Diffeomorphism-Equivariant Configuration Spaces with Twisted Partial Summable Labels

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Abstract

We construct the configuration space of points in a smooth manifold with twisted noncommutative partial summable labels. The theory of twisted operads is developed to encode twisted noncommutative summations. We also define a diffeomorphism-equivariant scanning map from the configuration space to a section space and show that it is a weak equivalence. To achieve the equivariance the configuration space is mapped to a section space over the space of all disks in the ambient manifold.

Contents

Chapter 1

Introduction

1.1 Motivation and results

Gather various settings to get the most general one The aim of this project is to construct the diffeomorphism-equivariant scanning map associated to the configuration spaces of manifolds with twisted partial summable labels. Our choice of generalised configuration spaces includes the generalisations studied in the following papers. Non-summable untwisted label was studied by Bödigheimer in [2]. Non-summable twisted label was studied by Hesselholt in [4], and by Manthorpe and Tillmann in [7]. Partial summable untwisted label was studied by Salvatore in [10]. We follow Salvatore's idea of noncommutative summation using E_n-algebra labels for n-dimensional ambient manifolds. Note that a non-summable label is a partial summable label with summation allowed to a minimum.

It is [7] that introduced the equivariance of scanning maps with respect to the diffeomorphism group of the ambient manifold. Since the smooth structure provides with "a scaffolding for scanning", the diffeomorphism-equivariance arises naturally, once we make the formula of the scanning map explicit. In particular our scanning map and that of [7] do not involve a choice of a metric for the ambient manifold. In fact the diffeomorphism-equivariance is a special case of the naturality of the scanning maps with respect to embeddings of manifolds of the same dimension preserving boundary. We will describe the naturality as a natural isomorphism between a non-commutative twisted homology theory for n-manifolds and corresponding cohomology theory. For the construction of configuration spaces with twisted partial summable labels, we have more general functoriality with respect to embeddings of manifolds of any dimensions and mapping between labels. We check for the configuration spaces axioms analogous to that for factorization homology introduced in [1]. As Salvatore's work [10] provides a concrete prototype for factorization homology and Lurie's "non-commutative Poincaré duality", we hope our generalisation to be a setting to ask concrete questions about homology theories of manifolds.

Concept of Configuration spaces First let us review the basic constructions of configuration spaces. For a given space M and non-negative integer k, the configuration space of ordered k-points of M, denoted by $\widetilde{C}_k(M)$, is defined by

$$\widetilde{C}_k(M) = \left\{ \ (x_i)_{i=1}^k \in M^k \mid x_i \text{ all distinct. } \right\}.$$

Similarly the configuration space of unordered k-points of M, denoted by $C_k(M)$, is defined by

$$C_k(M) = \left\{ \ \{x_i\}_{i=1}^k \subset M \mid x_i \text{ all distinct. } \right\} = \widetilde{C}_k(M)/\Sigma_k.$$

In general the physical intuition of configuration spaces is that each point of a configuration space of M represents a finite number of certain objects lying in M, distinct one another in some way. The most studied is when the objects are points or disks.

The condition that objects are distinct can take various formulations. For example Manthorpe and Tillmann in [7] required that a finite number of tubular neighbourhoods of points are chosen to be disjoint. We can also introduce labels to decorate particles with subordinate data. Let $\pi : E \to M$ be a map. The configuration space of unordered k-points of M with labels in E, denoted by $C_k(\pi)$, is the subspace of E^k/Σ_k consisting of k-points in E with distinct π-values. A point of $C_k(\pi)$ can be written as $\{(m_i, e_i)\}_{i=1}^k$, where $\{m_i\}_{i=1}^k \in C_k(M)$ and $\pi(e_i) = m_i$. We say that the particle m_i is labelled with e_i.

Another important type of variations is to allow fusions of objects when they collide. Let N be a closed subspace of M. The configuration space of points of M vanishing at N, denoted by $C(M, N)$, consists of such choices that we choose N as a whole by default and pick a finite number of distinct particles in $M \setminus N$:

$$C(M, N) = \coprod_{k=0}^{\infty} C_k(M) \Big/ \sim$$

where $a \sim b$ if and only if $a \cup N = b \cup N$. $C(M, N)$ encodes the idea that N subsumes particles colliding with N. Note that we automatically allow new particles to be generated near N.

The infinite symmetric product $\mathrm{SP}_\infty(M)$ of a based space $(M,*)$ is another example of configuration spaces allowing collisions. A point of $\mathrm{SP}_\infty(M)$ is represented by a finite number of distinct particles in M each of which is paired with a non-negative integer, and we forget particles at $*$ and perform a summation of labels whenever particles collide. More precisely $\mathrm{SP}_\infty(M)$ is the quotient

$$\mathrm{SP}_\infty(M) = \Big(\coprod_{k=0}^{\infty} M^k/\Sigma_k\Big) \Big/ \sim$$

modulo the equivalence relation $\sim$ generated by $(m_1,\dots,m_k) \sim (m_1,\dots,m_k,*)$. Note that each point of $\mathrm{SP}_\infty(M)$ can be rewritten as a finite number of distinct particles labelled with non-negative integers. The infinite symmetric product is the starting point of the idea of configuration spaces with summable labels.

Application of summable labels Configuration spaces with summable labels provide continuous models of algebraic operations. The Dold-Thom theorem asserts that $\pi_k\big(\mathrm{SP}_\infty(M)\big) \cong \widetilde{H}_k(M;\mathbb{Z})$ for a connected CW-complex M. Using summable labels Segal constructed a model of K-theory in [12] and the space of based rational maps of the Riemann sphere in [13].

Twisted noncommutative summable labels A noncommutative summable label is a non-trivial system to track collisions and decode them into ways of summations. Thus it involves the local structure of an ambient space M. As Salvatore considered in [10], if M is an n-dimensional smooth manifold, which is locally diffeomorphic to $\mathbb{R}^n$ by definition, the n-dimensional Fulton-MacPherson operad $\mathcal{F}_n$ and its action on an E_n-algebra X exemplifies a noncommutative summable label.

For twisted generalisation we replace $\mathcal{F}_n$ with a bundle $\mathcal{F}_M$ associated to TM with the typical fibre $\mathcal{F}_n$ and X with a fibre bundle η over M with the typical fibre X, and let $\mathcal{F}_M$ act on η fibrewise. We call η a twisted summable label over M.

To make the idea of twisted summable labels precise, we define the notion of twisted operads. A twisted operad over M is a sequence of bundles over M, $\big(E(k)\big)_{k\in\mathbb{N}}$, with a system of composition (bundle) maps

$$E(k) \times_M \big(E(i_1) \times_M \cdots \times_M E(i_k)\big) \to E(i_1 + \cdots + i_k)$$

covering the identity map of M, that can be locally identified with a fixed topological operad $\big(\,\mathrm{o}(k)\big)_{k\in\mathbb{N}}$.

Equivariant scanning maps It is well known that there exist weak homotopy equivalences, so called scanning maps, between configuration spaces and mapping spaces. For example, for a parallelizable connected closed smooth n-dimensional manifold M with basepoint $*$,

$$\mathrm{SP}_\infty(M) \simeq_w \mathrm{Map}\left(M \setminus *,\ B^{(n)}(\mathbb{Z}) \right),$$

where $B^{(n)}(\mathbb{Z})$ denotes the n-fold classifying space of $\mathbb{Z}$ and is homotopy equivalent to $\mathrm{SP}_\infty(S^n)$.

Taking π_i on both sides, and using the Dold-Thom theorem on the left hand side and the Mayer-Vietoris sequence on the right hand side, we obtain the Poincaré duality for M. In this way scanning maps can be interpreted as continuous Poincaré dualities. We generalise the duality to the case of non-parallelizable manifolds and twisted noncommutative partial summable labels, and also obtain the diffeomorphism equivariance of scanning maps.

The equivariant scanning map is constructed by considering the space of all tubular neighbourhoods of points of M and let those neighbourhoods restrict a given configuration. The space of all tubular neighbourhoods of points plays a similar role with that of the fundamental class in the ordinary Poincaré duality. As the naturality of the Poincaré duality holds only with respect to maps preserving fundamental classes, the naturality of the scanning map holds only for maps preserving the spaces of tubular neighbourhoods.

1.2 Content

Chapter 2 reviews basic definitions of ordinary configuration spaces.

Chapter 3 explains the twisted operad theory. Section 3.1 reviews the operad theory. Section 3.2 constructs the tubular configuration spaces with untwisted summable labels. Section 3.3 defines the twisted operads. Section 3.4 constructs the twisted little disks operads and the configuration spaces with twisted m-fold loop space labels.

Chapter 4 studies the configuration spaces with twisted summable labels. Section 4.1 constructs the twisted Fulton-MacPherson operads and the configuration spaces with twisted summable labels. Section 4.2 solves a technical problem relating to the diffeomorphism-equivariance. Section 4.3 formulates the configuration spaces with twisted summable labels as a functor and checks properties. Section 4.4 constructs the diffeomorphism-equivariant scanning map for configuration spaces with twisted summable labels and shows that it is a weak equivalence.

1.3 Conventions

1. A space means a compactly generated Hausdorff topological space. Every construction of spaces is assumed to be performed in a fixed category of compactly generated Hausdorff spaces: products, compact open topology, etc..

2. Every based space is well-pointed.

3. Every map is continuous.

4. A manifold is assumed to be second-countable, smooth, and possibly with boundary. A topological manifold which is not necessarily smooth will always be called a topological manifold.

5. An embedding means a smooth embedding. A topological embedding which is not necessarily smooth will always be called a topological embedding.

1.4 Notations and symbols

1. For maps $f : X \to Z$ and $g : Y \to Z$, $f \times_Z g : X \times_Z Y \to Z$ denotes the fibre product of X and Y over Z. $f^{(\times_Z)n} : X^{(\times_Z)n} \to Z$ denotes the n-fold fibre product of X over Z. For maps $f_j : X_j \to Z$, $j = 1, \dots, k$; $(\times_Z)_{j=1}^k f_j : (\times_Z)_{j=1}^k X_j \to Z$ denotes the fibre product of X_j over B.

2. $(m_1, \dots, \hat{m}_i, \dots, m_k)$ denotes $(m_j)_{j=1}^k$ with m_i deleted.

3. n in n-manifolds and n-submanifolds denotes the dimension of manifolds.

4. $\dim M$ denotes the dimension of M.

5. $\overline{X}$ denotes the closure of X.

6. $\operatorname{int} X$ denotes the topological interior of X.

7. id_X denotes the identity map of X. The subscript X is omitted when convenient and clear.

8. $\operatorname{im}(f)$ denotes the image of f.

9. A basepoint is denoted generically as $*$.

10. $\Delta \subset X^k$ denotes the diagonal: $\{(x_j)_{j=1}^k \in X^k |\ x_1 = \cdots = x_k.\}$.

11. For a homotopy $H : X \times I \to Y$, $H_t : X \to Y$ denotes the map at t-level.

12. $\mathrm{Diff}(X)$ denotes the space of self-diffeomorphisms of X with the Whitney C^∞ topology.

13. $\mathrm{Emb}(X, Y)$ denotes the space of embeddings from X to Y with the Whitney C^∞ topology.

14. $\mathrm{Emb}_{top}(X, Y)$ denotes the space of topological embeddings from X to Y with the compact open topology.

15. For a based space X, $\mathrm{Homeo}_0(X)$ denotes the space of basepoint preserving self-homeomorphisms of X with the compact open topology.

16. $\mathrm{Map}(X, Y)$ denotes the space of maps from X to Y with the compact open topology.

17. For a based space Y, $\mathrm{Map}_c(X, Y) = \left\{ f \in \mathrm{Map}(X, Y) \mid \overline{f^{-1}(Y \setminus *)} \text{ is compact.} \right\}$.

18. $\Gamma(\gamma)$ denotes the space of sections of γ with the compact open topology.

19. $\simeq$, $\simeq_w$ stand for a homotopy equivalence, a weak equivalence respectively. $\cong$ stands for a homeomorphism, diffeomorphism, or other isomorphisms depending on context.

20. $\mathrm{GL}(n)$ denotes the general linear group of degree n over $\mathbb{R}$.

21. D^n denotes the closed n-dimensional disk. $\mathring{D}^n$ denotes the open n-dimensional disk. S^n denotes the n-dimensional sphere.

22. $\widehat{T}M$ denotes the fibrewise one-point compactification of the tangent bundle TM of M. $\widehat{T}_pM$ denotes the fibre of $\widehat{T}M$ over p. $\widehat{\mathbb{R}^n}$ denotes the one-point compactification of $\mathbb{R}^n$.

23. $\mathcal{F}_n$ denotes the n-dimensional Fulton-MacPherson operad. $\mathcal{F}_M$ denotes the twisted Fulton-MacPherson operad over M.

24. $\mathrm{SP}_\infty(M)$ denotes the infinite symmetric product of M.

25. $\widetilde{C}_k(M)$ denotes the configuration space of ordered k-points of M.

26. $C_k(M)$ denotes the configuration space of unordered k-points of M.

27. $\widetilde{C}_k(M;Y)$ denotes the configuration space of ordered k-points of M with labels in Y.

28. $C_k(M;Y)$ denotes the configuration space of unordered k-points of M with labels in Y.

29. $\widetilde{C}(M,N;Y)$ denotes the configuration space of ordered points of M vanishing at N with labels in Y.

30. $C(M,N;Y)$ denotes the configuration space of unordered points of M vanishing at N with labels in Y.

31. $\widetilde{C}_k[M]$ denotes the compactified configuration space of ordered k-points of M.

32. $C_\Sigma\big[M,N;\eta\big]$ denotes the configuration space of M vanishing at N with twisted summable label η.

Chapter 2

Basic Examples of Configuration Spaces

The goal of this chapter is to introduce the basic variations of the concept of a configuration space, and to motivate the twisted Fulton-MacPherson operads as a tool to construct configuration spaces with twisted noncommutative summable labels.

2.1 Conventions for symmetric groups

Symmetric groups act naturally on configuration spaces. We need to fix conventions and notations for symmetric groups.

Notation 2.1.

- $\mathbb{N}$ *denotes the set of all non-negative integers and* $\mathbb{N}_{>0}$ *denotes the set of all positive integers.*
- *For* $n \in \mathbb{N}_{>0}$, $[n] = \{1, \dots, n\}$. $[0] = \emptyset$.
- *For* $n \in \mathbb{N}_{>0}$, Σ_n *is a symmetric group permuting* $[n]$. *For* σ, $\mu \in \Sigma_n$, *the group product* $\sigma\mu$ *is understood to be* $\mu \circ \sigma$ *as a composite of functions.*
- Σ_0 *is the trivial group.*

2.2 Configuration spaces of i-points

For a space M and a non-negative integer i, the **degree-i Cartesian product of** M, denoted by M^i, will be understood as follows:

- M^0 is a one-point space. In particular, when M comes with a basepoint $*$, $M^0 = \{*\}$. Σ_0 acts on the left of M^0 trivially.

- For $i \in \mathbb{N}_{>0}$, M^i is the space of all functions from $[i]$ to M with the product topology, and Σ_i acts on the left of M^i by precomposition:

$$\text{For } \sigma \in \Sigma_i \text{ and } \ m : [i] \to M, \ \ \sigma m = m \circ \sigma : [i] \to M.$$

The **configuration space of ordered i-points of M**, denoted by $\widetilde{C}_i(M)$, is the subspace of M^i defined as

- $\widetilde{C}_0(M) = M^0$.
- For $i \in \mathbb{N}_{>0}$,
$$\widetilde{C}_i(M) = \{\ m \in M^i \mid m \text{ is one-to-one.}\ \}.$$
- The action of Σ_i on M^i restricts to $\widetilde{C}_i(M)$.

The **configuration space of unordered i-points of M**, denoted by $C_i(M)$, is the quotient
$$C_i(M) = \widetilde{C}_i(M)/\Sigma_i.$$

Notation 2.2.
We will use the coordinate notation $m = (m_1, \ldots, m_i)$ not only for the element $(m_j)_{j=1}^i \in \widetilde{C}_i(M)$ but also for its orbit in $C_i(M)$. An element of various configuration spaces will be called a configuration.

2.3 Configuration spaces with space labels

For spaces M and Y, and a non-negative integer i, the **configuration space of ordered i-points of M with labels in Y**, denoted by $\widetilde{C}_i(M;Y)$, is the Σ_i-space defined as

- $\widetilde{C}_i(M;Y) = \widetilde{C}_i(M) \times Y^i$.
- Σ_i acts diagonally on the left of $\widetilde{C}_i(M;Y)$.

The **configuration space of unordered i-points of M with labels in Y**, denoted by $C_i(M;Y)$, is defined to be the quotient

$$C_i(M;Y) = \widetilde{C}_i(M;Y)/\Sigma_i.$$

Notation 2.3.
We will use the coordinate notation

$$(m;y) = (m_1,\dots,m_i;y_1,\dots,y_i) = \big((m_1;y_1),\dots,(m_i;y_i)\big) = \big((m_j)_{j=1}^i;(y_j)_{j=1}^i\big)$$

or the summation notation

$$(m;y) = \sum_{j=1}^{i} y_j m_j$$

to refer to an element $(m;y)$ of $\widetilde{C}_i(M;Y)$ or its orbit.

2.4 Relative configuration spaces

We define new configuration spaces from configuration spaces by forgetting points lying inside a closed subspace $N \subset M$.

Definition 2.1.
For a space M, a closed subspace $N \subset M$, and a based space (Y,y_0), the **(relative) configuration space of ordered points of M vanishing at N with labels in** *(Y,y_0), denoted by $\widetilde{C}(M,N;Y)$, is the quotient*

$$\Big(\coprod_{i\in\mathbb{N}} \widetilde{C}_i(M;Y)\Big)\Big/\sim$$

modulo the equivalence relation $\sim$ generated by

$$(m_1,\dots,m_{i+1};\ y_1,\dots,y_{i+1}) \sim (m_1,\dots,\hat{m}_j,\dots,m_{i+1};\ y_1,\dots,\hat{y}_j,\dots,y_{i+1})$$

for $(m_j;y_j) \in (N\times Y)\cup(M\times\{y_0\})$. We base $\widetilde{C}(M,N;Y)$ at the empty configuration $*$ represented by the unique element of $\widetilde{C}_0(M;Y)$.

Note that if (M,N) is an NDR pair, $\widetilde{C}(M,N;Y)$ has a filtration by closed subspaces

$$\Bigg(\Big(\coprod_{i=0}^{k} \widetilde{C}_i(M;Y)\Big)\Big/\sim\Bigg)_{k\in\mathbb{N}}.$$

Definition 2.2.
For a space M, a closed subspace $N \subset M$, and a based space (Y,y_0), the **(relative) configuration space (of unordered points) of M vanishing at N with labels in** *(Y,y_0), denoted by $C(M,N;Y)$, is the quotient*

$$\Big(\coprod_{i\in\mathbb{N}} C_i(M;Y)\Big)\Big/\sim$$

modulo the equivalence relation $\sim$ generated by

$$(m_1, \ldots, m_{i+1};\ y_1, \ldots, y_{i+1}) \sim (m_1, \ldots, m_i;\ y_1, \ldots, y_i)$$

for $(m_{i+1}; y_{i+1}) \in (N \times Y) \cup (M \times \{y_0\})$. We base $C(M, N; Y)$ at the empty configuration $*$ represented by the unique element of $C_0(M; Y)$.

Note that if (M, N) is an NDR pair, $C(M, N; Y)$ has a filtration by closed subspaces:

$$\left(\left(\coprod_{i=0}^{k} C_i(M; Y) \right) \Big/ \sim \right)_{k \in \mathbb{N}}.$$

Example 2.1.
$C(M, \emptyset; S^0) \cong \coprod_{i \in \mathbb{N}} C_i(M)$ *and* $\widetilde{C}(M, \emptyset; S^0) \cong \coprod_{i \in \mathbb{N}} \widetilde{C}_i(M)$.

Remark 2.1.
If (M, N) or Y is connected, then $\widetilde{C}(M, N; Y)$ and $C(M, N; Y)$ are connected.

2.5 Partial abelian monoids

We review the concept of partial abelian monoids, a simple example of summable labels.

Definition 2.3 (partial monoid)**.**
A **partial semigroup** *is a triple $(P, Comp, \times)$ such that*

1. *P is a space and $(P^2, Comp)$ is an NDR pair.*

2. *$\times$ is a map*
$$\times : Comp \to P, \ \ (x, y) \mapsto xy.$$

3. *For $(x, y, z) \in P^3$, if $(x, y), (xy, z) \in Comp$, then $(y, z), (x, yz) \in Comp$ and $x(yz) = (xy)z$.*

*The k-***composable set of** *P, denoted by $Comp_k$, is the set of all k-tuples $(p_j)_{j=1}^k$ of P such that the word $p_1 \cdots p_k$ is defined. In particular $Comp_2 = Comp$ and $Comp_1 = P$. A partial semigroup $(P, Comp, \times)$ is called a* **partial monoid***, if there exists an element e_P of P, called an* **identity***, such that $(x, e_P), (e_P, x) \in Comp$ and $xe_P = e_P x = x$ for every $x \in P$. A partial semigroup $(P, Comp, \times)$ is called* **abelian** *if the following hold:*

$$\text{If } (x, y) \in Comp, \text{ then } (y, x) \in Comp \text{ and } xy = yx.$$

A partial semigroup $(P, Comp, \times)$ will be simply referred to as P and be based at e_P if exists.

Example 2.2.
The morphism set of a small category is a partial semigroup with $\times$ the morphism composition. A vector bundle is a partial abelian semigroup with $\times$ the addition of vectors.

Example 2.3.
Let $(Y, *)$ be a based space. The **trivial partial monoid of** Y *is a partial abelian monoid $(Y, Comp, \times)$ with*

$$\times : Comp = Y \vee Y \to Y,$$
$$(y, *) \mapsto y,\ (*, y) \mapsto y.$$

$e_Y = *$.

2.6 Configuration spaces with labels in partial abelian monoids

We construct the configuration spaces with summable labels such that each particle comes with a label in a partial abelian monoid.

Definition 2.4.
For a space M, a closed subspace $N \subset M$, and a partial abelian monoid P, the **configuration space (of unordered points) of** M **vanishing at** N **with summable labels in** P*, denoted by $C_\Sigma(M, N; P)$, is constructed as follows:*

1. *For each $i \in \mathbb{N}_{>0}$, there exists a map*

$$\iota_i : C_i(M; \coprod_{k \in \mathbb{N}_{>0}} Comp_k) \to \coprod_{n \in \mathbb{N}} M^n \times_{\Sigma_n} P^n,$$
$$\big(m_1, \ldots, m_i; (p^1_j)_{j=1}^{k_1}, \ldots, (p^i_j)_{j=1}^{k_i}\big) \mapsto$$
$$\big((m_1)_{j=1}^{k_1}, \ldots, (m_i)_{j=1}^{k_i};\ (p^1_j)_{j=1}^{k_1}, \ldots, (p^i_j)_{j=1}^{k_i}\big).$$

2. *$C_\Sigma(M, N; P) = \big(\coprod_{i \in \mathbb{N}_{>0}} \mathrm{im}(\iota_i)\big)/\sim$, where the equivalence relation $\sim$ is generated by the following two relations:*

   (a)

$$(m_1, \ldots, m_{i+1};\ p_1, \ldots, p_{i+1}) \sim (m_1, \ldots, m_i;\ p_1, \ldots, p_i)$$

   for $(m_{i+1}; p_{i+1}) \in (N \times P) \cup (M \times \{e_P\})$.

(b)

$$(m_1, \ldots, m_{i+1};\ p_1, \ldots, p_{i+1}) \sim (m_1, \ldots, m_i;\ p_1, \ldots, p_{i-1}, p_i p_{i+1})$$

for $m_i = m_{i+1}$.

3. *For* $m \in M$, $(m; e_P)$ *represents the basepoint* $* \in C_\Sigma(M, N; P)$.

Remark 2.2.
In $C_\Sigma(M, N; P)$ *a collision of labelled particles is allowed to occur with the addition of the labels.*

The ordered case can be constructed by the same procedure, but the coordinates are ordered and P need not be abelian.

Definition 2.5.
For a space M*, a closed subspace* $N \subset M$*, and a partial monoid* P*, the* **configuration space of ordered points of M vanishing at N with summable labels in** P*, denoted by* $\widetilde{C}_\Sigma(M, N; P)$*, is constructed as follows:*

1. *For each* $i \in \mathbb{N}_{>0}$*, there exists a map*

$$\begin{aligned} \iota_i : \widetilde{C}_i(M; \coprod_{k \in \mathbb{N}_{>0}} Comp_k) &\to \coprod_{n \in \mathbb{N}} M^n \times P^n, \\ \big(m_1, \ldots, m_i; (p^1_j)^{k_1}_{j=1}, \ldots, (p^i_j)^{k_i}_{j=1}\big) &\mapsto \\ \big((m_1)^{k_1}_{j=1}, \ldots, (m_i)^{k_i}_{j=1};\ (p^1_j)^{k_1}_{j=1}, \ldots, (p^i_j)^{k_i}_{j=1}\big)&. \end{aligned}$$

2. $\widetilde{C}_\Sigma(M, N; P) = \big(\coprod_{i \in \mathbb{N}_{>0}} \mathrm{im}(\iota_i)\big)/\sim$*, where the equivalence relation* $\sim$ *is generated by the following two relations:*

(a)

$$(m_1, \ldots, m_j, \ldots, m_{i+1};\ p_1, \ldots, p_j, \ldots, p_{i+1}) \sim \\ (m_1, \ldots, \hat{m}_j, \ldots, m_{i+1};\ p_1, \ldots, \hat{p}_j, \ldots, p_{i+1})$$

for $(m_j; p_j) \in (N \times P) \cup (M \times \{e_P\})$.

(b)

$$(m_1, \ldots, m_j, m_{j+1}, \ldots, m_{i+1};\ p_1, \ldots, p_j, p_{j+1}, \ldots, p_{i+1}) \sim \\ (m_1, \ldots, \hat{m}_{j+1}, \ldots, m_{i+1};\ p_1, \ldots, p_{j-1}, p_j p_{j+1}, p_{j+2}, \ldots, p_{i+1})$$

for $m_j = m_{j+1}$.

3. For $m \in M$, $(m; e_P)$ represents the basepoint $* \in \widetilde{C}_\Sigma(M, N; P)$.

Remark 2.3.
Each non-basepoint configuration in $\widetilde{C}_\Sigma(M, N; P)$ is uniquely written as $\sum_{j=1}^{i} p_j m_j$, where $m_1, \ldots, m_i$ are all distinct and contained in $M \setminus N$, and $p_1, \ldots, p_i \neq e_P$. A similar uniqueness, up to order, holds for $C_\Sigma(M, N; P)$.

Example 2.4.
For a based space $(M, *)$, the infinite symmetric product $\mathrm{SP}_\infty(M)$ is homeomorphic to $C_\Sigma(M, \{*\}; \mathbb{N})$, and the James construction $J(M)$ is homeomorphic to $\widetilde{C}_\Sigma(M, \{*\}; \mathbb{N})$.

Proposition 2.1.
Let M be a space, $N \subset M$ be a closed subspace, and (Y, y_0) be a based space. Regard Y as a trivial partial monoid. Then

$$\widetilde{C}_\Sigma(M, N; Y) \cong \widetilde{C}(M, N; Y),$$
$$C_\Sigma(M, N; Y) \cong C(M, N; Y).$$

Example 2.5.
$C_\Sigma\big([0,1], \{1\}; \mathbb{N}\big)$ and $C\big([0,1], \{1\}; \mathbb{N}\big)$ are not homeomorphic. In $C_\Sigma([0,1], \{1\}; \mathbb{N})$,

$$\lim_{n \to \infty} \{(\frac{1}{n}; 1), (\frac{1}{n+1}; 1)\} = (0; 2)$$

but not in $C([0,1], \{1\}; \mathbb{N})$.

We mainly study configuration spaces of unordered points of manifolds with summable labels. The definition of $C_\Sigma(M, N; P)$ assumes that P is commutative. The commutative summation is the trivial way to translate collisions into summations.

If M is a parallelizable n-manifold, the n-dimensional Fulton-MacPherson operad $\mathcal{F}_n$ can encode both collisions of particles in M and noncommutative summations. Thus $\mathcal{F}_n$ provides a way to translate collisions into summations and to construct configuration spaces of M with noncommutative summable labels. For a general n-manifold M, we are motivated to define the bundle of n-dimensional Fulton-MacPherson operads associated to the frame bundle of M.

Chapter 3

Twisted Operads

In this chapter we first review basic definitions in operad theory. As an application of operads to configuration spaces we construct tubular configuration spaces of topological m-manifolds with m-fold loop space labels. Then we introduce the notion of a twisted operad, which is a bundle of operads. The twisted little disks operad over a manifold is presented as an example. As an application to configuration spaces, we construct configuration spaces of m-manifolds with twisted m-fold loop space labels.

3.1 Operads

We refer to [8] for the detail of basic operad theory. We only define the terminology.

First we introduce some notations for Σ_n.

