HIGHER RAMANUJAN EQUATIONS AND PERIODS OF ABELIAN VARIETIES

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Abstract. We describe higher dimensional generalizations of Ramanujan's classical differential relations satisfied by the Eisenstein series E_2, E_4, E_6. Such "higher Ramanujan equations" are given geometrically in terms of vector fields living on certain moduli stacks classifying abelian schemes equipped with suitable frames of their first de Rham cohomology. These vector fields are canonically constructed by means of the Gauss-Manin connection and the Kodaira-Spencer isomorphism. Using Mumford's theory of degenerating families of abelian varieties, we construct remarkable solutions of these differential equations generalizing (E_2, E_4, E_6), which are also shown to be defined over $\mathbf{Z}$.

This geometric framework taking account of integrality issues is mainly motivated by questions in Transcendental Number Theory regarding an extension of Nesterenko's celebrated theorem on the algebraic independence of values of Eisenstein series. In this direction, we discuss the precise relation between periods of abelian varieties and the values of the above referred solutions of the higher Ramanujan equations, thereby linking the study of such differential equations to Grothendieck's Period Conjecture. Working in the complex analytic category, we prove "functional" transcendence results, such as the Zariski-density of every leaf of the holomorphic foliation induced by the higher Ramanujan equations.

Contents

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0. Introduction

0.1. **Motivation.** The *higher Ramanujan equations* are higher dimensional generalizations of the classical Ramanujan differential relations between the Eisenstein series

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \quad E_4(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \quad E_6(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}.$$

In 1916 [69] Ramanujan proved that these formal series satisfy the system of algebraic differential equations

$$\theta E_2 = \frac{E_2^2 - E_4}{12}, \quad \theta E_4 = \frac{E_2E_4 - E_6}{3}, \quad \theta E_6 = \frac{E_2E_6 - E_4^2}{2}, \tag{R}$$

where $\theta \coloneqq q\frac{d}{dq}$. The study of equivalent forms of such differential equations actually predates Ramanujan. To the best of our knowledge, Jacobi [37] was the first to prove in 1848 that his *Thetanullwerte* satisfy a third order algebraic differential equation. Equivalent differential equations were also introduced by Darboux in 1878 and subsequently studied by Halphen and Brioschi; see the introduction of [34] and the references therein.

Further, in 1911, Chazy [14] considered a differential equation[1] satisfied by the Eisenstein series E_2 which plays an important role in his classification of differential equations of third order:

$$\theta^3E_2 = E_2\theta^2E_2 - \frac{3}{2}(\theta E_2)^2. \tag{C}$$

We refer to [62] for a thorough study of Jacobi's, Halphen's, and Chazy's equations, and the relations between them. Note that Ramanujan's and Chazy's equations concern level 1 (quasi)modular forms, whereas the equations of Jacobi and Halphen involve level 2 (quasi)modular forms.[2]

A higher dimensional generalization of Jacobi's equation concerning *Thetanullwerte* of complex abelian varieties of dimension 2 was first given by Ohyama [63] in 1996, and for any dimension

[1] In Chazy's original notation (cf. [14] (4)) the equation he considered is written as $y''' = 2yy'' - 3(y')^2$. If derivatives in this equation are with respect to a variable t, equation (C) is obtained from this one by the change of variables $q = e^{2t}$.

[2] The reader might also be familiar with the fact that the j-invariant $j = 1728\frac{E_4^3}{E_4^3 - E_6^2}$ (as any other elliptic modular function) satisfies an algebraic differential equation of the third order; this follows immediately from the Ramanujan equations, which show that the ring of quasimodular forms $\mathbf{Q}[E_2, E_4, E_6]$ is closed under θ (cf. [80]).

by Zudilin [81] in 2000; see also Bertrand-Zudilin [5]. In another direction, differential equations related to Hilbert modular forms were studied by Resnikoff [71] in 1972, and by Pellarin [68] in 2005.

This paper grew out from our attempt to obtain a more conceptual understanding of the Ramanujan equations and of their higher dimensional extensions, aiming to shed some light on their arithmetic and geometric properties. A key motivation for this program is the crucial role played by the original Ramanujan equations (R) and by the integrality properties of the series E_2, E_4, E_6 in Nesterenko's celebrated result on the transcendence of their values, when regarded as holomorphic functions on the complex unit disc $\mathbf{D} = \{q \in \mathbf{C} \mid |q| < 1\}$:

Theorem 0.1 (Nesterenko [60], 1996)**.** *For every $q \in \mathbf{D} \setminus \{0\}$,*

$$\operatorname{trdeg}_{\mathbf{Q}} \mathbf{Q}(q, E_2(q), E_4(q), E_6(q)) \geq 3.$$

Note that Zudilin's work on *Thetanullwerte* [81] and Pellarin's study of the differential properties of Hilbert modular forms [68] were also motivated by this same algebraic independence result.

In contrast with the concrete methods of Ohyama, Resnikoff, Zudilin, Bertrand, and Pellarin, relying on modular functions and their derivatives, we follow a geometric approach initially based on Movasati's reinterpretation of the Ramanujan equations as a vector field living on a suitable moduli space of elliptic curves (see [53], [54]).[3] Namely, we construct by purely algebraic methods some higher dimensional avatars of the system (R), involving suitable moduli spaces of abelian varieties enjoying remarkable smoothness properties over $\mathbf{Z}$. The definition of such moduli spaces presupposes the choice of a PEL moduli problem of abelian varieties, and we work out this theory in the Siegel and the Hilbert-Blumenthal cases.

Another distinguishing feature of our approach lies in our emphasis on integrality phenomena. Accordingly, it is imperative to work in "level 1", although it should be clear that we can also include higher level structures in the picture. This introduces certain representability issues, and naturally leads to the use of (Deligne-Mumford) algebraic stacks. As we shall explain below, the appearance of stacks is not a serious problem, since it is possible to recover a purely scheme-theoretic situation (preserving integrality) if needed.

Besides the construction of the higher Ramanujan equations and the study of some of their geometric properties, we take Nesterenko's theorem as a guiding example to explore the deep connections between such differential equations and the vast landscape of problems in the theory of transcendental numbers pertaining to Grothendieck's Period Conjecture, specially in relation with *periods of abelian varieties*. We also discuss future directions, and speculate on possible applications of our constructions to transcendental number theory, such as the algebraic independence of π, $\Gamma(1/5)$, and $\Gamma(2/5)$.

0.2. **Higher Ramanujan equations over Z; Siegel case.** We now explain our main results regarding the construction of the higher Ramanujan equations attached to a Siegel moduli problem. This suffices for the purposes of this introduction, since their Hilbert-Blumenthal counterparts are obtained through a similar yoga.

[3] One may argue that this point of view is already contained, although not explicitly in the form of a vector field on a moduli space, in the concept of Serre derivative of modular forms ([74] 1.4) and in its geometric interpretation in terms of the Gauss-Manin connection given by Deligne ([39] A1.4).

Fix an integer $g \geq 1$. Let k be a field, and (X, λ) be a principally polarized abelian variety over k of dimension g (here, λ denotes a suitable isomorphism from X onto the dual abelian variety X^t). The first algebraic de Rham cohomology $H^1_{\mathrm{dR}}(X/k)$ is a k-vector space of dimension $2g$ endowed with a canonical subspace $F^1(X/k) \cong H^0(X, \Omega^1_{X/k})$ of dimension g — the Hodge filtration — and a non-degenerate alternating k-bilinear form

$$\langle\ ,\ \rangle_\lambda : H^1_{\mathrm{dR}}(X/k) \times H^1_{\mathrm{dR}}(X/k) \longrightarrow k$$

induced by the principal polarization λ. By a *symplectic-Hodge basis* of (X, λ), we mean a basis $b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ of the k-vector space $H^1_{\mathrm{dR}}(X/k)$, such that

(1) each ω_i is in $F^1(X/k)$, and
(2) b is symplectic with respect to $\langle\ ,\ \rangle_\lambda$, that is, $\langle \omega_i, \omega_j\rangle_\lambda = \langle \eta_i, \eta_j\rangle_\lambda = 0$ and $\langle \omega_i, \eta_j\rangle_\lambda = \delta_{ij}$ for every $1 \leq i, j \leq g$.

The above notions generalize to abelian schemes over arbitrary base schemes (see Paragraph 2). We may thus consider a moduli stack $\mathcal{B}_g$ over $\operatorname{Spec} \mathbf{Z}$ classifying principally polarized abelian varieties of dimension g equipped with a symplectic-Hodge basis.

Let $\mathcal{A}_g$ denote the moduli stack of g-dimensional principally polarized abelian varieties, and P_g denote the Siegel parabolic subgroup of Sp_{2g}. Then, the stack $\mathcal{B}_g$ can be regarded as a "principal P_g-bundle" over $\mathcal{A}_g$ via the canonical forgetful map $\mathcal{B}_g \longrightarrow \mathcal{A}_g$. We shall deduce from this that $\mathcal{B}_g$ is a smooth Deligne-Mumford stack over $\operatorname{Spec} \mathbf{Z}$ of relative dimension $2g^2 + g$ (Theorem .41).

The Deligne-Mumford stack $\mathcal{B}_g$ is not representable by a scheme, or even an algebraic space. Nevertheless, we have the following representability theorem.

Theorem 0.2 (see Theorem .80)**.** *The Deligne-Mumford stack $\mathcal{B}_g \otimes \mathbf{Z}[1/2]$ is representable by a smooth quasi-affine scheme B_g over $\mathbf{Z}[1/2]$ of relative dimension $2g^2 + g$.*

This also answers a question of Movasati (see Paragraph 0.6.1 below). The representability of $\mathcal{B}_g \otimes \mathbf{Z}[1/2]$ by a scheme relies essentially on a theorem of Oda ([61] Corollary 5.11) relating $H^1_{\mathrm{dR}}(X/k)$ to the Dieudonné module associated to the p-torsion subscheme $X[p]$ when k is a perfect field of characteristic p.

Next, we study the tangent bundle $T_{\mathcal{B}_g/\mathbf{Z}}$. We show that the *Gauss-Manin connection* induces a canonical horizontal structure on $T_{\mathcal{B}_g/\mathbf{Z}}$ with respect to $\mathcal{B}_g \longrightarrow \mathcal{A}_g$. Namely, if ∇ denotes the Gauss-Manin connection on the de Rham cohomology of the universal abelian scheme over $\mathcal{B}_g$, and $b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ denotes the universal symplectic-Hodge basis over $\mathcal{B}_g$, then we have the following result.

Theorem 0.3 (see Theorem .50 and Definition .51)**.** *Let $\mathcal{R}_g$ be the subsheaf of $T_{\mathcal{B}_g/\mathbf{Z}}$ given by the vector fields v such that $\nabla_v \eta_j = 0$ for every $1 \leq j \leq g$. Then $\mathcal{R}_g$ is an integrable subbundle of $T_{\mathcal{B}_g/\mathbf{Z}}$ such that*

$$T_{\mathcal{B}_g/\mathcal{A}_g} \oplus \mathcal{R}_g = T_{\mathcal{B}_g/\mathbf{Z}}.$$

We then explain how the deformation theory of abelian schemes canonically yields a global trivialization $(v_{ij})_{1\leq i\leq j\leq g}$ of $\mathcal{R}_g$; these are the *higher Ramanujan vector fields* (see Section 5 for precise statements). Alternatively, these vector fields may be characterized by the following formulas.

Proposition 0.4 (see Proposition .61 and Remark .62)**.** *For every $1 \leq i \leq j \leq g$ we have*

(1) $\nabla_{v_{ij}}\omega_i = \eta_j$*,* $\nabla_{v_{ij}}\omega_j = \eta_i$*, and* $\nabla_{v_{ij}}\omega_k = 0$ *for every* $k \notin \{i, j\}$*,*
(2) $\nabla_{v_{ij}}\eta_k = 0$ *for every* $1 \leq k \leq g$*,*

and these equations completely determine v_{ij}*.*

Next, we explain in Section 6 how to construct a particular *integral solution of the higher Ramanujan equations.* Namely, for $1 \leq i \leq j \leq g$, let q_{ij} be a formal variable, and consider the ring

$$\mathbf{Z}(\!(q_{ij})\!) \coloneqq \mathbf{Z}[\![q_{11}, \ldots, q_{gg}]\!][(q_{11}\cdots q_{gg})^{-1}].$$

We obtain from Mumford's classical construction of degenerating families of abelian varieties [57], a principally polarized abelian scheme $(\hat{X}_g, \hat{\lambda}_g)$ over $\mathbf{Z}(\!(q_{ij})\!)$ which can be formally represented by the quotient

$$\hat{X}_g = \mathbf{G}_m^g / \langle (q_{1j}, \ldots, q_{gj}) \mid 1 \leq j \leq g \rangle,$$

and admits a canonical trivialization of $F^1(\hat{X}_g/\mathbf{Z}(\!(q_{ij})\!)) \cong H^0(\hat{X}_g, \Omega^1_{\hat{X}_g/\mathbf{Z}(\!(q_{ij})\!)})$ given by

$$\hat{\omega}_j = \frac{dt_j}{t_j}, \quad 1 \leq j \leq g,$$

where $t_1, \ldots, t_g$ denote the coordinates on $\mathbf{G}_m^g$.

Theorem 0.5 (see Theorem .71)**.** *Let* ∇ *be the Gauss-Manin connection on* $H^1_{\mathrm{dR}}(\hat{X}_g/\mathbf{Z}(\!(q_{ij})\!))$ *and, for* $1 \leq k \leq g$*, define*

$$\hat{\eta}_k = \nabla_{q_{kk}\frac{\partial}{\partial q_{kk}}} \hat{\omega}_k.$$

Then:

(1) the $2g$*-uple* $\hat{b}_g = (\hat{\omega}_1, \ldots, \hat{\omega}_g, \hat{\eta}_1, \ldots, \hat{\eta}_g)$ *is a symplectic-Hodge basis of* $(\hat{X}_g, \hat{\lambda}_g)$*, and*
(2) the morphism

$$\hat{\varphi}_g : \operatorname{Spec} \mathbf{Z}(\!(q_{ij})\!) \longrightarrow \mathcal{B}_g,$$

associated to $\hat{b}_g$ *by the universal property of* $\mathcal{B}_g$*, satisfies the differential equations*

$$q_{ij}\frac{\partial \hat{\varphi}_g}{\partial q_{ij}} = v_{ij} \circ \hat{\varphi}_g$$

for every $1 \leq i \leq j \leq g$*.*

In spite of the above result being purely algebraic, we shall actually prove it via analytic methods in Section 11.

At this point, let us briefly remark that it is possible to pass to a scheme-theoretic picture by considering the ring of global sections $\Gamma(\mathcal{B}_g, \mathcal{O}_{\mathcal{B}_g})$. Namely, the higher Ramanujan vector fields "extend" to derivations of $\Gamma(\mathcal{B}_g, \mathcal{O}_{\mathcal{B}_g})$, so that the composition of $\hat{\varphi}_g$ with the canonical map $\mathcal{B}_g \longrightarrow \operatorname{Spec}\Gamma(\mathcal{B}_g, \mathcal{O}_{\mathcal{B}_g})$ still satisfies the higher Ramanujan equations. Since $\mathcal{B}_g \otimes \mathbf{Z}[1/2]$ is representable by a quasi-affine scheme, little information is lost when replacing $\mathcal{B}_g$ by $\operatorname{Spec}\Gamma(\mathcal{B}_g, \mathcal{O}_{\mathcal{B}_g})$.

When $g = 1$, we shall recall how B_1 may be identified, by means of the classical theory of elliptic curves, with an open subscheme of $\mathbf{A}^3_{\mathbf{Z}[1/2]} = \operatorname{Spec} \mathbf{Z}[1/2, b_2, b_4, b_6]$. Under this isomorphism, the vector field v_{11} gets identified with

$$2b_4\frac{\partial}{\partial b_2} + 3b_6\frac{\partial}{\partial b_4} + (b_2b_6 - b_4^2)\frac{\partial}{\partial b_6}$$

(which is, up to scaling, the vector field associated to Chazy's equation (C)), and

$$\hat{\varphi}_1 = (E_2, \frac{1}{2}\theta E_2, \frac{1}{6}\theta^2 E_2).$$

We also show that $B_1 \otimes \mathbf{Z}[1/6]$ may be identified with the open subscheme Spec $\mathbf{Z}[1/6, e_2, e_4, e_6, (e_4^3 - e_6^2)^{-1}]$ of $\mathbf{A}^3_{\mathbf{Z}[1/6]}$, and that, under this isomorphism, the vector field v_{11} gets identified with the "original" vector field associated to the Ramanujan equations (R):

$$v = \frac{e_2^2 - e_4}{12}\frac{\partial}{\partial e_2} + \frac{e_2 e_4 - e_6}{3}\frac{\partial}{\partial e_4} + \frac{e_2 e_6 - e_4^2}{2}\frac{\partial}{\partial e_6}. \tag{0.1}$$

Naturally, under this identification, we have

$$\hat{\varphi}_1 = (E_2, E_4, E_6).$$

Remark 0.6. One might remark that our theory in $g = 1$ yields a curve $\hat{\varphi}_1$ with coefficients in $\mathbf{Z}(\!(q)\!)$, while Eisenstein series are actually regular at $q = 0$, i.e., $E_{2k} \in \mathbf{Z}[\![q]\!]$. To remedy this (with g arbitrary), one must work more generally with semi-abelian schemes, with logarithmic de Rham cohomology, and with smooth toroidal compactifications of $\mathcal{A}_g$, as developed in [24]. In this paper, we shall not elaborate further on this point.

0.3. **Interlude: Grothendieck's Period Conjecture.** As explained above, questions in Transcendental Number Theory constitute our main source of motivation for the study of these higher dimensional analogs of Ramanujan's equations. In order to fully motivate the precise statements of our next results, we now digress into a discussion of periods of abelian varieties and Grothendieck's conjecture on the algebraic relations between them.

Let X be an abelian variety defined over a subfield $k \subset \mathbf{C}$. By a *period* of X over k, we mean any complex number of the form

$$\int_\gamma \alpha$$

where α is an element of the first algebraic de Rham cohomology $H^1_{\mathrm{dR}}(X/k)$ and $\gamma \in H_1(X(\mathbf{C}), \mathbf{Z})$ is the class of a singular 1-cycle. We define the *field of periods* $\mathcal{P}(X/k)$ as the smallest subfield of $\mathbf{C}$ containing k and all the periods of X over k. Equivalently, $\mathcal{P}(X/k)$ may be regarded as the field of rationality of the comparison isomorphism

$$H^1_{\mathrm{dR}}(X/k) \otimes_k \mathbf{C} \xrightarrow{\sim} H^1(X(\mathbf{C}), \mathbf{C}) = \mathrm{Hom}(H_1(X(\mathbf{C}), \mathbf{Z}), \mathbf{C}).$$

A central problem in the theory of transcendental numbers is to determine, or simply to estimate, the transcendence degree over $\mathbf{Q}$ of the field of periods $\mathcal{P}(X/k)$.

In a first approach, one might observe that any algebraic cycle in some power $X^n = X \times_k \cdots \times_k X$ of X induces an algebraic relation between its periods (cf. [22] Proposition I.1.6). Broadly speaking, Grothendieck conjectured that *every* algebraic relation between periods of an abelian variety can be "explained" through algebraic cycles on its powers.

A convenient way of giving a precise formulation for Grothendieck's conjecture for abelian varieties is by means of Mumford-Tate groups. Let X be a complex abelian variety, and denote by H the $\mathbf{Q}$-Hodge structure of weight 1 with underlying $\mathbf{Q}$-vector space given by $H^1(X(\mathbf{C}), \mathbf{Q})$, and Hodge filtration F^1H given by $H^0(X, \Omega^1_{X/\mathbf{C}}) \subset H^1_{\mathrm{dR}}(X/\mathbf{C}) \cong H^1(X(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$. The decomposition $H_{\mathbf{C}} = F^1H \oplus \overline{F^1H}$ corresponds to the morphism of real algebraic groups

$$h : \mathbf{C}^\times \longrightarrow \mathrm{GL}(H_{\mathbf{R}}),$$

where $h(z)$ acts on F^1H by a homothety of ratio z^{-1}, and on $\overline{F^1H}$ by a homothety of ratio $\bar{z}^{-1}$. The *Mumford-Tate group* $\mathrm{MT}(X)$ of X is defined as the smallest $\mathbf{Q}$-algebraic subgroup of $\mathrm{GL}(H)$ such that h factors through $\mathrm{MT}(X)_{\mathbf{R}}$. It can also be interpreted as the smallest $\mathbf{Q}$-algebraic subgroup of $\mathrm{GL}(H) \times \mathbf{G}_{m,\mathbf{Q}}$ fixing all Hodge classes in twisted mixed tensor powers of the $\mathbf{Q}$-Hodge structure H (cf. [22] I.3).

The following formulation of *Grothendieck's Period Conjecture* (GPC) for abelian varieties is a specialization of the "Generalized Period Conjecture" proposed by André ([1] 23.4.1; see also [47] Historical Note pp. 40-44 and [29] footnote 10).

Conjecture 0.7 (Grothendieck-André)**.** *For any abelian variety X over a subfield $k \subset \mathbf{C}$, we have*

$$\mathrm{trdeg}_{\mathbf{Q}} \mathcal{P}(X/k) \overset{?}{\geq} \dim \mathrm{MT}(X_{\mathbf{C}}).$$

It follows from Deligne [20] (cf. [22] Corollary I.6.4) that we always have the upper bound

$$\mathrm{trdeg}_{\mathbf{Q}} \mathcal{P}(X/k) \leq \dim \mathrm{MT}(X_{\mathbf{C}}) + \mathrm{trdeg}_{\mathbf{Q}} k.$$

In particular, if k is contained in the field of algebraic numbers $\overline{\mathbf{Q}} \subset \mathbf{C}$ — the case originally considered by Grothendieck — the above conjectural inequality becomes the conjectural equality

$$\mathrm{trdeg}_{\mathbf{Q}} \mathcal{P}(X/k) \overset{?}{=} \dim \mathrm{MT}(X_{\mathbf{C}}).$$

In the case $\dim X = 1$, the Mumford-Tate group of a complex elliptic curve may be easily computed. Its dimension only depends on the existence or not of complex multiplication, and GPC predicts that

$$\mathrm{trdeg}_{\mathbf{Q}} \mathcal{P}(X/k) \overset{?}{\geq} \begin{cases} 2 & \text{if } X_{\mathbf{C}} \text{ has complex multiplication} \\ 4 & \text{otherwise.} \end{cases}$$

Even in this minimal case, GPC is not yet established in full generality — only the complex multiplication case is understood; see below. Nevertheless, an approach that has been proved fruitful for obtaining non-trivial lower bounds in the direction of GPC relies on a *modular description* of the fields of periods of elliptic curves, which we now recall.

Let E be a complex elliptic curve and let $j \in \mathbf{C}$ be its j-invariant. Then E admits a model

$$E: \quad y^2 = 4x^3 - g_2 x - g_3$$

with $g_2, g_3 \in \mathbf{Q}(j)$, and we can consider the algebraic differential forms defined over $\mathbf{Q}(j)$

$$\omega := \frac{dx}{y}, \qquad \eta := x\frac{dx}{y}.$$

They form a (symplectic-Hodge) basis of the first algebraic de Rham cohomology $H^1_{\mathrm{dR}}(E/\mathbf{Q}(j))$. If (γ, δ) is any basis of the first singular homology group $H_1(E(\mathbf{C}), \mathbf{Z})$, we may consider the periods

$$\omega_1 = \int_\gamma \omega, \quad \omega_2 = \int_\delta \omega, \quad \eta_1 = \int_\gamma \eta, \quad \eta_2 = \int_\delta \eta.$$

We may assume moreover that the basis (γ, δ) is oriented, in the sense that their topological intersection product $\gamma \cdot \delta = 1$.

The field of periods of E is given by

$$\mathcal{P}(E/\mathbf{Q}(j)) = \mathbf{Q}(j, \omega_1, \omega_2, \eta_1, \eta_2).$$

Now, observe that $\omega_1 \neq 0$ and let

$$\tau := \frac{\omega_2}{\omega_1}.$$

As the basis (γ, δ) of $H_1(E(\mathbf{C}), \mathbf{Z})$ is oriented, the complex number τ is in the Poincaré upper half-plane $\mathbf{H}$. By the classical theory of modular forms, we have

$$E_2(\tau) = 12\left(\frac{\omega_1}{2\pi i}\right)\left(\frac{\eta_1}{2\pi i}\right), \quad E_4(\tau) = 12 g_2 \left(\frac{\omega_1}{2\pi i}\right)^4, \quad E_6(\tau) = -216 g_3 \left(\frac{\omega_1}{2\pi i}\right)^6.$$

Here, we see the Eisenstein series E_{2k} as analytic functions on $\mathbf{H}$ via the change of variables $q = e^{2\pi i\tau}$.

Finally, Legendre's period relation and the definition of j show that $\mathcal{P}(E/\mathbf{Q}(j))$ is a finite extension of the field $\mathbf{Q}(2\pi i, \tau, E_2(\tau), E_4(\tau), E_6(\tau))$, and we obtain in particular

$$\operatorname{trdeg}_{\mathbf{Q}} \mathcal{P}(E/\mathbf{Q}(j)) = \operatorname{trdeg}_{\mathbf{Q}} \mathbf{Q}(2\pi i, \tau, E_2(\tau), E_4(\tau), E_6(\tau)). \tag{0.2}$$

In this way, the problem of estimating the transcendence degree of fields of periods of elliptic curves translates into the problem of estimating the transcendence degree of values of some analytic functions. Accordingly, the theorem of Nesterenko stated above asserts that, for any $\tau \in \mathbf{H}$,

$$\operatorname{trdeg}_{\mathbf{Q}} \mathbf{Q}(e^{2\pi i\tau}, E_2(\tau), E_4(\tau), E_6(\tau)) \geq 3.$$

As an immediate consequence, we obtain

$$\operatorname{trdeg}_{\mathbf{Q}} \mathbf{Q}(2\pi i, \tau, E_2(\tau), E_4(\tau), E_6(\tau)) \geq \operatorname{trdeg}_{\mathbf{Q}} \mathbf{Q}(E_2(\tau), E_4(\tau), E_6(\tau)) \geq 2$$

for any $\tau \in \mathbf{H}$. Equivalently, by equation (0.2), for any complex elliptic curve E, we obtain the uniform bound

$$\operatorname{trdeg}_{\mathbf{Q}} \mathcal{P}(E/\mathbf{Q}(j)) \geq 2,$$

which is sharp when E has complex multiplication. This last result had already been previously established by Chudnovsky (cf. [15]) via elliptic methods.[4]

0.4. Analytic higher Ramanujan equations, periods of abelian varieties, and transcendence.

In this paper, we also generalize the modular description (0.2). For this, we consider a complex analytic avatar of $\hat{\varphi}_g$: an analytic map

$$\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C}),$$

parametrized in the Siegel upper half-space

$$\mathbf{H}_g := \{\tau = (\tau_{kl})_{1 \leq k,l \leq g} \in M_{g\times g}(\mathbf{C}) \mid \tau^{\mathsf{T}} = \tau, \ \operatorname{Im} \tau > 0\},$$

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[4] We should also point out that the modular parameter $e^{2\pi i\tau}$, ignored in our discussion, can also be seen as a period. Namely, it is a period of a certain 1-motive naturally attached to E. We refer to [3] (cf. [1] 23.4.3) for further discussion on these matters.

which, loosely speaking, coincides with $\hat{\varphi}_g$ through the change of variables $q_{kl} = e^{2\pi i \tau_{kl}}$. For instance, under the above identification of $B_1 \otimes \mathbf{Z}[1/6]$ with an open subscheme of $\mathbf{A}^3_{\mathbf{Z}[1/6]}$, the analytic map $\varphi_1 : \mathbf{H}_1 = \mathbf{H} \longrightarrow B_1(\mathbf{C})$ is given by

$$\tau \longmapsto (E_2(\tau), E_4(\tau), E_6(\tau)).$$

In other words, $\hat{\varphi}_g$ should be regarded as the "q-expansion" of φ_g.

Now, for any $\tau \in \mathbf{H}_g$, let X_τ be the complex abelian variety given by the (polarizable) complex torus $\mathbf{C}^g/(\mathbf{Z}^g + \tau\mathbf{Z}^g)$. It admits a canonical principal polarization λ_τ induced by the Riemann form

$$\begin{aligned} \mathbf{C}^g \times \mathbf{C}^g &\longrightarrow \mathbf{R} \\ (v, w) &\longmapsto \operatorname{Im}(\overline{v}^{\mathsf{T}}(\operatorname{Im}\tau)^{-1} w). \end{aligned}$$

Let k_τ be the field of definition of (X_τ, λ_τ); formally, k_τ is the residue field of the point in the (coarse) moduli space of principally polarized abelian varieties A_g given by the isomorphism class of (X_τ, λ_τ).

Theorem 0.8 (see Theorem .149)**.** *For any $\tau \in \mathbf{H}_g$, the field of periods $\mathcal{P}(X_\tau/k_\tau)$ is a finite extension of $\mathbf{Q}(2\pi i, \tau, \varphi_g(\tau))$.*

Here, $\mathbf{Q}(2\pi i, \tau, \varphi_g(\tau))$ is defined as the residue field in $\mathbf{A}^1_{\mathbf{Q}} \times_{\mathbf{Q}} \operatorname{Sym}_{g,\mathbf{Q}} \times_{\mathbf{Q}} B_{g,\mathbf{Q}}$ of the complex point $(2\pi i, \tau, \varphi_g(\tau))$, where Sym_g denotes the group scheme of symmetric matrices of order $g \times g$.

It follows from the above theorem that

$$\operatorname{trdeg}_{\mathbf{Q}} \mathcal{P}(X_\tau/k_\tau) = \operatorname{trdeg}_{\mathbf{Q}} \mathbf{Q}(2\pi i, \tau, \varphi_g(\tau)).$$

This generalized modular description raises the question of whether it is possible to adapt Nesterenko's methods to this higher dimensional setting; see Paragraph 0.5 below. This problem leads us to the study of the *higher Ramanujan foliation*, namely, the holomorphic foliation on $B_g(\mathbf{C})$ generated by the higher Ramanujan vector fields. We prove the following result.

Theorem 0.9 (see Theorem .203)**.** *Every leaf of the higher Ramanujan foliation on $B_g(\mathbf{C})$ is Zariski-dense in $B_{g,\mathbf{C}}$.*

This property of a foliation plays an important role, at least in the case in which leaves are one dimensional (where it implies Nesterenko's D-property), in the "multiplicity estimates" appearing in applications of differential equations to transcendental number theory (cf. [6], [59], [60]).

The Zariski-density of the image of $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ in $B_{g,\mathbf{C}}$ also implies the *a priori* stronger result that its graph

$$\{(\tau, \varphi_g(\tau)) \in \operatorname{Sym}_g(\mathbf{C}) \times B_g(\mathbf{C}) \mid \tau \in \mathbf{H}_g\}$$

is Zariski-dense in $\operatorname{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} B_{g,\mathbf{C}}$. This can be interpreted as a "functional version" of GPC: roughly, it says that there is no algebraic relation simultaneously satisfied by the periods of every (principally polarized) abelian variety other than the relations given by the polarization data.[5]

We shall also use our Zariski-density result to establish a relation between our work and that of Bertrand and Zudilin [5] concerning derivatives of Siegel modular functions.

[5]Such "functional version" is an example of a statement that must hold if GPC is true. This follows from the existence of $\tau \in \mathbf{H}^g \cap \operatorname{Sym}_g(\overline{\mathbf{Q}})$ such that $\dim \operatorname{MT}(X_\tau) = 2g^2 + g + 1$ (or, equivalently, $\operatorname{MT}(X_\tau) = \operatorname{GSp}_{2g,\mathbf{Q}}$); cf. [73].

Proposition 0.10 (see Paragraph 15.4)**.** *The field of functions* $\mathbf{Q}(B_{g,\mathbf{Q}})$*, identified with a field of meromorphic functions on* $\mathbf{H}_g$ *via* φ_g*, is a finite extension of the differential field generated by the Siegel modular functions defined over* $\mathbf{Q}$*.*

In particular, the generalization of Mahler's result [49] on the algebraic independence of the holomorphic functions τ, $e^{2\pi i\tau}$, $E_2(\tau)$, $E_4(\tau)$, and $E_6(\tau)$, of $\tau \in \mathbf{H}$, obtained by Bertrand and Zudilin [4] in the context of Siegel modular functions, also holds in our context: the set

$$\{(\tau, q(\tau), \varphi_g(\tau)) \in \mathrm{Sym}_g(\mathbf{C}) \times \mathrm{Sym}_g(\mathbf{C}) \times B_g(\mathbf{C}) \mid \tau \in \mathbf{H}_g\}$$

is Zariski-dense in $\mathrm{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} \mathrm{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} B_{g,\mathbf{C}}$, where $q(\tau) := (e^{2\pi i \tau_{kl}})_{1\leq k,l\leq g}$.

Our proof of Theorem .9 will rely on a characterization of the leaves of the higher Ramanujan foliation in terms of an action by $\mathrm{Sp}_{2g}(\mathbf{C})$. In fact, from the complex analytic viewpoint, the complex manifold $B_g(\mathbf{C})$ and the higher Ramanujan vector fields admit a simple description in terms of Lie groups.

Namely, we shall explain in Section 14 how to realize $B_g(\mathbf{C})$ as a domain (in the analytic topology) of the quotient manifold $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$.

Theorem 0.11 (see Theorem .179)**.** *Under this identification:*

(1) The vector field v_{kl} *is induced by the left invariant holomorphic vector field on* $\mathrm{Sp}_{2g}(\mathbf{C})$ *associated to*

$$\frac{1}{2\pi i}\begin{pmatrix} 0 & \mathbf{E}^{kl} \\ 0 & 0 \end{pmatrix} \in \mathrm{Lie}\,\mathrm{Sp}_{2g}(\mathbf{C}).$$

(2) The map $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ *is given by*

$$\tau \longmapsto \left[\begin{pmatrix} \mathbf{1}_g & \tau \\ 0 & \mathbf{1}_g \end{pmatrix}\right] \in \mathrm{Sp}_{2g}(\mathbf{Z})\backslash\mathrm{Sp}_{2g}(\mathbf{C}).$$

In the above statement, $\mathbf{E}^{kl}$ is the symmetric matrix of order $g \times g$ whose entry in the kth row and lth column (resp. lth row and kth column) is 1, and whose all other entries are 0, and $\mathbf{1}_g$ denotes the identity matrix of order $g \times g$.

This result enables us to obtain every leaf of the higher Ramanujan foliation as the image of a holomorphic map $\varphi_\delta : U_\delta \longrightarrow B_g(\mathbf{C})$ defined on some explicitly defined open subset $U_\delta \subset \mathbf{H}_g$ obtained from φ_g via a "twist" by some element $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$.

In the case $g = 1$, the above twisting procedure may be illustrated as follows. Let

$$\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C}),$$

let $U_\delta = \{\tau \in \mathbf{H} \mid c\tau + d \neq 0\}$, and define a holomorphic map $\varphi_\delta : U_\delta \longrightarrow B_1(\mathbf{C}) \subset \mathbf{C}^3$ by

$$\varphi_\delta(\tau) = \left((c\tau+d)^2 E_2(\tau) + \frac{12c}{2\pi i}(c\tau+d), (c\tau+d)^4 E_4(\tau), (c\tau+d)^6 E_6(\tau)\right)$$

Then one may easily check that φ_δ satisfy the differential equation

$$\frac{1}{2\pi i}\frac{d\varphi_\delta}{d\tau} = (c\tau+d)^{-2} v \circ \varphi_\delta$$

where v is the classical Ramanujan vector field defined by (0.1).

0.5. **The Hilbert-Blumenthal case and an algebraic independence conjecture.** Parallel to the above geometric generalization of the Ramanujan equations in terms of a Siegel moduli problem, we may develop similar theories concerning polarized abelian varieties with extra endomorphism structure, which has the effect of producing moduli spaces with fewer dimensions. This might be advantageous for applications to transcendental numbers, which should necessarily take "special subvarieties" into account, as we shall explain below.

To illustrate this point, we consider abelian varieties with *real multiplication*. Namely, let F be a totally real number field of degree $g \geq 1$, and denote by R its ring of integers. Then, an R-multiplication (with Rapoport's condition) on a principally polarized abelian variety (X, λ) is a morphism of rings $m : R \longrightarrow \operatorname{End}_k X$ invariant by the Rosatti involution defined by λ, and for which $F^1(X/k)$ becomes a free $k \otimes_{\mathbf{Z}} R$-module of rank 1. The moduli problem of principally polarized abelian varieties endowed with an R-multiplication is an example of a Hilbert-Blumenthal moduli problem.

Accordingly, we shall also consider a smooth Deligne-Mumford moduli stack $\mathcal{B}_F$ over $\operatorname{Spec} \mathbf{Z}$ of relative dimension $3g$, classifying principally polarized abelian varieties with an R-multiplication and a symplectic-Hodge basis "compatible" with it. Here, we also have that $\mathcal{B}_F \otimes \mathbf{Z}[1/2]$ is representable by a quasi-affine smooth scheme B_F over $\mathbf{Z}[1/2]$.

As in the Siegel case, we shall also construct a family of higher Ramanujan vector fields on $\mathcal{B}_F$, and a canonical analytic solution

$$\varphi_F : \mathbf{H}^g \longrightarrow B_F(\mathbf{C})$$

with integral "q-expansion" $\hat{\varphi}_F$ (see Paragraphs 5.3, 5.7, 6.4, 11.5, and 11.6 for precise statements). Moreover, we shall also establish a precise relation between the values of φ_F with fields of periods of principally polarized abelian varieties with R-multiplication (Theorem .150).

Remark 0.12. The Siegel and Hilbert-Blumenthal higher Ramanujan equations are constructed by a similar procedure, and satisfy various natural compatibilities (see Remarks .40, .67, and .146). This observation hints to the existence of an underlying theory of higher Ramanujan equations attached to more general Shimura varieties (cf. Section 14). We refer to Movasati [55] for a Hodge-theoretic approach to these questions, which also allows to consider examples unrelated with abelian varieties (cf. Scholium 0.6.1 below).

In the case of abelian surfaces, we formulate the following algebraic independence conjecture.

Conjecture 0.13. *Let F be a real quadratic number field. Then, for every $\tau \in \mathbf{H}^2 \setminus \mathrm{HZ}_F$, we have*

$$\operatorname{trdeg}_{\mathbf{Q}} \mathbf{Q}(\varphi_F(\tau)) \overset{?}{\geq} 3.$$

Here, HZ_F is a countable union of certain special divisors of $\mathbf{H}^2$, first introduced and studied by Hirzebruch and Zagier (see Paragraph 13.1), classifying abelian surfaces with quaternionic multiplication.

The above statement is a higher dimensional analog of the uniform bound

$$\operatorname{trdeg}_{\mathbf{Q}} \mathbf{Q}(E_2(\tau), E_4(\tau), E_6(\tau)) \geq 2$$

for $\tau \in \mathbf{H}$, which can be obtained, as explained above, as a corollary of Nesterenko's theorem. Correspondingly, we shall prove that Conjecture .13 implies Grothendieck's Period Conjecture for complex multiplication abelian surfaces; for instance, by considering the Jacobian of the curve $y^2 = 1 - x^5$, we see that such conjecture for $F = \mathbf{Q}(\sqrt{5})$ contains the classical conjecture on the algebraic independence of π, $\Gamma(1/5)$, and $\Gamma(2/5)$ (see Paragraph 13.3).

A natural strategy to attack Conjecture .13 would consist in adapting Nesterenko's method to prove Theorem .1 to our geometric context, and in generalizing it in "two variables". A first step in this program was taken in [25], where we show that Nesterenko's method, still in one variable, can be cast in purely geometric terms, not relying on the Taylor expansion of explicitly defined analytic functions.

0.6. **Scholia.**

0.6.1. As acknowledged above, our definition of the moduli stack $\mathcal{B}_g$ was inspired by Movasati's point of view on the Ramanujan vector field in terms of the Gauss-Manin connection on the de Rham cohomology of the universal elliptic curve (cf. [54] 4.2), which corresponds to the case $g = 1$ of our construction.

After I completed a first version this article, H. Movasati has kindly indicated to me that a number of our results and constructions has some overlap with his article [55]. In this work, he considers complex analytic spaces U classifying lattices in maximal totally real subspaces of some given complex vector space V_0 (i.e., subgroups of V_0 generated by a $\mathbf{C}$-basis of V_0) satisfying suitable compatibility conditions with a fixed Hodge filtration $F_0^\bullet$ on V_0, and a fixed polarization ψ_0; these spaces come equipped with a natural analytic right action of the complex algebraic group

$$G_0 = \{g \in \mathrm{GL}(V_0) \mid gF_0^i = F_0^i \text{ for every } i, \text{ and } g^*\psi_0 = \psi_0\}.$$

For the particular case where $V_0 = \mathbf{C}^{2g}$,

$$F_0^\bullet = (F_0^0 = V_0 \supset F_0^1 = \mathbf{C}^g \times \{0\} \supset F_0^2 = 0),$$

and ψ_0 is the standard (complex) symplectic form ([55] 5.1), the space U becomes the analytic moduli space $B_g(\mathbf{C})$, investigated in the present article. Of course, the algebraic group G_0 coincides with our P_g, and the action of G_0 on U gets identified with the action of P_g on $B_g(\mathbf{C})$ under $U \cong B_g(\mathbf{C})$.

In [55] 3.2, Movasati also describes U as a quotient $\Gamma_{\mathbf{Z}} \backslash P$, where P is the space of "period matrices" and $\Gamma_{\mathbf{Z}}$ is some explicitly defined discrete group. In our particular case, P may be identified with our $\mathbf{B}_g$ (cf. Proposition .176) and $\Gamma_{\mathbf{Z}} = \mathrm{Sp}_{2g}(\mathbf{Z})$. Moreover, the map $\mathbf{H}_g \longrightarrow P$ defined in [55] p. 584 coincides with our $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ constructed via the universal property of $B_g(\mathbf{C})$.

In his article, Movasati explicitly states the problem of algebraizing U — i.e., of finding the algebraic variety T over $\overline{\mathbf{Q}}$, in his notation — and the action of G_0. This is solved "by definition" in our construction, where T is here called $B_{g,\overline{\mathbf{Q}}}$. Note that our methods also yield that $B_{g,\overline{\mathbf{Q}}}$ is quasi-affine, which was previously conjectured by Movasati. On his web page[6], Movasati also indicates a construction of what we call "higher Ramanujan vector fields" with slightly different normalizations.

[6]See "What is a Siegel quasi-modular form?" in http://w3.impa.br/~hossein/WikiHossein/WikiHossein.html.

0.6.2. The moduli stacks $\mathcal{B}_g$, or variants of it, have also appeared elsewhere in the literature in different contexts, most notably in relation with sheaf theoretic reformulations of Shimura's theory of *nearly holomorphic modular forms*, as in Urban [78] and Liu [48].

For instance, in [48], Paragraph 2.1, the parabolic subgroup $\mathbf{Q}$ of GSp_{2g}, and the $\mathbf{Q}$-torsor $T_{\mathcal{H}}^{\times}$, used in the definition of *automorphic sheaves* are "up to similitude" versions of our P_g and $\mathcal{B}_g$. Moreover, the definition of the polynomial q-expansions in [48], Paragraph 2.6, involves the construction of $(\omega_{\mathrm{can}}, \delta_{\mathrm{can}})$, which coincides with our $\hat{b}_g$ (see Theorem .5 above). In [48], it is stated that $(\omega_{\mathrm{can}}, \delta_{\mathrm{can}})$ belongs to $T_{\mathcal{H}}^{\times}$, and that this can be checked analytically; this is proved in details in Section 11 below.

The connections between the present work and the theory of nearly holomorphic modular forms should come as no surprise. Indeed, in the case $g = 1$, recall that the differential ring of quasimodular forms is isomorphic to the differential ring of nearly holomorphic modular forms endowed with the Maass-Shimura differential operator (cf. [80] 5). Using the results of [78], this can be explained geometrically as follows.

To fix ideas, we ignore the "condition at infinity", i.e., we work with "weakly holomorphic forms", although [78] does consider it; otherwise, see Remark .6 above. Let $\mathcal{H}$ be the first de Rham cohomology of the universal elliptic curve over $\mathcal{A}_{1,\mathbf{C}}$, and let $\mathcal{F}$ be its Hodge subbundle. It is shown in [78] that the ring of nearly holomorphic modular forms is isomorphic to $H^0(\mathcal{A}_{1,\mathbf{C}}, \mathrm{Sym}\,\mathcal{H})$, and that the Maass-Shimura operator corresponds to the $\mathbf{C}$-derivation ∂ on this ring induced by Gauss-Manin connection on $\mathcal{H}$ together with the Kodaira-Spencer isomorphism $\Omega^1_{\mathcal{A}_{1,\mathbf{C}}/\mathbf{C}} \cong \mathrm{Sym}^2 \mathcal{F}$. On the other hand, $H^0(\mathcal{A}_{1,\mathbf{C}}, \mathrm{Sym}\,\mathcal{H})$ can be shown to be isomorphic to $H^0(B_{1,\mathbf{C}}, \mathcal{O}_{B_{1,\mathbf{C}}})$, with ∂ being induced by the Ramanujan vector field v_{11} on $B_{1,\mathbf{C}}$ (see also [54] Sections 6 and 7).

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Terminology and conventions

0.8. By a *vector bundle* over a scheme U we mean a locally free sheaf $\mathcal{E}$ over U of finite rank. A *line bundle* is a vector bundle of rank 1. A *subbundle* of $\mathcal{E}$ is a subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that $\mathcal{F}$ and $\mathcal{E}/\mathcal{F}$ are also vector bundles, that is, $\mathcal{F}$ is locally a direct factor of $\mathcal{E}$. If $\mathcal{E}$ has constant rank r, by a *basis* of $\mathcal{E}$ over U we mean an ordered family of r global sections of $\mathcal{E}$ that generate this sheaf as an $\mathcal{O}_U$-module. The *dual* of a vector bundle $\mathcal{E}$ is the vector bundle $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_U}(\mathcal{E}, \mathcal{O}_U)$.

0.9. Let U be a scheme. By an *abelian scheme* over U, we mean a proper and smooth group scheme $p : X \longrightarrow U$ over U with geometrically connected fibers. The group law of X over U is commutative (cf. [58] Corollary 6.5) and will be denoted additively. A *morphism of abelian schemes* over U is a morphism of U-group schemes.

When p is projective, the relative Picard functor $\mathrm{Pic}_{X/U}$ is representable by a group scheme over U ([9] Chapter 8). Then, the open group subscheme X^t of $\mathrm{Pic}_{X/U}$, whose geometric points correspond to line bundles some power of which are algebraically equivalent to zero, is a projective abelian scheme over U, called the *dual abelian scheme*; we denote its structural morphism by $p^t : X^t \longrightarrow U$. There is a canonical biduality isomorphism $X \xrightarrow{\sim} X^{tt}$ (cf. [9] 8.4 Theorem 5). The formation of both the dual abelian scheme and the biduality isomorphism is compatible with every base change in U. The universal line bundle over $X \times_U X^t$, the so-called *Poincaré line bundle*, will be denoted by $\mathcal{P}_{X/U}$.

A *principal polarization* on a projective abelian scheme X over U is an isomorphism of U-group schemes $\lambda : X \longrightarrow X^t$ satisfying the equivalent conditions (cf. [58] 6.2 and [23] 1.4)

(1) λ is symmetric (i.e. $\lambda = \lambda^t$ under the biduality isomorphism $X \cong X^{tt}$) and $(\mathrm{id}_X, \lambda)^*\mathcal{P}_{X/U}$ is relatively ample over U.
(2) Étale locally over U, λ is *induced by a line bundle on* X (cf. [58] Definition 6.2) relatively ample over U.

A *principally polarized abelian scheme* over U is a couple (X, λ), where X is a projective abelian scheme over U and λ is a principal polarization on X.

0.10. If $X \longrightarrow S$ is a smooth morphism of schemes, the dual $\mathcal{O}_X$-module of the sheaf of relative differentials $\Omega^1_{X/S}$ (i.e. the sheaf of $\mathcal{O}_S$-derivations of $\mathcal{O}_X$) is denoted by $T_{X/S}$. It is a vector bundle over X whose rank is given by the relative dimension of $X \longrightarrow S$. If $S = \mathrm{Spec}\, R$ is affine, we denote $T_{X/S} = T_{X/R}$.

The *Lie bracket* $[\ ,\ ] : T_{X/S} \times T_{X/S} \longrightarrow T_{X/S}$ is defined on derivations by $[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - \theta_2 \circ \theta_1$.

If S is a scheme, and $f : X \longrightarrow Y$ is a morphism of smooth S-schemes, then there is a canonical morphism of $\mathcal{O}_X$-modules $f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S}$. Further, as $Y \longrightarrow S$ is smooth, the canonical morphism of $\mathcal{O}_X$-modules $f^*T_{Y/S} \longrightarrow (f^*\Omega^1_{Y/S})^\vee$ is an isomorphism. We denote by

$$Df : T_{X/S} \longrightarrow f^*T_{Y/S}$$

the dual $\mathcal{O}_X$-morphism of $f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S}$ after the identification $(f^*\Omega^1_{Y/S})^\vee \cong f^*T_{Y/S}$. If f is smooth, we have an exact sequence of vector bundles over X

$$0 \longrightarrow T_{X/Y} \longrightarrow T_{X/S} \xrightarrow{Df} f^*T_{Y/S} \longrightarrow 0.$$

0.11. If U is any scheme, the category of U-schemes (resp. U-group schemes) is denoted by $\mathsf{Sch}_{/U}$ (resp. $\mathsf{GpSch}_{/U}$). The category of sets is denoted by Set. If C is any category, its opposite category is denoted by C^{op}.

0.12. We shall use the language of *categories fibered in groupoids* and the elements of the theory of *Deligne-Mumford stacks* ([21] Paragraph 4). We follow the same conventions and terminology of [65]. In particular, if S is a scheme, whenever we talk about a *stack* over the category of S-schemes

$\mathsf{Sch}_{/S}$ (cf. [65] Definition 4.6.1), or simply a stack over S (or an S-stack), we shall always assume that $\mathsf{Sch}_{/S}$ is endowed with the *étale topology*.

In view of [65] Corollary 8.3.5, by an *algebraic space* over a scheme S we mean a Deligne-Mumford stack $\mathcal{X}$ over S such that for any S-scheme U the fiber category $\mathcal{X}(U)$ is discrete (i.e. any automorphism is the identity).

The *étale site* of a Deligne-Mumford stack $\mathcal{X}$ is denoted by $\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$ (cf. [65] Paragraph 9.1). We recall that the objects of the underlying category of $\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$ are *étale schemes over* $\mathcal{X}$, that is, pairs (U,u) where U is an S-scheme and $u : U \longrightarrow \mathcal{X}$ is an étale S-morphism; morphisms are given by couples $(f,f^b) : (U',u') \longrightarrow (U,u)$, where $f : U' \longrightarrow U$ is an S-morphism and $f^b : u' \longrightarrow u\circ f$ is an isomorphism of functors $U' \longrightarrow \mathcal{X}$. Coverings in $\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$ are given by families of morphisms $\{(f_i,f_i^b) : (U_i,u_i) \longrightarrow (U,u)\}_{i\in I}$ such that $\{f_i : U_i \longrightarrow U\}_{i\in I}$ is an étale covering of U.

The structural sheaf on $\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$, which to any (U,u) associates the ring $\Gamma(U,\mathcal{O}_U)$, is denoted by $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$. We recall that an $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$-module $\mathcal{F}$ is said to be *quasi-coherent* if $u^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module for any object (U,u) of $\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$.

By a *vector bundle* over a Deligne-Mumford stack $\mathcal{X}$, we mean a locally free $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$-module of finite rank. We define subbundles, bases, and duals as in 0.8.

0.13. Sheaves of differentials and tangent sheaves can also be defined for Deligne-Mumford stacks. If $\mathcal{X}$ is a Deligne-Mumford stack over S, we define a presheaf of $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$-modules $\Omega^1_{\mathcal{X}/S}$ on $\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$ by

$$\Gamma((U,u),\Omega^1_{\mathcal{X}/S}) \coloneqq \Gamma(U,\Omega^1_{U/S})$$

for any étale scheme (U,u) over $\mathcal{X}$; restriction maps are defined in the obvious way. Since, for any étale morphism of S-schemes $f : U' \longrightarrow U$, the induced morphism $f^*\Omega^1_{U/S} \longrightarrow \Omega^1_{U'/S}$ is an isomorphism of $\mathcal{O}_{U'}$-modules, and for any S-scheme U the sheaf $\Omega^1_{U/S}$ is a quasi-coherent $\mathcal{O}_U$-module, we see that $\Omega^1_{\mathcal{X}/S}$ is in fact a quasi-coherent sheaf over $\mathcal{X}$ (cf. [65] Lemma 4.3.3). Note that $u^*\Omega^1_{\mathcal{X}/S} = \Omega^1_{U/S}$ for any étale scheme (U,u) over $\mathcal{X}$.

Let $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of Deligne-Mumford stacks over S. If φ is representable by schemes, then there exists a unique morphism of $\mathcal{O}_{\mathcal{Y}}$-modules $\Omega^1_{\mathcal{Y}/S} \longrightarrow \varphi_*\Omega^1_{\mathcal{X}/S}$ inducing, for any étale scheme (V,v) over $\mathcal{Y}$, the canonical morphism $\Omega^1_{V/S} \longrightarrow \varphi'_*\Omega^1_{U/S}$, where (U,u) (resp. $\varphi' : U \longrightarrow V$) denotes the étale scheme over $\mathcal{X}$ (resp. the morphism of S-schemes) obtained from (V,v) (resp. φ) by base change. If, moreover, φ is quasi-compact and quasi-separated, by adjointness (cf. [65] Proposition 9.3.6), we obtain a morphism of $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$-modules

$$\varphi^*\Omega^1_{\mathcal{Y}/S} \longrightarrow \Omega^1_{\mathcal{X}/S}. \tag{0.3}$$

We then define a quasi-coherent $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$-module

$$\Omega^1_{\mathcal{X}/\mathcal{Y}} \coloneqq \operatorname{coker}(\varphi^*\Omega^1_{\mathcal{Y}/S} \longrightarrow \Omega^1_{\mathcal{X}/S}).$$

Recall that a Deligne-Mumford stack $\mathcal{X}$ over S is *smooth* if there exists a surjective étale S-morphism $u : U \longrightarrow \mathcal{X}$ such that U is smooth over S (see [21] page 100). In this case, $\Omega^1_{\mathcal{X}/S}$ is a vector bundle over $\mathcal{X}$. We define $T_{\mathcal{X}/S}$ as the dual $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$-module of $\Omega^1_{\mathcal{X}/S}$. If $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ is a morphism of smooth Deligne-Mumford stacks over S representable by smooth schemes, then $\Omega^1_{\mathcal{X}/\mathcal{Y}}$

is a vector bundle over $\mathcal{X}$, and its dual is denoted by $T_{\mathcal{X}/\mathcal{Y}}$. Moreover, in this case, the morphism in (0.3) is injective and induces a surjective morphism of $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$-modules $D\varphi : T_{\mathcal{X}/S} \longrightarrow \varphi^* T_{\mathcal{Y}/S}$. We thus obtain an exact sequence of quasi-coherent $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$-modules

$$0 \longrightarrow T_{\mathcal{X}/\mathcal{Y}} \longrightarrow T_{\mathcal{X}/S} \xrightarrow{D\varphi} \varphi^* T_{\mathcal{Y}/S} \longrightarrow 0.$$

0.14. Let M be a complex manifold. Every holomorphic vector bundle $\pi : V \longrightarrow M$ may be seen as a (commutative) relative complex Lie group over M. We shall occasionally identify V with its corresponding locally free sheaf of $\mathcal{O}_M$-modules of holomorphic sections of π.

0.15. If R is any ring, we denote the constant sheaf with values in R over some complex manifold M by R_M. A *local system* of R-modules over M is a locally constant sheaf L of R-modules over M. The *dual* of L is denoted by $L^\vee := \mathcal{H}om_R(L, R_M)$.

The *étalé space* of a local system of R-modules L over M will be denoted by $E(L)$; this is a topological covering space over M whose fiber at each $p \in M$ is naturally identified to L_p.

0.16. Let $m, n \geq 1$ be integers. The set of matrices of order $m \times n$ over a ring R is denoted by $M_{m\times n}(R)$. We shall frequently adopt a block notation for elements in $M_{2n\times 2n}(R)$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (A\ B\ ;\ C\ D),$$

where $A, B, C, D \in M_{n\times n}(R)$.

The transpose of a matrix $M \in M_{m\times n}(R)$ is denoted by $M^{\mathsf{T}} \in M_{n\times m}(R)$. For $1 \leq i \leq n$, $\mathbf{e}_i \in M_{n\times 1}(R)$ denotes for the column vector whose entry in the ith line is 1, and all the others are 0. The identity matrix in $M_{n\times n}(R)$ is denoted by $\mathbf{1}_n$. For every $1 \leq i \leq j \leq n$, we denote by $\mathbf{E}^{ij}$ the unique symmetric matrix $(\mathbf{E}^{ij}_{kl})_{1\leq k,l\leq n} \in M_{n\times n}(R)$ such that

$$\mathbf{E}^{ij}_{kl} = \begin{cases} 1 & \text{if } (k,l) = (i,j) \text{ or } (k,l) = (j,i) \\ 0 & \text{otherwise.} \end{cases}$$

The *symmetric group* Sym_n is the subgroup scheme of $M_{n\times n}$ consisting of symmetric matrices. The *symplectic group* Sp_{2n} is defined as the subgroup scheme of GL_{2n} such that for every affine scheme $V = \operatorname{Spec} R$

$$\mathrm{Sp}_{2g}(V) = \{M \in \mathrm{GL}_{2n}(R) \mid MJM^{\mathsf{T}} = J\}$$

where

$$J := \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}.$$

Remark 0.14. As $J^2 = -\mathbf{1}_{2n}$, the condition $MJM^{\mathsf{T}} = J$ is equivalent to $M^{-1} = -JM^{\mathsf{T}}J$; thus $MJM^{\mathsf{T}} = J$ if and only if $M^{\mathsf{T}}JM = J$. In particular, if we write

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n\times 2n}(R)$$

for some $A, B, C, D \in M_{n\times n}(R)$, then M is in $\mathrm{Sp}_{2n}(R)$ if and only if one of the following two conditions is satisfied

(1) $AB^{\mathsf{T}} = BA^{\mathsf{T}}$, $CD^{\mathsf{T}} = DC^{\mathsf{T}}$, and $AD^{\mathsf{T}} - BC^{\mathsf{T}} = \mathbf{1}_n$.
(2) $A^{\mathsf{T}}C = C^{\mathsf{T}}A$, $B^{\mathsf{T}}D = D^{\mathsf{T}}B$, and $A^{\mathsf{T}}D - C^{\mathsf{T}}B = \mathbf{1}_n$.

Finally, the *Siegel parabolic subgroup* P_n of Sp_{2n} consists of matrices $(A\ B\ ;\ C\ D)$ in Sp_{2n} such that $C = 0$.

