

# Configurations in abelian categories—I: Basic properties and moduli stacks

Dominic Joyce

*Lincoln College, Oxford, OX1 3DR, UK*

Received 13 October 2004; accepted 14 April 2005

Communicated by L. Katzarkov

Available online 21 July 2005

## Abstract

This is the first in a series of papers on *configurations* in an abelian category  $\mathcal{A}$ . Given a finite partially ordered set  $(I, \preceq)$ , an  $(I, \preceq)$ -*configuration*  $(\sigma, \iota, \pi)$  is a finite collection of objects  $\sigma(J)$  and morphisms  $\iota(J, K)$  or  $\pi(J, K) : \sigma(J) \rightarrow \sigma(K)$  in  $\mathcal{A}$  satisfying some axioms, where  $J, K$  are subsets of  $I$ . Configurations describe how an object  $X$  in  $\mathcal{A}$  decomposes into subobjects, and are useful for studying *stability conditions* on  $\mathcal{A}$ .

We define and motivate the idea of configurations, and explain some natural operations upon them—subconfigurations, quotient configurations, substitution, refinements and improvements. Then we study *moduli spaces* of  $(I, \preceq)$ -configurations in  $\mathcal{A}$ , and natural morphisms between them, using the theory of *Artin stacks*. We prove well-behaved moduli stacks exist when  $\mathcal{A}$  is the abelian category of coherent sheaves on a projective scheme  $P$ , or of representations of a quiver  $Q$ .

In the sequels, given a stability condition  $(\tau, T, \leq)$  on  $\mathcal{A}$ , we will show the moduli spaces of  $\tau$ -(semi)stable objects or configurations are constructible subsets in the moduli stacks of all objects or configurations. We associate infinite-dimensional algebras of constructible functions to a quiver  $Q$  using the method of Ringel–Hall algebras, and define systems of invariants of  $P$  that ‘count’  $\tau$ -(semi)stable coherent sheaves on  $P$  and satisfy interesting identities.

© 2005 Elsevier Inc. All rights reserved.

MSC: 14D20; 14A20; 18F20

Keywords: Configuration; Moduli space; Artin stack; Coherent sheaf; Quiver representation

---

E-mail address: [dominic.joyce@lincoln.ox.ac.uk](mailto:dominic.joyce@lincoln.ox.ac.uk)

## 1. Introduction

This is the first of a series of papers [10–12] developing the concept of *configuration* in an abelian category. Given an abelian category  $\mathcal{A}$  and a finite partially ordered set (poset)  $(I, \preceq)$ , we define an  $(I, \preceq)$ -*configuration*  $(\sigma, \iota, \pi)$  in  $\mathcal{A}$  to be a collection of objects  $\sigma(J)$  and morphisms  $\iota(J, K)$  or  $\pi(J, K) : \sigma(J) \rightarrow \sigma(K)$  in  $\mathcal{A}$  satisfying certain axioms, where  $J, K$  are subsets of  $I$ . Configurations are a tool for describing *how an object  $X$  in  $\mathcal{A}$  decomposes into subobjects*. They are especially useful for studying *stability conditions* on  $\mathcal{A}$ .

This paper introduces configurations, studies their basic properties, and develops the theory of *moduli stacks of configurations*. We begin in §2 with background material on abelian categories and Artin stacks. Section 3 refines the Jordan–Hölder theorem for abelian categories in the case when the simple factors  $S_1, \dots, S_n$  of  $X \in \mathcal{A}$  are nonisomorphic. We find that the set of all *subobjects* of  $X$  may be classified using a *partial order*  $\preceq$  on  $I = \{1, \dots, n\}$ , the indexing set for the simple factors of  $X$ . We also classify quotient objects and composition series for  $X$  using  $(I, \preceq)$ .

Motivated by this, §4 defines the notion of  $(I, \preceq)$ -*configuration*  $(\sigma, \iota, \pi)$  in  $\mathcal{A}$ , and proves that it captures the properties of the set of all subobjects of  $X \in \mathcal{A}$  when  $X$  has nonisomorphic simple factors  $\{S^i : i \in I\}$ . Section 5 considers some elementary operations on configurations. Given an  $(I, \preceq)$ -configuration we can make *sub-* and *quotient*  $(K, \trianglelefteq)$ -configurations, where  $(K, \trianglelefteq)$  comes from  $(I, \preceq)$  with  $K \subseteq I$  or using a surjective  $\phi : I \rightarrow K$ . We also construct new configurations by *substituting* one configuration into another.

Let  $\trianglelefteq, \preceq$  be partial orders on  $I$ , with  $i \preceq j$  implies  $i \trianglelefteq j$ . Then each  $(I, \preceq)$ -configuration  $(\sigma, \iota, \pi)$  has a quotient  $(I, \trianglelefteq)$ -configuration  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ . Call  $(\sigma, \iota, \pi)$  an  $(I, \preceq)$ -*improvement* of  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ . Call  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  *best* if it has no strict improvements. Section 6 shows that improvements can be divided into a sequence of *steps*, classifies *one step improvements*, and gives a criterion for best configurations in terms of whether short exact sequences split.

Fix an algebraically closed field  $\mathbb{K}$ , and a  $\mathbb{K}$ -linear abelian category  $\mathcal{A}$ . To form moduli spaces of configurations in  $\mathcal{A}$  we need some extra data, on algebraic families of objects and morphisms in  $\mathcal{A}$  parameterized by a base  $\mathbb{K}$ -scheme  $U$ . We encode this in a *stack in exact categories*  $\mathfrak{F}_{\mathcal{A}} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{exactcat})$ , which must satisfy conditions given in Assumptions 7.1 and 8.1 below.

Section 7 defines *moduli stacks of objects*  $\text{Obj}_{\mathcal{A}}$  and  $(I, \preceq)$ -*configurations*  $\mathfrak{M}(I, \preceq)_{\mathcal{A}}$  in  $\mathcal{A}$ , and *substacks*  $\text{Obj}_{\mathcal{A}}^{\alpha}, \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}$  of objects and configurations with prescribed classes in  $K(\mathcal{A})$ . There are many natural 1-*morphisms* between these stacks. Section 8 shows these are *algebraic* (Artin)  $\mathbb{K}$ -stacks, *locally of finite type*, and some of the 1-morphisms are *representable*, or of *finite type*.

We finish with some examples. Section 9 takes  $\mathcal{A}$  to be the abelian category  $\text{coh}(P)$  of *coherent sheaves* on a projective  $\mathbb{K}$ -scheme  $P$ , and §10 considers the abelian category  $\text{mod-}\mathbb{K}Q$  of *representations of a quiver*  $Q$  and some variants  $\text{nil-}\mathbb{K}Q, \text{mod-}\mathbb{K}Q/I, \text{nil-}\mathbb{K}Q/I$ , and the abelian category  $\text{mod-}A$  of representations of a *finite-dimensional*  $\mathbb{K}$ -*algebra*  $A$ . We define the data  $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$  and prove it satisfies Assumptions 7.1 and 8.1 in each example.

The second paper [10] defines and studies infinite-dimensional algebras of constructible functions on  $\mathcal{M}(I, \preceq)_{\mathcal{A}}$ , motivated by the idea of *Ringel–Hall algebras*. The sequels [11,12] concern *stability conditions*  $(\tau, T, \leq)$  on  $\mathcal{A}$ , such as Gieseker stability on  $\text{coh}(P)$ , or slope stability on  $\text{mod-}\mathbb{K}Q$ .

We shall regard the set  $\text{Obj}_{\text{ss}}^{\alpha}(\tau)$  of  $\tau$ -semistable objects in  $\mathcal{A}$  with class  $\alpha$  in  $K(\mathcal{A})$  not as a moduli scheme under S-equivalence, but as a *constructible subset* in the stack  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ . One of our goals is to understand the relationship between  $\text{Obj}_{\text{ss}}^{\alpha}(\tau)$  and  $\text{Obj}_{\text{ss}}^{\alpha}(\tilde{\tau})$  for *two different* stability conditions  $(\tau, T, \leq)$ ,  $(\tilde{\tau}, \tilde{T}, \leq)$ . Our key idea is that this is best done using the moduli stacks  $\mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}$ .

Write  $\mathcal{M}_{\text{ss}}(I, \preceq, \kappa, \tau)_{\mathcal{A}}$  for the subset of points  $[(\sigma, \iota, \pi)]$  in  $\mathcal{M}(I, \preceq, \kappa)_{\mathcal{A}}$  with  $\sigma(\{i\})$   $\tau$ -semistable for all  $i \in I$ . We shall express  $\text{Obj}_{\text{ss}}^{\alpha}(\tilde{\tau})$  and  $\mathcal{M}_{\text{ss}}(I, \preceq, \kappa, \tilde{\tau})_{\mathcal{A}}$  in terms of projections of  $\mathcal{M}_{\text{ss}}(K, \trianglelefteq, \mu, \tau)_{\mathcal{A}}$  for other finite posets  $(K, \trianglelefteq)$ . We will then define systems of invariants of  $\mathcal{A}$ ,  $(\tau, T, \leq)$  by taking weighted Euler characteristics of  $\mathcal{M}_{\text{ss}}(I, \preceq, \kappa, \tau)_{\mathcal{A}}$ , and determine identities the invariants satisfy, and their transformation laws as  $(\tau, T, \leq)$  changes.

## 2. Background material

We review *abelian categories* and *Artin stacks*. Some useful references for §§2.1 and 2.2 are Popescu [15] and Gelfand and Manin [3, §II.5–II.6], and for §2.3 are Gómez [4], Behrend et al. [1], and Laumon and Moret-Bailly [14].

### 2.1. Abelian and exact categories

Here is the definition of abelian category, taken from [3, §II.5].

**Definition 2.1.** A category  $\mathcal{A}$  is called *abelian* if

- (i)  $\text{Hom}(X, Y)$  is an abelian group for all  $X, Y \in \mathcal{A}$ , and composition of morphisms is biadditive.
- (ii) There exists a *zero object*  $0 \in \mathcal{A}$  such that  $\text{Hom}(0, 0) = 0$ .
- (iii) For any  $X, Y \in \mathcal{A}$  there exists  $Z \in \mathcal{A}$  and morphisms  $\iota_X : X \rightarrow Z$ ,  $\iota_Y : Y \rightarrow Z$ ,  $\pi_X : Z \rightarrow X$ ,  $\pi_Y : Z \rightarrow Y$  with  $\pi_X \circ \iota_X = \text{id}_X$ ,  $\pi_Y \circ \iota_Y = \text{id}_Y$ ,  $\iota_X \circ \pi_X + \iota_Y \circ \pi_Y = \text{id}_Z$  and  $\pi_X \circ \iota_Y = \pi_Y \circ \iota_X = 0$ . We write  $Z = X \oplus Y$ , the *direct sum* of  $X$  and  $Y$ .
- (iv) For any morphism  $f : X \rightarrow Y$  there is a sequence  $K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C$  in  $\mathcal{A}$  such that  $j \circ i = f$ , and  $K$  is the kernel of  $f$ , and  $C$  the cokernel of  $f$ , and  $I$  is both the cokernel of  $k$  and the kernel of  $c$ .

An abelian category  $\mathcal{A}$  is called  $\mathbb{K}$ -linear over a field  $\mathbb{K}$  if  $\text{Hom}(X, Y)$  is a  $\mathbb{K}$ -vector space for all  $X, Y \in \mathcal{A}$ , and composition maps are bilinear.

We will often use the following properties of abelian categories:

- If  $i \circ f = i \circ g$  and  $i$  is injective, then  $f = g$  ( $i$  is *left cancellable*).
- If  $f \circ \pi = g \circ \pi$  and  $\pi$  is surjective, then  $f = g$  ( $\pi$  is *right cancellable*).

- If  $f : X \rightarrow Y$  is injective and surjective, then it is an isomorphism.

In an abelian category  $\mathcal{A}$  we can define *exact sequences* [3, §II.6].

**Definition 2.2.** Let  $X \xrightarrow{f} Y \xrightarrow{g} Z$  be a sequence in  $\mathcal{A}$  with  $g \circ f = 0$ . Let  $k : K \rightarrow Y$  be the kernel of  $g$  and  $c : Y \rightarrow C$  the cokernel of  $f$ . Then there exist unique morphisms  $a : X \rightarrow K$  and  $b : C \rightarrow Z$  with  $f = k \circ a$  and  $g = b \circ c$ . We say  $X \xrightarrow{f} Y \xrightarrow{g} Z$  is *exact at Y* if  $a$  is surjective, or equivalently if  $b$  is injective.

A short exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\mathcal{A}$  is called *split* if there exists a compatible isomorphism  $X \oplus Z \rightarrow Y$ . The *Grothendieck group*  $K_0(\mathcal{A})$  of  $\mathcal{A}$  is the abelian group generated by  $\text{Obj}(\mathcal{A})$ , with a relation  $[Y] = [X] + [Z]$  for each short exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\mathcal{A}$ . Throughout the paper  $K(\mathcal{A})$  will mean *the quotient of  $K_0(\mathcal{A})$  by some fixed subgroup*.

*Exact categories* were introduced by Quillen [16, §2], and are discussed in Gelfand and Manin [3, Example IV.3.3, p. 275].

**Definition 2.3.** Let  $\hat{\mathcal{A}}$  be an abelian category, and  $\mathcal{A}$  be a full additive subcategory of  $\hat{\mathcal{A}}$ , which is closed under extensions. Let  $\mathcal{E}$  be the class of exact sequences  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\hat{\mathcal{A}}$  with  $X, Y, Z \in \mathcal{A}$ . Then the pair  $(\mathcal{A}, \mathcal{E})$  is called an *exact category*. Usually we refer to  $\mathcal{A}$  as the exact category, taking  $\mathcal{E}$  to be implicitly given. Quillen [16, §2] gives necessary and sufficient conditions on  $\mathcal{A}, \mathcal{E}$  for  $\mathcal{A}$  to be embedded in an abelian category  $\hat{\mathcal{A}}$  in this way, and we take this to be the *definition* of an exact category. An *exact functor*  $F : (\mathcal{A}, \mathcal{E}) \rightarrow (\mathcal{A}', \mathcal{E}')$  of exact categories is a functor  $F : \mathcal{A} \rightarrow \mathcal{A}'$  taking exact sequences  $\mathcal{E}$  in  $\mathcal{A}$  to exact sequences  $\mathcal{E}'$  in  $\mathcal{A}'$ .

## 2.2. Subobjects and the Jordan–Hölder theorem

*Subobjects* of objects in  $\mathcal{A}$  are analogous to subgroups of an abelian group.

**Definition 2.4.** Let  $\mathcal{A}$  be an abelian category. Two injective morphisms  $i : S \rightarrow X$ ,  $i' : S' \rightarrow X$  in  $\mathcal{A}$  are *equivalent* if there exists an isomorphism  $h : S \rightarrow S'$  with  $i = i' \circ h$ . Then  $h$  is unique. A *subobject* of  $X \in \mathcal{A}$  is an equivalence class of injective morphisms  $i : S \rightarrow X$ . Usually we refer to  $S$  as the subobject, and write  $S \subset X$  to mean  $S$  is a subobject of  $X$ . We write  $0, X$  for the subobjects of  $X$  which are equivalence classes of  $0 \rightarrow X$  and  $\text{id}_X : X \rightarrow X$ .

Similarly, surjective morphisms  $\pi : X \rightarrow Q$ ,  $\pi' : X \rightarrow Q'$  in  $\mathcal{A}$  are *equivalent* if there is an isomorphism  $h : Q \rightarrow Q'$  with  $\pi' = h \circ \pi$ . A *quotient object* of  $X \in \mathcal{A}$  is an equivalence class of surjective  $\pi : X \rightarrow Q$ . If  $S, T \subset X$  are represented by  $i : S \rightarrow X$  and  $j : T \rightarrow X$ , we write  $S \subset T \subset X$  if there exists  $a : S \rightarrow T$  with  $i = j \circ a$ . Then  $a$  fits into an exact sequence  $0 \rightarrow S \xrightarrow{a} T \xrightarrow{b} F \rightarrow 0$ . We write  $F = T/S$ , and call  $F$  a *factor* of  $X \in \mathcal{A}$ .

We define operations  $\cap, +$  on subobjects, following Popescu [15, §2.6]. The notation comes from the intersection and sum of *subgroups of abelian groups*.

**Definition 2.5.** Let  $\mathcal{A}$  be an abelian category, let  $X \in \mathcal{A}$ , and suppose injective maps  $i : S \rightarrow X$ ,  $j : T \rightarrow X$  define subobjects  $S, T$  of  $X$ . Apply Definition 2.1(iv) to  $f = i \circ \pi_S + j \circ \pi_T : S \oplus T \rightarrow X$ . This yields  $U, V \in \mathcal{A}$  and morphisms  $k : U \rightarrow S \oplus T$ ,  $l : S \oplus T \rightarrow V$  and  $e : V \rightarrow X$  such that  $i \circ \pi_S + j \circ \pi_T = e \circ l$ , and  $k$  is the kernel of  $i \circ \pi_S + j \circ \pi_T$ , and  $l$  is the cokernel of  $k$ , and  $e$  is the *image* (the kernel of the cokernel) of  $i \circ \pi_S + j \circ \pi_T$ .

Define  $a : U \rightarrow S$  by  $a = k \circ \pi_S$ , and  $b : U \rightarrow T$  by  $b = -k \circ \pi_T$  and  $c : S \rightarrow V$  by  $c = f \circ \iota_S$ , and  $d : T \rightarrow V$  by  $d = f \circ \iota_T$ . Then  $k = \iota_S \circ a - \iota_T \circ b$ ,  $l = c \circ \pi_S + d \circ \pi_T$ ,  $i = e \circ c$  and  $j = e \circ d$ . Now  $0 \rightarrow U \xrightarrow{k} S \oplus T \xrightarrow{l} V \rightarrow 0$  is exact. So  $i \circ a = j \circ b$ , and

$$0 \longrightarrow U \xrightarrow{\iota_S \circ a - \iota_T \circ b} S \oplus T \xrightarrow{c \circ \pi_S + d \circ \pi_T} V \longrightarrow 0 \quad \text{is exact.} \quad (1)$$

As  $i, a$  are injective  $i \circ a = j \circ b : U \rightarrow X$  is too, and defines a subobject  $S \cap T$  of  $X$ . Also  $e : V \rightarrow X$  is injective, and defines a subobject  $S + T$  of  $X$ . Then  $S \cap T, S + T$  depend only on  $S, T \subset X$ , with inclusions  $S \cap T \subset S, T \subset S + T \subset X$ . Popescu [15, Proposition 2.6.4, p. 39] gives canonical isomorphisms

$$S/(S \cap T) \cong (S + T)/T \quad \text{and} \quad T/(S \cap T) \cong (S + T)/S. \quad (2)$$

These operations  $\cap, +$  are *commutative* and *associative*, so we can form multiple sums and intersections. We write  $\sum_{j \in J} T_j$  for the multiple sum  $+$  of a finite set of subobjects  $T_j \subset X$ , in the obvious way.

**Definition 2.6.** We call an abelian category  $\mathcal{A}$  *artinian* if all descending chains of subobjects  $\cdots \subset A_2 \subset A_1 \subset X$  stabilize, that is,  $A_{n+1} = A_n$  for  $n \gg 0$ . We call  $\mathcal{A}$  *noetherian* if all ascending chains of subobjects  $A_1 \subset A_2 \subset \cdots \subset X$  stabilize. We call  $\mathcal{A}$  of *finite length* if it is artinian and noetherian.

A nonzero object  $X$  in  $\mathcal{A}$  is called *simple* if it has no nontrivial proper subobjects. Let  $X \in \mathcal{A}$  and consider *filtrations* of subobjects

$$0 = A_0 \subset A_1 \subset \cdots \subset A_n = X. \quad (3)$$

We call (3) a *composition series* if the factors  $S_k = A_k/A_{k-1}$  are all *simple*.

Here is the *Jordan–Hölder theorem* in an abelian category [17, Theorem 2.1].

**Theorem 2.7.** Let  $\mathcal{A}$  be an abelian category of finite length. Then every filtration  $0 = A_0 \subset A_1 \subset \cdots \subset A_n = X$  without repetitions can be refined to a composition series for  $X$ . Suppose  $0 = A_0 \subset A_1 \subset \cdots \subset A_m = X$  and  $0 = B_0 \subset B_1 \subset \cdots \subset B_n = X$  are two composition series for  $X \in \mathcal{A}$ , with simple factors  $S_k = A_k/A_{k-1}$

and  $T_k = B_k/B_{k-1}$ . Then  $m = n$ , and for some permutation  $\sigma$  of  $1, \dots, n$  we have  $S_k \cong T_{\sigma(k)}$  for  $k = 1, \dots, n$ .

### 2.3. Introduction to algebraic $\mathbb{K}$ -stacks

Fix an *algebraically closed field*  $\mathbb{K}$  throughout. There are four main classes of ‘spaces’ over  $\mathbb{K}$  used in algebraic geometry, in increasing order of generality:

$$\mathbb{K}\text{-varieties} \subset \mathbb{K}\text{-schemes} \subset \text{algebraic } \mathbb{K}\text{-spaces} \subset \text{algebraic } \mathbb{K}\text{-stacks}.$$

*Algebraic stacks* (also known as Artin stacks) were introduced by Artin, generalizing *Deligne–Mumford stacks*. For an introduction see Gómez [4], and for a thorough treatment see Behrend et al. [1] or Laumon and Moret-Bailly [14].

We write our definitions in the language of *2-categories* [4, Appendix B]. A 2-category has *objects*  $X, Y$ , *1-morphisms*  $f, g : X \rightarrow Y$  between objects, and *2-morphisms*  $\alpha : f \rightarrow g$  between 1-morphisms. An example to keep in mind is a *2-category of categories*, where *objects* are categories, *1-morphisms* are functors, and *2-morphisms* are isomorphisms (natural transformations) of functors.

As in Gómez [4, §2.1 and 2.2] there are two different but equivalent ways of defining  $\mathbb{K}$ -stacks. We shall work with the first [4, Definition 2.10], even though the second is more widely used, as it is more convenient for our applications.

**Definition 2.8.** A *groupoid* is a category with all morphisms isomorphisms. Let (groupoids) be the 2-category whose *objects* are groupoids, *1-morphisms* functors of groupoids, and *2-morphisms* natural transformations of functors.

Let  $\mathbb{K}$  be an algebraically closed field, and  $\text{Sch}_{\mathbb{K}}$  the *category of  $\mathbb{K}$ -schemes*. We make  $\text{Sch}_{\mathbb{K}}$  into a 2-category by taking 1-morphisms to be morphisms, and the only 2-morphisms to be identities  $\text{id}_f$  for each 1-morphism  $f$ . To define  $\mathbb{K}$ -stacks we need to choose a *Grothendieck topology* on  $\text{Sch}_{\mathbb{K}}$ , as in [4, Appendix A], and we choose the *étale topology*.

A *prestack in groupoids* on  $\text{Sch}_{\mathbb{K}}$  is a *contravariant 2-functor*  $\mathfrak{F} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{groupoids})$ . As in [4, Appendix B], this comprises the following data, satisfying conditions we shall not give

- For each object  $U$  in  $\text{Sch}_{\mathbb{K}}$ , an object (groupoid)  $\mathfrak{F}(U)$  in (groupoids).
- For each 1-morphism  $f : U \rightarrow V$  in  $\text{Sch}_{\mathbb{K}}$ , a 1-morphism (functor)  $\mathfrak{F}(f) : \mathfrak{F}(V) \rightarrow \mathfrak{F}(U)$  in (groupoids).
- For each 2-morphism  $\alpha : f \rightarrow f'$  in  $\text{Sch}_{\mathbb{K}}$ , a 2-morphism  $\mathfrak{F}(\alpha) : \mathfrak{F}(f') \rightarrow \mathfrak{F}(f)$ . As the only 2-morphisms in  $\text{Sch}_{\mathbb{K}}$  are  $\text{id}_f$ , this data is *trivial*.
- If  $f : U \rightarrow V$  and  $g : V \rightarrow W$  are 1-morphisms in  $\text{Sch}_{\mathbb{K}}$ , a *2-isomorphism*  $\varepsilon_{g,f} : \mathfrak{F}(f) \circ \mathfrak{F}(g) \rightarrow \mathfrak{F}(g \circ f)$ , that is, an *isomorphism of functors*. Thus,  $\mathfrak{F}$  only respects composition of 1-morphisms up to 2-isomorphism. The 2-isomorphisms  $\varepsilon_{g,f}$  are often omitted in proofs.

A  $\mathbb{K}$ -stack is a *stack in groupoids* on  $\text{Sch}_{\mathbb{K}}$ . That is, it is a prestack  $\mathfrak{F}$  satisfying the following axioms. Let  $\{f_i : U_i \rightarrow V\}_{i \in I}$  be an open cover of  $V$  in the site  $\text{Sch}_{\mathbb{K}}$ . Write

$U_{ij} = U_i \times_{f_i, V, f_j} U_j$  for the fibre product scheme and  $f_{ij} : U_{ij} \rightarrow V$ ,  $f_{ij,i} : U_{ij} \rightarrow U_i$ ,  $f_{ij,j} : U_{ij} \rightarrow U_j$  for the projections, and similarly for ‘triple intersections’  $U_{ijk}$ . Then

- (i) (Glueing of morphisms). If  $X, Y \in \text{Obj}(\mathfrak{F}(V))$  and  $\phi_i : \mathfrak{F}(f_i)X \rightarrow \mathfrak{F}(f_i)Y$  are morphisms for  $i \in I$  such that

$$\begin{aligned} & \varepsilon_{f_i, f_{ij,i}}(Y) \circ (\mathfrak{F}(f_{ij,i})\phi_i) \circ \varepsilon_{f_i, f_{ij,i}}(X)^{-1} \\ &= \varepsilon_{f_j, f_{ij,j}}(Y) \circ (\mathfrak{F}(f_{ij,j})\phi_j) \circ \varepsilon_{f_j, f_{ij,j}}(X)^{-1} \end{aligned} \quad (4)$$

in  $\text{Hom}(\mathfrak{F}(f_{ij})X, \mathfrak{F}(f_{ij})Y)$  for all  $i, j$ , then there exists a morphism  $\eta : X \rightarrow Y$  in  $\text{Mor}(\mathfrak{F}(V))$  with  $\mathfrak{F}(f_i)\eta = \phi_i$ .

- (ii) (Monopresheaf). If  $X, Y$  lie in  $\text{Obj}(\mathfrak{F}(V))$  and  $\phi, \psi : X \rightarrow Y$  in  $\text{Mor}(\mathfrak{F}(V))$  with  $\mathfrak{F}(f_i)\phi = \mathfrak{F}(f_i)\psi$  for all  $i \in I$ , then  $\phi = \psi$ .
- (iii) (Glueing of objects). If  $X_i \in \text{Obj}(\mathfrak{F}(U_i))$  and  $\phi_{ij} : \mathfrak{F}(f_{ij,j})X_j \rightarrow \mathfrak{F}(f_{ij,i})X_i$  are morphisms for all  $i, j$  satisfying the *cocycle condition*

$$\begin{aligned} & [\varepsilon_{f_{ij,i}, f_{ijk,ij}}(X_i) \circ (\mathfrak{F}(f_{ijk,ij})\phi_{ij}) \circ \varepsilon_{f_{ij,j}, f_{ijk,ij}}(X_j)^{-1}] \\ & \circ [\varepsilon_{f_{jk,j}, f_{ijk,jk}}(X_j) \circ (\mathfrak{F}(f_{ijk,jk})\phi_{jk}) \circ \varepsilon_{f_{jk,k}, f_{ijk,jk}}(X_k)^{-1}] \\ &= [\varepsilon_{f_{ik,i}, f_{ijk,ik}}(X_i) \circ (\mathfrak{F}(f_{ijk,ik})\phi_{ik}) \circ \varepsilon_{f_{ik,k}, f_{ijk,ik}}(X_k)^{-1}] \end{aligned} \quad (5)$$

in  $\text{Hom}(\mathfrak{F}(f_{ijk,k})X_k, \mathfrak{F}(f_{ijk,k})X_i)$  for all  $i, j, k$ , then there exists  $X \in \text{Obj}(\mathfrak{F}(V))$  and isomorphisms  $\phi_i : \mathfrak{F}(f_i)X \rightarrow X_i$  in  $\text{Mor}(\mathfrak{F}(U_i))$  such that

$$\phi_{ji} \circ (\mathfrak{F}(f_{ij,i})\phi_i) \circ \varepsilon_{f_i, f_{ij,i}}(X)^{-1} = (\mathfrak{F}(f_{ij,j})\phi_j) \circ \varepsilon_{f_j, f_{ij,j}}(X)^{-1} \quad (6)$$

in  $\text{Hom}(\mathfrak{F}(f_{ij})X, \mathfrak{F}(f_{ij,j})X_j)$  for all  $i, j$ .

Using notation from [14, §3], an *algebraic  $\mathbb{K}$ -stack* is a  $\mathbb{K}$ -stack  $\mathfrak{F}$  such that

- (i) The *diagonal*  $\Delta_{\mathfrak{F}}$  is representable, quasicompact and separated.
- (ii) There exists a scheme  $U$  and a smooth surjective 1-morphism  $u : U \rightarrow \mathfrak{F}$ . We call  $U, u$  an *atlas* for  $\mathfrak{F}$ .

As in [4, §2.2], [14, §3],  $\mathbb{K}$ -stacks form a 2-category. A 1-morphism  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  is a natural transformation between the 2-functors  $\mathfrak{F}, \mathfrak{G}$ . A 2-morphism  $\alpha : \phi \rightarrow \psi$  of 1-morphisms  $\phi, \psi : \mathfrak{F} \rightarrow \mathfrak{G}$  is an isomorphism of the natural transformations. *Representable* and *finite type* 1-morphisms are defined in [14, Definition 3.9, 4.16], and will be important in §8.2.

We define the set of  $\mathbb{K}$ -points of a stack.

**Definition 2.9.** Let  $\mathfrak{F}$  be a  $\mathbb{K}$ -stack. Regarding  $\text{Spec } \mathbb{K}$  as a  $\mathbb{K}$ -stack, we can form  $\text{Hom}(\text{Spec } \mathbb{K}, \mathfrak{F})$ . This is a *groupoid*, with *objects* 1-morphisms  $\text{Spec } \mathbb{K} \rightarrow \mathfrak{F}$ , and *morphisms* 2-morphisms. Define  $\mathfrak{F}(\mathbb{K})$  to be the set of *isomorphism classes of objects*

in  $\text{Hom}(\text{Spec } \mathbb{K}, \mathfrak{F})$ . Elements of  $\mathfrak{F}(\mathbb{K})$  are called  $\mathbb{K}$ -points, or *geometric points*, of  $\mathfrak{F}$ . If  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  is a 1-morphism of  $\mathbb{K}$ -stacks then composition with  $\phi$  yields a morphism of groupoids  $\text{Hom}(\text{Spec } \mathbb{K}, \mathfrak{F}) \rightarrow \text{Hom}(\text{Spec } \mathbb{K}, \mathfrak{G})$ , and therefore induces a map of sets  $\phi_* : \mathfrak{F}(\mathbb{K}) \rightarrow \mathfrak{G}(\mathbb{K})$ .

One important difference in working with 2-categories rather than ordinary categories is that in diagram-chasing one only requires 1-morphisms to be *2-isomorphic* rather than *equal*. The simplest kind of *commutative diagram* is

$$\begin{array}{ccc} & \mathfrak{G} & \\ \phi \nearrow & \Downarrow F & \searrow \psi \\ \mathfrak{F} & \xrightarrow{\chi} & \mathfrak{H}, \end{array}$$

by which we mean that  $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$  are  $\mathbb{K}$ -stacks,  $\phi, \psi, \chi$  are 1-morphisms, and  $F : \psi \circ \phi \rightarrow \chi$  is a 2-isomorphism. Sometimes we omit  $F$ , and mean that  $\psi \circ \phi \cong \chi$ .

**Definition 2.10.** Let  $\phi : \mathfrak{F} \rightarrow \mathfrak{H}, \psi : \mathfrak{G} \rightarrow \mathfrak{H}$  be 1-morphisms of  $\mathbb{K}$ -stacks. Define the *fibre product*  $\mathfrak{F} \times_{\phi, \mathfrak{H}, \psi} \mathfrak{G}$ , or  $\mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G}$  for short, such that  $(\mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G})(U)$  for  $U \in \text{Sch}_{\mathbb{K}}$  is the groupoid with *objects* triples  $(A, B, \alpha)$  for  $A \in \text{Obj}(\mathfrak{F}(U)), B \in \text{Obj}(\mathfrak{G}(U))$  and  $\alpha : \phi(U)A \rightarrow \psi(U)B$  in  $\text{Mor}(\mathfrak{H}(U))$ , and *morphisms*  $(\beta, \gamma) : (A, B, \alpha) \rightarrow (A', B', \alpha')$  for  $\beta : A \rightarrow A'$  in  $\text{Mor}(\mathfrak{F}(U))$  and  $\gamma : B \rightarrow B'$  in  $\text{Mor}(\mathfrak{G}(U))$  with  $\alpha' \circ (\phi(U)\beta) = (\psi(U)\gamma) \circ \alpha$  in  $\text{Mor}(\mathfrak{H}(U))$ .

The rest of the data to make a *contravariant 2-functor*  $\mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{groupoids})$  is defined in the obvious way. Then  $\mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G}$  is a  $\mathbb{K}$ -stack, with projection 1-morphisms  $\pi_{\mathfrak{F}} : \mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G} \rightarrow \mathfrak{F}, \pi_{\mathfrak{G}} : \mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G} \rightarrow \mathfrak{G}$  fitting into a *commutative diagram*:

$$\begin{array}{ccccc} & & \pi_{\mathfrak{F}} \nearrow & \mathfrak{F} & \searrow \phi \\ \mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G} & & & \Downarrow A & \\ & & \pi_{\mathfrak{G}} \searrow & \mathfrak{G} & \nearrow \psi \end{array} \quad (7)$$

where  $A : \phi \circ \pi_{\mathfrak{F}} \rightarrow \psi \circ \pi_{\mathfrak{G}}$  is a 2-isomorphism. Any commutative diagram

$$\begin{array}{ccccc} & & \theta \nearrow & \mathfrak{F} & \searrow \phi \\ \mathfrak{E} & & & \Downarrow B & \\ & & \eta \searrow & \mathfrak{G} & \nearrow \psi \end{array} \quad (8)$$



extends to a commutative diagram with  $\rho$  unique up to 2-isomorphism

$$\begin{array}{ccc}
 \mathfrak{E} & \xrightarrow{\theta} & \mathfrak{F} \\
 \downarrow \rho & \nearrow C & \downarrow \pi_{\mathfrak{F}} \\
 \mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G} & \xrightarrow{\pi_{\mathfrak{F}}} & \mathfrak{F} \\
 \downarrow D & \nearrow \pi_{\mathfrak{G}} & \downarrow A \\
 \mathfrak{G} & \xrightarrow{\psi} & \mathfrak{H}
 \end{array}
 \quad \text{with} \quad
 \begin{array}{ccc}
 \phi \circ \pi_{\mathfrak{F}} \circ \rho & \longrightarrow & \phi \circ \theta \\
 A \circ \rho \downarrow & \phi \circ C & \downarrow B \\
 \psi \circ \pi_{\mathfrak{G}} \circ \rho & \longrightarrow & \psi \circ \eta
 \end{array}
 \quad (9)$$

commuting 2-morphisms

We call (8) a *Cartesian square* if  $\rho$  in (9) is a 1-isomorphism, so that  $\mathfrak{E}$  is 1-isomorphic to  $\mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G}$ . Cartesian squares may also be characterized by a *universal property*, as in [1, §3]. By the *product*  $\mathfrak{F} \times \mathfrak{G}$  of  $\mathbb{K}$ -stacks  $\mathfrak{F}, \mathfrak{G}$  we mean their fibre product  $\mathfrak{F} \times_{\text{Spec } \mathbb{K}} \mathfrak{G}$  over  $\text{Spec } \mathbb{K}$ , using the natural projections  $\mathfrak{F}, \mathfrak{G} \rightarrow \text{Spec } \mathbb{K}$ .

Here are some properties of these that we will need later.

- If  $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$  are *algebraic*  $\mathbb{K}$ -stacks, then their fibre product  $\mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G}$  is also algebraic, by Laumon and Moret-Bailly [14, Proposition 4.5(i)]. Hence, in any Cartesian square (8), if  $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$  are algebraic then  $\mathfrak{E}$  is also algebraic, as it is 1-isomorphic to  $\mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G}$ . If also  $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$  are *locally of finite type*, then so are  $\mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G}$  and  $\mathfrak{E}$ .
- Let  $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$  in (7) be *algebraic*  $\mathbb{K}$ -stacks. We may think of  $\mathfrak{G}$  as a stack over a base  $\mathfrak{H}$ , and then  $\pi_{\mathfrak{F}} : \mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G} \rightarrow \mathfrak{F}$  is obtained from  $\psi : \mathfrak{G} \rightarrow \mathfrak{H}$  by *base extension*. Therefore, for any property  $P$  of morphisms of algebraic  $\mathbb{K}$ -stacks that is *stable under base extension*, if  $\psi$  has  $P$  then  $\pi_{\mathfrak{F}}$  has  $P$ . But in any Cartesian square (8),  $\mathfrak{E}$  is 1-isomorphic to  $\mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G}$ . Thus if  $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$  are algebraic and  $\psi$  has  $P$  in (8), then  $\theta$  has  $P$ , as in [14, Remark 4.14.1]. This holds when  $P$  is *of finite type* [14, Remark 4.17(2)], or *locally of finite type* [14, p. 33–4], or *representable* [14, Lemma 3.11].

### 3. Refining the Jordan–Hölder theorem

We shall study the following situation.

**Definition 3.1.** Let  $\mathcal{A}$  be an abelian category of finite length, and  $X \in \mathcal{A}$ . Then  $X$  admits a composition series  $0 = A_0 \subset A_1 \subset \cdots \subset A_n = X$  by Theorem 2.7, and the simple factors  $S_k = A_k/A_{k-1}$  for  $k = 1, \dots, n$  of  $X$  are independent of choices, up to isomorphism and permutation of  $1, \dots, n$ . Suppose  $S_k \not\cong S_l$  for  $1 \leq k < l \leq n$ . Then we say that  $X$  has *nonisomorphic simple factors*.

Let  $X$  have nonisomorphic simple factors, and let  $I$  be an *indexing set* for  $\{S_1, \dots, S_n\}$ , so that  $|I| = n$ , and write  $\{S_1, \dots, S_n\} = \{S^i : i \in I\}$ . Then Theorem 2.7 implies that for every composition series  $0 = B_0 \subset B_1 \subset \cdots \subset B_n = X$  for  $X$  with simple factors  $T_k = B_k/B_{k-1}$ , there exists a unique bijection  $\phi : I \rightarrow \{1, \dots, n\}$  such that  $S^i \cong T_{\phi(i)}$  for all  $i \in I$ .

Define a *partial order*  $\preceq$  on  $I$  by  $i \preceq j$  for  $i, j \in I$  if and only if  $\phi(i) \leq \phi(j)$  for all bijections  $\phi : I \rightarrow \{1, \dots, n\}$  constructed from a composition series  $0 = B_0 \subset B_1 \subset \dots \subset B_n = X$  for  $X$  as above. Then  $(I, \preceq)$  is a *partially ordered set*, or *poset* for short.