Notation 3.1.
Let $k \in \mathbb{N}_{>0}$, $n_1, \ldots, n_k \in \mathbb{N}$, and $n = \sum_{j=1}^{k} n_j$. $[n]$ admits an ordered partition κ into $[n_1], \ldots, [n_k]$:

$$\kappa = ([n_1], n_1 + [n_2], \ldots, \sum_{j=1}^{k-1} n_j + [n_k]).$$

Each Σ_{n_j} acts on the right of $[n]$ by permuting the j-th block of κ. This induces a group homomorphism

$$\Pi_{j=1}^{k} \Sigma_{n_j} \hookrightarrow \Sigma_n, \quad (\mu_j)_{j=1}^{k} \mapsto (\oplus_{j=1}^{k} \mu_j).$$

Σ_k acts on the right of $[n]$ by permuting κ blockwise and it induces a group homomorphism

$$\Sigma_k \hookrightarrow \Sigma_n, \quad \sigma \mapsto \sigma\big((n_j)_{j=1}^{k}\big).$$

Definition 3.1 (topological operad)**.**
A (topological) operad *is a sequence of left Σ_n-spaces $o = \big(o(n)\big)_{n\in\mathbb{N}}$ with a collection of continuous maps, called* **composition maps***:*

$$\begin{aligned} &o(k) \times \Pi_{j=1}^{k} o(n_j) \to o(\Sigma_{j=1}^{k} n_j), \qquad (k, n_j \in \mathbb{N}) \\ &\big(f; (f_j)_{j=1}^{k}\big) \mapsto f \circ (f_j)_{j=1}^{k} \end{aligned}$$

such that the following hold:

1. $o(0) = \{*\}$ *is a one-point space. If $f = *$, $f \circ (f_j)_{j=1}^{k}$ means $*$.*
2. $o(1)$ *contains an element* 1 *called the* **unit** *such that*
$$1 \circ f = f,\ f \circ (1)_{j=1}^{k} = f. \qquad \big(k \in \mathbb{N},\ f \in o(k)\big)$$
3. *Composition maps are associative and compatible with Σ_n-actions*
$$\begin{aligned} &\big(f \circ (f_j)_{j=1}^{k}\big) \circ \big((g_i^j)_{i=1}^{n_j}\big)_{j=1}^{k} = f \circ \big(f_j \circ (g_i^j)_{i=1}^{n_j}\big)_{j=1}^{k}, \\ &(\sigma f) \circ (f_j)_{j=1}^{k} = \sigma\big((n_j)_{j=1}^{k}\big)\big(f \circ (f_{\sigma^{-1}j})_{j=1}^{k}\big), \\ &f \circ (\mu_j f_j)_{j=1}^{k} = (\oplus_{j=1}^{k} \mu_j)\big(f \circ (f_j)_{j=1}^{k}\big) \end{aligned}$$
where (1) $k \in \mathbb{N}$, $f \in o(k)$, $\sigma \in \Sigma_k$, (2) $n_j \in \mathbb{N}$, $f_j \in o(n_j)$, $\mu_j \in \Sigma_{n_j}$, *and* (3) *each g belongs to a component of o, $o(n_g)$, dependent to g.*

A **suboperad of** *o is an operad $u = \big(u(n)\big)_{n\in\mathbb{N}}$ such that $u(n) \subset o(n)$ for $n \in \mathbb{N}$, and the composition maps and Σ_n-actions of u are restrictions of those of o on u.*

Definition 3.2 (Little X operad)**.**
Let X be a space. We give an operad structure to the sequence

$$\amalg_X = \big(\amalg_X(n) = \mathrm{Map}(\amalg_{i=1}^{n} X, X)\big)_{n\in\mathbb{N}}$$

such that

1. $\amalg_X(0)$ *is the one-point space containing the unique inclusion $\emptyset \hookrightarrow X$.*
2. *For $n \in \mathbb{N}$, Σ_n acts on the left of $\amalg_X(n)$ by permuting n of X's in the domain.*
3. *The composition maps are defined by arranging compositions of maps as follows:*
$$\begin{aligned} &\amalg_X(k) \times \big(\Pi_{j=1}^{k} \amalg_X(n_j)\big) \to \amalg_X(\Sigma_{j=1}^{k} n_j), \\ &\Big(f = \amalg_{j=1}^{k}(f_j : X \to X);\ (g_j)_{j=1}^{k}\Big) \mapsto \amalg_{j=1}^{k} f_j g_j. \end{aligned}$$

4. $id_X = 1$.

We call the operad $\amalg_X$ *the* **little** X **operad**.

Definition 3.3 (operad morphism)**.**
An **operad morphism** *between two operads* $o = \big(o(n)\big)_{n\in\mathbb{N}}$ *and* $p = \big(p(n)\big)_{n\in\mathbb{N}}$*, denoted by* $\varphi : o \to p$*, is a sequence of* Σ_n*-equivariant maps*

$$\varphi(n) : o(n) \to p(n) \qquad (n \in \mathbb{N})$$

such that $\varphi(1) : 1 \mapsto 1$ *and satisfies*

$$\varphi(\Sigma_{j=1}^k n_j) : f \circ (f_j)_{j=1}^k \mapsto \varphi(k)(f) \circ \big(\varphi(n_j)(f_j)\big)_{j=1}^k$$

for (1) $k \in \mathbb{N}, f \in o(k)$*, and* (2) $n_j \in \mathbb{N}, f_j \in o(n_j)$*.*

Example 3.1.
For an operad o*, there is an obvious operad morphism,* $id_o : o \to o$*, called the* **identity map of** o*, defined by* $\big(id(n) : o(n) = o(n)\big)_{n\in\mathbb{N}}$*. For two operad morphisms,* $\varphi : o \to p$*,* $\psi : p \to q$*, the composition* $\psi\circ\varphi$ *is defined by* $\big((\psi\circ\varphi)(n) = \psi(n)\circ\varphi(n)\big)_{n\in\mathbb{N}}$*. Obviously* $\psi \circ \varphi : o \to q$ *is an operad morphism. Thus we can form the* **category of operads**.

As a monoid acting on a space X encodes endomorphisms of X , an operad acting on X encodes a composable system of n-ary operations, $X^n \to X$ $(n \in \mathbb{N})$. To make this idea precise, and introduce the actions of operads only partially defined, we define a partial left algebra over an operad.

Definition 3.4 (partial left algebra over an operad)**.**
A **partial left algebra over an operad** $o = \big(o(n)\big)_{n\in\mathbb{N}}$ *is a based space* $(X, *)$ *with a partially-defined left action of* o*, which is a sequence of maps from* $Comp(n) \subset o(n) \times X^n$ *to* X *for* $n \in \mathbb{N}$*, denoted as*

$$Comp(n) \to X, \ \ \big(f; (x_k)_{k=1}^n\big) \mapsto f \circ (x_k)_{k=1}^n$$

satisfying the following:

1. $\big(o(n) \times X^n, Comp(n)\big)$ *is an NDR pair for* $n \in \mathbb{N}$*.*

2. $o(n) \times \{(*)_{k=1}^n\} \subset Comp(n)$*, and* $f \circ (*)_{k=1}^n = *$ *for* $n \in \mathbb{N}$ *and* $f \in o(n)$*. In particular,* $Comp(0) = o(0) \times X^0 = \{(*; *)\}$*, and* $* \circ * = * \in X$*.*

3. $\{1\} \times X \subset Comp(1)$*, and* $1 \circ x = x$ *for* $x \in X$*.*

4. *The following two terms:*

$$A = \left(f \circ (f_j)_{j=1}^k\right) \circ \left((x_i^j)_{i=1}^{n_j}\right)_{j=1}^k,$$
$$B = f \circ \left(f_j \circ (x_i^j)_{i=1}^{n_j}\right)_{j=1}^k$$

are both defined and coincide whenever either A or B is defined, where (1) $k \in \mathbb{N}$, $f \in o(k)$, (2) $n_j \in \mathbb{N}$, $f_j \in o(n_j)$, *and* (3) *each x belongs to X.*

5. *For $n \in \mathbb{N}$, $f \in o(n)$, $\sigma \in \Sigma_n$, and $(x_k)_{k=1}^n \in X^n$, the following two terms*

$$C = (\sigma f) \circ (x_k)_{k=1}^n,$$
$$D = f \circ (x_{\sigma^{-1}k})_{k=1}^n$$

are both defined and coincide whenever either C or D is defined.

If $Comp(n) = o(n) \times X^n$ for $n \in \mathbb{N}$, X is called a **left algebra over** *o, or we simply say that X is* **not partial**.

Example 3.2 (trivial summable label)**.**
Let $(X, *)$ be a based space and o be an operad. X is a partial left algebra over o with the following action:

$$Comp(n) = o(n) \times \bigvee_{k=1}^n X \to X,$$
$$\left(f; \left((*)_{j=1}^{k-1}, x, (*)_{j=1}^{n-k}\right)\right) \mapsto x.$$

We will call X with this action a **trivial summable label** *(with respect to o).*

Definition 3.5 (partial left algebra morphism)**.**
Let $\varphi : o \to p$ be an operad morphism, and $(X, *)$ be a partial left algebra over o and $(Y, *)$ be a partial left algebra over p. A based map $\beta : X \to Y$ is called a **φ-equivariant partial left algebra morphism**, *if the following hold:*

- *For $n \in \mathbb{N}$, $f \in o(n)$, and $(x_k)_{k=1}^n \in X^n$, if $f \circ (x_k)_{k=1}^n$ is defined, then $\varphi(n)(f) \circ \left(\beta(x_k)\right)_{k=1}^n$ is also defined and*

$$\beta : f \circ (x_k)_{k=1}^n \mapsto \varphi(n)(f) \circ \left(\beta(x_k)\right)_{k=1}^n.$$

When φ is the identity map of o, we say that β is **o-equivariant**.

Example 3.3.
Let $(X, *)$ be a partial left algebra over an operad o. Then id_X is o-equivariant. Assume that $\varphi : o \to o'$ and $\varphi' : o' \to o''$ are operad morphisms, $\beta : X \to X'$ is φ-equivariant, and $\beta' : X' \to X''$ is φ'-equivariant. Then $\beta' \circ \beta$ is $\varphi' \circ \varphi$-equivariant. Thus we can form the **category of pairs** (o, X) **of an operad and its partial left algebra** *such that morphisms from (o, X) to (o', X') are pairs (φ, β) of an operad morphism $\varphi : o \to o'$ and a φ-equivariant partial left algebra morphism $\beta : X \to X'$. If $(\varphi, \beta) : (o, X) \to (o', X')$ is an isomorphism in the category, we also say that β* **is a φ-equivariant isomorphism**.

Now we define a right module over an operad.

Definition 3.6 (right module over an operad)**.**
A **right module over an operad** $o = \big(o(n)\big)_{n\in\mathbb{N}}$ *is a sequence of left Σ_k-spaces, $C = \big(C(k)\big)_{k\in\mathbb{N}}$, with a right action of o, which is a collection of maps*

$$\begin{aligned}&C(k) \times \Pi_{j=1}^{k} o(n_j) \to C(\Sigma_{j=1}^{k} n_j), \qquad (k, n_j \in \mathbb{N})\\ &\big(c; (f_j)_{j=1}^{k}\big) \mapsto c \circ (f_j)_{j=1}^{k}\end{aligned}$$

such that the following hold:

1. *$C(0) = \{*\}$ is a one-point space. If $c = *$, $c \circ (f_j)_{j=1}^{k}$ means $*$.*
2. *For $k \in \mathbb{N}$ and $c \in C(k)$,*
$$c \circ (1)_{j=1}^{k} = c.$$
3. *The action of o on C is compatible with composition maps of o, Σ_n-actions on o, and Σ_k-actions on C*
$$\begin{aligned}&\big(c \circ (f_j)_{j=1}^{k}\big) \circ \big((g_i^j)_{i=1}^{n_j}\big)_{j=1}^{k} = c \circ \big(f_j \circ (g_i^j)_{i=1}^{n_j}\big)_{j=1}^{k},\\ &(\sigma c) \circ (f_j)_{j=1}^{k} = \sigma\big((n_j)_{j=1}^{k}\big)\big(c \circ (f_{\sigma^{-1}j})_{j=1}^{k}\big),\\ &c \circ (\mu_j f_j)_{j=1}^{k} = (\oplus_{j=1}^{k}\mu_j)\big(c \circ (f_j)_{j=1}^{k}\big)\end{aligned}$$
where (1) *$k \in \mathbb{N}$, $c \in C(k)$, $\sigma \in \Sigma_k$,* (2) *$n_j \in \mathbb{N}$, $f_j \in o(n_j)$, $\mu_j \in \Sigma_{n_j}$, and* (3) *each g belongs to a component of o, $o(n_g)$, dependent to g.*

Example 3.4.
An operad o is a right module over o itself.

Remark 3.1.
We can grade a left algebra to obtain a left module as we define a right module. A left module over an operad o is a sequence of based Σ_k-spaces, $X = \big(X(k)\big)_{k\in\mathbb{N}}$, with a left action of the form

$$o(n) \times \Pi_{j=1}^{n} X(k_j) \to X(\Sigma_{j=1}^{n} k_j).$$

In particular $X(0)$ need not be a one-point space, and a left algebra X is a left module concentrated on the 0-th component, $X = (X(0), \emptyset, \emptyset, \ldots)$. Partial modules can also be defined. However, we only use right modules and partial left algebras.

Definition 3.7 (right module morphism)**.**
Let $\varphi : o \to p$ be an operad morphism, and $C = \big(C(k)\big)_{k\in\mathbb{N}}$ be a right module over o, and $D = \big(D(k)\big)_{k\in\mathbb{N}}$ be a right module over p. A sequence of Σ_k-equivariant maps, $\alpha = \big(\alpha(k) : C(k) \to D(k)\big)_{k\in\mathbb{N}} : C \to D$, is called a φ**-equivariant right module morphism**, *if the following hold:*

- *For $k \in \mathbb{N}$, $c \in C(k)$, and $f_j \in o(n_j)$,*

$$\alpha(\Sigma_{j=1}^{k} n_j) : c \circ (f_j)_{j=1}^{k} \mapsto \alpha(k)(c) \circ \big(\varphi(n_j)(f_j)\big)_{j=1}^{k}.$$

When φ is the identity map of o, we say that α is o**-equivariant**.

Example 3.5.
Let $C = (C(k))_{k\in\mathbb{N}}$ be a right module over an operad o. Then the identity map of C

$$id_C = \Big(id_{C(k)} : C(k) \to C(k)\Big)_{k\in\mathbb{N}} : C \to C$$

is o-equivariant. Assume that $\varphi : o \to o'$ and $\varphi' : o' \to o''$ are operad morphisms, $\alpha : C \to C'$ is φ-equivariant, and $\alpha' : C' \to C''$ is φ'-equivariant. Then the composition

$$\alpha' \circ \alpha = \Big(\alpha'(k) \circ \alpha(k)\Big)_{k\in\mathbb{N}} : C \to C''$$

is $\varphi' \circ \varphi$-equivariant. Thus we can form the **category of pairs** (C, o) **of an operad** o **and its right module** C *such that morphisms from (C, o) to (C', o') are pairs (α, φ) of a φ-equivariant right module morphism $\alpha : C \to C'$ and an operad morphism $\varphi : o \to o'$. If $(\alpha, \varphi) : (C, o) \to (C', o')$ is an isomorphism in the category, we also say that α is a* φ**-equivariant isomorphism**.

We define a tensor product of a right module and a partial left algebra over an operad. This will be used later as an operation of decorating configurations with summable labels.

Definition 3.8 (tensor product over an operad)**.**
Let $o = \big(o(n)\big)_{n\in\mathbb{N}}$ be an operad, $C = \big(C(k)\big)_{k\in\mathbb{N}}$ be a right module over o, and $(X, *)$ be a partial left algebra over o.

The **tensor product of C and X over** *o is the quotient*

$$C \otimes_o X = \coprod_{k\in\mathbb{N}} (C \otimes_o X)(k) \Big/ \sim$$

where

$$\begin{aligned}
&(C \otimes_o X)(0) = C(0) \times X^0, \\
&(C \otimes_o X)(k) = \coprod_{n_1,\dots,n_k\in\mathbb{N}} \Big(C(k) \times \big(\Pi_{j=1}^k o(n_j)\big) \times X^{\sum_{j=1}^k n_j} \Big) \qquad (k \neq 0)
\end{aligned}$$

modulo the equivalence relation $\sim$ generated by the following relations.

Assume $k \neq 0$. Write an element of $(C \otimes_o X)(k)$ as $\big(c\,;\ (o_j)_j\,;\ (x_i)_i\big)$.

1.

$$\begin{aligned}
&\Big(c\,;\ (*)_j\,;\ *\Big) \sim \Big(*, *\Big) \in (C \otimes_o X)(0), \\
&\Big(c\,;\ (1)_j\,;\ (*)_j\Big) \sim \Big(*, *\Big) \in (C \otimes_o X)(0).
\end{aligned}$$

2. *For* (1) *$\sigma \in \Sigma_k$, $c \in C(k)$, $(x_i)_i \in X^{\sum_{j=1}^k n_j}$;* (2) *$\mu_j \in \Sigma_{n_j}$, $o_j \in o(n_j)$,*

$$\begin{aligned}
&\Big(c\,;\ (o_j)_j\,;\ (x_i)_i\Big) \sim \Big(\sigma c\,;\ (o_{\sigma j})_j\,;\ \sigma\big((n_j)_{j=1}^k\big)(x_i)_i\Big), \\
&\Big(c\,;\ (o_j)_j\,;\ (x_i)_i\Big) \sim \Big(c\,;\ (\mu_j o_j)_j\,;\ (\oplus_j \mu_j)(x_i)_i\Big).
\end{aligned}$$

3. *If $n_j \neq 0$ for some j,*

$$\Big(c\,;\ (o_j)_j\,;\ (x_i)_i\Big) \sim \Big(c \circ (o_j)_j\,;\ (1)_i\,;\ (x_i)_i\Big).$$

4. *If $n_1 \neq 0$ and the action $o_1 \circ (x_i)_{i=1}^{n_1}$ is defined*

$$\Big(c\,;\ (o_j)_j\,;\ (x_i)_i\Big) \sim \Big(c\,;\ 1, (o_j)_{j=2}^k\,;\ o_1 \circ (x_i)_{i=1}^{n_1}, (x_i)_{i=n_1+1}^{\sum_{j=1}^k n_j}\Big).$$

   If $n_1 = 0$, and $n_j \neq 0$ for some j,

$$\Big(c\,;\ (o_j)_j\,;\ (x_i)_i)\Big) = \Big(c\,;\ *, (o_j)_{j=2}^k\,;\ (x_i)_i\Big) \sim \Big(c\,;\ 1, (o_j)_{j=2}^k\,;\ *, (x_i)_i\Big).$$

The unique point in $(C \otimes_o X)(0)$ represents the basepoint of $C \otimes_o X$.

The construction of a tensor product over an operad is functorial in the following sense. Let $\varphi : o \to p$ be an operad morphism, $\alpha : C \to D$ be a φ-equivariant morphism between right modules, and $\beta : X \to Y$ be a φ-equivariant morphism between partial left algebras. Then we have an induced based map between tensor products

$$\begin{aligned}&(\alpha, \varphi, \beta)_* : C \otimes_o X \to D \otimes_p Y,\\ &\Big(c\,;\ (o_j)_j\,;\ (x_i)_i\Big) \mapsto \Big(\alpha(c)\,;\ \big(\varphi(o_j)\big)_j\,;\ \big(\beta(x_i)\big)_i\Big).\end{aligned}$$

Notation 3.2.
In the formula of $(\alpha, \varphi, \beta)_*$ above, the grading of φ and α are omitted. We will use this abuse of notation whenever convenient.

Proposition 3.1.
Let $\mathcal{C}$ be the category of triples (C, o, X) such that o is an operad, C is a right module over o, X is a partial left algebra over o. The morphisms from (C, o, X) to (C', o', X') are triples (α, φ, β) of a φ-equivariant morphism $\alpha : C \to C'$ between right modules, an operad morphism $\varphi : o \to o'$, and a φ-equivariant morphism $\beta : X \to X'$ between partial left algebras. The composition $(\alpha', \varphi', \beta') \circ (\alpha, \varphi, \beta)$ is defined by componentwise compositions. Then there is a functor from $\mathcal{C}$ to the category of based spaces defined by

$$\begin{aligned}(C, o, X) &\mapsto C \otimes_o X,\\ (\alpha, \varphi, \beta) &\mapsto (\alpha, \varphi, \beta)_*.\end{aligned}$$

Example 3.6 (infinite symmetric product)**.**
Let $o = \big(\{*_n\}\big)_{n\in\mathbb{N}}$ be the operad such that $o(n)$ is a one-point space for $n \in \mathbb{N}$. There exists an obvious left action of o on $\mathbb{N}$ such that $*_n$ encodes summations of n integers. Let $(X, *)$ and $(Y, *)$ be based spaces. The sequence of Cartesian products $C = (X^k)_{k\in\mathbb{N}}$ forms a right module over o such that

$$(x_j)_{j=1}^k \circ (*_{n_j})_{j=1}^k = \big((x_j)_{i=1}^{n_j}\big)_{j=1}^k.$$

Similarly $D = (Y^k)_{k\in\mathbb{N}}$ is a right module over o. Then $C \otimes_o \mathbb{N} \cong \mathrm{SP}_\infty(X)$, $D \otimes_o \mathbb{N} \cong \mathrm{SP}_\infty(Y)$, and a based map $f : X \to Y$ induces a map between tensor products $\mathrm{SP}_\infty(X) \to \mathrm{SP}_\infty(Y)$ defined by

$$\big((x_j)_{j=1}^k; (*_1)_{j=1}^k; (a_j)_{j=1}^k\big) \mapsto \Big(\big(f(x_j)\big)_{j=1}^k; (*_1)_{j=1}^k; (a_j)_{j=1}^k\Big).$$

3.2 Little disks operads and tubular configuration spaces with summable lables

We define the embedded m-dimensional open little disks operad and construct tubular configuration spaces with m-fold loop space labels by taking a tensor product over the operad.

Definition 3.9 (embedded open little disks operad)**.**
The **embedded m-dimensional open little disks operad** $\mathring{\mathcal{D}}^m$ *is the suboperad of* $\amalg_{\mathring{D}^m}$ *such that*

$$\mathring{\mathcal{D}}^m = \left(\mathring{\mathcal{D}}^m(n) = \mathrm{Emb}_{top}(\coprod_{k=1}^{n} \mathring{D}^m, \mathring{D}^m) \right)_{n \in \mathbb{N}}.$$

We define a left algebra and right module over $\mathring{\mathcal{D}}^m$ to be used for the construction of tubular configuration spaces with summable labels.

Definition 3.10 (m-fold loop space)**.**
*We formulate the m-***fold loop space of a based space** $(Y, *)$, *denoted by* $\Omega^m Y$, *as the mapping space*

$$\Omega^m Y = \mathrm{Map}_c\left(\mathring{D}^m, Y\right)$$

based at the constant map $*$ sending $\mathring{D}^m$ to $*$. $\Omega^m Y$ is a left algebra over $\mathring{\mathcal{D}}^m$ with the action

$$\begin{aligned} &\mathring{\mathcal{D}}^m(n) \times (\Omega^m Y)^n \to \Omega^m Y, \\ &\left(f = (f_k : \mathring{D}^m \to \mathring{D}^m)_{k=1}^n ; (g_k)_{k=1}^n\right) \mapsto f \circ (g_k)_{k=1}^n \end{aligned}$$

where $f \circ (g_k)_{k=1}^n : \mathring{D}^m \to Y$ is the union of the following $(n+1)$-maps:

$$\begin{aligned} &f_k(\mathring{D}^m) \xrightarrow{f_k^{-1}} \mathring{D}^m \xrightarrow{g_k} Y, \qquad (k = 1, \dots, n) \\ &\mathring{D}^m \setminus \mathrm{im}(f) \to \{*\} \hookrightarrow Y. \end{aligned}$$

Definition 3.11 (open little disks in M)**.**
For a topological m-manifold M, the **open little disks in** M *denoted by $\mathring{\mathcal{D}}_M$ is a right module over $\mathring{\mathcal{D}}^m$*

$$\mathring{\mathcal{D}}_M = \left(\mathring{\mathcal{D}}_M(n) = \mathrm{Emb}_{top}(\coprod_{k=1}^{n} \mathring{D}^m, M) \right)_{n \in \mathbb{N}}$$

such that

1. $\mathring{\mathcal{D}}_M(0)$ *is the one-point space containing the unique inclusion* $\emptyset \hookrightarrow M$.

2. For $n \in \mathbb{N}$, Σ_n *acts on the left of* $\mathring{\mathcal{D}}_M(n)$ *by permuting the disks in the domain.*

3. The right action of $\mathring{\mathcal{D}}^m$ *is defined as*

$$\mathring{\mathcal{D}}_M(k) \times \Pi_{j=1}^k \mathring{\mathcal{D}}^m(n_j) \to \mathring{\mathcal{D}}_M(\Sigma_{j=1}^k n_j),$$
$$\Big(f = (f_j : \mathring{D}^m \to M)_{j=1}^k;\ (g_j)_{j=1}^k\Big) \mapsto f \circ (g_j)_{j=1}^k = \Pi_{j=1}^k (f_j g_j).$$

Definition 3.12 (tubular configuration space with summable label)**.**
Let M *be a topological* m*-manifold,* $N \subset M$ *be a codimension-*0 *topological submanifold such that* $M \setminus N$ *is an open manifold, and* $(Y, *)$ *be a based space. The* **tubular configuration space of** M **vanishing at** N **with summable label** $\Omega^m Y$ *is the quotient*

$$E_\Sigma(M, N; \Omega^m Y) = (\mathring{\mathcal{D}}_M \otimes_{\mathring{\mathcal{D}}^m} \Omega^m Y)\Big/\!\sim$$

modulo the equivalence relation $\sim$ *generated by the following relation:*

- *For* $f = (f_j)_{j=1}^k \in \mathring{\mathcal{D}}_M(k)$*, and* $(h_j)_{j=1}^k \in \big(\Omega^m(Y)\big)^k$*, if* $\mathrm{im}(f_1) \subset N$,

$$\big((f_j)_{j=1}^k\,;\ (1)_{j=1}^k\,;\ (h_j)_{j=1}^k\big) \sim \big((f_j)_{j=2}^k\,;\ (1)_{j=2}^k\,;\ (h_j)_{j=2}^k\big).$$

An element of $E_\Sigma(M, N; \Omega^m Y)$

$$\big(f\,;\ (g_j)_j\,;\ (h_i)_i\big)$$

is determined by a finite number of disks $f \circ (g_j)_j$ in M labelled with m-fold loops $(h_i)_i$. Thus each element of $E_\Sigma(M, N; \Omega^m Y)$ can be represented as $\big((f_j)_{j=1}^k\,;\ (1)_{j=1}^k\,;\ (h_j)_{j=1}^k\big)$. If some of those disks are placed near together and identified as an element of $\mathring{\mathcal{D}}^m$ embedded in M, the labels of the disks are summed into a loop labelling a single larger disk. Thus the labels in $\Omega^m Y$ are actually summable in $E_\Sigma(M, N; \Omega^m Y)$.

Each element of $E_\Sigma(M, N; \Omega^m Y)$ induces a map from $M \setminus N$ to Y by taking union of pullbacks of loops.

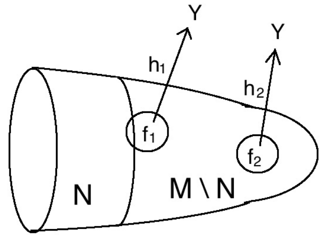


Figure 3.1: $E_\Sigma(M, N; \Omega^m Y)$

Proposition 3.2.

1. *There exists a map*

$$s : E_\Sigma(M, N; \Omega^m Y) \to \mathrm{Map}(M \setminus N, Y)$$

defined as

$$\begin{aligned}
&x = \big(f\,;\ (1)_{j=1}^k\,;\ (h_j)_{j=1}^k\big) \in E_\Sigma(M, N; \Omega^m Y),\\
&f = (f_j : \mathring{D}^m \to M)_{j=1}^k \in \mathring{\mathcal{D}}_M(k),\ \ h_j \in \Omega^m Y,\\
&s(x) : M \setminus N \to Y,\\
&s(x) : p \mapsto \left\{ \begin{array}{cl} * & \textit{if } p \notin \mathrm{im}(f) \\ h_j f_j^{-1}(p) & \textit{if } p \in \mathrm{im}(f_j) \end{array} \right. .
\end{aligned}$$

2. *Let* $\mathrm{Homeo}(M; M \setminus N, N)$ *denote the group of self-homeomorphisms of* M *sending* $M \setminus N$ *to* $M \setminus N$ *and sending* N *to* N*. The map* s *is* $\mathrm{Homeo}(M; M \setminus N, N)$*-equivariant with respect to the following actions:*

$$\begin{aligned}
&\varphi \in \mathrm{Homeo}(M; M \setminus N, N),\ \alpha \in \mathrm{Map}(M \setminus N, Y),\\
&\varphi \cdot \big(f\,;\ (g_j)_j\,;\ (h_i)_i\big) = \big(\varphi f\,;\ (g_j)_j\,;\ (h_i)_i\big),\\
&\varphi \cdot \alpha = \alpha \circ \varphi^{-1}.
\end{aligned}$$

Proof.

1. Check that s is continuous. It is equivalent to check that the map

$$E_\Sigma(M, N; \Omega^m Y) \times (M \setminus N) \to Y,\ \ (x, p) \mapsto \big(s(x)\big)(p)$$

is continuous.