0.17. Let K be a subfield of $\mathbf{C}$ and X be an algebraic variety over K (i.e. a reduced separated scheme of finite type over K). For any complex point $\overline{x} : \operatorname{Spec} \mathbf{C} \longrightarrow X$, if $x \in X$ denotes the point in the image of $\overline{x}$, and $k(x)$ denotes its residue field, we put

$$K(\overline{x}) \coloneqq k(x),$$

and we call it the *field of definition* of $\overline{x}$ in X. Let us remark that

$$\mathrm{trdeg}_K K(\overline{x}) = \min\{\dim Y \mid Y \text{ is an integral closed } K\text{-subscheme of } X \text{ such that } \overline{x} \in Y(\mathbf{C})\}.$$

List of frequently used notation

The following list describes several symbols that will be later used within the body of the article, as well as their first page of occurrence.

$\langle\ ,\ \rangle_\lambda$	symplectic form on $H^1_{\mathrm{dR}}(X/U)$ induced by a principal polarization $\lambda : X \longrightarrow X^t$ of an abelian scheme X over U, page 23
$\mathcal{A}_F$	moduli stack over Spec $\mathbf{Z}$ of principally polarized abelian schemes with R-multiplication, page 28
$\mathcal{A}_g$	moduli stack over Spec $\mathbf{Z}$ of principally polarized abelian schemes of relative dimension g, page 28
A_F	coarse moduli scheme over Spec $\mathbf{Z}$ of principally polarized abelian varieties with R-multiplication, page 91
A_g	coarse moduli scheme over Spec $\mathbf{Z}$ of principally polarized abelian varieties of dimension g, page 91
$\mathcal{B}_F$	moduli stack over Spec $\mathbf{Z}$ of principally polarized abelian schemes with R-multiplication endowed with a symplectic-Hodge basis (see Definition .34), page 30
$\mathcal{B}_g$	moduli stack over Spec $\mathbf{Z}$ of principally polarized abelian schemes of relative dimension g endowed with a symplectic-Hodge basis (see Definition .28), page 29
B_F	smooth quasi-affine scheme over Spec $\mathbf{Z}[1/2]$ representing $\mathcal{B}_F \otimes \mathbf{Z}[1/2]$, page 50
b_F	universal symplectic-Hodge basis over $\mathcal{B}_F$, page 43
B_g	smooth quasi-affine scheme over Spec $\mathbf{Z}[1/2]$ representing $\mathcal{B}_g \otimes \mathbf{Z}[1/2]$, page 50
b_g	universal symplectic-Hodge basis over $\mathcal{B}_g$, page 41
comp	comparison isomorphism between de Rham and Betti cohomology, page 67
D	different ideal of a totally real number field F of degree g over $\mathbf{Q}$, page 25
$\mathcal{F}_F$	Hodge subbundle of $\mathcal{H}_F$, page 40
$\mathcal{F}_g$	Hodge subbundle of $\mathcal{H}_g$, page 40
F	totally real number field of degree g over $\mathbf{Q}$, page 25
$F^1(X/U)$	Hodge subbundle of $H^1_{\mathrm{dR}}(X/U)$ for an abelian scheme X over U, page 23
$\hat{\varphi}_F$	solution of the higher Ramanujan equations over $\mathcal{B}_F$ defined on Spec $\mathbf{Z}(\!(q^{r_i})\!)$, page 49
$\hat{\varphi}_g$	solution of the higher Ramanujan equations over $\mathcal{B}_g$ defined on Spec $\mathbf{Z}(\!(q_{ij})\!)$, page 46
φ_F	analytic solution of the higher Ramanujan equations over $B_F(\mathbf{C})$ defined on $\mathbf{H}^g$, page 88
φ_g	analytic solution of the higher Ramanujan equations over $B_g(\mathbf{C})$ defined on $\mathbf{H}_g$, page 81
$\mathbf{H}^g$	gth Cartesian power of the Poincaré upper half-plane $\mathbf{H}$, page 70
$\mathbf{H}_g$	Siegel upper half-space, page 64
$\mathcal{H}^i_{\mathrm{dR}}(X/M)$	ith analytic de Rham cohomology sheaf of a complex torus X over M, page 66
$\mathcal{H}_F$	vector bundle over $\mathcal{A}_F$ given by the first de Rham cohomology of the "universal abelian scheme" over $\mathcal{A}_F$, page 36

$\mathcal{H}_g$	vector bundle over $\mathcal{A}_g$ given by the first de Rham cohomology of the "universal abelian scheme" over $\mathcal{A}_g$, page 34
$H^i_{\mathrm{dR}}(X/U)$	ith algebraic de Rham cohomology sheaf of an abelian scheme X over U, page 22
j_F	"uniformization map" from $\mathbf{H}^g$ to $A_F(\mathbf{C})$, page 91
j_g	"uniformization map" from $\mathbf{H}_g$ to $A_g(\mathbf{C})$, page 91
$\mathcal{P}(X/k)$	field of periods of an abelian variety X over $k \subset \mathbf{C}$, page 90
P_F	parabolic subgroup scheme of $\mathrm{Res}_{R/\mathbf{Z}}\,\mathrm{Aut}_{(M,\Psi)}$ fixing the Lagrangian $R \oplus 0 \subset M$ (see Paragraph 3.3), page 32
P_g	parabolic Siegel subgroup of the symplectic group Sp_{2g}, page 17
π_F	forgetful functor $\mathcal{B}_F \longrightarrow \mathcal{A}_F$, page 30
π_g	forgetful functor $\mathcal{B}_g \longrightarrow \mathcal{A}_g$, page 30
Ψ_λ	$\mathcal{O}_U \otimes R$-bilinear form on $H^1_{\mathrm{dR}}(X/U)$ with values in $\mathcal{O}_U \otimes D^{-1}$ satisfying $\mathrm{Tr}\,\Psi_\lambda = \langle\ ,\ \rangle_\lambda$, page 27
$\mathcal{R}_F$	Ramanujan subbundle of $T_{\mathcal{B}_F/\mathbf{Z}}$, page 37
$\mathcal{R}_g$	Ramanujan subbundle of $T_{\mathcal{B}_g/\mathbf{Z}}$, page 35
R	ring of integers of a totally real number field F of degree g over $\mathbf{Q}$, page 25
$R_1\pi_*\mathbf{Z}_X$	dual of the local system of abelian groups $R^1\pi_*\mathbf{Z}_X$ over M, where $\pi : X \longrightarrow M$ is a complex torus over a complex manifold M, page 63
Sp_{2g}	symplectic group scheme of order $2g$ over $\mathrm{Spec}\,\mathbf{Z}$, page 16
Sym_g	additive group scheme over $\mathrm{Spec}\,\mathbf{Z}$ of symmetric matrices of orger g, page 16
Tr	trace map $\mathrm{Tr}_{F/\mathbf{Q}} : F \longrightarrow \mathbf{Q}$, page 25
θ^{r_i}	either the derivation $q^{r_i}\frac{\partial}{\partial q^{r_i}}$ of the ring $\mathbf{Z}(\!(q^{r_i})\!)$ or the holomorphic vector field $\frac{1}{2\pi i}\sum_{j=1}^{g}\sigma_j(x_i)\frac{\partial}{\partial\tau_j}$ over $\mathbf{H}^g$, pages 47 and 88
θ_{ij}	either the derivation $q_{ij}\frac{\partial}{\partial q_{ij}}$ of the ring $\mathbf{Z}(\!(q_{ij})\!)$ or the holomorphic vector field $\frac{1}{2\pi i}\frac{\partial}{\partial\tau_{ij}}$ over $\mathbf{H}_g$, pages 45 and 81
v^{r_i}	vector field $v_F(1 \otimes x_i)$ over $\mathcal{B}_F$, page 88
v_F	higher Ramanujan vector field over $\mathcal{B}_F$, page 43
v_{ij}	higher Ramanujan vector field over $\mathcal{B}_g$, page 41
$(\hat{X}_F, \hat{\lambda}_F, \hat{m}_F)$	principally polarized abelian scheme with R-multiplication over $\mathrm{Spec}\,\mathbf{Z}(\!(q^{r_i})\!)$ given by Mumford's construction, page 49
$(\hat{X}_g, \hat{\lambda}_g)$	principally polarized abelian scheme of relative dimension g over $\mathrm{Spec}\,\mathbf{Z}(\!(q_{ij})\!)$ given by Mumford's construction, page 46
$(\mathbf{X}_F, E_F, m_F)$	"universal" principally polarized complex torus with R-multiplication over $\mathbf{H}^g$, page 71
$(\mathbf{X}_g, E_g)$	"universal" principally polarized complex torus of relative dimension g over $\mathbf{H}_g$, page 65
$\mathbf{Z}(\!(q^{r_i})\!)$	ring of formal Laurent power series over $\mathbf{Z}$ in the variables $q^{r_1}, \ldots, q^{r_g}$, page 47
$\mathbf{Z}(\!(q_{ij})\!)$	ring of formal Laurent power series over $\mathbf{Z}$ in the variables q_{ij}, for $1 \leq i \leq j \leq g$, page 44

Part 1. **The arithmetic theory of the higher Ramanujan equations**

1. Symplectic vector bundles over schemes

In this section we develop (or recall) some preliminary general material on vector bundles over schemes endowed with a symplectic bilinear form with values in some line bundle.

We fix once and for all a scheme U, and a line bundle $\mathcal{L}$ over U.

1.1. **Symplectic vector bundles.**
Let $\mathcal{E}$ be a vector bundle over U. An $\mathcal{O}_U$-bilinear form with values in $\mathcal{L}$

$$\langle\ ,\ \rangle : \mathcal{E} \otimes_{\mathcal{O}_U} \mathcal{E} \longrightarrow \mathcal{L}$$

is said to be

(1) *perfect* if the $\mathcal{O}_U$-morphism $e \longmapsto \langle\ \ , e\rangle$ from $\mathcal{E}$ to $\mathcal{L} \otimes_{\mathcal{O}_U} \mathcal{E}^\vee$ is an isomorphism,
(2) *alternating* if $\langle\ ,\ \rangle$ factors through $\mathcal{E} \otimes_{\mathcal{O}_U} \mathcal{E} \longrightarrow \bigwedge^2 \mathcal{E}$, i.e., if $\langle e, e\rangle = 0$ for every section e of $\mathcal{E}$.

Definition 1.1. An $\mathcal{L}$*-valued symplectic form* over $\mathcal{E}$ is a perfect alternating $\mathcal{O}_U$-bilinear form over $\mathcal{E}$ with values in $\mathcal{L}$. An $\mathcal{L}$*-symplectic vector bundle* over U is a couple $(\mathcal{E}, \langle\ ,\ \rangle)$, where $\mathcal{E}$ is a vector bundle over U and $\langle\ ,\ \rangle$ is an $\mathcal{L}$-valued symplectic form over $\mathcal{E}$.

When $\mathcal{L} = \mathcal{O}_U$, we write simply *symplectic form* and *symplectic vector bundle*.

By considering $\mathcal{O}_U$-linear morphisms preserving the $\mathcal{L}$-valued symplectic forms, we obtain a category of $\mathcal{L}$-symplectic vector bundles over U.

1.2. **Lagrangian subbundles.**
Let $(\mathcal{E}, \langle\ ,\ \rangle)$ be an $\mathcal{L}$-valued symplectic vector bundle over U and $\mathcal{F}$ be a subbundle of $\mathcal{E}$. We denote by $\mathcal{F}^\perp$ the subsheaf of $\mathcal{E}$ consisting of those sections e of $\mathcal{E}$ such that $\langle f, e\rangle = 0$ for every section f of $\mathcal{F}$.

Lemma 1.2. *We have an exact sequence of* $\mathcal{O}_U$*-modules*

$$0 \longrightarrow \mathcal{F}^\perp \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_U} \mathcal{F}^\vee \longrightarrow 0$$
$$e \longmapsto \langle\ \ , e\rangle|_{\mathcal{F}}$$

In particular, $\mathcal{F}^\perp$ *is a subbundle of* $\mathcal{E}$ *of rank* $\mathrm{rank}(\mathcal{E}) - \mathrm{rank}(\mathcal{F})$.

Proof. The sequence $0 \longrightarrow \mathcal{F}^\perp \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_U} \mathcal{F}^\vee$ is exact by definition. To see that $\mathcal{E} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_U} \mathcal{F}^\vee$ defined above is surjective, one may work locally. In this case, $\mathcal{F}$ is a direct factor of $\mathcal{E}$, and thus any $\mathcal{O}_U$-linear map $\mathcal{F} \longrightarrow \mathcal{L}$ can be extended to $\mathcal{E}$; we conclude by using that $\langle\ ,\ \rangle$ is perfect. ■

Definition 1.3. A subbundle $\mathcal{F}$ of $\mathcal{E}$ is said to be *isotropic* with respect to $\langle\ ,\ \rangle$ if $\mathcal{F} \subset \mathcal{F}^\perp$. An isotropic subbundle of $\mathcal{E}$ such that $\mathcal{F} = \mathcal{F}^\perp$ is said to be a *Lagrangian subbundle*.

The next result easily follows from Lemma .16.

Corollary 1.4. *Let* $\mathcal{F}$ *be an isotropic subbundle of* $\mathcal{E}$*. Then* $2\,\mathrm{rank}(\mathcal{F}) \leq \mathrm{rank}(\mathcal{E})$*. Moreover,* $\mathcal{F}$ *is Lagrangian if and only if* $2\,\mathrm{rank}(\mathcal{F}) = \mathrm{rank}(\mathcal{E})$*.* ■

The next lemma shows that Lagrangian subbundles exist locally for the Zariski topology over U. This implies in particular that the rank of every symplectic vector bundle is even.

Lemma 1.5. *Let $(\mathcal{E}, \langle\ ,\ \rangle)$ be an $\mathcal{L}$-valued symplectic vector bundle over U, and assume that U is the spectrum of a local ring. Then there exists a Lagrangian subbundle of $\mathcal{E}$.*

Proof. Let S be the set of isotropic subbundles of $\mathcal{E}$ ordered by inclusion. It is sufficient to prove that every maximal element in S is Lagrangian (maximal elements always exist: consider the rank).

We proceed by contraposition. Let $\mathcal{F}$ be an element of S that is not Lagrangian. As U is local, and both $\mathcal{F}$ and $\mathcal{F}^{\perp}$ are subbundles $\mathcal{E}$ (cf. Lemma .16), there exists an integer $k \geq 1$ and global sections $e_1, \ldots, e_k$ of $\mathcal{F}^{\perp}$ such that

$$\mathcal{F}^{\perp} = \mathcal{F} \oplus \mathcal{O}_U e_1 \oplus \cdots \oplus \mathcal{O}_U e_k.$$

In particular, $\mathcal{F} \oplus \mathcal{O}_U e_1$ is an element of S strictly containing $\mathcal{F}$; thus, $\mathcal{F}$ is not maximal. ■

Remark 1.6. The same statement (and the same proof) holds for every scheme U over which any vector bundle is trivializable, e.g., U the spectrum of a principal ideal domain or of a polynomial ring over a field.

1.3. **Symplectic bases.** In what follows, we take $\mathcal{L} = \mathcal{O}_U$. Let $(\mathcal{E}, \langle\ ,\ \rangle)$ be a symplectic vector bundle of constant rank $2n$ over U.

Definition 1.7. A *symplectic basis* of $(\mathcal{E}, \langle\ ,\ \rangle)$ over U is a basis of $\mathcal{E}$ over U of the form $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ with $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ and $\langle e_i, f_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$.

Remark 1.8. Equivalently, if the trivial vector bundle $\mathcal{O}_U^{2n}$ is given the *standard* symplectic form

$$\langle v, w \rangle_{\mathrm{std}} \coloneqq v^{\mathsf{T}} \begin{pmatrix} 0 & \mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix} w,$$

then a symplectic basis of $(\mathcal{E}, \langle\ ,\ \rangle)$ can be regarded as an isomorphism of symplectic vector bundles $(\mathcal{O}_U^{2n}, \langle\ ,\ \rangle_{\mathrm{std}}) \xrightarrow{\sim} (\mathcal{E}, \langle\ ,\ \rangle)$. This point of view turns out to be useful when dealing with symplectic vector bundles with real multiplication; see Section 3 below.

As Lagrangian subbundles exist locally by Lemma .19, the next proposition implies in particular that symplectic bases also exist locally.

Proposition 1.9. *Let U be an affine scheme, $(\mathcal{E}, \langle\ ,\ \rangle)$ be a symplectic vector bundle over U, and $\mathcal{E}_0$ be a Lagrangian subbundle of $\mathcal{E}$. Then*

(1) Every basis $(e_1, \ldots, e_n)$ of $\mathcal{E}_0$ over U can be completed to a symplectic basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ of $\mathcal{E}$ over U.

(2) If $\mathcal{F}$ is a Lagrangian subbundle of $\mathcal{E}$ such that $\mathcal{E}_0 \oplus \mathcal{F} = \mathcal{E}$, and $(f_1, \ldots, f_n)$ is a basis of $\mathcal{F}$ over U, then there exists a unique basis $(e_1, \ldots, e_n)$ of $\mathcal{E}_0$ over U such that $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ is a symplectic basis of $\mathcal{E}$ over U.

Proof. Consider the surjective morphism of $\mathcal{O}_U$-modules (cf. Lemma .16)

$$\begin{aligned} \mathcal{E} &\longrightarrow \mathcal{E}_0^{\vee} \\ e &\longmapsto \langle\ , e \rangle|_{\mathcal{E}_0}. \end{aligned}$$

Since U is affine, there exists a sequence $(f_1',\dots,f_n')$ of global sections of $\mathcal{E}$ lifting the dual basis of $(e_1,\dots,e_n)$ in $\mathcal{E}_0^\vee$, so that $\langle e_i, f_j'\rangle = \delta_{ij}$ for every $1\leq i,j\leq n$. As $\mathcal{E}_0$ is an isotropic subbundle of $\mathcal{E}$, to prove (1) it is sufficient to show the existence of global sections g_j of $\mathcal{E}_0$ such that

$$f_j \coloneqq f_j' + g_j$$

satisfy $\langle f_i, f_j\rangle = 0$ for every $1\leq i,j\leq n$.

Since the bilinear form $\langle\ ,\ \rangle$ is alternating, $A \coloneqq (\langle f_i', f_j'\rangle)_{1\leq i,j\leq n}$ is an antisymmetric matrix in $M_{n\times n}(\mathcal{O}_U(U))$. Thus, there exists a matrix $B=(b_{ij})_{1\leq i,j\leq g}$ in $M_{n\times n}(\mathcal{O}_U(U))$ such that $A = B - B^{\mathsf{T}}$. We put

$$g_i \coloneqq \sum_{j=1}^n b_{ij}e_j,$$

hence

$$\langle f_i, f_j\rangle = \langle f_i', f_j'\rangle + \langle g_i, f_j'\rangle - \langle g_j, f_i'\rangle = \langle f_i', f_j'\rangle + b_{ij} - b_{ji} = 0.$$

We now proceed to the proof of (2). As $\mathcal{F}$ is an isotropic subbundle of $\mathcal{E}$ satisfying $\mathcal{E}_0\oplus\mathcal{F} = \mathcal{E}$, and since $\langle\ ,\ \rangle$ is perfect, the morphism of $\mathcal{O}_U$-modules

$$\begin{aligned}\mathcal{F} &\longrightarrow \mathcal{E}_0^\vee\\ f &\longmapsto \langle\ \ , f\rangle|_{\mathcal{E}_0}\end{aligned}$$

is injective, thus an isomorphism since $\mathcal{F}$ and $\mathcal{E}_0^\vee$ have equal rank. The existence and unicity of $(e_1,\dots,e_n)$ follows from remarking that $(e_1,\dots,e_n,f_1,\dots,f_n)$ is a symplectic basis of $\mathcal{E}$ over U if and only if $(e_1,\dots,e_n)$ is the basis of $\mathcal{E}_0$ over U dual to the basis $(\langle\ \ ,f_1\rangle|_{\mathcal{E}_0},\dots,\langle\ \ ,f_n\rangle|_{\mathcal{E}_0})$ of $\mathcal{E}_0^\vee$. ■

2. Symplectic-Hodge bases of principally polarized abelian schemes

We start this section by recalling the definition of the de Rham cohomology of an abelian scheme and its main properties. We next recall how to associate to a principal polarization on an abelian scheme a symplectic form, as defined in Section 1, on its first de Rham cohomology. This leads us to the definition of *symplectic-Hodge bases*.

2.1. **De Rham cohomology of abelian schemes.** Let $p: X\longrightarrow U$ be an abelian scheme of relative dimension g.

Recall that, for any integer $i\geq 0$, the *i-th de Rham cohomology* sheaf of $\mathcal{O}_U$-modules associated to p is defined as the i-th left hyperderived functor of p_* applied to the complex of relative differential forms $\Omega^\bullet_{X/U}$:

$$H^i_{\mathrm{dR}}(X/U) \coloneqq \mathbf{R}^i p_*\Omega^\bullet_{X/U}.$$

If $\varphi : X\longrightarrow Y$ is a morphism of abelian schemes over U, we denote by $\varphi^* : H^i_{\mathrm{dR}}(Y/U)\longrightarrow H^i_{\mathrm{dR}}(X/U)$ the induced $\mathcal{O}_U$-morphism on de Rham cohomology.

One can prove that there is a canonical isomorphism given by cup product

$$\bigwedge^i H^1_{\mathrm{dR}}(X/U) \xrightarrow{\sim} H^i_{\mathrm{dR}}(X/U),$$

and that $H^1_{\mathrm{dR}}(X/U)$ is a vector bundle over U of rank $2g$. Moreover, the canonical $\mathcal{O}_U$-morphism $p_*\Omega^1_{X/U} \longrightarrow H^1_{\mathrm{dR}}(X/U)$ induces an isomorphism of $p_*\Omega^1_{X/U}$ with a rank g subbundle of $H^1_{\mathrm{dR}}(X/U)$, its *Hodge subbundle* $F^1(X/U)$. It fits into a canonical exact sequence of $\mathcal{O}_U$-modules:

$$0 \longrightarrow F^1(X/U) \longrightarrow H^1_{\mathrm{dR}}(X/U) \longrightarrow R^1p_*\mathcal{O}_X \longrightarrow 0. \tag{2.1}$$

The formation of $H^1_{\mathrm{dR}}(X/U)$, $F^1(X/U)$, $R^1p_*\mathcal{O}_X$, and the above exact sequence is compatible with every base change in U.

For a proof of all these facts, the reader may consult [2] 2.5.

2.2. **Symplectic form associated to a principal polarization.** Let $p : X \longrightarrow U$ be a projective abelian scheme of relative dimension g, and $\lambda : X \longrightarrow X^t$ be a principal polarization. In this paragraph, we recall how to associate to λ a canonical *symplectic* $\mathcal{O}_U$-bilinear form

$$\langle\ ,\ \rangle_\lambda : H^1_{\mathrm{dR}}(X/U) \otimes_{\mathcal{O}_U} H^1_{\mathrm{dR}}(X/U) \longrightarrow \mathcal{O}_U.$$

Recall that to any line bundle $\mathcal{L}$ on X we can associate its *first Chern class in de Rham cohomology* $c_{1,\mathrm{dR}}(\mathcal{L})$, namely the global section of $H^2_{\mathrm{dR}}(X/U)$ given by the image of the class of the line bundle $\mathcal{L}$ under the morphism of $\mathcal{O}_U$-modules

$$R^1p_*\mathcal{O}_X^\times \longrightarrow \mathbf{R}^1p_*\Omega^\bullet_{X/U}[1] \cong H^2_{\mathrm{dR}}(X/U)$$

induced by $\mathrm{dlog} : \mathcal{O}_X^\times \longrightarrow \Omega^\bullet_{X/U}[1]$.[7]

We apply the above construction to the Poincaré line bundle $\mathcal{P}_{X/U}$ on the projective abelian scheme $X \times_U X^t$ over U. Let

$$\phi_{X/U} : H^1_{\mathrm{dR}}(X/U)^\vee \longrightarrow H^1_{\mathrm{dR}}(X^t/U)$$

be the morphism of $\mathcal{O}_U$-modules given by the image of $c_{1,\mathrm{dR}}(\mathcal{P}_{X/U})$ in the Künneth component $H^1_{\mathrm{dR}}(X/U) \otimes_{\mathcal{O}_U} H^1_{\mathrm{dR}}(X^t/U)$ of $H^2_{\mathrm{dR}}(X/U)$. By [2] 5.1.3.1, $\phi_{X/U}$ is in fact an isomorphism.

Remark 2.1 (cf. [2] (5.1.3.3))**.** The isomorphisms $\phi_{X/U}$ are natural in the following sense. If $\varphi : X \longrightarrow Y$ is a morphism of projective abelian schemes over U, then the diagram of $\mathcal{O}_U$-modules

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(X/U)^\vee & \xrightarrow{\phi_{X/U}} & H^1_{\mathrm{dR}}(X^t/U) \\ {\scriptstyle (\varphi^*)^\vee}\downarrow & & \downarrow{\scriptstyle (\varphi^t)^*} \\ H^1_{\mathrm{dR}}(Y/U)^\vee & \xrightarrow[\phi_{Y/U}]{} & H^1_{\mathrm{dR}}(Y^t/U) \end{array}$$

commutes.

Consider the isomorphism of $\mathcal{O}_U$-modules

$$\lambda^* : H^1_{\mathrm{dR}}(X^t/U) \longrightarrow H^1_{\mathrm{dR}}(X/U)$$

[7] We adopt the same sign conventions of [2] 0.3 for the differentials of the shifted complex $\Omega^\bullet_{X/U}[1]$ and for the isomorphism $\mathbf{R}^1p_*\Omega^\bullet_{X/U}[1] \cong H^2_{\mathrm{dR}}(X/U)$.

induced by the principal polarization $\lambda : X \longrightarrow X^t$. For any sections γ and δ of $H^1_{\mathrm{dR}}(X/U)^\vee$, we set

$$E^{\mathrm{dR}}_\lambda(\gamma,\delta) := \delta \circ \lambda^* \circ \phi_{X/U}(\gamma).$$

It is clear that E^{dR}_λ defines an $\mathcal{O}_U$-bilinear form over $H^1_{\mathrm{dR}}(X/U)^\vee$. Since $\phi_{X/U}$ is an isomorphism, E^{dR}_λ is perfect. By duality, we can thus define a perfect bilinear form $\langle\ ,\ \rangle_\lambda$ over $H^1_{\mathrm{dR}}(X/U)$ via

$$\langle E^{\mathrm{dR}}_\lambda(\gamma,\ ), E^{\mathrm{dR}}_\lambda(\delta,\ )\rangle_\lambda := E^{\mathrm{dR}}_\lambda(\gamma,\delta),$$

where we identified $H^1_{\mathrm{dR}}(X/U)^{\vee\vee}$ with $H^1_{\mathrm{dR}}(X/U)$.

Lemma 2.2. *The perfect bilinear form $\langle\ ,\ \rangle_\lambda$ is alternating, thus symplectic.*

Proof. It suffices to prove that E^{dR}_λ is alternating. Since λ is a polarization, it is étale locally over U induced by a line bundle $\mathcal{L}$ over X relatively ample over U. We consider the first Chern class $c_{1,\mathrm{dR}}(\mathcal{L})$ in $H^2_{\mathrm{dR}}(X/U) \cong \bigwedge^2 H^1_{\mathrm{dR}}(X/U)$. Then, one can verify that E^{dR}_λ defined above coincides with the alternating form

$$(\gamma,\delta) \longmapsto \gamma \wedge \delta(c_{1,\mathrm{dR}}(\mathcal{L})).$$

We refer to [23], Section 1, for further details. ■

Thus we obtain a symplectic vector bundle $(H^1_{\mathrm{dR}}(X/U), \langle\ ,\ \rangle_\lambda)$ over U in the sense of Definition .15.

Lemma 2.3. *$F^1(X/U)$ is a Lagrangian subbundle of $H^1_{\mathrm{dR}}(X/U)$ with respect to the symplectic form $\langle\ ,\ \rangle_\lambda$.*

Proof. Since the rank of $H^1_{\mathrm{dR}}(X/U)$ is $2g$, and $F^1(X/U)$ is a rank g subbundle of $H^1_{\mathrm{dR}}(X/U)$, it suffices to prove that $F^1(X/U)$ is isotropic with respect to $\langle\ ,\ \rangle_\lambda$ (cf. Corollary .18). This follows immediately from the compatibility of $\phi_{X/U}$ with the exact sequence (2.1), that is, from the existence of canonical morphisms $\phi^0_{X/U}$ and $\phi^1_{X/U}$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (R^1p_*\mathcal{O}_X)^\vee & \longrightarrow & H^1_{\mathrm{dR}}(X/U)^\vee & \longrightarrow & F^1(X/U)^\vee & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi^0_{X/U}} & & \downarrow{\scriptstyle \phi_{X/U}} & & \downarrow{\scriptstyle \phi^1_{X/U}} & & \\ 0 & \longrightarrow & F^1(X^t/U) & \longrightarrow & H^1_{\mathrm{dR}}(X^t/U) & \longrightarrow & R^1p^t_*\mathcal{O}_{X^t} & \longrightarrow & 0 \end{array}$$

commute ([2] Lemme 5.1.4; the morphisms $\phi^0_{X/U}$ and $\phi^1_{X/U}$ are uniquely determined by this commutative diagram, and are isomorphisms). ■

Remark 2.4. It is clear from the above construction that the formation of the symplectic form $\langle\ ,\ \rangle_\lambda$ is compatible with base change. Namely, if $f : U' \longrightarrow U$ is a morphism of schemes, and (X',λ') denotes the principally polarized abelian scheme over U' obtained by base change via f, then $f^*\langle\ ,\ \rangle_\lambda$ coincides with $\langle\ ,\ \rangle_{\lambda'}$ under the base change isomorphism $f^*H^1_{\mathrm{dR}}(X/U) \xrightarrow{\sim} H^1_{\mathrm{dR}}(X'/U')$.

2.3. **Symplectic-Hodge bases of $H^1_{\mathrm{dR}}(X/U)$.** Let U be a scheme and (X,λ) be a principally polarized abelian scheme over U of relative dimension g.

Definition 2.5. A *symplectic-Hodge basis* of $(X,\lambda)_{/U}$ is a $2g$-uple $b = (\omega_1,\ldots,\omega_g,\eta_1,\ldots,\eta_g)$ global sections of $H^1_{\mathrm{dR}}(X/U)$ such that:

(1) $\omega_1,\ldots,\omega_g$ are sections of $F^1(X/U)$, and
(2) b is a symplectic basis of $(H^1_{\mathrm{dR}}(X/U), \langle\ ,\ \rangle_\lambda)$ (Definition .21).

Note that symplectic-Hodge bases may not exist globally, but such bases always exist locally for the Zariski topology over U by Proposition .23.

3. Abelian schemes with real multiplication

In this section, we introduce notation and analogs of the above basic notions for principally polarized abelian schemes with *real multiplication*.

From now on, we fix a totally real number field F of degree g over $\mathbf{Q}$, and we denote its ring of integers by R. Recall that the inverse different ideal $D^{-1} \subset F$ is a fractional ideal of F which can be identified with the $\mathbf{Z}$-dual of R via the trace form.

Tensor products without subscripts are taken over $\mathbf{Z}$.

3.1. **Symplectic vector bundles with real multiplication.** Let U be a scheme and $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_U$-module. An *R-multiplication* on $\mathcal{M}$ is a ring morphism $R \longrightarrow \mathrm{End}_{\mathcal{O}_U}(\mathcal{M})$; giving such a ring morphism amounts to giving $\mathcal{M}$ the structure of an $\mathcal{O}_U \otimes R$-module compatible with its structure of $\mathcal{O}_U$-module via $\mathcal{O}_U \longrightarrow \mathcal{O}_U \otimes R$.

Remark 3.1. Consider the natural projection $f : U_R := U \otimes_{\mathbf{Z}} R \longrightarrow U$. Observe that $f_*\mathcal{O}_{U_R} = \mathcal{O}_U \otimes R$. Since f is finite, thus affine, the functor

$$\mathcal{F} \longmapsto f_*\mathcal{F}$$

induces an equivalence between the category of quasi-coherent $\mathcal{O}_{U_R}$-modules and the category of quasi-coherent $\mathcal{O}_U$-modules with R-multiplication (i.e., quasi-coherent $\mathcal{O}_U \otimes R$-modules; cf. [30] Proposition 1.4.3).

Following [70] and [23], we denote the $\mathcal{O}_U \otimes R$-dual of a quasi-coherent $\mathcal{O}_U$-module with R-multiplication $\mathcal{M}$ by

$$\mathcal{M}^* := \mathcal{H}om_{\mathcal{O}_U\otimes R}(\mathcal{M}, \mathcal{O}_U \otimes R).$$

The trace map $\mathrm{Tr} := \mathrm{Tr}_{F/\mathbf{Q}} : F \longrightarrow \mathbf{Q}$ induces an isomorphism of quasi-coherent $\mathcal{O}_U$-modules with R-multiplication

$$\mathrm{Tr} : \mathcal{M}^* \otimes_R D^{-1} \xrightarrow{\sim} \mathcal{M}^\vee. \tag{3.1}$$

Remark 3.2. The above duality relation comes from the following general fact (cf. [23] 2.11). Let A be a commutative ring, M be an $A \otimes R$-module, and N be an A-module. Then the trace map induces an $A \otimes R$-isomorphism

$$\mathrm{Tr} : \mathrm{Hom}_{A\otimes R}(M, N \otimes D^{-1}) \xrightarrow{\sim} \mathrm{Hom}_A(M, N).$$

Remark 3.3. In the light of Remark .29, we may interpret (3.1) as a version of the Serre-Grothendieck duality for the *finite* morphism f. For a quasi-coherent $\mathcal{O}_U$-module $\mathcal{G}$, we define a quasi-coherent $\mathcal{O}_{U_R}$-module $f^!\mathcal{G}$ by $f_*f^!\mathcal{G} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_U \otimes R, \mathcal{G}) = \mathcal{G} \otimes \mathrm{Hom}_{\mathbf{Z}}(R, \mathbf{Z})$. We then have natural isomorphisms $f_*\mathcal{H}om_{\mathcal{O}_{U_R}}(\mathcal{F}, f^!\mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_U}(f_*\mathcal{F}, \mathcal{G})$ for any *quasi*-coherent sheaves $\mathcal{F}$ on U_R and $\mathcal{G}$ on U.

By a *vector bundle with R-multiplication* over U, we mean a quasi-coherent sheaf with R-multiplication $\mathcal{E}$ over U which is, locally over U, a free $\mathcal{O}_U \otimes R$-module of finite rank. Equivalently, under the notation of Remark .29, $\mathcal{E}$ is given by the direct image of a vector bundle over U_R. Clearly, $\mathcal{E}$ is also a vector bundle over U and we have

$$\mathrm{rank}_{\mathcal{O}_U}\mathcal{E} = g \cdot \mathrm{rank}_{\mathcal{O}_U \otimes R}\mathcal{E}.$$

By the *rank* of a symplectic vector bundle with R-multiplication, we mean its rank as a locally free $\mathcal{O}_U \otimes R$-module.

We say that an $\mathcal{O}_U$-bilinear form $\langle\ ,\ \rangle$ on the vector bundle with R-multiplication $\mathcal{E}$ over U is *compatible with the R-multiplication* if it factors through

$$\mathcal{E} \otimes_{\mathcal{O}_U \otimes R} \mathcal{E} \longrightarrow \mathcal{O}_U.$$

In this case, it follows from (3.1) that there exists a unique $\mathcal{O}_U \otimes R$-bilinear form

$$\Psi : \mathcal{E} \otimes_{\mathcal{O}_U \otimes R} \mathcal{E} \longrightarrow \mathcal{O}_U \otimes D^{-1}$$

such that

$$\langle\ ,\ \rangle = \mathrm{Tr}\,\Psi.$$

If, moreover, $\langle\ ,\ \rangle$ is symplectic, then Ψ is perfect and alternating — that is, if $\mathcal{E} = f_*\mathcal{F}$ under the notation of Remark .29, then Ψ is given by the direct image of a $\mathcal{O}_{U_R} \otimes_R D^{-1}$-valued symplectic form on $\mathcal{F}$. The couple $(\mathcal{E}, \Psi)$ is then said to be a *symplectic vector bundle with R-multiplication* over U.

3.2. **Principally polarized abelian schemes with real multiplication.** Let (X, λ) be a principally polarized abelian scheme over some scheme U. Then λ defines a Rosatti involution $\varphi \longmapsto \lambda^{-1} \circ \varphi^t \circ \lambda$ on the ring of abelian scheme endomorphisms $\mathrm{End}_U(X)$. We denote by $\mathrm{End}_U(X)^\lambda$ the subset of $\mathrm{End}_U(X)$ of elements fixed by the Rosatti involution.

Definition 3.4. A *principally polarized abelian scheme with R-multiplication* over U is a triple (X, λ, m), where (X, λ) is a principally polarized abelian scheme over U, and $m : R \longrightarrow \mathrm{End}_U(X)$ is a ring morphism such that:

(1) $m(R) \subset \mathrm{End}_U(X)^\lambda$, and
(2) m gives $F^1(X/U)$ the structure of a vector bundle with R-multiplication of rank 1 over U.

A morphism of principally polarized abelian schemes with R-multiplication is a morphism of principally polarized abelian schemes commuting with the R-multiplications.

The condition (2) above, which implies in particular that X is of relative dimension g over U, is due to Rapoport (cf. [70] Definition 1.1); it is automatically satisfied whenever the discriminant of R is invertible in U ([23] Corollaire 2.9).

Remark 3.5. For any non-zero $r \in R$, the endomorphism $m(r) : X \longrightarrow X$ is an isogeny over U, i.e., surjective and quasi-finite — which, in this case, is equivalent to finite and locally free. Indeed, if $N(r) \in \mathbf{Z} \setminus \{0\}$ denotes the norm of $r \in R$, then there exists $s \in R$ such that $rs = N(r)$. Thus, that $m(r)$ is an isogeny follows easily from the fact that and the composition $m(r) \circ m(s) : X \longrightarrow X$ is the multiplication by $N(r)$, which is an isogeny itself. In particular, m is always injective.

For a principally polarized abelian scheme with R-multiplication (X, λ, m) over U, it follows from [70], Lemme 1.3, that $H^1_{\mathrm{dR}}(X/U)$ is a rank 2 vector bundle with R-multiplication over U. Since the image of m lies in $\mathrm{End}_U(X)^\lambda \subset \mathrm{End}_U(X)$, we may check using the explicit construction given in Paragraph 2.2 that the symplectic form $\langle\ ,\ \rangle_\lambda$ is compatible with the R-multiplication. We denote by

$$\Psi_\lambda : H^1_{\mathrm{dR}}(X/U) \otimes_{\mathcal{O}_U \otimes R} H^1_{\mathrm{dR}}(X/U) \longrightarrow \mathcal{O}_U \otimes D^{-1}$$

the unique (perfect alternating) $\mathcal{O}_U \otimes R$-bilinear form for which

$$\operatorname{Tr} \Psi_\lambda = \langle\ ,\ \rangle_\lambda,$$

so that $(H^1_{\mathrm{dR}}(X/U), \Psi_\lambda)$ is a rank 2 symplectic vector bundle with R-multiplication over U.

Note that any rank 1 subbundle with R-multiplication of $H^1_{\mathrm{dR}}(X/U)$ is isotropic for Ψ_λ; this applies in particular to $F^1(X/U)$.

3.3. **Symplectic-Hodge bases.** Consider the rank 2 projective R-module $M \coloneqq R \oplus D^{-1}$ endowed with the standard D^{-1}-valued symplectic form

$$\Psi : M \times M \longrightarrow D^{-1}$$
$$((r, x), (r', x')) \longmapsto rx' - r'x.$$

For any scheme U, we obtain a rank 2 symplectic vector bundle with R-multiplication

$$(\mathcal{O}_U \otimes M, 1 \otimes \Psi)$$

over U.

Definition 3.6. Let U be a scheme and (X, λ, m) be a principally polarized abelian scheme with R-multiplication over U. A *symplectic-Hodge basis* of $(X, \lambda, m)_{/U}$ is an isomorphism of symplectic vector bundles with R-multiplication over U

$$b : (\mathcal{O}_U \otimes M, 1 \otimes \Psi) \xrightarrow{\sim} (H^1_{\mathrm{dR}}(X/U), \Psi_\lambda)$$

sending $\mathcal{O}_U \otimes (R \oplus 0) \subset \mathcal{O}_U \otimes M$ to $F^1(X/U) \subset H^1_{\mathrm{dR}}(X/U)$.

Note that $1 \otimes \Psi$ induces an $\mathcal{O}_U \otimes R$-isomorphism

$$\bigwedge\nolimits^2_{\mathcal{O}_U \otimes R} \mathcal{O}_U \otimes M \xrightarrow{\sim} \mathcal{O}_U \otimes D^{-1}$$

trivializing the $\mathcal{O}_U \otimes R$-module of alternating $\mathcal{O}_U \otimes R$-bilinear forms over $\mathcal{O}_U \otimes M$ with values in $\mathcal{O}_U \otimes D^{-1}$:

$$(*) \qquad \mathcal{O}_U \otimes R \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_U \otimes R}\left(\bigwedge\nolimits^2_{\mathcal{O}_U \otimes R} \mathcal{O}_U \otimes M, \mathcal{O}_U \otimes D^{-1}\right).$$

A symplectic-Hodge basis b of $(X, \lambda, m)_{/U}$ may be seen as an $\mathcal{O}_U \otimes R$-isomorphism

$$b = (\omega, \eta) : \mathcal{O}_U \otimes M \cong (\mathcal{O}_U \otimes R) \oplus (\mathcal{O}_U \otimes D^{-1}) \xrightarrow{\sim} H^1_{\mathrm{dR}}(X/U)$$

such that

(1) $\omega : \mathcal{O}_U \otimes R \longrightarrow H^1_{\mathrm{dR}}(X/U)$ factors through $F^1(X/U) \subset H^1_{\mathrm{dR}}(X/U)$, and
(2) $\Psi_\lambda(\omega, \eta) = 1$.

Here, $\Psi_\lambda(\omega, \eta)$ is regarded as the element of $\mathcal{O}_U \otimes R$ mapping to $b^*\Psi_\lambda$ via $(*)$.

Equivalently, if we regard Ψ_λ as an alternating $\mathcal{O}_U \otimes R$-bilinear form

$$\Psi_\lambda : H^1_{\mathrm{dR}}(X/U) \otimes_{\mathcal{O}_U \otimes R} H^1_{\mathrm{dR}}(X/U) \otimes_R D \longrightarrow \mathcal{O}_U \otimes R,$$

then a symplectic-Hodge basis of $(X, \lambda, m)_{/U}$ is a couple $b = (\omega, \eta)$, where ω is a global section of $F^1(X/U) \subset H^1_{\mathrm{dR}}(X/U)$ generating it as an $\mathcal{O}_U \otimes R$-module, η is a global section of $H^1_{\mathrm{dR}}(X/U) \otimes_R D$ whose image in $(H^1_{\mathrm{dR}}(X/U)/F^1(X/U)) \otimes_R D$ generates it as an $\mathcal{O}_U \otimes R$-module, and $\Psi_\lambda(\omega, \eta) = 1$.

Remark 3.7. Since Ψ_λ is perfect, if ω is an $\mathcal{O}_U \otimes R$-trivialization of $F^1(X/U)$, and η is any global section of $H^1_{\mathrm{dR}}(X/U) \otimes_R D$ satisfying $\Psi_\lambda(\omega, \eta) = 1$, then $b = (\omega, \eta)$ is a symplectic-Hodge basis.

Remark 3.8. If η is a global section of $H^1_{\mathrm{dR}}(X/U) \otimes_R D$ whose image in $(H^1_{\mathrm{dR}}(X/U)/F^1(X/U)) \otimes_R D$ generates it as an $\mathcal{O}_U \otimes R$-module, then there exists a unique $\mathcal{O}_U \otimes R$-trivialization ω of $F^1(X/U)$ such that (ω, η) is a symplectic-Hodge basis.

4. The moduli stacks $\mathcal{B}_g$ and $\mathcal{B}_F$

In this section we define for every integer $g \geq 1$ (resp. for every totally real number field F) a category $\mathcal{B}_g$ (resp. $\mathcal{B}_F$) fibered in groupoids over the category of schemes $\mathsf{Sch}_{/\mathbf{Z}}$ classifying principally polarized abelian schemes of relative dimension g (resp. principally polarized abelian schemes with R-multiplication) endowed with a symplectic-Hodge basis.

Using classical results on moduli stacks of abelian schemes, we then prove that $\mathcal{B}_g \longrightarrow \operatorname{Spec} \mathbf{Z}$ (resp. $\mathcal{B}_F \longrightarrow \operatorname{Spec} \mathbf{Z}$) is a smooth Deligne-Mumford stack over $\operatorname{Spec} \mathbf{Z}$ of relative dimension $2g^2 + g$ (resp. $3g$).

4.1. The moduli stacks $\mathcal{A}_g$ and $\mathcal{A}_F$.

Let $g \geq 1$ be an integer (resp. F be a totally real number field of degree g with ring of integers R). To fix ideas and notation we recall the definition of the moduli stack of principally polarized abelian schemes of relative dimension g (resp. principally polarized abelian schemes with R-multiplication).

For any scheme S, we define a category fibered in groupoids $\mathcal{A}_{g,S} \longrightarrow \mathsf{Sch}_{/S}$ (resp. $\mathcal{A}_{F,S} \longrightarrow \mathsf{Sch}_{/S}$) as follows.

(i) An object of $\mathcal{A}_{g,S}$ (resp. $\mathcal{A}_{F,S}$) is given by an S-scheme U and a principally polarized abelian scheme (X, λ) of relative dimension g (resp. a principally polarized abelian scheme with R-multiplication (X, λ, m)) over U; when U is not clear in the context, we shall incorporate it in the notation by writing $(X, \lambda)_{/U}$. A morphism $(X, \lambda)_{/U} \longrightarrow (Y, \mu)_{/V}$ (resp. $(X, \lambda, m)_{/U} \longrightarrow (Y, \mu, n)_{/V}$) in $\mathcal{A}_{g,S}$ (resp. $\mathcal{A}_{F,S}$), denoted $\varphi_{/f}$, is given by a Cartesian diagram of S-schemes

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow & \square & \downarrow \\ U & \xrightarrow[f]{} & V \end{array}$$

preserving the identity sections of the abelian schemes and identifying λ with the pullback of μ by $f : U \longrightarrow V$ (and satisfying $n(r) \circ \varphi = \varphi \circ m(r)$ for every $r \in R$, in the case of R-multiplication). We shall occasionally denote $\varphi_{/f}$ simply by φ when there will be no danger of confusion. We may also denote $(X, \lambda) = (Y, \mu) \times_U V$ (resp. $(X, \lambda, m) = (Y, \mu, n) \times_U V$).

(ii) The structural functor $\mathcal{A}_{g,S} \longrightarrow \mathsf{Sch}_{/S}$ (resp. $\mathcal{A}_{F,S} \longrightarrow \mathsf{Sch}_{/S}$) is given by sending an object $(X, \lambda)_{/U}$ of $\mathcal{A}_{g,S}$ (resp. $(X, \lambda, m)_{/U}$ of $\mathcal{A}_{F,S}$) to the S-scheme U, and a morphism $\varphi_{/f}$ to f.

If $S = \operatorname{Spec} \Lambda$ is affine, then we denote $\mathcal{A}_{g,S} =: \mathcal{A}_{g,\Lambda}$ (resp. $\mathcal{A}_{F,S} =: \mathcal{A}_{F,\Lambda}$). When $\Lambda = \mathbf{Z}$, we simply drop it from notation.

Recall that the category of S-schemes can be seen as a subcategory of the 2-category of categories fibered in groupoids over $\mathsf{Sch}_{/S}$ by sending each S-scheme U to the category $\mathsf{Sch}_{/U}$ endowed with its natural functor $\mathsf{Sch}_{/U} \longrightarrow \mathsf{Sch}_{/S}$. In the sequel, we shall adopt the standard convention of denoting $\mathsf{Sch}_{/U}$ simply by U when working in the context of categories fibered in groupoids. Then $\mathcal{A}_{g,S}$ (resp. $\mathcal{A}_{F,S}$) is canonically equivalent to $\mathcal{A}_g \times_{\mathbf{Z}} S$ (resp. $\mathcal{A}_F \times_{\mathbf{Z}} S$) as categories fibered in groupoids over S.

Theorem 4.1. *For any scheme S, $\mathcal{A}_{g,S}$ (resp. $\mathcal{A}_{F,S}$) is a smooth Deligne-Mumford stack over S of relative dimension $g(g+1)/2$ (resp. g).*

A proof that $\mathcal{A}_{g,S}$ is a Deligne-Mumford stack over S is essentially contained in [58] Theorem 7.9 (cf. [64] Theorem 2.1.11). Smoothness and relative dimension are obtained by a theorem of Grothendieck (cf. [66] Proposition 2.4.1). The case of real multiplication is treated in [70] Théorème 1.20; in Rapoport's notation, our $\mathcal{A}_F$ corresponds to $\mathcal{M}^L$ with $L = R$.

Remark 4.2. The stack $\mathcal{A}_g$ is often called a *Siegel moduli stack*, whereas $\mathcal{A}_F$ is known as a *Hilbert-Blumenthal moduli stack*.

Remark 4.3. Beware that there is a fundamental difference between the moduli stack $\mathcal{A}_g$ and the *coarse moduli scheme* A_g (see page 91), often referred in the literature simply as "the moduli space of principally polarized abelian varieties of dimension g" (and similarly for the case of real multiplication). Even over $\mathbf{C}$, the moduli stack $\mathcal{A}_{g,\mathbf{C}}$ is *not* representable by a scheme (or an algebraic space). Let us also remark that, while $\mathcal{A}_{g,\mathbf{C}}$ is smooth over $\operatorname{Spec} \mathbf{C}$ in the sense of Deligne-Mumford stacks for every $g \geq 1$,the coarse moduli scheme $A_{g,\mathbf{C}}$ is not a smooth scheme over $\operatorname{Spec} \mathbf{C}$ for $g \geq 3$ (see [67]).

4.2. **Definition of the moduli stacks $\mathcal{B}_g$ and $\mathcal{B}_F$.** We first treat the Siegel case. Let $\varphi_{/f}$: $(X, \lambda)_{/U} \longrightarrow (Y, \mu)_{/V}$ be a morphism in $\mathcal{A}_g$. By the compatibility with base change of the symplectic forms induced by principal polarizations (Remark .27), the pullback $\varphi^* b$ of every symplectic-Hodge basis b of $(Y, \mu)_{/V}$ is a symplectic-Hodge basis of $(X, \lambda)_{/U}$. We can thus define a functor

$$\underline{B}_g : \mathcal{A}_g^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

that sends every object $(X, \lambda)_{/U}$ of $\mathcal{A}_g$ to the set of symplectic-Hodge bases of $(X, \lambda)_{/U}$, and whose action on morphisms is given by pullbacks as above.

From the functor $\underline{B}_g$, we form a category fibered in groupoids

$$\mathcal{B}_g \longrightarrow \operatorname{Spec} \mathbf{Z}$$

as follows.

(i) An object of $\mathcal{B}_g$ is a "triple" $(X,\lambda,b)_{/U}$ where $(X,\lambda)_{/U}$ is an object of $\mathcal{A}_g$ and $b\in \underline{B}_g(X,\lambda)$. An arrow $(X,\lambda,b)_{/U}\longrightarrow (Y,\mu,c)_{/V}$ is given by a morphism $\varphi_{/f}:(X,\lambda)_{/U}\longrightarrow (Y,\mu)_{/V}$ in $\mathcal{A}_g$ such that $b=\varphi^*c$. We denote by

$$\pi_g:\mathcal{B}_g\longrightarrow \mathcal{A}_g$$

the forgetful functor $(X,\lambda,b)_{/U}\longmapsto (X,\lambda)_{/U}$.

(ii) The structural functor $\mathcal{B}_g\longrightarrow \operatorname{Spec}\mathbf{Z}$ is defined as the composition of π_g with the structural functor $\mathcal{A}_g\longrightarrow \operatorname{Spec}\mathbf{Z}$.

Analogously, in the Hilbert-Blumenthal case, we consider a functor

$$\underline{B}_F:\mathcal{A}_F^{\mathrm{op}}\longrightarrow \mathsf{Set}$$

sending a principally polarized abelian scheme with R-multiplication to the set of its symplectic-Hodge basis (Definition .34), and we derive from it a category fibered in groupoids

$$\mathcal{B}_F\longrightarrow \operatorname{Spec}\mathbf{Z}$$

whose objects over a scheme U are given by "quadruples" $(X,\lambda,m,b)_{/U}$. We denote by

$$\pi_F:\mathcal{B}_F\longrightarrow \mathcal{A}_F$$

the natural forgetful functor.

Remark 4.4 (Relating $\mathcal{B}_F$ with $\mathcal{B}_g$)**.** Consider the canonical morphism of stacks $f:\mathcal{A}_F\longrightarrow \mathcal{A}_g$ given by the forgetful functor. Let $(x_1,\ldots,x_g)$ be a $\mathbf{Z}$-basis of D^{-1}, and $(r_1,\ldots,r_g)$ be the corresponding dual $\mathbf{Z}$-basis of R, so that

$$t\coloneqq (r_1,\ldots,r_g,x_1,\ldots,x_g):(\mathbf{Z}^{2g},\langle\ ,\ \rangle_{\mathrm{std}})\xrightarrow{\sim}(M=R\oplus D^{-1},\operatorname{Tr}\Psi)$$

is an isomorphism of symplectic $\mathbf{Z}$-modules (notation as in Paragraph 3.3). Then it is easy to check that t induces a morphism of stacks

$$\begin{aligned} f_t:\mathcal{B}_F&\longrightarrow \mathcal{B}_g\\ (X,\lambda,m,b)_{/U}&\longmapsto (X,\lambda,b\circ\theta_U)_{/U}\end{aligned}$$

making the diagram

$$\begin{array}{ccc} \mathcal{B}_F & \xrightarrow{f_t} & \mathcal{B}_g\\ {\scriptstyle\pi_g}\downarrow & & \downarrow{\scriptstyle\pi_F}\\ \mathcal{A}_F & \xrightarrow[f]{} & \mathcal{A}_g\end{array} \tag{4.1}$$

commute.

The rest of this section is devoted to the proof of the next theorem.

Theorem 4.5. *The category fibered in groupoids $\mathcal{B}_g\longrightarrow \operatorname{Spec}\mathbf{Z}$ (resp. $\mathcal{B}_F\longrightarrow \operatorname{Spec}\mathbf{Z}$) is a smooth Deligne-Mumford stack over $\operatorname{Spec}\mathbf{Z}$ of relative dimension $2g^2+g$ (resp. $3g$).*

4.3. **Siegel parabolic subgroup and proof of Theorem .41 for $\mathcal{B}_g$.** Fix a scheme U and an object (X,λ) of $\mathcal{A}_g$ lying over U. Then we can define a functor

$$\underline{B}_{(X,\lambda)} : \mathsf{Sch}_{/U}^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

that sends a U-scheme U' to the set $\underline{B}_g((X,\lambda)\times_U U')$. It is clear that this functor defines a sheaf for the Zariski topology over $\mathsf{Sch}_{/U}$.

Let us now consider the *symplectic group* Sp_{2g}, namely the smooth affine group scheme over $\operatorname{Spec}\mathbf{Z}$ of relative dimension $2g^2+g$ such that for every affine scheme $V=\operatorname{Spec}\Lambda$

$$\mathrm{Sp}_{2g}(V) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g\times 2g}(\Lambda) \;\middle|\; \begin{array}{c} A,B,C,D \in M_{g\times g}(\Lambda) \text{ satisfy} \\ AB^{\mathsf{T}} = BA^{\mathsf{T}},\ CD^{\mathsf{T}} = DC^{\mathsf{T}}, \text{ and } AD^{\mathsf{T}} - BC^{\mathsf{T}} = \mathbf{1}_g \end{array} \right\}.$$

The *Siegel parabolic subgroup* P_g of Sp_{2g} is defined as the subgroup scheme of Sp_{2g} such that, for every affine scheme $V=\operatorname{Spec}\Lambda$,

$$P_g(V) = \left\{ \begin{pmatrix} A & B \\ 0 & (A^{\mathsf{T}})^{-1} \end{pmatrix} \in M_{2g\times 2g}(\Lambda) \;\middle|\; A \in \mathrm{GL}_g(\Lambda) \text{ and } B \in M_{g\times g}(\Lambda) \text{ satisfy } AB^{\mathsf{T}} = BA^{\mathsf{T}} \right\}.$$

Note that P_g is a smooth affine group scheme over $\operatorname{Spec}\mathbf{Z}$ of relative dimension $g(3g+1)/2$.

Let (X,λ,b) be an object of $\mathcal{B}_g$ lying over $V=\operatorname{Spec}\Lambda$ and consider $b = (\,\omega \;\; \eta\,)$ as a row vector of order $2g$ with coefficients in the R-module $H^1_{\mathrm{dR}}(X/V)$. For any

$$p = \begin{pmatrix} A & B \\ 0 & (A^{\mathsf{T}})^{-1} \end{pmatrix} \in P_g(V)$$

it easy to check that

$$b\cdot p := (\,\omega A \quad \omega B + \eta (A^{\mathsf{T}})^{-1}\,)$$

is a symplectic-Hodge basis of $(X,\lambda)_{/V}$. This defines a right action of $P_g(V)$ on $\underline{B}_g(X,\lambda)$:

$$\underline{B}_g(X,\lambda)\times P_g(V) \longrightarrow \underline{B}_g(X,\lambda).$$

Moreover, it is clear that if $V'\subset V$ is an affine open subscheme of V, then the natural diagram

$$\begin{array}{ccc} \underline{B}_g(X,\lambda)\times P_g(V) & \longrightarrow & \underline{B}_g(X,\lambda) \\ \downarrow & & \downarrow \\ \underline{B}_g(X',\lambda')\times P_g(V') & \longrightarrow & \underline{B}_g(X',\lambda') \end{array}$$

commutes, where $(X',\lambda') = (X,\lambda)\times_V V'$.

Thus, for any scheme U, and any object (X,λ) of $\mathcal{A}_g$ lying over U, we obtain a right action of the U-group scheme $P_{g,U} = P_g \times_{\mathbf{Z}} U$ on $\underline{B}_{(X,\lambda)}$.

Lemma 4.6. *The Zariski sheaf $\underline{B}_{(X,\lambda)}$ over $\mathsf{Sch}_{/U}$ is a right Zariski $P_{g,U}$-torsor for the above action.*

Proof. If V is any affine scheme over U such that $\underline{B}_{(X,\lambda)}(V)$ is non-empty, a routine computation shows that the action of $P_g(V)$ on $\underline{B}_{(X,\lambda)}(V)$ is free and transitive. Moreover, it was already remarked above that symplectic-Hodge bases exist locally for the Zariski topology. ∎

Since $P_{g,U}$ is affine, smooth, and of relative dimension $g(3g+1)/2$ over U, Lemma .42 immediately implies the following.

Corollary 4.7. *For every scheme U, and every object (X,λ) of $\mathcal{A}_g$ lying over U, the functor $\underline{B}_{(X,\lambda)}$ is representable by a smooth affine U-scheme $B(X,\lambda)$ of relative dimension $g(3g+1)/2$.* ∎

Remark 4.8. With the notation of the above corollary, recall that the principally polarized abelian scheme (X,λ) over U corresponds to a morphism $U \longrightarrow \mathcal{A}_g$, so that $B(X,\lambda)$ is a scheme representing $\mathcal{B}_g \times_{\mathcal{A}_g} U$.

