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On the logical structure of physics and continuous model theory

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Abstract

One of the main claims of the paper is that Dirac's calculus and broader theories of physics can be treated as theories written in the language of Continuous Logic. Establishing its true interpretation (model) is a core model-theoretic challenge. The paper introduces such a model for the fragment which covers "free theories", that is, physical theories with Gaussian (quadratic) potential. The model is pseudo-finite (equivalently, a limit of finite models), based on a pseudo-finite field in place of the field of complex numbers. The advantage of this unusual setting is that it treats quantum and statistical mechanics as just domains in the same model and explains Wick rotation as a natural transformation of the model corresponding to a shift in scales of physical units.

Keywords Continuous model theory · Hilbert space · Wick rotation

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1 Introduction

1.1 The success of model theory in its numerous applications in various areas of mathematics is due, in the first place, to its focus on the fundamental questions: what is the adequate mathematical language of the area? what is the structure that is being studied?

We start here by asking the same questions about physics or rather some parts of physics.

More precisely, our focus of interest lies in the foundations of quantum theory and statistical physics. These two areas, distinguished by their distinct physical processes and scales of magnitude, utilize different mathematical frameworks connected by the formal mathematical technique known as “the Wick rotation”, or “the Wick rotation trick” as it called in [1].

Statistical physics was properly established in the 19th century when it was noticed that the probability of a state (of a gas) with certain energy E and temperature T is proportional to $\mathrm{e}^{-\frac{E}{T}}$.

Several decades later quantum physics started with the basic observation that the state of a quantum system of energy E at time t can only be adequately observed by attributing to it the complex number of norm 1, e^{iEt}, called the probability amplitude.

These frameworks correspond to certain logics which in the first case one can recognise as a form of *probabilistic logic*. For quantum physics, a more general, in fact, framework emerges, which can be tentatively classified as *continuous logic with values in* $\mathbb{C}$.

Lukasiewicz and Tarski suggested a real-valued logic in the 1930’s. Chang and Kiesler in [2] introduced continuous model theory as a multivalued logic with values in an arbitrary Hausdorff compact set $\mathbb{S}$, n-ary logical connectives as continuous functions on $\mathbb{S}^n \to \mathbb{S}$. This logic has many quantifiers, in particular ones in the form of integral. Models of continuous logic in [2] have to be metric spaces but in recent research this and other assumptions have been weakened.

In the framework of quantum or statistical physics a typical n-ary predicate in such continulous logic (with continuous domain) would be of the form

$$\mathbb{U}^n \to \mathbb{C};\quad (x_1\mathfrak{u}, \dots, x_n\mathfrak{u}) \mapsto c \cdot \mathrm{e}^{f(x_1 \dots x_n)} \tag{1}$$

where $\mathbb{U}$ is the domain of physical units, $(x_1, \dots, x_n) = \bar{x} \in \mathbb{C}^n$, $\mathfrak{u} \in \mathbb{U}$, $f : \mathbb{C}^n \to \mathbb{C}$ some nice function, c a constant. (One may compare $\mathbb{U}$ to the 1-dimensional configuration space of Hamiltonian mechanics with coordinate p (the conjugate coordinate q omitted) taking real values. When presenting $\mathbb{U}$ as a vector space over $\mathbb{R}$ the above $\mathfrak{u}$ is the choice of a basis vector. In physics setting, one can consider $\mathfrak{u}$ as the choice of units, which consequently establishes the scale of physical processes modeled.)

The quantifiers are of the form

$$\mathrm{e}^{f(\bar{x},y)} \mapsto \int_{\mathbb{R}} \mathrm{e}^{f(\bar{x},y)} \cdot \mathrm{e}^{a(y)}\, dy$$

and in basic cases the right-hand side can be calculated to be of the form $b \cdot e^{g(\bar{x})}$ for some complex number b.

In statistical theory f and a are real functions. In quantum theory these are typically similar functions obtained by changing $f(x, y)$ and $a(y)$ to $if(x, y)$ and $ia(y)$. The quantum mechanical version is an instance of the Dirac calculus with specific agreements on improper integrals.

The Wick rotation effect is that the calculation of quantifiers for real f and a return the same result as for $if(x, y)$ and $ia(y)$ with respective changes from $b \cdot e^{g(\bar{x})}$ to $b\sqrt{i} \cdot e^{ig(\bar{x})}$ - rotation of complex plane moving the real axis to the imaginary one. The effect can be explained mathematically in many cases but the physical nature of the formal link between statistical and quantum theory remains a mystery, see [3].

1.2 In this paper we take seriously the assumption that the logical setting of physics is that of continuous logic (CL) and treat the laws of physics written in terms of CL-formulas as *axioms of a CL-theory*. The problem then is to find an *interpretation* of the axioms, that is a class of yet to be defined structures, which may be continuous, finite, or pseudo-finite. (In Hrushovski [4] solves a similar, albeit in fact inverse, problem of writing down a CL-theory for a class of finite and pseudo-finite structures.)

First of all we need to determine the universe $\mathbb{U}$ as in (1) and the structure on it.

A crucial question of significance in physics is that whether $\mathbb{U}$ is discrete or continuous (see e.g. [5] justifying the discrete picture).

It is also reasonable to assume the finiteness of the universe if we accept that the age of the universe is finite. These issues are being actively discussed in physics literature, see e.g. [5].

Now, if we accept that $\mathbb{U}$ is finite or pseudo-finite (in the model theory sense) then the range of the maps in (1) does not have to be a continuous field, in fact it makes sense to consider logical values in a finite or pseudo-finite field $\mathrm{F}_{\mathfrak{p}}$ in place of $\mathbb{C}$. However, there is an obvious difficulty of practical nature. R.Penrose in his book *The Road to Reality* [6] writes about the prospect of using finite fields in physics:

... It is unclear whether such things really have a significalnt role to play in physics, although the idea has been revived from time to time. If F_p*were to take the place of the real-number system, in any significant sense, then* p*has to be very large indeed. ... To my mind, a physical theory which depends fundamentally upon some absurdly enormous prime number would be a far more complicated (and improbable) theory than one that is able to depend on a simple notion of infinity. Nevertheless, it is of some interest to pursue these matters. ...*

One of the main results of our paper is that the setting with pseudo-finite $\mathrm{F}_{\mathfrak{p}}$ is quite easily convertible to the setting over the field $\mathbb{C}$. Conversely the continuous calculus over $\mathbb{C}$ (and $\mathbb{R}$) can be translated into some meaningful calculations over $\mathrm{F}_{\mathfrak{p}}$ without loss.

1.3 The mathematical tool which is behind the passage between the pseudo-finite setting and $\mathbb{C}$ is the following commutative diagram first established in [7] and worked out in the current paper (Sect. 4 and Theorem 4.11) in more detail:

$$\begin{array}{ccc} & \mathsf{lm}_{\mathbb{U}} & \\ \mathbb{U} & \longrightarrow & \bar{\mathbb{C}} \\ \exp_{\mathfrak{p}} \downarrow & & \downarrow \exp \\ & \mathsf{lm}_{\mathrm{F}} & \\ \mathrm{F}_{\mathfrak{p}} & \longrightarrow & \bar{\mathbb{C}} \end{array} \tag{2}$$

Here $\mathbb{U}$ is a pseudo-finite additive group and $\exp_{\mathfrak{p}}$ is a surjective homomorphism of the group onto the multiplicative group $\mathrm{F}_{\mathfrak{p}}^{\times}$. The horizontal arrows are "limit" maps $\mathsf{lm}_{\mathbb{U}}$ and lm_{F} respectively, which provide a *structural approximation* (see [8]) of the discrete structure $(\mathbb{U}, \exp_{\mathfrak{p}}, \mathrm{F}_{\mathfrak{p}})$ by the continuous structure $(\bar{\mathbb{C}}, \exp, \bar{\mathbb{C}})$, where $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the compactified field of complex numbers (Riemann sphere). Intuitively, limit maps establish continuous structure on $(\mathbb{U}, \exp_{\mathfrak{p}}, \mathrm{F}_{\mathfrak{p}})$ by determining that a pair of points u_1, u_2 in the discrete structure is "infinitesimally close" if $\mathsf{lm}\, u_1 = \mathsf{lm}\, u_2$. The approximation via $(\mathsf{lm}_{\mathbb{U}}, \mathsf{lm}_{\mathrm{F}})$ is "structural" in the sense that it preserves the structure, that is the relations on $(\mathbb{U}, \mathrm{F}_{\mathfrak{p}})$ given by equations are preserved under application of $(\mathsf{lm}_{\mathbb{U}}, \mathsf{lm}_{\mathrm{F}})$.

It is crucial that the limit maps are well-controlled, in particular certain natural multiplicative subgroups $'\mathbb{S}'$, $'\mathbb{R}'_+ \subset \mathrm{F}_{\mathfrak{p}}$ map onto the unit circle $\mathbb{S} \subset \bar{\mathbb{C}}$ and the non-negative reals $\mathbb{R}_+ \subset \mathbb{C}$ which allows to mimic polar coordinates $(\varphi, r) \in \mathbb{S} \times \mathbb{R}_+$ of $\mathbb{C}$ in $\mathrm{F}_{\mathfrak{p}}$ and thus develop a working analogue of continuous complex calculus. This model-theoretic technique, in effect, is an answer to the above Penrose caution against $\mathrm{F}_{\mathfrak{p}}$ in physics.

In essence, the main model theory result of the paper is the construction and description of structure $(\mathbb{U}, \mathrm{F}_{\mathfrak{p}})$ as a continuous-model ultraproduct of quite intricate finite structures (see 2.1) which include wave-functions, finite-dimensional "Hermitian and Euclidean Hilbert spaces" over subfields $'\mathbb{C}'$ and $'\mathbb{R}'$ of $\mathrm{F}_{\mathfrak{p}}$ respectively, linear operators on the spaces and other relevant constructs.

1.4 One of the main gains of the pseudo-finite setting in terms of foundations of physics is in explaining the effect of Wick rotation as the transformation/homomorphism of $\mathbb{U}$ caused by the multiplication

$$u \mapsto \mathbf{i} \cdot u$$

where $\mathbf{i}$ is a non-standard integer such that

$$\mathsf{lm}_{\mathbb{U}}(\mathbf{i} \cdot u) = i \cdot \mathsf{lm}_{\mathbb{U}}(u) \quad (i = \sqrt{-1}).$$

More precisely, in pseudo-finite setting, the additive group $\mathbb{U}$ acquires the structure of a $^*\mathbb{Z}$-module (the analogue of $\mathbb{C}$-vector space of (1)), where $^*\mathbb{Z}$ is the ring of pseudo-finite integers, a non-standard arithmetic, and $\mathbf{i} \in {}^*\mathbb{Z}$ is chosen to satisfy the property above. Such an integer can not be standard, in fact it has to be "huge", approximately equal to $\sqrt{\mathfrak{p} - 1}$.

The action of the "huge" integer $\mathbf{i}$ shifts a subdomain $\mathbb{V}_{\mathfrak{u}}$ (with unit $\mathfrak{u}$) of the universe $\mathbb{U}$ to the subdomain $\mathbb{V}_{\mathfrak{v}} = \mathbf{i} \cdot \mathbb{V}_{\mathfrak{u}}$ (with unit $\mathfrak{v} = \mathbf{i}\mathfrak{u}$) and thus a predicate $\mathrm{e}^{f(\bar{x})}$ as in (1) is shifted to a predicate $\mathrm{e}^{if(\bar{x})}$.

We associate $\mathbb{V}_{\mathfrak{u}}$ with physics at the scale of statistical mechanics and the Euclidean Hilbert space formalism, and $\mathbb{V}_{\mathfrak{v}}$ with quantum mechanics and Hermitian Hilbert spaces. Thus the shift by $\mathbf{i}$ is the mathematical form of the change of scales in physics which manifests itself as the Wick rotation.

2 Main results

2.1 We define $\mathbb{U} = \mathbb{U}_{\mathfrak{p},\mathfrak{l}}$ as the quotient of the additive group of non-standard integers ${}^*\mathbb{Z}$ by the ideal $\mathfrak{P} = (\mathfrak{p}-1)\mathfrak{l} \cdot {}^*\mathbb{Z}$

$$\mathbb{U} := {}^*\mathbb{Z}/\mathfrak{P}$$

for distinguished parameters: prime $\mathfrak{p}$ and a highly divisible $\mathfrak{l}$.

$$\exp_{\mathfrak{p}} : \mathbb{U} \to \mathrm{F}_{\mathfrak{p}}^{\times}$$

is a homomorphism with kernel

$$\ker \exp_{\mathfrak{p}} = (\mathfrak{p}-1) \cdot \mathbb{U}$$

of order $\mathfrak{l}$ and $\mathbf{i} \in {}^*\mathbb{Z}$ is assumed to divide $(\mathfrak{p}-1)$ and to be divisible by $\mathfrak{l}$ (see 3.1 for more detail).

2.2 Logical evaluations of $\mathbb{U}$ in $\mathrm{F}_{\mathfrak{p}}$. We think in terms of multivalued logic with values in $\mathrm{F}_{\mathfrak{p}}$, the domain of logical values. $\mathbb{U}$ and its cartesian powers $\mathbb{U}^{\mathrm{M}}$ are domains of physical units. We think of definable operations on $\mathrm{F}_{\mathfrak{p}}$ as logical connectives, where definable means interpretable in the non-standard model of arithmetic ${}^*\mathbb{Z}$ with distinguished parameters $\mathfrak{p}$, $\mathfrak{l}$ and the map $\exp_{\mathfrak{p}}$.

The map $\exp_{\mathfrak{p}} : \mathbb{U} \to \mathrm{F}_{\mathfrak{p}}$ gives rise to logical evaluations of "physical models" which take the form of a basic notion of the Hilbert space formalism of physics, a state (over $\mathrm{F}_{\mathfrak{p}}$):

$$\varphi : \mathbb{V}^{\mathrm{M}} \to \mathrm{F}_{\mathfrak{p}}$$

where $\mathbb{V} \subset \mathbb{U}$ is a specific subdomain, a pseudo-cyclic group generated by $\mathfrak{u}$. A basic state (basic predicate) φ has the form

$$\varphi(\bar{x}) = \exp_{\mathfrak{p}}(f(\bar{x}) \cdot \mathfrak{u}) = \exp_{\mathfrak{p}}(\mathfrak{u})^{f(\bar{x})}$$

where $f(\bar{x})$ is a polynomial over $\mathbb{Z}$, $\bar{x} = (x_1, \ldots, x_M) \in {}^*\mathbb{Z}^{\mathrm{M}}$ and $f(\bar{x}) \cdot \mathfrak{u}$ is the $f(\bar{x})$-multiple of $\mathfrak{u}$ in the additive group $\mathbb{V}$.

A general predicate is obtained from basic ones by using "logical connectives", polynomially defined functions in $\mathrm{F}_{\mathfrak{p}}$, and "quantifiers", sums $\sum_{r \in R} \varphi_r$ for definable families of predicates.

2.3 Let

$$\mathcal{N} := \#\exp_{\mathfrak{p}}(\mathbb{V})$$

the number of elements in the pseudo-cyclic group. Thus a basic state $\varphi(\bar{x})$ on $\mathbb{V}^{\mathrm{M}}$ is a function of period $\mathcal{N}$ in every variable x_i. So we may assume that the variables x_i run in $\mathrm{K}_{\mathcal{N}} := {}^*\mathbb{Z}/\mathcal{N}$, the residue ring modulo $\mathcal{N}$, equivalently in the quotient $\exp_{\mathfrak{p}}(\mathbb{V})$ rather than in $\mathbb{V}$. However, in physics setting we are interested in the structure on the full domain $\mathbb{V}$, which is accessible by tools which allow the inversion of the exponential map (see 5.2 and [9] for more detail).

For simplicity, we concentrate on the one-dimensional case, that is $\mathrm{M} = 1$, until claimed otherwise.

The states (which are of course a certain kind of coordinate functions) form an $\mathrm{F}_{\mathfrak{p}}$-linear space $\mathcal{H}_{\mathbb{V}}$ of pseudo-finite dimension, with natural choices of orthonormal bases and well-defined inner product with values in $\mathrm{F}_{\mathfrak{p}}$. Moreover, we consider definable linear maps on $\mathcal{H}_{\mathbb{V}}$, analogues of linear unitary operators playing an important role in physics (such as the Fourier transform and time evolution operators).

The definable family of position states

$$\mathcal{L}_{\mathbf{u}} = \{\mathbf{u}_r : r \in \mathrm{K}_{\mathcal{N}}\}; \quad \mathbf{u}_r(s) := \delta(r - s)$$

(Kronecker-delta) forms a basis of $\mathcal{H}_{\mathbb{V}}$. A definable injective linear operator $\mathcal{A}$ gives rise to other definable bases

$$\mathcal{L}_{\mathcal{A}\mathbf{u}} = \{\mathcal{A}\mathbf{u}_r : r \in \mathrm{K}_{\mathcal{N}}\}.$$

In regards to model-theoretic formalism, unlike the traditional approach (see e.g. [10]), $\mathcal{H}_{\mathbb{V}}$ is not considered to be a universe of a structure. Instead, we consider the universe to be multisorted, the union of definable sorts $\mathcal{L}_{\mathcal{A}\mathbf{u}}$ (written also as $\mathcal{L}_{\psi}$) which are all the definable bases above. The multisorted structure also comes with maps $\mathcal{A}$ between these, together with the $\mathrm{F}_{\mathfrak{p}}$-linear space $\mathcal{H}_{\mathbb{V}}$ interpretable in the sorts in the sense of definability as above.

2.4 In the current paper we restrict the study to so called Gaussian setting. This means that the $f(\bar{x})$ defining basic states φ are quadratic forms on the ring $\mathrm{K}_{\mathcal{N}} = {}^*\mathbb{Z}/\mathcal{N}$, and the operators are of the special form

$$\mathcal{A} : \varphi \mapsto \frac{1}{\sqrt{\mathcal{N}}} \sum_{r \in \mathrm{K}_{\mathcal{N}}} \exp_{\mathfrak{p}}(a(q, r) \cdot \mathfrak{u}) \cdot \varphi(r), \tag{3}$$

where $a(q, r)$ is a quadratic form. φ is function of $r \in \mathrm{K}_{\mathcal{N}}$ and the right-hand side of (3) is a function of $q \in \mathrm{K}_{\mathcal{N}}$. (In the more general case one has to write $\sqrt{\mathcal{N}}^{\mathrm{M}}$ and $\mathrm{K}_{\mathcal{N}}^{\mathrm{M}}$ in place of $\sqrt{\mathcal{N}}$ and $\mathrm{K}_{\mathcal{N}}$.)

These are discrete analogues of quantum mechanics unitary operators for the free particle and also includes the important case of the harmonic oscillator. We believe free field theories should be representable in the setting once we switch to considering domains $\mathbb{V}^{\mathrm{M}} \subset \mathbb{U}^{\mathrm{M}}$ for appropriate infinite pseudo-finite M.