The number of disks induces an open filtration of $E_\Sigma(M, N; \Omega^m Y)$

$$\begin{aligned}
&* = E_\Sigma^0 \subset E_\Sigma^1 \subset \cdots \subset E_\Sigma(M, N; \Omega^m Y),\\
&E_\Sigma^k = \coprod_{\substack{1 \le j \le k \\ n_1, \ldots, n_j \in \mathbb{N}}} \left(\mathring{\mathcal{D}}_M(j) \times (\mathring{\mathcal{D}}^m(n_1) \times \cdots \times \mathring{\mathcal{D}}^m(n_j)) \times \Omega^m Y^{\sum_{i=1}^j n_i}\right) \Big/ \sim
\end{aligned}$$

It is clear that small variations of p, k-disks and loops induce small variations in Y. The relation forgetting disks contained in N does not affect $s(x)$ defined on $M \setminus N$.

2. To prove the equivariance is a straightforward computation.

$$\begin{aligned}
x &= \big(f = (f_j)_{j=1}^k\,;\ (1)_{j=1}^k\,;\ (h_j)_{j=1}^k\big),\\
\varphi \cdot x &= \big((\varphi f_j)_{j=1}^k\,;\ (1)_{j=1}^k\,;\ (h_j)_{j=1}^k\big),\\
s(\varphi\cdot x) : p &\mapsto \begin{cases} * & \text{if } p \notin \operatorname{im}(\varphi f)\\ h_j(\varphi\circ f_j)^{-1}(p) & \text{if } p \in \operatorname{im}(\varphi\circ f_j)\end{cases}\\
&= \begin{cases} * & \text{if } \varphi^{-1}(p) \notin \operatorname{im}(f)\\ h_j f_j^{-1}\big(\varphi^{-1}(p)\big) & \text{if } \varphi^{-1}(p) \in \operatorname{im}(f_j)\end{cases}\\
&= s(x)\big(\varphi^{-1}(p)\big) = \big(\varphi\cdot(s(x))\big)(p).
\end{aligned}$$

□

Remark 3.2.
Note that a reparametrization of $\mathring{D}^m$ by a self-homeomorphism g of $\mathring{D}^m$ does not yield a different element of $E_\Sigma(M, N; \Omega^m Y)$. More precisely consider the following diagram:

$$\begin{tikzcd}
M & \mathring{D}^m \arrow[l, "f"'] \arrow[r, "h"] & Y\\
& \mathring{D}^m \arrow[u, "\cong"', "g"] &
\end{tikzcd}$$

$(fg; 1; hg) = (f \circ g; 1; g^{-1} \circ h) = (f; g \circ g^{-1}; h) = (f; 1; h)$ *in* $\mathring{\mathcal{D}}_M \otimes_{\mathring{\mathcal{D}}^m} \Omega^m Y$*, where* $\circ$ *denotes actions of the operad.*

Remark 3.3.
We conjecture that the map s is a weak equivalence for compact M if Y is m-connected, or (M, N) is connected and Y is $(m-1)$-connected. The first approach to try is to adjust the proof in [9] to tubular configurations.

3.3 Twisted operads

We develop the basic theory of twisted operads. A twisted operad over a space B is a sequence of fibre bundles over B such that the sequence of fibres over each $b \in B$ is isomorphic to a fixed operad in a locally trivial manner.

Notation 3.3.
To refer to a fibre bundle $(F \to E \xrightarrow{p} B)$, we will briefly write E for convenience as long as it is clear. For $b \in B$ the fibre of E over b is denoted by E_b. For $b \in B$ and

a bundle map

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$

ϕ_b denotes a restriction of ϕ over b, $\phi_b : E_b \to E'_{g(b)}$. For $U \subset B$, $E|_U$ denotes the restricted bundle of E over U.

Definition 3.13 (twisted operad)**.**
Let B be a space and $o = \big(o(n)\big)_{n\in\mathbb{N}}$ be an operad. A **twisted operad over B with the typical fibre** *o is a sequence of fibre bundles over B*

$$E = \Big(o(n) \to E(n) \xrightarrow{p(n)} B\Big)_{n\in\mathbb{N}}$$

together with (1) *fibre-preserving actions of Σ_n on the left of $E(n)$ for $n \in \mathbb{N}$,* (2) *a collection of bundle maps covering id_B, called* **composition maps***,*

$$\begin{aligned} &E(k) \times_B \big((\times_B)_{j=1}^{k} E(n_j)\big) \to E(\Sigma_{j=1}^{k} n_j), \qquad (k, n_j \in \mathbb{N}) \\ &\big(f; (f_j)_{j=1}^{k}\big) \mapsto f \circ (f_j)_{j=1}^{k} \end{aligned}$$

and (3) *a section of $E(1)$, called the* **unit***,*

$$1 : B \to E(1),\ b \mapsto 1_b$$

such that the following local triviality conditions hold:

1. *For each $b \in B$, if we restrict the Σ_n-actions and composition maps of E to the fibres of E over b and write those fibres as a sequence*

   $$E_b = \big(E(n)_b\big)_{n\in\mathbb{N}}$$

   then E_b forms an operad with the unit 1_b.

2. *There exists an open cover $\{U\}$ of B and for each U a sequence of local trivializations covering id_U*

   $$t_n : U \times o(n) \to E(n)|_U \qquad (n \in \mathbb{N})$$

   such that for each $u \in U$ the restriction of $(t_n)_{n\in\mathbb{N}}$

   $$\Big(t_n(u, -) : o(n) \to E(n)_u\Big)_{n\in\mathbb{N}}$$

   gives an isomorphism of operads between o and E_u.

Remark 3.4.

Let G be the topological group of operad isomorphisms of an operad o. The definition of a twisted operad

$$E = \Big(o(n) \to E(n) \xrightarrow{p(n)} B\Big)_{n\in\mathbb{N}}$$

is equivalent to saying that $(\times_B)_{n=0}^{\infty} E(n)$ is a fibre bundle over B with the typical fibre $\Pi_n o(n)$ and the structure group G.

Example 3.7.

For an operad o and a space B, the **trivial twisted operad over B with the typical fibre** *o, denoted by $o \times B$, is the twisted operad defined by*

$$o \times B = \Big(o(n) \to o(n) \times B \xrightarrow{p(n)} B\Big)_{n\in\mathbb{N}}$$

where each component of the sequence is the trivial bundle over B with the typical fibre $o(n)$. The actions of Σ_n and composition maps are the fibrewise copies of those of $o(n)$.

Example 3.8 (fibre product of twisted operads)**.**

For two twisted operads over B

$$E = \Big(o(n) \to E(n) \xrightarrow{p(n)} B\Big)_{n\in\mathbb{N}}, \quad E' = \Big(o'(n) \to E'(n) \xrightarrow{p'(n)} B\Big)_{n\in\mathbb{N}}$$

the sequence of fibre products

$$E \times_B E' = \Big(o(n) \times o'(n) \to E(n) \times_B E'(n) \xrightarrow{p(n)\times_B p'(n)} B\Big)_{n\in\mathbb{N}}$$

forms a twisted operad over B with diagonal Σ_n-actions and composition maps induced componentwise from E and E'. If E and E' are locally trivial over open covers $\{U\}$ and $\{U'\}$ respectively, then $E \times_B E'$ is locally trivial over the open cover $\{U \cap U'\}$.

Example 3.9 (associated twisted operad to a fibre bundle)**.**

Let $(F \to E \to B)$ be a fibre bundle with structure group G, and $o = \big(o(n)\big)_{n\in\mathbb{N}}$ be an operad with G-equivariant composition maps and G-equivariant Σ_n-actions in the sense that $o(n)$ for $n \in \mathbb{N}$ is a left G-space with the properties

$$\begin{aligned}
&(g \cdot f) \circ (g \cdot f_k)_{k=1}^n = g \cdot \big(f \circ (f_k)_{k=1}^n\big),\\
&g \cdot 1 = 1,\\
&g \cdot (\sigma f) = \sigma(g \cdot f)
\end{aligned}$$

for $g \in G$; $f \in o(n)$, $\sigma \in \Sigma_n$; $f_k \in o(n_k)$. In other words there exists a homomorphsm from G to the group of operad isomorphisms of o. Let $\{U_\alpha\}_\alpha$ be an open cover of B

such that E is locally trivial over U_α. Then we can form a twisted operad over B by taking the associated bundle to E with the typical fibre $\coprod_{n\in\mathbb{N}} o(n)$. It is constructed by gluing

$$\Big\{U_\alpha \times \coprod_{n\in\mathbb{N}} o(n)\Big\}_\alpha$$

with respect to transition maps extracted from E. We call the twisted operad the **associated twisted operad to E with the typical fibre** *o.*

Definition 3.14 (twisted operad morphism)**.**
A **twisted operad morphism** *between two twisted operads*

$$E = \Big(o(n) \to E(n) \xrightarrow{p(n)} B\Big)_{n\in\mathbb{N}}, \quad E' = \Big(o'(n) \to E'(n) \xrightarrow{p'(n)} B'\Big)_{n\in\mathbb{N}}$$

denoted by $(\varphi, g) : E \to E'$, consists of a map $g : B \to B'$ and a sequence of Σ_n-equivariant bundle maps covering g

$$\varphi = \Big(\varphi(n) : E(n) \to E'(n)\Big)_{n\in\mathbb{N}}$$

such that the restriction of φ over each $b \in B$ gives an operad morphism $\varphi_b : E_b \to E'_{g(b)}$,

$$\varphi_b = \Big(\varphi(n)_b : E(n)_b \to E'(n)_{g(b)}\Big)_{n\in\mathbb{N}}.$$

Example 3.10 (pullback of twisted operads)**.**
For a twisted operad $E = \big(o(n) \to E(n) \xrightarrow{p(n)} B\big)_{n\in\mathbb{N}}$ and a map $g : B' \to B$, we can define a pullback twisted operad consisting of pullback bundles $g^*E(n)$:

$$g^*E = \Big(o(n) \to g^*E(n) \xrightarrow{p'(n)} B'\Big)_{n\in\mathbb{N}}$$

with twisted operad structures induced from E in an obvious way. The sequence of pullback squares gives a twisted operad morphism $(\widetilde{g}, g) : g^*E \to E$:

$$(\widetilde{g}, g) = \left(\begin{tikzcd} g^*E(n) \arrow[r, "\widetilde{g}(n)"] \arrow[d, "p'(n)"'] & E(n) \arrow[d, "p(n)"] \\ B' \arrow[r, "g"] & B \end{tikzcd}\right)_{n\in\mathbb{N}}.$$

If $g : B' \hookrightarrow B$ is an inclusion, g^*E is also denoted as $E|_{B'}$ and called a restriction of E on B'.

Example 3.11.
For a twisted operad $E = \big(o(n) \to E(n) \xrightarrow{p(n)} B\big)_{n\in\mathbb{N}}$ there is an obvious twisted

operad morphism, $id_E = (\varphi, g) : E \to E$*, called the* **identity map of** E*, such that* $g = id_B$ *and* $\varphi(n) = id_{E(n)}$ *for* $n \in \mathbb{N}$*.*

We can compose two twisted operad morphisms $(\varphi, g) : E \to E'$ *and* $(\varphi', g') : E' \to E''$ *to obtain*

$$\begin{aligned}(\varphi', g') \circ (\varphi, g) &= (\varphi' \circ \varphi, g' \circ g) : E \to E'',\\ (\varphi' \circ \varphi)(n) &= \varphi'(n) \circ \varphi(n). \qquad (n \in \mathbb{N})\end{aligned}$$

Thus we can form the **category of twisted operads***.*

We define the twisted analogue of partial left algebras.

Definition 3.15 (twisted partial left algebra over a twisted operad)**.**
Let $E = \big(o(n) \to E(n) \xrightarrow{p(n)} B\big)_{n\in\mathbb{N}}$ *be a twisted operad, and* $(X, *)$ *be a partial left algebra over the operad* $o = \big(o(n)\big)_{n\in\mathbb{N}}$*. Recall that the action of* o *on* X *is written as a sequence of maps from* $Comp(n) \subset o(n) \times X^n$ *to* X*:*

$$Comp(n) \to X. \qquad\qquad (n \in \mathbb{N})$$

A **twisted partial left algebra over** E **with the typical fibre** X *is a fibre bundle over* B *with a basepoint section*

$$\eta = \Big((X, *) \to (\eta, \iota) \to B\Big)$$

together with a partially-defined left action of E *on* η*, which is a sequence of bundle maps from a subbundle* $Comp^\eta(n) \subset E(n) \times_B \eta^{(\times_B)n}$ *to* η *covering* id_B

$$\Big(E(n) \times_B \eta^{(\times_B)n} \supset Comp^\eta(n) \to \eta\Big)_{n\in\mathbb{N}}$$

such that the following local triviality conditions hold:

1. *For each* $b \in B$ *the based space* (η_b, ι_b) *is a partial left algebra over the operad* E_b *whose action*

   $$Comp^\eta(n)_b \to \eta_b \qquad\qquad (n \in \mathbb{N})$$

   is the restriction of that of E *on* η*.*

2. *There exists an open cover* $\{U\}$ *of* B *and for each* U *local trivializations*

   $$\begin{aligned}&s : U \times (X, *) \to (\eta|_U, \iota|_U),\\ &t_n : U \times o(n) \to E(n)|_U \qquad (n \in \mathbb{N})\end{aligned}$$

such that for each $u \in U$ the restriction of s over u

$$s(u, -) : X \to \eta_u$$

is an isomorphism of partial left algebras between X and η_u equivariant with respect to the operad isomorphism between o and E_u

$$\Big(t_n(u, -) : o(n) \to E(n)_u \Big)_{n \in \mathbb{N}}.$$

In particular the restriction of $t_n \times_U s^{(\times_U)n}$

$$U \times Comp(n) \hookrightarrow U \times o(n) \times X^n \xrightarrow{t_n \times_U s^{(\times_U)n}} \big(E(n) \times_B \eta^{(\times_B)n}\big)|_U$$

gives a local trivialization $U \times Comp(n) \cong Comp^\eta(n)|_U$ covering id_U.

If X is not partial, η is called **a twisted left algebra over** *E, or we simply say that η* **is not partial**.

Remark 3.5.
Let o be an operad and $(X, *)$ be a partial left algebra over o. Let G be the topological group consisting of pairs (φ, β) of an operad isomorphism $\varphi : o \to o$ and an φ-equivariant isomorphism $\beta : X \to X$. The definition of a twisted partial left algebra $\eta = \big((X, *) \to (\eta, \iota) \to B\big)$ over a twisted operad $E = \big(o(n) \to E(n) \xrightarrow{p(n)} B\big)_{n \in \mathbb{N}}$ is equivalent to saying that $\big((\times_B)_{n=0}^{\infty} E(n)\big) \times_B \eta$ is a fibre bundle over B with the typical fibre $\Pi_n o(n) \times X$ and the structure group G.

Remark 3.6.
The local triviality conditions of twisted operads and twisted partial left algebras are chosen to suit our purpose proving the weak equivalence of scanning maps. In other contexts it might be natural to drop the conditions so that a twisted operad means a sequence of bundles with composition maps given as bundle maps, and a twisted partial left algebra is a bundle on the left of which a twisted operad acts by bundle maps.

Example 3.12.
Let $o \times B$ be a trivial twisted operad over B with the typical fibre o, and $(X, *)$ be a partial left algebra over o. Then the trivial bundle $\big((X, *) \to (X, *) \times B \to B\big)$ is a twisted partial left algebra over $o \times B$ whose action is the fibrewise copies of the action of o on X.

Definition 3.16 (twisted partial left algebra morphism).
Let $(\varphi, g) : E \to E'$ be a twisted operad morphism written as

$$(\varphi, g) = \left(\begin{tikzcd} E(n) \arrow[r, "\varphi(n)"] \arrow[d, "p(n)"'] & E'(n) \arrow[d, "p'(n)"] \\ B \arrow[r, "g"] & B' \end{tikzcd} \right)_{n\in\mathbb{N}}$$

between two twisted operads

$$E = \Big(o(n) \to E(n) \xrightarrow{p(n)} B\Big)_{n\in\mathbb{N}}, \quad E' = \Big(o'(n) \to E'(n) \xrightarrow{p'(n)} B'\Big)_{n\in\mathbb{N}}.$$

Let η be a twisted partial left algebra over E with the typical fibre X:

$$\eta = \Big((X, *) \to (\eta, \iota) \to B\Big)$$

and η' be a twisted partial left algebra over E' with the typical fibre X':

$$\eta' = \Big((X', *) \to (\eta', \iota') \to B'\Big)$$

Then a bundle map $\beta : (\eta, \iota) \to (\eta', \iota')$ covering g

$$\begin{tikzcd} (\eta, \iota) \arrow[r, "\beta"] \arrow[d] & (\eta', \iota') \arrow[d] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$

*is called a (φ, g)-***equivariant twisted partial left algebra morphism***, if the restriction of β over each $b \in B$*

$$\beta_b : (\eta_b, \iota_b) \to (\eta'_{g(b)}, \iota'_{g(b)})$$

is a partial left algebra morphism equivariant with respect to the operad morphism

$$\varphi_b : E_b \to E'_{g(b)}.$$

*When (φ, g) is the identity map of E, we say that α is E-***equivariant***.*

Example 3.13 (pullback twisted partial left algebra).
Let $\big((X, *) \to (\eta, \iota) \to B\big)$ be a twisted partial left algebra over a twisted operad $E = \big(o(n) \to E(n) \xrightarrow{p(n)} B\big)_{n\in\mathbb{N}}$, and $g : B' \to B$ be a map. Then the pullback bundle $\big((X, *) \to (g^*\eta, g^*\iota) \to B'\big)$ is a twisted partial left algebra over g^*E with the action induced from that of E on η. Let $(\widetilde{g}, g) : g^*E \to E$ and $\hat{g} : g^*\eta \to \eta$ be the pullback maps. Then $\hat{g}$ is a $(\widetilde{g}, g)$-equivariant twisted partial left algebra morphism. If $g : B' \hookrightarrow B$ is an inclusion, $g^*\eta$ is also denoted as $\eta|_{B'}$ and called a restriction of η on B'.

Example 3.14.
For a twisted partial left algebra η over a twisted operad E, the identity map of η is E-equivariant. Let $(\varphi, g) : E \to E'$ and $(\varphi', g') : E' \to E''$ be twisted operad morphisms, and $\beta : \eta \to \eta'$ be a (φ, g)-equivariant twisted partial left algebra morphism, and $\beta' : \eta' \to \eta''$ be a (φ', g')-equivariant twisted partial left algebra morphism. Then the composition $\beta' \circ \beta : \eta \to \eta''$ is equivariant with respect to $(\varphi', g') \circ (\varphi, g) : E \to E''$. Thus we can form the **category of pairs** (E, η) **of a twisted operad and its twisted partial left algebra** *such that morphisms from (E, η) to (E', η') are pairs $\big((\varphi, g), \beta\big)$ of a twisted operad morphism $(\varphi, g) : E \to E'$ and a (φ, g)-equivariant twisted partial left algebra morphism $\beta : \eta \to \eta'$. If $\big((\varphi, g), \beta\big) : (E, \eta) \to (E', \eta')$ is an isomorphism in the category, we also say that β is a* (φ, g)**-equivariant isomorphism**.

It is a more difficult problem to find a right definition of twisted right modules than to define twisted operads and twisted partial left algebras. The motivating example of a twisted right module is a compactified configuration space of a manifold M where collisions of some particles in a configuration can occur, a collision of particles is represented by a single particle decorated with collision data and the decoration is encoded by the action of a twisted operad over M. Thus we define a twisted right module as right modules parametrized by $(M^k)_{k\in\mathbb{N}}$.

Definition 3.17 (twisted right module over a twisted operad)**.**
Let $E = \big(o(n) \to E(n) \xrightarrow{p(n)} B\big)_{n\in\mathbb{N}}$ be a twisted operad. A **twisted right module over** *E is a sequence of left Σ_k-spaces parametrized Σ_k-equivariantly by $(B^k)_{k\in\mathbb{N}}$*

$$\epsilon = \Big(\epsilon(k) \xrightarrow{\tau(k)} B^k\Big)_{k\in\mathbb{N}}$$

together with a right action of E on ϵ, which is a collection of maps

$$\begin{aligned} &\epsilon(k) \times_{B^k} \Pi_{j=1}^k E(n_j) \to \epsilon(\Sigma_{j=1}^k n_j), \qquad (k, n_j \in \mathbb{N}) \\ &\big(e; (f_j)_{j=1}^k\big) \mapsto e \circ (f_j)_{j=1}^k \end{aligned}$$

such that the following diagram commutes:

$$\begin{tikzcd} \epsilon(k) \times_{B^k} \Pi_{j=1}^k E(n_j) \arrow[r] \arrow[d] & \epsilon(\sum_{j=1}^k n_j) \arrow[d, "\tau(\sum_{j=1}^k n_j)"] \\ B^k \arrow[r, "\Delta\left((n_j)_{j=1}^k\right)"'] & B^{\sum_{j=1}^k n_j} \end{tikzcd}$$

where $\Delta\big((n_j)_{j=1}^k\big)$ *is the diagonal map*

$$\Delta\big((n_j)_{j=1}^k\big) : (b_j)_{j=1}^k \mapsto \big((b_j)_{i=1}^{n_j}\big)_{j=1}^k.$$

$\Delta\big((0)_{j=1}^k\big)$ *is understood to be the unique map from* B^k *to* B^0*, and if* $(n_j)_{j=1}^k \neq (0)_{j=1}^k$ *and* $n_j = 0$ *for some* j*, the coordinate* $(b_j)_{i=1}^{n_j}$ *is ignored.*

In addition ϵ *is required to satisfy the following:*

1. $\epsilon(0) = (* \xrightarrow{\tau(0)} B^0)$ *is a one-point space over* B^0*. If* $e = *$*,* $e \circ (f_j)_{j=1}^k$ *means* $*$*.*

2. *If* $\tau(k)(e) = (b_j)_{j=1}^k$*, then*
$$e \circ (1_{b_j})_{j=1}^k = e.$$

3. *The right action is compatible with composition maps of* E *and the actions of symmetric groups. More precisely,*
$$\begin{aligned}
&\big(e \circ (f_j)_{j=1}^k\big) \circ \big((g_i^j)_{i=1}^{n_j}\big)_{j=1}^k = e \circ \big(f_j \circ (g_i^j)_{i=1}^{n_j}\big)_{j=1}^k,\\
&(\sigma e) \circ (f_j)_{j=1}^k = \sigma\big((n_j)_{j=1}^k\big)\big(e \circ (f_{\sigma^{-1}j})_{j=1}^k\big),\\
&e \circ (\mu_j f_j)_{j=1}^k = (\oplus_{j=1}^k \mu_j)\big(e \circ (f_j)_{j=1}^k\big).
\end{aligned}$$
where (1) $k \in \mathbb{N}$*,* $e \in \epsilon(k)$*,* $\sigma \in \Sigma_k$*,* (2) $n_j \in \mathbb{N}$*,* $f_j \in E(n_j)$*,* $\mu_j \in \Sigma_{n_j}$*,* (3) *each* g *belongs to some component of* E*,* $E(n_g)$*, dependent to* g*, and* (4) *every* $\circ$ *in the equations is defined.*

Example 3.15.
Let $E = \big(o(n) \to E(n) \xrightarrow{p(n)} B\big)_{n\in\mathbb{N}}$ *be a twisted operad. Then the sequence*

$$\Big(E(k) \xrightarrow{p(k)} B \xrightarrow{\Delta(k)} B^k\Big)_{k\in\mathbb{N}}$$

is a twisted right module over E *with the action induced from the composition maps of* E*.*

Remark 3.7.
Note that we do not impose any local triviality condition for ϵ*. In some contexts, for example, we may impose a condition for* ϵ *that the restriction of* ϵ *on diagonals*

$$\Big(\tau(k)^{-1}(\Delta) \to \Delta \cong B\Big)_{k\in\mathbb{N}_{>0}}$$

with $(B \times * \to B)$ *for the* 0*-th component forms a bundle of right modules over* B *on which* E *acts by bundle maps in a locally trivial way.*

Since we focus on a few examples of twisted right modules, we are not motivated to tailor the definition in a more particular way.

Remark 3.8.
If ϵ satisfies

$$\tau(k)^{-1}(\Delta) \subset \overline{\Big(\tau(k)^{-1}\big(\widetilde{C}_k(B)\big)\Big)},$$

we can interpret each point over diagonals as a collision of k-particles in B.

Definition 3.18 (morphism between twisted right modules)**.**
Let $(\varphi, g) : E \to E'$ be a twisted operad morphism, $\epsilon = \big(\epsilon(k) \xrightarrow{\tau(k)} B^k\big)_{k\in\mathbb{N}}$ be a twisted right module over E, and $\epsilon' = \big(\epsilon'(k) \xrightarrow{\tau'(k)} B'^k\big)_{k\in\mathbb{N}}$ be a twisted right module over E'. Then a sequence of Σ_k-equivariant maps covering g^k

$$\alpha = \Big(\alpha(k) : \epsilon(k) \to \epsilon'(k)\Big)_{k\in\mathbb{N}}$$

*is called a (φ, g)-**equivariant twisted right module morphism** if the following hold:*

- *For $k \in \mathbb{N}$, $e \in \epsilon(k)$, and $f_j \in E(n_j)$,*

  $$\alpha(\Sigma_{j=1}^k n_j) : e \circ (f_j)_{j=1}^k \mapsto \alpha(k)(e) \circ \big(\varphi(n_j)(f_j)\big)_{j=1}^k$$

  *whenever the $\circ$ in the left hand side is defined. When (φ, g) is the identity map of E, we say that α is E-**equivariant**.*

Example 3.16.
Let $\epsilon = \big(\epsilon(k) \xrightarrow{\tau(k)} B^k\big)_{k\in\mathbb{N}}$ be a twisted right module over E. Then the identity map of ϵ defined by

$$id_\epsilon = \Big(id_{\epsilon(k)} : \epsilon(k) \to \epsilon(k)\Big)_{k\in\mathbb{N}} : \epsilon \to \epsilon$$

is E-equivariant. Assume that $(\varphi, g) : E \to E'$ and $(\varphi', g') : E' \to E''$ are twisted operad morphisms, $\alpha : \epsilon \to \epsilon'$ is a (φ, g)-equivariant twisted right module morphism, and $\alpha' : \epsilon' \to \epsilon''$ is a (φ', g')-equivariant twisted right module morphism. Then the composition

$$\alpha' \circ \alpha = \Big(\alpha'(k) \circ \alpha(k) : \epsilon(k) \to \epsilon''(k)\Big)_{k\in\mathbb{N}} : \epsilon \to \epsilon''$$

is $(\varphi', g') \circ (\varphi, g)$-equivariant. Thus we can form the **category of pairs (ϵ, E) of a twisted operad E and its twisted right module ϵ** *such that morphisms from (ϵ, E) to (ϵ', E') are pairs $\big(\alpha, (\varphi, g)\big)$ of a (φ, g)-equivariant twisted right module morphism $\alpha : \epsilon \to \epsilon'$ and a twisted operad morphism $(\varphi, g) : E \to E'$. If $\big(\alpha, (\varphi, g)\big)$ is an isomorphism in the category, we also say that α is a (φ, g)-**equivariant isomorphism**.*

We define the twisted analogue of a tensor product over an operad. This will be used to construct configuration spaces with twisted summable labels.