Proof of Theorem .41 for $\mathcal{B}_g$. Recall that for any scheme U and any abelian scheme X over U, $H^1_{\mathrm{dR}}(X/U)$ is a quasi-coherent sheaf over U, and that any quasi-coherent sheaf over U induces a sheaf over $\mathsf{Sch}_{/U}$ endowed with the fppf topology ([65] Lemma 4.3.3). Since the étale topology is coarser than the fppf topology, this shows in particular that $H^1_{\mathrm{dR}}(X/U)$ induces a sheaf over $\mathsf{Sch}_{/U}$ endowed with the étale topology; this immediately implies that $\mathcal{B}_g \longrightarrow \operatorname{Spec}\mathbf{Z}$ is a stack over $\operatorname{Spec}\mathbf{Z}$.

It follows in particular from Corollary .43 that the morphism $\pi_g : \mathcal{B}_g \longrightarrow \mathcal{A}_g$ is representable by smooth schemes (Remark .44). Hence, as $\mathcal{A}_g \longrightarrow \operatorname{Spec}\mathbf{Z}$ is a Deligne-Mumford stack over $\operatorname{Spec}\mathbf{Z}$, the same holds for $\mathcal{B}_g \longrightarrow \operatorname{Spec}\mathbf{Z}$ ([65] Proposition 10.2.2). The smoothness of $\mathcal{B}_g \longrightarrow \operatorname{Spec}\mathbf{Z}$ follows by composition from that of $\mathcal{A}_g \longrightarrow \operatorname{Spec}\mathbf{Z}$ and that of π_g. Finally, we can compute the relative dimension of $\mathcal{B}_g \longrightarrow \operatorname{Spec}\mathbf{Z}$ as the sum of that of $\mathcal{A}_g \longrightarrow \operatorname{Spec}\mathbf{Z}$ and that of π_g:

$$\frac{g(g+1)}{2} + \frac{g(3g+1)}{2} = 2g^2 + g.$$

∎

4.4. **Proof of Theorem .41 for $\mathcal{B}_F$.** Let M and Ψ be as in Paragraph 3.3, and consider the affine group scheme $\mathrm{Aut}_{(M,\Psi)}$ over $\operatorname{Spec} R$ of R-automorphisms of M preserving Ψ. It contains a (Borel) subgroup scheme $\mathrm{Aut}_{(M,\Psi,R\oplus 0)}$ of those automorphisms fixing the Lagrangian $R\oplus 0 \subset M = R\oplus D^{-1}$. We set

$$P_F \coloneqq \mathrm{Res}_{R/\mathbf{Z}}\mathrm{Aut}_{(M,\Psi,R\oplus 0)}.$$

This is a smooth affine group scheme of relative dimension $2g$ over $\operatorname{Spec}\mathbf{Z}$. If $V = \operatorname{Spec}\Lambda$ is an affine scheme, then

$$P_F(V) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}_2(\Lambda\otimes F) \,\middle|\, a \in (\Lambda\otimes R)^\times,\, b \in \Lambda\otimes D \right\}$$

where $D \subset R$ denotes the different ideal.

Arguing as above, for an object $(X,\lambda,m)_{/U}$ of $\mathcal{B}_F$, we see that the Zariski sheaf

$$\underline{B}_{(X,\lambda,m)} : \mathsf{Sch}_{/U} \longrightarrow \mathsf{Set}$$

sending an U-scheme U' to $\underline{B}_F((X,\lambda,m)\times_U U')$ is a right Zariski $P_{F,U}$-torsor. This implies that $\pi_F : \mathcal{B}_F \longrightarrow \mathcal{A}_F$ is relatively representable by smooth affine schemes of relative dimension $2g$. We conclude, as in the proof for $\mathcal{B}_g$, with an application of Theorem .37. ∎

5. The tangent bundles of $\mathcal{B}_g$ and $\mathcal{B}_F$; higher Ramanujan vector fields

This section is devoted the study of the tangent bundles $T_{\mathcal{B}_g/\mathbf{Z}}$ and $T_{\mathcal{B}_F/\mathbf{Z}}$.

We shall first explain how the Gauss-Manin connection on the first de Rham cohomology of abelian schemes induces a canonical decomposition

$$T_{\mathcal{B}_g/\mathbf{Z}} = T_{\mathcal{B}_g/\mathcal{A}_g} \oplus \mathcal{R}_g \text{ (resp. } T_{\mathcal{B}_F/\mathbf{Z}} = T_{\mathcal{B}_F/\mathcal{A}_F} \oplus \mathcal{R}_F);$$

$\mathcal{R}_g \subset T_{\mathcal{B}_g/\mathbf{Z}}$ and $\mathcal{R}_F \subset T_{\mathcal{B}_F/\mathbf{Z}}$ are called *Ramanujan subbundles.*

Then, we show that the deformation theory of abelian varieties, in the guise of the Kodaira-Spencer morphism, allows us to canonically trivialize the Ramanujan subbundles. These trivializations are the *higher Ramanujan vector fields.*

5.1. **Horizontal subbundles and linear connections.** We briefly review Ehresmann's point of view on connections over vector bundles. In the context of differential geometry, this is standard material; for a more general discussion in the algebraic setting, we refer to [10] 6.1.

Let S be a scheme, X be a smooth S-scheme, and $\pi : E \longrightarrow X$ be a smooth scheme over X.

Definition 5.1. A subbundle $\mathcal{F}$ of $T_{E/S}$ is said to be *horizontal* (with respect to $\pi : E \longrightarrow X$) if $T_{E/S} = T_{E/X} \oplus \mathcal{F}$.

As $T_{E/X} = \ker(T\pi : T_{E/S} \longrightarrow \pi^* T_{X/S})$, a horizontal subbundle is a splitting of the exact sequence

$$0 \longrightarrow T_{E/X} \longrightarrow T_{E/S} \xrightarrow{T\pi} \pi^* T_{X/S} \longrightarrow 0.$$

In particular, $T\pi$ restricts to an isomorphism $\mathcal{F} \xrightarrow{\sim} \pi^* T_{X/S}$.

Assume now that $\mathcal{E}$ is a vector bundle over X, and that $\pi : E = \mathbf{V}(\mathcal{E}^\vee) \longrightarrow X$ is its associated space over X. Then, to any $\mathcal{O}_S$-linear connection on $\mathcal{E}$

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

there is attached a canonical horizontal subbundle of $T_{E/S}$.

Indeed, observe first that there is a canonical identification

$$T_{E/X} \xrightarrow{\sim} \pi^* \mathcal{E} \tag{5.1}$$

given by the dual of $\pi^* \mathcal{E}^\vee \xrightarrow{\sim} \Omega^1_{E/X}$, defined locally (on X) by $1 \otimes f \longmapsto df$ (cf. [32] Corollaire 16.4.9).

Lemma 5.2. *Let $e \in \Gamma(E, \pi^*\mathcal{E})$ be the "universal section" of $\pi^*\mathcal{E}$, and $\pi^*\nabla$ be the pullback of ∇ to $\pi^*\mathcal{E}$. The $\mathcal{O}_E$-morphism*

$$\begin{aligned} P_\nabla : T_{E/S} &\longrightarrow \pi^*\mathcal{E} \\ \theta &\longmapsto (\pi^*\nabla)_\theta e \end{aligned}$$

restricts to the isomorphism (5.1) on $T_{E/X} \subset T_{E/S}$. ■

It follows that the subbundle $\ker P_\nabla \subset T_{E/S}$ is horizontal: under the identification (5.1), P_∇ becomes a projection of $T_{E/S}$ onto the subbundle $T_{E/X}$. This is the horizontal subbundle attached to ∇.

Remark 5.3. If ∇ is integrable, then $\ker P_\nabla$ is an integrable subbundle of $T_{E/S}$.

It is not difficult to transpose the above considerations to the case of smooth Deligne-Mumford stacks (cf. 0.13).

5.2. **The Ramanujan subbundle $\mathcal{R}_g \subset T_{\mathcal{B}_g/\mathbf{Z}}$.**

5.2.1. Fix a base scheme S and let $p : X \longrightarrow U$ be a projective abelian scheme, with U a *smooth* S-scheme. Then there is defined an integrable S-connection over the de Rham cohomology sheaves ([41]; see also [38]), the *Gauss-Manin connection*

$$\nabla : H^i_{\mathrm{dR}}(X/U) \longrightarrow H^i_{\mathrm{dR}}(X/U) \otimes_{\mathcal{O}_U} \Omega^1_{U/S}, \tag{5.2}$$

whose formation is compatible with every base change $U' \longrightarrow U$, where U' is a smooth S-scheme.

We next construct a "universal" version of Gauss-Manin connection over $\mathcal{A}_g$. Consider the presheaf $\mathcal{H}_g$ of $\mathcal{O}_{\mathcal{A}_{g,\text{ét}}}$-modules on $\text{Ét}(\mathcal{A}_g)$ defined as follows. Let (U,u) be an étale scheme over $\mathcal{A}_g$, and (X,λ) be the principally polarized abelian scheme over U corresponding to $u : U \longrightarrow \mathcal{A}_g$. We put

$$\Gamma((U,u),\mathcal{H}_g) \coloneqq \Gamma(U, H^1_{\mathrm{dR}}(X/U))$$

If $(f,f^b) : (U',u') \longrightarrow (U,u)$ is a morphism in $\text{Ét}(\mathcal{A}_g)$, the restriction map is given by the base change morphism $f^*H^1_{\mathrm{dR}}(X/U) \longrightarrow H^1_{\mathrm{dR}}(X'/U')$, where $(X',\lambda') = (X,\lambda)\times_U U'$. As the base change morphism is actually an isomorphism (i.e., the formation of $H^1_{\mathrm{dR}}(X/U)$ is compatible with base change), and $H^1_{\mathrm{dR}}(X/U)$ is quasi-coherent, $\mathcal{H}_g$ is a quasi-coherent sheaf over $\mathcal{A}_g$ (cf. 0.12 and [65] Lemma 4.3.3). We finally remark that $\mathcal{H}_g$ is actually a vector bundle of rank $2g$ over $\mathcal{A}_g$.

Remark 5.4. The sheaf $\mathcal{H}_g$ should be thought as the first de Rham cohomology of the "universal abelian scheme" over $\mathcal{A}_g$.

For any scheme S, let $\mathcal{H}_{g,S}$ be the vector bundle over $\mathcal{A}_{g,S}$ obtained from $\mathcal{H}_g$ by the base change $\mathcal{A}_{g,S} \longrightarrow \mathcal{A}_g$. Since the formation of the Gauss-Manin connection is compatible with base change, we have an S-connection on $\mathcal{H}_{g,S}$

$$\nabla : \mathcal{H}_{g,S} \longrightarrow \mathcal{H}_{g,S} \otimes_{\mathcal{O}_{\mathcal{A}_{g,S,\text{ét}}}} \Omega^1_{\mathcal{A}_{g,S}/S}$$

defined by (5.2) over every étale S-scheme (U,u) over $\mathcal{A}_{g,s}$ as above.

5.2.2. Consider the morphism of coherent $\mathcal{O}_{\mathcal{B}_{g,\text{ét}}}$-modules

$$\pi_g^*\mathcal{H}_g^{\oplus g} \longrightarrow M_{g\times g}(\mathcal{O}_{\mathcal{B}_{g,\text{ét}}}) \tag{5.3}$$

given on an étale scheme (U,u) over $\mathcal{B}_g$ corresponding to $(X,\lambda,b)_{/U}$, $b = (\omega_1,\ldots,\omega_g,\eta_1,\ldots,\eta_g)$, by

$$\begin{aligned} H^1_{\mathrm{dR}}(X/U)^{\oplus g} &\longrightarrow M_{g\times g}(\mathcal{O}_U) \\ (\alpha_1,\ldots,\alpha_g) &\longmapsto (\langle \alpha_i,\eta_j\rangle_\lambda)_{1\le i,j\le g}, \end{aligned}$$

and let $\mathcal{S}_g$ be the subbundle of $\pi_g^*\mathcal{H}_g^{\oplus g}$ defined as the inverse image of the subbundle of symmetric matrices $\mathrm{Sym}_g(\mathcal{O}_{\mathcal{B}_{g,\text{ét}}}) \subset M_{g\times g}(\mathcal{O}_{\mathcal{B}_{g,\text{ét}}})$ by (5.3).

Remark 5.5. Note that (5.3) is surjective: for a given matrix $(a_{ij})_{1\leq i,j\leq g}$ in $M_{g\times g}(\mathcal{O}_U)$, take $\alpha_i := \sum_{j=1}^{g} a_{ij}\omega_j$. In particular, $\mathcal{S}_g$ is a subbundle of $\pi_g^*\mathcal{H}_g^{\oplus g}$ of rank $g^2 + g(g+1)/2 = g(3g+1)/2$.

Theorem 5.6. *Consider the morphism of quasi-coherent $\mathcal{O}_{\mathcal{B}_{g,\text{ét}}}$-modules*

$$P : T_{\mathcal{B}_g/\mathbf{Z}} \longrightarrow \pi_g^*\mathcal{H}_g^{\oplus g}$$

defined by

$$\begin{aligned} T_{U/\mathbf{Z}} &\longrightarrow H^1_{\mathrm{dR}}(X/U)^{\oplus g} \\ \theta &\longmapsto (\nabla_\theta\eta_1, \ldots, \nabla_\theta\eta_g) \end{aligned}$$

for every étale scheme (U, u) over $\mathcal{B}_g$ corresponding to the object $(X, \lambda, b)_{/U}$ of $\mathcal{B}_g(U)$, where $b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$, and ∇ denotes the Gauss-Manin connection on $H^1_{\mathrm{dR}}(X/U)$. Then the morphism P

(1) factors through $\mathcal{S}_g \subset \pi_g^*\mathcal{H}_g^{\oplus g}$, and
(2) restricts to an isomorphism $P : T_{\mathcal{B}_g/\mathcal{A}_g} \xrightarrow{\sim} \mathcal{S}_g$.

Definition 5.7. With the above notation, the *Ramanujan subbundle* of $T_{\mathcal{B}_g/\mathbf{Z}}$ is the horizontal subbundle with respect to $\pi_g : \mathcal{B}_g \longrightarrow \mathcal{A}_g$ defined by $\mathcal{R}_g := \ker P$.

We now proceed to the proof of Theorem .50.

5.2.3. Consider the associated space of the vector bundle $\mathcal{H}_g^{\oplus g}$ (cf. [65] 10.2)

$$\mathcal{V}_g := \mathbf{V}((\mathcal{H}_g^{\oplus g})^\vee) = \underline{\mathrm{Spec}}_{\mathcal{A}_g}\mathcal{S}ym((\mathcal{H}_g^{\oplus g})^\vee).$$

This is a Deligne-Mumford stack over $\operatorname{Spec}\mathbf{Z}$ whose objects lying over a scheme U are given by "$(g+2)$-uples"

$$(X, \lambda, \alpha_1, \ldots, \alpha_g)_{/U},$$

where $(X, \lambda)_{/U}$ is an object of $\mathcal{A}_g(U)$, and α_i is a global section of $H^1_{\mathrm{dR}}(X/U)$ for every $1 \leq i \leq g$. Note that the forgetful functor

$$\tilde{\pi}_g : \mathcal{V}_g \longrightarrow \mathcal{A}_g$$

defines a morphism of stacks representable by smooth affine schemes.

We define a morphism of stacks

$$i_g : \mathcal{B}_g \longrightarrow \mathcal{V}_g$$

as follows. Let $(X, \lambda, b)_{/U}$ be an object of $\mathcal{B}_g$ and denote $b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$. Then i_g sends $(X, \lambda, b)_{/U}$ to the object

$$(X, \lambda, \eta_1, \ldots, \eta_g)_{/U}$$

of $\mathcal{V}_g$. The action of i_g on morphisms is evident. Note that the diagram of morphisms of stacks

$$\begin{array}{ccc} \mathcal{B}_g & \xrightarrow{i_g} & \mathcal{V}_g \\ & {\scriptstyle\pi_g}\searrow \quad \swarrow{\scriptstyle\tilde{\pi}_g} & \\ & \mathcal{A}_g & \end{array}$$

is (strictly) commutative.

Lemma 5.8. *The morphism $i_g : \mathcal{B}_g \longrightarrow \mathcal{V}_g$ is an immersion of stacks.*

Proof. Let U be a scheme and $U \longrightarrow \mathcal{V}_g$ be a morphism corresponding to the object $(X, \lambda, \alpha_1, \ldots, \alpha_g)_{/U}$ of $\mathcal{V}_g(U)$. Then the fiber product $\mathcal{B}_g \times_{\mathcal{V}_g} U$ can be naturally identified with the locally closed subscheme of U defined by the equations

$$\overline{\alpha}_1 \wedge \cdots \wedge \overline{\alpha}_g \neq 0$$
$$\langle \alpha_i, \alpha_j \rangle_\lambda = 0, \quad \forall i, j$$

where $\overline{\alpha}_i$ denotes the image of α_i in $H^1_{\mathrm{dR}}(X/U)/F^1(X/U)$ (cf. Proposition .23 (2)). ■

Proof of Theorem .50. To prove (1), let (U, u) be an étale scheme over $\mathcal{B}_g$ corresponding to the object $(X, \lambda, b)_{/U}$ of $\mathcal{B}_g(U)$, with $b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$, and let θ be a section of $T_{U/\mathbf{Z}}$. As $\langle \eta_i, \eta_j \rangle_\lambda = 0$, we obtain

$$0 = \nabla_\theta \langle \eta_i, \eta_j \rangle_\lambda = \langle \nabla_\theta \eta_i, \eta_j \rangle_\lambda + \langle \eta_i, \nabla_\theta \eta_j \rangle_\lambda = \langle \nabla_\theta \eta_i, \eta_j \rangle_\lambda - \langle \nabla_\theta \eta_j, \eta_i \rangle_\lambda.$$

We now prove (2). Observe that $\mathcal{H}_g^{\oplus g}$ is endowed with an integrable connection ∇ given by the sum of the "universal" Gauss-Manin connection on each factor. As $\tilde{\pi}_g : \mathcal{V}_g \longrightarrow \mathcal{A}_g$ is the space associated to $\mathcal{H}_g^{\oplus g}$, we obtain from Lemma .46 a morphism of $\mathcal{O}_{\mathcal{V}_{g,\text{ét}}}$-modules

$$P_\nabla : T_{\mathcal{V}_g/\mathbf{Z}} \longrightarrow \tilde{\pi}_g^* \mathcal{H}_g^{\oplus g}$$

inducing an isomorphism

$$T_{\mathcal{V}_g/\mathcal{A}_g} \xrightarrow{\sim} \tilde{\pi}_g^* \mathcal{H}_g^{\oplus g}.$$

The morphism P is simply the restriction of P_∇ to $T_{\mathcal{B}_g/\mathbf{Z}}$ via the immersion $i_g : \mathcal{B}_g \longrightarrow \mathcal{V}_g$. In particular, as Ti_g identifies $T_{\mathcal{B}_g/\mathcal{A}_g}$ with a subbundle of $i_g^* T_{\mathcal{V}_g/\mathcal{A}_g}$, the induced the morphism

$$P_\nabla = P : T_{\mathcal{B}_g/\mathcal{A}_g} \longrightarrow \mathcal{S}_g$$

is injective; since both vector bundles have the same rank (cf. Remark .49), this must be an isomorphism. ■

Remark 5.9. It follows from the above proof and from Remark .47 that the Ramanujan subbundle $\mathcal{R}_g \subset T_{\mathcal{B}_g/\mathbf{Z}}$ is integrable.

5.3. **The Ramanujan subbundle $\mathcal{R}_F \subset T_{\mathcal{B}_F/\mathbf{Z}}$.** Let S be a scheme, U be a smooth S-scheme, and (X, λ, m) be a principally polarized abelian scheme with R-multiplication over U.

Since, for every $r \in R$, the endomorphism $m(r) : X \longrightarrow X$ is an isogeny (Remark .33), the action of R on $H^1_{\mathrm{dR}}(X/U)$ induced by m is horizontal for the Gauss-Manin connection ∇ (cf. [52] Proposition 2.2). In particular, by linearity, ∇ induces a connection on $H^1_{\mathrm{dR}}(X/U) \otimes_R D$; by abuse, we denote it by the same symbol:

$$\nabla : H^1_{\mathrm{dR}}(X/U) \otimes_R D \longrightarrow (H^1_{\mathrm{dR}}(X/U) \otimes_R D) \otimes_{\mathcal{O}_U} \Omega^1_{U/S}.$$

By the same reasoning of (5.2.1), we define a universal first de Rham cohomology $\mathcal{H}_F$ over $\mathcal{A}_F$. For any scheme S, we denote by $\mathcal{H}_{F,S}$ the vector bundle over $\mathcal{A}_{F,S}$ obtained from $\mathcal{A}_F$ by base change. We also have a universal Gauss-Manin connection

$$\nabla : \mathcal{H}_{F,S} \longrightarrow \mathcal{H}_{F,S} \otimes_{\mathcal{O}_{\mathcal{A}_{F,S,\text{ét}}}} \Omega^1_{\mathcal{A}_{F,S}/S}.$$

Note that the vector bundle $\mathcal{H}_{F,S}$ over $\mathcal{A}_{F,S}$ is endowed with a canonical R-multiplication which is horizontal for the universal Gauss-Manin connection above. In particular, we also have a connection

$$\nabla : \mathcal{H}_{F,S} \otimes_R D \longrightarrow (\mathcal{H}_{F,S} \otimes_R D) \otimes_{\mathcal{O}_{\mathcal{A}_{F,S,\text{ét}}}} \Omega^1_{\mathcal{A}_{F,S}/S}.$$

Theorem 5.10. *Consider the morphism of quasi-coherent $\mathcal{O}_{\mathcal{B}_{F,\text{ét}}}$-modules*

$$P : T_{\mathcal{B}_F/\mathbf{Z}} \longrightarrow \pi_F^* \mathcal{H}_F \otimes_R D$$

defined by

$$\begin{aligned} T_{U/\mathbf{Z}} &\longrightarrow H^1_{\text{dR}}(X/U) \otimes_R D \\ \theta &\longmapsto \nabla_\theta \eta \end{aligned}$$

for every étale scheme (U, u) over $\mathcal{B}_F$ corresponding to the object $(X, \lambda, m, b)_{/U}$ of $\mathcal{B}_F(U)$, where $b = (\omega, \eta)$, and ∇ denotes the Gauss-Manin connection on $H^1_{\text{dR}}(X/U) \otimes_R D$. Then the morphism P restricts to an isomorphism

$$P : T_{\mathcal{B}_F/\mathcal{A}_F} \xrightarrow{\sim} \pi_F^* \mathcal{H}_F \otimes_R D. \tag{5.4}$$

The proof below is analogous to the case $g = 1$ of Theorem .50.

Proof. Consider the stack

$$\mathcal{V}_F := \mathbf{V}((\mathcal{H}_F \otimes_R D)^\vee),$$

and denote by $\tilde{\pi}_F : \mathcal{V}_F \longrightarrow \mathcal{A}_F$ the natural projection. Let ∇ be the universal Gauss-Manin connection on $\mathcal{H}_F \otimes_R D$, and let

$$P_\nabla : T_{\mathcal{V}_F/\mathbf{Z}} \longrightarrow \tilde{\pi}_F^* \mathcal{H}_F \otimes_R D$$

be defined as in Lemma .46, so that it induces an isomorphism

$$T_{\mathcal{V}_F/\mathcal{A}_F} \xrightarrow{\sim} \tilde{\pi}_F^* \mathcal{H}_F \otimes_R D.$$

It follows from Remark .36 that the morphism

$$i_F : \mathcal{B}_F \longrightarrow \mathcal{V}_F$$

over $\mathcal{A}_F$ given by $(X, \lambda, m, b = (\omega, \eta))_{/U} \longmapsto (X, \lambda, m, \eta)_{/U}$ is an *open immersion* of stacks. We conclude by remarking that the morphism P is simply the restriction of the above P_∇ to $T_{\mathcal{B}_F/\mathbf{Z}}$ via i_F. ∎

Definition 5.11. With the above notation, the *Ramanujan subbundle* of $T_{\mathcal{B}_F/\mathbf{Z}}$ is the horizontal subbundle with respect to $\pi_F : \mathcal{B}_F \longrightarrow \mathcal{A}_F$ defined by $\mathcal{R}_F := \ker P$.

Observe that the Ramanujan subbundle $\mathcal{R}_F \subset T_{\mathcal{B}_F/\mathbf{Z}}$ is integrable by Remark .47.

Remark 5.12. The morphism $f_t : \mathcal{B}_F \longrightarrow \mathcal{B}_g$ defined in Remark .40 preserves the decomposition of the tangent bundles of $\mathcal{B}_F$ and $\mathcal{B}_g$ induced by the Ramanujan subbundles. Observe first that the commutativity of the diagram (4.1) implies that $Tf_t : T_{\mathcal{B}_F/\mathbf{Z}} \longrightarrow f_t^* T_{\mathcal{B}_g/\mathbf{Z}}$ preserves the vertical subbundles:

$$Tf_t(T_{\mathcal{B}_F/\mathcal{A}_F}) \subset f_t^* T_{\mathcal{B}_g/\mathcal{A}_g}.$$

Now, it follows from the definition of the Ramanujan subbundles that $\mathcal{R}_g$ (resp. $\mathcal{R}_F$) is given by the equations $\nabla_v \eta_i = 0$ (resp. $\nabla_v \eta_F(1 \otimes x_i) = 0$), for $1 \leq i \leq g$, where $(\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ (resp.

(ω_F, η_F)) denotes the "universal" symplectic Hodge basis over $\mathcal{B}_g$ (resp. $\mathcal{B}_F$). Since, by definition of f_t, we have $f_t^*\eta_i = \eta_F(1\otimes x_i)$, and since the formation of the Gauss-Manin connection commutes with base change, we deduce that

$$Tf_t(\mathcal{R}_F) \subset f_t^*\mathcal{R}_g.$$

5.4. **Recollections on the Kodaira-Spencer morphism.**

5.4.1. Fix a base scheme S and let $p : X \longrightarrow U$ be a projective abelian scheme, with U a smooth S-scheme. The Gauss-Manin connection on $H^1_{\mathrm{dR}}(X/U)$ induces a morphism

$$\begin{aligned} T_{U/S} &\longrightarrow \mathcal{H}om_{\mathcal{O}_S}(H^1_{\mathrm{dR}}(X/U), H^1_{\mathrm{dR}}(X/U)) \\ \theta &\longmapsto \nabla_\theta(\ \ ). \end{aligned}$$

Restricting to $F^1(X/U)$ and passing to the quotient (cf. exact sequence (2.1)), we obtain an $\mathcal{O}_U$-morphism

$$T_{U/S} \longrightarrow \mathcal{H}om_{\mathcal{O}_U}(F^1(X/U), R^1p_*\mathcal{O}_X) \cong F^1(X/U)^\vee \otimes_{\mathcal{O}_U} R^1p_*\mathcal{O}_X.$$

Applying the inverse of the canonical isomorphism $\phi^1_{X^t/U} : F^1(X^t/U)^\vee \xrightarrow{\sim} R^1p_*\mathcal{O}_X$ (cf. proof of Lemma .26, where we identified X with X^{tt} via the canonical biduality isomorphism), we obtain an $\mathcal{O}_U$-morphism

$$\delta : T_{U/S} \longrightarrow F^1(X/U)^\vee \otimes_{\mathcal{O}_U} F^1(X^t/U)^\vee.$$

This is, possibly up to a sign, the dual of ρ defined in [24] III.9.[8]

5.4.2. With the same notation as above, let $\lambda : X \longrightarrow X^t$ be a principal polarization. The Gauss-Manin connection ∇ on $H^1_{\mathrm{dR}}(X/U)$ is compatible with the symplectic form $\langle\ ,\ \rangle_\lambda$ in the following sense. For every sections θ of $T_{U/S}$, and α and β of $H^1_{\mathrm{dR}}(X/U)$, we have

$$\theta\langle\alpha,\beta\rangle_\lambda = \langle\nabla_\theta\alpha,\beta\rangle_\lambda + \langle\alpha,\nabla_\theta\beta\rangle_\lambda. \tag{5.5}$$

This can be deduced from the fact that the first Chern class in $H^2_{\mathrm{dR}}(X\times_U X^t/U)$ of the Poincaré line bundle $\mathcal{P}_{X/U}$ is horizontal for the Gauss-Manin connection, since it actually comes from a class in $H^2_{\mathrm{dR}}(X\times_U X^t/S)$.

By composing δ with $((\lambda^*)^\vee)^{-1} : F^1(X^t/U)^\vee \xrightarrow{\sim} F^1(X/U)^\vee$, we obtain a morphism

$$\kappa : T_{U/S} \longrightarrow F^1(X/U)^\vee \otimes_{\mathcal{O}_U} F^1(X/U)^\vee. \tag{5.6}$$

This is the *Kodaira-Spencer morphism* associated to $(X,\lambda)_{/U}$ over S. It follows from the compatibility (5.5) that κ factors through the second divided power $\Gamma^2(F^1(X/U)^\vee)$, i.e., the submodule of symmetric tensors in $F^1(X/U)^\vee \otimes_{\mathcal{O}_U} F^1(X/U)^\vee$.

[8] With notation as in the proof of Lemma .26, there are two natural ways of identifying $R^1p_*\mathcal{O}_X$ with $F^1(X^t/U)^\vee$: one by $(\phi^0_{X/U})^\vee$, and another by $\phi^1_{X^t/U}$. These produce the same isomorphisms up to a sign. In [24] this choice is not specified.

Remark 5.13. As $\phi^\vee_{X^t/U} = -\phi_{X/U}$ under the canonical biduality isomorphism $X \cong X^{tt}$ (cf. [2] Lemme 5.1.5), one may verify that the composition

$$R^1p_*\mathcal{O}_X \stackrel{(\phi^1_{X^t/U})^{-1}}{\longrightarrow} F^1(X^t/U)^\vee \stackrel{((\lambda^*)^\vee)^{-1}}{\longrightarrow} F^1(X/U)^\vee$$

considered above is given by the isomorphism of vector bundles $H^1_{\mathrm{dR}}(X/U)/F^1(X/U) \xrightarrow{\sim} F^1(X/U)^\vee$ induced by (cf. Lemma .16)

$$\begin{aligned} H^1_{\mathrm{dR}}(X/U) &\longrightarrow H^1_{\mathrm{dR}}(X/U)^\vee \\ \alpha &\longmapsto \langle\ \ ,\alpha\rangle_\lambda. \end{aligned}$$

Thus, if $(\omega_1, \dots, \omega_g)$ is a trivialization of $F^1(X/U)$, κ admits the following explicit description:

$$\kappa(\theta) = \sum_{i=1}^g \omega_i^\vee \otimes \langle\ \ , \nabla_\theta \omega_i\rangle_\lambda.$$

Finally, we observe that the Kodaira-Spencer morphism is natural in the following sense. Let U' be a smooth scheme over S and let $F_{/f} : (X', \lambda')_{/U'} \longrightarrow (X, \lambda)_{/U'}$ be a morphism in $\mathcal{A}_{g,S}$. Denote by κ (resp. κ') the Kodaira-Spencer morphism associated to $(X, \lambda)_{/U}$ (resp. $(X', \lambda')_{/U'}$) over S. Then the diagram

$$\begin{array}{ccc} T_{U'/S} & \xrightarrow{\quad Df \quad} & f^*T_{U/S} \\ {\scriptstyle \kappa'}\downarrow & & \downarrow{\scriptstyle f^*\kappa} \\ \Gamma^2(F^1(X'/U')^\vee) & \xrightarrow[(f^*)^\vee \otimes (f^*)^\vee]{} & \Gamma^2(f^*F^1(X/U)^\vee) \end{array}$$

commutes.

5.4.3. We keep the above notation and we further assume that $(X, \lambda)_{/U}$ is endowed with an R-multiplication $m : R \longrightarrow \mathrm{End}_U(X)^\lambda$.

Since the action of R on $H^1_{\mathrm{dR}}(X/U)$ is horizontal for the Gauss-Manin connection, we obtain an $\mathcal{O}_U$-morphism

$$\begin{aligned} T_{U/S} &\longrightarrow \mathcal{H}om_{\mathcal{O}_U \otimes R}(F^1(X/U), R^1p_*\mathcal{O}_X) \\ \theta &\longmapsto \nabla_\theta(\ \ ) \mod F^1(X/U). \end{aligned}$$

By combining this with the $\mathcal{O}_U \otimes R$-isomorphism induced by Ψ_λ

$$\begin{aligned} R^1p_*\mathcal{O}_X = H^1_{\mathrm{dR}}(X/U)/F^1(X/U) &\xrightarrow{\sim} F^1(X/U)^* \otimes_R D^{-1} \\ \alpha \mod F^1(X/U) &\longmapsto \Psi_\lambda(\ \ ,\alpha) \end{aligned}$$

we obtain a Kodaira-Spencer morphism (of $\mathcal{O}_U$-modules)

$$\kappa : T_{U/S} \longrightarrow \Gamma^2_{\mathcal{O}_U \otimes R}(F^1(X/U)^*) \otimes_R D^{-1}$$

associated to $(X, \lambda, m)_{/U}$ over S.

Remark 5.14. If ω is an $\mathcal{O}_U \otimes R$-trivialization of $F^1(X/U)$, then

$$\kappa(\theta) = \omega^* \otimes \Psi_\lambda(\ \ , \nabla_\theta \omega) = \Psi_\lambda(\omega, \nabla_\theta\omega)\, \omega^* \otimes \omega^*.$$

Remark 5.15. By the natural duality between second divided powers Γ^2 and second symmetric powers S^2, we get the following canonical isomorphisms (cf. Remark .30)

$$\Gamma^2_{\mathcal{O}_U\otimes R}(F^1(X/U)^*)\otimes_R D^{-1}\cong S^2_{\mathcal{O}_U\otimes R}(F^1(X/U))^*\otimes_R D^{-1}\overset{\mathrm{Tr}}{\cong} S^2_{\mathcal{O}_U\otimes R}(F^1(X/U))^\vee.$$

Under these identifications, the $\mathcal{O}_U$-dual of κ is given explicitly by

$$\kappa^\vee : S^2_{\mathcal{O}_U\otimes R}(F^1(X/U))\longrightarrow \Omega^1_{U/S}$$
$$\omega\otimes\omega\longmapsto \langle\omega,\nabla\omega\rangle_\lambda.$$

5.5. The Kodaira-Spencer isomorphism for $\mathcal{A}_g$ and $\mathcal{A}_F$.

5.5.1. Just like we defined a universal first de Rham cohomology $\mathcal{H}_g$ over $\mathcal{A}_g$, we may define a universal Hodge subbundle $\mathcal{F}_g$: for any étale scheme (U,u) over $\mathcal{A}_g$ corresponding to the object $(X,\lambda)_{/U}$ of $\mathcal{A}_g(U)$ we have $u^*\mathcal{F}_g = F^1(X/U)$.

Let S be a scheme, and denote by $\mathcal{F}_{g,S}$ the rank g vector bundle over $\mathcal{A}_{g,S}$ obtained from $\mathcal{F}_g$ by base change. The naturality of the Kodaira-Spencer morphism permits us to construct a "universal" Kodaira-Spencer morphism

$$\kappa : T_{\mathcal{A}_{g,S}}\longrightarrow \Gamma^2(\mathcal{F}^\vee_{g,S}).$$

We remark that κ is actually an *isomorphism* of $\mathcal{O}_{\mathcal{A}_{g,S,\text{ét}}}$-modules by [24] Theorem 5.7.(3) (cf. [46] 2.3.5).

Let $\mathcal{U}$ be a smooth Deligne-Mumford stack over S and $u:\mathcal{U}\longrightarrow\mathcal{A}_{g,S}$ be a quasi-compact and quasi-separated morphism of S-stacks representable by schemes. Then, the Gauss-Manin connection over $(\mathcal{U},u)$, or simply over $\mathcal{U}$ if u is implicit,

$$\nabla : u^*\mathcal{H}_{g,S}\longrightarrow u^*\mathcal{H}_{g,S}\otimes_{\mathcal{O}_{\mathcal{U}_{\text{ét}}}}\Omega^1_{\mathcal{U}/S}$$

is defined by pulling back the universal Gauss-Manin connection on $\mathcal{A}_{g,S}$. Further, we may define a Kodaira-Spencer morphism over $(\mathcal{U},u)$ as the composition

$$\kappa_u : T_{\mathcal{U}/S}\xrightarrow{Tu} u^*T_{\mathcal{A}_{g,S}/S}\xrightarrow{u^*\kappa}\Gamma^2(u^*\mathcal{F}^\vee_{g,S}).$$

5.5.2. Analogously, we define a Hodge subbundle $\mathcal{F}_F\subset\mathcal{H}_F$ endowed with a canonical R-multiplication. For any scheme S, we also have a "universal" Kodaira-Spencer *isomorphism* (cf. [70] 1.5 and [46] 2.3.5)

$$\kappa : T_{\mathcal{A}_{F,S}}\xrightarrow{\sim}\Gamma^2_{\mathcal{O}_{\mathcal{A}_{F,S,\text{ét}}}\otimes R}(\mathcal{F}^*_{F,S})\otimes_R D^{-1}.$$

For a smooth Deligne-Mumford stack $\mathcal{U}$ over S endowed with a quasi-compact and quasi-separated morphism of S-stacks representable by schemes $u:\mathcal{U}\longrightarrow\mathcal{A}_{F,S}$, we can also associate a Gauss-Manin connection

$$\nabla : u^*\mathcal{H}_{F,S}\longrightarrow u^*\mathcal{H}_{F,S}\otimes_{\mathcal{O}_{\mathcal{U}_{\text{ét}}}}\Omega^1_{\mathcal{U}/S}$$

and a Kodaira-Spencer morphism

$$\kappa_u : T_{\mathcal{U}/S}\longrightarrow\Gamma^2_{\mathcal{O}_{\mathcal{U}_{\text{ét}}}\otimes R}(u^*\mathcal{F}^*_{F,S})\otimes_R D^{-1}.$$

5.6. **The higher Ramanujan vector fields on $\mathcal{B}_g$.** Recall that the Ramanujan subbundle $\mathcal{R}_g \subset T_{\mathcal{B}_g/\mathbf{Z}}$ is a horizontal subbundle with respect to $\pi_g : \mathcal{B}_g \longrightarrow \mathcal{A}_g$. In particular, the tangent map

$$T\pi_g : \mathcal{R}_g \longrightarrow \pi_g^* T_{\mathcal{A}_g/\mathbf{Z}}$$

is an isomorphism. By composing it with (the pullback by π_g of) the Kodaira-Spencer isomorphism for $\mathcal{A}_g$, we obtain an isomorphism

$$\kappa_{\pi_g} : \mathcal{R}_g \xrightarrow{\sim} \Gamma^2(\pi_g^*\mathcal{F}_g^\vee). \tag{5.7}$$

Consider the "universal" symplectic-Hodge basis over $\mathcal{B}_g$

$$b_g = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g);$$

that is, the basis of the vector bundle $\pi_g^*\mathcal{H}_g$ such that for every étale scheme (U, u) over $\mathcal{B}_g$ corresponding to the object $(X, \lambda, b)_{/U}$ of $\mathcal{B}_g(U)$ we have $u^*b_g = b$. In particular, $(\omega_1, \ldots, \omega_g)$ trivializes $\pi_g^*\mathcal{F}_g$, and its dual basis induces an isomorphism

$$\Gamma^2(\pi_g^*\mathcal{F}_g^\vee) \xrightarrow{\sim} \Gamma^2(\mathcal{O}_{\mathcal{B}_{g,\text{ét}}}^{\oplus g}) = \mathcal{O}_{\mathcal{B}_{g,\text{ét}}} \otimes \Gamma^2(\mathbf{Z}^g).$$

By composing the above isomorphism with (5.7), we obtain

$$\mathcal{R}_g \xrightarrow{\sim} \Gamma^2(\mathcal{O}_{\mathcal{B}_{g,\text{ét}}}^{\oplus g}) = \mathcal{O}_{\mathcal{B}_{g,\text{ét}}} \otimes \Gamma^2(\mathbf{Z}^g). \tag{5.8}$$

Definition 5.16. For every $1 \leq i \leq j \leq g$, we define the *higher Ramanujan vector field* v_{ij} as being the unique global section of $\mathcal{R}_g \subset T_{\mathcal{B}_g/\mathbf{Z}}$ such that

$$v_{ij} \longmapsto \begin{cases} \mathbf{e}_i \otimes \mathbf{e}_i & i = j \\ \mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i & i < j \end{cases}$$

under the isomorphism (5.8).

Alternatively, let

$$\langle\ ,\ \rangle : \pi_g^*\mathcal{H}_g \times \pi_g^*\mathcal{H}_g \longrightarrow \mathcal{O}_{\mathcal{B}_{g,\text{ét}}}$$

be the symplectic $\mathcal{O}_{\mathcal{B}_{g,\text{ét}}}$-bilinear form given, for each étale scheme (U, u) over $\mathcal{B}_g$ corresponding to the object $(X, \lambda, b)_{/U}$ of $\mathcal{B}_g(U)$, by

$$u^*\langle\ ,\ \rangle := \langle\ ,\ \rangle_\lambda : H^1_{\text{dR}}(X/U) \times H^1_{\text{dR}}(X/U) \longrightarrow \mathcal{O}_U.$$

This is well-defined by Remark .27. Then the higher Ramanujan vector fields satisfy

$$\kappa_{\pi_g}(v_{ij}) = \begin{cases} \langle\ \ ,\eta_i\rangle \otimes \langle\ \ ,\eta_i\rangle & i = j \\ \langle\ \ ,\eta_i\rangle \otimes \langle\ \ ,\eta_j\rangle + \langle\ \ ,\eta_j\rangle \otimes \langle\ \ ,\eta_i\rangle & i < j \end{cases}$$

The next proposition characterizes the higher Ramanujan vector fields in terms of the "universal" Gauss-Manin connection over $\mathcal{B}_g$ (cf. Paragraph 5.5):

$$\nabla : \pi_g^*\mathcal{H}_g \longrightarrow \pi_g^*\mathcal{H}_g \otimes_{\mathcal{O}_{\mathcal{B}_{g,\text{ét}}}} \Omega^1_{\mathcal{B}_g/\mathbf{Z}}.$$

Proposition 5.17. *Let us regard b_g as a row vector of order $2g$. Then, the higher Ramanujan vector fields are the unique global sections v_{ij} of $T_{\mathcal{B}_g/\mathbf{Z}}$ such that*

$$\nabla_{v_{ij}} b_g = b_g \begin{pmatrix} 0 & 0 \\ \mathbf{E}^{ij} & 0 \end{pmatrix}$$

for every $1 \leq i \leq j \leq g$.

Remark 5.18. The matricial equation above is equivalent to conditions (1) and (2) below

(1) $\nabla_{v_{ij}} \omega_i = \eta_j$, $\nabla_{v_{ij}} \omega_j = \eta_i$, and $\nabla_{v_{ij}} \omega_k = 0$ for $k \notin \{i, j\}$.
(2) $\nabla_{v_{ij}} \eta_k = 0$, for every $1 \leq k \leq g$.

Proof of Proposition .61. The vector fields v_{ij} satisfy (2) in the above remark by definition of $\mathcal{R}_g$. Moreover, using the explicit expression of the Kodaira-Spencer morphism in Remark .57, we see that

$$\sum_{k=1}^{g} \langle \ \ , \eta_k \rangle \otimes \langle \ \ , \nabla_{v_{ij}} \omega_k \rangle = \begin{cases} \langle \ \ , \eta_i \rangle \otimes \langle \ \ , \eta_i \rangle & i = j \\ \langle \ \ , \eta_i \rangle \otimes \langle \ \ , \eta_j \rangle + \langle \ \ , \eta_j \rangle \otimes \langle \ \ , \eta_i \rangle & i < j \end{cases} \tag{5.9}$$

in $\Gamma^2(\pi_g^* \mathcal{F}_g^\vee)$ for every $1 \leq i \leq j \leq g$. As b_g is symplectic with respect to $\langle \ , \ \rangle$, by evaluating the second factors at η_l for every $1 \leq l \leq g$ in the above equation, we see that $\nabla_{v_{ij}} \omega_k$ lies in the subbundle of $\pi_g^* \mathcal{H}_g$ generated by $\eta_1, \ldots, \eta_g$, for every $1 \leq i \leq j \leq g$ and $1 \leq k \leq g$.

Thus, to prove that the vector fields v_{ij} satisfy (1), it is sufficient to prove that

$$\langle \omega_l, \nabla_{v_{ij}} \omega_i \rangle = \delta_{lj}, \ \langle \omega_l, \nabla_{v_{ij}} \omega_j \rangle = \delta_{li}, \text{ and } \langle \omega_l, \nabla_{v_{ij}} \omega_k \rangle = 0 \text{ for } k \notin \{i, j\} \tag{5.10}$$

for every $1 \leq l \leq g$. This in turn follows immediately from (5.9) by evaluating the second factors at ω_l.

To prove unicity, let $(w_{ij})_{1 \leq i \leq j \leq g}$ be a family of vector fields on $\mathcal{B}_g$ satisfying (1) and (2). It follows immediately from (2) that each w_{ij} is a section of $\mathcal{R}_g$. Moreover, by the explicit expression of the Kodaira-Spencer morphism in Remark .57, the equations in (1) imply that

$$\kappa_{\pi_g}(w_{ij}) = \begin{cases} \langle \ \ , \eta_i \rangle \otimes \langle \ \ , \eta_i \rangle & i = j \\ \langle \ \ , \eta_i \rangle \otimes \langle \ \ , \eta_j \rangle + \langle \ \ , \eta_j \rangle \otimes \langle \ \ , \eta_i \rangle & i < j \end{cases}$$

Since $\kappa_{\pi_g} : \mathcal{R}_g \longrightarrow \Gamma^2(\pi_g^* \mathcal{F}_g^\vee)$ is an isomorphism, we must have $w_{ij} = v_{ij}$. ■

Lemma 5.19. *Let S be a scheme, and θ be a section of $T_{\mathcal{B}_{g,S}/S}$ such that $\nabla_\theta \omega_i = \nabla_\theta \eta_i = 0$ for every $1 \leq i \leq g$. Then $\theta = 0$.*

Proof. Let θ be as in the statement. Note that θ is in the subbundle $\mathcal{R}_{g,S}$ of $T_{\mathcal{B}_{g,S}/S}$; thus, there exist sections $(f_{ij})_{1 \leq i \leq j \leq g}$ of $\mathcal{O}_{\mathcal{B}_{g,S,\text{ét}}}$ such that

$$\theta = \sum_{1 \leq i \leq j \leq g} f_{ij} v_{ij}.$$

We prove that each $f_{ij} = 0$ by induction on i. For $i = 1$, we have by Proposition .61

$$0 = \nabla_\theta \omega_1 = \sum_{1 \leq i \leq j \leq g} f_{ij} \nabla_{v_{ij}} \omega_1 = \sum_{j=1}^{g} f_{1j} \eta_j,$$

thus $f_{1j} = 0$ for every $1 \leq j \leq g$. Let $2 \leq i_0 \leq g$, and assume that $f_{ij} = 0$ for every $i < i_0$ and $i \leq j \leq g$. From

$$0 = \nabla_\theta \omega_{i_0} = \sum_{i_0 \leq i \leq j \leq g} f_{ij} \nabla_{v_{ij}} \omega_{i_0} = \sum_{j=i_0}^{g} f_{i_0 j} \eta_j$$

we conclude that $f_{i_0 j} = 0$ for every $i_0 \leq j \leq g$. ■

Let $[\ ,\ ]$ denote the Lie bracket in $T_{\mathcal{B}_g/\mathbf{Z}}$.

Corollary 5.20. *The higher Ramanujan vector fields commute. That is,*

$$[v_{ij}, v_{i'j'}] = 0$$

for any $1 \leq i \leq j \leq g$ *and* $1 \leq i' \leq j' \leq g$.

Proof. We already remarked that $\mathcal{R}_g$ is integrable (Remark .53). In particular, for any $1 \leq i \leq j \leq g$ and any $1 \leq i' \leq j' \leq g$, the vector field $\theta \coloneqq [v_{ij}, v_{i'j'}]$ is a section of $\mathcal{R}_g$. By Lemma .63, to prove that $\theta = 0$, it is sufficient to prove that $\nabla_\theta \omega_k = 0$ for every $1 \leq k \leq g$.

We have

$$\nabla_\theta \omega_k = \nabla_{v_{ij}}(\nabla_{v_{i'j'}} \omega_k) - \nabla_{v_{i'j'}}(\nabla_{v_{ij}} \omega_k).$$

It follows from Proposition .61 that $\nabla_{v_{i'j'}} \omega_k$ (resp. $\nabla_{v_{ij}} \omega_k$) is an element of $\{0, \eta_1, \ldots, \eta_g\}$; hence $\nabla_{v_{ij}}(\nabla_{v_{i'j'}} \omega_k) = 0$ (resp. $\nabla_{v_{i'j'}}(\nabla_{v_{ij}} \omega_k) = 0$). ■

5.7. **The higher Ramanujan vector fields on $\mathcal{B}_F$.** We argue as in the Siegel case: since the Ramanujan subbundle $\mathcal{R}_F \subset T_{\mathcal{B}_F/\mathbf{Z}}$ is horizontal with respect to $\pi_F : \mathcal{B}_F \longrightarrow \mathcal{A}_F$, the tangent map

$$T\pi_F : \mathcal{R}_F \longrightarrow \pi_F^* T_{\mathcal{A}_F/\mathbf{Z}}$$

is an isomorphism. By composing it with the Kodaira-Spencer isomorphism for $\mathcal{A}_F$, we obtain an isomorphism

$$\kappa_{\pi_F} : \mathcal{R}_F \xrightarrow{\sim} \Gamma^2_{\mathcal{O}_{\mathcal{A}_{F,\text{ét}}} \otimes R}(\pi_F^* \mathcal{F}_F^*) \otimes_R D^{-1}.$$

Let $b_F \coloneqq (\omega_F, \eta_F)$ be the "universal" symplectic-Hodge basis over $\mathcal{B}_F$. By duality, the trivialization of $\pi_F^* \mathcal{F}_F$ as a (rank 1) $\mathcal{O}_{\mathcal{B}_{F,\text{ét}}} \otimes R$-module given by ω_F induces a trivialization of $\pi_F^* \mathcal{F}_F^*$. As the $\mathbf{Z}$-module $\Gamma^2(\mathbf{Z})$ may be canonically identified with $\mathbf{Z}$, we then obtain an isomorphism (of $\mathcal{O}_{\mathcal{B}_{F,\text{ét}}}$-modules)

$$\mathcal{R}_F \xrightarrow{\sim} \mathcal{O}_{\mathcal{B}_{F,\text{ét}}} \otimes D^{-1}. \tag{5.11}$$

Definition 5.21. The *higher Ramanujan vector field* over $\mathcal{B}_F$ is the $\mathcal{O}_{\mathcal{B}_{F,\text{ét}}}$-isomorphism

$$v_F : \mathcal{O}_{\mathcal{B}_{F,\text{ét}}} \otimes D^{-1} \xrightarrow{\sim} \mathcal{R}_F$$

given by the inverse of (5.11).

Strictly speaking, v_F is not a vector field on $\mathcal{B}_F$, but for any fixed choice of $\mathbf{Z}$-basis of D^{-1} it determines g *bona fide* vector fields trivializing $\mathcal{R}_F$.

If we endow the tangent bundle $T_{\mathcal{B}_F/\mathbf{Z}} = T_{\mathcal{B}_F/\mathcal{A}_F} \oplus \mathcal{R}_F$ with the R-multiplication induced by the isomorphisms (5.4) and (5.11), then v_F is $\mathcal{O}_{\mathcal{B}_{F,\text{ét}}} \otimes R$-linear, and can be thought as a global section of $T_{\mathcal{B}_F/\mathbf{Z}} \otimes_R D$.

As the Gauss-Manin connection on $\pi_F^*\mathcal{H}_F$ is R-linear, it induces, for any fractional ideal $I \subset F$, an $\mathcal{O}_{\mathcal{B}_{F,\text{ét}}} \otimes R$-morphism

$$T_{\mathcal{B}_F/\mathbf{Z}} \otimes_R I \longrightarrow \mathcal{H}om_R(\pi_F^*\mathcal{H}_F, \pi_F^*\mathcal{H}_F \otimes_R I).$$

We omit the proof of the analogous of Proposition .61.

Proposition 5.22. *The higher Ramanujan vector field v_F is the unique global section of $T_{\mathcal{B}_F/\mathbf{Z}} \otimes_R D$ such that $\nabla_{v_F}\omega = \eta$ and $\nabla_{v_F}\eta = 0$.* ■

Remark 5.23. As an application of Propositions .61 and .66, we can compute the effect of the morphism $f_t : \mathcal{B}_F \longrightarrow \mathcal{B}_g$ of Remark .40 on the higher Ramanujan vector fields. Namely, one may check that the following diagram commutes

$$\begin{array}{ccc} T_{\mathcal{B}_F/\mathbf{Z}} & \xrightarrow{\quad Tf_T \quad} & f_t^* T_{\mathcal{B}_g/\mathbf{Z}} \\ {\scriptstyle v_F}\uparrow & & \uparrow{\scriptstyle f_t^*(v_{ij})_{1\leq i\leq j\leq g}} \\ \mathcal{O}_{\mathcal{B}_F} \otimes D^{-1} & \longrightarrow & f_t^*(\mathcal{O}_{\mathcal{B}_g} \otimes \mathrm{Sym}_g(\mathbf{Z})) \end{array}$$

where the bottom arrow is induced by the morphism of abelian groups

$$\begin{aligned} D^{-1} &\longrightarrow \mathrm{Sym}_g(\mathbf{Z}) \\ x &\longmapsto (\mathrm{Tr}(r_i r_j x))_{1\leq i,j\leq g}. \end{aligned}$$

6. Integral solution of the higher Ramanujan equations

In this section, we define the *higher Ramanujan equations* over $\mathcal{B}_g$ and $\mathcal{B}_F$, and we construct particular solutions of such differential equations defined over $\mathbf{Z}$. The definition of these solutions is based on Mumford's construction of degenerating families of abelian varieties, which we shall not recall in detail. Besides Mumford's original paper [57], the reader may consult [11] 2.3 and [24] III as general references.

Our main theorems here, whose statement are purely algebraic, are immediate corollaries of their analytic counterparts to be proved in Section 11.

6.1. **Higher Ramanujan equations over $\mathcal{B}_g$.** Let $1 \leq i \leq j \leq g$, and q_{ij} be formal variables. For any commutative ring Λ, we denote the ring of formal power series in the variables q_{ij} with coefficients in Λ by

$$\Lambda[\![q_{ij}]\!] \coloneqq \Lambda[\![q_{ij}\ ;\ 1 \leq i \leq j \leq g]\!].$$

We set

$$\Lambda(\!(q_{ij})\!) \coloneqq \Lambda[\![q_{ij}]\!][(\prod_{1\leq i\leq j\leq g} q_{ij})^{-1}].$$

Recall that every Λ-derivation of $\Lambda[\![q_{ij}]\!]$ is continuous for the linear topology given by the ideal generated by the q_{ij}, and that $\mathrm{Der}_\Lambda(\Lambda[\![q_{ij}]\!])$ is freely generated by $\frac{\partial}{\partial q_{ij}}$. In particular, as each q_{ij} is invertible in $\Lambda(\!(q_{ij})\!)$, the derivations

$$\theta_{ij} := q_{ij}\frac{\partial}{\partial q_{ij}}, \quad 1 \leq i \leq j \leq g$$

of $\Lambda(\!(q_{ij})\!)$ form a basis of the $\Lambda(\!(q_{ij})\!)$-module $\mathrm{Der}_\Lambda(\Lambda(\!(q_{ij})\!))$.

Definition 6.1. A *solution of the higher Ramanujan equations over $\mathcal{B}_g$ defined over* Λ is a Λ-morphism (of Deligne-Mumford stacks over Λ)

$$\hat{\varphi} : \mathrm{Spec}\,\Lambda(\!(q_{ij})\!) \longrightarrow \mathcal{B}_{g,\Lambda}$$

such that

$$T\hat{\varphi}(\theta_{ij}) = \hat{\varphi}^* v_{ij}, \quad 1 \leq i \leq j \leq g.$$

A morphism $\hat{\varphi} : \mathrm{Spec}\,\Lambda(\!(q_{ij})\!) \longrightarrow \mathcal{B}_{g,\Lambda}$ as above corresponds to a principally polarized abelian scheme (X,λ) over $\Lambda(\!(q_{ij})\!)$ endowed with a symplectic-Hodge basis b. Let ∇ be the Gauss-Manin connection over $H^1_{\mathrm{dR}}(X/\Lambda(\!(q_{ij})\!))$.

Proposition 6.2. *With the above notation, $\hat{\varphi} : \mathrm{Spec}\,\Lambda(\!(q_{ij})\!) \longrightarrow \mathcal{B}_{g,\Lambda}$ is a solution of the higher Ramanujan equations over $\mathcal{B}_g$ defined over Λ if and only if*

$$\nabla_{\theta_{ij}} b = b\begin{pmatrix} 0 & 0 \\ \mathbf{E}^{ij} & 0 \end{pmatrix}$$

for every $1 \leq i \leq j \leq g$.

Proof. For any (formally) smooth scheme U over Λ and any object $(X,\lambda,b)_{/U}$ of $\mathcal{B}_g(U)$, with $b = (\omega_1,\ldots,\omega_g,\eta_1,\ldots,\eta_g)$, we may consider the $\mathcal{O}_U$-morphism

$$\begin{aligned} \rho : T_{U/\Lambda} &\longrightarrow \Gamma^2(F^1(X/U)^\vee) \oplus H^1_{\mathrm{dR}}(X/U)^{\oplus g} \\ \theta &\longmapsto (\kappa(\theta), \nabla_\theta\eta_1, \ldots, \nabla_\theta\eta_g). \end{aligned}$$

This construction is compatible with base change in U; in particular, if $u : U \longrightarrow \mathcal{B}_{g,\Lambda}$ is the morphism associated to $(X,\lambda,b)_{/U}$, we get a commutative diagram

$$\begin{array}{ccc} T_{U/\Lambda} & \xrightarrow{\quad Tu \quad} & u^*T_{\mathcal{B}_{g,\Lambda}/\Lambda} \\ {\scriptstyle\rho}\downarrow & & \downarrow \\ \Gamma^2(F^1(X/U)^\vee) \oplus H^1_{\mathrm{dR}}(X/U)^{\oplus g} & \xrightarrow{\sim} & u^*(\Gamma^2(\pi^*_{g,\Lambda}\mathcal{F}^\vee_{g,\Lambda}) \oplus \pi^*_{g,\Lambda}\mathcal{H}^{\oplus g}_{g,\Lambda}) \end{array}$$

where the arrow on the right is the pullback by u of the morphism

$$(\kappa_{\pi_g}, P) : T_{\mathcal{B}_{g,\Lambda}/\Lambda} \longrightarrow \Gamma^2(\pi^*_{g,\Lambda}\mathcal{F}^\vee_{g,\Lambda}) \oplus \pi^*_{g,\Lambda}\mathcal{H}^{\oplus g}_{g,\Lambda}$$

which identifies $T_{\mathcal{B}_{g,\Lambda}/\Lambda}$ with the subbundle $\Gamma^2(\pi^*_{g,\Lambda}\mathcal{F}^\vee_{g,\Lambda}) \oplus \mathcal{S}_{g,\Lambda}$ of $\Gamma^2(\pi^*_{g,\Lambda}\mathcal{F}^\vee_{g,\Lambda}) \oplus \pi^*_{g,\Lambda}\mathcal{H}^{\oplus g}_{g,\Lambda}$ by Theorem .50 (cf. Paragraph 5.6).