The point of this definition is to treat all the Jordan–Hölder composition series  $0 = B_0 \subset B_1 \subset \dots \subset B_n = X$  for  $X$  on an equal footing. Writing the simple factors of  $X$  as  $S_1, \dots, S_n$  gives them a preferred order, and favours one composition series over the rest. So instead we write the simple factors as  $S^i$  for  $i \in I$ , some arbitrary indexing set. Here is some more notation.

**Definition 3.2.** Let  $(I, \preceq)$  be a finite poset. Define  $J \subseteq I$  to be

- (i) an *s-set* if  $i \in I$ ,  $j \in J$  and  $i \preceq j$  implies  $i \in J$ ,
- (ii) a *q-set* if  $i \in I$ ,  $j \in J$  and  $j \preceq i$  implies  $i \in J$ , and
- (iii) an *f-set* if  $i \in I$  and  $h, j \in J$  and  $h \preceq i \preceq j$  implies  $i \in J$ .

The motivation for this is that below s-sets will correspond to *subobjects*  $S \subset X$ , q-sets to *quotient objects*  $X/S$ , and f-sets to *factors*  $T/S$  for  $S \subset T \subset X$ .

Here are some properties of s-sets, q-sets and f-sets.

**Proposition 3.3.** Let  $(I, \preceq)$  be a finite poset. Then

- (a)  $I$  and  $\emptyset$  are s-sets. If  $J, K$  are s-sets then  $J \cap K$  and  $J \cup K$  are s-sets.
- (b)  $J$  is an s-set if and only if  $I \setminus J$  is a q-set.
- (c) If  $J \subset K$  are s-sets then  $K \setminus J$  is an f-set. Every f-set is of this form.
- (d) If  $J, K$  are f-sets then  $J \cap K$  is an f-set, but  $J \cup K$  may not be an f-set.

The proof is elementary, and left as an exercise. For the last part of (c), if  $F \subseteq I$  is an f-set, define  $K = \{i \in I : i \preceq j \text{ for some } j \in F\}$  and  $J = K \setminus F$ . Then  $J \subset K$  are s-sets with  $F = K \setminus J$ . Note that (a) and (b) imply the collections of s-sets and q-sets are both *topologies* on  $I$ , but (d) shows the f-sets may not be. Also  $\preceq$  can be reconstructed from the set of s-sets on  $I$ , as  $i \preceq j$  if and only if  $i \in J$  for every s-set  $J \subset I$  with  $j \in J$ .

**Lemma 3.4.** In the situation of Definition 3.1, suppose  $S \subset X$  is a subobject. Then there exists a unique s-set  $J \subseteq I$  such that the simple factors in any composition series for  $S$  are isomorphic to  $S^i$  for  $i \in J$ .

**Proof.** Let  $0 = B_0 \subset \dots \subset B_l = S$  be a composition series for  $S$ , with simple factors  $T_k = B_k/B_{k-1}$  for  $k = 1, \dots, l$ . Then  $0 = B_0 \subset \dots \subset B_l \subset X$  is a filtration of  $X$  without repetitions, and can be refined to a composition series by Theorem 2.7. As  $T_k$  is simple, no extra terms are inserted between  $B_{k-1}$  and  $B_k$ . Thus  $X$  has a composition series  $0 = B_0 \subset \dots \subset B_l \subset \dots \subset B_n = X$ , with simple factors  $T_k = B_k/B_{k-1}$  for  $k = 1, \dots, n$ .

By Definition 3.1 there is a unique bijection  $\phi : I \rightarrow \{1, \dots, n\}$  such that  $S^i \cong T_{\phi(i)}$  for all  $i \in I$ . Define  $J = \phi^{-1}(\{1, \dots, l\})$ . Then  $J \subseteq I$ , and the simple factors  $T_k$  of the composition series  $0 = B_0 \subset \dots \subset B_l = S$  are isomorphic to  $S^i$  for  $i \in J$ . Theorem 2.7 then implies that the simple factors in *any* composition series for  $S$  are isomorphic to  $S^i$  for  $i \in J$ .

Uniqueness of  $J$  is now clear, as a different  $J$  would give different simple factors for  $S$ . Suppose  $j \in J$  and  $i \in I \setminus J$ . Then  $1 \leq \phi(j) \leq l$  and  $l+1 \leq \phi(i) \leq n$ , so  $\phi(j) < \phi(i)$ , which implies that  $i \not\leq j$  by Definition 3.1. Hence if  $j \in J$  and  $i \in I$  with  $i \leq j$  then  $i \in J$ , and  $J$  is an s-set.  $\square$

**Lemma 3.5.** *Suppose  $S, T \subset X$  correspond to s-sets  $J, K \subseteq I$ , as in Lemma 3.4. Then  $S \cap T$  corresponds to  $J \cap K$ , and  $S + T$  corresponds to  $J \cup K$ .*

**Proof.** Let  $S \cap T$  correspond to the s-set  $L \subseteq I$ , and  $S + T$  to the s-set  $M$ . We must show  $L = J \cap K$  and  $M = J \cup K$ . By Theorem 2.7, we may refine the filtration  $0 \subset S \cap T \subset S$  to a composition series for  $S$  containing one for  $S \cap T$ . Thus the simple factors of  $S$  contain those of  $S \cap T$ , and  $L \subseteq J$ . Similarly  $L \subseteq K$ , so  $L \subseteq J \cap K$ , and  $J \cup K \subseteq M$  as  $S, T \subseteq S \cap T$ .

Now the simple factors of  $S/(S \cap T)$  are  $S^i$  for  $i \in J \setminus L$ , and the simple factors of  $(S + T)/T$  are  $S^i$  for  $i \in M \setminus K$ . As  $S/(S \cap T) \cong (S + T)/T$  by (2) we see that  $J \setminus L = M \setminus K$ . Together with  $L \subseteq J \cap K$  and  $J \cup K \subseteq M$  this implies that  $L = J \cap K$  and  $M = J \cup K$ .  $\square$

**Lemma 3.6.** *Suppose  $S, T \subset X$  correspond to s-sets  $J, K \subseteq I$ . Then  $J \subseteq K$  if and only if  $S \subset T \subset X$ , and  $J = K$  if and only if  $S = T$ .*

**Proof.** If  $S \subset T \subset X$  we can refine  $0 \subset S \subset T \subset X$  to a composition series  $0 = B_0 \subset B_1 \subset \dots \subset B_n = X$  with  $S = B_k$  and  $T = B_l$  for  $0 \leq k \leq l \leq n$ . Let  $T_m = B_m/B_{m-1}$ . Then the simple factors of  $S$  are  $T_1, \dots, T_k$  and of  $T$  are  $T_1, \dots, T_l$ . Hence  $J \subseteq K$ , as  $k \leq l$ . This proves the first ‘if’.

Now suppose  $S, T \subset X$  and  $J \subseteq K$ . Then  $J \cap K = J$ , so  $S \cap T$  corresponds to the s-set  $J$ . But  $S \cap T \subset S$ , so  $S/(S \cap T)$  has no simple factors, and  $S = S \cap T$ . Thus  $S \subset T$ , proving the first ‘only if’. The second part is immediate.  $\square$

**Lemma 3.7.** *Let  $j \in I$  and define  $J^j = \{i \in I : i \leq j\}$ . Then  $J^j$  is an s-set, and there exists a subobject  $D^j \subset X$  corresponding to  $J^j$ .*

**Proof.** Clearly  $J^j$  is an s-set. By Definition 3.1 each composition series  $0 = B_0 \subset B_1 \subset \dots \subset B_n = X$  for  $X$  gives a bijection  $\phi : I \rightarrow \{1, \dots, n\}$ . Let  $\phi_1, \dots, \phi_r$  be the distinct bijections  $\phi : I \rightarrow \{1, \dots, n\}$  realized by composition series for  $X$ . For each  $k = 1, \dots, r$  choose a composition series  $0 = B_0 \subset B_1 \subset \dots \subset B_n = X$  with bijection  $\phi_k$ , and define  $C_k$  to be the subobject  $B_{\phi_k(j)} \subset X$ .

This defines subobjects  $C_1, \dots, C_r \subset X$ , where  $C_k$  corresponds to the s-set  $\phi_k^{-1}(\{1, \dots, \phi_k(j)\}) \subseteq I$ . Define  $D^j = C_1 \cap C_2 \cap \dots \cap C_r$ . Then  $S \subset X$ , and Lemma 3.5

shows that  $D^j$  corresponds to the s-set

$$\begin{aligned} & \bigcap_{k=1}^r \phi_k^{-1}(\{1, \dots, \phi_k(j)\}) \\ &= \bigcap_{k=1}^r \{i \in I : \phi_k(i) \leq \phi_k(j)\} \\ &= \{i \in I : \phi_k(i) \leq \phi_k(j) \text{ for all } k = 1, \dots, r\} = \{i \in I : i \preceq j\} = J^j, \end{aligned}$$

by definition of  $\preceq$ .  $\square$

We can now *classify subobjects* of  $X$  in terms of s-sets.

**Proposition 3.8.** *In the situation of Definitions 3.1 and 3.2, for each s-set  $J \subseteq I$  there exists a unique subobject  $S \subset X$  such that the simple factors in any composition series for  $S$  are isomorphic to  $S^i$  for  $i \in J$ . This defines a 1-1 correspondence between subobjects  $S \subset X$  and s-sets  $J \subseteq I$ .*

**Proof.** For each  $j \in J$  define  $J^j$  and  $D^j$  as in Lemma 3.7. Then  $j \in J^j \subseteq J$ , so  $J = \bigcup_{j \in J} J^j$ . Set  $S = \sum_{j \in J} D^j$ . Then  $S \subset X$  corresponds to the s-set  $\bigcup_{j \in J} J^j = J$  by Lemma 3.5. Uniqueness follows from Lemma 3.6.  $\square$

The dual proof classifies *quotient objects* of  $X$  in terms of q-sets.

**Proposition 3.9.** *In the situation of Definitions 3.1 and 3.2, for each q-set  $K \subseteq I$  there exists a unique quotient object  $Q = X/S$  of  $X$  such that the simple factors in any composition series for  $Q$  are isomorphic to  $S^i$  for  $i \in K$ . This defines a 1-1 correspondence between quotient objects and q-sets.*

We can also classify *composition series* for  $X$ .

**Proposition 3.10.** *In the situation of Definition 3.1, for each bijection  $\phi : I \rightarrow \{1, \dots, n\}$  there exists a unique composition series  $0 = B_0 \subset \dots \subset B_n = X$  with  $S^i \cong B_{\phi(i)}/B_{\phi(i)-1}$  for all  $i \in I$  if and only if  $i \preceq j$  implies  $\phi(i) \leq \phi(j)$ .*

**Proof.** The ‘only if’ part follows from Definition 3.1. For the ‘if’ part, let  $\phi : I \rightarrow \{1, \dots, n\}$  be a bijection for which  $i \preceq j$  implies that  $\phi(i) \leq \phi(j)$ . Then  $\phi^{-1}(\{1, \dots, k\})$  is an s-set for each  $k = 0, 1, \dots, n$ . Let  $B_k \subset X$  be the unique subobject corresponding to  $\phi^{-1}(\{1, \dots, k\})$ , which exists by Proposition 3.8. It easily follows that  $0 = B_0 \subset B_1 \subset \dots \subset B_n = X$  is the unique composition series with  $B_k/B_{k-1} \cong S^{\phi^{-1}(k)}$  for  $k = 1, \dots, n$ , and the result follows.  $\square$

This implies that composition series for  $X$  up to isomorphism are in 1-1 correspondence with *total orders* on  $I$  compatible with the partial order  $\preceq$ . In Definition 3.1 we

defined the partial order  $\preceq$  on  $I$  to be the intersection of all the total orders on  $I$  coming from composition series for  $X$ . We now see that *every* total order on  $I$  compatible with  $\preceq$  comes from a composition series.

#### 4. Posets $(I, \preceq)$ and $(I, \preceq)$ -configurations in $\mathcal{A}$

Although a *subobject* of  $X$  is an equivalence class of injective  $i : S \rightarrow X$ , in §3 we for simplicity suppressed the morphisms  $i$ , and just wrote  $S \subset X$ . We shall now change our point of view, and investigate the natural morphisms between the factors  $T/S$  of  $X$ . Therefore we adopt some new notation, which stresses morphisms between objects. The following definition encodes the properties we expect of the factors of  $X$ , and their natural morphisms.

**Definition 4.1.** Let  $(I, \preceq)$  be a finite poset, and use the notation of Definition 3.2. Define  $\mathcal{F}_{(I, \preceq)}$  to be the set of f-sets of  $I$ . Define  $\mathcal{G}_{(I, \preceq)}$  to be the subset of  $(J, K) \in \mathcal{F}_{(I, \preceq)} \times \mathcal{F}_{(I, \preceq)}$  such that  $J \subseteq K$ , and if  $j \in J$  and  $k \in K$  with  $k \preceq j$ , then  $k \in J$ . Define  $\mathcal{H}_{(I, \preceq)}$  to be the subset of  $(J, K) \in \mathcal{F}_{(I, \preceq)} \times \mathcal{F}_{(I, \preceq)}$  such that  $K \subseteq J$ , and if  $j \in J$  and  $k \in K$  with  $k \preceq j$ , then  $j \in K$ . It is easy to show that  $\mathcal{G}_{(I, \preceq)}$  and  $\mathcal{H}_{(I, \preceq)}$  have the following properties:

- (a)  $(J, K)$  lies in  $\mathcal{G}_{(I, \preceq)}$  if and only if  $(K, K \setminus J)$  lies in  $\mathcal{H}_{(I, \preceq)}$ .
- (b) If  $(J, K) \in \mathcal{G}_{(I, \preceq)}$  and  $(K, L) \in \mathcal{G}_{(I, \preceq)}$  then  $(J, L) \in \mathcal{G}_{(I, \preceq)}$ .
- (c) If  $(J, K) \in \mathcal{H}_{(I, \preceq)}$  and  $(K, L) \in \mathcal{H}_{(I, \preceq)}$  then  $(J, L) \in \mathcal{H}_{(I, \preceq)}$ .
- (d) If  $(J, K) \in \mathcal{G}_{(I, \preceq)}$ ,  $(K, L) \in \mathcal{H}_{(I, \preceq)}$  then  $(J, J \cap L) \in \mathcal{H}_{(I, \preceq)}$ ,  $(J \cap L, L) \in \mathcal{G}_{(I, \preceq)}$ .

Let  $\mathcal{A}$  be an *abelian category*, or more generally an *exact category*, as in §2.1. Define an  $(I, \preceq)$ -*configuration*  $(\sigma, \iota, \pi)$  in  $\mathcal{A}$  to be maps  $\sigma : \mathcal{F}_{(I, \preceq)} \rightarrow \text{Obj}(\mathcal{A})$ ,  $\iota : \mathcal{G}_{(I, \preceq)} \rightarrow \text{Mor}(\mathcal{A})$ , and  $\pi : \mathcal{H}_{(I, \preceq)} \rightarrow \text{Mor}(\mathcal{A})$ , where

- (i)  $\sigma(J)$  is an object in  $\mathcal{A}$  for  $J \in \mathcal{F}_{(I, \preceq)}$ , with  $\sigma(\emptyset) = 0$ .
- (ii)  $\iota(J, K) : \sigma(J) \rightarrow \sigma(K)$  is *injective* for  $(J, K) \in \mathcal{G}_{(I, \preceq)}$ , and  $\iota(J, J) = \text{id}_{\sigma(J)}$ .
- (iii)  $\pi(J, K) : \sigma(J) \rightarrow \sigma(K)$  is *surjective* for  $(J, K) \in \mathcal{H}_{(I, \preceq)}$ , and  $\pi(J, J) = \text{id}_{\sigma(J)}$ .

These should satisfy the conditions

- (A) Let  $(J, K) \in \mathcal{G}_{(I, \preceq)}$  and set  $L = K \setminus J$ . Then the following is exact in  $\mathcal{A}$ :

$$0 \longrightarrow \sigma(J) \xrightarrow{\iota(J, K)} \sigma(K) \xrightarrow{\pi(K, L)} \sigma(L) \longrightarrow 0. \quad (10)$$

- (B) If  $(J, K) \in \mathcal{G}_{(I, \preceq)}$  and  $(K, L) \in \mathcal{G}_{(I, \preceq)}$  then  $\iota(J, L) = \iota(K, L) \circ \iota(J, K)$ .
- (C) If  $(J, K) \in \mathcal{H}_{(I, \preceq)}$  and  $(K, L) \in \mathcal{H}_{(I, \preceq)}$  then  $\pi(J, L) = \pi(K, L) \circ \pi(J, K)$ .
- (D) If  $(J, K) \in \mathcal{G}_{(I, \preceq)}$  and  $(K, L) \in \mathcal{H}_{(I, \preceq)}$  then

$$\pi(K, L) \circ \iota(J, K) = \iota(J \cap L, L) \circ \pi(J, J \cap L). \quad (11)$$

Note that (A)–(D) make sense because of properties (a)–(d), respectively.

A morphism  $\alpha : (\sigma, \iota, \pi) \rightarrow (\sigma', \iota', \pi')$  of  $(I, \preceq)$ -configurations in  $\mathcal{A}$  is a collection of morphisms  $\alpha(J) : \sigma(J) \rightarrow \sigma'(J)$  for each  $J \in \mathcal{F}_{(I, \preceq)}$  satisfying

$$\begin{aligned} \alpha(K) \circ \iota(J, K) &= \iota'(J, K) \circ \alpha(J) && \text{for all } (J, K) \in \mathcal{G}_{(I, \preceq)} \text{ and} \\ \alpha(K) \circ \pi(J, K) &= \pi'(J, K) \circ \alpha(J) && \text{for all } (J, K) \in \mathcal{H}_{(I, \preceq)}. \end{aligned} \quad (12)$$

It is an *isomorphism* if  $\alpha(J)$  is an isomorphism for all  $J \in \mathcal{F}_{(I, \preceq)}$ . Morphisms compose in the obvious way.

We now show that Definition 4.1 captures the properties of the families of subobjects  $S^J \subset X$  considered in §3.

**Theorem 4.2.** *Let  $(I, \preceq)$  be a finite poset,  $\mathcal{A}$  an abelian category, and  $X \in \mathcal{A}$ . Suppose that for each s-set  $J \subseteq I$  we are given a subobject  $S^J \subset X$ , such that*

$$S^\emptyset = 0, \quad S^I = X, \quad S^A \cap S^B = S^{A \cap B} \quad \text{and} \quad S^A + S^B = S^{A \cup B} \quad (13)$$

*for all s-sets  $A, B \subseteq I$ . Then there exists an  $(I, \preceq)$ -configuration  $(\sigma, \iota, \pi)$  in  $\mathcal{A}$  with  $\sigma(I) = X$  such that  $\iota(J, I) : \sigma(J) \rightarrow X$  represents  $S^J \subset X$  for each s-set  $J \subseteq I$ . This  $(\sigma, \iota, \pi)$  is unique up to canonical isomorphism in  $\mathcal{A}$ .*

**Proof.** Throughout (i)–(iii) and (A)–(D) will refer to Definition 4.1. We divide the proof into the following seven steps:

*Step 1:* Define  $\sigma$  and  $\iota$  on s-sets, and prove (B) for s-sets.

*Step 2:* For  $J, K$  s-sets with  $J \cap K = \emptyset$ , show  $\sigma(J \cup K) \cong \sigma(J) \oplus \sigma(K)$ .

*Step 3:* Define  $\sigma$  on f-sets and  $\pi(J, L)$  for s-sets  $J$ .

*Step 4:* Complete the definitions of  $\iota, \pi$ , and prove (A).

*Step 5:* Prove partial versions of (C), (D), mixing s-sets and f-sets.

*Step 6:* Prove (B), (C), and  $\iota(J, J) = \pi(J, J) = \text{id}_{\sigma(J)}$  in (ii) and (iii).

*Step 7:* Prove (D).

*Step 1:* For each s-set  $J \subseteq I$ , choose  $\sigma(J) \in \mathcal{A}$  and an injective morphism  $\iota(J, I) : \sigma(J) \rightarrow X$  representing  $S^J \subset X$ . Then  $\sigma(J)$  and  $\iota(J, I)$  are unique up to canonical isomorphism. In particular, choose  $\sigma(\emptyset) = 0$  as in (i),  $\sigma(I) = X$ , and  $\iota(I, I) = \text{id}_X$ . Suppose  $J \subseteq K$  are s-sets. Then (13) implies that  $S^J \subset S^K \subset X$ . Hence there exists a unique, injective  $\iota(J, K) : \sigma(J) \rightarrow \sigma(K)$  such that

$$\iota(J, I) = \iota(K, I) \circ \iota(J, K) \quad \text{for } J \subseteq K \text{ s-sets, as in (B).} \quad (14)$$

By uniqueness the two definitions of  $\iota(J, I)$  coincide, and  $\iota(J, J) = \text{id}_{\sigma(J)}$ .

Suppose  $J \subseteq K \subseteq L$  are s-sets. Applying (14) to  $(K, L)$ ,  $(J, K)$ ,  $(J, L)$  gives

$$\iota(L, I) \circ \iota(K, L) \circ \iota(J, K) = \iota(K, I) \circ \iota(J, K) = \iota(J, I) = \iota(L, I) \circ \iota(J, L).$$

Since  $\iota(L, I)$  is injective we can cancel it from both sides, so that

$$\iota(J, L) = \iota(K, L) \circ \iota(J, K), \quad \text{for } J \subseteq K \subseteq L \text{ s-sets, as in (B).} \quad (15)$$

*Step 2:* Let  $J, K$  be s-sets with  $J \cap K = \emptyset$ . We shall show that

$$\iota(J, J \cup K) \circ \pi_{\sigma(J)} + \iota(K, J \cup K) \circ \pi_{\sigma(K)} : \sigma(J) \oplus \sigma(K) \rightarrow \sigma(J \cup K) \quad (16)$$

is an *isomorphism*. Apply Definition 2.5 with  $\iota(J, I) : \sigma(J) \rightarrow X$  in place of  $i : S \rightarrow X$ , and  $\iota(K, I) : \sigma(K) \rightarrow X$  in place of  $j : T \rightarrow X$ . By (13) we may take  $U = \sigma(I \cap J) = \sigma(\emptyset) = 0$ ,  $V = \sigma(J \cup K)$  and  $e = \iota(J \cup K, I)$ . The definition gives  $c : \sigma(J) \rightarrow \sigma(J \cup K)$  with  $\iota(J, I) = \iota(J \cup K, I) \circ c$ , so  $c = \iota(J, J \cup K)$  by (14) and injectivity of  $\iota(J \cup K, I)$ . Similarly  $d = \iota(K, J \cup K)$ . Thus (16) is the second map in (1). As  $U = 0$ , exactness implies (16) is an isomorphism.

*Step 3:* Let  $L \subseteq I$  be an f-set which is not an s-set or a q-set, and define  $J' = \{i \in I \setminus L : l \nmid i \text{ for all } l \in L\}$  and  $K' = J' \cup L$ . Then  $J' \subset K'$  are s-sets with  $L = K' \setminus J'$ . Choose  $\sigma(L) \in \mathcal{A}$  and a surjective  $\pi(K', L) : \sigma(K') \rightarrow \sigma(L)$  which is a *cokernel* for  $\iota(J', K') : \sigma(J') \rightarrow \sigma(K')$ . Then  $\sigma(L)$ ,  $\pi(K', L)$  are unique up to canonical isomorphism.

If  $L$  is an s-set then  $J', L$  are s-sets with  $J' \cap L = \emptyset$ , and Step 2 shows that  $\sigma(K') \cong \sigma(J') \oplus \sigma(L)$ , and we take  $\pi(K', L)$  to be the natural projection with  $\pi(K', L) \circ \iota(L, K') = \text{id}_{\sigma(L)}$ .

Now let  $J \subset K$  be s-sets in  $I$  with  $K \setminus J = L$ . Then  $J \subseteq J'$  and  $K \subseteq K'$ , as  $J', K'$  are defined to be as large as possible, and  $J' \cap K = J$ ,  $J' \cup K' = K'$ . Let  $c : \sigma(K) \rightarrow C$  be a cokernel for  $\iota(J, K)$ , and consider the commutative diagram with rows short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \sigma(J) & \xrightarrow{\iota(J, K)} & \sigma(K) & \xrightarrow{c} & C \longrightarrow 0 \\ & & \downarrow \iota(J, J') & & \downarrow \iota(K, K') & & \downarrow h \\ 0 & \longrightarrow & \sigma(J') & \xrightarrow{\iota(J', K')} & \sigma(K') & \xrightarrow{\pi(K', L)} & \sigma(L) \longrightarrow 0, \end{array} \quad (17)$$

where  $h$  is not yet constructed. The first square of (17) commutes by (15), so

$$0 = \pi(K', L) \circ \iota(J', K') \circ \iota(J, J') = \pi(K', L) \circ \iota(K, K') \circ \iota(J, K).$$

But  $c$  is the cokernel of  $\iota(J, K)$ , so there exists a unique  $h : C \rightarrow \sigma(L)$  such that  $h \circ c = \pi(K', L) \circ \iota(K, K')$ , that is, the second square in (17) commutes.

As  $S^J = S^{J'} \cap S^K$  and  $S^{K'} = S^{J'} + S^K$  by (13), Eq. (1) implies that

$$0 \rightarrow \sigma(J) \xrightarrow{\iota_{\sigma(J')} \circ \iota(J, J') - \iota_{\sigma(K)} \circ \iota(J, K)} \sigma(J') \oplus \sigma(K) \xrightarrow{\iota(J', K') \circ \pi_{\sigma(J')} + \iota(K, K') \circ \pi_{\sigma(K)}} \sigma(K') \rightarrow 0$$

is exact. As the composition of the first map with  $c \circ \pi_{\sigma(K)}$  is zero we see that

$$c \circ \pi_{\sigma(K)} = l \circ (\iota(J', K') \circ \pi_{\sigma(J')} + \iota(K, K') \circ \pi_{\sigma(K)})$$

for some unique  $l : \sigma(K') \rightarrow C$ , by definition of cokernel. Composing with  $\iota_{\sigma(J')}$  gives  $l \circ \iota(J', K') = 0$ , so  $l = m \circ \pi(K', L)$  for some unique  $m : \sigma(L) \rightarrow C$ , by exactness of the bottom line of (17).

Then  $m \circ h \circ c = m \circ \pi(K', L) \circ \iota(K, K') = l \circ \iota(K, K') = c = \text{id}_C \circ c$ , so as  $c$  is surjective we have  $m \circ h = \text{id}_C$ . Also  $\pi(K', L) = h \circ l$ , since

$$\begin{aligned} h \circ l \circ (\iota(J', K') \circ \pi_{\sigma(J')} + \iota(K, K') \circ \pi_{\sigma(K)}) \\ = h \circ c \circ \pi_{\sigma(K)} \\ = \pi(K', L) \circ \iota(K, K') \circ \pi_{\sigma(K)} = \pi(K', L) \circ (\iota(J', K') \circ \pi_{\sigma(J')} + \iota(K, K') \circ \pi_{\sigma(K)}), \end{aligned}$$

and  $\iota(J', K') \circ \pi_{\sigma(J')} + \iota(K, K') \circ \pi_{\sigma(K)}$  is surjective. Hence  $h \circ m \circ \pi(K', L) = h \circ l = \pi(K', L) = \text{id}_{\sigma(L)} \circ \pi(K', L)$ , and  $h \circ m = \text{id}_{\sigma(L)}$  as  $\pi(K', L)$  is surjective. Thus  $m = h^{-1}$ , and  $h$  is an isomorphism.

Define  $\pi(K, L) = \pi(K', L) \circ \iota(K, K')$ , so that  $\pi(K, L) = h \circ c$  as (17) is commutative. As  $h$  is an isomorphism and  $c$  is a cokernel for  $\iota(J, K)$ , we see that  $\pi(K, L) : \sigma(K) \rightarrow \sigma(L)$  is a cokernel for  $\iota(J, K) : \sigma(J) \rightarrow \sigma(K)$ . Hence  $\pi(K, L)$  is surjective, and (10) is exact when  $J \subseteq K$  are s-sets.

Suppose now that  $J, K$  are s-sets and  $L$  is an f-set with  $(J, K) \in \mathcal{G}_{(I, \preccurlyeq)}$  and  $(J, L), (K, L) \in \mathcal{H}_{(I, \preccurlyeq)}$ . Define  $K'$  using  $L$  as above. Then  $J \subseteq K \subseteq K'$  and

$$\pi(J, L) = \pi(K', L) \circ \iota(J, K') = \pi(K', L) \circ \iota(K, K') \circ \iota(J, K) = \pi(K, L) \circ \iota(J, K),$$

by (15) and the definitions of  $\pi(J, L), \pi(K, L)$ . Hence

$$\pi(J, L) = \pi(K, L) \circ \iota(K, J) \quad \text{when } J, K \text{ are s-sets.} \quad (18)$$

*Step 4:* Let  $(J, K) \in \mathcal{G}_{(I, \preccurlyeq)}$ , and define  $A' = \{i \in I \setminus K : k \not\preccurlyeq i \text{ for all } k \in K\}$ ,  $B' = A' \cup J$  and  $C' = A' \cup K$ . Then  $A' \subseteq B' \subseteq C'$  are s-sets with  $J = B' \setminus A'$ ,  $K = C' \setminus A'$ , and  $K \setminus J = C' \setminus B'$ , and they are the largest s-sets with this property.



Consider the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \sigma(A') & \xrightarrow{\iota(A', B')} & \sigma(B') & \xrightarrow{\pi(B', J)} & \sigma(J) \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 & & \text{id}_{\sigma(A')} & & \iota(B', C') & & \iota(J, K) \\
 0 & \longrightarrow & \sigma(A') & \xrightarrow{\iota(A', C')} & \sigma(C') & \xrightarrow{\pi(C', K)} & \sigma(K) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \pi(C', K \setminus J) & & \pi(K, K \setminus J) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \sigma(K \setminus J) & \xlongequal{\text{id}_{\sigma(K \setminus J)}} & \sigma(K \setminus J) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array} \tag{19}$$

Here solid arrows ‘ $\rightarrow$ ’ have already been defined, and dashed arrows ‘ $\dashrightarrow$ ’ remain to be constructed. The left-hand square commutes by (15).

Now  $\pi(C', K \setminus J) \circ \iota(A', C') = \pi(C', K \setminus J) \circ \iota(B', C') \circ \iota(A', B') = 0$  as the middle column is exact. Since  $\pi(C', K)$  is the cokernel of  $\iota(A', C')$ , there exists a unique  $\pi(K, K \setminus J) : \sigma(K) \rightarrow \sigma(K \setminus J)$  with  $\pi(K, K \setminus J) \circ \pi(C', K) = \pi(C', K \setminus J)$ . As  $\pi(C', K \setminus J)$  is surjective,  $\pi(K, K \setminus J)$  is surjective as in (iii). Thus in (19) the lower dashed arrow exists, and the lower square commutes.

Suppose  $f : D \rightarrow \sigma(B')$  with  $\pi(C', K) \circ \iota(B', C') \circ f = 0$ . Then there is a unique  $h : D \rightarrow \sigma(A')$  with  $\iota(A', C') \circ h = \iota(B', C') \circ f$ , as  $\iota(A', C')$  is the kernel of  $\pi(C', K)$ . But then  $\iota(B', C') \circ \iota(A', B') \circ h = \iota(B', C') \circ f$ , so  $\iota(A', B') \circ h = f$  as  $\iota(B', C')$  is injective. Thus  $\iota(A', B')$  is the kernel of  $\pi(C', K) \circ \iota(B', C')$ . Similarly,  $\pi(K, K \setminus J)$  is the cokernel of  $\pi(C', K) \circ \iota(B', C')$ .

Now apply Definition 2.1(iv) to  $\pi(C', K) \circ \iota(B', C')$ . As it has kernel  $\iota(A', B')$  and cokernel  $\pi(K, K \setminus J)$ , and  $\pi(B', J)$  is the cokernel of  $\iota(A', B')$ , this gives a unique  $\iota(J, K) : \sigma(J) \rightarrow \sigma(K)$  with  $\iota(J, K) \circ \pi(B', J) = \pi(C', K) \circ \iota(B', C')$ , such that  $\iota(J, K)$  is the kernel of  $\pi(K, K \setminus J)$ . Thus  $\iota(J, K)$  is injective, as in (ii), and in (19) the upper dashed arrow exists, the upper right square commutes, and the right-hand column is exact, proving (A).

We should also check that if  $J, K$  are s-sets, the definition above gives the same answer for  $\iota(J, K)$  as Step 1, and for  $\pi(K, K \setminus J)$  as Step 3. If  $J, K$  are s-sets then  $A', J, K$  are s-sets with  $A' \cap J = A' \cap K = \emptyset$ , so Step 2 gives  $\sigma(B') \cong \sigma(A') \oplus \sigma(J)$  and  $\sigma(C') \cong \sigma(A') \oplus \sigma(K)$ . Substituting these into (19), we find the definitions are consistent.

*Step 5:* Let  $C$  be an s-set and  $D, E$  f-sets with  $(C, D), (C, E), (D, E) \in \mathcal{H}_{(I, \leq)}$ , so that  $C \supseteq D \supseteq E$ . Apply Step 4 with  $J = D \setminus E$  and  $K = D$ . This gives  $C'$  which is

the largest s-set with  $(C', D) \in \mathcal{H}_{(I, \preccurlyeq)}$ , so  $C \subseteq C'$ . Therefore

$$\pi(C, E) = \pi(C', E) \circ \iota(C, C') = \pi(D, E) \circ \pi(C', D) \circ \iota(C', C) = \pi(D, E) \circ \pi(C, D),$$

using (18) for the first and third steps, and commutativity of the bottom square in (19) for the second. Hence

$$\pi(C, E) = \pi(D, E) \circ \pi(C, D) \quad \text{for } C \text{ an s-set, as in (C).} \quad (20)$$

Suppose  $J, K$  are s-sets and  $L$  an f-set with  $(J, K) \in \mathcal{G}_{(I, \preccurlyeq)}$  and  $(K, L) \in \mathcal{H}_{(I, \preccurlyeq)}$ . Then  $J \cap L$  is an s-set with  $(J, J \cap L) \in \mathcal{H}_{(I, \preccurlyeq)}$  and  $(J \cap L, L) \in \mathcal{G}_{(I, \preccurlyeq)}$ . As in Step 4 with  $J, K$  replaced by  $J \cap L, L$ , define  $A' = \{i \in I \setminus L : l \not\preccurlyeq i \text{ for all } l \in L\}$ ,  $B' = A' \cup (J \cap L)$  and  $C' = A' \cup L$ . Then  $J \subseteq B'$  and  $K \subseteq C'$ . Therefore

$$\begin{aligned} \iota(J \cap L, L) \circ \pi(J, J \cap L) &= \iota(J \cap L, L) \circ \pi(B', J \cap L) \circ \iota(J, B') \\ &= \pi(C', L) \circ \iota(B', C') \circ \iota(J, B') = \pi(C', L) \circ \iota(B, C') \\ &= \pi(C', L) \circ \iota(K, C') \circ \iota(J, K) \\ &= \pi(K, L) \circ \iota(J, K), \end{aligned}$$

using (18) at the first and fifth steps, commutativity of the upper right square in (19) at the second, and (15) at the third and fourth. This proves

$$\pi(K, L) \circ \iota(J, K) = \iota(J \cap L, L) \circ \pi(J, J \cap L) \quad \text{for } J, K \text{ s-sets, as in (D).} \quad (21)$$

*Step 6:* Suppose  $(J, K), (K, L) \in \mathcal{G}_{(I, \preccurlyeq)}$ , and define  $D = \{i \in I : i \preccurlyeq l \text{ for some } l \in L\}$ ,  $A = D \setminus L$ ,  $B = A \cup J$ , and  $C = A \cup K$ . Then  $A \subseteq B \subseteq C \subseteq D$  are s-sets, with  $J = B \setminus A$ ,  $K = C \setminus A$ , and  $L = D \setminus A$ . Therefore

$$\begin{aligned} \iota(K, L) \circ \iota(J, K) \circ \pi(B, J) &= \iota(K, L) \circ \pi(C, K) \circ \iota(B, C) \\ &= \pi(D, L) \circ \iota(C, D) \circ \iota(B, C) \\ &= \pi(D, L) \circ \iota(B, D) = \iota(J, L) \circ \pi(B, J), \end{aligned}$$

using (21) at the first, second and fourth steps with  $J = B \cap K$ ,  $K = C \cap L$  and  $J = B \cap L$ , respectively, and (15) at the third. As  $\pi(B, J)$  is surjective this implies that  $\iota(K, L) \circ \iota(J, K) = \iota(J, L)$ , proving (B).

Similarly, suppose  $(J, K), (K, L) \in \mathcal{H}_{(I, \preccurlyeq)}$ , and define  $D = \{i \in I : i \preccurlyeq j \text{ for some } j \in J\}$ ,  $A = D \setminus J$ ,  $B = D \setminus K$ , and  $C = D \setminus L$ . Then  $A \subseteq B \subseteq C \subseteq D$  are s-sets, with  $J = D \setminus A$ ,  $K = D \setminus B$ , and  $L = D \setminus C$ . Therefore

$$\pi(J, L) \circ \pi(D, J) = \pi(D, L) = \pi(K, L) \circ \pi(D, K) = \pi(K, L) \circ \pi(J, K) \circ \pi(D, J),$$

using (20) three times. As  $\pi(D, J)$  is surjective, this proves (C). Applying (B), (C) with  $J = K = L$  gives  $\iota(J, J) = \pi(J, J) = \text{id}_{\sigma(J)}$ , as in (ii) and (iii).

*Step 7:* Suppose  $(J, K) \in \mathcal{G}_{(I, \preceq)}$  and  $(K, L) \in \mathcal{H}_{(I, \preceq)}$ , and define  $C = \{i \in I : i \preceq k \text{ for some } k \in K\}$ ,  $A = C \setminus K$  and  $B = A \cup J$ . Then  $A \subseteq B \subseteq C$  are s-sets, with  $J = B \setminus A$  and  $K = C \setminus A$ . Therefore

$$\begin{aligned} \pi(K, L) \circ \iota(J, K) \circ \pi(B, J) &= \pi(K, L) \circ \pi(C, K) \circ \iota(B, C) = \pi(C, L) \circ \iota(B, C) \\ &= \iota(J \cap L, L) \circ \pi(B, J \cap L) \\ &= \iota(J \cap L, L) \circ \pi(J, J \cap L) \circ \pi(B, J), \end{aligned}$$

using (21) at the first and third steps with  $J = B \cap K$ ,  $B \cap L = J \cap L$ , respectively, and (C) at the second and fourth. As  $\pi(B, J)$  is surjective this proves (D).