Definition 3.19 (tensor product over a twisted operad)**.**
Let $E = \big(o(n) \to E(n) \to B\big)_{n\in\mathbb{N}}$ *be a twisted operad,* $\eta = \big((X,*) \to (\eta,\iota) \to B\big)$ *be a twisted partial left algebra over* E*, and* $\epsilon = \big(\epsilon(k) \xrightarrow{\tau(k)} B^k\big)_{k\in\mathbb{N}}$ *be a twisted right module over* E*.*

For $0 < k$*, let* $(\epsilon \otimes_E \eta)\big(k;(n_j)_{j=1}^k\big)$ *denote the pullback*

$$\epsilon(k) \times_{B^k} \Big(\Pi_{j=1}^k \big(E(n_j) \times_B \eta^{(\times_B)n_j}\big)\Big).$$

The **tensor product of ϵ and η over** E *is the quotient*

$$\epsilon \otimes_E \eta = \Bigg(\coprod_{\substack{k, n_j \in \mathbb{N} \\ k \neq 0}} (\epsilon \otimes_E \eta)\big(k;(n_j)_{j=1}^k\big) \Bigg) \amalg \epsilon(0) \Big/ \sim ,$$

modulo the equivalence relation $\sim$ *generated by the following relations.*

Assume $0 < k$*. Write an element of* $(\epsilon \otimes_E \eta)\big(k;(n_j)_{j=1}^k\big)$ *simply as*

$$\Big(e\,;\,(f_j)_j\,;\,(g_i)_i\Big)$$

so that $e \in \epsilon(k)$*,* $f_j \in E(n_j)$*, and* $g_i \in \eta$*. Let* $\tau(k)(e) = (b_j)_{j=1}^k$*.*

1.

$$\Big(e\,;\,(*_{b_j})_j\,;\,*\Big) \sim * \in \epsilon(0), \qquad\qquad (*_{b_j} \in E(0)_{b_j})$$
$$\Big(e\,;\,(1_{b_j})_j\,;\,\big(\iota(b_j)\big)_j\Big) \sim * \in \epsilon(0).$$

2. For (1) $\sigma \in \Sigma_k$*, and* (2) $\mu_j \in \Sigma_{n_j}$*,*

$$\Big(e\,;\,(f_j)_j\,;\,(g_i)_i\Big) \sim \Big(\sigma e\,;\,(f_{\sigma j})_j,\,;\,\sigma\big((n_j)_{j=1}^k\big)(g_i)_i\Big),$$
$$\Big(e\,;\,(f_j)_j\,;\,(g_i)_i\Big) \sim \Big(e\,;\,(\mu_j f_j)_j\,;\,(\oplus_j \mu_j)(g_i)_i\Big).$$

3. If $n_1 \neq 0$*,*

$$\Big(e\,;\,(f_j)_j\,;\,(g_i)_i\Big) \sim \Big(e \circ \big(f_1,(1_{b_j})_{j=2}^k\big)\,;\,\big((1_{b_1})_{i=1}^{n_1},(f_j)_{j=2}^k\big)\,;\,(g_i)_i\Big).$$

4. If $n_1 \neq 0$ and the action $f_1 \circ (g_i)_{i=1}^{n_1}$ is defined,

$$\Big(e\,;\ (f_j)_j\,;\ (g_i)_i\Big) \sim \Big(e\,;\ 1_{b_1}, (f_j)_{j=2}^{k}\,;\ f_1 \circ (g_i)_{i=1}^{n_1}, (g_i)_{i=n_1+1}^{\sum_{j=1}^{k} n_j}\Big).$$

If $n_1 = 0$, and $n_j \neq 0$ for some j,

$$\Big(e\,;\ (f_j)_j\,;\ (g_i)_i)\Big) = \Big(e\,;\ *_{b_1}, (f_j)_{j=2}^{k}\,;\ (g_i)_i\Big) \sim \Big(e\,;\ 1_{b_1}, (f_j)_{j=2}^{k}\,;\ \iota(b_1), (g_i)_i\Big).$$

The unique point in $\epsilon(0)$ represents the basepoint of $\epsilon \otimes_E \eta$.

The construction of a tensor product over a twisted operad is functorial in the following sense. Let $(\varphi, g) : E \to E'$ be a twisted operad morphism, $\alpha : \epsilon \to \epsilon'$ be a (φ, g)-equivariant morphism between twisted right modules, and $\beta : \eta \to \eta'$ be a (φ, g)-equivariant morphism between twisted partial left algebras. Then we have an induced based map between tensor products defined by:

$$\begin{aligned}
&\big(\alpha; (\varphi, g), \beta\big)_* : \epsilon \otimes_E \eta \to \epsilon' \otimes_{E'} \eta', \\
&\Big(e\,;\ (f_j)_j\,;\ (g_i)_i\Big) \mapsto \Big(\alpha(e)\,;\ \big(\varphi(f_j)\big)_j\,;\ \big(\beta(g_i)\big)_i\Big).
\end{aligned}$$

Thus we obtain the following proposition.

Proposition 3.3.
Let $\mathcal{B}$ be the category of triples (ϵ, E, η) such that E is a twisted operad, ϵ is a twisted right module over E, η is a twisted partial left algebra over E. The morphisms from (ϵ, E, η) to (ϵ', E', η') are triples $\big(\alpha, (\varphi, g), \beta\big)$ of a (φ, g)-equivariant twisted right module morphism $\alpha : \epsilon \to \epsilon'$, a twisted operad morphism $(\varphi, g) : E \to E'$, and a (φ, g)-equivariant twisted partial left algebra morphism $\beta : \eta \to \eta'$. The composition $\big(\alpha', (\varphi', g'), \beta'\big) \circ \big(\alpha, (\varphi, g), \beta\big)$ is defined by componentwise compositions. Then there is a functor from $\mathcal{B}$ to the category of based spaces defined by

$$\begin{aligned}
&(\epsilon, E, \eta) \mapsto \epsilon \otimes_E \eta, \\
&\big(\alpha, (\varphi, g), \beta\big) \mapsto \big(\alpha, (\varphi, g), \beta\big)_*.
\end{aligned}$$

3.4 Configuration spaces with twisted m-fold loop space labels

In this section we give an example of a twisted operad. We define the twisted little disks operad $\mathcal{E}_M$ over a manifold M as the bundle of little disks operads associated to TM. By taking a tensor product over $\mathcal{E}_M$ we obtain configuration spaces of M with summable labels in a twisted m-fold loop space.

Twisted embedded little disks operad over manifold Let M be an m-manifold, and $\gamma = \big((F,*) \to (\gamma,\iota) \to M\big)$ be a fibre bundle with the structure group $G = \mathrm{Homeo}_0(F)$.

We construct a twisted operad over M such that a fibre over $p \in M$ is an operad consisting of embeddings $\amalg_{k=1}^{n} T_pM \hookrightarrow T_pM$. We first define the typical fibre.

Definition 3.20 (Embedded little $\mathbb{R}^m$ operad)**.**
$\mathcal{R}^m$ is the operad such that

$$\mathcal{R}^m(n) = \mathrm{Emb}(\coprod_{k=1}^{n} \mathbb{R}^m, \mathbb{R}^m). \qquad (n \in \mathbb{N})$$

The composition maps and Σ_n-actions are induced from the operad $\amalg_{\mathbb{R}^m}$.

$\mathrm{GL}(m)$ acts on $\mathcal{R}^m$ by reparametrization.

Proposition 3.4 ($\mathrm{GL}(m)$-equivariance of $\mathcal{R}^m$)**.**
The action of $\mathrm{GL}(m)$ on $\mathcal{R}^m$

$$\begin{aligned}&\mathrm{GL}(m) \times \mathcal{R}^m(n) \to \mathcal{R}^m(n),\\ &(A,f) \mapsto A \cdot f = A \circ f \circ (\underbrace{A^{-1} \amalg \cdots \amalg A^{-1}}_{n \text{ times}})\end{aligned}$$

induces a homomorphism from $\mathrm{GL}(m)$ to the group of operad isomorphisms of $\mathcal{R}^m$.

Proof. The equivariance of Σ_n-actions is obvious. It suffices to check the equivariance of composition maps. We show that

$$(A \cdot f) \circ \big((A \cdot g_1), \ldots, (A \cdot g_k)\big) = A \cdot \big(f \circ (g_1, \ldots, g_k)\big)$$

for $A \in \mathrm{GL}(m)$, $f \in \mathcal{R}^m(k)$, $g_j \in \mathcal{R}^m(n_j)$. Write $f = (f_j : \mathbb{R}^m \to \mathbb{R}^m)_{j=1}^{k}$.

$$\begin{aligned}&(A \cdot f) \circ \big((A \cdot g_1), \ldots, (A \cdot g_k)\big) = \big((A \cdot f_1) \circ (A \cdot g_1)\big) \amalg \cdots \amalg \big((A \cdot f_k) \circ (A \cdot g_k)\big)\\ &= \big(A \circ f_1 \circ g_1 \circ (\underbrace{A^{-1} \amalg \cdots \amalg A^{-1}}_{n_1 \text{ times}})\big) \amalg \cdots \amalg \big(A \circ f_k \circ g_k \circ (\underbrace{A^{-1} \amalg \cdots \amalg A^{-1}}_{n_k \text{ times}})\big)\\ &= A \circ \big((f_1 \circ g_1) \amalg \cdots \amalg (f_k \circ g_k)\big) \circ (\underbrace{A^{-1} \amalg \cdots \amalg A^{-1}}_{\sum_{j=1}^{k} n_j \text{ times}})\\ &= A \cdot \big(f \circ (g_1, \ldots, g_k)\big).\end{aligned}$$

□

Thus we can associate $\mathcal{R}^m$ fibrewise to TM.

Definition 3.21 (twisted embedded little disks operad over M)**.**
Let $\mathcal{E}_M$ denote the associated twisted operad to TM with the typical fibre $\mathcal{R}^m$ and the structure group $\mathrm{GL}(m)$.

The fibre of $\mathcal{E}_M(n)$ over p can be identified with $\mathrm{Emb}(\amalg_{k=1}^{n} T_pM, T_pM)$.

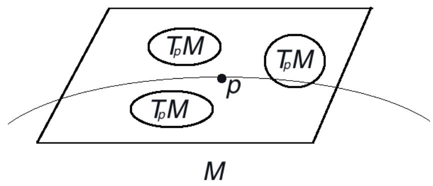


Figure 3.2: $\mathcal{E}_M$

Twisted loop space of γ over M We construct the twisted left algebra $\Omega_M(\gamma)$ over $\mathcal{E}_M$ with the typical fibre $\mathrm{Map}_c(\mathbb{R}^m, F)$ such that the fibre over $p \in M$ is $\mathrm{Map}_c(T_pM, \gamma_p)$. $\mathrm{Map}_c(\mathbb{R}^m, F)$ is based at the constant map $* : \mathbb{R}^m \mapsto * \in F$.

$\mathrm{GL}(m) \times G$ acts on the left of $\mathrm{Map}_c(\mathbb{R}^m, F)$ by

$$\begin{aligned} \mathrm{GL}(m) \times G \times \mathrm{Map}_c(\mathbb{R}^m, F) &\rightarrow \mathrm{Map}_c(\mathbb{R}^m, F), \\ (A, a, x) &\mapsto a \circ x \circ A^{-1}. \end{aligned}$$

$\big(\mathrm{Map}_c(\mathbb{R}^m, F), *\big)$ has a left algebra structure over $\mathcal{R}^m$ equivariant with respect to $\mathrm{GL}(m) \times G$. as follows.

Proposition 3.5.

1. *$\big(\mathrm{Map}_c(\mathbb{R}^m, F), *\big)$ is a left algebra over $\mathcal{R}^m$ with the action defined by*

$$\begin{aligned} &\mathcal{R}^m(n) \times \mathrm{Map}_c(\mathbb{R}^m, F)^n \rightarrow \mathrm{Map}_c(\mathbb{R}^m, F), \\ &(f = (f_k : \mathbb{R}^m \rightarrow \mathbb{R}^m)_{k=1}^n; x_1, \ldots, x_n) \mapsto f \circ (x_1, \ldots, x_n), \end{aligned}$$

where $f \circ (x_1, \ldots, x_n) : \mathbb{R}^m \rightarrow F$ is the union of the following $(n+1)$-maps

$$\begin{aligned} &f_k(\mathbb{R}^m) \xrightarrow{f_k^{-1}} \mathbb{R}^m \xrightarrow{x_k} F, \qquad (k = 1, \ldots, n) \\ &\mathbb{R}^m \setminus \mathrm{im}(f) \rightarrow \{*\} \hookrightarrow F. \end{aligned}$$

2. *The action of $\mathcal{R}^m$ is $\big(\mathrm{GL}(m) \times G\big)$-equivariant:*

$$(A, a) \cdot \big(f \circ (x_1, \ldots, x_n)\big) = (A \cdot f) \circ \Big((A, a) \cdot x_1, \ldots, (A, a) \cdot x_n\Big).$$

Proof.

1. This is the same as the standard proof that an m-fold loop space is an algebra over the little disks operad of dimension m.

2. This is a straightforward computation.

$\square$

Definition 3.22 (Twisted loop space of γ over M)**.**
Let $\Omega_M(\gamma)$ be the associated bundle to $TM \times_M \gamma$ with the typical fibre $\mathrm{Map}_c(\mathbb{R}^m, F)$ and the structure group $\mathrm{GL}(m) \times G$. Since the basepoint $* \in \mathrm{Map}_c(\mathbb{R}^m, F)$ is fixed by the action of $\mathrm{GL}(m) \times G$, $\Omega_M(\gamma)$ has the basepoint section $*$. We call the bundle

$$\Omega_M(\gamma) = \Big(\big(\mathrm{Map}_c(\mathbb{R}^m, F), * \big) \to \big(\Omega_M(\gamma), * \big) \to M \Big)$$

the **twisted loop space of γ over** M*.*

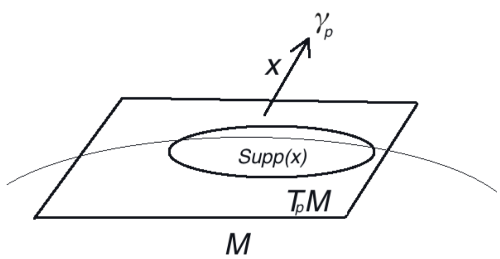


Figure 3.3: $x \in \Omega_M(\gamma)$

Proposition 3.6.
$\Omega_M(\gamma)$ is a twisted left algebra over $\mathcal{E}_M$ with the action locally defined by

$$\begin{aligned} &U \times \mathcal{R}^m(n) \times \mathrm{Map}_c(\mathbb{R}^m, F)^n \to U \times \mathrm{Map}_c(\mathbb{R}^m, F), \qquad (n \in \mathbb{N}) \\ &\big(u, f, (x_k)_{k=1}^n\big) \mapsto \big(u, f \circ (x_k)_{k=1}^n\big). \end{aligned}$$

Proof. Choose an open cover $\{U\}$ of M and local trivializations of $TM \times_M \gamma$ over $\{U\}$

$$t_U : U \times (\mathbb{R}^m \times F) \to (TM \times_M \gamma)|_U.$$

A transition between $\{t_U\}$ transforms f to $A \cdot f$, and x_k to $(A, a) \cdot x_k$ for some $(A, a) \in \mathrm{GL}(m) \times G$. By the equivariance

$$(A, a) \cdot \big(f \circ (x_k)_{k=1}^n\big) = (A \cdot f) \circ \big((A, a) \cdot x_k\big)_{k=1}^n$$

the two local representaions of the action are compatible. $\square$

Right action of $\mathcal{E}_M$ We define a twisted right module over $\mathcal{E}_M$ consisting of finite number of particles in M with mutually disjoint labels in $\mathcal{E}_M(1)$.

Definition 3.23 (ordered tangential configuration space of M)**.**
Let $\widetilde{D}(M) = \big(\widetilde{D}_k(M) \xrightarrow{\tau(k)} M^k\big)_{k\in\mathbb{N}}$ be the sequence of spaces parametrized by M^k such that

1. *$\widetilde{D}_k(M) \subset \mathcal{E}_M(1)^k$.*
2. *For $(f_j : T_{p_j}M \hookrightarrow T_{p_j}M)_{j=1}^k \in \mathcal{E}_M(1)^k$, $(f_j)_{j=1}^k \in \widetilde{D}_k(M)$ if and only if $\mathrm{im}(f_j)$ are mutually disjoint.*
3. *The parametrization is given by $(f_j)_{j=1}^k \mapsto (p_j)_{j=1}^k$.*

We call $\widetilde{D}(M)$ the **ordered tangential configuration space of** *M.*

The composition maps of $\mathcal{E}_M$ induce a right action on $\widetilde{D}(M)$

$$\begin{gathered}\widetilde{D}_k(M) \times_{M^k} \Pi_{j=1}^k \mathcal{E}_M(n_j) \to \widetilde{D}_{\sum_{j=1}^k n_j}(M),\\ \Big((f_j)_{j=1}^k,\ \big((g_i^j)_{i=1}^{n_j}\big)_{j=1}^k\Big) \mapsto \big((f_j \circ g_i^j)_{i=1}^{n_j}\big)_{j=1}^k.\end{gathered}$$

With this action $\widetilde{D}(M)$ is a twisted right module over $\mathcal{E}_M$.

Configuration space with twisted m-fold loop space labels We can take a tensor product $\widetilde{D}(M) \otimes_{\mathcal{E}_M} \Omega_M(\gamma)$.

Definition 3.24.
Let $N \subset M$ be a closed subspace. The **configuration space of M vanishing at N with summable labels in $\Omega_M(\gamma)$** *is the quotient*

$$\widetilde{D}\big(M, N; \Omega_M(\gamma)\big) = \widetilde{D}(M) \otimes_{\mathcal{E}_M} \Omega_M(\gamma)\Big/\sim$$

modulo the equivalence relation $\sim$ generated by

$$\Big((f_j)_{j=1}^k\,;\ (1_{p_j})_{j=1}^k\,;\ (x_j)_{j=1}^k\Big) \sim \Big((f_j)_{j=2}^k\,;\ (1_{p_j})_{j=2}^k\,;\ (x_j)_{j=2}^k\Big)$$

for $\tau(1)(f_1) \in N$.

Diffeomorphism-equivariance of $\widetilde{D}\big(M,N;\Omega_M(\gamma)\big)$ For $i=1,2$, let M_i be an m-manifold, $N_i \subset M_i$ be a closed subspace, and $\big((F_i,*)\to(\gamma_i,*)\to M_i\big)$ be a fibre bundle with the basepoint section. Let $\widetilde{\varphi}:\gamma_1\to\gamma_2$ be a basepoint preserving bundle map covering an embedding $\varphi: M_1\to M_2$ such that $\varphi(N_1)\subset N_2$.

Then $(\widetilde{\varphi},\varphi)$ induces a twisted operad morphism $(\mathcal{E}_\varphi,\varphi):\mathcal{E}_{M_1}\to\mathcal{E}_{M_2}$, whose restriction to the fibre over p is given by

$$\begin{aligned}&\mathcal{E}_\varphi(n)_p:\mathrm{Emb}\left(\amalg_{k=1}^n T_pM_1, T_pM_1\right)\to \mathrm{Emb}\left(\amalg_{k=1}^n T_{\varphi(p)}M_2, T_{\varphi(p)}M_2\right),\\ &(f_j:T_pM\to T_pM)_{j=1}^k\mapsto \big(D\varphi(p)\circ f_j\circ D\varphi(p)^{-1}\big)_{j=1}^k,\end{aligned}$$

and an $(\mathcal{E}_\varphi,\varphi)$-equivariant twisted left algebra morphism $\Omega(\varphi):\Omega_{M_1}(\gamma_1)\to\Omega_{M_2}(\gamma_2)$, whose restriction to the fibre over p is

$$\begin{aligned}&\Omega(\varphi)_p:\mathrm{Map}_c\left(T_pM_1,(\gamma_1)_p\right)\to\mathrm{Map}_c\left(T_{\varphi(p)}M_2,(\gamma_2)_{\varphi(p)}\right),\\ &x\mapsto\widetilde{\varphi}\circ x\circ D\varphi(p)^{-1},\end{aligned}$$

and an $(\mathcal{E}_\varphi,\varphi)$-equivariant twisted right module morphism $\widetilde{D}(\varphi):\widetilde{D}(M_1)\to\widetilde{D}(M_2)$, whose k-th component $\widetilde{D}(\varphi)(k):\widetilde{D}_k(M_1)\to\widetilde{D}_k(M_2)$ is a restriction of

$$\big(\mathcal{E}_{M_1}(1)\big)^k\xrightarrow{\mathcal{E}_\varphi(1)^k}\big(\mathcal{E}_{M_2}(1)\big)^k.$$

Thus $(\widetilde{\varphi},\varphi)$ induces a map between configuration spaces

$$\big(\widetilde{D}(\varphi),\ (\mathcal{E}_\varphi,\varphi),\ \Omega(\varphi)\big)_*:\widetilde{D}\big(M_1,N_1;\Omega_{M_1}(\gamma_1)\big)\to\widetilde{D}\big(M_2,N_2;\Omega_{M_2}(\gamma_2)\big).$$

Chapter 4

Configuration Spaces with Twisted Summable Labels

In this chapter we study configuration spaces of manifolds with twisted summable labels. We first define the twisted Fulton-MacPherson operad over a manifold and construct configuration spaces as a tensor product over the twisted operad. Some properties of the configuration spaces are checked comparable to Eilenberg-Steenrod axioms for singular homology. In particular we obtain the Mayer-Vietoris sequence for two manifolds glued along common boundaries. We also define the diffeomorphism-equivariant scanning map and show that it is a weak equivalence.

4.1 Construction and the twisted Fulton-MacPherson operads

In this section we observe a bundle of the Fulton-MacPherson operads in the compactification of configuration spaces of manifolds introduced in [14], and formulate it as a twisted operad. By taking a tensor product over the twisted Fulton-MacPherson operad we construct configuration spaces with twisted summable labels. We examine the functoriality of the twisted Fulton-MacPherson operads with respect to immersions, and the functoriality of configuration spaces with twisted summable labels with respect to embeddings. We also define the compactification of configuration spaces determined by an immersion so that an immersion induces a map between configuration spaces.

Compactification of configuration spaces We follow the compactification technique introduced in [14]. Let $\rho : M \hookrightarrow \mathbb{R}^N$ be an n-manifold embedded in $\mathbb{R}^N$.

Consider the embedding of $\widetilde{C}_k(M)$

$$
\begin{aligned}
&I : \widetilde{C}_k(M) \to M^k \times (S^{N-1})^{2\binom{k}{2}} \times [0,\infty]^{6\binom{k}{3}},\\
&I = \rho^k \times \left(\pi_{hi}\right)_{(h,i)\in\widetilde{C}_2([k])} \times \left(s_{hij}\right)_{(h,i,j)\in\widetilde{C}_3([k])},\\
&\pi_{hi} : (m_j)_{j=1}^k \mapsto \frac{m_h - m_i}{\|m_h - m_i\|},\quad s_{hij} : (m_j)_{j=1}^k \mapsto \frac{\|m_h - m_i\|}{\|m_h - m_j\|}.
\end{aligned}
$$

π measures relative directions between particles in $(m_j)_{j=1}^k$, and s measures ratios of distances between particles in $(m_j)_{j=1}^k$. By taking the closure of the image of I, new points are added to $\widetilde{C}_k(M)$ representing infinitesimal data of collisions of particles.

Definition 4.1 (compactified configuration space, [14]).
The closure of the image of I is called the **compactified configuration space of ordered k-points in** *M and denoted by $\widetilde{C}_k[M]$.*

The topology of $\widetilde{C}_k[M]$ is thoroughly studied in [14]. We quote the part of the results in [14] relevant to our goal to construct the diffeomorphism-equivariant configuration spaces with twisted summable labels.

Theorem 4.1 ([14]).

1. *If M is compact, $\widetilde{C}_k[M]$ is compact.*
2. *$\widetilde{C}_k[M]$ is a manifold with corners whose interior is $\widetilde{C}_k(M)$, and its diffeomorphism type is independent to choices of $\rho : M \hookrightarrow \mathbb{R}^N$.*
3. *The inclusion $\widetilde{C}_k(M) \hookrightarrow \widetilde{C}_k[M]$ induced by I is a homotopy equivalence, the Σ_k-action on $\widetilde{C}_k(M)$ extends to a free action on $\widetilde{C}_k[M]$, and $\widetilde{C}_k(M) \hookrightarrow M^k$ extends to a surjective map $b_k : \widetilde{C}_k[M] \to M^k$ sending $\widetilde{C}_k[M] \setminus \widetilde{C}_k(M)$ onto the union of all diagonals of M^k.*

In [5] Kontsevich gives a nice representation of $\widetilde{C}_k[M]$ as a set using a collection of trees. Different formulations of the same idea are also found in [10] and [14]. We first fix the terminology for trees.

Definition 4.2 (tree).
A **tree** *is a finite connected acyclic graph such that*

1. *Each edge e is directed. We say that e is directed from its initial vertex v to terminal vertex w, and e is an* **incoming edge** *of w and an* **outgoing edge** *of v.*

2. *The* **valence** *of a vertex v is the number of incoming edges of v.*

3. *A vertex with valence* 0 *is called* **a leaf**.

4. *Every nonempty tree has a unique vertex with no outgoing edge. We call the vertex the* **root**.

5. *A vertex which is neither the root nor a leaf is called an* **internal vertex**.

6. *Every vertex but the root has exactly one outgoing edge.*

7. *There is no vertex with valence* 1.

8. *The incoming edges of each vertex are ordered.*

Remark 4.1.
The trees as above are planar. We allow the empty tree. The smallest nonempty tree is the tree consisting of a single vertex which is the root and a leaf at the same time. There is no tree with only one edge.

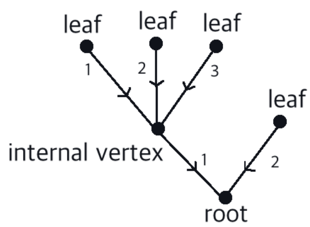


Figure 4.1: a tree

Theorem 4.2 (tree representation of $\widetilde{C}_k[M]$, [5], [10], [14])**.**
Let M be an n-manifold. $\widetilde{C}_k[M]$ as a set has a bijective representation such that each element corresponds to a finite collection of trees with vertices labelled with additional data as follows. A representation of an element of $\widetilde{C}_k[M]$ consists of a finite number of trees $t = \{t_i\}_{i=1}^{j}$ for some $j \in \mathbb{N}$ and labels on the vertices of t such that

1. *The set of leaves of trees in t are ordered by $[k]$. In particular it has the cardinality k. The Σ_k-action on $\widetilde{C}_k[M]$ permutes the order of leaves.*

2. *Each root of trees in t is labelled with a point in M, and the labels are mutually distinct. We call such labels the* **macroscopic locations of the configuration**.

3. *If v is a vertex of t_i with valence $h > 1$ and the root of t_i is labelled with m, then v is labelled with an element of*

$$\widetilde{C}_h(T_m M) \Big/ \mathrm{Aff}_+(n)$$

where $\mathrm{Aff}_+(n)$ is the subgroup of the affine transformation group of $\mathbb{R}^n$ generated by translations and positive scalar multiplications. We call such labels the **collision data of the configuration**.

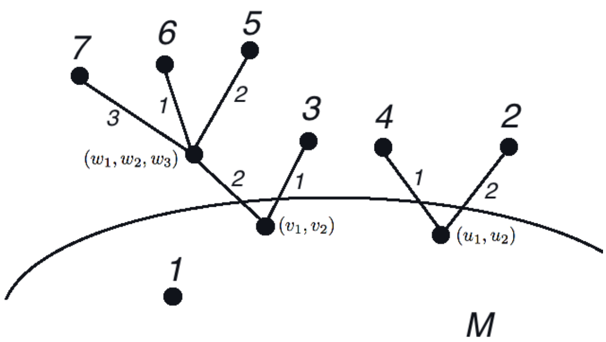


Figure 4.2: $\widetilde{C}_7[M]$

Remark 4.2.
A root can have two labels, the macroscopic location and collision data, or the macroscopic location and order. The macroscopic location of a root of a tree can be interpreted as the location of the collision that the tree represents. Let $(w_1, \ldots, w_h)$ be the collision data of a vertex v. Recall that the incoming edges of v come with an order. w_g can be interpreted as the direction of what is represented by the g-th incoming edge of v when the collision represented by v occurs. The direction is measured by the π and s-component of I. If the g-th incoming edge of v is the outgoing edge of the l-th leaf for some $l \in [k]$, we can imagine that the l-th particle in a configuration participates in the collision that v represents. As the existence of internal vertices indicates, some collisions are subordinate to some other collisions and always to the

collision represented by the root. Such hierarchies are measured by the value 0 *and* ∞ *of the* s*-component of* I*.*

A point in $\widetilde{C}_k(M)$ *is represented by* k *of single vertex trees. The unique point of* $\widetilde{C}_0[M]$ *is represented by the empty tree.*

The tree representation reacts to embeddings by transforming its labels.

Theorem 4.3 ([14])**.**
The construction $M \mapsto \widetilde{C}_k[M]$ *is functorial to embeddings of manifolds so that an embedding between manifolds* $\varphi : M \to N$ *corresponds to a map*

$$\widetilde{C}_k[\varphi] : \widetilde{C}_k[M] \to \widetilde{C}_k[N]$$

transforming trees by applying (1) φ *to macroscopic locations and* (2) $D\varphi$ *to collision data:*

$$m \in M \mapsto \varphi(m), \tag{1}$$

$$(v_1, \dots, v_h) \in \widetilde{C}_h(T_m M) \Big/ \mathrm{Aff}_+(n) \mapsto \Big(D\varphi(v_1), \dots, D\varphi(v_h)\Big). \tag{2}$$

Twisted Fulton-MacPherson operads We define the Fulton-MacPherson operads and its twisted analogue. Let $b_k : \widetilde{C}_k[M] \to M^k$ be the map extending $\widetilde{C}_k(M) \hookrightarrow M^k$. Let $\Delta_k : M \to M^k$ be the diagonal map. Consider the pullback

$$\begin{tikzcd} \mathcal{F}_M(k) \arrow[r] \arrow[d, "p(k)"'] & \widetilde{C}_k[M] \arrow[d, "b_k"] \\ M \arrow[r, "\Delta_k"] & M^k \end{tikzcd}.$$

In particular $\mathcal{F}_M(0)$ is a bundle $(* \to M \times \{*\} \xrightarrow{p(0)} M)$. Note that each element of $\mathcal{F}_M(k)$ is represented by a single tree with k-leaves.

Proposition 4.1 (n-dimensional Fulton-MacPherson operad, [10])**.**
Let $M = \mathbb{R}^n$*. Then the sequence of the inverse images of* 0 *along* $p(k)$

$$\mathcal{F}_n = \Big(\mathcal{F}_n(k) = p(k)^{-1}(0) \subset \mathcal{F}_{\mathbb{R}^n}(k)\Big)_{k \in \mathbb{N}}$$

forms an operad with composition maps given by tree merging. The composition $t \circ (t_j)_{j=1}^k$ *is obtained by gluing the root of* t_j *with the* j*-th leaf of* t *for each* j*. In particular* $t \circ (t_j)_{j=1}^k$ *inherits collision data and orders of incoming edges from* $t, t_1, \dots, t_k$*. In addition the order of leaves of* $t \circ (t_j)_{j=1}^k$ *is given by reading lexicographically the order of edges in the unique path from the root to each leaf.*

We clarify the effect of merging the empty tree $*$. $*$ merged to a single vertex tree yields $*$. Suppose that $*$ is merged to a leaf v and the outgoing edge of v is the i-th incoming edge of v'. If the valence of v' is 2, then the effect is to delete all the incoming edges of v' and make v' a new leaf. Otherwise, delete the i-th incoming edge of v' and the i-th coordinate of the collision data labelling v'.