We are in particular interested in studying relationship between two domains $\mathbb{V}_{\mathfrak{u}}$ and $\mathbb{V}_{\mathfrak{v}}$ in $\mathbb{U}$ determined by the choices of units, $\mathfrak{u}$ and $\mathfrak{v}$ respectively, of different scales: $\mathfrak{v} = \mathbf{i} \cdot \mathfrak{u}$, for $\mathbf{i}$ described above.

2.5 Our first main result, Theorem 6.11, establishes for the discrete pseudo-finite model a unifying treatment of "physics" over the two domains:

In fact,

$$\mathbb{V}_{\mathfrak{v}} \subset \mathbb{V}_{\mathfrak{u}}$$

both equipped with non-standard-valued metrics defined in terms of units $\mathfrak{v}$ or $\mathfrak{u}$ respectively.

The multiplication by $\mathbf{i}$ determines the projection

$$\mathbf{i}: \ \mathbb{V}_{\mathfrak{u}} \twoheadrightarrow \mathbb{V}_{\mathfrak{v}} = \mathbf{i}\mathbb{V}_{\mathfrak{u}}.$$

Given a state φ on $\mathbb{V}_{\mathfrak{u}}$, its restriction to $\mathbb{V}_{\mathfrak{v}}$ is a state on $\mathbb{V}_{\mathfrak{v}}$, which we denote $\varphi^{\mathbf{i}}$. In fact, if $\varphi = \exp_{\mathfrak{p}}(f(r)\mathfrak{u})$ then $\varphi^{\mathbf{i}} = \exp_{\mathfrak{p}}(\mathbf{i}f(r)\mathfrak{u})$.

Respectively the action of a linear operator $\mathcal{A}$ on $\varphi \in \mathcal{H}_{\mathbb{V}_{\mathfrak{u}}}$ becomes the action of some well-defined linear operator $\mathcal{A}^{\mathbf{i}}$ on $\varphi^{\mathbf{i}} \in \mathcal{H}_{\mathbb{V}_{\mathfrak{v}}}$,

$$\mathcal{A}^{\mathbf{i}}\varphi^{\mathbf{i}} = (\mathcal{A}\varphi)^{\mathbf{i}}.$$

A formal inner product on the spaces transforms correspondingly

$$\langle\varphi^{\mathbf{i}}|\psi^{\mathbf{i}}\rangle = \langle\varphi|\psi\rangle^{\mathbf{i}},$$

where we consider both a formal-Euclidean and a formal-Hermitian versions of inner product.

This gives us the isomorphism of the structures

$$\{\}^{\mathbf{i}}: \ \mathcal{H}_{\mathbb{V}_{\mathfrak{u}}} \to \mathcal{H}_{\mathbb{V}_{\mathfrak{v}}} \tag{4}$$

Note that a tensor product power $\mathcal{H}_{\mathbb{V}}^{\otimes \mathrm{M}}$ for pseudo-finite M is interpretable in $\mathcal{H}_{\mathbb{V}}$ (or rather in the underlying structure $(\mathbb{U}; \mathrm{F}_{\mathfrak{p}})$). Thus the picture can be generalised to pseudo-finite-dimensional setting

$$\mathbb{V}_{\mathfrak{v}}^{\mathrm{M}} \subset \mathbb{V}_{\mathfrak{u}}^{\mathrm{M}} \subset \mathbb{U}^{\mathrm{M}}$$

with the Hilbert spaces replaced by tensor product powers $\mathcal{H}_{\mathbb{V}_{\mathfrak{v}}}^{\otimes \mathrm{M}}$ and $\mathcal{H}_{\mathbb{V}_{\mathfrak{u}}}^{\otimes \mathrm{M}}$ respectively.

2.6 Our final task is to recast the pseudo-finite ultraproduct of finite structures underlying the formal Hilbert spaces $\mathcal{H}_{\mathbb{V}_{\mathfrak{u}}}$, $\mathcal{H}_{\mathbb{V}_{\mathfrak{v}}}$ as a continuous logic (CL) structure with values in the complex numbers. The key to the construction are the limit maps $\mathsf{lm}_{\mathbb{U}}$ and lm_{F} shown in diagram (2). The construction and the properties of the limit maps is explained in Sect. 4 and in Theorem 4.11. These are slight improvements of those in [7].

Now, according to CL-principles, we have to convert the domains $\mathbb{V}_{\mathfrak{u}}$ and $\mathbb{V}_{\mathfrak{v}}$ into metric spaces presentable as countable unions of finite-diameter subspaces. The non-standard-valued metric is determined in $\mathbb{V}_{\mathfrak{u}}$ by the distance

$$\mathrm{dist}(0, n\mathfrak{u}) := n|\mathfrak{u}|, \ \ n \in \mathrm{K}_{\mathcal{N}}$$

where $|\mathfrak{u}|$ is specifically chosen non-standard rational number (and respectively for $\mathbb{V}_{\mathfrak{v}}$). The diameters of $\mathbb{V}_{\mathfrak{u}}$ and $\mathbb{V}_{\mathfrak{v}}$ turn out to be large non-standard numbers. However we use these metrics and define, for $\mathbb{V} := \mathbb{V}_{\mathfrak{u}}$ and $\mathbb{V} := \mathbb{V}_{\mathfrak{v}}$ respectively,

$$\mathbb{V}_{|n} = \{u \in \mathbb{V} : \ \mathrm{dist}(0, u) \leq n\},$$

$$\mathbb{V}^{\mathsf{lm}}_{|n} := \mathsf{lm}_{\mathbb{U}}(\mathbb{V}_{|n}) \text{ and } \mathbb{V}^{\mathsf{lm}} := \bigcup_{n \in \mathbb{N}} \mathbb{V}^{\mathsf{lm}}_{|n}$$

It turns out that

$$\mathbb{V}^{\mathsf{lm}}_{\mathfrak{u}} = \mathbb{R} \text{ and } \mathbb{V}^{\mathsf{lm}}_{\mathfrak{v}} = \mathrm{i}\mathbb{R}$$

For a state φ as above we define

$$\varphi^{\mathsf{lm}} : \ \mathbb{V}^{\mathsf{lm}} \to \mathbb{C}; \ \ \varphi^{\mathsf{lm}}(r^{\mathsf{lm}}) := \mathsf{lm}_{\mathrm{F}}(\sqrt{\mathcal{N}}\varphi(r))$$

Note that we have to use a normalising coefficient, an infinite pseudo-finite number $\sqrt{\mathcal{N}}$ ($\mathcal{N}$ is $\mathcal{N}_{\mathfrak{u}}$ or $\mathcal{N}_{\mathfrak{v}}$, respectively) in order to produce a meaningful wave-function $\mathbb{R} \to \mathbb{C}$. As a result, inner products also have to be renormalised, which agrees with the Dirac delta-function renormalisation of respective integral formulas.

Crucially, in case of $\mathbb{V}_{\mathfrak{u}}$ we find it technically necessary to choose the Euclidean inner product, and in case of $\mathbb{V}_{\mathfrak{v}}$ the Hermitian inner product.

Finally, the linear operator $\mathcal{A}$ of the form (3) becomes the integral operator on the Hilbert space of functions ϕ,

$$\mathcal{A}^{\mathsf{lm}} : \phi \mapsto \int_{\mathbb{R}} \mathrm{e}^{\alpha(s,x)} \cdot \phi(x)\, dx. \tag{5}$$

In terms of continuous logic, this is a quantifier analogous to the bounded existential quantifier in the formula

$$\exists x\, A(s, x) \ \& \ \varphi(x)$$

of first order logic.

We refer to the two structures as

$$\mathcal{H}_{\mathbb{R}} := \mathcal{H}^{\mathsf{lm}}_{\mathbb{V}_{\mathfrak{u}}} \text{ and } \mathcal{H}_{\imath\mathbb{R}} := \mathcal{H}^{\mathsf{lm}}_{\mathbb{V}_{\mathfrak{v}}}$$

with Euclidean and Hermitian inner products respectively.

2.7 The final result presented in Sect. 8 can be summarised as the following Theorem comparing the two continuous logic structures:

The map (4) *passes to a morphism of CL-structures*

$$\{\}^{\mathbf{i}}: \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\imath\mathbb{R}}$$

realised by the bijection on the CL-position states sorts

$$\mathcal{L}^{\mathsf{Im}}_{\mathbf{u}} \to \mathcal{L}^{\mathsf{Im}}_{\mathbf{u}^{\mathbf{i}}}; \quad \mathbf{u}_r \mapsto \mathbf{u}_{\imath r}, \quad r \in \mathbb{R}$$

and general sorts

$$\mathcal{L}^{\mathsf{Im}}_{\psi} \to \mathcal{L}^{\mathsf{Im}}_{\psi^{\mathbf{i}}}; \quad \psi_r \mapsto \psi_{\imath r}, \quad r \in \mathbb{R}$$

which commute, as we prove, with the transformation of the integral operators (5)

$$\int_{\mathbb{R}} \mathrm{e}^{\alpha(s,x)} \cdot \phi(x)\, dx \mapsto \int_{\mathbb{R}} \mathrm{e}^{\imath\alpha(s,x)} \cdot \phi^{\mathbf{i}}(x)\, dx$$

and induces the $\mathbb{R}$*-bilinear bijective maps between inner products*

$$\langle \psi_r | \varphi_s \rangle_{\mathrm{E}} \mapsto \langle \psi_r^{\mathbf{i}} | \varphi_s^{\mathbf{i}} \rangle_{\mathrm{H}}; \quad r, s \in \mathbb{R}$$

(Euclidean on the left hand side and Hermitian on the right hand side).

This gives a full account on the Wick rotation in Gaussian setting.

Also, treating the integral operators as quantifiers, the structure allows quantifier elimination.

2.8 Future directions. The current presentation is essentially schematic and limited to presenting a model-theoretic perspective on physics. However the author believes that the scheme can be developed into a useful framework for physics.

The extension of the Gaussian setting to free fields theories appears quite feasible. The more general setting which necessitates the use of perturbation methods presents a greater challenge. However, it does not seem insurmountable, as the tools of real and complex analysis are readily available within the $\mathrm{F}_{\mathfrak{p}}$-setting.

3 Definability and scales in $\mathbb{U}$

3.1 As in [7], let

$${}^*\mathbb{C} = \mathbb{C}^P/D, \quad {}^*\mathbb{Z} = \mathbb{Z}^P/D$$

be ultrapowers of the field of complex numbers and of the ring of integers by a non-principal ultrafilter on the set of prime numbers P.

We essentially use the fact that these ultrapowers are ω_1-saturated and apply some rather basic model theory analysis avoiding possible deeper questions.

Recall that by definition we have a representation of $\mathbb{U}$ and $\mathrm{F}_{\mathfrak{p}}$ together with $\exp_{\mathfrak{p}}$ in $({}^*\mathbb{Z}; +, \cdot, \mathfrak{p}, \mathbf{i}, \mathfrak{l})$ as quotients

$$\mathbb{U} = {}^*\mathbb{Z}/\mathfrak{P}; \quad \mathrm{F}_{\mathfrak{p}} = {}^*\mathbb{Z}/\mathfrak{p}$$

($\mathfrak{P}$ and $\mathfrak{p}$ here are ideals in ${}^*\mathbb{Z}$ and also positive integers generating the ideals, $\mathfrak{P} = (\mathfrak{p}-1)\cdot\mathfrak{l}$) and

$$\exp_{\mathfrak{p}} : \eta\cdot\hat{1} \mapsto \epsilon^{\eta}, \text{ for some } \epsilon\in \mathrm{F}_{\mathfrak{p}} \text{ and each } \eta\in {}^*\mathbb{Z}.$$

We also distinguish some positive $\mathfrak{m}$, $\mathbf{j}\in {}^*\mathbb{Z}$ such that

$$\bigwedge_{n\in\mathbb{N}} n|\mathfrak{m} \;\&\; \mathfrak{m}^2=\mathfrak{l} \;\&\; \mathbf{m}|\mathbf{j} \;\&\; \mathbf{j}^2=\mathbf{i} \;\&\; \mathbf{i}|(\mathfrak{p}-1) \tag{6}$$

Note that $\mathfrak{m}$ and $\mathbf{j}$ are uniquely defined from $\mathfrak{l}$ and $\mathbf{i}$, and (6) also postulates that $\mathfrak{m}$ is divisible by all standard integers (highly divisible) as well as $\mathfrak{l}$, $\mathbf{i}$ and $\mathfrak{p}-1$ are.

We assume that ϵ is a generator of the pseudo-finite group $\mathrm{F}_{\mathfrak{p}}^{\times}$ which means

$$\epsilon^{\eta}=1 \bmod \mathfrak{p} \text{ iff } (\mathfrak{p}-1)|\eta \tag{7}$$

Notation

$$\mathrm{acl}(X) = \text{ the algebraic closure of subset } X\subset \mathrm{F}_{\mathfrak{p}} \text{ in } \mathrm{F}_{\mathfrak{p}}.$$

$${}^*\mathbb{Z}[N:M] := \{z\in{}^*\mathbb{Z}:\ N\le z\le M\};\quad {}^*\mathbb{Z}[N:M) := \{z\in{}^*\mathbb{Z}:\ N\le z< M\}$$

Lemma 3.2 *We may assume in addition to (6) that*

$$\boldsymbol{i}^2\neq \mathfrak{p}-1 \Rightarrow \bigwedge_{a_1,\dots,a_k\in\mathbb{Z}} \boldsymbol{i}^k + a_1\boldsymbol{i}^{k-1}+\cdots+a_k\neq 0\, mod\, \mathfrak{p}$$

Proof The set of conditions on $\mathbf{i}$ on the right hand side of $\Rightarrow$ is over parameter $\mathfrak{p}$ and countable as is the set of conditions in (6). Thus these can be realised together with $\mathbf{j}$ and $\mathfrak{l}$ in the ω_1-saturated structure ${}^*\mathbb{Z}$. □

Lemma 3.3 *Asssuming $\mathfrak{p}, \boldsymbol{i}$ satisfying (6) and 3.2 are fixed, there is an $\epsilon\in F_{\mathfrak{p}}$ which along with (7) satisfies*

$$\epsilon^{\frac{\mathfrak{p}-1}{i}} \notin acl(\mathbb{Q}[\boldsymbol{i}]) \tag{8}$$

that is

$$\alpha := \epsilon^{\frac{\mathfrak{p}-1}{i}} = \exp_{\mathfrak{p}}\left(\frac{\mathfrak{p}-1}{\boldsymbol{i}}\right)$$

is transcendental in $F_{\mathfrak{p}}$ over $\boldsymbol{i}$.

Proof Indeed, $\epsilon^{\frac{\mathfrak{p}-1}{\mathbf{i}}}$ is an element of (non-standard) order $\mathbf{i}$ in the multiplicative group $\mathrm{F}_{\mathfrak{p}}^{\times}$ of order $\mathfrak{p}-1$. The set of integers $r\in{}^*\mathbb{Z}[0:\mathbf{i}]$ co-prime with $\mathfrak{p}-1$ is an infinite definable set and for each such r the element ϵ^r satisfies (7) in place of ϵ. At the same time $(\epsilon^r)^{\frac{\mathfrak{p}-1}{\mathbf{i}}}$ takes different values for distinct r. Since ${}^*\mathbb{Z}$ is ${}_1$-saturated, there is an r such that $(\epsilon^r)^{\frac{\mathfrak{p}-1}{\mathbf{i}}} \notin \mathrm{acl}(\mathbb{Q}[\mathbf{i}])$. Taking ϵ to be ϵ^r proves the claim.

3.4 We write $(^*\mathbb{Z}; \Omega_\mathfrak{p})$ for the structure of nonstandard arithmetic with extra parameters $\mathfrak{p}, \epsilon, \mathbf{i}$.

Note that the constants in $\Omega_\mathfrak{p}$ together with $\mathfrak{l}$ also determine $\mathfrak{m}$ and $\mathbf{j}$, when (6) is satisfied.

Note that $^*\mathbb{Z}$ in the language $\Omega_\mathfrak{p}$ is not an elementary extension of $\mathbb{Z}$. Only its reduct to the ring language is.

Call a k-tuple $\mathsf{L} \in {}^*\mathbb{Z}^k$ finite-generic with respect to $\Omega_\mathfrak{p}$ (f,-g. for short) if for any $\Omega_\mathfrak{p}$-formula $\Phi(\bar{x})$

$$\text{For each } \bar{l} \in \mathbb{N}^k,\ (^*\mathbb{Z}; \Omega_\mathfrak{p}) \vDash \Phi(\bar{l}) \quad \Rightarrow \quad (^*\mathbb{Z}; \Omega_\mathfrak{p}) \vDash \Phi(\mathsf{L}) \tag{9}$$

Note that the set of formulas satisfying the antecedent of (9) is a k-type. We call a formula Φ as in (9) an f.-g. formula.

3.5 Properties.

1. $\bar{n} \in \mathbb{N}^k$ then $\bar{n}$ is f.-g.

2. (L_1, L_2) is f.-g. then L_1 and L_2 are f.-g.

3. $L \in {}^*\mathbb{Z}^k$ f.-g. and $f(X)$ an $\Omega_\mathfrak{p}$-definable function such that, $f(\mathbb{N}^k) \subseteq \mathbb{N}$. Then $f(L)$ is f.-g. and $f(L) \geq 0$.

4. Let β be a non-standard definable elemenent in $(^*\mathbb{Z}; \Omega_\mathfrak{p})$. If $x = L$ is a solution of a non-trivial equation $p(x, \beta) = 0 \mod \mathfrak{p}$, for $p(x, y) \in \mathbb{Z}[x, y]$, then L is not f.-g.

Proof 1. and 2. immediate by definition.

3. Let $\Phi(x)$ be an f.-g. formula. Then $\Phi(f(\bar{y}))$ is f.-g. since $(^*\mathbb{Z}; \Omega_\mathfrak{p}) \vDash \Phi(f(\bar{n}))$ for all $\bar{n} \in \mathbb{N}^k$. Thus $\vDash \Phi(f(L))$. So $f(L)$ is f.-g.. Note that $x \geq 0$ is a f.-g. formula, so $f(L) \geq 0$.

4. There are only finitely many solutions of $p(x, \beta) = 0 \mod \mathfrak{p}\ \&\ 0 \leq x < \mathfrak{p}$, so L not f.-g. □

3.6 We would need to consider also the structure on the field $\mathbb{C}$ with a predicate distinguishing $\mathbb{Z}$ and constants symbols $\mathbf{i}$, ϵ and $\mathfrak{p}$, call it $(\mathbb{C}, \mathbb{Z}, \Omega_\mathfrak{p})$, together with its non-standard version $(^*\mathbb{C}, {}^*\mathbb{Z}; \Omega_\mathfrak{p})$.