Hence  $(\sigma, \iota, \pi)$  is an  $(I, \preceq)$ -configuration, in the sense of Definition 4.1. It remains only to show that  $(\sigma, \iota, \pi)$  is *unique up to canonical isomorphism* in  $\mathcal{A}$ . At each stage in the construction the objects and morphisms were determined either uniquely up to canonical isomorphism, or uniquely. Thus, if  $(\sigma, \iota, \pi)$ ,  $(\sigma', \iota', \pi')$  both satisfy the conditions of the theorem, one can go through the steps above and construct a canonical isomorphism between them.  $\square$

Applying the theorem to the situation of §3 gives

**Corollary 4.3.** *Let  $\mathcal{A}$ ,  $X$ ,  $I$  and  $\preceq$  be as in Definition 3.1. Then there exists an  $(I, \preceq)$ -configuration  $(\sigma, \iota, \pi)$  in  $\mathcal{A}$  with  $\sigma(I) = X$  and  $\sigma(\{i\}) \cong S^i$  for  $i \in I$ , such that  $\iota(J, I) : \sigma(J) \rightarrow X$  represents the subobject of  $X$  corresponding to  $J$  under the 1-1 correspondence of Proposition 3.8 for each s-set  $J \subseteq I$ . This  $(\sigma, \iota, \pi)$  is unique up to canonical isomorphism in  $\mathcal{A}$ .*

As the s-sets of  $(\{1, \dots, n\}, \leq)$  are  $\{1, \dots, j\}$  for  $0 \leq j \leq n$ , the corresponding subobjects  $S^j$  form a *filtration*  $0 = A_0 \subset \dots \subset A_n = X$ , so Theorem 4.2 gives

**Corollary 4.4.** *Let  $0 = A_0 \subset A_1 \subset \dots \subset A_n = X$  be a filtration in an abelian category  $\mathcal{A}$ . Then there is a  $(\{1, \dots, n\}, \leq)$ -configuration  $(\sigma, \iota, \pi)$  in  $\mathcal{A}$ , unique up to canonical isomorphism, such that  $\iota(\{1, \dots, j\}, \{1, \dots, n\}) : \sigma(\{1, \dots, j\}) \rightarrow X$  represents  $A_j \subset X$  for  $j = 0, \dots, n$ .*

This shows we can regard  $(I, \preceq)$ -configurations as *generalized filtrations*. Here is the converse to Theorem 4.2.

**Theorem 4.5.** *Let  $(\sigma, \iota, \pi)$  be an  $(I, \preceq)$ -configuration in an abelian category  $\mathcal{A}$ . Define  $X = \sigma(I)$ , and let  $S^J \subset X$  be represented by  $\iota(J, I) : \sigma(J) \rightarrow X$  for each s-set  $J \subseteq I$ . Then the  $S^J$  satisfy (13).*

**Proof.** The first two equations of (13) are obvious. So suppose  $A, B \subseteq I$  are s-sets. Definition 2.5 with  $S = \sigma(A)$ ,  $T = \sigma(B)$ ,  $i = \iota(A, I)$  and  $j = \iota(B, I)$  gives  $U \in \mathcal{A}$

and  $a : U \rightarrow \sigma(A)$ ,  $b : U \rightarrow \sigma(B)$  with  $i \circ a = j \circ b$ , such that  $i \circ a : U \rightarrow X$  represents  $S^A \cap S^B$ . As  $i \circ \iota(A \cap B, A) = \iota(A \cap B, I) = j \circ (A \cap B, B)$ , by exactness in (1) there is a unique  $h : \sigma(A \cap B) \rightarrow U$  with

$$(\iota_{\sigma(A)} \circ a - \iota_{\sigma(B)} \circ b) \circ h = \iota_{\sigma(A)} \circ \iota(A \cap B, A) - \iota_{\sigma(B)} \circ \iota(A \cap B, B).$$

Composing  $\pi_{\sigma(A)}$ ,  $\pi_{\sigma(B)}$  gives  $\iota(A \cap B, A) = a \circ h$  and  $\iota(A \cap B, B) = b \circ h$ . Now

$$\iota(A \cup B, I) \circ \iota(A, A \cup B) \circ a = i \circ a = j \circ b = \iota(A \cup B, I) \circ \iota(B, A \cup B) \circ b,$$

by Definition 4.1(B). Thus  $\iota(A, A \cup B) \circ a = \iota(B, A \cup B) \circ b$ , as  $\iota(A \cup B, I)$  is injective. Hence

$$\begin{aligned} \pi(A, A \setminus B) \circ a &= \iota(A \setminus B, A \setminus B) \circ \pi(A, A \setminus B) \circ a \\ &= \pi(A \cup B, A \setminus B) \circ \iota(A, A \cup B) \circ a \\ &= \pi(A \cup B, A \setminus B) \circ \iota(B, A \cup B) \circ b = 0, \end{aligned}$$

using Definition 4.1(D) at the second step and exactness in (A) at the fourth.

But  $\iota(A \cap B, A)$  is the kernel of  $\pi(A, A \setminus B)$ , so there is a unique  $h' : U \rightarrow \sigma(A \cap B)$  with  $a = \iota(A \cap B, A) \circ h'$ . As  $\iota(A \cap B, A) = a \circ h$  and  $a, \iota(A \cap B, A)$  are injective we see that  $h, h'$  are *inverse*, so  $h$  is *invertible*. This implies that

$$\iota(A \cap B, I) = \iota(A, I) \circ \iota(A \cap B, A) = \iota(A, I) \circ a \circ h : \sigma(A \cap B) \rightarrow X$$

represents  $S^A \cap S^B$ , so that  $S^{A \cap B} = S^A \cap S^B$ . We prove  $S^{A \cup B} = S^A + S^B$  in a similar way.  $\square$

Combining Theorems 4.2 and 4.5 we deduce:

**Corollary 4.6.** *For  $(I, \preceq)$  a finite poset and  $\mathcal{A}$  an abelian category, there is an equivalence of categories between the groupoid of  $(I, \preceq)$ -configurations in  $\mathcal{A}$ , and the groupoid of collections  $(X \in \mathcal{A}, \text{ subobjects } S^J \subset X \text{ for } s\text{-sets } J \subset I)$  satisfying (13), with the obvious notion of isomorphism.*

Finally, for an  $(I, \preceq)$ -configuration  $(\sigma, \iota, \pi)$  we show how the classes  $[\sigma(J)]$  in the Grothendieck group  $K_0(\mathcal{A})$  are related.

**Proposition 4.7.** *Let  $(\sigma, \iota, \pi)$  be an  $(I, \preceq)$ -configuration in an abelian category  $\mathcal{A}$ . Then there exists a unique map  $\kappa : I \rightarrow K_0(\mathcal{A})$  such that  $[\sigma(J)] = \sum_{j \in J} \kappa(j)$  in  $K_0(\mathcal{A})$  for all  $f$ -sets  $J \subseteq I$ .*

**Proof.** Combining Definitions 2.2 and 3.1(A) shows that

$$[\sigma(K)] = [\sigma(J)] + [\sigma(K \setminus J)] \quad \text{for all } (J, K) \in \mathcal{G}_{(I, \preccurlyeq)}. \quad (22)$$

Define  $\kappa : I \rightarrow K_0(\mathcal{A})$  by  $\kappa(i) = [\sigma(\{i\})]$ . As  $\{i\} \in \mathcal{F}_{(I, \preccurlyeq)}$  for all  $i \in I$  this is unique and well-defined. Suppose  $K \in \mathcal{F}_{(I, \preccurlyeq)}$  with  $|K| \geq 1$ . Let  $j \in K$  be  $\preccurlyeq$ -minimal. Then  $(\{j\}, K) \in \mathcal{G}_{(I, \preccurlyeq)}$ , so (22) gives  $[\sigma(K)] = \kappa(j) + [\sigma(K \setminus \{j\})]$ . Thus  $[\sigma(J)] = \sum_{j \in J} \kappa(j)$  for all  $J \in \mathcal{F}_{(I, \preccurlyeq)}$  by induction on  $|J|$ , completing the proof.  $\square$

## 5. New $(I, \preccurlyeq)$ -configurations from old

Let  $(\sigma, \iota, \pi)$  be an  $(I, \preccurlyeq)$ -configuration in an abelian or exact category  $\mathcal{A}$ . Then we can derive  $(K, \trianglelefteq)$ -configurations  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  in  $\mathcal{A}$  from  $(\sigma, \iota, \pi)$  for other, simpler finite posets  $(K, \trianglelefteq)$ , by forgetting some of the information in  $(\sigma, \iota, \pi)$ . The next two definitions give two ways to do this. We use the notation of §§3 and 4.

**Definition 5.1.** Let  $(I, \preccurlyeq)$  be a finite poset and  $J \in \mathcal{F}_{(I, \preccurlyeq)}$ . Then  $(J, \preccurlyeq)$  is also a finite poset, and  $K \subseteq J$  is an f-set in  $(J, \preccurlyeq)$  if and only if it is an f-set in  $(I, \preccurlyeq)$ . Hence  $\mathcal{F}_{(J, \preccurlyeq)} \subseteq \mathcal{F}_{(I, \preccurlyeq)}$ . We also have  $\mathcal{G}_{(J, \preccurlyeq)} = \mathcal{G}_{(I, \preccurlyeq)} \cap (\mathcal{F}_{(J, \preccurlyeq)} \times \mathcal{F}_{(J, \preccurlyeq)})$  and  $\mathcal{H}_{(J, \preccurlyeq)} = \mathcal{H}_{(I, \preccurlyeq)} \cap (\mathcal{F}_{(J, \preccurlyeq)} \times \mathcal{F}_{(J, \preccurlyeq)})$ , so that  $\mathcal{G}_{(J, \preccurlyeq)} \subseteq \mathcal{G}_{(I, \preccurlyeq)}$  and  $\mathcal{H}_{(J, \preccurlyeq)} \subseteq \mathcal{H}_{(I, \preccurlyeq)}$ .

Let  $(\sigma, \iota, \pi)$  be an  $(I, \preccurlyeq)$ -configuration in an abelian or exact category  $\mathcal{A}$ , and define  $\sigma' : \mathcal{F}_{(J, \preccurlyeq)} \rightarrow \text{Obj}(\mathcal{A})$ ,  $\iota' : \mathcal{G}_{(J, \preccurlyeq)} \rightarrow \text{Mor}(\mathcal{A})$  and  $\pi' : \mathcal{H}_{(J, \preccurlyeq)} \rightarrow \text{Mor}(\mathcal{A})$  by  $\sigma' = \sigma|_{\mathcal{F}_{(J, \preccurlyeq)}}$ ,  $\iota' = \iota|_{\mathcal{G}_{(J, \preccurlyeq)}}$  and  $\pi' = \pi|_{\mathcal{H}_{(J, \preccurlyeq)}}$ . Then (A)–(D) of Definition 4.1 for  $(\sigma, \iota, \pi)$  imply (A)–(D) for  $(\sigma', \iota', \pi')$ , so  $(\sigma', \iota', \pi')$  is a  $(J, \preccurlyeq)$ -configuration in  $\mathcal{A}$ . We call  $(\sigma', \iota', \pi')$  a *subconfiguration* of  $(\sigma, \iota, \pi)$ .

**Definition 5.2.** Let  $(I, \preccurlyeq), (K, \trianglelefteq)$  be finite posets, and  $\phi : I \rightarrow K$  be surjective with  $\phi(i) \trianglelefteq \phi(j)$  when  $i, j \in I$  with  $i \preccurlyeq j$ . If  $J \subseteq K$  is an f-set in  $K$  then  $\phi^{-1}(J) \subseteq I$  is an f-set in  $I$ . Hence  $\phi^*(\mathcal{F}_{(K, \trianglelefteq)}) \subseteq \mathcal{F}_{(I, \preccurlyeq)}$ , where  $\phi^*$  pulls back subsets of  $K$  to subsets of  $I$ . Similarly,  $\phi^*(\mathcal{G}_{(K, \trianglelefteq)}) \subseteq \mathcal{G}_{(I, \preccurlyeq)}$  and  $\phi^*(\mathcal{H}_{(K, \trianglelefteq)}) \subseteq \mathcal{H}_{(I, \preccurlyeq)}$ .

Let  $(\sigma, \iota, \pi)$  be an  $(I, \preccurlyeq)$ -configuration in an abelian or exact category  $\mathcal{A}$ , and define  $\tilde{\sigma} : \mathcal{F}_{(K, \trianglelefteq)} \rightarrow \text{Obj}(\mathcal{A})$ ,  $\tilde{\iota} : \mathcal{G}_{(K, \trianglelefteq)} \rightarrow \text{Mor}(\mathcal{A})$  and  $\tilde{\pi} : \mathcal{H}_{(K, \trianglelefteq)} \rightarrow \text{Mor}(\mathcal{A})$  by  $\tilde{\sigma}(A) = \sigma(\phi^{-1}(A))$ ,  $\tilde{\iota}(A, B) = \iota(\phi^{-1}(A), \phi^{-1}(B))$ , and  $\tilde{\pi}(A, B) = \pi(\phi^{-1}(A), \phi^{-1}(B))$ . Then  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  is a  $(K, \trianglelefteq)$ -configuration in  $\mathcal{A}$ , the *quotient configuration* of  $(\sigma, \iota, \pi)$ . We also call  $(\sigma, \iota, \pi)$  a *refinement* of  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ .

Compositions of these constructions all behave in the obvious ways. Next we explain a method to *glue two configurations*  $(\sigma', \iota', \pi')$ ,  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  together, to get  $(\sigma, \iota, \pi)$  containing  $(\sigma', \iota', \pi')$  as a *subconfiguration*, and  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  as a *quotient configuration*. Consider the following situation.

**Definition 5.3.** Let  $(J, \preccurlyeq), (K, \trianglelefteq)$  be finite posets and  $L \subset K$  an f-set, with  $J \cap (K \setminus L) = \emptyset$ . Suppose  $\psi : J \rightarrow L$  is a surjective map with  $\psi(i) \trianglelefteq \psi(j)$  when  $i, j \in J$  with

$i \lesssim j$ . Set  $I = J \cup (K \setminus L)$ , and define a binary relation  $\preccurlyeq$  on  $I$  by

$$i \preccurlyeq j \quad \text{for } i, j \in I \quad \text{if} \quad \begin{cases} i \lesssim j, & i, j \in J, \\ i \trianglelefteq j, & i, j \in K \setminus L, \\ \psi(i) \trianglelefteq j, & i \in J, \quad j \in K \setminus L, \\ i \trianglelefteq \psi(j), & i \in K \setminus L, \quad j \in J. \end{cases}$$

One can show  $\preccurlyeq$  is a partial order on  $I$ , and  $J \subseteq I$  an f-set in  $(I, \preccurlyeq)$ . The restriction of  $\preccurlyeq$  to  $J$  is  $\lesssim$ . Define  $\phi : I \rightarrow K$  by  $\phi(i) = \psi(i)$  if  $i \in J$  and  $\phi(i) = i$  if  $i \in K \setminus L$ . Then  $\phi$  is surjective, with  $\phi(i) \trianglelefteq \phi(j)$  when  $i, j \in I$  with  $i \preccurlyeq j$ .

An  $(I, \preccurlyeq)$ -configuration gives the same  $(L, \trianglelefteq)$ -configuration in two ways.

**Lemma 5.4.** *In the situation of Definition 5.3, suppose  $(\sigma, \iota, \pi)$  is an  $(I, \preccurlyeq)$ -configuration in an abelian or exact category. Let  $(\sigma', \iota', \pi')$  be its  $(J, \lesssim)$ -subconfiguration, and  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  its quotient  $(K, \trianglelefteq)$ -configuration from  $\phi$ . Let  $(\hat{\sigma}, \hat{\iota}, \hat{\pi})$  be the quotient  $(L, \trianglelefteq)$ -configuration from  $(\sigma', \iota', \pi')$  and  $\psi$ , and  $(\check{\sigma}, \check{\iota}, \check{\pi})$  the  $(L, \trianglelefteq)$ -subconfiguration from  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ . Then  $(\hat{\sigma}, \hat{\iota}, \hat{\pi}) = (\check{\sigma}, \check{\iota}, \check{\pi})$ .*

Our third construction is a kind of converse to Lemma 5.4. In categorical notation, the last part says there is an *equivalence of categories* between the category of  $(I, \preccurlyeq)$ -configurations, and the *fibred product* of the categories of  $(J, \lesssim)$ -configurations and  $(K, \trianglelefteq)$ -configurations over  $(L, \trianglelefteq)$ -configurations.

**Theorem 5.5.** *In the situation of Definition 5.3, let  $\mathcal{A}$  be an abelian or exact category,  $(\sigma', \iota', \pi')$  a  $(J, \lesssim)$ -configuration in  $\mathcal{A}$ , and  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  a  $(K, \trianglelefteq)$ -configuration in  $\mathcal{A}$ . Define  $(\hat{\sigma}, \hat{\iota}, \hat{\pi})$  to be the quotient  $(L, \trianglelefteq)$ -configuration from  $(\sigma', \iota', \pi')$  and  $\psi$ , and  $(\check{\sigma}, \check{\iota}, \check{\pi})$  to be the  $(L, \trianglelefteq)$ -subconfiguration from  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ .*

*Suppose  $(\hat{\sigma}, \hat{\iota}, \hat{\pi}) = (\check{\sigma}, \check{\iota}, \check{\pi})$ . Then there exists an  $(I, \preccurlyeq)$ -configuration  $(\sigma, \iota, \pi)$  in  $\mathcal{A}$ , unique up to canonical isomorphism, such that  $(\sigma', \iota', \pi')$  is its  $(J, \lesssim)$ -subconfiguration, and  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  its quotient  $(K, \trianglelefteq)$ -configuration from  $\phi$ .*

*More generally, given  $\alpha : (\check{\sigma}, \check{\iota}, \check{\pi}) \xrightarrow{\cong} (\hat{\sigma}, \hat{\iota}, \hat{\pi})$  there is an  $(I, \preccurlyeq)$ -configuration  $(\sigma, \iota, \pi)$  in  $\mathcal{A}$ , unique up to canonical isomorphism, with  $(J, \lesssim)$ -subconfiguration isomorphic to  $(\sigma', \iota', \pi')$ , and quotient  $(K, \trianglelefteq)$ -configuration from  $\phi$  isomorphic to  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ , such that the equality in Lemma 5.4 corresponds to  $\alpha$ .*

**Proof.** Let  $\mathcal{A}$  be an abelian category. We prove the first part in five steps:

*Step 1:* Characterize  $(I, \preccurlyeq)$  s-sets.

*Step 2:* Define  $\sigma(B)$  for all  $(I, \preccurlyeq)$  s-sets, and some morphisms  $\iota(B, C)$ .

*Step 3:* Define  $\iota(B, B')$  for all  $(I, \preccurlyeq)$  s-sets  $B \subseteq B' \subseteq I$ , and prove  $\iota = \iota \circ \iota$ .

*Step 4:* Let  $S^B$  be the subobject represented by  $\iota(B, I) : \sigma(B) \rightarrow \sigma(I) = X$  for all  $(I, \preccurlyeq)$  s-sets  $B$ . Show that the  $S^B$  satisfy (13).

Step 5: Apply Theorem 4.2 to construct  $(\sigma, \iota, \pi)$ , and complete the proof.

Step 1: The proof of the next lemma is elementary, and left as an exercise.

**Lemma 5.6.** *In the situation above, let  $B \subseteq I$  be an  $s$ -set in  $(I, \preceq)$ . Define  $P = \{k \in K : \text{if } i \in I \text{ and } \phi(i) \leq k, \text{ then } i \in B\}$ , and  $R = \{k \in K : k \leq \phi(i) \text{ for some } i \in B\}$ . Define  $A = \phi^{-1}(P)$  and  $C = \phi^{-1}(R)$ . Then  $P \subseteq R$  are  $(K, \leq)$   $s$ -sets, and  $A \subseteq B \subseteq C$  are  $(I, \preceq)$   $s$ -sets, with  $P \setminus L = R \setminus L = B \setminus J$ . Define  $D = A \cap J$ ,  $E = B \cap J$  and  $F = C \cap J$ . Then  $D \subseteq E \subseteq F$  are  $(J, \lesssim)$   $s$ -sets, with  $(P, R) \in \mathcal{G}_{(K, \leq)}$  and  $(D, E), (D, F), (E, F) \in \mathcal{G}_{(J, \lesssim)}$ . Define  $U = P \cap L$  and  $W = R \cap L$ . Then  $U \subseteq W$  are  $(L, \leq)$   $s$ -sets with  $\phi^{-1}(U) = \psi^{-1}(U) = D$  and  $\phi^{-1}(W) = \psi^{-1}(W) = F$ . Hence*

$$\begin{aligned} \sigma'(D) &= \hat{\sigma}(U) = \check{\sigma}(U) = \tilde{\sigma}(U), \text{ and similarly } \sigma'(F \setminus D) = \tilde{\sigma}(W \setminus U), \\ \sigma'(F) &= \tilde{\sigma}(W), \quad \iota'(D, F) = \tilde{\iota}(U, W), \quad \pi'(F, F \setminus D) = \tilde{\pi}(W, W \setminus U). \end{aligned} \quad (23)$$

Here  $P, R$  are the largest, smallest  $(K, \leq)$   $s$ -sets with  $\phi^{-1}(P) \subseteq B \subseteq \phi^{-1}(R)$ .

Step 2: Let  $B$  be an  $(I, \preceq)$   $s$ -set, and use the notation of Lemma 5.6. As  $R \setminus P = W \setminus U$  we have  $\tilde{\sigma}(R \setminus P) = \tilde{\sigma}(W \setminus U) = \sigma'(F \setminus D)$  by (23). Consider

$$\pi'(F \setminus D, F \setminus E) \circ \tilde{\pi}(R, R \setminus P) : \tilde{\sigma}(R) \rightarrow \sigma'(F \setminus E). \quad (24)$$

Choose  $\sigma(B) \in \mathcal{A}$  and  $\iota(B, C) : \sigma(B) \rightarrow \sigma(C) = \tilde{\sigma}(R)$  to be a kernel for (24). If  $B = \phi^{-1}(Q)$  for some  $(K, \leq)$   $s$ -set  $Q$  then  $B = C$  and (24) is zero, and we choose  $\sigma(B) = \tilde{\sigma}(R)$  and  $\iota(B, C) = \text{id}_{\sigma(B)}$ . Define  $\iota(B, I) = \tilde{\iota}(R, K) \circ \iota(B, C)$ .

Step 3: Let  $B \subseteq B'$  be  $(I, \preceq)$   $s$ -sets. Use the notation of Lemma 5.6 for  $B$ , and  $P', \dots, F'$  for  $B'$ . Then  $P \subseteq P', R \subseteq R'$ , and so on. We have

$$\begin{aligned} &\pi'(F' \setminus D', F' \setminus E') \circ \tilde{\pi}(R', R' \setminus P') \circ \tilde{\iota}(R, R') \circ \iota(B, C) \\ &= \pi'(F' \setminus D', F' \setminus E') \circ \tilde{\iota}(R \setminus P', R' \setminus P') \circ \tilde{\pi}(R, R \setminus P') \circ \iota(B, C) \\ &= \pi'(F' \setminus D', F' \setminus E') \circ \iota'(F \setminus D', F' \setminus D') \circ \tilde{\pi}(R \setminus P, R \setminus P') \circ \tilde{\pi}(R, R \setminus P) \circ \iota(B, C) \\ &= \iota'(F \setminus E', F' \setminus E') \circ \pi'(F \setminus D', F \setminus E') \circ \pi'(F \setminus D, F \setminus D') \circ \tilde{\pi}(R, R \setminus P) \circ \iota(B, C) \\ &= \iota'(F \setminus E', F' \setminus E') \circ \pi'(F \setminus E, F \setminus E') \circ \pi'(F \setminus D, F \setminus E) \circ \tilde{\pi}(R, R \setminus P) \circ \iota(B, C) = 0, \end{aligned}$$

using Definition 4.1(C), (D), and the definition of  $\iota(B, C)$ .

Thus, as  $\iota(B', C')$  is the kernel of  $\pi'(F' \setminus D', F' \setminus E') \circ \tilde{\pi}(R', R' \setminus P')$ , there exists a unique  $\iota(B, B')$  with  $\tilde{\iota}(R, R') \circ \iota(B, C) = \iota(B', C') \circ \iota(B, B')$ . Hence

$$\begin{aligned} \iota(B', I) \circ \iota(B, B') &= \tilde{\iota}(R', K) \circ \iota(B', C') \circ \iota(B, B') \\ &= \tilde{\iota}(R', K) \circ \tilde{\iota}(R, R') \circ \iota(B, C) = \tilde{\iota}(R, K) \circ \iota(B, C) = \iota(B, I). \end{aligned}$$

The proof of (15) from (14) then gives

$$\iota(B, B'') = \iota(B', B'') \circ \iota(B, B') \quad \text{when } B \subseteq B' \subseteq B'' \text{ are } (I, \preccurlyeq) \text{ s-sets.} \quad (25)$$

*Step 4:* Set  $X = \sigma(I) = \tilde{\sigma}(K)$ , and for each  $(I, \preccurlyeq)$  s-set  $B$  let  $S^B \subset X$  be subobject represented by  $\iota(B, I) : \sigma(B) \rightarrow \sigma(I) = X$ . We must prove that these  $S^B$  satisfy (13). The first two equations of (13) are immediate. Let  $B', B''$  be  $(I, \preccurlyeq)$  s-sets, and  $B = B' \cap B''$ . We shall show that  $S^B = S^{B'} \cap S^{B''}$ .

Use the notation of Lemma 5.6 for  $B$ , and  $P', R', \dots$  for  $B'$  and  $P'', R'', \dots$  for  $B''$  in the obvious way. Apply Definition 2.5 with  $i = \iota(B', I)$  and  $j = \iota(B'', I)$ , giving  $U, V \in \mathcal{A}$  and morphisms  $a, b, c, d, e$  with  $i \circ a = j \circ b$ ,  $i = e \circ c$  and  $j = e \circ d$ , such that  $i \circ a : U \rightarrow X$  represents  $S^{B'} \cap S^{B''}$ .

Set  $\hat{C} = C' \cap C''$ ,  $\hat{R} = R' \cap R''$  and  $\hat{F} = F' \cap F''$ , so that  $C \subseteq \hat{C}$ ,  $R \subseteq \hat{R}$ ,  $F \subseteq \hat{F}$ ,  $\hat{C} = \phi^{-1}(\hat{R})$ , and  $\hat{F} = \hat{C} \cap J$ . Define  $\hat{a} : U \rightarrow \sigma(C')$  and  $\hat{b} : U \rightarrow \sigma(C'')$  by  $\hat{a} = \iota(B', C') \circ a$  and  $\hat{b} = \iota(B'', C'') \circ b$ . Then

$$\begin{aligned} \tilde{\iota}(R', K) \circ \hat{a} &= \tilde{\iota}(R', K) \circ \iota(B', C') \circ a = \iota(B', I) \circ a = i \circ a \\ &= j \circ b = \iota(B'', I) \circ b = \tilde{\iota}(R'', K) \circ \iota(B'', C'') \circ b = \tilde{\iota}(R'', K) \circ \hat{b}. \end{aligned}$$

As  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  is a configuration we see that  $S^{\hat{C}} = S^{C'} \cap S^{C''}$  by Theorem 4.5. Thus there is a unique  $\hat{h} : U \rightarrow \tilde{\sigma}(\hat{R})$  with  $\hat{a} = \tilde{\iota}(\hat{R}, R') \circ \hat{h}$  and  $\hat{b} = \tilde{\iota}(\hat{R}, R'') \circ \hat{h}$ .

As  $\iota(B', C')$  is the kernel of  $\pi'(F' \setminus D', F' \setminus E') \circ \tilde{\pi}(R', R' \setminus P')$ , we have

$$\begin{aligned} 0 &= \pi'(F' \setminus D', F' \setminus E') \circ \tilde{\pi}(R', R' \setminus P') \circ \iota(B', C') \circ a \\ &= \pi'(F' \setminus D', F' \setminus E') \circ \tilde{\pi}(R', R' \setminus P') \circ \hat{a} \\ &= \pi'(F' \setminus D', F' \setminus E') \circ \tilde{\pi}(R', R' \setminus P') \circ \tilde{\iota}(\hat{R}, R') \circ \hat{h} \\ &= \dots = \iota'(\hat{F} \setminus E', F' \setminus E') \circ \pi'(\hat{F} \setminus E, \hat{F} \setminus E') \circ \pi'(F \setminus D, \hat{F} \setminus E) \circ \tilde{\pi}(\hat{R}, \hat{R} \setminus P) \circ \hat{h}, \end{aligned}$$

by Definition 4.1(C), (D). Since  $\iota'(\hat{F} \setminus E', F' \setminus E')$  is injective this gives

$$\pi'(\hat{F} \setminus E, \hat{F} \setminus E') \circ \pi'(\hat{F} \setminus D, \hat{F} \setminus E) \circ \tilde{\pi}(\hat{R}, \hat{R} \setminus P) \circ \hat{h} = 0$$

and

$$\pi'(\hat{F} \setminus E, \hat{F} \setminus E'') \circ \pi'(\hat{F} \setminus D, \hat{F} \setminus E) \circ \tilde{\pi}(\hat{R}, \hat{R} \setminus P) \circ \hat{h} = 0, \quad (26)$$

proving the second equation in the same way using  $B'', C'', \dots$

As  $(\sigma', \iota', \pi')$  is a configuration and  $E = E' \cap E''$ , one can show that

$$\iota_{\sigma'(\hat{F} \setminus E')} \circ \pi'(\hat{F} \setminus E, \hat{F} \setminus E') + \iota_{\sigma'(\hat{F} \setminus E'')} \circ \pi'(\hat{F} \setminus E, \hat{F} \setminus E'')$$



is an injective morphism  $\sigma'(\hat{F} \setminus E) \rightarrow \sigma'(\hat{F} \setminus E') \oplus \sigma'(\hat{F} \setminus E'')$ . Therefore

$$\pi'(\hat{F} \setminus D, \hat{F} \setminus E) \circ \tilde{\pi}(\hat{R}, \hat{R} \setminus P) \circ \hat{h} = 0, \quad (27)$$

by (26). Composing (27) with  $\pi'(\hat{F} \setminus E, \hat{F} \setminus F)$  and using Definition 4.1(C) shows that  $\tilde{\pi}(\hat{R}, \hat{R} \setminus R) \circ \hat{h} = 0$ . But  $\tilde{\iota}(R, \hat{R})$  is the kernel of  $\tilde{\pi}(\hat{R}, \hat{R} \setminus R)$ , so  $\hat{h} = \tilde{\iota}(R, \hat{R}) \circ \tilde{h}$  for some unique  $\tilde{h} : U \rightarrow \tilde{\sigma}(R) = \sigma(C)$ .

Substituting  $\hat{h} = \tilde{\iota}(R, \hat{R}) \circ \tilde{h}$  into (27) and using Definition 4.1(D) gives

$$\iota'(F \setminus E, \hat{F} \setminus E) \circ \pi'(F \setminus D, F \setminus E) \circ \tilde{\pi}(R, R \setminus P) \circ \tilde{h} = 0.$$

Hence  $\pi'(F \setminus D, F \setminus E) \circ \tilde{\pi}(R, R \setminus P) \circ \tilde{h} = 0$ , as  $\iota'(F \setminus E, \hat{F} \setminus E)$  is injective. Thus, as  $\iota(B, C)$  is the kernel of (24), there is a unique  $h : U \rightarrow \sigma(B)$  with  $\tilde{h} = \iota(B, C) \circ h$ . Then

$$\begin{aligned} \iota(B', C') \circ a &= \hat{a} = \tilde{\iota}(\hat{R}, R') \circ \hat{h} = \tilde{\iota}(\hat{R}, R') \circ \tilde{\iota}(R, \hat{R}) \circ \tilde{h} \\ &= \tilde{\iota}(R, R') \circ \iota(B, C) \circ h = \iota(B, C') \circ h = \iota(B', C') \circ \iota(B, B') \circ h \end{aligned}$$

by (25), so  $a = \iota(B, B') \circ h$  as  $\iota(B', C')$  is injective, and similarly  $b = \iota(B, B'') \circ h$ .

Recall the definition of  $i, j, U, V, a, \dots, e$  above. By (25) we have

$$\begin{aligned} e \circ c \circ \iota(B, B') &= i \circ \iota(B, B') = \iota(B', I) \circ \iota(B, B') = \iota(B, I) \\ &= \iota(B'', I) \circ \iota(B, B'') = j \circ \iota(B, B'') = e \circ d \circ \iota(B, B''). \end{aligned}$$

Since  $e$  is injective this gives  $c \circ \iota(B, B') = d \circ \iota(B, B'')$ , and hence

$$(c \circ \pi_{\sigma(B')} \oplus d \circ \pi_{\sigma(B'')}) \circ (\iota_{\sigma(B')} \circ \iota(B, B') - \iota_{\sigma(B'')} \circ \iota(B, B'')) = 0,$$

factoring via  $\sigma(B') \oplus \sigma(B'')$ . So by (1) there is a unique  $m : \sigma(B) \rightarrow U$  with

$$\iota_{\sigma(B')} \circ \iota(B, B') - \iota_{\sigma(B'')} \circ \iota(B, B'') = (\iota_{\sigma(B')} \circ a - \iota_{\sigma(B'')} \circ b) \circ m.$$

Composing with  $\pi_{\sigma(B')}$  gives  $\iota(B, B') = a \circ m$ . As  $a = \iota(B, B') \circ h$  and  $a, \iota(B, B')$  are injective, we see  $m$  and  $h$  are inverse, so  $h$  is an isomorphism.

Since  $S^{B'} \cap S^{B''}$  is represented by  $\iota(B', I) \circ a = \iota(B', I) \circ \iota(B, B') \circ h = \iota(B, I) \circ h$  and  $S^B$  by  $\iota(B, I)$ , this proves that  $S^B = S^{B'} \cap S^{B''}$  for all  $(I, \preccurlyeq)$  s-sets  $B', B''$  and  $B = B' \cap B''$ . A similar proof shows that  $S^B = S^{B'} + S^{B''}$  when  $B = B' \cup B''$ . Hence the  $S^B$  satisfy (13).

*Step 5:* Theorem 4.2 now constructs an  $(I, \preccurlyeq)$ -configuration  $(\sigma, \iota, \pi)$ , unique up to canonical isomorphism, from the  $S^B$ . It follows from the construction of the  $S^B$  that the  $(J, \lesssim)$ -subconfiguration of  $(\sigma, \iota, \pi)$  is canonically isomorphic to  $(\sigma', \iota', \pi')$ , and the

quotient  $(K, \trianglelefteq)$ -configuration from  $\phi$  is canonically isomorphic to  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ . It is not difficult to see that we can choose  $(\sigma, \iota, \pi)$  so that these sub- and quotient configurations are equal to  $(\sigma', \iota', \pi')$  and  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ .

For the last part, define  $\beta : \mathcal{F}_{(K, \trianglelefteq)} \rightarrow \text{Mor}(\mathcal{A})$  by  $\beta(A) = \alpha(A)$  if  $A \in \mathcal{F}_{(L, \trianglelefteq)}$ , and  $\beta(A) = \text{id}_{\tilde{\sigma}(A)}$  if  $A \notin \mathcal{F}_{(L, \trianglelefteq)}$ . Define a  $(K, \trianglelefteq)$ -configuration  $(\dot{\sigma}, \dot{\iota}, \dot{\pi})$  by

$$\dot{\sigma}(A) = \begin{cases} \hat{\sigma}(A) & A \in \mathcal{F}_{(L, \trianglelefteq)} \\ \tilde{\sigma}(A) & A \notin \mathcal{F}_{(L, \trianglelefteq)} \end{cases} \quad \text{and} \quad \begin{aligned} \dot{\iota}(A, B) &= \beta(B)^{-1} \circ \tilde{\iota}(A, B) \circ \beta(A), \\ \dot{\pi}(A, B) &= \beta(B)^{-1} \circ \tilde{\pi}(A, B) \circ \beta(A). \end{aligned}$$

Then  $\beta : (\tilde{\sigma}, \tilde{\iota}, \tilde{\pi}) \rightarrow (\dot{\sigma}, \dot{\iota}, \dot{\pi})$  is an isomorphism. The  $(L, \trianglelefteq)$ -subconfiguration of  $(\dot{\sigma}, \dot{\iota}, \dot{\pi})$  is  $(\hat{\sigma}, \hat{\iota}, \hat{\pi})$ , so we may apply the first part with  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  replaced by  $(\dot{\sigma}, \dot{\iota}, \dot{\pi})$ . This proves Theorem 5.5 when  $\mathcal{A}$  is an *abelian category*.

If  $\mathcal{A}$  is only an *exact category* we have more work to do, as Steps 2,5 above involve choosing kernels and cokernels, which may not exist in  $\mathcal{A}$ . So suppose  $\mathcal{A}$  is an exact category, contained in an abelian category  $\hat{\mathcal{A}}$  as in §2.1. In the situation of the first part of the theorem, the proof above yields an  $(I, \preceq)$ -configuration  $(\sigma, \iota, \pi)$  in  $\hat{\mathcal{A}}$  with the properties we want. We must show  $\sigma(J) \in \text{Obj}(\mathcal{A})$  for all  $J \in \mathcal{F}_{(I, \preceq)}$ , so that  $(\sigma, \iota, \pi)$  is a configuration in  $\mathcal{A}$ .

As  $(\sigma', \iota', \pi')$  is the  $(J, \lesssim)$ -subconfiguration of  $(\sigma, \iota, \pi)$  we have  $\sigma(\{i\}) = \sigma'(\{i\}) \in \text{Obj}(\mathcal{A})$  for  $i \in J$ . And as  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  is the quotient  $(K, \trianglelefteq)$ -configuration of  $(\sigma, \iota, \pi)$  from  $\phi$  we have  $\sigma(\{i\}) = \tilde{\sigma}(\{i\}) \in \text{Obj}(\mathcal{A})$  for  $i \in I \setminus J = K \setminus L$ . Hence  $\sigma(\{i\}) \in \text{Obj}(\mathcal{A})$  for all  $i \in I$ . Also  $\sigma(\emptyset) = 0 \in \text{Obj}(\mathcal{A})$ .

Suppose by induction that  $\sigma(A) \in \text{Obj}(\mathcal{A})$  for all  $A \in \mathcal{F}_{(I, \preceq)}$  with  $|A| \leq k$ , for  $1 \leq k < |I|$ . Let  $B \in \mathcal{F}_{(I, \preceq)}$  with  $|B| = k + 1$ , let  $i$  be  $\preceq$ -maximal in  $B$ , and set  $A = B \setminus \{i\}$ . Then  $(A, B) \in \mathcal{G}_{(I, \preceq)}$ , so (10) gives a short exact sequence  $0 \rightarrow \sigma(A) \rightarrow \sigma(B) \rightarrow \sigma(\{i\}) \rightarrow 0$ . Now  $\sigma(A) \in \text{Obj}(\mathcal{A})$  by induction,  $\sigma(\{i\}) \in \text{Obj}(\mathcal{A})$  from above, and  $\mathcal{A}$  is closed under extensions in  $\hat{\mathcal{A}}$ , so  $\sigma(B) \in \text{Obj}(\mathcal{A})$ . Thus by induction  $\sigma(A) \in \text{Obj}(\mathcal{A})$  for all  $A \in \mathcal{F}_{(I, \preceq)}$ , and  $(\sigma, \iota, \pi)$  is an  $(I, \preceq)$ -configuration in  $\mathcal{A}$ . This proves the first part for  $\mathcal{A}$  an exact category, and the last part follows as above.  $\square$

The case when  $L = \{l\}$  is one point will be particularly useful.

