parts three and seven. And, indeed, in the case of all numbers you never will find such a boundary, common to any two parts for the parts remain ever distinct. Thus is number discrete, not continuous…Each, indeed, is distinct from the rest. A line is, however, continuous. Here we discover the limit of which we have just now been speaking. This limit or term is a point. So it is with a plane or a solid. Their parts also have such a limit—a line in the case of the former, a line or a plane in the latter.

Again time and space are continuous. Time is a whole and continuous, the present, past, future are linked. Space is also this kind of a quantity. For seeing the parts of a solid themselves occupy so much space and these parts have a limit in common, it follows the parts of space also, which those parts themselves occupy, have exactly the same common limit or term as the parts of the solid. As is time, so is space, then, continuous the parts meet at one common boundary." Aristotle, "Categories," 6.4b-5a.

interval is decomposed into two contiguous sets of points, both sets cannot contain the boundary value. Because the boundary value is a point and the line is composed of points, the boundary value must belong to one set or the other.[253] But in traditional continuity a line decomposes into continuous subsegments. The boundaries are points which are dimensionally distinct from the continuous subsegments. For this reason, when two continuous line segments overlap their boundary points fuse.

The theme of welding together is clearly important to Peirce because he mentions it elsewhere regarding a continuum:

> In the calculus and theory of functions it is assumed that between any two rational points (or points at distances along the line expressed by rational fractions) there are rational points and that further for every convergent series of such fractions (such as 3.1, 3.14, 3.141, 3.1415, 3.14159 etc.) there is just one limiting point; and such a collection of points is called continuous. But this does not seem to be the common sense idea of continuity. It is only a collection of independent points. Breaking grains of sand more and more will only make the sand more broken. It will not weld the grains into unbroken continuity.[254]

In this discussion Peirce likens a multitude of real numbers to grains of sand. Breaking them into smaller bits, or further subdividing them (e.g., distinguishing 3.140 from 3.141), does not fuse them.[255] A continuous entity needs to be a united entity not a mere collection. And if two discrete things like line segments are to become a continuous entity they must be capable of fusion.

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[253] See also footnote 259 on Dedekind Cuts.

[254] Peirce, *Philosophy of Mathematics*, 139.

[255] Peirce here is likely referring to a Cauchy sequence for the real number, π.

This quote above also displays that Peirce has a more traditional notion of the real numbers which stems from his more traditional notion of the continuum.[256] Even if the real numbers somehow correspond to points on a line, that does not mean that the points constitute the line. In this he is acknowledging the basic Aristotelian continuum and disagreeing with the Cantor-Dedekind axiom described in chapter two.

Boundary points are key to Peirce's philosophy of continuity. He speaks of them at various times. It is helpful to understand the link in Peirce's mind between continuity and possibility. A quote regarding the end of a line, a boundary, is instructive:

> The end of a line might burst into any discrete multitude of points whatever, and they would all have been one point before the explosion. Points might fly off, in multitude and order like all the real irrational quantities from 0 to 1; and they might all have had that order of succession in the line and yet all have been at one point.[257]

In the picture above, the continuum is generative. It begets innumerable points, as many as can be imagined. It is for this reason that continuity seems to be a source of indeterminacy and modality in Peirce's thought. While the above quote does not explicitly discuss the fusion of boundary points, it indicates the generative capacity of a boundary point. If boundary points are fused when joined, then they should be generative and 'unfused' when separated.

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[256] Francisco Vargas and Matthew E. Moore, "The Peircean Continuum," in *The History of Continua: Philosophical and Mathematical Perspectives*, ed. Stewart Shapiro and Geoffrey Hellman (Oxford University Press, 2020), 328–46.

[257] Charles S. Peirce, "The Logic of Relatives," in *Reasoning and the Logic of Things: The Cambridge Conferences Lectures of 1898*, with Kenneth Laine Ketner (Cambridge, Mass ; Harvard University Press, 1993), 160.

In Peirce's mature philosophy boundary properties play a role that appears central to his thought:

> Thus, a blot is made on the sheet. Then every point of the sheet is unblackened or blackened. But there are points on the boundary line, and those points are insusceptible of being unblackened or of being blackened, since these predicates refer to the area about S and a line has no area about any point of it.[258]

Peirce claims that the points on the boundary line are not areas and hence are not able to be blackened. The status of boundaries is ambiguous. The principle of the excluded middle does not apply to boundaries or boundary points in the way it does to the areas themselves, because discrete points or lines of infinitesimal thinness are not the same sort of thing as continuous areas. The areas do not consist of points. On Peirce's traditional view of continuity the continua cannot be neatly separated by a Dedekind cut as discussed in chapter two.[259] Some points are determinable within an area, but boundary points are not because they belong to both areas simultaneously. This is similar to Aristotle's notion of fused boundary points on a one-dimensional line.

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[258] Peirce as cited by Robert Lane, "Peirce's Triadic Logic Revisited," *Transactions of the Charles S. Peirce Society* (Buffalo, NY) 35, no. 2 (1999): 294, 2.

[259] Dedekind's most important concept, the "Dedekind cut" is predicated on the assumption that boundary points are not shared by two line segments but are excluded by one. "I find the essence of continuity… in the following principle: "If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions." Richard Dedekind, *Essays on the Theory of Numbers* (Chicago: Open Court Publishing Co, 1901), 5. As Dedekind continues, he seems to realize that there are two ways to get this boundary point which makes the cut. "We can then say that every rational number 'a' produces one cut or, strictly speaking, two cuts, which, however, we shall not look upon as essentially different;" But he does not seem to recognize that these two cuts means that the point itself at which the cut occurs belongs to both classes simultaneously.

Peirce writes often and extensively about discreteness and continuity. But these passages above suffice to illustrate his conviction that both discrete and continuous quantity should be treated as mutually irreducible in mathematics and formal systems. How does this conviction affect the way Peirce suggests formal systems should signify the world? To understand this, first Peirce's emphasis on discreteness and continuity in persistent metaphysical concepts should be considered.

Discrete and Continuous Aspects of Persistent Metaphysical Concepts in Peirce

Peirce was a key influence on the rise of the new philosophical logic and a champion of traditional metaphysical thought, so examining his work in these areas may shed light on the topic of how quantity signifies the world through metaphysical concepts. How did Peirce champion traditional metaphysical thought? He sometimes described himself as a "scholastic realis[t]." [260] And because of Aristotle's influence on scholasticism and Peirce's frequent references to both scholasticism and Aristotle, it may be helpful to consider aspects of Aristotelian thought related to Peirce's thought for comparison. The key persistent metaphysical concepts to be examined are first, identity-over-time especially as related to notions of substance and persons and second, an expansive notion of causality which includes not only efficient cause but also final, formal, and material cause.

[260] Charles S. Peirce, "Issues of Pragmatism," in *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks, Past Masters (Charlottesville, Va: InteLex Corporation, 1994), 5.453.

Before examining specific metaphysical topics, a general introduction to Peirce's metaphysical framework may prove instructive. As seen in the last section, Peirce's approach to mathematics embraces both discrete quantity and continuous quantity. Peirce's opposition to the arithmeticization project's exclusive use of discrete quantity finds a corollary in his metaphysics, "I like to call my theory Synechism, because it rests on the study of continuity." [261] In addition to the term, 'synechism', he also refers to his philosophy as 'thirdness', which for Peirce is another way to reference continuity. Peirce emphasizes the importance of three different aspects of being in which the third was the most particular to his thought:

> The First is that whose being is simply in itself, not referring to anything nor lying behind anything. The second is that which is what it is by force of something to which it is second. The third is that which is what it is owing to things between which it mediates and which it brings into relation to each other.[262]

Peirce's categories of firstness and secondness are sufficient to establish the idea of discrete disjoint things. "First and Second, Agent and Patient, Yes and No, are categories which enable us roughly to describe the facts of experience, and they satisfy the mind for a very long time." [263] Nonetheless, the first and second are not enough. "But at last they are found

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[261] Charles S. Peirce, "The Logic of Continuity," in *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks, Past Masters (Charlottesville, Va: InteLex Corporation, 1994), 6.202.

[262] Charles S. Peirce, "A Guess at the Riddle," in *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks, Past Masters (Charlottesville, Va: InteLex Corporation, 1994), 1.356.

[263] Peirce, "A Guess at the Riddle," 1.359.

inadequate, and the Third is the conception which is then called for." [264] The first and second allow discrete categories, "Yes and No", binary opposition. But, "that which is really…the characteristic of my doctrine, namely, that I chiefly insist upon [is] continuity, or Thirdness." [265] Thus, in Peirce's central metaphysical doctrines, he highlights the role of both the discrete and the continuous, concepts which pervade his thought.

A passage from Peirce illustrates his emphasis on continuity and discreteness in other aspects of metaphysics:

> The extraordinary disposition of the human mind to think of every thing under the difficult and almost incomprehensible form of a continuum can only be explained by supposing that each one of us is in his own real nature a continuum. I will not trouble you with any disquisition on the extreme form of realism which I myself entertain that every true universal, every continuum, is a living and conscious being, but I will content myself with saying that the only things valuable, even here in this life, are the continuities.
>
> The zero collection is bare, abstract, germinal possibility. The continuum is concrete, developed possibility. The whole universe of true and real possibilities forms a continuum, upon which this Universe of Actual Existence is, by virtue of the essential Secondness of Existence, a discontinuous mark—like a line figure drawn on the area of the blackboard.[266]

Peirce claims above that "human mind[s]…think of every thing [in the]…form of a continuum", that "living conscious being[s]" are continua, the "only things valuable…are the continuities", and "the whole universe of…real possibilities forms a continuum". Quite

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[264] Peirce, "A Guess at the Riddle," 1.359.

[265] Peirce, "The Logic of Continuity," 6.202.

[266] Peirce, *The New Elements of Mathematics*, 4.345.

quickly the pervasiveness of continuity in Peirce's metaphysics is manifest. But he also notes a role for discreteness in the passage above. "[T]he Universe of Actual Existence is, by virtue of the essential Secondness of Existence, a discontinuous mark." Secondness, as seen above, emphasizes "yes and no", binary opposition. "Existence… [is] a discontinuous mark" on a realm of continuous possibility. What is discretely actualized limits what is possible.[267] Both discreteness and continuity appear in the passage above.

While continuity and discreteness are evident in Peirce's thought, how does he relate them to traditional metaphysics which was heavily influenced by Aristotle? Consider now traditional metaphysical concepts starting first with identity-over-time including notions of substance and persons.

Peirce says, "identity is a continuous relation." [268] This appears to be a departure from the position of Hume codified in Russell's set theoretic approach. To understand more clearly, consider whether Peirce's conception of identity as a continuous relation has any precedent in traditional metaphysical thought.

Common sense notions of substance and person rely on an implicit conception of identity-over-time. To identify something like a frog or a man as a substance is also to claim that the substance has identity-over-time. But today, since the concept of substance is

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[267] "The whiteness or blackness, the Firstness, is essentially indifferent as to continuity. It lends itself readily to generalization but is not itself general. The limit between the whiteness and blackness is essentially discontinuous, or antigeneral. It is insistently *this here.* The original potentiality is essentially continuous, or general." Peirce, "The Logic of Continuity," 6.205.

[268] Peirce, *The New Elements of Mathematics*, 4.330.

either banished or highly contended, conceptions of identity-over-time must rest on their own merits.[269] As the notion of person for many medieval philosophers also invoked substance, the concept of person has also lost this reference.[270] With neither a stable conception of substance nor a stable conception of identity-over-time, the concept of person is difficult to explain. While it is plain to see that identity depends on discreteness, a more subtle reason why these concepts are difficult to maintain today is because of the loss of the traditional notion of continuity.

Although Aristotle only once explicitly addresses persistent identity-over-time, Aristotle's notion of substance allows him to talk about identity-through-change.[271] A substance has non-changing essential properties and changeable accidental properties. Nonetheless, a substance is a unity, "the things that are primarily called one are those whose substance is one, and one either in continuity or in form or in definition." [272] Aristotle mentions that the unity of a substance may depend upon its continuity.

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[269] For example, by the early 20th century Bertrand Russell can say, "the notion of substance, in the sense of a permanent entity with changing states, is no longer applicable to the world." Russell, *An Outline of Philosophy*, 327–28.

[270] Robert Spaemann, *Persons: The Difference between "someone" and "Something,"* Oxford Studies in Theological Ethics (Oxford: University Press, 2006), 31.

[271] "Although Aristotle gives ample attention in the Physics to the Zenonian paradoxes of motion, he makes no mention of puzzles that focus mainly on problems of persistence, except for one brief reference in Physics Book 4, chapter 11." John Bowin, "Aristotle on Identity and Persistence," *Apeiron (Clayton)* (Edmonton, AB) 41, no. 1 (2008): 63–88, https://doi.org/10.1515/APEIRON.2008.41.1.63.

[272] Aristotle, *Metaphysics*, trans. Ross, 1016b. See Book V:6:15

Aristotle repeats this theme about the continuity of a substance in various places. He says,

> of the things that are thought to be substances, most are only potencies—both the parts of animals…and earth and fire and air; for none of them is a unity, but as it were a mere heap, till they are worked up and some unity is made out of them…Yet all the parts must exist only potentially, when they are one and continuous by nature…[273]

In an organic unity like an animal there are many potential parts ("potencies") which are "one and continuous by nature." These are not actual parts, as in a machine, but potential parts. While they may be separated in theory, they exist together as a continuous unity. An animal, for example, is a unified continuous substance which is made of continuous potential parts.

This notion extends to other organic unities:

> Organic unity differs from contact; for in the latter case there need not be anything besides the contact, but in organic unities there is something identical in both parts, which makes them grow together instead of merely touching, and be one in respect of continuity and quantity, though not of quality…[274]

Contact or contiguity does not suffice for true continuity according to Aristotle. In a continuous thing, the boundaries of its parts fuse or weld together. This is a key difference between the Cantor-Dedekind conception of continuity and the Aristotelian conception.

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[273] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1040b15. See Book VII:16

[274] Aristotle, *Metaphysics*, trans. Ross, 1014b. See Book V:4

The importance of the distinction is clear in the context of an organism. A blood vessel which has two parts that are merely contiguous is not a functioning vessel. The vessel must be continuous. Its parts are potential and as a unity they are actually fused together and only potentially separable. "There is a resemblance between the osseous and the vascular systems; for each has a central part in which it begins, and each forms a continuous whole." [275]

This continuity of parts is an important aspect of an organic substance. But also important is its continuous identity through time which allows accidental change but not a change to the unifying nature of the substance. While there are many potential parts in an organic substance like a frog, it is considered one discrete thing. Thus, as seen by the examples above, for Aristotle, both discrete quantity and continuous quantity are required to signify a substance.

The medieval conception of persons does not need to deal with identity-over-time as a separate topic from substance because it can rest on the notion of substance for identity-over-time. Accidental changes do not affect a person's identity. In his book *Persons* Robert Spaemann summarizes the view of Aquinas on how a person is a substance:

> [Aquinas] follows Boethius's definition in designating a person as a 'substance', but stipulates that it is a *substantia prima*, Aristotle's 'first *ousia*', i.e. a concrete individual. While the word 'man' denotes a species, a natural kind defined by the

[275] Aristotle, "Parts of Animals," in *Complete Works of Aristotle, Volume 1*, vol. 192, trans. William Ogle, with Jonathan Barnes (Princeton: University Press, 2019), 2.9, https://doi.org/10.1515/9781400835843-025.

> specific predicates of its members, 'person' does not denote the kind, but the member of the kind, and not *as* a member of the kind, but as an individual.[276]

As seen above, important medieval thinkers rely on substance to define persons as *substantia prima*. Because of this dependence, to establish personal identity these medieval philosophers do not need a separate conception of identity-over-time distinct from the notion of substance. As long as substance remains a way of invoking identity-over-time and persons remain substances, there is no separate puzzle of identity-over-time.[277] But by the time of Hume or Russell, the philosophical conception of substance was no longer widely held. Therefore, the disappearance of the notion of substance means the disappearance of a philosophical notion of personal identity.

Peirce does not often address the notion of substance but instead addresses personal identity directly. As seen earlier, Peirce holds continuity as a central part of metaphysics, and this comes out in Peirce's notion of identity-over-time:

> In order to illustrate what it would mean to say that Identity is a Continuous Relation,—that is, continuous in meaning,—we may compare it with another. To say that who commanded the French in the battle of Leipzig commanded them in the final battle of Waterloo, is not merely a statement of identity: it is a statement of Becoming. There is an existential continuity in time between the two events. But so understood the statement asserts no Significative Identity, inasmuch as the intervening continuum is a continuum of Assertoric Truths. Now, upon a continuous line there are no points (where the line is continuous). There is only room for points—possibilities of points. Yet it is through that continuum, that line of

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[276] Spaemann, *Persons*, 31.

[277] While the story of the ship of Theseus, which inquires about identity-over-time for the ship, comes from Plutarch. It was not considered a philosophical puzzle until the time of Thomas Hobbes in the early modern era as the doctrine of substance was being challenged.

> generalization of possibilities, that the actual point at one extremity necessarily leads to the actual point at the other extremity.[278]

As can be seen here, identity for Peirce is not merely the synchronic identity, i.e. the possession of identical properties at an instant as it is for Hume or Russell. There is a persistence of identity that clearly for Peirce depends upon continuity over time, diachronic identity. "Yet it is through that continuum, that line of generalizations of possibilities, that the actual point at one extremity [Leipzig Napoleon] necessarily leads to the actual point at the other extremity [Waterloo Napoleon]." The being and becoming are linked by continuous identity.

Peirce elsewhere addresses how "the barbaric conception of personal identity must be broadened". He says,

> All communication from mind to mind is through continuity of being. A man is capable of having assigned to him a *rôle* in the drama of creation, and so far as he loses himself in that *rôle,* — no matter how humble it may be, — so far he identifies himself with its Author.[279]

Here Peirce underscores the way that humans are connected to—even have a "continuity of being" with—other humans. This is especially seen through "communication from mind to mind". How can humans speak intelligibly to other humans? Humans are connected to other humans through mind and continuous being. Thus, even though Peirce emphasizes

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[278] Peirce, *The New Elements of Mathematics*, 4.330.

[279] Charles S. Peirce, "Synechism and Immortality," in *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks, Past Masters (Charlottesville, Va: InteLex Corporation, 1994), 7.570-572.

the "discontinuity" of the "act of existence", there still remains a continuity in a particular human's personal identity and between humans as human beings when they communicate mind to mind.[280]

As a philosophical category Peirce's synechism or continuity has broad implications. It influences his notion of humans, of community, even his notion of man's relation to God. "[S]o far as he loses himself in that *rôle* [in the drama of creation]…so far he identifies himself with its Author." The continuity of identity for Peirce is compatible with both the discreteness of an individual and with the possibility of many different discrete experiences over time. A continuous white line is discrete against a black chalkboard background, and as Peirce explains above, the line has the possibilities of discrete points upon it. This notion of identity-over-time relies on simultaneous unity in multiplicity. From one perspective the many experiences of Napoleon are like the sets of Russell's Socrates as possible points on a line. But from another perspective the Leipzig Napoleon is the same and continuous in identity with the Waterloo Napoleon. Thus, for Peirce the persistent metaphysical concept of identity-over-time central to recovering traditional notions of personal identity and substance requires both discreteness and continuity.

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[280] See also the development of joint attention in Andrew Pinsent, *The Second-Person Perspective in Aquinas's Ethics: Virtues and Gifts*, Routledge Studies in Ethics and Moral Theory 17 (London: Routledge, 2012), 41–49.

Second, consider the persistent metaphysical concept of causality, as seen in its expanded conception as efficient, final, material and formal cause. Peirce offers a quick summary of the expanded conception of causality along with its supposed meaning. "Aristotle himself…recognizes four distinct kinds of cause, which go to determining a fact: the matter to which it owes its existence, the form to which it owes its nature, the efficient cause which acts upon it from past time, and the final cause which acts upon it from future time." [281] But before delving further into Peirce's own views consider first the thought of the ancients.

For Plato and Aristotle there is a link between the unity and multiplicity of a thing and causality. Aristotle says, "In all things which have a plurality of parts, and which are not a total aggregate but a whole of some sort distinct from the parts, there is some cause; inasmuch as even in bodies sometimes contact is the cause of their unity, and sometimes viscosity or some other such quality." [282] Aristotle distinguishes here an aggregate (or multitude) from a whole (which retains some form of connectedness or continuity). How is causality related to wholeness, continuity, and unity?

Plato explains existence: 'to be' is 'to be able to act or be acted upon'. In the mouth of the Eleatic Stranger Plato puts it like this, "anything which possesses any sort of power

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[281] Charles S. Peirce, "Physical Causation," in *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks, Past Masters (Charlottesville, Va: InteLex Corporation, 1994), 6.66.

[282] Aristotle, *Metaphysics*, trans. Ross, 1045a. See Book VIII:6.

to affect another, or to be affected by another…has real existence; and I hold that the definition of being is simply power." [283] The notion that being is the power to affect or be affected by something is sometimes called the Eleatic principle.[284] Aristotle slightly modifies this notion to claim that "being…is act". [285] If the capacity to act or act upon something could be called a causal power, then both Plato's and Aristotle's definitions confirm a close connection between being and causality.[286] If multiple things work together

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[283] Consider the definition of being given by the Eleatic Stranger in Plato's *Sophist*. "My notion would be, that anything which possesses any sort of power to affect another, or to be affected by another, if only for a single moment, however trifling the cause and however slight the effect, has real existence; and I hold that the definition of being is simply power." Plato, "Sophist," in *The Dialogues of Plato*, trans. Benjamin Jowett (Chicago: Encyclopaedia Britannica, 2003), lines 247d–247e.

[284] "For Aristotle, ontological commitment is guided by the Eleatic principle: 8 Proposition 2.2. A thing exists just in case that thing has the causal power to affect and/or to be affected, to bring about change and/ or to suffer change." William M. R. Simpson, "From Quantum Physics to Classical Metaphysics," in *Neo-Aristotelian Metaphysics and the Theology of Nature* (Routledge, 2021), 25.

[285] The Greek reads, "ὥστε φανερὸν ὅτι ἡ οὐσία καὶ τὸ εἶδος ἐνέργειά ἐστιν." Aristotle, *Metaphysics*, 1050b See Book IX:8:14 Emerson Buchanan offers his own translation of this statement, "So that it is evident that being or form is act." Buchanan also contrasts Plato's definition with Aristotle's. "Plato suggests as a definition of being that is *dynamis*, i.e., power, or potentiality, or capacity. It is a plausible hypothesis that Aristotle's assertion that *ousia* is *energeia*, that Being is act or activity, was made in conscious opposition to the suggestion in the *Sophist*." Here the contrast is between power to act and action itself. Because both Plato and Aristotle hold that these mutually involve each other in many cases (i.e. potency and act), it is mostly a matter of priority. Is act prior in being? Or is potential to act prior in being? Emerson Buchanan, *Aristotle's Theory of Being*, Greek, Roman, and Byzantine Monographs ; No. 2 (University, Mississippi, 1962), 54.

[286] This debate which first occurred between Aristotle and Plato as detailed in the previous footnote is of contemporary relevance to quantum physics. Consider this comment in an article which claims that only particles are real and fields are not. "A quantum 'field' can certainly be understood in the abstract sense of a 'field of potentialities,'—a field of potential properties that can possibly be actualized (in our three dimensional space) through measurement processes (interactions with measuring apparatuses). This transition from a 'potential' mode of being to an 'actual' mode of being, where each time only one among countless different possibilities is selected, is at the heart of a quantum measurement and certainly needs to be considered if one wants to clarify the true nature of quantum entities." Massimiliano Sassoli de Bianchi, "Quantum Fields Are Not Fields; Comment on 'There Are No Particles, There Are Only Fields,' by Art Hobson [Am. J. Phys. 81(3), 211–223 (2013)]," *American Journal of Physics* 81, no. 9 (2013): 707–8, https://doi.org/10.1119/1.4811240.

to cause something as do the various organs of a body or the parts of a tree, then this also means that causality and notions of unity and multiplicity are related.

Aristotle's broad notion of causality influenced thinkers for thousands of years. And while Peirce seems to approve of the Aristotelian notion of four causes, he acknowledges that there are limits to this presentation, though not for the usual reasons. Peirce notes that in his own era, "[i]t is generally held that the word 'cause' has simply been narrowed to that one of the four Aristotelian causes which was named from the circumstance that it alone produces an effect." [287] Causality has been reduced to only efficient cause, it is claimed. But Peirce disagrees with this characterization of cause.

What Peirce believes is missing is a certain kind of continuity between the past and the future that true causality must keep together. "[T]here is an essential thirdness, which the principle of causality fails to recognize…" [288] Remembering that thirdness for Peirce generally means continuity provides the key to his concern. His reasoning has to do with the nature of kinematical quantities (such as acceleration) and dynamical quantities (such as force). In his discussion of Newton's second law (often given as F=ma), Peirce says this:

> The second proposition, that the cause precedes the effect in time, is equally false. The effect is the acceleration. The cause which produces this effect under the law of force is, according to the doctrine of the conservation of energy, the relative positions of the particles. Now the acceleration which the position requires does not come later than the assumption of that position. It is, on the contrary, absolutely

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[287] Peirce, "Physical Causation," 6.66.

[288] Peirce, "Physical Causation," 6.69.

> simultaneous with it. Thus, the second proposition of the principle of causation is false.[289]

In the case above, Peirce wishes to show that the relative positions of the particles (which is supposedly the cause when using least action or conservation of energy principles) cannot be before the effect (which is supposedly the acceleration), because while the initial position of the particle is before the acceleration the final position is after the acceleration.[290] Thus the relative positions of the particles are both before and after the acceleration. This would make the cause both before and after the effect.

This need for a formula to "know" the final position of a particle before its dynamics (or path) can be determined is a puzzling aspect of Euler-LaGrange mechanics and the principle of least action.[291] Euler himself made mention of final causality in his papers on the principle of least action and considered the link with Leibniz' mechanics

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[289] Peirce, "Physical Causation," 6.69.

[290] While Peirce references Newton's second law he also mentions conservation of energy. Chaining the ideas together, he probably means something like $-\frac{dU}{dx} = F = ma$. The object's potential energy (U) is based on its location in a field. And the forces impinging upon it are a result of differentials in the field based on location. But given the context, Peirce is also likely thinking of the principle of least action which is a more general way to get force from locational differences.

[291] "The very enunciation of the principle of least action is objectionable. To move from one point to another, a material molecule, acted upon by no force, but compelled to move on a surface, will take as its path the geodesic line—i.e., the shortest path. This molecule seems to know the point to which we want to take it, to foresee the time that it will take it to reach it by such a path, and then to know how to choose the most convenient path. The enunciation of the principle presents it to us, so to speak, as a living and free entity. It is clear that it would be better to replace it by a less objectionable enunciation, one in which, as philosophers would say, final effects do not seem to be substituted for acting causes." Poincaré, *Science and Hypothesis*, 144.

which emphasized final cause.[292] While this was not an unknown point, its metaphysical significance is hard to discern. Peirce makes it a keystone of his critique of the disconnectedness of before and after in cause and effect. Instead of a disconnected sequence of events in time, Peirce wishes to establish continuity through time between cause and effect to prove his philosophy of synechism. Peirce does not dismiss that there is also secondness or discreteness in cause and effect. He wants to add "an essential thirdness" or continuity which he believes is overlooked.[293] Russell and Hume envisioned causation as contiguous sequences of events in a relation.[294] But to this multiplicity, Peirce also sees within them the unity of continuity. In this, Peirce depends upon both discrete and continuous quantity to signify the unity in multiplicity of causation.

In addition to recovering a role for final cause as connected to efficient cause as seen above, continuity and final cause also appear in Peirce's discussions of scholastic

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[292] See Leonhard Euler, "On the Principle of Least Action," in *Opera Omnia Leonhard Euler*, vol. 5, trans. Michael Saclolo, 2 (1753), 179–93, https://scholarlycommons.pacific.edu/euler-works/198/.