Lemma. *A subset $P \subseteq {}^*\mathbb{Z}^n$ is definable in $(^*\mathbb{C}, {}^*\mathbb{Z}; \Omega_\mathfrak{p})$ if and only if it is definable in $(^*\mathbb{Z}; \Omega_\mathfrak{p})$.*

Proof Recall the following Theorem of Jonsson theory, see [11]: Let κ be an infinite cardinal satisfying

$$\kappa = \sum_{\lambda<\kappa} 2^\lambda.$$

Such arbitrary large cardinals exists in ZFC and, for any complete first order theory T with infinite models, there exists a model $\mathcal{M}$ of T of cardinality κ which is universal and cf(κ)-homogeneous, called *special* model.

Now let $\mathrm{T} = \mathrm{Th}(^*\mathbb{C}, {}^*\mathbb{Z}; \Omega_\mathfrak{p})$ and $(\mathbb{C}', \mathbb{Z}'; \Omega_\mathfrak{p})$ be a special model of T of cardinality κ containing $(^*\mathbb{C}, {}^*\mathbb{Z}; \Omega_\mathfrak{p})$.

Assume that P is not definable in $(^*\mathbb{Z}; \Omega_\mathfrak{p})$ and so in $(\mathbb{Z}'; \Omega_\mathfrak{p})$. Then by ω-universality of $(\mathbb{Z}'; \Omega_\mathfrak{p})$ there is are n-tuples $a, b \in \mathbb{Z}'$ satisfying the same $\Omega_\mathfrak{p}$-type with $a \in P$ and $b \notin P$.

By cf(κ)-homogeneity there is $\sigma \in \mathrm{Aut}(\mathbb{Z}', \Omega_\mathfrak{p})$ such that $\sigma : a \mapsto b$. This is a ring-automorphism.

Let $\sigma' : \mathrm{acl}(\mathbb{Z}') \to \mathrm{acl}(\mathbb{Z}')$ be the extension of σ to $\mathrm{acl}(\mathbb{Z}')$ where acl is field-theoretic algebraic closure. Clearly, $\mathrm{acl}(\mathbb{Z}')$ is algebraically closed subfield of $\mathbb{C}'$. Hence σ' can be extended to an automorphism σ'' of $\mathbb{C}'$. Thus we showed that our assumption implies that there exists $\sigma'' \in \mathrm{Aut}(\mathbb{C}', \mathbb{Z}'; \Omega_\mathfrak{p})$ which does not preserve P, so P is not definable in $(\mathbb{C}', \mathbb{Z}'; \Omega_\mathfrak{p})$. Lemma follows. □

Let $\mathcal{F}$ be the set of all $\Omega_\mathfrak{p}$-definable functions $f : {}^*\mathbb{Z} \to {}^*\mathbb{Z}$ such that $f(\mathbb{N}) \subseteq \mathbb{N}$.

Remark. We don't know if $\mathcal{F}$ is a proper extension of $\mathcal{F}_0$, the set of all functions 0-definable in arithmetic.

However, in more general setting, there is a nonstandard $\mathfrak{q} \in {}^*\mathbb{Z}$ such that the set $\mathcal{F}_\mathfrak{q}$ of all functions $g : {}^*\mathbb{Z}^k \to {}^*\mathbb{Z}$ defined in $({}^*\mathbb{Z}; +, \cdot, \mathfrak{q})$ and satisfying the condition $g(\mathbb{N}^k) \subset \mathbb{N}$ is bigger than $\mathcal{F}_0$. Indeed, set

$$g(n) := \begin{cases} p_n, & \text{if } p_n | \mathfrak{q} (n\text{-th prime}) \\ 0, & \text{otherwise} \end{cases}$$

Using ω_1-saturation one can, for any set Q of standard primes $p_n \in \mathbb{N}$, find $\mathfrak{q} \in {}^*\mathbb{Z}$ such that, for all $n \in \mathbb{N}$,

$$p_n \in Q \Leftrightarrow p_n | \mathfrak{q}$$

Clearly, there are $Q \subset \mathbb{N}$ which are not 0-definable in arithmetic. So there exists $\mathfrak{q}$ and $g \in \mathcal{F}_\mathfrak{q} \setminus \mathcal{F}_0$.

Proposition 3.7 *Assume for* $\mathfrak{p}, \boldsymbol{i}$ *and* ϵ *there exist* $\mathfrak{l}, \mathfrak{m}$ *and* $\boldsymbol{j}$ *such that (6)–(8). Assume* $\boldsymbol{i}^2 + 1 \neq \mathfrak{p}$.

Then there exists a finite-generic $\mathfrak{l}$ *satisfying (6)–(8) together with the condition: for each* $f \in \mathcal{F}$

$$\forall a_0 \ldots a_k \in {}^*\mathbb{Z}[-f(\mathfrak{l}) : f(\mathfrak{l})]\, a_0 \neq 0 \to \; a_0 \boldsymbol{i}^k + a_1 \boldsymbol{i}^{k-1} + \cdots + a_k \neq 0 \, mod \, \mathfrak{p} \quad (10)$$

In particular, independently on the condition $\mathbf{i}^2 + 1 \neq \mathfrak{p}$, *for each* $f \in \mathcal{F}$:

$$\boldsymbol{i} > f(\mathfrak{l}) \;\; and \;\; \frac{\mathfrak{p} - 1}{\boldsymbol{i}} > f(\mathfrak{l}) \quad (11)$$

Proof Let, for $M \in \mathbb{N}$,

$$P_M(l) := \bigwedge_{n \leq M} \exists m, j : \; n | m \;\&\; m^2 = l \;\&\; m | j \;\&\; j^2 = \mathbf{i} \;\&\; \mathbf{i} | (\mathfrak{p} - 1).$$

Clearly $\bigwedge_{M \in \mathbb{N}} P_M(l)$ is a type whose realisation $\mathfrak{l}$ satisfies (6). One can see that for any standard $n \geq M$,

$$\left({}^*\mathbb{Z}; \Omega_\mathfrak{p} \right) \vDash P_M((n!)^2)$$

That is the $\Omega_\mathfrak{p}$-formula

$$\Psi_M(n) := \; n \geq M \to P_M((n!)^2)$$

holds for all $n \in \mathbb{N}$, that is Ψ_M is an f.-g,-formula. Let $\mathfrak{n} \in {}^*\mathbb{Z}$ be an infinite f.-.g. number. In particular, $({}^*\mathbb{Z}; \Omega_\mathfrak{p}) \vDash \bigwedge_{M \in \mathbb{N}} \Psi_M(\mathfrak{n})$. Let $\mathfrak{l} := (\mathfrak{n}!)^2$, which is also an infinite f.-.g. number by 3. of 3.5.

Let $\Phi_f(x)$ be the $\Omega_\mathfrak{p}$-formula stating (10) when $x := \mathfrak{l}$. Lemma 3.2 states that this is a f.-g. formula. Then $({}^*\mathbb{Z}; \Omega_\mathfrak{p}) \vDash \Phi_f(\mathfrak{l})$ by our definition of $\mathfrak{l}$, which proves (10).

In case $\mathbf{i}^2 + 1 \neq \mathfrak{p}$ (11) follows immediately. Otherwise we use the fact that $\mathbf{i} > f(n)$ & $\frac{\mathfrak{p}-1}{\mathbf{i}} > f(n)$ for all $n \in \mathbb{N}$. □

3.8 Notation/Corollary. We choose $\mathfrak{l}$, ϵ and $\mathbf{i}$ so that (6)–(8) and (10) hold and $\mathfrak{l}$ is finite-generic.

Denote

$$\mathrm{O}(\mathcal{F}) := \bigcup_{f \in \mathcal{F}} {}^*\mathbb{Z}[-f(\mathfrak{l}) : f(\mathfrak{l})],$$

$$\mathrm{O}(\mathfrak{l}) := \bigcup_{m \in \mathbb{N}} {}^*\mathbb{Z}[-m\mathfrak{l} : m\mathfrak{l}],$$

$$\mathfrak{u} := \frac{\mathfrak{p}-1}{\mathbf{i}\mathfrak{l}} \text{ and } \mathfrak{v} := \frac{\mathfrak{p}-1}{\mathfrak{l}} = \mathbf{i}\mathfrak{u}.$$

Corollary 3.9 *$O(\mathcal{F})$ is a convex subring of ${}^*\mathbb{Z}$ containing $O(\mathfrak{l})$ and closed under every $f \in \mathcal{F}$. In particular,*

$$O(\mathcal{F}) \prec {}^*\mathbb{Z}$$

in the language of rings.

Also

$$O(\mathcal{F})_+ < \; O(\mathcal{F})_+ \cdot \mathfrak{u} < \; O(\mathcal{F})_+ \cdot \mathfrak{v}$$

for $O(\mathcal{F})_+ := \{x \in O(\mathcal{F}) : \; x > 0\}$ (where $X \; < \; Y$ for $X, Y \subset {}^*\mathbb{Z}$ means that $\forall x \in X \forall y \in Y \; x < y$).

Proof The first follows from the definition and the fact that $\mathrm{O}(\mathcal{F})$ is closed under all 0-definable maps in the structure $({}^*\mathbb{Z}; +, \cdot)$ with definable Skolem functions.

The second is a corollary of (11). □

Definition 3.10

$$'\mathbb{S}' := \exp_\mathfrak{p}\{\mathrm{O}(\mathfrak{l}) \cdot \mathfrak{v}\}$$

$$'\mathbb{R}'_+ := \exp_\mathfrak{p}\{\mathrm{O}(\mathfrak{l}) \cdot \mathfrak{u}\}$$

Note that since $\exp_\mathfrak{p}((n + \mathfrak{l}) \cdot \mathfrak{v}) = \exp_\mathfrak{p}(n \cdot \mathfrak{v})$

$$'\mathbb{S}' := \exp_\mathfrak{p}\{{}^*\mathbb{Z}[0 : \mathfrak{l}) \cdot \mathfrak{v}\}$$

4 Embedding into $^*\mathbb{C}$

4.1 Let $\pi' \in {}^*\mathbb{C}$ stand for a real number, possibly non-standard, such that $\mathrm{e}^{\pi'}$ is transcendental and π' is infinitesimally close to π (assuming Schanuel's conjecture we can take $\pi' := \pi$).

We consider $\mathbb{U}$ as an $\mathrm{O}(\mathfrak{l})$-module. Note that $^*\mathbb{Z} \subset {}^*\mathbb{C}$ and thus $\mathrm{O}(\mathcal{F}) \subset {}^*\mathbb{C}$ and acts on $^*\mathbb{C}$ by multiplication.

First we define the map

$$\mathrm{I}_{\mathbb{U}} : \mathbb{U} \to {}^*\mathbb{C}$$

on the 2-elements set $\{\mathfrak{u}, \mathfrak{v}\} \subset \mathbb{U}$ (see 3.8):

$$\mathrm{I}_{\mathbb{U}} : \quad \mathfrak{u} \mapsto -\frac{2\pi'}{\mathfrak{l}}; \quad \mathfrak{v} \mapsto -\frac{2\pi \mathrm{i}}{\mathfrak{l}}$$

where $\mathrm{i} = \sqrt{-1}$.

For $\alpha, \beta \in \mathrm{O}(\mathfrak{l})$ define

$$\mathrm{I}_{\mathbb{U}} : \ (\alpha \cdot \mathfrak{u} + \beta \cdot \mathfrak{v}) \mapsto -\frac{1}{\mathfrak{l}}(2\pi'\alpha + 2\pi \mathrm{i}\beta). \tag{12}$$

Note that by 3.7, $\mathfrak{u}$ and $\mathfrak{v}$ are linearly independent over $\mathrm{O}(\mathfrak{l})$). Hence the map is well-defined and invertible.

Let $\mathbb{U}(\mathfrak{l})$ be the 2-dimensional $\mathrm{O}(\mathfrak{l})$-submodule of $\mathbb{U}$ spanned by $\{\mathfrak{u}, \mathfrak{v}\}$ and let $^*\mathbb{C}(\mathfrak{l}) \subset {}^*\mathbb{C}$ be the $\mathrm{O}(\mathfrak{l})$-submodule spanned by $\{\frac{2\pi'}{\mathfrak{l}}, \frac{2\pi \mathrm{i}}{\mathfrak{l}}\}$

(12) defines

$$\mathrm{I}_{\mathbb{U}} : \ \mathbb{U}(\mathfrak{l}) \to {}^*\mathbb{C}(\mathfrak{l})$$

as an $\mathrm{O}(\mathfrak{l})$-linear isomorphism.

4.2 Consider elements

$$1, \ \exp_{\mathfrak{p}}(\mathfrak{u}), \ \exp_{\mathfrak{p}}(\mathfrak{v}) \in \mathrm{F}_{\mathfrak{p}}$$

(1 is $1_{\mathrm{mod}\,\mathfrak{p}}$ of $\mathrm{F}_{\mathfrak{p}}$) and define a partial map $\mathrm{I}_{\mathrm{F}} : \mathrm{F}_{\mathfrak{p}} \to {}^*\mathbb{C}$ on the three points:

$$\mathrm{I}_{\mathrm{F}} : \ 1 \mapsto 1, \ \exp_{\mathfrak{p}}(\mathfrak{u}) \mapsto \mathrm{e}^{-\frac{2\pi'}{\mathfrak{l}}}, \ \ \exp_{\mathfrak{p}}(\mathfrak{v}) \mapsto \mathrm{e}^{-\frac{2\pi \mathrm{i}}{\mathfrak{l}}}$$

and further, for any $a \in \mathrm{O}(\mathcal{F})$ and $s, r \in \mathrm{O}(\mathfrak{l})$,

$$\mathrm{I}_{\mathrm{F}} : \ a \cdot 1 \mapsto a \cdot 1, \ \exp_{\mathfrak{p}}(r\mathfrak{u}) \mapsto \mathrm{e}^{-\frac{2r\pi'}{\mathfrak{l}}}, \ \ \exp_{\mathfrak{p}}(s\mathfrak{v}) \mapsto \mathrm{e}^{-\frac{2s\pi \mathrm{i}}{\mathfrak{l}}}.$$

This is internally definable over respective elements by our assumptions. Moreover, if $a = a_i, s = s_i$ and $r = r_i$ represent elements of internally definable sequences, $i \in I \subset {}^*\mathbb{Z}[0 : \mathfrak{l}^m]$ then one can definably extend to sequences

$$\mathrm{I}_{\mathrm{F}} : \ a_i \cdot 1 \mapsto a_i \cdot 1, \ \exp_{\mathfrak{p}}(r_i\mathfrak{u}) \mapsto \mathrm{e}^{-\frac{2r_i\pi'}{\mathfrak{l}}}, \ \ \exp_{\mathfrak{p}}(s_i\mathfrak{v}) \mapsto \mathrm{e}^{-\frac{2s_i\pi \mathrm{i}}{\mathfrak{l}}}.$$

Let

$$\mathrm{O(F)} := \left\{ \sum_{i\in I} a_i \exp_{\mathfrak{p}}(r_i \mathfrak{u}) \exp_{\mathfrak{p}}(s_i \mathfrak{v}) : \ I \subseteq {}^*\mathbb{Z}[0 : \mathfrak{l}^m] \text{ definable} \right\}$$

$$\mathrm{O}({}^*\mathbb{C}) := \left\{ \sum_{i\in I} a_i \mathrm{e}^{\frac{-2r_i\pi'}{\mathfrak{l}}} \mathrm{e}^{\frac{-2s_i\pi \mathrm{i}}{\mathfrak{l}}} : \ I \subseteq {}^*\mathbb{Z}[0 : \mathfrak{l}^m] \text{ definable} \right\}$$

These are rings containing $\mathrm{O}(\mathcal{F})$ and closed under internally definable summation.

Extend

$$\mathrm{I_F} : \mathrm{O(F)} \to \mathrm{O}({}^*\mathbb{C})$$

accordingly.

Lemma 4.3 *I_F is well-defined and bijective.*

Proof It is enough to prove that

$$\sum_{i\in I} a_i \exp_{\mathfrak{p}}(r_i \mathfrak{u}) \exp_{\mathfrak{p}}(s_i \mathfrak{v}) = 0 \text{ iff } \sum_{i\in I} a_i \mathrm{e}^{\frac{-2r_i\pi'}{\mathfrak{l}}} \mathrm{e}^{\frac{-2s_i\pi \mathrm{i}}{\mathfrak{l}}} = 0 \tag{13}$$

when the left hand side sums run through all elements of O(F), equivalently, the right hand side sums run through all elements of $\mathrm{O}({}^*\mathbb{C})$.

Note that the left hand side equality can be expressed by a formula $\Psi(\mathfrak{l}, I)$ in $({}^*\mathbb{Z}; \Omega_{\mathfrak{p}})$.

Similarly, the right hand side equality can be expressed by a formula $\dot{\Psi}(\mathfrak{l}, I)$ in $({}^*\mathbb{C}, {}^*\mathbb{Z}; \Omega_{\mathfrak{p}})$.