*Σ_k acts on $\mathcal{F}_n(k)$ by permuting the order of leaves. The single vertex tree represents the unit of $\mathcal{F}_n$. We call $\mathcal{F}_n$ the n-**dimensional Fulton-MacPherson operad**.*

Proposition 4.2 (twisted Fulton-MacPherson operad)**.**
The sequence

$$\mathcal{F}_M = \left(\mathcal{F}_n(k) \to \mathcal{F}_M(k) \xrightarrow{p(k)} M\right)_{k \in \mathbb{N}}$$

*forms a twisted operad over M with composition maps given by tree merging. We call $\mathcal{F}_M$ **the twisted Fulton-MacPherson operad over** M.*

Proof. First note that $\mathcal{F}_{\mathbb{R}^n}(k)$ is a trivial bundle over $\mathbb{R}^n$ with fibre $\mathcal{F}_n(k)$. This can be checked by observing the translation map

$$\mathbb{R}^n \times \mathbb{R}^n, \;\; (r, s) \to r + s$$

induces the map

$$\mathbb{R}^n \times \widetilde{C}_k[\mathbb{R}^n] \to \widetilde{C}_k[\mathbb{R}^n]$$

translating the roots.

Due to the fact that the compactification $\widetilde{C}_k[\cdot]$ is functorial to embeddings and the nature of the functoriality represented by trees, each local parametrization $\varphi : \mathbb{R}^n \hookrightarrow M$ induces local trivializations

$$\begin{tikzcd} \mathcal{F}_{\mathbb{R}^n}(k) \arrow[r] \arrow[d] & \mathcal{F}_M(k) \arrow[d] \\ \mathbb{R}^n \arrow[r, "\varphi"] & M \end{tikzcd} \qquad (k \in \mathbb{N})$$

preserving tree merging operations. $\square$

Remark 4.3.
$\mathrm{GL}(n)$ acts on $\mathcal{F}_n$ by transforming collision data. $\mathcal{F}_M$ can be defined alternatively as the associated twisted operad to a frame bundle of M with the typical fibre $\mathcal{F}_n$. The alternative definition reduces the structure group of $\mathcal{F}_M$.

We examine the naturality of $\mathcal{F}_M$.

Proposition 4.3.
An immersion $\varphi : M \to M'$ induces a twisted operad morphism

$$(\mathcal{F}_\varphi, \varphi) = \left(\begin{tikzcd} \mathcal{F}_M(k) \arrow[r, "\mathcal{F}_\varphi(k)"] \arrow[d] & \mathcal{F}_{M'}(k) \arrow[d] \\ M \arrow[r, "\varphi"] & M' \end{tikzcd} \right)_{k\in\mathbb{N}}$$

applying φ to macroscopic locations and $D\varphi$ to collision data.

Proof. It is obvious that $\mathcal{F}_\varphi$ is compatible with tree merging. We check the continuity of $\mathcal{F}_\varphi(k)$. Note that φ is a local embedding. For $m \in M$, let U be an open neighbourhood of m such that $\varphi|_U$ is an embedding. The value of $\mathcal{F}_\varphi(k)$ over U coincides with $\widetilde{C}_k[\varphi|_U]$. Thus $\mathcal{F}_\varphi(k)$ is continuous. $\square$

Configuration spaces with twisted summable labels

Proposition 4.4.
The sequence of spaces parametrized by M^k

$$\widetilde{C}[M] = \left(\widetilde{C}_k[M] \xrightarrow{b_k} M^k\right)_{k\in\mathbb{N}}$$

is a twisted right module over $\mathcal{F}_M$ with the action given by tree merging.

Proof. It suffices to show the continuity of the action. Let $m \in \widetilde{C}_k[M]$. m has a tree representation $t = \{t_i\}_i$. There is an open neighbourhood of m homeomorphic to the product $\Pi_i \widetilde{C}_{h_i}[\mathbb{R}^n]$ where h_i is the number of leaves of t_i. Thus we only need to check the continuity of the action of the form

$$\begin{array}{rcl} \widetilde{C}_k[\mathbb{R}^n] \times \Pi_{j=1}^k \mathcal{F}_n(l_j) & \to & \widetilde{C}_{\sum_{j=1}^k l_j}[\mathbb{R}^n], \\ \left(m, (f_j)_{j=1}^k\right) & \mapsto & m \circ (f_j)_{j=1}^k \end{array}$$

at a point $m \in \mathcal{F}_n(k) \subset \widetilde{C}_k[\mathbb{R}^n]$. Suppose that a sequence $(m_\alpha)_\alpha$ in $\widetilde{C}_k(\mathbb{R}^n)$ converges to the collision m, and a sequence $(f_j^\alpha)_\alpha$ in $\mathcal{F}_n(l_j)$ converges to f_j for each j. Then the sequence of trees $\left(m_\alpha \circ (f_j^\alpha)_{j=1}^k\right)_\alpha$ converges to the tree $m \circ (f_j)_{j=1}^k$. This can be seen by checking the convergence of the values of the embedding I. For the more detailed description of the local topology of $\widetilde{C}_k[\mathbb{R}^n]$ we refer to [14].

Since both $\widetilde{C}_k[\mathbb{R}^n] \times \Pi_{j=1}^k \mathcal{F}_n(l_j)$ and $\widetilde{C}_{\sum_{j=1}^k l_j}[\mathbb{R}^n]$ are metric spaces and $\widetilde{C}_k(\mathbb{R}^n) \subset \widetilde{C}_k[\mathbb{R}^n]$ is dense, the sequence argument shows that the action is continuous at m.

$\square$

We examine the naturality of the construction $\widetilde{C}[M]$.

Proposition 4.5.
Let $\varphi : M \to M'$ be an embedding and $\mathcal{F}_\varphi : \mathcal{F}_M \to \mathcal{F}_{M'}$ be the twisted operad morphism induced by φ

$$(\mathcal{F}_\varphi, \varphi) = \left(\begin{tikzcd} \mathcal{F}_M(k) \arrow[r, "\mathcal{F}_\varphi(k)"] \arrow[d] & \mathcal{F}_{M'}(k) \arrow[d] \\ M \arrow[r, "\varphi"] & M' \end{tikzcd} \right)_{k\in\mathbb{N}}.$$

The sequence of maps $\widetilde{C}[\varphi]$ defined by

$$\widetilde{C}[\varphi] = \left(\widetilde{C}_k[M] \xrightarrow{\widetilde{C}_k[\varphi]} \widetilde{C}_k[M'] \right)_{k\in\mathbb{N}}$$

is a $(\mathcal{F}_\varphi, \varphi)$-equivariant twisted right module morphism from $\widetilde{C}[M]$ to $\widetilde{C}[M']$.

Proof. We only need to check the equivariance. Let $m \in \widetilde{C}_k[M]$, and $f_j \in \mathcal{F}_M(n_j)$ for $j = 1, \dots, k$ such that $m \circ (f_j)_{j=1}^k$ is defined. We need to show the equality

$$\widetilde{C}_{\sum_{j=1}^k n_j}[\varphi]\Big(m \circ (f_j)_{j=1}^k\Big) = \Big(\widetilde{C}_k[\varphi](m)\Big) \circ \Big(\mathcal{F}_\varphi(n_j)(f_j)\Big)_{j=1}^k.$$

Both hand sides have the same macroscopic locations given by φ of roots of m, and the same collision data given by $D\varphi$ of collision data of m and $(f_j)_{j=1}^k$. $\square$

Now we construct configuration spaces with twisted summable labels by taking a tensor product over $\mathcal{F}_M$.

Definition 4.3 (configuration spaces with twisted summable labels)**.**
Let M be an n-manifold, $N \subset M$ be a closed subspace, and $\big((X, *) \to (\eta, \iota) \to M\big)$ be a twisted partial left algebra over $\mathcal{F}_M$. The **configuration space of** *M* **vanishing at** *N* **with twisted summable labels in** *η, denoted as $C_\Sigma\big[M, N; \eta\big]$, is the quotient*

$$\widetilde{C}[M] \otimes_{\mathcal{F}_M} \eta \Big/ \sim$$

modulo the equivalence relation $\sim$ generated by

$$\Big((m_j)_{j=1}^k \,;\, (f_j)_{j=1}^k \,;\, \big((g_i^j)_{i=1}^{n_j}\big)_{j=1}^k\Big) \sim \Big((m_j)_{j=2}^k \,;\, (f_j)_{j=2}^k \,;\, \big((g_i^j)_{i=1}^{n_j}\big)_{j=2}^k\Big)$$

for (1) $k, n_j \in \mathbb{N}$*;* (2) $(m_j)_{j=1}^k \in \widetilde{C}_k(M)$, $m_1 \in N$*;* (3) $f_j \in \mathcal{F}_M(n_j)_{m_j}$*;* (4) $g_i^j \in \eta_{m_j}$. *The relation deletes trees rooted in N.*

The tree representation of $\widetilde{C}[M]$ induces the tree representation of $C_\Sigma\big[M,N;\eta\big]$.

Proposition 4.6 (tree representation of $C_\Sigma\big[M,N;\eta\big]$)**.**
$C_\Sigma\big[M,N;\eta\big]$ as a set has a bijective representation such that each element corresponds to a finite collection of trees with vertices labelled with additional data as follows. A representation of an element of $C_\Sigma\big[M,N;\eta\big]$ consists of trees $t=\{t_i\}_{i=1}^{j}$ for some $j\in\mathbb{N}$ and labels on the vertices of t such that

1. *Each root of trees in t is labelled with a point in $M\setminus N$, and the labels are mutually distinct. We call such labels the* **macroscopic locations of the configuration**.

2. *If v is a vertex of t_i with valence $h>1$ and the root of t_i is labelled with m, then v is labelled with an element of*
$$\widetilde{C}_h(T_mM)\Big/\mathrm{Aff}_+(n)\ .$$
We call such labels the **collision data of the configuration**.

3. *If v is a leaf of t_i and the root of t_i is labelled with m, then v is labelled with an element e of $\eta_m\setminus\iota_m$. We call such labels the* **summable labels of the configuration.** *We also say that v is a* **particle** *located at m labelled with e.*

4. *Each tree is required to be not reducible by computing an operad action represented by a subtree and replacing the subtree with the value of the operad action.*

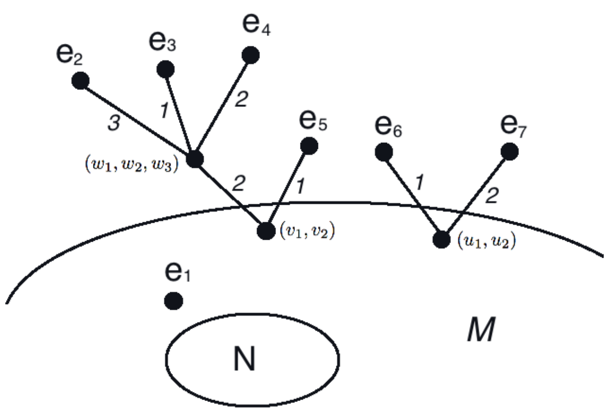


Figure 4.3: $C_\Sigma\big[M,N;\eta\big]$

From the functoriality of $\widetilde{C}[M]$, $\mathcal{F}_M$, and $\otimes_{\mathcal{F}_M}$, the construction of configuration spaces with twisted summable labels can be formulated as a functor. Let $\mathcal{U}$ denote the category of based spaces.

Proposition 4.7.
We define the category $\mathcal{M}$ whose objects are triples $\big(M, N; (\eta, \iota)\big)$ of a manifold M, a closed subspace $N \subset M$, and a twisted partial left algebra over $\mathcal{F}_M$, (η, ι). The morphism set from $\big(M, N; (\eta, \iota)\big)$ to $\big(M', N'; (\eta', \iota')\big)$ is the collection of pairs $(\widetilde{\varphi}, \varphi)$ such that

1. *$\varphi : (M, N) \to (M', N')$ is a map that restricts to an embedding*
$$\varphi_0 : \varphi^{-1}(M' \setminus N') \to M'.$$
2. *$\widetilde{\varphi} : (\eta, \iota)|_{\varphi^{-1}(M' \setminus N')} \to (\eta', \iota')$ is a $(\mathcal{F}_{\varphi_0}, \varphi_0)$-equivariant morphism between twisted partial left algebra.*

$$\begin{tikzcd}
(\eta, \iota)|_{\varphi^{-1}(M' \setminus N')} \arrow[r, "\widetilde{\varphi}"] \arrow[d] & (\eta', \iota') \arrow[d] \\
\varphi^{-1}(M' \setminus N') \arrow[r, "\varphi_0"] & M'
\end{tikzcd}$$

We have the functor
$$C^1_\Sigma : \mathcal{M} \to \mathcal{U}$$
given by the correspondence
$$\begin{aligned}
\big(M, N; (\eta, \iota)\big) &\mapsto C_\Sigma\big[M, N; \eta\big], \\
(\widetilde{\varphi}, \varphi) &\mapsto C^1_\Sigma(\widetilde{\varphi}, \varphi),
\end{aligned}$$
where $C^1_\Sigma(\widetilde{\varphi}, \varphi)$ is defined as follows:

1. *Let F be the forgetting map induced by $id_M : (M, N) \to \big(M, \varphi^{-1}(N')\big)$:*

$$\begin{tikzcd}
\widetilde{C}[M] \otimes_{\mathcal{F}_M} \eta \arrow[r, "="] \arrow[d, "/\sim"'] & \widetilde{C}[M] \otimes_{\mathcal{F}_M} \eta \arrow[d, "/\sim"] \\
C_\Sigma\big[M, N; \eta\big] \arrow[r, "F"] & C_\Sigma\big[M, \varphi^{-1}(N'); \eta\big]
\end{tikzcd}$$

2. *Let $G : C_\Sigma\big[M, \varphi^{-1}(N'); \eta\big] \to C_\Sigma\big[M', N'; \eta'\big]$ be the map sending $*$ to $*$, and for other configurations applying φ to macroscopic locations, $D\varphi$ to collision data, and $\widetilde{\varphi}$ to summable labels.*
3. *$C^1_\Sigma(\widetilde{\varphi}, \varphi) = G \circ F$.*

Remark 4.4.
Let M be an n-manifold, $N \subset M$ be a closed subspace, and $\eta = \big((X,*) \to (\eta,\iota) \to M\big)$ be a bundle over M such that $\eta_{M\setminus N}$ is a twisted partial left algebra over $\mathcal{F}_{M\setminus N}$. We can still define $C_\Sigma\big[M,N;\eta\big]$ as follows. First treat $(X,*)$ as a trivial summable label over $\mathcal{F}_n$ as we defined in **Example** *3.2. Then we can regard η as a twisted partial left algebra η_0 over $\mathcal{F}_M$ with the trivial action, and take the tensor product $\widetilde{C}[M] \otimes_{\mathcal{F}_M} \eta_0$. Define $C_\Sigma\big[M,N;\eta\big]$ to be the quotient*

$$C_\Sigma\big[M,N;\eta\big] = \widetilde{C}[M] \otimes_{\mathcal{F}_M} \eta_0 \Big/ \sim$$

modulo the equivalence relation $\sim$ generated by relations: (1) *forgetting trees rooted in N, and* (2) *reducing trees representing the actions of $\mathcal{F}_{M\setminus N}$. The similar functoriality with C^1_Σ holds.*

We check two basic properties of C^1_Σ.

Proposition 4.8 (product)**.**
*Let $\big(M,N,(\eta,\iota)\big)$ be an object of $\mathcal{M}$. Let M_1, M_2 be codimension-*0 *submanifolds of M such that $M_1 \cup M_2 = M$, and $M_1 \cap M_2 \subset N$. Then we have a decomposition*

$$C_\Sigma\big[M,N;\eta\big] \cong C_\Sigma\big[M_1, M_1\cap N; \eta|_{M_1}\big] \times C_\Sigma[M_2, M_2 \cap N; \eta|_{M_2}].$$

Proof. Since $M_i \subset M$ has codimension-0, $\mathcal{F}_{M_i}$ is the restriction of $\mathcal{F}_M$ on M_i. Thus we have the decomposition given by sorting trees by macroscopic locations into M_1 or M_2 . Those falling into $M_1 \cap M_2$ vanish. □

Proposition 4.9 (excision)**.**
Let $\big(M,N;(\eta,\iota)\big)$ be an object of $\mathcal{M}$, and $K \subset \operatorname{int} N$. Assume that

$$\big(M\setminus K, N\setminus K; (\eta,\iota)|_{M\setminus K}\big)$$

is an object of $\mathcal{M}$. Then

$$C_\Sigma\big[M,N;\eta\big] \cong C_\Sigma\big[M\setminus K, N\setminus K; \eta|_{M\setminus K}\big].$$

Proof. The homeomorphism is given by reading tree representations from the domain as trees in the range. The condition $K \subset \operatorname{int} N$ guarantees that a tree vanishing at N also vanishes at $N \setminus K$. □

Map between configuration spaces induced by an immersion Let $\big(M, N; (\eta, \iota)\big)$, $\big(M', N'; (\eta', \iota')\big)$ be objects of $\mathcal{M}$ with M, M' compact, and $(\widetilde{\varphi}, \varphi)$ be an $(\mathcal{F}_\varphi, \varphi)$-equivariant twisted partial left algebra morphism

$$\begin{tikzcd}
(\eta, \iota) \arrow[r, "\widetilde{\varphi}"] \arrow[d] & (\eta', \iota') \arrow[d] \\
M \arrow[r, "\varphi"] & M'
\end{tikzcd}$$

such that φ is an immersion satisfying

$$\varphi(N) \subset N',\ \varphi^{-1}\big(\varphi(N)\big) = N.$$

Since φ is not necessarily injective, we cannot define $\widetilde{C}_k[\varphi] : \widetilde{C}_k[M] \to \widetilde{C}_k[M']$ as we did for an embedding. φ fails to determine collision data for the roots of trees in $C_k[M]$ possibly sent to the same macroscopic location in M'.

We obtain a map between configuration spaces induced from $(\widetilde{\varphi}, \varphi)$ by considering compactifications of configuration spaces dependent to φ.

Let $\widetilde{C}^{\varphi}_k(M) \subset \widetilde{C}_k[M]$ be the subspace consisting of configurations whose macroscopic locations have mutually distinct images along φ. There exists a map

$$\widetilde{C}_k[\varphi] : \widetilde{C}^{\varphi}_k(M) \hookrightarrow \widetilde{C}_k[M']$$

applying φ to macroscopic locations and $D\varphi$ to collision data.

Let $\lambda_k : \widetilde{C}^{\varphi}_k(M) \hookrightarrow \widetilde{C}_k[M]$ denote the inclusion. We have the following commutative diagram:

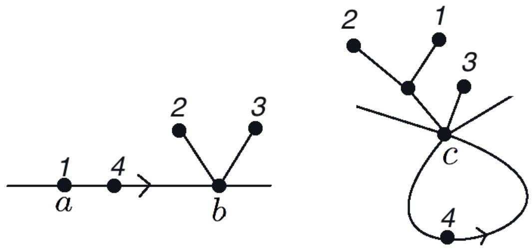
$$\begin{tikzcd}
\widetilde{C}^{\varphi}_k(M) \arrow[rr, "\widetilde{C}_k[\varphi]"] \arrow[dr] \arrow[dd, "\lambda_k"'] & & \widetilde{C}_k[M'] \arrow[dd, "b'_k"] \\
& \widetilde{C}_k[M] \times_{M'^k} \widetilde{C}_k[M'] \arrow[ur] \arrow[dl, "r_k"'] & \\
\widetilde{C}_k[M] \arrow[r, "b_k"] & M^k \arrow[r, "\varphi^k"] & M'^k
\end{tikzcd}$$

where the right lower triangle is a pullback diagram, and b_k, b'_k are extensions of $\widetilde{C}_k(M) \hookrightarrow M^k$, $\widetilde{C}_k(M') \hookrightarrow M'^k$ respectively.

Definition 4.4 (φ-compactified configuration space)**.**
Let $\widetilde{C}^{\varphi}_k[M]$ be the closure of the image of

$$(\lambda_k, \widetilde{C}_k[\varphi]) : \widetilde{C}^{\varphi}_k(M) \to \widetilde{C}_k[M] \times \widetilde{C}_k[M'].$$

We call $\widetilde{C}^{\varphi}_k[M]$ the φ**-compactified configuration space of ordered k-points in** *M.*



$$\mathbb{R} \xrightarrow{\varphi} \mathbb{R}^2 \qquad a, b \mapsto c$$

Figure 4.4: $\widetilde{C}_4^{\varphi}[\mathbb{R}]$

Proposition 4.10.

$$\widetilde{C}_k^{\varphi}[M] \subset \widetilde{C}_k[M] \times_{M'^k} \widetilde{C}_k[M'].$$

Proof. The image of $(\lambda_k, \widetilde{C}_k[\varphi])$ is contained in $\widetilde{C}_k[M] \times_{M'^k} \widetilde{C}_k[M']$. Thus it suffices to show that $\widetilde{C}_k[M] \times_{M'^k} \widetilde{C}_k[M']$ is closed in $\widetilde{C}_k[M] \times \widetilde{C}_k[M']$. $\widetilde{C}_k[M] \times_{M'^k} \widetilde{C}_k[M']$ is the inverse image of the diagonal along the map

$$(\varphi^k b_k) \times b'_k : \widetilde{C}_k[M] \times \widetilde{C}_k[M'] \to (M'^k)^2.$$

Thus $\widetilde{C}_k[M] \times_{M'^k} \widetilde{C}_k[M']$ is closed. $\square$

Remark 4.5.
Each element of $\widetilde{C}_k^{\varphi}[M]$ is represented by a finite number of trees $t = \{t_i\}_{i=1}^{j}$ for some $j \in \mathbb{N}$ and labels on the vertices of t such that

1. *Each root of trees in t is labelled with a point in $\varphi(M)$, and the labels are mutually distinct. We call such labels the* **macroscopic locations of the configuration**.

2. *If v is a leaf of t_i and the root of t_i is labelled with m', then v is labelled with an element $m \in \varphi^{-1}(m')$. We call such labels the* **origins of the configuration**.

3. *If v is a vertex of t_i with valence $h > 1$, the root of t_i is labelled with m', and the leaves whose unique path to the root passes through v have origins in $A_v \subset \varphi^{-1}(m')$, then v is labelled with an element of*

$$[w_1, \ldots, w_h] \in \widetilde{C}_h(T_{m'}M') \Big/ \mathrm{Aff}_+(\dim M') \, .$$

such that $w_1, \ldots, w_h$ can be chosen to be vectors generated by $\cup_{a \in A_v} D\varphi(T_a M)$.

We call such labels the **collision data of the configuration**.

4. The leaves of t are ordered by $[k]$.

However, it is not obvious whether every representative is realized by an element of $\widetilde{C}_k^{\varphi}[M]$. It is a research project to investigate the topology of $\widetilde{C}_k^{\varphi}[M]$.

The twisted operad morphism $(\mathcal{F}_{\varphi}, \varphi) : \mathcal{F}_M \to \mathcal{F}_{M'}$ induces an diagonal action of $\mathcal{F}_M$ on $\left(\widetilde{C}_k[M] \times_{M'^k} \widetilde{C}_k[M'] \xrightarrow{b_k r_k} M^k\right)_{k\in\mathbb{N}}$.

Proposition 4.11.
The diagonal action of $\mathcal{F}_M$ on $\left(\widetilde{C}_k[M] \times_{M'^k} \widetilde{C}_k[M'] \to M^k\right)_{k\in\mathbb{N}}$ restricts to an action of $\mathcal{F}_M$ on $\widetilde{C}^{\varphi}[M] = \left(\widetilde{C}_k^{\varphi}[M] \to M^k\right)_{k\in\mathbb{N}}$. Thus $\widetilde{C}^{\varphi}[M]$ is a twisted right module over $\mathcal{F}_M$.

Proof. Let $(m, m') \in \widetilde{C}_k^{\varphi}[M]$, $m \in \widetilde{C}_k[M]$, and $m' \in \widetilde{C}_k[M']$. Let $t = (t_j)_{j=1}^k \in \Pi_{j=1}^k \mathcal{F}_M(n_j)$. Let $t' = \left(\mathcal{F}_{\varphi}(n_j)(t_j)\right)_{j=1}^k$. Let $n = \sum_{j=1}^k n_j$. Suppose that $m \circ t$ is defined. Then $m' \circ t'$ is defined, and $(m, m') \circ t = (m \circ t, m' \circ t')$.

Let $(m_\alpha)_\alpha$ be a sequence in $\widetilde{C}_k^{\varphi}(M)$ such that $(m_\alpha)_\alpha$ converges to m in $\widetilde{C}_k[M]$, and $\left(\widetilde{C}_k[\varphi](m_\alpha)\right)_\alpha$ converges to m' in $\widetilde{C}_k[M']$. We may assume that for each j every j-th particle in the sequence $(m_\alpha)_\alpha$ is contained in a neighbourhood where $\mathcal{F}_M(n_j)$ is locally trivial. Thus we can choose local trivializations of $\mathcal{F}_M(n_j)$ over such neighbourhoods and translate t to m_α according to those local trivializations. $(m_\alpha \circ t)_\alpha$ is a sequence in $\widetilde{C}_n^{\varphi}(M)$ converging to $m \circ t$ in $\widetilde{C}_n[M]$, and its image along $\widetilde{C}_n[\varphi]$

$$\widetilde{C}_n[\varphi](m_\alpha \circ t) = \widetilde{C}_k[\varphi](m_\alpha) \circ t'$$

converges to $m' \circ t'$ in $\widetilde{C}_n[M']$. Thus $(m, m') \circ t \in \widetilde{C}_n^{\varphi}[M]$. □

Definition 4.5 (φ-immersed configuration space with twisted summable labels)**.**
Since $\widetilde{C}_k^{\varphi}[M] \subset \widetilde{C}_k[M] \times \widetilde{C}_k[M']$, we can regard $\widetilde{C}^{\varphi}[M] \otimes_{\mathcal{F}_M} \eta$ as a subset of

$$C_\Sigma\big[M, \emptyset; \eta\big] \times \Big(\coprod_{k\in\mathbb{N}} \widetilde{C}_k[M']\Big).$$

Then each element of $\widetilde{C}^{\varphi}[M] \otimes_{\mathcal{F}_M} \eta$ is represented by a pair of tree representations (t, t'). Let $C_\Sigma^{\varphi}\big[M, N; \eta\big]$ be the quotient

$$\left(\widetilde{C}^{\varphi}[M] \otimes_{\mathcal{F}_M} \eta\right)\Big/\!\sim$$

modulo the equivalence relation $\sim$ forgetting the trees in t rooted in N and the trees in t' rooted in $\varphi(N)$. Since $\varphi^{-1}(\varphi(N)) = N$ by assumption, the new representative obtained from each (t, t') by forgetting trees actually belongs to $\widetilde{C}^{\varphi}[M] \otimes_{\mathcal{F}_M} \eta$.

We call $C_\Sigma^{\varphi}\big[M, N; \eta\big]$ the **φ-immersed configuration space of M vanishing at N with twisted summable labels in η.**

Let

$$\big(p(k), q(k)\big) : \widetilde{C}_k^{\varphi}[M] \to \widetilde{C}_k[M] \times_{M'^k} \widetilde{C}_k[M']$$

denote the inclusion extending $(\lambda_k, \widetilde{C}_k[\varphi])$.

Proposition 4.12.
The sequence of maps $q = \Big(\widetilde{C}_k^{\varphi}[M] \xrightarrow{q(k)} \widetilde{C}_k[M']\Big)_{k\in\mathbb{N}}$ *is an* $(\mathcal{F}_{\varphi}, \varphi)$*-equivariant twisted right module morphism* $q : \widetilde{C}^{\varphi}[M] \to \widetilde{C}[M']$.

Proof. We need to check the equivariance. Let $(m, m') \in \widetilde{C}_k^{\varphi}[M]$, $m \in \widetilde{C}_k[M]$, and $m' \in \widetilde{C}_k[M']$. Let $t = (t_j)_{j=1}^k \in \Pi_{j=1}^k \mathcal{F}_M(n_j)$. Suppose that $m \circ t$ is defined. Let $t' = \big(\mathcal{F}_{\varphi}(n_j)(t_j)\big)_{j=1}^k$.

$$\begin{aligned} q(\sum_{j=1}^{k} n_j)\big((m, m') \circ t\big) = q(\sum_{j=1}^{k} n_j)(m \circ t, m' \circ t') = \\ m' \circ t' = \big(q(k)(m, m')\big) \circ t'. \end{aligned}$$

□

Thus $(\widetilde{\varphi}, \varphi)$ induces a map

$$\big(q,\ (\mathcal{F}_{\varphi}, \varphi),\ \widetilde{\varphi}\big)_* : \widetilde{C}^{\varphi}[M] \otimes_{\mathcal{F}_M} \eta \to \widetilde{C}[M'] \otimes_{\mathcal{F}_{M'}} \eta'.$$

$\big(q,\ (\mathcal{F}_{\varphi}, \varphi),\ \widetilde{\varphi}\big)_*$ descends to a map

$$C_{\Sigma}(\widetilde{\varphi}, \varphi) : C_{\Sigma}^{\varphi}\big[M, N; \eta\big] \to C_{\Sigma}\big[M', N'; \eta'\big].$$

One example of summable labels without local triviality condition We finish this section with an example of configuration spaces whose summable labels are not locally trivial. The example gives a model for summing configurations from two manifolds in an A_∞ way when the two manifolds have common boundaries.