[293] A complement to Peirce's concerns can be seen in Pinsent's fascinating analysis of the role of final cause in the complex dynamics of chaos theory which also underscores the role of matter as a principle of continuity. "The final cause (also equated with what Aristotle calls the formal cause in the case of biological systems) is the strange attractor; the efficient cause is the process by which the attractor is generated and the matter is the principle of continuity between the initial and final states…" Andrew Pinsent, "Why Matter Matters to Neo-Aristotelian Teleology in Mechanics," *Anuario Filosófico* 46, no. 2 (2013): 358, 2.

[294] "[W]hen I say 'causal' I do not mean exactly what the word naturally conveys. There must be no idea of compulsion or 'force', neither the force of contact which we imagine we see between billiard-balls nor the action at a distance which was formerly supposed to constitute gravitation. There is merely an observed law of succession from next to next. An event at one moment is succeeded by an event at a neighbouring moment, which, to the first order of small quantities, can be calculated from the earlier event. This enables us to construct a string of events, each, approximately, growing out of a slightly earlier event according to an intrinsic law." Russell, *An Outline of Philosophy*, 123–24.

realism. Peirce's perspective can be seen in his concept of natural kinds as explained by a commentator:

> But whereas Peirce was neither a 'radical ontological pluralist' nor an 'anarchistic pluralist,' he most certainly defended a causal pluralism. There are many different levels of natural classes (physical entities, chemical entities, biological entities, sociological entities, artifacts, etc.), and therefore also many different levels of causation. *Each class is characterized by a distinctive final cause, and the objects belonging to a natural class do so by virtue of their ability to exert a type of real causal influence.* Thus, it is the task of science to determine exactly what natural classes there are.[295]

Peirce's perspective holds that unity and causality are related. A "class is characterized by a distinctive final cause." Unlike Hume, causal connectedness or causal continuity is a key for Peirce as is continuity in general. The final cause binds together the multiple entities into a unity, or perhaps a unity in multiplicity. These ideas will affect his notion of the relative individual regarding flexible hierarchical classes in logic, a topic to be examined later.

How does Peirce conceive of material cause, especially since material and formal cause have often been held together in the notion of a substance? Peirce says, "[t]he original potentiality is the Aristotelian matter or indeterminacy from which the universe is formed." [296] What does Peirce mean by invoking Aristotelian matter? In the *Physics* Aristotle says regarding matter and form,

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[295] Menno Hulswit, "Peirce's Teleological Approach to Natural Classes," *Transactions of the Charles S. Peirce Society* 33, no. 3 (1997): 764, 3.

[296] Peirce, "The Logic of Continuity," 6.206.

> [I]f we think of a thing's place as its 'dimensionality' or 'room-occupancy' (to be distinguished from the thing itself, as a concrete quantum) we must then regard it as 'matter' rather than as 'form,' for matter is the factor that is bounded and determined by the form, as a surface, or other limit, moulds and determines; for it is just that which is in itself undetermined, but capable of being determined, that we mean by matter.[297]

In this passage, Aristotle expresses the important doctrine that form delimits and discretizes matter. "For [matter] is just that which is in itself undetermined." The form makes the extension discrete so as to avoid an actual infinity. "For Aristotle the underlying matter is simply formless, unqualified, space-filling stuff. Its extension is not an attribute, quality or predicate, but is as primary as matter itself." [298] Here lies the original inspiration which the Cartesian *res extensa* took as its background. Matter and extension are equally primary. And given that matter is extended in space, matter is continuous.[299] Potentiality is also inherent in matter. What is "undetermined" may still yet be determined.[300] Thus, Aristotle associates the pair form and matter with discreteness and continuity.

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[297] Aristotle, *Physics*, trans. Wicksteed and Cornford, 209b. See Book IV:2

[298] Sokolowski, "Matter, Elements and Substance in Aristotle," 279.

[299] "their matter (either primary or ultimate) is formally divisible; others because the definitions of their essence are more than one." Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1017a6. See Book V:6:23

[300] Thomas Aquinas makes a very similar point and connects it explicitly with discrete and continuous quantity, "I call that matter intelligible which is considered in the nature of a continuum, and sensible, that which is physical matter….I call matter designated if it is considered together will the determination of its dimensions…In this connection, it must be noted that designated matter is the principle of individuation, from which every intellect abstracts…The intellect of the mathematician…abstracts entirely from sensible matter…Hence, it is clear that abstraction, which is common to all intellects, makes a form universal." Aquinas, *The Disputed Questions on Truth*, 2.6.

Having looked briefly at the Aristotelian background, now consider again Peirce's metaphysical conception of matter and form. Return to the quote from Peirce on Aristotelian matter, "[t]he original potentiality is the Aristotelian matter or indeterminacy from which the universe is formed." [301] First of all Peirce links matter with potentiality, which resembles an Aristotelian and medieval conception of matter. Matter has the potential to be determined by various forms. But Peirce goes further. "This habit is a generalizing tendency, and as such a generalization, and as such a general, and as such a continuum or continuity. It must have its origin in the original continuity which is inherent in potentiality. Continuity, as generality, is inherent in potentiality, which is essentially general." [302] Peirce chains together potentiality and continuity. Thus the above statement about matter could easily have been read, 'the original potentiality which is continuous is the indeterminate, continuous Aristotelian matter.' The quote above indicates a link between the traditional pre-Cartesian notion of matter and Peirce's conception of matter.[303]

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[301] Peirce, "The Logic of Continuity," 6.206.

[302] Peirce, "The Logic of Continuity," 6.204.

[303] Recovering the link between matter and continuity may also benefit the contemporary understanding of complex systems. "Such behavior [of chaotic systems] does require, however, a principle of continuity between initial and final states, a principle that is not formal but corresponds to the Aristotelian notion of matter. Hence matter matters." Pinsent, "Why Matter Matters to Neo-Aristotelian Teleology in Mechanics," 363.

While Peirce does not use the term 'form' very often, and only occasionally discusses formal cause, he often insists upon the reality of classes and real generals.[304] He is a strong defender of, "the scholastic doctrine of realism":

> This [doctrine of scholastic realism] is usually defined as the opinion that there are real objects that are general, among the number being the modes of determination of existent singulars, if, indeed, these be not the only such objects. But the belief in this can hardly escape being accompanied by the acknowledgment that there are, besides, real *vagues,* and especially real possibilities.[305]

Peirce defends himself not merely as a realist in contrast to Kantian or Hegelian idealism, but as a traditional medieval-style scholastic realist. The scholastic doctrine of realism was hardly intelligible without a notion of formal cause.

By some accounts, a real general is the same as formal cause. Peirce invokes many of the traditional ideas around formal cause as that which the mind abstracts from matter in a manner very similar to Thomas Aquinas. For example, Peirce references "first intentions" which correlate to real generals and natural kinds both in nature and in our minds.[306] Peirce

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[304] See one mention of the term 'form' in Peirce, "Physical Causation," 6.66.

[305] Peirce, "Issues of Pragmatism," 5.453.

[306] First intentions are concepts grasped from sensible experience. Second intentions are gained by abstraction or reflection. Signs can be first or second intentions, but symbols are always second intentions. Peirce finds these ideas important for philosophical logic as seen from the Peirce-Schröder correspondence. "We have to distinguish in the universe of substantive logical possibility the hecceities, the first intentions, and the second intentions. Let indices of the Italic alphabet denote hecceities, those of the Greek alphabet first intentions, and those that are accents and other such not alphabetic characters second intentions." Nathan Houser, "The Schröder-Peirce Correspondence," *Modern Logic* 1, nos. 2–3 (January 1991): 222. Nonetheless,

understands matter in terms of potentiality and hence continuity, and Peirce understands "real objects that are general" as having participated in the "discontinuous" act of "existence" or as "modes of determination for existent singulars". This suggests a parallel as Aristotle pairs matter and form with continuity and discreteness, and Peirce pairs matter and real generals with continuity and discreteness.

The persistent metaphysical concepts explored in this section such as identity-over-time (including substance and person) as well as the expansive notion of causality (including efficient, final, material and formal cause) have discrete and continuous aspects. Given this background now consider how Peirce incorporates aspects of irreducibly discrete and irreducibly continuous quantity in the formal systems he helps develop.

The Peircean Notion of Discreteness and Continuity in Triadic Logic

Do the above observations make any difference in the way formal systems signify the world? Must one consider these metaphysical issues when constructing a system of formal logic? The way one constructs a formal system of logic touches upon notions of one

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the distinction between first intentions and second intentions is hard to understand. Consider this further explanation from another scholar. "The mind turns and reflects on its own knowing when it is directed to the idea of number as a multitude of units, and, in so doing, it no longer apprehends the multitude of units directly, in the "performed act" (*actus exercitus*) and thus as object of its first intention, but rather indirectly, in the "signified act" (*actus signatus*), as object of its second intention. Thus, notwithstanding the fact that what is being conceived by the intellect is a multitude of units, the intellect's immediate apprehension of its own conceiving as something, as one and therefore as a being, has the effect of transforming the multitude belonging to the number into a seemingly independent being, albeit a being that is only a "rational being" (*ens rationis*)." Burt C. Hopkins, "Husserl and Jacob Klein," *The European Legacy, toward New Paradigms* 21, nos. 5–6 (2016): 548, 5–6, https://doi.org/10.1080/10848770.2016.1173480.

and many, of unity and multiplicity, of individuals and classes. Topics discussed above such as classes (or kinds) and relations (like identity) are topics which formal systems attempt to signify. The next three sections each consider a different manner by which discreteness and continuity are signified in formal systems. First, the role of discreteness and continuity in Peirce's triadic logic is explored. Second, the Peirce-Schröder tradition of formal logic which uses classes to create relative individuals is examined. Last, how recent attempts to signify wholeness and substance in mereology require discreteness and continuity are examined in an echo of Peircean themes. These three examples consider whether irreducibly discrete quantity and irreducibly continuous quantity are both needed in formal systems to signify persistent metaphysical concepts.

First, consider triadic logic which is a sort of logic that has three truth values. Peirce's triadic logic used the notation of V (*veritas* for true), F (false), and L (indeterminate). Peirce's development of triadic logic predates the otherwise first discovery of it by Jan Łukasiewicz by ten years.[307] Is there a link between Peirce's triadic logic, discreteness, and continuity?

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[307] While Peirce had similar insights to Łukasiewicz, Peirce's papers on triadic logic were not published in Peirce's lifetime. Nonetheless, Peirce is the first documented thinker in the Western tradition to develop a rigorous foundation for what are now known as many-valued logics. Many-valued logic and modal logic have also been extended by others into the controversial but highly-discussed paraconsistent logic. See Graham Priest, "Many-Valued Modal Logics: A Simple Approach," *The Review of Symbolic Logic* (New York, USA) 1, no. 2 (2008): 190–203, https://doi.org/10.1017/S1755020308080179. Also see A. S. Karpenko, "Paraconsistent Structure inside of Many-Valued Logic," *Synthese* 66, no. 1 (1986): 63–69. Also see F. G. Asenjo, "A Calculus of Antinomies," *Notre Dame Journal of Formal Logic* (Notre Dame, Ind) 7, no. 1 (1966): 103–5, https://doi.org/10.1305/ndjfl/1093958482.

Peirce's motivation for developing a triadic logic may have roots in topics explored in the previous sections. Peirce says, "there are points on the boundary line, and those points are insusceptible of being unblackened or of being blackened..." [308] If boundary points share the properties of both of the adjacent continua, then their status regarding a host of otherwise binary questions will be indeterminate. For the inkblot most points on the page will fall determinately in one area or the other, and hence they will be black or white. But the values on the boundary line will be indeterminate. Thus, the statement, "the point on the boundary line is black" cannot receive a truth value of "true" or "false" it must receive a truth-value of "indeterminate." [309]

An excellent account of the development and significance of Peirce's triadic logic is given by Max Fisch and Atwell Turquette. Not only did Peirce develop truth tables thirty years before Wittgenstein, but they reveal that he also applied them to triadic logic as did Emil Leon Post ten years after Peirce. "Peirce not only originated a matrix method [truth-table method] for two-valued logic, but he also originated a matrix method for three-valued logic." [310] For students of logic a truth table with two-values, true and false, is met in an introductory class. But since three-valued logic is not as commonly seen, consider the idea

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[308] Peirce as cited by Lane, "Peirce's Triadic Logic Revisited," 294.

[309] Paraconsistent logic instead makes the truth value "both true and false" instead of "indeterminate" which seems to have substantial and problematic metaphysical implications. Asenjo, "A Calculus of Antinomies."

[310] Max Fisch and Atwell Turquette, "Peirce's Triadic Logic," *Transactions of the Charles S. Peirce Society* (Buffalo, N. Y) 2, no. 2 (1966): 72.

reproduced below for some common operators where T is true, F is false, and L is indeterminate.

Negation

¬	
T	F
L	L
F	T

And

∧	T	L	F
T	T	L	F
L	L	L	F
F	F	F	F

Or

∨	T	L	F
T	T	T	T
L	T	L	L
F	T	L	F

For the "or" operator Peirce used the Greek letter theta; for the "and" operator he used the Greek letter zeta; and for negation he used an apostrophe over the variable. From just these operators and these matrices he had the fundamentals of a complete triadic logic such as that developed ten years later by Jan Łukasiewicz and Emil Leon Post.

While his accomplishment in conceiving this idea is alone remarkable, the philosophical links between triadic logic, potentiality, and continuity are the present interest. While many-valued logics have garnered attention over the past century, their philosophical significance is not always clear. Peirce's comments offer a unique opportunity to consider how in one of their earliest conceptions these logics might complement traditional logic.

What are Peirce's opinions regarding the role and purpose of triadic logic? First, Peirce thinks that triadic logic is somehow more universal than dyadic logic. In the manuscript where he develops it, he writes, "Triadic Logic is universally true—But Dyadic

Logic is not absolutely it is only L."[311] Here L presumably means indeterminate, or potentially true or false. In other words, there are some things that have not yet been determined, they have the potential to be determined later and at that time will be subject to dyadic logic. But at present they are not determined. Peirce's assertion that dyadic logic is not universally true anticipates the way that later logicians such as both Gödel and Wittgenstein saw significant limitations in formal systems. It also resembles reservations expressed by Aristotle about the limits of logic.[312]

Second, the recovery of potentiality in logic is a great insight. Peirce thought of his system of triadic logic in terms of the being of its objects, where potentiality is a different mode of being than actuality:

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[311] Peirce, Charles S. From "Peirce's Logic Notebook (Ms 339 Plate 3) as cited by Fisch and Turquette, "Peirce's Triadic Logic," 75.

[312] Aristotle too had thought that ordinary logic (effectively two-valued logic) did not apply to some things, namely future contingents. For events that had already passed, what had happened is determinately fixed and what can be said about them is subject to the categories true and false. But Aristotle held that things that are still in the future are indeterminate and therefore it cannot yet be said whether they are true or false.

Aristotle considers whether one can say a thing will be white or not in the future:

> Nor, however, can we say that neither is true – that it neither will be [white] nor will not be so. For, firstly, though the affirmation is false the negation is not true, and though the negation is false the affirmation, on this view, is not true. Moreover, if it is true to say that something is white and large, both have to hold of it, and if true that they will hold tomorrow, they will have to hold tomorrow; and if it neither will be nor will not be the case tomorrow, then there is no 'as chance has it'. Take a sea-battle: it would have neither to happen nor not to happen.
>
> These and others like them are the absurdities that follow if it is necessary for every affirmation and negation either about universals spoken of universally or about particulars, that one of the opposites be true and the other false, and that nothing of what happens is as chance has it, but everything is and happens of necessity.

Aristotle, "De Interpretatione," in *The Complete Works of Aristotle: The Revised Oxford Translation*, vol. 1, trans. J. L. Ackrill, with Jonathan Barnes, Bollingen Series 71 (Princeton ; Guildford: Princeton University Press, 1984), 18b26-19a8. See Book IX.

> That every proposition, S is P, is either true, or false, or else S has a lower mode of being such that it can neither be determinately P, nor determinately not-P, but is at the limit between P and not P.[313]

A statement S is P may not be true or false because S is in a lower mode of being than actuality. If S is only potentially P then it is in a lower mode of being and true or false cannot yet be said of the statement 'S is P'. If Peirce indeed developed his triadic logic in response to the notion of potentiality as is argued by Fisch and Turquette, then he would not be the only thinker to do so. Quoting Alonzo Church, they note that Jan Łukasiewicz followed the same path, "Łukasiewicz first introduced a three-valued propositional calculus in 1920. He was led to this by ideas about modality, according to which a third truth-value—possibility, or better, contingency—has to be considered in addition to truth and falsehood". [314] Thus, modality may be a key motivator in the development of three-valued logic.

Modal reasoning in logic formalizes the concept of potentiality, a key topic in Peirce's metaphysics as seen above. A commentator believes that both potentiality and continuity as generality are keys to understanding why Peirce pursues triadic logic, a logic without the principle of the excluded middle:

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[313] Peirce as reproduced on MS 339 Plate 3 by Fisch and Turquette, "Peirce's Triadic Logic," 75.

[314] Alonzo Church as cited by Fisch and Turquette, "Peirce's Triadic Logic," 72.

> Since the definitions of otherness and of identity proper presuppose a universe of individuals, Peirce considers that in the original continuum, "…the principle of excluded middle, or that of contradiction, ought to be regarded as violated". [315]

In this quote Peirce seems to think the real metaphysical continuum violates the law of non-contradiction and the principle of the excluded middle. How would one then signify this with a formal system? For Peirce logic and metaphysics are linked and continuity is a necessary topic in both of them. A metaphysical violation of "the principle of the excluded middle" is of interest to logic. Peirce does not believe that all aspects of reality are actualized. Recall Peirce's statement, "the continuum is all that is possible, in whatever dimension it be Continuous". [316] Peirce insists that real possibilities exist. The potentiality inherent in the world requires a way to be signified.

Because Peirce believes signs are inherently connected to reality so also logic is connected to reality.[317] Thus, Peirce prefers to refer to logic as semiotics.[318] And if reality has continuous aspects, then logic as semiotics must also have elements of continuity because it is connected to reality.

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[315] Havenel, "Peirce's Clarifications of Continuity," 115–16. *NEM* is C.S Peirce, 1976, The New Elements of Mathematics. Peirce, *The New Elements of Mathematics*, 3.747.

[316] Peirce, *Philosophy of Mathematics*, 174.

[317] "In the next place a sign must have some real connection with the thing it signifies so that when the object is present or is so as the sign signifies it to be, the sign shall so signify it and otherwise not. What I mean will best be understood by illustration. A weathercock is a sign of the direction of the wind. It would not be so unless the wind made it turn round. There is to be such a physical connection between every sign and its object." Peirce, *The Writings of Charles S. Peirce, a Chronological Edition [Electronic Resource]*., 3.66-68.

[318] See Francesco Bellucci, "'Logic, Considered as Semeiotic': On Peirce's Philosophy of Logic," *Transactions of the Charles S. Peirce Society* 50, no. 4 (2014): 523–47, https://doi.org/10.2979/trancharpeirsoc.50.4.523.

For Peirce, these possibilities of points on the continuum are real multiplicity just as the line is a real unity. And hence a logic capable of including the middle, not excluding the middle, is crucial for its ability to signify the world. The multiple potential points and the unity of the line can coexist though in different ways.

Does the potentiality in Peirce's triadic logic link with continuity? That Peirce himself connects potentiality to continuity is clear. As noted above, the "original potentiality is essentially continuous, or general". [319] "The whole universe of true and real possibilities forms a continuum…" [320] And because "in the original continuum, '…the principle of excluded middle, or that of contradiction, ought to be regarded as violated'" another version of logic which does not limit questions only to binary opposition of true and false, firstness and secondness, must apply.[321] That Peirce develops a logic that can identify the contingency or the potentiality of some statements makes triadic logic substantially more universal. And this emphasis on potentiality, a notion related to the concept of causality and the Eleatic principle, is additionally related to the metaphysical concept of causality.

Peirce's approach to triadic logic resembles a statement from Iain McGilchrist who has drawn inspiration from the Peircean legacy of pragmatism, "The principle for division

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[319] Peirce, "The Logic of Continuity," 6.205.

[320] Peirce, *The New Elements of Mathematics*, 4.345.

[321] Peirce, *The New Elements of Mathematics*, 3.747. with Havenel, "Peirce's Clarifications of Continuity," 115–16.

and the principle for union need to be brought together, not divided. We need not *either* both/and *or* either/or, but *both* both/and *and* either/or. We need not non-duality only, but the non-duality of duality and non-duality." [322] The duality of two-valued logic is good and often sufficient, but as Peirce says and McGilchrist seems to indicate, a three valued logic which also allows for non-duality is more universal. A principle of union brings things together as one, a principle for division allows for multiplicity. Peirce's triadic logic creates a logic that can signify both unity and multiplicity in consonance with Peirce's philosophy of thirdness.

In summary, for Peirce reality is composed of both discrete quantities and continuous quantities. Dyadic logic can only signify discrete quantities, firsts and seconds, and therefore is not universally able to signify of the world. Triadic logic can signify both irreducibly discrete quantities and irreducibly continuous quantities with indeterminate boundary points and indeterminate unactualized potentials and therefore is more universal.

The Relative Individual in Peirce-Schröder Logic

While seeing the role of continuity and discreteness in triadic logic is instructive, such logic is not currently a core topic of contemporary formal systems. In order to gain deeper insight into the role of discrete and continuous quantity in formal systems consider two major traditions in formal systems: the Frege-Russell tradition with links to the

[322] McGilchrist, *The Matter with Things*, 833.

predicate logic and the Peirce-Schröder tradition with links to both the predicate logic and mereology.[323] The Peirce-Schröder formal system contains the notion of a relative individual, which from one perspective is discrete as in set theory and from another perspective is continuous as in mereology. Does the relative individual involve both irreducibly discrete and irreducibly continuous quantity?

C.S. Peirce had a great influence on the logician Ernst Schröder, and the two of them influenced many European and American mathematical logicians.[324] While Russell was aware of the mathematical logic of Peirce and Schröder, he preferred Gottlob Frege's hitherto unpopular approach to theirs and built upon Frege's logic.[325]

Frege-Russell logic and Peirce-Schröder logic have two different notions of classes or sets. Russell, following Giuseppe Peano, emphasizes the centrality of set membership. For Russell the discrete members of the set are the basic objects of set theory. In addition to members and sets there can also be subsets of sets. But Peirce does not use a membership operator (e.g., $x \in Y$), instead Peirce uses an operation he calls illation which is more

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[323] One commentator goes so far as to say, "As a matter of fact Schröder belongs to the German–Austrian (and Polish) tradition of Mereology, according to which the totality is more important than its elements." Davide Bondoni, "Structural Features in Ernst Schröder's Work. I," *Logic and Logical Philosophy* 20 (January 2011): 341, https://doi.org/10.12775/LLP.2012.014.

[324] Putnam, "Peirce the Logician," 297–98. See also footnote 55 in chapter one.

[325] Russell comments on Peirce and Schröder's work, "The calculus of relations is a more modern subject than the calculus of classes. Although a few hints for it are to be found in De Morgan, the subject was first developed by C. S. Peirce…Peirce and Schröder have realized the great importance of the subject, but unfortunately their methods, being based, not on Peano, but on the older Symbolic Logic derived (with modifications) from Boole, are so cumbrous and difficult that most of the applications which ought to be made are practically not feasible." Bertrand Russell, *The Principles of Mathematics* (Cambridge University Press, 1903), 23–24.

general than membership. Peirce's illation operation covered the equivalent of class inclusion, implication, and set membership. Distinguishing which operation was strictly intended was clear by the context of the operator.[326] Important to recognize is that treating class inclusion on the same level as membership creates a parallel between the mereological approach and the set-theoretic approach as shown in the work of David Lewis.[327]

Schröder, like Peirce who was his acknowledged inspiration, treats parthood as in mereology at the same level of basicness as the membership of set theory.[328] This makes sense given the ease with which mereology can formalize traditional continuity and C.S. Peirce's insistence on the importance of traditional continuity. Nonetheless, this treatment

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[326] "What Russell and Peano failed to appreciate was that Peirce intended his illation to serve as a generalized, nontransitive, copula, whose interpretation, as class inclusion, implication, or set elementhood, was determined strictly by the context in which it applied. Reciprocally, Peirce criticized Russell for failure on Russell's part to distinguish material implication and truth-functional implication (conditionality) and for his erroneous attempt to treat classes, in function-theoretic terms, as individual entities." Irving H. Anellis, "How Peircean Was the "Fregean' Revolution" in Logic?," *Logicheskie Issledovaniya* 18 (2012): 18–19, https://cspeirce.iupui.edu/menu/library/aboutcsp/ANELLIS/csp-frege-revolu.pdf.

[327] Consider the parallel David Lewis suggests between mereology and set theory. A set theoretic form of mereology can be created by treating class inclusion as the counterpart to parthood in mereology. David K. Lewis, *Parts of Classes* (Oxford: Basil Blackwell, 1991).

[328] Bondoni, "Structural Features in Ernst Schröder's Work. I," 341.

of class inclusion and membership as equally basic was the original sin of the Peirce-Schröder approach according to Frege and Russell.[329]

Both the Frege-Russell approach and the Peirce-Schröder approach agree that what belongs to a collection (or set) are individuals. But the Frege-Russell approach holds that those individuals or set elements are absolutely indivisible and discrete. The Peirce-Schröder approach holds that those individuals are only indivisible or discrete as seen from one vantage point. As seen from another vantage point they are potentially made of parts. The Peirce-Schröder approach holds that the individuals are both discrete in one way and continuous in another, both one and many, though not in the same aspect.

These differences played out in the way these mathematicians thought about sets or in Schröder's terms, classes. For Schröder a class (or collection or set) may also be considered as an individual. "But also any class that itself contains many individuals can again be represented as an object of thought and accordingly as an individual (in a broader

[329] "The fact that Peirce applied one logical connective, which he called illation, to serve as a copula holding between terms, classes, and relata, was a basis for one of the severest criticisms levelled against his logic by Bertrand Russell and others, who argued that a signal weakness of Peirce's logic was that he failed to distinguish between implication and class inclusion (see [Russell 1901c; 1903, 187]), referring presumably to Peirce's [1870], while both Russell and Peano criticized Peirce's lack of distinction between class inclusion and set membership. Indeed, prior to 1885, Peirce made no distinction between sets and classes, so that Russell's criticism that Peirce failed to distinguish between class inclusion and set membership is irrelevant in any event." Anellis, "How Peircean Was the "Fregean' Revolution" in Logic?," 18–19.

sense, e.g. as a 'relative' individual in regard to higher classes)" [330] Frege comments on Schröder, "Here there is indeed talk of individuals; but we also see straightaway that the distinction between individual and class is treated as a fluctuating one. It is apparent that from several passages that a class is to be treated as a totality of objects, a collective unity."[331] Frege also notes, "in Schröder's use of language 'system' stands for a collective unity." [332] Peirce scholar Randall Dipert explains the notion of relative individual in Peirce and Schröder:

> [While] Schröder remarks that divisions of the extension always lead back to…individuals, described as things which are not further divisible... Peirce, Peirce's student O.H. Mitchell, and Schröder considered entities to have individuality or indivisibility only from one perspective; from another perspective…what had been considered individual can be considered plural.[333]

According to Peirce, O.H. Mitchell, and Schröder the concept of individual, unlike the concept of element, is only a relative one. "[E]ntities…have individuality or indivisibility…only from one perspective."

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[330] "Auch jene Klasse aber, die selber eine Menge von Individuen umfasst, kann wieder als ein Gedankending und demgemäss auch als ein „Individuum" (im weiteren Sinne, z. B. „relativ" in Bezug auf höhere Klassen) hingestellt werden." Ernst Schröder, *Vorlesungen über die Algebra der Logik (Exakte Logik)*, with Jakob Lüroth and Karl Eugen Müller (Leipzig: Druck und Verlag von B.G. Teubner, 1890), 148. Translation given by Gottlob Frege, "A Critical Elucidation of Some Points in E. Schroeder's Vorlesungen Ueber Die Algebra Der Logik," in *Translations from the Philosophical Writings of Gottlob Frege*, 3rd ed., ed. P. T. Geach and Max Black (Oxford: Basil Blackwell, 1980), 87.

[331] Frege, "A Critical Elucidation of Some Points in E. Schroeder's Vorlesungen Ueber Die Algebra Der Logik," 88.