By taking $u = \hat{\varphi} : \operatorname{Spec} \Lambda(\!(q_{ij})\!) \longrightarrow \mathcal{B}_{g,\Lambda}$ in the above construction, we observe that $\hat{\varphi}$ is a solution of the higher Ramanujan equations if and only if

$$\rho(\theta_{ij}) = \hat{\varphi}^*(\kappa_{\pi_g}(v_{ij}), P(v_{ij}))$$

for every $1 \leq i \leq j \leq g$. By the definition of ρ, our statement now follows from Proposition .61. ■

6.2. **Integral solution of the higher Ramanujan equations; Siegel case.** Let $K \coloneqq \operatorname{Frac} \mathbf{Z}[\![q_{ij}]\!]$. Consider the "period subgroup"

$$Y \coloneqq \langle (q_{1j}, \ldots, q_{gj}) \mid 1 \leq j \leq g \rangle \subset \mathbf{G}_m^g(K),$$

and let

$$\phi : Y \longrightarrow \mathbf{Z}^g (\cong \operatorname{Hom}_{\mathsf{GpSch}}(\mathbf{G}_m^g, \mathbf{G}_m))$$

be the unique group isomorphism such that

$$\phi(q_{1j}, \ldots, q_{gj}) = \mathbf{e}_j, \quad 1 \leq j \leq g.$$

Then, Mumford's construction [57] (cf. [11] 2.3, [24] V.1) canonically attaches to $(\mathbf{G}_m^g, Y, \phi)$ a principally polarized semi-abelian scheme (G, λ) over $\mathbf{Z}[\![q_{ij}]\!]$ of relative dimension g. The restriction of (G, λ) to $\mathbf{Z}(\!(q_{ij})\!)$ is a principally polarized abelian scheme that we denote by $(\hat{X}_g, \hat{\lambda}_g)$.

If we denote $\mathbf{G}_m^g = \operatorname{Spec} \mathbf{Z}[t_1^{\pm 1}, \ldots, t_g^{\pm 1}]$, then the Hodge subbundle $F^1(\hat{X}_g/\mathbf{Z}(\!(q_{ij})\!))$ is canonically trivialized by

$$\hat{\omega}_k \coloneqq \frac{dt_k}{t_k}, \quad 1 \leq k \leq g.$$

Remark 6.3. For $g = 1$, $\hat{X}_1$ is known as the *Tate elliptic curve* over $\mathbf{Z}(\!(q)\!)$, and $\hat{\omega} = dt/t$ is its "canonical differential form". See [19], Paragraph 8, for an explicit algebraic equation of $\hat{X}_1$.

Theorem 6.4. *Let ∇ be the Gauss-Manin connection on $H^1_{\mathrm{dR}}(\hat{X}_g/\mathbf{Z}(\!(q_{ij})\!))$ and, for $1 \leq k \leq g$, define*

$$\hat{\eta}_k \coloneqq \nabla_{\theta_{kk}} \hat{\omega}_k \in H^1_{\mathrm{dR}}(\hat{X}_g/\mathbf{Z}(\!(q_{ij})\!)).$$

Then:

(1) The $2g$-uple $\hat{b}_g \coloneqq (\hat{\omega}_1, \ldots, \hat{\omega}_g, \hat{\eta}_1, \ldots, \hat{\eta}_g)$ is a symplectic-Hodge basis of $(\hat{X}_g, \hat{\lambda}_g)_{/\mathbf{Z}(\!(q_{ij})\!)}$.

(2) The morphism of Deligne-Mumford stacks

$$\hat{\varphi}_g : \operatorname{Spec} \mathbf{Z}(\!(q_{ij})\!) \longrightarrow \mathcal{B}_g$$

given by $(\hat{X}_g, \hat{\lambda}_g, \hat{b}_g)_{/\mathbf{Z}(\!(q_{ij})\!)}$ is a solution of the higher Ramanujan equations over $\mathcal{B}_g$ defined over $\mathbf{Z}$.

■

This result follows directly from its complex analytic counterpart (Theorem .134); see Paragraph 11.4.

Remark 6.5. For concreteness, we have chosen to work with the "coordinates" q_{ij} as above. We refer to [24], p. 138-139, for a discussion on how to generalize some of the above constructions to more general coordinate rings $\mathbf{Z}[\![S^2(\mathbf{Z}^g) \cap \sigma^\vee]\!]$ associated to a rational polyhedral cone σ in the cone of positive definite symmetric bilinear forms on $\mathbf{R}^g$.

Remark 6.6. Note that Mumford's construction yields a semi-abelian scheme over $\operatorname{Spec}\mathbf{Z}[\![q_{ij}]\!]$ which only becomes an abelian scheme (so that it fits into our framework) after inverting q_{ij}. This explains why our solution $\hat{\varphi}_g$ of the higher Ramanujan equations is only defined over $\operatorname{Spec}\mathbf{Z}(\!(q_{ij})\!)$. See also Remark .6.

6.3. **Higher Ramanujan equations over $\mathcal{B}_F$.** From now on, for simplicity, we fix a $\mathbf{Z}$-basis $(x_1,\ldots,x_g)$ of D^{-1} with each x_i totally positive — that is, $\sigma_j(x_i)>0$ for every $1\leq i,j\leq g$ —, and we let $(r_1,\ldots,r_g)$ be its dual $\mathbf{Z}$-basis of R with respect to the trace form.

Let $q^{r_1},\ldots,q^{r_g}$ be formal variables. For any commutative ring Λ, we set

$$\Lambda[\![q^{r_i}]\!]\coloneqq\Lambda[\![q^{r_1},\ldots,q^{r_g}]\!]$$

and

$$\Lambda(\!(q^{r_i})\!)\coloneqq\Lambda[\![q^{r_1},\ldots,q^{r_g}]\!][(\prod_{i=1}^{g}q^{r_i})^{-1}].$$

For every $r\in R$, we denote

$$q^r\coloneqq\prod_{i=1}^{g}(q^{r_i})^{\operatorname{Tr}(rx_i)}\in\Lambda(\!(q^{r_i})\!).$$

As in the Siegel case, note that

$$\theta^{r_i}\coloneqq q^{r_i}\frac{\partial}{\partial q^{r_i}},\quad 1\leq i\leq g$$

form a basis of the $\Lambda(\!(q^{r_i})\!)$-module $\operatorname{Der}_\Lambda(\Lambda(\!(q^{r_i})\!))$. We consider the following isomorphism of $\Lambda(\!(q^{r_i})\!)$-modules:

$$\begin{aligned}\theta_F:\Lambda(\!(q^{r_i})\!)\otimes D^{-1}&\longrightarrow\operatorname{Der}_\Lambda(\Lambda(\!(q^{r_i})\!))\\ 1\otimes x&\longmapsto\sum_{i=1}^{g}\operatorname{Tr}(r_ix)\theta^{r_i}.\end{aligned}$$

Definition 6.7. A *solution of the higher Ramanujan equations over $\mathcal{B}_F$ defined over* Λ is a Λ-morphism of (Deligne-Mumford stacks over Λ)

$$\hat{\varphi}:\operatorname{Spec}\Lambda(\!(q^{r_i})\!)\longrightarrow\mathcal{B}_{F,\Lambda}$$

such that

$$T\hat{\varphi}\circ\theta_F=\hat{\varphi}^*v_F,$$

that is, such that the diagram

$$\begin{array}{ccc}\Lambda(\!(q^{r_i})\!)\otimes D^{-1} & \xrightarrow{\theta_F} & \operatorname{Der}_\Lambda(\Lambda(\!(q^{r_i})\!))\\ \cong\downarrow & & \downarrow T\hat{\varphi}\\ \hat{\varphi}^*(\mathcal{O}_{\mathcal{B}_{F,\Lambda}}\otimes D^{-1}) & \xrightarrow[v_F]{} & \hat{\varphi}^*T_{\mathcal{B}_{F,\Lambda}/\Lambda}\end{array}$$

commutes.

More concretely, if we denote $v_F(1\otimes x_i) =: v^{r_i} \in \Gamma(\mathcal{B}_F, T_{\mathcal{B}_F/\mathbf{Z}})$ for every $1\leq i\leq g$, then $\hat{\varphi}$ is a solution of the higher Ramanujan equations over $\mathcal{B}_F$ if and only if it satisfies

$$T\hat{\varphi}\,(\theta^{r_i}) = \hat{\varphi}^* v^{r_i}, \quad 1\leq i\leq g.$$

A morphism $\hat{\varphi} : \operatorname{Spec}\Lambda(\!(q^{r_i})\!) \longrightarrow \mathcal{B}_{F,\Lambda}$ as above corresponds to a principally polarized abelian scheme with R-multiplication (X,λ,m) over $\Lambda(\!(q^{r_i})\!)$ endowed with a symplectic-Hodge basis b. Let ∇ be the Gauss-Manin connection over $H^1_{\mathrm{dR}}(X/\Lambda(\!(q^{r_i})\!))$.

Proposition 6.8 (cf. Proposition .69)**.** *With the above notation, $\hat{\varphi} : \operatorname{Spec}\Lambda(\!(q^{r_i})\!) \longrightarrow \mathcal{B}_{F,\Lambda}$ is a solution of the higher Ramanujan equations over $\mathcal{B}_F$ defined over Λ if and only if*

$$\nabla_{\theta_F} b = b\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}.$$

■

By considering the $\mathcal{O}_{\mathcal{B}_{F,\Lambda,\text{ét}}}$-*isomorphism* (cf. Theorem .54 and Paragraph 5.5)

$$(\kappa_{\pi_F}, P) : T_{\mathcal{B}_{F,\Lambda}} \xrightarrow{\sim} (\Gamma^2_{\mathcal{O}_{\mathcal{B}_{F,\Lambda,\text{ét}}}\otimes R}(\pi^*_{F,\Lambda}\mathcal{F}^*_{F,\Lambda}) \otimes_R D^{-1}) \oplus (\pi^*_{F,\Lambda}\mathcal{H}_{F,\Lambda}\otimes_R D)$$

the proof of the above proposition is analogous to that of Proposition .69.

6.4. **Integral solution of the higher Ramanujan equations; Hilbert-Blumenthal case.** Let $K := \operatorname{Frac}\mathbf{Z}[\![q^{r_i}]\!]$. Consider the split torus $\mathbf{G}_m\otimes D^{-1}$ over $\operatorname{Spec}\mathbf{Z}$ defined by

$$(\mathbf{G}_m\otimes D^{-1})(\Lambda) = \Lambda^\times \otimes_{\mathbf{Z}} D^{-1}$$

for any commutative ring Λ. Note that the $\mathbf{Z}$-basis $(x_1,\ldots,x_g)$ of D^{-1} induces an isomorphism of group schemes

$$\mathbf{G}_m\otimes D^{-1} \xrightarrow{\sim} \mathbf{G}_m^g \tag{6.1}$$

given on points by

$$t\otimes x \longmapsto (t^{\operatorname{Tr}(r_1 x)},\ldots,t^{\operatorname{Tr}(r_g x)}).$$

To define the period subgroup, consider the morphism of abelian groups

$$\begin{aligned} R &\longrightarrow K^\times \\ r &\longmapsto q^r. \end{aligned}$$

By Remark .30, there exists a unique R-linear morphism

$$\varpi : R\longrightarrow K^\times\otimes D^{-1}$$

such that $\operatorname{Tr}(\varpi(r)) = q^r$ for every $r\in R$. Set

$$Y := \varpi(R)\subset (\mathbf{G}_m\otimes D^{-1})(K).$$

Observe that, since ϖ is injective, it induces an isomorphism of R onto Y. We let

$$\phi := \varpi^{-1} : Y \xrightarrow{\sim} R\left(\cong \operatorname{Hom}_{\mathsf{GpSch}}(\mathbf{G}_m\otimes D^{-1},\mathbf{G}_m)\right).$$

Then Mumford's construction [57] canonically attaches to $(\mathbf{G}_m\otimes D^{-1}, Y, \phi)$ a principally polarized semi-abelian scheme (G,λ) over $\mathbf{Z}[\![q^{r_i}]\!]$ of relative dimension g. The restriction of (G,λ) to $\mathbf{Z}(\!(q^{r_i})\!)$ is a principally polarized abelian scheme that we denote by $(\hat{X}_F,\hat{\lambda}_F)$. Moreover, the canonical action of R on $\mathbf{G}_m \otimes D^{-1}$, which preserves the period subgroup Y and is compatible with the polarization ϕ, induces an R-multiplication $\hat{m}_F : R \longrightarrow \mathrm{End}_{\mathbf{Z}(\!(q^{r_i})\!)}(\hat{X}_F)^{\hat{\lambda}_F}$; we thus obtain a principally polarized abelian scheme with R-multiplication $(\hat{X}_F, \hat{\lambda}_F, \hat{m}_F)$ over $\mathbf{Z}(\!(q^{r_i})\!)$.

Since $\mathrm{Lie}\,\hat{X}_F$ is canonically isomorphic to $\mathrm{Lie}(\mathbf{G}_{m,\mathbf{Z}(\!(q^{r_i})\!)} \otimes D^{-1}) \cong \mathbf{Z}(\!(q^{r_i})\!) \otimes D^{-1}$, we obtain by duality a canonical isomorphism of $\mathbf{Z}(\!(q^{r_i})\!) \otimes R$-modules

$$\mathbf{Z}(\!(q^{r_i})\!) \otimes R \cong F^1(\hat{X}_F/\mathbf{Z}(\!(q^{r_i})\!));$$

we let $\hat{\omega}_F$ be the $\mathbf{Z}(\!(q^{r_i})\!) \otimes R$-generator of $F^1(\hat{X}_F/\mathbf{Z}(\!(q^{r_i})\!))$ corresponding to the above trivialization.

Remark 6.9. If we identify $\mathbf{G}_m \otimes D^{-1} \xrightarrow{\sim} \mathrm{Spec}\,\mathbf{Z}[(t^{r_1})^{\pm 1}, \ldots, (t^{r_g})^{\pm 1}]$ via (6.1), then the canonical $\mathbf{Z}(\!(q^{r_i})\!) \otimes R$-trivialization of $F^1(\hat{X}_F/\mathbf{Z}(\!(q^{r_i})\!))$ is given by

$$\begin{aligned} \mathbf{Z}(\!(q^{r_i})\!) \otimes R &\xrightarrow{\sim} F^1(\hat{X}_F/\mathbf{Z}(\!(q^{r_i})\!)) \\ 1 \otimes r &\longmapsto \sum_{i=1}^{g} \mathrm{Tr}(r x_i) \frac{dt^{r_i}}{t^{r_i}}, \end{aligned}$$

so that

$$\hat{\omega}_F = \sum_{i=1}^{g} \mathrm{Tr}(x_i) \frac{dt^{r_i}}{t^{r_i}}.$$

Theorem 6.10. *Let ∇ be the Gauss-Manin connection on $H^1_{\mathrm{dR}}(\hat{X}_F/\mathbf{Z}(\!(q^{r_i})\!))$ and denote*

$$\hat{\eta}_F := \nabla_{\theta_F} \hat{\omega}_F \in H^1_{\mathrm{dR}}(\hat{X}_F/\mathbf{Z}(\!(q^{r_i})\!)) \otimes D.$$

Then:

(1) *The couple $\hat{b}_F := (\hat{\omega}_F, \hat{\eta}_F)$ is a symplectic-Hodge basis of $(\hat{X}_F, \hat{\lambda}_F, \hat{m}_F)_{/\mathbf{Z}(\!(q^{r_i})\!)}$.*

(2) *The morphism of Deligne-Mumford stacks*

$$\hat{\varphi}_F : \mathrm{Spec}\,\mathbf{Z}(\!(q^{r_i})\!) \longrightarrow \mathcal{B}_F$$

given by $(\hat{X}_F, \hat{\lambda}_F, \hat{m}_F, \hat{b}_F)_{/\mathbf{Z}(\!(q^{r_i})\!)}$ is a solution of the higher Ramanujan equations over $\mathcal{B}_F$ defined over $\mathbf{Z}$.

■

As in the Siegel case, this result follows directly from its complex analytic counterpart (Theorem .145)); see Paragraph 11.6.

Remark 6.11. Here again, we have chosen to work with explicit "coordinates" q^{r_i} induced by a fixed $\mathbf{Z}$-basis of D^{-1} (cf. Remark .72). We refer to [27] 5.2 for an exposition on how to work with more general coordinate rings.

Remark 6.12. We have the following compatibility between $\hat{\varphi}_F$ and $\hat{\varphi}_g$. Let $(x_1, \ldots, x_g)$ be as above, and $f_t : \mathcal{B}_F \longrightarrow \mathcal{B}_g$ be the corresponding morphism as defined in Remark .40. Define a morphism

$$\hat{h}_t : \mathrm{Spec}\,\mathbf{Z}(\!(q^{r_i})\!) \longrightarrow \mathrm{Spec}\,\mathbf{Z}(\!(q_{ij})\!)$$

by

$$\hat{h}_t^*(q_{ij}) = q^{r_i r_j}.$$

Then, the diagram

$$\begin{array}{ccc} \operatorname{Spec} \mathbf{Z}(\!(q^{r_i})\!) & \xrightarrow{\hat{\varphi}_F} & \mathcal{B}_F \\ \hat{h}_t \downarrow & & \downarrow f_t \\ \operatorname{Spec} \mathbf{Z}(\!(q_{ij})\!) & \xrightarrow[\hat{\varphi}_g]{} & \mathcal{B}_g \end{array}$$

commutes. This can be checked directly using the above constructions; it also follows from the corresponding complex analytic statement (see Remark .146).

7. Representability of $\mathcal{B}_g$ and $\mathcal{B}_F$ by a scheme

It is easy to see that if S is a scheme over $\mathbf{F}_2$, then $\mathcal{B}_g \times_{\mathbf{Z}} S \longrightarrow S$ is not representable. Indeed, if $(X, \lambda, b)_{/U}$ is an object of $\mathcal{B}_g$ lying over a scheme U over $\mathbf{F}_2$, then the involution $[-1] : P \longmapsto -P$ on X defines a non-trivial automorphism $[-1]_{/\mathrm{id}_U} : (X, \lambda)_{/U} \longrightarrow (X, \lambda)_{/U}$ in $\mathcal{A}_g(U)$ such that

$$[-1]^* b = -b = b,$$

thus a non-trivial automorphism of $(X, \lambda, b)_{/U}$ in $\mathcal{B}_g(U)$. This same argument applies to $\mathcal{B}_F$.

For any commutative ring Λ, let us denote $\mathcal{B}_{g,\Lambda} := \mathcal{B}_g \otimes_{\mathbf{Z}} \Lambda$ (resp. $\mathcal{B}_{F,\Lambda} := \mathcal{B}_F \otimes_{\mathbf{Z}} \Lambda$). In this section we prove the following theorem.

Theorem 7.1. *The stack $\mathcal{B}_{g,\mathbf{Z}[1/2]} \longrightarrow \operatorname{Spec} \mathbf{Z}[1/2]$ (resp. $\mathcal{B}_{F,\mathbf{Z}[1/2]} \longrightarrow \operatorname{Spec} \mathbf{Z}[1/2]$) is representable by a smooth quasi-affine scheme B_g (resp. B_F) over $\mathbf{Z}[1/2]$ of relative dimension $2g^2 + g$ (resp. $3g$).*

For the sake of concision, we shall only treat in detail the case of $\mathcal{B}_g$; there should be no difficulty in translating our arguments to obtain the analogous statement for $\mathcal{B}_F$ (see Remark .90).

7.1. **Representability by an algebraic space.** Let Λ be a commutative ring. The following terminology has been borrowed from [40] 4.4.

Definition 7.2. We say that the functor $\underline{B}_g$ (cf. Paragraph 4.2) is *rigid* over Λ if, for every Λ-scheme U, and every object (X, λ) of $\mathcal{A}_g$ lying over U, the action of $\mathrm{Aut}_U(X, \lambda)$ on $\underline{B}_g((X, \lambda)_{/U})$ is free.

Note that $\underline{B}_g$ is rigid over Λ if and only if the fiber categories of $\mathcal{B}_{g,\Lambda} \longrightarrow \operatorname{Spec} \Lambda$ are discrete. As $\mathcal{B}_g$ is a Deligne-Mumford stack over $\operatorname{Spec} \mathbf{Z}$, this amounts to saying that $\mathcal{B}_{g,\Lambda} \longrightarrow \operatorname{Spec} \Lambda$ is an algebraic space over $\operatorname{Spec} \Lambda$ (cf. 0.12).

Lemma 7.3. *Let k be a field of characteristic 0. Then $\underline{B}_g$ is rigid over k.*

Proof. Let (X, λ, b) be an object of $\mathcal{B}_g$ lying over k and $\varphi : X \longrightarrow X$ be a k-automorphism of (X, λ) such that $\varphi^* b = b$; we must show that $\varphi = \mathrm{id}_X$.

We claim that it is sufficient to treat the case $k = \mathbf{C}$. In fact, as X is of finite type over k, by "elimination of Noetherian hypothesis" (cf. [31] 8.8, 8.9, 8.10, 12.2.1, and [32] 17.7.9), there exists a subfield k_0 of k, of finite type over $\mathbf{Q}$, and a principally polarized abelian variety (X_0, λ_0)

over k_0 endowed with a symplectic-Hodge basis b_0 and a k_0-automorphism φ_0 of (X_0, λ_0) satisfying $\varphi_0^* b_0 = b_0$, such that (X, λ, b) (resp. φ) is obtained from (X_0, λ_0, b_0) (resp. φ_0) by the base change $\operatorname{Spec} k \longrightarrow \operatorname{Spec} k_0$. After fixing an embedding of k_0 in $\mathbf{C}$, we finally remark that if $\varphi_{0,\mathbf{C}}$ is the identity over $X_0 \otimes_{k_0} \mathbf{C}$, then the same holds for φ_0, and thus also for φ.

Let then $k = \mathbf{C}$. It is sufficient to prove that the induced automorphism of complex Lie groups $\varphi^{\mathrm{an}} : X^{\mathrm{an}} \longrightarrow X^{\mathrm{an}}$ is the identity. As X^{an} is a complex torus, the exponential $\exp : \operatorname{Lie} X \longrightarrow X^{\mathrm{an}}$ is a surjective morphism of complex Lie groups. Therefore, it follows from the commutative diagram

$$\begin{array}{ccc} \operatorname{Lie} X & \xrightarrow{\operatorname{Lie}\varphi} & \operatorname{Lie} X \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ X^{\mathrm{an}} & \xrightarrow[\varphi^{\mathrm{an}}]{} & X^{\mathrm{an}} \end{array}$$

that it sufficient to prove that $\operatorname{Lie}\varphi = \mathrm{id}_{\operatorname{Lie} X}$. Now, if φ preserves symplectic-Hodge basis of (X, λ), then in particular the $\mathbf{C}$-linear map $\varphi^* : H^0(X, \Omega^1_{X/\mathbf{C}}) \longrightarrow H^0(X, \Omega^1_{X/\mathbf{C}})$ is the identity, and thus its dual $\operatorname{Lie}\varphi : \operatorname{Lie} X \longrightarrow \operatorname{Lie} X$ is also the identity. ■

We now treat the case of positive characteristic. Let us briefly recall some notions in Dieudonné theory and its relation with abelian varieties.

Let k be a perfect field of characteristic $p > 0$. We denote by $W(k)$ the ring of Witt vectors over k, and by σ the unique ring automorphism of $W(k)$ lifting the absolute Frobenius $x \longmapsto x^p$ of k. We can then define a $W(k)$-algebra $D(k)$ generated by elements F and V subject to the relations

$$FV = VF = p, \quad Fx = \sigma(x)F, \quad xV = V\sigma(x)$$

for any $x \in W(k)$.

The theory of Dieudonné (cf. [61] Definition 3.12) provides an additive contravariant functor

$$G \longmapsto M(G) \tag{7.1}$$

from the category of commutative finite k-group schemes of p-power order to the category of left $D(k)$-modules. This functor is shown to be faithful and its essential image is given by the category of left $D(k)$-modules of finite $W(k)$-length: $M(G)$ is of $W(k)$-length r if and only if G is of order p^r ([61] Corollary 3.16).

Let X be an abelian variety over k and consider the k-vector space $H^1_{\mathrm{dR}}(X/k)$ as a $W(k)$-module via the canonical map $W(k) \longrightarrow k$. Then one can endow $H^1_{\mathrm{dR}}(X/k)$ with the structure of a $D(k)$-module, the action of F (resp. V) being induced by the relative Frobenius on X (resp. the Cartier operator in degree 1); we refer to [61] Definition 5.3 and Definition 5.6 for further details. This construction is functorial in the sense that for any morphism $\varphi : X \longrightarrow Y$ of abelian varieties over k, if we endow $H^1_{\mathrm{dR}}(X/k)$ and $H^1_{\mathrm{dR}}(Y/k)$ with the preceding $D(k)$-module structure, then the induced morphism on de Rham cohomology $\varphi^* : H^1_{\mathrm{dR}}(Y/k) \longrightarrow H^1_{\mathrm{dR}}(X/k)$ is $D(k)$-linear.

In the next statement, for any abelian variety X over k, we regard $H^1_{\mathrm{dR}}(X/k)$ with the above $D(k)$-module structure, and we denote its p-torsion subscheme by $X[p]$. Note that $X[p]$ is a commutative finite k-group scheme of order $p^{2\dim X}$.

Theorem 7.4 (Oda, [61] Corollary 5.11[9]). *The contravariant functors $X \longmapsto M(X[p])$ and $X \longmapsto H^1_{\mathrm{dR}}(X/k)$ from the category of abelian varieties over k to the category of (p-torsion) $D(k)$-modules of finite $W(k)$-length are naturally equivalent.*

Lemma 7.5. *Let k be a perfect field of characteristic $p > 2$. Then $\underline{B}_g$ is rigid over k.*

Proof. Let (X, λ) be a principally polarized abelian variety over k of dimension g and $\varphi : X \longrightarrow X$ be a k-automorphism of (X, λ).

If φ preserves a symplectic-Hodge basis of $(X, \lambda)_{/k}$, then in particular $\varphi^* : H^1_{\mathrm{dR}}(X/k) \longrightarrow H^1_{\mathrm{dR}}(X/k)$ is the identity; a fortiori, φ induces the identity on $H^1_{\mathrm{dR}}(X/k)$ regarded as a $D(k)$-module. Then, by Theorem .83, φ induces the identity on the $D(k)$-module $M(X[p])$. As the functor $G \longmapsto M(G)$ in (7.1) is faithful, φ restricts to the identity on the p-torsion subscheme $X[p]$ of X. As φ preserves, in addition, the polarization λ on X, and since $p \geq 3$, then necessarily $\varphi = \mathrm{id}_X$ by a lemma of Serre (cf. [56] IV.21, Theorem 5). ■

Recall the following version of the classical "rigidity lemma" for abelian schemes which follows from the arguments in the proof of Proposition 6.1 in [58].

Lemma 7.6. *Let A be a local Artinian ring, and X be an abelian scheme over* $\operatorname{Spec} A$*. If an abelian scheme endomorphism $\varphi \in \operatorname{End}_A(X)$ restricts to the identity on the closed fiber of $X \longrightarrow \operatorname{Spec} A$, then $\varphi = \mathrm{id}_X$.* ■

Proposition 7.7. *The functor $\underline{B}_g$ is rigid over* $\mathbf{Z}[1/2]$*.*

Proof. Let U be a $\mathbf{Z}[1/2]$-scheme, (X, λ) be an object of $\mathcal{A}_g$ lying over U, and φ be an automorphism of (X, λ) in the fiber category $\mathcal{A}_g(U)$ preserving an element b of $\underline{B}_g(X, \lambda)$. We must show that $\varphi = \mathrm{id}_X$. This being a local property over U, we can assume that U is affine.

Suppose that U is Noetherian. By Lemmas .82 and .84, for every geometric point $\overline{u}$ of U, we have $\varphi_{X_{\overline{u}}} = \mathrm{id}_{X_{\overline{u}}}$. Let Z be the closed subscheme of U where $\varphi = \mathrm{id}$. Then Z contains every closed point of U. By Lemma .85, and Krull's intersection theorem, Z is also an open subscheme of U; hence $Z = U$, which amounts to saying that $\varphi = \mathrm{id}_X$.

In general, by "elimination of Noetherian hypothesis" (cf. [31], 8.8, 8.9, 8.10, 12.2.1, and [32], 17.7.9), there exists an affine Noetherian scheme U_0 under U, and a principally polarized abelian scheme (X_0, λ_0) over U_0 endowed with a symplectic-Hodge basis b_0, and with an U_0-automorphism φ_0, such that $\varphi_0^* b_0 = b_0$, and (X, λ) (resp. b, resp. φ) is deduced from (X_0, λ_0) (resp. b_0, resp. φ_0) by the base change $U \longrightarrow U_0$. The preceding paragraph shows that $\varphi_0 = \mathrm{id}_{X_0}$, hence $\varphi = \mathrm{id}_X$. ■

7.2. **Representability of $\mathcal{B}_{g,\mathbf{Z}[1/2]}$ by a quasi-projective scheme B_g.** We briefly recollect some facts on quotients of schemes by actions of finite groups.

Let S be a scheme and Γ be a finite constant group scheme over S, that is, an S-group scheme associated to a finite abstract group $|\Gamma|$.

For any S-scheme X, an S-action of Γ on X is equivalent to a morphism of groups $|\Gamma| \longrightarrow \operatorname{Aut}_S(X)$. If X is an S-scheme, we say that an action of Γ on X is *free* if the action of $\Gamma(U)$ on $X(U)$ is free for any S-scheme U.

[9] Oda's theorem can be seen nowadays as part of the much more general Grothendieck-Messing theory; see for instance the introduction of B. Mazur, W. Messing, *Universal Extensions and One Dimensional Crystalline Cohomology*, Lecture Notes in Mathematics 370, Springer-Verlag.

The next lemma easily follows from [33] V and [44] IV.1.

Lemma 7.8. *Let S be an affine Noetherian scheme and X be a quasi-projective S-scheme equipped with an S-action of a finite constant group scheme Γ over S. Then*

(1) Up to isomorphism, there exists a unique quasi-projective S-scheme Y together with a Γ-invariant finite surjective morphism $p : X \longrightarrow Y$ such that the natural morphism of sheaves of rings over Y

$$\mathcal{O}_Y \longrightarrow (p_*\mathcal{O}_X)^{|\Gamma|}$$

is an isomorphism. If X is affine (resp. quasi-affine) over S, so is Y. We denote $Y =: X/\Gamma$.

(2) If moreover the action of Γ on X is free, then p is étale and

$$\begin{aligned} \Gamma \times_S X &\longrightarrow X \times_Y X \\ (\gamma, x) &\longmapsto (x, \gamma \cdot x) \end{aligned}$$

is an isomorphism.

Remark 7.9. Part (2) in the above lemma implies that, when the action of Γ on X is free, then the stacky quotient $[X/\Gamma]$ (cf. [65] Example 8.1.12) is representable by the scheme X/Γ.

For clarity, we split the proof of Theorem .80 for $\mathcal{B}_g$ in two parts; see Remark .90 for $\mathcal{B}_F$.

Proof of Theorem .80, part 1. Recall from [58] Theorem 7.9 (cf. [64] proof of Theorem 2.1.11) that there exists a quasi-projective scheme A over $\mathbf{Z}[1/2]$ endowed with an action by the constant finite group scheme Γ over $\mathbf{Z}[1/2]$ given by $|\Gamma| = \mathrm{GL}_g(\mathbf{Z}/4\mathbf{Z})$, and with a surjective étale morphism $A \longrightarrow \mathcal{A}_{g,\mathbf{Z}[1/2]}$ inducing an isomorphism of the stacky quotient $[A/\Gamma]$ with $\mathcal{A}_{g,\mathbf{Z}[1/2]}$; namely, A is the fine moduli scheme classifying of principally polarized abelian schemes with a full level 4 structure.

As the morphism of Deligne-Mumford stacks over $\operatorname{Spec} \mathbf{Z}$

$$\pi_g : \mathcal{B}_g \longrightarrow \mathcal{A}_g$$

is representable by smooth affine schemes (Remark .123), the fiber product

$$A \times_{\mathcal{A}_{g,\mathbf{Z}[1/2]}} \mathcal{B}_{g,\mathbf{Z}[1/2]} \longrightarrow A$$

is representable by a smooth affine scheme B over A. In particular, B is affine and of finite type over A. Since A is quasi-projective over $\mathbf{Z}[1/2]$, it follows that B is a quasi-projective $\mathbf{Z}[1/2]$-scheme.

The action of Γ on A naturally induces an action of Γ on the fiber product B; as $\mathcal{B}_{g,\mathbf{Z}[1/2]}$ is an algebraic space by Proposition .86 (cf. remark following Definition .81), this action is free. Moreover, by the compatibility of quotients of stacks by group actions with base change (cf. [72] Proposition 2.6), the second projection $B \longrightarrow \mathcal{B}_{g,\mathbf{Z}[1/2]}$ induces an isomorphism of the stacky quotient $[B/\Gamma]$ with $\mathcal{B}_{g,\mathbf{Z}[1/2]}$. Finally, by Lemma .87 and Remark .88, we conclude that $\mathcal{B}_{g,\mathbf{Z}[1/2]}$ is representable by the quasi-projective $\mathbf{Z}[1/2]$-scheme B/Γ. ∎

7.3. B_g **is quasi-affine over** $\mathbf{Z}[1/2]$**.** Our proof that B_g is quasi-affine over $\mathbf{Z}[1/2]$ is based on the following elementary fact from algebraic geometry.

Lemma 7.10. *Let S be an affine Noetherian scheme, X be a separated S-scheme of finite type, and $\mathcal{L}$ be an* ample *or* anti-ample *(i.e., the dual $\mathcal{L}^\vee$ is ample) line bundle over X. Let $T(\mathcal{L}) \longrightarrow X$ be the $\mathbf{G}_{m,S}$-torsor associated to $\mathcal{L}$. Then $T(\mathcal{L})$ is a quasi-affine S-scheme.*

Proof. Assume first that $\mathcal{L}^\vee$ is very ample over S. Then there exists $n \in \mathbf{N}$ and an S-immersion $i : X \longrightarrow \mathbf{P}^n_S = \operatorname{Proj} \mathcal{O}_S(S)[X_0, \ldots, X_n]$ such that $\mathcal{L}^\vee = i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$. Let $\varphi^j = i^*X_j \in \Gamma(X, \mathcal{L}^\vee)$ for $0 \leq j \leq n$, and denote by $p : T(\mathcal{L}) \longrightarrow X$ the canonical projection. The morphism of S-schemes

$$\begin{aligned} i_{\mathcal{L}} : T(\mathcal{L}) &\longrightarrow \mathbf{A}^{n+1}_S \setminus \{0\} = T(\mathcal{O}_{\mathbf{P}^n_S}(-1)) \\ \ell &\longmapsto (\varphi^0_{p(\ell)}(\ell), \ldots, \varphi^n_{p(\ell)}(\ell)) \end{aligned}$$

is an immersion, since it fits into the Cartesian square

$$\begin{array}{ccc} T(\mathcal{L}) & \xrightarrow{i_{\mathcal{L}}} & \mathbf{A}^{n+1}_S \setminus \{0\} \\ {\scriptstyle p}\big\downarrow & \square & \big\downarrow \\ X & \xrightarrow[i]{\qquad} & \mathbf{P}^n_S \end{array}$$

Thus $T(\mathcal{L})$ is a quasi-affine S-scheme.

If $\mathcal{L}^\vee$ is only ample, then we consider some very ample tensor power $(\mathcal{L}^\vee)^{\otimes k} = (\mathcal{L}^{\otimes k})^\vee$ of $\mathcal{L}^\vee$. Since the k-th power map $T(\mathcal{L}) \longrightarrow T(\mathcal{L}^{\otimes k})$ is a finite morphism of S-schemes, and $T(\mathcal{L}^{\otimes k})$ is quasi-affine over S by the above reasoning, $T(\mathcal{L})$ is also quasi-affine over S.

If $\mathcal{L}$ is ample, then $T(\mathcal{L}^\vee)$ is a quasi-affine S-scheme. By duality, $T(\mathcal{L})$ is isomorphic to $T(\mathcal{L}^\vee)$ as an S-scheme, thus $T(\mathcal{L})$ is quasi-affine over S. ■

To conclude, we apply the above lemma and the fact that the determinant of the Hodge bundle on a fine moduli space of principally polarized abelian varieties with level structure is ample (see [24] or [46]):

Proof of Theorem .80, part 2. Let $T(\det \mathcal{F}_g)$ be the category fibered in groupoids over $\operatorname{Spec} \mathbf{Z}$ whose objects over a scheme U are triples (X, λ, t), where (X, λ) is a principally polarized abelian scheme over U of relative dimension g, and t is a trivialization of the line bundle $\det F^1(X/U)$ over U — in other words, $T(\det \mathcal{F}_g)$ is the $\mathbf{G}_m$-torsor associated to the determinant of the universal Hodge bundle $\mathcal{F}_g$ over $\mathcal{A}_g$. Then $\pi_g : \mathcal{B}_g \longrightarrow \mathcal{A}_g$ factors through the forgetful functor $T(\det \mathcal{F}_g) \longrightarrow \mathcal{A}_g$ via

$$f : \mathcal{B}_g \longrightarrow T(\det \mathcal{F}_g),$$

given by $(X, \lambda, (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g))_{/U} \longmapsto (X, \lambda, \omega_1 \wedge \cdots \wedge \omega_g)_{/U}$.

We keep the notation of the first part of this proof. Let (X, λ) the principally polarized abelian scheme over A corresponding to the finite étale covering $A \longrightarrow \mathcal{A}_{g,\mathbf{Z}[1,2]} \cong [A/\Gamma]$, then it follows from [24] Theorem V.2.5 (cf. [46] Theorem 7.2.4.1 (2)) that $\det F^1(X/A)$ is an ample line bundle over A. By the above lemma, $T(\det F^1(X/A))$ is a quasi-affine $\mathbf{Z}[1/2]$-scheme.

Consider now the following commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & B_g \\ \downarrow & & \downarrow f \\ T(\det F^1(X/A)) & \longrightarrow & T(\det \mathcal{F}_g)_{\mathbf{Z}[1/2]} \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathcal{A}_{g,\mathbf{Z}[1/2]} \end{array}$$

in which every square is Cartesian. As f is relatively representable by affine schemes, B is affine over $T(\det F^1(X/A))$, thus quasi-affine over $\mathbf{Z}[1/2]$. Since $B_g \cong B/\Gamma$, we conclude by the part (1) of Lemma .87. ■

Remark 7.11. By considering level structures on principally polarized abelian schemes with R-multiplication and the ampleness of the determinant of the Hodge bundle ([46] Theorem 7.2.4.1 (2)), virtually the same proof can be applied to the case of $\mathcal{B}_F$.

8. The case of elliptic curves: explicit equations

When $g = 1$ (or, equivalently, $F = \mathbf{Q}$), we can compute explicit equations for $B_1 = B_{\mathbf{Q}}$, for the Ramanujan vector field, and for the integral solution $\hat{\varphi}_1$ of the Ramanujan equation.

8.1. **Explicit equation for the universal elliptic curve X_1 over B_1 and its universal symplectic-Hodge basis.** Fix a scheme U. Let us recall that every *elliptic curve* E over U (namely, an abelian scheme of relative dimension 1) has a canonical unique principal polarization $\lambda_E : E \longrightarrow E^t$ given, for any U-scheme V and any point $P \in E(V)$, by

$$\lambda_E(P) = \mathcal{O}_E([P] - [O])$$

where $O \in E(V)$ denotes the identity section and $\mathcal{O}_E([P] - [O])$ denotes the class in $E^t(V)$ of the inverse of the ideal sheaf defined by the relative Cartier divisor $[P] - [O]$.

Therefore, the functor

$$E \longmapsto (E, \lambda_E)$$

defines an equivalence between the category of elliptic curves over U and that of principally polarized elliptic curves over U. We can thus "forget" the principal polarization: an elliptic curve E will always be assumed to be endowed with its canonical principal polarization λ_E. In particular, an object of $\mathcal{B}_1$ will be denoted simply by a "couple" $(E, b)_{/U}$.

Remark 8.1. The symplectic form induced by λ_E coincides with the composition of the cup product in de Rham cohomology $H^1_{\mathrm{dR}}(E/U) \times H^1_{\mathrm{dR}}(E/U) \longrightarrow H^2_{\mathrm{dR}}(E/U)$ with the trace map $H^2_{\mathrm{dR}}(E/U) \longrightarrow \mathcal{O}_U$.

Theorem 8.2. *Let*

$$B_1 \coloneqq \operatorname{Spec} \mathbf{Z}[1/2, b_2, b_4, b_6, \Delta^{-1}]$$

where

$$\Delta := \frac{b_2^2(b_4^2 - b_2b_6)}{4} - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 = 16\,\mathrm{disc}\left(x^3 + \frac{b_2}{4}x^2 + \frac{b_4}{2}x + \frac{b_6}{4}\right),$$

and let X_1 be the elliptic curve over B_1 given by the equation

$$y^2 = x^3 + \frac{b_2}{4}x^2 + \frac{b_4}{2}x + \frac{b_6}{4}.$$

Then $b_1 = (\omega_1, \eta_1)$ defined by

$$\omega_1 := \frac{dx}{2y}, \qquad \eta_1 := x\frac{dx}{2y}$$

is a symplectic-Hodge basis of X_{1/B_1} and the morphism $B_1 \longrightarrow \mathcal{B}_1$ corresponding to $(X_1, b_1)_{/B_1}$ induces an isomorphism of B_1 with the $\mathbf{Z}[1/2]$-stack $\mathcal{B}_{1,\mathbf{Z}[1/2]}$.

In other words, if $(X_1, b_1)_{/B_1}$ is defined as above, then for any $\mathbf{Z}[1/2]$-scheme U, and any elliptic curve E over U endowed with a symplectic-Hodge basis b, there exists a unique morphism $F_{/f} : E_{/U} \longrightarrow X_{1/B_1}$ in $\mathcal{A}_{1,\mathbf{Z}[1/2]}$ such that $F^*b_1 = b$.

Proof. It is classical that ω_1 so defined is in $F^1(X_1/B_1)$. To prove that $\langle \omega_1, \eta_1 \rangle_{\lambda_E} = 1$ one can, for instance, use the compatibility with base change to reduce this statement to an analogous statement concerning an elliptic curve over $\mathbf{C}$, and then apply the classical residue formula (cf. [22] pp. 23-25).

Let U be a $\mathbf{Z}[1/2]$-scheme and $(E, b)_{/U}$ be an object of $\mathcal{B}_1(U)$, with $b = (\omega, \eta)$. It is sufficient to prove that, locally for the Zariski topology over U, there exists a unique morphism $(E, b)_{/U} \longrightarrow (X_1, b_1)_{/B_1}$ in $\mathcal{B}_{1,\mathbf{Z}[1/2]}$.

We follow essentially the same steps in [40] 2.2 to find a Weierstrass equation for an elliptic curve. Let us denote by $O : U \longrightarrow E$ the identity section of the elliptic curve E over U and by $p : E \longrightarrow U$ its structural morphism. Locally for the Zariski topology on U we can find a formal parameter t in the neighborhood of O such that ω has a formal expansion in t of the form

$$\omega = (1 + O(t))dt,$$

where $O(t)$ stands for a formal power series in t of order ≥ 1. Up to replacing U by an open subscheme, we can and shall assume from now on that t exists globally over U.

There exist bases $(1, x)$ of $p_*\mathcal{O}_E(2[O])$, and $(1, x, y)$ of $p_*\mathcal{O}_E(3[O])$, such that

$$x = \frac{1}{t^2}(1 + O(t)) \quad \text{and} \quad y = \frac{1}{t^3}(1 + O(t)). \tag{8.1}$$

Then the rational functions x and y necessarily satisfy an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where a_i are uniquely defined global sections of $\mathcal{O}_U$. Since 2 is invertible in U, the above equation is equivalent to

$$\left(y + \frac{a_1}{2}x + \frac{a_3}{2}\right)^2 = x^3 + \left(\frac{a_1^2 + 4a_2}{4}\right)x^2 + \left(\frac{a_1a_3 + 2a_4}{2}\right)x + \frac{a_3^2 + 4a_6}{4}.$$

Therefore, after the change of coordinates $(x,y) \longmapsto (x, y + \frac{a_1}{2}x + \frac{a_3}{2})$, we can assume that x and y satisfy

$$y^2 = x^3 + \frac{b_2}{4}x^2 + \frac{b_4}{2}x + \frac{b_6}{4},$$

where b_i are global sections of $\mathcal{O}_U$. Put differently, we obtain a morphism $F_{/f} : E_{/U} \longrightarrow X_{1/B_1}$ in $\mathcal{A}_{1,\mathbf{Z}[1/2]}$.

By considering formal expansions in t, we see that $F^*\omega_1 = \omega$. In particular,

$$(\omega, F^*\eta_1) = F^*b_1$$

is a symplectic-Hodge basis of $E_{/U}$, and there exists a section s of $\mathcal{O}_U$ such that $\eta = F^*\eta_1 + s\omega$. Thus, after the change of coordinates $(x,y) \longmapsto (x+s, y)$, we have $F^*b_1 = b$. Therefore, we have constructed a morphism $F_{/f} : (E,b)_{/U} \longrightarrow (X_1, b_1)_{/B_1}$ in $\mathcal{B}_{1,\mathbf{Z}[1/2]}$.

We now prove that the morphism $F_{/f}$ is unique. Let $F'_{/f'} : (E,b)_{/U} \longrightarrow (X_1, b_1)_{/B_1}$ be any morphism in $\mathcal{B}_{1,\mathbf{Z}[1/2]}$. If $f' = (b_2', b_4', b_6')$ are the coordinates of f', then F' is given by a basis $(1, x', y')$ of $p_*\mathcal{O}_E(3[O])$ satisfying

$$(*) \qquad (y')^2 = (x')^3 + \frac{b_2'}{4}(x')^2 + \frac{b_4'}{2}x' + \frac{b_6'}{4}.$$

As both $(1,x,y)$ and $(1,x',y')$ (resp. $(1,x)$ and $(1,x')$) are a basis of $p_*\mathcal{O}_E(3[O])$ (resp. $p_*\mathcal{O}_E(2[O])$), then there exists global sections c_1, c_2, c_3 of $\mathcal{O}_U$ (resp. u, v of $\mathcal{O}_U^\times$) such that

$$\begin{aligned} x' &= u(x + c_1) \\ y' &= v(y + c_2x + c_3). \end{aligned}$$

Note that equation $(*)$ implies that $u^3 = v^2$.

Now, as $(F')^*\omega_1 = F^*\omega_1$, we obtain

$$\frac{dx'}{2y'} = \frac{dx}{2y} \iff \frac{u}{v}\frac{dx}{2(y + c_2x + c_3)} = \frac{dx}{2y},$$

thus $c_2x + c_3 = 0$ and $u = v$. Since $u^3 = v^2$, we obtain $u = v = 1$ and $(x', y') = (x + c_1, y)$. Finally, as $(F')^*\eta_1 = F^*\eta_1$, we have

$$x'\frac{dx'}{2y'} = \frac{dx}{2y} \iff x\frac{dx}{2y} + c_1\frac{dx}{2y} = x\frac{dx}{2y},$$

hence $c_1 = 0$. Thus $(x', y') = (x, y)$ and this also implies that $f = f'$. ■

Remark 8.3. By considering the change of variables

$$\begin{cases} b_2 = e_2 \\ b_4 = (e_2^2 - e_4)/24 \\ b_6 = (4e_2^3 - 12e_2e_4 + 8e_6)/1728 \end{cases} \iff \begin{cases} e_2 = b_2 \\ e_4 = b_2^2 - 24b_4 \\ e_6 = b_2^3 - 36b_2b_4 + 216b_6 \end{cases}$$

we see that $B_1 \otimes_{\mathbf{Z}[1/2]} \mathbf{Z}[1/6]$ is isomorphic to

$$\operatorname{Spec} \mathbf{Z}[1/6, e_2, e_4, e_6, (e_4^3 - e_6^2)^{-1}].$$

Under this identification, the universal elliptic curve X_1 is given by the equation

$$y^2 = 4\left(x + \frac{e_2}{12}\right)^3 - \frac{e_4}{12}\left(x + \frac{e_2}{12}\right) + \frac{e_6}{216},$$

and the universal symplectic-Hodge basis b_1 by $(dx/y, xdx/y)$.

8.2. **Explicit formulas for the Ramanujan vector field.** It is also possible to give an explicit formula for the Ramanujan vector field v_{11} over B_1. Indeed, consider the global section of $T_{B_1/\mathbf{Z}[1/2]}$ given by

$$v := 2b_4\frac{\partial}{\partial b_2} + 3b_6\frac{\partial}{\partial b_4} + (b_2b_6 - b_4^2)\frac{\partial}{\partial b_6}.$$

One may easily verify using the expression for the Gauss-Manin connection on $H^1_{\mathrm{dR}}(X_1/B_1)$ given in A.3 that

$$\nabla_v \begin{pmatrix} \omega_1 & \eta_1 \end{pmatrix} = \begin{pmatrix} \omega_1 & \eta_1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

By Proposition .61, v is the Ramanujan vector field v_{11} over B_1.

Remark 8.4. Under the isomorphism $B_1 \otimes_{\mathbf{Z}[1/2]} \mathbf{Z}[1/6] \cong \mathbf{Z}[1/6, e_2, e_4, e_6, (e_4^3 - e_6^2)^{-1}]$ of Remark .93, v gets identified with the vector field associated to the classical Ramanujan equations:

$$v = \frac{e_2^2 - e_4}{12}\frac{\partial}{\partial e_2} + \frac{e_2e_4 - e_6}{3}\frac{\partial}{\partial e_4} + \frac{e_2e_6 - e_4^2}{2}\frac{\partial}{\partial e_6}.$$

8.3. **Explicit formulas for $\hat{\varphi}_1$.** We now explicitly describe the integral solution

$$\hat{\varphi}_1 : \operatorname{Spec} \mathbf{Z}(\!(q)\!) \longrightarrow \mathcal{B}_1$$

constructed in Section 6.

Recall that we denote $\theta := \theta_{11} = q\frac{d}{dq}$, and

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},\;\; E_4(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},\;\; E_6(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} \in \mathbf{Z}[\![q]\!].$$

Proposition 8.5. *We have:*

(1) under the identification $B_1 \cong \operatorname{Spec}\mathbf{Z}[1/2, b_2, b_4, b_6, \Delta^{-1}]$ of Theorem .92,

$$\hat{\varphi}^*_{1,\mathbf{Z}[1/2]}(b_2, b_4, b_6) = \left(E_2(q), \frac{1}{2}\theta E_2(q), \frac{1}{6}\theta^2 E_2(q)\right) \in (\mathbf{Z}(\!(q)\!) \otimes \mathbf{Z}[1/2])^3;$$

(2) under the identification $B_{1,\mathbf{Z}[1/6]} \cong \operatorname{Spec}\mathbf{Z}[1/6, e_2, e_4, e_6, (e_4^3 - e_6^2)^{-1}]$ of Remark .93, we have

$$\hat{\varphi}^*_{1,\mathbf{Z}[1/6]}(e_2, e_4, e_6) = (E_2(q), E_4(q), E_6(q)) \in (\mathbf{Z}(\!(q)\!) \otimes \mathbf{Z}[1/6])^3.$$

Proof. By the change-of-coordinates formulas in Remark .93, it is sufficient to prove (2).

It is classical that the Tate curve $\hat{X}_{1,\mathbf{Z}[1/6]}$ over $\mathbf{Z}(\!(q)\!) \otimes \mathbf{Z}[1/6]$ is given by the equation

$$y^2 = 4x^3 - \frac{E_4(q)}{12}x + \frac{E_6(q)}{216},$$

with canonical differential $\hat{\omega}_1 = \frac{dx}{y}$. This can be deduced from its analytic counterpart (see Paragraph 11.4), which implies moreover that

$$\hat{\eta_1} := \nabla_\theta \omega_1 = \hat{\omega}_1 - \frac{E_2(q)}{12}\hat{\omega}_1;$$

cf. equation (A.1) in Appendix A.

Let $\varphi : \mathbf{Z}(\!(q)\!) \otimes \mathbf{Z}[1/6] \longrightarrow B_{1,\mathbf{Z}[1/6]}$ be defined by

$$\varphi^*(e_2, e_4, e_6) = (E_2(q), E_4(q), E_6(q)).$$

Observe that we have a morphism in $\mathcal{A}_{1,\mathbf{Z}[1/6]}$

$$\begin{array}{ccc} \hat{X}_{1,\mathbf{Z}[1/6]} & \xrightarrow{\Phi} & X_{1,\mathbf{Z}[1/6]} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbf{Z}(\!(q)\!) \otimes \mathbf{Z}[1/6] & \xrightarrow[\varphi]{} & B_{1,\mathbf{Z}[1/6]} \end{array}$$

where the top arrow is defined by

$$\Phi^*(x, y) = \left(x - \frac{E_2(q)}{12}, y \right).$$

By the universal property of $B_{1,\mathbf{Z}[1/6]}$, to prove that $\hat{\varphi}_{1,\mathbf{Z}[1/6]} = \varphi$, it is sufficient to prove that $\Phi^* b_1 = \hat{b}_1$, i.e., that

$$\Phi^*\omega_1 = \hat{\omega}_1 \quad \text{ and } \Phi^*\eta_1 = \hat{\eta}_1.$$

This, in turn, is a simple computation using the explicit formulas for Φ and $\hat{\omega}_1$, $\hat{\eta}_1$ above, and the formulas for ω_1 and η_1 in Remark .93. ■

Note that, by the explicit formulas given at the beginning of this paragraph, we know beforehand that the coefficients of E_2, E_4, E_6 (and of $\frac{1}{2}\theta E_2$ and $\frac{1}{6}\theta^2 E_2$ as well) are integral, but our explicit expression for $\hat{\varphi}$ in terms of Eisenstein series relies on a base change to $\mathbf{Z}[1/2]$ or $\mathbf{Z}[1/6]$ (so that $\mathcal{B}_1$ becomes representable).

As hinted in Paragraph 0.2 of our introductory section, in order to remain in a purely integral situation, we should consider the ring of global sections $\Gamma(\mathcal{B}_1, \mathcal{O}_{\mathcal{B}_{1,\text{ét}}})$. Let $E_{/U}$ be an elliptic curve endowed with a symplectic-Hodge basis $b = (\omega, \eta)$. Arguing as in the proof of Theorem .92, we see that locally over U the elliptic curve E admits a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $\omega = \frac{dx}{y}$ and $\eta = x\frac{dx}{y}$. If we set, as in Tate's classical formulas (cf. [19] 1.4),

$$b_2 := a_1^2 + 4a_2, \ \ b_4 := a_1a_3 + 2a_4, \ \ b_6 := a_3^2 + 4a_6, \ \ b_8 := -a_1a_3a_4 - a_4^2 + a_1^2a_6 + a_2a_3^2 + 4a_2a_6,$$

then we check that $b_4^2 - b_2b_6 + 4b_8 = 0$, and that b_2, b_4, b_6, and b_8 do not depend on the choice of the particular Weierstrass equation for which $\omega = \frac{dx}{y}$ and $\eta = x\frac{dx}{y}$. In particular, they define global sections of $\mathcal{O}_{\mathcal{B}_{1,\text{ét}}}$. In this sense, Theorem .92 simply says that the morphism

$$(b_2, b_4, b_6) : \mathcal{B}_1 \longrightarrow \mathbf{A}^3_{\mathbf{Z}}$$

induces, after base change to $\mathbf{Z}[1/2]$, an isomorphism of $\mathcal{B}_{1,\mathbf{Z}[1/2]}$ with the open affine subscheme of $\mathbf{A}^3_{\mathbf{Z}[1/2]}$ defined by $\Delta \neq 0$, and it follows from Proposition .95 (1) that

$$\hat{\varphi}_1^*(b_2, b_4, b_6) = \left(E_2(q), \frac{1}{2}\theta E_2(q), \frac{1}{6}\theta^2 E_2(q)\right) \in \mathbf{Z}(\!(q)\!)^3.$$

Analogously, the formulas for e_2, e_4, and e_6 in Remark .93 also define global sections of $\mathcal{O}_{\mathcal{B}_{1,\text{ét}}}$, so that the components of $\hat{\varphi}_1^*$ in the "coordinates" (e_2, e_4, e_6), namely $E_2(q)$, $E_4(q)$, and $E_6(q)$, are in $\mathbf{Z}(\!(q)\!)$.

Remark 8.6. The ring $\Gamma(\mathcal{B}_1, \mathcal{O}_{\mathcal{B}_{1,\text{ét}}})$, which can be shown to be isomorphic to

$$\mathbf{Z}[b_2, b_4, b_6, b_8, \Delta^{-1}]/(b_4^2 - b_2 b_6 + 4b_8)$$

by arguments similar to [19], Paragraph 6, can be thought as the ring of "integral weakly holomorphic quasimodular forms", i.e., integral quasimodular forms which are only meromorphic at infinity (cf. 0.6.2).