Consider the formula

$$\Phi(\mathfrak{l}) := \forall I \subset {}^*\mathbb{Z}[0 : \mathfrak{l}^m] \ \Psi(\mathfrak{l}, I) \leftrightarrow \dot{\Psi}(\mathfrak{l}, I).$$

We claim that, for all $n \in \mathbb{N}$,

$$({}^*\mathbb{C}; {}^*\mathbb{Z}, \Omega_{\mathfrak{p}}) \vDash \Phi(n).$$

Indeed, the internally definable sequences on $\mathbb{Z}[0 : n^m]$ have finite values of parameters a_i, r_i, s_i and $\mathfrak{m}$ and thus $\Phi(n)$ expresses the fact that the algebraic dependence of some $\mathrm{e}^{\frac{2r_i\pi'}{k}}$, $\mathrm{e}^{\frac{2s_i\pi\mathrm{i}}{k}}$ with coefficients a_i takes place if and only if the algebraic dependence with coefficients a_i of $\exp_{\mathfrak{p}}(\frac{r_i(\mathfrak{p}-1)}{k\mathbf{i}})$, $\exp_{\mathfrak{p}}(\frac{s_i(\mathfrak{p}-1)}{k})$ takes place. And note that both $\mathrm{e}^{\frac{2s_i\pi\mathrm{i}}{k}}$ and $\exp_{\mathfrak{p}}(\frac{s_i(\mathfrak{p}-1)}{k})$ are standard roots of unity in respective fields while $\mathrm{e}^{\frac{2r_i\pi'}{k}}$, $\exp_{\mathfrak{p}}(\frac{r_i(\mathfrak{p}-1)}{k\mathbf{i}})$, are transcendental elements. It follows that the equalities on the both side of (13) can only happen when all $a_i = 0$. Which proves $\Phi(n)$ and hence $\Phi(\mathfrak{l})$. □

Remark 4.4

$$\mathrm{acl}(\mathrm{O}({}^*\mathbb{C})) \neq {}^*\mathbb{C}.$$

Indeed, elements of $\mathrm{O}({}^*\mathbb{C}))$ are definable over $\mathfrak{l}$, $\mathrm{e}^{\frac{\pi'}{\mathfrak{l}}}$ and $\mathrm{e}^{\frac{\pi i}{\mathfrak{l}}}$ in $({}^*\mathbb{C}; {}^*\mathbb{Z}, \Omega_{\mathfrak{p}})$ and thus can be reduced to elements of the ring $\mathrm{O}_l \subset \mathbb{C}$ generated by roots of unity of order l and elements of the form $\mathrm{e}^{\frac{2\pi' i}{l}}$ over $l \in \mathbb{N}$ in a structure of the form $(\mathbb{C}; \mathbb{Z}, \Omega_p)$. Hence any element of $\mathrm{acl}(\mathrm{O}({}^*\mathbb{C}))$ is in the ultraproduct of $\mathrm{acl}(\mathrm{O}_l)$ such that l are the restrictions of $\mathfrak{l}$ to the index set of the ultraproduct. The statement follows from the fact that $\mathrm{acl}(\mathrm{O}_l) \neq \mathbb{C}$. □

The Lemma together with the statement in 4.1 prove:

Theorem 4.5 *The maps*

$$I_{\mathbb{U}} : \mathbb{U}(\mathfrak{l}) \to {}^*\mathbb{C}(\mathfrak{l})$$

and

$$I_F : O(F) \to O({}^*\mathbb{C})$$

are bijections preserving internally definable summation and commute with respective exponentiation maps

$$\exp_{\mathfrak{p}} : \mathbb{U}(\mathfrak{l}) \to O(F) \text{ and } \exp : {}^*\mathbb{C}(\mathfrak{l}) \to O({}^*\mathbb{C})$$

4.6 Define $\mathrm{F} \subset \mathrm{F}_{\mathfrak{p}}$ to be the fraction field of $\mathrm{O}(\mathrm{F})$.

Theorem 4.5 implies that I_{F} extends to the embedding

$$\mathrm{I}_{\mathrm{F}} : \mathrm{F} \hookrightarrow {}^*\mathbb{C} \tag{14}$$

Corollary 4.7 *There is an internally definable notion of complex conjugation $z \mapsto \bar{z}$, $\bar{z} = z^{-1}$, for $z \in {}'\mathbb{S}'$, and $\bar{y} = y$ for $y \in {}'\mathbb{R}'_+$ and this determines an automorphism $x \mapsto \bar{x}$ on F.*

Moreover,

$$I_F(\bar{x}) = \overline{I_F(x)}$$

(complex conjugation in ${}^*\mathbb{C}$ on the right).

Proof I_{F} maps ${}'\mathbb{S}'$ to $\mathbb{S}$ and ${}'\mathbb{R}'_+$ to positive reals of ${}^*\mathbb{C}$, thus complex conjugation is correctly defined on ${}'\mathbb{S}'$ and ${}'\mathbb{R}'_+$. The extension to F is by internally definable summation and hence reduces readily to finite sums, which satisfies the algebraic identities of complex conjugation. □

Set

$$\mathsf{Im}_{\mathrm{F}} := \mathrm{st} \circ \mathrm{I}_{\mathrm{F}}$$

Lemma 4.8

$$\mathsf{Im}_F : F \twoheadrightarrow \bar{\mathbb{C}}$$

Proof The surjectivity of Im_{F} follows from the facts:

$$\mathsf{Im}_{\mathrm{F}} : \{\exp_{\mathfrak{p}}(r\mathfrak{u}) : r \in \mathrm{O}(\mathfrak{l})\} = \mathbb{R}_+$$

$$\mathsf{Im}_{\mathrm{F}} : \{\exp_{\mathfrak{p}}(r\mathfrak{v}) : r \in \mathrm{O}(\mathfrak{l})\} = \mathbb{S}$$

and

$$\mathsf{Im}_{\mathrm{F}}(\mathfrak{l}) = \infty.$$

□

Lemma 4.9 *Let* $\hat{\boldsymbol{i}} = \boldsymbol{i}_{mod_{\mathfrak{p}}} \in F_{\mathfrak{p}}$. *Then*

$$\hat{\boldsymbol{i}}^2 = -1 \text{ or } \hat{\boldsymbol{i}} \notin acl(F).$$

Proof Suppose $\hat{\mathbf{i}}^2 \neq -1$ and suppose towards a contradiction that $\hat{\mathbf{i}} \in \mathrm{acl(F)}$. It implies that

$$c_0\hat{\mathbf{i}}^k + c_1\hat{\mathbf{i}}^{k-1} + \cdots + c_k = 0 \tag{15}$$

for some $c_0, c_1, \ldots c_k \in \mathrm{O(F)}$, $c_0 \neq 0$, that is of the form $\sum_{i\in I} a_i \exp_{\mathfrak{p}}(r_i\mathfrak{u}) \exp_{\mathfrak{p}}(s_i\mathfrak{v})$ each, i.e. $c_i = c_i(\mathfrak{l})$ internally definable in $({}^*\mathbb{Z}; \Omega_{\mathfrak{p}})$ over $\mathfrak{l}$. Note that if we substitute $n \in \mathbb{N}$ in place of $\mathfrak{l}$, $c(n)$ is in the ring generated by $\exp_{\mathfrak{p}}(\frac{r_i(\mathfrak{p}-1)}{k\mathbf{i}})$, $\exp_{\mathfrak{p}}(\frac{s_i(\mathfrak{p}-1)}{k})$ with finite r_i, s_i and k. That is

$$c_0(n), \ldots, c_k(n) \in \mathbb{Z}[\zeta_n, \alpha^{\frac{1}{n}}], \quad \alpha = \exp_{\mathfrak{p}}\left(\frac{\mathfrak{p}-1}{\mathbf{i}}\right) : \ \zeta_n \text{ the } n\text{-th root of unity.}$$

It follows that, for all $n \in \mathbb{N}$:

$$c_0(n)\hat{\mathbf{i}}^k + c_1(n)\hat{\mathbf{i}}^{k-1} + \cdots + c_k(n) \neq 0 \ \vee \bigwedge_{0\leq i\leq k} c_i(n) = 0 \tag{16}$$

for otherwise $\hat{\mathbf{i}} \in \mathrm{acl}(\alpha)$ while $\hat{\mathbf{i}} \neq -1$, in contradiction with 3.3 and 3.7.

Since $\mathfrak{l}$ is finite-generic (16) implies the negation of (15), the contradiction which proves our statement. □

Recall that

$$\mathrm{i} = \mathrm{e}^{\frac{\pi i}{2}}.$$

Corollary 4.10 *For some* $\mathrm{i}' \in {}^*\mathbb{C}$ *such that* $\mathrm{i}' - i$ *is an infinitesimal in* ${}^*\mathbb{C}$, *the embedding* I_F *of (14) extends to the embedding*

$$I_F : F_{\mathfrak{p}} \to {}^*\mathbb{C}; \quad \text{so that } \hat{\boldsymbol{i}} \mapsto i'$$

Thus,

$$\mathsf{Im}_F : \ F_{\mathfrak{p}} \twoheadrightarrow \bar{\mathbb{C}} \tag{17}$$

$$\mathsf{Im}_F : \hat{\boldsymbol{i}} \mapsto e^{\frac{\pi i}{2}} \text{ and } \hat{\boldsymbol{j}} \mapsto e^{\frac{\pi i}{4}}. \tag{18}$$

Proof If $\hat{\mathbf{i}}^2 = -1$ then set $\mathrm{i}' = \mathrm{i}$. Otherwise, pick $\mathrm{i}' \in {}^*\mathbb{C}$ in the infinitesimal neighborhood of i but not in $\mathrm{I_F(acl(F))}$, which exists because of 4.4.

The extension of $\mathrm{I_F}$ to an embedding of $\mathrm{F}_{\mathfrak{p}}$ is by the routine algebraic construction using the fact that

$$\mathrm{tr.deg_F F_{\mathfrak{p}}} \leq \mathrm{tr.deg_F}\, {}^*\mathbb{C}$$

and ${}^*\mathbb{C}$ is algebraically closed. □

Remark. Note that in terms of the embedding ${}^*\mathbb{Z}[0 : \mathfrak{p}-1] \hookrightarrow \mathrm{F}_\mathfrak{p}$ one may identify $\hat{\mathbf{i}} = \mathbf{i}$, $\hat{\mathbf{j}} = \mathbf{j}$ and

$$\mathsf{Im}_\mathrm{F} : \mathbf{i} \mapsto \mathrm{e}^{\frac{\pi i}{2}} \text{ and } \mathbf{j} \mapsto \mathrm{e}^{\frac{\pi i}{4}}.$$

Remark An immediate consequence of the properties of the standard part map is that Im_F as defined in (17) is a place of fields, that is there is a local ring F^0 of F such that the restriction

$$\mathsf{Im}_\mathrm{F} : \mathrm{F}^0 \twoheadrightarrow \mathbb{C}$$

is a homomorphism of rings and, for $x \in \mathrm{F} \setminus \mathrm{F}^0$, $\mathsf{Im}_\mathrm{F}(x) = \infty$.

Along with the next theorem this completes the construction and summarises properties of the crucial diagram (2).

Theorem 4.11 *There is an additive surjective homomorphism*

$$\mathsf{Im}_\mathbb{U} : \mathbb{U} \twoheadrightarrow \mathbb{C} \cup \{\infty\}, \quad \mathsf{Im}_\mathbb{U} : \mathbb{U}(\mathfrak{l}) \twoheadrightarrow \mathbb{C}$$

such that, for $u \in \mathbb{U}(\mathfrak{l})$

$$\mathsf{Im}_\mathbb{U}(u) = st \circ I_\mathbb{U}(u)$$

$$\mathsf{Im}_\mathbb{U}(\boldsymbol{i}u) = i\, \mathsf{Im}_\mathbb{U}(u)$$

and for any and $x \in F$

$$\mathsf{Im}_F(x) = st \circ I_F(x).$$

For any $u \in \mathbb{U}$

$$\exp(\mathsf{Im}_\mathbb{U}(u)) = \mathsf{Im}_F(\exp_\mathfrak{p}(u)) \tag{19}$$

where

$$\exp(\infty) = \infty.$$

Proof Set for $u \in \mathbb{U}(\mathfrak{l})$

$$\mathsf{Im}_\mathbb{U}(u) := \mathrm{st} \circ \mathrm{I}_\mathbb{U}(u)$$

which is well-defined and satisfies (19) by 4.5.

(12) implies $\mathsf{Im}_\mathbb{U}(\mathbf{i}u) = \mathrm{i}\, \mathsf{Im}_\mathbb{U}(u)$.

$\mathsf{Im}_\mathbb{U}(\mathbb{U}(\mathfrak{l})) = \mathbb{C}$ follows from the fact that

$$\mathrm{st}({}^*\mathbb{C}(\mathfrak{l})) = \mathbb{C}$$

since by construction

$$\mathrm{st}\left(\frac{2\pi' \cdot \mathrm{O}(\mathfrak{l})}{\mathfrak{l}}\right) = \mathbb{R} \text{ and } \mathrm{st}\left(\frac{2\pi i \cdot \mathrm{O}(\mathfrak{l})}{\mathfrak{l}}\right) = \mathrm{i}\mathbb{R}.$$

Lemma 4.8 proves the properties of Im_F on F.

In order to defines $\mathsf{Im}_\mathbb{U}$ on $\mathbb{U} \setminus \mathbb{U}(\mathfrak{l})$ in agreement with Im_F on $\mathrm{F}_\mathfrak{p}$ defined by (17), consider

$$\mu := \{x \in \mathrm{F}_\mathfrak{p}^\times : \mathsf{Im}_\mathrm{F} x = 0\}$$

$$\mu^{-1} := \{x \in \mathrm{F}_{\mathfrak{p}}^{\times} : \mathrm{Im}_{\mathrm{F}} x = \infty\}$$

which by definitions are multiplicative semigroups with the property

$$\mathrm{F}_{\mathfrak{p}}^{\times} \cdot \mu = \mu \text{ and } \mathrm{F}_{\mathfrak{p}}^{\times} \cdot \mu^{-1} = \mu^{-1}.$$

Note that

$$1 + \mu = \{x \in \mathrm{F}_{\mathfrak{p}}^{\times} : \mathrm{Im}_{\mathrm{F}} x = 1\}$$

and is a subgroup of $\mathrm{F}_{\mathfrak{p}}^{\times}$.

Let

$$\ln(1+\mu) = \{u \in \mathbb{U} : \exp_{\mathfrak{p}}(u) \in 1 + \mu\}.$$

This is an additive subgroup of $\mathbb{U}$. Set

$$\mathrm{Im}_{\mathbb{U}}\, v := 0 \text{ for all } v \in \ln(1+\mu),$$

which implies $\exp(\mathrm{Im}_{\mathbb{U}} v) = 1 = \mathrm{Im}_{\mathrm{F}}(\exp_{\mathfrak{p}}(v))$. Thus (19) holds for all $u \in \mathbb{U}(\mathfrak{l}) + \ln(1+\mu)$.

For $w \in \ln(\mu^{-1}) := \{u \in \mathbb{U} : \exp_{\mathfrak{p}}(u) \in \mu^{-1}\}$, set $\mathrm{Im}_{\mathbb{U}}\, w := \infty$ hence $\exp(\mathrm{Im}_{\mathbb{U}} w) = \infty = \mathrm{Im}_{\mathrm{F}}(\exp_{\mathfrak{p}}(w))$.

Claim.

$$\mathbb{U} \setminus \ln(\mu^{-1}) = \mathbb{U}(\mathfrak{l}) + \ln(1+\mu).$$

Indeed, $u \in \mathbb{U} \setminus \ln(\mu^{-1}) \Leftrightarrow \exp_{\mathfrak{p}}(u) = x \in \mathrm{F}_{\mathfrak{p}}^{\times} \setminus \mu^{-1}$. On the other hand $\exp_{\mathfrak{p}}(\mathbb{U}(\mathfrak{l})) \subseteq \mathrm{F}_{\mathfrak{p}}^{\times} \setminus \mu^{-1}$ and $\mathrm{Im}_{\mathrm{F}}(\exp_{\mathfrak{p}}(\mathbb{U}(\mathfrak{l}))) = \mathbb{C} = \mathrm{Im}_{\mathrm{F}}(\mathrm{F}_{\mathfrak{p}} \setminus \mu^{-1})$, which implies that there is $v \in \mathbb{U}(\mathfrak{l})$ such that $\mathrm{Im}_{\mathrm{F}}(\exp_{\mathfrak{p}} v) = \mathrm{Im}_{\mathrm{F}}\, x$ that is $x^{-1} \cdot \exp_{\mathfrak{p}}(v) \in (1+\mu)$. Equivalently, $\exp_{\mathfrak{p}}(v-u) \in (1+\mu)$, $v - u \in \ln(1+\mu)$. Thus $u \in \mathbb{U}(\mathfrak{l}) + \ln(1+\mu)$, which proves the claim and implies (19) for all $u \in \mathbb{U}$. □

4.12 Order, distance and continuity. Recall that by definition of F and (14) we may assume

$$\mathrm{F} \subset {}^*\mathbb{C}. \tag{20}$$

This allows us from now on to consider the inequality $\leq$ on the reals of F, the restriction of the internally definable relation $\leq$ on the reals of ${}^*\mathbb{C}$.

More generally, suppose $X \subset \mathbb{U}^{\mathrm{M}}$ is a definable set with the structure of a ${}^*\mathbb{Q}_+$-valued non-standrad length-metric, namely there are internally definable ternary predicates on ${}^*\mathbb{Q}_+ \times X^2$, written as $d_q(x, y)$, which are interpreted as "the distance between x and y is $\leq q$". Since X is pseudo-finite the distance $\mathrm{dist}(x, y)$, equal to the minimum length of the paths between the two points, is a well-defined value in ${}^*\mathbb{Q}_+$.

A map $g : X \to \mathrm{F}$ will be called (Lipschitz) pseudo-continuous (with derivative bounded by c) if there exists positive $c \in \mathbb{Q}$ such that for any $x_1, x_2 \in X$,

$$d_{\frac{1}{\mathfrak{l}}}(x_1, x_2) \to |g(x_1) - g(x_2)| \leq \frac{c}{\mathfrak{l}}.$$

Lemma 4.13 *Let*

$$g : X \to F$$

be pseudo-continuous with derivative bounded by c. Then for all $M \in O(\mathcal{F})$ *for all* $z_1, z_2 \in X$

$$d_{\frac{M}{\mathfrak{l}}}(z_1, z_2) \rightarrow |g(z_1) - g(z_2)| \leq c\frac{M}{\mathfrak{l}}.$$

Proof Immediate from definition by induction on M. □

5 States and the Hilbert space

5.1 We will assume that $\mathbb{V} \subseteq \mathbb{U}$ generated by a unit $\mathfrak{u}$ introduced in 2.2 is a group with some family $\Omega_{\mathbb{V}}$ of internally definable relations on it. In particular, $\Omega_{\mathbb{V}}$ contains the predicates for the (non-standard) $^*\mathbb{Q}$-metric structure on $\mathbb{V}$: this is a family of binary predicates $d_q(x, y)$, $0 < q \in {}^*\mathbb{Q}$, with intended interpretation "the distance between x and y in units $\mathfrak{u}$ is $\leq q$".

We denote

$$\mathcal{N} := \# \exp_{\mathfrak{p}}(\mathbb{V}),$$

the (non-standard) number of elements in the group $\exp_{\mathfrak{p}}(\mathbb{V}) \subseteq \mathrm{F}_{\mathfrak{p}}^{\times}$.

We say that $\mathbb{V}$ is tame if there is an internally definable embedding

$$\mathbb{V} \hookrightarrow {}^*\mathbb{Z}[0 : \mathfrak{l}^m].$$

It follows that

$$\mathcal{N} \leq \mathfrak{l}^m$$

for tame $\mathbb{V}$.

5.2 The main part of $\mathbb{V}$ is defined to be

$$\mathbb{V}^0 := {}^*\mathbb{Z}\Big[-\frac{\mathcal{N}}{2} : \frac{\mathcal{N}}{2}\Big) \cdot \mathfrak{u} = \Big\{r \in \mathbb{V} : \mathrm{dist}(0, r) \leq \frac{\mathcal{N}}{2},\ r \neq \frac{\mathcal{N}}{2}\Big\}$$

Recall the natural bijective correspondence

$$^*\mathbb{Z}\Big[-\frac{\mathcal{N}}{2} : \frac{\mathcal{N}}{2}\Big) \cong \mathrm{K}_{\mathcal{N}}$$

and the decomposition

$$\mathbb{V} = \bigcup_{u \in \mathbb{V}^0} (u + \mathbb{V} \cap \ker \exp_{\mathfrak{p}})$$

due to the fact that $\mathbb{V}^0$ is a well-defined set of representatives of $\mathbb{V}/\mathbb{V} \cap \ker \exp_{\mathfrak{p}}$.

Commentary. It is the special structure of the lattice which, unlike in the continuous setting, allows a definable set of reprsentatives for $\mathbb{V}/\mathbb{V} \cap \ker \exp_{\mathfrak{p}}$ and motivates the introduction of $\mathbb{V}^0$.