Let M be an n-manifold obtained by gluing two manifolds along the collars of common boundaries:

$$\begin{aligned} &M = \Big(M_{-1} \amalg \big(M_0 \times [-1, 1]\big) \amalg M_1\Big)\Big/\!\sim\ , \\ &\partial M_{-1} = \partial' M_{-1} \amalg \partial'' M_{-1},\ \partial' M_{-1} = M_0 \times \{-1\}, \\ &\partial M_1 = \partial' M_1 \amalg \partial'' M_1,\ \partial' M_1 = M_0 \times \{1\}. \end{aligned}$$

Let f be the smooth height map defined by

$$f : M \to [-1, 1], \quad M_{-1} \mapsto -1,\ M_0 \times \{t\} \mapsto t,\ M_1 \mapsto 1.$$

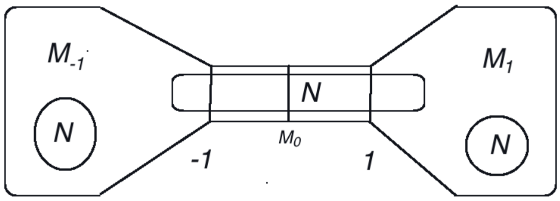


Figure 4.5: M

Let $N \subset M$ be a closed submanifold such that $N \cap \left(M_0 \times [-1,1]\right) = N_0 \times [-1,1]$ for a closed submanifold $N_0 \subset M_0$, and (η, ι) be a twisted partial left algebra over $\mathcal{F}_M$.

We consider the subspace C^f_Σ of $C_\Sigma\left[M, N; \eta\right]$ consisting of configurations each of which is represented by a finite number of trees whose roots have the same f-value.

Let f_0 be the map taking the f-value of roots

$$f_0 : C^f_\Sigma \to [-1,1].$$

C^f_Σ is not necessarily a bundle over $[-1,1]$, but we can treat it as a partial summable label over $[-1,1]$ with special behaviour at the boundary $\{-1,1\}$. We define the action of $\mathcal{F}_{[-1,1]} = \mathcal{F}_1 \times [-1,1]$ on C^f_Σ as the following three maps. Let $-1 < t < 1$. The maps

$$\begin{aligned}
\mathcal{F}_1(k) \times \left(f_0^{-1}(t)\right)^k &\to f_0^{-1}(t), \quad \left(g, (h_j)_{j=1}^k\right) \mapsto g \circ (h_j)_{j=1}^k, \\
\mathcal{F}_1(k) \times \left(f_0^{-1}(-1) \times \left(f_0^{-1}(0)\right)^{k-1}\right) &\to f_0^{-1}(-1), \quad \left(g, (h_j)_{j=1}^k\right) \mapsto g \circ (h_j)_{j=1}^k, \\
\mathcal{F}_1(k) \times \left(\left(f_0^{-1}(0)\right)^{k-1} \times f_0^{-1}(1)\right) &\to f_0^{-1}(1), \quad \left(g, (h_j)_{j=1}^k\right) \mapsto g \circ (h_j)_{j=1}^k
\end{aligned}$$

define $g \circ (h_j)_{j=1}^k$ to be the collision of $(h_j)_{j=1}^k$ along g. More precisely $g \circ (h_j)_{j=1}^k$ is obtained by the following process:

1. Plant g all over M_0, where $M_0 \times \{t\}$ is identified with M_0 for the first map, $\partial' M_{-1}$ is identified with M_0 for the second map, and $\partial' M_1$ is identified with M_0 for the third map.

2. Regard configurations in $f_0^{-1}(0)$ as configurations rooted in M_0 for the second and third map. For a tree in h_j, say its root is labelled with m. If $m \notin M_0$, just plant it over m. If $m \in M_0$, plant it over the j-th leaf of the g rooted over m.

3. Label the remaining leaves of g's with ι, and perform the action of $\mathcal{F}_M$.

C^f_Σ as a partial summable label over $[-1,1]$ is partial since the action of $\mathcal{F}_{[-1,1]}$ are only partially defined at -1 and 1.

We define the configuration space A as the subspace

$$A \subset \widetilde{C}\big[[-1,1]\big] \otimes_{\mathcal{F}_{[-1,1]}} C^f_\Sigma$$

consisting of configurations each of which is represented by a finite number of single vertex trees rooted over $[-1,1]$. In other words we exclude configurations coming with an undefined action of $\mathcal{F}_{[-1,1]}$ on C^f_Σ. A tree containing an undefined action is rooted at -1 or 1. $\otimes$ here is abuse of notation since C^f_Σ is not a twisted partial left algebra over $\mathcal{F}_{[-1,1]}$.

Suppose that two trees a, b in $C_\Sigma\big[M, N; \eta\big]$ approach towards one macroscopic location in a slice $M_0 \times \{t\}$ along directions asymptotically parallel to M_0. a, b cannot collide when they are regarded as a configuration in A. However, if they approach each other along directions asymptotically parallel to $[-1,1]$, they can collide.

4.2 Spaces of local trivializations over tubular neighbourhoods of points

This section deals with a technical problem relating to the diffeomorphism-equivariance of twisted summable labels. Let W be an n-manifold without boundary and $\big((X,*) \to (\eta,\iota) \to W\big)$ be a twisted partial left algebra over $\mathcal{F}_W$. We define and study the space of local trivializations of η over tubular neighbourhoods of points in W, denoted by $\mathrm{Tub}^\eta(W)$. The range of diffeomorphism-equivariant scanning maps will be a section space over $\mathrm{Tub}^\eta(W)$ instead of W. The goal of this section is to show that $\mathrm{Tub}^\eta(W) \simeq_w W$.

Additional assumption for twisted partial summable label We first introduce a condition for a partial left algebra over $\mathcal{F}_n$.

Definition 4.6 (framed partial left algebra over $\mathcal{F}_n$).
Let $(X,*)$ be a partial left algebra over $\mathcal{F}_n$. We say that X is **framed** *if it is equipped with an action of* $\mathrm{GL}(n)$ *on* X

$$\begin{aligned} \mathrm{GL}(n) \times (X,*) &\to (X,*), \\ (A,x) &\mapsto A \cdot x, \\ (A,*) &\mapsto * \end{aligned}$$

such that $A\cdot : X \to X,\ x \mapsto A \cdot x$ *is a partial left algebra morphism equivariant with respect to* $\mathcal{F}_A : \mathcal{F}_n \to \mathcal{F}_n$ *for every* $A \in \mathrm{GL}(n)$.

If $(X, *)$ is a framed partial left algebra over $\mathcal{F}_n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is an embedding, then the map

$$\widetilde{\varphi} : \mathbb{R}^n \times X \to \mathbb{R}^n \times X, \qquad (r, x) \mapsto \big(\varphi(r), D\varphi(r) \cdot x\big)$$

is an $(\mathcal{F}_\varphi, \varphi)$-equivariant twisted partial left algebra morphism. It induces a map

$$C^1_\Sigma(\widetilde{\varphi}, \varphi) : C_\Sigma\big[\mathbb{R}^n, \emptyset; \mathbb{R}^n \times X\big] \to C_\Sigma\big[\mathbb{R}^n, \emptyset; \mathbb{R}^n \times X\big].$$

Proposition 4.13.
Let X *be a framed partial left algebra over* $\mathcal{F}_n$. *Let* G *denote the topological group such that*

$$(g_1, g_2) \in G \subset \mathrm{GL}(n) \times \mathrm{Homeo}_0(X)$$

if and only if $g_1 \times g_2$ *is an* $(\mathcal{F}_{g_1}, g_1)$*-equivariant twisted partial left algebra isomorphism*

$$\begin{tikzcd}
\mathbb{R}^n \times (X, *) \arrow[r, "g_1 \times g_2"] \arrow[d] & \mathbb{R}^n \times (X, *) \arrow[d] \\
\mathbb{R}^n \arrow[r, "g_1"] & \mathbb{R}^n
\end{tikzcd}$$

Let $\mathrm{Iso}(X)$ *denote the topological group of* $id_{\mathcal{F}_n}$*-equivariant isomorphisms of* X. *Then we have an isomorphism of topological groups*

$$G \cong \mathrm{GL}(n) \times \mathrm{Iso}(X).$$

Proof. The isomorphism is given by

$$\begin{aligned}
G \to \mathrm{GL}(n) \times \mathrm{Iso}(X), &\quad (g_1, g_2) \mapsto (e_1, e_2) = \big(g_1, (g_1\cdot)^{-1} g_2\big), \\
\mathrm{GL}(n) \times \mathrm{Iso}(X) \to G, &\quad (e_1, e_2) \mapsto (g_1, g_2) = \big(e_1, (e_1\cdot) e_2\big).
\end{aligned}$$

□

Now we define a special class of twisted partial left algebras over $\mathcal{F}_W$.

Definition 4.7 (twisted partial summable labels over W).
Let W *be an* n*-manifold without boundary. Let* $\big((X, *) \to (\eta, \iota) \to W\big)$ *be a twisted partial left algebra over* $\mathcal{F}_W$.

η *is called a* **twisted partial summable label over** W*, if the following hold:*

1. X *is framed.*

2. $TW \times_W \eta$ *has a structure group G where G acts on the typical fibre $\mathbb{R}^n \times X$ as*

$$(g_1, g_2) \in G \subset \mathrm{GL}(n) \times \mathrm{Homeo}_0(X), \quad (r, x) \in \mathbb{R}^n \times X,$$
$$(g_1, g_2) \cdot (r, x) = \big((g_1(r), g_2(x)\big).$$

Proposition 4.14.
Let $\varphi : W_1 \to W_2$ be an embedding between n-manifolds without boundary, and $\big((X_i, *) \to (\eta_i, \iota_i) \to W_i\big)$ be a twisted partial summable label for $i = 1, 2$. Assume that there exists a $\mathrm{GL}(n)$-equivariant map $\alpha : X_1 \to X_2$ which is also an $\mathcal{F}_n$-equivariant partial left algebra morphism. Then there exists a twisted partial left algebra morphism $\widetilde{\varphi} : \eta_1 \to \eta_2$ equivariant with respect to $(\mathcal{F}_\varphi, \varphi)$.

Proof. Gather local representations

$$\mathbb{R}^n \times X_1 \to \mathbb{R}^n \times X_2, \quad (r, x) \mapsto (\varphi(r), D\varphi(r) \cdot \alpha(x)).$$

$\square$

We will mainly work for twisted partial summable labels in the rest of the paper rather than general twisted partial left algebras.

Remark 4.6.
If $\big((X, *) \to (\eta, \iota) \to W\big)$ is a bundle of spaces, η can be regarded as a twisted partial summable label over W with the trivial action of $\mathrm{GL}(n)$ on X and the action of $\mathcal{F}_n$ on X defined as

$$Comp(k) = \mathcal{F}_n(k) \times \bigvee_{j=1}^{k} X \to X, \qquad (k \in \mathbb{N})$$
$$\Big(f; \big((*)_{i=1}^{j-1}, x, (*)_{i=1}^{k-j}\big)\Big) \mapsto x.$$

Space of local trivializations We define $\mathrm{Tub}^\eta(W)$.

Definition 4.8 (space of local trivializations of η)**.**
Let W be an n-manifold without boundary and $\big((X, *) \to (\eta, \iota) \to W\big)$ be a twisted partial summable label over $\mathcal{F}_W$.

1. *Let* $\mathrm{Emb}^\eta(\mathbb{R}^n, W)$ *be the space such that*

$$(\hat{f}, f) \in \mathrm{Emb}^\eta(\mathbb{R}^n, W) \subset \mathrm{Map}(\mathbb{R}^n \times X, \eta) \times \mathrm{Emb}(\mathbb{R}^n, W)$$

   if and only if the following diagram commutes and gives a $(\mathcal{F}_f, f)$*-equivariant twisted partial left algebra isomorphism*

$$\begin{tikzcd} \mathbb{R}^n \times (X, *) \arrow[r, "\hat{f}"] \arrow[d] & \big(\eta|_{\mathrm{im}(f)}, \iota|_{\mathrm{im}(f)}\big) \arrow[d] \\ \mathbb{R}^n \arrow[r, "f"] & \mathrm{im}(f) \end{tikzcd}$$

   $\mathrm{Emb}^\eta(\mathbb{R}^n, W)$ *is nonempty by the stronger local triviality condition for* η *assumed in Definition 4.7. In fact every contractible open subspace of* W *can be covered by an element of* $\mathrm{Emb}^\eta(\mathbb{R}^n, W)$.

2. $\mathrm{GL}(n)$ *acts on* $\mathrm{Emb}(\mathbb{R}^n, W)$ *by precomposition. Let* $\mathrm{Tub}(W)$ *denote the quotient*

$$\mathrm{Emb}(\mathbb{R}^n, W)/\,\mathrm{GL}(n).$$

3. G *acts on* $\mathrm{Emb}^\eta(\mathbb{R}^n, W)$ *by precomposition:*

$$\begin{aligned} &(g_1, g_2) \in G \subset \mathrm{GL}(n) \times \mathrm{Homeo}_0(X), \\ &(\hat{f}, f) \in \mathrm{Emb}^\eta(\mathbb{R}^n, W), \\ &(\hat{f}, f) \cdot (g_1, g_2) = \big(\hat{f}(g_1 \times g_2), f g_1\big). \end{aligned}$$

   Let $\mathrm{Tub}^\eta(W)$ *be the quotient*

$$\mathrm{Emb}^\eta(\mathbb{R}^n, W)/G.$$

We have a useful representation of $\mathrm{Tub}^\eta(W)$ relating it to tubular neighbourhoods of points in W. A tubular neighbourhood of a point m in W means an embedding

$$h : T_m W \to W$$

such that $Dh(0) = id_{T_m M}$. By the correspondence

$$\begin{tikzcd} T_{f(0)}W \times \big(\eta_{f(0)}, \iota_{f(0)}\big) \arrow[d] & \mathbb{R}^n \times (X, *) \arrow[l, "\cong"', "Df(0)\times \hat{f}(0,\cdot)"] \arrow[r, "\hat{f}"] \arrow[d] & (\eta, \iota) \arrow[d] \\ T_{f(0)}W & \mathbb{R}^n \arrow[l, "\cong"', "Df(0)"] \arrow[r, "f"] & W \end{tikzcd}$$

$$(\hat{f}, f) \quad \mapsto \quad (\hat{h}, h) = (\hat{f}, f) \circ \Big(Df(0) \times \hat{f}(0, \cdot), Df(0)\Big)^{-1}$$

we can think of $(\hat{f}, f)$ as a tubular neighbourhood h of $f(0)$ in W and an $(\mathcal{F}_h, h)$-equivariant isomorphism $\hat{h}$ normalized as $\hat{h}(0, \cdot) = id_{\eta_{f(0)}}$.

Proposition 4.15.
$(\hat{f}, f) \mapsto (\hat{h}, h)$ is the one-to-one correspondence.

Proof. For $(\hat{h}, h)$, choose $(\hat{g}, g) \in \mathrm{Tub}^{\eta}(W)$ such that $\mathrm{im}(g) = \mathrm{im}(h)$ and $g(0) = h(0)$

$$\begin{tikzcd}
T_mW \times (\eta_m, \iota_m) \arrow[r, "\hat{h}"] \arrow[d] & (\eta, \iota) \arrow[d] & \mathbb{R}^n \times (X, *) \arrow[l, "\hat{g}"'] \arrow[d] \\
T_mW \arrow[r, "h"] & W & \mathbb{R}^n \arrow[l, "g"']
\end{tikzcd}$$

This choice exists since $\mathrm{im}(h)$ is contractible. Take $(\hat{f}, f) = (\hat{h}, h) \circ \big(Dg(0) \times \hat{g}(0, \cdot), Dg(0)\big)$. $\square$

Proposition 4.16.
The map

$$\pi : \mathrm{Tub}^{\eta}(W) \to \mathrm{Tub}(W), \quad (\hat{h}, h) \mapsto h$$

is surjective.

Proof. For a tubular neighbourhood h, choose $(\hat{g}, g) \in \mathrm{Tub}^{\eta}(W)$ such that $\mathrm{im}(g) = \mathrm{im}(h)$ and $g(0) = h(0)$

$$\begin{tikzcd}
 & (\eta, \iota) \arrow[d] & \mathbb{R}^n \times (X, *) \arrow[l, "\hat{g}"'] \arrow[d] \\
T_mW \arrow[r, "h"] & W & \mathbb{R}^n \arrow[l, "g"]
\end{tikzcd}$$

Let $F = (g^{-1}h) \circ Dg(0) : \mathbb{R}^n \to \mathbb{R}^n$. Then the map:

$$\hat{F} : \mathbb{R}^n \times X \to \mathbb{R}^n \times X, \quad (r, x) \mapsto \big(F(r), DF(r) \cdot x\big)$$

is an $(\mathcal{F}_F, F)$-equivariant isomorphism.

$$\begin{tikzcd}
\mathbb{R}^n \times X \arrow[rr, "\hat{F}"] \arrow[d] & & \mathbb{R}^n \times X \arrow[r, "\hat{g}"] \arrow[d] & \eta \arrow[d] \\
\mathbb{R}^n \arrow[r, "Dg(0)"] & T_mM \arrow[r, "g^{-1}h"] & \mathbb{R}^n \arrow[r, "g"] & W
\end{tikzcd}$$

If we take $(\hat{f}, f) = (\hat{g}, g) \circ (\hat{F}, F)$, it corresponds to $(\hat{h}, h) \in \mathrm{Tub}^{\eta}(W)$. $\square$

Proposition 4.17.
π_1 and π_2 are fibre bundles with contractible fibres

$$\begin{tikzcd} \mathrm{Tub}^\eta(W) \arrow[d, "\pi"] \arrow[dr, "\pi_1"] & \\ \mathrm{Tub}(W) \arrow[r, "\pi_2"] & W \end{tikzcd}$$

$$\pi_1 : (\hat{f}, f) \mapsto f(0),\ \pi_2 : f \mapsto f(0).$$

Thus they allow global sections.

Proof.

1. We first show that they are fibre bundles. Let m be a point in W. Choose $(\hat{e}, e) \in \mathrm{Tub}^\eta(W)$

$$\begin{tikzcd} \mathbb{R}^n \times X \arrow[r, "\hat{e}"] \arrow[d] & \eta \arrow[d] \\ \mathbb{R}^n \arrow[r, "e"] & W \end{tikzcd}$$

such that $e(0) = m$. Let $B_1, B_{1/2}$ denote the open balls in $\mathbb{R}^n$ around 0 with radii $1, 1/2$ respectively. Choose a map

$$k : B_{1/2} \to \mathrm{Diff}(\mathbb{R}^n),\ b \mapsto k_b$$

such that k_b sends 0 to b and fixes points outside B_1.

Let $\widetilde{k_b}$ be the map

$$\widetilde{k_b} : \mathbb{R}^n \times X \to \mathbb{R}^n \times X,\ \ (r, x) \mapsto \big(k_b(r), Dk_b(r) \cdot (x)\big).$$

Then $\widetilde{k_b}$ is an $(\mathcal{F}_{k_b}, k_b)$-equivariant isomorphism

$$\begin{tikzcd} \mathbb{R}^n \times X \arrow[r, "\widetilde{k_b}"] \arrow[d] & \mathbb{R}^n \times X \arrow[d] \\ \mathbb{R}^n \arrow[r, "k_b"] & \mathbb{R}^n \end{tikzcd}$$

Since $\widetilde{k_b}$ fixes points outside $B_1 \times X$, we can glue the map $\big(\hat{e} \circ \widetilde{k_b} \circ \hat{e}^{-1}\big)|_{\mathrm{im}(\hat{e})}$ and $id_\eta|_{\eta \setminus \hat{e}(B_1 \times X)}$, and obtain a continuous family of bundle isomorphisms parametrized by $B_{1/2}$

$$\begin{tikzcd} \eta \arrow[r, "\widetilde{\varphi}_b"] \arrow[d] & \eta \arrow[d] \\ W \arrow[r, "\varphi_b"] & W \end{tikzcd}$$

such that $\widetilde{\varphi}_b$ is an $(\mathcal{F}_{\varphi_b}, \varphi_b)$-equivariant isomorphism. Note that $\varphi_b(m) = e(b)$. Now we obtain a local trivialization of π_1 over $e(B_{1/2})$

$$t_1 : B_{1/2} \times \pi_1^{-1}(m) \to \pi_1^{-1}\big(e(B_{1/2})\big), \ \ \big(b, (\hat{f}, f)\big) \mapsto (\widetilde{\varphi}_b \circ \hat{f}, \varphi_b \circ f)$$

$$\begin{tikzcd}
B_{1/2} \times \pi_1^{-1}(m) \arrow[r, "t_1"] \arrow[d] & \pi_1^{-1}\big(e(B_{1/2})\big) \arrow[d] \\
B_{1/2} \arrow[r, "e"] & e(B_{1/2})
\end{tikzcd}$$

with an inverse

$$(\hat{g}, g) \mapsto \Big(b = e^{-1}\big(g(0)\big), \big(\widetilde{\varphi}_b^{-1} \circ \hat{g}, \varphi_b^{-1} \circ g\big)\Big).$$

Since G acts by precomposition, the formula for t_1 gives the same value for representatives from the same orbit. The same holds for the inverse of t_1. The second coordinate of t_1 gives a local trivialization $t_2 : (b, f) \mapsto \varphi_b \circ f$ of π_2 and the following diagram commutes:

$$\begin{tikzcd}
B_{1/2} \times \pi_1^{-1}(m) \arrow[r, "t_1", "\cong"'] \arrow[d, "id\times\pi"'] & \pi_1^{-1}\big(e(B_{1/2})\big) \arrow[d, "\pi"] \\
B_{1/2} \times \pi_2^{-1}(m) \arrow[r, "t_2", "\cong"'] & \pi_2^{-1}\big(e(B_{1/2})\big)
\end{tikzcd}$$

2. It is a classical result that the space of tubular neighbourhoods of m, $\pi_2^{-1}(m)$, is contractible. We show that $\pi_1^{-1}(m)$ is contractible. Choose a smooth shrinking homotopy through embeddings

$$H : \mathbb{R}^n \times I \to \mathbb{R}^n$$

such that $H_0 = id_{\mathbb{R}^n}$, $H_1 : \mathbb{R}^n \cong \mathring{D}^n \hookrightarrow \mathbb{R}^n$, $DH_t(0) = id_{\mathbb{R}^n}$, and $H_t(0) = 0$.

Fix a linear isomorphism $A : \mathbb{R}^n \cong T_mM$ and an $(\mathcal{F}_A, A)$-equivariant isomorphism $B : (X, *) \cong (\eta_m, \iota_m)$. $\pi_1^{-1}(m)$ is homeomorphic to the space

$$\mathrm{tub}^{\eta}(m) = \big\{(\hat{f}, f) \in \mathrm{Emb}^{\eta}(\mathbb{R}^n, W) \mid f(0) = m,\, Df(0) = A,\, \hat{f}(0, \cdot) = B\big\}$$

and $\pi_2^{-1}(m)$ is homeomorphic to the space

$$\begin{aligned}
\mathrm{tub}(m) &= \big\{f \in \mathrm{Emb}(\mathbb{R}^n, W) \mid (\hat{f}, f) \in \mathrm{tub}^{\eta}(m)\big\} \\
&= \big\{f \in \mathrm{Emb}(\mathbb{R}^n, W) \mid f(0) = m,\, Df(0) = A\big\}.
\end{aligned}$$

Let

$$p : \mathrm{tub}^\eta(m) \to \mathrm{tub}(m),\ (\hat{f}, f) \mapsto f.$$

Let $\mathrm{dub}^\eta(m) \subset \mathrm{tub}^\eta(m)$ be the subspace such that

$$(\hat{f}, f) \in \mathrm{dub}^\eta(m) \subset \mathrm{tub}^\eta(m)$$

if and only if $\mathrm{im}(f)$ is contained in a closed disk in W. Let $\mathrm{dub}(m) = p\big(\mathrm{dub}^\eta(m)\big)$. Consider a weak deformation retract K of $\mathrm{tub}^\eta(m)$ to $\mathrm{dub}^\eta(m)$

$$\begin{aligned}
&K : \mathrm{tub}^\eta(m) \times I \to \mathrm{tub}^\eta(m),\\
&K_t : \mathrm{tub}^\eta(m) \to \mathrm{tub}^\eta(m), \quad (\hat{f}, f) \mapsto (\hat{f}, f) \circ (\widetilde{H}_t, H_t),\\
&\widetilde{H}_t : (r, x) \mapsto \big(H_t(r), DH_t(r) \cdot x\big),
\end{aligned}$$

$$\begin{tikzcd}
\mathbb{R}^n \times X \arrow[r, "\widetilde{H}_t"] \arrow[d] & \mathbb{R}^n \times X \arrow[r, "\hat{f}"] \arrow[d] & \eta \arrow[d] \\
\mathbb{R}^n \arrow[r, "H_t"] & \mathbb{R}^n \arrow[r, "f"] & W
\end{tikzcd}$$

Since $DH_t(0) = id_{\mathbb{R}^n}$, $DH_t(0) \cdot x = id_{\mathbb{R}^n} \cdot x = x$, and $H_t(0) = 0$, we actually have $(\hat{f}, f) \circ (\widetilde{H}_t, H_t) \in \mathrm{tub}^\eta(m)$. Thus

$$\mathrm{tub}^\eta(m) \simeq \mathrm{dub}^\eta(m).$$

K_t descends along p to a weak deformation retract of $\mathrm{tub}(m)$ to $\mathrm{dub}(m)$, $f \mapsto f \circ H_t$. Thus $\mathrm{dub}(m)$ is contractible.

Now it suffices to show that $\mathrm{dub}^\eta(m) \xrightarrow{p} \mathrm{dub}(m)$ is a fibre bundle with a contractible fibre. Let $f \in \mathrm{dub}(m)$. Choose $(\hat{c}, c) \in \mathrm{dub}^\eta(m)$ such that $\mathrm{im}(c)$ contains a closed disk containing $\mathrm{im}(f)$. Consider an open neighbourhood U of f in $\mathrm{dub}(m)$ such that

$$d \in U \subset \mathrm{dub}(m)$$

if and only if $\mathrm{im}(d)$ is contained in a closed disk in $\mathrm{im}(c)$. Then each d determines a canonical restriction d_c of $(\hat{c}, c)$:

$$\begin{aligned}
&d \mapsto d_c = (\hat{c}, c) \circ (\widetilde{d}, c^{-1}d)\\
&\widetilde{d} : \mathbb{R}^n \times X \to \mathbb{R}^n \times X, \qquad (r, x) \mapsto \big(c^{-1}d(r), D(c^{-1}d)(r) \cdot x\big)
\end{aligned}$$

$$\begin{tikzcd}
\mathbb{R}^n \times X \arrow[r, "\widetilde{d}"] \arrow[d] & \mathbb{R}^n \times X \arrow[r, "\hat{c}"] \arrow[d] & \eta \arrow[d] \\
\mathbb{R}^n \arrow[r, "c^{-1}d"] & \mathbb{R}^n \arrow[r, "c"] & W
\end{tikzcd}$$

We have a local trivialization of $\mathrm{dub}^\eta(m) \xrightarrow{p} \mathrm{dub}(m)$ over U

$$\begin{aligned}
&U \times \mathrm{Map}\Big((\mathbb{R}^n, 0),\ \big(\mathrm{Iso}(X), id\big)\Big) \to p^{-1}(U),\\
&\big(d,\ \alpha : \mathbb{R}^n \to \mathrm{Iso}(X)\big) \mapsto (d_\alpha, d),\\
&d_\alpha : \mathbb{R}^n \times X \to \eta, \quad (r, x) \mapsto d_c\big(r, \alpha(r)(x)\big).
\end{aligned}$$

$\mathrm{Map}\Big((\mathbb{R}^n, 0),\ \big(\mathrm{Iso}(X), id\big)\Big)$ is contractible by the deformation retract J

$$\big(J_t(\alpha)\big)(r) = \alpha\big((1-t)r\big).$$

□

Proposition 4.18.
$\mathrm{Tub}^\eta(W) \xrightarrow{\pi_1} W$ *is a weak equivalence.*

Proof. $\mathrm{Tub}^\eta(W)$ is a fibre bundle over W with contractible fibre. □

4.3 Homology theoretic properties of configuration spaces

Factorization algebra in the sense of John Francis and David Ayala conceptualized in [1] is designed to be a homology theory defined on topological n-manifolds and topological embeddings rather than general spaces and maps. On the other hand configuration spaces have been known to satisfy properties similar with singular homology. The infinite symmetric products inducing singular homology theory is an example of the connection between configuration spaces and homology theory. Configuration spaces with summable labels by Salvatore in [10] can be viewed as a generalisation of infinite symmetric products making sense only for smooth manifolds and functorial only to smooth embeddings. Thus it is worth revisiting the construction of configuration spaces with twisted summable labels in the perspective of a homology theory for manifolds.