A. Gauss-Manin connection on some elliptic curves

A.1. **The Weierstrass elliptic curve.** Let

$$W := \operatorname{Spec} \mathbf{C}[g_2, g_3, \Delta^{-1}]$$

where

$$\Delta := g_2^3 - 27g_3^2.$$

Then we can define an elliptic curve E over W by the classical Weierstrass equation

$$y^2 = 4x^3 - g_2 x - g_3.$$

Further, we define a symplectic-Hodge basis (ω, η) of $E_{/W}$ by the formulas

$$\omega := \frac{dx}{y}, \quad \eta := x\frac{dx}{y}.$$

Lemma A.1. *With the above notations, the Gauss-Manin connection* ∇ *on* $H^1_{\mathrm{dR}}(E/W)$ *is given by*

$$\nabla \begin{pmatrix} \omega & \eta \end{pmatrix} = \begin{pmatrix} \omega & \eta \end{pmatrix} \otimes \frac{1}{\Delta} \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

where

$$\begin{aligned}
\Omega_{11} &= -\frac{1}{4}g_2^2\, dg_2 + \frac{9}{2}g_3\, dg_3 \\
\Omega_{12} &= \frac{3}{8}g_2 g_3\, dg_2 - \frac{1}{4}g_2^2\, dg_3 \\
\Omega_{21} &= -\frac{9}{2}g_3\, dg_2 + 3g_2\, dg_3 \\
\Omega_{22} &= -\Omega_{11}.
\end{aligned}$$

Let us briefly explain how these expressions follow from the description given in [39] A1.3 of the Gauss-Manin connection on the relative first de Rham cohomology of the universal elliptic curve $\mathbf{E}$ over the Poincaré half-plane $\mathbf{H}$ (whose fiber at each $\tau \in \mathbf{H}$ is given by the complex torus $\mathbf{E}_\tau = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$; in the notation of Example .105, we have $\mathbf{E} = \mathbf{X}_1$).[10]

We first remark that for any $u \in \mathbf{C}^\times$ we can define an automorphism $M_{u/\mu_u} : E_{/W} \longrightarrow E_{/W}$ in the category $\mathcal{A}_{1,\mathbf{C}}$ by

$$\mu_u(g_2, g_3) = (u^{-4}g_2, u^{-6}g_3), \quad M_u(x,y) = (u^{-2}x, u^{-3}y).$$

Using that the Gauss-Manin connection commutes with base change and admits regular singularities, we deduce by homogeneity that there exists constants $c_1, \ldots, c_8$ in $\mathbf{C}$ such that

$$\Omega_{11} = c_1 g_2^2\, dg_2 + c_2 g_3\, dg_3, \quad \Omega_{12} = c_3 g_2 g_3\, dg_2 + c_4 g_2^2\, dg_3,$$
$$\Omega_{21} = c_5 g_3\, dg_2 + c_6 g_2\, dg_3, \quad \Omega_{22} = c_7 g_2^2\, dg_2 + c_8 g_3\, dg_3.$$

To determine these constants, we consider the Cartesian diagram in the category of complex analytic spaces

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{\Psi} & E(\mathbf{C}) \\ \downarrow & \square & \downarrow \\ \mathbf{H} & \xrightarrow[\psi]{} & W(\mathbf{C}) \end{array}$$

given by the classical Weierstrass theory:

$$\psi(\tau) = (g_2(\tau), g_3(\tau)), \quad \Psi_\tau(z) = (\wp_\tau(z), \wp'_\tau(z))$$

Finally, we apply once again that that the formation of the Gauss-Manin connection (now in the complex analytic category) commutes with base change, and we use the formulas in [39] A1.3:

(A.1)

$$\nabla \begin{pmatrix} dz & \wp_\tau(z)dz \end{pmatrix} = \begin{pmatrix} dz & \wp_\tau(z)dz \end{pmatrix} \otimes \frac{1}{2\pi i} \begin{pmatrix} -(2\pi i)^2 E_2(\tau)/12 & -(2\pi i)^4 E_4(\tau)/144 \\ 1 & (2\pi i)^2 E_2(\tau)/12 \end{pmatrix} d\tau$$

A.2. **The elliptic curve $X_{/B}$ over $\mathbf{Z}[1/6]$.** Let

$$B \coloneqq \operatorname{Spec} \mathbf{Z}[1/6, e_2, e_4, e_6, \Delta^{-1}]$$

where

$$\Delta \coloneqq e_4^3 - e_6^2.$$

We define an elliptic curve X over B by

$$y^2 = 4\left(x + \frac{e_2}{12}\right)^3 - \frac{e_4}{12}\left(x + \frac{e_2}{12}\right) + \frac{e_6}{216}.$$

We define a symplectic-Hodge basis (ω, η) of $X_{/B}$ by the formulas

$$\omega \coloneqq \frac{dx}{y}, \quad \eta \coloneqq x\frac{dx}{y}.$$

[10] A direct algebraic approach is also possible. See for instance [41] 3, [42] 3.4, and [54] 3.4.

Note that there is a morphism $F_{/f}:(X_{\mathbf{C}})_{/B_{\mathbf{C}}}\longrightarrow E_{/W}$ in $\mathcal{A}_{1,\mathbf{C}}$ given by

$$f(e_2,e_4,e_6)=\left(\frac{e_4}{12},-\frac{e_6}{216}\right),\quad F(x,y)=\left(x+\frac{e_2}{12},y\right).$$

By pulling back the Gauss-Manin connection on $H^1_{\mathrm{dR}}(E/W)$ described in Lemma .97 by the morphism $F_{/f}$, we obtain that the Gauss-Manin connection ∇ on $H^1_{\mathrm{dR}}(X/B)$ over $\mathbf{Z}[1/6]$ is given by

$$\nabla\begin{pmatrix}\omega & \eta\end{pmatrix}=\begin{pmatrix}\omega & \eta\end{pmatrix}\otimes\frac{1}{\Delta}\begin{pmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{pmatrix}$$

where

$$\begin{aligned}
\Omega_{11}&=\left(\frac{e_2e_6-e_4^2}{4}\right)de_4+\left(\frac{e_6-e_2e_4}{6}\right)de_6\\
\Omega_{12}&=-\frac{\Delta}{12}de_2-\left(\frac{e_4e_6-2e_2e_4^2+e_2^2e_6}{48}\right)de_4+\left(\frac{e_4^2-2e_2e_6+e_2^2e_4}{72}\right)de_6\\
\Omega_{21}&=3e_6de_4-2e_4de_6\\
\Omega_{22}&=-\Omega_{11}.
\end{aligned}$$

A.3. **The universal elliptic curve X_{1/B_1} over $\mathbf{Z}[1/2]$.** Consider the elliptic curve X_1 over B_1 defined in Theorem .92 and let $\Phi_{/\varphi}:(X_{1,\mathbf{Z}[1/6]})_{/B_{1,\mathbf{Z}[1/6]}}\longrightarrow X_{/B}$ be the isomorphism in $\mathcal{A}_{1,\mathbf{Z}[1/6]}$ given by

$$\varphi(b_2,b_4,b_6)=(b_2,b_2^2-24b_4,b_2^3-36b_2b_4+216b_6),\quad \Phi(x,y)=(x,2y).$$

If (ω_1,η_1) denotes de symplectic-Hodge basis of X_{1/B_1} defined in Theorem .92, then by pulling back the Gauss-Manin connection on $H^1_{\mathrm{dR}}(X/B)$ described in A.2 by the isomorphism $\Phi_{/\varphi}$, we obtain that the Gauss-Manin connection ∇ on $H^1_{\mathrm{dR}}(X_1/B_1)$ over $\mathbf{Z}[1/2]$ is given by

$$\nabla\begin{pmatrix}\omega_1 & \eta_1\end{pmatrix}=\begin{pmatrix}\omega_1 & \eta_1\end{pmatrix}\otimes\frac{1}{\Delta}\begin{pmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{pmatrix}$$

where

$$\begin{aligned}
\Omega_{11}&=\frac{b_2^2b_6-6b_4b_6-b_2b_4^2}{8}db_2+\frac{4b_4^2-3b_2b_6}{2}db_6+\frac{18b_6-b_2b_4}{4}db_6\\
\Omega_{12}&=\frac{2b_4^3+9b_6^2-2b_2b_4b_6}{4}db_2+\frac{b_2^2b_6-b_2b_4^2-6b_4b_6}{4}db_6+\frac{4b_4^2-3b_2b_6}{4}db_6\\
\Omega_{21}&=\frac{3b_2b_6-4b_4^2}{4}db_2+\frac{b_2b_4-18b_6}{2}db_4+\frac{24b_4-b_2^2}{4}db_6\\
\Omega_{22}&=-\Omega_{11}.
\end{aligned}$$

Part 2. The analytic higher Ramanujan equations and periods of abelian varieties

9. Analytic families of complex tori, abelian varieties, and their uniformization

In this section we briefly transpose some of the standard theory of complex tori to a relative situation, that is, we shall consider analytic families of complex tori. To both simplify and shorten our exposition, we shall assume that the parameter space is smooth (i.e., a complex manifold); this largely suffices for our needs.

Most of the material included in here, and in the following section, is well known to experts — and may be even considered as "classical" — but we could not find a convenient reference in the literature.

9.1. Relative complex tori.

Let M be a complex manifold.

Definition 9.1. A *(relative) complex torus over* M is a relative complex Lie group $\pi : X \longrightarrow M$ over M such that π is proper with connected fibers. A morphism of complex tori over M is a morphism of relative complex Lie groups over M.

As any compact connected complex Lie group is a complex torus, every fiber of π in the above definition is a complex torus.

In general, for any relative complex Lie group $\pi : X \longrightarrow M$ over M, we may consider its *relative Lie algebra* $\mathrm{Lie}_M X$; this is a holomorphic vector bundle over M whose fiber at each $p \in M$ is the Lie algebra $\mathrm{Lie}\, X_p$ of the Lie group $X_p := \pi^{-1}(p)$. Moreover, there exists a canonical morphism of complex manifolds over M

$$\exp : \mathrm{Lie}_M X \longrightarrow X$$

restricting to the usual exponential map of complex Lie groups at each fiber.

Lemma 9.2. *Let $\pi : X \longrightarrow M$ be a complex torus over M. Then* $\exp : \mathrm{Lie}_M X \longrightarrow X$ *is a surjective and submersive morphism of relative complex Lie groups over M. Moreover, the sheaf of sections of the relative complex Lie group* $\ker(\exp)$ *over M is canonically isomorphic to*

$$R_1\pi_*\mathbf{Z}_X := (R^1\pi_*\mathbf{Z}_X)^\vee.$$

■

This follows from the classical case where M is a point via a fiber-by-fiber consideration (cf. [56] I.1). Note that $R_1\pi_*\mathbf{Z}_X$ is a local system of free abelian groups over M whose fiber at $p \in M$ is given by the first singular homology group $H_1(X_p, \mathbf{Z})$.

Definition 9.3. Let V be a holomorphic vector bundle of rank g over M. By a *lattice* in V, we mean a subsheaf of abelian groups L of $\mathcal{O}_M(V)$ such that

(1) L is a local system of free abelian groups of rank $2g$,
(2) for each $p \in M$, the quotient V_p/L_p is compact.

It follows from Lemma .99 that, for any complex torus $\pi : X \longrightarrow M$ of relative dimension g, $R_1\pi_*\mathbf{Z}_X$ may be canonically identified to a lattice in $\mathrm{Lie}_M X$.

Conversely, if V is a holomorphic vector bundle of rank g over M and L is a lattice in V, then the étalé space $E(L)$ of L is a relative complex Lie subgroup of V over M and $X := V/E(L)$ is a complex torus over M of relative dimension g. Furthermore, the relative Lie algebra $\mathrm{Lie}_M X$ gets canonically identified with V and, under this identification, $E(L)$ is the kernel of the exponential map $\exp : \mathrm{Lie}_M X \longrightarrow X$.

Remark 9.4. The above reasoning actually proves that the category of complex tori over M of relative dimension g is equivalent to the category of couples (V, L) where V is a holomorphic vector bundle of rank g over M and L is a lattice in V; a morphism $(V, L) \longrightarrow (V', L')$ in this category is given by a morphism of holomorphic vector bundles $\varphi : V \longrightarrow V'$ such that $\varphi(E(L)) \subset E(L')$.

In what follows, we shall drop the notation $E(L)$ and identify a local system with its étalé space.

9.2. **Riemann forms and principally polarized complex tori.** Let M be a complex manifold and $\pi : X \longrightarrow M$ be a complex torus over M.

Definition 9.5. A *Riemann form* over X is a C^∞ Hermitian metric[11] H on the vector bundle $\mathrm{Lie}_M X$ over M such that

$$E := \mathrm{Im}\, H$$

takes integral values on $R_1\pi_*\mathbf{Z}_X$.

Observe that E is an alternating $\mathbf{R}$-bilinear form. We also remark that the Hermitian metric H is completely determined by E: for any sections v and w of $\mathrm{Lie}_M X$ we have $H(v, w) = E(v, iw) + iE(v, w)$. In particular, by abuse, we may also say that E is Riemann form over X.

Definition 9.6. With the above notation, we say that the Riemann form E is *principal* if the induced morphism of local systems

$$\begin{aligned} R_1\pi_*\mathbf{Z}_X &\longrightarrow (R_1\pi_*\mathbf{Z}_X)^\vee \cong R^1\pi_*\mathbf{Z}_X \\ \gamma &\longmapsto E(\gamma,\ ) \end{aligned}$$

is an isomorphism.

Definition 9.7. Let M be a complex manifold. A *principally polarized complex torus* over M of relative dimension g is a couple (X, E), where X is a complex torus over M of relative dimension g and E is a principal Riemann form over X.

Example 9.8. Let $g \geq 1$ and consider the Siegel upper half-space

$$\mathbf{H}_g := \{\tau \in M_{g\times g}(\mathbf{C}) \mid \tau = \tau^{\mathsf{T}},\ \mathrm{Im}\,\tau > 0\}.$$

If $g = 1$, we denote $\mathbf{H} := \mathbf{H}_1$; this is the Poincaré upper half-plane. Let us consider the trivial vector bundle $V := \mathbf{C}^g \times \mathbf{H}_g$ over $\mathbf{H}_g$ and let L be the subsheaf of $\mathcal{O}_{\mathbf{H}_g}(V)$ given by the image of the morphism of sheaves of abelian groups

$$\begin{aligned} (\mathbf{Z}^g \oplus \mathbf{Z}^g)_{\mathbf{H}_g} &\longrightarrow \mathcal{O}_{\mathbf{H}_g}(V) = \mathcal{O}_{\mathbf{H}_g}^{\oplus g} \\ (m, n) &\longmapsto m + \tau n \end{aligned}$$

[11]Our convention is that Hermitian forms are anti-linear on the first coordinate and linear on the second.

where m and n are considered as column vectors of order g. Then L is a lattice in V and we denote by

$$\boldsymbol{p}_g : \mathbf{X}_g \longrightarrow \mathbf{H}_g$$

the corresponding complex torus over $\mathbf{H}_g$ of relative dimension g (cf. Remark .101). Let E_g be imaginary part of the Hermitian metric over V given by

$$(v, w) \longmapsto \overline{v}^{\mathsf{T}}(\operatorname{Im}\tau)^{-1}w.$$

One may easily verify that E_g takes integral values on L and that $\gamma \longmapsto E_g(\gamma,\ )$ induces an isomorphism $L \xrightarrow{\sim} L^{\vee}$. We thus obtain a principally polarized complex torus $(\mathbf{X}_g, E_g)$ over $\mathbf{H}_g$ of relative dimension g.

9.3. **The category $\mathcal{A}_g^{\mathrm{an}}$ of principally polarized complex tori of relative dimension g.** Let $\mathsf{Man}_{/\mathbf{C}}$ denote the category of complex manifolds. We define a category $\mathcal{A}_g^{\mathrm{an}}$ fibered in groupoids over $\mathsf{Man}_{/\mathbf{C}}$ as follows.

(1) An object of the category $\mathcal{A}_g^{\mathrm{an}}$ consists in a complex manifold M and a principally polarized complex torus (X, E) over M of relative dimension g; we denote such an object by $(X, E)_{/M}$.
(2) Let $(X, E)_{/M}$ and $(X', E')_{/M'}$ be objects of $\mathcal{A}_g^{\mathrm{an}}$. A morphism

$$\varphi_{/f} : (X', E')_{/M'} \longrightarrow (X, E)_{/M}$$

in $\mathcal{A}_g^{\mathrm{an}}$ is a Cartesian diagram of complex manifolds

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ \downarrow & \square & \downarrow \\ M' & \xrightarrow[f]{} & M \end{array}$$

preserving the identity sections of the complex tori and such that $E' = f^*E$ under the isomorphism of holomorphic vector bundles $\operatorname{Lie}_{M'} X' \xrightarrow{\sim} f^* \operatorname{Lie}_M X$ induced by φ. We may also denote $(X', E') = (X, E) \times_M M'$.
(3) The structural functor $\mathcal{A}_g^{\mathrm{an}} \longrightarrow \mathsf{Man}_{/\mathbf{C}}$ sends an object $(X, E)_{/M}$ of $\mathcal{A}_g^{\mathrm{an}}$ to the complex manifold M, and a morphism $\varphi_{/f}$ as above to f.

Example 9.9. We define an action of $\mathrm{Sp}_{2g}(\mathbf{Z})$ on the object $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ of $\mathcal{A}_g^{\mathrm{an}}$

$$\begin{aligned} \mathrm{Sp}_{2g}(\mathbf{Z}) &\longrightarrow \operatorname{Aut}_{\mathcal{A}_g^{\mathrm{an}}}\left((\mathbf{X}_g, E_g)_{/\mathbf{H}_g}\right) \\ \gamma &\longmapsto \varphi_{\gamma/f_\gamma} \end{aligned}$$

as follows. Recall that an element $\gamma = (A\ B\ ;\ C\ D) \in \mathrm{Sp}_{2g}(\mathbf{R})$ acts on $\mathbf{H}_g$ by

$$\begin{aligned} f_\gamma : \mathbf{H}_g &\longrightarrow \mathbf{H}_g \\ \tau &\longmapsto \gamma \cdot \tau := (A\tau + B)(C\tau + D)^{-1}. \end{aligned}$$

For γ as above, consider the holomorphic map

$$\begin{aligned} \widetilde{\varphi}_\gamma : \mathbf{C}^g \times \mathbf{H}_g &\longrightarrow \mathbf{C}^g \times \mathbf{H}_g \\ (z, \tau) &\longmapsto ((j(\gamma, \tau)^{\mathsf{T}})^{-1}z, \gamma \cdot \tau) \end{aligned}$$

where

$$j(\gamma,\tau) := C\tau + D \in \mathrm{GL}_g(\mathbf{C}).$$

If $\gamma \in \mathrm{Sp}_{2g}(\mathbf{Z})$, then for every $\tau \in \mathbf{H}_g$ we have

$$\widetilde{\varphi}_{\gamma,\tau}(\mathbf{Z}^g + \tau\mathbf{Z}^g) = \mathbf{Z}^g + (\gamma\cdot\tau)\mathbf{Z}^g,$$

so that $\widetilde{\varphi}_\gamma$ induces a holomorphic map $\varphi_\gamma : \mathbf{X}_g \longrightarrow \mathbf{X}_g$. One easily verifies that

$$\begin{array}{ccc} \mathbf{X}_g & \xrightarrow{\varphi_\gamma} & \mathbf{X}_g \\ {\scriptstyle \boldsymbol{p}_g}\downarrow & & \downarrow{\scriptstyle \boldsymbol{p}_g} \\ \mathbf{H}_g & \xrightarrow[f_\gamma]{} & \mathbf{H}_g \end{array}$$

is a Cartesian diagram of complex manifolds preserving the identity sections and the Riemann forms E_g, i.e., it defines a morphism $\varphi_{\gamma/f_\gamma} : (\mathbf{X}_g, E_g)_{/\mathbf{H}_g} \longrightarrow (\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ in $\mathcal{A}_g^{\mathrm{an}}$. Finally, the formula

$$j(\gamma_1\gamma_2,\tau) = j(\gamma_1, \gamma_2\cdot\tau)j(\gamma_2,\tau)$$

implies that $\varphi_{\gamma/f_\gamma}$ is in fact an automorphism of $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ in $\mathcal{A}_g^{\mathrm{an}}$ and that $\gamma \longmapsto \varphi_{\gamma/f_\gamma}$ is a morphism of groups.[12]

9.4. **De Rham cohomology of complex tori.** Let M be a complex manifold and $\pi : X \longrightarrow M$ be a complex torus over M of relative dimension g.

9.4.1. For any integer $i \geq 0$, we define the ith *analytic de Rham cohomology* sheaf of $\mathcal{O}_M$-modules by

$$\mathcal{H}^i_{\mathrm{dR}}(X/M) := \mathbf{R}^i\pi_*\Omega^\bullet_{X/M},$$

where $\Omega^\bullet_{X/M}$ is the complex of relative holomorphic differential forms. If M is a point, we denote $\mathcal{H}^i_{\mathrm{dR}}(X) := \mathcal{H}^i_{\mathrm{dR}}(X/M)$.

Remark 9.10. If M is a point, then the analytic de Rham cohomology $\mathcal{H}^i_{\mathrm{dR}}(X)$ is canonically isomorphic to the quotient of the complex vector space of C^∞ closed i-forms over X with values in $\mathbf{C}$ by the subspace of exact i-forms (cf. [22] I.1 p. 16).

The arguments in [2] 2.5 prove, *mutatis mutandis*, that there is a canonical isomorphism of $\mathcal{O}_M$-modules given by cup product

$$\bigwedge^i \mathcal{H}^1_{\mathrm{dR}}(X/M) \xrightarrow{\sim} \mathcal{H}^i_{\mathrm{dR}}(X/M),$$

and that $\mathcal{H}^1_{\mathrm{dR}}(X/M)$ is (the sheaf of sections of) a holomorphic vector bundle over M of rank $2g$. Moreover, the canonical $\mathcal{O}_M$-morphism $\pi_*\Omega^1_{X/M} \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(X/M)$ induces an isomorphism of $\pi_*\Omega^1_{X/M}$ onto a rank g subbundle of $\mathcal{H}^1_{\mathrm{dR}}(X/M)$ that we denote by $\mathcal{F}^1(X/M)$.

[12]Actually, it follows from Proposition .119 below (see also Remark .120) that $\gamma \longmapsto \varphi_{\gamma/f_\gamma}$ is an *isomorphism* of groups.

Analogously, it follows from the arguments of [41] that $\mathcal{H}^1_{\mathrm{dR}}(X/M)$ is equipped with a canonical integrable holomorphic connection

$$\nabla : \mathcal{H}^1_{\mathrm{dR}}(X/M) \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(X/M) \otimes_{\mathcal{O}_M} \Omega^1_M,$$

the *Gauss-Manin connection.*

Furthermore, the formation of $\mathcal{H}^1_{\mathrm{dR}}(X/M)$ (resp. $\mathcal{F}^1(X/M)$, resp. ∇) is compatible with every base change in M.

9.4.2. There is a canonical *comparison isomorphism* of holomorphic vector bundles

$$\text{comp} : \mathcal{H}^1_{\mathrm{dR}}(X/M) \xrightarrow{\sim} \mathcal{H}om_{\mathbf{Z}}(R_1\pi_*\mathbf{Z}_X, \mathcal{O}_M) \cong \mathcal{O}_M \otimes_{\mathbf{Z}} R^1\pi_*\mathbf{Z}_X \tag{9.1}$$

identifying the subsheaf of $\mathcal{H}^1_{\mathrm{dR}}(X/M)$ consisting of horizontal sections for the Gauss-Manin connection with the local system of $\mathbf{C}$-vector spaces $\mathcal{H}om_{\mathbf{Z}}(R_1\pi_*\mathbf{Z}_X, \mathbf{C}_M) \cong R^1\pi_*\mathbf{C}_X$ ([17] I Proposition 2.28 and II 7.6-7.7). The induced pairing

$$\mathcal{H}^1_{\mathrm{dR}}(X/M) \otimes_{\mathbf{Z}} R_1\pi_*\mathbf{Z}_X \longrightarrow \mathcal{O}_M$$
$$\alpha \otimes \gamma \longmapsto \text{comp}(\alpha)(\gamma) =: \int_\gamma \alpha$$

is given at each fiber by "integration of differential forms" (cf. Remark .107).

Remark 9.11. In particular, for any section γ of $R_1\pi_*\mathbf{Z}_X$, any C^∞ section α of the vector bundle $\mathcal{H}^1_{\mathrm{dR}}(X/M)$, and any holomorphic vector field θ on M, we have

$$\theta\left(\int_\gamma \alpha\right) = \int_\gamma \nabla_\theta \alpha.$$

Remark 9.12. In the absolute case (where M is a point), the comparison isomorphism can be written

$$\text{comp} : \mathcal{H}^1_{\mathrm{dR}}(X) \xrightarrow{\sim} \mathbf{C} \otimes_{\mathbf{Z}} H^1(X, \mathbf{Z})$$

where X is a complex torus. If X is now an abelian variety over a subfield k of $\mathbf{C}$, then the associated analytic space $X^{\mathrm{an}}_{\mathbf{C}}$ is a complex torus and we have a canonical isomorphism $\mathbf{C} \otimes_k H^1_{\mathrm{dR}}(X/k) \cong \mathcal{H}^1_{\mathrm{dR}}(X^{\mathrm{an}}_{\mathbf{C}})$. In this case, we also write

$$\text{comp} : \mathbf{C} \otimes_k H^1_{\mathrm{dR}}(X/k) \xrightarrow{\sim} \mathbf{C} \otimes_{\mathbf{Z}} H^1(X, \mathbf{Z})$$

for the composition of comp with the above canonical identification.

Recall that $R_1\pi_*\mathbf{Z}_X$ may be naturally identified with a lattice in the holomorphic vector bundle $\mathrm{Lie}_M X$. Accordingly, the dual bundle $(\mathrm{Lie}_M X)^\vee$ gets naturally identified with a holomorphic subbundle of $\mathcal{H}om_{\mathbf{Z}}(R_1\pi_*\mathbf{Z}_X, \mathcal{O}_M)$.

Lemma 9.13. *With notation as above, the comparison isomorphism (9.1) induces an isomorphism of the holomorphic vector bundle $\mathcal{F}^1(X/M)$ onto $(\mathrm{Lie}_M X)^\vee$.* ■

This also follows from a fiber-by-fiber argument: if M is a point, by identifying $\mathcal{H}^1_{\mathrm{dR}}(X)$ with the C^∞ de Rham cohomology with values in $\mathbf{C}$ (Remark .107), the subspace $\mathcal{F}^1(X)$ gets identified with the space of $(1,0)$-forms in $\mathcal{H}^1_{\mathrm{dR}}(X)$, and these correspond to $\mathrm{Hom}_{\mathbf{C}}(\mathrm{Lie}\, X, \mathbf{C})$ under the de Rham isomorphism (cf. [7] Theorem 1.4.1).

9.4.3. If X admits a principal Riemann form E, then, by linearity, we may define a holomorphic symplectic form $\langle\ ,\ \rangle_E$ on the holomorphic vector bundle $\mathcal{H}^1_{\mathrm{dR}}(X/M)$ over M by

$$\langle E(\gamma,\ ), E(\delta,\ )\rangle_E \coloneqq \frac{1}{2\pi i}E(\gamma,\delta)$$

for any sections γ and δ of $R_1\pi_*\mathbf{Z}_X$, where $E(\gamma,\ )$ and $E(\delta,\ )$ are regarded as sections of $\mathcal{H}^1_{\mathrm{dR}}(X/M)$ via the comparison isomorphism (9.1).

Since every section of $R^1\pi_*\mathbf{Z}_X$ is horizontal for the Gauss-Manin connection ∇ on $\mathcal{H}^1_{\mathrm{dR}}(X/M)$ under the comparison isomorphism (9.1), the symplectic form $\langle\ ,\ \rangle_E$ is compatible with ∇: for every sections α,β of $\mathcal{H}^1_{\mathrm{dR}}(X/M)$, and every holomorphic vector field θ on M, we have

$$\theta\langle\alpha,\beta\rangle_E = \langle\nabla_\theta\alpha,\beta\rangle_E + \langle\alpha,\nabla_\theta\beta\rangle_E. \tag{9.2}$$

9.5. **Relative uniformization of complex abelian schemes.** Let U be a smooth separated $\mathbf{C}$-scheme of finite type and (X,λ) be a principally polarized abelian scheme over U of relative dimension g. Denote by $p : X \longrightarrow U$ its structural morphism. Then the associated analytic space U^{an} is a complex manifold, and the analytification $p^{\mathrm{an}} : X^{\mathrm{an}} \longrightarrow U^{\mathrm{an}}$ of p is a complex torus over U^{an} of relative dimension g.

Since the analytification of the coherent $\mathcal{O}_U$-module $H^1_{\mathrm{dR}}(X/U)$ is canonically isomorphic to $\mathcal{H}^1_{\mathrm{dR}}(X^{\mathrm{an}}/U^{\mathrm{an}})$, the symplectic form $\langle\ ,\ \rangle_\lambda$ on $H^1_{\mathrm{dR}}(X/U)$ induces a symplectic form $\langle\ ,\ \rangle_\lambda^{\mathrm{an}}$ on the holomorphic vector bundle $\mathcal{H}^1_{\mathrm{dR}}(X^{\mathrm{an}}/U^{\mathrm{an}})$ over U^{an}.

Lemma 9.14. *Let γ and δ be sections of $R_1p^{\mathrm{an}}_*\mathbf{Z}_{X^{\mathrm{an}}}$, and let α and β be sections of $\mathcal{H}^1_{\mathrm{dR}}(X^{\mathrm{an}}/U^{\mathrm{an}})$ such that $\gamma = \langle\ \ ,\alpha\rangle_\lambda^{\mathrm{an}}$ and $\delta = \langle\ \ ,\beta\rangle_\lambda^{\mathrm{an}}$ under (the dual of) the comparison isomorphism (9.1). Then*

(1) The formula

$$E_\lambda(\gamma,\delta) \coloneqq \frac{1}{2\pi i}\langle\alpha,\beta\rangle_\lambda^{\mathrm{an}}$$

defines a Riemann form over X^{an}.

(2) The holomorphic symplectic forms $\langle\ ,\ \rangle_{E_\lambda}$ and $\langle\ ,\ \rangle_\lambda^{\mathrm{an}}$ over $\mathcal{H}^1_{\mathrm{dR}}(X^{\mathrm{an}}/U^{\mathrm{an}})$ coincide.

Proof. We can assume $U = \operatorname{Spec}\mathbf{C}$, so that (X,λ) is a principally polarized complex abelian variety.

Recall from Paragraph 2.2 that we have constructed an alternating bilinear form E_λ^{dR} on $H^1_{\mathrm{dR}}(X/\mathbf{C})^\vee$, and that the bilinear form $\langle\ ,\ \rangle_\lambda$ over $H^1_{\mathrm{dR}}(X/\mathbf{C})$ is obtained from E_λ^{dR} by duality. Therefore, to prove (1), it is sufficient to prove that, under the identification of $H_1(X^{\mathrm{an}},\mathbf{Z})$ with an abelian subgroup of $H^1_{\mathrm{dR}}(X/\mathbf{C})^\vee$ via (the dual of) the comparison isomorphism (9.1), for any elements γ and δ of $H_1(X^{\mathrm{an}},\mathbf{Z})$,

$$E_\lambda(\gamma,\delta) \coloneqq \frac{1}{2\pi i}E_\lambda^{\mathrm{dR}}(\gamma,\delta)$$

is in $\mathbf{Z}$, and that the induced morphism

$$(*) \qquad \begin{aligned} H_1(X^{\mathrm{an}},\mathbf{Z}) &\longrightarrow \operatorname{Hom}(H_1(X^{\mathrm{an}},\mathbf{Z}),\mathbf{Z}) \\ \gamma &\longmapsto E_\lambda(\gamma,\ ) \end{aligned}$$

is an isomorphism of abelian groups.

Note that, with this definition, (2) is automatic, since for any $\gamma, \delta \in H_1(X^{\mathrm{an}}, \mathbf{Z})$ we have

$$\begin{aligned}\langle E_\lambda(\gamma,\ ), E_\lambda(\delta,\ )\rangle_{E_\lambda} &= \frac{1}{2\pi i} E_\lambda(\gamma,\delta) = \frac{1}{(2\pi i)^2} E_\lambda^{\mathrm{dR}}(\gamma,\delta) = \frac{1}{(2\pi i)^2} \langle E_\lambda^{\mathrm{dR}}(\gamma,\ ), E_\lambda^{\mathrm{dR}}(\delta,\ )\rangle_\lambda^{\mathrm{an}} \\ &= \langle \frac{1}{2\pi i} E_\lambda^{\mathrm{dR}}(\gamma,\ ), \frac{1}{2\pi i} E_\lambda^{\mathrm{dR}}(\delta,\ )\rangle_\lambda^{\mathrm{an}} = \langle E_\lambda(\gamma,\ ), E_\lambda(\delta,\ )\rangle_\lambda^{\mathrm{an}}.\end{aligned}$$

where we identified the vector space $H^1_{\mathrm{dR}}(X/\mathbf{C})$ with $\mathcal{H}^1_{\mathrm{dR}}(X^{\mathrm{an}})$ via the canonical analytification isomorphism.

Now, the topological Chern class $c_{1,\mathrm{top}} : \mathrm{Pic}(X) \longrightarrow H^2(X^{\mathrm{an}}, \mathbf{Z})$, defined via the exponential sequence

$$\begin{aligned}0 \longrightarrow \mathbf{Z}_{X^{\mathrm{an}}} \longrightarrow \mathcal{O}_{X^{\mathrm{an}}} &\longrightarrow \mathcal{O}^\times_{X^{\mathrm{an}}} \longrightarrow 0 \\ f &\longmapsto \exp(2\pi i f)\end{aligned}$$

and the de Rham Chern class $c_{1,\mathrm{dR}} : \mathrm{Pic}(X) \longrightarrow H^2_{\mathrm{dR}}(X/\mathbf{C})$ (cf. Paragraph 2.2) are related by the following commutative diagram (cf. [18] 2.2.5.2)

$$\begin{array}{ccc} \mathrm{Pic}(X) & \xrightarrow{c_{1,\mathrm{dR}}} & H^2_{\mathrm{dR}}(X/\mathbf{C}) \\ {\scriptstyle c_{1,\mathrm{top}}}\downarrow & & \downarrow \\ H^2(X^{\mathrm{an}}, \mathbf{Z}) & \xrightarrow[-2\pi i]{} & H^2(X^{\mathrm{an}}, \mathbf{C}) \end{array}$$

where the arrow $H^2_{\mathrm{dR}}(X/\mathbf{C}) \longrightarrow H^2(X^{\mathrm{an}}, \mathbf{C}) \cong \mathrm{Hom}(H_2(X^{\mathrm{an}}, \mathbf{Z}), \mathbf{C})$ is given by the comparison isomorphism.

If $\mathcal{L}$ is an ample line bundle on X inducing λ, then $E_\lambda^{\mathrm{dR}} = c_{1,\mathrm{dR}}(\mathcal{L})$ under the identification $H^2_{\mathrm{dR}}(X/\mathbf{C})$ with the vector space of alternating bilinear forms on $H^1_{\mathrm{dR}}(X/\mathbf{C})^\vee$ (cf. proof of Lemma .25). By the commutativity of the above diagram, we see that $E_\lambda = -c_{1,\mathrm{top}}(\mathcal{L})$ under the identification of $H^2(X^{\mathrm{an}}, \mathbf{Z})$ with the module of alternating (integral) bilinear forms on $H_1(X^{\mathrm{an}}, \mathbf{Z})$. This proves that E_λ takes integral values.

To prove that $(*)$ is an isomorphism, we simply use the fact that λ^{an} is an isomorphism of X^{an} onto its dual torus, hence the determinant of the bilinear form on $H_1(X^{\mathrm{an}}, \mathbf{Z})$ induced by $c_{1,\mathrm{top}}(\mathcal{L})$ is 1 (cf. [7] 2.4.9). ■

Thus, for any smooth separated $\mathbf{C}$-scheme of finite type U and any principally polarized abelian scheme (X, λ) over U of relative dimension g, the above construction gives a principally polarized complex torus $(X^{\mathrm{an}}, E_\lambda)$ over U^{an} of relative dimension g.

Let $\mathsf{SmVar}_{/\mathbf{C}}$ be the full subcategory of $\mathsf{Sch}_{/\mathbf{C}}$ consisting of smooth separated $\mathbf{C}$-schemes of finite type, and $\mathcal{A}^{\mathrm{sm}}_{g,\mathbf{C}}$ be the full subcategory of $\mathcal{A}_{g,\mathbf{C}}$ consisting of objects $(X,\lambda)_{/U}$ of $\mathcal{A}_{g,\mathbf{C}}$ such that U is an object of $\mathsf{SmVar}_{/\mathbf{C}}$.

We can summarize this paragraph by remarking that we have constructed a "relative uniformization functor" $\mathcal{A}_{g,\mathbf{C}}^{\mathrm{sm}} \longrightarrow \mathcal{A}_g^{\mathrm{an}}$ making the diagram

$$\begin{array}{ccc} \mathcal{A}_{g,\mathbf{C}}^{\mathrm{sm}} & \longrightarrow & \mathcal{A}_g^{\mathrm{an}} \\ \downarrow & & \downarrow \\ \mathsf{SmVar}_{/\mathbf{C}} & \longrightarrow & \mathsf{Man}_{/\mathbf{C}} \end{array}$$

(strictly) commutative, where $\mathsf{SmVar}_{/\mathbf{C}} \longrightarrow \mathsf{Man}_{/\mathbf{C}}$ is the classical analytification functor $U \longmapsto U^{\mathrm{an}}$.

Remark 9.15. One can prove that the above diagram is "Cartesian" in the sense that it induces an equivalence of categories between $\mathcal{A}_{g,\mathbf{C}}^{\mathrm{sm}}$ and the full subcategory of $\mathcal{A}_g^{\mathrm{an}}$ formed by the objects lying above the essential image of the analytification functor $\mathsf{SmVar}_{/\mathbf{C}} \longrightarrow \mathsf{Man}_{/\mathbf{C}}$ (cf. [18] Rappel 4.4.3 and [8] Theorem 3.10). In particular, for any object U of $\mathsf{SmVar}_{/\mathbf{C}}$ and any principally polarized complex torus (X', E) over U^{an} of relative dimension g, there exists up to isomorphism a unique principally polarized abelian scheme (X, λ) over U of relative dimension g such that $(X', E)_{/U^{\mathrm{an}}}$ is isomorphic to $(X^{\mathrm{an}}, E_\lambda)_{/U^{\mathrm{an}}}$ in $\mathcal{A}_g^{\mathrm{an}}(U^{\mathrm{an}})$. In this paper, we shall only need this algebraization result when $U = \operatorname{Spec} \mathbf{C}$, which is classical (cf. [56] Corollary p. 35).

9.6. **Principally polarized complex tori with real multiplication.** Recall that F denotes a totally real number field of degree g with ring of integers R and inverse different ideal D^{-1}.

For a complex manifold M, we may also consider *principally polarized complex tori with R-multiplication over M*. By this we mean a triple $(X, E, m)_{/M}$, where (X, E) is a principally polarized complex torus of relative dimension g over M, and $m : R \longrightarrow \operatorname{End}_M(X)$ is a ring morphism such that, for every $r \in R$, and every sections v, w of $\operatorname{Lie}_M X$,

$$E(\operatorname{Lie} m(r)(v), w) = E(v, \operatorname{Lie} m(r)(w)).$$

Example 9.16. Consider the complex manifold

$$\mathbf{H}^g = \{\tau = (\tau_1, \ldots, \tau_g) \in \mathbf{C}^n \mid \operatorname{Im} \tau_j > 0,\ 1 \leq j \leq g\}.$$

Let $V \coloneqq \mathbf{C}^g \times \mathbf{H}^g$ be the trivial vector bundle over $\mathbf{H}^g$, and L be the subsheaf of $\mathcal{O}_{\mathbf{H}^g}(V)$ given by the image of the morphism of sheaves of abelian groups

$$\begin{aligned} (D^{-1} \oplus R)_{\mathbf{H}^g} &\longrightarrow \mathcal{O}_{\mathbf{H}^g}(V) = \mathcal{O}_{\mathbf{H}^g}^{\oplus g} \\ (x, y) &\longmapsto x + \tau y \coloneqq (\sigma_j(x) + \tau_j \sigma_j(y))_{1 \leq j \leq g} \end{aligned}$$

where $\sigma_1, \ldots, \sigma_g$ are the field embeddings of F into $\mathbf{C}$. Then L is a lattice in V and we denote by

$$\boldsymbol{p}_F : \mathbf{X}_F \longrightarrow \mathbf{H}^g$$

the corresponding complex torus over $\mathbf{H}^g$ of relative dimension g. Let E_F be the imaginary part of the Hermitian metric over V given by

$$(v, w) \longmapsto \sum_{j=1}^{g} \frac{\overline{v}_j w_j}{\operatorname{Im} \tau_j}.$$

Then E_F defines a principal Riemann form on $\mathbf{X}_F$. The action of R on L given by its natural action on $D^{-1} \oplus R$ via the above isomorphism induces an R-multiplication m_F on the principally polarized complex torus $(\mathbf{X}_F, E_F)$. We thus obtain a principally polarized complex torus with R-multiplication $(\mathbf{X}_F, E_F, m_F)_{/\mathbf{H}^g}$.

Let $(X, E, m)_{/M}$ be a principally polarized complex torus with R-multiplication, with structural morphism $\pi : X \longrightarrow M$. Then m induces an action of R on the holomorphic vector bundle $\mathrm{Lie}_M X$ making its sheaf of holomorphic sections a locally free $\mathcal{O}_M \otimes R$-module of rank 1 (that is, Rapoport's condition is automatically satisfied; see the remark following Definition .32). We denote by

$$\Phi_E : \mathrm{Lie}_M X \times \mathrm{Lie}_M X \longrightarrow \mathbf{R}_M \otimes D^{-1}$$

the unique R_M-bilinear form such that $\mathrm{Tr}\,\Phi_E = E$ (cf. Remark .30). We also have a compatible action of R on the lattice $R_1\pi_*\mathbf{Z}_X$, making it a locally free R_M-module of rank $2g$; the restriction of Φ_E to $R_1\pi_*\mathbf{Z}_X$ is a D_M^{-1}-valued integral R_M-bilinear symplectic form.

Let Ψ_E be the unique D^{-1}-valued $\mathcal{O}_M \otimes R$-bilinear symplectic form on $\mathcal{H}^1_{\mathrm{dR}}(X/M)$ satisfying $\mathrm{Tr}\,\Psi_E = \langle\ ,\ \rangle_E$. By unicity, Ψ_E satisfies

$$\Psi_E(\Phi_E(\gamma,\ ), \Phi_E(\delta,\ )) = \frac{1}{2\pi i}\Phi_E(\gamma, \delta)$$

for every sections γ, δ of $R_1\pi_*\mathbf{Z}_X$; here, we use that the comparison isomorphism (9.1) is R-linear, and we regard $\Phi_E(\gamma,\ ), \Phi_E(\delta,\ )$ as sections of $\mathcal{H}^1_{\mathrm{dR}}(X/M)$.

The category fibered in groupoids over $\mathsf{Man}_{/\mathbf{C}}$ of principally polarized complex tori with R-multiplication, defined in an obvious way, is denoted by $\mathcal{A}_F^{\mathrm{an}}$.

Example 9.17. Let

$$\mathrm{SL}(D^{-1} \oplus R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \,\middle|\, a, d \in R,\, b \in D^{-1},\, c \in D \right\}.$$

Alternatively, $\mathrm{SL}(D^{-1} \oplus R)$ can be defined as $\mathrm{Res}_{R/\mathbf{Z}}\,\mathrm{Aut}_{(D^{-1}\oplus R, \Phi)}(\mathbf{Z})$, where Φ denotes the standard D^{-1}-valued R-bilinear symplectic form on $D^{-1} \oplus R$. As in Example .106, we may define a group action

$$\begin{aligned} \mathrm{SL}(D^{-1} \oplus R) &\longrightarrow \mathrm{Aut}_{\mathcal{A}_F^{\mathrm{an}}}((\mathbf{X}_F, E_F, m_F)_{/\mathbf{H}^g}) \\ \gamma &\longmapsto \varphi_{\gamma/\gamma} \end{aligned}$$

by the following explicit formulas: the left action of $\mathrm{SL}(D^{-1} \oplus R)$ on $\mathbf{H}^g$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \left(\frac{\sigma_1(a)\tau_1 + \sigma_1(b)}{\sigma_1(c)\tau_1 + \sigma_1(d)}, \ldots, \frac{\sigma_g(a)\tau_g + \sigma_g(b)}{\sigma_g(c)\tau_g + \sigma_g(d)} \right)$$

where $\sigma_1, \ldots, \sigma_g$ denote the field embeddings of F into $\mathbf{C}$, and, for $\tau \in \mathbf{H}^g$, the isomorphism

$$\varphi_{\gamma,\tau} : \mathbf{X}_{F,\tau} \xrightarrow{\sim} \mathbf{X}_{F,\gamma\cdot\tau}$$

is induced by

$$\begin{aligned} \tilde{\varphi}_{\gamma,\tau} : \mathbf{C}^g &\longrightarrow \mathbf{C}^g \\ z &\longmapsto \frac{z}{c\tau + d} := \left(\frac{z_1}{\sigma_1(c)\tau_1 + \sigma_1(d)}, \ldots, \frac{z_g}{\sigma_g(c)\tau_g + \sigma_g(d)} \right). \end{aligned}$$

Finally, to a principally polarized abelian scheme with R-multiplication $(X, \lambda, m)_{/U}$, with U a smooth separated $\mathbf{C}$-scheme of finite type, we may functorially associate the object $(X^{\mathrm{an}}, E_\lambda, m^{\mathrm{an}})_{/U^{\mathrm{an}}}$ of $\mathcal{A}_F^{\mathrm{an}}$. We remark that by Lemma .111, and by the unicity of the D^{-1}-valued $\mathcal{O}_{U^{\mathrm{an}}} \otimes R$-bilinear symplectic forms, we have

$$\Psi_{E_\lambda} = \Psi_\lambda^{\mathrm{an}}.$$

10. Analytic moduli spaces of complex abelian varieties with a symplectic-Hodge basis

In this section we consider some moduli problems of principally polarized complex tori, regarded as functors

$$(\mathcal{A}_g^{\mathrm{an}})^{\mathrm{op}} \longrightarrow \mathsf{Set} \text{ (resp. } (\mathcal{A}_F^{\mathrm{an}})^{\mathrm{op}} \longrightarrow \mathsf{Set})$$

where $\mathcal{A}_g^{\mathrm{an}}$ (resp. $\mathcal{A}_F^{\mathrm{an}}$) is the category fibered in groupoids over the category of complex manifolds $\mathsf{Man}_{/\mathbf{C}}$ defined in Paragraph 9.3 (resp. 9.6). As usual, we provide a detailed account for the Siegel case $\mathcal{A}_g^{\mathrm{an}}$, and merely indicate the necessary modifications to treat the Hilbert-Blumenthal case $\mathcal{A}_F^{\mathrm{an}}$.

10.1. **Descent of principally polarized complex tori.** Let M be a complex manifold and (X, E) be a principally polarized complex torus over M of relative dimension g.

If M_0 is another complex manifold and $M \longrightarrow M_0$ is a holomorphic map, we say that (X, E) *descends* to M_0 if there exists a principally polarized complex torus (X_0, E_0) over M_0 and a morphism $(X, E)_{/M} \longrightarrow (X_0, E_0)_{/M_0}$ in $\mathcal{A}_g^{\mathrm{an}}$.

Lemma 10.1. *With the above notation, suppose that there exists a proper and free left action of a discrete group Γ on M. If the action of Γ on M lifts to an action of Γ on $(X, E)_{/M}$ in the category $\mathcal{A}_g^{\mathrm{an}}$, then $(X, E)_{/M}$ descends to a principally polarized complex torus over the quotient $\Gamma \backslash M$.*

Sketch of the proof. Consider X as a pair (V, L), where V is a holomorphic vector bundle over M of rank g, and L is a lattice in V (cf. Remark .101). Then, to every $\gamma \in \Gamma$ there is associated a holomorphic map $\varphi_\gamma : V \longrightarrow V$ making the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi_\gamma} & V \\ \downarrow & & \downarrow \\ M & \xrightarrow[\gamma]{} & M \end{array}$$

commute, and compatible with the vector bundle structures. It follows from the commutativity of this diagram that the action of Γ on V is also proper and free. Thus, there exists a unique holomorphic vector bundle structure on the complex manifold $\Gamma \backslash V$ over $\Gamma \backslash M$ such that the canonical holomorphic map $V \longrightarrow \Gamma \backslash V$ induces a vector bundle isomorphism of V onto the pullback to M of the vector bundle $\Gamma \backslash V$ over $\Gamma \backslash M$.

Analogously, one descends the lattice L to a lattice in $\Gamma \backslash V$ (consider the étalé space, for instance), and the bilinear form E on V to a bilinear form on $\Gamma \backslash V$, which is seen to be a principal polarization *a posteriori.* ∎

Remark 10.2. It is not difficult to check that an analogous statement holds for principally polarized complex tori with R-multiplication: if a proper and free action of a discrete group Γ on a complex manifold M lifts to an action on a principally polarized complex torus with R-multiplication (X, E, m) over M, then $(X, E, m)_{/M}$ descends to a principally polarized complex torus with R-multiplication over $\Gamma\backslash M$.

10.2. **Integral symplectic bases over principally polarized complex tori.** Let M be a complex manifold and (X, E) be a principally polarized complex torus over M of relative dimension g. We denote by $\pi : X \longrightarrow M$ its structural morphism.

Definition 10.3. An *integral symplectic basis* of $(X, E)_{/M}$ is a trivializing $2g$-uple $(\gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g)$ of global sections of $R_1\pi_*\mathbf{Z}_X$ which is symplectic with respect to the Riemann form E, that is,

$$E(\gamma_i, \gamma_j) = E(\delta_i, \delta_j) = 0 \quad \text{and} \quad E(\gamma_i, \delta_j) = \delta_{ij}$$

for any $1 \leq i, j \leq g$.

Example 10.4. Consider the principally polarized complex torus $(\mathbf{X}_g, E_g)$ over $\mathbf{H}_g$ of Example .105 and recall that a section of $R_1\boldsymbol{p}_{g*}\mathbf{Z}_{\mathbf{X}_g}$ is given by a column vector of holomorphic functions on $\mathbf{H}_g$ of the form $\tau \longmapsto m + \tau n$, for some sections (m, n) of $(\mathbf{Z}^g \oplus \mathbf{Z}^g)_{\mathbf{H}_g}$. We can thus define an integral symplectic basis

$$\beta_g = (\gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g)$$

of $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ by

$$\gamma_i(\tau) \coloneqq \mathbf{e}_i \quad \text{and} \quad \delta_i(\tau) \coloneqq \tau\mathbf{e}_i$$

for any $\tau \in \mathbf{H}_g$.

Let $(X', E')_{/M'}$ and $(X, E)_{/M}$ be objects of $\mathcal{A}_g^{\text{an}}$ with structural morphisms $\pi' : X' \longrightarrow M'$ and $\pi : X \longrightarrow M$. If $\varphi_{/f} : (X', E')_{/M'} \longrightarrow (X, E)_{/M}$ is a morphism in $\mathcal{A}_g^{\text{an}}$, then the isomorphism of vector bundles

$$\text{Lie}_{M'}X' \xrightarrow{\sim} f^*\text{Lie}_M X \tag{10.1}$$

induced by φ identifies the lattice $R_1\pi'_*\mathbf{Z}_{X'}$ with $f^*R_1\pi_*\mathbf{Z}_X$. If γ is a section of $R_1\pi_*\mathbf{Z}_X$, we denote by $\varphi^*\gamma$ the section of $R_1\pi'_*\mathbf{Z}_{X'}$ mapping to $f^*\gamma$ under (10.1). As the isomorphism (10.1) also preserves the corresponding Riemann forms, for any integral symplectic basis $(\gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g)$ of $(X, E)_{/M}$, the $2g$-uple of global sections of $R_1\pi'_*\mathbf{Z}_{X'}$ given by

$$\varphi^*\beta \coloneqq (\varphi^*\gamma_1, \ldots, \varphi^*\gamma_g, \varphi^*\delta_1, \ldots, \varphi^*\delta_g)$$

is an integral symplectic basis of $(X', E')_{/M'}$.

Proposition 10.5 (cf. [7] Proposition 8.1.2)**.** *The functor* $(\mathcal{A}_g^{\text{an}})^{\text{op}} \longrightarrow \mathsf{Set}$ *sending an object* $(X, E)_{/M}$ *of* $\mathcal{A}_g^{\text{an}}$ *to the set of integral symplectic bases of* $(X, E)_{/M}$ *is representable by* $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$*, with universal integral symplectic basis* β_g *defined in Example .118.*

Proof. Let $(X, E)_{/M}$ be an object of $\mathcal{A}_g^{\text{an}}$ with structural morphism $\pi : X \longrightarrow M$, and let $\beta = (\gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g)$ be an integral symplectic basis of $(X, E)_{/M}$. Let W be the real subbundle of $\text{Lie}_M X$ generated by $\gamma_1, \ldots, \gamma_g$. Since E is the imaginary part of a Hermitian metric, for any nontrivial section γ of W, we have $E(\gamma, i\gamma) \neq 0$. As W is isotropic with respect to E, it follows that $\mathrm{Lie}_M X = W \oplus iW$ as a real vector bundle. In particular, $\gamma \coloneqq (\gamma_1, \dots, \gamma_g)$ trivializes $\mathrm{Lie}_M X$ as a holomorphic vector bundle. Hence, if $\delta \coloneqq (\delta_1, \dots, \delta_g)$, then there exists a unique holomorphic map $\tau : M \longrightarrow \mathrm{GL}_g(\mathbf{C})$ such that $\delta = \gamma\tau$, where γ and δ are regarded as row vectors of global holomorphic sections of $\mathrm{Lie}_M X$.

Let $A \coloneqq (E(\gamma_k, i\gamma_l))_{1\le k,l\le g} \in M_{g\times g}(\mathbf{C})$. Since

$$\delta = \gamma \operatorname{Re}\tau + i\gamma \operatorname{Im}\tau,$$

the matrix of E in the basis β is given by

$$\begin{pmatrix} 0 & A \operatorname{Im}\tau \\ -(A\operatorname{Im}\tau)^{\mathsf{T}} & (\operatorname{Re}\tau)^{\mathsf{T}} A \operatorname{Im}\tau - (\operatorname{Im}\tau)^{\mathsf{T}} A^{\mathsf{T}} \operatorname{Re}\tau \end{pmatrix}.$$

Using that β is symplectic with respect to E, and that A is symmetric and positive-definite (recall that E is the imaginary part of a Hermitian metric), we conclude that τ factors through $\mathbf{H}_g \subset \mathrm{GL}_g(\mathbf{C})$.

Finally, writing X as the quotient of $\mathrm{Lie}_M X$ by $R_1\pi_*\mathbf{Z}_X$, we see that τ lifts to a unique morphism in $\mathcal{A}_g^{\mathrm{an}}$

$$\varphi_{/\tau} : (X, E)_{/M} \longrightarrow (\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$$

satisfying $\varphi^*\beta_g = \beta$. ■

Remark 10.6. We may define a *left* action of the group $\mathrm{Sp}_{2g}(\mathbf{Z})$ on the functor $(\mathcal{A}_g^{\mathrm{an}})^{\mathrm{op}} \longrightarrow \mathsf{Set}$ of integral symplectic bases, considered in the above proposition, as follows. Let $(X, E)_{/U}$ be an object of $\mathcal{A}_g^{\mathrm{an}}$ and β be an integral symplectic basis of $(X, E)_{/U}$. Let $\gamma = (A\ \ B\ ;\ C\ D) \in \mathrm{Sp}_{2g}(\mathbf{Z})$, and consider $\beta = (\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g)$ as a row vector of order $2g$; then we define

$$\gamma \cdot \beta \coloneqq \begin{pmatrix} \gamma_1 & \cdots & \gamma_g & \delta_1 & \cdots & \delta_g \end{pmatrix} \begin{pmatrix} D^{\mathsf{T}} & B^{\mathsf{T}} \\ C^{\mathsf{T}} & A^{\mathsf{T}} \end{pmatrix}$$

The morphism

$$\varphi_{\gamma/f_\gamma} : (\mathbf{X}_g, E_g)_{/\mathbf{H}_g} \longrightarrow (\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$$

defined in Example .106 is the unique morphism in $\mathcal{A}_g^{\mathrm{an}}$ satisfying

$$\varphi_\gamma^*\beta_g = \gamma \cdot \beta_g.$$

10.3. **Principal (symplectic) level structures.**

10.3.1. Let U be a scheme, and X be an abelian scheme over U. Recall that, for any integer $n \geq 1$, we may define a natural pairing, the so-called *Weil pairing*,

$$X[n] \times X^t[n] \longrightarrow \mu_{n,U},$$

where $\mu_{n,U}$ denotes the U-group scheme of nth roots of unity (cf. [56] IV.20).

Fix an integer $n \geq 1$, and let $\zeta_n \in \mathbf{C}$ be the nth root of unity $e^{\frac{2\pi i}{n}}$. For any scheme U over $\mathbf{Z}[1/n, \zeta_n]$, and any principally polarized abelian scheme (X, λ) over U of relative dimension g, by identifying $X^t[n]$ with $X[n]$ via λ, and $\mu_{n,U}$ with $(\mathbf{Z}/n\mathbf{Z})_U$ via ζ_n, we obtain a pairing

$$e_n^\lambda : X[n] \times X[n] \longrightarrow (\mathbf{Z}/n\mathbf{Z})_U.$$

The formation of e_n^λ is compatible with every base change in U. Moreover, e_n^λ is skew-symmetric and non-degenerate (cf. [56] IV.23).

Since, for any integer $n \geq 3$, there exists a fine moduli space $A_{g,1,n}$ over $\mathbf{Z}[1/n]$ for principally polarized abelian varieties of dimension g endowed with a full level n-structure (see [58] Theorem 7.9, and the following remark; see also [51] Théorème VII.3.2), there also exists a fine moduli space $A_{g,n}$ over $\mathbf{Z}[1/n, \zeta_n]$ for principally polarized abelian varieties (X, λ) of dimension g endowed with a symplectic basis of $X[n]$ for the pairing e_n^λ (cf. [24] IV.6). The scheme $A_{g,n}$ is quasi-projective and smooth over $\mathbf{Z}[1/n, \zeta_n]$, with connected fibers.

In the sequel, we denote the universal principally polarized abelian scheme over $A_{g,n}$ by $(X_{g,n}, \lambda_{g,n})$, and the universal symplectic basis of $X_{g,n}[n]$ by $\alpha_{g,n}$.

10.3.2. Let $(X, E)_{/M}$ be an object of $\mathcal{A}_g^{\mathrm{an}}$ with structural morphism $\pi : X \longrightarrow M$. For any integer $n \geq 1$, by an *integral symplectic basis modulo n* of $(X, E)_{/M}$, we mean a $2g$-uple of global sections of the local system of $\mathbf{Z}/n\mathbf{Z}$-modules

$$R_1\pi_*(\mathbf{Z}/n\mathbf{Z})_X = R_1\pi_*\mathbf{Z}_X/nR_1\pi_*\mathbf{Z}_X$$

which is symplectic with respect to the alternating $\mathbf{Z}/n\mathbf{Z}$-linear form on $R_1\pi_*(\mathbf{Z}/n\mathbf{Z})_X$ induced by E.

Remark 10.7. Every integral symplectic basis of $(X, E)_{/M}$ induces an integral symplectic basis modulo n of $(X, E)_{/M}$. Conversely, since the natural map $\mathrm{Sp}_{2g}(\mathbf{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z})$ is surjective, locally on M, every integral symplectic basis modulo n of $(X, E)_{/M}$ can be lifted to an integral symplectic basis of $(X, E)_{/M}$.

The notion of integral symplectic bases modulo n is compatible with the notion of principal level n structures of 10.3.1 in the following sense. Let $(X, \lambda)_{/U}$ be an object of $\mathcal{A}_{g,\mathbf{C}}^{\mathrm{sm}}$ (see Paragraph 9.5) with structural morphism $p : X \longrightarrow U$. The étalé space of the local system $R_1 p_*^{\mathrm{an}}(\mathbf{Z}/n\mathbf{Z})_{X^{\mathrm{an}}}$ is canonically isomorphic to the n-torsion Lie subgroup $X^{\mathrm{an}}[n]$ of X^{an}. Under this identification, the pairing e_n^λ on $X[n]$ coincides, up to a sign, with the reduction modulo n of the Riemann form E_λ (cf. [56] IV.23 and IV.24), and thus an integral symplectic basis modulo n of $(X^{\mathrm{an}}, E_\lambda)_{/U^{\mathrm{an}}}$ canonically corresponds to a symplectic trivialization of $X^{\mathrm{an}}[n]$ with respect to e_n^λ.

10.3.3. Let $\Gamma(n)$ the kernel of the natural map $\mathrm{Sp}_{2g}(\mathbf{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z})$. Recall that for any $n \geq 3$ the induced action of $\Gamma(n)$ on $\mathbf{H}_g$ is free ([56] IV.21 Theorem 5) and proper.