[332] Frege, "A Critical Elucidation of Some Points in E. Schroeder's Vorlesungen Ueber Die Algebra Der Logik," 88.

[333] Randall Dipert, "Individuals and Extensional Logic in Schröder's Vorlesungen Über Die Algebra Der Logik," *The Review of Modern Logic* 1 (January 1991): 150–51.

While Frege considered this relativeness of the individual to be a fault of the Peirce-Schröder approach, it also has an interesting advantage. "Both Peirce and Schröder used this technique [of the relative individual] to accomplish in their 'single-rank' class theory what is done in set theory with multiple-type or multiple-rank sets of sets, and Schröder developed a formal account of this theory in his domain calculus." [334] The use of the relative individual by Peirce and Schröder anticipated a move that Russell would be forced to make to fix his predicate logic, the introduction of types. When Russell uncovered the Russell paradox in his *Principia Mathematica*, he introduced a theory of types which he used to fix it.[335] In other words, the precise weakness in Russell's system was that he did not have a notion of the relative individual. If he had, then he could have avoided the Russell paradox.[336] Thus, what Frege criticizes as the problem of the "relative individual"

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[334] Dipert, "Individuals and Extensional Logic in Schröder's Vorlesungen Über Die Algebra Der Logik," 150–51.

[335] "The tendency intensions have to proliferate also means that identity is different for intensions than for extensions, and no adequate criterion for establishing the identity of intensions seems to exist. This foundational difference between objects and what can be predicated of them forced the failure of Basic Law V." Claire Ortiz Hill, *Word and Object in Husserl, Frege and Russell: Roots of Twentieth-Century Philosophy*, Series in Continental Thought (Athens, Ohio: Ohio University Press, 1991), 172.

[336] In a section entitled "Possible Solutions to the Paradox of Classes or Sets" one solution explains Russell's response. "The Theory of Types for Classes: It was mentioned earlier that Russell advocated a more comprehensive theory of types than Frege's distinction of levels, one that divided not only properties or concepts into various types, but classes as well. Russell divided classes into classes of individuals, classes of classes of individuals, and so on. Classes were not taken to be individuals, and classes of classes of individuals were not taken to be classes of individuals. A class is never of the right type to have itself as member. Therefore, there is no such thing as the class of all classes that are not members of themselves, because for any class, the question of whether it is in itself is a violation of type. Once again, here the challenge is to explain the metaphysics of classes or sets in order to explain the philosophical grounds of the type-division." Thus, the problem in Russell's approach is what seems like an arbitrary *ad hoc* fix. Why kind of metaphysics of sets justifies this division? Kevin C. Clement, "Russell's Paradox," in *Internet Encyclopedia of Philosophy*, accessed November 15, 2024, https://iep.utm.edu/par-russ/.

in Schröder presages a parallel move that the Frege-Russell logic will ultimately have to make in order to avoid inconsistency. Peirce and Schröder's system, by making use of the relative individual, avoided the problem in the first place.

The way that the Peirce-Schröder logic could implement a relative individual was especially through the notion of concept-words. Concept-words, like "forest" or "men" in Schröder's mathematical logic are a natural correlate to universals, similar to the notion of form, and a key topic in the scholastic realism which Peirce upheld. Chapter one highlighted the role of counting words as "count nouns" and chapter two discussed "*arithmos*" for traditional mathematics before the 17th century. Ancient and medieval mathematicians held that one, not zero, was the principle (*arche*) of number because a counting unit or measure was needed in order to have numbers.

An 'apple' is a necessary concept in order to identify that there are ten apples on the table. And these apples could be counted differently if a different count-noun were employed. In the material sense, 'these ten apples on the table' can also be counted as some number of apple seeds, perhaps 'eighty-five apple seeds on the table'. A different count-noun when applied to the same material yields a different number, e.g. fifteen apple stem

leaves. Peirce and Schröder recognize this same issue and Schröder includes the notion of concept-words in his system of mathematical logic.[337]

In a section entitled, "What Makes Counting Possible?" Schröder says,

> It is reasonable to demand a count only when we are presented with objects that seem clearly distinguishable, for example objects separate in space or time or objects demarcated from one another…*We can count things only if we conceive them* (through distinguishing marks) *as separate from one another;* they must be—as they say in logic books—*disjoint things*.
>
> Furthermore, an almost self-evident precondition of counting is that the objects to be counted be given in their entirety and be individually *determined*; that is, one must know *what*, in fact, is to be counted.[338]

Schröder asks a question similar to those thinkers before him. What are the conditions for counting? Both a distinctness ("*disjoint things*") and a similarity ("one must know *what*, in fact, is to be counted") are required in order for the counting numbers to be used. What is it? This question, *ti estin?*, for Aristotle was answered by a thing's form. This accounts for the close link between form (or universals) and count-nouns throughout ancient and medieval mathematics. Schröder writes similarly. "We will, however, feel compelled to count things only if they have something in *common*, some shared feature through which

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[337] Peirce talks about numerical identity regarding the count noun or *arithmos*. "It is therefore absurd to speak of the collection of points upon a line; because a collection consists of units, which a possibility being essentially general in its nature, has no self-identity, or, as we say in the terminology of logic, no numerical identity (ἑν ἀριθμω)." From "The Logic of History" in Charles S. Peirce, *Mathematical Philosophy*, with Carolyn Eisele, The New Elements of Mathematics; Volume 4 (Berlin ; Boston: De Gruyter, 2011), 7.

[338] Ernst Schröder, *Ernst Schröder on Algebra and Logic*, ed. Stephen Pollard, Synthese Library : Studies in Epistemology, Logic, Methodology, and Philosophy of Science, Volume 465 (Cham: Springer, 2022), 3. Italics original here and throughout.

they present or suggest themselves to us as objects of our counting." [339] Schröder elaborates, "they are usually characterized by a shared property." [340]

Schröder has a notion very similar to count-nouns. He calls it the "appellation":

> First of all, we must emphasize that, before we can even discuss the numerousness of a thing, the name of the thing must be a *common noun*, *a word denoting a general concept (notitia communio)*…
>
> That common noun or concept is called the *appellation* (*Benennung)* of the number formed in the way indicated above. It captures the essence of that number's units. For example, if we count the coins in a roll and find there are nine, then the concept "coins in that roll" is the *appellation* of the number nine.
>
> So the appellation of a number is always the result of an abstraction. It is a 'general concept allowing various modes of specification.' (Riemann 1867, p.135) All the things to be counted fall under this concept uniting the properties they alone share (through which they alone are characterized as objects of the counting), excluding as inessential the remaining properties that distinguish between them and mark them as individuals.[341]

It is clear that Schröder's notion of appellation is very similar to the traditional idea associated with count-nouns or the *arithmos*. "[T]he name of the thing must be a *common noun*…*a general concept*…the things to be counted fall under this concept uniting the properties they alone share." There is a distinction between the thing as counted, which becomes an object, and the thing itself. As counted, the thing is an instantiation of certain properties, but as the thing itself it may have many more distinguishing characteristics

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[339] Schröder, *Ernst Schröder on Algebra and Logic*, 4.

[340] Schröder, *Ernst Schröder on Algebra and Logic*, 4.

[341] Schröder, *Ernst Schröder on Algebra and Logic*, 6.

which are disregarded for the purposes of counting. Thus, it is crucial to recognize the mind's role of abstraction in counting.

As seen in concept words, Peirce and Schröder consider "entities to have individuality or indivisibility only from one perspective; from another perspective…what had been considered individual can be considered plural." And the Peirce-Schröder formal system accommodates this metaphysical position. The framework for the formal systems of Peirce and Schröder is better able to signify wholes than the framework for the formal systems of Frege and Russell. In Russell's type theory, while the classes can take quantifiers and therefore have properties, they are not the same sort of thing as the elements of a class. The elements for Frege and Russell are still more fundamental to set theory than a class. Thus, for a set theorist to claim that a class can be a whole because it can take properties is not the same as to say that a class can be an element. And because without elements classes are oddly empty in the Frege-Russell system, there is an asymmetric relationship between the elements as wholes and the classes as wholes. In the Frege-Russell system there is an atomic hypothesis injected into each discourse, which implies that the classes as wholes are not true wholes in the same way elements are.

The Peirce-Schröder framework, on the other hand, does not have the same inherent bias. While a property of some individual might be that it is absolutely indivisible, the formal system does not demand an indivisible foundation in the system. Because the Peirce-Schröder system uses relative individuals, all the levels of individuation are equal. Consider using philosophical logic to represent tissues in a mammalian body. The logic would not need to posit atoms in order to begin the signification of molecules. The molecules can be the individuals needed to discuss tissues. But later if one wished to

discuss the composition of molecules, the Peirce-Schröder system would provide a manner to do so. The Frege-Russell system does not allow this relative individuation because it holds that in each system one must begin either with some indivisible elements or the empty set. One can build up from this atomic foundation, but the individuals in set theory are absolutely indivisible.

Because the Peirce-Schröder system can unprejudicially signify how a whole can be made up of parts from top to bottom or how parts can compose a whole from bottom to top, it can signify unity in multiplicity. Through the illation operation Peirce's system treats the discreteness of set membership as equally basic as the continuity of the parthood or class inclusion relation. Through the concept of the relative individual it shows that individuals may have potential parts that are both continuous in one sense and potentially discrete in another sense. In addition, the Peirce-Schröder system has a complex understanding of how to signify the unity in multiplicity of reality through its formal system especially through the use of concept-words and classes which allow for discrete counting. The Peirce-Schröder approach thus treats discreteness and continuity as equally basic allowing their formal system to signify the world through the metaphysical concept of identity-over-time which is crucial for notions of wholeness, individuality, and potential parthood.

Implications of C.S. Peirce's Approach for Mereology and the Signification of Substance

As mentioned above C.S. Peirce and Ernst Schröder can be thought of as forerunners of the mereological tradition, a tradition which as examined in chapter two models its parthood operator on irreducible continuity. Though their work is often

compared to predicate logic and set theory, their conception of the relative individual has mereological aspects especially when seen in light of David Lewis' proposal in *Parts of Classes*. Many contemporary metaphysicians work in the formal system of mereology and some have revived themes that resonate with Peircean conceptions of topics like identity-over-time. Peirce thought that identity-over-time and by extension substance would have both discrete aspects and continuous aspects. This differs from the set theoretic approach to signifying identity which has only discrete aspects. But on account of the Peirce-Schröder use of discrete appellations it also differs from the mereological approach which emphasizes continuity.

In the Humean approach to identity formalized by Bertrand Russell, Socrates has the same metaphysical status as sets. "[I]f you are going to hold Piccadilly [or Socrates] as real, you must hold that series of classes are real, and whatever sort of metaphysical status you assign to them, you must assign to it. As you know, I believe that series and classes are of the nature of logical fictions…" [342] For Russell, Socrates and Piccadilly station are merely a "series of classes of material entities…which, at varying times, occupy that portion of the earth's surface". The series of discrete classes are not connected or continuous through time. This manner of signifying the world as absolutely discrete quantity comports with Russell's vision of reality, "I think the universe is all spots and jumps, without unity, without continuity". Though on Russell's view admitting to

[342] Russell, *The Philosophy of Logical Atomism*, 9.

'signifying the world' may already go to far. "Indeed, there is little but prejudice and habit to be said for the view that there is a world at all." [343]

Peirce disagrees with the above notions about discrete quantity and identity. "To say that who commanded the French in the battle of Leipzig commanded them in the final battle of Waterloo, is not merely a statement of identity: it is a statement of Becoming. There is an existential continuity in time between the two events." [344] The "existential continuity" between younger Napoleon and older Napoleon rejects the idea that identity is just a sequence of discrete classes.

The Peircean conception of identity is closer to the typical four-dimensionalist presentation of perdurance. David Lewis says, "Something *perdures* iff it exists by having different temporal parts, or stages, at different times, though no one part of it is wholly present at more than one time…" [345] Regarding these temporal parts, their relation to identity-over-time is more unified and continuous. According to Ted Sider, "a four dimensionalist will say that my current temporal part is, atemporally, sitting, 69 inches tall, and wearing a hat; and a four dimensionalist will say that this temporal part is, atemporally, part of the larger space-time worm that *is* me." [346] The various parts can be conceived as part of a continuous unity which is the identity of the person. This more closely resembles

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[343] Russell, *The Scientific Outlook*, 98.

[344] Peirce, *The New Elements of Mathematics*, 4.330.

[345] David K. Lewis, *On the Plurality of Worlds* (Oxford: Basil Blackwell, 1986), 202.

[346] Sider, "Four-Dimensionalism," 199.

the Peircean claim that there is an "existential continuity" between the younger and older Napoleon.

But even though David Lewis can depict the "existential continuity" by using mereology in ways a set theorist cannot, a serious problem for signifying identity in mereology remains. In short, mereology has a principle of continuity for identity, but does it have an adequate principle of discreteness? The problem becomes apparent when David Lewis suggests a new category of "many but almost one" wholes. Consider a question concerning the identity of a cat named Tibbles.[347] There are an indefinitely large number of different ways to take one molecule out of Tibbles the cat, but is the cat-minus-one-molecule the same cat? These differing-by-one-molecule versions of the cat would be perceptually identical to the naked eye but according to classical extensional mereology, in which wholes or 'mereological sums' are nothing but the sum of their parts, they would be distinct mereological wholes. Lewis refrains from saying that there is something essential about Tibbles that does not change, something discrete, versus other aspects of Tibbles which are accidental and can change like the cat's fur length.

Kathrin Koslicki criticizes Lewis and suggests that it reveals a weakness in mereological representations of identity. She argues that simply enumerating the parts to create a mereological sum is not enough to designate a whole. Something like corporate

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[347] The original example for Lewis is inspired by Peter Geach. David K. Lewis, "Many, but Almost One," in *Ontology, Causality and Mind: Essays in Honor of D.M. Armstrong*, ed. John Bacon et al. (Cambridge: University Press, 1993), 23–38.

causality or causal continuity additionally must be exhibited in order to display unity. She describes what is needed as "team-work-requiring capacities" (TWR-capacities). Koslicki says,

> an integrated whole derives its unity from the way in which its parts are able to interact with other parts of the same whole; the interplay between these activities carried out by the parts of an integral whole in turn allows the whole as well to manifest certain of its capacities, namely, those whose manifestation by the whole requires "team work" among its parts.[348]

In Koslicki's argument that a whole exhibits "team-work-requiring capacities" one witnesses the centrality of causality in describing unity, the Eleatic principle. The capacity to act or be acted upon since the time of Plato and Aristotle has often been recognized as a marker of being or unity as noted earlier.

This working together to activate some causal ability is similar to how Peirce thinks of classes. As a commentator on Peirce's doctrine notes, "*Each class is characterized by a distinctive final cause, and the objects belonging to a natural class do so by virtue of their ability to exert a type of real causal influence.* Thus, it is the task of science to determine exactly what natural classes there are." [349] The TWR-capacities Koslicki describes allow the various parts of an object to work together for a distinct, discrete end. This capacity for united continuous action draws them together into a continuous unity in a manner similar to formal cause. This continuous action resembles the continuous motion Aristotle references,

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[348] Kathrin Koslicki, *Form, Matter, Substance*, First edition., Oxford Scholarship Online (Oxford: University Press, 2018), 210.

[349] Hulswit, "Peirce's Teleological Approach to Natural Classes," 764.

"a [bundle of sticks] is made continuous by its string, [because] 'continuous' means that whose motion is essentially one". [350]

Koslicki's capacities and Peirce's interest in final cause emphasize the role of causality in determining substances. But contemporary set theory and predicate logic tend to either ignore causality or to reduce causality to some sort of implication relation or function.[351] But reducing causality to functions biases set theory towards a Humean notion of causality which is really no causality at all. Russell seems to acknowledge as much when he says, "[W]hen I say 'causal' I do not mean what the word naturally conveys… There is merely an observed law of succession from next to next. An event at one moment is succeeded by an event at a neighbouring moment". [352]

The Humean aspect of function language is exacerbated by the metaphors used to explain the concept. The concept of a 'function' is often referred to as a black box 'function machine' into which one inserts an input and consequently an output emerges. For example, when defining a function Ted Sider says, "A function 'takes in' an object or objects (in a certain order), and 'spits out' a further object." [353] What happens 'inside' the function is unknown. The function machine "takes in" one thing and "spits out" another

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[350] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1015b. See Book V:6.

[351] "the reason why the old 'law of causality' has so long continued to pervade the books of philosophers is simply that the idea of a function is unfamiliar to most of them." Bertrand Russell, "On the Notion of Cause," *Proceedings of the Aristotelian Society* 13 (1912): 14.

[352] Russell, *An Outline of Philosophy*, 123–24.

[353] Theodore Sider, *Logic for Philosophy* (Oxford, England: Oxford University Press, 2010), 15.

thing with no intelligible connection between the input and output. When relations, including the relation of causality, are reduced to functions, then sequencing is preserved but causal connection is lost.[354] If true causality is lost, then Koslicki's capacities and Peirce's final cause are no longer relevant for the determination of substances. Being is reduced to unintelligible mechanistic relations.

Koslicki does not argue that all things exhibit TWR-capacities. But things that do exhibit TWR-capacities, mirroring the Eleatic principle, are likely candidates for being a substance. As a part of her solution Koslicki argues for a principle of "restricted composition" which helps her conceptualize a notion of substance: [355]

> According to this approach, all entities which qualify as wholes at all would be classified as both structured and unified, and as unified to the same extent, since in

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[354] Bergmann describes the hidden nominalism in the notion of function. It opposes true causal explanations based in the character of things. "Still another circumstance subtly undermines the ontological status of functions. What mathematicians say about the latter often has a subjective ring. As they speak, it is they who do the mapping or, as it is often put, establish the correlation. 'Rule' itself, in most of its uses, has the same ring or tinge… The realist pulls the fangs of this potentially dangerous talk by construing functional expressions as, in the Principia sense, indefinite descriptions, either mathematical or otherwise (the square of x, the father of x.) Descriptions in turn contain expressions standing for characters. And there is nothing subjective about characters (including, of course, relations), least of all in the case of numbers. As Russell put it, in a justly celebrated passage about the order relation, "we can no more 'arrange' the natural numbers than we can the starry heavens." Notice, though, that even if starting from numbers in one sense, one who accepts this clarification starts in another sense from the realist's paradigm… while within his system at least Frege succeeded in securing full ontological status for his odd objects, he did not so succeed, even within the system, in the case of functions. That is what I mean by his hidden nominalism. Notice that I call it hidden or implicit." Gustav Bergmann, "Frege's Hidden Nominalism," *The Philosophical Review* 67, no. 4 (1958): 445, https://doi.org/10.2307/2182943.

[355] "(RCP) Restricted Composition Principle: Some objects, m_1 , …, m_n , compose an object, O, of kind, K, at a time t just in case m_1 , …, m_n satisfy at t the constraints dictated by some formal components […], f_1 , …, f_n , associated with objects of kind, K. I construe the constraints that are dictated by an object's formal components as primarily structural…Structural constraints: Requirements concerning (1) the type or variety of material components; (2) their configuration, i.e., manner of arrangement; and, in some cases, (3) the number of material components of which objects of kind, K, may be composed. " Koslicki summarizes here an argument she first made in *The Structure of Objects* in 2008. Koslicki, *Form, Matter, Substance*, 192.

> order for something to be a whole at all (i.e., in order for mereological composition to take place successfully) the entity in question must be structured in the sense specified by (RCP) [Restricted Composition Principle] in conjunction with (1) and (2). I will refer to this proposal as (WSU), short for 'Wholeness=Structure=Unity.' As it stands, then, (WSU) apparently does not make room for the possibility of wholes which are less unified than matter-form compounds.[356]

Koslicki underscores that there is a difference between a whole and a mere multitude. A whole must be "structured and unified". While classical extensional mereology can do a good job of designating the extension of a thing, i.e. its many molecules, it can not do an equally good job on its own of designating identity. It needs a discrete principle like formal cause in order to bring structure and unity to the multiplicity.

In the traditional metaphysics of Aristotle examined earlier, matter is continuous but form makes the continuous thing discrete. Koslicki's argument reiterates this Aristotelian pattern. Lewis can designate the matter of a cat, but without some additional properties like TWR-capacities he cannot designate a cat as having persisting identity-over-time. This parallels the matter-form concept of Aristotle as Koslicki points out. "[The] Wholeness=Structure=Unity [proposal]… apparently does not make room for the possibility of wholes which are less unified than matter-form compounds."

Mereology can do a good job at designating matter which has often been understood as continuous, but Koslicki identifies a way to inject discreteness into mereology in order to obtain persistent identity-over-time. She does so through a Restricted Composition

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[356] Koslicki, *Form, Matter, Substance*, 193.

Principle which depends upon continuous causality to establish a distinct formal principle. Without the distinct formal principle of TWR-capacities that Koslicki invokes, Tibbles-the-cat-minus-one-molecule is a completely different whole from Tibbles-the-cat that retains the molecule.

This use of a distinct formal principle alongside an otherwise four-dimensionalist continuous material-spatial-temporal identity shows that both discrete quantity and continuous quantity are needed in order for formal systems to signify the persistent metaphysical concept of identity-over-time. The use of a discrete principle and a continuous principle allows mereology to show how from one perspective Tibbles is an individual with identity, but from another perspective, Tibbles has potential mereological parts.

Koslicki's Restricted Composition Principle in effect revives the notion of a relative individual and concept-words used in the formal system of Peirce and Schröder. In this case, the distinctness of a form signified by the concept word brings together all of the multiple mereological wholes in Lewis' 'many but almost one' doctrine. Thus by using TWR-capacities to appropriate the discreteness of form and by using the many mereological sums of Tibbles continuous parts over time, identity-over-time can be signified.

In summary, irreducibly discrete and irreducibly continuous quantity are both necessary for formal systems to signify the world through persistent metaphysical concepts. Recovering continuity as a co-principle with discreteness played a central role for Charles Sanders Peirce in solving philosophical problems such as identity-over-time and causality. He could then signify those solutions in either triadic logic or his formal system using the

relative individual. Koslicki shows that team-work-requiring capacities can help signify the wholeness and discreteness of a substance which along with the multiple mereological sums demonstrates the continuity of the parts. Together these approaches demonstrate how formal systems need both irreducibly discrete and irreducibly continuous quantity in order to signify the world through persistent metaphysical concepts.

To conclude this chapter, philosopher Mary Hesse's analysis of the alternation between "atomicity and continuity" in the history of physical science explains why natural science needs a mathematics with both irreducibly discrete and irreducibly continuous quantity to signify the world. In addition, the mathematician, logician, and philosopher Charles Sanders Peirce's study of persistent metaphysical concepts and how to convey them reveals the same conclusion about any notion of quantity adequate to signify the world. Two distinct lines of argument in this chapter thus support the same conclusion reached by Aristotle, namely that quantity that signifies the world is irreducibly discrete and irreducibly continuous.

Chapter Four: Discrete and Continuous Quantity and the Unity in Multiplicity of the World as a *Vestigium Trinitatis*

This thesis has raised the question: is quantity that signifies the world irreducibly discrete and irreducibly continuous, exhibiting unity in multiplicity and, if so, can such quantity be regarded as a *vestigium Trinitatis*? The previous chapter concluded from two different lines of reasoning, one from Mary Hesse and one from C.S. Peirce that quantity which signifies the world is both irreducibly discrete and irreducibly continuous. This chapter considers whether irreducibly discrete and irreducibly continuous quantity that signifies the world exhibits unity in multiplicity and can therefore be regarded as a *vestigium Trinitatis*.

The last chapter described how quantity that signifies the world in our best scientific theories is irreducibly discrete and irreducibly continuous first by following the work of Mary Hesse. The chapter then described how quantity that signifies the world in formal systems through metaphysical concepts such as identity-over-time, substance, person, and causality is irreducibly discrete and irreducibly continuous with special attention to the thought of C.S. Peirce.

But some questions still remain, particularly those regarding the relation between quantity and the *vestigia Trinitatis*. First, does Augustine's reference to measure, number, and weight suggest that quantity that signifies the world may be a *vestigium Trinitatis*? Second, is unity in multiplicity a mark of many prominent formulations of the *vestigia Trinitatis*, and if so does quantity that signifies the world exhibit this mark? Finally, is irreducibly discrete and irreducibly continuous quantity, exhibiting unity in multiplicity,

important for signifying aspects of the *vestigia Trinitatis* such as substance, persons, and causality?

This last question has important consequences. If a) one wishes to signify the *vestigia Trinitatis* through mathematics or formal systems, and b) this requires both irreducibly discrete and irreducibly continuous quantity, then c) mathematics and formal systems that reduce all quantity to either discrete quantity or continuous quantity will not be able to signify the world let alone the *vestigia Trinitatis*. This would suggest that foundational choices in one's mathematics and formal systems limit the kinds of entities that mathematics and formal systems can signify. Furthermore, this means that Christians who research at the seam between mathematics, logic, natural science, philosophy, and theology should attend carefully to the formal systems of signs in which they discourse. This issue will be explored more fully in the final concluding chapter which follows the replies to objections later in this chapter.

The Unity in Multiplicity of Quantity that Signifies the World and the *Vestigia Trinitatis*

A number of reasons arise for why quantity that signifies the world exhibits unity in multiplicity and could thus be regarded as a *vestigium Trinitatis*. First, though Augustine showcases human persons in his discussion of the *vestigia Trinitatis*, he also references quantity in *De Trinitate* especially as a way through which persons think. He suggests that considering our thoughts on measure, number, and weight allows us to "trace this same

trinity in the inner man". [357] Augustine appears to conceive measure, number, and weight as a trace of the Trinity, even if for him it is more faint than other traces. Augustine holds that measure, number, and weight are connected to vision and the "outer man". The trinity of measure, number, and weight is less manifest and satisfactory than the "more manifest trinity" of the "inner man", but he nonetheless references it as a trinity.[358] Because number and measure are closely related to irreducibly discrete and irreducibly continuous quantity that signifies the world, it makes such quantity a good candidate for being regarded as a *vestigium Trinitatis*. Bonaventure extends the centrality of this Augustinian theme for the *vestigia Trinitatis* highlighting the role measure, number, and weight play in causal metaphysical principles.

A second reason to consider why irreducibly discrete and irreducibly continuous quantity which signifies the world may be regarded as a *vestigium Trinitatis* is because if such quantity exhibits unity in multiplicity that would provide a quantitative corollary to Herman Bavinck's conception of the *vestigia Trinitatis* as unity in diversity. Given that, as I have argued, quantity does have these mutually irreducible characteristics, then quantity that signifies the world is a possible candidate for a *vestigium Trinitatis*.

A final consideration is whether irreducibly discrete quantity and irreducibly continuous quantity, exhibiting unity in multiplicity, are both helpful to signify other aspects of the *vestigia Trinitatis* as conceived by Augustine, Aquinas, Bonaventure, and

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[357] Augustine of Hippo, *On the Holy Trinity*, 11.11.

[358] Augustine of Hippo, *On the Holy Trinity*, 15.3.

Baxter such as substance, persons, and causality. The *vestigia Trinitatis* tradition has developed through the centuries and unity in multiplicity may be implicit in its many formulations, whether in Augustine's memory, understanding, and will or Bonaventure's three-fold causality. Aristotle says that "in a sense unity means the same thing as being". [359] If as Augustine, Bonaventure, and Baxter argue these *vestigia Trinitatis* exhibit unity in multiplicity, then a non-instrumentalist quantitative system that signifies their being must also exhibit unity-in-multiplicity. All of these reasons suggest that irreducibly discrete and irreducibly continuous quantity that signifies the world exhibits unity in multiplicity and, thus, may indeed be a good candidate for a *vestigium Trinitatis*.

Measure, Number, and Weight as *Vestigia Trinitatis* and Quantity that Signifies the World

At the end of Book XI in *De Trinitate* Augustine references the apocryphal (or deuterocanonical) book, the Wisdom of Solomon, which in verse 11:20 says of God, "you have arranged all things by measure and number and weight." [360] Augustine explores both in *De Trinitate* and elsewhere the implications of measure, number, and weight for the *vestigia Trinitatis*. He says, "Measure, number, and weight…are to be perceived in all other things". [361] This is a curious statement that deserves attention. To what degree might

[359] Aristotle, *Metaphysics*, trans. Ross, 1054a. See Book X:2:9

[360] (New Revised Standard Version)

[361] Augustine of Hippo, *On the Holy Trinity*, 11.11.