**Definition 5.7.** Let  $(J, \lesssim)$  and  $(K, \trianglelefteq)$  be nonempty finite posets with  $J \cap K = \emptyset$ , and  $l \in K$ . Set  $I = J \cup (K \setminus \{l\})$ , and define a partial order  $\preceq$  on  $I$  by

$$i \preceq j \quad \text{for } i, j \in I \quad \text{if} \quad \begin{cases} i \lesssim j, & i, j \in J, \\ i \trianglelefteq j, & i, j \in K \setminus \{l\}, \\ l \trianglelefteq j, & i \in J, \quad j \in K \setminus \{l\}, \\ i \trianglelefteq l, & i \in K \setminus \{l\}, \quad j \in J, \end{cases}$$

and a surjective map  $\phi : I \rightarrow K$  by  $\phi(i) = l$  if  $i \in J$ , and  $\phi(i) = i$  if  $i \in K \setminus \{l\}$ .

Let  $\mathcal{A}$  be an abelian or exact category,  $(\sigma', \iota', \pi')$  a  $(J, \lesssim)$ -configuration in  $\mathcal{A}$ , and  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  a  $(K, \trianglelefteq)$ -configuration in  $\mathcal{A}$  with  $\sigma'(J) = \tilde{\sigma}(\{l\})$ . Then by Theorem 5.5 there

exists an  $(I, \preceq)$ -configuration  $(\sigma, \iota, \pi)$  in  $\mathcal{A}$ , unique up to canonical isomorphism, such that  $(\sigma', \iota', \pi')$  is its  $(J, \preceq)$ -subconfiguration, and  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  its quotient  $(K, \sqsubseteq)$ -configuration from  $\phi$ . We call  $(\sigma, \iota, \pi)$  the *substitution of  $(\sigma', \iota', \pi')$  into  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$* .

## 6. Improvements and best configurations

We now study *quotient configurations* from  $(I, \preceq)$ ,  $(K, \sqsubseteq)$  when  $\phi : I \rightarrow K$  is a *bijection*. So we identify  $I, K$  and regard  $\preceq, \sqsubseteq$  as two partial orders on  $I$ .

**Definition 6.1.** Let  $I$  be a finite set and  $\preceq, \sqsubseteq$  partial orders on  $I$  such that if  $i \preceq j$  then  $i \sqsubseteq j$  for  $i, j \in I$ . Then we say that  $\sqsubseteq$  *dominates*  $\preceq$ , and  $\sqsubseteq$  *strictly dominates*  $\preceq$  if  $\preceq, \sqsubseteq$  are distinct. Let  $s$  be the number of pairs  $(i, j) \in I \times I$  with  $i \sqsubseteq j$  but  $i \not\preceq j$ . Then we say that  $\sqsubseteq$  *dominates  $\preceq$  by  $s$  steps*. Also

$$\mathcal{F}_{(I, \sqsubseteq)} \subseteq \mathcal{F}_{(I, \preceq)}, \quad \mathcal{G}_{(I, \sqsubseteq)} \subseteq \mathcal{G}_{(I, \preceq)} \quad \text{and} \quad \mathcal{H}_{(I, \sqsubseteq)} \subseteq \mathcal{H}_{(I, \preceq)}. \quad (28)$$

For each  $(I, \preceq)$ -configuration  $(\sigma, \iota, \pi)$  in an abelian or exact category we have a quotient  $(I, \sqsubseteq)$ -configuration  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ , as in Definition 5.2 with  $\phi = \text{id} : I \rightarrow I$ . We call  $(\sigma, \iota, \pi)$  an *improvement* or an  $(I, \preceq)$ -*improvement* of  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ , and a *strict improvement* if  $\preceq \neq \sqsubseteq$ . We call an  $(I, \sqsubseteq)$ -configuration  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  *best* if there exists no strict improvement  $(\sigma, \iota, \pi)$  of  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ . Improvements are a special kind of *refinement*, in the sense of Definition 5.2.

Our first result is simple. An  $(I, \sqsubseteq)$ -configuration  $(\sigma, \iota, \pi)$  cannot have an infinite sequence of strict improvements, as  $I$  has finitely many partial orders. So after finitely many improvements we reach a *best* configuration, giving:

**Lemma 6.2.** Let  $(\sigma, \iota, \pi)$  be an  $(I, \sqsubseteq)$ -configuration in an abelian or exact category. Then  $(\sigma, \iota, \pi)$  can be improved to a best  $(I, \preceq)$ -configuration  $(\sigma', \iota', \pi')$ , for some partial order  $\preceq$  on  $I$  dominated by  $\sqsubseteq$ .

After some preliminary results on partial orders in §6.1, Section 6.2 proves a criterion for *best configurations* in terms of *split* short exact sequences.

### 6.1. Partial orders $\sqsubseteq, \preceq$ where $\sqsubseteq$ dominates $\preceq$

We study partial orders  $\sqsubseteq, \preceq$  on  $I$  where  $\sqsubseteq$  strictly dominates  $\preceq$ .

**Lemma 6.3.** Let  $\sqsubseteq, \preceq$  be partial orders on a finite set  $I$ , where  $\sqsubseteq$  strictly dominates  $\preceq$ . Then there exist  $i, j \in I$  with  $i \sqsubseteq j$  and  $i \not\preceq j$ , such that there exists no  $k \in I$  with  $i \neq k \neq j$  and  $i \sqsubseteq k \sqsubseteq j$ . Also  $(\{j\}, \{i, j\}) \in \mathcal{G}_{(I, \preceq)} \setminus \mathcal{G}_{(I, \sqsubseteq)}$  and  $(\{i, j\}, \{i\}) \in \mathcal{H}_{(I, \preceq)} \setminus \mathcal{H}_{(I, \sqsubseteq)}$ .

**Proof.** As  $\sqsubseteq$  strictly dominates  $\preceq$  there exist  $i, j \in I$  with  $i \sqsubseteq j$  and  $i \not\preceq j$ . Suppose there exists  $k \in I$  with  $i \neq k \neq j$  and  $i \sqsubseteq k \sqsubseteq j$ . Then as  $i \not\preceq j$  either (a)  $i \not\preceq k$ , or (b)  $k \not\preceq j$ . In case (a) we replace  $j$  by  $k$ , and in case (b) we replace  $i$  by  $k$ . Then the new  $i, j$  satisfy the original conditions, but are ‘closer together’ than the old  $i, j$ . After finitely many steps we reach  $i, j$  satisfying the lemma.  $\square$

This implies that if  $\sqsubseteq$  strictly dominates  $\preceq$  then  $\mathcal{G}_{(I, \sqsubseteq)} \subseteq \mathcal{G}_{(I, \preceq)}$  and  $\mathcal{H}_{(I, \sqsubseteq)} \subseteq \mathcal{H}_{(I, \preceq)}$  in (28) are *strict* inclusions. But  $\mathcal{F}_{(I, \sqsubseteq)} \subseteq \mathcal{F}_{(I, \preceq)}$  need not be strict. For example, if  $I = \{1, 2\}$  and  $\sqsubseteq = \leq$  then  $\mathcal{F}_{(I, \sqsubseteq)} = \mathcal{F}_{(I, \preceq)}$  is the set of subsets of  $I$ .

The following elementary lemma characterizes  $\sqsubseteq, \preceq$  differing by one step.

**Lemma 6.4.** *Let  $(I, \sqsubseteq)$  be a finite poset, and suppose  $i \neq j \in I$  with  $i \sqsubseteq j$  but there exists no  $k \in I$  with  $i \neq k \neq j$  and  $i \sqsubseteq k \sqsubseteq j$ . Define  $\preceq$  on  $I$  by  $a \preceq b$  if and only if  $a \sqsubseteq b$  and  $a \neq i, b \neq j$ . Then  $\preceq$  is a partial order and  $\sqsubseteq$  dominates  $\preceq$  by one step. Conversely, if  $\preceq$  is a partial order and  $\sqsubseteq$  dominates  $\preceq$  by one step then  $\preceq$  arises as above for some unique  $i, j \in I$ .*

If  $\sqsubseteq$  dominates  $\preceq$ , we can interpolate a chain of partial orders differing by one step. The proof is elementary, using Lemmas 6.3 and 6.4.

**Proposition 6.5.** *Let  $I$  be a finite set and  $\preceq, \sqsubseteq$  partial orders on  $I$ , where  $\sqsubseteq$  dominates  $\preceq$  by  $s$  steps. Then there exist partial orders  $\sqsubseteq = \lesssim_0, \lesssim_1, \dots, \lesssim_s = \preceq$  on  $I$  such that  $\lesssim_{r-1}$  dominates  $\lesssim_r$  by one step, for  $r = 1, \dots, s$ .*

## 6.2. Best $(I, \preceq)$ -configurations and split sequences

We now prove a criterion for *best*  $(I, \sqsubseteq)$ -configurations. First we decompose certain objects  $\sigma(J \cup K)$  as *direct sums*  $\sigma(J) \oplus \sigma(K)$ .

**Proposition 6.6.** *Suppose  $(\sigma, \iota, \pi)$  is an  $(I, \preceq)$ -configuration in an abelian or exact category. Let  $J, K \in \mathcal{F}_{(I, \preceq)}$  with  $j \not\preceq k$  and  $k \not\preceq j$  for all  $j \in J$  and  $k \in K$ . Then  $J \cup K \in \mathcal{F}_{(I, \preceq)}$  is an  $f$ -set and there is a canonical isomorphism  $\sigma(J) \oplus \sigma(K) \cong \sigma(J \cup K)$  identifying  $\iota_{\sigma(J)}, \iota_{\sigma(K)}, \pi_{\sigma(J)}, \pi_{\sigma(K)}$  with  $\iota(J, J \cup K), \iota(K, J \cup K), \pi(J \cup K, J), \pi(J \cup K, K)$ , respectively. Hence*

$$\iota(J, J \cup K) \circ \pi(J \cup K, J) + \iota(K, J \cup K) \circ \pi(J \cup K, K) = \text{id}_{\sigma(J \cup K)}.$$

**Proof.** The conditions on  $J, K$  imply that  $J \cap K = \emptyset$  and  $J \cup K \in \mathcal{F}_{(I, \preceq)}$  with  $(J, J \cup K), (K, J \cup K) \in \mathcal{G}_{(I, \preceq)}$  and  $(J \cup K, J), (J \cup K, K) \in \mathcal{H}_{(I, \preceq)}$ . Definition 4.1(A) applied to  $(J, J \cup K)$  shows that  $\pi(J \cup K, K) \circ \iota(J, J \cup K) = 0$ , and similarly  $\pi(J \cup K, J) \circ \iota(K, J \cup K) = 0$ . Parts (ii), (iii) and (D) of Definition 4.1 with  $J, J \cup K, J$  in place of  $J, K, L$  give  $\pi(J \cup K, J) \circ \iota(J, J \cup K) = \iota(J, J) \circ \pi(J, J) = \text{id}_{\sigma(J)}$ , and similarly  $\pi(J \cup K, K) \circ \iota(K, J \cup K) = \text{id}_{\sigma(K)}$ . The proposition then follows from Popescu [15, Corollary 2.7.4, p. 48].  $\square$

Recall that a short exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\mathcal{A}$  is called *split* if there is a compatible isomorphism  $Y \cong X \oplus Z$ .

**Proposition 6.7.** *Suppose  $(\sigma, \iota, \pi)$  is an  $(I, \trianglelefteq)$ -configuration in an abelian or exact category, which is not best. Then there exist  $i \neq j \in I$  with  $i \trianglelefteq j$  but there exists no  $k \in I$  with  $i \neq k \neq j$  and  $i \trianglelefteq k \trianglelefteq j$ , such that the following short exact sequence is split:*

$$0 \longrightarrow \sigma(\{i\}) \xrightarrow{\iota(\{i\}, \{i, j\})} \sigma(\{i, j\}) \xrightarrow{\pi(\{i, j\}, \{j\})} \sigma(\{j\}) \longrightarrow 0. \quad (29)$$

**Proof.** As  $(\sigma, \iota, \pi)$  is not best it has a strict  $(I, \preccurlyeq)$ -improvement  $(\sigma', \iota', \pi')$ , for some  $\preccurlyeq$  dominated by  $\trianglelefteq$ . Let  $i, j$  be as in Lemma 6.3. Then  $i \neq j$  as  $i \not\preccurlyeq j$ , and there exists no  $k \in I$  with  $i \neq k \neq j$  and  $i \preccurlyeq k \preccurlyeq j$ . As  $i \not\preccurlyeq j$ ,  $j \not\preccurlyeq i$  Proposition 6.6 shows that  $\sigma'(\{i, j\}) \cong \sigma'(\{i\}) \oplus \sigma'(\{j\})$ . But  $\sigma'(\{i\}) = \sigma(\{i\})$ ,  $\sigma'(\{i, j\}) = \sigma(\{i, j\})$ ,  $\sigma'(\{j\}) = \sigma(\{j\})$ , so  $\sigma(\{i, j\}) \cong \sigma(\{i\}) \oplus \sigma(\{j\})$ .

Proposition 6.6 and equalities between  $\iota, \iota'$  and  $\pi, \pi'$  show that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \sigma(\{i\}) & \xrightarrow{\quad \quad \quad} & \sigma(\{i\}) \oplus \sigma(\{j\}) & \xrightarrow{\quad \quad \quad} & \sigma(\{j\}) \longrightarrow 0 \\ & & \text{id}_{\sigma(\{i\})} \downarrow & & h \downarrow & & \downarrow \text{id}_{\sigma(\{j\})} \\ 0 & \longrightarrow & \sigma(\{i\}) & \xrightarrow{\iota(\{i\}, \{i, j\})} & \sigma(\{i, j\}) & \xrightarrow{\pi(\{i, j\}, \{j\})} & \sigma(\{j\}) \longrightarrow 0 \end{array}$$

commutes, where  $h = \iota(\{i\}, \{i, j\}) \circ \pi_{\sigma(\{i\})} + \iota'(\{j\}, \{i, j\}) \circ \pi_{\sigma(\{j\})}$  is an isomorphism. Therefore the short exact sequence (10) is split.  $\square$

We classify improvements for a two point indexing set  $K = \{i, j\}$ . The 1-1 correspondence below is *not canonical*, but depends on a choice of base point; canonically, the  $(K, \preccurlyeq)$ -improvements form a  $\text{Hom}(\sigma(\{j\}), \sigma(\{i\}))$ -torsor.

**Lemma 6.8.** *Define partial orders  $\trianglelefteq, \preccurlyeq$  on  $K = \{i, j\}$  by  $i \trianglelefteq i$ ,  $i \trianglelefteq j$ ,  $j \trianglelefteq j$ ,  $i \preccurlyeq i$  and  $j \preccurlyeq j$ . Let  $(\sigma, \iota, \pi)$  be a  $(K, \trianglelefteq)$ -configuration in an abelian or exact category. Then there exists a  $(K, \preccurlyeq)$ -improvement  $(\sigma', \iota', \pi')$  of  $(\sigma, \iota, \pi)$  if and only if the short exact sequence (29) is split, and then such  $(K, \preccurlyeq)$ -improvements  $(\sigma', \iota', \pi')$  are in 1-1 correspondence with  $\text{Hom}(\sigma(\{j\}), \sigma(\{i\}))$ .*

**Proof.** If there exists a  $(K, \preccurlyeq)$ -improvement of  $(\sigma, \iota, \pi)$  then (29) is split by Proposition 6.7, which proves the ‘only if’ part. For the ‘if’ part, suppose (29) is split. Then we can choose morphisms  $\iota'(\{j\}, K) : \sigma(\{j\}) \rightarrow \sigma(K)$  and  $\pi'(K, \{i\}) : \sigma(K) \rightarrow \sigma(\{i\})$

with

$$\pi'(K, \{i\}) \circ \iota(\{i\}, K) = \text{id}_{\sigma(\{i\})} \quad \text{and} \quad \pi(K, \{j\}) \circ \iota'(\{j\}, K) = \text{id}_{\sigma(\{j\})}. \quad (30)$$

Defining  $\sigma' = \sigma$ ,  $\iota'|_{\mathcal{G}_{(K, \trianglelefteq)}} = \iota$ ,  $\pi'|_{\mathcal{H}_{(K, \trianglelefteq)}} = \pi$  then gives a  $(K, \lesssim)$ -improvement  $(\sigma', \iota', \pi')$  of  $(\sigma, \iota, \pi)$ , proving the ‘if’ part.

Finally, fix  $\iota'_0(\{j\}, K)$ ,  $\pi'_0(K, \{i\})$  satisfying (30). We can easily prove that every  $(K, \lesssim)$ -improvement  $(\sigma', \iota', \pi')$  of  $(\sigma, \iota, \pi)$  is defined uniquely by  $\sigma' = \sigma$ ,  $\iota'|_{\mathcal{G}_{(K, \trianglelefteq)}} = \iota$ ,  $\pi'|_{\mathcal{H}_{(K, \trianglelefteq)}} = \pi$  and

$$\iota'(\{j\}, K) = \iota'_0(\{j\}, K) + \iota(\{i\}, K) \circ f, \quad \pi'(K, \{i\}) = \pi'_0(K, \{i\}) - f \circ \pi(K, \{j\})$$

for some unique  $f \in \text{Hom}(\sigma(\{j\}), \sigma(\{i\}))$ , and every  $f \in \text{Hom}(\sigma(\{j\}), \sigma(\{i\}))$  gives a  $(K, \lesssim)$ -improvement. This establishes a 1-1 correspondence between  $(K, \lesssim)$ -improvements  $(\sigma', \iota', \pi')$  and  $f \in \text{Hom}(\sigma(\{j\}), \sigma(\{i\}))$ .  $\square$

Here is the converse to Proposition 6.7.

**Proposition 6.9.** *Suppose  $(\sigma, \iota, \pi)$  is an  $(I, \trianglelefteq)$ -configuration in an abelian or exact category. Let  $i \neq j \in I$  with  $i \trianglelefteq j$  but there exists no  $k \in I$  with  $i \neq k \neq j$  and  $i \trianglelefteq k \trianglelefteq j$ , such that (29) is split. Define  $\preccurlyeq$  on  $I$  by  $a \preccurlyeq b$  if  $a \trianglelefteq b$  and  $a \neq i$ ,  $b \neq j$ , so that  $\trianglelefteq$  dominates  $\preccurlyeq$  by one step. Then there exists an  $(I, \preccurlyeq)$ -improvement  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  of  $(\sigma, \iota, \pi)$ . Such improvements up to canonical isomorphism are in 1-1 correspondence with  $\text{Hom}(\sigma(\{j\}), \sigma(\{i\}))$ .*

**Proof.** Set  $K = \{i, j\}$ , and let  $(\check{\sigma}, \check{\iota}, \check{\pi})$  be the  $(K, \trianglelefteq)$ -subconfiguration of  $(\sigma, \iota, \pi)$ . As (29) is split, Lemma 6.8 shows that there exists a  $(K, \lesssim)$ -improvement  $(\sigma', \iota', \pi')$  of  $(\check{\sigma}, \check{\iota}, \check{\pi})$ . Then  $(\sigma, \iota, \pi)$  and  $(\sigma', \iota', \pi')$  satisfy the conditions of Theorem 5.5 with  $\phi = \text{id}$ ,  $I$  in place of  $K$ , and  $K$  in place of both  $J$  and  $L$ . Therefore Theorem 5.5 gives the  $(I, \preccurlyeq)$ -improvement  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  that we want.

For the last part, note that every  $(I, \preccurlyeq)$ -improvement  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  of  $(\sigma, \iota, \pi)$  may be constructed this way, taking  $(\sigma', \iota', \pi')$  to be the  $(K, \preccurlyeq)$ -subconfiguration of  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ . Thus, uniqueness up to canonical isomorphism in Theorem 5.5 shows that such improvements  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  up to canonical isomorphism are in 1-1 correspondence with  $(K, \lesssim)$ -improvements  $(\sigma', \iota', \pi')$  of  $(\check{\sigma}, \check{\iota}, \check{\pi})$ . But Lemma 6.8 shows that these are in 1-1 correspondence with  $\text{Hom}(\sigma(\{j\}), \sigma(\{i\}))$ .  $\square$

Propositions 6.7 and 6.9 imply a criterion for best configurations:

**Theorem 6.10.** *An  $(I, \preccurlyeq)$ -configuration  $(\sigma, \iota, \pi)$  in an abelian or exact category is best if and only if for all  $i \neq j \in I$  with  $i \preccurlyeq j$  but there exists no  $k \in I$  with  $i \neq k \neq j$  and  $i \preccurlyeq k \preccurlyeq j$ , the short exact sequence (29) is not split.*

If this criterion holds, it also holds for any subconfiguration of  $(\sigma, \iota, \pi)$ , giving:

**Corollary 6.11.** *Suppose  $(\sigma, \iota, \pi)$  is a best  $(I, \preceq)$ -configuration in an abelian or exact category. Then all subconfigurations of  $(\sigma, \iota, \pi)$  are also best.*

## 7. Moduli stacks of configurations

Let  $\mathcal{A}$  be an abelian category. We wish to study *moduli stacks of configurations*  $\mathfrak{M}(I, \preceq)_{\mathcal{A}}$ ,  $\mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}$  in  $\mathcal{A}$ . To do this we shall need some *extra structure* on  $\mathcal{A}$ , which is described in Assumption 7.1 below, and encodes information about *families of objects and morphisms* in  $\mathcal{A}$  over a *base scheme*  $U$ .

This section will construct  $\mathfrak{M}(I, \preceq)_{\mathcal{A}}$ ,  $\mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}$  just as  $\mathbb{K}$ -stacks, and some 1-morphisms of  $\mathbb{K}$ -stacks between them. But this is not enough to do algebraic geometry with. So under some additional conditions Assumption 8.1, Section 8 will prove  $\mathfrak{M}(I, \preceq)_{\mathcal{A}}$ ,  $\mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}$  are *algebraic*  $\mathbb{K}$ -stacks, and various morphisms between them are *representable* or *of finite type*.

### 7.1. Stacks in exact categories and stacks of configurations

Here is our first assumption, which uses ideas from Definition 2.8.

**Assumption 7.1.** Fix an algebraically closed field  $\mathbb{K}$ , and let  $\mathcal{A}$  be an abelian category with  $\mathrm{Hom}(X, Y) = \mathrm{Ext}^0(X, Y)$  and  $\mathrm{Ext}^1(X, Y)$  finite-dimensional  $\mathbb{K}$ -vector spaces for all  $X, Y \in \mathcal{A}$ , and all composition maps  $\mathrm{Ext}^i(Y, Z) \times \mathrm{Ext}^j(X, Y) \rightarrow \mathrm{Ext}^{i+j}(X, Z)$  bilinear for  $i, j, i+j = 0$  or  $1$ . Let  $K(\mathcal{A})$  be the quotient of the Grothendieck group  $K_0(\mathcal{A})$  by some fixed subgroup. Suppose that if  $X \in \mathrm{Obj}(\mathcal{A})$  with  $[X] = 0$  in  $K(\mathcal{A})$  then  $X \cong 0$ .

Let  $(\mathrm{exactcat})$  be the 2-category whose objects are exact categories, as in Definition 2.3, 1-morphisms are exact functors between exact categories, and 2-morphisms are natural transformations between these functors. Regard  $\mathrm{Sch}_{\mathbb{K}}$  as a 2-category as in Definition 2.8, and also as a *site* with the *étale topology*.

Suppose  $\mathfrak{F}_{\mathcal{A}} : \mathrm{Sch}_{\mathbb{K}} \rightarrow (\mathrm{exactcat})$  is a *contravariant 2-functor* which is a *stack in exact categories* on  $\mathrm{Sch}_{\mathbb{K}}$  with its Grothendieck topology, that is, Definition 2.8(i)–(iii) hold for  $\mathfrak{F}_{\mathcal{A}}$ , satisfying the following conditions:

- (i)  $\mathfrak{F}_{\mathcal{A}}(\mathrm{Spec} \mathbb{K}) = \mathcal{A}$ .
- (ii) Let  $\{f_i : U_i \rightarrow V\}_{i \in I}$  be an open cover of  $V$  in the site  $\mathrm{Sch}_{\mathbb{K}}$ . Then a sequence  $0 \rightarrow X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \rightarrow 0$  is exact in  $\mathfrak{F}_{\mathcal{A}}(V)$  if its images under  $\mathfrak{F}_{\mathcal{A}}(f_i)$  in  $\mathfrak{F}_{\mathcal{A}}(U_i)$  are exact for all  $i \in I$ .
- (iii) For all  $U \in \mathrm{Sch}_{\mathbb{K}}$  and  $X \in \mathrm{Obj}(\mathfrak{F}_{\mathcal{A}}(U))$ , the map  $\mathrm{Hom}(\mathrm{Spec} \mathbb{K}, U) \rightarrow K(\mathcal{A})$  given by  $u \mapsto [\mathfrak{F}_{\mathcal{A}}(u)X]$  is *locally constant* in the *Zariski topology* on  $\mathrm{Hom}(\mathrm{Spec} \mathbb{K}, U)$ . Here  $\mathfrak{F}_{\mathcal{A}}(u)X \in \mathrm{Obj}(\mathfrak{F}_{\mathcal{A}}(\mathrm{Spec} \mathbb{K})) = \mathrm{Obj}(\mathcal{A})$  by (i), so  $[\mathfrak{F}_{\mathcal{A}}(u)X]$  is well-defined in  $K(\mathcal{A})$ .
- (iv) Let  $X, Y \in \mathcal{A}$ , and regard  $\mathrm{Hom}(X, Y)$  as an *affine  $\mathbb{K}$ -scheme*, with projection morphism  $\pi : \mathrm{Hom}(X, Y) \rightarrow \mathrm{Spec} \mathbb{K}$ . Then there should exist a *tautological morphism*  $\theta_{X,Y} : \mathfrak{F}_{\mathcal{A}}(\pi)X \rightarrow \mathfrak{F}_{\mathcal{A}}(\pi)Y$  in  $\mathfrak{F}_{\mathcal{A}}(\mathrm{Hom}(X, Y))$  such that if  $f \in \mathrm{Hom}(X, Y)$  and  $\iota_f : \mathrm{Spec} \mathbb{K} \rightarrow \mathrm{Hom}(X, Y)$  is the corresponding morphism then the following

commutes in  $\mathcal{A}$ :

$$\begin{array}{ccc}
 \mathfrak{F}_{\mathcal{A}}(\iota_f) \circ \mathfrak{F}_{\mathcal{A}}(\pi)X & \xrightarrow{\quad} & X \\
 \downarrow \mathfrak{F}_{\mathcal{A}}(\iota_f)\theta_{X,Y} & \begin{array}{c} \varepsilon_{\pi, \iota_f}(X) \\ \varepsilon_{\pi, \iota_f}(Y) \end{array} & \downarrow f \\
 \mathfrak{F}_{\mathcal{A}}(\iota_f) \circ \mathfrak{F}_{\mathcal{A}}(\pi)Y & \xrightarrow{\quad} & Y.
 \end{array} \tag{31}$$

Here is some explanation of all this.

- The 2-functor  $\mathfrak{F}_{\mathcal{A}}$  contains information about families of objects and morphisms in  $\mathcal{A}$ . For  $U \in \text{Sch}_{\mathbb{K}}$ , objects  $X$  in  $\mathfrak{F}_{\mathcal{A}}(U)$  should be interpreted as *families of objects*  $X_u$  in  $\mathcal{A}$  parameterized by  $u \in U$ , which are *flat over*  $U$ . Morphisms  $\phi : X \rightarrow Y$  in  $\mathfrak{F}_{\mathcal{A}}(U)$  should be interpreted as *families of morphisms*  $\phi_u : X_u \rightarrow Y_u$  in  $\mathcal{A}$  parameterized by  $u \in U$ .
- For nontrivial  $U$ , the condition that objects of  $\mathfrak{F}_{\mathcal{A}}(U)$  be *flat over*  $U$  means that  $\mathfrak{F}_{\mathcal{A}}(U)$  is not an abelian category, but only an *exact category*, as (co)kernels of morphisms between flat families may not be flat.
- Families of objects and morphisms parameterized by  $U = \text{Spec } \mathbb{K}$  are just objects and morphisms in  $\mathcal{A}$ , so we take  $\mathfrak{F}_{\mathcal{A}}(\text{Spec } \mathbb{K}) = \mathcal{A}$  in (i).
- Part (ii) is necessary for the 2-functor of *groupoids of exact sequences* in  $\mathfrak{F}_{\mathcal{A}}(U)$  to be a  $\mathbb{K}$ -stack.
- Part (iii) requires algebraic families of elements of  $\mathcal{A}$  to have locally constant classes in  $K(\mathcal{A})$ . Roughly, this means the kernel of  $K_0(\mathcal{A}) \rightarrow K(\mathcal{A})$  includes all continuous variations, and so cannot be ‘too small’.

The condition that  $[X] = 0$  in  $K(\mathcal{A})$  implies  $X \cong 0$  and Assumption 8.1 both mean the kernel of  $K_0(\mathcal{A}) \rightarrow K(\mathcal{A})$  cannot be ‘too large’.

- Part (iv) says if we pull  $X, Y \in \mathcal{A}$  back to *constant families*  $\pi^*(X), \pi^*(Y)$  over the base scheme  $\text{Hom}(X, Y)$ , then there is a *tautological morphism*  $\theta_{X,Y} : \pi^*(X) \rightarrow \pi^*(Y)$  taking the value  $f$  over each  $f \in \text{Hom}(X, Y)$ . It will be needed in [11, §6] to ensure families of configurations with constant objects  $\sigma(J)$  but varying morphisms  $\iota, \pi(J, K)$  behave as expected.
- *Examples* of data  $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$  satisfying Assumption 7.1 will be given in §§9 and 10. Readers may wish to refer to them at this point.

## 7.2. Moduli stacks of configurations

We can now define *moduli stacks*  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$  of  $(I, \preccurlyeq)$ -configurations, and two other stacks  $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{Exact}_{\mathcal{A}}$ . We will show they are  $\mathbb{K}$ -stacks in Theorem 7.5.

**Definition 7.2.** We work in the situation of Assumption 7.1. Define *contravariant 2-functors*  $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{Exact}_{\mathcal{A}} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{groupoids})$  as follows. For  $U \in \text{Sch}_{\mathbb{K}}$ , let  $\mathfrak{Obj}_{\mathcal{A}}(U)$  be the groupoid with *objects*  $X \in \text{Obj}(\mathfrak{F}_{\mathcal{A}}(U))$ , and *morphisms*  $\phi : X \rightarrow Y$  isomorphisms in  $\text{Mor}(\mathfrak{F}_{\mathcal{A}}(U))$ .



Let  $\mathfrak{Exact}_{\mathcal{A}}(U)$  be the groupoid with *objects*  $(X, Y, Z, \phi, \psi)$  for short exact sequences  $0 \rightarrow X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \rightarrow 0$  in  $\mathfrak{F}_{\mathcal{A}}(U)$ . Let the *morphisms* in  $\mathfrak{Exact}_{\mathcal{A}}(U)$  be  $(\alpha, \beta, \gamma) : (X, Y, Z, \phi, \psi) \rightarrow (X', Y', Z', \phi', \psi')$  for  $\alpha : X \rightarrow X'$ ,  $\beta : Y \rightarrow Y'$ ,  $\gamma : Z \rightarrow Z'$  isomorphisms in  $\mathfrak{F}_{\mathcal{A}}(U)$  with  $\phi' \circ \alpha = \beta \circ \phi$ ,  $\psi' \circ \beta = \gamma \circ \psi$ .

If  $f : U \rightarrow V$  is a 1-morphism in  $\text{Sch}_{\mathbb{K}}$  then  $\mathfrak{F}_{\mathcal{A}}(f) : \mathfrak{F}_{\mathcal{A}}(V) \rightarrow \mathfrak{F}_{\mathcal{A}}(U)$  induces functors  $\mathfrak{Obj}_{\mathcal{A}}(f) : \mathfrak{Obj}_{\mathcal{A}}(V) \rightarrow \mathfrak{Obj}_{\mathcal{A}}(U)$  and  $\mathfrak{Exact}_{\mathcal{A}}(f) : \mathfrak{Exact}_{\mathcal{A}}(V) \rightarrow \mathfrak{Exact}_{\mathcal{A}}(U)$  in the obvious way, since  $\mathfrak{F}_{\mathcal{A}}(f)$  is an exact functor. If  $f : U \rightarrow V$  and  $g : V \rightarrow W$  are scheme morphisms,  $\varepsilon_{g,f} : \mathfrak{F}_{\mathcal{A}}(f) \circ \mathfrak{F}_{\mathcal{A}}(g) \rightarrow \mathfrak{F}_{\mathcal{A}}(g \circ f)$  induces isomorphisms of functors  $\varepsilon_{g,f} : \mathfrak{Obj}_{\mathcal{A}}(f) \circ \mathfrak{Obj}_{\mathcal{A}}(g) \rightarrow \mathfrak{Obj}_{\mathcal{A}}(g \circ f)$  and  $\varepsilon_{g,f} : \mathfrak{Exact}_{\mathcal{A}}(f) \circ \mathfrak{Exact}_{\mathcal{A}}(g) \rightarrow \mathfrak{Exact}_{\mathcal{A}}(g \circ f)$ .

As in §2.3, this data defines the 2-functors  $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{Exact}_{\mathcal{A}}$ . It is easy to verify  $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{Exact}_{\mathcal{A}}$  are *contravariant 2-functors*. We call  $\mathfrak{Obj}_{\mathcal{A}}$  the *moduli stack of objects in  $\mathcal{A}$* , and  $\mathfrak{Exact}_{\mathcal{A}}$  the *moduli stack of short exact sequences in  $\mathcal{A}$* .

Let  $(I, \preceq)$  be a finite poset. Define a *contravariant 2-functor*  $\mathfrak{M}(I, \preceq)_{\mathcal{A}} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{groupoids})$  as follows. For  $U \in \text{Sch}_{\mathbb{K}}$ , let  $\mathfrak{M}(I, \preceq)_{\mathcal{A}}(U)$  be the groupoid with *objects*  $(I, \preceq)$ -configurations  $(\sigma, \iota, \pi)$  in  $\mathfrak{F}_{\mathcal{A}}(U)$ , and *morphisms* isomorphisms of configurations  $\alpha : (\sigma, \iota, \pi) \rightarrow (\sigma', \iota', \pi')$  in  $\mathfrak{F}_{\mathcal{A}}(U)$ .

If  $f : U \rightarrow V$  is a scheme morphism  $\mathfrak{F}_{\mathcal{A}}(f) : \mathfrak{F}_{\mathcal{A}}(V) \rightarrow \mathfrak{F}_{\mathcal{A}}(U)$  is an exact functor, and so takes  $(I, \preceq)$ -configurations to  $(I, \preceq)$ -configurations, and isomorphisms of them to isomorphisms. Thus  $\mathfrak{F}_{\mathcal{A}}(f)$  induces a functor  $\mathfrak{M}(I, \preceq)_{\mathcal{A}}(f) : \mathfrak{M}(I, \preceq)_{\mathcal{A}}(V) \rightarrow \mathfrak{M}(I, \preceq)_{\mathcal{A}}(U)$ .

If  $f : U \rightarrow V$  and  $g : V \rightarrow W$  are scheme morphisms,  $\varepsilon_{g,f} : \mathfrak{F}_{\mathcal{A}}(f) \circ \mathfrak{F}_{\mathcal{A}}(g) \rightarrow \mathfrak{F}_{\mathcal{A}}(g \circ f)$  induces  $\varepsilon_{g,f} : \mathfrak{M}(I, \preceq)_{\mathcal{A}}(f) \circ \mathfrak{M}(I, \preceq)_{\mathcal{A}}(g) \rightarrow \mathfrak{M}(I, \preceq)_{\mathcal{A}}(g \circ f)$  in the obvious way. It is easy to verify  $\mathfrak{M}(I, \preceq)_{\mathcal{A}}$  is a *contravariant 2-functor*. We call  $\mathfrak{M}(I, \preceq)_{\mathcal{A}}$  the *moduli stack of  $(I, \preceq)$ -configurations in  $\mathcal{A}$* .

It is usual in algebraic geometry to study moduli spaces not of *all* coherent sheaves on a variety, but of sheaves with a fixed Chern character or Hilbert polynomial. The analogue for configurations  $(\sigma, \iota, \pi)$  is to fix the classes  $[\sigma(J)]$  in  $K(\mathcal{A})$  for  $J \in \mathcal{F}_{(I, \preceq)}$ . To do this we introduce  $(I, \preceq, \kappa)$ -configurations.

**Definition 7.3.** We work in the situation of Assumption 7.1. Define

$$\bar{C}(\mathcal{A}) = \{[X] \in K(\mathcal{A}) : X \in \mathcal{A}\} \subset K(\mathcal{A}).$$

That is,  $\bar{C}(\mathcal{A})$  is the collection of classes in  $K(\mathcal{A})$  of objects  $X \in \mathcal{A}$ . Note that  $\bar{C}(\mathcal{A})$  is *closed under addition*, as  $[X \oplus Y] = [X] + [Y]$ . In [11,12] we shall make much use of  $C(\mathcal{A}) = \bar{C}(\mathcal{A}) \setminus \{0\}$ . We think of  $C(\mathcal{A})$  as the ‘positive cone’ and  $\bar{C}(\mathcal{A})$  as the ‘closed positive cone’ in  $K(\mathcal{A})$ , which explains the notation. For  $(I, \preceq)$  a finite poset and  $\kappa : I \rightarrow \bar{C}(\mathcal{A})$ , define an  $(I, \preceq, \kappa)$ -configuration to be an  $(I, \preceq)$ -configuration  $(\sigma, \iota, \pi)$  with  $[\sigma(\{i\})] = \kappa(i)$  in  $K(\mathcal{A})$  for all  $i \in I$ .

We will also use the following shorthand: we *extend  $\kappa$  to the set of subsets of  $I$*  by defining  $\kappa(J) = \sum_{j \in J} \kappa(j)$ . Then  $\kappa(J) \in \bar{C}(\mathcal{A})$  for all  $J \subseteq I$ , as  $\bar{C}(\mathcal{A})$  is closed under

addition. If  $(\sigma, \iota, \pi)$  is an  $(I, \preceq, \kappa)$ -configuration then  $[\sigma(J)] = \kappa(J)$  for all  $J \in \mathcal{F}_{(I, \preceq)}$ , by Proposition 4.7.

Here is the generalization of Definition 7.2 to  $(I, \preceq, \kappa)$ -configurations.

**Definition 7.4.** For  $\alpha \in \bar{C}(\mathcal{A})$ , define  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{groupoids})$  as follows. For  $U \in \text{Sch}_{\mathbb{K}}$ , let  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(U)$  be the full subcategory of  $\mathfrak{Obj}_{\mathcal{A}}(U)$  with objects  $X \in \mathfrak{Obj}_{\mathcal{A}}(U)$  such that  $[\mathfrak{Obj}_{\mathcal{A}}(p)X] = \alpha \in K(\mathcal{A})$  for all  $p : \text{Spec } \mathbb{K} \rightarrow U$ , so that  $\mathfrak{Obj}_{\mathcal{A}}(p)X \in \text{Obj}(\mathfrak{F}_{\mathcal{A}}(\text{Spec } \mathbb{K})) = \text{Obj}(\mathcal{A})$ .