Proposition 10.8 (cf. [7] Theorem 8.3.2)**.** *For any integer $n \geq 3$, the complex manifold $A_{g,n}(\mathbf{C}) = A_{g,n,\mathbf{C}}^{\mathrm{an}}$ is canonically biholomorphic to the quotient of $\mathbf{H}_g$ by $\Gamma(n)$, and the functor $(\mathcal{A}_g^{\mathrm{an}})^{\mathrm{op}} \longrightarrow \mathsf{Set}$ sending an object $(X, E)_{/M}$ of $\mathcal{A}_g^{\mathrm{an}}$ to the set of integral symplectic bases modulo n of $(X, E)_{/M}$ is representable by $(X_{g,n,\mathbf{C}}^{\mathrm{an}}, E_{\lambda_{g,n}})_{/A_{g,n,\mathbf{C}}^{\mathrm{an}}}$.*

Proof. As the action of $\Gamma(n)$ on $\mathbf{H}_g$ is proper and free, the quotient

$$\mathbf{A}_{g,n} := \Gamma(n)\backslash\mathbf{H}_g$$

is a complex manifold, and the canonical holomorphic map $\mathbf{H}_g \longrightarrow \mathbf{A}_{g,n}$ is a covering map with Galois group $\Gamma(n)$. Moreover, since the action of $\Gamma(n)$ on $\mathbf{H}_g$ lifts to an action of $\Gamma(n)$ on $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$

in the category $\mathcal{A}_g^{\mathrm{an}}$, the principally polarized complex torus $(\mathbf{X}_g, E_g)$ over $\mathbf{H}_g$ descends to a principally polarized complex torus $(\mathbf{X}_{g,n}, E_{g,n})$ over $\mathbf{A}_{g,n}$ (Lemma .115).

Let $\overline{\beta}_g$ be the integral symplectic basis modulo n of $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ obtained from β_g by reduction modulo n. Then $\overline{\beta}_g$ is invariant under the action of $\Gamma(n)$, and thus it descends to an integral symplectic basis modulo n of $(\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$, say $\beta_{g,n}$.

The object $(\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$ of $\mathcal{A}_g^{\mathrm{an}}$ so constructed represents the functor in the statement with $\beta_{g,n}$ serving as universal symplectic basis modulo n. Indeed, let $(X,E)_{/M}$ be an object of $\mathcal{A}_g^{\mathrm{an}}$, and β be an integral symplectic basis modulo n of $(X,E)_{/M}$. By Remark .121, there exists an open covering $M = \bigcup_{i\in I} U^i$ and, for each $i \in I$, an integral symplectic basis β^i of $(X,E)_{/U^i}$ lifting β. By Proposition .119, we obtain for each $i \in I$ a morphism $\varphi^i_{/f^i} : (X,E)_{/U^i} \longrightarrow (\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ in $\mathcal{A}_g^{\mathrm{an}}$ satisfying $(\varphi^i)^*\beta_g = \beta^i$. Finally, by construction, for any $i, j \in I$, the compositions of $\varphi^i_{/f^i}$ and $\varphi^j_{/f^j}$ with the projection $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g} \longrightarrow (\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$ agree over the intersection $U^i \cap U^j$; hence they glue to a morphism

$$\varphi_{/f} : (X,E)_{/M} \longrightarrow (\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$$

satisfying $\varphi^*\beta_{g,n} = \beta$, and uniquely determined by this property.

To finish the proof, it is sufficient to show that $(X^{\mathrm{an}}_{g,n,\mathbf{C}}, E_{\lambda_{g,n}})_{/A^{\mathrm{an}}_{g,n,\mathbf{C}}}$ is isomorphic to $(\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$ in the category $\mathcal{A}_g^{\mathrm{an}}$. By the compatibility of principal level n structures with integral symplectic bases modulo n, there exists a unique morphism in $\mathcal{A}_g^{\mathrm{an}}$

$$\varphi_{/f} : (X^{\mathrm{an}}_{g,n,\mathbf{C}}, E_{\lambda_{g,n}})_{/A^{\mathrm{an}}_{g,n,\mathbf{C}}} \longrightarrow (\mathbf{X}_{g,n}, E_{g,n})_{/\mathbf{A}_{g,n}}$$

such that $\varphi^*\beta_{g,n}$ is the integral symplectic basis modulo n of $(X^{\mathrm{an}}_{g,n,\mathbf{C}}, E_{\lambda_{g,n}})_{/A^{\mathrm{an}}_{g,n,\mathbf{C}}}$ associated to $\alpha_{g,n}$ (the universal principal level n structure of $(X_{g,n}, \lambda_{g,n})_{/A_{g,n}}$). Since complex tori (over a point) endowed with a principal Riemann form are algebraizable (cf. Remark .112), the holomorphic map

$$f : A_{g,n}(\mathbf{C}) = A^{\mathrm{an}}_{g,n,\mathbf{C}} \longrightarrow \mathbf{A}_{g,n}$$

is bijective. As the complex manifolds $\mathbf{A}_{g,n}$ and $A_{g,n}(\mathbf{C})$ have same dimension, f is necessarily a biholomorphism ([28] p. 19). ■

10.4. **Symplectic-Hodge bases over complex tori.**

10.4.1. Let M be a complex manifold and (X,E) be a principally polarized complex torus over M of relative dimension g. As in Definition .28, by a *symplectic-Hodge basis* of $(X,E)_{/M}$, we mean a $2g$-uple $b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ of global sections of the holomorphic vector bundle $\mathcal{H}^1_{\mathrm{dR}}(X/M)$ such that $\omega_1, \ldots, \omega_g$ are sections of the subbundle $\mathcal{F}^1(X/M)$, and b is symplectic with respect to the holomorphic symplectic form $\langle\ ,\ \rangle_E$.

It follows from Lemma .111 that this notion of symplectic-Hodge basis is compatible with its algebraic counterpart via the "relative uniformization functor" in Paragraph 9.5.

10.4.2. Consider Siegel parabolic subgroup of $\mathrm{Sp}_{2g}(\mathbf{C})$

$$P_g(\mathbf{C}) = \left\{ \begin{pmatrix} A & B \\ 0 & (A^{\mathsf{T}})^{-1} \end{pmatrix} \in M_{2g\times 2g}(\mathbf{C}) \;\middle|\; A \in \mathrm{GL}_g(\mathbf{C}) \text{ and } B \in M_{g\times g}(\mathbf{C}) \text{ satisfy } AB^{\mathsf{T}} = BA^{\mathsf{T}} \right\}.$$

Note that $P_g(\mathbf{C})$ is a complex Lie group of dimension $g(3g+1)/2$.

Let (X,E) be a principally polarized complex torus of dimension g. If $b=(\omega\ \eta)$ is a symplectic-Hodge basis of (X,E), seen as a row vector of order $2g$ with coefficients in $\mathcal{H}^1_{\mathrm{dR}}(X)$, and $p=(A\ B\ ;\ 0\ (A^{\mathsf{T}})^{-1})\in P_g(\mathbf{C})$, then we put

$$b\cdot p := (\ \omega A \quad \omega B+\eta(A^{\mathsf{T}})^{-1}\ ).$$

It is easy to check that $b\cdot p$ is a symplectic-Hodge basis of (X,E), and that the above formula defines a free and transitive action of $P_g(\mathbf{C})$ on the set of symplectic-Hodge bases of (X,E) (cf. Lemma .42).

10.4.3. For a complex manifold M, let us denote by $\mathsf{Man}_{/M}$ the category of complex manifolds endowed with a holomorphic map to M.

Lemma 10.9 (cf. Corollary .43)**.** *Let M be a complex manifold and (X,E) be a principally polarized complex torus over M of relative dimension g. The functor*

$$\begin{aligned}\mathsf{Man}^{\mathrm{op}}_{/M} &\longrightarrow \mathsf{Set}\\ M' &\longmapsto \{\textit{symplectic-Hodge bases of } (X,E)\times_M M'\}\end{aligned}$$

is representable by a principal $P_g(\mathbf{C})$-bundle $B(X,E)$ over M.

Proof. Let us denote by $\pi: V\longrightarrow M$ the holomorphic vector bundle $\mathcal{H}^1_{\mathrm{dR}}(X/M)^{\oplus g}$ over M. For any $p\in M$, the fiber $\pi^{-1}(p)=V_p$ is the vector space of g-uples $(\alpha_1,\dots,\alpha_g)$, with each $\alpha_i\in\mathcal{H}^1_{\mathrm{dR}}(X_p)$. Let B be the locally closed analytic subspace of V consisting of points $v=(\alpha_1,\dots,\alpha_g)$ of V such that

$$L:=\mathbf{C}\alpha_1+\cdots+\mathbf{C}\alpha_g$$

is a Lagrangian subspace of $\mathcal{H}^1_{\mathrm{dR}}(X_{\pi(v)})$ with respect to $\langle\ ,\ \rangle_{E_{\pi(v)}}$ satisfying

$$\mathcal{F}^1(X_{\pi(v)})\oplus L=\mathcal{H}^1_{\mathrm{dR}}(X_{\pi(v)}).$$

By Proposition .23 (2), a symplectic-Hodge basis $(\omega_1,\dots,\omega_g,\eta_1,\dots,\eta_g)$ of a principally polarized complex torus is uniquely determined by $(\eta_1,\dots,\eta_g)$. In particular, for each $p\in M$, the fiber $B_p=B\cap V_p$ may be naturally identified with the set of symplectic-Hodge bases of (X_p,E_p).

Thus, it follows from 10.4.2 that B is a principal $P_g(\mathbf{C})$-bundle over M; in particular, it is a complex manifold. We also conclude from the above paragraph that B represents the functor in the statement. ■

Remark 10.10. The above construction is compatible, under analytification, with its algebraic counterpart. Namely, let U be a smooth separated $\mathbf{C}$-scheme of finite type, and (X,λ) be a principally polarized abelian scheme over U. The complex manifold $B(X^{\mathrm{an}},E_\lambda)$ over U^{an} constructed in Lemma .123 is canonically isomorphic to the analytification of the scheme $B(X,\lambda)$ over U constructed in Corollary .43.

Recall that we denote by (X_g,λ_g) the universal principally polarized abelian scheme over B_g, and by b_g the universal symplectic-Hodge basis of $(X_g,\lambda_g)_{/B_g}$.

Proposition 10.11. *The functor $(\mathcal{A}^{\mathrm{an}}_g)^{\mathrm{op}}\longrightarrow\mathsf{Set}$ sending an object $(X,E)_{/M}$ of $\mathcal{A}^{\mathrm{an}}_g$ to the set of symplectic-Hodge bases of $(X,E)_{/M}$ is representable by $(X^{\mathrm{an}}_{g,\mathbf{C}},E_{\lambda_g})_{/B^{\mathrm{an}}_{g,\mathbf{C}}}$, with universal symplectic-Hodge basis b_g.*

Proof. By Lemma .123, there exists a complex manifold $\mathbf{B}_g := B(\mathbf{X}_g, E_g)$ over $\mathbf{H}_g$ representing the functor

$$\begin{aligned} \mathsf{Man}^{\mathrm{op}}_{/\mathbf{H}_g} &\longrightarrow \mathsf{Set} \\ M &\longmapsto \{\text{symplectic-Hodge bases of } (\mathbf{X}_g, E_g) \times_{\mathbf{H}_g} M\} \end{aligned}$$

Let $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g}) = (\mathbf{X}_g, E_g) \times_{\mathbf{H}_g} \mathbf{B}_g$. Note that the principally polarized complex torus $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})$ over $\mathbf{B}_g$ is equipped with a universal symplectic-Hodge basis $b_{\mathbf{B}_g}$, and with an integral symplectic basis $\beta_{\mathbf{B}_g}$ obtained by pullback from β_g via the canonical morphism $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})_{/\mathbf{B}_g} \longrightarrow (\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ in $\mathcal{A}_g^{\mathrm{an}}$.

We now remark that $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})_{/\mathbf{B}_g}$ represents the functor $(\mathcal{A}_g^{\mathrm{an}})^{\mathrm{op}} \longrightarrow \mathsf{Set}$ sending an object $(X, E)_{/M}$ of $\mathcal{A}_g^{\mathrm{an}}$ to the Cartesian product of the set of symplectic-Hodge bases of $(X, E)_{/M}$ with the set of integral symplectic bases of $(X, E)_{/M}$, with $(b_{\mathbf{B}_g}, \beta_{\mathbf{B}_g})$ serving as a universal object. Thus, for any element $\gamma \in \mathrm{Sp}_{2g}(\mathbf{Z})$, there exists a unique automorphism $\Psi_{\gamma/\psi_\gamma}$ of $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})_{/\mathbf{B}_g}$ in $\mathcal{A}_g^{\mathrm{an}}$ such that $\Psi_\gamma^* b_{\mathbf{B}_g} = b_{\mathbf{B}_g}$ and $\Psi_\gamma^* \beta_{\mathbf{B}_g} = \gamma \cdot \beta_{\mathbf{B}_g}$ (where the left action of $\mathrm{Sp}_{2g}(\mathbf{Z})$ on integral symplectic bases is defined as in Remark .120).

As the functor $\underline{B}_g : \mathcal{A}_g^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is rigid over $\mathbf{C}$ (Lemma .82), we see that

(1) $\gamma \longmapsto \Psi_{\gamma/\psi_\gamma}$ is in fact an action of $\mathrm{Sp}_{2g}(\mathbf{Z})$ on $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})_{/\mathbf{B}_g}$ in the category $\mathcal{A}_g^{\mathrm{an}}$, and
(2) the action $\gamma \longmapsto \psi_\gamma$ of $\mathrm{Sp}_{2g}(\mathbf{Z})$ on the complex manifold $\mathbf{B}_g$ is free; it is also proper since it lifts the action on $\mathbf{H}_g$.

Let M be the quotient manifold $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathbf{B}_g$ and descend $(\mathbf{X}_{\mathbf{B}_g}, E_{\mathbf{B}_g})$ to a principally polarized complex torus (X, E) over M. Since $b_{\mathbf{B}_g}$ is invariant under the action of $\mathrm{Sp}_{2g}(\mathbf{Z})$, we can descend it to a symplectic-Hodge basis b of $(X, E)_{/M}$. As in the proof of Proposition .122, we may check that $(X, E)_{/M}$ represents the functor in the statement, with b serving as universal symplectic-Hodge basis.

To finish the proof, we must prove that $(X, E)_{/M}$ is isomorphic to $(X^{\mathrm{an}}_{g,\mathbf{C}}, E_{\lambda_g})_{/B^{\mathrm{an}}_{g,\mathbf{C}}}$ in $\mathcal{A}_g^{\mathrm{an}}$. For this, it is sufficient to remark that, by the universal property of $(X, E)_{/M}$, there exists a unique morphism in $\mathcal{A}_g^{\mathrm{an}}$

$$\varphi_{/f} : (X^{\mathrm{an}}_{g,\mathbf{C}}, E_{\lambda_g})_{/B^{\mathrm{an}}_{g,\mathbf{C}}} \longrightarrow (X, E)_{/M}$$

satisfying $\varphi^* b = b_g$, and that the holomorphic map

$$f : B_g(\mathbf{C}) = B^{\mathrm{an}}_{g,\mathbf{C}} \longrightarrow M$$

is bijective since principally polarized complex tori (over a point) are algebraizable (cf. Remark .112); then f is necessarily a biholomorphism ([28] p. 19). ■

10.5. **The Hilbert-Blumenthal case.** In this paragraph we state without proof the R-multiplication counterparts of the above results.

10.5.1. Let M be a complex manifold, $(X, E, m)_{/M}$ be a principally polarized complex torus with R-multiplication over M, and denote by $\pi : X \longrightarrow M$ the structural morphism.

Consider the local system of abelian groups $(D^{-1} \oplus R)_M := \mathbf{Z}_M \otimes (D^{-1} \oplus R)$ over M, endowed with its natural R-multiplication, and with the standard D^{-1}-valued R-bilinear symplectic form Φ.

Definition 10.12. An *integral symplectic basis* of $(X,E,m)_{/M}$ is an R-linear isomorphism

$$\beta : ((D^{-1}\oplus R)_M,\Phi) \xrightarrow{\sim} (R_1\pi_*\mathbf{Z}_X,\Phi_E).$$

Equivalently, we may think of an integral symplectic basis as a couple $\beta=(\gamma,\delta)$, where γ (resp. δ) is a global section of $R_1\pi_*\mathbf{Z}_X\otimes D$ (resp. $R_1\pi_*\mathbf{Z}_X$), satisfying $\Phi_E(\gamma,\delta)=1$. Here, we see Φ_E as an R-bilinear map

$$\Phi_E : (R_1\pi_*\mathbf{Z}_X\otimes D)\times R_1\pi_*\mathbf{Z}_X \longrightarrow R_M.$$

Example 10.13. The principally polarized complex torus with R-multiplication $(\mathbf{X}_F,E_F,m_F)_{/\mathbf{H}^g}$ constructed in Example .113 is equipped with a canonical integral symplectic basis β_F given by the defining isomorphism $(D^{-1}\oplus R)_{\mathbf{H}^g}\xrightarrow{\sim} L$ and the natural identification $L\cong R_1\boldsymbol{p}_{F*}\mathbf{Z}_{\mathbf{X}_F}$.

We then have the analogous of Proposition .119.

Proposition 10.14. *The functor $(\mathcal{A}_F^{\mathrm{an}})^{\mathrm{op}}\longrightarrow \mathsf{Set}$ sending an object $(X,E,m)_{/M}$ of $\mathcal{A}_F^{\mathrm{an}}$ to the set of integral symplectic bases of $(X,E,m)_{/M}$ is representable by $(\mathbf{X}_F,E_F,m_F)_{/\mathbf{H}^g}$, with universal integral symplectic basis β_F.* ■

Remark 10.15. As in Remark .120, we define a *left* action of $\mathrm{SL}(D^{-1}\oplus R)$ (cf. Example .114) on the functor $(\mathcal{A}_F^{\mathrm{an}})^{\mathrm{op}}\longrightarrow \mathsf{Set}$ considered in the above proposition: if $\gamma=(a\ b\ ;\ c\ d)\in\mathrm{SL}(D^{-1}\oplus R)$, and $\beta=\begin{pmatrix}\gamma_1 & \delta_1\end{pmatrix}$ is an integral symplectic basis, then

$$\gamma\cdot\beta := \begin{pmatrix}\gamma_1 & \delta_1\end{pmatrix}\cdot\begin{pmatrix} d & b\\ c & a\end{pmatrix} = \begin{pmatrix} d\gamma+c\delta & b\gamma+a\delta\end{pmatrix}.$$

The morphism

$$\varphi_{\gamma/\gamma} : (\mathbf{X}_F,E_F,m_F)_{/\mathbf{H}^g}\longrightarrow (\mathbf{X}_F,E_F,m_F)_{/\mathbf{H}^g}$$

defined in Example .114 is the unique morphism in $\mathcal{A}_F^{\mathrm{an}}$ satisfying

$$\varphi_\gamma^*\beta_F=\gamma\cdot\beta_F.$$

Remark 10.16. Let $t:(\mathbf{Z}^{2g},\langle\ ,\ \rangle_{\mathrm{std}})\xrightarrow{\sim}(R\oplus D^{-1},\mathrm{Tr}\,\Psi)$ be the trivialization of the symplectic $\mathbf{Z}$-module $(R\oplus D^{-1},\mathrm{Tr}\,\Psi)$ as defined in Remark .40, so that $(t^\vee)^{-1}$ is a trivialization of $(D^{-1}\oplus R,\mathrm{Tr}\,\Phi)$. Then we can use Propositions .119 and .128 to see that t induces a holomorphic map

$$h_t : \mathbf{H}^g\longrightarrow \mathbf{H}_g$$

given, under the moduli theoretic interpretation, by

$$(X,E,m,\beta)\longmapsto (X,E,\beta\circ(t^\vee)^{-1}).$$

It follows from the construction in the proof of Proposition .119 that h_t is given in coordinates by

$$(\tau_1,\ldots,\tau_g)\longmapsto (\sigma_i(x_j))_{1\le i,j\le g}^{-1}\mathrm{diag}(\tau_1,\ldots,\tau_g)(\sigma_i(r_j))_{1\le i,j\le g}$$

Note that h_t actually lifts to a morphism in $\mathcal{A}_g^{\mathrm{an}}$

$$(\mathbf{X}_F,E_F)_{/\mathbf{H}^g}\longrightarrow(\mathbf{X}_g,E_g)_{/\mathbf{H}_g}$$

given on the fiber of $\tau\in\mathbf{H}^g$ by

$$\begin{aligned}\mathbf{X}_{F,\tau}&\longrightarrow \mathbf{X}_{g,h_t(\tau)}\\ z&\longmapsto (\sigma_i(x_j))_{1\le i,j\le g}^{-1}\cdot z.\end{aligned}$$

Finally, let us remark that, by definition of r_i and x_i, we have

$$(\sigma_i(x_j))^{-1}_{1\leq i,j\leq g} = (\sigma_j(r_i))_{1\leq i,j\leq g} = (\sigma_i(r_j))^{\mathsf{T}}_{1\leq i,j\leq g}.$$

10.5.2. Let $n\geq 1$ be an integer, and $(X,\lambda,m)_{/U}$ be a principally polarized abelian scheme with R-multiplication. Clearly, the action of R on X preserves the n-torsion subscheme $X[n]$. If U is a $\mathbf{Z}[1/n,\zeta_n]$-scheme, then there exists a perfect alternating R-bilinear pairing (see Remark .30)

$$\epsilon_n^\lambda : X[n]\times X[n] \longrightarrow (D^{-1}/nD^{-1})_U$$

such that

$$\operatorname{Tr}\epsilon_n^\lambda = e_n^\lambda.$$

If $n\geq 3$, then there exists a fine moduli scheme $A_{F,n}$ over $\mathbf{Z}[1/n,\zeta_n]$ classifying principally polarized abelian schemes with R-multiplication $(X,\lambda,m)_{/U}$ equipped with an R-trivialization of $(X[n],\epsilon_n^\lambda)$, i.e., an R-isomorphism

$$(((D^{-1}/nD^{-1})\oplus(R/nR))_U,\Phi_n) \stackrel{\sim}{\longrightarrow} (X[n],\epsilon_n^\lambda),$$

where Φ_n denotes the standard symplectic form modulo n. We denote the universal principally polarized abelian scheme with R-multiplication over $A_{F,n}$ by $(X_{F,n},\lambda_{F,n},m_{F,n})$, and its universal symplectic R-trivialization by $\alpha_{F,n}$.

In the analytic category $\mathcal{A}_F^{\mathrm{an}}$, we may consider the notion of an "integral symplectic basis modulo n" of a principally polarized complex torus with R-multiplication $(X,E,m)_{/M}$; namely, an R-linear isomorphism

$$(((D^{-1}/nD^{-1})\oplus(R/nR))_U,\Phi_n) \stackrel{\sim}{\longrightarrow} (R_1\pi_*\mathbf{Z}_X/nR_1\pi_*\mathbf{Z}_X,\Phi_{E,n}),$$

where $\Phi_{E,n}$ denotes the reduction modulo n of R-bilinear symplectic form Φ_E. This notion coincides with its algebraic counterpart, since for a principally polarized abelian scheme with R-multiplication $(X,\lambda,m)_{/U}$, with U a smooth separated $\mathbf{C}$-scheme of finite type, the R-symplectic modules $(R_1p_*^{\mathrm{an}}\mathbf{Z}_{X^{\mathrm{an}}}/nR_1p_*^{\mathrm{an}}\mathbf{Z}_{X^{\mathrm{an}}},\Phi_{E_\lambda,n})$ and $(X^{\mathrm{an}}[n],\epsilon_n^\lambda)$ are naturally isomorphic.

For any integer $n\geq 1$, let $\Gamma_F(n)$ be the kernel of the "reduction modulo n" map $\mathrm{SL}(D^{-1}\oplus R)\longrightarrow \mathrm{SL}((D^{-1}/nD^{-1})\oplus(R/nR))$. If $n\geq 3$, then $\Gamma_F(n)$ acts properly and freely on $\mathbf{H}^g$.

Proposition 10.17 (cf. Proposition .122)**.** *For any integer $n\geq 3$, the complex manifold $A_{F,n}(\mathbf{C}) = A^{\mathrm{an}}_{F,n,\mathbf{C}}$ is canonically biholomorphic to the quotient of $\mathbf{H}^g$ by $\Gamma_F(n)$, and the functor $(\mathcal{A}_F^{\mathrm{an}})^{\mathrm{op}}\longrightarrow \mathsf{Set}$ sending an object $(X,E,m)_{/M}$ of $\mathcal{A}_F^{\mathrm{an}}$ to the set of integral symplectic bases modulo n of $(X,E,m)_{/M}$ is representable by $(X^{\mathrm{an}}_{F,n,\mathbf{C}},E_{F,n},m_{F,n})_{/A^{\mathrm{an}}_{F,n,\mathbf{C}}}$.* ■

10.5.3. Finally, we define *symplectic-Hodge bases* of principally polarized complex tori as in Paragraph 3.3 (cf. 10.4.1).

Let (X_F,λ_F,m_F) be the universal principally polarized abelian scheme with R-multiplication over B_F, and let b_F be its universal symplectic-Hodge basis.

Proposition 10.18. *The functor $(\mathcal{A}_F^{\mathrm{an}})^{\mathrm{op}}\longrightarrow \mathsf{Set}$ sending an object $(X,E,m)_{/M}$ of $\mathcal{A}_F^{\mathrm{an}}$ to the set of symplectic-Hodge bases of $(X,E,m)_{/M}$ is representable by $(X^{\mathrm{an}}_{F,\mathbf{C}},E_{\lambda_F},m_F^{\mathrm{an}})_{/B^{\mathrm{an}}_{F,\mathbf{C}}}$, with universal symplectic-Hodge basis b_F.* ■

11. The analytic higher Ramanujan equations

In this section we consider the complex analytic avatars of the higher Ramanujan equations introduced in Section 6.

We shall then construct particular solutions φ_g and φ_F of these differential equations, defined on $\mathbf{H}_g$ in the Siegel case, and on $\mathbf{H}^g$ in the Hilbert-Blumenthal case. The "q-expansions" of these solutions coincide with the previously defined integral solutions $\hat{\varphi}_g$ and $\hat{\varphi}_F$.

11.1. **Definition of φ_g and statement of our main theorem in the Siegel case.** Let us first define the analytic higher Ramanujan equations. Consider the holomorphic coordinate system $(\tau_{kl})_{1\leq k\leq l\leq g}$ on the complex manifold $\mathbf{H}_g$, where $\tau_{kl} : \mathbf{H}_g \longrightarrow \mathbf{C}$ associates to any $\tau \in \mathbf{H}_g$ its entry in the kth row and lth column. To this system of coordinates is attached a family $(\theta_{kl})_{1\leq k\leq l\leq g}$ of holomorphic vector fields on $\mathbf{H}_g$, defined by

$$\theta_{kl} \coloneqq \frac{1}{2\pi i}\frac{\partial}{\partial \tau_{kl}}.$$

Let $(v_{kl})_{1\leq k\leq l\leq g}$ be the family of holomorphic vector fields on $B_g(\mathbf{C})$ induced by the higher Ramanujan vector fields on $\mathcal{B}_g$ defined in Section 5.

Definition 11.1. Let U be an open subset of $\mathbf{H}_g$. We say that a holomorphic map $u : U \longrightarrow B_g(\mathbf{C})$ is an *analytic solution of the higher Ramanujan equations* over $\mathcal{B}_g$ if

$$Tu(\theta_{kl}) = u^* v_{kl}$$

for every $1 \leq k \leq l \leq g$.

We now construct a global holomorphic solution

$$\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$$

of the higher Ramanujan equations. In view of the universal property of the moduli space $B_g(\mathbf{C})$ (Proposition .125), the holomorphic map φ_g will be induced by a certain symplectic-Hodge basis of the principally polarized complex torus $(\mathbf{X}_g, E_g)$ over $\mathbf{H}_g$.

Recall that the comparison isomorphism (9.1) identifies the holomorphic vector bundle $(\mathrm{Lie}_{\mathbf{H}_g}\mathbf{X}_g)^\vee$ over $\mathbf{H}_g$ with $\mathcal{F}^1(\mathbf{X}_g/\mathbf{H}_g)$ (Lemma .110). Moreover, it follows from the construction of $\mathbf{X}_g$ in Example .105 that $\mathrm{Lie}_{\mathbf{H}_g}\mathbf{X}_g$ is canonically isomorphic to the trivial vector bundle $\mathbf{C}^g \times \mathbf{H}_g$ over $\mathbf{H}_g$. Under this isomorphism, we define the holomorphic frame

$$(dz_1, \ldots, dz_g)$$

of $\mathcal{F}^1(\mathbf{X}_g/\mathbf{H}_g)$ as the dual of the canonical holomorphic frame of $\mathbf{C}^g \times \mathbf{H}_g$.

Theorem 11.2. *For each $1 \leq k \leq g$, consider the global sections of $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$*

$$\boldsymbol{\omega}_k \coloneqq 2\pi i\, dz_k, \quad \boldsymbol{\eta}_k \coloneqq \nabla_{\theta_{kk}}\boldsymbol{\omega}_k,$$

where ∇ denotes the Gauss-Manin connection on $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$. Then,

(1) The $2g$-uple

$$\boldsymbol{b}_g \coloneqq (\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_g, \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_g)$$

of holomorphic global sections of $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$ *is a symplectic-Hodge basis of the principally polarized complex torus* $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$.

(2) *The holomorphic map*

$$\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$$

corresponding to $\boldsymbol{b}_g$ *by the universal property of* $B_g(\mathbf{C})$ *is a solution of the higher Ramanujan equations (Definition .133).*

The main idea in our proof is to compute with a C^∞ trivialization of the vector bundle $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$; in the next subsection we develop some preliminary background.

11.2. **Preliminary results.** Consider the *complex conjugation*, seen as a C^∞ morphism of real vector bundles over $\mathbf{H}_g$,

$$\begin{aligned} \mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g) &\longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g) \\ \alpha &\longmapsto \overline{\alpha} \end{aligned}$$

induced by the comparison isomorphism (9.1), and denote $d\bar{z}_k := \overline{dz_k}$ for every $1 \leq k \leq g$. We may check fiber by fiber that the $2g$-uple of C^∞ global sections of $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$

$$(dz_1, \ldots, dz_g, d\bar{z}_1, \ldots, d\bar{z}_g)$$

trivializes $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$ as a C^∞ complex vector bundle over $\mathbf{H}_g$.

For $1 \leq i \leq j \leq g$ and $1 \leq k \leq g$, let us define

$$\boldsymbol{\eta}_k^{ij} := \nabla_{\theta_{ij}} \boldsymbol{\omega}_k,$$

so that

$$\boldsymbol{\eta}_k = \boldsymbol{\eta}_k^{kk}.$$

Proposition 11.3. *Consider the notations in 0.16. For every* $1 \leq i \leq j \leq g$ *and* $1 \leq k \leq g$, *we have*

$$\boldsymbol{\eta}_k^{ij} = \sum_{l=1}^{g} \mathbf{e}_k^{\top} \mathbf{E}^{ij} (\operatorname{Im} \tau)^{-1} \mathbf{e}_l \operatorname{Im} dz_l$$

as a C^∞ *section of* $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$, *where* $\operatorname{Im} dz_l := (dz_l - d\bar{z}_l)/2i$.

Proof. For $1 \leq i \leq j \leq g$ and $1 \leq k, l \leq g$, let λ_{kl}^{ij} and μ_{kl}^{ij} be the C^∞ functions on $\mathbf{H}_g$ with values in $\mathbf{C}$ defined by the equation

$$\boldsymbol{\eta}_k^{ij} = \sum_{l=1}^{g} (\lambda_{kl}^{ij} dz_l + \mu_{kl}^{ij} d\bar{z}_l).$$

We must prove that $\lambda_{kl}^{ij} + \mu_{kl}^{ij} = 0$ and that $\lambda_{kl}^{ij} = \frac{1}{2i} \mathbf{e}_k^{\top} \mathbf{E}^{ij} (\operatorname{Im} \tau)^{-1} \mathbf{e}_l$.

Let us consider the integral symplectic basis $\beta_g = (\gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g)$ of $R_1 \boldsymbol{p}_{g*} \mathbf{Z}_{\mathbf{X}_g}$ defined in Example .118. For every $1 \leq i \leq j \leq g$ and $1 \leq k, l \leq g$, we have (cf. Remark .108)

$$\int_{\gamma_l} \boldsymbol{\eta}_k^{ij} = \int_{\gamma_l} \nabla_{\frac{\partial}{\partial \tau_{ij}}} dz_k = \frac{\partial}{\partial \tau_{ij}} \int_{\gamma_l} dz_k = \frac{\partial}{\partial \tau_{ij}} \delta_{kl} = 0$$

and

$$\int_{\delta_l} \boldsymbol{\eta}_k^{ij} = \int_{\delta_l} \nabla_{\frac{\partial}{\partial \tau_{ij}}} dz_k = \frac{\partial}{\partial \tau_{ij}} \int_{\delta_l} dz_k = \frac{\partial}{\partial \tau_{ij}} \tau_{kl} = \mathbf{E}_{kl}^{ij}.$$

Thus, by definition of λ_{kl}^{ij} and μ_{kl}^{ij}, we obtain

$$0 = \int_{\gamma_l} \boldsymbol{\eta}_k^{ij} = \sum_{m=1}^{g} \left(\lambda_{km}^{ij} \int_{\gamma_l} dz_m + \mu_{km}^{ij} \int_{\gamma_l} d\bar{z}_m \right) = \lambda_{kl}^{ij} + \mu_{kl}^{ij}$$

and

$$\mathbf{E}_{kl}^{ij} = \int_{\delta_l} \boldsymbol{\eta}_k^{ij} = \sum_{m=1}^{g} \left(\lambda_{km}^{ij} \int_{\delta_l} dz_m + \mu_{km}^{ij} \int_{\delta_l} d\bar{z}_m \right) = \sum_{m=1}^{g} \lambda_{km}^{ij} (\tau_{ml} - \overline{\tau_{ml}}) = 2i \sum_{m=1}^{g} \lambda_{km}^{ij} (\operatorname{Im} \tau)_{ml}.$$

In matricial notation, if we put $\lambda^{ij} := (\lambda_{kl}^{ij})_{1 \leq k,l \leq g} \in M_{g \times g}(\mathbf{C})$, then we have shown that

$$2i\lambda^{ij} \operatorname{Im} \tau = \mathbf{E}^{ij}$$

The assertion follows. ■

Specializing to the case $i = j = k$ in the above proposition, we obtain the following formulas.

Corollary 11.4. *For any $1 \leq k \leq g$, we have*

$$\boldsymbol{\eta}_k = \sum_{l=1}^{g} ((\operatorname{Im} \tau)^{-1})_{kl} \operatorname{Im} dz_l.$$

In particular, $\boldsymbol{\eta}_k$ is the unique global section of $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$ satisfying

$$\int_{\gamma_l} \boldsymbol{\eta}_k = 0 \quad \text{and} \quad \int_{\delta_l} \boldsymbol{\eta}_k = \delta_{kl}$$

for every $1 \leq l \leq g$. In other words, $\boldsymbol{\eta}_k$ may be identified with $E_g(\gamma_k, \ )$ under the comparison isomorphism (9.1).

Since every section of $R^1 \boldsymbol{p}_{g*} \mathbf{Z}_{\mathbf{X}_g} = (R_1 \boldsymbol{p}_{g*} \mathbf{Z}_{\mathbf{X}_g})^\vee$, seen as a section of $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$ via the comparison isomorphism (9.1), is horizontal for the Gauss-Manin connection, we obtain the next corollary.

Corollary 11.5. *For any $1 \leq k \leq g$, the global section $\boldsymbol{\eta}_k$ of $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$ is horizontal for the Gauss-Manin connection:*

$$\nabla \boldsymbol{\eta}_k = 0.$$

Our next goal is to use the duality given by the Riemann form E_g to express dz_l in terms of C^∞ sections of $\operatorname{Lie}_{\mathbf{H}_g} \mathbf{X}_g$.

Lemma 11.6. *Let $1 \leq k \leq g$, and denote by τ_k the k-th column of $\tau \in \mathbf{H}_g$. Then*

$$dz_k = -E_g(i \operatorname{Im} \tau_k, \ ) + iE_g(\operatorname{Im} \tau_k, \ )$$

as a C^∞ section of $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$ under the comparison isomorphism (9.1).

Proof. Note that $\operatorname{Im}\tau_k = (\operatorname{Im}\tau)\mathbf{e}_k$. Let γ be a section of $R_1 \boldsymbol{p}_{g*}\mathbf{Z}_{\mathbf{X}_g}$. As $\operatorname{Im}\tau$ is symmetric and $\gamma = \operatorname{Re}\gamma + i\operatorname{Im}\gamma$, we have

$$\begin{aligned}
-E_g(i\operatorname{Im}\tau_k,\gamma) + iE_g(\operatorname{Im}\tau_k,\gamma) &= -\operatorname{Im}(\overline{i\operatorname{Im}\tau_k}^{\mathsf{T}}(\operatorname{Im}\tau)^{-1}\gamma) + i\operatorname{Im}(\overline{\operatorname{Im}\tau_k}^{\mathsf{T}}(\operatorname{Im}\tau)^{-1}\gamma)\\
&= \operatorname{Im}(i\mathbf{e}_k^{\mathsf{T}}(\operatorname{Im}\tau)(\operatorname{Im}\tau)^{-1}\gamma) + i\operatorname{Im}(\mathbf{e}_k^{\mathsf{T}}(\operatorname{Im}\tau)(\operatorname{Im}\tau)^{-1}\gamma)\\
&= \operatorname{Re}(\mathbf{e}_k^{\mathsf{T}}\gamma) + i\operatorname{Im}(\mathbf{e}_k^{\mathsf{T}}\gamma)\\
&= \mathbf{e}_k^{\mathsf{T}}\gamma = dz_k(\gamma).
\end{aligned}$$

■

11.3. **Proof of Theorem .134.** We prove parts (1) and (2) separately.

Proof of Theorem .134 (1). As each $\boldsymbol{\omega}_k$ is by definition a section of $\mathcal{F}^1(\mathbf{X}_g/\mathbf{H}_g)$, to prove that $\boldsymbol{b}_g$ is a symplectic-Hodge basis of $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ it is sufficient to show that it is a symplectic trivialization of $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$ with respect to the holomorphic symplectic form $\langle\ ,\ \rangle_{E_g}$. For this, we claim that it is enough to prove that

$$(*) \qquad \langle\boldsymbol{\omega}_i, \boldsymbol{\eta}_j\rangle_{E_g} = \delta_{ij}$$

for every $1 \leq i \leq j \leq g$. Indeed, by Corollary .137 and by the compatibility (9.2), equation $(*)$ implies that $\langle\boldsymbol{\eta}_i, \boldsymbol{\eta}_j\rangle_{E_g} = 0$ (apply $\nabla_{\theta_{ii}}$). Since we already know that $\mathcal{F}^1(\mathbf{X}_g/\mathbf{H}_g)$ is Lagrangian, this proves indeed that $\boldsymbol{b}_g$ is a symplectic trivialization of $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_g/\mathbf{H}_g)$.

Fix $1 \leq i \leq j \leq g$. By Corollary .136, we have

$$\boldsymbol{\eta}_j = \sum_{l=1}^{g}((\operatorname{Im}\tau)^{-1})_{jl}\operatorname{Im} dz_l,$$

thus

$$\langle\boldsymbol{\omega}_i, \boldsymbol{\eta}_j\rangle_{E_g} = 2\pi i\sum_{l=1}^{g}((\operatorname{Im}\tau)^{-1})_{jl}\langle dz_i, \operatorname{Im} dz_l\rangle_{E_g}.$$

Now, using Lemma .138, we obtain

$$\begin{aligned}
\langle dz_i, \operatorname{Im} dz_l\rangle_{E_g} &= \langle -E_g(i\operatorname{Im}\tau_i,\ ) + iE_g(\operatorname{Im}\tau_i,\ ), E_g(\operatorname{Im}\tau_l,\ )\rangle_{E_g}\\
&= -\langle E_g(i\operatorname{Im}\tau_i,\ ), E_g(\operatorname{Im}\tau_l,\ )\rangle_{E_g} + i\langle E_g(\operatorname{Im}\tau_i,\ ), E_g(\operatorname{Im}\tau_l,\ )\rangle_{E_g}\\
&= \frac{1}{2\pi i}\left(-E_g(i\operatorname{Im}\tau_i, \operatorname{Im}\tau_l) + iE_g(\operatorname{Im}\tau_i, \operatorname{Im}\tau_l)\right)\\
&= \frac{1}{2\pi i}\operatorname{Im}(i\operatorname{Im}\tau_i^{\mathsf{T}}(\operatorname{Im}\tau)^{-1}\operatorname{Im}\tau_l)\\
&= \frac{1}{2\pi i}\mathbf{e}_i^{\mathsf{T}}(\operatorname{Im}\tau)\mathbf{e}_l = \frac{1}{2\pi i}(\operatorname{Im}\tau)_{il}.
\end{aligned}$$

Therefore, since $\operatorname{Im}\tau$ is symmetric,

$$\langle\boldsymbol{\omega}_i, \boldsymbol{\eta}_j\rangle_{E_g} = \sum_{l=1}^{g}((\operatorname{Im}\tau)^{-1})_{jl}(\operatorname{Im}\tau)_{li} = \delta_{ij}.$$

■

Part (2) in Theorem .134 will be an easy consequence of the following analytic analog of Proposition .69.

Proposition 11.7. *Let $U \subset \mathbf{H}_g$ be an open subset and $u : U \longrightarrow B_g(\mathbf{C})$ be the holomorphic map corresponding to a principally polarized complex torus (X, E) over U endowed with some symplectic-Hodge basis $b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$. Then the following are equivalent:*

(1) u is a solution of the higher Ramanujan equations.

(2) For every $1 \leq i \leq j \leq g$, we have

$$\nabla_{\theta_{ij}} b = b \begin{pmatrix} 0 & 0 \\ \mathbf{E}^{ij} & 0 \end{pmatrix}$$

that is,

(i) $\nabla_{\theta_{ij}} \omega_i = \eta_j$, $\nabla_{\theta_{ij}} \omega_j = \eta_i$, and $\nabla_{\theta_{ij}} \omega_k = 0$, for $k \notin \{i, j\}$

(ii) $\nabla_{\theta_{ij}} \eta_k = 0$, for $1 \leq k \leq g$.

■

Proof of Theorem .134 (2). By Proposition .139, it is sufficient to prove that, for every $1 \leq i \leq j \leq g$, we have

(i) $\nabla_{\theta_{ij}} \boldsymbol{\omega}_i = \boldsymbol{\eta}_j$, $\nabla_{\theta_{ij}} \boldsymbol{\omega}_j = \boldsymbol{\eta}_i$, and $\nabla_{\theta_{ij}} \boldsymbol{\omega}_k = 0$, for $k \notin \{i, j\}$

(ii) $\nabla_{\theta_{ij}} \boldsymbol{\eta}_k = 0$, for $1 \leq k \leq g$.

Now, (i) follows directly from Proposition .135, and (ii) is the content of Corollary .137. ■

11.4. **Compatibility of φ_g with $\hat{\varphi}_g$.** Recall that we have constructed in Section 6 a morphism of stacks $\hat{\varphi}_g : \operatorname{Spec} \mathbf{Z}(\!(q_{ij})\!) \longrightarrow \mathcal{B}_g$. Let us briefly explain how Theorem .71, which claims that $\hat{\varphi}_g$ is an integral solution of the higher Ramanujan equations on $\mathcal{B}_g$, follows from Theorem .134 above.

Recall that the group of $g \times g$ integral symmetric matrices $\operatorname{Sym}_g(\mathbf{Z})$ is isomorphic to the subgroup

$$\left\{ \begin{pmatrix} \mathbf{1}_g & N \\ 0 & \mathbf{1}_g \end{pmatrix} \in M_{g \times g}(\mathbf{Z}) \;\middle|\; N \in \operatorname{Sym}_g(\mathbf{Z}) \right\}$$

of $\mathrm{Sp}_{2g}(\mathbf{Z})$, so that it acts on the object $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ of $\mathcal{A}_g^{\mathrm{an}}$ by Example .106; its action on the base manifold $\mathbf{H}_g$ is given by translations:

$$N \cdot \tau = \tau + N,$$

hence it is proper and free.

By Lemma .115, the principally polarized complex torus $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ descends to a principally polarized complex torus (X, E) over the quotient $\operatorname{Sym}_g(\mathbf{Z}) \backslash \mathbf{H}_g$. Moreover, since the symplectic-Hodge basis $\boldsymbol{b}_g$ is easily checked to be invariant under the action of $\operatorname{Sym}_g(\mathbf{Z})$, it also descends to a symplectic-Hodge basis b on $(X, E)_{/\operatorname{Sym}_g(\mathbf{Z}) \backslash \mathbf{H}_g}$. It follows that the holomorphic map $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ defined in Theorem .134 factors through a map

$$\psi : \operatorname{Sym}_g(\mathbf{Z}) \backslash \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$$

associated to the principally polarized complex torus with symplectic-Hodge basis (X, E, b) over $\operatorname{Sym}_g(\mathbf{Z}) \backslash \mathbf{H}_g$.

Observe that

$$\begin{aligned}\mathbf{H}_g &\longrightarrow \mathrm{Sym}_g(\mathbf{C})\\ \tau &\longmapsto q(\tau) = (q_{kl}(\tau))_{1\leq k,l\leq g} \coloneqq (e^{2\pi i \tau_{kl}})_{1\leq k,l\leq g}\end{aligned}$$

induces a biholomorphism of the quotient $\mathrm{Sym}_g(\mathbf{Z})\backslash\mathbf{H}_g$ onto an open submanifold $\mathbf{D}_g$ of $\mathrm{Sym}_g(\mathbf{C})$. Under this identification, we have $\frac{1}{2\pi i}\frac{\partial}{\partial\tau_{kl}} = q_{kl}\frac{\partial}{\partial q_{kl}}$, and one may check that (X,E,b) corresponds formally to $(\hat{X}_g, \hat{\lambda}_g, \hat{b}_g)$ defined in Paragraph 6.2 (that is, (X,E) is obtained by the Mumford construction performed in the analytic category, and b is defined as b_g). For instance, for $q = (q_{ij})_{1\leq i,j\leq g} \in \mathbf{D}_g$, we have

$$X_q = (\mathbf{C}^\times)^g/\langle (q_{1j},\ldots,q_{g,j}) \mid 1\leq j\leq g\rangle,$$

and the isomorphism $\mathbf{X}_{g,\tau} \xrightarrow{\sim} X_{q(\tau)}$ is induced by

$$z = (z_1,\ldots,z_g) \longmapsto (t_1(z),\ldots,t_g(z)) \coloneqq (e^{2\pi i z_1},\ldots,e^{2\pi i z_g})$$

so that

$$\boldsymbol{\omega}_k = \frac{dt_k}{t_k}.$$

It follows that $\hat{\varphi}_g$ is the "Taylor expansion" of ψ in the variables q_{kl}. In particular, Theorem .71 is an immediate corollary of Theorem .134.

Remark 11.8. A rigorous construction of such correspondence requires the theory of toroidal compactification and completion at components at infinity; we refer to [24] p. 141-142 for further details.

11.5. **Analytic Higher Ramanujan equations over $\mathcal{B}_F$.** Let $(\mathbf{X}_F, E_F, m_F)$ be the principally polarized complex torus with R-multiplication over $\mathbf{H}^g$ constructed in Example .113. As $\mathrm{Lie}_{\mathbf{H}^g}\mathbf{X}_F$ is canonically isomorphic to the trivial vector bundle $\mathbf{C}^g\times\mathbf{H}^g$ over $\mathbf{H}^g$, we may define a global section of $\mathcal{F}^1(\mathbf{X}_F/\mathbf{H}^g) = (\mathrm{Lie}_{\mathbf{H}^g}\mathbf{X}_F)^\vee$ by the formula

$$\boldsymbol{\omega}_F \coloneqq 2\pi i\sum_{j=1}^g dz_j.$$

It is easy to check that $\boldsymbol{\omega}_F$ trivializes $\mathcal{F}^1(\mathbf{X}_F/\mathbf{H}^g)$ as a $\mathcal{O}_{\mathbf{H}^g}\otimes R$-module.

Proposition 11.9. *The dual of the Kodaira-Spencer morphism*

$$\kappa^\vee : S^2_{\mathcal{O}_{\mathbf{H}^g}\otimes R}(\mathcal{F}^1(\mathbf{X}_F/\mathbf{H}^g)) \longrightarrow \Omega^1_{\mathbf{H}^g}$$

is an isomorphism of $\mathcal{O}_{\mathbf{H}^g}$-modules satisfying

$$\kappa^\vee(\boldsymbol{\omega}_F) = 2\pi i\sum_{j=1}^g d\tau_j. \tag{11.1}$$

Proof. Recall from Remark .59 that

$$\kappa^\vee(\boldsymbol{\omega}_F) = \langle\boldsymbol{\omega}_F, \nabla\boldsymbol{\omega}_F\rangle_{E_F}$$

where ∇ denotes the Gauss-Manin connection on $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_F/\mathbf{H}^g)$. Thus, (11.1) is equivalent to

$$\langle \boldsymbol{\omega}_F, \nabla_{\frac{1}{2\pi i}\frac{\partial}{\partial\tau_j}} \boldsymbol{\omega}_F\rangle_{E_F} = 1, \quad 1 \leq j \leq g. \tag{11.2}$$

To prove this, we may argue as in Paragraph 11.2, to which we refer for further details on the computations:

(1) we have, for any $1 \leq i, j \leq g$,

$$\nabla_{\frac{\partial}{\partial\tau_j}} dz_i = \begin{cases} 0 & i \neq j \\ \frac{\operatorname{Im} dz_j}{\operatorname{Im}\tau_j} & i = j \end{cases}$$

as C^∞ global sections of $\mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_F/\mathbf{H}^g)$;

(2) under the comparison isomorphism (9.1), we may write

$$dz_j = -E_F(i \operatorname{Im}\tau_j \mathbf{e}_j,\ ) + iE_F(\operatorname{Im}\tau_j \mathbf{e}_j,\ ),$$

and we deduce from the definition of $\langle\ ,\ \rangle_{E_F}$ (9.4.3) that

$$\langle dz_j, \operatorname{Im} dz_j\rangle_{E_F} = \frac{\operatorname{Im}\tau_j}{2\pi i}.$$

The equation (11.2) now easily follows from (1) and (2) above.

If we endow $\Omega^1_{\mathbf{H}^g}$ with the unique R-multiplication satisfying $r\cdot d\tau_j = \sigma_j(r)d\tau_j$ for every $1 \leq j \leq g$, then $\kappa^\vee$ becomes $\mathcal{O}_{\mathbf{H}^g} \otimes R$-linear. Since $2\pi i\sum_{j=1}^g d\tau_j$ trivializes $\Omega^1_{\mathbf{H}^g}$ as an $\mathcal{O}_{\mathbf{H}^g} \otimes R$-module, we conclude from (11.1) that $\kappa^\vee$ is an isomorphism.[13] ∎

By composing the Kodaira-Spencer isomorphism

$$\kappa : T_{\mathbf{H}^g} \xrightarrow{\sim} \Gamma^2_{\mathcal{O}_{\mathbf{H}^g}\otimes R}(\mathcal{F}^1(\mathbf{X}_F/\mathbf{H}^g)) \otimes_R D^{-1}$$

with the trivialization of $\Gamma^2_{\mathcal{O}_{\mathbf{H}^g}\otimes R}(\mathcal{F}^1(\mathbf{X}_F/\mathbf{H}^g))$ induced by $\boldsymbol{\omega}_F$, we obtain an isomorphism

$$T_{\mathbf{H}^g} \xrightarrow{\sim} \mathcal{O}_{\mathbf{H}^g} \otimes D^{-1}.$$

We denote the inverse of this isomorphism by

$$\theta_F : \mathcal{O}_{\mathbf{H}^g} \otimes D^{-1} \xrightarrow{\sim} T_{\mathbf{H}^g}.$$

Remark 11.10. Explicitly, we deduce from Proposition .141 that, for any $x \in D^{-1}$,

$$\theta_F(1\otimes x) = \frac{1}{2\pi i}\sum_{j=1}^g \sigma_j(x)\frac{\partial}{\partial\tau_j}.$$

Definition 11.11. Let $U \subset \mathbf{H}^g$ be an open subset, and $u : U \longrightarrow B_F(\mathbf{C})$ be a holomorphic map. We say that u is an *analytic solution of the higher Ramanujan equations over* $\mathcal{B}_F$ if

$$Tu \circ \theta_F = u^* v_F, \tag{11.3}$$

[13] Alternatively, we might deduce that $\kappa^\vee$ is an isomorphism from the corresponding fact on the universal Kodaira-Spencer morphism over $\mathcal{A}_F$ (cf. Paragraph 5.5 and Proposition .131).

that is, if the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbf{H}^g} \otimes D^{-1} & \xrightarrow{\theta_F} & T_{\mathbf{H}^g} \\ \cong\downarrow & & \downarrow Tu \\ u^*(\mathcal{O}_{B_F(\mathbf{C})} \otimes D^{-1}) & \xrightarrow[v_F]{} & u^*T_{B_F(\mathbf{C})} \end{array}$$

commutes.

Let $(x_1, \ldots, x_g)$ be a $\mathbf{Z}$-basis of D^{-1}, and let $(r_1, \ldots, r_g)$ be the dual $\mathbf{Z}$-basis of R. If we denote $\theta^{r_j} = \theta_F(1 \otimes x_j)$ (resp. $v^{r_j} = v_F(1 \otimes x_j)$), then the higher Ramanujan equations acquire the more concrete form

$$Tu(\theta^{r_j}) = u^* v^{r_j}, \quad 1 \leq j \leq g.$$

To construct an analytic solution of the higher Ramanujan equations over $\mathcal{B}_F$ defined on $\mathbf{H}^g$, we proceed as in the Siegel case. The proof of the next result is analogous to its Siegel counterpart.

Proposition 11.12 (cf. Proposition .139)**.** *Let $U \subset \mathbf{H}^g$ be an open subset and $u : U \longrightarrow B_F(\mathbf{C})$ be the holomorphic map corresponding to a principally polarized complex torus with R-multiplication (X, E, m) over U endowed with some symplectic-Hodge basis $b = (\omega, \eta)$. Then the following are equivalent:*

(1) u is an analytic solution of the higher Ramanujan equations over $\mathcal{B}_F$.
(2) We have

$$\nabla_{\theta_F} b = b \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

■

Theorem 11.13. *Let*

$$\boldsymbol{\eta}_F := \nabla_{\theta_F} \boldsymbol{\omega}_F \in \Gamma(\mathbf{H}^g, \mathcal{H}^1_{\mathrm{dR}}(\mathbf{X}_F/\mathbf{H}^g) \otimes_R D).$$

Then:

(1) The couple $\mathbf{b}_F := (\boldsymbol{\omega}_F, \boldsymbol{\eta}_F)$ is a symplectic-Hodge basis of $(\mathbf{X}_F, E_F, m_F)_{/\mathbf{H}^g}$.
(2) The holomorphic map

$$\varphi_F : \mathbf{H}^g \longrightarrow B_F(\mathbf{C})$$

induced by $\mathbf{b}_F$ is an analytic solution of the higher Ramanujan equations over $\mathcal{B}_F$.

Proof. In view of Remark .35, to prove (1) it suffices to prove that $\Psi_{E_F}(\boldsymbol{\omega}_F, \boldsymbol{\eta}_F) = 1$, i.e., that the $\mathcal{O}_{\mathbf{H}^g} \otimes R$-linear morphism

$$\begin{aligned} \mathcal{O}_{\mathbf{H}^g} \otimes D^{-1} &\longrightarrow \mathcal{O}_{\mathbf{H}^g} \otimes D^{-1} \\ 1 \otimes x &\longmapsto \Psi_{E_F}(\boldsymbol{\omega}_F, \nabla_{\theta_F(1\otimes x)} \boldsymbol{\omega}_F) \end{aligned}$$

is the identity. By Remark .30, this is yet equivalent to proving that, for every $x \in D^{-1}$,

$$\langle \boldsymbol{\omega}_F, \nabla_{\theta_F(1\otimes x)} \boldsymbol{\omega}_F \rangle_{E_F} = \mathrm{Tr}(x).$$

This follows immediately from Remark .142 and from formula (11.2) in the proof of Proposition .141:

$$\langle \boldsymbol{\omega}_F, \nabla_{\theta_F(1\otimes x)}\boldsymbol{\omega}_F\rangle_{E_F} = \sum_{j=1}^{g} \sigma_j(x)\langle \boldsymbol{\omega}_F, \nabla_{\frac{1}{2\pi i}\frac{\partial}{\partial \tau_j}}\boldsymbol{\omega}_F\rangle_{E_F} = \sum_{j=1}^{g}\sigma_j(x) = \mathrm{Tr}(x).$$

To prove (2), we apply Proposition .144: the equation $\nabla_{\theta_F}\boldsymbol{\omega}_F = \boldsymbol{\eta}_F$ holds by definition, whereas $\nabla_{\theta_F}\boldsymbol{\eta}_F = 0$ is equivalent to asserting that

$$\theta_F(1\otimes x)\theta_F(1\otimes y)\int_\gamma \boldsymbol{\omega}_F = 0$$

for every $x, y \in D^{-1}$ and γ local section of $R_1\boldsymbol{p}_{F*}\mathbf{Z}_{\mathbf{X}_F}$; this, in turn, is an easy consequence of Remark .142 and of the explicit definition of $\boldsymbol{\omega}_F$. ■

Remark 11.14. Consider the morphism of stacks $f_t : \mathcal{B}_F \longrightarrow \mathcal{B}_g$ of Remark .40, and the holomorphic map $h_t : \mathbf{H}^g \longrightarrow \mathbf{H}_g$ of Remark .130. One may check using the characterization in Corollary .136 that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{H}^g & \xrightarrow{\varphi_F} & B_F(\mathbf{C}) \\ {\scriptstyle h_t}\downarrow & & \downarrow{\scriptstyle f_t} \\ \mathbf{H}_g & \xrightarrow[\varphi_g]{} & B_g(\mathbf{C}) \end{array}$$

11.6. **Compatibility of φ_F and $\hat{\varphi}_F$.** Analogously to the Siegel case, φ_F and $\hat{\varphi}_F$ are compatible.

To see this, we first recall that the abelian group D^{-1} can be seen as a subgroup of $\mathrm{SL}(D^{-1}\oplus R)$ via $x \longmapsto (1\ x\ ;\ 0\ 1)$, so that it acts on the object $(\mathbf{X}_F, E_F, m_F)_{/\mathbf{H}^g}$ of $\mathcal{A}_F^{\mathrm{an}}$ by Example .114. The action of D^{-1} on the base manifold $\mathbf{H}^g$ is given by translations:

$$x\cdot\tau = \tau + (\sigma_j(x))_{1\le j\le g},$$

so that it is proper and free.