Augustine consider the triad of "measure, number, and weight" merely a poetic device and to what degree does he consider the above a technical statement?

Consider first measure. In a parallel passage in a commentary on Genesis Augustine says, "measure sets a limit to everything". [362] Measurement is often applied to delimit the boundaries of seemingly continuous things. Comparing Augustine's words with those of other authors provides insight. Aristotle says,

> Measure is that by which quantity is known, and quantity qua quantity is known either by unity or by number, and all number is known by unity…Hence in other cases too "measure" means that by which each thing is primarily known, and the measure of each thing is a unit—in length, breadth, depth, weight and speed.[363]

Aristotle says, "measure is that by which quantity is known". For Aristotle, quantity can be either irreducibly discrete or irreducibly continuous. Discrete number is measured by discrete units. "But not every unit is indivisible in the same sense—*e.g.* the foot and the arithmetical unit; but the latter is absolutely indivisible, and the former must be classed as indivisible with respect to our power of perception…since presumably everything which is

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[362] "measure sets a limit to everything, and number gives everything its specific form, and weight draws everything to rest and stability" Augustine of Hippo, "The Literal Meaning of Genesis," in *On Genesis: On Genesis: A Refutation of the Manichees, Unfinished Literal Commentary on Genesis, the Literal Meaning of Genesis*, with Edmund Hill et al., Works of Saint Augustine (Hyde Park, New York: New City Press, 2005), 4.3.7.

[363] "Measure is that by which quantity is known, and quantity qua quantity is known either by unity or by number, and all number is known by unity. Therefore all quantity qua quantity is known by unity, and that by which quantities are primarily known is absolute unity. Thus unity is the starting point of number qua number. Hence in other cases too :measure" means that by which each thing is primarily known, and the measure of each thing is a unit—in length, breadth, depth, weight and speed." Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1052b. See Book X:1.

continuous is divisible." [364] For continuous magnitudes a measure is used such as 'a foot' and is indivisible *qua* 'a foot', even though the magnitude is itself divisible. Augustine's claim that "measure sets a limit to everything" comports with Aristotle's claim that the "measure of each thing is a unit—in length, depth, breadth, weight, and speed," all of which are perceptually continuous quantities. Thus, the concept of measure is particularly well suited to quantify continuous entities.

In the Genesis commentary, Augustine proceeds to say, "[N]umber gives everything its specific form". [365] What is the link between number and form? Aristotle acknowledges that the concept of unity is polyvalent. "a) Continuity establishes one kind of unity; b) indivisibility another; c) identity of definition [*logos,* λογος] and of constituent characteristics yet a third." [366] This last point, identity of definition [λογος] relates to Augustine's conception that number gives everything its form. The previous chapter also quoted Aristotle on the relation between continuous matter and individuating form. "[F]or matter is the factor that is bounded and determined by the form, as a surface, or other limit, moulds and determines". [367]

Thomas Aquinas draws from both the Augustinian and Aristotelian themes. Though Augustine does not explicitly distinguish between discrete quantity and continuous quantity

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[364] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1053a. See Book X:1.

[365] Augustine of Hippo, "The Literal Meaning of Genesis," 4.3.7.

[366] Aristotle, *Physics*, trans. Wicksteed and Cornford, 185b. See Book I:2.

[367] Aristotle, *Physics*, trans. Wicksteed and Cornford, 209b. See Book IV:2.

in his writings, Aquinas does distinguish them because he explicitly follows Aristotle in his commentary on the *Physics*.[368] Thus when as examined above Aquinas offers an exposition of designated matter which sounds similar to Augustine, one might hear echoes of Aristotle as well:

> The intellect of the mathematician…abstracts entirely from sensible [physical] matter…I call matter designated if it is considered together with the determination of its dimensions… designated matter is the principle of individuation, from which every intellect abstracts inasmuch as it is said to abstract from the here and now… abstraction, which is common to all intellects, makes a form universal.[369]

Mathematicians abstract form from sensible physical matter. But they only do so from designated matter, matter "considered together with the determination of its dimensions". This sounds both like Augustine and Aristotle. For Augustine, "measure sets a limit to everything, and number gives everything its specific form." [370] For Aristotle, "matter is the factor that is bounded and determined by the form, as a surface, or other limit, moulds and determines". [371] And for Aquinas, the mathematician abstracts form from "designated matter". It is the form of a thing which allows a mathematician to count it or to compare it with a measure.

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[368] For example, in his commentary on Aristotle's *Physics* Aquinas says, "just as being is said in many ways, so also is one. And so we must consider in what way they say that all things are one. For 'one' is used in three ways: either as the continuous is one, such as a line or a body, or as the indivisible is one, such as a point, or as those things are said to be one whose nature [*ratio*] or definition is one, as drink and wine are said to be one." Thomas Aquinas, *Commentary on Aristotle's Physics*, trans. Richard J. Blackwell and Richard J. Spath, with W. Edmund Thirlkel, Rare Masterpieces of Philosophy and Science. (London: Routledge & Kegan Paul, 1963), 1.3.22.

[369] Aquinas, *The Disputed Questions on Truth*, 2.26.

[370] Augustine of Hippo, "The Literal Meaning of Genesis," 4.3.7.

[371] Aristotle, *Physics*, trans. Wicksteed and Cornford, 209b. See Book IV:2

Finally, consider Augustine's explanation of why weight is to "be perceived in all other things":

> But the will which unites and regulates these things, and combines them into a certain unity, and does not quietly rest its desire of [sensuously] perceiving or of conceiving, except in those things from whence the visions are formed, resembles weight…these three things, measure, number, weight…are to be perceived in all other things also…[In addition,] the will is the uniter of the visible thing and of the vision…[372]

In this passage from *De Trinitate* Augustine likens weight to the will. He compares the will which "does not quietly rest its desire" to weight moving heavy things downwards towards their natural place of the earth. In his commentary on Genesis Augustine says, "weight draws everything to rest and stability…[God] is the weight which guides all things" [373] The common principle seems to be final cause.

Augustine is not satisfied resting with his discovery of measure, number, and weight. Speaking of them he says, "time admonishes us to seek for this same trinity in the inner man, and to strive to pass inwards from that animal and carnal and (as he is called) outward man, of whom I have so long spoken. And here we hope to be able to find an image of God according to the Trinity." [374] Though Augustine does indeed refer to measure, number, and weight as a trinity and is thus for him a trace of the trinity, he

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[372] Augustine of Hippo, *On the Holy Trinity*, 11.11.

[373] Augustine of Hippo, "The Literal Meaning of Genesis," 245–46.

[374] Augustine of Hippo, *On the Holy Trinity*, 11.11.

searches for a stronger one. He reserves the phrase "image of God" or "image of the Trinity" only for man as a whole or the inner man and this is where he looks next.

Augustine develops the notion that measure, number, and weight point further to the trinity in the inner man through a complex relation between the world, sensation, and mind. "[M]easure appears in the memory, number in the vision of things…" [375] Augustine draws from his trinity of memory, understanding, and will developed in book X and deployed in book XI to describe the process. Measure links to memory, number to understanding, and weight to will. By linking these notions to the mind, he claims to have "found this trinity to pertain to the outer man". [376] Measure, number, and weight are neither fully outside man, neither are they merely in his senses, nor are they only in his mind. "Measure, number, and weight…are to be perceived in all other things". They are how all things are perceived, and this may be why they are often associated with three of the mathematical arts of the

[375] "But because those things which are impressed on the memory singly, can be conceived according to number, measure seems to belong to the memory, but number to the vision; because, although the multiplicity of such visions is innumerable, yet a limit not to be transgressed is prescribed for each in the memory. Therefore, measure appears in the memory, number in the vision of things: as there is some measure in visible bodies themselves, to which measure the sense of those who see is most numerously adjusted, and from one visible object is formed the vision of many beholders, so that even a single person sees commonly a single thing under a double appearance, on account of the number of his two eyes, as we have laid down above. Therefore there is some measure in those things whence visions are copied, but in the visions themselves there is number." Augustine of Hippo, *On the Holy Trinity*, 11.11.

[376] Augustine of Hippo, *On the Holy Trinity*, 15.3.

Quadrivium, respectively geometry, arithmetic, and music. [377] Not only are they how all things are to be perceived, but this perception reflects a divine order within God. "[B]y the words, 'You have arranged all things in measure and number and weight' nothing else was being said but 'You have arranged all things in yourself'". [378] It is this divine ordering which allows contemplating these things to lead to contemplating the Trinity. Augustine

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[377] The fourth mathematical art is astronomy. See, for example Cassiodorus on the link between measure, number, and weight and the Quadrivial arts. "Secular writers maintain that arithmetic is the first mathematical discipline because arithmetic is essential to explain the excellences of music, geometry, and astronomy. For example, the relation of the single to the double that music involves needs arithmetic; astronomy also requires arithmetic since it considers the numbers of positions in the movement of the stars. Arithmetic, however, can exist without music, geometry, or astronomy. Arithmetic is the source and mother of these other disciplines. Pythagoras regarded this science so highly that he remarked that everything created by God has number and measure. He said that some things were fashioned in motion and other things in place in such a way however that only those disciplines of which we have spoken received substance. I believe this, and take my start as many philosophers have done from that statement of the prophet, that God arranged everything according to number, measure, and weight." Cassiodorus, "Institutions of Divine and Secular Learning," in *Cassiodorus: "Institutions of Divine and Secular Learning" and "On the Soul,"* trans. James W. Halporn and Barbara Halporn (Liverpool: Liverpool University Press, 2011), chap. 4.

[378] "[L]et us call to mind what is said to him in another place in scripture: *You have arranged all things in measure and number and weight* (Wis 11:20). And then, after calling upon God to help by his inspiration and encouragement, let the soul which can do so think about whether these three, measure, number, weight, in which, as it is written, God has arranged all things, were somewhere or other before the whole natural cosmos was created, or whether they too were created; and if they existed beforehand, where were they? After all, before creation there was nothing except the creator. Therefore they were in him. But how? I mean, we read that these other things that have been created are also in him; so are these three identical with him, while the others are in him by whom they are governed and directed?

And how are these identical with him? God, after all, is neither measure nor number nor weight, nor all of them together. Or rather, as we ordinarily understand measure in the things we measure, and number in the things we number or count, and weight in the things we weigh, no, God is not these things; but insofar as measure sets a limit to everything, and number gives everything its specific form, and weight draws everything to rest and stability, he is the original, true and unique measure which defines for all things their bounds, the number which forms all things, the weight which guides all things; so are we to understand that by the words You have arranged all things in measure and number and weight nothing else was being said but "You have arranged all things in yourself"?"

Augustine of Hippo, "The Literal Meaning of Genesis," 245–46.

even seems to describe the Father somewhat cryptically as the "supreme measure". [379] He appears to relate the Son with number, by implying that number is consubstantial with wisdom.[380] And in *Confessions* he strongly associates the Holy Spirit with weight.[381] In each of these three references he underscores the interrelatedness of the persons, reflectively underscoring the interrelatedness of these concepts, measure, number, and weight, in the Godhead.

Augustine does not say that number alone is a trace of the Trinity, but instead Augustine invokes the triad measure, number, and weight. Is there a particular reason why measure and weight are mentioned along with number? Consider again the words of A.N.

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[379] "[T]he Son of God is nothing but the wisdom of God, and the Son of God is truly God… The truth, however, receives its being through a supreme measure, from which it emanates and into which it is converted when perfected…'Of course, the supreme measure also must be a true measure. But, just as the truth is engendered through measure, so measure is recognized in truth. Thus, neither has truth ever been without measure, nor measure without truth. 'Who is the Son of God? It has been said: The Truth. Who is it that has no father? Who other than the supreme measure?" Augustine of Hippo, *The Happy Life*, 1st ed., vol. 5, trans. Ludwig Schopp, The Fathers of the Church (New York: CIMA Pub, 2008), 34.

[380] "[E]nough of wisdom's being found inferior in comparison with number! They are the same, but this calls for an eye able to discern it. now one senses the brightness and the heat in a fire as 'consubstantial…'" Augustine of Hippo, *On the Free Choice of the Will*, trans. Peter King (Cambridge: Cambridge University Press, 2010), 2.11.32. See also the previous footnote, "[T]he Son of God is nothing but the wisdom of God…"

[381] "[T]hen did both Father, Son, and Holy Ghost Move over the waters. Why therefore is this said of thy Spirit only? Why in his case only is a sort of place, where he should be mentioned (which, however, is not a place), why in his case, of whom alone it is said that he is thy gift? In thy gift we rest; then we enjoy thee. Our rest is thy gift, our life's place. Love lifts us up thither, and thy good spirit advances our lowliness, from the gates of death. In thy good pleasure lies our peace. Our body with its lumpishness strives towards its own place. Weight makes not downward only, but to his own place also. The fire mounts upward, a stone sinks downward. All things pressed by their own weight go towards their proper places. Oil poured in the bottom of the water, is raised above it: water poured upon oil, sinks to the bottom of the oil. They are driven by their own weights, to seek their own places. Things a little out of their places become unquiet: put them in their order again, and they are quieted. My weight is my love: by that am I carried, whithersoever I be carried." Augustine of Hippo, *Confessions*, trans. William Watts, Loeb Classical Library 27 (Cambridge, MA: Harvard University Press, 2014), 13.9.

Whitehead in what seems to be a surprise discovery he conveys to Bertrand Russell, "[t]he modern arithmeticization of mathematics is an entire mistake". [382] Arithmeticization creates a "limitation of proof" which "leaves the whole theory of applied mathematics (measurement etc.) unproved." [383] Whitehead links the application of mathematics to the world with measurement. In this he appears to agree with Aristotle's claim that " 'measure' is that by which quantity is known". One might say, quantity that signifies the world requires measure. Thus Augustine's use of the word measure may indeed be a technical phrase indicating not just number but indicating 'quantity which signifies the world'.

But does measure involve both irreducibly discrete and irreducibly continuous quantity? A.N. Whitehead above says measure requires irreducible continuity to augment the previous emphasis of his and Russell's project on irreducibly discrete quantity in the arithmeticization of mathematics. Whitehead does not suggest that they discard all of their previous work, merely that they augment it. It seems unlikely that Whitehead would be interested only in irreducibly continuous quantity, instead he is encouraging Russell to consider both irreducibly continuous and irreducibly discrete quantity.

In addition, Frege, also a supporter of irreducibly discrete quantity, refers to measure as a way to signify the world:

> What then is the substance of the assertion that number-signs designate quantities? Let us look at the applications of arithmetical laws in geometry, astronomy, physics.

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[382] Whitehead, "On Quantity—the Whole Theory of Applied Mathematics Is Unproved.," 1.

[383] Whitehead, "On Quantity—the Whole Theory of Applied Mathematics Is Unproved.," 1.

> Here, indeed, numbers occur in relation to magnitudes, such as length, mass, light intensity, electric charge … [A] number-sign cannot on its own designate a length, a force, and so on, but only in connection with the designation of a measure, a unit, such as metre, gramme, and so on. What in that case does a number-sign refer to on its own? Clearly a magnitude-ratio. And this suggests itself so strongly that we must not be surprised to encounter this insight early. If we now understand by "number" the reference of a number-sign, then real number is the same as magnitude-ratio … [W]e will say that a length can have the same ratio to another length, as a mass to another mass, as a light intensity to another light intensity; and this same ratio is the same number and can be designated by the same number-sign. [384]

Thus, it appears that Frege and Whitehead both acknowledge that a mathematics which is applied to the world requires measure which involves continuous quantity and magnitudes in order to identify discrete number in the world. In this they support the notion that Augustine's invocation of measure as well as number suggests the use of both irreducibly discrete and irreducibly continuous quantity.

The emphasis on weight seems more puzzling. It could be that weight is merely one kind of measure for which Augustine offers an example. Weight is a species of force, and above Frege mentions, "a number-sign cannot on its own designate…force". But might it be that this tradition which Augustine invokes is not interested in weight merely as one sort of measure but is interested in weight as a manner by which motions are coordinated and brought into unity? As mentioned in chapter three, because of Newton's third law of equal and opposite reactions, every force law binds together the particles described into a

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[384] Gottlob Frege, *Basic Laws of Arithmetic, Derived Using Concept-Script: Volumes I & II*, vol. 2, trans. Philip A. Ebert and Marcus Rossberg, with Crispin Wright (Oxford, U.K.: Oxford University Press, 2013), 84.

coordinated motion. Augustine even mentions a similar theme when he compares weight to "the will [which] is the uniter of the visible thing and of the vision". [385]

Moreover, if Augustine's invocation of weight is aptly likened to final cause then that links to the physical analysis of chapter three as well. Peirce defended the role of final cause in physics especially as discerned in the principle of least action. Augustine likens weight both to the will as "uniter" and to God, "the weight who guides all things". Physics which only uses distance and time measurements is called kinematics. While physics that explores the force or energy which causes motion is called dynamics. Augustine may wish to invoke the principles underlying dynamics, or the causes of motion, and not just motion itself. Augustine may be interested not only in the designated matter of things which require discrete and continuous quantity, but Augustine may also be interested in their final cause. All these reasons suggest that Augustine's use of weight may not merely be poetic, but it may be technical.

It is in this manner that Bonaventure seems to interpret Augustine. Augustine emphasizes the relation between measure, number, and weight in the mind. But Bonaventure reiterates this theme and uses it as a key to his more metaphysical scheme which is exhibited throughout creation. When this fuller background is considered for a quote from the third objection, it becomes clear that Bonaventure understands his threefold causality in terms of Augustine's measure, number, and weight:

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[385] " Augustine of Hippo, *On the Holy Trinity*, 11.11.

> [W]hen we say in measure, number, and weight, we indicate that the creature is an effect of the creating Trinity by virtue of a three-fold causality: efficient, through which there is in the creature unity, mode, and measure; exemplary, from which the creature derives truth, form, and number; and final, from which it is endowed with goodness, order, and weight. These, as vestiges of the Creator, are found in all creatures, whether corporeal, spiritual, or composites of both.[386]

Bonaventure claims that "the creature is an effect of the creating Trinity" and "these, [including measure, number, and weight] as vestiges of the Creator, are found in all creatures." The way measure, number, and weight order reality is through Bonaventure's threefold causality. The question for Bonaventure is similar to the question for Augustine. Do measure, number, and weight exist both in the things themselves and in our minds? Or do measure, number, and weight only exist in our minds? Or do they exist in our minds, in God's mind, and in the things themselves? If the last answer is the correct one, this would explain how the unity in multiplicity which is really in the world is also exhibited when humans signify the world with quantity. Because it is "an effect of the creating Trinity" the created world exhibits unity in multiplicity. But the unity in multiplicity of the created world can only be properly recognized when it is signified by quantity which is irreducibly discrete and irreducibly continuous, by "measure, number, and weight".

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[386] Bonaventure, *Breviloquium*, 2.2. See also Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, 1.11.

Bonaventure has hereby expanded Augustine's program. And Aquinas agrees with Bonaventure and Augustine that number, weight and measure are among the *vestigia Trinitatis*.[387] So no longer are the vestiges only to be found in man made in the image of God, an image of the Trinity, but now the vestiges of the Trinity are to be "found in all creatures". It even seems by this that Bonaventure means all of creation, not just living creatures. This does not appear unique to Bonaventure as Aquinas says the same, "in all creatures there is found the trace of the Trinity." [388]

In Bonaventure's book *The Journey of the Mind into God* (hereafter *Itinerarium*) he speaks even more strongly about the relationship between the vestiges of the Creator and number:

> Therefore, since all things are beautiful and in some way delightful; and since there is no beauty or delight without proportion; and since proportion resides first of all in numbers; it is necessary that all things involve number. From this we conclude that number is the principal exemplar in the mind of the Creator, and in creatures it is the principal vestige leading to wisdom. Since number is most evident to all and is closest to God, it leads us very close to God by its seven-fold distinction; and it

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[387] "And therefore Augustine says that the trace of the Trinity is found in every creature, according 'as it is one individual,' and according 'as it is formed by a species,' and according as it 'has a certain relation of order.' And to these also are reduced those three, 'number,' 'weight,' and 'measure,' mentioned in the Book of Wisdom. For 'measure' refers to the substance of the thing limited by its principles, 'number' refers to the species, 'weight' refers to the order. And to these three are reduced the other three mentioned by Augustine, 'mode,' 'species,' and 'order,' and also those he mentions: 'that which exists; whereby it is distinguished; whereby it agrees.' For a thing exists by its substance, is distinct by its form, and agrees by its order. Other similar expressions may be easily reduced to the above." Aquinas, *Summa Theologica*, 1.45.7.

[388] "But in all creatures there is found the trace of the Trinity, inasmuch as in every creature are found some things which are necessarily reduced to the divine Persons as to their cause." Aquinas, *Summa Theologica*, 1.45.7.

> makes God known in all bodily and sensible things when we become aware of numerical realities, when we take delight in numerical proportions, and when we come to make irrefutable judgments by means of the laws of numerical proportions.[389]

Bonaventure emphasizes numbers as vestiges. "[N]umber… is the principal vestige leading to wisdom." Bonaventure's comment on seven types of number is a reference to Augustine's book *On Music*, another text in which Augustine advances the thesis Bonaventure describes: that numbers can lead one to God.[390]

In summary, Augustine refers to measure, number, and weight as "this same trinity" which he wishes to trace to the inner man.[391] In this it appear that Augustine conceives measure, number, and weight as a trace of the Trinity, even if only faintly. Bonaventure, on the other hand extends this Augustinian insight highlighting the role measure, number, and weight play in causal metaphysical principles. Bonaventure explicitly includes measure, number, and weight as *vestigia Trinitatis* prominently explaining how the contemplation of these things leads to God. Quantity that signifies the world involves both measure and number, irreducibly continuous quantity and irreducibly discrete quantity, as shown by the thought of Whitehead and Frege. Moreover, the linking of weight to force and final cause

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[389] Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, 2.10.

[390] Augustine spends many chapters discussing reacting numbers, judicial numbers, memorial numbers and others. He eventually says, "And what more do you think there's for me to say when Holy Scripture, in so many volumes endowed with such authority and holiness, exhorts us only to love our God and Lord with all our heart, with all our soul, and with all our mind, and to love our neighbor as ourself ? If, then, we refer all those motions and numbers of human action to this end, we shall certainly be cleansed. Isn't it so?" Augustine of Hippo, "On Music," in *The Immortality of the Soul: The Magnitude of the Soul ; On Music ; The Advantage of Believing ; On Faith in Things Unseen*, trans. Robert Catesby Taliaferro (New York: Catholic University of America Press, 1947), 6.14.43.

[391] Augustine of Hippo, *On the Holy Trinity*, 11.11.

suggests that all of these together reflect a unity in multiplicity which will be discussed next. For all of these reasons, irreducibly discrete and irreducibly continuous quantity that signifies the world is a good candidate for a *vestigium Trinitatis*.

Is Unity in Multiplicity a General Mark of *Vestigia Trinitatis* and is it Exhibited by Irreducibly Discrete and Irreducibly Continuous Quantity that Signifies the World?

Another reason to consider why irreducibly discrete and irreducibly continuous quantity that signifies the world should be regarded as a *vestigium Trinitatis* is because such quantity seems to exhibit unity in multiplicity. Unity in multiplicity is the quantitative corollary to Herman Bavinck's conception of the *vestigia Trinitatis* as unity in diversity. An advantage of this formulation is that existing conceptions of the *vestigia Trinitatis* may also exhibit unity in multiplicity revealing this theme as an implicit aspect of the tradition. For example, since unity in triadicity implies unity in multiplicity, if unity in triadicity is an aspect of the *vestigia Trinitatis*, then is unity in multiplicity, at least in some restricted sense, implied as an aspect of the *vestigia Trinitatis* as well? Whether irreducibly discrete and irreducibly continuous quantity that signifies the world actually exhibits unity in multiplicity should be carefully considered. If such quantity does exhibit this mark and if unity in multiplicity is indeed a mark of all *vestigia Trinitatis*, then it makes quantity that signifies the world a more likely candidate for a *vestigium Trinitatis*. Consider first the argument drawing from Bavinck that unity in multiplicity can be regarded as a *vestigium Trinitatis*.

As encountered in the third objection, in the 16th century Richard Fowns offered a novel consequence of Bonaventure's creating Trinity. Fowns claimed the "Trinity in unity hath so created things" that their principles are "differing, but not divided: sundry, and yet

the same." [392] Similar to Fowns' notion, unity in diversity is the hallmark conception of the *vestigia Trinitatis* for the Dutch theologian Herman Bavinck (1854-1921) who would write three hundred years later. Like Baxter and Bonaventure, Bavinck does not disparage the search for triads completely, and his approach to the *vestigia Trinitatis* is more metaphysical. But Bavinck does not envision the basic principles of the cosmos in terms of triadic structure as Bonaventure and Baxter do. Instead, Bavinck believes that the unity in essence and diversity in relations existing among the three persons of the Divine Trinity is reflected in creation wherever there is unity in diversity. Bavinck's deemphasis on triads in favor of a more general unity in diversity is a new conception of the *vestigia Trinitatis*.

Bavinck's discussion of the *vestigia Trinitatis* carefully recounts details of the inquiry going back to Augustine and makes fine distinctions along the way. While Bavinck is sympathetic to past thinkers, he wishes to place the doctrine on firmer ground. Bavinck concludes his analysis of the *vestigia Trinitatis* tradition and 19th century triadic philosophies with a measure of dissatisfaction but not without some optimism:

> Nevertheless, the arguments advanced to shed light on the dogma of the Trinity are not devoid of all value. In the first place, Scripture itself gives us the freedom to use them when it says that the entire creation and especially humankind is a work of the triune God. Certainly, all God's works ad extra are undivided and common to all three persons. Prominent in these works, therefore, is the oneness of God rather than the distinction of the persons. In this unity, however, the diversity cannot be lacking. For Scripture itself points to this truth by saying that all created beings will show these imprints and human beings will exhibit the image of the triune God.[393]

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[392] Klinck, "Vestigia Trinitatis in Man and His Works in the English Renaissance," 16.

[393] Bavinck, *Reformed Dogmatics. Volume 2, God and Creation [Electronic Resource]*, 255–56.

In this, Bavinck introduces his generalized approach to the *vestigia Trinitatis* tradition. "In this unity, however, the diversity cannot be lacking…all created beings will show these imprints and human beings will exhibit the image of the triune God." Like the medieval scholars Bavinck restricts the image of the Trinity to humans but acknowledges the traces or imprints of the Trinity upon all created beings.

Bavinck also echoes a theme common to Augustine, Aquinas, Baxter, and most orthodox Christian thinkers: proofs for the Trinity must rest on Scripture. Trinitarian analogies or especially traces of the trinity are only *a posteriori* demonstrations of the reasonableness of the doctrine of the Trinity, not *a priori* proofs for it. "Attempts have been made to infer the Trinity from God's thinking and willing, from his love, goodness, and perfection, and so forth. Intended as philosophical construals of the doctrine of the Trinity, these attempts have been anything but satisfactory." [394] The *vestigia Trinitatis* can show how the doctrine of the Trinity comports with reason, but they cannot prove the Holy Trinity.

Bavinck develops what he contends is the better understanding of the traces of the Trinity:

> The Trinity reveals God to us as the fullness of being, the true life, eternal beauty. In God, too, there is unity in diversity, diversity in unity. Indeed, this order and this harmony is present in him absolutely. In the case of creatures we see only a faint analogy of it. Either the unity or the diversity does not come into its own. Creatures exist in time and space, exist side by side, and do not interpenetrate each other [like the persons in the Trinity]. Among us unity exists only by attraction, by the will and the disposition of the will; it is a moral unity that is fragile and unstable. And where

[394] Bavinck, *Reformed Dogmatics. Volume 2, God and Creation [Electronic Resource]*, 257.