If  $f : U \rightarrow V$  and  $g : V \rightarrow W$  are scheme morphisms,  $\mathfrak{Obj}_{\mathcal{A}}(f)$  restricts to a functor  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(f) : \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(V) \rightarrow \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(U)$  and  $\varepsilon_{g,f} : \mathfrak{Obj}_{\mathcal{A}}(f) \circ \mathfrak{Obj}_{\mathcal{A}}(g) \rightarrow \mathfrak{Obj}_{\mathcal{A}}(g \circ f)$  restricts to  $\varepsilon_{g,f} : \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(f) \circ \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(g) \rightarrow \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(g \circ f)$ . We call  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$  the *moduli stack of objects in  $\mathcal{A}$  with class  $\alpha$* .

For  $\alpha, \beta, \gamma \in \bar{C}(\mathcal{A})$  with  $\beta = \alpha + \gamma$ , define  $\mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{groupoids})$  as follows. For  $U \in \text{Sch}_{\mathbb{K}}$ , let  $\mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma}(U)$  be the full subgroupoid of  $\mathfrak{Exact}_{\mathcal{A}}(U)$  with objects  $(X, Y, Z, \phi, \psi)$  for  $X \in \text{Obj}(\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(U))$ ,  $Y \in \text{Obj}(\mathfrak{Obj}_{\mathcal{A}}^{\beta}(U))$  and  $Z \in \text{Obj}(\mathfrak{Obj}_{\mathcal{A}}^{\gamma}(U))$ . Define  $\mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma}(f)$  and  $\varepsilon_{g,f}$  by restriction from  $\mathfrak{Exact}_{\mathcal{A}}$ . We call  $\mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma}$  the *moduli stack of short exact sequences in  $\mathcal{A}$  with classes  $\alpha, \beta, \gamma$* .

Now let  $(I, \preceq)$  be a finite poset and  $\kappa : I \rightarrow \bar{C}(\mathcal{A})$ . Define  $\mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{groupoids})$  as follows. For  $U \in \text{Sch}_{\mathbb{K}}$ , let  $\mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}(U)$  be the full subgroupoid of  $\mathfrak{M}(I, \preceq)_{\mathcal{A}}(U)$  with *objects*  $(\sigma, \iota, \pi)$  with  $\sigma(J) \in \text{Obj}(\mathfrak{Obj}_{\mathcal{A}}^{\kappa(J)}(U))$  for all  $J \in \mathcal{F}_{(I, \preceq)}$ . Define  $\mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}$  on morphisms and  $\varepsilon_{g,f}$  by restricting  $\mathfrak{M}(I, \preceq)_{\mathcal{A}}$ . Then  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}, \mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma}, \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}$  are *contravariant 2-functors*. We call  $\mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}$  the *moduli stack of  $(I, \preceq, \kappa)$ -configurations in  $\mathcal{A}$* .

The basic idea here is that  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$  contains information on *families of objects*  $X_u$  in  $\mathcal{A}$  with  $[X_u] = \alpha$  in  $K(\mathcal{A})$  for  $u$  in  $U$ , and isomorphisms between such families. We prove the 2-functors of Definitions 7.2 and 7.4 are  $\mathbb{K}$ -stacks.

**Theorem 7.5.**  $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{Exact}_{\mathcal{A}}, \mathfrak{M}(I, \preceq)_{\mathcal{A}}$  above are  $\mathbb{K}$ -stacks, as in Definition 2.8, and  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}, \mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma}, \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}$  are open and closed  $\mathbb{K}$ -substacks of them, so that we have the disjoint unions

$$\begin{aligned} \mathfrak{Obj}_{\mathcal{A}} &= \coprod_{\alpha \in \bar{C}(\mathcal{A})} \mathfrak{Obj}_{\mathcal{A}}^{\alpha}, & \mathfrak{Exact}_{\mathcal{A}} &= \coprod_{\substack{\alpha, \beta, \gamma \in \bar{C}(\mathcal{A}) \\ \beta = \alpha + \gamma}} \mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma} \text{ and} \\ \mathfrak{M}(I, \preceq)_{\mathcal{A}} &= \coprod_{\kappa : I \rightarrow \bar{C}(\mathcal{A})} \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}. \end{aligned} \quad (32)$$

**Proof.** For the first part, we already know  $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{Exact}_{\mathcal{A}}, \mathfrak{M}(I, \preceq)_{\mathcal{A}}$  are contravariant 2-functors  $\text{Sch}_{\mathbb{K}} \rightarrow (\text{groupoids})$ , and we must show they are *stacks in groupoids*, that is, that Definition 2.8(i)–(iii) hold. For  $\mathfrak{Obj}_{\mathcal{A}}$  this follows immediately from  $\mathfrak{F}_{\mathcal{A}}$  being a stack in exact categories, as  $\mathfrak{Obj}_{\mathcal{A}}$  comes from  $\mathfrak{F}_{\mathcal{A}}$  by omitting morphisms which are not isomorphisms.

The proofs for  $\mathfrak{Exact}_{\mathcal{A}}, \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$  are similar, and we give only that for  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ . Let  $\{f_a : U_a \rightarrow V\}_{a \in A}$  be an open cover of  $V$  in the site  $\text{Sch}_{\mathbb{K}}$ . For Definition 2.8(i), let  $(\sigma, \iota, \pi), (\sigma', \iota', \pi') \in \text{Obj}(\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(V))$  and morphisms

$$\phi_a : \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(f_a)(\sigma, \iota, \pi) \rightarrow \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(f_a)(\sigma', \iota', \pi') \text{ for } a \in A$$

satisfy (4). Then for  $J \in \mathcal{F}_{(I, \preccurlyeq)}$ , applying Definition 2.8(i) for  $\mathfrak{F}_{\mathcal{A}}$  to the family of morphisms  $\phi_a(J) : \mathfrak{F}_{\mathcal{A}}(f_a)(\sigma(J)) \rightarrow \mathfrak{F}_{\mathcal{A}}(f_a)(\sigma'(J))$  for  $a \in A$  gives  $\eta(J) : \sigma(J) \rightarrow \sigma'(J)$  in  $\text{Mor}(\mathfrak{F}_{\mathcal{A}}(V))$  with  $\mathfrak{F}_{\mathcal{A}}(f_a)\eta(J) = \phi_a(J)$  for all  $a \in A$ .

Moreover, Definition 2.8(ii) implies  $\eta(J)$  is unique. Since the  $\phi_a(J)$  are isomorphisms, gluing the  $\phi_a(J)^{-1}$  in the same way yields an inverse for  $\eta(J)$ , so  $\eta(J)$  is an isomorphism in  $\mathfrak{F}_{\mathcal{A}}(V)$ . By (12) and functoriality of  $\mathfrak{F}_{\mathcal{A}}(f_a)$  we have

$$\begin{aligned} \mathfrak{F}_{\mathcal{A}}(f_a)(\eta(K) \circ \iota(J, K)) &= \phi_a(K) \circ \mathfrak{F}_{\mathcal{A}}(f_a)(\iota(J, K)) = \mathfrak{F}_{\mathcal{A}}(f_a)(\iota'(J, K)) \circ \phi_a(J) \\ &= \mathfrak{F}_{\mathcal{A}}(f_a)(\iota'(J, K) \circ \eta(J)), \quad \text{for all } a \in A, \text{ and } (J, K) \in \mathcal{G}_{(I, \preccurlyeq)}. \end{aligned}$$

Therefore  $\eta(K) \circ \iota(J, K) = \iota'(J, K) \circ \eta(J)$  for all  $(J, K) \in \mathcal{G}_{(I, \preccurlyeq)}$  by Definition 2.8(ii) for  $\mathfrak{F}_{\mathcal{A}}$ . Similarly  $\eta(K) \circ \pi(J, K) = \pi'(J, K) \circ \eta(J)$  for all  $(J, K) \in \mathcal{H}_{(I, \preccurlyeq)}$ , so  $\eta : (\sigma, \iota, \pi) \rightarrow (\sigma', \iota', \pi')$  is an isomorphism of configurations by (12).

That is,  $\eta \in \text{Mor}(\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(V))$ . Clearly  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(f_a)\eta = \phi_a$  for all  $a \in A$ . This proves Definition 2.8(i) for  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ . For (ii), let  $\eta, \zeta : (\sigma, \iota, \pi) \rightarrow (\sigma', \iota', \pi')$  lie in  $\text{Mor}(\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(V))$  with  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(f_a)\eta = \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(f_a)\zeta$  for all  $a \in A$ . Then for  $J \in F_{(I, \preccurlyeq)}$  we have  $\mathfrak{F}_{\mathcal{A}}(f_a)\eta(J) = \mathfrak{F}_{\mathcal{A}}(f_a)\zeta(J)$  for  $a \in A$ , so Definition 2.8(ii) for  $\mathfrak{F}_{\mathcal{A}}$  gives  $\eta(J) = \zeta(J)$ , and thus  $\eta = \zeta$ .

A more complicated proof using Definition 2.8(i)–(iii) for  $\mathfrak{F}_{\mathcal{A}}$  shows (iii) holds for  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ , so  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$  is a  $\mathbb{K}$ -stack. We find  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha, \beta, \gamma}, \mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma}, \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  are  $\mathbb{K}$ -substacks of  $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{Exact}_{\mathcal{A}}, \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$  by the same methods.

If  $U$  is a connected, nonempty  $\mathbb{K}$ -scheme and  $X \in \mathfrak{Obj}_{\mathcal{A}}(U)$  then by Assumption 7.1(iii) the map  $\text{Hom}(\text{Spec } \mathbb{K}, U) \rightarrow K(\mathcal{A})$  given by  $u \mapsto [\mathfrak{F}_{\mathcal{A}}(u)X]$  is locally constant on  $\text{Hom}(\text{Spec } \mathbb{K}, U)$ , so  $[\mathfrak{F}_{\mathcal{A}}(u)X] \equiv \alpha$  for some  $\alpha \in \bar{C}(\mathcal{A})$ , and therefore  $X \in \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(U)$ . Hence  $\mathfrak{Obj}_{\mathcal{A}}(U) = \coprod_{\alpha \in \bar{C}(\mathcal{A})} \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(U)$ . It follows easily that  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$  is an open and closed  $\mathbb{K}$ -substack of  $\mathfrak{Obj}_{\mathcal{A}}$ , and the first equation of (32) holds. The proofs for  $\mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma}$  and  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  are similar.  $\square$

Write  $\mathcal{M}(I, \preccurlyeq)_{\mathcal{A}} = \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(\mathbb{K})$ ,  $\mathcal{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} = \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}(\mathbb{K})$  for the sets of geometric points of  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$  and  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$ , as in Definition 2.9. We show that these are the sets of isomorphism classes of  $(I, \preccurlyeq)$ - and  $(I, \preccurlyeq, \kappa)$ -configurations in  $\mathcal{A}$ . This justifies calling  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}, \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  moduli stacks of  $(I, \preccurlyeq)$ - and  $(I, \preccurlyeq, \kappa)$ -configurations.

**Proposition 7.6.** *In the situation above,  $\mathcal{M}(I, \preccurlyeq)_{\mathcal{A}}$  and  $\mathcal{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  are the sets of isomorphism classes of  $(I, \preccurlyeq)$ - and  $(I, \preccurlyeq, \kappa)$ -configurations in  $\mathcal{A}$ .*

**Proof.** By definition,  $\mathcal{M}(I, \preccurlyeq)_{\mathcal{A}}$  is the isomorphism classes in the groupoid  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$  ( $\text{Spec } \mathbb{K}$ ). By Assumption 7.1(i), objects of  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(\text{Spec } \mathbb{K})$  are  $(I, \preccurlyeq)$ -configurations in  $\mathcal{A}$ , and morphisms are isomorphisms of configurations. Thus  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(\mathbb{K}) = \mathcal{M}(I, \preccurlyeq)_{\mathcal{A}}$  is the set of isomorphism classes of  $(I, \preccurlyeq)$ -configurations. Similarly,  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}(\mathbb{K})$  has objects  $(I, \preccurlyeq, \kappa)$ -configurations and morphisms their isomorphisms, and the result follows.  $\square$

### 7.3. Morphisms of moduli stacks

We shall now define families of natural 1-morphisms between the  $\mathbb{K}$ -stacks of §7.2. As in §2.3, a 1-morphism of  $\mathbb{K}$ -stacks  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  is a *natural transformation* between the 2-functors  $\mathfrak{F}, \mathfrak{G}$ . For each  $U \in \text{Sch}_{\mathbb{K}}$  we must provide a functor  $\mathfrak{F}(U) \rightarrow \mathfrak{G}(U)$ , satisfying some obvious natural conditions. In all our examples these conditions hold trivially, as each  $\phi(U)$  is a ‘forgetful functor’ omitting part of the structure, so we shall not bother to verify them.

**Definition 7.7.** Define 1-morphisms  $b, m, e : \mathfrak{Exact}_{\mathcal{A}} \rightarrow \mathfrak{Obj}_{\mathcal{A}}$  as follows. For  $U \in \text{Sch}_{\mathbb{K}}$ , let  $b(U), m(U), e(U) : \mathfrak{Exact}_{\mathcal{A}}(U) \rightarrow \mathfrak{Obj}_{\mathcal{A}}(U)$  act on  $(X, Y, Z, \phi, \psi)$  in  $\text{Obj}(\mathfrak{Exact}_{\mathcal{A}}(U))$  and  $(\alpha, \beta, \gamma)$  in  $\text{Mor}(\mathfrak{Exact}_{\mathcal{A}}(U))$  by

$$\begin{aligned} b(U) : (X, Y, Z, \phi, \psi) &\mapsto X, & b(U) : (\alpha, \beta, \gamma) &\mapsto \alpha, & m(U) : (X, Y, Z, \phi, \psi) &\mapsto Y, \\ m(U) : (\alpha, \beta, \gamma) &\mapsto \beta, & e(U) : (X, Y, Z, \phi, \psi) &\mapsto Z & \text{ and } e(U) : (\alpha, \beta, \gamma) &\mapsto \gamma. \end{aligned}$$

Then  $b, m, e$  project to the *beginning*, *middle* and *end* objects  $X, Y, Z$ , respectively, in  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ . For  $\alpha, \beta, \gamma \in \bar{C}(\mathcal{A})$  with  $\beta = \alpha + \gamma$ , these restrict to

$$b : \mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma} \rightarrow \mathfrak{Obj}_{\mathcal{A}}^{\alpha}, \quad m : \mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma} \rightarrow \mathfrak{Obj}_{\mathcal{A}}^{\beta}, \quad e : \mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma} \rightarrow \mathfrak{Obj}_{\mathcal{A}}^{\gamma}.$$

For  $(I, \preccurlyeq)$  a finite poset,  $\kappa : I \rightarrow \bar{C}(\mathcal{A})$ , and  $J \in \mathcal{F}_{(I, \preccurlyeq)}$ , define a 1-morphism  $\sigma(J) : \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}} \rightarrow \mathfrak{Obj}_{\mathcal{A}}$ , where  $\sigma(J)(U) : \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(U) \rightarrow \mathfrak{Obj}_{\mathcal{A}}(U)$  acts as  $\sigma(J)(U) : (\sigma, \iota, \pi) \mapsto \sigma(J)$  on objects and  $\sigma(J)(U) : \alpha \mapsto \alpha(J)$  on morphisms, for  $U \in \text{Sch}_{\mathbb{K}}$ . It restricts to  $\sigma(J) : \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Obj}_{\mathcal{A}}^{\kappa(J)}$ .

As in Definition 5.1 we have  $\mathcal{F}_{(J, \preccurlyeq)} \subseteq \mathcal{F}_{(I, \preccurlyeq)}$ ,  $\mathcal{G}_{(J, \preccurlyeq)} \subseteq \mathcal{G}_{(I, \preccurlyeq)}$ ,  $\mathcal{H}_{(J, \preccurlyeq)} \subseteq \mathcal{H}_{(I, \preccurlyeq)}$ . Define a 1-morphism  $S(I, \preccurlyeq, J) : \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}} \rightarrow \mathfrak{M}(J, \preccurlyeq)_{\mathcal{A}}$  by, for  $U \in \text{Sch}_{\mathbb{K}}$ ,

$$S(I, \preccurlyeq, J)(U) : (\sigma, \iota, \pi) \mapsto (\sigma|_{\mathcal{F}_{(J, \preccurlyeq)}}, \iota|_{\mathcal{G}_{(J, \preccurlyeq)}}, \pi|_{\mathcal{H}_{(J, \preccurlyeq)}}), \quad \alpha \mapsto \alpha|_{\mathcal{F}_{(J, \preccurlyeq)}}$$

on  $(\sigma, \iota, \pi) \in \text{Obj}(\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(U))$  and  $\alpha \in \text{Mor}(\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(U))$ . It restricts to  $S(I, \preccurlyeq, J) : \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{M}(J, \preccurlyeq, \kappa|_J)_{\mathcal{A}}$ .

Now let  $(I, \preccurlyeq)$  and  $(K, \trianglelefteq)$  be finite posets, and  $\phi : I \rightarrow K$  a surjective map with  $i \preccurlyeq j$  implies  $\phi(i) \trianglelefteq \phi(j)$  for  $i, j \in I$ . As in Definition 5.2, the pull-back  $\phi^*$  of subsets of  $K$  to subsets of  $I$  gives injective maps  $\phi^* : \mathcal{F}_{(K, \trianglelefteq)} \rightarrow \mathcal{F}_{(I, \preccurlyeq)}$ ,  $\phi^* : \mathcal{G}_{(K, \trianglelefteq)} \rightarrow \mathcal{G}_{(I, \preccurlyeq)}$

and  $\phi^* : \mathcal{H}_{(K, \trianglelefteq)} \rightarrow \mathcal{H}_{(I, \preccurlyeq)}$ . Define a 1-morphism  $Q(I, \preccurlyeq, K, \trianglelefteq, \phi) : \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}} \rightarrow \mathfrak{M}(K, \trianglelefteq)_{\mathcal{A}}$  by, for  $U \in \text{Sch}_{\mathbb{K}}$ ,

$$Q(I, \preccurlyeq, K, \trianglelefteq, \phi)(U) : (\sigma, \iota, \pi) \mapsto (\sigma \circ \phi^*, \iota \circ \phi^*, \pi \circ \phi^*), \quad \alpha \mapsto \alpha \circ \phi^*$$

on  $(\sigma, \iota, \pi) \in \text{Obj}(\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(U))$  and  $\alpha \in \text{Mor}(\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(U))$ .

In the special case when  $I = K$  and  $\phi : I \rightarrow I$  is the identity map  $\text{id}_I$ , write  $Q(I, \preccurlyeq, \trianglelefteq) = Q(I, \preccurlyeq, I, \trianglelefteq, \text{id}_I)$ . Given  $\kappa : I \rightarrow \bar{C}(\mathcal{A})$ , define  $\mu : K \rightarrow \bar{C}(\mathcal{A})$  by  $\mu(k) = \kappa(\phi^{-1}(k))$ . Then  $Q(I, \preccurlyeq, K, \trianglelefteq, \phi)$  restricts to  $Q(I, \preccurlyeq, K, \trianglelefteq, \phi) : \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{M}(K, \trianglelefteq, \mu)_{\mathcal{A}}$ . When  $I = K$  and  $\phi = \text{id}_I$  we have  $\mu = \kappa$ , so  $Q(I, \preccurlyeq, \trianglelefteq) : \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{M}(I, \trianglelefteq, \kappa)_{\mathcal{A}}$ .

Each of these 1-morphisms  $\psi$  induces a map  $\psi_*$  on the sets of geometric points of the  $\mathbb{K}$ -stacks, as in Definition 2.9. Following Proposition 7.6, it is easy to show these do the obvious things.

**Proposition 7.8.** *In the situation above, the induced maps act as*

$$\begin{aligned} \sigma(J)_* : \mathcal{M}(I, \preccurlyeq)_{\mathcal{A}} &\rightarrow \mathfrak{Sbj}_{\mathcal{A}}(\mathbb{K}), \quad S(I, \preccurlyeq, J)_* : \mathcal{M}(I, \preccurlyeq)_{\mathcal{A}} \rightarrow \mathcal{M}(J, \preccurlyeq)_{\mathcal{A}}, \\ Q(I, \preccurlyeq, K, \trianglelefteq, \phi)_* : \mathcal{M}(I, \preccurlyeq)_{\mathcal{A}} &\rightarrow \mathcal{M}(K, \trianglelefteq)_{\mathcal{A}}, \quad \sigma(J)_* : [(\sigma, \iota, \pi)] \mapsto [\sigma(J)], \\ S(I, \preccurlyeq, J)_* : [(\sigma, \iota, \pi)] &\mapsto [(\sigma', \iota', \pi')], \quad Q(I, \preccurlyeq, K, \trianglelefteq, \phi)_* : [(\sigma, \iota, \pi)] \mapsto [(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})] \end{aligned}$$

on geometric points, where  $(\sigma, \iota, \pi)$  is an  $(I, \preccurlyeq)$ -configuration in  $\mathcal{A}$ ,  $(\sigma', \iota', \pi')$  its  $(J, \preccurlyeq)$ -subconfiguration, and  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  its quotient  $(K, \trianglelefteq)$ -configuration from  $\phi$ .

#### 7.4. 1-isomorphisms of moduli stacks

We conclude this section by proving that a number of 1-morphisms above are 1-isomorphisms. To show  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  is a 1-isomorphism, we must show that the functor  $\phi(U) : \mathfrak{F}(U) \rightarrow \mathfrak{G}(U)$  is an equivalence of categories for each  $U \in \text{Sch}_{\mathbb{K}}$ . That is, we must prove two things:

- (a)  $\phi(U) : \text{Obj}(\mathfrak{F}(U)) \rightarrow \text{Obj}(\mathfrak{G}(U))$  induces a bijection between isomorphism classes of objects in  $\mathfrak{F}(U)$  and  $\mathfrak{G}(U)$ ; and
- (b)  $\phi(U) : \text{Mor}(\mathfrak{F}(U)) \rightarrow \text{Mor}(\mathfrak{G}(U))$  induces for all  $X, Y \in \text{Obj}(\mathfrak{F}(U))$  a bijection  $\text{Hom}(X, Y) \rightarrow \text{Hom}(\phi(U)X, \phi(U)Y)$ .

**Proposition 7.9.** (i) *Let  $(I, \bullet)$  be a finite poset with  $i \bullet j$  if and only if  $i = j$ , and  $\kappa : I \rightarrow \bar{C}(\mathcal{A})$ . Then the following are 1-isomorphisms:*

$$\prod_{i \in I} \sigma(\{i\}) : \mathfrak{M}(I, \bullet)_{\mathcal{A}} \rightarrow \prod_{i \in I} \mathfrak{Sbj}_{\mathcal{A}}, \quad \prod_{i \in I} \sigma(\{i\}) : \mathfrak{M}(I, \bullet, \kappa)_{\mathcal{A}} \rightarrow \prod_{i \in I} \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(i)}.$$

(ii) Let  $(\{i, j\}, \preceq)$  be a poset with  $i \preceq j$  and  $\kappa : \{i, j\} \rightarrow \tilde{C}(\mathcal{A})$ . Define 1-morphisms  $\Pi : \mathfrak{M}(\{i, j\}, \preceq)_{\mathcal{A}} \rightarrow \mathfrak{Exact}_{\mathcal{A}}$  and  $\Pi : \mathfrak{M}(\{i, j\}, \preceq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Exact}_{\mathcal{A}}^{\kappa(i), \kappa(\{i, j\}), \kappa(j)}$  by

$$\Pi(U) : (\sigma, \iota, \pi) \mapsto (\sigma(\{i\}), \sigma(\{i, j\}), \sigma(\{j\}), \iota(\{i\}, \{i, j\}), \pi(\{i, j\}, \{j\})) \quad \text{and}$$

$$\Pi(U) : \alpha \mapsto (\alpha(\{i\}), \alpha(\{i, j\}), \alpha(\{j\}))$$

for all  $(\sigma, \iota, \pi) \in \text{Obj}(\mathfrak{M}(\{i, j\}, \preceq)_{\mathcal{A}}(U))$ ,  $\alpha \in \text{Mor}(\mathfrak{M}(\{i, j\}, \preceq)_{\mathcal{A}}(U))$  and  $U \in \text{Sch}_{\mathbb{K}}$ . Both of these  $\Pi$  are 1-isomorphisms.

**Proof.** For (i) the proof for both 1-morphisms is the same, so we consider only the first. Let  $U \in \text{Sch}_{\mathbb{K}}$ , and  $(\sigma, \iota, \pi) \in \text{Obj}(\mathfrak{M}(I, \bullet)_{\mathcal{A}}(U))$ . Suppose  $J \subseteq K \subseteq I$ , and set  $L = K \setminus J$ . Then  $J, K, L \in \mathcal{F}_{(I, \bullet)}$ , and  $(J, K), (L, K) \in \mathcal{G}_{(I, \bullet)}$ . Considering the diagram

$$\begin{array}{ccccccc} 0 & \rightleftarrows & \sigma(J) & \xrightleftharpoons[\pi_{\sigma(J)}]{\iota_{\sigma(J)}} & \sigma(J) \oplus \sigma(L) & \xrightleftharpoons[\pi_{\sigma(L)}]{\iota_{\sigma(L)}} & \sigma(L) \rightleftarrows 0 \\ & & \downarrow \text{id}_{\sigma(J)} & & \downarrow & & \downarrow \text{id}_{\sigma(L)} \\ 0 & \rightleftarrows & \sigma(J) & \xrightleftharpoons[\pi(K, J)]{\iota(J, K)} & \sigma(K) & \xrightleftharpoons[\iota(L, K)]{\pi(K, L)} & \sigma(L) \rightleftarrows 0, \end{array} \quad (33)$$

and using Definitions 2.1(iii) and 4.1 gives a canonical isomorphism  $\sigma(J) \oplus \sigma(L) \rightarrow \sigma(K)$  making (33) commute. By induction on  $|J|$  we construct canonical isomorphisms  $\bigoplus_{i \in J} \sigma(\{i\}) \rightarrow \sigma(J)$  for all  $J \subseteq I$ , with  $\iota(J, K), \pi(J, K)$  corresponding to projections from or to subfactors in the direct sums.

Let  $(X_i)_{i \in I}$  lie in  $\text{Obj}(\prod_{i \in I} \mathfrak{Ob}_{\mathcal{A}}(U))$ . Setting  $\sigma(J) = \bigoplus_{i \in J} X_i$ , so that  $\sigma(\{i\}) = X_i$ , and taking  $\iota(J, K), \pi(J, K)$  to be the natural projections gives  $(\sigma, \iota, \pi)$  in  $\text{Obj}(\mathfrak{M}(I, \bullet)_{\mathcal{A}}(U))$  with  $\prod_{i \in I} \sigma(\{i\})(U)(\sigma, \iota, \pi) = (X_i)_{i \in I}$ . Hence

$$\prod_{i \in I} \sigma(\{i\})(U) : \text{Obj}(\mathfrak{M}(I, \bullet)_{\mathcal{A}}(U)) \rightarrow \text{Obj}(\prod_{i \in I} \mathfrak{Ob}_{\mathcal{A}}(U)) \quad (34)$$

is surjective. Suppose  $(\sigma, \iota, \pi), (\sigma', \iota', \pi')$  lie in  $\text{Obj}(\mathfrak{M}(I, \bullet)_{\mathcal{A}}(U))$  with images  $(X_i)_{i \in I}, (X'_i)_{i \in I}$  under  $\prod_{i \in I} \sigma(\{i\})(U)$ . A morphism  $(f_i)_{i \in I} : (X_i)_{i \in I} \rightarrow (X'_i)_{i \in I}$  is a collection of isomorphisms  $f_i : X_i \rightarrow X'_i$  in  $\mathfrak{F}_{\mathcal{A}}(U)$ .

Any such  $(f_i)_{i \in I}$  extends uniquely to a morphism  $\alpha : (\sigma, \iota, \pi) \rightarrow (\sigma', \iota', \pi')$ , where  $\alpha(\{i\}) = f_i$  for  $i \in I$ , and more generally  $\alpha(J)$  corresponds to  $\bigoplus_{i \in J} f_i : \bigoplus_{i \in J} X_i \rightarrow \bigoplus_{i \in J} X'_i$  under the canonical isomorphisms  $\bigoplus_{i \in J} X_i \rightarrow \sigma(J)$  and  $\bigoplus_{i \in J} X'_i \rightarrow \sigma'(J)$ .

Therefore the following map is a bijection:

$$\prod_{i \in I} \sigma(\{i\})(U) : \text{Hom}((\sigma, \iota, \pi), (\sigma', \iota', \pi')) \rightarrow \text{Hom}((X_i)_{i \in I}, (X'_i)_{i \in I}).$$

Together with surjectivity of (34), this shows  $\prod_{i \in I} \sigma(\{i\})$  is a 1-isomorphism.

For (ii), both functors  $\Pi(U)$  are actually *isomorphisms of categories*, not just equivalences. This is because the only data ‘forgotten’ by  $\Pi(U)$  is  $\sigma(\emptyset)$ ,  $\iota(\emptyset, J)$  and  $\pi(J, \emptyset)$  for  $J \subseteq \{i, j\}$  on objects  $(\sigma, \iota, \pi)$ , and  $\alpha(\emptyset)$  on morphisms  $\alpha$ . But by definition  $\sigma(\emptyset) = 0$ , so  $\iota(\emptyset, J) = 0$ ,  $\pi(J, \emptyset) = 0$  and  $\alpha(\emptyset) = 0$ , and there are unique choices for the forgotten data. Thus  $\Pi$  is a 1-isomorphism.

(Note: this assumes there is a *prescribed zero object* 0 in  $\mathfrak{F}_{\mathcal{A}}(U)$ , and that  $\sigma(\emptyset) = 0$  is part of the definition of configuration in  $\mathfrak{F}_{\mathcal{A}}(U)$ . If instead 0 in  $\mathfrak{F}_{\mathcal{A}}(U)$  is defined *only up to isomorphism*, then  $\Pi(U)$  ‘forgets’ a choice of 0 in  $\sigma(\emptyset)$ , but is still an equivalence of categories.)  $\square$

Our final result extends Theorem 5.5 to moduli stacks of configurations.

**Theorem 7.10.** *Let  $(J, \lesssim)$  and  $(K, \trianglelefteq)$  be finite posets and  $L \in \mathcal{F}_{(K, \trianglelefteq)}$ , with  $J \cap (K \setminus L) = \emptyset$ . Suppose  $\psi : J \rightarrow L$  is surjective with  $i \lesssim j$  implies  $\psi(i) \trianglelefteq \psi(j)$ . Set  $I = J \cup (K \setminus L)$ , and define a partial order  $\preceq$  on  $I$  by*

$$i \preceq j \text{ for } i, j \in I \text{ if } \begin{cases} i \lesssim j, & i, j \in J, \\ i \trianglelefteq j, & i, j \in K \setminus L, \\ \psi(i) \trianglelefteq j, & i \in J, \quad j \in K \setminus L, \\ i \trianglelefteq \psi(j), & i \in K \setminus L, \quad j \in J. \end{cases}$$

*Then  $J \in \mathcal{F}_{(I, \preceq)}$  with  $\preceq|_J = \lesssim$ . Define  $\phi : I \rightarrow K$  by  $\phi(i) = \psi(i)$  if  $i \in J$  and  $\phi(i) = i$  if  $i \in K \setminus L$ . Then  $\phi$  is surjective, with  $i \preceq j$  implies  $\phi(i) \trianglelefteq \phi(j)$ .*

*Let  $\kappa : I \rightarrow \tilde{C}(\mathcal{A})$ , and define  $\mu : K \rightarrow K(\mathcal{A})$  by  $\mu(k) = \kappa(\phi^{-1}(k))$ . Then the following 1-morphism diagrams commute and are Cartesian squares:*

$$\begin{array}{ccc} \mathfrak{M}(I, \preceq)_{\mathcal{A}} & \longrightarrow & \mathfrak{M}(J, \lesssim)_{\mathcal{A}} & \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} & \longrightarrow & \mathfrak{M}(J, \lesssim, \kappa)_{\mathcal{A}} \\ \downarrow Q(I, \preceq, K, \trianglelefteq, \phi) & S(I, \preceq, J) & \downarrow Q(J, \lesssim, L, \trianglelefteq, \psi) & \downarrow Q(I, \preceq, K, \trianglelefteq, \phi) & S(I, \preceq, J) & \downarrow Q(J, \lesssim, L, \trianglelefteq, \psi) \\ \mathfrak{M}(K, \trianglelefteq)_{\mathcal{A}} & \longrightarrow & \mathfrak{M}(L, \trianglelefteq)_{\mathcal{A}} & \mathfrak{M}(K, \trianglelefteq, \mu)_{\mathcal{A}} & \longrightarrow & \mathfrak{M}(L, \trianglelefteq, \mu)_{\mathcal{A}} \end{array} \quad (35)$$

**Proof.** We give the proof for the first square of (35) only, as the second is the same. Let  $U \in \text{Sch}_{\mathbb{K}}$  and  $(\sigma, \iota, \pi) \in \text{Obj}(\mathfrak{M}(I, \preceq)_{\mathcal{A}}(U))$ . Write

$$\begin{aligned} (\tilde{\sigma}, \tilde{\iota}, \tilde{\pi}) &= Q(I, \preceq, K, \trianglelefteq, \phi)(U)(\sigma, \iota, \pi), \quad (\check{\sigma}, \check{\iota}, \check{\pi}) = S(K, \trianglelefteq, L)(U)(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi}), \\ (\sigma', \iota', \pi') &= S(I, \preceq, J)(U)(\sigma, \iota, \pi), \quad (\hat{\sigma}, \hat{\iota}, \hat{\pi}) = Q(J, \lesssim, L, \trianglelefteq, \psi)(U)(\sigma', \iota', \pi'). \end{aligned} \quad (36)$$

Then  $(\sigma, \iota, \pi)$  is an  $(I, \preccurlyeq)$ -configuration in  $\mathfrak{F}_{\mathcal{A}}(U)$ ,  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  its quotient  $(K, \trianglelefteq)$ -configuration from  $\phi$ , and  $(\sigma', \iota', \pi')$  its  $(J, \lesssim)$ -subconfiguration. Lemma 5.4 gives  $(\check{\sigma}, \check{\iota}, \check{\pi}) = (\hat{\sigma}, \hat{\iota}, \hat{\pi})$ , and the analogue for morphisms holds. Thus

$$S(K, \trianglelefteq, L)(U) \circ Q(I, \preccurlyeq, K, \trianglelefteq, \phi)(U) = Q(J, \lesssim, L, \trianglelefteq, \psi)(U) \circ S(I, \preccurlyeq, J)(U)$$

as functors  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}(U) \rightarrow \mathfrak{M}(L, \trianglelefteq)_{\mathcal{A}}(U)$ . Since this holds for all  $U \in \text{Sch}_{\mathbb{K}}$ , the first square in (35) commutes. In fact it *strictly commutes*, that is, the 1-morphisms  $S(K, \trianglelefteq, L) \circ Q(I, \preccurlyeq, K, \trianglelefteq, \phi)$  and  $Q(J, \lesssim, L, \trianglelefteq, \psi) \circ S(I, \preccurlyeq, J)$  are not just 2-isomorphic, but equal.

As in Definition 2.10 we now get a 1-morphism

$$\rho : \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}} \rightarrow \mathfrak{M}(K, \trianglelefteq)_{\mathcal{A}} \times_{\mathfrak{M}(L, \trianglelefteq)_{\mathcal{A}}} \mathfrak{M}(J, \lesssim)_{\mathcal{A}},$$

unique up to 2-isomorphism, making (9) commute, and the first square of (35) is Cartesian if  $\rho$  is a 1-isomorphism. From Definition 2.10, we deduce that *objects* of  $(\mathfrak{M}(K, \trianglelefteq)_{\mathcal{A}} \times_{\mathfrak{M}(L, \trianglelefteq)_{\mathcal{A}}} \mathfrak{M}(J, \lesssim)_{\mathcal{A}})(U)$  are triples

$$((\tilde{\sigma}, \tilde{\iota}, \tilde{\pi}), (\sigma', \iota', \pi'), \alpha : (\check{\sigma}, \check{\iota}, \check{\pi}) \xrightarrow{\cong} (\hat{\sigma}, \hat{\iota}, \hat{\pi})),$$

where  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi}) \in \text{Obj}(\mathfrak{M}(K, \trianglelefteq)_{\mathcal{A}}(U))$ ,  $(\sigma', \iota', \pi') \in \text{Obj}(\mathfrak{M}(J, \lesssim)_{\mathcal{A}}(U))$ , and  $(\check{\sigma}, \check{\iota}, \check{\pi})$ ,  $(\hat{\sigma}, \hat{\iota}, \hat{\pi})$  are as in (36), and  $\alpha \in \text{Mor}(\mathfrak{M}(L, \trianglelefteq)_{\mathcal{A}}(U))$ .

We can define  $\rho(U)$  explicitly on objects  $(\sigma, \iota, \pi)$  and morphisms  $\alpha$  by

$$\begin{aligned} \rho(U) : (\sigma, \iota, \pi) &\longmapsto ((\tilde{\sigma}, \tilde{\iota}, \tilde{\pi}), (\sigma', \iota', \pi'), \text{id}_{(\check{\sigma}, \check{\iota}, \check{\pi})}), \\ \rho(U) : \alpha &\longmapsto (\alpha \circ \phi^*, \alpha|_{\mathcal{F}_{(J, \lesssim)}}), \end{aligned}$$

where  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$ ,  $(\sigma', \iota', \pi')$  and  $(\check{\sigma}, \check{\iota}, \check{\pi})$  are as in (36), and  $(\check{\sigma}, \check{\iota}, \check{\pi}) = (\hat{\sigma}, \hat{\iota}, \hat{\pi})$ . Now Theorem 5.5 shows  $\rho(U)$  is an *equivalence of categories*. As this holds for all  $U \in \text{Sch}_{\mathbb{K}}$ ,  $\rho$  is a 1-isomorphism, so the first square of (35) is Cartesian.  $\square$

Proposition 7.9 and Theorem 7.10 will be used in §8.1 to prove  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$  and  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  are *algebraic*  $\mathbb{K}$ -stacks.

## 8. Algebraic $\mathbb{K}$ -stacks of configurations

So far we have only shown that the moduli stacks  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ ,  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  of §7.2 are  $\mathbb{K}$ -stacks, which is quite a weak, categorical concept. We now impose some additional assumptions, which will enable us to prove that  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ ,  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  are *algebraic*  $\mathbb{K}$ -stacks *locally of finite type*, and that various morphisms between them are *representable* or *of finite type*.