In this section we formulate the construction of configuration spaces with twisted summable labels as a functor from the category of n-manifolds with labels, and we verify properties of the functor that can be compared to the Eilenberg-Steenrod axioms for singular homology theory.

Configuration spaces as a functor We first define the domain.

Definition 4.9 (category of n-manifolds with twisted partial summable labels)**.** *We define the* **category of n-manifolds with twisted partial summable labels** *denoted by $(\mathcal{M}^n, \amalg)$. $\mathcal{M}^n$ is the symmetric monoidal category whose tensor product is disjoint union $\amalg$, and objects are generated by triples $\big(M, N; (\eta, \iota)\big)$ such that*

1. *M is a connected n-manifold.*
2. *$N \subset M$ is a closed codimension-0 submanifold. We allow the case $N = \emptyset$.*
3. *If $\partial M = \emptyset$, (η, ι) is a twisted partial summable label over M.*
4. *If $\partial M \neq \emptyset$, let $W_M = M \cup_{\partial M} \partial M \times [0, 1)$ be an open manifold containing M obtained by extending ∂M. (η, ι) is a twisted partial summable label over W_M.*

Thus an object of $\mathcal{M}^n$ is written as

$$\amalg_{i=1}^{l} \big(M_i, N_i; (\eta_i, \iota_i)\big).$$

The morphism set from $\big(M, N; (\eta, \iota)\big)$ to $\big(M', N'; (\eta', \iota')\big)$ is the collection of pairs $(\widetilde{\varphi}, \varphi)$ such that

1. *$\varphi : (M, N) \to (M', N')$ is an embedding.*
2. *$\widetilde{\varphi} : (\eta, \iota) \to (\eta', \iota')$ is a $(\mathcal{F}_\varphi, \varphi)$-equivariant twisted partial left algebra morphism.*

$$\begin{tikzcd}
(\eta, \iota) \arrow[r, "\widetilde{\varphi}"] \arrow[d] & (\eta', \iota') \arrow[d] \\
M \arrow[r, "\varphi"] & M'
\end{tikzcd}$$

The morphism set from $\amalg_{i=1}^{l} \big(M_i, N_i; (\eta_i, \iota_i)\big)$ to $\amalg_{j=1}^{l'} \big(M'_j, N'_j; (\eta'_j, \iota'_j)\big)$ is an l-tuple $(\widetilde{\varphi}_i, \varphi_i)_{i=1}^{l}$ such that

1. *$(\widetilde{\varphi}_i, \varphi_i)$ is a morphism from $\big(M_i, N_i; (\eta_i, \iota_i)\big)$ to the $j(i)$-th component of the range, $\big(M'_{j(i)}, N'_{j(i)}; (\eta'_{j(i)}, \iota'_{j(i)})\big)$.*
2. *The map $\amalg_{i=1}^{l} \varphi_i$ is injective.*

The composition is defined to be the ordinary composition of maps.

Let $(\mathcal{U}, \times)$ denote the symmetric monoidal category of based spaces with tensor product $\times$, and $(\mathcal{G}, \times)$ denote the symmetric monoidal category of groups with tensor product $\times$.

We have the functor from $\mathcal{M}^n$ to $\mathcal{U}$ taking configuration spaces

$$\begin{aligned} C^2_\Sigma : (\mathcal{M}^n, \amalg) &\to (\mathcal{U}, \times), \\ \amalg_{i=1}^l \big(M_i, N_i; (\eta_i, \iota_i)\big) &\mapsto \Pi_{i=1}^l C_\Sigma \big[M_i, N_i; \eta_i\big] \end{aligned}$$

such that a morphism between generators

$$(\widetilde{\varphi}, \varphi) : \big(M, N; (\eta, \iota)\big) \to \big(M', N'; (\eta', \iota')\big)$$

is sent to the map

$$C^2_\Sigma(\widetilde{\varphi}, \varphi) = C^1_\Sigma(\widetilde{\varphi}, \varphi) : C_\Sigma\big[M, N; \eta\big] \to C_\Sigma\big[M', N'; \eta'\big]$$

applying φ to macroscopic locations, $D\varphi$ to collision data, $\widetilde{\varphi}$ to summable labels. Consequently the morphism

$$(\widetilde{\varphi}_i, \varphi_i)_{i=1}^l : \amalg_{i=1}^l \big(M_i, N_i; (\eta_i, \iota_i)\big) \to \amalg_{j=1}^{l'} \big(M'_j, N'_j; (\eta'_j, \iota'_j)\big)$$

is sent to the map

$$\begin{aligned} C^2_\Sigma\big((\widetilde{\varphi}_i, \varphi_i)_{i=1}^l\big) : \Pi_{i=1}^l C_\Sigma\big[M_i, N_i; \eta_i\big] &\to \Pi_{j=1}^{l'} C_\Sigma\big[M'_j, N'_j; \eta'_j\big] \\ (m_i)_{i=1}^l &\mapsto \Big(\cup_{j(i)=j}\, C^2_\Sigma(\widetilde{\varphi}_i, \varphi_i)(m_i)\Big)_{j=1}^{l'} \end{aligned}$$

whose j-th summand is given by the union of configurations mapped into $C_\Sigma\big[M'_j, N'_j; \eta'_j\big]$.

We also have the functor $\mathcal{H}_k$ from $\mathcal{M}^n$ to $\mathcal{G}$ composing π_k with C^2_Σ

$$\begin{aligned} \mathcal{H}_k : (\mathcal{M}^n, \amalg) &\to (\mathcal{G}, \times), \\ \amalg_{i=1}^l \big(M_i, N_i; (\eta_i, \iota_i)\big) &\mapsto \pi_k\Big(\Pi_{i=1}^l C_\Sigma\big[M_i, N_i; \eta_i\big]\Big) \cong \Pi_{i=1}^l \pi_k C_\Sigma\big[M_i, N_i; \eta_i\big]. \end{aligned}$$

Filtration We first describe the filtration topology of $C_\Sigma[M, N; \eta]$.

Proposition 4.19 (filtration)**.**
Let $(M, N; (\eta, \iota))$ be an object of $\mathcal{M}^n$ with one component. For $k \in \mathbb{N}$, let $C_k \subset C_\Sigma[M, N; \eta]$ be the subspace consisting of configurations with at most k-leaves each of which is rooted outside N and not labelled with ι. $C_\Sigma[M, N; \eta]$ has the weak topology with respect to the filtration $\{C_k\}_{k \in \mathbb{N}}$.

Proof. For the untwisted case, refer to proposition 4.13 and 5.5 in [10].

Let D_k denote $\left(\widetilde{C}_k[M] \times_{M^k} \eta^k\right) \Big/ \Sigma_k$, where Σ_k acts diagonally. Regard each element of D_k as a representative of C_k, in other words, as a finite number of trees with macroscopic locations, collision data and summable labels. Let $R_k \subset D_k$ be the subspace consisting of those representing configurations in C_{k-1}. Let h_k be the quotient map sending representatives to configurations:

$$h_k : (D_k, R_k) \to (C_k, C_{k-1}).$$

Then

$$h_k : D_k \setminus R_k \cong C_k \setminus C_{k-1}.$$

We will show that (D_k, R_k) is an NDR pair. This implies that (C_k, C_{k-1}) is an NDR pair by 8.4 in [15]. Consequently, by 9.2 and 9.4 in [15], we can conclude that $C_\Sigma\big[M, N; \eta\big]$ is compactly generated. In particular $C_\Sigma[M, N; \eta]$ has the weak topology with respect to the filtration $\{C_k\}_{k \in \mathbb{N}}$.

Recall that $Comp^\eta(j) \subset \mathcal{F}_M(j) \times_M \eta^{(\times_M)j}$ is a subbundle and the domain of partially defined action of $\mathcal{F}_M(j)$. Consider $S_k, S'_k \subset R_k$ such that S_k is the space of Σ_k-orbits in D_k of the elements of the following set:

$$\widetilde{C}_j[M] \times_{M^j} \big(Comp^\eta(k-j+1) \times \eta^{j-1}\big),$$

and S'_k is the space of Σ_k-orbits in D_k of the elements of the following set:

$$\widetilde{C}_k[M] \times_{M^k} (\iota \times \eta^{k-1}).$$

Then $T_k = R_k \setminus (S_k \cup S'_k)$ consists of configurations containing a tree rooted in N. By definition of partial left algebras $\big(\mathcal{F}_M(j) \times_M \eta^{(\times_M)j}, Comp^\eta(j)\big)$ is an NDR pair for all j, and (η, ι) is also an NDR pair. Thus it suffices to show that (D_k, T_k) is an NDR pair. We find an open neighbourhood of V_k of T_k in D_k and a deformation retract of V_k to T_k.

Let $\partial' N$ denote the topological boundary of N. First choose some settings.

- Choose local trivializations of η around each $m \in \partial' N$ in a continuously varying way. The choice can be made by restricting a section of $\mathrm{Tub}^\eta(M)$ if $\partial M = \emptyset$, or $\mathrm{Tub}^\eta(W_M)$ if $\partial M \neq \emptyset$. Let v_m denote the open neighbourhood of m over which the trivialization chosen at m is defined.

- Choose an open collared neighbourhood U of N in M such that

1. $U \cong N \cup \big(\partial' N \times (-1,1)\big)$, where $\partial' N \times (-1,0]$ is an inner collar of $\partial' N$ in N.
2. For $m \in \partial' N$, the ray $m \times (-1,1)$ is contained in v_m.

- Choose a smooth isotopy $f : (M,N) \times [0,1] \to (M,N)$ such that
  1. $f_0 = id_M$. $f_1\Big(N \cup \big(\partial' N \times [0,1/2)\big)\Big) \subset N$.
  2. f_t fixes points outside $\partial' N \times (-1,1)$.
  3. For $m \in \partial' N$, $f_t(m \times (-1,1)) \subset m \times (-1,1)$. If m is fixed and t increases, the $(-1,1)$-coordinate decreases.

Let $V_k \subset D_k$ be the subspace consisting of configurations containing a tree rooted in $N \cup \partial' N \times [0,1/2)$. We have a deformation retract of V_k to T_k moving trees along f_t until the trees represent a configuration in C_{k-1}.

The only potential problem is to determine the variations of summable labels of trees moving on $\partial' N \times (-1,1)$ along f_t. Note that f_t preserves each ray $m \times (-1,1)$ and the ray is contained in the domain v_m of a trivialization. Thus we can set the summable labels of trees moving on $m \times (-1,1)$ to be constant according to the trivialization chosen at m.

□

Additivity, excision, dimension properties By definition C^2_Σ and $\mathcal{H}_k$ are symmetric monoidal. Thus we can say that the additivity axiom holds.

Proposition 4.20 (additivity)**.**
C^2_Σ and $\mathcal{H}_k$ are symmetric monoidal.

In section 4.1 we checked the excision property for C^1_Σ. We adjust the statement to C^2_Σ.

Proposition 4.21 (excision)**.**
Let $A = \amalg_{i=1}^{l}\big(M_i, N_i; (\eta_i, \iota_i)\big)$ be an object of $\mathcal{M}^n$, and $K_i \subset \operatorname{int} N_i$. Assume that $B = \amalg_{i=1}^{l}\big(M_i \setminus K_i, N_i \setminus K_i; (\eta_i|_{M\setminus K_i}, \iota_i|_{M\setminus K_i})\big)$ is an object of $\mathcal{M}^n$ after rewriting it as a disjoint union of components. Then

$$C^2_\Sigma(A) \cong C^2_\Sigma(B).$$

We also verify what can be compared to the dimension axiom.

Proposition 4.22 (dimension)**.**
Let $\big((X,*) \to (\eta,\iota) \to D^n\big)$ be a twisted partial summable label over D^n. Then we have the homotopy equivalence

$$C_\Sigma\big[D^n, \emptyset; \eta\big] \simeq \mathcal{F}_n \otimes_{\mathcal{F}_n} X.$$

In particular if X is not partial we have

$$C_\Sigma\big[D^n, \emptyset; \eta\big] \simeq X, \;\; \mathcal{H}_k\big(D^n, \emptyset; (\eta,\iota)\big) \cong \pi_k(X).$$

Proof. Since D^n is contractible, $\mathcal{F}_{D^n} \cong D^n \times \mathcal{F}_n$ and $\eta \cong D^n \times X$. For the homotopy equivalence, take the deformation retract forcing particles collide at the center along radial directions. □

Quasifibration The quasifibrations between configuration spaces induce long exact sequences of homotopy groups. It corresponds to sequences of pairs in the singular homology theory. The quasifibration also plays the central role in the proof of the weak equivalence of scanning maps.

We first introduce the notion of homotopy inverse of a partial left algebra over $\mathcal{F}_n$.

Definition 4.10 (homotopy inverse)**.**
Let $(X,*)$ be a partial left algebra over $\mathcal{F}_n$. X is said to have a **homotopy inverse** *if there exists a map*

$$C_\Sigma\big[\mathbb{R}^n, \emptyset; X\big] \to C_\Sigma[\mathbb{R}^n, \emptyset; X], \quad m \mapsto m^{-1}$$

such that

1. *m and m^{-1} have distinct macroscopic locations.*
2. *The map taking union of m and m^{-1}*

$$C_\Sigma\big[\mathbb{R}^n, \emptyset; X\big] \to C_\Sigma\big[\mathbb{R}^n, \emptyset; X\big], \quad m \mapsto m \cup m^{-1}$$

is null-homotopic.

The following proposition is the twisted version of lemma 6.1 in [10].

Theorem 4.4 (quasifibration)**.**
Let $\big(M, N; (\eta, \iota)\big)$ be an object of $\mathcal{M}^n$ with one component. Let $(X, *)$ be the typical fibre of η. Let $M' \subset M$ be a closed codimension-0 submanifold such that

$$\big(M, N \cup M'; (\eta, \iota)\big), \text{ and } \big(M', N \cap M'; (\eta|_{M'}, \iota|_{M'})\big)$$

are objects of $\mathcal{M}^n$ after rewriting the latter as a disjoint union of components if necessary.

Suppose that either $(M', N \cap M')$ is connected, or X has a homotopy inverse. Then the following inclusions:

$$(M', N \cap M') \to (M, N) \xrightarrow{q} (M, N \cup M')$$

induce a quasifibration

$$C_\Sigma\big[M', N \cap M'; \eta|_{M'}\big] \to C_\Sigma\big[M, N; \eta\big] \xrightarrow{Q} C_\Sigma\big[M, N \cup M'; \eta\big].$$

Proof. The proof is a modification of lemma 6.1 in [10], which following the proof of proposition 3.2 in [9]. For convenience we will omit η in the notation of configuration spaces. We first do some preparations:

- Let $\partial' M'$ denote the topological boundary of M'. Choose local trivializations of η around each $m \in \partial' M'$ in a continuously varying way. The choice can be made by restricting a section of $\mathrm{Tub}^\eta(M)$ if $\partial M = \emptyset$, or $\mathrm{Tub}^\eta(W_M)$ if $\partial M \neq \emptyset$. Let v_m denote the open neighbourhood of m over which the trivialization chosen at m is defined.

- Choose an open collared neighbourhood U of M' in M such that

  1. $U \cong M' \cup \big(\partial' M' \times (-1, 1)\big)$, where $\partial' M' \times (-1, 0]$ is an inner collar of $\partial' M'$ in M'.
  2. For $m \in \partial' M'$, the ray $m \times (-1, 1)$ is contained in v_m.
  3. If $(m, t) \in N$ and $s < t$, $(m, s) \in N$.

- Choose a smooth isotopy $f : (M, M') \times [0, 1] \to (M, M')$ such that

  1. $f_0 = id_M$. $f_1\Big(M' \cup \big(\partial' M' \times [0, 1/2)\big)\Big) \subset M'$.
  2. f_t fixes points outside $\partial' M' \times (-1, 1)$.
  3. For $m \in \partial' M'$, $f_t(m \times (-1, 1)) \subset m \times (-1, 1)$. If m is fixed and t increases, the $(-1, 1)$-coordinate decreases.

- Let $\{C_k\}_k$ be a filtration of $C_\Sigma\big[M, N \cup M'\big]$ such that C_k comprises those configurations with at most k-leaves each of which is rooted outside $N \cup M'$ and not labelled with ι. We can choose an open neighbourhood V_k of C_k in C_{k+1} and a deformation retract h_t of V_k to C_k such that trees rooted in $\partial' M' \times [0,1)$ moves along f_t.

Now we show the following two statements:

1. Q is a quasifibration over $C_{k+1} \setminus C_k$ and $\big(C_{k+1} \setminus C_k\big) \cap V_k$.

2. If Q is a quasifibration over C_k, then Q is a quasifibration over V_k.

By the property of quasifibrations, if Q is a quasifibration over $C_{k+1} \setminus C_k$, V_k, and $\big(C_{k+1} \setminus C_k\big) \cap V_k$, then it is a quasifibration over $\big(C_{k+1} \setminus C_k\big) \cup V_k = C_{k+1}$. If Q is a quasifibration over C_k for $k \in \mathbb{N}$, then it is a quasifibration over $\cup_{k \in \mathbb{N}} C_k$. ([3]) Thus the proof of the proposition follows from the two statements by induction.

- The first statement is clear from the homeomorphism

$$Q^{-1}(C_{k+1} - C_k) \cong (C_{k+1} - C_k) \times C_\Sigma\big[M', N \cap M'\big]$$

  sorting trees by macroscopic locations.

- For the second statement, it suffices to find a weak deformation retract H_t of $Q^{-1}(V_k)$ to $Q^{-1}(C_k)$ covering h_t

$$\begin{tikzcd} Q^{-1}(V_k) \arrow[r, "H_t"] \arrow[d, "Q"'] & Q^{-1}(C_k) \arrow[d, "Q"] \\ V_k \arrow[r, "h_t"'] & C_k \end{tikzcd}$$

  such that the map between fibres

$$H_1|_{Q^{-1}(a)} : Q^{-1}(a) \to Q^{-1}\big(h_1(a)\big) \qquad (a \in V_k)$$

  is a weak equivalence. h_t can be lifted to H_t by letting trees rooted inside M' move along f_t and letting trees entering into M' along f_t keep moving along f_t. The only potential problem is to determine the variations of summable labels of trees moving on $\partial' M' \times (-1, 0]$ along f_t. Note that f_t preserves each ray $m \times (-1, 1)$ and the ray is contained in the domain v_m of a trivialization. Thus we can set the summable labels of trees moving on $m \times (-1, 1)$ to be determined by the action of Df_t when observed according to the trivialization chosen at m.

- The last thing to show is that $H_1|_{Q^{-1}(a)}$ is a weak equivalence. We can identify both $Q^{-1}(a)$ and $Q^{-1}\big(h_1(a)\big)$ with $C_\Sigma\big[M', N \cap M'\big]$. $H_1|_{Q^{-1}(a)}$ is the map

$$H_1|_{Q^{-1}(a)} : C_\Sigma\big[M', N \cap M'\big] \to C_\Sigma\big[M', N \cap M'\big], \quad b \mapsto p(b) \cup a_0$$

pushing the input configuration b away from $\partial' M'$ to obtain $p(b)$, and add a foreign configuration a_0, which is the part of a pushed into M' reaching somewhere in the inner collar of $\partial' M'$. Thus we only need to push back $p(b)$ and get rid of a_0.

- If $(M', N \cap M')$ is connected, we can push back $p(b)$ and at the same time lead a_0 to $N \cap M'$ to vanish. If X has a homotopy inverse, then we can define a homotopy inverse of $H_1|_{Q^{-1}(a)}$ as follows. Let $c \in Q^{-1}(h_1(a))$. Push c slightly away from $\partial' M'$ so that there is no tree rooted in $\partial' M'$, and for each tree t in a_0, say rooted in the ray $m \times (-1, 0]$, obtain t^{-1} according to the trivialization over v_m, let t^{-1} collide into a single tree along radial directions, and root it at $(m, 0)$.

□

Mayer-Vietoris sequence The quasifibration property induces a long exact sequence of $\mathcal{H}_k$ for two manifolds glued along common boundaries. It can be compared to the Mayer-Vietoris sequence. Our formulation of gluing is a modification of 3.13 in [1].

Let $\big(M, N; (\eta, \iota)\big)$ be an object of $\mathcal{M}^n$ with one component. Let

$$f : M \to [-1, 1]$$

be a map. Denote $f^{-1}(J)$ as M_J and $M_J \cap N$ as N_J for an interval J, and denote $f^{-1}(j)$ as M_j and $M_j \cap N$ as N_j for $j \in [-1, 1]$. In particular $M = M_{[-1,1]}$. Suppose the following:

1. M_J and M_j are submanifolds of M for all J and j.

2. N_J is a closed submanifold of M_J and N_j is a closed submanifold of M_j.

3. M_{-1} and M_1 are connected n-manifolds.

4. $(M_{(-1,1)}, N_{(-1,1)})$ is a manifold bundle over $(-1,1)$ with the typical fibre (M_0, N_0)

$$\begin{tikzcd} (M_{(-1,1)}, N_{(-1,1)}) \arrow[d, "\cong"] \arrow[dr, "f"] & \\ (M_0, N_0) \times (-1,1) \arrow[r] & (-1,1) \end{tikzcd}$$

5. N_0 is nonempty, or N_{-1} and N_1 are nonempty, or the typical fibre X of η has a homotopy inverse.

Proposition 4.23.
Let A be the pullback of the following digram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C_\Sigma\big[M_{[-\frac{1}{2},1]}, M_{-\frac{1}{2}} \amalg N_{[-\frac{1}{2},1]}; \eta\big] \arrow[d] \\ C_\Sigma\big[M_{[-1,\frac{1}{2}]}, N_{[-1,\frac{1}{2}]} \amalg M_{\frac{1}{2}}; \eta\big] \arrow[r] & C_\Sigma\big[M_{[-\frac{1}{2},\frac{1}{2}]}, M_{-\frac{1}{2}} \amalg N_{[-\frac{1}{2},\frac{1}{2}]} \amalg M_{\frac{1}{2}}; \eta\big] \end{tikzcd}$$

where the two maps to the lower right corner are the map forgetting particles outside $M_{[-\frac{1}{2},\frac{1}{2}]}$. Then

$$A \cong C_\Sigma\big[M, N; \eta\big].$$

Proof. It suffices to prove that $C_\Sigma\big[M, N; \eta\big]$ has the universal property indicated by the diagram. Note that the diagram commutes if we put $C_\Sigma\big[M, N; \eta\big]$ in the place of A and define all the new maps to be particle forgetting maps. We only need to find a unique map

$$A \to C_\Sigma\big[M, N; \eta\big].$$

The map is given by gluing configurations from

$$C_\Sigma\big[M_{[-1,\frac{1}{2}]}, N_{[-1,\frac{1}{2}]} \amalg M_{\frac{1}{2}}; \eta\big], \quad C_\Sigma\big[M_{[-\frac{1}{2},1]}, M_{-\frac{1}{2}} \amalg N_{[-\frac{1}{2},1]}; \eta\big]$$

along a configuration from

$$C_\Sigma\big[M_{[-\frac{1}{2},\frac{1}{2}]}, M_{-\frac{1}{2}} \amalg N_{[-\frac{1}{2},\frac{1}{2}]} \amalg M_{\frac{1}{2}}; \eta\big].$$

□

Proposition 4.24 (Mayer-Vietoris sequence for configuration spaces)**.**
Denote the pullback diagram above as

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, "h"] & D \end{tikzcd}$$

Then we have the long exact sequence

$$\ldots \to \pi_n(\Omega D) \to \pi_n(A) \to \pi_n(B) \times \pi_n(C) \to \pi_{n-1}(\Omega D) \to \cdots \to \pi_0(B) \times \pi_0(C).$$

Proof. First we observe that the particle forgetting maps e, f, g, h are all quasifibrations by Theorem 4.4:

$$\begin{aligned}
C_\Sigma\big[M, N; \eta\big] &\xrightarrow{e} C_\Sigma\big[M_{[-\frac{1}{2},1]}, M_{-\frac{1}{2}} \amalg N_{[-\frac{1}{2},1]}; \eta\big],\\
C_\Sigma\big[M, N; \eta\big] &\xrightarrow{f} C_\Sigma\big[M_{[-1,\frac{1}{2}]}, N_{[-1,\frac{1}{2}]} \amalg M_{\frac{1}{2}}; \eta\big],\\
C_\Sigma\big[M_{[-\frac{1}{2},1]}, M_{-\frac{1}{2}} \amalg N_{[-\frac{1}{2},1]}; \eta\big] &\xrightarrow{g} C_\Sigma\big[M_{[-\frac{1}{2},\frac{1}{2}]}, M_{-\frac{1}{2}} \amalg N_{[-\frac{1}{2},\frac{1}{2}]} \amalg M_{\frac{1}{2}}; \eta\big],\\
C_\Sigma\big[M_{[-1,\frac{1}{2}]}, N_{[-1,\frac{1}{2}]} \amalg M_{\frac{1}{2}}; \eta\big] &\xrightarrow{h} C_\Sigma\big[M_{[-\frac{1}{2},\frac{1}{2}]}, M_{-\frac{1}{2}} \amalg N_{[-\frac{1}{2},\frac{1}{2}]} \amalg M_{\frac{1}{2}}; \eta\big].
\end{aligned}$$

We check for e. e is the same map as the following particle forgetting map:

$$C_\Sigma\big[M, N; \eta\big] \xrightarrow{e} C_\Sigma\big[M, N \cup M_{[-1,-\frac{1}{2}]}; \eta\big]$$

Since $M_{[-1,-\frac{1}{2}]}$ is connected, if N_{-1} or N_0 is nonempty, $\big(M_{[-1,-\frac{1}{2}]}, M_{[-1,-\frac{1}{2}]} \cap N\big)$ is connected. Otherwise X was assumed to have a homotopy inverse. e is a quasifibration.

Now we show the exactness by diagram chasing. Denote the four long exact sequences of homotopy groups induced by the four quasifibrations f, g, h, i using reduced notations as follows:

$$\begin{tikzcd}
 & & F_n \arrow[r, "\cong"] \arrow[d, "i_n^f"] & F_n \arrow[d, "i_n^g"] & \\
 & G_n \arrow[r, "i_n^e"] \arrow[d, "\cong"] & A_n \arrow[r, "e_n"] \arrow[d, "f_n"] & B_n \arrow[r] \arrow[d, "g_n"] & G_{n-1} \arrow[d, "\cong"] \\
D_{n+1} \arrow[r, "\partial_{n+1}^h"] & G_n \arrow[r, "i_n^h"] & C_n \arrow[r, "h_n"] \arrow[d, "\partial_n^f"] & D_n \arrow[r] \arrow[d, "\partial_n^g"] & G_{n-1} \\
 & & F_{n-1} \arrow[r, "\cong"] \arrow[d, "i_{n-1}^f"] & F_{n-1} & \\
 & & A_{n-1} & &
\end{tikzcd}$$

where the subscripts n of spaces stand for π_n of the spaces, and F and G are homotopy fibres. We show that the following sequence is exact up to $B_1 \times C_1$

$$D_{n+1} \xrightarrow{i_n^e \circ \partial_{n+1}^h} A_n \xrightarrow{(e_n, f_n)} B_n \times C_n \xrightarrow{g_n p_1 - h_n p_2} D_n \xrightarrow{i_{n-1}^f \circ \partial_n^g} A_{n-1}$$

where p_i is the projection to the i-th coordinate.

1. Check the exactness at A_n for $n \geq 1$. $e_n(i^e_n \partial^h_{n+1}) = (e_n i^e_n)\partial^h_{n+1} = 0$. $f_n(i^e_n \partial^h_{n+1}) = i^h_n \partial^h_{n+1} = 0$. Let $x \in \ker(e_n, f_n)$. Then $i^e_n(y) = x$ for some $y \in G_n$. $i^h_n(y) = f_n(x) = 0$. $y \in \operatorname{im} \partial^h_{n+1}$. $x \in \operatorname{im} i^e_n \circ \partial^h_{n+1}$.

2. Check the exactness at $B_n \times C_n$ for $n \geq 2$. $(g_n p_1 - h_n p_2) \circ (e_n, f_n) = g_n e_n - h_n f_n = 0$. Let $(x, y) \in B_n \times C_n$ and $g_n(x) - h_n(y) = 0$. $\partial^f_n(y) = \partial^g_n h_n(y) = \partial^g_n g_n(x) = 0$. $y = f_n(a)$ for some $a \in A_n$. $g_n(e_n(a) - x) = 0$. $e_n(a) - x = i^g_n(b)$ for some $b \in F_n$. $(e_n, f_n)(a - i^f_n(b)) = (x, y)$.

3. Check the exactness at D_n for $n \geq 2$. Let $(x, y) \in B_n \times C_n$. $\big((i^f_{n-1}\partial^g_n) g_n\big)(x) = \big(i^f_{n-1}(\partial^g_n g_n)\big)(x) = 0$. $\big((i^f_{n-1}\partial^g_n) h_n\big)(y) = i^f_{n-1}\partial^f_n(y) = 0$. Thus $(i^f_{n-1}\partial^g_n) \circ (g_n p_1 - h_n p_2) = 0$ Let $d \in D_n$ and $i^f_{n-1}\partial^g_n(d) = 0$. $\partial^g_n(d) \in \ker i^f_{n-1}$. $\partial^g_n(d) = \partial^f_n(c)$ for some $c \in C_n$. $d - h_n(c) \in \ker \partial^g_n$. $d - h_n(c) = g_n(b)$ for some $b \in B_n$. $d = (g_n p_1 - h_n p_2)(b, -c)$.