> there is a more profound physical unity as, say, between the capacities of a single substance, there is no independence, and the unity swallows up the diversity. But in God both are present: absolute unity as well as absolute diversity. It is one selfsame being sustained by three hypostases.[395]

In this passage, Bavinck claims that unity in diversity is the common aspect to the entire tradition of the analogies of the Trinity and the *vestigia Trinitatis*. God exhibits unity in diversity and diversity in unity. The cosmos imperfectly exhibits unity in diversity and diversity in unity. It is because God created the cosmos that it is analogously like him.

Scholar James Eglinton argues that unity in diversity is Bavinck's distinct contribution to the tradition of the *vestigia Trinitatis*:

> Bavinck was not against triads per se. Rather, he was strictly opposed to speculation and natural theology's use of trinitarian analogies as 'proofs' for God's triunity. His key development within the idea of *vestigia trinitatis*, however, lies in the establishment of the non-numerical paradigm of unity-in-diversity (rather than the medieval triad form) as the norm in terms of triniform hallmarks.[396]

The notion of God's unity in diversity is reflected throughout Bavinck's doctrine of God as can be seen in the quote from Bavinck above. For example, even Bavinck's treatment of God's unique name along with a diversity of names reflecting his attributes, point to this

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[395] The quote continues, "This results in the most perfect kind of community, a community of the same beings; at the same time it results in the most perfect diversity, a diversity of divine persons. Therefore, if God is triune, the three persons can only be conceived as being "consubstantial" (ομοουσιοι)." Bavinck, *Reformed Dogmatics. Volume 2, God and Creation [Electronic Resource]*, 257.

[396] Eglinton, *Trinity and Organism*, 86.

paradigm of unity in diversity.[397] And because Bavinck believes that creation has the imprint of its maker, creation also exhibits unity in diversity. Thus, without dismissing previous conceptions, Bavinck expands the conception of the *vestigia Trinitatis*.

How is unity in diversity related to unity in multiplicity? Unity in diversity is a broad category, a flower can be united but have diverse colors. The flower's petals are blue, green, and yellow but it is one flower. The above is a qualitative description of the flower's unity in diversity. But if one counts the number of colors present in the flower, the description is quantitative. It is one flower possessing three distinct colors, which are multiple. Unity in diversity is a qualitative way of expressing an idea which can also be expressed quantitatively as unity in multiplicity. Thus, Bavinck's expansion of the *vestigia Trinitatis* to include unity in diversity is also an implicit inclusion of unity in multiplicity in the doctrine of the *vestigia Trinitatis*.

Unity in triadicity has been a hallmark of the *vestigia Trinitatis* since Augustine initiated this theological inquiry.[398] The number three represents three unities or, in other

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[397] Not only does Bavinck's treatment of names reflect unity in diversity, but he also emphasizes the role of the divine names as signs (similar to the notion of traces or *vestigia*). "Etymologically, Bavinck regards 'name' in the Biblical languages (ὀνομα, *onoma* etc.) as rooted in the concepts of 'sign' and 'distinguishing mark'. A name, in the true sense, is no mere arbitrary tag. For Bavinck, a name is closely bound to the character of its bearer. Interestingly, Bavinck laments the recreation of name as a purely aesthetic device in the modern world." Eglinton, *Trinity and Organism*, 105.

[398] For example, see Augustine's emphasis on unity in triadicity in this passage examined earlier: "Since, then, these three, memory, understanding, will, are not three lives, but one life; nor three minds, but one mind;

words, multiple unities. The number four also represents multiple unities. Triadicity is one particular form of multiplicity. Thus, unity in triadicity is one particular form of unity in multiplicity. To claim that the world exhibits unity in triadicity would necessarily mean that the world exhibits unity in multiplicity as well. It would be possible for the world to exhibit unity in multiplicity by only exhibiting unity in duality or unity in quaternity. But it is not logically possible to say that the world exhibits unity in triadicity and not unity in multiplicity. Therefore, a claim that the world exhibits unity in multiplicity is a more general claim which is in some way necessarily asserted by the more specific claim that the world exhibits unity in triadicity.

Because all of the other versions of the *vestigia Trinitatis* exhibit unity in triadicity, they also therefore exhibit unity in multiplicity. For example, the mind is a unity but since it can also be considered as memory, understanding, and will, it is a unity in triadicity. But memory, understanding, and will are also multiple, so the mind could also be described as a unity in multiplicity. Bonaventure describes one causality in three aspects: efficient cause, exemplary cause, and final cause. If this can be described as unity in triadicity, then it can also be described, with less accuracy but still truly, as unity in multiplicity. Any unity in triadicity logically entails that unity in multiplicity is present. And therefore if all previous

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it follows certainly that neither are they three substances, but one substance…But they are three, in that wherein they are mutually referred to each other; and if they were not equal, and this not only each to each, but also each to all, they certainly could not mutually contain each other; for not only is each contained by each, but also all by each…And, therefore, while all are mutually comprehended by each, and as wholes, each as a whole is equal to each as a whole, and each as a whole at the same time to all as wholes; and these three are one, one life, one mind, one essence." Augustine of Hippo, *On the Holy Trinity*, 10.11.

conceptions of the *vestigia Trinitatis* exhibit unity in triadicity they also exhibit unity in multiplicity in some way.

A caution should be noted before proceeding. When Augustine first uses the phrase *vestigia Trinitatis*, he claims that we must understand the Creator, "by the things that are made [and thus] we must needs understand the Trinity of whom there appear traces in the creature". [399] So, Augustine sanctions a positive case for theological understanding of the Trinity by its traces. Nonetheless a theologian should not expect any language regarding quantity such as 'one in three' or 'unity in multiplicity' to apply in exactly the same way for created things as it does for the mystery of the Trinity. Augustine continues, "here in corporeal things, one thing alone is not as much as three together, and two are something more than one; but in that highest Trinity one is as much as the three together, nor are two anything more than one." [400] Augustine uses notions of unity and multiplicity, one, two, and three, but he distinguishes the corporeal understanding of three from the way the term three refers to God. To say that the divine persons are three is a meaningful statement, but it does not mean exactly the same thing as saying that a corporeal quantity is three. Thus, to explore the notion of unity in multiplicity as a *vestigium Trinitatis* by no means suggests that unity in multiplicity means exactly the same thing in reference to creation and to the Holy Trinity.

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[399] Augustine of Hippo, *On the Holy Trinity*, 6.10.12.

[400] Augustine of Hippo, *On the Holy Trinity*, 6.10.12.

If one accepts unity in multiplicity as a *vestigium Trinitatis*, it does not demonstrate that quantity that signifies the world exhibits such unity in multiplicity. So, what reasons are there to conclude that irreducibly discrete and irreducibly continuous quantity that signifies the world exhibits unity in multiplicity? Unity in multiplicity requires more than just conceptions of unity *and* multiplicity. Unity *in* multiplicity requires that a thing is *simultaneously* both one and many. But in order for unity and multiplicity to coexist simultaneously in one thing, the thing has to be one and many in different ways.

Socrates is discrete and indivisible as a man, but he is a continuous volume as a body. Socrates *qua* man really is one discrete entity, but Socrates *qua* magnitude really is composed of many continuous potential parts. He is one and indivisible in the aspect of his humanness, but he is many in the divisibility of his body. If his hair is shorn or he loses a finger, he is still Socrates. Socrates is simultaneously one and many but not in the same way. In this formulation, his simultaneous unity and multiplicity do not challenge the law of non-contradiction. As Aristotle noted,

> the fact is that one thing obviously can be many, either in the sense of being several conceptually distinct things at once…or in the sense in which a whole can be regarded as being the sum of the parts into which it can be divided. [Our philosophers] had to admit multiplicity in unity, unwillingly, as though it were a paradox. And so, of course, it would be, if 'one' and 'many' were used in the sense

> in which they contradict each other, but not if, for instance, we are regarding the 'one' both in its actual unity and its potential multiplicity. [401]

Because there are multiple senses of unity, Aristotle claims that it is not a contradiction to say a thing exhibits unity in multiplicity. But there are different ways that a thing can exhibit unity in multiplicity. Aristotle says a thing may be "several conceptually distinct things at once". One thing, many concepts. For example, one thing, Fido, is simultaneously multiple concepts: a dog, a mammal, or a quadruped. Another way Aristotle says something can exhibit unity in multiplicity is that the whole "can be regarded as being the sum of the parts into which it can be divided". One whole, many parts. One whole, Fido, is multiple parts: head, trunk, legs and tail.[402]

Now these sorts of unity in multiplicity show that a thing may be regarded as one and many but only by referencing different senses of unity. Thus, Socrates is both one and not-one but in different ways. But importantly both of these aspects of Socrates are required to

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[401] "Indeed some of the later ancients were themselves disturbed by the danger of finding themselves admitting that the same thing is both one and many; and therefore some of them, like Lycophron, banned the word 'is' altogether…[The] nervousness resulted from their equating ' is ' with 'exists' and ignoring the different senses of 'one' or of 'being. But the fact is that one thing obviously can be many, either in the sense of being several conceptually distinct things at once (for though it is one thing to be pale-complexioned and another to be cultivated, yet one and the same man can be both, so that ' one' is ' many'); or in the sense in which a whole can be regarded as being the sum of the parts into which it can be divided. This latter case brought our philosophers to a stand; and they had to admit multiplicity in unity, unwillingly, as though it were a paradox. And so, of course, it would be, if 'one' and 'many' were used in the sense in which they contradict each other, but not if, for instance, we are regarding the 'one' both in its actual unity and its potential multiplicity." Aristotle, *Physics*, trans. Wicksteed and Cornford, 185b27-186a4. See Book I:2

[402] Interestingly there is no one unique privileged division of parts. Fido is: 1) head, trunk, legs and tail; 2) circulatory system, nervous system, muscular system, skeletal system; or 3) flesh and bones. There are many different ways to identify the 'parts' of Fido.

explain the existence of Socrates. His unity in multiplicity is not trivial, but the manner in which Socrates exists is inextricably connected to his simultaneous unity in multiplicity. Previous sections have explored how form has often been associated with the discrete indivisibility of a thing and matter with its divisibility.

The unity in multiplicity of a basketball team is of a different sort than the unity in multiplicity of Socrates. The multitude of discrete team members create a multiplicity. But their ability to exert team-work-requiring capacities and act as one displays continuous causality. They are able to move in one harmonious motion, as when Aristotle says "a bundle [of sticks] is made continuous by its string". [403] The discrete unities of the players display multiplicity and the concerted continuous action towards a purpose or towards a final cause display the unity. In this example, the unity in multiplicity requires the one purposeful unified causality as well as the many members. The team exhibits simultaneous conjoined unity and multiplicity. As a "whole" the team plays other teams. "In the sense that a whole can be described as the parts into which it is divided," the team has members who each have unique skills.

Unity in multiplicity requires conceptions of discreteness and continuity because it requires at least two different quantitative notions of unity when referencing the same thing in order not to posit a contradiction. It is false to say that Socrates is one man and that he is many men. That would be to invoke the same notion of unity and to say contradictory things about it. Likewise, it is false to say that the basketball team is one basketball team

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[403] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1015b. See Book V:6.

and many basketball teams. Socrates can only be simultaneously one and many because he is one as a discrete man and many as an extended body with continuous potential parts. A team can be simultaneously one and many because the team has one continuous causality but many discrete members.

But in the realm of quantity, discrete quantity and continuous quantity are incommensurable. It is because of this property of incommensurability that discrete and continuous quantity can signify two different aspects of reality without contradicting each other. According to Peirce, "A collection is a whole whose being consists in the independent being of its members; a line, on the contrary, has a being from which the being of its points is derived and in which they, as possibilities, are involved." [404] A set of points is a set of irreducibly discrete entities. A line, on the other hand, has the possibility of other lines intersecting it and thereby determining possible points. Irreducibly discrete and irreducibly continuous quantity can signify a line as a continuous unity and a potential multiplicity of points: a unity in multiplicity. But in the Dedekind-Cantor notion of continuity a line is merely a multiplicity because it is composed of a multitude of points. To say that a line is one point and many points would be false. Under the Cantor-Dedekind conception a line is only many points. It is for this reason that Henri Poincaré says of the arithmetical continuum, "[m]ultiplicity alone subsists, unity has disappeared…" But on the other hand, the true irreducibly continuous "continuum is unity in multiplicity". [405]

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[404] Peirce, *The New Elements of Mathematics*, 2.531.

[405] Poincaré, *Science and Hypothesis*, 23.

Another relevant Aristotelian notion of unity was mentioned in the previous chapter, "things are one…analogically…which have the same relation as something else to some third object." [406] A unity of proportion is to have the same kind of relation, and having the same kind of relation is what underlies the force laws which show coordinated motion. But if in Aristotle's conception there can be a unity of relations, then by corollary there may also be a multiplicity of relations. For example, the quadratic relation between distance and time for an object falling from rest is not the same relation as the linear relation between distance and time for an object moving at constant velocity. These are two distinct relations.

Unity and the possibility of multiplicity in relations are, of course, also important for Trinitarian theology because the one and only distinction in the Godhead is the multiplicity of the three relations of the persons, Father, Son, and Holy Spirit. Gregory of Nyssa says "that while we confess the invariable character of the nature, we do not deny the difference in respect of cause (*aition*), and that which is caused, by which alone we apprehend that one Person is distinguished from another". [407] The word cause (*aitia*) is a kind of relation. The distinction of relations, Fatherhood, Sonship, and Spiration, is the one distinction in the Godhead. Thus, Aquinas says, "But in God the relations themselves are the persons

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[406] Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1016b31-35. See Book V:6:21-22

[407] Gregory of Nyssa, "On 'Not Three Gods,'" in *Nicene and Post-Nicene Fathers: Gregory of Nyssa: Dogmatic Treatises, Etc*, ed. Philip Schaff and Henry Wace, trans. H. A. Wilson (Hendrickson Publishers, 1995).

subsisting in one nature." [408] Thus, the unity or multiplicity of relations is a key notion not only for the *vestigia Trinitatis* but in Trinitarian theology proper. In this a different sense of unity is displayed which allows for a different sense of unity in multiplicity.

To signify things simultaneously as discrete and continuous does not mean that those entities are in all ways discrete and continuous nor are they equally discrete and continuous. Conceiving entities as particles interacting through force at a distance is to conceive them similar to Aristotle's discrete sticks individually moving though also moving in a bundle in a coordinated motion, in unity of proportion. Conceiving entities as continuous wholes is to conceive them as bundles having internal continuous parts made continuous by causal linkages. Coordinated action is an important aspect of the *vestigia Trinitatis* according to Augustine in Sermon 52 in which he says, "[y]ou are searching within creation for three somethings which can both be indicated separately and act inseparably." [409] In *De Trinitate*, Augustine says, "the Father, and the Son, and the Holy Spirit, as they are indivisible, so

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[408] Aquinas refers to the persons of the Trinity as "subsistent relations". Aquinas, *Summa Theologica*, 1.42.3ad2.

[409] Augustine of Hippo, "Augustine of Hippo, Sermon 52," in *The Cambridge Edition of Early Christian Writings: Volume 1: God*, vol. 1, ed. Andrew Radde-Gallwitz, trans. Mark DelCogliano, The Cambridge Edition of Early Christian Writings (Cambridge: Cambridge University Press, 2017), 309–10, https://doi.org/10.1017/9781107449596.023.

work indivisibly." [410] This coordinated action of discrete things is referred to by Aristotle both as a kind of continuity and could also be understood as a unity of proportion.

Thus, in a variety of ways irreducibly discrete and irreducibly continuous quantity that signifies the world exhibits unity in multiplicity. Such quantity exhibits unity in multiplicity by signifying Socrates as simultaneously indivisible (irreducibly discrete) and divisible (irreducibly continuous with multiple potential parts). Such quantity also can signify individuals in coordinated action as in a basketball team, who while discrete members are unified through their will and their continuous causal action. And this unity in multiplicity resembles the use of discrete and continuous quantity in physics as represented by Hesse's work in the previous chapter on forces and fields. Individual electrons may be understood as coordinated into a continuous unified action through force pairs. Or a continuous electric field can be understood to have multiple potential parts some of which might be singular excitations conceived of as discrete particles. Thus, irreducibly discrete and irreducibly continuous quantity that signifies the world does indeed exhibit unity in multiplicity.

To summarize, unity in multiplicity does seem to appear in all conceptions of the *vestigia Trinitatis*. Unity in triadicity entails unity in multiplicity. And since all traditional

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[410] Augustine of Hippo, *On the Holy Trinity*, 1.4.7. The Latin is, "*quamvis Pater et Filius et Spiritus sanctus, sicut inseparabiles sunt, ita inseparabiliter operentur.*" Augustine of Hippo, *De Trinitate*, 42:1.4.7. Also, "the Trinity works indivisibly in everything that God works…" Augustine of Hippo, *On the Holy Trinity*, 1.5.8. The Latin is, "*inseparabiliter operari Trinitatem in omni re quam Deus operator*…" Augustine of Hippo, *De Trinitate*, 42:1.5.8.

forms of the *vestigia Trinitatis* exhibit unity in triadicity they also exhibit unity in multiplicity. Unity in multiplicity is also entailed by Bavinck's conception of the *vestigia Trinitatis* as unity in diversity, making unity in multiplicity a trace of the Trinity, even if more faint than unity in triadicity. Because irreducibly discrete and irreducibly continuous quantity that signifies the world exhibits unity in multiplicity, this suggests that this metaphysically rich conception of quantity may itself be a *vestigium Trinitatis*.

Key Aspects of the *Vestigia Trinitatis* like Substance, Persons, and Causal Metaphysical Principles Exhibit Unity in Multiplicity and should be Signified with Mutually Irreducible Discrete and Continuous Quantity

Consider now some key aspects of previous *vestigia Trinitatis* such as substance, persons, and causal metaphysical principles and how these might be signified by quantity. Is irreducibly discrete and irreducibly continuous quantity important to signifying these conceptions? Augustine says that all things are perceived through "measure and number and weight". And by this he suggests that these three are traces of the Trinity. What if this is akin to saying that all things are perceived through irreducibly discrete and irreducibly continuous quantity?

By examining how some key aspects of traditional *vestigia Trinitatis* are signified it allows us to consider carefully the role of irreducibly discrete quantity and irreducibly continuous quantity in signifying those things traditionally held to exhibit unity in multiplicity. Augustine focused his search for the *vestigia Trinitatis* on human persons. When considering which aspect of persons might point to the Trinity he looked for multiple things but of one substance, which he found in the mind. And Bonaventure and Baxter

when extending the conception of the *vestigia Trinitatis* to all things focused on causal metaphysical principles. Thus if these three things: persons, substance, and causality can be well signified with irreducibly discrete and irreducibly continuous quantity, then it is likely that irreducibly discrete and irreducibly continuous quantity are similarly helpful to signifying other conceptions of the *vestigia Trinitatis*.

When Augustine initiated an inquiry into the *vestigia Trinitatis,* substance and persons were among the core concepts he explored. Consider the role of substance and persons in the following passage:

> But in these three, when the mind knows itself and loves itself, a trinity remains: the mind, love, and knowledge; and there is no confusion through any commingling, although each is a substance in itself, and all are found mutually in all, whether each one in each two, or each two in each one. Consequently, all are in all.[411]

Augustine identifies mind, love, and knowledge each as substances. Yet they mutually cohere in each other. But these substances only exist within a person. In fact, Augustine judges that what counts as *vestigia Trinitatis* are almost exclusively limited to aspects of persons. As considered in the previous chapter, both persons and substance involve some notion of identity-over-time. Thus, consider the role of irreducibly discrete and irreducibly continuous quantity in signifying identity-over-time.

Identity-over-time is necessary to signify either a substance or a person and identity-over-time shows unity in multiplicity. When Peirce discusses identity-over-time he notes, "Identity is a Continuous Relation,—that is, continuous in meaning…Yet it is through that

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[411] Augustine of Hippo, *On the Trinity*, 9.5.

continuum, that line of generalization of possibilities, that the actual [discrete] point at one extremity necessarily leads to the actual [discrete] point at the other extremity."[412] At any moment in time "Assertoric Truths" can be stated of an entity but the truths at these discrete moments are also connected to the ongoing identity of the thing.[413]

Peirce's view contrasts sharply to the claim of David Hume that "[t]he identity, which we ascribe to the mind of man, is only a fictitious one, and of a like kind with that which we ascribe to vegetables and animal bodies."[414] This is because unlike Hume Peirce holds that there is a real continuity, a real 'connexion' in identity-over-time. Discrete points or instants may be determined upon the continuum and assertoric truths may be stated of those moments. But those moments do not compose the flow of identity, they are only possibilities on the continuous identity of a thing.

In order for identity-over-time not to be made fictitious, it must display this unity-in-multiplicity. The identity of a thing is one continuous whole, but there are real possibilities of an infinite number of moments at which assertoric truths may be stated. The discrete aspects of the thing with identity and the continuous aspects of the thing, both coexist but in different ways that do not contradict each other. They are incommensurable not contradictory.

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[412] Peirce, *The New Elements of Mathematics*, 4.330.

[413] Peirce, *The New Elements of Mathematics*, 4.330.

[414] Hume, *David Hume*, 1.4.6.16.

Second, consider the importance of unity-in-multiplicity for a substance. Since a substance displays identity-over-time, a substance displays the unity in multiplicity discussed above. But how else does a substance display unity-in-multiplicity? Koslicki proposes that a true whole exhibits team-work-requiring capacities. "[A]n integrated whole derives its unity from the way in which its parts are able to interact with other parts of the same whole…as well to manifest certain of its capacities, namely, those whose manifestation by the whole requires 'team work' among its parts."[415] When the whole and the parts work together to create one continuous motion or action there is unity in multiplicity.

To manifest a capacity is to activate a causal ability. According to the Eleatic principle, to be is to be able to act or be acted upon. This leads to Koslicki's assertion of "Restricted Composition" in mereology. Not all things which are sets, classes, or sequences are true unities or wholes but only those things that have causal team-work-requiring capacities. If causation is a continuous relation as argued by Peirce, then the discrete existence of a substance is established by a continuous relation. In addition, for organic structures Aristotle claims that an "[o]rganic unity differs from contact…[the] parts…are

[415] Koslicki, *Form, Matter, Substance*, 210.

one in respect of continuity and quantity." [416] Thus, an organic substance exhibits this additional kind of unity in multiplicity.[417]

Finally, consider the unity in multiplicity of persons. Like substances, persons too exhibit the unity in multiplicity of identity-over-time. Since persons can be considered as organic substances, they also exhibit the unity in multiplicity described in the Restricted Composition Principle and the continuity of parts described above. But persons exhibit other kinds of unity in multiplicity as well. Both Augustine and Peirce consider the centrality of mind to persons. As quoted above, Augustine considers mind, love, and knowledge as simultaneously one in three. Peirce says more about the connectedness between minds. "All communication from mind to mind is through continuity of being." [418] That discrete humans have the capacity to communicate to each other and deeply understand each other relies on a connectedness—a "continuity of being". Though humans are discrete persons, they have the ability to connect with each other in a unique way because of their minds. The sentiment that the inner man connects humans is echoed by Bavinck though in a less optimistic way. "Among [creatures] unity exists only by attraction, by the will and the disposition of the will; it is a moral unity that is fragile and

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[416] "Organic unity differs from contact; for in the latter case there need not be anything besides the contact, but in organic unities there is something identical in both parts, which makes them grow together instead of merely touching, and be one in respect of continuity and quantity, though not of quality…" Aristotle, *Metaphysics*, trans. Ross, 1014b. See Book V:4.

[417] The role of continuity in organic substances may relate to Bavinck's use of the organic metaphor. The connection between the organic metaphor in Bavinck and the *vestigia Trinitatis* can be be seen in Eglinton, *Trinity and Organism*.

[418] Peirce, "Synechism and Immortality," 7.570-572.

unstable." [419] Through our minds, wills, and understanding humans are connected to each other into community.[420] And these unities of persons in community have both discrete and continuous aspects. Thus, persons exhibit an additional form of unity in multiplicity based on inner, soulish realities and not just bodily realities.

If both discrete quantity and continuous quantity are required to signify the unity in multiplicity of substances and human persons, then discrete and continuous quantity are required to signify fully the unity in multiplicity of the Augustinian conception of the *vestigia Trinitatis*.

Causal metaphysical principles have also been associated with the *vestigia Trinitatis*. Both Bonaventure and Richard Baxter link the unity in multiplicity of causality explicitly with the *vestigia Trinitatis*. Augustine emphasized the relation between measure, number, and weight in the mind and in perception, Bonaventure emphasizes this triad in creation itself on account of causal principles. Bonaventure understands measure, number, and weight to be present in "all creatures, whether corporeal, spiritual, or composites of both" as an evidence of God's "three-fold causality". Bonaventure also links this idea to the broader metaphysical notions of efficient, exemplary, and final causality. Like Bonaventure, Aristotle, Augustine, and Aquinas all linked number to formal cause as well.

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[419] Bavinck, *Reformed Dogmatics. Volume 2, God and Creation [Electronic Resource]*, 257.

[420] Also consider the work of Andrew Pinsent on second person relatedness and joint attention. Pinsent, *The Second-Person Perspective in Aquinas's Ethics*, 41–49.

In addition, Richard Baxter also suggested replacing Aristotle's fourfold causal scheme with three causes: efficient cause, constitutive cause (subsuming both material and formal cause), and final cause. Baxter associates these causes with characteristics of God, particularly the "virtue of power, the light of wisdom, and love or goodness." [421] Baxter considers these three principles: power, wisdom, and goodness, as a trinity of essentialities spread throughout reality.[422] Baxter asserts that "this Trinity of essentialities or primalities shows its footsteps [*vestigia*] through the whole nature of things." [423]

Both Baxter and Bonaventure associate causality with the *vestigia Trinitatis*. But causality itself as a metaphysical principle has been highly contended since the time of David Hume. Hume claimed, "the understanding never observes any real connexion among objects, and that even the union of cause and effect, when strictly examin'd, resolves itself into a customary association of ideas." [424] Because of the claim that there is no "real connexion" among events or objects Hume dismissed causality as merely a habit of the mind "in the imagination". [425]

In order to signify adequately the broader metaphysical principles of causality considered as *vestigia Trinitatis* by Bonaventure and Baxter, formal systems need both

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[421] Baxter, *Methodus theologiae Christianæ*, 1.4.

[422] Baxter, *Methodus theologiae Christianæ*, 1.4-6.

[423] Baxter, *Methodus theologiae Christianæ*, 1.121. Translation given by Sytsma, *Richard Baxter and the Mechanical Philosophers*, 126.

[424] Hume, *David Hume*, 1.4.6.17.

[425] Hume, *David Hume*, 1.4.6.17.

discrete and continuous quantity. Peirce claimed that the typical presentation of efficient and final cause left the impression that the former was completely before the event and the latter was completely after the event. Peirce believed that the two causes were actually continuous in the same manner in which time is continuous. The push from before and the pull from after cannot be completely separated. "[T]here is an essential thirdness, which the principle of causality fails to recognize…" [426] Thus, while there is a sequence of discrete events, those events are still connected. In contrast to Hume, Peirce argued that there is a real connection in causality, and thus both discrete and continuous quantity are needed to signify it.

Neither Bonaventure nor Baxter acknowledge a separate category for material cause. Bonaventure describes an "exemplary cause" which is related to form, and Baxter describes "constitutive cause". Baxter puts matter and form together as aspects of constitutive cause and in one instance he explicitly links them.[427] In Christian theology, not all beings have matter, i.e., angels, but for scholastic realists those beings that do have matter also have form. This linking of matter and form simply points back to the discussion of substance from earlier. In the Aristotelian tradition matter had been understood as a continuous potentiality which is indefinitely divisible. And an object's form is discrete, making a thing countable. Thus, for terrestrial things the exemplary cause of Bonaventure and the constitutive cause of Baxter both require discrete and continuous quantity to signify them.

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[426] Peirce, "Physical Causation," 6.69.

[427] Baxter, *Methodus theologiae Christianæ*, 1.5.

Various aspects of causality have been considered as *vestigia Trinitatis*. Bonaventure and Baxter both considered efficient and final cause to be *vestigia Trinitatis*. They also considered some sort of exemplary or constitutive cause to be an aspect of the *vestigia Trinitatis*. But these causal concepts need both true discreteness and true continuity to signify the simultaneous unity in multiplicity of these notions of causality. Thus, mathematics and formal systems that do not treat discrete and continuous quantity as mutually irreducible cannot signify the unity in multiplicity of causality as *vestigia Trinitatis*.