Therefore, by Lemma .115 and Remark .116, $(\mathbf{X}_F, E_F, m_F)_{/\mathbf{H}^g}$ descends to a principally polarized complex torus with R-multiplication (X, E, m) over the quotient $D^{-1}\backslash\mathbf{H}^g$. Since $\boldsymbol{b}_F$ is invariant under the action of D^{-1}, it also descends to a symplectic-Hodge basis b of $(X, E, m)_{/D^{-1}\backslash\mathbf{H}^g}$, so that $\varphi_F : \mathbf{H}^g \longrightarrow B_F(\mathbf{C})$ factors through an analytic map

$$\psi : D^{-1}\backslash\mathbf{H}^g \longrightarrow B_F(\mathbf{C}).$$

To check that $\hat{\varphi}_F$ is the formal version of ψ, we let $(x_1, \ldots, x_g)$ be the same $\mathbf{Z}$-basis of D^{-1} considered in Paragraph 6.3, and we observe that $D^{-1}\backslash\mathbf{H}^g$ can be identified with an open submanifold $\mathbf{D}_F$ of $\mathbf{C}^g$ via

$$\tau \longmapsto q(\tau) = (q^{r_1}(\tau), \ldots, q^{r_g}(\tau)) := (e^{2\pi i\,\mathrm{Tr}(r_1\tau)}, \ldots, e^{2\pi i\,\mathrm{Tr}(r_g\tau)}) \in \mathbf{C}^g,$$

where, for $r \in R$, we denote $\mathrm{Tr}(r\tau) := \sum_{j=1}^{g}\sigma_j(r)\tau_j$.

If we identify $\mathbf{C}^g$ with $\mathbf{C}\otimes D^{-1}$ via the field embeddings $\sigma_i : F \longrightarrow \mathbf{C}$, then

$$\mathbf{X}_{F,\tau} = \mathbf{C}\otimes D^{-1}/(D^{-1} + \tau R)$$

and the natural isomorphism

$$\mathbf{X}_{F,\tau} \xrightarrow{\sim} X_{q(\tau)}$$

is induced by $z \otimes x \longmapsto e^{2\pi i z} \otimes x$. We deduce from this that, for $q = (q^{r_1}, \ldots, q^{r_g}) \in \mathbf{D}_F$, we have

$$X_q = \mathbf{C}^\times \otimes D^{-1}/Y_q,$$

where Y_q is the image of the unique R-linear map $R \longrightarrow \mathbf{C}^\times \otimes D^{-1}$ whose trace $R \longrightarrow \mathbf{C}^\times$ is given by $r \longmapsto q^r(\tau) \coloneqq e^{2\pi i \operatorname{Tr}(r\tau)}$ (cf. Remark .30). This shows that $\hat{X}_F$ is the formal analog of X, and we may argue similarly for the principal polarization and the R-multiplication.

To see that $\hat{b}_F$ coincides with b, we consider the identification of $\mathbf{C}^\times \otimes D^{-1}$ with $\mathbf{C}^g$ given by $(x_1, \ldots, x_g)$, so that $\mathbf{X}_{F,\tau} \xrightarrow{\sim} Y_{q(\tau)}$ is induced by

$$\tau \longmapsto (t^{r_1}(z), \ldots, t^{r_g}(z)) \coloneqq (e^{2\pi i \operatorname{Tr}(r_1 z)}, \ldots, e^{2\pi i \operatorname{Tr}(r_g z)})$$

where, for $r \in R$, we define $\operatorname{Tr}(rz) \coloneqq \sum_{j=1}^g r_j z_j$. Thus (cf. Remark .76)

$$\boldsymbol{\omega}_F = 2\pi i \sum_{j=1}^g dz_j = \sum_{i=1}^g \operatorname{Tr}(x_i) \frac{dt^{r_i}}{t^{r_i}}.$$

Also, if $q^{r_i} : \mathbf{H}^g \longrightarrow \mathbf{C}$ is defined as above, a computation shows that, for $x \in D^{-1}$, the vector field $\theta_F(1 \otimes x)$ defined in Paragraph 11.5 (cf. Remark .142) is given by

$$\theta_F(1 \otimes x) = \sum_{i=1}^g \operatorname{Tr}(r_i x) q^{r_i} \frac{\partial}{\partial q^{r_i}}.$$

It follows from these formulas that Theorem .77 is an immediate corollary of Theorem .145 (see also Remark .140).

12. Values of φ_g and φ_F; periods of abelian varieties

In this section we show that the values of the analytic maps $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ (resp. $\varphi_F : \mathbf{H}^g \longrightarrow B_F(\mathbf{C})$) defined in Theorem .134 (resp. Theorem .145) can be used to "compute", up to a finite extension, the fields generated by the periods of principally polarized abelian varieties (resp. principally polarized abelian varieties with real multiplication). In particular, the transcendence degree of such fields of periods can be read from the analytic maps φ_g and φ_F.

12.1. **Fields of periods of abelian varieties and statement of our main theorems.** Let X be a complex abelian variety (resp. a complex torus). A *field of definition* of X is a subfield k of $\mathbf{C}$ for which there exists an abelian variety X_0 over k such that X is isomorphic to $X_0 \otimes_k \mathbf{C}$ as a complex abelian variety (resp. isomorphic to $X_0(\mathbf{C})$ as a complex torus); we say that X_0 is a k-model of X.

Definition 12.1. Let X be a complex abelian variety, k be a field of definition of X, and fix a k-model X_0 of X. The *field of periods* $\mathcal{P}(X/k)$ of X over k is defined as the smallest subfield of $\mathbf{C}$

containing k and the image of pairing

$$\begin{aligned} H^1_{\mathrm{dR}}(X_0/k) \otimes H_1(X_0(\mathbf{C}), \mathbf{Z}) &\longrightarrow \mathbf{C} \\ \alpha \otimes \gamma &\longmapsto \int_\gamma \alpha \end{aligned}$$

given by "integration of differential forms" (cf. 9.4.2).

The field $\mathcal{P}(X/k)$ does not depend on the choice of X_0.

Remark 12.2. Alternatively, the field of periods $\mathcal{P}(X/k)$ can be regarded as the "field of rationality" of the comparison isomorphism (see Remark .109)

$$\mathrm{comp} : \mathbf{C} \otimes_k H^1_{\mathrm{dR}}(X/k) \xrightarrow{\sim} \mathbf{C} \otimes_{\mathbf{Q}} H^1(X(\mathbf{C}), \mathbf{Q}),$$

that is, the field of definition (cf. 0.17) of the complex point comp of the k-variety

$$\mathrm{Isom}(H^1_{\mathrm{dR}}(X/k), k \otimes_{\mathbf{Q}} H^1(X(\mathbf{C}), \mathbf{Q})).$$

Let A_g be the coarse moduli space associated to the Deligne-Mumford stack $\mathcal{A}_g \longrightarrow \operatorname{Spec} \mathbf{Z}$ (which exists as an algebraic space by the Keel-Mori theorem, cf. [65] Theorem 11.1.2). We recall that A_g is a quasi-projective scheme over $\operatorname{Spec} \mathbf{Z}$ (cf. [51] VII Théorème 4.2) endowed with a canonical morphism $\mathcal{A}_g \longrightarrow A_g$ inducing, for every algebraically closed field k, a bijection of $A_g(k)$ with the set of isomorphism classes of principally polarized abelian varieties over k.

Since any principally polarized complex torus (X, E) of dimension g is algebraizable, (X, E) defines an isomorphism class in the category $\mathcal{A}_g(\mathbf{C})$ that we shall denote $[(X, E)]$. Let

$$\begin{aligned} j_g : \mathbf{H}_g &\longrightarrow A_g(\mathbf{C}) \\ \tau &\longmapsto [(\mathbf{X}_{g,\tau}, E_{g,\tau})]. \end{aligned}$$

Observe that, for any $\tau \in \mathbf{H}_g$, the field $\mathbf{Q}(j_g(\tau)) \subset \mathbf{C}$ (see 0.17) is a field of definition of $\mathbf{X}_{g,\tau}$.

This section is devoted to the proof of the following theorem.

Theorem 12.3. *With notation as in Example .105 and Theorem .134, for every $\tau \in \mathbf{H}_g$ the field of periods $\mathcal{P}(\mathbf{X}_{g,\tau}/\mathbf{Q}(j_g(\tau)))$ is a finite field extension of $\mathbf{Q}(2\pi i, \tau, \varphi_g(\tau))$. In particular,*

$$\mathrm{trdeg}_{\mathbf{Q}} \mathbf{Q}(2\pi i, \tau, \varphi_g(\tau)) = \mathrm{trdeg}_{\mathbf{Q}} \mathcal{P}(\mathbf{X}_{g,\tau}/\mathbf{Q}(j_g(\tau))).$$

Here, we see $(2\pi i, \tau, \varphi_g(\tau))$ as a complex point of the $\mathbf{Q}$-variety $\mathbf{A}^1_{\mathbf{Q}} \times_{\mathbf{Q}} \mathrm{Sym}_{g,\mathbf{Q}} \times B_{g,\mathbf{Q}}$, and $\mathbf{Q}(2\pi i, \tau, \varphi_g(\tau))$ denotes its field of definition; see 0.17.

The above result also admits a Hilbert-Blumenthal analog, and we indicate at the end of this section, without proofs, how to obtain it. As above, we denote by A_F the coarse moduli space associated to $\mathcal{A}_F$, and we consider a map

$$\begin{aligned} j_F : \mathbf{H}^g &\longrightarrow A_F(\mathbf{C}) \\ \tau &\longmapsto [(\mathbf{X}_{F,\tau}, E_{F,\tau}, m_{F,\tau})]. \end{aligned}$$

Theorem 12.4. *With notation as in Example .113 and Theorem .145, for every $\tau \in \mathbf{H}^g$ the field of periods $\mathcal{P}(\mathbf{X}_{F,\tau}/\mathbf{Q}(j_F(\tau)))$ is a finite field extension of $\mathbf{Q}(2\pi i, \tau, \varphi_F(\tau))$. In particular,*

$$\mathrm{trdeg}_{\mathbf{Q}} \mathbf{Q}(2\pi i, \tau, \varphi_F(\tau)) = \mathrm{trdeg}_{\mathbf{Q}} \mathcal{P}(\mathbf{X}_{F,\tau}/\mathbf{Q}(j_F(\tau))).$$

12.2. **Period matrices.** Let us consider the *general symplectic group* (or the group of "symplectic similitudes"); namely, the subgroup scheme GSp_{2g} of GL_{2g} over $\operatorname{Spec}\mathbf{Z}$ such that, for every affine scheme $V = \operatorname{Spec}\Lambda$, we have

$$\mathrm{GSp}_{2g}(V) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g\times 2g}(\Lambda) \;\middle|\; \begin{array}{c} A, B, C, D \in M_{g\times g}(\Lambda) \text{ satisfy} \\ AB^{\mathsf{T}} = BA^{\mathsf{T}},\ CD^{\mathsf{T}} = DC^{\mathsf{T}}, \text{ and } AD^{\mathsf{T}} - BC^{\mathsf{T}} \in \Lambda^{\times}\mathbf{1}_g \end{array} \right\}.$$

Then we have the canonical character

$$\nu : \mathrm{GSp}_{2g} \longrightarrow \mathbf{G}_m$$

defined as follows: if $s = (A\ B\ ;\ C\ D) \in \mathrm{GSp}_{2g}(V)$, then $\nu(s) \in R^{\times}$ satisfies $AD^{\mathsf{T}} - BC^{\mathsf{T}} = \nu(s)\mathbf{1}_g$. Note that Sp_{2g} is the kernel of ν.

We denote by GSp_{2g}^{*} the open subscheme of GSp_{2g} defined by the condition $A \in \mathrm{GL}_g(\Lambda)$ in the above notation.

Let (X, E) be a principally polarized complex torus of dimension g, and $b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ (resp. $\beta = (\gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g)$) be a symplectic-Hodge basis (resp. an integral symplectic basis) of (X, E).

Definition 12.5. The *period matrix* of (X, E) with respect to b and β is defined by

$$P(X, E, b, \beta) := \begin{pmatrix} \Omega_1 & N_1 \\ \Omega_2 & N_2 \end{pmatrix} \in M_{2g\times 2g}(\mathbf{C}),$$

where

$$(\Omega_1)_{ij} := \int_{\gamma_i} \omega_j \qquad (N_1)_{ij} := \int_{\gamma_i} \eta_j$$
$$(\Omega_2)_{ij} := \int_{\delta_i} \omega_j \qquad (N_2)_{ij} := \int_{\delta_i} \eta_j.$$

Note that $P(X, E, b, \beta)$ is simply the matrix of the comparison isomorphism (9.1) with respect to the bases b of $\mathcal{H}^1_{\mathrm{dR}}(X)$ and $(E(\ \ , \delta_1), \ldots, E(\ \ , \delta_g), E(\gamma_1, \ \ ), \ldots, E(\gamma_g, \ \ ))$ of $\mathrm{Hom}(H_1(X, \mathbf{Z}), \mathbf{C})$.

Remark 12.6. In particular, let (X, λ) be a principally polarized complex abelian variety, k be a field of definition of X, and X_0 be a k-model of X. Assume moreover that λ descends to a principal polarization λ_0 on X_0. Then, if b is any symplectic-Hodge basis of (X_0, λ_0), and β is any integral symplectic basis of $(X^{\mathrm{an}}, E_{\lambda})$, the field of periods $\mathcal{P}(X/k)$ of X is generated over k by the coefficients of the period matrix $P(X^{\mathrm{an}}, E_{\lambda}, b, \beta)$ (cf. Remark .148).

Lemma 12.7. *For any (X, E, b, β) as above, we have*

(1) $P(X, E, b, \beta) \in \mathrm{GSp}_{2g}(\mathbf{C})$ and $\nu(P(X, E, b, \beta)) = 2\pi i$,
(2) $\Omega^1 \in \mathrm{GL}_g(\mathbf{C})$ (i.e., $P(X, E, b, \beta) \in \mathrm{GSp}^{*}_{2g}(\mathbf{C})$) and $\Omega_2\Omega_1^{-1} \in \mathbf{H}_g$.

Observe that $\Omega_2\Omega_1^{-1}$ is the point of $\mathbf{H}_g$ corresponding to (X, E, β) via Proposition .119.

Proof. Knowing that $P(X, E, b, \beta)$ is a base change matrix with respect to symplectic bases, (1) is simply a reformulation of Lemma .111; (2) is a particular case of the classical *Riemann relations* (cf. proof of Proposition .119). ∎

12.3. **Auxiliary lemmas.** We shall need the following auxiliary results.

Lemma 12.8. *The morphism of schemes*

$$\begin{aligned} \mathrm{GSp}_{2g}^* &\longrightarrow \mathbf{G}_m \times_{\mathbf{Z}} \mathrm{Sym}_g \times_{\mathbf{Z}} P_g \\ s &\longmapsto (\nu(s), \tau(s), p(s)) \end{aligned}$$

where

$$\tau \begin{pmatrix} A & B \\ C & D \end{pmatrix} := CA^{-1} \quad \text{and} \quad p \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} A^{-1} & -B^{\mathsf{T}} \\ 0 & A^{\mathsf{T}} \end{pmatrix}$$

is an isomorphism.

Proof. We simply remark that

$$\left(\lambda, Z, \begin{pmatrix} X & Y \\ 0 & (X^{\mathsf{T}})^{-1} \end{pmatrix}\right) \longmapsto \begin{pmatrix} X^{-1} & -Y^{\mathsf{T}} \\ ZX^{-1} & (\lambda \mathbf{1}_g - ZX^{-1}Y)X^{\mathsf{T}} \end{pmatrix}$$

is an inverse to the morphism defined in the statement. ■

A straightforward computation yields the following result.

Lemma 12.9. *Let $\varphi : (X, E) \longrightarrow (X', E')$ be an isomorphism of principally polarized complex tori of dimension g, $\beta = (\gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g)$ be an integral symplectic basis of (X, E) and b' be a symplectic-Hodge basis of (X', E'). We denote by $\varphi_*\beta$ the integral symplectic basis of (X', E') given by pushforward in singular homology. Then the symplectic-Hodge basis*

$$b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g) := \varphi^* b' \cdot p\left(\frac{1}{2\pi i} P(X', E', b', \varphi_*\beta)\right)$$

of (X, E) satisfies

$$\int_{\gamma_i} \eta_j = 0, \ \int_{\delta_i} \eta_j = \delta_{ij}$$

for every $1 \leq i, j \leq g$. ■

12.4. **Proof of Theorem .149.** Let $\varpi_g : B_{g,\mathbf{Q}} \longrightarrow A_{g,\mathbf{Q}}$ be the map obtained by composition of $\pi_g : B_{g,\mathbf{Q}} \cong \mathcal{B}_{g,\mathbf{Q}} \longrightarrow \mathcal{A}_{g,\mathbf{Q}}$ with the natural map $\mathcal{A}_{g,\mathbf{Q}} \longrightarrow A_{g,\mathbf{Q}}$; for a field $k \supset \mathbf{Q}$, it acts on k-points by sending the isomorphism class $[(X, \lambda, b)]$ of a principally polarized abelian variety with symplectic-Hodge basis $(X, \lambda, b)_{/k}$ to the isomorphism class $[(X, \lambda)]$.

Note that ϖ_g is invariant under the right action of $P_{g,\mathbf{Q}}$ on $B_{g,\mathbf{Q}}$ and that each fiber of ϖ_g is a $P_{g,\mathbf{Q}}$-homogeneous space.

Lemma 12.10. *Let $k \supset \mathbf{Q}$ be a field, $y \in B_{g,\mathbf{Q}}(k)$, and denote $x = \varpi_g(y) \in A_{g,\mathbf{Q}}(k)$. Then the orbit map $P_{g,k} \longrightarrow \varpi_g^{-1}(x) = B_{g,\mathbf{Q}} \times_{\mathbf{Q}} x$ associated to y is a finite and surjective morphism of k-schemes.*

Proof. Let G be the stabilizer of y, seen as a k-subgroup scheme of $P_{g,k}$; it is sufficient to prove that G is a finite k-group scheme.

Let (X,λ,b) be a principally polarized abelian variety with symplectic-Hodge basis over k for which $y=[(X,\lambda,b)]$. For any k-algebra Λ, we may define a antihomomorphism of groups

$$h:\mathrm{Aut}((X,\lambda)\otimes_k\Lambda)\longrightarrow P_{g,k}(\Lambda)$$

by sending σ to the unique element $p\in P_{g,k}(\Lambda)$ such that $\sigma^*b=b\cdot p$. By definition of G, the image of h is precisely $G(\Lambda)$.

Now, if Λ is a field, then $\mathrm{Aut}((X,\lambda)\otimes_k\Lambda)$ is finite ([56] IV.21 Theorem 5). Since G is an (affine) algebraic group over k, this implies that G is finite. ■

Proof of Theorem .149. Fix $\tau\in\mathbf{H}_g$, let $k=\mathbf{Q}(j_g(\tau))$, and let $(X,\lambda)_{/k}$ be a k-model of $(\mathbf{X}_{g,\tau},E_{g,\tau})$. Fix an isomorphism

$$F:(\mathbf{X}_{g,\tau},E_{g,\tau})\xrightarrow{\sim}(X(\mathbf{C}),E_\lambda),$$

and a symplectic-Hodge basis b of $(X,\lambda)_{/k}$.

We set

$$s:=\frac{1}{2\pi i}P(X(\mathbf{C}),E_\lambda,b,F_*\beta_{g,\tau})\in\mathrm{GSp}^*_{2g}(\mathbf{C}).$$

If $f:P_{g,k}\longrightarrow\varpi_g^{-1}([(X,\lambda)])$ denotes the orbit map associated to $[(X,\lambda,b)]\in B_{g,\mathbf{Q}}(k)$, then it follows from Lemma .155 and Corollary .136 that

$$f(p(s))=[(X_{\mathbf{C}},\lambda_{\mathbf{C}},b\cdot p(s))]=[(\mathbf{X}_{g,\tau},E_{g,\tau},F^*b\cdot p(s))]=[(\mathbf{X}_{g,\tau},E_{g,\tau},\boldsymbol{b}_{g,\tau})]=\varphi_g(\tau).$$

Thus, by Lemma .156, $k(p(s))$ is a finite field extension of $k(\varphi_g(\tau))$. But $k(\varphi_g(\tau))=\mathbf{Q}(\varphi_g(\tau))$, since $\mathbf{Q}(\varphi_g(\tau))$ is the field of definition of $\varphi_g(\tau)$ in $B_{g,\mathbf{Q}}$, which maps to $j_g(\tau)$ via ϖ_g.

By Lemma .153, we have $\nu(s)=\frac{1}{2\pi i}$, and $\tau(s)=\tau$. Thus, it follows from Remark .152 and Lemma .154 that

$$\mathcal{P}(\mathbf{X}_{g,\tau}/k)=k(s)=k(2\pi i,\tau,p(s)).$$

Finally, we conclude from the last paragraph that $\mathcal{P}(\mathbf{X}_{g,\tau}/k)$ is a finite field extension of

$$k(2\pi i,\tau,\varphi_g(\tau))=\mathbf{Q}(2\pi i,\tau,\varphi_g(\tau)).$$

■

Remark 12.11. For latter use, let us remark that with notation as in the above proof, if we denote

$$s=\begin{pmatrix}\Omega_1 & N_1\\ \Omega_2 & N_2\end{pmatrix},$$

then we have actually showed that

$$\mathbf{Q}(j_g(\tau),\Omega_1,N_1)\supset\mathbf{Q}(\varphi_g(\tau))$$

is a finite field extension.

12.5. **Periods of abelian varieties with real multiplication.** As in Paragraph 3.3, consider the R-module $M := R \oplus D^{-1}$ endowed with its standard D^{-1}-valued R-bilinear symplectic form Ψ. The $\mathbf{Z}$-dual of M is given by $M^\vee = D^{-1} \oplus R$, and we denote by Φ its standard D^{-1}-valued R-bilinear symplectic form (cf. Example .114).

Let (X, E, m) be a principally polarized complex torus with R-multiplication (over a point). In order to define period matrices for (X, E, m), it is convenient to adopt the following slightly more abstract approach.

Recall that a symplectic-Hodge basis b of (X, E, m) is a $\mathbf{C} \otimes R$-linear isomorphism

$$b : \mathbf{C} \otimes M \xrightarrow{\sim} \mathcal{H}^1_{\mathrm{dR}}(X)$$

such that $b^*\Psi_E = 1 \otimes \Psi$ and $b(\mathbf{C} \otimes (R \oplus 0)) = \mathcal{F}^1(X)$; an integral symplectic basis of (X, E, m) is an R-linear isomorphism

$$\beta : M^\vee \xrightarrow{\sim} H_1(X, \mathbf{Z})$$

satisfying $\beta^*\Phi_E = \Phi$, so that β induces a $\mathbf{C} \otimes R$-linear isomorphism

$$(\beta^\vee_{\mathbf{C}})^{-1} : \mathbf{C} \otimes M \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(H_1(X, \mathbf{Z}), \mathbf{C}).$$

Since the comparison isomorphism

$$\mathrm{comp} : \mathcal{H}^1_{\mathrm{dR}}(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(H_1(X, \mathbf{Z}), \mathbf{C})$$

is $\mathbf{C} \otimes R$-bilinear, we obtain a $\mathbf{C} \otimes R$-linear isomorphism

$$\mathrm{comp}^{-1} \circ (\beta^\vee_{\mathbf{C}})^{-1} : \mathbf{C} \otimes M \xrightarrow{\sim} \mathcal{H}^1_{\mathrm{dR}}(X).$$

Definition 12.12. The *period matrix* of (X, E, m) with respect to b and β is defined as the unique element $P(X, E, m, b, \beta)$ of $\mathrm{Aut}_{\mathbf{C}\otimes R}(\mathbf{C} \otimes M) = (\mathrm{Res}_{R/\mathbf{Z}} \mathrm{Aut}_M)(\mathbf{C})$ such that

$$\mathrm{comp}^{-1} \circ (\beta^\vee_{\mathbf{C}})^{-1} \circ P(X, E, m, b, \beta) = b.$$

Remark 12.13. It follows from Remark .148 that, if $k \subset \mathbf{C}$ is a subfield, $(X, \lambda, m)_{/k}$ is a principally polarized abelian variety with R-multiplication over k, b is a symplectic-Hodge basis of $(X, \lambda, m)_{/k}$, and β is an integral symplectic basis of $(X(\mathbf{C}), E_\lambda, m^{\mathrm{an}})$, then

$$\mathcal{P}(X/k) = k(P(X(\mathbf{C}), E_\lambda, m^{\mathrm{an}}, b, \beta)),$$

where $k(P(X(\mathbf{C}), E_\lambda, m^{\mathrm{an}}, b, \beta))$ is the field of definition of the complex point $P(X(\mathbf{C}), E_\lambda, m_{\mathbf{C}}, b, \beta)$ of the k-variety $k \otimes \mathrm{Res}_{R/\mathbf{Z}} \mathrm{Aut}_M$ (cf. 0.17).

In order to realize $P(X, E, m, b, \beta)$ as an actual matrix we remark that, for every commutative ring Λ, if $V = \mathrm{Spec}\,\Lambda$, then we have the natural identification

$$(\mathrm{Res}_{R/\mathbf{Z}}\mathrm{Aut}_M)(V) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\Lambda \otimes R) \,\middle|\, a, d \in \Lambda \otimes R,\ b \in \Lambda \otimes D,\ c \in \Lambda \otimes D^{-1} \right\},$$

so that we can write

$$P(X, E, m, b, \beta) = \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix} \in (\mathrm{Res}_{R/\mathbf{Z}}\mathrm{Aut}_M)(\mathbf{C}).$$

Remark 12.14. The coefficients of $P(X, E, m, b, \beta)$ in the above presentation can be understood as follows. With the above notation, since the comparison isomorphism is $\mathbf{C} \otimes R$-linear, and since the trace form induces a natural identification $(\mathbf{C} \otimes H_1(X, \mathbf{Z}))^* \otimes_R D^{-1} \cong \mathrm{Hom}_{\mathbf{Z}}(H_1(X, \mathbf{Z}), \mathbf{C})$, we obtain an R-bilinear pairing

$$\begin{aligned} \mathcal{H}^1_{\mathrm{dR}}(X) \times H_1(X, \mathbf{Z}) &\longrightarrow \mathbf{C} \otimes D^{-1} \\ (\alpha, \gamma) &\longmapsto I_\gamma \alpha \end{aligned}$$

satisfying

$$\mathrm{Tr}\, I_\gamma \alpha = \int_\gamma \alpha.$$

Then, if we write $b = (\omega, \eta)$, and $\beta = (\gamma, \delta)$, we have

$$P(X, E, m, b, \beta) = \begin{pmatrix} I_\gamma \omega & I_\gamma \eta \\ I_\delta \omega & I_\delta \eta \end{pmatrix} \in (\mathrm{Res}_{R/\mathbf{Z}} \mathrm{Aut}_M)(\mathbf{C}).$$

Consider the subgroup scheme G_F of $\mathrm{Res}_{R/\mathbf{Z}}\mathrm{Aut}_M$ defined, for every affine scheme $V = \mathrm{Spec}\,\Lambda$, by

$$G_F(V) = \left\{ s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (\mathrm{Res}_{R/\mathbf{Z}}\mathrm{Aut}_M)(V) \,\middle|\, \det(s) = ad - bc \in \Lambda^\times \subset (\Lambda \otimes R)^\times \right\}.$$

We denote by G_F^* the open subscheme of G_F given by the condition $a \in (\Lambda \otimes R)^\times$.

In the next lemma we see $\mathbf{H}^g$ inside the $\mathbf{C}$-vector space $\mathbf{C} \otimes D^{-1}$ via the identification $\mathbf{C} \otimes D^{-1} \xrightarrow{\sim} \mathbf{C}^g$ given by $1 \otimes x \longmapsto (\sigma_1(x), \ldots, \sigma_g(x))$.

Lemma 12.15 (cf. Lemma .153)**.** *For any (X, E, m, b, β) as above, we have*

(1) $P(X, E, m, b, \beta) \in G_F(\mathbf{C})$ and $\det P(X, E, m, b, \beta) = 2\pi i$,
(2) $I_\gamma \omega \in (\mathbf{C} \otimes R)^\times$ (i.e., $P(X, E, m, b, \beta) \in G_F^*(\mathbf{C})$) and $(I_\delta \omega)(I_\gamma \omega)^{-1} \in \mathbf{H}^g$. ■

Next, we state the analogous auxiliary lemmas.

Lemma 12.16 (cf. Lemma .154)**.** *The morphism of schemes*

$$\begin{aligned} G_F^* &\longrightarrow \mathbf{G}_m \times_{\mathbf{Z}} \mathrm{Res}_{R/\mathbf{Z}} \mathbf{A}^1_R \times_{\mathbf{Z}} P_F \\ s &\longmapsto (\det(s), \tau(s), p(s)) \end{aligned}$$

where

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} \coloneqq ca^{-1} \quad \text{and} \quad p \begin{pmatrix} a & b \\ c & d \end{pmatrix} \coloneqq \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix}$$

is an isomorphism. ■

Lemma 12.17 (cf. Lemma .155)**.** *Let $\varphi : (X, E, m) \longrightarrow (X', E', m')$ be an isomorphism of principally polarized complex tori with R-multiplication, $\beta = (\gamma, \delta)$ be an integral symplectic basis of (X, E, m) and b' be a symplectic-Hodge basis of (X', E', m'). We denote by $\varphi_*\beta$ the integral symplectic basis of (X', E', m') given by pushforward in singular homology. Then the symplectic-Hodge* *basis*

$$b = (\omega, \eta) := \varphi^* b' \cdot p \left(\frac{1}{2\pi i} P(X', E', b', m', \varphi_* \beta) \right)$$

of (X, E, m) satisfies (cf. Remark .160)

$$I_\gamma \eta = 0, \ I_\delta \eta = 1.$$

■

Using the above preliminary results, the proof of Theorem .150 is completely analogous to that of Theorem .149.

13. An algebraic independence conjecture on the values of φ_F

In this paragraph, we use the analytic maps φ_F, for F real quadratic, to formulate a transcendence conjecture containing Grothendieck's Period Conjecture (GPC) for abelian surfaces with complex multiplication, much like Nesterenko's theorem on $\varphi_{\mathbf{Q}} = (E_2, E_4, E_6)$ allows to recover GPC for complex multiplication elliptic curves.

In such higher dimensional versions of Nesterenko-type statements, it is necessary to take into account the presence of "special subvarieties" of positive dimension of the corresponding moduli problem of abelian varieties. In the case of A_F, for F quadratic, these are given by the *Hirzebruch-Zagier* divisors.

13.1. **Hirzebruch-Zagier divisors and statement of the conjecture.**

Let F be a real quadratic number field, and let σ the non-trivial element of $\mathrm{Gal}(F/\mathbf{Q})$. The next definition is due to Kudla and Rapoport [45] (cf. [35] Chapter 3).

Definition 13.1. A *special endomorphism* of a principally polarized abelian scheme with R-multiplication $(X, \lambda, m)_{/U}$ is an element $j \in \mathrm{End}_U(X)^\lambda$ such that

$$j \circ m(r) = m(r^\sigma) \circ j \tag{13.1}$$

for every $r \in R$.

For every integer $N \geq 1$, let $\mathcal{T}_F(N)$ be the moduli stack classifying principally polarized abelian schemes with R-multiplication endowed with a special endomorphism j satisfying $j^2 = N$. These are Deligne-Mumford stacks over $\mathrm{Spec}\, \mathbf{Z}$; moreover, as shown in [35] Paragraph 3.3, the forgetful functor $\mathcal{T}_F(N) \longrightarrow \mathcal{A}_F$ is finite and unramified, and its image defines an effective Cartier divisor in the stack $\mathcal{A}_F$.

For every $N \geq 1$, we denote by $T_F(N)$ the divisor on the $\mathbf{C}$-scheme $A_{F,\mathbf{C}}$ induced by $\mathcal{T}_F(N)_{\mathbf{C}} \longrightarrow \mathcal{A}_{F,\mathbf{C}}$. These are known as *Hirzebruch-Zagier divisors*, or "modular curves" (cf. [26] Chapter V), on the Hilbert modular surface $A_{F,\mathbf{C}}$.

Recall that Nesterenko's theorem [60] states that, for every $\tau \in \mathbf{H}$, we have

$$\mathrm{trdeg}_{\mathbf{Q}} \mathbf{Q}(e^{2\pi i \tau}, E_2(\tau), E_4(\tau), E_6(\tau)) \geq 3.$$

As a corollary, we get

$$\mathrm{trdeg}_{\mathbf{Q}} \mathbf{Q}(\varphi_{\mathbf{Q}}(\tau)) \geq 2.$$

We next state the conjectural analog of the above lower bound for a real quadratic number field F.

Conjecture 13.2. *Let F be a real quadratic number field. Then, for every $\tau \in \mathbf{H}^2 \setminus \bigcup_{N=1}^{\infty} j_F^{-1}(T_F(N))$, we have*

$$\mathrm{trdeg}_{\mathbf{Q}} \mathbf{Q}(\varphi_F(\tau)) \overset{?}{\geq} 3.$$

In the following paragraphs, we explain the precise relation between the above conjecture and Grothendieck's Period Conjecture for abelian surfaces.

13.2. **Periods in the presence of complex multiplication.** In this paragraph, we let F be a totally real number field of any degree $g \geq 1$. Recall that we denote by $\sigma_1, \ldots, \sigma_g$ the field embeddings of F into $\mathbf{C}$.

Let k be an algebraically closed subfield of $\mathbf{C}$, and (X, λ, m) be a principally polarized abelian variety with R-multiplication over k. We have already remarked that $m : R \longrightarrow \mathrm{End}(X)^{\lambda}$ is injective, and that each element in its image is an isogeny (Remark .33); we thus obtain an embedding of $\mathbf{Q}$-algebras $m : F \longrightarrow \mathrm{End}^0(X) := \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{End}(X)$.

Definition 13.3. We say that (X, λ, m) has *complex multiplication*, or that it is *CM*, if there exists a totally imaginary quadratic extension E of F, and an embedding of $\mathbf{Q}$-algebras $E \longrightarrow \mathrm{End}^0(X)$ extending m.

If X is a *simple* abelian variety, then $\mathrm{End}^0(X)$ is a division algebra acting faithfully on the $\mathbf{Q}$-vector space $H_1(X(\mathbf{C}), \mathbf{Q})$, so that $\dim_{\mathbf{Q}} \mathrm{End}^0(X)$ divides $2g$; in particular, the map $E \longrightarrow \mathrm{End}^0(X)$ in the above definition is necessarily an isomorphism of $\mathbf{Q}$-algebras.

We say that a point $\tau \in \mathbf{H}^g$ is CM if $(\mathbf{X}_{F,\tau}, E_{F,\tau}, m_{F,\tau})$ is CM. We shall need the following well known fact.

Lemma 13.4. *If $\tau \in \mathbf{H}^g$ is CM, then $\tau \in (\overline{\mathbf{Q}} \cap \mathbf{H})^g$ and $j_F(\tau) \in A_F(\overline{\mathbf{Q}})$.* ■

The classical proof for the case $F = \mathbf{Q}$ (see, for instance, [76] 4.4-4.6) generalizes to any totally real F. Here, as in the case of elliptic curves, if F is seen as a subring of $\mathbf{C}^g$ via $(\sigma_1, \ldots, \sigma_g)$, then $\tau \in \mathbf{H}^g \subset \mathbf{C}^g$ satisfies a *quadratic* equation with coefficients in F.

Although not necessary for the sequel, let us mention that the converse of the above result is also true, thus providing a characterization of CM points by a "bi-algebraicity" property. This characterization actually holds in a much broader framework (see [75] and [16]).

Proposition 13.5. *Let (X, λ, m) be a simple CM principally polarized abelian variety with R-multiplication over $\overline{\mathbf{Q}}$, $b = (\omega, \eta)$ be a symplectic-Hodge basis of $(X, \lambda, m)_{/\overline{\mathbf{Q}}}$, and $\beta = (\gamma, \delta)$ be an integral symplectic basis of $(X(\mathbf{C}), E_{\lambda}, m^{\mathrm{an}})$. Then*

$$\mathcal{P}(X/\overline{\mathbf{Q}}) = \overline{\mathbf{Q}}(I_{\gamma}\omega, I_{\gamma}\eta).$$

The notation $I_{\gamma}(\cdot)$ was introduced in Remark .160. Concretely, by identifying $\mathbf{C} \otimes F$ with $\mathbf{C}^g$ via $(\sigma_1, \ldots, \sigma_g)$, the element $I_{\gamma}\omega \in \mathbf{C} \otimes R$ (resp. $I_{\gamma}\eta \in \mathbf{C} \otimes D$) defines g complex numbers; the field $\overline{\mathbf{Q}}(I_{\gamma}\omega, I_{\gamma}\eta)$ is obtained from $\overline{\mathbf{Q}}$ by adjoining these $2g$ numbers.

Proof. Let φ be any element of $\mathrm{End}^0(X) \setminus m(F)$. Since the right R-module of symmetric morphisms $\mu : X \longrightarrow X^t$ satisfying $m(r)^t \circ \mu = \mu \circ m(r)$ for every $r \in R$ is projective of rank 1 (see, for instance,

[70] Proposition 1.17), and since φ commutes with every element of $m(F)$, there exists $u \in F^\times$ such that
$$\varphi^t \circ \lambda \circ \varphi = \lambda \circ m(u).$$
It follows that φ induces an automorphism of $\overline{\mathbf{Q}} \otimes F$-modules
$$\varphi^* : H^1_{\mathrm{dR}}(X/\overline{\mathbf{Q}}) \longrightarrow H^1_{\mathrm{dR}}(X/\overline{\mathbf{Q}})$$
preserving $F^1(X/\overline{\mathbf{Q}})$, and satisfying
$$\Psi_\lambda(\varphi^*\alpha, \varphi^*\beta) = u\Psi_\lambda(\alpha, \beta)$$
for every $\alpha, \beta \in H^1_{\mathrm{dR}}(X/\overline{\mathbf{Q}})$. In particular, there exists a $\overline{\mathbf{Q}} \otimes F$-automorphism of $\overline{\mathbf{Q}} \otimes (R \oplus D^{-1}) = (\overline{\mathbf{Q}} \otimes F)^{\oplus 2}$ of the form
$$A = \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} \in M_{2\times 2}(\overline{\mathbf{Q}} \otimes F)$$
with $rt = u$ such that
$$\varphi^* b = b \cdot A.$$

Analogously, φ induces an automorphism of F-vector spaces
$$\varphi_* : H_1(X(\mathbf{C}), \mathbf{Q}) \longrightarrow H_1(X(\mathbf{C}), \mathbf{Q})$$
such that
$$\Phi_{E_\lambda}(\varphi_*\gamma, \varphi_*\delta) = u\Phi_{E_\lambda}(\gamma, \delta)$$
for every $\gamma, \delta \in H_1(X(\mathbf{C}), \mathbf{Q})$. Thus, there exists a F-automorphism of $\mathbf{Q} \otimes (D^{-1} \oplus R) = F^{\oplus 2}$ of the form
$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(F)$$
with $ad - bc = u$ such that
$$\varphi_*\beta = \beta \cdot B.$$

It follows from the commutativity of the diagram of $\mathbf{C} \otimes F$-isomorphisms (given by the naturality of the comparison isomorphism)
$$\begin{array}{ccc} \mathbf{C} \otimes H^1_{\mathrm{dR}}(X/\overline{\mathbf{Q}}) & \xrightarrow{\text{comp}} & \mathrm{Hom}_{\mathbf{Q}}(H_1(X(\mathbf{C}), \mathbf{Q}), \mathbf{C}) \\ \varphi^* \downarrow & & \downarrow \varphi_*^\vee \\ \mathbf{C} \otimes H^1_{\mathrm{dR}}(X/\overline{\mathbf{Q}}) & \xrightarrow[\text{comp}]{} & \mathrm{Hom}_{\mathbf{Q}}(H_1(X(\mathbf{C}), \mathbf{Q}), \mathbf{C}) \end{array}$$
and from the definition of the period matrix $P = P(X(\mathbf{C}), E_\lambda, m^{\mathrm{an}}, b, \beta)$ (Definition .158) that
$$B^{\mathsf{T}} P = PA,$$
that is,
$$\begin{pmatrix} aI_\gamma\omega + cI_\delta\omega & aI_\gamma\eta + cI_\delta\eta \\ bI_\gamma\omega + dI_\delta\omega & bI_\gamma\eta + dI_\delta\eta \end{pmatrix} = \begin{pmatrix} rI_\gamma\omega & sI_\gamma\omega + tI_\gamma\eta \\ rI_\delta\omega & sI_\delta\omega + tI_\delta\eta \end{pmatrix}.$$
We claim that $c \neq 0$. By contradiction, if $c = 0$, then by comparing the $(1,1)$ entries, we obtain $a = r$ (recall that $I_\gamma\omega \in (\mathbf{C} \otimes R)^\times$ by Lemma .161). This also implies that $d = t$, since $ad = u = rt$. Now, by comparing $(2,1)$ entries, we obtain $bI_\gamma\omega = (a-d)I_\delta\omega$; since $(I_\delta\omega)(I_\gamma\omega)^{-1} \in \mathbf{H}^g \subset \mathbf{C} \otimes D^{-1} =$

$\mathbf{C}\otimes F$ (cf. Lemma .161), this is only possible if $a-d=b=0$. In particular, $\varphi_*=m(a)_*$, but since X is simple, the action of $\mathrm{End}^0(X)$ on $H_1(X(\mathbf{C}),\mathbf{Q})$ is faithful, so that $\varphi=m(a)$. This contradicts the fact that $\varphi\notin m(F)$.

By comparing $(1,2)$ entries, we obtain the linear equation in $\mathbf{C}\otimes F$

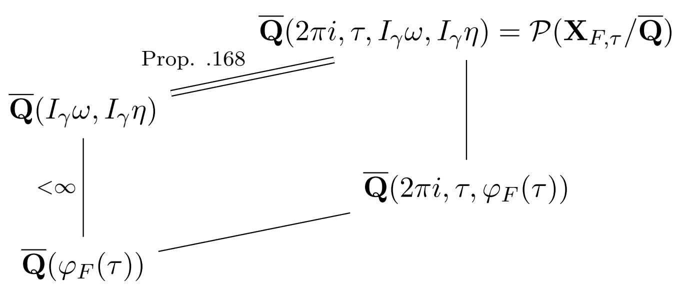
$$I_\delta\eta = sc^{-1}I_\gamma\omega + (t-a)c^{-1}I_\gamma\eta,$$

so that $I_\delta\eta\in\overline{\mathbf{Q}}(I_\gamma\omega,I_\gamma\eta)$. As (X,λ,m) is CM, we have $\tau\coloneqq(I_\delta\omega)(I_\gamma\omega)^{-1}\in\overline{\mathbf{Q}}\otimes F$ by Lemma .167[14]; thus $I_\delta\omega\in\overline{\mathbf{Q}}(I_\gamma\omega,I_\gamma\eta)$. To conclude, it is enough to recall that $\mathcal{P}(X/\overline{\mathbf{Q}})=\overline{\mathbf{Q}}(I_\gamma\omega,I_\delta\omega,I_\gamma\eta,I_\delta\eta)$ (Remarks .159 and .160). ■

Corollary 13.6. *Let $\tau\in\mathbf{H}^g$ be a CM point, and assume that $\mathbf{X}_{F,\tau}$ is simple. Then*

$$\mathrm{trdeg}_{\mathbf{Q}}\mathbf{Q}(2\pi i,\tau,\varphi_F(\tau)) = \mathrm{trdeg}_{\mathbf{Q}}\mathbf{Q}(\varphi_F(\tau)).$$

Proof. Let (X,λ,m) be a model of $(\mathbf{X}_{F,\tau},E_{F,\tau},m_{F,\tau})$ over $\overline{\mathbf{Q}}$, b be a symplectic-Hodge basis of $(X,\lambda,m)_{/\overline{\mathbf{Q}}}$, and β be an integral symplectic basis of $(\mathbf{X}_{F,\tau},E_{F,\tau},m_{F,\tau})$. Our statement follows immediately from the diagram of field extensions



where the finiteness of $\overline{\mathbf{Q}}(I_\gamma\omega,I_\gamma\eta)\supset\overline{\mathbf{Q}}(\varphi_F(\tau))$ derives from a Hilbert-Blumenthal analog of Remark .157. ■

13.3. **Grothendieck's Period Conjecture for abelian surfaces with real multiplication.** In this paragraph we assume that F is a real *quadratic* number field.

Lemma 13.7 (cf. [13] Lemma 6, [26] Proposition IX.1.2)**.** *Let k be an algebraically closed field of characteristic 0, and (X,λ,m) be a principally polarized abelian variety with R-multiplication over k. If X is simple, then* $\mathrm{End}^0(X)$ *is a division algebra over* $\mathbf{Q}$ *isomorphic to one of the following:*

(S1) F,
(S2) $E\supset F$ totally imaginary quadratic extension (CM case),
(S3) $B\supset F$ indefinite quaternion algebra over $\mathbf{Q}$*.*

If X is not simple, then X is necessarily isogenous to $Y\times_k Y$ for some elliptic curve Y over k, and $\mathrm{End}^0(X)=M_{2\times2}(\mathrm{End}^0(Y))$*; in particular,* $\mathrm{End}^0(X)$ *is a* $\mathbf{Q}$*-algebra isomorphic to*

(N1) $M_{2\times2}(\mathbf{Q})$, if Y is not CM,
(N2) $M_{2\times2}(K)$, where $K=\mathrm{End}^0(Y)$ is an imaginary quadratic field if Y is CM. ■

[14]Actually, the quadratic equation satisfied by τ (see the remark following Lemma .167) is obtained by dividing the $(2,1)$ entries by the $(1,1)$ entries.

Proposition 13.8. *Let k be an algebraically closed field of characteristic 0, and (X, λ, m) be a principally polarized abelian variety with R-multiplication over k. The $\mathbf{Q}$-algebra $\operatorname{End}^0(X)$ is isomorphic to a (commutative) field if and only if X does not admit a non-trivial special endomorphism.*

Proof. If $\operatorname{End}^0(X)$ is commutative, then the condition (13.1) in the definition of special endomorphisms is clearly only satisfied by $j = 0$. Conversely, let us prove that, if $\operatorname{End}^0(X)$ is not commutative, i.e., cases (S3), (N1), and (N2) in Lemma .170, then X admits a non-trivial special endomorphism.

Let us identify F with a subalgebra of $\operatorname{End}^0(X)$ via m, and denote by $\varphi \longmapsto \varphi^\dagger$ the Rosatti involution on $\operatorname{End}^0(X)$ defined by λ. By hypothesis, for every $x \in F$, we have $x^\dagger = x$. Up to multiplication by a convenient integer, it is sufficient to prove the existence of $j \in \operatorname{End}^0(X) \setminus \{0\}$ such that $j^\dagger = j$, and

$$jx = x^\sigma j$$

for every $x \in F$. If we write $F = \mathbf{Q}(\rho)$, for some $\rho \in F$ satisfying $\rho^2 \in \mathbf{Q}_{>0}$, then it is enough to check that $j^\dagger = j$ and $j\rho = -\rho j$.

(1) Let us assume that $\operatorname{End}^0(X) = B$ is a quaternion algebra over $\mathbf{Q}$ (cases (S3) and (N1)). Since B is indefinite, the positive involution $b \longmapsto b^\dagger$ cannot coincide with the canonical involution $b \longmapsto \overline{b}$ of B. By [43] Proposition 2.21, there exists $u \in B^\times$ such that $\overline{u} = -u$ and

$$b^\dagger = u^{-1}\overline{b}u$$

for every $b \in B$. Note that $b \longmapsto \overline{b}$ restricts to σ on F, so that $\overline{\rho} = -\rho$ and the condition $\rho^\dagger = \rho$ means that $\rho u = -u\rho$. Thus, we can take $j = \rho u$.

(2) Suppose that $\operatorname{End}^0(X) = M_{2\times 2}(K)$, where $K = \mathbf{Q}(\theta)$, with $\theta^2 \in \mathbf{Q}_{<0}$ (case (N2)). Since $\dagger$ is positive, it must restrict to the unique non-trivial automorphism of K (embedded diagonally in $M_{2\times 2}(K)$), i.e., $\theta^\dagger = -\theta$. By [43] Proposition 2.22, there exists a unique quaternion $\mathbf{Q}$-subalgebra $B \subset M_{2\times 2}(K)$ such that $B \otimes_{\mathbf{Q}} K = M_{2\times 2}(K)$ and

$$(b \otimes (s + t\theta))^\dagger = \overline{b} \otimes (s - t\theta)$$

for every $b \in B$, $s, t \in \mathbf{Q}$, where $b \longmapsto \overline{b}$ denotes the canonical involution of B. Write $\rho = b \otimes 1 + c \otimes \theta$. Using that $\rho^\dagger = \rho$ and $\rho^2 \in \mathbf{Q}$, we get $b = 0$ and $\overline{c} = -c$. By the Skolem-Noether theorem (cf. [43] Theorem 1.4), there exists $d \in B^\times$ such that $dc = -cd$; in particular, the reduced trace of d is zero, so that $d^2 \in \mathbf{Q}$. Thus, we can take $j = d \otimes \theta$.

■

Since any special endomorphism j of an abelian surface with R-multiplication necessarily satisfies $j^2 = N$ for some integer $N > 0$ (see [35] Corollary 3.1.4), we obtain the following corollary.

Corollary 13.9. *Let $\tau \in \mathbf{H}^2$. If $j_F(\tau) \notin \bigcup_{N=1}^{\infty} T_F(N)$, then $\mathbf{X}_{F,\tau}$ is simple and the $\mathbf{Q}$-division algebra $\operatorname{End}^0(\mathbf{X}_{F,\tau})$ is isomorphic to*

(1) $E \supset F$ a totally imaginary quadratic extension of F, if τ is CM;
(2) F otherwise. ■

We are now in position to relate Conjecture .165 with Grothendieck's Period Conjecture.

Let $\tau \in \mathbf{H}^2 \setminus \bigcup_{N=1}^{\infty} j_F^{-1}(T_F(N))$. Set $d \coloneqq \dim \mathrm{MT}(\mathbf{X}_{F,\tau})$, and $t \coloneqq \mathrm{trdeg}_{\mathbf{Q}} \mathcal{P}(\mathbf{X}_{F,\tau}/\mathbf{Q}(j_F(\tau)))$. It follows from the above corollary, and from the list of possible Mumford-Tate groups of abelian surfaces (see [50] 2.2), that $d = 3$ if τ is CM and $d = 7$ otherwise.

Recall from the introduction that the generalized Grothendieck's Period Conjecture asserts that $t \geq d$, i.e., that $t \geq 3$ if τ is CM and $t \geq 7$ otherwise. By Theorem .150, we have $t = \mathrm{trdeg}_{\mathbf{Q}} \mathbf{Q}(2\pi i, \tau, \varphi_F(\tau))$. Thus, when τ is CM, it follows from Corollary .169 that Conjecture .165 is *equivalent* to Grothendieck's Period Conjecture for the abelian variety $\mathbf{X}_{F,\tau}$. If τ is not CM, then, since

$$\mathrm{trdeg}_{\mathbf{Q}(\varphi_F(\tau))} \mathbf{Q}(2\pi i, \tau, \varphi_F(\tau)) \leq 3,$$

Corollary .169 is simply a weaker (but still non-trivial) statement than Grothendieck's Period conjecture for $\mathbf{X}_{F,\tau}$.

Despite being generally weaker than Grothendieck's Period Conjecture, our statement in Conjecture .165 already contains some classical transcendence problems, such as the algebraic independence of π, $\Gamma(1/5)$, and $\Gamma(2/5)$, if $F = \mathbf{Q}(\sqrt{5})$. Indeed, it is classical (see [79] Paragraph 4, and references therein) that π, $\Gamma(1/5)$, and $\Gamma(2/5)$, are generators of the field of periods over $\overline{\mathbf{Q}}$ of the Jacobian $J(C)$ of the hyperelliptic curve C over $\mathbf{Q}$ given by the affine equation

$$C : y^2 = 1 - x^5;$$

observe that $\mu_5 = \{\zeta \in \overline{\mathbf{C}} \mid \zeta^5 = 1\}$ acts on C via

$$\zeta \cdot (x, y) = (\zeta x, y),$$

so that $J(C)$, with its canonical principal polarization, admits a real multiplication by $R = \mathbf{Z}[(1 + \sqrt{5})/2]$ and is actually CM, with CM field $\mathbf{Q}(\mu_5)$.

14. Group-theoretic description of the higher Ramanujan vector fields

This section is devoted to an alternative description of the complex manifold $B_g(\mathbf{C})$ (resp. $B_F(\mathbf{C})$) as a domain in the quotient of some Lie group by a discrete subgroup. Under this analytic description, we also give explicit formulas for the higher Ramanujan vector fields and for the solution $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ (resp. $\varphi_F : \mathbf{H}^g \longrightarrow B_F(\mathbf{C})$) of the higher Ramanujan equations.

These results will be applied in Section 15 to obtain explicit parametrization of every analytic leaf of the Ramanujan foliation $\mathcal{R}_g^{\mathrm{an}}$ on $B_g(\mathbf{C})$ (resp. $\mathcal{R}_F^{\mathrm{an}}$ on $B_F(\mathbf{C})$).

14.1. **Realization of $B_g(\mathbf{C})$ as an open submanifold of $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$.** Let $\mathbf{B}_g = B(\mathbf{X}_g, E_g)$ be the principal $P_g(\mathbf{C})$-bundle over $\mathbf{H}_g$ associated to the principally polarized complex torus $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ as defined in Lemma .123, so that the fiber of $\mathbf{B}_g \longrightarrow \mathbf{H}_g$ over $\tau \in \mathbf{H}_g$ is given by the set of symplectic-Hodge bases of $(\mathbf{X}_{g,\tau}, E_{g,\tau})$.

We shall first realize $\mathbf{B}_g$ as a "period domain" in $\mathrm{Sp}_{2g}(\mathbf{C})$. For this, let us introduce the following convenient modification of period matrices (Definition .151).

Definition 14.1. Let (X, E) be a principally polarized complex torus of dimension g, and b (resp. β) be a symplectic-Hodge basis (resp. an integral symplectic basis) of (X, E). Let

$$P(X, E, b, \beta) = \begin{pmatrix} \Omega_1 & N_1 \\ \Omega_2 & N_2 \end{pmatrix} \in \mathrm{GSp}_{2g}(\mathbf{C})$$

be the period matrix of (X,E) with respect to b and β. We define

$$\Pi(X,E,b,\beta) := \begin{pmatrix} N_2 & \frac{1}{2\pi i}\Omega_2 \\ N_1 & \frac{1}{2\pi i}\Omega_1 \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{C})$$

Observe that this matrix is indeed symplectic by Lemma .153.

We define a holomorphic map $\Pi : \mathbf{B}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$ as follows. Let q be a point in $\mathbf{B}_g$ lying above $\tau \in \mathbf{H}_g$, and corresponding to a symplectic-Hodge basis b of $(\mathbf{X}_{g,\tau}, E_{g,\tau})$, then

$$\Pi(q) := \Pi(\mathbf{X}_{g,\tau}, E_{g,\tau}, b, \beta_{g,\tau})$$

where β_g is the integral symplectic basis of $(\mathbf{X}_g, E_g)_{/\mathbf{H}_g}$ defined in Example .118.

Remark 14.2. Alternatively, recall that $\mathbf{H}_g$ may be regarded as the moduli space for principally polarized complex tori of dimension g endowed with an integral symplectic basis (Proposition .119). In particular, as already remarked in the proof of Proposition .125, points in $\mathbf{B}_g$ correspond to isomorphism classes $[(X,E,b,\beta)]$ of quadruples (X,E,b,β), where (X,E) is a principally polarized complex torus of dimension g, and b (resp. β) is a symplectic-Hodge basis (resp. integral symplectic basis) of (X,E). Under this identification, the map $\Pi : \mathbf{B}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$ is given by $[(X,E,b,\beta)] \longmapsto \Pi(X,E,b,\beta)$.

Let us consider the moduli-theoretic interpretation of $\mathbf{B}_g$ of the above remark, and recall that $\mathbf{B}_g$ is endowed with a natural left action of the discrete group $\mathrm{Sp}_{2g}(\mathbf{Z})$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot [(X,E,b,\beta)] = \left[\left(X,E,b,\beta \cdot \begin{pmatrix} D^{\mathsf{T}} & B^{\mathsf{T}} \\ C^{\mathsf{T}} & A^{\mathsf{T}} \end{pmatrix}\right)\right]$$

(cf. Remark .120), and a right action of the Siegel parabolic subgroup $P_g(\mathbf{C}) \leq \mathrm{Sp}_{2g}(\mathbf{C})$ given by

$$[(X,E,b,\beta)] \cdot p = [(X,E,b\cdot p,\beta)],$$

where both β and b are regarded as row vectors of order $2g$.

Let us denote by P'_g the subgroup scheme of Sp_{2g} consisting of matrices $(A\ B\ ; C\ D)$ such that $B = 0$. A simple computation proves the following equivariance properties of $\Pi : \mathbf{B}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$.

Lemma 14.3. *Consider the isomorphism of groups*

$$P_g(\mathbf{C}) \xrightarrow{\sim} P'_g(\mathbf{C})$$
$$p = \begin{pmatrix} A & B \\ 0 & (A^{\mathsf{T}})^{-1} \end{pmatrix} \longmapsto p' := \begin{pmatrix} (A^{\mathsf{T}})^{-1} & 0 \\ 2\pi i B & A \end{pmatrix}.$$

Then, for any $q \in \mathbf{B}_g$, $\gamma \in \mathrm{Sp}_{2g}(\mathbf{Z})$, and $p \in P_g(\mathbf{C})$, we have

$$\Pi(\gamma \cdot q) = \gamma \Pi(q) \quad \textit{and} \quad \Pi(q\cdot p) = \Pi(q)p'$$

in $\mathrm{Sp}_{2g}(\mathbf{C})$.

Let us now consider the *Lagrangian Grassmannian*, namely the smooth and quasi-projective $\mathbf{C}$-scheme of dimension $g(g+1)/2$ obtained as the quotient of complex affine algebraic groups

$$L_g := \mathrm{Sp}_{2g,\mathbf{C}}/P'_{g,\mathbf{C}}.$$

The complex manifold $L_g(\mathbf{C}) = \mathrm{Sp}_{2g}(\mathbf{C})/P_g'(\mathbf{C})$ may be naturally identified with the quotient of

$$M \coloneqq \{(Z_1, Z_2) \in M_{g\times g}(\mathbf{C}) \times M_{g\times g}(\mathbf{C}) \mid Z_1^{\mathsf{T}} Z_2 = Z_2^{\mathsf{T}} Z_1, \, \mathrm{rank}(Z_1 \; Z_2) = g\}$$

by the right action of $\mathrm{GL}_g(\mathbf{C})$ defined by matrix multiplication:

$$(Z_1, Z_2) \cdot S \coloneqq (Z_1 S, Z_2 S).$$

We denote the class in $L_g(\mathbf{C})$ of a point $(Z_1, Z_2) \in M$ by $(Z_1 : Z_2)$. The canonical map

$$\pi : \mathrm{Sp}_{2g,\mathbf{C}} \longrightarrow L_g$$

is then given on complex points by

$$\pi \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (B : D).$$

Proposition 14.4. *Let* $\iota : \mathbf{H}_g \longrightarrow L_g(\mathbf{C})$ *be the open embedding given by* $\iota(\tau) = (\tau : \mathbf{1}_g)$*. Then the diagram of complex manifolds*

$$\begin{array}{ccc} \mathbf{B}_g & \xrightarrow{\Pi} & \mathrm{Sp}_{2g}(\mathbf{C}) \\ \downarrow & & \downarrow \pi \\ \mathbf{H}_g & \xrightarrow[\iota]{} & L_g(\mathbf{C}) \end{array}$$

is Cartesian. That is, $\Pi : \mathbf{B}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$ *induces a biholomorphism of* $\mathbf{B}_g$ *onto the open submanifold*

$$\pi^{-1}(\iota(\mathbf{H}_g)) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{C}) \middle| \, D \in \mathrm{GL}_g(\mathbf{C}), \, BD^{-1} \in \mathbf{H}_g \right\}$$

of $\mathrm{Sp}_{2g}(\mathbf{C})$*, and makes the above diagram commute.*

Proof. The commutativity of the diagram in the statement is easy (cf. proof of Proposition .119). In particular, if $q, q' \in \mathbf{B}_g$ satisfy $\Pi(q) = \Pi(q')$, then they lie above the same point $\tau \in \mathbf{H}_g$. Let b (resp. b') be the symplectic-Hodge basis of $(\mathbf{X}_{g,\tau}, E_{g,\tau})$ corresponding to q (resp. q'). Since period matrices are base change matrices for the comparison isomorphism, and

$$\Pi(\mathbf{X}_{g,\tau}, E_{g,\tau}, b, \beta_{g,\tau}) = \Pi(\mathbf{X}_{g,\tau}, E_{g,\tau}, b', \beta_{g,\tau}),$$

it is clear that $b = b'$. This proves that Π is injective.