**Assumption 8.1.** Let Assumption 7.1 hold for  $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$ . Suppose the  $\mathbb{K}$ -stacks  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}, \mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma}$  of §7.2 are *algebraic (Artin)  $\mathbb{K}$ -stacks, locally of finite type*. Suppose the following 1-morphisms of §7.3 are of *finite type*:

$$m : \mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma} \rightarrow \mathfrak{Obj}_{\mathcal{A}}^{\beta}, \quad b \times e : \mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma} \rightarrow \mathfrak{Obj}_{\mathcal{A}}^{\alpha} \times \mathfrak{Obj}_{\mathcal{A}}^{\gamma}. \quad (37)$$

This list of assumptions is motivated firstly because they hold for the examples the author is interested in, given in §§9 and 10, and secondly as the results of §§8.1–8.3 that we use them to prove will be essential for the theory of invariants ‘counting’ (semi)stable configurations in  $\mathcal{A}$  to be developed in [11,12].

### 8.1. Moduli stacks of configurations are algebraic $\mathbb{K}$ -stacks

The moduli stacks  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}, \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  of §7.2 are *algebraic  $\mathbb{K}$ -stacks*.

**Theorem 8.2.** *Let Assumptions 7.1 and 8.1 hold for  $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$ . Then the  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}, \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  of §7.2 are algebraic  $\mathbb{K}$ -stacks, locally of finite type.*

**Proof.** The  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$  case follows from the  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  case by (32), so we prove the  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  case. When  $|I| = 0, 1$  or  $2$  there are four cases:

- (a)  $I = \emptyset$ . Then  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is 1-isomorphic to  $\mathrm{Spec} \mathbb{K}$ .
- (b)  $I = \{i\}$ . By Proposition 7.9(i),  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is 1-isomorphic to  $\mathfrak{Obj}_{\mathcal{A}}^{\kappa(i)}$ .
- (c)  $I = \{i, j\}$  with  $a \preccurlyeq b$  if and only if  $a = b$ . Then by Proposition 7.9(i)  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is 1-isomorphic to  $\mathfrak{Obj}_{\mathcal{A}}^{\kappa(i)} \times \mathfrak{Obj}_{\mathcal{A}}^{\kappa(j)}$ .
- (d)  $I = \{i, j\}$  with  $i \preccurlyeq j$ . By Proposition 7.9(ii)  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is 1-isomorphic to  $\mathfrak{Exact}_{\mathcal{A}}^{\kappa(i), \kappa(\{i, j\}), \kappa(j)}$ .

In each case  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is an *algebraic  $\mathbb{K}$ -stack locally of finite type*, by Assumption 8.1 in (b)–(d). So the theorem holds when  $|I| \leq 2$ .

Next we prove the case that  $\preccurlyeq$  is a *total order*, that is,  $i \preccurlyeq j$  or  $j \preccurlyeq i$  for all  $i, j \in I$ . Then  $(I, \preccurlyeq)$  is canonically isomorphic to  $(\{1, \dots, n\}, \leq)$  for  $n = |I|$ . Suppose by induction that  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is an algebraic  $\mathbb{K}$ -stack locally of finite type for all total orders  $(I, \preccurlyeq)$  with  $|I| \leq n$ . From above this holds for  $n = 2$ , so take  $n \geq 1$ . Let  $(I, \preccurlyeq)$  be a total order with  $|I| = n + 1$ .

Let  $i$  be  $\preccurlyeq$ -minimal in  $I$ , and  $j$  be  $\preccurlyeq$ -minimal in  $I \setminus \{i\}$ , which defines  $i, j$  uniquely as  $\preccurlyeq$  is a total order. Let  $J = \{i, j\}$  and  $L = \{l\}$  be a one point set with  $l \notin I$ , set  $K = \{l\} \cup I \setminus \{i, j\}$ , and define  $\leq$  on  $K$  by  $a \leq b$  if either  $a = l$ , or  $a, b \in I \setminus \{i, j\}$  with  $a \preccurlyeq b$ . Then  $\leq$  is a total order on  $K$ , with minimal element  $l$ . Define  $\mu : K \rightarrow \bar{C}(\mathcal{A})$  by  $\mu(l) = \kappa(i) + \kappa(j)$ , and  $\mu(a) = \kappa(a)$  for  $a \in I \setminus \{i, j\}$ . Define  $\phi : I \rightarrow K$  by  $\phi(a) = l$  for  $a = i, j$  and  $\phi(a) = a$  otherwise. Define  $\psi : \{i, j\} \rightarrow \{l\}$  by  $\psi(a) = l$ .

Theorem 7.10 now applies, and shows that the diagram

$$\begin{array}{ccc}
 \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} & \xrightarrow{\quad} & \mathfrak{M}(\{i, j\}, \preccurlyeq, \kappa|_{\{i, j\}})_{\mathcal{A}} \\
 \downarrow Q(I, \preccurlyeq, K, \trianglelefteq, \phi) & \begin{array}{c} S(I, \preccurlyeq, \{i, j\}) \\ S(K, \trianglelefteq, \{l\}) \end{array} & \downarrow Q(\{i, j\}, \preccurlyeq, \{l\}, \trianglelefteq, \psi) \\
 \mathfrak{M}(K, \trianglelefteq, \mu)_{\mathcal{A}} & \xrightarrow{\quad} & \mathfrak{M}(\{l\}, \trianglelefteq, \mu|_{\{l\}})_{\mathcal{A}}
 \end{array} \quad (38)$$

is commutative, and a Cartesian square. The two right-hand corners are algebraic  $\mathbb{K}$ -stacks locally of finite type by (b), (d) above, and the bottom left-hand corner is by induction as  $\trianglelefteq$  is a total order and  $|K| = n$ . Hence  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is an *algebraic  $\mathbb{K}$ -stack locally of finite type*, by properties of Cartesian squares in §2.3. By induction, this holds whenever  $\preccurlyeq$  is a *total order*.

Now let  $(I, \preccurlyeq)$  be a finite poset. Define  $S_{\preccurlyeq} = \{(i, j) \in I \times I : i \not\preccurlyeq j \text{ and } j \not\preccurlyeq i\}$ , and let  $n_{\preccurlyeq} = |S_{\preccurlyeq}|$ . Let  $\lesssim$  be a total order on  $I$  which dominates  $\preccurlyeq$ . Then  $\lesssim$  dominates  $\preccurlyeq$  by  $n_{\preccurlyeq}$  steps. If  $n_{\preccurlyeq} > 0$ , by Proposition 6.5 there exists  $\trianglelefteq$  on  $I$  such that  $\lesssim$  dominates  $\trianglelefteq$  by  $n_{\preccurlyeq} - 1$  steps, so that  $n_{\trianglelefteq} = n_{\preccurlyeq} - 1$ , and  $\trianglelefteq$  dominates  $\preccurlyeq$  by one step. By Lemma 6.3, there exist unique  $i, j \in I$  with  $i \trianglelefteq j$  but  $i \not\preccurlyeq j$ .

Suppose by induction that  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is an algebraic  $\mathbb{K}$ -stack locally of finite type whenever  $n_{\preccurlyeq} \leq n$ , for  $n \geq 0$ . When  $n = 0$  this implies  $\preccurlyeq$  is a *total order*, so the first step  $n = 0$  holds from above. Let  $(I, \preccurlyeq), \kappa$  have  $n_{\preccurlyeq} = n + 1$ . Then from above there is a partial order  $\trianglelefteq$  on  $I$  dominating  $\preccurlyeq$  by one step, so that  $n_{\trianglelefteq} = n$ , and unique  $i, j \in I$  with  $i \trianglelefteq j$  but  $i \not\preccurlyeq j$ . Define  $K = I$ ,  $J = L = \{i, j\}$ ,  $\phi : I \rightarrow K$  and  $\psi : J \rightarrow L$  to be the identity maps, and  $\mu = \kappa$ .

Theorem 7.10 now applies, and shows that the diagram

$$\begin{array}{ccc}
 \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} & \xrightarrow{\quad} & \mathfrak{M}(\{i, j\}, \preccurlyeq, \kappa|_{\{i, j\}})_{\mathcal{A}} \\
 \downarrow Q(I, \preccurlyeq, \trianglelefteq) & \begin{array}{c} S(I, \preccurlyeq, \{i, j\}) \\ S(I, \trianglelefteq, \{i, j\}) \end{array} & \downarrow Q(\{i, j\}, \preccurlyeq, \trianglelefteq) \\
 \mathfrak{M}(I, \trianglelefteq, \kappa)_{\mathcal{A}} & \xrightarrow{\quad} & \mathfrak{M}(\{i, j\}, \trianglelefteq, \kappa|_{\{i, j\}})_{\mathcal{A}}
 \end{array} \quad (39)$$

is commutative, and a *Cartesian square*. The two right-hand corners are algebraic  $\mathbb{K}$ -stacks locally of finite type by (c), (d) above, and the bottom left-hand corner is by induction, as  $n_{\trianglelefteq} = n$ . Hence  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is an *algebraic  $\mathbb{K}$ -stack locally of finite type*. By induction, this completes the proof.  $\square$

The underlying idea in this proof is that  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is 1-isomorphic to a complicated *multiple fibre product*, constructed from many copies of the  $\mathfrak{Ob}_{\mathcal{A}}^{\alpha, \beta, \gamma}$  and  $\mathfrak{Ex}_{\mathcal{A}}^{\alpha, \beta, \gamma}$ . As the class of algebraic  $\mathbb{K}$ -stacks locally of finite type is *closed under 1-isomorphisms and fibre products*, and  $\mathfrak{Ob}_{\mathcal{A}}^{\alpha, \beta, \gamma}$  and  $\mathfrak{Ex}_{\mathcal{A}}^{\alpha, \beta, \gamma}$  lie in this class by Assumption 8.1, we see that  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  also lies in this class.

## 8.2. Representable and finite type morphisms

Next we show some 1-morphisms from §7.3 are *representable*, or of *finite type*. We begin with 1-morphisms involving *two-point posets*.

**Proposition 8.3.** *In the situation above, let  $\preceq, \trianglelefteq$  be partial orders on  $\{i, j\}$  with  $a \preceq b$  only if  $a = b$  and  $i \trianglelefteq j$ , and let  $\kappa : \{i, j\} \rightarrow \bar{C}(\mathcal{A})$ . Then*

- (a)  $Q(\{i, j\}, \preceq, \trianglelefteq) : \mathfrak{M}(\{i, j\}, \preceq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{M}(\{i, j\}, \trianglelefteq, \kappa)_{\mathcal{A}}$  is representable and of finite type.
- (b)  $\sigma(\{i, j\}) : \mathfrak{M}(\{i, j\}, \trianglelefteq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Ob}_{\mathcal{A}}^{\kappa(\{i, j\})}$  is representable and of finite type.
- (c)  $\sigma(\{i\}) \times \sigma(\{j\}) : \mathfrak{M}(\{i, j\}, \trianglelefteq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Ob}_{\mathcal{A}}^{\kappa(i)} \times \mathfrak{Ob}_{\mathcal{A}}^{\kappa(j)}$  is of finite type.

**Proof.** For (a), consider the equality of 1-morphisms

$$\sigma(\{i\}) \times \sigma(\{j\}) = (\sigma(\{i\}) \times \sigma(\{j\})) \circ Q(\{i, j\}, \preceq, \trianglelefteq) \quad (40)$$

acting  $\mathfrak{M}(\{i, j\}, \preceq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Ob}_{\mathcal{A}}^{\kappa(i)} \times \mathfrak{Ob}_{\mathcal{A}}^{\kappa(j)}$  or  $\mathfrak{M}(\{i, j\}, \preceq)_{\mathcal{A}} \rightarrow \mathfrak{Ob}_{\mathcal{A}} \times \mathfrak{Ob}_{\mathcal{A}}$ . By Proposition 7.9(i), the l.h.s. of (40) is a 1-isomorphism, and so is representable and finite type. But if  $\mathfrak{F} \xrightarrow{\phi} \mathfrak{G} \xrightarrow{\psi} \mathfrak{H}$  are 1-morphisms of algebraic  $\mathbb{K}$ -stacks and  $\psi \circ \phi$  is representable and finite type, then  $\phi$  is too by Laumon and Moret [14, Lemma 3.12(c)(ii) and Remark 4.17(1)]. Hence  $Q(\{i, j\}, \preceq, \trianglelefteq)$  is representable and of finite type.

In (b) it is easy to see that  $\sigma(\{i, j\}) = m \circ \Pi$ , where  $\Pi$  is the 1-isomorphism of Proposition 7.9(ii) with  $\trianglelefteq$  in place of  $\preceq$ , and

$$m : \mathfrak{Exact}_{\mathcal{A}}^{\kappa(i), \kappa(i)+\kappa(j), \kappa(j)} \longrightarrow \mathfrak{Ob}_{\mathcal{A}}^{\kappa(i)+\kappa(j)} \quad (41)$$

is as in Definition 7.7. Since  $\Pi$  is a 1-isomorphism, as (41) is of finite type by Assumption 8.1,  $\sigma(\{i, j\})$  in (b) is of finite type. To show it is representable, we must show (41) is representable.

From [14, Corollary 8.1.1] we deduce the following necessary and sufficient condition for a 1-morphism  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  of algebraic  $\mathbb{K}$ -stacks to be *representable*: for all  $U \in \text{Sch}_{\mathbb{K}}$  and all  $X \in \text{Obj}(\mathfrak{F}(U))$ , the map

$$\phi(U) : \text{Hom}(X, X) \rightarrow \text{Hom}(\phi(U)X, \phi(U)X) \quad (42)$$

induced by the functor  $\phi(U) : \mathfrak{F}(U) \rightarrow \mathfrak{G}(U)$  should be injective. Since (42) is a group homomorphism, it is enough that  $\phi(U)\alpha = \text{id}_{\phi(U)X}$  implies  $\alpha = \text{id}_X$ .

Let  $(\alpha, \beta, \gamma) : (X, Y, Z, \phi, \psi) \rightarrow (X, Y, Z, \phi, \psi)$  in  $\text{Mor}(\mathfrak{Exact}_{\mathcal{A}}^{\kappa(i), \kappa(\{i, j\}), \kappa(j)}(U))$  or  $\text{Mor}(\mathfrak{Exact}_{\mathcal{A}}(U))$ . Then  $m(U)(X, Y, Z, \phi, \psi) = Y$  and  $m(U)(\alpha, \beta, \gamma) = \beta$ , so to show  $m$  is representable we must prove that  $\beta = \text{id}_Y$  implies  $\alpha = \text{id}_X$  and  $\gamma = \text{id}_Z$ . By definition of  $\text{Mor}(\mathfrak{Exact}_{\mathcal{A}}(U))$  we have

$$\phi \circ \alpha = \beta \circ \phi = \text{id}_Y \circ \phi = \phi \circ \text{id}_X \quad \text{and} \quad \text{id}_Z \circ \psi = \psi \circ \text{id}_Y = \psi \circ \beta = \gamma \circ \psi.$$

As  $\phi$  is injective and  $\psi$  surjective these imply  $\alpha = \text{id}_X$  and  $\gamma = \text{id}_Z$ . Thus (41) is representable, proving (b). Finally, in (c) we have  $\sigma(\{i\}) \times \sigma(\{j\}) = (b \times e) \circ \Pi$ . But

$b \times e$  is of finite type by Assumption 8.1, and  $\Pi$  is a 1-isomorphism. So  $\sigma(\{i\}) \times \sigma(\{j\})$  is of finite type.  $\square$

Using this and inductive methods as in Theorem 8.2, we show

**Theorem 8.4.** *If Assumptions 7.1 and 8.1 hold then*

(a) *In Definition 7.7, the following are representable and of finite type*

$$Q(I, \preceq, \trianglelefteq) : \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{M}(I, \trianglelefteq, \kappa)_{\mathcal{A}}, \quad (43)$$

$$Q(I, \preceq, \trianglelefteq) : \mathfrak{M}(I, \preceq)_{\mathcal{A}} \rightarrow \mathfrak{M}(I, \trianglelefteq)_{\mathcal{A}}, \quad (44)$$

$$Q(I, \preceq, K, \trianglelefteq, \phi) : \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{M}(K, \trianglelefteq, \mu)_{\mathcal{A}} \quad (45)$$

and also  $Q(I, \preceq, K, \trianglelefteq, \phi) : \mathfrak{M}(I, \preceq)_{\mathcal{A}} \rightarrow \mathfrak{M}(K, \trianglelefteq)_{\mathcal{A}}$  is representable.

(b)  $\sigma(I) : \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(I)}$  is representable and of finite type.

(c)  $\sigma(I) : \mathfrak{M}(I, \preceq)_{\mathcal{A}} \rightarrow \mathfrak{Sbj}_{\mathcal{A}}$  is representable.

(d)  $\prod_{i \in I} \sigma(\{i\}) : \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} \rightarrow \prod_{i \in I} \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(i)}$  and  $\prod_{i \in I} \sigma(\{i\}) : \mathfrak{M}(I, \preceq)_{\mathcal{A}} \rightarrow \prod_{i \in I} \mathfrak{Sbj}_{\mathcal{A}}$  are of finite type.

**Proof.** For (43), first suppose  $\trianglelefteq$  dominates  $\preceq$  by one step. Then as in the proof of Theorem 8.2,  $Q(I, \preceq, \trianglelefteq)$  fits into a Cartesian square (39). The right-hand morphism  $Q(\{i, j\}, \preceq, \trianglelefteq)$  in (39) is representable and finite type by Proposition 8.3(a). Hence, the left-hand morphism (43) is representable and finite type.

When  $\trianglelefteq$  dominates  $\preceq$  by  $s$  steps, by Proposition 6.5 we may write  $Q(I, \preceq, \trianglelefteq)$  as the composition of  $s$  1-morphisms  $Q(I, \preceq_r, \preceq_{r-1})$  with  $\preceq_{r-1}$  dominating  $\preceq_r$  by one step. By Laumon and Moret-Bailly [14, Lemma 3.12(b)] compositions of representable or finite type 1-morphisms are too, so (43) is representable and finite type by the first part.

From (32) we have  $\mathfrak{M}(I, \trianglelefteq)_{\mathcal{A}} = \coprod_{\kappa} \mathfrak{M}(I, \trianglelefteq, \kappa)_{\mathcal{A}}$ , and (44) coincides with (43) over the open substack  $\mathfrak{M}(I, \trianglelefteq, \kappa)_{\mathcal{A}}$  of  $\mathfrak{M}(I, \trianglelefteq)_{\mathcal{A}}$ . For a 1-morphism  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  to be representable or finite type is a local condition on  $\mathfrak{G}$ . Thus, (44) is representable and finite type as (43) is.

Next we prove (b). Suppose by induction that (b) holds whenever  $(I, \preceq)$  is a *total order* with  $|I| \leq n$ . Follow the middle part of the proof of Theorem 8.2. Now  $Q(\{i, j\}, \preceq, \{l\}, \trianglelefteq, \psi)$  in (38) is identified with  $\sigma(\{i, j\})$  in Proposition 8.3(b) by the 1-isomorphism  $\mathfrak{M}(\{l\}, \trianglelefteq, \mu|_{\{l\}})_{\mathcal{A}} \rightarrow \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(\{l, j\})}$  of Proposition 7.9(i). Thus  $Q(\{i, j\}, \preceq, \{l\}, \trianglelefteq, \psi)$  in (38) is representable and finite type as  $\sigma(\{i, j\})$  is, so  $Q(I, \preceq, K, \trianglelefteq, \phi)$  in (38) is representable and finite type.

But  $\sigma(I) = \sigma(K) \circ Q(I, \preceq, K, \trianglelefteq, \phi) : \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(I)}$ , where  $\sigma(K)$  is representable and finite type by induction, and  $Q(I, \preceq, K, \trianglelefteq, \phi)$  is representable and finite type from above. Thus  $\sigma(I)$  is representable and finite type by Laumon and Moret-Bailly [14, Lemma 3.12(b)]. By induction, this proves (b) whenever  $\preceq$  is a total order.

For the general case, let  $(I, \preceq)$  be a finite poset,  $\trianglelefteq$  a total order on  $I$  dominating  $\preceq$ , and  $\kappa : I \rightarrow \bar{C}(\mathcal{A})$ . Then  $\sigma(I) = \sigma(I) \circ Q(I, \preceq, \trianglelefteq) : \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(I)}$ , where the second  $\sigma(I)$  acts on  $\mathfrak{M}(I, \trianglelefteq, \kappa)_{\mathcal{A}}$ , and is representable and finite type as  $\trianglelefteq$  is a total order. But  $Q(I, \preceq, \trianglelefteq)$  is representable and finite type by (43) in (a). Therefore the composition  $\sigma(I)$  is, giving (b).

For  $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$  to be representable is a local condition on both  $\mathfrak{F}$  and  $\mathfrak{G}$ . By (32) we see that  $\sigma(I) : \mathfrak{M}(I, \preceq)_{\mathcal{A}} \rightarrow \mathfrak{Sbj}_{\mathcal{A}}$  locally coincides with  $\sigma(I) : \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(I)}$ , which is representable by (b). This proves (c).

We can now complete (a). If  $\mathfrak{F} \xrightarrow{\phi} \mathfrak{G} \xrightarrow{\psi} \mathfrak{H}$  are 1-morphisms of algebraic  $\mathbb{K}$ -stacks and  $\psi \circ \phi$  is representable and finite type, then  $\phi$  is too by Laumon and Moret-Bailly [14, Lemma 3.12(c)(ii) and Remark 4.17(1)]. For (45) we have

$$\sigma(I) = \sigma(K) \circ Q(I, \preceq, K, \trianglelefteq, \phi) : \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(I)}.$$

As  $\sigma(I)$  is representable and finite type by (b), we see (45) is. Similarly,  $Q(I, \preceq, K, \trianglelefteq, \phi) : \mathfrak{M}(I, \preceq)_{\mathcal{A}} \rightarrow \mathfrak{M}(K, \trianglelefteq)_{\mathcal{A}}$  is representable using (c).

Finally we prove (d). When  $|I| = 0, 1$  the  $\prod_{i \in I} \sigma(\{i\})$  are 1-isomorphisms, so of finite type. Suppose by induction that the first line of (d) is finite type whenever  $(I, \preceq)$  is a total order with  $|I| \leq n$ , for  $n \geq 1$ . Let  $(I, \preceq)$  be a total order with  $|I| = n + 1$ , and define  $J, K, L, \trianglelefteq, \mu$  as in the proof of Theorem 8.2. Since (38) is Cartesian and  $\sigma(\{a\}) = \sigma(\{a\}) \circ Q(I, \preceq, K, \trianglelefteq, \phi) : \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(a)}$  for all  $a \in I \setminus \{i, j\}$ , we have a Cartesian square

$$\begin{array}{ccc} \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} & \xrightarrow{\quad} & \mathfrak{M}(\{i, j\}, \preceq, \kappa|_{\{i, j\}})_{\mathcal{A}} \times \prod_{a \in I \setminus \{i, j\}} \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(a)} \\ \downarrow Q(I, \preceq, K, \trianglelefteq, \phi) & \begin{array}{c} S(I, \preceq, \{i, j\}) \times \prod_{a \in I \setminus \{i, j\}} \sigma(\{a\}) \\ S(K, \trianglelefteq, \{l\}) \times \prod_{a \in I \setminus \{i, j\}} \sigma(\{a\}) \end{array} & \begin{array}{c} Q(\{i, j\}, \preceq, \{l\}, \trianglelefteq, \psi) \times \\ \prod_{a \in I \setminus \{i, j\}} \text{id}_{\mathfrak{Sbj}_{\mathcal{A}}^{\kappa(a)}} \end{array} \downarrow \\ \mathfrak{M}(K, \trianglelefteq, \mu)_{\mathcal{A}} & \xrightarrow{\quad} & \mathfrak{M}(\{l\}, \trianglelefteq, \mu|_{\{l\}})_{\mathcal{A}} \times \prod_{a \in I \setminus \{i, j\}} \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(a)}. \end{array} \quad (46)$$

Now  $\sigma(\{l\}) : \mathfrak{M}(\{l\}, \trianglelefteq, \mu|_{\{l\}})_{\mathcal{A}} \rightarrow \mathfrak{Sbj}_{\mathcal{A}}^{\mu(l)}$  is a 1-isomorphism by Proposition 7.9(i). Composing with this identifies the bottom morphism of (46) with  $\prod_{a \in K} \sigma(\{a\})$ , which is finite type by induction. Hence the bottom morphism in (46) is finite type, so the top morphism in (46) is too. But  $\sigma(\{i\}) \times \sigma(\{j\})$  is finite type by Proposition 8.3(c). Composing with the top morphism of (46) gives  $\prod_{a \in I} \sigma(\{a\})$ , which is therefore finite type. So by induction, the first line of (d) is finite type for  $(I, \preceq)$  a total order.

Suppose  $(I, \preceq)$  is a finite poset,  $\trianglelefteq$  a total order on  $I$  dominating  $\preceq$ , and  $\kappa : I \rightarrow \bar{C}(\mathcal{A})$ . Then  $\prod_{i \in I} \sigma(\{i\}) = \prod_{i \in I} \sigma(\{i\}) \circ Q(I, \preceq, \trianglelefteq) : \mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} \rightarrow \prod_{i \in I} \mathfrak{Sbj}_{\mathcal{A}}^{\kappa(i)}$ , where the second  $\prod_{i \in I} \sigma(\{i\})$  acts on  $\mathfrak{M}(I, \trianglelefteq, \kappa)_{\mathcal{A}}$ , and is finite type as  $\trianglelefteq$  is a total order. As  $Q(I, \preceq, \trianglelefteq)$  is finite type by (a), composition gives the first line of (d), and the second line follows as for (44).  $\square$

### 8.3. The moduli spaces $\mathfrak{M}(X, I, \preccurlyeq)_{\mathcal{A}}$ and $\mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$

We can now form two further classes of moduli spaces of  $(I, \preccurlyeq)$ -configurations  $(\sigma, \iota, \pi)$  with  $\sigma(I) = X$ , for  $X$  a fixed object in  $\mathcal{A}$ .

**Definition 8.5.** In the situation above, let  $X \in \text{Obj } \mathcal{A}$ . Assumption 7.1(i) identifies  $X$  with a 1-morphism  $X : \text{Spec } \mathbb{K} \rightarrow \mathfrak{Ob}_{\mathcal{A}}^{[X]}$  or  $\mathfrak{Ob}_{\mathcal{A}}$ . For  $(I, \preccurlyeq)$  a finite poset and  $\kappa : I \rightarrow \bar{C}(\mathcal{A})$  with  $\kappa(I) = [X]$  in  $\bar{C}(\mathcal{A})$ , define

$$\begin{aligned}\mathfrak{M}(X, I, \preccurlyeq)_{\mathcal{A}} &= \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}} \times_{\sigma(I), \mathfrak{Ob}_{\mathcal{A}}, X} \text{Spec } \mathbb{K} \text{ and} \\ \mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}} &= \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} \times_{\sigma(I), \mathfrak{Ob}_{\mathcal{A}}^{\kappa(I)}, X} \text{Spec } \mathbb{K}.\end{aligned}\quad (47)$$

As  $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ ,  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  are algebraic  $\mathbb{K}$ -stacks locally of finite type by Theorems 8.2 and 8.4(b) implies that  $\mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is *represented by an algebraic  $\mathbb{K}$ -space of finite type*, and Theorem 8.4(c) that  $\mathfrak{M}(X, I, \preccurlyeq)_{\mathcal{A}}$  is *represented by an algebraic  $\mathbb{K}$ -space locally of finite type*. Write  $\Pi_X : \mathfrak{M}(X, I, \preccurlyeq)_{\mathcal{A}} \rightarrow \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$  and  $\Pi_X : \mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  for the 1-morphisms of stacks from the fibre products. Write  $\mathcal{M}(X, I, \preccurlyeq)_{\mathcal{A}} = \mathfrak{M}(X, I, \preccurlyeq)_{\mathcal{A}}(\mathbb{K})$  and  $\mathcal{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}} = \mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}(\mathbb{K})$  for their sets of geometric points.

For the examples of §§9 and 10 the  $\mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$  are actually *represented by quasiprojective  $\mathbb{K}$ -schemes*. The reason for this is that (37) are *quasiprojective* 1-morphisms. Replacing finite type with quasiprojective 1-morphisms in the proofs of §8.2 shows that  $\sigma(I) : \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{Ob}_{\mathcal{A}}^{\kappa(I)}$  is *representable and quasiprojective*, implying that  $\mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is represented by a quasiprojective  $\mathbb{K}$ -scheme in Definition 8.5. Here is the analogue of Proposition 7.6.

**Theorem 8.6.** In Definition 8.5,  $\mathcal{M}(X, I, \preccurlyeq)_{\mathcal{A}}$  and  $\mathcal{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$  are naturally identified with the sets of isomorphism classes of  $(I, \preccurlyeq)$ - and  $(I, \preccurlyeq, \kappa)$ -configurations  $(\sigma, \iota, \pi)$  in  $\mathcal{A}$  with  $\sigma(I) = X$ , modulo isomorphisms  $\alpha : (\sigma, \iota, \pi) \rightarrow (\sigma', \iota', \pi')$  of  $(I, \preccurlyeq)$ -configurations with  $\alpha(I) = \text{id}_X$ .

**Proof.** By Definition 2.10, we find the groupoid  $\mathfrak{M}(X, I, \preccurlyeq)_{\mathcal{A}}(\text{Spec } \mathbb{K})$  has objects  $((\tilde{\sigma}, \tilde{\iota}, \tilde{\pi}), \text{id}_{\text{Spec } \mathbb{K}}, \phi)$ , for  $(\tilde{\sigma}, \tilde{\iota}, \tilde{\pi})$  an  $(I, \preccurlyeq)$ -configuration in  $\mathcal{A}$ , and  $\phi : \tilde{\sigma}(I) \rightarrow X$  an isomorphism in  $\mathcal{A}$ . Given such a  $((\tilde{\sigma}, \tilde{\iota}, \tilde{\pi}), \text{id}_{\text{Spec } \mathbb{K}}, \phi)$ , define an  $(I, \preccurlyeq)$ -configuration  $(\sigma, \iota, \pi)$  by  $\sigma(I) = X$ ,  $\sigma(J) = \tilde{\sigma}(J)$  for  $J \neq I$ , and

$$\iota(J, K) = \begin{cases} \tilde{\iota}(J, K), & J \neq I \neq K, \\ \phi \circ \tilde{\iota}(J, K), & J \neq I = K, \\ \text{id}_X, & J = I = K, \end{cases} \quad \pi(J, K) = \begin{cases} \tilde{\pi}(J, K), & J \neq I \neq K, \\ \tilde{\pi}(J, K) \circ \phi^{-1}, & J = I \neq K, \\ \text{id}_X, & J = I = K. \end{cases}$$

Define  $\beta : (\tilde{\sigma}, \tilde{\iota}, \tilde{\pi}) \rightarrow (\sigma, \iota, \pi)$  by  $\beta(J) = \text{id}_{\tilde{\sigma}(J)}$  if  $I \neq J \in \mathcal{F}_{(I, \preccurlyeq)}$ , and  $\beta(I) = \phi$ . Then  $\beta$  is an isomorphism of  $(I, \preccurlyeq)$ -configurations. Moreover,

$$(\beta, \text{id}_{\text{id}_{\text{Spec } \mathbb{K}}}) : ((\tilde{\sigma}, \tilde{\iota}, \tilde{\pi}), \text{id}_{\text{Spec } \mathbb{K}}, \phi) \longrightarrow ((\sigma, \iota, \pi), \text{id}_{\text{Spec } \mathbb{K}}, \text{id}_X)$$

is an isomorphism in  $\mathfrak{M}(X, I, \preccurlyeq)_{\mathcal{A}}(\text{Spec } \mathbb{K})$ . So each object of  $\mathfrak{M}(X, I, \preccurlyeq)_{\mathcal{A}}(\text{Spec } \mathbb{K})$  is isomorphic to some  $((\sigma, \iota, \pi), \text{id}_{\text{Spec } \mathbb{K}}, \text{id}_X)$  for an  $(I, \preccurlyeq)$ -configuration  $(\sigma, \iota, \pi)$  with  $\sigma(I) = X$ . Isomorphisms

$$(\alpha, \text{id}_{\text{id}_{\text{Spec } \mathbb{K}}}) : ((\sigma, \iota, \pi), \text{id}_{\text{Spec } \mathbb{K}}, \text{id}_X) \longrightarrow ((\sigma', \iota', \pi'), \text{id}_{\text{Spec } \mathbb{K}}, \text{id}_X)$$

between two elements of this form come from isomorphisms  $\alpha : (\sigma, \iota, \pi) \rightarrow (\sigma', \iota', \pi')$  of  $(I, \preccurlyeq)$ -configurations with  $\alpha(I) = \text{id}_X$ . Hence,  $\mathcal{M}(X, I, \preccurlyeq)_{\mathcal{A}}$  is naturally identified with the set of isomorphism classes of  $(I, \preccurlyeq)$ -configurations  $(\sigma, \iota, \pi)$  in  $\mathcal{A}$  with  $\sigma(I) = X$ , modulo isomorphisms  $\alpha : (\sigma, \iota, \pi) \rightarrow (\sigma', \iota', \pi')$  with  $\alpha(I) = \text{id}_X$ . The proof for  $\mathcal{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is the same.  $\square$

## 9. Coherent sheaves on a projective scheme

Let  $\mathbb{K}$  be an algebraically closed field,  $P$  a projective  $\mathbb{K}$ -scheme, and  $\text{coh}(P)$  the abelian category of coherent sheaves on  $P$ . We shall apply the machinery of §§7 and 8 to  $\mathcal{A} = \text{coh}(P)$ . Section 9.1 defines the data  $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$  required by Assumption 7.1. Then §9.2 proves that Assumption 7.1 holds, and §§9.3 and 9.4 that Assumption 8.1 holds, for this data.

Thus by §§7 and 8, we have well-defined moduli stacks of configurations of coherent sheaves  $\mathfrak{M}(I, \preccurlyeq)_{\text{coh}(P)}, \mathfrak{M}(I, \preccurlyeq, \kappa)_{\text{coh}(P)}$ , which are algebraic  $\mathbb{K}$ -stacks, locally of finite type, and many 1-morphisms between them, some of which are representable or of finite type. For background on coherent and quasicoherent sheaves see Hartshorne [8, §II.5] or Grothendieck [6, §I.0.5].

### 9.1. Definition of the data $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$

Our first two examples define the data  $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$  of Assumption 7.1 for the abelian category of coherent sheaves on a smooth projective  $\mathbb{K}$ -scheme  $P$ . The assumption that  $P$  is smooth will be relaxed in Example 9.2.

**Example 9.1.** Let  $\mathbb{K}$  be an algebraically closed field and  $P$  a smooth projective  $\mathbb{K}$ -scheme, and take  $\mathcal{A}$  to be the abelian category  $\text{coh}(P)$  of coherent sheaves on  $P$ . Then one may define the Chern character  $\text{ch} : K_0(\mathcal{A}) \rightarrow H^{\text{even}}(P, \mathbb{Z})$ , a homomorphism of abelian groups. Let  $K(\mathcal{A})$  be the quotient of  $K_0(\mathcal{A})$  by  $\text{Ker}(\text{ch})$ . Then  $\text{ch}$  identifies  $K(\mathcal{A})$  with a subgroup of  $H^{\text{even}}(P, \mathbb{Z})$ .

Motivated by Laumon and Moret-Bailly [14, §2.4.4], for  $U \in \text{Sch}_{\mathbb{K}}$  define  $\mathfrak{F}_{\mathcal{A}}(U) = \mathfrak{F}_{\text{coh}(P)}(U)$  to be the exact category of finitely presentable quasicoherent sheaves on



$P \times U$ , as in [6, §I.0.5], which are *flat over*  $U$ , as in [6, I.0.6.7]. This is a full additive subcategory of the abelian category  $\mathrm{qcoh}(P \times U)$  of quasicoherent sheaves on  $P \times U$ , closed under extensions by Grothendieck [6, Proposition IV.2.1.8], so it is an exact category.

If  $f : U \rightarrow V$  is a morphism in  $\mathrm{Sch}_{\mathbb{K}}$  then so is  $\mathrm{id}_P \times f : P \times U \rightarrow P \times V$ . Define a functor  $\mathfrak{F}_{\mathrm{coh}(P)}(f) : \mathfrak{F}_{\mathrm{coh}(P)}(V) \rightarrow \mathfrak{F}_{\mathrm{coh}(P)}(U)$  by pullback  $(\mathrm{id}_P \times f)^*$  of sheaves and their morphisms along  $\mathrm{id}_P \times f$ . That is, if  $X \in \mathfrak{F}_{\mathrm{coh}(P)}(V)$  then  $\mathfrak{F}_{\mathrm{coh}(P)}(f)X$  is the *inverse image sheaf*  $(\mathrm{id}_P \times f)^*(X)$  on  $P \times U$ , as in [8, p. 110].

Then  $\mathfrak{F}_{\mathrm{coh}(P)}(f)X = (\mathrm{id}_P \times f)^*(X)$  is quasicoherent by Grothendieck [6, I.0.5.1.4] or [8, Proposition II.5.8(a)], finitely presentable by Grothendieck [6, I.0.5.2.5], and flat over  $U$  by Grothendieck [6, Proposition IV.2.1.4]. Thus  $\mathfrak{F}_{\mathrm{coh}(P)}(f)X \in \mathfrak{F}_{\mathrm{coh}(P)}(U)$ , as we need. Also,  $(\mathrm{id}_P \times f)^*$  takes exact sequences of quasicoherent sheaves on  $P \times V$  flat over  $V$  to exact sequences of quasicoherent sheaves on  $P \times U$  flat over  $U$  by Grothendieck [6, Proposition IV.2.1.8(i)]. Hence  $\mathfrak{F}_{\mathrm{coh}(P)}(f)$  is an exact functor.

Here is a slightly subtle point. Inverse images come from a universal construction, and so are given not uniquely, but only *up to canonical isomorphism*. So there could be *many possibilities* for  $(\mathrm{id}_P \times f)^*(X)$ . To define  $\mathfrak{F}_{\mathrm{coh}(P)}(f)X$  we *choose* an inverse image  $(\mathrm{id}_P \times f)^*(X)$  in an arbitrary way for each  $X \in \mathfrak{F}_{\mathrm{coh}(P)}(U)$ , using the axiom of choice. Let  $f : U \rightarrow V$ ,  $g : V \rightarrow W$  be scheme morphisms, and  $X \in \mathfrak{F}_{\mathrm{coh}(P)}(W)$ . Then  $(\mathrm{id}_P \times f)^*((\mathrm{id}_P \times g)^*(X))$  and  $(\mathrm{id}_P \times (g \circ f))^*(X)$  are both inverse images of  $X$  on  $P \times U$ , which are canonically isomorphic, but *may not* be the same.

That is,  $\mathfrak{F}_{\mathrm{coh}(P)}(f) \circ \mathfrak{F}_{\mathrm{coh}(P)}(g)(X)$  and  $\mathfrak{F}_{\mathrm{coh}(P)}(g \circ f)(X)$  may be different, but there is a canonical isomorphism  $\varepsilon_{g,f}(X) : \mathfrak{F}_{\mathrm{coh}(P)}(f) \circ \mathfrak{F}_{\mathrm{coh}(P)}(g)(X) \rightarrow \mathfrak{F}_{\mathrm{coh}(P)}(g \circ f)(X)$  for all  $X \in \mathfrak{F}_{\mathrm{coh}(P)}(W)$ . These make up an isomorphism of functors  $\varepsilon_{g,f} : \mathfrak{F}_{\mathrm{coh}(P)}(f) \circ \mathfrak{F}_{\mathrm{coh}(P)}(g) \rightarrow \mathfrak{F}_{\mathrm{coh}(P)}(g \circ f)$ , that is, a 2-morphism in  $(\mathrm{exactcat})$ . From §2.3, these 2-morphisms  $\varepsilon_{g,f}$  are the last piece of data we need to define the 2-functor  $\mathfrak{F}_{\mathrm{coh}(P)}$ . It is straightforward to show the definition of a *contravariant 2-functor* [4, Appendix B] is satisfied.