In summary, irreducibly discrete quantity and irreducibly continuous quantity are required to signify identity-over-time, substance, persons, and causality because each of these exhibits true unity in multiplicity and requires a conception of quantity that can signify this unity in multiplicity.

Above, we have considered three different reasons why quantity that signifies the world exhibits unity in multiplicity and could thus be regarded as a *vestigium Trinitatis*. First, Augustine prominently references quantity, through the phrase "measure and number and weight," even referring to it as a trinity, though for him it is more faint than the *vestigium Trinitatis* of the inner man. Bonaventure extends the role that measure, number, and weight play in causal metaphysical principles, clearly highlighting these as *vestigia Trinitatis*. The close relationship between measure, number, and weight and irreducibly discrete and irreducibly continuous quantity that signifies the world, makes such quantity a good candidate for being regarded as a *vestigium Trinitatis*.

Second, unity in multiplicity is the quantitative corollary to Herman Bavinck's conception of the *vestigia Trinitatis* as unity in diversity. Unity in multiplicity is also

entailed by the unity in triadicity common to traditional conceptions of the *vestigia Trinitatis*. Claims of unity *in* multiplicity require two different kinds of unities, discrete unity and continuous unity so that the law of non-contradiction is not challenged when both are asserted. The fact that quantity that signifies the world does exhibit simultaneous unity and multiplicity, or unity in multiplicity, makes irreducibly discrete and irreducibly continuous quantity that signifies the world a good candidate for a *vestigium Trinitatis*.

Finally, how irreducibly discrete quantity and irreducibly continuous quantity signify key aspects of the traditional *vestigia Trinitatis* such as substance, persons, and causality was considered. Identity-over-time, substance, persons, and causality are concepts key to discussions of *vestigia Trinitatis*, and each involve the unity in multiplicity which requires signification by using both irreducibly discrete and irreducibly continuous quantity. Whether in the Augustinian formulation emphasizing persons and substance or in Bonaventure's or Baxter's formulation emphasizing three-fold causality, all of these notions of the *vestigia Trinitatis* require irreducibly discrete and irreducibly continuous quantity to signify them. If these aspects of reality, these *vestigia Trinitatis*, exhibit unity in multiplicity then quantity which signifies them must be able to exhibit unity in multiplicity as well.

This conclusion comports with Aristotle's conception that "in a sense unity means the same thing as being". [428] Thus, using formal systems of quantity to signify unity in multiplicity suggests claims about the nature of the entities under consideration. Are they

[428] Aristotle, *Metaphysics*, trans. Ross, 1054a. See Book X:2:9

one? Are they many? Or could they be unities-in-multiplicities? After replying to objections, the conclusion will evaluate whether these insights have ramifications for the foundational choices scientists and philosophers make when establishing their systems.

Reply to Objections

After this extensive review of the issues, the objections raised to the thesis in chapter two are substantial and deserve a direct answer. First, some object that continuous quantity can be reduced to discrete quantity, therefore continuous quantity should not be treated as an irreducible primitive. Second, others object that discrete quantity can be reduced to continuous quantity, therefore discrete quantity should not be treated as an irreducible primitive. And finally, some theologians deny that either of these reductions of quantity are relevant to the *vestigia Trinitatis*. Each of these objections will be answered in turn.

Reply to First Objection: Mathematics and Formal Systems Need More than just Discrete Quantity as an Irreducible Primitive in order to Signify the World

As examined in chapter two, a forceful objection to treating discrete and continuous quantity as mutually irreducible primitives is the claim that continuous quantity can be reduced to discrete quantity. Many phenomena, such as a black line drawn on a piece of white paper, look continuous, but under closer examination are full of skips and jumps. The famous 1665 *Micrographia* written by Robert Hooke detailed the results of microscopic examination of various things to make this very point, “the most curious and smoothly

engraven strokes and *points*, looking but as so many *furrows* and *holes*." [429] Nonetheless, does the observation of discrete aspects of a perceptually continuous phenomenon necessarily mean that continuous quantity is not needed in order to signify that phenomenon?

Consider three replies to the claim that continuous quantity can be reduced to discrete quantity in mathematics and formal systems. First, the Cantor-Dedekind axiom which identifies the real numbers with discrete points on a line is an unproven assumption to which an alternative can be conceived. Second, using only discrete quantity in mathematics results in a measurement problem which is acutely seen in discussions of applied mathematics. Third, using only discrete quantity in mathematics and formal systems to signify the world leads to a metaphysical bias which is not inherent in the phenomena.

First, consider whether the unproven nature of the Cantor-Dedekind axiom suggests that alternative conceptions of continuity which resist reduction to discreteness may be possible. As described in chapter two, the Cantor-Dedekind axiom is the identification of the discrete points on a line with discrete numbers or numerical magnitudes. The axiom assumes that a line is composed of discrete points, not composed of continuous magnitudes. Of this axiom Dedekind admitted that he was "unable to adduce any proof of its correctness, nor has anyone the power," but it is "by which we attribute to the line its continuity". [430] A critic of this axiom, Paul du Bois-Reymond, described why many

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[429] Robert Hooke, *Micrographia* (London: Martyn and Allestry, 1665), 3.

[430] Dedekind, "Continuity and Irrational Numbers," 771–72.

disagree with Dedekind. Du Bois-Reymond wrote in 1882, "The resting, solid idea of space will never create the image of a precise, uniform line from a series of dots, no matter how dense. Because points are dimensionless, and therefore an arbitrarily dense sequence of points can never become a distance, but the distance will always be viewed as the sum of the distances between the points." [431]

Du Bois-Reymond simply echoed what has been a consistent position in the philosophy of mathematics since the time of Euclid and Aristotle. Aristotle wrote, "no continuum can be made up of indivisibles, as for instance a line out of points." [432] Peirce said,

> In the calculus and theory of functions it is assumed that…such a collection of points [as corresponding to the real numbers] is called continuous. But this does not seem to be the common sense idea of continuity. It is only a collection of independent points. Breaking grains of sand more and more will only make the sand more broken. It will not weld the grains into unbroken continuity.[433]

For Aristotle, Du-Bois Reymond, and Peirce, discrete things do not become continuous by putting more of them next to each other. The Aristotelian alternative conception of a continuous line as being composed of subregions not points reveals that the Cantor-Dedekind axiom is not necessarily self-evident nor is it the only possible way to conceive of continuity. Peirce even describes it as not being "the common sense idea of continuity."

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[431] du Bois-Reymond, *Die allgemeine Functionentheorie*, 66. Translated by Jan Bentz. See alternate translation given in Ehrlich, *Real Numbers, Generalizations of the Reals, and Theories of Continua*, x.

[432] Aristotle, *Physics*, trans. Wicksteed and Cornford, 231a. See Book VI:1.

[433] Peirce, *Philosophy of Mathematics*, 139.

Alternative ideas of continuity have been recovered in the past decade by philosophers of math as shown in Hellman and Shapiro's books, *Varieties of Continua* and *History of Continua*. [434] Because there are other mathematically robust versions of continuity which have properties that differ from Dedekind-Cantor continuity, not all coherent conceptions of continuity can be reduced to discrete quantity.

Second, even Gottlob Frege, Bertrand Russell, and A.N. Whitehead seemed to have misgivings about the arithmeticization project which claimed to need only discrete quantity as an irreducible primitive. While Gottlob Frege is usually considered an arithmeticizer especially on account of his logicism, he does not follow either Cantor's nor Dedekind's derivation of the real numbers. Instead in his *Grundgesetze der Arithmetik* Frege develops the real numbers in a manner akin to Burali-Forti and Newton as ratios of magnitudes. "What in that case does a number-sign refer to on its own? Clearly a magnitude-ratio…If we now understand by 'number' the reference of a number-sign, then real number is the same as magnitude-ratio." [435] Similar to Bigelow and Pargetter, Frege says there are continuous magnitudes in nature, and when we put these magnitudes in ratios with each other, they yield the real numbers. "[R]eal number is the same as magnitude-ratio" and "a length can have the same ratio to another length, as a mass to another mass, as a light intensity to another light intensity." [436]

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[434] Hellman and Shapiro, *Varieties of Continua*. Shapiro and Hellman, *The History of Continua*.

[435] Frege, *Basic Laws of Arithmetic, Derived Using Concept-Script*, 2:84.

[436] Frege, *Basic Laws of Arithmetic, Derived Using Concept-Script*, 2:84.

To be clear Frege has a very strong notion of discrete number which he develops in a purely logical fashion. But in his *Grundgesetze* cited above he does not develop real numbers used for applied mathematics from his definition of discrete number. Instead he develops them separately just as Euclid did in Books V and VII.

Whitehead and Russell followed a similar path. In the remarkable letter that Whitehead wrote to Russell, Whitehead acknowledged, "The modern arithmeticization of mathematics is an entire mistake". [437] He describes the problem arithmeticization creates as a "limitation of proof" which "leaves the whole theory of applied mathematics (measurement etc.) unproved." [438] Using only discrete quantity as an irreducible primitive results in a loss of ability to signify the world—applied mathematics. Thus, if mathematics has no interest in signifying the world, then perhaps there is no need to prove applied mathematics. But if one wishes to use mathematics to signify the world, then mathematics needs continuous quantity as an irreducible primitive in addition to discrete quantity as Whitehead finally concluded. This led Russell and Whitehead to import the reasoning of Burali-Forti into their system by treating real numbers as ratios, though this is an aspect of their project that has garnered little attention. Even Russell himself does not seem to consider this accommodation of deep significance, since it is more than a decade later when

[437] Whitehead, "On Quantity—the Whole Theory of Applied Mathematics Is Unproved.," 1.

[438] Whitehead, "On Quantity—the Whole Theory of Applied Mathematics Is Unproved.," 1.

he writes, "I think the universe is all spots and jumps, without unity, without continuity…"[439]

The approach Russell and Whitehead were ultimately forced to use was the approach used by Euclid, by the 17th century algebraists, by Burali-Forti, and by Bigelow and Pargetter: numbers can be understood as ratios of magnitudes. Whitehead admits that for the conceptions of continuity and applied mathematics used in the final part of the *Principia Mathematica*, "our antecedents are Euclid and Burali-Forti." [440] Thus, the approach to continuity which undergirds measurement and allows mathematics to signify the world is closer to the Aristotelian-Euclidean approach to continuity than the Cantor-Dedekind approach. The Aristotelian-Euclidean approach to continuity cannot be reduced to discrete quantity since an Aristotelian continuum is not composed of points. Something is lost when trying to model an Aristotelian continuum as an arithmetic continuum. And this affects the ability of mathematics to signify the world as noted above by A.N. Whitehead. Thus, in order to signify the world mathematicians need to treat continuous quantity as irreducibly basic alongside of discrete quantity.

The second part of the first objection was that philosophical ideas involving apparent continuity can be signified fully using formal systems relying on discrete quantity alone. Bertrand Russell who popularized philosophical logic joined with set theory said of his

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[439] Russell, *The Scientific Outlook*, 98.

[440] Alfred North Whitehead, "Whitehead Makes Suggestions for the Preface to Volume 3 of Principia.," January 28, 1913, RA1, document number 057466, p. 2, Bertrand Russell Archives, McMaster University Libraries, https://bracers.mcmaster.ca/81685.

approach to logic and set theory, "my premisses are simply assumed; but the fact that they allow mathematics to be true, which most current philosophies do not, is surely a powerful argument in their favour." [441] Russell says his newfound approach to logic and set theory, which is grounded in discrete quantity alone, liberated him from his inability to "to construct any philosophy of arithmetic…". [442]

While Russell's assumptions in logic and set theory allow him to create a philosophy of discrete arithmetic which comports with his views, it also introduces atomistic assumptions into the philosophical topics he signifies with it. In particular the topics of identity-over-time, substance, and persons are tinged with atomistic assumptions when set theory is exclusively used in philosophical logic. Russell acknowledges that identity dissolves "into a fiction" and that substance should be "shut out" from the minds of contemporary philosophers.[443] So while Russell's philosophical logic claims to be able to designate these ideas, it so drastically defaces them that these persistent metaphysical concepts are essentially destroyed. Instead of wishing to signify a common world but differing on the best signs to use to accomplish the task, he claims that "there is little but prejudice and habit to be said for the view that there is a world at all." [444] Therefore formal

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[441] Bertrand Russell, *The Principles of Mathematics*, 2nd ed. (London: George Allen & Unwin, 1937), xviii.

[442] Russell, *The Principles of Mathematics*, 2nd ed., xviii.

[443] Russell, *An Outline of Philosophy*, 268–69.

[444] Russell, *The Scientific Outlook*, 98.

systems need both discrete and continuous quantity in order to signify persistent metaphysical concepts.

It might be contended that all formal systems destroy the metaphysical concepts of identity-over-time, substance, and persons. While there is some truth to this for both set theory and mereology, it is not a necessary result of formal systems. The Peirce-Schröder logic upheld a notion of the relative individual, which is only indivisible "from one perspective". Their logic also used appellations or "count-nouns" to offer a unit making property. The Peirce-Schröder approach allows a formalization of both discreteness and continuity within one formal system. This would allow Peirce to signify the reality of identity-over-time, actualized generals (analogous to substances), and persons in his formal system. None of these notions: persons, substances, or identity-over-time, are fictions for Peirce.

In another formal system, Kathrin Koslicki offers an example of how to augment the notion of identity in four-dimensionalist mereology. Through Koslicki's requirement of team-work-requiring capacities, she provides a way to establish a discrete formal principle in the continuous-dominated formal system of mereology. By joining together the discrete and the continuous in mereology she recovers a notion of substance as real and not a fiction. Establishing an account of substance then allows accounts of persons or identity-over-time to employ the concept of substance as a starting point. This shows that formal systems do not inherently need to disintegrate identity-over-time, substance, or persons into fictions. The treating of these concepts as fictions is a bias which is particularly strong in philosophical logic and set theory.

There is a *prima facie* defense available to set theorists accused of a nominalistic metaphysical bias by resorting to a claim that they are Platonists and thus realists of one sort or another. For example, Russell says, "I imagined all the numbers sitting in a row in a Platonic heaven." [445] But when Russell describes his realism he associates it with pluralism. "I should therefore prefer to call the philosophy which I advocate 'pluralism' rather than 'realism,'".[446] This statement reveals that Russell's Platonism is really his otherwise called logical atomism developed from the philosophy of G.E. Moore.[447] This philosophy blurs the distinction between the real and the ideal.[448] Traditional Platonists were realists and fierce foes of traditional atomists because Platonists believed in real transcendent forms

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[445] Russell, *My Philosophical Development*, 62.

[446] Bertrand Russell, "The Basis of Realism," *The Journal of Philosophy, Psychology and Scientific Methods* (New York) 8, no. 6 (1911): 160, https://doi.org/10.2307/2013237.

[447] "On fundamental questions of philosophy, my position, in all its chief features, is derived from Mr G. E. Moore. I have accepted from him the non-existential nature of propositions (except such as happen to assert existence) and their independence of any knowing mind; also the pluralism which regards the world, both that of existents and that of entities, as composed of an infinite number of mutually independent entities, with relations which are ultimate, and not reducible to adjectives of their terms or of the whole which these compose. Before learning these views from him, I found myself completely unable to construct any philosophy of arithmetic, whereas their acceptance brought about an immediate liberation from a large number of difficulties which I believe to be otherwise insuperable." Russell, *The Principles of Mathematics*, 2nd ed., xviii.

[448] "[A] true proposition is itself a piece of reality; it does not correspond to a fact, it is (the Moorean equivalent of) a fact. In Bradleyan terms, Moore denies the distinction between the real and the ideal." Peter Hylton, *Russell, Idealism, and the Emergence of Analytic Philosophy* (Oxford: Clarendon Press, 1990), 134.

which offered unity in multiplicity. Atomists believed these forms are fictions. Thus, Russell's atomistic realism is very different from Plato's realism.[449]

These reasons show that in order to signify the world fully, mathematics and formal systems need both discrete quantity and continuous quantity as mutually irreducible primitives. The unprovable nature of the Cantor-Dedekind axiom reveals that point-set continuity is only one version of continuity which is not equivalent to Aristotelian-Euclidean continuity. Frege develops the real numbers with magnitude-ratios. A.N. Whitehead and Bertrand Russell incorporate Euclid and Burali-Forti's conception of continuity which rejects the arithmeticization project at a late stage in their own writing. In doing this to establish applied mathematics and measurement, they show that significatory power is lost with the arithmeticization of all mathematics. Persistent metaphysical concepts such as identity-over-time, substance, and person are denounced as fictions in a discrete-quantity-only formal system. This occurs because such systems are unable to adequately signify unity in multiplicity, and this limitation introduces a metaphysical bias which should be avoided. On the other hand, quantity that signifies the world well is

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[449] Peter Hylton calls this remix of Platonism and atomism in Russell's thought "Platonic atomism". While Russell may consider himself a realist of a sort about numbers, he is certainly not a Platonic realist about metaphysics in general. In fact, many mathematical Platonists following Russell and Frege by objectifying the modern number concept, have created a metaphysics largely at odds with Plato's metaphysics. Thus, the moniker 'mathematical Platonist' does not mean realist in anything like the same sense meant when Peirce claims to defend "the doctrine of scholastic realism." The sort of realism endorsed by Russell and other mathematical Platonists dissolves identity-over-time, substance, and persons into fictions. Thus, the charge stands. The use of exclusively discrete quantity in philosophical logic leads to metaphysical bias. See Part II "Platonic Atomism" in Hylton, *Russell, Idealism, and the Emergence of Analytic Philosophy*.

irreducibly discrete and irreducibly continuous, exhibiting unity in multiplicity, and is thus a good candidate for a *vestigium Trinitatis*.

Reply to Second Objection: Mathematics and Formal Systems Need More than just Continuous Quantity as an Irreducible Primitive in order to Signify the World

A second major objection to treating discrete and continuous quantity as mutually irreducible primitives is the claim that mathematics and formal systems need only continuous quantity as an irreducible primitive. While it is true that discrete numbers can arise from ratios of continuous quantities, an additional element is added to the continuum when deriving discrete numbers from continua: thought. As Nicolas of Cusa said, "Mind alone counts; if mind is removed, distinct numbers do not exist. Because mind grasps one and the same thing individually and in signs." [450] The particular way that discrete numbers arise from continuous magnitudes shows that in order to have discreteness, additional thought is always included. If numbers are signs of mental conceptions, this extra thought is consistent with the ordinary process of numbering things.

First, though recent thinkers, such as Burali-Forti, derive discrete number from continuous magnitude, they need ratios, an aspect of thought, to do so. Second, mereology, a formal system emphasizing continuity, is not foundationally robust with regards to arithmetic without extra postulates. And third, the problem of reference in mereology can

[450] Nicolas of Cusa as cited by Reuben Hersh, *What Is Mathematics, Really?* (Oxford; New York: Oxford University Press, 1997), 108–9.

only be satisfactorily solved by the introduction of a discrete formal principle as is done with Kathrin Koslicki's TWR-capacities.

First consider the role of ratio, an aspect of thought, in developing discrete number from continuous magnitudes. Some thinkers in early modern algebra like Simon Stevin rejected the need for a distinct notion of discrete number. "Number is some thing in magnitude…" [451] Since algebra signifies the unity in multiplicity in the world in many sciences, this suggests that only continuous quantity, such as indicated in the variables x and y, is needed as a basic irreducible primitive to signify the world. But in response to this objection, consider that the early modern number concept is not merely based on continuous magnitudes. The early modern number concept is primarily based on ratios. These ratios involve discrete conceptions in thought.

Discrete numbers are not inherent in continua themselves. Continua plus discrete mental conceptions lead to discrete number. Mental conceptions are at the heart of how Schröder defines a number, "*A natural number is a sign that expresses how often a thing is conceived as repeatedly present*—whether already present in reality or yet to be or even just existing as an idea." [452] All it takes to get a counting number is a distinct conception "even just existing as an idea." Thus, the discreteness of the number "three" arising from the ratio "triple the length of" is neither in the property of the first length nor in the property

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[451] Stevin, "L'Arithmetique," 501.

[452] Schröder, *Ernst Schröder on Algebra and Logic*, 9.

of the compared length, but the discreteness of the number "three" arises on account of the mind's distinct conception of relating the two lengths together in one ratio.[453]

Consider the significance of counting in both ratios of discrete numbers and ratios of continuous magnitudes. First, consider counting and ratios of discrete quantity. Imagine a scenario in which a person walks around her block and counts the number of dogs and the number of cats owned by residents of her block. In order to count how many dogs there are, first she needs to know what a dog is. Is a German Shepherd a dog or a wolf? Is a

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[453] The question of whether this is a real relation or a mental relation alone is a chief concern debated in mathematics. Peirce distinguishes real relations and relations of reason, "[a] real relation subsists in virtue of a fact which would be totally impossible were either of the related objects destroyed; while a relation of reason subsists in virtue of two facts, only one of which would disappear on the annihilation of either of the relates". Leibniz describes a ratio as possibly an ideal or mental relation alone. "The ratio or proportion between two lines L and M may be conceived three several ways; as a ratio of the greater L to the lesser M; as a ratio of the lesser M to the greater L; and lastly, as something abstracted from both, that is, as the ratio between L and M, without considering which is the antecedent, or which the consequent; which the subject, and which the object. And thus it is that proportions are considered in music. In the first way of considering them, L the greater; in the second M the lesser is the subject of that accident which philosophers call relation. But which of them will be the subject, in the third way of considering, them? It cannot be said that both of them, L and M together, are the subject of such an accident; for if so, we should have an accident in two subjects, with one leg in one, and the other in the other; which is contrary to the notion of accidents. Therefore we must say that this relation, in this third way of considering it, is indeed out of the subjects; but being neither a substance, nor an accident, it must be a mere ideal thing, the consideration of which is nevertheless useful." And Russell opposes Leibniz notion of ideal relations, though he recognizes the influence of these sorts of relations in mathematics. "I found the doctrine of internal relations [as advanced by Gottfried Leibniz] particularly inapplicable in the case of 'asymmetrical' relations—i.e. relations which, if they hold between A and B, do not hold between B and A. Let us take again the relation earlier. If A is earlier than B, then B is not earlier than A. If you try to express the relation of A to B by means of adjectives of A and B, you will have to make the attempt by means of dates. You may say that the date of A is a property of A and the date of B is a property of B, but that will not help you because you will have to go on to say that the date of A is earlier than the date of B, so that you will have found no escape from the relation. If you adopt the plan of regarding the relation as a property of the whole composed of A and B, you are in a still worse predicament, for in that whole A and B have no order and therefore you cannot distinguish between 'A is earlier than B' and 'B is earlier than A'. As asymmetrical relations are essential in most parts of mathematics, this doctrine was important." Peirce, "A Guess at the Riddle," 1.365. Gottfried Wilhelm Leibniz, "Five Letters to Samuel Clarke, 1716," in *The Philosophical Works of Leibnitz*, ed. George M. Duncan (New Haven: Tuttle, Morehouse, and Taylor, 1890), 266–67, https://www.loc.gov/item/11019544/. Russell, *My Philosophical Development*, 57.

Chihuahua a dog or another small mammal? If she counts 13 dogs, then she disregards particularities of each dog, and understands them merely as dogs. Dog is a general concept, a count noun, a unit. She must have a different concept of a cat. Cats have different properties. These are mental conceptions of real individual things. Which are the key unit-making properties of a cat that distinguish cat from dog but unite all cats? If she counts 10 cats, she has depended on particular unit-making properties of those 10 cats. She can now make a report. There are 13 dogs and 10 cats on the block. There are more dogs on the block than cats. The ratio of dogs to cats is 1.3.

Next, consider counting and ratios of continuous quantity. Imagine how a ratio of 1.3 might arise from a ratio of continuous magnitudes instead of from a ratio of discrete numbers. An arborist fells two very straight trees, tree A and tree B, and removes their branches leaving only their trunks. When laid side by side it is clear that the second tree (tree B) is longer than the first tree (tree A). But how much longer? In order to get a ratio between the two lengths, first a common measure must be established; a unit making property is required. To get this unit, the arborist goes through an ancient mathematical process.[454]

First, the arborist cuts the excess length off tree B which extends beyond the length of tree A. This excess length is region C. Then the arborist lays region C next to tree A. It so happens that region C is smaller than tree A. By picking up region C and laying it down

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[454] Euclid, *The Thirteen Books of Euclid's Elements*, 2nd ed., rev. with additions., trans. Thomas Little Heath, with J. L. Heiberg (Cambridge: University Press, 1956), 10.2-3.

again end-to-end, he realizes that region C's length can fit three times into tree A's length. But then there is still some excess left over where tree A's length exceeds 3 lengths of region C. The leftover part of tree A's length is called region D. The arborist cuts off region D.

The arborist then repeats the process but with a slight difference. Now he checks to see if region D measures region C without excess. It does. Region D's length fits exactly three times into region C's length. This means that Region D can measure both tree A and tree B without excess. Tree B is recomposed (region C is put back at the top of the tree). Region D is used to measure tree B. 13 placements end-to-end of region D measure tree B perfectly. The remainder of tree A is measured by region D. Tree A (including the previously removed region D) measures exactly 10 placements end-to-end of region D. This is because region D's length is exactly 1/10 of tree A's length. The length of tree B is 13 lengths of region D; the length of tree A is 10 lengths of region D. Therefore the ratio of tree B to tree A is 13 D-lengths:10 D-lengths, or 13:10, or 1.3.

Compare the two ways of obtaining ratios above, a ratio of discrete quantities vs. the ratio of continuous quantities. In the second process, when laying the two trees next to each other the first impression the arborist obtains is not a number. The first impression that he obtains is that tree B is longer than tree A. This 'greater than' or 'less than' comparative is obvious with lengths. But 'how much' greater than or less than requires measuring. In order to measure the lengths, a common unit must be obtained (unless the objects are incommensurable or not co-measurable). The common unit or common measure for this example was region D. Once the common unit was found, then the number of times the

common unit was laid end-to-end to measure each tree could be counted. Only then could the ratio be established.

In the example with discrete quantity, the unit required for counting was a dog in one case or a cat in another, a count-noun or appellation. A ratio of 1.3 can be established by comparing two discrete quantities. And a ratio of 1.3 can be established by comparing two continuous magnitudes. But in either case, units are required to measure the quantities. The unit-making-property is only obtained with thought. Two magnitudes can be compared without being counted. The length of tree B is longer than the length of tree A. But to obtain a ratio, a discrete quantity—some sort of unit or measure—is required. Therefore, even though ratios can be obtained by comparing continuous magnitudes, they cannot be obtained without measuring units. And these measuring units require mental conceptions. These mental conceptions are what Schröder described as numbers, "how often a thing is conceived as repeatedly present". [455] Thus, it is false that one can obtain discrete numbers with only continuous magnitude without anything else. In order to obtain ratios, in addition to continuous magnitudes units or measures are needed which are signs of discrete mental conceptions.

The ratios which are the objects of thought in the early modern number concept can be ratios of discrete counting numbers (rational numbers), or these ratios may be ratios of magnitudes which are not commensurable (irrational numbers like π). While an irrational number like π does not equal a ratio of discrete numbers, it is still a sign. It points to a ratio

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[455] Schröder, *Ernst Schröder on Algebra and Logic*, 9.

between two different kinds of magnitudes. In the case of π, it points to a ratio between the circumference and the diameter of a circle.[456] In this way, ratios could be either ratios of discrete numbers like 1.3 or ratios of continuous magnitudes like π.

These ratios are worth studying and can offer a very interesting basis for all sorts of mathematical computation. There is no need to discard the manifold sorts of modern mathematics that today exist based off of ratio as a foundation. But these ratios are not the same as discrete number, and they create problems for signification if this point is not understood. This is why Aristotle and Euclid did not advance a theory that number is ratio (that is, number as the object of the general analytic). In order to count and measure to begin with, notions of discrete quantity and continuous quantity are both necessary. Identifying 4 students or 2 students requires a conception of a discrete thing—a concept word. Measuring requires a discrete number of repeated marks made with mental effort on a perceptually continuous ruler. A measure can only measure by a proportion or a ratio (5 cm to 1 cm, or 5 units to 1 unit). Thus it is impossible to develop these ratios (either measures or ratios of discrete quantity) without first having notions of discrete quantity. Therefore, more than continuous quantity is needed to signify the world.