Note that the arrows are functions between sets from $B_1 \times C_1$ on:

$$A_1 \xrightarrow{(e_1, f_1)} B_1 \times C_1 \xrightarrow{(g_1 p_1) \cdot (h_1 p_2)^{-1}} D_1 \xrightarrow{i^f_0 \partial^g_1} A_0 = 0 \xrightarrow{(e_0, f_0)} B_0 \times C_0 = 0$$

By similar argument the sequence is exact also at the tail. □

4.4 Diffeomorphism-equivariant scanning maps

In this section we give a functorial construction of scanning maps for configuration spaces with twisted partial summable labels. In particular the scanning maps are diffeomorphism-equivariant. We show that the scanning map is a weak equivalence.

The weak equivalence holds for two types of $\big(M, N; (\eta, \iota)\big)$. Thus we define a category for each type.

Definition 4.11 (category of closed n-manifolds for scanning maps)**.**
We define the category $\mathcal{M}^n_1$ as follows. The objects consist of triples $\big(M, N; (\eta, \iota)\big)$ such that

1. *M is a closed n-manifold.*

2. *(η, ι) is a twisted partial summable label over M whose typical fibre $(X, *)$ has a homotopy inverse.*

3. *$N = \emptyset$.*

The morphism set from $\big(M, N; (\eta, \iota)\big)$ *to* $\big(M', N'; (\eta', \iota')\big)$ *is the collection of pairs* $(\widetilde{\varphi}, \varphi)$ *such that*

1. $\varphi : M \to M'$ *is an embedding.*
2. $\widetilde{\varphi} : (\eta, \iota) \to (\eta'|_{\operatorname{im}\varphi}, \iota'|_{\operatorname{im}\varphi})$ *is an* $(\mathcal{F}_\varphi, \varphi)$*-equivariant isomorphism between twisted partial left algebras.*

The composition is defined to be the ordinary composition of maps.

Definition 4.12 (category of compact n-manifolds with boundary for scanning maps)**.** *We define the category* $\mathcal{M}_2^n$ *as follows. The objects consist of triples* $\big(M, N; (\eta, \iota)\big)$ *such that*

1. M *is a compact n-manifold with nonempty boundary.*
2. *Let* $W_M = M \cup_{\partial M} \partial M \times [0, 1)$ *be an open manifold containing* M *obtained by extending* ∂M*.* (η, ι) *is a twisted partial summable label over* W_M*.*
3. $N \subset M$ *is a closed codimension-*0 *submanifold such that* (M, N) *is connected.*

The morphism set from $\big(M, N; (\eta, \iota)\big)$ *to* $\big(M', N'; (\eta', \iota')\big)$ *is the collection of pairs* $(\widetilde{\varphi}, \varphi)$ *such that*

1. $\varphi : (W_M, N) \to (W_{M'}, N')$ *is an embedding such that*
$$\varphi(M) \subset M',\ \varphi(W_M \setminus M) \subset W_{M'} \setminus M',\ \varphi(M \setminus N) \subset M' \setminus N'.$$
2. $\widetilde{\varphi} : (\eta, \iota) \to (\eta'|_{\operatorname{im}\varphi}, \iota'|_{\operatorname{im}\varphi})$ *is an* $(\mathcal{F}_\varphi, \varphi)$*-equivariant isomorphism between twisted partial left algebras.*

The composition is defined to be the ordinary composition of maps.

For both $\mathcal{M}_1^n$ and $\mathcal{M}_2^n$, as before, we can construct a functor to the category of based spaces

$$C_\Sigma : \mathcal{M}_i^n \to \mathcal{U},\ \ \big(M, N; (\eta, \iota)\big) \mapsto C_\Sigma\big[M, N; \eta\big],\ \ (\widetilde{\varphi}, \varphi) \mapsto C_\Sigma(\widetilde{\varphi}, \varphi)$$

taking configuration spaces. We do not repeat the detail.

Formula of the scanning map Let $\big(M, N; (\eta, \iota)\big)$ be an object of $\mathcal{M}_i^n$. For $i = 1$, let W_M denote M. Let M_0 denote $W_M \setminus N$. In particular, if $i = 1$, $M_0 = M$.

As standard we want a so-called scanning map s from the configuration space $C_\Sigma[M, N; \eta]$ to a section space

$$s : C_\Sigma\big[M, N; \eta\big] \to \Gamma_\partial(\alpha).$$

Recall that we defined in Definition 4.8 a bundle of local trivializations of η over tubular neighbourhoods of points in M_0

$$\pi_1 : \mathrm{Tub}^\eta(M_0) \simeq_w M_0.$$

α is a fibre bundle over $\mathrm{Tub}^\eta(M_0)$ such that the fibre over $(\hat{f}, f)$ is the restriction of $C_\Sigma[M, N; \eta]$ into the closure of the image of f:

$$F_{(\hat{f},f)}(\alpha) = C_\Sigma\big[W_M, W_M \setminus \mathrm{im}(f); \eta\big].$$

We will use the notation $F_{(\hat{f},f)}(\alpha) = C_\Sigma[\overline{f}, \partial; \eta]$. The topology of α will be later defined.

Definition 4.13 (scanning map)**.**
$\Gamma_\partial(\alpha)$ *denotes the subspace of* $\Gamma(\alpha)$ *consisting of those sections sending* $(\hat{f}, f)$ *to the basepoint if* $\mathrm{im}(f) \subset W_M \setminus M$. *The* **scanning map** s *is defined by*

$$s : C_\Sigma\big[M, N; \eta\big] \to \Gamma_\partial(\alpha), \quad s : m \;\mapsto\; \big(s_m : (\hat{f}, f) \;\mapsto\; m|_{\overline{\mathrm{im}(f)}}\big)$$

where $m|_{\overline{\mathrm{im}(f)}}$ *denotes the image of* m *along the forgetting map*

$$C_\Sigma\big[M, N; \eta\big] \to C_\Sigma\big[W_M, W_M \setminus \mathrm{im}(f); \eta\big] = C_\Sigma\left[\,\overline{f}, \partial; \eta\,\right].$$

Since m lies inside M, the section s_m actually belongs to $\Gamma_\partial(\alpha)$.

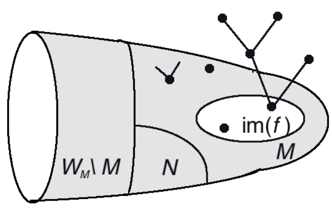


Figure 4.6: $s_m : (\hat{f}, f) \;\mapsto\; m|_{\overline{\mathrm{im}(f)}}$

Topology of α We give a commutative diagram displaying the whole process of the construction:

$$\begin{tikzcd}
\coprod C_\Sigma\left[\overline{f},\partial;\eta\right] \arrow[r, "\cong", "F"'] \arrow[d, "\pi(\alpha)"'] & \coprod C_\Sigma\left[\widehat{T}_m M_0,*;\eta_m\right] \arrow[r, "Q"] \arrow[d, "\pi(\beta)"'] & \coprod C_\Sigma\left[\widehat{T}_m M_0,*;\eta_m\right] \arrow[r, "\subset"] \arrow[d, "\pi(\gamma)"'] & C^V_\Sigma[\widehat{T}M_0,*;p^*\eta] \\
\mathrm{Tub}^\eta(M_0) \arrow[r, equal] & \mathrm{Tub}^\eta(M_0) \arrow[r, "\pi_1"] & M_0 &
\end{tikzcd}$$

1. The disjoint unions in the first row are taken over the corresponding spaces in the second row. $\pi(\alpha),\pi(\beta),\pi(\gamma)$ are obvious projections. The right square is a pullback diagram.

2. $\widehat{T}_m M_0$ denotes the fibre over m of $\widehat{T}M_0$, the fibrewise one-point compactification of TM_0. $*$ denotes points at infinity. η_m in $C_\Sigma\left[\widehat{T}_m M_0,*;\eta_m\right]$ denotes the trivial bundle $\widehat{T}_m M_0\times\eta_m\to\widehat{T}_m M_0$.

3. F is a one-to-one correspondence gathering the fibrewise defined maps:

$$C_\Sigma\left[W_M, W_M\setminus \mathrm{im}(f);\eta\right] \xrightarrow{C^1_\Sigma(\hat{f}^{-1},f^{-1})} C_\Sigma\left[\widehat{T}_{f(0)}M_0,*;\eta_{f(0)}\right]$$

   where f^{-1} by abuse of notation denotes the map combining the inverse of $f : T_{f(0)}M_0\to\mathrm{im}(f)$ and $W_M\setminus\mathrm{im}(f)\to\{*\}$. $\hat{f}^{-1}$ denotes the inverse of $\hat{f} : T_{f_0}M_0\times\eta_{f(0)}\to\mathrm{im}(\hat{f})$. F will be forced to be a homeomorphism.

4. $C^V_\Sigma[\widehat{T}M_0,*;p^*\eta]$ is the configuration space of $\widehat{T}M_0$ vanishing at the set of points at infinity with the label $p^*\eta$, where p is the projection $\widehat{T}M_0\to M_0$, such that the summations are allowed only for vertical collisions. The detail will follow.

The topology of α will be induced in the following order. Define the topology of γ, let β be the pullback of γ along π_1, and identify α with β. Thus what it remains to do is to define the following inclusion precisely:

$$\coprod_{m\in M_0} C_\Sigma\left[\widehat{T}_m M_0,*;\eta_m\right]\subset C^V_\Sigma[\widehat{T}M_0,*;p^*\eta].$$

Vertical Collision First we clarify what a vertical collision means. The subbundle of the double tangent bundle TTM_0 consisting of vertical vectors is the kernel of the following vector bundle map:

$$\begin{tikzcd}
TTM_0 \arrow[r, "Dp"] \arrow[d] & TM_0 \arrow[d, "p"] \\
TM_0 \arrow[r, "p"] & M_0
\end{tikzcd}$$

The kernel is decomposed as

$$\ker(Dp) = \coprod_{x\in M_0} \coprod_{v_x\in T_xM_0} T_{v_x}T_xM_0.$$

Gathering the canonical isomorphisms $D(-v_x) : T_{v_x}T_xM_0 \cong T_xM_0$ induced by the affine map

$$-v_x : T_xM_0 \to T_xM_0,\ v \mapsto v - v_x$$

we obtain the following pullback diagram:

$$\begin{tikzcd} \ker Dp \arrow[r] \arrow[d] & TM_0 \arrow[d, "p"] \\ TM_0 \arrow[r, "p"] & M_0 \end{tikzcd}$$

Now we can define $C^V_\Sigma[\widehat{T}M_0, *; p^*\eta]$ by specifying the necessary and sufficient condition for summations to occur.

Let

$$t \in \mathcal{F}_{TM_0}(k) \times_{TM_0} \left((p^*\eta)|_{TM_0}\right)^{(\times_{TM_0})k}.$$

Consider the tree representation of t. We perform the summation represented by t if and only if the following hold:

1. All the collision data in t consist of vertical vectors.

2. Let $v_x \in T_xM$ be the macroscopic location of t. If t has collision data all vertical, they can be read as an element of

$$\frac{\widetilde{C}_i(T_xM)}{\mathrm{Aff}_+(n)}$$

and t can be regarded as a tree t' with macroscopic location x. We require that the summation represented by t' is defined in $C_\Sigma\left[M, \emptyset; \eta\right]$.

Thus we obtain the following inclusion

$$\coprod_{m\in M_0} C_\Sigma\left[\widehat{T}_mM_0, *; \eta_m\right] \subset C^V_\Sigma[\widehat{T}M_0, *; p^*\eta].$$

Proposition 4.25.
$\pi(\gamma) : \coprod_{m\in M_0} C_\Sigma\left[\widehat{T}_mM_0, *; \eta_m\right] \to M_0$ is a fibre bundle.

Proof. Let G denote the topological group such that

$$(g_1, g_2) \in G \subset \mathrm{GL}(n) \times \mathrm{Homeo}_0(X)$$

if and only if $g_1 \times g_2 : \mathbb{R}^n \times X \to \mathbb{R}^n \times X$ is an $(\mathcal{F}_{g_1}, g_1)$-equivariant twisted partial left algebra morphism. γ is the associated bundle to $TM_0 \times_{M_0} \eta$ with the typical fibre $C_\Sigma\big[\widehat{\mathbb{R}^n}, *; X\big]$ and the structure group G. $\square$

γ induces the bundle α.

Continuity of the scanning map To check the continuity, write the scanning map as

$$\begin{aligned}
&\mathrm{Tub}^\eta(W) \times C_\Sigma\big[M, N; \eta\big] \to \alpha \xrightarrow{F} \beta \xrightarrow{Q} \gamma \subset C^V_\Sigma[\widehat{T}M_0, *; p^*\eta],\\
&\Big((\hat{f}, f),\ m\Big) \mapsto QF\big(s_m(\hat{f}, f)\big).
\end{aligned}$$

We can observe that the output configuration in $C^V_\Sigma[\widehat{T}M_0, *; p^*\eta]$ varies continuously with $\Big((\hat{f}, f),\ m\Big)$.

Diffeomorphism-equivariance By the construction of the scanning map, the equivariance directly follows. We define the contravariant functor

$$\Gamma_\partial : \mathcal{M}^n_i \to \mathcal{U}, \quad \big(M, N; (\eta, \iota)\big) \mapsto \Gamma_\partial(\beta)$$

such that a morphism

$$(\widetilde{\varphi}, \varphi) : \big(M, N; (\eta, \iota)\big) \to \big(M', N'; (\eta', \iota')\big)$$

is sent to a map $\Gamma_\partial(\widetilde{\varphi}, \varphi)$ defined as follows. Write $\Gamma_\partial : \big(M', N'; (\eta', \iota')\big) \mapsto \Gamma_\partial(\beta')$. Let $M'_0 = W_{M'} \setminus N'$. The map

$$\Gamma_\partial(\widetilde{\varphi}, \varphi) : \Gamma_\partial(\beta') \to \Gamma_\partial(\beta), \quad \sigma \mapsto \Gamma_\partial(\widetilde{\varphi}, \varphi)(\sigma)$$

takes the pullback of section σ along the following bundle map (R, L):

$$\begin{tikzcd}
\mathrm{Tub}^{\eta'}(M'_0) \arrow[r, "\sigma"] & \beta' = \coprod C_\Sigma[\widehat{T}_{f'(0)} M'_0, *; \eta'_{f'(0)}] \\
\mathrm{Tub}^{\eta}(M_0) \arrow[u, "L"] \arrow[r, "\Gamma_\partial(\widetilde{\varphi},\varphi)(\sigma)"] & \beta = \coprod C_\Sigma[\widehat{T}_{f(0)} M_0, *; \eta_{f(0)}] \arrow[u, "R"']
\end{tikzcd}$$

such that

1. L sends $(\hat{f}, f)$ to the horizontal compositions from left to right of the following diagram:

$$\begin{tikzcd}
T_{\varphi f(0)}M'_0 \times \eta'_{\varphi f(0)} \arrow[d] & & T_{f(0)}M_0 \times \eta_{f(0)} \arrow[ll, "\cong"', "D\varphi\times\widetilde{\varphi}"] \arrow[r, "\hat{f}"] \arrow[d] & \eta \arrow[r, "\widetilde{\varphi}"] \arrow[d] & \eta' \arrow[d] \\
T_{\varphi f(0)}M'_0 & & T_{f(0)}M_0 \arrow[ll, "\cong"', "D\varphi"] \arrow[r, "f"] & M_0 \arrow[r, "\varphi"] & M'_0
\end{tikzcd}$$

Note that by our assumption $\widetilde{\varphi}_{f(0)}$ is an isomorphism. We denote the map as

$$L : (\hat{f}, f) \mapsto (\widetilde{\varphi}_* \hat{f}, \varphi_* f).$$

2. R is the map whose restriction on each fibre $(\hat{f}, f)$ is the map:

$$C_\Sigma\left[\widehat{T}_{f(0)}M_0, *; \eta_{f(0)}\right] \cong C_\Sigma\left[\widehat{T}_{\varphi f(0)}M'_0, *; \eta'_{\varphi f(0)}\right]$$

induced by the left square of the diagram above.

Check that $\Gamma_\partial(\widetilde{\varphi}, \varphi)$ is well-defined. Since R is a homeomorphism on each fibre, σ can be pulled back. If $\mathrm{im}(f) \subset W_M \setminus M$, then $\mathrm{im}(\varphi_* f) \in W_{M'} \setminus M'$ and

$$\Big(\Gamma_\partial(\widetilde{\varphi}, \varphi)(\sigma)\Big)(\hat{f}, f) = R^{-1}\sigma(\widetilde{\varphi}_*\hat{f}, \varphi_* f) = *.$$

Thus $\Gamma_\partial(\widetilde{\varphi}, \varphi)(\sigma) \in \Gamma_\partial(\beta)$.

Since both L and R are functorial to $(\widetilde{\varphi}, \varphi)$, Γ_∂ is a contravariant functor.

Proposition 4.26 (naturality of scanning maps)**.**
Write the scanning maps as

$$\begin{aligned} C_\Sigma\big[M, N; \eta\big] &\xrightarrow{s} \Gamma_\partial(\beta), \\ C_\Sigma\big[M', N'; \eta'\big] &\xrightarrow{s'} \Gamma_\partial(\beta'). \end{aligned}$$

The diagram commutes

$$\begin{tikzcd}
C_\Sigma\big[M, N; \eta\big] \arrow[r, "s"] \arrow[d, "C_\Sigma(\widetilde{\varphi},\varphi)"'] & \Gamma_\partial(\beta) \\
C_\Sigma\big[M', N'; \eta'\big] \arrow[r, "s'"] & \Gamma_\partial(\beta') \arrow[u, "\Gamma_\partial(\widetilde{\varphi},\varphi)"']
\end{tikzcd}$$

Proof. We rewrite the diagram as

$$\begin{tikzcd}
\mathrm{Tub}^{\eta}(M_0) \times C_\Sigma\big[M, N; \eta\big] \arrow[r] \arrow[d] & \alpha \arrow[d] \\
\mathrm{Tub}^{\eta'}(M'_0) \times C_\Sigma\big[M', N'; \eta'\big] \arrow[r] & \alpha'
\end{tikzcd}$$

The right vertical arrow is the bundle map defined fibrewise as

$$C_\Sigma(\widetilde{\varphi},\varphi) : C_\Sigma\big[W_M, W_M \setminus \mathrm{im}(f);\ \eta\big] \to C_\Sigma\big[W_{M'}, W_{M'} \setminus \mathrm{im}(\varphi f);\ \eta'\big].$$

The diagram gives the following elementwise correspondence:

$$\begin{tikzcd}
(\hat{f}, f), m \arrow[r] \arrow[d] & m|_{\overline{\mathrm{im}\, f}} \in C_\Sigma\Big[\overline{\mathrm{im}(f)}, \partial; \eta_{f(0)}\Big] \arrow[d, leftrightarrow] \\
(\widetilde{\varphi}_* \hat{f}, \varphi_* f), C_\Sigma(\widetilde{\varphi},\varphi)(m) \arrow[r] & C_\Sigma(\widetilde{\varphi},\varphi)(m)|_{\overline{\mathrm{im}(\varphi_* f)}} \in C_\Sigma\Big[\overline{\mathrm{im}(\varphi_* f)}, \partial; \eta'_{\varphi f(0)}\Big]
\end{tikzcd}$$

The commutativity is clear. We check the right vertical arrow.

$$C_\Sigma(\widetilde{\varphi},\varphi)(m)|_{\overline{\mathrm{im}(\varphi_* f)}} = C_\Sigma(\widetilde{\varphi},\varphi)(m|_{\overline{\mathrm{im}\, f}})|_{\overline{\mathrm{im}(\varphi_* f)}} = C_\Sigma(\widetilde{\varphi},\varphi)(m|_{\overline{\mathrm{im}\, f}}) \mapsto m|_{\overline{\mathrm{im}\, f}}.$$

□

Weak equivalence of scanning maps We show that the scanning map s is a weak equivalence. The proof follows the standard method comparing the quasifibration sequence ? configuration spaces and the fibration sequence of section spaces.([2], [4], [9], [10])

First consider the following pullback:

$$\begin{tikzcd}
\coprod C_\Sigma[\widehat{T}_{f(0)} M_0, *; \eta_{f(0)}] \arrow[r] \arrow[d, "\pi(\beta)"] & \coprod C_\Sigma[\widehat{T}_m M_0, *; \eta_m] \arrow[d, "\pi(\gamma)"] \\
\mathrm{Tub}^\eta(M_0) \arrow[r] & M_0
\end{tikzcd}$$

Fix a section $\mu : M_0 \simeq_w \mathrm{Tub}^\eta(M_0)$. We choose μ such that each m in $\partial M \times [1/2, 1)$ is sent to $(\hat{f}, f)$ with $\mathrm{im}(f) \subset W_M \setminus M$. μ induces $\Gamma_\partial(\beta) \simeq \Gamma_\partial(\gamma)$, where $\Gamma_\partial(\gamma)$ denotes the space of sections of γ vanishing at $\partial M \times [1/2, 1)$. The composition

$$S : C_\Sigma\big[M, N; \eta\big] \xrightarrow{s} \Gamma_\partial(\alpha) \cong \Gamma_\partial(\beta) \simeq \Gamma_\partial(\gamma)$$

coincides with the construction of scanning maps in [10] if η is a trivial bundle.

Theorem 4.5 (Weak equivalence of scanning maps)**.**
For an object $\big(M, N; (\eta, \iota)\big)$ *of* $\mathcal{M}_i^n$,

$$s : C_\Sigma[M, N; \eta] \simeq_w \Gamma_\partial(\alpha).$$

Proof. We omit η in the notation of configuration spaces. Let $I = [0,1]$. If $(M,N) = (I^n, \emptyset)$, or $(M,N) = \big(I^n, \partial(I^k)\times I^{n-k}\big)$ for $k = 1,\dots,n$, then η is a trivial bundle over M. The proof for those cases is done in proposition 6.2 in [10]. Thus we have the weak equivalences for handles of any index. Let H denote a handle. For a submanifold $K \subset M_0$, let $\Gamma_\partial(\alpha)|_K$ denote the space of sections of the restriction of α over

$$\Big\{(\hat{f}, f) \in \mathrm{Tub}^\eta(M_0) \mid \mathrm{im}(f) \subset \Big(K \cup \big((K\cap \partial M)\times [0,1)\big)\Big)\Big\}.$$

vanishing at

$$\Big\{(\hat{f}, f) \in \mathrm{Tub}^\eta(M_0) \mid \mathrm{im}(f) \in W_M \setminus M\Big\}.$$

We proved the quasifibration property only for the case that all the manifolds have the same dimension. Thus some manifolds below should be understood after being thickened to obtain quasifibrations. We also assume that all the corners of manifolds below are smoothified.

1. First, let $\big(M, N; (\eta,\iota)\big)$ be either an object of $\mathcal{M}_1^n$, or an object of $\mathcal{M}_2^n$ such that N is an inner collar of ∂M. Choose a handle decomposition of M and use induction. Assume that M_{i+1} is obtained by attaching H to M_i. Consider the following three rows induced by the formula of s.

$$\begin{tikzcd}
 & C_\Sigma\big[H, \overline{\partial H \setminus M_i}\big] \arrow[r, "\simeq_w"] \arrow[d] & \Gamma_\partial(\alpha)|_{H\setminus\overline{\partial H\setminus M_i}} \arrow[d] \\
C_\Sigma\big[M_{i+1}, \partial M_{i+1}\big] \arrow[r, phantom, "\cong"] & C_\Sigma\big[M_i \cup H, \partial(M_i\cup H)\big] \arrow[r] \arrow[d] & \Gamma_\partial(\alpha)|_{M_{i+1}\setminus\partial M_{i+1}} \arrow[d] \\
C_\Sigma\big[M_i, \partial M_i\big] \arrow[r, phantom, "\cong"] & C_\Sigma\big[M_i \cup H, (\partial M_i)\cup H)\big] \arrow[r, "\simeq_w"] & \Gamma_\partial(\alpha)|_{M_i\setminus\partial M_i}
\end{tikzcd}$$

   The middle column is a quasifibration induced by inclusions of manifolds and the right column is a fibration induced by restrictions of sections. By induction the second row is a weak equivalence. By the isotopy invariance, $C_\Sigma\big[M, \partial M\big] \cong C_\Sigma\big[M, N\big]$. Thus the second row at the final step gives the desired equivalence.

2. For the remaining cases, remove an open set V of N so that $N' = N \setminus V$ is a codimension-0 submanifold of $\partial(M\setminus V)$. Let $M' = M\setminus V$. Choose a submanifold $L \subset \partial M'$ such that $L \cup N' = \partial M'$ and $L \cap N' = \partial L = \partial N'$. Let $L \times I, N' \times I$

be the thickenings of L, N' in M respectively.

$$
\begin{array}{ccccc}
 & & C_\Sigma\big[L\times I, \partial L\times I\big] & \longrightarrow & \Gamma_\partial(\alpha)|_{(L\setminus \partial L)\times I} \\
 & & \downarrow & & \downarrow \\
C_\Sigma\big[M,N\big] & \cong & C_\Sigma\big[M', N'\times I\big] & \longrightarrow & \Gamma_\partial(\alpha)|_{M'\setminus(N'\times I)} \\
 & & \downarrow & & \downarrow \\
C_\Sigma\big[M, \partial M\times I\big] & \cong & C_\Sigma\big[M', (N'\cup L)\times I\big] & \xrightarrow{\simeq_w} & \Gamma_\partial(\alpha)|_{M'\setminus(\partial M'\times I)}
\end{array}
$$

The $\cong$'s in the second and third row are by the isotopy invariance and excision property. The $\simeq_w$ in the third row follows from the first case. The middle column is a quasifibration and the right column is a fibration. We only need to show that the first row is a weak equivalence.

3.

$$
\begin{array}{ccccc}
 & & C_\Sigma\big[L\times I, \partial L\times I\big] & \longrightarrow & \Gamma_\partial(\alpha)|_{(L\setminus \partial L)\times I} \\
 & & \downarrow & & \downarrow \\
* & \simeq & C_\Sigma\Big[L\times[0,2], \big(\partial L\times[0,2]\big)\cup\big(L\times\{2\}\big)\Big] & \longrightarrow & \Gamma_\partial(\alpha)|_{(L\times[0,2])\setminus((\partial L\times[0,2])\cup(L\times\{2\}))} \\
 & & \downarrow & & \downarrow \\
 & & C_\Sigma\Big[L\times[1,2], \partial\big(L\times[1,2]\big)\Big] & \xrightarrow{\simeq_w} & \Gamma_\partial(\alpha)|_{(L\times[1,2])\setminus\partial(L\times[1,2])}
\end{array}
$$

The spaces in the second row are contractible. The $\simeq_w$ in the third row follows from the first case. The middle column is a quasifibration and the right column is a fibration. The first row is a weak equivalence and this finishes the proof of the theorem.

$\square$

Bibliography

[1] David Ayala and John Francis, *Factorization homology of topological manifolds*, J. Topology **8** (2015), no. 4, 1045-1084.

[2] Carl-F. Bödigheimer, *Stable splittings of mapping spaces*, Algebraic Topology (Proc. Seattle, 1985), Lecture Notes in Math. **1286**, Springer, Berlin, 1987, 174-187.

[3] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2001, 479-481.

[4] Lars Hesselholt, *A homotopy theoretical derivation of* $Q\operatorname{Map}(K,-)_+$, Math. Scand. **70** (1992), no. 2, 193-203.

[5] Maxim Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Math. **2**, Progr. Math. **120**, Birkhäuser, Basel, 1994, 97-121.

[6] Jacob Lurie, *Higher Algebra*, preprint, http://www.math.harvard.edu/ lurie/

[7] Richard Manthorpe and Ulrike Tillmann, *Tubular configurations: equivariant scanning and splitting*, J. London Math. Soc. **90** (2014), no. 3, 940-962.

[8] John P. May, *The geometry of iterated loop spaces*, Lecture Notes in Math. **271**, Springer, Berlin, 1972.

[9] Dusa McDuff, *Configuration spaces of positive and negative particles*, Topology **14** (1975), 91-107.

[10] Paolo Salvatore, *Configuration spaces with summable labels*, Cohomological methods in homotopy theory, Progr. Math. **196**, Birkhäuser, Basel, 2001, 375-395.

[11] Graeme Segal, *Configuration-spaces and iterated loop-spaces*, Invent. Math. **21** (1973), no. 3, 213-221.

[12] Graeme Segal, *K-homology theory and algebraic K-theory*, K-Theory and Operator Algebras, Lecture Notes in Math. **575**, Springer, Berlin, 1977, 113-127.

[13] Graeme Segal, *The topology of spaces of rational functions*, Acta Math. **143** (1979), no. 1, 39-72.

[14] Dev P. Sinha, *Manifold-theoretic compactification of configuration spaces*, Selec. Math. **10** (2004), no. 3, 391-428.

[15] Norman Steenrod, *A convenient category of topological spaces*, Mich. Math. J. **14** (1967), 133-152.