Observe that $\mathbf{B}_g$ and $\mathrm{Sp}_{2g}(\mathbf{C})$ are complex manifolds of same dimension. Thus, to finish our proof, it suffices to check that $\Pi(\mathbf{B}_g) = \pi^{-1}(\iota(\mathbf{H}_g))$ ([28] p. 19). Let $s \in \pi^{-1}(\iota(\mathbf{H}_g))$, and let $\tau \in \mathbf{H}_g$ be such that $\iota(\tau) = \pi(s)$. Fix any $q \in \mathbf{B}_g$ lying above $\tau \in \mathbf{H}_g$. Then, there exists a unique $p' \in P_g'(\mathbf{C})$ such that $s = \Pi(q)p'$. Hence, by Lemma .175, $s = \Pi(q \cdot p) \in \Pi(\mathbf{B}_g)$. ■

Remark 14.5. In other words, through period matrices, one can realize the moduli space $\mathbf{B}_g$ as an open submanifold of $\mathrm{Sp}_{2g}(\mathbf{C})$ given by some positivity condition. For a more direct Hodge-theoretic approach, we refer to [55] Section 4.1.

Recall from Proposition .125 that the canonical map

$$\begin{aligned} \mathbf{B}_g &\longrightarrow B_g(\mathbf{C}) \\ [(X,E,b,\beta)] &\longmapsto [(X,E,b)] \end{aligned} \tag{14.1}$$

induces a biholomorphism

$$\mathrm{Sp}_{2g}(\mathbf{Z})\backslash\mathbf{B}_g \xrightarrow{\sim} B_g(\mathbf{C}).$$

Furthermore, note that Lemma .175 implies that the action of $\mathrm{Sp}_{2g}(\mathbf{Z})$ on $\mathrm{Sp}_{2g}(\mathbf{C})$ by left multiplication preserves the open subset $\Pi(\mathbf{B}_g)$.

Corollary 14.6. *The map $\Pi : \mathbf{B}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$ induces a biholomorphism of $B_g(\mathbf{C})$ onto the open submanifold of $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$*

$$\mathrm{Sp}_{2g}(\mathbf{Z}) \setminus \Pi(\mathbf{B}_g) = \{\mathrm{Sp}_{2g}(\mathbf{Z})s \in \mathrm{Sp}_{2g}(\mathbf{Z})\backslash\mathrm{Sp}_{2g}(\mathbf{C}) \mid \pi(s) \in \iota(\mathbf{H}_g)\}.$$

■

14.2. **Explicit analytic description of the higher Ramanujan vector fields v_{ij} and of φ_g.** Recall that the Lie algebra of $\mathrm{Sp}_{2g}(\mathbf{C})$ is given by

$$\mathrm{Lie}\,\mathrm{Sp}_{2g}(\mathbf{C}) = \left\{ \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \in M_{2g\times 2g}(\mathbf{C}) \,\middle|\, B^{\mathsf{T}} = B,\, C^{\mathsf{T}} = C,\, D = -A^{\mathsf{T}} \right\}.$$

For $1 \leq k \leq l \leq g$, let us consider the left invariant holomorphic vector field $\tilde{V}_{kl}$ on $\mathrm{Sp}_{2g}(\mathbf{C})$ corresponding to

$$\frac{1}{2\pi i}\left(\begin{array}{cc} 0 & \mathbf{E}^{kl} \\ 0 & 0 \end{array} \right) \in \mathrm{Lie}\,\mathrm{Sp}_{2g}(\mathbf{C});$$

it descends to a holomorphic vector field V_{kl} on the quotient $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$.

Theorem 14.7. *Let $(v_{kl})_{1\leq k\leq l\leq g}$ be the higher Ramanujan vector fields on $B_g(\mathbf{C})$. Under the identification of $B_g(\mathbf{C})$ with an open submanifold of $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$ of Corollary .178, we have:*

(1) For every $1 \leq k \leq l \leq g$,

$$v_{kl} = V_{kl}|_{B_g(\mathbf{C})}.$$

(2) The analytic solution of the higher Ramanujan equations $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ is given by

$$\varphi_g(\tau) = \mathrm{Sp}_{2g}(\mathbf{Z}) \left(\begin{array}{cc} \mathbf{1}_g & \tau \\ 0 & \mathbf{1}_g \end{array} \right) \in \mathrm{Sp}_{2g}(\mathbf{Z})\backslash\mathrm{Sp}_{2g}(\mathbf{C}).$$

As an example of application, we can prove the following easy consequence of the above theorem.

Corollary 14.8. *The image of $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ is closed for the analytic topology.*

Proof. Consider the subgroup

$$U_g(\mathbf{C}) := \left\{ \left(\begin{array}{cc} \mathbf{1}_g & Z \\ 0 & \mathbf{1}_g \end{array} \right) \in M_{2g\times 2g}(\mathbf{C}) \,\middle|\, Z^{\mathsf{T}} = Z \right\} \leq \mathrm{Sp}_{2g}(\mathbf{C}).$$

The statement is equivalent to asserting that the image of $U_g(\mathbf{C}) \subset \mathrm{Sp}_{2g}(\mathbf{C})$ in the quotient $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$ is closed, or, equivalently, that $\mathrm{Sp}_{2g}(\mathbf{Z}) \cdot U_g(\mathbf{C}) \subset \mathrm{Sp}_{2g}(\mathbf{C})$ is closed. Let us consider the (holomorphic) map

$$f : \mathrm{Sp}_{2g}(\mathbf{C}) \longrightarrow M_{g\times g}(\mathbf{C}) \times M_{g\times g}(\mathbf{C})$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto (A, C).$$

Now, one simply remarks that

$$\mathrm{Sp}_{2g}(\mathbf{Z}) \cdot U_g(\mathbf{C}) = f^{-1}(f(\mathrm{Sp}_{2g}(\mathbf{Z}))).$$

Since $f(\mathrm{Sp}_{2g}(\mathbf{Z})) \subset M_{g\times g}(\mathbf{Z}) \times M_{g\times g}(\mathbf{Z})$, and $M_g(\mathbf{Z}) \times M_g(\mathbf{Z})$ is a closed discrete subset of $M_{g\times g}(\mathbf{C}) \times M_{g\times g}(\mathbf{C})$ for the analytic topology, we conclude that $\mathrm{Sp}_{2g}(\mathbf{Z}) \cdot U_g(\mathbf{C})$ is closed in $\mathrm{Sp}_{2g}(\mathbf{C})$. ■

We prove parts (1) and (2) of Theorem .179 separately.

Proof of Theorem .179 (1). It is sufficient to prove that the solutions of the differential equations defined by v_{kl} and by V_{kl} coincide. More precisely, let U be a simply connected open subset of $\mathbf{H}_g$, and $u : U \longrightarrow B_g(\mathbf{C})$ be a solution of the higher Ramanujan equations (Definition .133); we shall prove that, for any lifting

$$\begin{array}{ccc} & \overset{\tilde{u}}{\nearrow} & \mathbf{B}_g \\ & & \downarrow \\ U & \xrightarrow[u]{} & B_g(\mathbf{C}) \end{array}$$

of u, the holomorphic map $h \coloneqq \Pi \circ \tilde{u} : U \longrightarrow \mathrm{Sp}_{2g}(\mathbf{C})$ is a solution of the differential equations

$$\theta_{kl} h = \tilde{V}_{kl} \circ h, \quad 1 \leq k \leq l \leq g. \tag{14.2}$$

where $\theta_{kl} = \frac{1}{2\pi i}\frac{\partial}{\partial \tau_{kl}}$.

By the universal property of $\mathbf{B}_g$, the holomorphic map $\tilde{u}$ corresponds to a principally polarized complex torus (X, E) over U, of relative dimension g, endowed with a symplectic-Hodge basis $b = (\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ and an integral symplectic basis $\beta = (\gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g)$. For $\tau \in U$, let us write

$$h(\tau) = \begin{pmatrix} N_2(\tau) & \frac{1}{2\pi i}\Omega_2(\tau) \\ N_1(\tau) & \frac{1}{2\pi i}\Omega_1(\tau) \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{C})$$

where $\Omega_1, \Omega_2, N_1, N_2 : U \longrightarrow M_{g\times g}(\mathbf{C})$ are holomorphic.

Now, since u is a solution of the higher Ramanujan equations, it follows from Proposition .139 (3) that, for every $1 \leq i \leq j \leq g$,

(i) $\theta_{ij}\Omega_1 = N_1 \mathbf{E}^{ij}$, $\theta_{ij}\Omega_2 = N_2 \mathbf{E}^{ij}$
(ii) $\theta_{ij} N_1 = 0$, $\theta_{ij} N_2 = 0$.

As U is connected, (ii) implies that N_1 and N_2 are constant. Thus, (i) implies that $\frac{1}{2\pi i}\Omega_1 - N_1\tau$ and $\frac{1}{2\pi i}\Omega_2 - N_2\tau$ are also constant. In other words, there exists a unique element $s \in \mathrm{Sp}_{2g}(\mathbf{C})$ such that

$$h(\tau) = s\begin{pmatrix} \mathbf{1}_g & \tau \\ 0 & \mathbf{1}_g \end{pmatrix}$$

for every $\tau \in U$. Finally, since each $\tilde{V}_{kl}$ is left invariant, it is easy to see that h is a solution of the differential equations (14.2). ■

Lemma 14.9. *For any $\tau \in \mathbf{H}_g$, we have*

$$\Pi(\mathbf{X}_{g,\tau}, E_{g,\tau}, \boldsymbol{b}_{g,\tau}, \beta_{g,\tau}) = \begin{pmatrix} \mathbf{1}_g & \tau \\ 0 & \mathbf{1}_g \end{pmatrix}.$$

Proof. Let us write

$$\Pi(\mathbf{X}_{g,\tau}, E_{g,\tau}, \boldsymbol{b}_{g,\tau}, \beta_{g,\tau}) = \begin{pmatrix} N_2(\tau) & \frac{1}{2\pi i}\Omega_2(\tau) \\ N_1(\tau) & \frac{1}{2\pi i}\Omega_1(\tau) \end{pmatrix}.$$

By definition of β_g and of $\boldsymbol{b}_g$, it is clear that $\Omega_1(\tau) = 2\pi i\mathbf{1}_g$ and that $\Omega_2(\tau) = 2\pi i\tau$. That $N_1(\tau) = 0$ and $N_2(\tau) = \mathbf{1}_g$ is a reformulation of Corollary .136. ■

Proof of Theorem .179 (2). By definition, φ_g is given by the composition of

$$\begin{aligned} \mathbf{H}_g &\longrightarrow \mathbf{B}_g \\ \tau &\longmapsto [(\mathbf{X}_{g,\tau}, E_{g,\tau}, \boldsymbol{b}_{g,\tau}, \beta_{g,\tau})] \end{aligned}$$

with the canonical map $\mathbf{B}_g \longrightarrow B_g(\mathbf{C})$. The result now follows from Lemma .181. ■

14.3. **Group-theoretic description of $\mathcal{B}_F$, v_F, and φ_F.** In this paragraph, we consider the Hilbert-Blumenthal analogs of the above results. As usual, most proofs here are omitted due to their similarity to those concerning the Siegel case.

Recall that we have defined in Paragraph 12.5 a subgroup scheme G_F of $\mathrm{Res}_{R/\mathbf{Z}}\mathrm{Aut}_M$, where $M = R \oplus D^{-1}$. We set

$$S_F := \ker(\det : G_F \longrightarrow \mathbf{G}_m) = \mathrm{Res}_{R/\mathbf{Z}}\mathrm{Aut}_{(M,\Psi)},$$

where Ψ is the standard D^{-1}-valued R-bilinear symplectic form on M. Observe that

$$P_F = \mathrm{Res}_{R/\mathbf{Z}}\mathrm{Aut}_{(M,\Psi,R\oplus 0)}$$

defined in Paragraph 4.4 is a parabolic subgroup of S_F.

We shall also need the dual counterparts of S_F and P_F. Namely, consider the $\mathbf{Z}$-dual $M^\vee = D^{-1} \oplus R$, with its standard D^{-1}-valued R-bilinear symplectic form Φ, and set

$$S'_F := \mathrm{Res}_{R/\mathbf{Z}}\mathrm{Aut}_{(M^\vee,\Phi)}, \quad P'_F := \mathrm{Res}_{R/\mathbf{Z}}\mathrm{Aut}_{(M,\Phi,0\oplus R)}.$$

For a commutative ring Λ, if $V = \operatorname{Spec}\Lambda$, and $S_F(V), P_F(V), S'_F(V), P'_F(V)$ are regarded as subgroups of $\mathrm{GL}_2(\Lambda \otimes F)$, then $S'_F(V)$ (resp. $P'_F(V)$) is simply the image of $S_F(V)$ (resp. $P_F(V)$) under the operation of matrix transposition $s \longmapsto s^\mathsf{T}$.

Also, observe that $S'_F(\mathbf{Z})$ is the group $\mathrm{SL}(D^{-1} \oplus R)$ considered in Example .114 and Remark .129.

Definition 14.10. Let (X, E, m) be a principally polarized complex torus with R-multiplication, and b (resp. β) be a symplectic-Hodge basis (resp. an integral symplectic basis) of (X, E, m). Let

$$P(X, E, m, b, \beta) = \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix} \in G_F(\mathbf{C})$$

be the period matrix of (X, E, m) with respect to b and β, as defined in Paragraph 12.5. We set

$$\Pi(X, E, m, b, \beta) := \begin{pmatrix} \eta_2 & \frac{1}{2\pi i} \cdot \omega_2 \\ \eta_1 & \frac{1}{2\pi i} \cdot \omega_1 \end{pmatrix} \in S'_F(\mathbf{C})$$

Observe that $\Pi(X, E, m, b, \beta)$ indeed belongs to $S'_F(\mathbf{C})$ by Lemma .161.

Let $\mathbf{B}_F = B(\mathbf{X}_F, E_F, m_F)$ be the principal P_F-bundle over $\mathbf{H}^g$ associated to the principally polarized torus with R-multiplication $(\mathbf{X}_F, E_F, m_F)_{/\mathbf{H}^g}$. The manifold $\mathbf{B}_F$ can also be regarded as the moduli space of principally polarized complex tori with R-multiplication equipped with a symplectic-Hodge basis and an integral symplectic basis, so that we have a holomorphic map

$$\Pi : \mathbf{B}_F \longrightarrow S'_F(\mathbf{C})$$
$$[(X, E, m, b, \beta)] \longmapsto \Pi(X, E, m, b, \beta).$$

The space $\mathbf{B}_F$ is endowed with a left action of $S'_F(\mathbf{Z})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [(X, E, m, b, \beta)] = \left[\left(X, E, m, b, \beta \cdot \begin{pmatrix} d & b \\ c & a \end{pmatrix}\right)\right]$$

and a right action of $P_F(\mathbf{C})$ given by

$$[(X, E, m, b, \beta)] \cdot p = [(X, E, m, b \cdot p, \beta)].$$

Lemma 14.11 (cf. Lemma .175)**.** *Consider the isomorphism of groups*

$$P_F(\mathbf{C}) \xrightarrow{\sim} P'_F(\mathbf{C})$$
$$p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \longmapsto p' := \begin{pmatrix} a^{-1} & 0 \\ 2\pi i \cdot b & a \end{pmatrix}.$$

Then, for any $q \in \mathbf{B}_F$, $\gamma \in S'_F(\mathbf{Z})$, and $p \in P_F(\mathbf{C})$, we have

$$\Pi(\gamma \cdot q) = \gamma \Pi(q) \quad \text{and} \quad \Pi(q \cdot p) = \Pi(q)p'$$

in $S'_F(\mathbf{C})$. ■

Consider the smooth quasi-projective $\mathbf{C}$-scheme of dimension g obtained as the quotient of complex affine algebraic groups

$$L_F := S'_{F,\mathbf{C}}/P'_{F,\mathbf{C}}.$$

Observe that for any fractional ideal I of F we have $\mathbf{C} \otimes I = \mathbf{C} \otimes R = \mathbf{C} \otimes_{\mathbf{Q}} F$. In particular, $S'_F(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C} \otimes R)$, and $L_F(\mathbf{C})$ may be identified with $\mathbf{P}^1(\mathbf{C} \otimes R)$; the quotient map

$$\pi : S'_{F,\mathbf{C}} \longrightarrow L_F$$

is then given at complex points by

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (b : d).$$

In the next proposition, we identify $\mathbf{H}^g$ with an open submanifold of $\mathbf{C} \otimes D^{-1} = \mathbf{C} \otimes R$ via $(\sigma_1, \ldots, \sigma_g) : \mathbf{C} \otimes R \xrightarrow{\sim} \mathbf{C}^g$.

Proposition 14.12 (cf. Proposition .176)**.** *Let $\iota : \mathbf{H}^g \longrightarrow L_F(\mathbf{C})$ be the open embedding given by $\iota(\tau) = (\tau : 1)$. Then the diagram of complex manifolds*

$$\begin{array}{ccc} \mathbf{B}_F & \xrightarrow{\Pi} & S'_F(\mathbf{C}) \\ \big\downarrow & & \big\downarrow \pi \\ \mathbf{H}^g & \xrightarrow[\iota]{} & L_F(\mathbf{C}) \end{array}$$

is Cartesian. That is, $\Pi : \mathbf{B}_F \longrightarrow S'_F(\mathbf{C})$ induces a biholomorphism of $\mathbf{B}_F$ onto the open submanifold

$$\pi^{-1}(\iota(\mathbf{H}^g)) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in S'_F(\mathbf{C}) \,\middle|\, d \in (\mathbf{C} \otimes R)^\times,\ bd^{-1} \in \mathbf{H}^g \right\}$$

of $S'_F(\mathbf{C})$, and makes the above diagram commute. ■

Since the canonical map

$$\begin{aligned} \mathbf{B}_F &\longrightarrow B_F(\mathbf{C}) \\ [(X, E, m, b, \beta)] &\longmapsto [(X, E, m, b)] \end{aligned}$$

induces a biholomorphism

$$S'_F(\mathbf{Z)}\backslash \mathbf{B}_F \xrightarrow{\sim} B_F(\mathbf{C}),$$

we obtain the next corollary.

Corollary 14.13 (cf. Corollary .178)**.** *The map $\Pi : \mathbf{B}_F \longrightarrow S'_F(\mathbf{C})$ induces a biholomorphism of $B_F(\mathbf{C})$ onto the open submanifold of $S'_F(\mathbf{Z})\backslash S'_F(\mathbf{C})$*

$$S'_F(\mathbf{Z}) \setminus \Pi(\mathbf{B}_F) = \{S'_F(\mathbf{Z})s \in S'_F(\mathbf{Z})\backslash S'_F(\mathbf{C}) \mid \pi(s) \in \iota(\mathbf{H}^g)\}.$$

■

The Lie algebra of $S'_F(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C} \otimes R)$ is given by

$$\mathrm{Lie}\, S'_F(\mathbf{C}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_{2\times 2}(\mathbf{C} \otimes R) \,\middle|\, a + d = 0 \right\}.$$

Let

$$\tilde{V}_F : \mathcal{O}_{S'_F(\mathbf{C})} \otimes D^{-1} \longrightarrow T_{S'_F(\mathbf{C})}$$

be the unique $\mathcal{O}_{S'_F(\mathbf{C})}$-morphism such that, for every $x \in D^{-1}$, $\tilde{V}_F(1 \otimes x)$ is the left invariant holomorphic vector field over $T_{S'_F(\mathbf{C})}$ corresponding to

$$\frac{1}{2\pi i} \left(\begin{array}{cc} 0 & 1 \otimes x \\ 0 & 0 \end{array} \right) \in S'_F(\mathbf{C}).$$

Note that $\tilde{V}_F$ descends to a $\mathcal{O}_{S'_F(\mathbf{Z})\backslash S'_F(\mathbf{C})}$-morphism

$$V_F : \mathcal{O}_{S'_F(\mathbf{Z})\backslash S'_F(\mathbf{C})} \otimes D^{-1} \longrightarrow T_{S'_F(\mathbf{Z})\backslash S'_F(\mathbf{C})}.$$

Theorem 14.14 (cf. Theorem .179)**.** *Let $v_F : \mathcal{O}_{B_F(\mathbf{C})} \otimes D^{-1} \longrightarrow T_{B_F(\mathbf{C})}$ be the higher Ramanujan vector field on $B_F(\mathbf{C})$. Under the identification of $B_F(\mathbf{C})$ with an open submanifold of $S'_F(\mathbf{Z})\backslash S'_F(\mathbf{C})$ of Corollary .185:*

(1) We have

$$v_F = V_F|_{B_F(\mathbf{C})}.$$

(2) The analytic solution of the higher Ramanujan equations $\varphi_F : \mathbf{H}^g \longrightarrow B_F(\mathbf{C})$ is given by

$$\varphi_F(\tau) = S'_F(\mathbf{Z})\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \in S'_F(\mathbf{Z})\backslash S'_F(\mathbf{C}).$$

■

The proof of this theorem is, as usual, similar to that of the analogous Theorem .179, but it deserves some comments. To prove (1), it is enough to show that, for any analytic solution of the higher Ramanujan equations over $\mathcal{B}_F$ (Definition .143) defined on a connected open subset $U \subset \mathbf{H}^g$, say $u : U \longrightarrow B_F(\mathbf{C})$, and any lifting $\tilde{u} : U \longrightarrow \mathbf{B}_F$ of u, the composition $h \coloneqq \Pi \circ \tilde{u} : U \longrightarrow S'_K(\mathbf{C})$ satisfies

$$Th \circ \theta_F|_U = h^*\tilde{V}_F, \tag{14.3}$$

where $\theta_F : \mathcal{O}_{\mathbf{H}^g} \otimes D^{-1} \longrightarrow T_{\mathbf{H}^g}$ is the $\mathcal{O}_{\mathbf{H}^g}$-morphism defined in Paragraph 11.5.

For any $x \in D^{-1}$, we may extend the derivation $\theta_F(1\otimes x)$ of $\mathcal{O}_{\mathbf{H}^g}$ to a derivation of $\mathcal{O}_{\mathbf{H}^g}\otimes D^{-1} = \mathcal{O}_{\mathbf{H}^g} \otimes R$ by requiring that $\theta_F(1 \otimes x)(1 \otimes r) = 0$ for every $r \in R$.

Lemma 14.15. *Let us regard the standard coordinate $\tau = (\tau_1, \ldots, \tau_g)$ of $\mathbf{H}^g$ as a global section of $\mathcal{O}_{\mathbf{H}^g} \otimes D^{-1}$ via the identification $(\sigma_1, \ldots, \sigma_g) : \mathcal{O}_{\mathbf{H}^g} \otimes D^{-1} \xrightarrow{\sim} \mathcal{O}_{\mathbf{H}^g}^{\oplus g}$. Then, for every $x \in D^{-1}$,*

$$\theta_F(1 \otimes x)(\tau) = \frac{1}{2\pi i} \otimes x.$$

Proof. Follows immediately from Remark .142. ■

We deduce from the above lemma and from the left invariance of $\tilde{V}_F(1\otimes x)$ that equation (14.3) is equivalent to asserting the existence of $s \in S'_F(\mathbf{C})$ such that

$$h = s\begin{pmatrix} 1 & \tau \\ 0 & 0 \end{pmatrix} \in S'_F(\mathcal{O}_U(U)).$$

For this, we write

$$h = \begin{pmatrix} \eta_2 & \frac{1}{2\pi i}\cdot\omega_2 \\ \eta_1 & \frac{1}{2\pi i}\cdot\omega_1 \end{pmatrix} \in S'_F(\mathcal{O}_U(U))$$

and we remark that, as in the proof of Theorem .179, it suffices to prove that η_j and $\frac{1}{2\pi i}\omega_j - \eta_j\tau$ are constant for $j = 1, 2$; equivalently, we must prove that, for any $x \in D^{-1}$, $\theta_F(1\otimes x)(\eta_j) = 0$ and (by Lemma .187) $\theta_F(1 \otimes x)(\omega_j) = \eta_j 1 \otimes x$. This, in turn, is a simple consequence of Proposition .144 and of the next lemma.

Lemma 14.16. *Let M be a complex manifold, and $(\pi : X \longrightarrow M, E, m)$ be a principally polarized complex torus with R-multiplication over M. Consider the F-linear pairing*

$$\begin{aligned}\mathcal{H}^1_{\mathrm{dR}}(X/M) \times R_1\pi_*\mathbf{Q}_X &\longrightarrow \mathcal{O}_M \otimes_{\mathbf{Q}} F\\ (\alpha, \gamma) &\longmapsto I_\gamma \alpha\end{aligned}$$

defined as in Remark .160. Then, for any section γ of $R_1\pi_*\mathbf{Q}_X$, α of $\mathcal{H}^1_{\mathrm{dR}}(X/M)$, and any holomorphic vector field θ on M, we have

$$\theta(I_\gamma\alpha) = I_\gamma(\nabla_\theta \alpha).$$

Proof. Use the corresponding result result for $\int$ and apply Remark .30. ■

This concludes the proof of (1). The proof of (2) is a simple computation using Lemma .187. The proof of the next corollary is completely analogous to that of Corollary .180.

Corollary 14.17 (cf. Corollary .180)**.** *The image of $\varphi_F : \mathbf{H}^g \longrightarrow B_F(\mathbf{C})$ is closed for the analytic topology.* ■

15. Zariski-density of leaves of the higher Ramanujan foliation

Let $\mathcal{R}^{\mathrm{an}}_g$ be the integrable subbundle of the holomorphic tangent bundle $T_{B_g(\mathbf{C})}$ induced by the Ramanujan subbundle $\mathcal{R}_g \subset T_{\mathcal{B}_g/\mathbf{Z}}$ introduced in Section 5. By the holomorphic Frobenius Theorem, $\mathcal{R}^{\mathrm{an}}_g$ induces a holomorphic foliation on $B_g(\mathbf{C})$; we call it the *higher Ramanujan foliation*.

In this section, we prove that every leaf of the higher Ramanujan foliation is Zariski-dense in $B_{g,\mathbf{C}}$. In particular, we obtain that the image of the solution of the higher Ramanujan equations $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ defined in Section 11 is Zariski-dense in $B_{g,\mathbf{C}}$. We can actually derive from this the *a priori* stronger result that the graph $\{(\tau, \varphi_g(\tau)) \in \mathrm{Sym}_g(\mathbf{C}) \times B_g(\mathbf{C}) \mid \tau \in \mathbf{H}_g\}$ is Zariski-dense in $\mathrm{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} B_{g,\mathbf{C}}$.

We apply our Zariski-density results to relate our work to that of Bertrand and Zudilin [5]. Namely, using φ_g, we prove that the function field of $\mathcal{B}_{g,\mathbf{Q}}$ is a finite extension of the field generated by derivatives of Siegel modular functions defined over $\mathbf{Q}$.

Using what has been developed so far in paragraphs 12.5 and 14.3, the above results can be easily carried over to the Hilbert-Blumenthal case. We provide precise statements below, but we omit proofs.

15.1. Characterization of the leaves of the higher Ramanujan foliation.

15.1.1. Let U_g be the unipotent subgroup scheme of Sp_{2g} defined by

$$U_g(R) = \left\{ \begin{pmatrix} \mathbf{1}_g & Z \\ 0 & \mathbf{1}_g \end{pmatrix} \in M_{2g\times 2g}(R) \middle| Z^{\mathsf{T}} = Z \right\}$$

for any ring R.

The Lie algebra of $U_g(\mathbf{C})$ is given by

$$\mathrm{Lie}\, U_g(\mathbf{C}) = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \in M_{2g\times 2g}(\mathbf{C}) \middle| Z^{\mathsf{T}} = Z \right\},$$

and admit as a basis the vectors

$$\frac{1}{2\pi i}\begin{pmatrix} 0 & \mathbf{E}^{kl} \\ 0 & 0 \end{pmatrix} \in \operatorname{Lie} U_g(\mathbf{C}), \quad 1 \leq k \leq l \leq g,$$

inducing the higher Ramanujan vector fields on the quotient $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$ (Section 14). In particular, under the realization of $B_g(\mathbf{C})$ as an open submanifold of $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$ of Corollary .178, the higher Ramanujan foliation on $B_g(\mathbf{C})$ is induced by the foliation on $\mathrm{Sp}_{2g}(\mathbf{C})$ defined by $U_g(\mathbf{C})$, i.e. the foliation whose leaves are left cosets of $U_g(\mathbf{C})$ in $\mathrm{Sp}_{2g}(\mathbf{C})$.

It follows from the above discussion that, under the identification of $\mathbf{B}_g$ (resp. $B_g(\mathbf{C})$) with an open submanifold of $\mathrm{Sp}_{2g}(\mathbf{C})$ (resp. $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$) via Π (cf. Proposition .176 and Corollary .178), for any leaf L of the higher Ramanujan foliation on $B_g(\mathbf{C})$, there exists $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$ such that L is a connected component of the image of $\delta U_g(\mathbf{C}) \cap \mathbf{B}_g$ in $B_g(\mathbf{C})$ under the quotient map $\mathrm{Sp}_{2g}(\mathbf{C}) \longrightarrow \mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$. We shall provide a more precise result in Proposition .193.

15.1.2. We may also obtain an explicit *parametrization* of every leaf. For this, let us consider $\mathrm{Sym}_g(\mathbf{C}) = \{Z \in M_{g\times g}(\mathbf{C}) \mid Z^{\mathsf{T}} = Z\}$ as an open subset of the Lagrangian Grassmannian $L_g(\mathbf{C})$ (cf. discussion preceding Proposition .176) via

$$\begin{aligned} \mathrm{Sym}_g(\mathbf{C}) &\longrightarrow L_g(\mathbf{C}) \\ Z &\longmapsto (Z : \mathbf{1}_g), \end{aligned}$$

so that the embedding $\iota : \mathbf{H}_g \longrightarrow L_g(\mathbf{C})$ defined in Proposition .176 is given by the restriction of $\mathrm{Sym}_g(\mathbf{C}) \longrightarrow L_g(\mathbf{C})$ to $\mathbf{H}_g$. Furthermore, let

$$\begin{aligned} \psi : \mathrm{Sym}_g(\mathbf{C}) &\longrightarrow \mathrm{Sp}_{2g}(\mathbf{C}) \\ Z &\longmapsto \begin{pmatrix} \mathbf{1}_g & Z \\ 0 & \mathbf{1}_g \end{pmatrix}. \end{aligned}$$

Remark 15.1. Under the obvious identification of $\mathrm{Sym}_g(\mathbf{C})$ with $\operatorname{Lie} U_g(\mathbf{C})$, the map ψ is simply the exponential $\exp : \operatorname{Lie} U_g(\mathbf{C}) \longrightarrow U_g(\mathbf{C}) \subset \mathrm{Sp}_{2g}(\mathbf{C})$.

Now, the action of $\mathrm{Sp}_{2g}(\mathbf{C})$ on itself by left multiplication descends to a left action of $\mathrm{Sp}_{2g}(\mathbf{C})$ on $L_g(\mathbf{C})$ given explicitly by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot (Z_1 : Z_2) = (AZ_1 + BZ_2 : CZ_1 + DZ_2).$$

For any $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, let us define

$$\begin{aligned} \psi_\delta : \delta^{-1} \cdot \mathrm{Sym}_g(\mathbf{C}) \subset L_g(\mathbf{C}) &\longrightarrow \mathrm{Sp}_{2g}(\mathbf{C}) \\ p &\longmapsto \delta^{-1}\psi(\delta \cdot p). \end{aligned}$$

Then ψ_δ induces a biholomorphism of $\delta^{-1} \cdot \mathrm{Sym}_g(\mathbf{C})$ onto the closed submanifold $\delta^{-1}U_g(\mathbf{C}) \subset \mathrm{Sp}_{2g}(\mathbf{C})$.

We put

$$U_\delta \coloneqq \{\tau \in \mathbf{H}_g \mid \delta \cdot (\tau : 1) \in \mathrm{Sym}_g(\mathbf{C}) \subset L_g(\mathbf{C})\} = (\delta^{-1} \cdot \mathrm{Sym}_g(\mathbf{C})) \cap \mathbf{H}_g.$$

Equivalently, if $\delta = (A\ B\ ;\ C\ D)$, then

$$U_\delta = \{\tau \in \mathbf{H}_g \mid C\tau + D \in \mathrm{GL}_g(\mathbf{C})\}.$$

Definition 15.2. For any $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, we define a holomorphic map $\varphi_\delta : U_\delta \longrightarrow B_g(\mathbf{C}) \subset \mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$ by

$$\varphi_\delta(\tau) \coloneqq \mathrm{Sp}_{2g}(\mathbf{Z})\psi_\delta(\tau)$$

for any $\tau \in U_\delta$.

Note that $\psi_\delta(U_\delta) = \delta^{-1}U_g(\mathbf{C}) \cap \mathbf{B}_g \subset \mathrm{Sp}_{2g}(\mathbf{C})$ by Proposition .176. In particular, the image of φ_δ is indeed in $B_g(\mathbf{C})$. Moreover, if $\delta \in U_g(\mathbf{C})$, then $U_\delta = \mathbf{H}_g$ and $\varphi_\delta = \varphi_g$ (cf. Theorem .179 (2)).

Lemma 15.3. *For any $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, U_δ is a dense connected open subset of $\mathbf{H}_g$.*

Proof. Let $\delta = (A\ B\ ;\ C\ D) \in \mathrm{Sp}_{2g}(\mathbf{C})$. By definition, U_δ is the complement in $\mathbf{H}_g$ of the codimension 1 analytic subset $\{\tau \in \mathbf{H}_g \mid \det(C\tau + D) = 0\}$. It is thus a dense open subset of $\mathbf{H}_g$. Since $\mathbf{H}_g$ is a connected open subset of an affine space, it follows from Riemann's extension theorem (cf. [36] Proposition 1.1.7) that U_δ is connected. ■

Proposition 15.4. *For every $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, the image of the map $\varphi_\delta : U_\delta \longrightarrow B_g(\mathbf{C})$ is a leaf of the higher Ramanujan foliation on $B_g(\mathbf{C})$, and coincides with the image of $\delta^{-1}U_g(\mathbf{C}) \cap \mathbf{B}_g$ in $B_g(\mathbf{C})$ under the quotient map $\mathrm{Sp}_{2g}(\mathbf{C}) \longrightarrow \mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$. Moreover, every leaf is of this form.*

Proof. Let $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$. It was already remarked above that $\psi_\delta(U_\delta) = \delta^{-1}U_g(\mathbf{C}) \cap \mathbf{B}_g$; by definition, $\varphi_\delta(U_\delta)$ is the image of $\psi_\delta(U_\delta)$ under the quotient map $\mathrm{Sp}_{2g}(\mathbf{C}) \longrightarrow \mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$. In particular, since the higher Ramanujan foliation on $B_g(\mathbf{C})$ is induced by the foliation on $\mathrm{Sp}_{2g}(\mathbf{C})$ defined by $U_g(\mathbf{C})$ (cf. 15.1.1), to prove that $\varphi_\delta(U_\delta)$ is a leaf of the higher Ramanujan foliation it is sufficient to prove that it is connected. This is an immediate consequence Lemma .192.

Conversely, if $L \subset B_g(\mathbf{C})$ is a leaf of the higher Ramanujan foliation, then it follows from 15.1.1 that there exists $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$ such that L is a connected component of the image of $\delta^{-1}U_g(\mathbf{C}) \cap \mathbf{B}_g$ in $B_g(\mathbf{C})$ under the quotient map $\mathrm{Sp}_{2g}(\mathbf{C}) \longrightarrow \mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$. By the last paragraph, $\delta^{-1}U_g(\mathbf{C}) \cap \mathbf{B}_g = \psi_\delta(U_\delta)$ is connected, and we conclude that $L = \varphi_\delta(U_\delta)$. ■

Remark 15.5. The holomorphic maps $\varphi_\delta : U_\delta \longrightarrow B_g(\mathbf{C})$ are immersive but not injective in general. For instance, if $\delta = \mathbf{1}_{2g}$, then one easily verifies that $\varphi_g(\tau) = \varphi_g(\tau')$ if and only if $\tau' \in U_g(\mathbf{Z}) \cdot \tau$. Thus φ_g induces a biholomorphism of the quotient $U_g(\mathbf{Z})\backslash\mathbf{H}_g$ onto the closed submanifold $\varphi_g(\mathbf{H}_g)$ of $B_g(\mathbf{C})$.

Remark 15.6. There exist non-closed leaves of the higher Ramanujan foliation on $B_g(\mathbf{C})$. Take for instance

$$\delta = \begin{pmatrix} x\mathbf{1}_g & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}$$

where $x \in \mathbf{R} \smallsetminus \mathbf{Q}$. Using the classical fact that the orbit of $(x, 1)$ in $\mathbf{R}^2$ under the obvious left action of $\mathrm{SL}_2(\mathbf{Z})$ is dense in $\mathbf{R}^2$, one may easily deduce that the leaf $L \subset B_g(\mathbf{C})$ given by the image of $\delta U_g(\mathbf{C}) \cap \mathbf{B}_g$ under the quotient map $\mathrm{Sp}_{2g}(\mathbf{C}) \longrightarrow \mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{C})$ has a limit point in $B_g(\mathbf{C}) \smallsetminus L$. In particular, the "space of leaves" of the higher Ramanujan foliation on $B_g(\mathbf{C})$, which may be identified with $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash\mathrm{Sp}_{2g}(\mathbf{C})/U_g(\mathbf{C})$ by Proposition .193, is not a Hausdorff topological space.

The dynamics of the higher Ramanujan foliation in the case $g=1$ was thoroughly studied by Movasati in [53].

15.1.3. In the sequel, it will be useful to obtain a description of φ_δ purely in terms of the universal property of $B_g(\mathbf{C})$. Let $\delta=(A\ B\ ;\ C\ D)\in\mathrm{Sp}_{2g}(\mathbf{C})$ and define a holomorphic map $p_\delta : U_\delta \longrightarrow P_g(\mathbf{C})$ by

$$p_\delta(\tau)=p_{\delta,\tau}:=\begin{pmatrix}(C\tau+D)^{-1} & -\frac{1}{2\pi i}C^{\mathsf{T}}\\ 0 & (C\tau+D)^{\mathsf{T}}\end{pmatrix}\in P_g(\mathbf{C}).$$

The proof of the next lemma is a straightforward computation using the equations defining the symplectic group (cf. Remark .14).

Lemma 15.7. *For every $\tau\in U_\delta\subset\mathbf{H}_g$, we have*

$$\psi_\delta(\tau)=\psi(\tau)p'_{\delta,\tau}$$

in $\mathrm{Sp}_{2g}(\mathbf{C})$, where $p'_{\delta,\tau}$ denotes the image of $p_{\delta,\tau}$ in $P'_g(\mathbf{C})$ under the isomorphism defined in Lemma .175. ■

In particular, by Lemma .175 and Lemma .181, if $\mathbf{B}_g$ is regarded as the moduli space of principally polarized complex tori of dimension g equipped with a symplectic-Hodge basis and an integral symplectic basis, we have

$$\psi_\delta(\tau)=[(\mathbf{X}_{g,\tau},E_{g,\tau},\boldsymbol{b}_{g,\tau}\cdot p_{\delta,\tau},\beta_{g,\tau})]\in\mathbf{B}_g \tag{15.1}$$

for every $\tau\in U_\delta$. Composing with the canonical map $\mathbf{B}_g\longrightarrow B_g(\mathbf{C})$, we obtain

$$\varphi_\delta(\tau)=[(\mathbf{X}_{g,\tau},E_{g,\tau},\boldsymbol{b}_{g,\tau}\cdot p_{\delta,\tau})]\in B_g(\mathbf{C}) \tag{15.2}$$

for every $\tau\in U_\delta$.

15.2. **Auxiliary results.** Our next objective is to prove that the leaves of the higher Ramanujan foliation on $B_g(\mathbf{C})$ are Zariski-dense in $B_{g,\mathbf{C}}$. We collect in this subsection some auxiliary results. In the last analysis, our proof is a reduction to the fact that $\mathrm{Sp}_{2g}(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sp}_{2g,\mathbf{C}}$ (Lemma .199).

Recall that for every $\tau\in\mathbf{H}_g$ and

$$\delta=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{Sp}_{2g}(\mathbf{C})$$

we put

$$j(\delta,\tau):=C\tau+D\in M_{g\times g}(\mathbf{C}),$$

so that $U_\delta=\{\tau\in\mathbf{H}_g\mid j(\delta,\tau)\in\mathrm{GL}_g(\mathbf{C})\}$.

The proof of the next lemma is a simple computation.

Lemma 15.8. *For $\delta_1,\delta_2\in\mathrm{Sp}_{2g}(\mathbf{C})$, we have $j(\delta_1\delta_2,\tau)=j(\delta_1,\delta_2\cdot\tau)j(\delta_2,\tau)$. In particular, if $\tau\in U_{\delta_2}$ and $\delta_2\cdot\tau\in U_{\delta_1}$, then $\tau\in U_{\delta_1\delta_2}$.* ■

Lemma 15.9. *Let $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, $\gamma \in \mathrm{Sp}_{2g}(\mathbf{Z})$, and $\tau \in U_{\delta\gamma} \subset \mathbf{H}_g$. Then $\gamma \cdot \tau \in U_\delta$ and $\varphi_{\delta\gamma}(\tau) = \varphi_\delta(\gamma \cdot \tau)$.*

Proof. That $\gamma \cdot \tau \in U_\delta$ is a direct consequence of Lemma .197 and the fact that $j(\gamma, \tau) \in \mathrm{GL}_g(\mathbf{C})$ (this is true for any $\gamma \in \mathrm{Sp}_{2g}(\mathbf{R})$ and $\tau \in \mathbf{H}_g$). Under the group-theoretic interpretation, we have

$$\begin{aligned}\varphi_{\delta\gamma}(\tau) &= \mathrm{Sp}_{2g}(\mathbf{Z})\psi_{\delta\gamma}(\tau) = \mathrm{Sp}_{2g}(\mathbf{Z})(\delta\gamma)^{-1}\psi((\delta\gamma)\cdot\tau) \\ &= \mathrm{Sp}_{2g}(\mathbf{Z})\delta^{-1}\psi(\delta\cdot(\gamma\cdot\tau)) = \mathrm{Sp}_{2g}(\mathbf{Z})\psi_\delta(\gamma\cdot\tau) = \varphi_\delta(\gamma\cdot\tau).\end{aligned}$$

■

Lemma 15.10. *The set $\mathrm{Sp}_{2g}(\mathbf{Z}) \subset \mathrm{Sp}_{2g}(\mathbf{C})$ is Zariski-dense in $\mathrm{Sp}_{2g,\mathbf{C}}$.*

Proof. Let Sp_{2g}^* be the open subscheme of Sp_{2g} defined by $\mathrm{Sp}_{2g}^*(R) = \{(A\ B\ ;\ C\ D) \in \mathrm{Sp}_{2g}(R) \mid A \in \mathrm{GL}_g(R)\}$ for any ring R. We may define an isomorphism of schemes $\mathrm{Sp}_{2g}^* \xrightarrow{\sim} \mathrm{Sym}_g \times_{\mathbf{Z}} \mathrm{Sym}_g \times_{\mathbf{Z}} \mathrm{GL}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto (CA^{-1}, AB^{\mathsf{T}}, A).$$

Since $\mathrm{Sym}_g \times_{\mathbf{Z}} \mathrm{Sym}_g \times_{\mathbf{Z}} \mathrm{GL}_g$ may be identified to an open subscheme of the affine space $\mathbf{A}_{\mathbf{Z}}^{2g^2+g}$, we see that $\mathrm{Sym}_g(\mathbf{Z}) \times \mathrm{Sym}_g(\mathbf{Z}) \times \mathrm{GL}_g(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} \mathrm{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} \mathrm{GL}_{g,\mathbf{C}}$. Thus $\mathrm{Sp}_{2g}^*(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sp}_{2g,\mathbf{C}}^*$. Finally, since $\mathrm{Sp}_{2g,\mathbf{C}}$ is an irreducible scheme, we conclude that $\mathrm{Sp}_{2g}(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sp}_{2g,\mathbf{C}}$. ■

Lemma 15.11. *Let $\tau \in \mathbf{H}_g$ and $p \in P_g(\mathbf{C})$. Then there exists $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$ such that $\tau \in U_\delta$ and $p = p_{\delta,\tau}$.*

Proof. Let $A \in \mathrm{GL}_g(\mathbf{C})$ and $B \in M_{g\times g}(\mathbf{C})$ such that

$$p = \begin{pmatrix} A & B \\ 0 & (A^{\mathsf{T}})^{-1} \end{pmatrix}.$$

One easily verifies, using the equation $AB^{\mathsf{T}} = BA^{\mathsf{T}}$, that

$$\delta := \begin{pmatrix} A^{\mathsf{T}} & -A^{\mathsf{T}}\tau \\ -2\pi i\, B^{\mathsf{T}} & A^{-1} + 2\pi i\, B^{\mathsf{T}}\tau \end{pmatrix} \in M_{2g\times 2g}(\mathbf{C})$$

is in $\mathrm{Sp}_{2g}(\mathbf{C})$ and satisfies the required conditions in the statement. ■

Lemma 15.12. *For every $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$ and $\tau \in \mathbf{H}_g$, the subset*

$$S_{\delta,\tau} := \{p_{\delta\gamma,\tau} \in P_g(\mathbf{C}) \mid \gamma \in \mathrm{Sp}_{2g}(\mathbf{Z}) \text{ such that } j(\delta\gamma, \tau) \in \mathrm{GL}_g(\mathbf{C})\}$$

of $P_g(\mathbf{C})$ is Zariski-dense in $P_{g,\mathbf{C}}$.

Proof. Let V be the unique open subscheme of $\mathrm{Sp}_{2g,\mathbf{C}}$ such that

$$V(\mathbf{C}) = \{\gamma \in \mathrm{Sp}_{2g}(\mathbf{C}) \mid j(\delta\gamma, \tau) \in \mathrm{GL}_g(\mathbf{C})\}$$

and let $h : V \longrightarrow P_{g,\mathbf{C}}$ be the morphism of $\mathbf{C}$-schemes given on complex points by $h(\gamma) = p_{\delta\gamma,\tau}$ (note that V and $P_{g,\mathbf{C}}$ are reduced separated $\mathbf{C}$-schemes of finite type). It follows from Lemma .200 that h is surjective on complex points, thus a dominant morphism of schemes.

Now, we remark that $S_{\delta,\tau} = h(\mathrm{Sp}_{2g}(\mathbf{Z}) \cap V)$. Since $\mathrm{Sp}_{2g,\mathbf{C}}$ is irreducible and $\mathrm{Sp}_{2g}(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sp}_{2g,\mathbf{C}}$ by Lemma .199, $\mathrm{Sp}_{2g}(\mathbf{Z}) \cap V$ is also Zariski-dense in $\mathrm{Sp}_{2g,\mathbf{C}}$. Hence, as h is dominant and continuous for the Zariski topology, $S_{\delta,\tau}$ is Zariski-dense in $P_{g,\mathbf{C}}$. ■

15.3. **Statement and proof of our Zariski-density results.** Recall that we denote the coarse moduli scheme of $\mathcal{A}_g$ by A_g, and that we have a canonical map $j_g : \mathbf{H}_g \longrightarrow A_g(\mathbf{C})$ associating to each $\tau \in \mathbf{H}_g$ the isomorphism class of the principally polarized complex torus $(\mathbf{X}_{g,\tau}, E_{g,\tau})$.

The proof of our Zariski-density results will rely on the following elementary lemma.

Lemma 15.13 (Fibration method)**.** *Let $p : X \longrightarrow S$ be a morphism of separated $\mathbf{C}$-schemes of finite type and let $E \subset X(\mathbf{C})$ be a subset. If, for every $s \in p(E)$, the set $E \cap X_s$ is Zariski-dense in $X_s := p^{-1}(s)$, and one of the following conditions is satisfied,*

(i) $p(E) = S(\mathbf{C})$,

(ii) p is open (in the Zariski topology) and $p(E)$ is Zariski-dense in S,

then E is Zariski-dense in X.

Proof. Let U be a non-empty Zariski open subset of X; we must show that $E \cap U$ is non-empty. In both cases (i) and (ii) above, there exists a closed point $s \in p(E) \cap p(U)$. Since $E \cap X_s$ is Zariski-dense in X_s and $U \cap X_s$ is a non-empty open subset of X_s, there exists a closed point $x \in E \cap U \cap X_s \subset E \cap U$. ■

Theorem 15.14. *Every leaf $L \subset B_g(\mathbf{C})$ of the higher Ramanujan foliation is Zariski-dense in $B_{g,\mathbf{C}}$, that is, for every closed subscheme Y of $B_{g,\mathbf{C}}$, if $Y(\mathbf{C})$ contains L, then $Y(\mathbf{C}) = B_g(\mathbf{C})$.*

Proof. By Proposition .193, we must prove that, for every $\delta \in \mathrm{Sp}_{2g}(\mathbf{C})$, the image of $\varphi_\delta : U_\delta \longrightarrow B_g(\mathbf{C})$ is Zariski-dense in $B_{g,\mathbf{C}}$.

Let $\varpi_g : B_{g,\mathbf{C}} \longrightarrow A_{g,\mathbf{C}}$ be as in Paragraph 12.4. By Lemma .202, we are reduced to proving that, for every $x \in A_g(\mathbf{C})$, the set

$$\varphi_\delta(U_\delta) \cap \varpi_g^{-1}(x)$$

is Zariski-dense in $\varpi_g^{-1}(x) \subset B_{g,\mathbf{C}}$. Indeed, by surjectivity of ϖ_g on the level of complex points, this proves in particular that $\varpi_g(\varphi_\delta(U_\delta)) = A_g(\mathbf{C})$ (cf. condition (i) in Lemma .202).

Let (X, λ) be a representative of the isomorphism class x. Recall that the set of complex points of the $\mathbf{C}$-scheme $\varpi_g^{-1}(x)$ can be identified with the set of isomorphism classes of objects of the category $\mathcal{B}_g(\mathbf{C})$ lying over (X, λ) — we denote these isomorphism classes by $[(X, \lambda, b)]$ —, and that the $\mathbf{C}$-group scheme $P_{g,\mathbf{C}}$ acts transitively on $\varpi_g^{-1}(x)$ by

$$[(X, \lambda, b)] \cdot p := [(X, \lambda, b \cdot p)].$$

Thus, if $\tau \in \mathbf{H}_g$ satisfies $j_g(\tau) = x$, we can define a surjective morphism of $\mathbf{C}$-schemes (cf. Lemma .156)

$$\begin{aligned} f_\tau : P_{g,\mathbf{C}} &\longrightarrow \varpi_g^{-1}(x) \\ p &\longmapsto \varphi_g(\tau) \cdot p. \end{aligned}$$

Now, let $\gamma \in \mathrm{Sp}_{2g}(\mathbf{Z})$ be such that $j(\delta\gamma, \tau) \in \mathrm{GL}_g(\mathbf{C})$. By Lemma .198, we have $\gamma \cdot \tau \in U_\delta$ and $\varphi_{\delta\gamma}(\tau) = \varphi_\delta(\gamma \cdot \tau)$. Thus, by formula (15.2), we obtain

$$f_\tau(p_{\delta\gamma,\tau}) = \varphi_g(\tau) \cdot p_{\delta\gamma,\tau} = \varphi_{\delta\gamma}(\tau) = \varphi_\delta(\gamma \cdot \tau).$$

This proves that

$$S_{\delta,\tau} = \{p_{\delta\gamma,\tau} \in P_g(\mathbf{C}) \mid \gamma \in \mathrm{Sp}_{2g}(\mathbf{Z}) \text{ such that } j(\delta\gamma, \tau) \in \mathrm{GL}_g(\mathbf{C})\} \subset f_\tau^{-1}(\varphi_\delta(U_\delta) \cap \varpi_g^{-1}(x)).$$

By Lemma .201, $S_{\delta,\tau}$ is Zariski-dense in $P_{g,\mathbf{C}}$. Hence, as f_τ is surjective and continuous for the Zariski topology, we conclude that $\varphi_\delta(U_\delta) \cap \varpi_g^{-1}(x)$ is Zariski-dense in $\varpi_g^{-1}(x)$. ■

Corollary 15.15. *The set* $\{(\tau, \varphi_g(\tau)) \in \mathrm{Sym}_g(\mathbf{C}) \times B_g(\mathbf{C}) \mid \tau \in \mathbf{H}_g\}$ *is Zariski-dense in* $\mathrm{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} B_{g,\mathbf{C}}$.

Proof. It is clear that $\mathrm{Sym}_g(\mathbf{Z})$ is Zariski-dense in $\mathrm{Sym}_{g,\mathbf{C}}$. Thus, by Theorem .203 and Lemma .202 (ii) applied to the projection on the second factor

$$\mathrm{Sym}_{g,\mathbf{C}} \times_{\mathbf{C}} B_{g,\mathbf{C}} \longrightarrow B_{g,\mathbf{C}},$$

it suffices to prove that for every $N \in \mathrm{Sym}_g(\mathbf{Z})$ and $\tau \in \mathbf{H}_g$ we have $\varphi_g(\tau + N) = \varphi_g(\tau)$. This was already observed in Remark .194. ■

We now state the analogous results for the Hilbert-Blumenthal case, which can be proved *mutatis mutandis* by the same method.

Theorem 15.16. *Every leaf* $L \subset B_F(\mathbf{C})$ *of the higher Ramanujan foliation (i.e., the holomorphic foliation given by the integrable subbundle* $\mathcal{R}_F^{\mathrm{an}}$ *of* $T_{B_F(\mathbf{C})}$ *generated by the image of* $v_F : \mathcal{O}_{B_F(\mathbf{C})} \otimes D^{-1} \longrightarrow T_{B_F(\mathbf{C})}$*) is Zariski-dense in* $B_{F,\mathbf{C}}$. ■

Corollary 15.17. *The set* $\{(\tau, \varphi_F(\tau)) \in (\mathrm{Res}_{R/\mathbf{Z}}\, \mathbf{A}^1_R)(\mathbf{C}) \times B_F(\mathbf{C}) \mid \tau \in \mathbf{H}^g\}$ *is Zariski-dense in* $(\mathrm{Res}_{R/\mathbf{Z}}\, \mathbf{A}^1_R)_{\mathbf{C}} \times_{\mathbf{C}} B_{F,\mathbf{C}}$. ■

15.4. **Derivatives of modular functions and B_g.** We next explain how the moduli space B_g and the holomorphic map $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ relate with derivatives of Siegel modular functions and the work of Bertrand-Zudilin [5].

Recall that a (level 1) *Siegel modular function* of genus g is a meromorphic function on $\mathbf{H}_g$ which is invariant under the action of $\mathrm{Sp}_{2g}(\mathbf{Z})$ on $\mathbf{H}_g$. In particular, a Siegel modular function f is invariant under $U_g(\mathbf{Z})$, so that it admits a Laurent expansion

$$f(\tau) = \sum_\alpha c_\alpha \prod_{1 \le i \le j \le g} q_{ij}(\tau)^{\alpha_{ij}},$$

where $q_{ij}(\tau) = e^{2\pi i \tau_{ij}}$ (cf. Paragraph 11.4). Here, we denote $\alpha = (\alpha_{ij})_{1 \le i \le j \le g}$ with $\alpha_{ij} \in \mathbf{Z}$ for every $1 \le i \le j \le g$. We say that f is *defined over a subfield k of* $\mathbf{C}$ if each c_α is in k.

From now on, let us fix a subfield k of $\mathbf{C}$, and let us denote by K_g the field of modular functions of genus g defined over k. It is classical that $j_g : \mathbf{H}_g \longrightarrow A_g(\mathbf{C})$ identifies the K_g with $k(A_{g,k})$, the function field of $A_{g,k}$ (see, for instance, [77] VI.25).

Since the image of $\varphi_g : \mathbf{H}_g \longrightarrow B_g(\mathbf{C})$ is Zariski-dense by Theorem .203, the function field $k(B_{g,k})$ can be identified with a subfield, say L_g, of the field of meromorphic functions on $\mathbf{H}_g$.

From the commutativity of the diagram

$$\begin{array}{ccc} & \overset{\varphi_g}{\nearrow} & B_g(\mathbf{C}) \\ & & \downarrow{\scriptstyle \pi_g} \\ \mathbf{H}_g & \xrightarrow[j_g]{} & A_g(\mathbf{C}) \end{array}$$

it follows that K_g is a subfield of L_g.

Lemma 15.18. *The field L_g is stable under the derivations $\theta_{ij} = \frac{1}{2\pi i}\frac{\partial}{\partial \tau_{ij}}$, $1 \leq i \leq j \leq g$.*

Proof. This follows from the fact that φ_g is a solution of the higher Ramanujan equations (Theorem .134): if f is a rational function on $B_{g,k}$, then

$$\theta_{ij}(\varphi_g^* f) = \varphi_g^*(v_{ij}(f)).$$

■

It follows from the above lemma that, if M_g denotes the differential field generated by K_g and θ_{ij}, $1 \leq i \leq j \leq g$, then L_g contains M_g.

Theorem 15.19 (Bertrand-Zudilin, [5])**.** *The field M_g has transcendence degree $2g^2 + g$ over k.*

Now, L_g being isomorphic to the function field of the k-variety $B_{g,k}$, it is a finitely generated extension of k of transcendence degree $\dim B_{g,k} = 2g^2 + g$. We conclude that L_g is a *finite* field extension of M_g.

Remark 15.20. When $g = 1$, we have $K_1 = k(j)$ and $L_1 = k(E_2, E_4, E_6)$ (cf. Proposition .95). The explicit formulas

$$E_2 = 6\frac{\theta^2 j}{\theta j} - 4\frac{\theta j}{j} - 3\frac{\theta j}{j - 1728}, \quad E_4 = \frac{(\theta j)^2}{j(j-1728)}, \quad E_6 = -\frac{(\theta j)^3}{j^2(j-1728)}$$

actually show that $M_1 = L_1$. We do not know whether M_g should be equal to L_g for $g \geq 2$.

Remark 15.21. Note that the methods of Bertrand and Zudilin can be adapted to deal with the case of Hilbert-Blumenthal modular functions (see [4] Remark 3; see also [68] 6.5). Working as above, we can prove that $k(B_{F,k})$ is a finite extension of the differential field generated by the Hilbert-Blumenthal modular functions defined over k for the group $\mathrm{SL}(D^{-1} \oplus R)$.

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