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[456] The ratio π can have an upper bound which is a rational number and a lower bound which is a rational number. This practice of approximating irrational numbers like π and $\sqrt{2}$ has been used since ancient times. This practice parallels the Dedekind definition of an irrational number. Dedekind understood an irrational number to be a cut, which did not include the closest approximation to the right or the closest approximation to the left. Likewise, the ancients understood that irrational numbers like π were neither their lower bound approximation nor their upper bound approximation but a stable ratio of two continuous magnitudes which had no exact value as a ratio of discrete numbers.

Second, in order for mereology to become foundationally robust regarding arithmetic additional postulates, such as the inclusion of a singleton operator, are necessary. In a recent article Joel David Hamkins asks, "Can set-theoretic mereology serve as a foundation of mathematics?" [457] His initial responses is, no. "[W]e conclude that this form of set-theoretic mereology cannot by itself serve as a foundation of mathematics." [458] Hamkins argues, in a manner compatible with David Lewis' proposal in *Parts of Classes*, that there are strong parallels between mereology and a set theory excluding the membership operator, $\in$. Instead "set theoretic mereology" primarily uses the operator for subset, $\subseteq$, sometimes called the inclusion relation in set theory as an analog to the parthood relation in mereology. Nonetheless, without membership and with only the inclusion relation mereology is not robust regarding the foundations of mathematics. "The particular understanding of set-theoretic mereology via the inclusion relation $\subseteq$ cannot adequately serve by itself as a foundation of mathematics." [459] As noted in earlier chapters, continuity is more basic in the parthood relation and discreteness is more basic in the membership relation. The above conclusion from Hamkins reinforces the idea that mereology is not an adequate foundation for arithmetic without additional postulates.

On the other hand as argued by David Lewis in *Parts of Classes*, incorporating a singleton operator into mereology can supply set theoretic properties, even providing in an

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[457] Joel David Hamkins and Makoto Kikuchi, "Set-Theoretic Mereology," *arXiv.Org* (Ithaca), ahead of print, 2016, 1, https://doi.org/10.48550/arxiv.1601.06593.

[458] Hamkins and Kikuchi, "Set-Theoretic Mereology," 1.

[459] Hamkins and Kikuchi, "Set-Theoretic Mereology," 2.

otherwise mereological system a robust foundation for arithmetic and mathematics.[460] When mereology introduces singletons it quite elegantly allows the definition of a membership operator within mereology. This incorporation of a singleton operator into mereology is equivalent to establishing a discrete quantity operator into a continuous quantity system. It allows mereology now to treat both discrete quantity and continuous quantity as equally basic or as mutually irreducible primitives. Mereology with singletons could provide an approach to mathematical logic which can signify the world fully by means of discrete and continuous quantity. This need to augment mereology in order to signify arithmetical notions shows that more than continuous quantity is needed to signify the world.

Finally, the second objection also contends that mereology needs only continuous quantity to represent wholes. But while mereologists have sought to represent unities without having discreteness as a basic concept, they have struggled to do so. Classical extensional mereology as a formal system lacks a conception of discrete quantity as basic as just noted. It treats the property of continuous quantity (divisibility or parthood) as basic

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[460] "Theorem 13. Every model of membership-based set theory <V, ∈> is interdefinable with the corresponding singleton-expanded mereological model <V, ⊆, s>.

Proof. For the one direction, we can easily define ⊆ and the singleton operator s using ∈ as follows:

$$u \subseteq v \longleftrightarrow \forall x\,(x \in u \rightarrow x \in v)$$

$$y = s(x) \longleftrightarrow \forall z\,(z \in y \longleftrightarrow z = x).$$

Conversely, we may define ∈ from ⊆ and s via

$$x \in y \longleftrightarrow s(x) \subseteq y.$$

So the theorem is proved."

Hamkins and Kikuchi, "Set-Theoretic Mereology," 16.

but not the property of discrete quantity (indivisibility or membership). The name "mereology" accurately represents its subject matter, not the study of parts and wholes as is commonly claimed, but as its name indicates just the study of parts (Greek, *meros*). Unlike set theory whose set elements are discrete, in classical extensional mereology it is always possible to make more parts of a mereological part.

The lack of ability to describe true wholes means mereology has a problem of reference for wholes. In classical extensional mereology, the dog Lassie is not the same as the dog Lassie after a shave. These two things are different mereological sums and one cannot refer to the two sums as possessing the same identity. The problem of organic wholes and reference, or identity-over-time, cannot be solved without an appeal to a principle of discrete unity. This has led some philosophers such as Kathrin Koslicki to suggest adding other properties to mereology in order to describe true wholes. These properties are analogous to the traditional notions of discreteness, like Aristotle's conception of form. Thus, mereology needs additional discrete conceptions in order to adequately solve the problem of reference for wholes.

In addition, because wholes or mereological sums are merely the sum of their parts, any sum is a putative whole. But these mereological wholes do not seem to have the metaphysical properties that other wholes do. For example, most organic wholes when their parts are arbitrarily separated into two will generally lose their properties. For example, if a fish is divided in half, it soon loses sentience. But the properties of a mereological sum (or whole) depend only upon the properties of its parts. Thus the organic whole (the fish) does not seem to behave in the same way that a mereological whole does.

Mereology is largely associated with nominalism. Thus, realists who use it often must invoke another principle similar to Aristotelian form in order to secure the indivisibility of things. This Aristotelian form is a discrete conception added to the phenomenon, and it is this discrete conception from which the unity in mereology arises. Thus the formal system of mereology is not able to signify the world fully without conceptions of discreteness. Therefore, formal systems need more than just continuous quantity in order to signify the world.

Reply to Third Objection: Irreducibly Discrete and Irreducibly Continuous Quantity that Signifies the World can be Regarded as a *Vestigium Trinitatis*.

It is true that Augustine's notion of the *vestigia Trinitatis* focuses on a qualitative exploration of human persons and their capacities and that Augustine's exploration scarcely mentions anything mathematical. Nonetheless, this disregard does not mean that mathematical and logical conceptions of the *vestigia Trinitatis* are irrelevant to the doctrine. First, even if Augustine did not treat quantity in any way as a *vestigium Trinitatis*, that does not mean that others should disregard it as well. Second, Augustine does mention one aspect of the *vestigia Trinitatis* that relates to mathematics and to discrete and continuous quantity in particular: "measure, number, and weight".[461] And in the medieval period other theologians such as Bonaventure and Aquinas develop this theme prominently. Third, irreducibly discrete and irreducibly continuous quantity is implicit in many traditional

[461] Augustine of Hippo, *On the Holy Trinity*, 11.11.

notions of the *vestigia Trinitatis* such as persons, substance, and causality. All of these reasons make irreducibly discrete and irreducibly continuous quantity that signifies the world a good candidate for a *vestigium Trinitatis*.

First, even though Augustine disregarded quantity as a *vestigium Trinitatis*, the current conception of the *vestigia Trinitatis* relies on a long tradition including medieval thinkers like Thomas Aquinas and Bonaventure as well as modern thinkers like Herman Bavinck. Augustine does establish the trajectory for this doctrine, but his teaching does not necessarily constitute the final authority for the doctrine. These other thinkers contend that the non-human world as well as human persons exhibit *vestigia Trinitatis*. Since many theologians have held that non-human aspects of the world are *vestigia Trinitatis,* there is no reason to rule out *a priori* non-human aspects of the *vestigia Trinitatis* based solely on Augustine's authority. A theological practice like this if used broadly would rule out any post-Augustinian theological reflection altogether. Thus, while Augustine neglects the importance of mutually irreducible discrete and continuous quantity for the doctrine, his neglect does not mean that it is of no relevance for the doctrine. As the doctrine develops it indeed has further points of contact with discrete and continuous quantity.

Second, Augustine does highlight measure, number, and weight as an aspect of the *vestigia Trinitatis*. As described earlier in this chapter Augustine would likely be aware of the philosophical context for the terms measure, number, and weight. For example, he connects number and form an approach similar to that of Aristotle and Aquinas. Numbers at the time were understood to signify discrete quantity, while weights and measures could signify continuous quantity. Aquinas and Bonaventure both consider and deepen this Augustinian theme. Bonaventure especially foregrounds "measure, number, and weight" as

vestigia Trinitatis and explicitly connect mathematics and the Trinity.[462] Aquinas at various times in his writing highlights notions of discrete quantity as distinct from continuous quantity, linking discrete quantity with number and form as noted earlier. Thus, through their increased emphasis on measure, number, and weight Bonaventure and Aquinas elevate this Augustinian theme and recognize the significance of discrete quantity and continuous quantity for describing the world.

In addition, while Augustine does not consider measure, number, and weight as the most manifest *vestigia Trinitatis*, he does seem to think of these as some sort of *vestigia Trinitatis*. The word *vestigia* can also be translated as tracks or footprints. And just as some footprints are clear and distinct, other footprints are more faint. Augustine seems to rank the clarity of the *vestigia Trinitatis*. In Book IX he says "we found a kind of trinity, i.e. the mind, and the knowledge whereby the mind knows itself, and the love whereby it loves both itself and its knowledge of itself." But in Book X he says "we found in the mind a still more manifest trinity of the mind, viz., in memory, and understanding, and will." Crucially, he uses the phrase "more manifest trinity" for some *vestigia Trinitatis* over others. This suggests that all of the things he refers to as a trinity are tracks on the trail. But some tracks are fainter and others are more manifest. Thus, even though Augustine might not consider measure, number, and weight as among the more manifest *vestigia Trinitatis*, he does seem to consider them as traces along the way. In this same way irreducibly discrete quantity and

[462] Bonaventure, *Breviloquium*, 2.2. See also Bonaventure, *The Journey of the Soul into God. Itinerarium Mentis in Deum*, 1.11.

irreducibly continuous quantity that signify the world may not be the most manifest *vestigia Trinitatis*, but nonetheless they can still be regarded as traces of the Trinity in the Augustinian paradigm.

Third, irreducibly discrete and irreducibly continuous quantity is implicit in many traditional conceptions of the *vestigia Trinitatis* which exhibit unity in multiplicity, such as conceptions regarding persons, substance, and causality. That discrete and continuous quantity are implicit in the traditional *vestigia Trinitatis* is most easily seen by linking two observations. First, Aristotle said "in a sense unity means the same thing as being". [463] Second, as argued earlier the traditional *vestigia Trinitatis* have both irreducibly discrete and irreducibly continuous aspects thereby exhibiting unity in multiplicity. To say, as has been previously argued, that the *vestigia Trinitatis* have aspects that are both discrete and continuous therefore is to say two things simultaneously: 1) that the being of the *vestigia Trinitatis* is both discrete and continuous and 2) that the conceptions of unity invoked by the *vestigia Trinitatis* are both discrete and continuous. One signifies continuous unity with continuous quantity, one signifies discrete unities with discrete quantity. Consider a few examples from previous sections.

Essential to the conception of persons is the notion of personal identity-over-time. Peirce conceives of "assertoric truths" or individual experiences (such as Russell's sets of experiences) as points residing on the continuous line of identity. The discrete sets of experiences do not compose the persistent identity, but they reside in the persistent identity

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[463] Aristotle, *Metaphysics*, trans. Ross, 1054a. See Book X:2:9

in a manner similar to the way sense impressions reside in the continuity of memory. There are discrete and continuous aspects to this conception of personal identity-over-time and thus both discrete and continuous quantity are implicit in it.

Compare this Peircean notion of identity with Augustine's understanding of how the sense impression resides in memory. "[T]he image of the bodily object…is in the memory." These correlate to Peirce's individual experiences. Augustine considers, "the impression formed therefrom when the mind's eye of the thinker is turned to it…" [464] Augustine does not maintain that the memory of sense experiences composes the person's identity the way sets of experiences do for Russell. Instead the existential continuity of the mind by its will turns its attention to the memory. The memories better resemble the points residing upon on a continuous line of identity as described by Peirce, rather than the sets of discrete disunified experiences described by Russell. Why? The memories are discrete in one sense and yet residing as possibilities within the continuity of the mind in another sense. Augustine's conception of personal identity described here is also, like Peirce's, discrete in one way and continuous in another. Thus, both discrete and continuous quantity seem implicit in this traditional Augustinian conception.

Augustine also invokes the concept of substance regarding the *vestigia Trinitatis* in which notions of discrete and continuous quantity are implicit. He discovers "a kind of trinity, in three somewhats of one substance." [465] But substances also have persistent

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[464] Introduction to book 11 in Augustine of Hippo, *On the Holy Trinity*, 11.

[465] Introduction to book 11 in Augustine of Hippo, *On the Holy Trinity*, 11.

identity-over-time, thus also having irreducibly discrete and irreducibly continuous aspects as just considered. Substances also act as a continuous causal unities, though each substance is discrete from others. As Augustine says, "[y]ou are searching within creation for three somethings which can both be indicated separately and act inseparably." [466] While in one way a substance is a discrete entity, in another way a substance has potentially many continuous parts and team-work-requiring capacities and therefore is a continuous entity as argued in chapter three. The elimination of either irreducible continuity or irreducible discreteness obscures the unity in multiplicity of a substance and therefore obscures the *vestigia Trinitatis* in the world. Thus, mutually irreducible continuous and discrete quantity is implicit in the traditional notion of a substance.

For a last example, consider that the causal principles of Bonaventure regarding the *vestigia Trinitatis* also reflect unity in multiplicity and thus imply discrete and continuous quantity. As seen in chapter three Peirce argues that continuity as well as discreteness is needed in order to signify true causality. For example, in the principle of least action, there is a surprising continuity between efficient and final cause. How a particle can 'know' its end from the beginning in order to minimize the action to determine its path presents a metaphysical puzzle for physical scientists. While Hume claims that there is no 'real connexion' in causality and Russell argues that it can be signified simply with sequencing and functions, Peirce disagrees. These reasons and more cause Peirce to conclude that there is an essential 'thirdness' or continuity in causality which cannot be ignored, especially

[466] Augustine of Hippo, "Augustine of Hippo, Sermon 52."

between efficient and final cause. Therefore, since Bonaventure depends on the traditional metaphysical conceptions of efficient and final cause, irreducibly discrete quantity and irreducibly continuous quantity are implicit in these conceptions.

Thus, not only can irreducibly discrete and irreducibly continuous quantity that signifies the world be regarded as a *vestigium Trinitatis*, one could even say that this conception has always been implicit in the doctrine and is the likely reason why the phrase measure, number, and weight has so often recurred in it. Notions of persons, substance, or causality exhibiting unity in multiplicity might be easily dismissed as incoherent if the formal systems which express them use exclusively discrete quantity or exclusively continuous quantity. In such formal systems unity in multiplicity cannot be exhibited, and therefore phenomena such as the traditional *vestigia Trinitatis* which exhibit such unity in multiplicity cannot properly be signified. Thus, because the unity in multiplicity of persons, substance, and causality has always been a part of the *vestigia Trinitatis* tradition, irreducibly discrete and irreducibly continuous quantity has been a latent implicit aspect of the tradition.

To summarize, some hold that Augustine's conception of the *vestigia Trinitatis* is the final form of the doctrine, but that kind of thinking would invalidate all post-Augustinian theological reflection. There are more voices in the tradition of the *vestigia Trinitatis* including Bonaventure, Aquinas, and Bavinck. In addition, even to signify concepts crucial for Augustine, such as persons and substance, both discrete and continuous quantity are needed. Augustine himself seems to acknowledge this point in some way when claiming that all things possess "measure, number, and weight." Moreover, the traditional formulations of the *vestigia Trinitatis* found in persons, substance, and causal metaphysical

principles also imply discrete and continuous quantity. Thus, in order to conceptualize the various notions of unity in multiplicity represented in the traditional conceptions of the *vestigia Trinitatis*, both discrete quantity and continuous quantity are required. Therefore, irreducibly discrete and irreducibly continuous quantity that signifies the world does exhibit unity in multiplicity and thus should be regarded as a *vestigium Trinitatis*.

Chapter Five: Conclusion

This thesis has explored how the choice of foundational notions in mathematics and formal systems gives rise to unavoidable limitations on what kinds of beings mathematics and formal systems can signify, which in turn has implications for the discernment of the *vestigia Trinitatis* ('traces of the Trinity') in creation. In chapter two, I presented an argument for why mathematics and formal systems need only discrete quantity to signify the world, as when Bertrand Russell imagines a world of logical atoms that he wishes to mathematicise. I followed with the contrary argument for why only continuous quantity is needed in mathematics and formal systems to signify the world, as in early modern algebra and in mereology. In both cases, I used reasoning that has been persuasive for many contemporary mathematicians.

But in chapter three, drawing from the thought of Mary Hesse I argued that things in the world as signified by our best scientific theories are, at least perceptually, neither exclusively discrete nor exclusively continuous. I also drew from the thought of C.S. Peirce to argue that things in the world as signified by persistent metaphysical concepts such as substance, persons, and causality are neither exclusively discrete nor exclusively continuous. The perspectives of these two thinkers reveal that while exclusively discrete quantity in a formal system can provide an important account of reality, it cannot present a complete univocal account of reality. Exclusively discrete quantity cannot provide a complete univocal account of reality because discrete quantity cannot signify adequately

the aspects of reality which are either perceptually or actually continuous.[467] Similarly, exclusively continuous quantity cannot present a complete univocal account of reality, because systems relying on continuous quantity alone cannot signify adequately those aspects of reality which are perceptually and perhaps actually discrete. In practice, both discrete quantity and continuous quantity are needed by mathematics and formal systems to signify a world that has both unitary and discrete characteristics, a reality which recalls Poincaré's "unity in multiplicity". [468] Thus, in order to signify fully the unity in multiplicity of the world, both discrete quantity and continuous quantity are needed by mathematics and formal systems. One consequence of this, I have argued, is that the use of both irreducibly discrete and irreducibly continuous quantity in contemporary mathematics and formal systems is consistent with Aristotle's conception of quantitative signification as analogical.[469]

These lessons in the philosophy of mathematics have important implications for any attempt to describe a world which reveals *vestigia Trinitatis*. The most obvious limitation

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[467] Aristotle warned against those philosophers that treat the meaning of quantity univocally "ignoring the different senses of 'one' or of 'being." On account of ignoring the different senses of one, Aristotle's contemporaries thought "'one' and 'many' were used in the sense in which they contradict each other…" But because there are both discrete senses of unity and continuous senses of unity mathematics can say, "the fact is that one thing obviously can be many". Aristotle, *Physics*, trans. Wicksteed and Cornford, 185b27-186a4. See Book I:2

[468] Poincaré, *Science and Hypothesis*, 23.

[469] The arithmetician considers, "man, *qua* man, [as] one indivisible thing". But "the geometrician considers man neither *qua* man nor *qua* indivisible, but *qua* something solid." Man is like an idealized hypothetical discrete entity, and man is like an idealized hypothetical continuous entity. Neither idealization signifies all the quantitative characteristics of man. Each idealization can only quantitatively signify man analogically. Aristotle, *Metaphysics*, trans. Hugh Tredennick and G. Cyril Armstrong, 1078a23-28. See Book XIII:3:9

that the above insights from the philosophy of mathematics reveal is that an intrinsic gulf exists between mathematical statements about reality and a complete account of reality. This conclusion underpins how one should expect an even greater gulf to exist between any purported *vestigia Trinitatis* in mathematics and statements about the Trinity *per se*, the immanent Trinity. This apophaticism comports with the broader theological tradition according to which accounts of the Trinity *per se* are obtained only on authority from God's self-revelation as discussed in chapter one. Much of the history of Trinitarian theology consists in attempts to rationalize the Trinity by reducing the Godhead to one or another human conception in opposition to God's self-revelation. In both mathematics and in Trinitarian theology a careful exploration of the foundations reveals the need for both an apophatic dimension and epistemic humility. Indeed, the fact that this humility is needed in mathematics suggests it should be at least as important in Trinitarian theology.

This project also makes a positive contribution to the doctrine of the *vestigia Trinitatis* in two ways. First, it has considered how unity in multiplicity can itself be considered as a kind of *vestigia Trinitatis.* An argument for the Trinity from its traces is an abductive argument for the reasonableness of a cause from its effects. This sort of argument is distinct from an argument from analogy. Thus the *vestigia Trinitatis* tradition is not concerned that every trace of the Trinity be a likeness. A footprint as an impression or trace reveals only a faint hint of the creature that made it. While those things that exhibit unity in triadicity may offer analogies and likenesses of the Trinity, the unity in multiplicity of the world explored in this thesis is merely claimed to be a trace of the Trinity.

Nonetheless, as a sign or trace the irreducible, non-finite, and unitary nature of the continuum can point to the irreducible, non-finite, and unitary nature of the Trinity, where

there is no distinction between 'here' and 'there' or 'now' and 'then', consistent with the doctrine of divine simplicity. And there is still one, and only one, irreducible distinction in God acknowledged by traditional Trinitarian theology, namely the distinction of persons. Thus, while the distinction between the persons of the Godhead is unlike any distinction in creation, nonetheless in God's self-revelation it is revealed as a true distinction. The Father is not the Son, nor are either the Father or the Son the Spirit.[470] The irreducibility of discrete quantity in mathematics allows for discussions of distinct things, distinctions which cannot merely be subsumed as an aspect of a continuous unity. And hence, discrete quantity in mathematical creation is consistent with the reality of the distinction between the persons of the Creator God. Mathematics describes an ontology without contradiction involving both kinds of mutually irreducible realities, what one could call a unity (the continuum) in multiplicity (the discrete points). Trinitarian theology, like mathematics, describes a unity in multiplicity without contradiction: one substance and three persons. Thus, irreducibly discrete and irreducibly continuous quantity that signifies the world and exhibits unity in multiplicity may indeed serve positively as a *vestigium Trinitatis*.[471]

Second, this project contributes positively to the doctrine of the *vestigia Trinitatis* by defending traditional formulations of the *vestigia Trinitatis* from philosophical attacks based on arguments from mathematics. Not all of the worlds that mathematicians imagine

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[470] Aquinas refers to the persons of the Trinity as "subsistent relations". "But in God the relations themselves are the persons subsisting in one nature." Aquinas, *Summa Theologica*, 1.42.3ad2.

[471] Indeed, these *vestigia Trinitatis* may be more expressive of these characteristics of the Godhead than anything else of which we are yet aware in the created order, except perhaps the wave-particle duality of light, one of the privileged metaphors of God in scripture (1 John 1:5).

and signify with mathematics are compatible with traditional Christian metaphysical categories. For example, Bertrand Russell dismisses personal identity as a metaphysical fiction and questions whether there is even a "world at all" to signify.[472] Nonetheless, traditional metaphysical categories such as personal identity and causality are crucial to traditional formulations of the doctrine of the *vestigia Trinitatis* as described earlier. Augustine, for example, emphasizes human persons with a persistent personal identity in his doctrine of the *vestigia Trinitatis*, and Bonaventure and Richard Baxter emphasize a threefold causality in their doctrines of the *vestigia Trinitatis*. But as shown in chapter three, quantity that adequately signifies personal identity and causality is irreducibly discrete and irreducibly continuous. Mathematics and formal systems which do not include both discrete quantity and continuous quantity will find it difficult to signify these traditional doctrines of the *vestigia Trinitatis* and, perhaps more seriously, will obstruct the human imagination in attempts to embrace all aspects of Trinitarian revelation. Instead, these doctrines may be condemned as irrelevant and obscure. If different sorts of mathematics carry different metaphysical implications when used to signify the world, then Christians should consider carefully foundational choices in their systems when signifying the world through mathematics and logic.

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[472] "Indeed, there is little but prejudice and habit to be said for the view that there is a world at all." Russell, *The Scientific Outlook*, 98.

I have offered an apophatic and a positive case relating mathematics and theology which have been mutually reinforcing.[473] The Trinity is a mystery of faith, but the doctrine is confessed as true by Christians as being consistent with the content of God's self-revelation. Similarly, as one studies the mathematical sciences finer distinctions emerge even as logical tensions remain. Neither in mathematics nor in theology should what is commonly termed mystery suggest obfuscation but should instead indicate ever-further depths to be sounded. The tradition of the *vestigia Trinitatis* reveals how the truth of a doctrine and its mystery can be a rich and generative source of thought. Truth does not illumine by dispelling mystery; truth illumines best by embracing mystery. This exploration of the *vestigia Trinitatis* has been a small attempt to embrace twin notions that are often perceived as contradictory because of inadequate metaphysical frameworks but in fact have deep unity in creation and in the Christian understanding of the Godhead.

[473] Karen Kilby who related mathematics and theology makes a similar argument along the lines of "clarity and mystery" Karen Kilby, "Beauty and Mystery in Mathematics and Theology," in *God, Evil and the Limits of Theology* (London: Bloomsbury T&T Clark, 2020), 140.

Bibliography

Anellis, Irving H. "How Peircean Was the "Fregean' Revolution" in Logic?" *Logicheskie Issledovaniya* 18 (2012). https://cspeirce.iupui.edu/menu/library/aboutcsp/ANELLIS/csp-frege-revolu.pdf.

Aquinas, Thomas. *Commentary on Aristotle's Physics*. Translated by Richard J. Blackwell and Richard J. Spath. With W. Edmund Thirlkel. Rare Masterpieces of Philosophy and Science. London: Routledge & Kegan Paul, 1963.

———. *Summa Theologica*. Translated by Fathers of the English Dominican Province. London: T. Baker, 1920.

———. *The Disputed Questions on Truth*. Translated by Robert W. Mulligan. Library of Living Catholic Thought. Chicago: Henry Regnery, 1952.

Aristotle. "Categories." In *The categories: On interpretation*, translated by Harold P. Cooke. Loeb Library. Cambridge, Mass: Harvard University Press, 1938.

———. "De Interpretatione." In *The Complete Works of Aristotle: The Revised Oxford Translation*, vol. 1, translated by J. L. Ackrill, with Jonathan Barnes. Bollingen Series 71. Princeton ; Guildford: Princeton University Press, 1984.

———. *Metaphysics*. Translated by Hugh Tredennick and G. Cyril Armstrong. Loeb Library. Cambridge, Mass.: Harvard University Press, 1935.

———. *Metaphysics*. Translated by W. D. Ross. Oxford scholarly editions online. Oxford: University Press, 2020.

———. "Parts of Animals." In *Complete Works of Aristotle, Volume 1*, vol. 192, translated by William Ogle, with Jonathan Barnes, 994–1086. Princeton: University Press, 2019. https://doi.org/10.1515/9781400835843-025.

———. "Physics." In *The Complete Works of Aristotle: The Revised Oxford Translation*, vol. 1, translated by Jonathan Barnes. Bollingen Series 71. Princeton ; Guildford: Princeton University Press, 1984.

———. *Physics*. Translated by Philip H. Wicksteed and Francis Macdonald Cornford. With Harvard University Press. Loeb Classical Library 228. Cambridge, MA: Harvard University Press, 1934.

———. "Posterior Analytics." In *Posterior Analytics Topica*, translated by Hugh Tredennick, with Aristotle, E. S. Forster, and Harvard University Press. Loeb Classical Library. Cambridge, Mass: Harvard University Press, 2014.

Asenjo, F. G. "A Calculus of Antinomies." *Notre Dame Journal of Formal Logic* (Notre Dame, Ind) 7, no. 1 (1966): 103–5. https://doi.org/10.1305/ndjfl/1093958482.

Augustine of Hippo. "Augustine of Hippo, Sermon 52." In *The Cambridge Edition of Early Christian Writings: Volume 1: God*, vol. 1, edited by Andrew Radde-Gallwitz,

translated by Mark DelCogliano, 309–10. The Cambridge Edition of Early Christian Writings. Cambridge: Cambridge University Press, 2017. https://doi.org/10.1017/9781107449596.023.

———. *Augustine: On the Trinity Books 8-15*. Edited by Gareth B. Matthews. Translated by Stephen McKenna. Cambridge Texts in the History of Philosophy. Cambridge: Cambridge University Press, 2002. https://doi.org/10.1017/CBO9781139164658.