An important reason for the use of $\mathbb{V}^0$, rather than working with $\mathbb{V}/\mathbb{V}\cap\ker \exp_{\mathfrak{p}} \hookrightarrow \mathrm{F}_{\mathfrak{p}}$, is that while it is natural to work with unitary operators such as $e^{i\mathrm{P}}$ and $e^{i\mathrm{Q}}$ on the space of states (see 5.10 and 7.1 below), it is essential for physics to work with self-adjoint operators P and Q (momentum and position operators), which requires to invert the exponential map. (The way P and Q emerge from $e^{i\mathrm{P}}$ and $e^{i\mathrm{Q}}$ in the context of structural approximation is explained in [9], subsection 8.7).

5.3 A state on $\mathbb{V}$ is an internally definable $\mathcal{N}$-periodic map

$$\varphi : \mathbb{V} \to \mathrm{F},\ \varphi(r + \mathcal{N} \cdot \mathfrak{u}) = \varphi(r)$$

A ket-state on $\mathbb{V}$ is a pseudo-continuous state on $\mathbb{V}$.

Set $\mathcal{H}_{\mathbb{V}}^{-}(\mathrm{F})$ to be the multisorted structure each sort being a definable set of states on $\mathbb{V}$. Thus the union of all sorts of $\mathcal{H}_{\mathbb{V}}^{-}(\mathrm{F})$ is the set of all states on $\mathbb{V}$.

5.4 Define now a special kind of states, the position states

$$\mathbf{u}[r] :\ \mathbb{V} \to \{0, 1\} \subset {}'\mathbb{S}';\quad r \in \mathbb{V}^0$$

$$\mathbf{u}[r](x) = \delta(r - x) := \begin{cases} 1, & \text{if } r = x \quad \text{mod } \mathcal{N} \\ 0, & \text{otherwise} \end{cases}$$

Define, for a ket-state φ the inner product with $\mathbf{u}[r]$

$$\langle \varphi | \mathbf{u}[r] \rangle := \varphi(r) \tag{21}$$

and also

$$\langle \mathbf{u}[x] | \mathbf{u}[r] \rangle := \delta(r - x) \tag{22}$$

It is immediate from the definition that

Lemma 5.5 *Assume $\mathbb{V}$ is tame. Then $\mathcal{H}_{\mathbb{V}}^{-}$ can be given the structure of F-linear space with an inner product defined as*

$$\langle \varphi | \psi \rangle := \sum_{r \in \mathbb{V}^0} \varphi(r) \cdot \bar{\psi}(r) \in F \tag{23}$$

The definition is consistent with φ or/and ψ being position states.

The product is Hermitian, that is satisfies the sesquilinearity condition and is positive definite.

Proof By assumptions $r \mapsto \varphi(r) \cdot \bar{\psi}(r)$ can be identified as an internally definable sequence from a subset of ${}^*\mathbb{Z}[0 : \mathfrak{l}^m]$. Thus the sum (23) is well-defined and belongs to F. The rest is immediate by definitions. □

Define the square of the norm

$$|\varphi|^2 := \langle \varphi | \varphi \rangle \in \mathbb{R}_+(\mathrm{F}).$$

From the logic point of view the intended meaning of the inner product, in the "logic with values in F", is to estimate the equality "$\varphi = \psi$" for φ, ψ of norm 1. Indeed, "$\psi = \psi$" is given value 1.

Set the support of φ to be:

$$\mathrm{Supp}(\varphi) := \{r \in \mathbb{V}^0 : \langle \varphi | \mathbf{u}[r] \rangle \neq 0\}$$

Lemma 5.6 *Suppose $|\varphi(r)| \leq \eta$ for all $r \in \mathbb{V}$. Then*

$$|\varphi|^2 := \langle \varphi | \varphi \rangle \leq \eta^2 \cdot \# Supp(\varphi) \tag{24}$$

Proof By definition

$$\langle \varphi | \varphi \rangle = \sum_{r \in \mathbb{V}^0} |\varphi(r)|^2$$

and since $\mathbb{V}$ is tame and the sum is internally definable in ${}^*\mathbb{C}$ we can lift the required inequality from the finite to pseudo-finite summation. ◻

5.7 Let $\{\psi_i : s \in I\}$ be a definable family of states on $\mathbb{V}$ over a definable set I. Then the sum

$$S = \sum_{i \in I} \psi_i$$

is definable, the value S of the sum is an element of the ultraproduct $\prod_{p \in I} \mathrm{F}_p / D$ such that $S(p) = \sum_{i \in I(p)} \psi_i$ (finite sum) in F_p along the ultrafilter.

It follows from definitions that the set of all position states of $\mathbb{V}$,

$$\mathcal{L}_{\mathbf{u}} := \{\mathbf{u}[r] : r \in \mathbb{V}^0\}$$

is definable.

Define $\mathcal{H}_{\mathbb{V}}(\mathrm{F})$ to be the smallest F-linear subspace of $\mathcal{H}_{\mathbb{V}}^{-}(\mathrm{F})$ containing $\mathcal{L}_{\mathbf{u}}$ and closed under taking definable sums.

Lemma 5.8 *The set of position states forms a basis of $\mathcal{H}_{\mathbb{V}}$ with regards to definable summation. This basis is orthonormal.*

$$\dim \mathcal{H}_{\mathbb{V}} = \mathcal{N}$$

Proof The orthonormality of position states is by the definition (22). Recall that an arbitrary state $\psi \in \mathcal{H}_{\mathbb{V}}$ is an $\mathcal{N}$-periodic map

$$r \mapsto \psi(r);\ \ \mathbb{V} \to \mathrm{F}$$

which can equivalently be written as

$$\psi = \sum_{r \in \mathbb{V}^0} \psi(r) \cdot \mathbf{u}[r].$$

and

$$\psi(r) = \langle \psi | \mathbf{u}[r] \rangle.$$

Which presents ψ as a definable F-linear combination of position states. □

Corollary 5.9 *The definition of inner product in (23) is applicable to any pair of states in $\mathcal{H}_{\mathbb{V}}(F)$ with tame $\mathbb{V}$.*

From now on we consider $\mathcal{H}_{\mathbb{V}} := \mathcal{H}_{\mathbb{V}}(\mathrm{F})$ as an $\mathcal{N}$-dimensional inner vector space.

5.10 Linear unitary operators on $\mathcal{H}_{\mathbb{V}}$.
We are going to consider linear operators

$$\mathcal{A} : \mathcal{H}_{\mathbb{V}_1} \to \mathcal{H}_{\mathbb{V}_2}$$

for $\mathbb{V}_1, \mathbb{V}_2 \subset \mathbb{U}$ definable domains.
Call such an operator definable if

$$\{r \mapsto \mathcal{A}\mathbf{u}[r] : \ r \in \mathbb{V}_1^0\}$$

is a definable family of states on $\mathbb{V}_2$.
By definition (which agrees with (3) because of 5.2)

$$\mathcal{A}\mathbf{u}[r] = \sum_{s \in \mathbb{V}_2^0} a(r, s) \cdot \mathbf{u}[s] \tag{25}$$

with unitarity condition, for all $r_1, r_2 \in \mathbb{V}_2^0$

$$\sum_{s \in \mathbb{V}_2^0} a(r_1, s) \cdot \bar{a}(r_2, s) = \delta(r_1 - r_2) \tag{26}$$

We say that $\mathcal{A}$ is pseudo-continuous if the map

$$s \mapsto a(r, s); \quad \mathbb{V}_2 \to \mathrm{F}$$

be pseudo-continuous.

Proposition 5.11 *Let $\mathcal{L}_\psi := \{\psi_r : r \in \mathbb{V}_1^0\}$ be a definable basis of $\mathcal{H}_{\mathbb{V}_2}$. Then*
- there is a definable operator $\mathcal{A}$ such that

$$\mathcal{A}\boldsymbol{u}[r] = \psi_r; \ r \in \mathbb{V}_1^0$$

- for any definable operator B on $\mathcal{H}_{\mathbb{V}_2}$ the set

$$\mathcal{B}\mathcal{L}_\psi := \{\mathcal{B}\psi_r : \ r \in \mathbb{V}_1^0\}$$

is definable.
Assuming that $\mathcal{A}$ and $\mathcal{B}$ are unitary, the product operator $\mathcal{B} \cdot \mathcal{A}$ is unitary.

Proof The statement can be equivalently reformulated as a property of finite dimensional vector spaces $\mathcal{H}(p),\ p \in P$, along the ultrafilter D. In this form it is obvious and

$$\mathcal{B}\,\psi_s = (\mathcal{B}\cdot\mathcal{A})\,\mathbf{u}[s]$$

the product of matrices. □

We only consider definable linear operators on $\mathcal{H}_{\mathbb{V}}$. In applications (as in 7.1) we do not assume that the operators are invertible but most of the time we deal with operators such as the image of $\mathcal{L}_{\mathbf{u}}$ which is of size $\frac{\mathcal{N}}{k}$ for some finite k. Moreover we have to deal with operators whose domain is a proper subset of $\mathcal{L}_{\mathbf{u}}$ of size $\frac{\mathcal{N}}{k}$ for some finite k.

Remark 5.12 1. Looking at states ψ_r and definable set of states $\mathcal{L}_\psi := \{\psi_r : r \in \mathbb{V}^0\}$ as structures, an operator

$$\mathcal{A} : \mathbf{u}[r] \mapsto \psi_r$$

can be seen as an interpretation of $\mathcal{L}_\psi$ in $\mathcal{L}_{\mathbf{u}}$.

Sums of states $\varphi + \psi$ are just objects interpretable in $\mathcal{L}_\varphi \dot{\cup} \mathcal{L}_\psi$ (with the intended logical meaning "φ or ψ").

2. In regards to model-theoretic formalism, $\mathcal{H}_{\mathbb{V}}$ is not considered to be a universe of a structure. Instead, we consider the multisorted structure on sorts $\mathcal{L}_\psi$ with linear maps between these, together with the $\mathrm{F}_\mathfrak{p}$-linear space $\mathcal{H}_{\mathbb{V}}$ interpretable in the sorts.

3. The multisorted structure $\mathcal{H}_{\mathbb{V}}$ is by construction interpretable in $(\mathbb{U}, \mathrm{F}_\mathfrak{p}; \Omega_\mathfrak{p})$.

Proposition 5.13 *(a) A bijective transformation $\sigma : \mathbb{V} \to \mathbb{V}$ induces the linear unitary transformation of $\mathcal{H}_{\mathbb{V}}$:*

$$U_\sigma : \psi \mapsto \psi^\sigma;\quad \psi^\sigma(r) = \psi(\sigma(r))$$

(b) Let

$$G(\mathbb{V}) := Aut(\mathbb{V}) \text{ and } G(\mathcal{H}_{\mathbb{V}}) := \{U_\sigma : \sigma \in G(\mathbb{V})\}$$

Then $G(\mathcal{H}_{\mathbb{V}}) \subseteq SU_{\mathcal{N}}(F) \subset GL_{\mathcal{N}}(F)$ is the unitary linear group and

$$\sigma \mapsto U_\sigma$$

is an injection.

In other words, the structure on $\mathbb{V}$ is reflected in the algebra of linear operators on $\mathcal{H}_{\mathbb{V}}$.

Proof (a) Linearity: for $a_1, a_2 \in \mathrm{F},\ \psi_1, \psi_2 \in \mathcal{H}_{\mathbb{V}}$,

$$(a_1\cdot\psi_1 + a_2\cdot\psi_2)^\sigma(r) = (a_1\cdot\psi_1 + a_2\cdot\psi_2)(\sigma(r)) = (a_1\cdot\psi_1^\sigma(r) + a_2\cdot\psi_2^\sigma(r).$$

Unitarity: it is enough to prove it for basis $\mathbf{u}[r] : r \in \mathbb{V}^0$.

$$\langle\mathbf{u}[r]|\mathbf{u}[s]\rangle = \delta_{r,s} = \delta_{\sigma(r),\sigma(s)} = \langle\mathbf{u}^\sigma[r]|\mathbf{u}^\sigma[s]\rangle.$$

(b) It is clear that $\sigma \mapsto U_\sigma$ is a homomorphism. Suppose that σ is in the kernel. Then $\mathbf{u}^\sigma[r] = \mathbf{u}[r]$, which implies by definition of $\mathbf{u}$ that $\sigma(r) = r$, for all $r \in \mathbb{V}^0$, that is $\sigma = \mathrm{id}_{\mathbb{V}}$. □

5.14 The dual to $\mathcal{H}_{\mathbb{V}}$ and model-theoretic $\mathcal{H}_{\mathbb{V}}^{\mathrm{eq}}$.
Every $\psi \in \mathcal{H}_{\mathbb{V}}$ gives rise to an F-linear map

$$L_\psi : \mathbf{x} \mapsto \langle \mathbf{x} | \psi \rangle, \quad \mathcal{H}_{\mathbb{V}} \to \mathrm{F}$$

By definition, L_ψ is uniquely determined by its values on a basis, that is

$$L_\psi^{\mathbf{u}} : \mathbf{u}[r] \mapsto \langle \mathbf{u}[r] | \psi \rangle, \quad r \in \mathbb{V}^0$$

determines the linear map L_ψ. Clearly, $L_\psi^{\mathbf{u}}$ is definable and thus we can treat L_ψ (otherwise given by the Dirac delta-function) as a definable, or interpretable in $\mathcal{H}_{\mathbb{V}}$.

Set $\mathcal{H}_{\mathbb{V}}^*$ to be the F-vector space of interpretable F-linear maps with a naturally induced Hermitian inner product structure. By the definition of $\mathcal{H}_{\mathbb{V}}^{\mathrm{eq}}$ as the structure containing all interpretable objects

$$\mathcal{H}_{\mathbb{V}}^* \subset \mathcal{H}_{\mathbb{V}}^{\mathrm{eq}}.$$

6 Gaussian Hilbert space

6.1 Here we consider states over a domain $\mathbb{V}$ which is not assumed to be tame. Until the last subsection of this section our domain is “one-dimensional” that is $\mathbb{V} \subseteq \mathbb{U}$. In the last subsection we allow domains $\mathbb{V}^{\mathrm{M}} \subseteq \mathbb{U}^{\mathrm{M}}$ for $\mathrm{M} \in \mathrm{O}(\mathcal{F})$ and then the respective Hilbert space is the M-th tensor power of the Hilbert space for the present case.

We require, as in 5.2, $\mathbb{V}$ to be of the form

$$\mathbb{V} = \mathbb{V}^0 + \ker \exp_{\mathfrak{p}}; \quad \mathbb{V}^0 = \mathfrak{u} \cdot {}^*\mathbb{Z}\left[-\frac{\mathcal{N}}{2} : \frac{\mathcal{N}}{2} \right),$$

for some $\mathfrak{u} \in \mathbb{U}$ and $\mathcal{N} \in {}^*\mathbb{Z}$ such that $\mathcal{N} | (\mathfrak{p} - 1)$, $\sqrt{\mathcal{N}} \in {}^*\mathbb{Z}$ and $\mathfrak{m} | \sqrt{\mathcal{N}}$.

Let

$$\mathrm{e}\left(\frac{n}{2\mathcal{N}} \right) := \exp_{\mathfrak{p}} \left((\mathfrak{p} - 1) \frac{n}{2\mathcal{N}} \right).$$

be roots of unity of order $2\mathcal{N}$.

We consider quadratic forms $f(x, y) = ax^2 + 2bxy + cy^2$, $a, b, c \in \mathbb{Z}$, where x, y run in ${}^*\mathbb{Z}$. It is easy to check that

$$\mathrm{e}\left(\frac{f(x, y)}{2\mathcal{N}} \right) = \mathrm{e}\left(\frac{f(x', y')}{2\mathcal{N}} \right) \text{ if } x = x' \;\&\; y = y' \mod \mathcal{N}.$$

Therefore, we may assume that x, y run in the ring $\mathrm{K}_{\mathcal{N}}$ often represented by ${}^*\mathbb{Z}[-\frac{\mathcal{N}}{2} : \frac{\mathcal{N}}{2})$.

(The assumption that f is over $\mathbb{Z}$ rather than over $\mathbb{Q}$ is not restrictive because we can deal with $\mathrm{e}\left(\frac{f(x,y)}{k\mathcal{N}}\right)$ by switching from $\mathcal{N}$ to $k\mathcal{N}$ and so assuming x, y run in $\mathrm{K}_{k\mathcal{N}}$.)

Denote

$$\mathrm{e}\left(\frac{f(x,y)}{2\mathcal{N}}\right) := \exp_{\mathfrak{p}}\left(\frac{\mathfrak{p}-1}{2\mathcal{N}} f(x,y)\right) \in \mathcal{G}(\mathcal{N}), \text{ for } x\mathfrak{u}, y\mathfrak{u} \in \mathbb{V}.$$

The following set of Gaussian coefficients play an important role:

$$\mathrm{O}\mathcal{G}(\mathcal{N}) := \left\{\frac{c}{\sqrt{\mathcal{N}}} \cdot \mathrm{e}\left(\frac{n}{2\mathcal{N}}\right) : n\mathfrak{u} \in \mathbb{V}, \ kc \in \mathrm{O}(\mathcal{F}), \text{ for } k \in \mathbb{Z}_{>0}\right\}$$

6.2 Gauss quadratic sums in F_p. Let $M \in \mathbb{N}$, even, $p > 2$ and $\zeta := \mathrm{e}^{\frac{\pi i}{M}} \in \mathbb{C}$, primitive root of unity of order $2M$. The classical Gauss quadratic sums formula (the basic form) is

$$\sum_{0<n\leq M} \zeta^{n^2} = \sqrt{M}\,\mathrm{e}^{\frac{\pi i}{4}}.$$

Suppose in addition M is divisible by 4. Then

$$\zeta^{\frac{M}{4}} = \mathrm{e}^{\frac{\pi i}{4}} \text{ and } \sum_{0<n\leq M} \zeta^{n^2} = \sqrt{M}\,\zeta^{\frac{M}{4}}$$

Let $\mathbb{Z}[\zeta] \subset \mathbb{C}$ be the ring generated by ζ. By the above $\sqrt{M} \in \mathbb{Z}[\zeta]$. Let $\mathrm{F}_p := \mathbb{Z}/p$ and let $\xi \in \mathrm{F}_p$ be a primitive $2\,M$-root of 1 in $\mathrm{F}_{\mathfrak{p}}$, that is

$$\xi^{2M} = \hat{1} := 1_{\mathrm{mod}\,p}, \ \mathrm{ord}\,\xi = 2M$$

Then there is a ring homomorphism

$$h : \mathbb{Z}[\zeta] \to \mathrm{F}_p \text{ such that } \zeta \mapsto \xi$$

Clearly,

$$h : 1 \mapsto \hat{1}, \ M \mapsto \hat{M}, \ \sqrt{M} \mapsto \sqrt{\hat{M}}$$

and

$$\sum_{0<n\leq M} \xi^{n^2} = \sqrt{\hat{M}} \cdot \mathrm{e}\left(\frac{1}{8}\right), \text{ where } \mathrm{e}\left(\frac{1}{8}\right) := \xi^{\frac{M}{4}}. \tag{27}$$

(Formula (27) was proved in [7],4.1 by direct calculations in case M is an exact square, for all characteristics).