Here are some remarks on this definition:

- The most obvious way to define  $\mathfrak{F}_{\mathrm{coh}(P)}(U)$  would be to use *coherent sheaves* on  $P \times U$  flat over  $U$ . However, this turns out to be a bad idea, as coherence is not well-behaved over non-noetherian schemes. In particular, *inverse images* of coherent sheaves may not be coherent, so the functors  $\mathfrak{F}_{\mathrm{coh}(P)}(f)$  would not be well-defined. Instead, we use *finitely presentable quasicoherent sheaves*, which are the same as coherent sheaves on noetherian  $\mathbb{K}$ -schemes, and behave well under inverse images, etc.
- If  $f : U \rightarrow V$  is not flat then  $(\mathrm{id}_P \times f)^* : \mathrm{qcoh}(P \times V) \rightarrow \mathrm{qcoh}(P \times U)$  is *not* an exact functor. We only claim above that exact sequences on  $P \times V$  *flat over*  $V$ , lift to exact sequences on  $P \times U$ .

We supposed  $P$  *smooth* in Example 9.1 to make the *Chern character* well-defined for coherent sheaves on  $P$ . For  $P$  not smooth there may be problems with this, so we need a different choice for  $K(\mathrm{coh}(P))$ . We cannot take  $K(\mathrm{coh}(P)) = K_0(\mathrm{coh}(P))$ , as Assumption 7.1(iii) would not hold. Instead, in the next example we define  $K(\mathrm{coh}(P))$  using *Hilbert polynomials*.



**Example 9.2.** Let  $\mathbb{K}$  be an algebraically closed field,  $P$  a projective  $\mathbb{K}$ -scheme, not necessarily smooth, and  $\mathcal{O}_P(1)$  a *very ample invertible sheaf* on  $P$ , so that  $(P, \mathcal{O}_P(1))$  is a *polarized  $\mathbb{K}$ -scheme*. Following [9, §1.2] and [8, Examples III.5.1 and III.5.2], define the *Hilbert polynomial*  $p_X$  for  $X \in \text{coh}(P)$  by

$$p_X(n) = \sum_{i=0}^{\dim P} (-1)^i \dim_{\mathbb{K}} H^i(P, X(n)) \quad \text{for } n \in \mathbb{Z},$$

where  $X(n) = X \otimes \mathcal{O}_P(1)^n$ , and  $H^*(P, \cdot)$  is sheaf cohomology on  $P$ . Then

$$p_X(n) = \sum_{i=0}^{\dim P} a_i n^i / i! \quad \text{for } a_0, \dots, a_{\dim P} \in \mathbb{Z}, \quad (48)$$

by Huybrechts and Lehn [9, p. 10]. So  $p_X(t)$  is a polynomial with rational coefficients, written  $p_X(t) \in \mathbb{Q}[t]$ , with degree no more than  $\dim P$ .

If  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  is exact then the long exact sequence in sheaf cohomology implies that  $p_Y = p_X + p_Z$ . Therefore the map  $X \mapsto p_X$  factors through the Grothendieck group  $K_0(\text{coh}(P))$ , and there is a unique group homomorphism  $p : K_0(\text{coh}(P)) \rightarrow \mathbb{Q}[t]$  with  $p([X]) = p_X$  for all  $X \in \text{coh}(P)$ . Set  $\mathcal{A} = \text{coh}(P)$ , and let  $K(\mathcal{A})$  be the quotient of  $K_0(\text{coh}(P))$  by the kernel of  $p$ . Then  $K(\mathcal{A})$  is isomorphic to the image of  $p$  in  $\mathbb{Q}[t]$ , which by (48) lies in a sublattice of  $\mathbb{Q}[t]$  isomorphic to  $\mathbb{Z}^{1+\dim P}$ . Now define  $\mathfrak{F}_{\text{coh}(P)}$  as in Example 9.1. Since this does not use  $K(\mathcal{A})$  or the fact that  $P$  is smooth, no changes are needed, and  $\mathfrak{F}_{\mathcal{A}} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{exactcat})$  is a *contravariant 2-functor*.

## 9.2. Verifying Assumption 7.1

We now show that the examples of §9.1 satisfy Assumption 7.1. The main point to verify is that  $\mathfrak{F}_{\text{coh}(P)}$  is a *stack in exact categories*. The proof uses results of Grothendieck [7], as in Laumon and Moret-Bailly [14, §3.4.4].

**Theorem 9.3.** *Examples 9.1 and 9.2 satisfy Assumption 7.1.*

**Proof.** Let  $\mathbb{K}$  be an algebraically closed field, and  $P$  a projective  $\mathbb{K}$ -scheme, not necessarily smooth. We first verify the condition that if  $X \in \text{Obj}(\mathcal{A})$  and  $[X] = 0$  in  $K(\mathcal{A})$  then  $X \cong 0$ . Let  $K(\text{coh}(P))$  be as in either Example 9.1 or Example 9.2, and suppose  $X \in \text{coh}(P)$  with  $[X] = 0$  in  $K(\mathcal{A})$ . In both cases, this implies the Hilbert polynomial  $p_X$  of  $X$  is zero. Now Serre's vanishing theorem shows that for  $n \gg 0$ , the tautological map  $H^0(X(n)) \otimes \mathcal{O}_P(-n) \rightarrow X$  is surjective, and  $\dim H^0(X(n)) = p_X(n) = 0$ , so  $X \cong 0$ .

We have shown  $\mathfrak{F}_{\text{coh}(P)}$  is a *contravariant 2-functor*  $\text{Sch}_{\mathbb{K}} \rightarrow (\text{exactcat})$ . We shall prove it is a *stack in exact categories*, that is, that Definition 2.8(i)–(iii) hold. Grothendieck [7, Corollary VIII.1.2] proves Definition 2.8(i), (ii) hold, using only the assumption that the sheaves involved are *quasicohherent*.

Let  $\{f_i : U_i \rightarrow V\}_{i \in I}$  be an open cover of  $V$  in the site  $\text{Sch}_{\mathbb{K}}$ , and let  $X_i \in \text{Obj}(\mathfrak{F}_{\text{coh}(P)}(U_i))$  and  $\phi_{ij} : \mathfrak{F}_{\text{coh}(P)}(f_{ij,j})X_j \rightarrow \mathfrak{F}_{\text{coh}(P)}(f_{ij,i})X_i$  be as in Definition

2.8(iii). Grothendieck [7, Corollary VIII.1.3] constructs  $X \in \mathrm{qcoh}(P \times V)$  and isomorphisms  $\phi_i : (\mathrm{id}_P \times f_i)^*(X) \rightarrow X_i$  in  $\mathrm{Mor}(\mathrm{qcoh}(P \times U_i))$  satisfying Definition 2.8(iii). This  $X$  is finitely presentable by Gorthendieck [7, Proposition VIII.1.10] and flat over  $V$  by Gorthendieck [6, Corollary IV.2.2.11(iii)], so  $X \in \mathfrak{F}_{\mathrm{coh}(P)}(V)$ , Definition 2.8(iii) holds for  $\mathfrak{F}_{\mathrm{coh}(P)}$ , and  $\mathfrak{F}_{\mathrm{coh}(P)}$  is a stack in exact categories.

It remains to verify Assumption 7.1(i)–(iii). As  $P \cong P \times \mathrm{Spec} \mathbb{K}$  is *noetherian*, Hartshorne [8, Proposition II.5.7] implies that  $X \in \mathrm{qcoh}(P \times \mathrm{Spec} \mathbb{K})$  is coherent if and only if it is finitely presentable, and flatness over  $\mathrm{Spec} \mathbb{K}$  is trivial. Hence  $\mathfrak{F}_{\mathrm{coh}(P)}(\mathrm{Spec} \mathbb{K}) = \mathrm{coh}(P)$ , identifying  $P$  and  $P \times \mathrm{Spec} \mathbb{K}$ , and Assumption 7.1(i) holds. Part (ii) follows from [6, Proposition IV.2.2.7].

Let  $U \in \mathrm{Sch}_{\mathbb{K}}$  and  $X \in \mathrm{Obj}(\mathfrak{F}_{\mathrm{coh}(P)}(U))$ , and write  $X_u = \mathfrak{F}_{\mathrm{coh}(P)}(u)X$  in  $\mathrm{coh}(P)$  for  $u \in \mathrm{Hom}(\mathrm{Spec} \mathbb{K}, U)$ . Then we can regard  $X$  as a flat family of  $X_u \in \mathrm{coh}(P)$ , depending on  $u \in \mathrm{Hom}(\mathrm{Spec} \mathbb{K}, U)$ . But Chern classes in Example 9.1, and by Gorthendieck [6, Proposition III.7.9.11] Hilbert polynomials in Example 9.2, are both locally constant in flat families. So Assumption 7.1(iii) holds.

Finally, let  $X, Y \in \mathrm{coh}(P)$ . Choose a basis  $e_1, \dots, e_n$  for  $\mathrm{Hom}(X, Y)$ , and let  $z_1, \dots, z_n : \mathrm{Hom}(X, Y) \rightarrow \mathbb{K}$  be the corresponding coordinates. Regarding  $\mathrm{Hom}(X, Y)$  as an *affine*  $\mathbb{K}$ -scheme,  $z_1, \dots, z_n$  become sections of  $\mathcal{O}_{\mathrm{Hom}(X, Y)}$ . Write  $\pi : \mathrm{Hom}(X, Y) \rightarrow \mathrm{Spec} \mathbb{K}$ ,  $\pi_1 : P \times \mathrm{Hom}(X, Y) \rightarrow P$  and  $\pi_2 : P \times \mathrm{Hom}(X, Y) \rightarrow \mathrm{Hom}(X, Y)$  for the natural projections. Define

$$\theta_{X,Y} = \sum_{i=1}^n \pi_2^*(z_i) \cdot \pi_1^*(e_i) : \pi_1^*(X) \longrightarrow \pi_1^*(Y).$$

Here  $\pi_1^*(X) = \mathfrak{F}_{\mathrm{coh}(P)}(\pi)X$  and  $\pi_1^*(Y) = \mathfrak{F}_{\mathrm{coh}(P)}(\pi)Y$  are coherent sheaves on  $P \times \mathrm{Hom}(X, Y)$ ,  $\pi_1^*(e_i) : \pi_1^*(X) \rightarrow \pi_1^*(Y)$  is a morphism of sheaves on  $P \times \mathrm{Hom}(X, Y)$ ,  $\pi_2^*(z_i)$  is a section of  $\mathcal{O}_{P \times \mathrm{Hom}(X, Y)}$ , and ‘ $\cdot$ ’ multiplies sections of  $\mathcal{O}_{P \times \mathrm{Hom}(X, Y)}$  and sheaf morphisms on  $P \times \mathrm{Hom}(X, Y)$ . Thus  $\theta_{X,Y} : \mathfrak{F}_{\mathrm{coh}(P)}(\pi)X \rightarrow \mathfrak{F}_{\mathrm{coh}(P)}(\pi)Y$  is a morphism in  $\mathfrak{F}_{\mathrm{coh}(P)}(\mathrm{Hom}(X, Y))$ , and clearly (31) holds for all  $f \in \mathrm{Hom}(X, Y)$ . This verifies Assumption 7.1(iv).  $\square$

### 9.3. Showing stacks are algebraic and locally of finite type

Here and in §9.4 we show that Examples 9.1, 9.2 satisfy Assumption 8.1.

**Theorem 9.4.** *In Examples 9.1 and 9.2,  $\mathfrak{Obj}_{\mathrm{coh}(P)}^\alpha$ ,  $\mathfrak{Exact}_{\mathrm{coh}(P)}^{\alpha,\beta,\gamma}$  are algebraic  $\mathbb{K}$ -stacks, locally of finite type, for all  $\alpha, \beta, \gamma$ .*

**Proof.** Laumon and Moret-Bailly [14, Theorem 4.6.2.1] prove  $\mathfrak{Obj}_{\mathrm{coh}(P)}$  is an algebraic  $\mathbb{K}$ -stack, locally of finite type. The corresponding result for  $\mathfrak{Obj}_{\mathrm{coh}(P)}^\alpha$  then follows from Theorem 7.5. An important part of the proof is to construct an *atlas* for  $\mathfrak{Obj}_{\mathrm{coh}(P)}$ , and we sketch how this is done using *Quot-schemes*.

For  $N \geq 0$ , write  $\mathrm{Quot}_P(\oplus^N \mathcal{O}_P)$  for Grothendieck’s *Quot-scheme* [5, §3.2], [9, §2.2]. This is the *moduli scheme of quotient sheaves*  $(Z, \psi)$  of the coherent sheaf  $\oplus^N \mathcal{O}_P$  on  $P$ , where  $\psi : \oplus^N \mathcal{O}_P \rightarrow Z$  is surjective. By Assumption 7.1(iii) it may be written as a

disjoint union

$$\mathrm{Quot}_P(\oplus^N \mathcal{O}_P) = \coprod_{\alpha \in \tilde{C}(\mathrm{coh}(P))} \mathrm{Quot}_P(\oplus^N \mathcal{O}_P, \alpha),$$

where  $\mathrm{Quot}_P(\oplus^N \mathcal{O}_P, \alpha)$  is the subscheme of  $(Z, \psi)$  with  $[Z] = \alpha$  in  $K(\mathrm{coh}(P))$ .

Considered as a  $\mathbb{K}$ -stack, for  $U \in \mathrm{Sch}_{\mathbb{K}}$ ,  $\mathrm{Quot}_P(\oplus^N \mathcal{O}_P, \alpha)(U)$  has *objects* pairs  $(Z, \psi)$  for  $Z \in \mathfrak{Sbj}_{\mathrm{coh}(P)}^{\alpha}(U)$  and  $\psi : \oplus^N \mathcal{O}_{P \times U} \rightarrow Z$  a surjective morphism in  $\mathfrak{F}_{\mathrm{coh}(P)}(U)$ , that is, the r.h.s. of some short exact sequence. *Morphisms*  $\beta : (Z, \psi) \rightarrow (Z', \psi')$  are isomorphisms  $\beta : Z \rightarrow Z'$  in  $\mathfrak{F}_{\mathrm{coh}(P)}(U)$  with  $\psi' = \beta \circ \psi$ .

Grothendieck [5, §3.2] shows  $\mathrm{Quot}_P(\oplus^N \mathcal{O}_P, \alpha)$  is represented by a projective  $\mathbb{K}$ -scheme. Let  $\mathrm{Quot}_P^{\circ}(\oplus^N \mathcal{O}_P, \alpha)$  be the open  $\mathbb{K}$ -substack of  $\mathrm{Quot}_P(\oplus^N \mathcal{O}_P, \alpha)$  with objects  $(Z, \psi)$  over  $U \in \mathrm{Sch}_{\mathbb{K}}$  such that  $(\pi_U)_*(\psi) : \oplus^N \mathcal{O}_U \rightarrow (\pi_U)_*(Z)$  is an isomorphism and  $R^p(\pi_U)_*(Z) = 0$  for all  $p > 0$ , in the notation of Hartshorne [8, §III.8], where  $\pi_U : P \times U \rightarrow U$  is the projection. Then  $\mathrm{Quot}_P^{\circ}(\oplus^N \mathcal{O}_P, \alpha)$  is represented by a quasiprojective  $\mathbb{K}$ -scheme, and so is of finite type.

Write  $\alpha \mapsto \alpha(n)$  for the unique automorphism of  $K(\mathrm{coh}(P))$  such that if  $X \in \mathrm{coh}(P)$  with  $[X] = \alpha$  in  $K(\mathrm{coh}(P))$ , then  $[X(n)] = \alpha(n)$ . For all  $N, n \geq 0$ , define a 1-morphism  $\Pi_{N,n} : \mathrm{Quot}_P^{\circ}(\oplus^N \mathcal{O}_P, \alpha(n)) \rightarrow \mathfrak{Sbj}_{\mathrm{coh}(P)}^{\alpha}$  by  $\Pi_{N,n} : (Z, \psi) \mapsto Z(-n)$  on objects and  $\Pi_{N,n} : \beta \mapsto \beta(-n)$  on morphisms, where  $Z(-n)$  and  $\beta(-n) : Z(-n) \rightarrow Z'(-n)$  are the twists of  $Z$  and  $\beta : Z \rightarrow Z'$  by the lift of the invertible sheaf  $\mathcal{O}_P(-n)$  to  $P \times U$ . Define a 1-morphism

$$\Pi^{\alpha} = \coprod_{N,n \geq 0} \Pi_{N,n} : \coprod_{N,n \geq 0} \mathrm{Quot}_P^{\circ}(\oplus^N \mathcal{O}_P, \alpha(n)) \longrightarrow \mathfrak{Sbj}_{\mathrm{coh}(P)}^{\alpha}.$$

As in [14, p. 30],  $\Pi^{\alpha}$  is smooth and surjective, so it is an atlas for  $\mathfrak{Sbj}_{\mathrm{coh}(P)}^{\alpha}$ . Since  $\coprod_{N,n \geq 0} \mathrm{Quot}_P^{\circ}(\oplus^N \mathcal{O}_P, \alpha(n))$  is locally of finite type, so is  $\mathfrak{Sbj}_{\mathrm{coh}(P)}^{\alpha}$ .

The proof for  $\mathfrak{Exact}_{\mathrm{coh}(P)}^{\alpha, \beta, \gamma}$  is mostly a straightforward generalization of that for  $\mathfrak{Sbj}_{\mathrm{coh}(P)}$ . The difficult part is to construct an atlas for  $\mathfrak{Exact}_{\mathrm{coh}(P)}^{\alpha, \beta, \gamma}$ , which we do by a method explained to me by Bernd Siebert. The important point is that Quot-schemes work not just for  $\mathbb{K}$ -schemes, but for  $S$ -schemes over a general locally noetherian base scheme  $S$ .

As above  $\mathrm{Quot}_P^{\circ}(\oplus^N \mathcal{O}_P, \beta)$  is represented by a quasiprojective  $\mathbb{K}$ -scheme  $S_N^{\beta}$ . Thus, the groupoid  $\mathrm{Quot}_P(\oplus^N \mathcal{O}_P, \beta)(S_N^{\beta})$  is equivalent to  $\mathrm{Hom}(S_N^{\beta}, S_N^{\beta})$  in  $\mathrm{Sch}_{\mathbb{K}}$ . Let  $(Y_N^{\beta}, \xi_N^{\beta})$  in  $\mathrm{Quot}_P(\oplus^N \mathcal{O}_P, \beta)(S_N^{\beta})$  be identified with  $\mathrm{id}_{S_N^{\beta}}$ . Then  $(Y_N^{\beta}, \xi_N^{\beta})$  is the *universal quotient sheaf* of  $\oplus^N \mathcal{O}_P$  with class  $\beta$ .

Now  $Y_N^{\beta}$  is a quasicohherent sheaf on  $P \times S_N^{\beta}$ , flat over  $S_N^{\beta}$ . Regard  $P \times S_N^{\beta}$  with the projection  $P \times S_N^{\beta} \rightarrow S_N^{\beta}$  as a *projective  $S_N^{\beta}$ -scheme*. Then  $Y_N^{\beta}$  becomes a *coherent sheaf* on the  $S_N^{\beta}$ -scheme  $P \times S_N^{\beta}$ . By Grothendieck [5, §3.2] we can therefore form the Quot-scheme  $\mathrm{Quot}_{P \times S_N^{\beta}/S_N^{\beta}}(Y_N^{\beta}, \gamma)$ . It is the *moduli stack of quotient sheaves* of  $Y_N^{\beta}$  on the  $S_N^{\beta}$ -scheme  $P \times S_N^{\beta}$  with class  $\gamma \in K(\mathrm{coh}(P))$ , and is represented by a *projective  $S_N^{\beta}$ -scheme*.

Interpreting  $\mathrm{Quot}_{P \times S_N^\beta / S_N^\beta}(Y_N^\beta, \gamma)$  as a  $\mathbb{K}$ -stack rather than an  $S_N^\beta$ -stack,  $\mathrm{Quot}_{P \times S_N^\beta / S_N^\beta}(Y_N^\beta, \gamma)(U)$  for  $U \in \mathrm{Sch}_{\mathbb{K}}$  has *objects*  $(g, Z, \psi)$ , where  $g : U \rightarrow S_N^\beta$  is a morphism in  $\mathrm{Sch}_{\mathbb{K}}$ ,  $Z \in \mathfrak{Obj}_{\mathrm{coh}(P)}^\gamma(U)$  and  $\psi : \mathfrak{F}_{\mathrm{coh}(P)}(g)Y_N^\beta \rightarrow Z$  is surjective in  $\mathrm{Mor}(\mathfrak{F}_{\mathrm{coh}(P)}(U))$ , and *morphisms*  $\tau : (g, Z, \psi) \rightarrow (g, Z', \psi')$ , where  $\tau : Z \rightarrow Z'$  is an isomorphism in  $\mathrm{Mor}(\mathfrak{F}_{\mathrm{coh}(P)}(U))$  with  $\psi' = \tau \circ \psi$ .

For  $N, n \geq 0$ , define  $\Pi_{N,n} : \mathrm{Quot}_{P \times S_N^{\beta(n)} / S_N^{\beta(n)}}(Y_N^{\beta(n)}, \gamma(n)) \rightarrow \mathfrak{Exact}_{\mathrm{coh}(P)}^{\alpha, \beta, \gamma}$  on *objects* by  $\Pi_{N,n}(U) : (g, Z, \psi) \mapsto (X, (\mathfrak{F}_{\mathrm{coh}(P)}(g)Y_N^{\beta(n)})(-n), Z(-n), \phi, \psi(-n))$ , where  $\phi : X \rightarrow (\mathfrak{F}_{\mathrm{coh}(P)}(g)Y_N^{\beta(n)})(-n)$  is a kernel for  $\psi(-n)$ . This involves a *choice* of  $X, \phi$ , but one which is *unique up to canonical isomorphism*. If  $\tau : (g, Z, \psi) \rightarrow (g, Z', \psi')$  in  $\mathrm{Quot}_{P \times S_N^\beta / S_N^\beta}(Y_N^\beta, \gamma)$  and  $X, \phi, X', \phi'$  are choices made for  $\Pi_{N,n}(U)(g, Z, \psi), \Pi_{N,n}(U)(g, Z', \psi')$ , we define  $\Pi_{N,n}$  on *morphisms* by  $\Pi_{N,n}(U) : \tau \mapsto (\zeta, \mathrm{id}_{(\mathfrak{F}_{\mathrm{coh}(P)}(g)Y_N^{\beta(n)})(-n)}, \tau(-n))$ , where  $\zeta : X \rightarrow X'$  is the unique isomorphism with  $\phi = \phi' \circ \zeta$ . Then  $\Pi_{N,n}$  is a 1-morphism. Define

$$\Pi^{\alpha, \beta, \gamma} = \coprod_{N, n \geq 0} \Pi_{N,n} : \coprod_{N, n \geq 0} \mathrm{Quot}_{P \times S_N^{\beta(n)} / S_N^{\beta(n)}}(Y_N^{\beta(n)}, \gamma(n)) \rightarrow \mathfrak{Exact}_{\mathrm{coh}(P)}^{\alpha, \beta, \gamma}.$$

As in [14, p. 30],  $\Pi^{\alpha, \beta, \gamma}$  is smooth and surjective, so it is an atlas.  $\square$

#### 9.4. Showing 1-morphisms are of finite type

Next we prove that the 1-morphisms of (37) are of finite type. For the 1-morphisms  $m$  in the next proposition, this is because the fibres of  $m$  are essentially Quot-schemes of quotient sheaves with fixed Hilbert polynomial. Thus by Grothendieck's construction they are *projective*  $\mathbb{K}$ -schemes.

**Proposition 9.5.** *In Examples 9.1 and 9.2,  $m : \mathfrak{Exact}_{\mathrm{coh}(P)}^{\alpha, \beta, \gamma} \rightarrow \mathfrak{Obj}_{\mathrm{coh}(P)}^\beta$  is a finite type 1-morphism.*

**Proof.** Consider the commutative diagram

$$\begin{array}{ccc} \coprod_{N, n \geq 0} \mathrm{Quot}_{P \times S_N^{\beta(n)} / S_N^{\beta(n)}}(X_N^{\beta(n)}, \gamma(n)) & \xrightarrow{\quad \quad \quad} & \mathfrak{Exact}_{\mathrm{coh}(P)}^{\alpha, \beta, \gamma} \\ \downarrow \pi_{N,n}^{\beta, \gamma} & \searrow \Pi^{\alpha, \beta, \gamma} & \downarrow m \\ \coprod_{N, n \geq 0} S_N^{\beta(n)} = \coprod_{N, n \geq 0} \mathrm{Quot}_P^\circ(\oplus^N \mathcal{O}_P, \beta(n)) & \xrightarrow{\quad \Pi^\beta \quad} & \mathfrak{Obj}_{\mathrm{coh}(P)}^\beta, \end{array} \quad (49)$$

where  $\pi_{N,n}^{\beta, \gamma}$  is the morphism making  $\mathrm{Quot}_{P \times S_N^{\beta(n)} / S_N^{\beta(n)}}(X_N^{\beta(n)}, \gamma(n))$  into an  $S_N^{\beta(n)}$ -scheme. It is *projective*, as  $\mathrm{Quot}_{P \times S_N^{\beta(n)} / S_N^{\beta(n)}}(X_N^{\beta(n)}, \gamma(n))$  is projective over  $S_N^{\beta(n)}$ . Thus  $\pi_{N,n}^{\beta, \gamma}$  is

of finite type, and so is  $\coprod_{N,n \geq 0} \pi_{N,n}^{\beta,\gamma}$  in (49). As  $\Pi^{\alpha,\beta,\gamma}, \Pi^\beta$  are atlases, (49) implies  $m$  is of finite type.  $\square$

For the 1-morphism  $b \times e$  of (37), the fibre over  $(X, Z)$  in  $\text{coh}(P) \times \text{coh}(P)$  is the stack of *isomorphism classes of exact sequences*  $0 \rightarrow X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \rightarrow 0$  in  $\text{coh}(P)$ . Such sequences are classified by  $\text{Ext}^1(Z, X)$ , so the fibre of  $b \times e$  over  $(X, Z)$  should be the *quotient stack*  $[\text{Ext}^1(Z, X)/\text{Aut}(X) \times \text{Aut}(Z)]$ .

As  $\text{Ext}^1(Z, X)$  is a finite-dimensional  $\mathbb{K}$ -vector space, this fibre is finite type, so  $b \times e$  should be of finite type. The proof below does not use this argument, but depends on facts about Quot-schemes which encode the same ideas.

**Theorem 9.6.** *In Examples 9.1 and 9.2, the 1-morphism  $b \times e : \mathfrak{E}xact_{\text{coh}(P)}^{\alpha,\beta,\gamma} \rightarrow \mathfrak{D}bj_{\text{coh}(P)}^\alpha \times \mathfrak{D}bj_{\text{coh}(P)}^\gamma$  is of finite type.*

**Proof.** Use the notation of Theorem 9.4. Then  $\mathfrak{D}bj_{\text{coh}(P)}^\alpha \times \mathfrak{D}bj_{\text{coh}(P)}^\gamma$  is covered by open substacks of the form

$$V_{M,N,n}^{\alpha,\gamma} = (\Pi_{M,n}^\alpha \times \Pi_{N,n}^\gamma)(\text{Quot}_P^\circ(\oplus^M \mathcal{O}_P, \alpha(n)) \times \text{Quot}_P^\circ(\oplus^N \mathcal{O}_P, \gamma(n))) \quad (50)$$

for  $M, N, n \geq 0$ . Let  $W_{M,N,n}^{\alpha,\beta,\gamma} = (b \times e)^{-1}(V_{M,N,n}^{\alpha,\gamma})$  be the inverse image of  $V_{M,N,n}^{\alpha,\gamma}$  in  $\mathfrak{E}xact_{\text{coh}(P)}^{\alpha,\beta,\gamma}$  under  $b \times e$ . As  $\text{Quot}_P^\circ(\oplus^M \mathcal{O}_P, \alpha(n)) \times \text{Quot}_P^\circ(\oplus^N \mathcal{O}_P, \gamma(n))$  is an atlas for  $V_{M,N,n}^{\alpha,\gamma}$  represented by a quasiprojective scheme,  $V_{M,N,n}^{\alpha,\gamma}$  is of finite type. Also  $S_{M+N}^{\beta(n)}$  is a quasiprojective scheme representing  $\text{Quot}_P^\circ(\oplus^{M+N} \mathcal{O}_P, \beta(n))$ , and  $(Y_{M+N}^{\beta(n)}, \zeta_{M+N}^{\beta(n)})$  is the universal quotient sheaf of  $\oplus^{M+N} \mathcal{O}_P$  on  $S_{M+N}^{\beta(n)}$ .

Form the Quot-scheme  $\mathcal{Q}_{M+N}^{\beta(n),\gamma(n)} = \text{Quot}_{P \times S_{M+N}^{\beta(n)}/S_{M+N}^{\beta(n)}}(Y_{M+N}^{\beta(n)}, \gamma(n))$ , as in Theorem 9.4. It is projective over  $S_{M+N}^{\beta(n)}$ , and  $S_{M+N}^{\beta(n)}$  is of finite type, so  $\mathcal{Q}_{M+N}^{\beta(n),\gamma(n)}$  is of finite type. We have a projection

$$\Pi_{M+N,n}^{\alpha,\beta,\gamma} : \mathcal{Q}_{M+N}^{\beta(n),\gamma(n)} \longrightarrow \mathfrak{E}xact_{\text{coh}(P)}^{\alpha,\beta,\gamma}, \quad (51)$$

forming part of an atlas for  $\mathfrak{E}xact_{\text{coh}(P)}^{\alpha,\beta,\gamma}$ . We shall show (51) covers  $W_{M,N,n}^{\alpha,\beta,\gamma}$ .

Let  $U \in \text{Sch}_{\mathbb{K}}$ , and let  $(X, Y, Z, \phi, \psi) \in W_{M,N,n}^{\alpha,\beta,\gamma}(U)$ . By definition this means that  $(X, Y, Z, \phi, \psi) \in \mathfrak{E}xact_{\text{coh}(P)}^{\alpha,\beta,\gamma}(U)$  and  $(X, Z) \in V_{M,N,n}^{\alpha,\gamma}(U)$ . As  $V_{M,N,n}^{\alpha,\gamma}$  is defined as an *image* in (50), roughly speaking this means that  $X$  lifts to  $\text{Quot}_P^\circ(\oplus^M \mathcal{O}_P, \alpha(n))(U)$  and  $Z$  lifts to  $\text{Quot}_P^\circ(\oplus^N \mathcal{O}_P, \gamma(n))(U)$ .

However, these lifts need exist only *locally* in the étale topology on  $U$ . That is, there exists an open cover  $\{f_i : U_i \rightarrow U\}_{i \in I}$  of  $U$  in the site  $\text{Sch}_{\mathbb{K}}$  and objects  $(X_i, \lambda_i) \in \text{Quot}_P^\circ(\oplus^M \mathcal{O}_P, \alpha(n))(U_i)$  and  $(Z_i, v_i) \in \text{Quot}_P^\circ(\oplus^N \mathcal{O}_P, \gamma(n))(U_i)$  with  $X_i(-n) = \mathfrak{F}_{\text{coh}(P)}(f_i)(X)$ ,  $Z_i(-n) = \mathfrak{F}_{\text{coh}(P)}(f_i)(Z)$  for all  $i \in I$ . Set  $Y_i = (\mathfrak{F}_{\text{coh}(P)}(f_i)Y)(n)$ ,  $\phi_i = (\mathfrak{F}_{\text{coh}(P)}(f_i)\phi)(n)$ ,  $\psi_i = (\mathfrak{F}_{\text{coh}(P)}(f_i)\psi)(n)$ .

Refining the cover  $\{f_i : U_i \rightarrow U\}_{i \in I}$  if necessary, we can construct  $\mu_i$  to make a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \oplus^M \mathcal{O}_{P \times U_i} & \longrightarrow & \oplus^{M+N} \mathcal{O}_{P \times U_i} & \longrightarrow & \oplus^N \mathcal{O}_{P \times U_i} \longrightarrow 0 \\
 & & \downarrow \lambda_i & & \downarrow \mu_i & & \downarrow \nu_i \\
 0 & \longrightarrow & X_i & \xrightarrow{\phi_i} & Y_i & \xrightarrow{\psi_i} & Z_i \longrightarrow 0
 \end{array}$$

in  $\mathrm{qcoh}(P \times U_i)$ , with exact rows and surjective columns, such that  $(Y_i, \mu_i) \in \mathrm{Quot}_P^\circ(\oplus^{M+N} \mathcal{O}_P, \beta(n))(U_i)$ , for all  $i \in I$ .

As the  $\mathbb{K}$ -scheme  $S_{M+N}^{\beta(n)}$  represents  $\mathrm{Quot}_P^\circ(\oplus^{M+N} \mathcal{O}_P, \beta(n))$  with universal quotient sheaf  $(Y_{M+N}^{\beta(n)}, \zeta_{M+N}^{\beta(n)})$ , there is a unique morphism  $g_i : U_i \rightarrow S_{M+N}^{\beta(n)}$  and an isomorphism  $\eta_i : Y_i \rightarrow \mathfrak{F}_{\mathrm{coh}(P)}(g_i) Y_{M+N}^{\beta(n)}$  with  $\eta_i \circ \mu_i = \mathfrak{F}_{\mathrm{coh}(P)}(g_i) \zeta_{M+N}^{\beta(n)}$ . One can then show that the image of  $(g_i, Z_i, \psi_i \circ \eta_i^{-1})$  under  $\Pi_{N,n}^{\alpha,\beta,\gamma}(U_i)$  is isomorphic to  $(X_i(-n), Y_i(-n), Z_i(-n), \phi_i(-n), \psi_i(-n)) = \mathfrak{Exact}_{\mathrm{coh}(P)}^{\alpha,\beta,\gamma}(f_i)(X, Y, Z, \phi, \psi)$ .

We have shown that for any  $U \in \mathrm{Sch}_{\mathbb{K}}$  and  $(X, Y, Z, \phi, \psi) \in W_{M,N,n}^{\alpha,\beta,\gamma}(U)$ , there exists an open cover  $\{f_i : U_i \rightarrow U\}_{i \in I}$  of  $U$  in the site  $\mathrm{Sch}_{\mathbb{K}}$  and objects  $(g_i, Z_i, \psi_i \circ \eta_i^{-1}) \in Q_{M+N}^{\beta(n),\gamma(n)}(U_i)$  such that

$$\Pi_{N,n}^{\beta,\gamma}(U_i)(g_i, Z_i, \psi_i \circ \eta_i^{-1}) \cong \mathfrak{Exact}_{\mathrm{coh}(P)}^{\alpha,\beta,\gamma}(f_i)(X, Y, Z, \phi, \psi)$$

in  $\mathfrak{Exact}_{\mathrm{coh}(P)}^{\alpha,\beta,\gamma}(U_i)$ , for all  $i \in I$ . This proves that (51) covers  $W_{M,N,n}^{\alpha,\beta,\gamma}$ .

But from above  $Q_{M+N}^{\beta(n),\gamma(n)}$  is of finite type, so  $W_{M,N,n}^{\alpha,\beta,\gamma}$  is of finite type, and therefore the 1-morphism  $b \times e : W_{M,N,n}^{\alpha,\beta,\gamma} \rightarrow V_{M,N,n}^{\alpha,\gamma}$  is of finite type. As the  $V_{M,N,n}^{\alpha,\gamma}$  cover  $\mathfrak{Obj}_{\mathrm{coh}(P)}^\alpha \times \mathfrak{Obj}_{\mathrm{coh}(P)}^\gamma$  and  $W_{M,N,n}^{\alpha,\beta,\gamma}$  is the inverse image of  $V_{M,N,n}^{\alpha,\gamma}$  under  $b \times e$ , this shows  $b \times e$  is of finite type, and the proof is complete.  $\square$

The last three results now prove:

**Theorem 9.7.** Examples 9.1 and 9.2 satisfy Assumption 8.1.

Theorems 9.3 and 9.7 show that we may apply the results of §§7 and 8 to Examples 9.1 and 9.2. This yields large classes of *moduli stacks of  $(I, \preccurlyeq)$ -configurations of coherent sheaves*  $\mathfrak{M}(I, \preccurlyeq)_{\mathrm{coh}(P)}$ ,  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathrm{coh}(P)}$  on a projective  $\mathbb{K}$ -scheme  $P$ , which are *algebraic  $\mathbb{K}$ -stacks, locally of finite type*. It also gives many 1-morphisms  $S(I, \preccurlyeq, J)$ ,  $Q(I, \preccurlyeq, K, \trianglelefteq, \phi)$ , ... between these moduli stacks, various of which are *representable or of finite type*.

## 10. Representations of quivers and algebras

Finally we consider configurations in some more large classes of examples of abelian categories, *representations of quivers*  $Q$  and of *finite-dimensional algebras*  $A$ . After

introducing quivers and their representations in §10.1, section 10.2 defines the data  $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$  of Assumption 7.1 for five related families of examples. Sections 10.3 and 10.4 then prove that Assumptions 7.1 and 8.1 hold for each case, so that the results of §§7 and 8 apply.

### 10.1. Introduction to quivers

Here are the basic definitions in quiver theory, taken from Benson [2, §4.1]. We fix an algebraically closed field  $\mathbb{K}$  throughout.

**Definition 10.1.** A *quiver*  $Q$  is a finite directed graph. That is,  $Q$  is a quadruple  $(Q_0, Q_1, b, e)$ , where  $Q_0$  is a finite set of *vertices*,  $Q_1$  is a finite set of *arrows*, and  $b, e : Q_1 \rightarrow Q_0$  are maps giving the *beginning* and *end* of each arrow.

The *path algebra*  $\mathbb{K}Q$  is an associative algebra over  $\mathbb{K}$  with basis all *paths of length*  $k \geq 0$ , that is, sequences of the form

$$v_0 \xrightarrow{a_1} v_1 \rightarrow \cdots \rightarrow v_{k-1} \xrightarrow{a_k} v_k, \quad (52)$$

where  $v_0, \dots, v_k \in Q_0$ ,  $a_1, \dots, a_k \in Q_1$ ,  $b(a_i) = v_{i-1}$  and  $e(a_i) = v_i$ . Multiplication is given by composition of paths *in reverse order*.

Each  $v \in Q_0$  determines a basis element (52) with  $k = 0$ ,  $v_0 = v$ , and the identity in  $\mathbb{K}Q$  is  $1 = \sum_{v \in Q_0} v$ . Each  $a \in Q_1$  determines a basis element  $b(a) \xrightarrow{a} e(a)$  in  $\mathbb{K}Q$  with  $k = 1$ . For brevity we refer to this element as  $a$ . Note that  $\mathbb{K}Q$  is finite-dimensional if and only if  $Q$  has *no oriented cycles*.