———. *Confessions*. Translated by William Watts. Loeb Classical Library 27. Cambridge, MA: Harvard University Press, 2014.

———. *De Trinitate*. Vol. 42, edited by J. P. Migne. Patrologia Latina. France, 1864.

———. *On Christian Doctrine*. Edited by Philip Schaff. Translated by J. F. Shaw. A Select Library of the Nicene and Post-Nicene Church Fathers. Grand Rapids, Michigan: Wm. B. Eerdmans Publishing Company, 1978.

———. "On Music." In *The Immortality of the Soul: The Magnitude of the Soul ; On Music ; The Advantage of Believing ; On Faith in Things Unseen*, translated by Robert Catesby Taliaferro. New York: Catholic University of America Press, 1947.

———. *On the Free Choice of the Will*. Translated by Peter King. Cambridge: Cambridge University Press, 2010.

———. *On the Holy Trinity*. Edited by Philip Schaff. Translated by Arthur W. Haddan. A Select Library of the Nicene and Post-Nicene Church Fathers. Grand Rapids, Michigan: Wm. B. Eerdmans Publishing Company, 1978.

———. *The Happy Life*. 1st ed. Vol. 5, translated by Ludwig Schopp. The Fathers of the Church. New York: CIMA Pub, 2008.

———. "The Literal Meaning of Genesis." In *On Genesis: On Genesis: A Refutation of the Manichees, Unfinished Literal Commentary on Genesis, the Literal Meaning of Genesis*, with Edmund Hill, John E. Rotelle, of Hippo Augustine, and Augustinian Heritage Institute. Works of Saint Augustine. Hyde Park, New York: New City Press, 2005.

Ayres, Lewis. *Augustine and the Trinity*. With Cambridge University Press. Cambridge Core. Cambridge: University Press, 2010.

Baggott, Jim. *The Quantum Story: A History in 40 Moments*. 1st edition. Oxford: Oxford University Press, 2011.

Bavinck, Herman. *Reformed Dogmatics. Volume 2, God and Creation [Electronic Resource]*. With John Bolt and John Vriend. Grand Rapids, Michigan: Baker Academic, 2015.

Baxter, Richard. *Methodus theologiae Christianæ*. London: White and Snowden, 1681. http://archive.org/details/bim_early-english-books-1641-1700_methodus-theologiae-chr_baxter-richard_1681.

———. *The Reasons of the Christian Religion*. London: R. White, 1667.

Bell, J. S. “Six Possible Worlds of Quantum Mechanics.” In *Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy*, 2nd ed., 181–95. Cambridge: Cambridge University Press, 2004. https://doi.org/10.1017/CBO9780511815676.022.

Bellucci, Francesco. “‘Logic, Considered as Semeiotic’: On Peirce’s Philosophy of Logic.” *Transactions of the Charles S. Peirce Society* 50, no. 4 (2014): 523–47. https://doi.org/10.2979/trancharpeirsoc.50.4.523.

Bergmann, Gustav. “Frege’s Hidden Nominalism.” *The Philosophical Review* 67, no. 4 (1958): 437–59. https://doi.org/10.2307/2182943.

Bigelow, J., and R. Pargetter. “Quantities.” *Philosophical Studies* (Heidelberg) 54, no. 3 (1988): 287–304. https://doi.org/10.1007/BF00646273.

Bois-Reymond, Paul du. *Die allgemeine Functionentheorie*. Tubingen: Verlag, 1882. http://archive.org/details/dieallgemeinefu00boisgoog.

Bonaventure. *Breviloquium*. Translated by Dominic Monti. Works of St. Bonaventure 9. St. Bonaventure, New York: Franciscan Institute Publications, 2005.

———. *The Journey of the Soul into God. Itinerarium Mentis in Deum*. Translated by Zachary Hayes. Works of St. Bonaventure ; II. St. Bonaventure, New York: Franciscan Institute Publications, 2002.

Bondoni, Davide. “Structural Features in Ernst Schröder’s Work. I.” *Logic and Logical Philosophy* 20 (January 2011). https://doi.org/10.12775/LLP.2012.014.

Bowin, John. “Aristotle on Identity and Persistence.” *Apeiron (Clayton)* (Edmonton, AB) 41, no. 1 (2008): 63–88. https://doi.org/10.1515/APEIRON.2008.41.1.63.

Brachtendorf, Johannes. “Augustine on Self-Knowledge and Human Subjectivity.” In *Self-Knowledge: A History*, edited by Ursula Renz, 0. Oxford University Press, 2017. https://doi.org/10.1093/acprof:oso/9780190226411.003.0007.

Buchanan, Emerson. *Aristotle’s Theory of Being*. Greek, Roman, and Byzantine Monographs ; No. 2. University, Mississippi, 1962.

Burali-Forti, C. “Les Propriétés Formales Des Opérations Logiques.” Translated by Sébastien Gandon. *Rivista Di Matematica* 6 (1899): 141–77.

Cantor, Georg. “Cantor’s Grundlagen.” In *From Kant to Hilbert Volume 2*, edited by William Bragg Ewald. Oxford, United Kingdom: Oxford University Press, 2007. http://ebookcentral.proquest.com/lib/oxford/detail.action?docID=422427.

———. “Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen.” *Mathematische annalen* 5, no. 1 (1872): 123–32. https://doi.org/10.1007/BF01446327.

Cassiodorus. “Institutions of Divine and Secular Learning.” In *Cassiodorus: “Institutions of Divine and Secular Learning” and “On the Soul,”* translated by James W. Halporn and Barbara Halporn. Liverpool: Liverpool University Press, 2011.

Champollion, Lucas, and Manfred Krifka. “Mereology.” In *The Cambridge Handbook of Formal Semantics*, edited by Maria Aloni and Paul Dekker, 369–88. Cambridge Handbooks in Language and Linguistics. Cambridge: Cambridge University Press, 2016. https://doi.org/10.1017/CBO9781139236157.014.

Clarke, W. Norris. *The One and the Many: A Contemporary Thomistic Metaphysics*. Notre Dame, Ind.: University of Notre Dame Press, 2001.

Clement, Kevin C. “Russell’s Paradox.” In *Internet Encyclopedia of Philosophy*. Accessed November 15, 2024. https://iep.utm.edu/par-russ/.

Dedekind, Richard. “Continuity and Irrational Numbers.” In *From Kant to Hilbert Volume 2*, translated by William Bragg Ewald. Oxford, United Kingdom: Oxford University Press, 2007. http://ebookcentral.proquest.com/lib/oxford/detail.action?docID=422427.

———. *Essays on the Theory of Numbers*. Chicago: Open Court Publishing Co, 1901.

Deely, John N. *Four Ages of Understanding: The First Postmodern Survey of Philosophy from Ancient Times to the Turn of the Twenty-First Century*. Toronto Studies in Semiotics. Toronto, Ontario: University of Toronto Press, 2001.

Dipert, Randall. “Individuals and Extensional Logic in Schröder’s Vorlesungen Über Die Algebra Der Logik.” *The Review of Modern Logic* 1 (January 1991).

Donne, John. *The Oxford Edition of the Sermons of John Donne*. With Peter McCullough, David Colclough, Katrin Ettenhuber, and Mary Ann Lund. Oxford: University Press, 2013.

Eco, Umberto. “Latratus Canis.” *Tijdschrift Voor Filosofie* 47, no. 1 (1985): 3–14.

Eglinton, James Perman. *Trinity and Organism: Towards a New Reading of Herman Bavinck’s Organic Motif*. T & T Clark Studies in Systematic Theology. London: T & T Clark, 2012.

Ehrlich, Philip. *Real Numbers, Generalizations of the Reals, and Theories of Continua*. Synthese Library ; v. 242. Dordrecht ; London: Kluwer Academic, 1994.

Einstein, Albert. “Maxwell’s Influence on the Development of the Conception of Physical Reality.” In *A Dynamical Theory of the Electromagnetic Field*, edited by Thomas F. Torrance, 29–32. Eugene, OR: Wipf & Stock, 1996.

Euclid. *The Thirteen Books of Euclid’s Elements*. 2nd ed., rev. With additions. Translated by Thomas Little Heath. With J. L. Heiberg. Cambridge: University Press, 1956.

Euler, Leonhard. “On the Principle of Least Action.” In *Opera Omnia Leonhard Euler*, vol. 5, translated by Michael Saclolo, 179–93. 2. 1753. https://scholarlycommons.pacific.edu/euler-works/198/.

Ewald, William Bragg, Jr., and William Bragg Ewald. *From Kant to Hilbert Volume 2*. Oxford, United Kingdom: Oxford University Press, 2007. http://ebookcentral.proquest.com/lib/oxford/detail.action?docID=422427.

Fedoseev, Vitaly, Hanzhen Lin, Yu-Kun Lu, Yoo Kyung Lee, Jiahao Lyu, and Wolfgang Ketterle. “Coherent and Incoherent Light Scattering by Single-Atom Wave Packets.” *Physical Review Letters* 135, no. 4 (July 2025): 043601. https://doi.org/10.1103/zwhd-1k2t.

Fisch, Max H. “Peirce’s Arisbe: The Greek Influence in His Later Philosophy.” *Transactions of the Charles S. Peirce Society* (Buffalo, N. Y) 7, no. 4 (1971): 187–210.

Fisch, Max, and Atwell Turquette. “Peirce’s Triadic Logic.” *Transactions of the Charles S. Peirce Society* (Buffalo, N. Y) 2, no. 2 (1966): 71–85.

Fisher, George Park. “The Theology of Richard Baxter.” *Bibliotecha Sacra* 9, no. 33 (January 1852): 135–69.

Franklin, James. *An Aristotelian Realist Philosophy of Mathematics: Mathematics as the Science of Quantity and Structure*. London: Palgrave Macmillan UK, 2014.

Frege, Gottlob. “A Critical Elucidation of Some Points in E. Schroeder’s Vorlesungen Ueber Die Algebra Der Logik.” In *Translations from the Philosophical Writings of Gottlob Frege*, 3rd ed., edited by P. T. Geach and Max Black. Oxford: Basil Blackwell, 1980.

———. *Basic Laws of Arithmetic, Derived Using Concept-Script: Volumes I & II*. Vol. 2, translated by Philip A. Ebert and Marcus Rossberg, with Crispin Wright. Oxford, U.K.: Oxford University Press, 2013.

Gandon, Sébastien. “On Quantity and Number in Principia Mathematica: A Plea for an Ontological Interpretation of the Application Constraint.” In *The Palgrave Centenary Companion to Principia Mathematica*, 413–34. History of Analytic Philosophy. London: Palgrave Macmillan UK, n.d. https://doi.org/10.1057/9781137344632_15.

———. “Which Arithmeticisation for Which Logicism? Russell on Relations and Quantities in The Principles of Mathematics.” *History and Philosophy of Logic* 29, no. 1 (2008): 1–30.

Grattan-Guinness, Ivor. *Landmark Writings in Western Mathematics 1640-1940*. Chantilly, Netherlands: Elsevier Science & Technology, 2005. http://ebookcentral.proquest.com/lib/oxford/detail.action?docID=269620.

Gregory of Nyssa. “On ‘Not Three Gods.’” In *Nicene and Post-Nicene Fathers: Gregory of Nyssa: Dogmatic Treatises, Etc*, edited by Philip Schaff and Henry Wace, translated by H. A. Wilson. Hendrickson Publishers, 1995.

Haffner, Emmylou, and Dirk Schlimm. “Dedekind on Continuity.” In *The History of Continua: Philosophical and Mathematical Perspectives*, edited by Stewart Shapiro and Geoffrey Hellman, 0. Oxford University Press, 2020. https://doi.org/10.1093/oso/9780198809647.003.0011.

Hamkins, Joel David, and Makoto Kikuchi. “Set-Theoretic Mereology.” *arXiv.Org* (Ithaca), ahead of print, 2016. https://doi.org/10.48550/arxiv.1601.06593.

Hardy, G. H. *A Mathematician’s Apology*. Canto Classics. Cambridge: Cambridge University Press, 2012.

Harris, Mark. “Quantum Fundamentalism and Theological Liberty.” *Zygon: Journal of Religion and Science* 59, no. 3 (October 2024). https://doi.org/10.16995/zygon.17702.

Havenel, Jérôme. “Peirce’s Clarifications of Continuity.” *Transactions of the Charles S. Peirce Society* (Bloomington, IN) 44, no. 1 (2008): 86–133.

Hecht, Eugene. “There Is No Really Good Definition of Mass.” *The Physics Teacher* 44, no. 1 (January 2006): 40–45. https://doi.org/10.1119/1.2150758.

Hellman, Geoffrey. “Mereology in the Philosophy of Mathematics.” In *Handbook of Mereology*, edited by Stamatios Gerogiorgakis, Guido Imaguire, Johanna Seibt, and Hans Burkhardt. Analytica. München: Philosophia, 2017.

Hellman, Geoffrey, and Stewart Shapiro. *Varieties of Continua: From Regions to Points and Back*. Oxford, New York: Oxford University Press, 2018.

Hersh, Reuben. *What Is Mathematics, Really?* Oxford; New York: Oxford University Press, 1997.

Hesse, Mary B. *Forces and Fields. The Concept of Action at a Distance in the History of Physics.* Nelson: London, 1961.

———. *Models and Analogies in Science*. 2. print. Philosophy. Notre Dame, Ind: Univ. of Notre Dame Press, 1970.

Hill, Claire Ortiz. *Word and Object in Husserl, Frege and Russell: Roots of Twentieth-Century Philosophy*. Series in Continental Thought. Athens, Ohio: Ohio University Press, 1991.

Hooke, Robert. *Micrographia*. London: Martyn and Allestry, 1665.

Hopkins, Burt C. “Husserl and Jacob Klein.” *The European Legacy, toward New Paradigms* 21, nos. 5–6 (2016): 5–6. https://doi.org/10.1080/10848770.2016.1173480.

Houser, Nathan. “The Schröder-Peirce Correspondence.” *Modern Logic* 1, nos. 2–3 (January 1991): 206–36.

Hulswit, Menno. “Peirce’s Teleological Approach to Natural Classes.” *Transactions of the Charles S. Peirce Society* 33, no. 3 (1997): 3.

Hume, David. *David Hume: A Treatise of Human Nature*. Second edition. With L. A. Selby-Bigge and P. H. Nidditch. Clarendon Edition of the Works of David Hume. Oxford: University Press, 2014.

Hylton, Peter. *Russell, Idealism, and the Emergence of Analytic Philosophy*. Oxford: Clarendon Press, 1990.

Jain, Ravi Scott. “Science and Theology as Abductive Responses to Reality.” *Science and Christian Belief* 35, no. 2 (October 2023): 158–65.

Jammer, Max. *Concepts of Mass: In Classical and Modern Physics*. Cambridge, Mass: Harvard University Press, 1961.

Kanamori, Akihiro. “Cantor and Continuity.” In *The History of Continua: Philosophical and Mathematical Perspectives*, edited by Stewart Shapiro and Geoffrey Hellman, 0. Oxford University Press, 2020. https://doi.org/10.1093/oso/9780198809647.003.0010.

Karpenko, A. S. “Paraconsistent Structure inside of Many-Valued Logic.” *Synthese* 66, no. 1 (1986): 63–69.

Kilby, Karen. “Beauty and Mystery in Mathematics and Theology.” In *God, Evil and the Limits of Theology*, 139–54. London: Bloomsbury T&T Clark, 2020.

Klein, Jacob. *Greek Mathematical Thought and the Origin of Algebra*. 1st edition. Translated by Eva Brann and J. Winfree Smith. With Vieta. M.I.T. Press, 1968.

Klinck, Dennis R. “Vestigia Trinitatis in Man and His Works in the English Renaissance.” *Journal of the History of Ideas* (Philadelphia, Pa., etc) 42, no. 1 (1981): 13–27. https://doi.org/10.2307/2709415.

Koslicki, Kathrin. *Form, Matter, Substance*. First edition. Oxford Scholarship Online. Oxford: University Press, 2018.

Lane, Robert. “Peirce’s Triadic Logic Revisited.” *Transactions of the Charles S. Peirce Society* (Buffalo, NY) 35, no. 2 (1999): 2.

Laudan, Larry. “Aim-Less Epistemology?” *Studies in History and Philosophy of Science. Part A* (Oxford) 21, no. 2 (1990): 315–22. https://doi.org/10.1016/0039-3681(90)90029-8.

Leibniz, Gottfried Wilhelm. “Five Letters to Samuel Clarke, 1716.” In *The Philosophical Works of Leibnitz*, edited by George M. Duncan. New Haven: Tuttle, Morehouse, and Taylor, 1890. https://www.loc.gov/item/11019544/.

———. *The Monadology and Other Philosophical Writings*. Translated by Robert Latta. Oxford: Clarendon Press, 1898. http://archive.org/details/monadologyandot01lattgoog.

Leśniewski, Stanisław, Stanisław J. Surma, Jan T. S. Srzednicki, and D. I. Barnett. *Collected Works*. Vol. 1. Nijhoff International Philosophy Series. Dordrecht: Kluwer Academic Publishers, 1992.

Lewis, David K. “Many, but Almost One.” In *Ontology, Causality and Mind: Essays in Honor of D.M. Armstrong*, edited by John Bacon, Keith Campbell, Lloyd Reinhardt, and D. M. Armstrong, 23–38. Cambridge: University Press, 1993.

———. *On the Plurality of Worlds*. Oxford: Basil Blackwell, 1986.

———. *Parts of Classes*. Oxford: Basil Blackwell, 1991.

Martin, R. M. “On Individuality and Quantification in Peirce’s Published Logic Papers, 1867-1885.” *Transactions of the Charles S. Peirce Society* 12, no. 3 (1976): 231–45.

McGilchrist, Iain. *The Matter with Things: Our Brains, Our Delusions, and the Unmaking of the World*. London: Perspectiva Press, 2021.

McGrath, Alister E. *The Science of God: An Introduction to Scientific Theology*. Grand Rapids, Mich.: William B. Eerdmans Pub., 2004.

Neal, Katherine. *From Discrete to Continuous: The Broadening of Number Concepts in Early Modern England*. Australasian Studies in History and Philosophy of Science, 16. Dordrecht: Kluwer, 2002.

Newton, Isaac. *Mathematical Principles of Natural Philosophy*. Vol. 32. Great Books of the Western World. Chicago; London: Encyclopaedia Britannica, 1990.

———. *Universal Arithmetick: Or, A Treatise of Arithmetical Composition and Resolution*. With Mr (Samuel) Cunn, Joseph Raphson, and Theaker Wilder. London: Printed for W. Johnston, 1769. http://archive.org/details/universalarithm00wildgoog.

O’Hear, Anthony. *The Element of Fire: Science, Art and the Human World*. Oxford, England: Taylor & Francis Group, 2013. http://ebookcentral.proquest.com/lib/oxford/detail.action?docID=1713396.

O’Rourke, Fran. *Aristotelian Interpretations*. Co. Kildare, Ireland: Irish Academic Press, 2016.

Peirce, Charles S. “A Guess at the Riddle.” In *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks. Past Masters. Charlottesville, Va: InteLex Corporation, 1994.

———. “Fallibilism, Continuity, and Evolution.” In *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks. Past Masters. Charlottesville, Va: InteLex Corporation, 1994.

———. “Issues of Pragmatism.” In *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks. Past Masters. Charlottesville, Va: InteLex Corporation, 1994.

———. *Mathematical Philosophy*. With Carolyn Eisele. The New Elements of Mathematics; Volume 4. Berlin ; Boston: De Gruyter, 2011.

———. “On the Logic of Number.” In *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks. Past Masters. Charlottesville, Va: InteLex Corporation, 1994.

———. *Philosophy of Mathematics: Selected Writings*. With Matthew E. Moore. Bloomington, Ind.: Indiana University Press, 2010.

———. “Physical Causation.” In *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks. Past Masters. Charlottesville, Va: InteLex Corporation, 1994.

———. “Synechism and Immortality.” In *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks. Past Masters. Charlottesville, Va: InteLex Corporation, 1994.

———. “The Logic of Continuity.” In *The Collected Papers of Charles Sanders Peirce. Electronic Edition*, with Charles Hartshorne, Paul Weiss, and Arthur Walter Burks. Past Masters. Charlottesville, Va: InteLex Corporation, 1994.

———. “The Logic of Relatives.” In *Reasoning and the Logic of Things: The Cambridge Conferences Lectures of 1898*, with Kenneth Laine Ketner. Cambridge, Mass ; Harvard University Press, 1993.

———. *The New Elements of Mathematics*. Hague: Mouton Publishers, 1976.

———. *The Writings of Charles S. Peirce, a Chronological Edition [Electronic Resource]*. With Max H. Fisch, Edward C. Moore, and Nathan Houser. Past Masters (InteLex Corporation : Online). Charlottesville, Va.: InteLex Corporation, 2008.

Pelletier, Francis Jeffry, Tibor Kiss, and Halima Husić. “Editorial Introduction: Background to the Count–Mass Distinction.” In *Things and Stuff: The Semantics of the Count-Mass Distinction*, edited by Francis Jeffry Pelletier, Halima Husić, and Tibor Kiss, 1–18. Cambridge: Cambridge University Press, 2021. https://doi.org/10.1017/9781108937979.002.

Philip Schaff, tran. “Athanasian Creed.” In *Creeds of Christendom, Volume 2: The Creeds of the Greek and Latin Churches*. New York: Harper & Brothers Franklin Square, 1877. http://archive.org/details/creedsofchristen0002unse_g3q6.

Pinsent, Andrew. *The Second-Person Perspective in Aquinas’s Ethics: Virtues and Gifts*. Routledge Studies in Ethics and Moral Theory 17. London: Routledge, 2012.

———. “Why Matter Matters to Neo-Aristotelian Teleology in Mechanics.” *Anuario Filosófico* 46, no. 2 (2013): 2.

Plato. “Sophist.” In *The Dialogues of Plato*, translated by Benjamin Jowett. Chicago: Encyclopaedia Britannica, 2003.

Poincaré, Henri. *Science and Hypothesis*. Translated by W. J. Greenstreet. With Joseph Larmor. London: Scott, 1905. http://catalog.hathitrust.org/api/volumes/oclc/2156620.html.

Priest, Graham. “Many-Valued Modal Logics: A Simple Approach.” *The Review of Symbolic Logic* (New York, USA) 1, no. 2 (2008): 190–203. https://doi.org/10.1017/S1755020308080179.

Putnam, Hilary. “Peirce the Logician.” *Historia Mathematica* 9, no. 3 (1982): 3. https://doi.org/10.1016/0315-0860(82)90123-9.

Recorde, Robert. *Pathway to Knowledge*. London: R. Wolfe, 1551.

Robinson, Andrew John Nottage, and Christopher Southgate. “Semiotics as a Metaphysical Framework for Christian Theology.” *Zygon* 45, no. 3 (September 2010): 689–712.

Russell, Bertrand. *An Outline of Philosophy*. Routledge Classics. Abingdon, Oxon, England ; Routledge, 2009.

———. *My Philosophical Development*. London: G. Allen & Unwin, 1959.

———. “On the Notion of Cause.” *Proceedings of the Aristotelian Society* 13 (1912): 1–26.

———. “The Basis of Realism.” *The Journal of Philosophy, Psychology and Scientific Methods* (New York) 8, no. 6 (1911): 158–61. https://doi.org/10.2307/2013237.

———. *The Philosophy of Logical Atomism*. Minneapolis: Dept. of Philosophy, University of Minnesota, 1918.

———. *The Principles of Mathematics*. Cambridge University Press, 1903.

———. *The Principles of Mathematics*. 2nd ed. London: George Allen & Unwin, 1937.

———. *The Scientific Outlook*. Second edition. London: George Allen & Unwin, 1949.

———. *Wisdom of the West: A Historical Survey of Western Philosophy in Its Social and Political Setting*. New York, NY: Doubleday, 1959.

Sassoli de Bianchi, Massimiliano. “Quantum Fields Are Not Fields; Comment on ‘There Are No Particles, There Are Only Fields,’ by Art Hobson [Am. J. Phys. 81(3), 211–223 (2013)].” *American Journal of Physics* 81, no. 9 (2013): 707–8. https://doi.org/10.1119/1.4811240.

Schröder, Ernst. *Ernst Schröder on Algebra and Logic*. Edited by Stephen Pollard. Synthese Library : Studies in Epistemology, Logic, Methodology, and Philosophy of Science, Volume 465. Cham: Springer, 2022.

———. *Vorlesungen über die Algebra der Logik (Exakte Logik)*. With Jakob Lüroth and Karl Eugen Müller. Leipzig: Druck und Verlag von B.G. Teubner, 1890.

Schrödinger, Erwin. *Collected Papers on Wave Mechanics*. London: Blackie and Sons, 1928.

Sciamanda, Robert J., and Art Hobson. “There Are No Particles, and There Are No Fields.” *American Journal of Physics* 81, no. 9 (2013): 645–645. https://doi.org/10.1119/1.4812316.

Shankar, Ramamurti. *Principles of Quantum Mechanics*. 2nd ed. New York: Springer Science Business Media, 1994.

Shapiro, Stewart, and Geoffrey Hellman. *The History of Continua: Philosophical and Mathematical Perspectives*. Oxford: University Press, Incorporated, 2021. https://doi.org/10.1093/oso/9780198809647.001.0001.

Sider, Theodore. “Four-Dimensionalism.” *The Philosophical Review* (Boston) 106, no. 2 (1997): 197–231. https://doi.org/10.2307/2998357.

———. *Logic for Philosophy*. Oxford, England: Oxford University Press, 2010.

Simons, Peter. “Leśniewski and Mereology.” In *The Lvov-Warsaw School. Past and Present*, edited by Ángel Garrido and Urszula Wybraniec-Skardowska, 337–59. Cham: Springer International Publishing, 2018. https://doi.org/10.1007/978-3-319-65430-0_26.

Simpson, William M. R. “From Quantum Physics to Classical Metaphysics.” In *Neo-Aristotelian Metaphysics and the Theology of Nature*. Routledge, 2021.

Sokolowski, Robert. “Matter, Elements and Substance in Aristotle.” *Journal of the History of Philosophy* (Berkeley, Calif) 8, no. 3 (1970): 263–88. https://doi.org/10.1353/hph.2008.1672.

Spaemann, Robert. *Persons: The Difference between “someone” and “Something.”* Oxford Studies in Theological Ethics. Oxford: University Press, 2006.

Stevin, Simon. “L’Arithmetique.” In *The Principal Works of Simon Stevin*, vol. 2, edited by Crone Ernst, translated by Dikshoorn C. Amsterdam: C. V. Swets & Zeitlinger, 1955.

Sytsma, David S. *Richard Baxter and the Mechanical Philosophers*. Oxford Studies in Historical Theology. New York: Oxford University Press, 2017.

Tarski, Alfred. “On the Calculus of Relations.” *The Journal of Symbolic Logic* (New York, USA) 6, no. 3 (1941): 73–89. https://doi.org/10.2307/2268577.

Tertullian. “Against Praxeas.” In *Latin Christianity: Its Founder, Tertullian*, vol. 3, edited by Alexander Roberts, James Donaldson, and A Cleveland Cox, translated by Peter Holmes. Nicene and Post-Nicene Fathers. Buffalo, NY: Christian Literature Co., 1885.

Vargas, Francisco, and Matthew E. Moore. “The Peircean Continuum.” In *The History of Continua: Philosophical and Mathematical Perspectives*, edited by Stewart Shapiro and Geoffrey Hellman, 328–46. Oxford University Press, 2020.

Weisheipl, James A. “The Commentary of St. Thomas on the De Caelo of Aristotle.” In *Thomas Aquinas: Contemporary Philosophical Perspectives*, edited by Brian Davies. Oxford University Press, 2002.

Whitehead, Alfred North. “On Quantity—the Whole Theory of Applied Mathematics Is Unproved.” September 14, 1909. Russell Letter no. 0246, RA1, document number 057422. Bertrand Russell Archives, McMaster University Libraries. https://bracers.mcmaster.ca/81610.

———. “Whitehead Makes Suggestions for the Preface to Volume 3 of Principia.” January 28, 1913. RA1, document number 057466. Bertrand Russell Archives, McMaster University Libraries. https://bracers.mcmaster.ca/81685.