Simple algebraic manipulation produce from (27) a more general version of Gauss quadratic sums both for characteristic zero and characteristic p, for $a, b \in \mathbb{Z}$, assuming

$\pm a > 0$ and $4a|M$,

$$\sum_{0<n\leq \frac{M}{|a|}} \mathrm{e}\left(\frac{an^2+2bn}{2M}\right) = \begin{cases} \sqrt{\frac{\hat{M}}{|a|}}\mathrm{e}\left(\pm\frac{1}{8}\right)\mathrm{e}\left(-\frac{b^2}{2aM}\right) \text{ if } a|b \\ 0, \text{ otherwise} \end{cases} \tag{28}$$

Note that the function $\mathrm{e}\left(\frac{an^2+2bn}{2M}\right)$ of n has period $\frac{M}{|a|}$ and thus we can assume n runs in $\mathrm{K}_{\frac{M}{a}}$.

6.3 Gaussian summation over $\mathrm{K}_{\mathcal{N}}$.

In the context of states on $\mathbb{V}$ and notations (3) we can write (28) equivalently, for $\pm A > 0$,

$$\sum_{n\in \mathrm{K}_{\frac{\mathcal{N}}{A}}} \mathrm{e}\left(\frac{An^2+2Bn}{2\mathcal{N}}\right) = \begin{cases} \mathrm{e}\left(\pm\frac{1}{8}\right)\sqrt{\frac{\mathcal{N}}{|A|}}\cdot \mathrm{e}\left(-\frac{B^2}{2A\mathcal{N}}\right), \text{ if } A|B \\ 0, \text{ otherwise} \end{cases} \tag{29}$$

which can be interpreted as a calculations over $\mathbb{V}_{\mathrm{u}}$ in units of scales $A\mathrm{u}$, replacing $\mathbb{V}_{\mathrm{u}}$ by $A\mathbb{V}_{\mathrm{u}} \subseteq \mathbb{V}_{\mathrm{u}}$.

We say that a quadratic form over $\mathbb{Z}$, $f(u,v) = Au^2+2Buv+Cv^2$, is admissible if $A, C \leq 0$

Consider

$$f_1(x,y) = A_1x^2+2B_1xy+C_1y^2 \text{ and } f_2(x,y) = A_2x^2+2B_2xy+C_2y^2,$$

quadratic forms over $\mathbb{Z}$. Assume $A = A_1 + A_2 \neq 0$.

We have:

$$\begin{aligned} &\sum_{nA\mathrm{u}\in\mathbb{V}^0} \mathrm{e}\left(\frac{f_1(n,p_1)}{2\mathcal{N}}\right)\cdot \mathrm{e}\left(\frac{f_2(n,p_2)}{2\mathcal{N}}\right) \\ &= \mathrm{e}\left(\frac{C_1p_1^2+C_2p_2^2}{2\mathcal{N}}\right)\cdot \sum_{nA\mathrm{u}\in\mathbb{V}^0} \mathrm{e}\left(\frac{(A_1+A_2)n^2+2(B_1p_1+B_2p_2)n}{2\mathcal{N}}\right) \\ &= \mathrm{e}\left(\frac{C}{2\mathcal{N}}\right) \sum_{nA\mathrm{u}\in\mathbb{V}^0} \mathrm{e}\left(\frac{An^2+2Bn}{2\mathcal{N}}\right) \end{aligned}$$

where $B = B_1p_1 + B_2p_2$ and $C = C_1p_1^2 + C_2p_2^2$.

This can be rewritten with normalising coefficients as

$$\sum_{nA\mathrm{u}\in\mathbb{V}^0} \frac{1}{\sqrt{\mathcal{N}}}\mathrm{e}\left(\frac{f_1(n,p_1)}{2\mathcal{N}}\right)\cdot \frac{1}{\sqrt{\mathcal{N}}}\mathrm{e}\left(\frac{f_2(n,p_2)}{2\mathcal{N}}\right) = \frac{c}{\sqrt{\mathcal{N}}}\cdot \mathrm{e}\left(\frac{C-\frac{B^2}{A}}{2\mathcal{N}}\right) \in \mathrm{O}\mathcal{G}(\mathcal{N}) \tag{30}$$

where

$$c = \begin{cases} \mathrm{e}\Big(\pm\frac{1}{8}\Big)\sqrt{\frac{1}{|A|}}, \text{ if } A|B \\ 0, \text{ otherwise} \end{cases}$$

6.4 Gaussian-ket states on $\mathbb{V}$ are definable sequences of elements of $\mathrm{F}_\mathfrak{p}$ of the form

$$\mathbf{s}_f[p] := \left\{\mathbf{c_s}\cdot\mathrm{e}\left(\frac{f(r,p)}{2\mathcal{N}}\right) : r\mathrm{u}\in\mathbb{V}^0\right\}, \text{ where } p\mathrm{u}\in\mathbb{V}^0,$$

where $f(r,p)$ is an admissible quadratic form over $\mathbb{Z}$ and $c_\mathbf{s}\in\mathrm{O}\mathcal{G}(\mathcal{N})$.

Equivalently these can be written as symbolic expressions of the form

$$\mathbf{s}_f[p] := \mathbf{c_s}\sum_{r\mathrm{u}\in\mathbb{V}^0}\mathrm{e}\left(\frac{f(r,p)}{2\mathcal{N}}\right)\mathbf{u}[r], \quad p\mathrm{u}\in\mathbb{V}$$

A Gaussian ket-sort is

$$\mathcal{L}_f := \{\mathbf{s}_f[p] : p\mathrm{u}\in\mathbb{V}^0\}$$

where $\mathbf{s}_f$ is as above.

We also consider the positions sort

$$\mathcal{L}_\mathbf{u} := \{\mathbf{u}[r] : r\mathrm{u}\in\mathbb{V}^0\}$$

Note that the definition of ket-states allows multiplication by elements of F. We consider $\mathcal{H}_\mathbb{V}$ as the F-linear space finitely generated by elements of Gaussian ket-sorts $\mathcal{L}_f$ and position sort $\mathcal{L}_\mathbf{u}$.

6.5 Now we consider two possible formal inner product with values in $\mathrm{F}_\mathfrak{p}$ beween Gaussian ket-states are defined using the fact that the states are defined in terms of the orthonormal basis $\mathcal{L}_\mathbf{u}$ with

$$\langle\mathbf{u}[r]|\mathbf{u}[s]\rangle = \delta(r-s).$$

Let $f_1(x,y)$ and $f_2(x,y)$ be quadratic forms as defined in 6.3.

formal-Euclidean inner product of ket-states is defined as

$$\langle\mathbf{s}_{f_1}[p_1]\,|\,\mathbf{s}_{f_2}[p_2]\rangle_\mathrm{E} = \mathbf{c}_1\mathbf{c}_2\cdot\sum_{r\mathrm{u}\in\mathbb{V}^0}\mathrm{e}\left(\frac{f_1(r,p_1)}{2\mathcal{N}}\right)\cdot\mathrm{e}\left(\frac{f_2(r,p_2)}{2\mathcal{N}}\right)$$

formal-Hermitian inner product of ket-states is defined as

$$\langle\mathbf{s}_{f_1}[p_1]\,|\,\mathbf{s}_{f_2}[p_2]\rangle_\mathrm{E} = \mathbf{c}_1\bar{\mathbf{c}}_2\cdot\sum_{r\mathrm{u}\in\mathbb{V}^0}\mathrm{e}\left(\frac{f_1(r,p_1)}{2\mathcal{N}}\right)\cdot\mathrm{e}\left(-\frac{f_2(r,p_2)}{2\mathcal{N}}\right)$$

The Euclidean and Hermitian inner products between a ket-state and a position state are both defined as

$$\langle \mathbf{s}_f[p] \,|\, \mathbf{u}[r] \rangle := \mathbf{c_s} \cdot \mathrm{e}\left(\frac{f(r,p)}{2\mathcal{N}} \right)$$

Lemma 6.6

$$\langle \boldsymbol{s}_{f_1}[p_1] \,|\, \boldsymbol{s}_{f_2}[p_2] \rangle \in O\mathcal{G}(\mathcal{N})$$
$$\langle \boldsymbol{s}_f[p] \,|\, \boldsymbol{u}[r] \rangle \in O\mathcal{G}(\mathcal{N})$$

both for Hermitian and Euclidean inner product.

Proof Follows directly from (30) and calculations above. □

Lemma 6.7 *Let $A := A_1 + A_2$ and $B = B_1 + B_2$ in the Euclidean case and let $A := A_1 - A_2$ and $B = B_1 - B_2$ in the Hermitian case. Suppose $A = 0$. Then*

$$\langle \boldsymbol{s}_{f_1}[p_1] \,|\, \boldsymbol{s}_{f_2}[p_2] \rangle_E = \begin{cases} 0, & \textit{if } B \neq 0 \\ \boldsymbol{c}_1 \boldsymbol{c}_2, & \textit{if } B = 0 \end{cases}$$

and

$$\langle \boldsymbol{s}_{f_1}[p_1] \,|\, \boldsymbol{s}_{f_2}[p_2] \rangle_H = \begin{cases} 0, & \textit{if } B \neq 0 \\ \boldsymbol{c}_1 \bar{\boldsymbol{c}}_2, & \textit{if } B = 0 \end{cases}$$

Proof In both cases the calculation is easily reducible to calculating the sum

$$\frac{1}{\mathcal{N}} \sum_{0 \le n < \mathcal{N}} \mathrm{e}\left(\frac{B}{\mathcal{N}} \right) = \begin{cases} 0, & \text{if } B \neq 0 \mod \mathcal{N} \\ 1, & \text{if } B = 0 \mod \mathcal{N} \end{cases}$$

The fact that $B << \mathcal{N}$ finishes the proof. □

6.8 Consider a linear unitary operator $\mathcal{A}$ which is defined on $\mathcal{L}_{\mathbf{u}}$ as

$$\mathcal{A} : \mathbf{u}[q] \mapsto \frac{1}{\sqrt{\mathcal{N}}} \sum_{r\mathrm{u} \in \mathbb{V}^0} \mathrm{e}\left(\frac{a(q,r)}{2\mathcal{N}} \right) \mathbf{u}[r]$$

$a(u, v)$ a quadratic form over $\mathbb{Z}$ and $\mathbf{c}_g \in \mathrm{O}\mathcal{G}(\mathcal{N})$. It acts on $\mathcal{L}_f$ by pseudo-finite linearity

$$\mathcal{A} : \mathbf{s}_f[p] \mapsto \frac{\mathbf{c_s}}{\sqrt{\mathcal{N}}} \sum_{q\mathrm{u} \in \mathbb{V}^0} \sum_{r\mathrm{u} \in \mathbb{V}^0} \mathrm{e}\left(\frac{f(q,p)}{2\mathcal{N}} \right) \cdot \mathrm{e}\left(\frac{a(q,r)}{2\mathcal{N}} \right) \mathbf{u}[r] \tag{31}$$

Remark. In general the image $\mathcal{A}\mathbf{s}_f[p]$ may fail to be pseudo-continuous on $\mathbb{V}$. Namely, the r-coordinate of $\mathcal{A}\mathbf{s}_f[p]$

$$\mathcal{A}\mathbf{s}_f[p](r) = \frac{\mathbf{c_s}}{\sqrt{\mathcal{N}}} \sum_{q\mathrm{u} \in \mathbb{V}^0} \mathrm{e}\left(\frac{f(q,p)}{2\mathcal{N}} \right) \cdot \mathrm{e}\left(\frac{a(q,r)}{2\mathcal{N}} \right)$$

can happen to be zero outside $k\mathbb{V}+d$, for some $k=k(\mathcal{A},f), d=d(\mathcal{A},f,r)\in\mathbb{Z}$ as investigated in [9]. These k and d can be easily calculated from the coefficients for q^2-terms in $a(q,r)$ and $f(p,q)$ by formula (29). The same calculations also prove:

Either $k(\mathcal{A},f)=0$*and* $\mathcal{A}\mathbf{s}_f[p]$*is a finite linear combination of position states, or* $k(\mathcal{A},f)\neq 0$*and* $\mathcal{A}\mathbf{s}_f[p]$ *is pseudo-continuous on* $k\mathbb{V}+d$*for some* $0\leq d<k$.

6.9 A (basic) Euclidean/Hermitian Gaussian Hilbert space $\mathcal{H}_{\mathbb{V}}$ is the F-linear space finitely generated by Gaussian ket-states. and position states. The inner product is defined as the formal Euclidean, respectively, Hermitian inner product between ket-states and ket-states and position states and extends uniquely to their linear combination by bi-linearity law.

A general Euclidean/Hermitian Gaussian Hilbert space is a tensor power $\mathcal{H}_{\mathbb{V}}^{\otimes N}$ of the basic Gaussian Hilbert space.

6.10 Corrsepondence of structures over $\mathfrak{u}$-and $\mathfrak{v}$-domains.

Let $\mathfrak{u}:=\frac{\mathfrak{p}-1}{\mathbf{i}\mathfrak{l}}$ and $\mathfrak{v}:=\frac{\mathfrak{p}-1}{\mathfrak{l}}=\mathbf{i}\mathfrak{u}$ as defined in 3.8.

Set

$$\mathbb{V}_{\mathfrak{v}}:=\mathfrak{v}\cdot{}^*\mathbb{Z};\quad \mathbb{V}_{\mathfrak{v}}^0:=\mathfrak{v}\cdot{}^*\mathbb{Z}\Big[-\frac{\mathfrak{l}}{2}:\frac{\mathfrak{l}}{2}\Big)$$

and

$$\mathbb{V}_{\mathfrak{u}}:=\mathfrak{u}\cdot{}^*\mathbb{Z};\quad \mathbb{V}_{\mathfrak{u}}^0=\mathfrak{u}\cdot{}^*\mathbb{Z}\Big[-\frac{\mathfrak{l}\mathbf{i}}{2}:\frac{\mathfrak{l}\mathbf{i}}{2}\Big)$$

Note that

$$\mathbb{V}_{\mathfrak{v}}\subset\mathbb{V}_{\mathfrak{u}}\text{ and }\mathbb{V}_{\mathfrak{v}}=\mathbf{i}\cdot\mathbb{V}_{\mathfrak{u}}. \tag{32}$$

Define

$$\mathcal{N}_{\mathfrak{u}}:=\#\exp_{\mathfrak{p}}(\mathbb{V}_{\mathfrak{u}})=\mathbf{i}\mathfrak{l},\quad \mathcal{N}_{\mathfrak{v}}:=\#\exp_{\mathfrak{p}}(\mathbb{V}_{\mathfrak{v}})=\mathfrak{l}$$

and note that

$$\mathcal{N}_{\mathfrak{u}}=\mathbf{i}\mathcal{N}_{\mathfrak{v}}\text{ and }\mathrm{e}\Big(\frac{\mathbf{i}f(r,p)}{2\mathcal{N}_{\mathfrak{u}}}\Big)=\mathrm{e}\Big(\frac{f(r,p)}{2\mathcal{N}_{\mathfrak{v}}}\Big).$$

The embedding of domains agrees with the correspondence between the Gaussian states

$$\{\}^{\mathbf{i}}:\ \mathbf{s}_f[p]\mapsto\mathbf{s}^{\mathbf{i}}_f[p]$$

where

$$\mathbf{s}_f[p]=\mathbf{c}_{\mathbf{s}}\sum_{r\mathfrak{u}\in\mathbb{V}_{\mathfrak{u}}}\mathrm{e}\Big(\frac{f(r,p)}{2\mathcal{N}_{\mathfrak{u}}}\Big)\mathbf{u}[r]\text{ and }\mathbf{s}^{\mathbf{i}}_f[p]=\mathbf{j}\mathbf{c}_{\mathbf{s}}\sum_{r\mathfrak{v}\in\mathbb{V}_{\mathfrak{v}}}\mathrm{e}\Big(\frac{\mathbf{i}f(r,p)}{2\mathcal{N}_{\mathfrak{u}}}\Big)\mathbf{u}[r]$$

(recall $\mathbf{j}=\sqrt{\mathbf{i}}$).

This can be extended to the F-linear surjective map

$$\{\}^{\mathbf{i}}:\mathcal{H}_{\mathbb{V}_{\mathfrak{u}}}\to\mathcal{H}_{\mathbb{V}_{\mathfrak{v}}}.$$

Consider also the related map

$$\{\}^{\mathbf{i}} : \mathrm{O}\mathcal{G}(\mathcal{N}_{\mathfrak{u}}) \to \mathrm{O}\mathcal{G}(\mathcal{N}_{\mathfrak{v}}); \quad \frac{c}{\sqrt{\mathcal{N}_{\mathfrak{u}}}} \cdot \mathrm{e}\left(\frac{f(r,p)}{2\mathcal{N}_{\mathfrak{u}}}\right) \mapsto \frac{c}{\sqrt{\mathcal{N}_{\mathfrak{v}}}} \cdot \mathrm{e}\left(\frac{f(r,p)}{2\mathcal{N}_{\mathfrak{v}}}\right)$$

Now for formal inner products calculation (30) gives:

$$\langle \mathbf{s}^{\mathbf{i}}_{f_1}[p_1] \,|\, \mathbf{s}^{\mathbf{i}}_{f_2}[p_2] \rangle = \{\langle \mathbf{s}_{f_1}[p_1] \,|\, \mathbf{s}_{f_2}[p_2] \rangle\}^{\mathbf{i}} \tag{33}$$

where the inner product is Euclidean on both sides or Hermitian on both sides.

For a linear operator $\mathcal{A}$ of the form (31) on $\mathcal{H}_{\mathbb{V}_{\mathfrak{u}}}$ define the operator on $\mathcal{H}_{\mathbb{V}_{\mathfrak{v}}}$

$$\mathcal{A}^{\mathbf{i}} : \; \mathbf{s}^{\mathbf{i}}_{f_1}[p] \mapsto \{\mathcal{A}\,\mathbf{s}_{f_1}[p]\}^{\mathbf{i}}$$

Theorem 6.11 *Under assumptions 6.10 the embedding*

$$\mathbb{V}_{\mathfrak{v}} \subset \mathbb{V}_{\mathfrak{u}}$$

gives rise to a canonical surjective homomorphism of Euclidean/Hermitian Gaussian Hilbert spaces equipped with Gaussian linear maps

$$\{\}^{i} : \mathcal{H}_{\mathbb{V}_{\mathfrak{u}}} \to \mathcal{H}_{\mathbb{V}_{\mathfrak{v}}}.$$

The map can be uniquely and definably extended to M-dimensional versions of domains

$$\mathbb{V}_{\mathfrak{v}} \subset \mathbb{V}_{\mathfrak{u}} \subset \mathbb{U}^{M}, \quad M \in O(\mathcal{F})$$

and to M-tensor-power version of Hilbert spaces

$$\{\}^{i} : \mathcal{H}^{\otimes M}_{\mathbb{V}_{\mathfrak{u}}} \to \mathcal{H}^{\otimes M}_{\mathbb{V}_{\mathfrak{v}}}.$$

Proof The construction and the argument for the first statement is in 6.10. The second statement is just consequence of the algebraic property of finite tensor products. □

7 Examples

7.1 Example: 1-dimensional QM.

Set

$$\mathbb{V} := \mathbb{V}_{\mathfrak{v}}$$

as in 6.10.