For  $n \geq 0$ , write  $\mathbb{K}Q_{(n)}$  for the vector subspace of  $\mathbb{K}Q$  with basis all paths of length  $k \geq n$ . Then  $\mathbb{K}Q_{(n)}$  is a *two-sided ideal* in  $\mathbb{K}Q$ . A *quiver with relations*  $(Q, I)$  is defined to be a quiver  $Q$  together with a two-sided ideal  $I$  in  $\mathbb{K}Q$  such that  $I \subseteq \mathbb{K}Q_{(2)}$ . Then  $\mathbb{K}Q/I$  is an associative  $\mathbb{K}$ -algebra.

**Definition 10.2.** Let  $Q = (Q_0, Q_1, b, e)$  be a quiver. A *representation* of  $Q$  consists of finite-dimensional  $\mathbb{K}$ -vector spaces  $X_v$  for each  $v \in Q_0$ , and linear maps  $\rho_a : X_{b(a)} \rightarrow X_{e(a)}$  for each  $a \in Q_1$ . Representations of  $Q$  are in 1-1 correspondence with *finite-dimensional left  $\mathbb{K}Q$ -modules*  $(X, \rho)$ , as follows.

Given  $X_v, \rho_a$ , define  $X = \bigoplus_{v \in Q_0} X_v$ , and a linear  $\rho : \mathbb{K}Q \rightarrow \text{End}(X)$  taking (52) to the linear map  $X \rightarrow X$  acting as  $\rho_{a_k} \circ \rho_{a_{k-1}} \circ \cdots \circ \rho_{a_1}$  on  $X_{v_0}$ , and 0 on  $X_v$  for  $v \neq v_0$ . Then  $(X, \rho)$  is a left  $\mathbb{K}Q$ -module. Conversely, any such  $(X, \rho)$  comes from a unique representation of  $Q$ , taking  $X_v$  for  $v \in Q_0$  to be the 1-eigenspace of  $\rho(v)$  in  $X$ , and  $\rho_a$  for  $a \in Q_1$  to be the restriction of  $\rho(a) : X \rightarrow X$  to  $X_{b(a)}$ .

We generally write representations of  $Q$  as left  $\mathbb{K}Q$ -modules  $(X, \rho)$ . A *morphism of representations*  $\phi : (X, \rho) \rightarrow (Y, \sigma)$  is a linear map  $\phi : X \rightarrow Y$  with  $\phi \circ \rho(\gamma) = \sigma(\gamma) \circ \phi$  for all  $\gamma \in \mathbb{K}Q$ . Equivalently,  $\phi$  defines linear maps  $\phi_v : X_v \rightarrow Y_v$  for all  $v \in Q_0$  with  $\phi_{e(a)} \circ \rho_a = \sigma_a \circ \phi_{b(a)}$  for all  $a \in Q_1$ .

A representation  $(X, \rho)$  of  $Q$  is called *nilpotent* if  $\rho(\mathbb{K}Q_{(n)}) = \{0\}$  in  $\text{End}(X)$  for some  $n \geq 0$ . Let  $(Q, I)$  be a *quiver with relations*. A *representation* of  $(Q, I)$  is a



representation  $(X, \rho)$  of  $Q$  with  $\rho(I) = \{0\}$ . Then  $X$  is a representation of the quotient algebra  $\mathbb{K}Q/I$ .

Write  $\text{mod-}\mathbb{K}Q$  for the category of representations of  $Q$ , and  $\text{nil-}\mathbb{K}Q$  for the full subcategory of *nilpotent* representations of  $Q$ . If  $(Q, I)$  is a *quiver with relations*, write  $\text{mod-}\mathbb{K}Q/I$  for the category of representations of  $(Q, I)$ , and  $\text{nil-}\mathbb{K}Q/I$  for the full subcategory of *nilpotent* representations of  $(Q, I)$ . It is easy to show all of these are abelian categories, of finite length. If  $Q$  has *no oriented cycles* then  $\text{mod-}\mathbb{K}Q = \text{nil-}\mathbb{K}Q$ , since  $\mathbb{K}Q_{(n)} = 0$  for  $n > |Q_1|$ . If  $\mathbb{K}Q_{(n)} \subseteq I$  for some  $n \geq 2$  then  $\text{mod-}\mathbb{K}Q/I = \text{nil-}\mathbb{K}Q/I$ .

We consider the *Grothendieck groups* of  $\text{mod-}\mathbb{K}Q, \dots, \text{nil-}\mathbb{K}Q/I$ .

**Definition 10.3.** Let  $Q = (Q_0, Q_1, b, e)$  be a quiver and  $(X, \rho)$  a representation of  $Q$ . Write  $\mathbb{N}^{Q_0}$  and  $\mathbb{Z}^{Q_0}$  for the sets of maps  $Q_0 \rightarrow \mathbb{N}$  and  $Q_0 \rightarrow \mathbb{Z}$ . Define the *dimension vector*  $\mathbf{dim}(X, \rho) \in \mathbb{N}^{Q_0} \subset \mathbb{Z}^{Q_0}$  of  $(X, \rho)$  by  $\mathbf{dim}(X, \rho) : v \mapsto \dim_{\mathbb{K}} X_v$ . This induces a surjective group homomorphism  $\mathbf{dim} : K_0(\text{mod-}\mathbb{K}Q) \rightarrow \mathbb{Z}^{Q_0}$ . The same applies to  $\text{nil-}\mathbb{K}Q, \dots, \text{nil-}\mathbb{K}Q/I$ . As  $\text{mod-}\mathbb{K}Q, \dots, \text{nil-}\mathbb{K}Q/I$  have finite length  $K_0(\text{mod-}\mathbb{K}Q), \dots, K_0(\text{nil-}\mathbb{K}Q/I)$  are the free abelian groups with bases isomorphism classes of simple objects in  $\text{mod-}\mathbb{K}Q, \dots, \text{mod-}\mathbb{K}Q/I$ .

A nilpotent representation  $(X, \rho)$  is simple if  $X_v \cong \mathbb{K}$  for some  $v \in Q_0$ , and  $X_w = 0$  for  $w \neq v$ , and  $\rho_a = 0$  for all  $a \in Q_1$ . So simple objects in  $\text{nil-}\mathbb{K}Q, \text{nil-}\mathbb{K}Q/I$  up to isomorphism are in 1-1 correspondence with  $Q_0$ , and  $\mathbf{dim} : K_0(\text{nil-}\mathbb{K}Q), K_0(\text{nil-}\mathbb{K}Q/I) \rightarrow \mathbb{Z}^{Q_0}$  is an isomorphism. When  $Q$  has *oriented cycles*, there are usually *many* simple objects in  $\text{mod-}\mathbb{K}Q, \text{mod-}\mathbb{K}Q/I$ , and  $K_0(\text{mod-}\mathbb{K}Q), K_0(\text{mod-}\mathbb{K}Q/I)$  are much larger than  $\mathbb{Z}^{Q_0}$ .

Quivers are used to study the representations of *finite-dimensional algebras*.

**Definition 10.4.** Let  $A$  be a *finite-dimensional*  $\mathbb{K}$ -algebra, and  $\text{mod-}A$  the category of *finite-dimensional left*  $A$ -modules  $(X, \rho)$ , where  $X$  is a finite-dimensional  $\mathbb{K}$ -vector space and  $\rho : A \rightarrow \text{End}(X)$  an algebra morphism. Then  $\text{mod-}A$  is an abelian category of finite length. Following Benson [2, §2.2, Definition 4.1.6, Proposition 4.1.7] one defines a quiver with relations  $(Q, I)$  called the *Ext-quiver* of  $A$ , whose vertices  $Q_0$  correspond to isomorphism classes of simple objects in  $\text{mod-}A$ , with a natural *equivalence of categories* between  $\text{mod-}A$  and  $\text{mod-}\mathbb{K}Q/I$ . So, the representations of  $A$  can be understood in terms of those of  $(Q, I)$ .

## 10.2. Definition of the data $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$

In five examples we define the data of Assumption 7.1 for the abelian categories  $\text{mod-}\mathbb{K}Q, \text{nil-}\mathbb{K}Q, \text{mod-}\mathbb{K}Q/I, \text{nil-}\mathbb{K}Q/I, \text{mod-}A$  of §10.1, respectively. The main ideas are all in Example 10.5, with minor variations in Examples 10.6–10.9. We fix an *algebraically closed field*  $\mathbb{K}$  throughout.

**Example 10.5.** Let  $Q = (Q_0, Q_1, b, e)$  be a quiver. Take  $\mathcal{A} = \text{mod-}\mathbb{K}Q$ , the abelian category of representations of  $Q$ . Define  $K(\text{mod-}\mathbb{K}Q)$  to be the quotient of



$K_0(\text{mod-}\mathbb{K}Q)$  by the kernel of  $\mathbf{dim} : K_0(\text{mod-}\mathbb{K}Q) \rightarrow \mathbb{Z}^{Q_0}$ . Then  $\mathbf{dim}$  induces an isomorphism  $K(\text{mod-}\mathbb{K}Q) \cong \mathbb{Z}^{Q_0}$ . We shall identify  $K(\text{mod-}\mathbb{K}Q)$  and  $\mathbb{Z}^{Q_0}$ , so that for  $X \in \text{mod-}\mathbb{K}Q$  the class  $[X]$  in  $K(\text{mod-}\mathbb{K}Q)$  is  $\mathbf{dim} X$ .

Motivated by King [13, Definition 5.1], for  $U \in \text{Sch}_{\mathbb{K}}$  define  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U)$  to be the category with *objects*  $(X, \rho)$  for  $X$  a locally free sheaf of finite rank on  $U$  and  $\rho : \mathbb{K}Q \rightarrow \text{Hom}(X, X)$  a  $\mathbb{K}$ -algebra homomorphism, and *morphisms*  $\phi : (X, \rho) \rightarrow (Y, \sigma)$  to be morphisms of sheaves  $\phi : X \rightarrow Y$  with  $\phi \circ \rho(\gamma) = \sigma(\gamma) \circ \phi$  in  $\text{Hom}(X, Y)$  for all  $\gamma \in \mathbb{K}Q$ .

Now define  $\mathcal{A}_U$  to be the category with objects  $(X, \rho)$  for  $X$  a quasicoherent sheaf on  $U$  and  $\rho : \mathbb{K}Q \rightarrow \text{Hom}(X, X)$  a  $\mathbb{K}$ -algebra homomorphism, and morphisms  $\phi$  as above. It is easy to show  $\mathcal{A}_U$  is an abelian category, and  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U)$  an exact subcategory of  $\mathcal{A}_U$ . Thus  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U)$  is an exact category.

For  $f : U \rightarrow V$  in  $\text{Mor}(\text{Sch}_{\mathbb{K}})$ , define a functor  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f) : \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(V) \rightarrow \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U)$  by  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f) : (X, \rho) \mapsto (f^*(X), f^*(\rho))$  on objects  $(X, \rho)$  and  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f) : \phi \mapsto f^*(\phi)$  on morphisms  $\phi : (X, \rho) \rightarrow (Y, \sigma)$ , where  $f^*(X)$  is the *inverse image sheaf* and  $f^*(\rho)(\gamma) = f^*(\rho(\gamma)) : f^*(X) \rightarrow f^*(X)$  for  $\gamma \in \mathbb{K}Q$  and  $f^*(\phi) : f^*(X) \rightarrow f^*(Y)$  are pullbacks of morphisms between inverse images.

Since  $f^*(\mathcal{O}_V) \cong \mathcal{O}_U$ , inverse images of locally free sheaves of finite rank are also locally free of finite rank, so  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f)$  is a functor  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(V) \rightarrow \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U)$ . As locally free sheaves on  $V$  are flat over  $V$ , Grothendieck [6, Proposition IV.2.1.8(i)] implies  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f)$  is an exact functor.

As in Example 9.1, defining  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f)(X, \rho)$  involves making a *choice* for  $f^*(X)$ , arbitrary up to canonical isomorphism. When  $f : U \rightarrow V$ ,  $g : V \rightarrow W$  are morphisms in  $\text{Sch}_{\mathbb{K}}$ , the canonical isomorphisms yield an isomorphism of functors  $\varepsilon_{g,f} : \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f) \circ \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(g) \rightarrow \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(g \circ f)$ , that is, a 2-morphism in  $(\text{exactcat})$ . The  $\varepsilon_{g,f}$  complete the definition of  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}$ , and one can readily show that the definition of a 2-functor [4, Appendix B] holds.

Here is how to extend this to *nilpotent* representations.

**Example 10.6.** Take  $\mathcal{A} = \text{nil-}\mathbb{K}Q$ , the abelian category of nilpotent representations of a quiver  $Q$ . Then  $K_0(\text{nil-}\mathbb{K}Q) \cong \mathbb{Z}^{Q_0}$ . Set  $K(\text{nil-}\mathbb{K}Q) = K_0(\text{nil-}\mathbb{K}Q)$ .

For  $U \in \text{Sch}_{\mathbb{K}}$  let  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q}(U)$  be the full exact subcategory of  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U)$  with objects  $(X, \rho)$  such that there exists an open cover  $\{f_n : U_n \rightarrow U\}_{n \in \mathbb{N}}$  of  $U$  in the site  $\text{Sch}_{\mathbb{K}}$  for which  $(X_n, \rho_n) = \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f_n)(X, \rho)$  satisfies  $\rho_n(\mathbb{K}Q_{(n)}) = \{0\}$  in  $\text{Hom}(X_n, X_n)$ , for all  $n \in \mathbb{N}$ . Define  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q}(f)$  and  $\varepsilon_{g,f}$  as in Example 10.5, but restricting to  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q}(U)$ ,  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q}(V)$ ,  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q}(W)$ . Then  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{exactcat})$  is a *contravariant 2-functor*.

The point here is that  $(X, \rho) \in \text{mod-}\mathbb{K}Q$  is nilpotent if  $\rho(\mathbb{K}Q_{(n)}) = 0$  for some  $n \in \mathbb{N}$ . But in a *family* of nilpotent representations  $(X_u, \rho_u)$  parameterized by  $u$  in a base scheme  $U$ , this number  $n$  could vary with  $u$ , and might be *unbounded* on  $U$ . Thus it is not enough to define  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q}(U)$  as the subcategory of  $(X, \rho)$  in  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U)$  with  $\rho(\mathbb{K}Q_{(n)}) = \{0\}$  for some  $n \in \mathbb{N}$ . Instead, we cover  $U$  by open sets  $U_n$  with  $\rho(\mathbb{K}Q_{(n)}) = \{0\}$  over  $U_n$  for  $n = 0, 1, \dots$

The extension of Example 10.5 to quivers with relations  $(Q, I)$  is trivial.

**Example 10.7.** Let  $(Q, I)$  be a *quiver with relations*. Take  $\mathcal{A} = \text{mod-}\mathbb{K}Q/I$ , and define  $K(\text{mod-}\mathbb{K}Q/I) \cong \mathbb{Z}^{Q_0}$  as in Example 10.5. For  $U \in \text{Sch}_{\mathbb{K}}$  define  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q/I}(U)$  to be the full exact subcategory of  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U)$  with objects  $(X, \rho)$  such that  $\rho(I) = \{0\}$  in  $\text{Hom}(X, X)$ . Define  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q/I}(f)$  and  $\varepsilon_{g,f}$  as in Example 10.5, but restricting to  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q/I}(U), \dots, \mathfrak{F}_{\text{mod-}\mathbb{K}Q/I}(W)$ . Then  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q/I} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{exactcat})$  is a *contravariant 2-functor*.

We can also combine Examples 10.6 and 10.7.

**Example 10.8.** Let  $(Q, I)$  be a *quiver with relations*. Take  $\mathcal{A} = \text{nil-}\mathbb{K}Q/I$ . Then  $K_0(\text{nil-}\mathbb{K}Q/I) \cong \mathbb{Z}^{Q_0}$ . Set  $K(\text{nil-}\mathbb{K}Q/I) = K_0(\text{nil-}\mathbb{K}Q/I)$ . For  $U \in \text{Sch}_{\mathbb{K}}$  define  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q/I}(U)$  to be the intersection of  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q}(U)$  and  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q/I}(U)$  in  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U)$ , which is a full exact subcategory of  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U)$ . Let  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q/I}(f)$  and  $\varepsilon_{g,f}$  be as in Example 10.5, but restricting to  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q/I}(U), \dots, \mathfrak{F}_{\text{nil-}\mathbb{K}Q/I}(W)$ . Then  $\mathfrak{F}_{\text{nil-}\mathbb{K}Q/I} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{exactcat})$  is a *contravariant 2-functor*.

Here is the generalization to finite-dimensional algebras  $A$ .

**Example 10.9.** Let  $A$  be a *finite-dimensional  $\mathbb{K}$ -algebra*, with *Ext-quiver*  $Q$ . Take  $\mathcal{A} = \text{mod-}A$ , so that  $K_0(\text{mod-}A) \cong \mathbb{Z}^{Q_0}$ . Set  $K(\text{mod-}A) = K_0(\text{mod-}A)$ . Define a *contravariant 2-functor*  $\mathfrak{F}_{\text{mod-}A} : \text{Sch}_{\mathbb{K}} \rightarrow (\text{exactcat})$  as in Example 10.5, replacing  $\mathbb{K}Q$  by  $A$  throughout.

### 10.3. Verifying Assumption 7.1

We show the examples of §10.2 satisfy Assumption 7.1, following §9.2.

**Theorem 10.10.** Examples 10.5–10.9 satisfy Assumption 7.1.

**Proof.** In each example, if  $(X, \rho) \in \text{Obj}(\mathcal{A})$  then  $[(X, \rho)] \in K(\mathcal{A})$  corresponds to  $\mathbf{dim}(X, \rho) \in \mathbb{Z}^{Q_0}$ , so  $[(X, \rho)] = 0$  implies  $\mathbf{dim}(X, \rho) = 0$  and hence  $\dim X = 0$ , so that  $X = \{0\}$  and  $(X, \rho) \cong 0$  in  $\mathcal{A}$ . This proves the condition on  $K(\mathcal{A})$  in Assumption 7.1. For the rest of the proof, we do Example 10.5 first.

We must prove Definition 2.8(i)–(iii) for  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}$ . Let  $\{f_i : U_i \rightarrow V\}_{i \in I}$  be an open cover of  $V$  in the site  $\text{Sch}_{\mathbb{K}}$ . For (i), let  $(X, \rho), (Y, \sigma) \in \text{Obj}(\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(V))$  and  $\phi_i : \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f_i)(X, \rho) \rightarrow \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f_i)(Y, \sigma)$  for  $i \in I$  satisfy (4). Applying [7, Corollary VIII.1.2] to the family of sheaf morphisms  $\phi_i : f_i^*(X) \rightarrow f_i^*(Y)$  gives a unique morphism  $\eta : X \rightarrow Y$  with  $f_i^*(\eta) = \phi_i$ . Let  $\gamma \in \mathbb{K}Q$ . Then

$$\begin{aligned} f_i^*(\eta \circ \rho(\gamma)) &= f_i^*(\eta) \circ f_i^*(\rho(\gamma)) = \phi_i \circ f_i^*(\rho)(\gamma) \\ &= f_i^*(\sigma)(\gamma) \circ \phi_i = f_i^*(\sigma(\gamma)) \circ f_i^*(\eta) = f_i^*(\sigma(\gamma) \circ \eta), \end{aligned} \quad (53)$$

since  $\phi_i : (f_i^*(X), f_i^*(\rho)) \rightarrow (f_i^*(Y), f_i^*(\sigma))$  is a morphism in  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U_i)$ . Using (53), uniqueness in [7, Corollary VIII.1.2] implies that  $\eta \circ \rho(\gamma) = \sigma(\gamma) \circ \eta$ . As this holds for all  $\gamma \in \mathbb{K}Q$ ,  $\eta : (X, \rho) \rightarrow (Y, \sigma)$  lies in  $\text{Mor}(\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(V))$ . This proves Definition 2.8(i). Part (ii) follows from [7, Corollary VIII.1.2].

For (iii), let  $(X_i, \rho_i) \in \text{Obj}(\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U_i))$  and  $\phi_{ij} : \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f_{ij,j})(X_j, \rho_j) \rightarrow \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(f_{ij,i})(X_i, \rho_i)$  for  $i, j \in I$  satisfy (5). Then [7, Corollary VIII.1.3] constructs  $X \in \text{qcoh}(V)$  and isomorphisms  $\phi_i : f_i^*(X) \rightarrow X_i$  satisfying (6), and [7, Proposition VIII.1.10] implies  $X$  is locally free of finite rank. Using [7, Corollary VIII.1.2] we construct  $\rho$  from the  $\rho_i$  such that  $(X, \rho) \in \text{Obj}(\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(V))$  satisfies Definition 2.8(iii). Thus  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}$  is a *stack in exact categories*.

As a locally free sheaf of finite rank on the point  $\text{Spec } \mathbb{K}$  is just a finite-dimensional  $\mathbb{K}$ -vector space,  $\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(\text{Spec } \mathbb{K}) = \text{mod-}\mathbb{K}Q$ , and Assumption 7.1(i) holds. Part (ii) follows from [6, Proposition IV.2.2.7] as in Theorem 9.3.

An object  $(X, \rho) \in \mathfrak{F}_{\text{mod-}\mathbb{K}Q}(U)$  is equivalent to vector bundles  $X_v$  on  $U$  for  $v \in Q_0$  and morphisms  $\phi_a : X_{b(a)} \rightarrow X_{e(a)}$  for  $a \in Q_1$ . For each  $u \in \text{Hom}(\text{Spec } \mathbb{K}, U)$  the class  $[\mathfrak{F}_{\text{mod-}\mathbb{K}Q}(u)(X, \rho)]$  in  $K(\text{mod-}\mathbb{K}Q) = \mathbb{Z}^{Q_0}$  is the map  $Q_0 \rightarrow \mathbb{Z}$  taking  $v \in Q_0$  to the rank of  $X_v$  at  $u$ . Clearly, this is a locally constant function of  $u$ , so Assumption 7.1(iii) holds. Part (iv) can be verified as in Theorem 9.3. Thus, the data of Example 10.5 satisfies Assumption 7.1. The modifications for Examples 10.6–10.9 are all more-or-less trivial.  $\square$

#### 10.4. Verifying Assumption 8.1

Here is the analogue of Theorem 9.4 for Examples 10.5–10.9. Note, however that we prove  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}, \mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma}$  are of *finite type*, not just *locally* so.

**Theorem 10.11.** *In Examples 10.5–10.9,  $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$  and  $\mathfrak{Exact}_{\mathcal{A}}^{\alpha, \beta, \gamma}$  are algebraic  $\mathbb{K}$ -stacks of finite type for all  $\alpha, \beta, \gamma$ .*

**Proof.** We begin with Example 10.5, so that  $\mathcal{A} = \text{mod-}\mathbb{K}Q$ . Fix  $\alpha \in \mathbb{N}^{Q_0}$ . For each  $v \in Q_0$  choose a  $\mathbb{K}$ -vector space  $A_v$  with  $\dim A_v = \alpha(v)$ . Define  $L = \prod_{a \in Q_1} A_{b(a)}^* \otimes A_{e(a)}$ . Then  $L$  is a finite-dimensional  $\mathbb{K}$ -vector space, and thus an *affine  $\mathbb{K}$ -scheme*. Write  $GL(A_v)$  for the group of automorphisms of  $A_v$ . Then  $G = \prod_{v \in Q_0} GL(A_v)$  acts naturally on  $L = \prod_{a \in Q_1} A_{b(a)}^* \otimes A_{e(a)}$ , so we may form the *quotient stack*  $[L/G]$ .

Let  $U \in \text{Sch}_{\mathbb{K}}$  and  $(X, \rho) \in \mathfrak{Obj}_{\text{mod-}\mathbb{K}Q}^{\alpha}(U)$ . Then  $X$  decomposes naturally as  $\bigoplus_{v \in Q_0} X_v$ , for  $X_v$  a locally free sheaf over  $U$  of rank  $\alpha(v)$ . As locally free sheaves are locally trivializable we may choose an open cover  $\{f_i : U_i \rightarrow U\}_{i \in I}$  of  $U$  in the site  $\text{Sch}_{\mathbb{K}}$  such that  $f_i^*(X_v)$  is trivial of rank  $\alpha(v)$  on  $U_i$  for all  $i \in I$ ,  $v \in Q_0$ . So we may choose isomorphisms  $f_i^*(X_v) \cong A_v \times U_i$ , unique up to the action of  $\text{Hom}(U, GL(A_v))$ . Using this and the definitions of  $\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q}^{\alpha}$  and  $[L/G]$ , one can construct a 1-isomorphism  $\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q}^{\alpha} \cong [L/G]$ . Thus  $\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q}^{\alpha}$  is an *algebraic  $\mathbb{K}$ -stack of finite type*, as  $[L/G]$  is.

Now fix  $\alpha, \gamma$  and  $\beta = \alpha + \gamma$  in  $\mathbb{N}^{Q_0}$ . For all  $v \in Q_0$ , choose  $\mathbb{K}$ -vector spaces  $A_v, B_v, C_v$  with  $\dim A_v = \alpha(v)$ ,  $\dim B_v = \beta(v)$ , and  $\dim C_v = \gamma(v)$ . Define

$$M = \left\{ \begin{aligned} & \left( \Pi_a x_a, \Pi_a y_a, \Pi_a z_a, \Pi_v p_v, \Pi_v q_v \right) \in \prod_{a \in Q_1} (A_{b(a)}^* \otimes A_{e(a)}) \\ & \times \prod_{a \in Q_1} (B_{b(a)}^* \otimes B_{e(a)}) \times \prod_{a \in Q_1} (C_{b(a)}^* \otimes C_{e(a)}) \times \prod_{v \in Q_0} (A_v^* \otimes B_v) \times \prod_{v \in Q_0} (B_v^* \otimes C_v) : \\ & y_a \circ p_{b(a)} = p_{e(a)} \circ x_a \text{ and } z_a \circ q_{b(a)} = q_{e(a)} \circ y_a \text{ for all } a \in Q_1, \\ & \text{and } 0 \rightarrow A_v \xrightarrow{p_v} B_v \xrightarrow{q_v} C_v \rightarrow 0 \text{ is exact for all } v \in Q_0 \end{aligned} \right\}. \quad (54)$$

This defines  $M$  as a subset of a finite-dimensional  $\mathbb{K}$ -vector space,  $N$  say. The third line of (54) is finitely many quadratic equations in  $N$ . Exactness in the fourth line is equivalent to  $q_v \circ p_v = 0$ , more quadratic equations, together with injectivity of  $p_v$  and surjectivity of  $q_v$ , which are open conditions. Thus,  $M$  is a Zariski open subset of the zeroes of finitely many polynomials in  $N$ , and is a *quasiaffine*  $\mathbb{K}$ -scheme. Define  $H = \prod_{v \in Q_0} (GL(A_v) \times GL(B_v) \times GL(C_v))$ , as an *algebraic*  $\mathbb{K}$ -group. Then  $H$  has an obvious action on  $M$ , so we can form the *quotient stack*  $[M/H]$ . A similar proof to the  $\mathfrak{Ob}_{\text{mod-}\mathbb{K}Q}^\alpha$  case gives a 1-isomorphism  $\mathfrak{Exact}_{\text{mod-}\mathbb{K}Q}^{\alpha, \beta, \gamma} \cong [M/H]$ . Thus  $\mathfrak{Exact}_{\text{mod-}\mathbb{K}Q}^{\alpha, \beta, \gamma}$  is an *algebraic*  $\mathbb{K}$ -stack of *finite type*, as  $[M/H]$  is. This proves the theorem for Example 10.5.

Next we do Example 10.7, so let  $(Q, I)$  be a *quiver with relations*. Let  $A_v, L$  and  $G$  be as above, and set  $A = \bigoplus_{v \in Q_0} A_v$ . For each  $\Pi_a x_a$  in  $L$ , define  $\rho_{\Pi_a x_a} : \mathbb{K}Q \rightarrow \text{End}(A)$  to be the unique algebra homomorphism such that  $\rho(v) = 1$  on  $A_v$  and 0 on  $A_w$  for  $v \neq w \in Q_0$ , and  $\rho(a) = x_a : A_{b(a)} \rightarrow A_{e(a)}$  on  $A_{b(a)}$ ,  $\rho(a) = 0$  on  $A_v$  for  $b(a) \neq v \in Q_0$ , and all  $a \in Q_1$ . Define

$$L_I = \{ \Pi_a x_a \in \prod_{a \in Q_1} (A_{b(a)}^* \otimes A_{e(a)}) : \rho_{\Pi_a x_a}(i) = 0 \text{ for all } i \in I \}.$$

Each  $i \in I$  is a finite linear combination of basis elements (52) of  $\mathbb{K}Q$ , so  $\rho_{\Pi_a x_a}$  is a finite linear combination of the corresponding  $x_{a_k} \circ \dots \circ x_{a_1}$  in  $\text{End}(A)$ . Thus for fixed  $i \in I$ , the map  $\Pi_a x_a \mapsto \rho_{\Pi_a x_a}(i)$  is a polynomial on  $L$  with values in  $\text{End}(A)$ . Hence  $L_I$  is the zeroes of a collection of polynomials on  $L$ , and so is a  $G$ -invariant affine  $\mathbb{K}$ -scheme. Modifying the proof above shows  $\mathfrak{Ob}_{\text{mod-}\mathbb{K}Q/I}^\alpha$  is 1-isomorphic to  $[L_I/G]$ , and so is *algebraic* and of *finite type*.

Similarly, for  $\mathfrak{Exact}_{\text{mod-}\mathbb{K}Q/I}^{\alpha, \beta, \gamma}$  we replace  $M$  in (54) by  $M_I$ , where we add extra conditions  $\rho_{\Pi_a x_a}(i) = 0$  in  $\text{End}(\bigoplus_{v \in Q_0} A_v)$ ,  $\rho_{\Pi_a y_a}(i) = 0$  in  $\text{End}(\bigoplus_{v \in Q_0} B_v)$  and  $\rho_{\Pi_a z_a}(i) = 0$  in  $\text{End}(\bigoplus_{v \in Q_0} C_v)$  on  $(\Pi_a x_a, \dots, \Pi_v q_v)$  for all  $i \in I$ . Then  $M_I$  is a quasiaffine  $\mathbb{K}$ -scheme invariant under  $H$ , and  $\mathfrak{Exact}_{\text{mod-}\mathbb{K}Q/I}^{\alpha, \beta, \gamma}$  is 1-isomorphic to  $[M_I/H]$ , so it is *algebraic* and of *finite type*.

For Examples 10.6 and 10.8, suppose  $(X, \rho) \in \text{Obj}(\text{nil-}\mathbb{K}Q)$ . Then the vector subspaces  $X_k = \rho(\mathbb{K}Q_{(k)})X$  of  $X$  must decrease strictly in dimension until they become zero. Hence  $\rho(\mathbb{K}Q_{(m)})X = \{0\}$  for  $m = \dim X$ . Now let  $U \in \text{Sch}_{\mathbb{K}}$  and  $(X, \rho) \in \mathfrak{D}\mathfrak{b}\mathfrak{j}_{\text{mod-}\mathbb{K}Q}^{\alpha}(U)$ . The same proof shows  $(X, \rho) \in \mathfrak{D}\mathfrak{b}\mathfrak{j}_{\text{nil-}\mathbb{K}Q}^{\alpha}(U)$  if and only if  $\rho(\mathbb{K}Q_{(m)}) = 0$  in  $\text{End}(X)$  for  $m = \sum_{v \in Q_0} \alpha(v)$ . By a similar argument for  $\mathfrak{E}\mathfrak{x}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\text{mod-}\mathbb{K}Q}^{\alpha, \beta, \gamma}$ , we see that

$$\begin{aligned} \mathfrak{D}\mathfrak{b}\mathfrak{j}_{\text{nil-}\mathbb{K}Q}^{\alpha} &= \mathfrak{D}\mathfrak{b}\mathfrak{j}_{\text{mod-}\mathbb{K}Q/\mathbb{K}Q_{(m)}}^{\alpha}, & \mathfrak{D}\mathfrak{b}\mathfrak{j}_{\text{nil-}\mathbb{K}Q/I}^{\alpha} &= \mathfrak{D}\mathfrak{b}\mathfrak{j}_{\text{mod-}\mathbb{K}Q/(I+\mathbb{K}Q_{(m)})}^{\alpha}, \\ \mathfrak{E}\mathfrak{x}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\text{nil-}\mathbb{K}Q}^{\alpha, \beta, \gamma} &= \mathfrak{E}\mathfrak{x}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\text{mod-}\mathbb{K}Q/\mathbb{K}Q_{(m)}}^{\alpha, \beta, \gamma}, & \mathfrak{E}\mathfrak{x}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\text{nil-}\mathbb{K}Q/I}^{\alpha, \beta, \gamma} &= \mathfrak{E}\mathfrak{x}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\text{mod-}\mathbb{K}Q/(I+\mathbb{K}Q_{(n)})}^{\alpha, \beta, \gamma}, \end{aligned}$$

for  $m = \sum_{v \in Q_0} \alpha(v)$  and  $n = \sum_{v \in Q_0} \beta(v)$ . Thus the theorem for Examples 10.6 and 10.8 follows from the Example 10.7 case, with  $\mathbb{K}Q_{(n)}$ ,  $\mathbb{K}Q_{(m)}$ ,  $I + \mathbb{K}Q_{(m)}$  or  $I + \mathbb{K}Q_{(n)}$  in place of the ideal  $I$ .

Finally, in the situation of Definition 10.4 and Example 10.9, the equivalence of categories between  $\text{mod-}A$  and  $\text{mod-}\mathbb{K}Q/I$  extends easily to give an equivalence of 2-functors  $\mathfrak{F}_{\text{mod-}A} \rightarrow \mathfrak{F}_{\text{mod-}\mathbb{K}Q/I}$ , and hence 1-isomorphisms  $\mathfrak{D}\mathfrak{b}\mathfrak{j}_{\text{mod-}A}^{\alpha} \rightarrow \mathfrak{D}\mathfrak{b}\mathfrak{j}_{\text{mod-}\mathbb{K}Q/I}^{\alpha}$  and  $\mathfrak{E}\mathfrak{x}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\text{mod-}A}^{\alpha, \beta, \gamma} \rightarrow \mathfrak{E}\mathfrak{x}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\text{mod-}\mathbb{K}Q/I}^{\alpha, \beta, \gamma}$ . Thus, the theorem for Example 10.9 again follows from the Example 10.7 case.  $\square$

In each of Examples 10.5–10.9, Theorem 10.12 shows  $\mathfrak{D}\mathfrak{b}\mathfrak{j}_{\mathcal{A}}^{\alpha}$ ,  $\mathfrak{E}\mathfrak{x}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\mathcal{A}}^{\alpha, \beta, \gamma}$  are algebraic  $\mathbb{K}$ -stacks of *finite type*, and hence *locally* of finite type. Since  $\mathfrak{E}\mathfrak{x}\mathfrak{a}\mathfrak{c}\mathfrak{t}_{\mathcal{A}}^{\alpha, \beta, \gamma}$  is of finite type, it follows immediately that the 1-morphisms (37) are of *finite type*. Thus we prove

**Theorem 10.12.** *Examples 10.5–10.9 satisfy Assumption 8.1.*

We can also strengthen Theorem 8.2 for the examples of §10.2.

**Theorem 10.13.** *Let  $\mathcal{A}$ ,  $K(\mathcal{A})$ ,  $\mathfrak{F}_{\mathcal{A}}$  be as in any of Examples 10.5–10.9. Then  $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$  is an algebraic  $\mathbb{K}$ -stack of finite type for all  $(I, \preccurlyeq, \kappa)$ .*

**Proof.** This follows as  $\sigma(I) : \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} \rightarrow \mathfrak{D}\mathfrak{b}\mathfrak{j}_{\mathcal{A}}^{\kappa(I)}$  is of finite type by Theorem 8.4, and  $\mathfrak{D}\mathfrak{b}\mathfrak{j}_{\mathcal{A}}^{\kappa(I)}$  is of finite type by Theorem 10.11.  $\square$

## Acknowledgements

I thank Tom Bridgeland for many inspiring conversations and for being interested, Frances Kirwan and Burt Totaro for help with moduli spaces and stacks, and Bernd Siebert for explaining Quot-schemes over a base. I also thank Ian Grojnowski, Alastair King, Andrew Kresch, Paul Seidel, and Richard Thomas for useful conversations. I was supported by an EPSRC Advanced Research Fellowship whilst writing this paper.

## References

- [1] K. Behrend, D. Edidin, B. Fantechi, W. Fulton, L. Göttsche, A. Kresch, Introduction to Stacks, to appear.
- [2] D.J. Benson, Representations and Cohomology I, Cambridge University Press, Cambridge, 1991.
- [3] S.I. Gelfand, Y.I. Manin, Methods of Homological Algebra, second ed., Springer Monographs in Mathematics, Springer, Berlin, 2003.
- [4] T.L. Gómez, Algebraic stacks, Proc. Indian Acad. Sci. Math. Sci. 111 (2001) 1–31 [math.AG/9911199](#).
- [5] A. Grothendieck, Techniques de construction et théorèmes d'existence en géométrie algébrique, IV: les schémas de Hilbert, Sémin. Bourbaki, vol. 221, 1961.
- [6] A. Grothendieck, Elements de Géométrie Algébrique, part I, Publ. Math. IHES 4 (1960), part II, Publ. Math. IHES 8 (1961), part III, Publ. Math. IHES 11 (1960), 17 (1963), part IV, Publ. Math. IHES 20 (1964), 24 (1965), 28 (1966), 32 (1967).
- [7] A. Grothendieck, Revêtements Étales et Groupe Fondamental (SGA1), Springer Lecture Notes, vol. 224, Springer, Berlin, 1971.
- [8] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, vol. 52, Springer, New York, 1977.
- [9] D. Huybrechts, M. Lehn, The Geometry of Moduli Spaces of Sheaves, Aspects of Mathematics, vol. E31, Vieweg, Braunschweig, 1997.
- [10] D.D. Joyce, Configurations in abelian categories, II: Ringel–Hall algebras, [math.AG/0503029](#), Version 3, 2005.
- [11] D.D. Joyce, Configurations in abelian categories, III: stability conditions and invariants, [math.AG/0410267](#), version 4, 2005.
- [12] D.D. Joyce, Configurations in abelian categories, IV: changing stability conditions, [math.AG/0410268](#), version 4, 2005.
- [13] A.D. King, Moduli of representations of finite-dimensional algebras, Quart. J. Math. Oxford 45 (1994) 515–530.
- [14] G. Laumon, L. Moret-Bailly, Champs Algébriques, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 39, Springer, Berlin, 2000.
- [15] N. Popescu, Abelian Categories with Applications to Rings and Modules, L.M.S. Monographs, vol. 3, Academic Press, London, 1973.
- [16] D. Quillen, Higher algebraic  $K$ -theory. I, H. Bass (Ed.), Algebraic  $K$ -theory. I, Springer Lecture Notes, vol. 341, Springer, Berlin, 1973, pp. 85–147.
- [17] C.S. Seshadri, Space of unitary vector bundles on a compact Riemann surface, Ann. of Math. 85 (1967) 303–336.