Respectively we have position states

$$\mathcal{L}_{\mathbf{u}} = \{\mathbf{u}[r] : r\mathfrak{v} \in \mathbb{V}^{0}\}.$$

Define, for $p\mathfrak{v} \in \mathbb{V}^0$ the momentum state

$$\mathbf{v}[p] : r \mapsto \frac{1}{\sqrt{\mathcal{N}}}\mathrm{e}\left(-\frac{rp}{\mathcal{N}}\right)$$

Hence

$$\mathbf{v}[p] := \frac{1}{\sqrt{\mathcal{N}}}\sum_{r\mathfrak{v}\in\mathbb{V}^0}\mathrm{e}\left(-\frac{rp}{\mathcal{N}}\right)\mathbf{u}[r]$$

Clearly,

$$\mathbf{u}[r] = \frac{1}{\sqrt{\mathcal{N}}}\sum_{p\mathfrak{v}\in\mathbb{V}^0}\mathrm{e}\left(\frac{rp}{\mathcal{N}}\right)\mathbf{v}[p]$$

and we consider the definable sort

$$\mathcal{L}_{\mathbf{v}} = \{\mathbf{v}[p] : p\mathfrak{v} \in \mathbb{V}^0\}.$$

Thus $\mathcal{H}_{\mathbb{V}}$ is generated by both the orthogonal systems $\mathcal{L}_{\mathbf{u}}$ and $\mathcal{L}_{\mathbf{v}}$ which are Fourier dual of each other:

One considers the unitary operators U and V that in continuous setting can be written as

$$U = \mathrm{e}^{i\mathrm{Q}} \text{ and } V = \mathrm{e}^{i\mathrm{P}}$$

for the self-adjoint unbounded operators Q (position) and P (momentum).

The operators in our (discrete) setting are defined by their action

$$U : \mathbf{v}[p] \mapsto \mathbf{v}[p-1]$$

and so acts on $\mathcal{L}_{\mathbf{u}}$ by linearity as

$$U : \mathbf{u}[r] \mapsto \frac{1}{\sqrt{\mathcal{N}}}\sum_{p\mathfrak{v}\in\mathbb{V}^0}\mathrm{e}(\frac{rp}{\mathcal{N}})\mathbf{v}[p-1] = \frac{1}{\sqrt{\mathcal{N}}}\sum_{p\mathfrak{v}\in\mathbb{V}^0}\mathrm{e}\left(\frac{rp+r}{\mathcal{N}}\right)\mathbf{v}[p] = \mathrm{e}\left(\frac{r}{\mathcal{N}}\right)\mathbf{u}[r]$$

and thus the $\mathbf{u}[r]$ are eigenvectors of the operator.

Similarly the unitary operator

$$V : \mathbf{u}[r] \mapsto \mathbf{u}[r+1], \quad \mathbf{v}[p] \mapsto \mathrm{e}\left(\frac{p}{\mathcal{N}}\right)\mathbf{v}[p]$$

has the $\mathbf{v}[p] \in \mathcal{L}_{\mathbf{v}}$ as its eigenvectors.

It is easy to check that

$$UV = qVU, \text{ for } q = \mathrm{e}\left(\frac{1}{\mathcal{N}}\right) = \exp_{\mathfrak{p}}\left(\frac{\mathfrak{p}-1}{\mathcal{N}}\right), \ q^{\mathcal{N}} = 1.$$

Free particle. The time evolution operator for the free particle is $e^{\frac{iP^2}{2}t}$, where $t \in {}^*\mathbb{Z}/\mathcal{N}$, the unitary operator with the action on $\mathcal{L}_{\mathbf{v}}$ defined as

$$e^{\frac{iP^2}{2}t}\mathbf{v}[p] := e\left(\frac{p^2}{2\mathcal{N}}t\right)\mathbf{v}[p].$$

One can calculate

$$\begin{aligned} e^{\frac{iP^2}{2}t}\mathbf{u}[r] &= e^{\frac{iP^2}{2}t}\frac{1}{\sqrt{\mathcal{N}}}\sum_{p\in\mathbb{V}^0} e\left(\frac{rp}{\mathcal{N}}\right)\mathbf{v}[p] = \frac{1}{\sqrt{\mathcal{N}}}\sum_{p\in\mathbb{V}^0} e\left(\frac{p^2t+2rp}{2\mathcal{N}}\right)\mathbf{v}[p] \\ &= \frac{1}{\sqrt{\mathcal{N}}}\sum_{p\in\mathbb{V}^0} e\left(\frac{p^2t+2rp}{2\mathcal{N}}\right)\frac{1}{\sqrt{\mathcal{N}}}\sum_{s\in\mathbb{V}^0} e\left(-\frac{sp}{\mathcal{N}}\right)\mathbf{u}[s] \\ &= \frac{1}{\mathcal{N}}\sum_{s\in\mathbb{V}^0}\left(\sum_{p\in\mathbb{V}^0} e(\frac{p^2t+2p(r-s)}{2\mathcal{N}})\right)\mathbf{u}[s] \end{aligned}$$

By Gauss' quadratic sums formula one gets

$$\sum_{p\mathfrak{v}\in\mathbb{V}^0} e(\frac{p^2t+2p(r-s)}{2\mathcal{N}} = \begin{cases} e\left(\frac{1}{8}\right)\sqrt{\frac{\mathcal{N}}{t}}e\left(-\frac{(r-s)^2}{2t\mathcal{N}}\right), & \text{if } t|(r-s) \\ 0, & \text{otherwise} \end{cases}$$

(note that by 4.1 $e\left(\frac{1}{8}\right) = e^{-\frac{\pi i}{4}}$ when identifying $F \subset {}^*\mathbb{C}$).

Thus

$$e^{\frac{iP^2}{2}t}\mathbf{u}[r] = \frac{1}{\sqrt{t\mathcal{N}}}e\left(\frac{1}{8}\right)\sum_{(r-s)\mathfrak{v}\in t\mathbb{V}^0} e\left(-\frac{(r-s)^2}{2t\mathcal{N}}\right)\mathbf{u}[s] \tag{34}$$

Note that the support of the state

$$\text{Supp}\left(e^{\frac{iP^2}{2}t}\mathbf{u}[r]\right) = t\mathbb{V} - r$$

is a proper subset of $\mathbb{V}$, a coset of a "dense" subgroup.

The example of $e^{\frac{iP^2}{2}t}$ and similar example $e^{i\frac{P^2+\omega Q^2}{2}t}$ for the quantum harmonic oscillator (particles with quadratic potential) can be found in [9], sections 11 and 12, and in a more detailed form in [12], section 6.

7.2 Example SM. (Statistical Mechanics) The setting of SM is even more relevant to the theory above since by definition its physics setting is an extremely large but finite model $\mathbb{V}$.)

In general, models of statistical mechanics are good analogues of models of *quantum field theory*, QFT, rather than quantum mechanics. The similarity becomes apparent once one replaces QFT and QM expressions like $\mathrm{e}^{iS(x)}$ by $\mathrm{e}^{S(x)}$ (Wick rotation). See [3] for a detailed discussion on the topic.

We single out a more specific SM-setting of Zinn-Justin [13], chapter 4, *Classical statistical physics: One dimension*. A more general setting which leads to a real Hilbert space formalism uses *probability density matrices* can also be found in [14], chapter 9, and indeed in many other sources.

Zinn-Justin [13] introduces a Hilbert space formalism in SM-context, position and momentum operators (position states but not momentum states) and the Gaussian (that is the quadratic case) transfer matrix, the propagator between states q' and q''

$$T(q', q'') := \mathrm{e}^{S(q', q'')}$$

T a real symmetric operator acting on a real Hilbert space $\mathcal{H}$ of dimension $\mathcal{N}$.

The continuous form of $T(q', q'')$ presented in Sect. 4.6 of [13] is in full analogy with QM analogue (49) which we discuss below in Sect. 8.10.

We introduce here the domain $\mathbb{V} := \mathbb{V}_{1\mathrm{SM}} := \mathbb{V}_{\mathfrak{u}}$ (as in 6.10) of the 1-dimensional statistical mechanics model which agrees with Zinn-Justin's in direct analogy with 7.1 and note that $\mathcal{N} := \#\mathbb{V} = \mathfrak{l}\mathbf{i} = (\mathfrak{m}\mathbf{j})^2$ is not an element of F, so $\mathbb{V}_{1\mathrm{SM}}$ is not tame.

Now we introduce the ingredients of pseudo-real Gaussian Hilbert space over $\mathbb{V}_{1\mathrm{SM}}$.

Define, for $r\mathfrak{u} \in \mathbb{V}$

$$\mathbf{u}[r](x) = \delta(x - r).$$

We set, for $p\mathfrak{u} \in \mathbb{V}$

$$\mathbf{v}[p] = \frac{1}{\sqrt{\mathcal{N}}} \sum_{r\mathfrak{u}\in\mathbb{V}^0} \mathrm{e}\Big(-\frac{rp}{\mathcal{N}}\Big)\mathbf{u}[r]$$

This makes $\mathbf{v}[p]$ Fourier-dual to $\mathbf{u}[r]$ and furnishes two orthonormal bases of $\mathcal{H}_{\mathbb{V}}$ both of size $\mathcal{N} = \mathfrak{l}\mathbf{i}$. (Note that the above definition of momentum states and its Fourier-duality to position states requires periodicity of $\mathbb{V}$, which in our setting is isomorphic to the group of period $\mathfrak{l}\mathbf{i}$, much larger scale than $\mathcal{N}$ of example 7.1.)

The Gauss quadaratic sums formula appropriate for the given $\mathbb{V}$ is presented in (28) of Proposition 4.1 of [7], assuming the current system of notations and moving $\mathfrak{m}$ to LHS

$$\frac{1}{\mathfrak{m}} \sum_{\eta\in a\mathbb{V}^0} \mathrm{e}\Big(-\frac{a\eta^2}{2}\Big) = \mathrm{e}\Big(\frac{1}{8}\Big)\frac{\mathbf{j}}{\sqrt{a}} \tag{35}$$

(bear in mind that $\mathbf{j}$ specialises by lm_{F} into $\mathrm{e}(-\frac{1}{8})$ and this brings the RHS to the purely real value $\frac{1}{\sqrt{a}}$.)

8 Continuous logic setting for Gaussian $\mathcal{H}_{\mathbb{V}}$

8.1 A formal setting for continuous model theory was established in the 1960's monograph by Chang and Kiesler [2] (see also [10] for a more recent presentation). It was defined as a model theory for a multivalued logic with values in a compact set. The universe of a model is supposed to be a metric space, the metric replacing the equality of the discrete framework. However, subsequent developments have introduced greater flexibility into the framework. Hrushovski in [4] considers the case where "metric is unnessary and even impossible" yet continuous logic remains applicable and effective. Our continuous model theory setting similarly deviates from the canonical metric setting by narrowing the domain where a standard metric and continuous predicates are defined. The effect of this procedure is somewhat similar to the effect of the "cut-off procedure" in quantum physics.

8.2 $\mathbb{V}$ in continuous logic CL.

We use $\mathsf{lm}_{\mathbb{U}} : \mathbb{U} \to \bar{\mathbb{C}}$ and $\mathsf{lm}_{\mathrm{F}} : \mathrm{F} \to \bar{\mathbb{C}}$ to move the domain $\mathbb{V}$ of states to a locally bounded metric space $\mathbb{V}^{\mathsf{lm}}$ and the domain F of logical values from F to $\mathbb{C}$. Our $\mathbb{V}$ in this section is either $\mathbb{V}_{\mathfrak{u}}$ or $\mathbb{V}_{\mathfrak{v}}$ of 6.1.

As in 5.1 $\mathbb{V}$ is assumed to be given with predicates determining non-standard distance between points.

We define metric predicates on the domains assuming that both $\mathfrak{u}$ and $\mathfrak{v}$ generate group lattices with the spacing evaluated as $\frac{1}{\mathfrak{l}}$.

Thus

$$\mathrm{dist}(0, l \cdot \mathfrak{u}) := \frac{l}{\mathfrak{l}}$$

$$\mathrm{dist}(0, l \cdot \mathfrak{v}) := \frac{l}{\mathfrak{l}}$$

In line with the standard boundedness assumptions [10] we distinguish the family of subdomains $\mathbb{V}_m \subset \mathbb{V}$, $m \in \mathbb{N}$,

$$\mathbb{V}_m = \{x \in \mathbb{V} : \ \mathrm{dist}(0, x) \leq m\}.$$

Set

$$\mathbb{V}^{\mathsf{lm}}_m = \mathsf{lm}_{\mathbb{U}}(\mathbb{V}_m) \text{ and } \mathbb{V}^{\mathsf{lm}} = \bigcup_{m \in \mathbb{N}} \mathbb{V}^{\mathsf{lm}}_m.$$

Thus $\mathbb{V}^{\mathsf{lm}}$ is covered by a family $\mathbb{V}^{\mathsf{lm}}_m \subseteq \mathbb{V}^{\mathsf{lm}}$ of bounded complete metric subspaces $\mathbb{V}^{\mathsf{lm}}_m$. Note that in general $\mathbb{V} \neq \bigcup_{m \in \mathbb{N}} \mathbb{V}_m$.

It follows

$$\mathbb{V}^{\mathsf{lm}}_{\mathfrak{u}} \cong \mathbb{V}^{\mathsf{lm}}_{\mathfrak{v}} \cong \mathbb{R}$$

as metric spaces homeomorphisms, and $\mathbb{V}^{\mathsf{lm}}_m$ in both cases corresponds to $\mathbb{R} \cap [-m : m]$. However, note that since by our choices of parameters

$$\mathbb{V}_{\mathfrak{v}} = \mathbf{i}\mathbb{V}_{\mathfrak{u}} \text{ and } \mathsf{lm}_{\mathbb{U}}(\mathbf{i}u) = \mathrm{i}\,\mathsf{lm}_{\mathbb{U}}(u),$$

more accurately, we have isomorphisms

$$\mathbb{V}^{\mathsf{lm}}_{\mathfrak{u}} \cong \mathbb{R} \text{ and } \mathbb{V}^{\mathsf{lm}}_{\mathfrak{v}} \cong \mathrm{i}\mathbb{R} \quad (\mathrm{i} = \sqrt{-1}) \tag{36}$$

We use the fact that $\mathbb{V}$ *has a structure of a K-module*, for a subring $K \subseteq {}^*\mathbb{Z}$ of our choice. For most purposes we can consider $K = \mathbb{Z}$ or slightly bigger subring of $\mathrm{O}(\mathcal{F})$. For $k \in K$, $k\mathbb{V}$ is a submodule and, for $r \in \mathbb{V}$, $k\mathbb{V} + r$ is a coset of the submodule,

$$\#(k\mathbb{V} + r) = \frac{\mathcal{N}}{k} \tag{37}$$

and at the same time

$$\mathsf{Im}_{\mathbb{U}}\,(k\mathbb{V} + r) = \mathbb{V}^{\mathsf{Im}} \tag{38}$$

(we say that $k\mathbb{V}$ is dense in $\mathbb{V}$).

Let $\varphi : \mathbb{V} \to \mathrm{F}$ be a pseudo-continuous definable state on $\mathbb{V}$. Set for $x \in \mathbb{V}_m$,

$$x^{\mathsf{Im}} := \mathsf{Im}_{\mathbb{U}}(x) \text{ and } \varphi^{\mathsf{Im}}(x^{\mathsf{Im}}) := \mathsf{Im}_{\mathrm{F}}(\sqrt{\mathcal{N}}\varphi(x)). \tag{39}$$

This is well-defined because of pseudo-continuity and defines a map

$$\varphi^{\mathsf{Im}} : \mathbb{V}^{\mathsf{Im}} \to \mathbb{C}.$$

In the more general case we consider φ which is pseudo-continuous on $k\mathbb{V} + r$ as above and apply the definition (39) to $\varphi_{|k}$, the restriction to $k\mathbb{V}+r$. This is well-defined.

Lemma 8.3 *Let φ be a ket state on $\mathbb{V}$ of norm 1 and $\varphi_{|k}$ its restriction on $k\mathbb{V}$ normalised to norm 1. Then*

$$\varphi_{|k}^{\mathsf{Im}}(x^{\mathsf{Im}}) = \varphi^{\mathsf{Im}}(x^{\mathsf{Im}}) \textit{ for } x \in k\mathbb{V} + r. \tag{40}$$

More generally, the equality holds for any ket state φ on $\mathbb{V}$ when setting $\varphi_{|k} : (k\mathbb{V} + r) \to F$

$$\varphi_{|k}(x) := \sqrt{k} \cdot \varphi_{|k}(x).$$

Proof We may assume $r = 0$.

By definition

$$\varphi = \sum_{s \in \mathbb{V}^0} \varphi(s)\mathbf{u}[s],$$

$$\varphi_{|k} = \sqrt{k} \sum_{s \in k\mathbb{V}^0} \varphi(s)\mathbf{u}[s]$$

and so

$$\varphi_{|k}(s) = \sqrt{k}\varphi(s).$$

Thus by (39), for $s \in k\mathbb{V}$,

$$\varphi_{|k}^{\mathsf{Im}}(s^{\mathsf{Im}}) = \mathsf{Im}_{\mathrm{F}}\left(\sqrt{\frac{\mathcal{N}}{k}}\sqrt{k}\,\varphi(s)\right) = \varphi^{\mathsf{Im}}(s^{\mathsf{Im}}).$$

□

The CL-state of the form $\varphi^{\mathsf{lm}}_{|k}$ we call CL-ket states. Lemma 8.3 allows us always to make reductions, if needed, to appropriate dense subdomains.

Define inner product for CL-ket-states on $\mathbb{V}^{\mathsf{lm}}$, for φ, ψ, $\varphi \neq \psi$

$$\langle \varphi^{\mathsf{lm}} | \psi^{\mathsf{lm}} \rangle = \lim_{m\to\infty} \int_{\mathbb{V}^{\mathsf{lm}}_m} \varphi^{\mathsf{lm}}(x) \bar{\psi}^{\mathsf{lm}}(x)\, dx \tag{41}$$

if limit exists and is finite. In case the condition is not satisfied we define the value below in 8.5.

If $\varphi = \psi$ then set

$$\langle \psi^{\mathsf{lm}} | \psi^{\mathsf{lm}} \rangle := \mathsf{lm}_{\mathrm{F}} |\psi|^2. \tag{42}$$

8.4 Consider the states φ and ψ of norm 1 represented in the form as in 6.4. We get

$$\varphi(r) = \frac{1}{\sqrt{\mathcal{N}}} \mathrm{e}\left(-\frac{A_\phi r^2 + 2B_\varphi r}{2\mathcal{N}} \right) \text{ and } \psi(r) = \frac{1}{\sqrt{\mathcal{N}}} \mathrm{e}\left(-\frac{A_\psi r^2 + 2B_\psi r}{2\mathcal{N}} \right)$$

where $A_\varphi > 0$, $\mathcal{A}_\psi > 0$ and B_φ and B_ψ depend on parameters p running in $\mathbb{V}^0$.

Respectively, for $x = r^{\mathsf{lm}}$, in the Euclidean case, that is for $\mathbb{V} = \mathbb{V}_{\mathrm{u}}$,

$$\varphi^{\mathsf{lm}}(x) = \mathrm{e}^{-\pi(A_\varphi x^2 + 2B_\varphi x)} \text{ and } \psi^{\mathsf{lm}}(x) = \mathrm{e}^{-\pi(A_\psi x^2 + 2B_\psi x)}$$

and so we have

$$\lim_{m\to\infty} \int_{\mathbb{V}^{\mathsf{lm}}_m} \varphi^{\mathsf{lm}}(x) \bar{\psi}^{\mathsf{lm}}(x)\, dx = \int_{x\in\mathbb{R}} \varphi^{\mathsf{lm}} \cdot \bar{\psi}^{\mathsf{lm}}\, dx = \int_{x\in\mathbb{R}} \mathrm{e}^{-\pi(Ax^2+2Bx)} dx = \sqrt{\frac{1}{A}}$$

for $A = A_\varphi + \mathcal{A}_\psi \neq 0$, the classical converging Gauss integral.

In the Hermitian case, that is for $\mathbb{V} = \mathbb{V}_{\mathrm{u}}$,

$$\varphi^{\mathsf{lm}}(x) = \mathrm{e}^{-\pi i(A_\varphi x^2 + 2B_\varphi x)} \text{ and } \psi^{\mathsf{lm}}(x) = \mathrm{e}^{-\pi i(A_\psi x^2 + 2B_\psi x)}$$

and so for each m, we see the classical Fresnel integral

$$\int_{x\in\mathbb{V}^{\mathsf{lm}}_m} \varphi^{\mathsf{lm}} \cdot \bar{\psi}^{\mathsf{lm}} dx = \int_{-m\le x\le m} \mathrm{e}^{-\pi i(Ax^2+2Bx)} dx$$

and assuming $A = A_\varphi - \mathcal{A}_\psi > 0$,

$$\lim_{m\to\infty} \int_{-m\le x\le m} \mathrm{e}^{-\pi i(Ax^2+2Bx)} dx = \mathrm{e}^{\frac{\pi i}{4}} \sqrt{\frac{1}{A}}$$

(in case $A < 0$ one has to change the coeffcient to $\mathrm{e}^{-\frac{\pi i}{4}}$).

8.5 In case $A = 0$ one defines axiomatically, for φ, ψ of norm 1,

$$\langle \varphi^{\mathsf{Im}} | \psi^{\mathsf{Im}} \rangle = \begin{cases} 0, & \text{if } B \neq 0 \\ 1, & \text{if } B = 0 \end{cases}$$

This is in accordance with the pseudo-finite setting and 6.7.

Remark. The integral calculations do not converge in this case and in quantum mechanics one uses the Dirac delta-function value $\delta_{\mathrm{Dir}}(B)$ to evaluate the inner product.

Proposition 8.6 *In the notations of 8.4, suppose the $A \neq 0$. Then the value of $\langle \varphi^{\mathsf{Im}} | \psi^{\mathsf{Im}} \rangle$, for $\mathbb{V} = \mathbb{V}_{\mathfrak{u}}$ and $\mathbb{V} = \mathbb{V}_{\mathfrak{v}}$, is well-defined finite and does not depend on B_φ and B_ψ. More precisely*

$$\langle \varphi^{\mathsf{Im}} | \psi^{\mathsf{Im}} \rangle = \begin{cases} \mathsf{Im}_F(\mathfrak{m}\langle \varphi | \psi \rangle_E), & \text{if } \langle \varphi | \psi \rangle_E \neq 0 \ \&\ \mathbb{V} = \mathbb{V}_{\mathfrak{u}} \\ \mathsf{Im}_F(\mathfrak{m}\langle \varphi | \psi \rangle_H), & \text{if } \langle \varphi | \psi \rangle_H \neq 0 \ \&\ \mathbb{V} = \mathbb{V}_{\mathfrak{v}} \end{cases} \tag{43}$$

The condition $\langle \varphi | \psi \rangle_{E/H} \neq 0$ depend on B only and has non-zero value if $A|B$, which is the case for a dense subsets of pairs of states φ, ψ on $\mathbb{V}^0 \times \mathbb{V}^0$.

In the case $A = 0$,

$$\langle \varphi^{\mathsf{Im}} | \psi^{\mathsf{Im}} \rangle = \begin{cases} \mathsf{Im}_F(\mathfrak{m}\langle \varphi | \psi \rangle_E), & \text{if } \mathbb{V} = \mathbb{V}_{\mathfrak{u}} \\ \mathsf{Im}_F(\mathfrak{m}\langle \varphi | \psi \rangle_H), & \text{if } \mathbb{V} = \mathbb{V}_{\mathfrak{v}} \end{cases} \tag{44}$$

Proof The first statement has been proved above in 8.4.

Now assume that $\mathbb{V} = \mathbb{V}_{\mathfrak{u}}$ and $\langle \varphi | \psi \rangle_{\mathrm{E}} \neq 0$ ($\mathcal{N} = \mathcal{N}_{\mathfrak{u}} = \mathfrak{l}\mathbf{i}$). By formula (29) $B = B_\phi + B_\psi$ is divisible by A and

$$\langle \varphi | \psi \rangle_{\mathrm{E}} = \frac{1}{\mathcal{N}} \sum_{rA\mathfrak{u} \in \mathbb{V}^0} \mathrm{e}\Big(-\frac{Ar^2 + 2Br}{2\mathcal{N}} \Big) = \frac{1}{\sqrt{\mathcal{N}}} \mathrm{e}\Big(-\frac{1}{8} \Big) \sqrt{\frac{1}{A}}\, \mathrm{e}\Big(-\frac{\frac{B^2}{2A}}{\mathcal{N}} \Big)$$

and so, since $\sqrt{\mathcal{N}} = \mathfrak{m}\mathbf{j}$,

$$\begin{aligned} \mathfrak{m} \cdot \langle \varphi | \psi \rangle_{\mathrm{E}} &= \mathbf{j}^{-1} \mathrm{e}\Big(-\frac{1}{8} \Big) \sqrt{\frac{1}{A}}\, \mathrm{e}\Big(-\frac{\frac{B^2}{2A}}{\mathcal{N}} \Big) = \mathbf{j}^{-1} \mathrm{e}\Big(-\frac{1}{8} \Big) \sqrt{\frac{1}{A}} \exp_{\mathfrak{p}} \Big(-\frac{\frac{B^2}{2A}(\mathfrak{p}-1)}{\mathfrak{l}\mathbf{i}} \Big) \\ &= \mathbf{j}^{-1} \mathrm{e}\Big(-\frac{1}{8} \Big) \sqrt{\frac{1}{A}} \exp_{\mathfrak{p}} \Big(-\frac{B^2}{2A}\mathfrak{u} \Big) \end{aligned}$$

Hence, since by definition (see 4.1–4.10)

$$\mathsf{Im}_{\mathrm{F}} : \ \mathbf{j} \mapsto \mathrm{e}^{i\frac{\pi}{4}}, \ \ \mathrm{e}\Big(-\frac{1}{8} \Big) \mapsto \mathrm{e}^{i\frac{\pi}{4}}, \ \ \exp_{\mathfrak{p}} \Big(-\frac{B^2}{2A}\mathfrak{u} \Big) \mapsto 1,$$

we obtain

$$\mathsf{Im}_{\mathrm{F}}\, \mathfrak{m} \cdot \langle \varphi | \psi \rangle_{\mathrm{E}} = \sqrt{\frac{1}{A}}$$

Comparing with calculations in 8.4 this proves the Euclidean case.

Now assume that $\mathbb{V} = \mathbb{V}_{\mathfrak{v}}$ and $\langle\varphi|\psi\rangle_{\mathrm{H}} \neq 0$ ($\mathcal{N} = \mathcal{N}_{\mathfrak{v}} = \mathfrak{l}$). Again by (29) B is divisible by A and (assuming for simplicity $A > 0$)

$$\langle\varphi|\psi\rangle_{\mathrm{H}} = \frac{1}{\mathcal{N}} \sum_{rA\mathfrak{u}\in\mathbb{V}^0} \mathrm{e}\Big(-\frac{Ar^2+2Br}{2\mathcal{N}}\Big) = \frac{1}{\sqrt{\mathcal{N}}}\mathrm{e}\Big(-\frac{1}{8}\Big)\sqrt{\frac{1}{A}}\mathrm{e}\Big(-\frac{B^2}{2A\mathcal{N}}\Big)$$

and so, since $\sqrt{\mathcal{N}} = \mathfrak{m}$,

$$\mathfrak{m}\cdot\langle\varphi|\psi\rangle_{\mathrm{H}} = \mathrm{e}\Big(-\frac{1}{8}\Big)\sqrt{\frac{1}{A}}\mathrm{e}\Big(-\frac{B^2}{2A\mathcal{N}}\Big) = \mathrm{e}\Big(-\frac{1}{8}\Big)\sqrt{\frac{1}{A}}\exp_{\mathfrak{p}}\Big(-\frac{B^2}{2A}\frac{\mathfrak{p}-1}{\mathfrak{l}}\Big) =$$

$$= \mathrm{e}\Big(-\frac{1}{8}\Big)\sqrt{\frac{1}{A}}\exp_{\mathfrak{p}}\Big(-\frac{B^2}{2A}\mathfrak{v}\Big)$$

Since

$$\mathsf{lm}_{\mathrm{F}}:\ \exp_{\mathfrak{p}}\Big(-\frac{B^2}{2A}\mathfrak{v}\Big)\mapsto 1$$

we obtain

$$\mathsf{lm}_{\mathrm{F}}\ \mathfrak{m}\cdot\langle\varphi|\psi\rangle_{\mathrm{H}} = \mathrm{e}^{\frac{\pi i}{4}}\sqrt{\frac{1}{A}}$$

This proves the Hermitian case. □

8.7 The treatment for the CL-version of position states $\mathbf{u}[r]$ and more generally bra-states will be following the known Hilbert spaces duality theory, see [1] for references.

Define the CL-version of ket-sorts in accordance with 6.4,

$$\mathcal{L}_f^{\mathsf{lm}} := \{\mathbf{s}_f^{\mathsf{lm}}[p]:\ p\in\mathbb{V}^{\mathsf{lm}}\}.$$

Recall that for every ket-state on $\mathbb{V}$

$$\langle\varphi|\mathbf{u}[r]\rangle = \varphi(r);\quad x = r^{\mathsf{lm}}\in\mathbb{V}^{\mathsf{lm}}. \tag{45}$$

In particular,

$$\langle\mathbf{s}_f[p]|\mathbf{u}[r]\rangle = \mathbf{s}_f[p](r)$$

Let $x = r^{\mathsf{lm}}\in\mathbb{V}^{\mathsf{lm}}$. Consider the collection of maps defined on all CL-ket states, sort by sort,

$$\mathbf{s}_f^{\mathsf{lm}}[p]\mapsto\mathbf{s}_f^{\mathsf{lm}}[p](x);\ \mathcal{L}_f^{\mathsf{lm}}\to\mathbb{C}.$$

This functional is CL-interpretable in the collection of all CL-ket sorts. We call the functional CL-position state $\mathbf{u}^{\mathsf{lm}}[x]$ and define the inner product with arbitrary CL-ket state φ^{lm} as

$$\langle\varphi^{\mathsf{lm}}|\mathbf{u}^{\mathsf{lm}}[x]\rangle := \varphi^{\mathsf{lm}}(x) \tag{46}$$

and for a pair of position states

$$\langle \mathbf{u}^{\mathsf{Im}}[x_1] | \mathbf{u}^{\mathsf{Im}}[x_2] \rangle := \delta(x_1 - x_2). \tag{47}$$

Note that the functional $\mathbf{u}^{\mathsf{Im}}[x]$ is interpretable in the multisorted structure with all the CL-ket-sorts and the inner product.

8.8 Define the Hermitian/Euclidean CL-Hilbert space $\mathcal{H}_{\mathbb{V}^{\mathsf{Im}}}$ as multisorted structure with CL-ket-sorts $\mathcal{L}_{\psi}^{\mathsf{Im}}$ with inner product defined between sorts defined in (41),(42).

Now it is natural to consider $\mathcal{H}_{\mathbb{V}^{\mathsf{Im}}}^{\mathrm{eq}}$, the extension of $\mathcal{H}_{\mathbb{V}^{\mathsf{Im}}}$ by *imaginary* (interpretable) elements. This contains finite linear combinations of elements of $\mathcal{H}_{\mathbb{V}^{\mathsf{Im}}}$, the CL-position state $\mathbf{u}^{\mathsf{Im}}$ as described in 8.7, as well as all the functionals $\mathcal{L}_{\psi}^{\mathsf{Im}} \to \mathbb{C}$ defined in terms of inner product. A properly defined analogue of the *rigged Hilbert space*, containing the space dual to $\mathcal{H}_{\mathbb{V}^{\mathsf{Im}}}$, would be a substructure of $\mathcal{H}_{\mathbb{V}^{\mathsf{Im}}}^{\mathrm{eq}}$.

We leave it to the reader to check that the map

$$\varphi \mapsto \varphi^{\mathsf{Im}}$$

is a "weak sort-by-sort homomorphism"

$$\mathcal{H}_{\mathbb{V}} \to \mathcal{H}_{\mathbb{V}^{\mathsf{Im}}};\ \mathcal{L}_{\psi} \to \mathcal{L}_{\psi}^{\mathsf{Im}}$$

which respects inner product.

8.9 Linear unitary operators on $\mathcal{H}_{\mathbb{V}^{\mathsf{Im}}}$ have form

$$\mathcal{A}^{\mathsf{Im}} : \mathcal{L}_{\psi}^{\mathsf{Im}} \to \mathcal{L}_{\varphi}^{\mathsf{Im}}$$

where

$$A : \mathcal{L}_{\psi} \to \mathcal{L}_{\varphi}$$

is an operator on $\mathcal{H}_{\mathbb{V}}$ as described in 5.10 and 5.11 as

$$A : \psi_r \mapsto \sum_{r \in \mathbb{V}^0} \left(\sum_{s \in \mathbb{V}^0} \psi(r, s) \cdot \alpha(s, r) \right) \mathbf{u}[r]$$

(here $\psi_r : s \mapsto \psi(r, s)$) and

$$\mathcal{A}^{\mathsf{Im}} : \psi_r^{\mathsf{Im}} \mapsto \left(\sum_{r \in \mathbb{V}^0} \left(\sum_{s \in \mathbb{V}^0} \psi(r, s) \cdot \alpha(s, r)) \right) \mathbf{u}[r] \right)^{\mathsf{Im}}.$$

Note that

$$r \mapsto \sum_{s \in \mathbb{V}^0} \psi(r, s) \cdot \alpha(s, r) = \langle \psi_r | \bar{\alpha}_r \rangle$$

and thus the integral expression (43) for inner product brings us in CL-setting to

$$\mathcal{A}^{\mathsf{Im}} : \psi_r^{\mathsf{Im}} \mapsto \int_{\mathbb{V}^{\mathsf{Im}}} \psi^{\mathsf{Im}}(r, s) \cdot \alpha^{\mathsf{Im}}(s, r)\, ds \tag{48}$$

the integral as a CL-quantifier bounded by condition α.

8.10 Example. One-dimensional quantum mechanics
We follow 7.1. Note that we assume $\hbar = 1$ and suppress some normalising coefficients.

Respectively

$$\mathcal{L}_{\mathbf{v}} = \{\mathbf{v}[p] : \quad p \in \mathbb{R}\}$$

A position state is a map (that is a CL-unary predicate):

$$\mathbf{u}[r] : \mathcal{L}_{\mathbf{v}} \to \mathbb{S} \subset \mathbb{C}; \quad \mathbf{v}[p] \mapsto \langle \mathbf{v}[p] | \mathbf{u}[r] \rangle = \mathbf{v}[p](r) = \mathrm{e}^{ipr}$$

and the sort $\mathcal{L}_{\mathbf{u}}$ with the pairing (binary CL-predicate)

$$\langle \mathbf{v}[p] | \mathbf{u}[r] \rangle : \mathcal{L}_{\mathbf{v}} \times \mathcal{L}_{\mathbf{u}} \to \mathbb{S} \subset \mathbb{C}.$$

Both position and momentum are binary CL-predicates on $\mathbb{R}$ as are general CL-states.

8.11 In [9] and [12] a number of calculations in Gaussian setting were carried out. In [12] we calculated the propagator for the quantum harmonic oscillator with frequency ω, the CL-value of reaching the position x from position x_0 in time t:

$$\mathrm{e}^{-i\frac{\pi}{4}} \sqrt{\frac{\omega}{2\pi\hbar|\sin\omega t|}} \exp\, i\omega \frac{(x^2 + x_0^2)\cos\omega t - 2xx_0}{2\hbar \sin\omega t} \tag{49}$$

This was demonstrated therein both by path integral calculation (Sect. 9) and by a more direct calculation (Sects. 7, 7.13).

8.12 Conclusions. Equalities of the form

$$\mathcal{A}^{\mathsf{Im}}(\psi^{\mathsf{Im}}) = \varphi^{\mathsf{Im}},$$

for Gaussian ψ, φ and $\mathcal{A}$, obtained as the result of the calculus defined by (41)–(48) are CL-sentences which form a CL-theory with the interpretation by pseudo-finite structures based on $\mathbb{V}_{\mathfrak{v}}$ or $\mathbb{V}_{\mathfrak{u}}$.

This theory has quantifier-elimination to Gaussian predicates (states) since by Gaussian summation formula the application of an $\mathcal{A}$ to a ψ always results in a Gaussian state φ.

Wick rotation

$$\{\}^{\mathbf{i}} : \mathcal{H}_{\mathbb{V}_{\mathfrak{u}}} \to \mathcal{H}_{\mathbb{V}_{\mathfrak{v}}}$$

described by 6.11 establishes an equaivalence between the theories based on $\mathbb{V}_{\mathfrak{v}}$ or $\mathbb{V}_{\mathfrak{u}}$.

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