

Mathematical Models of Two-Dimensional Sheets and Foundations



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Preface

The research described in this thesis was performed in the Oxford Centre of Industrial and Applied Mathematics at the Mathematical Institute, University of Oxford between October 2017 and May 2021, and was supervised by Professor Dominic Vella.

This thesis describes my own original work, except where specifically indicated in the text and acknowledgements. In particular, the experimental data presented in Figs. 2.3–2.5 as well as §2.4.1, §2.4.2, and §5.2 constitute review material. No part of this thesis has been submitted previously for a degree of the University or elsewhere.

At the time of writing, two papers based on the work presented in this thesis have been published, co-authored with D. Vella. A paper based on the indentation of suspended two-dimensional solids, presented in Chapters 2 and 3, has been published in the *Journal of the Mechanics and Physics of Solids* (Chandler and Vella, 2020a) and the validity of Winkler’s mattress model for thin elastomeric layers, presented in Chapter 4, has been published in the *Proceedings of the Royal Society A* (Chandler and Vella, 2020b). In both publications, I performed the theoretical analysis, while D. Vella supervised the research.

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Abstract

The ubiquity of thin elastic materials in biology and engineering has motivated the development of many reduced models to describe their elastic deformation. This thesis is concerned with the validity of some common reduced models to problems motivated by recent experiments, as well as the derivation of appropriate alternatives.

The first problem is motivated by the indentation of suspended elastic sheets, as is often done to characterize ultra-thin solids (including truly two-dimensional materials like graphene). While this is a convenient means of measuring properties such as the stretching modulus of these materials, experiments on ostensibly similar systems have reported material properties that differ by more than an order of magnitude. We demonstrate that such reported differences may arise from the inappropriate use of asymptotic results as well as commonly neglected effects, namely the indenter geometry and non-Hookean material behaviour. In particular, we present a modelling study of this indentation process assuming linear elasticity and implement a model that accounts for large strains and plate rotations.

The second problem comes from the deformation of a thin elastic foundation. A linear force–deflection relationship (known as Winkler’s mattress model) is often used as a simplified model to understand how a thin elastic layer deforms when subject to a distributed normal load. For an incompressible material, however, the model predicts infinite resistance to deformation and, hence, breaks down. We derive a model that describes the deformation of thin elastic layers in response to pressure and shear loading, and that holds for both incompressible and compressible materials alike. We find that the applicability of Winkler’s model is not determined by the value of the Poisson’s ratio alone, but by a compressibility parameter that combines the Poisson’s ratio with a measure of the layer’s slenderness. We illustrate the application of our combined foundation model to three example problems.

Finally, we consider the macroscopic elastic properties of thin cellular structures. Internal cellular pressure (turgor) is known to provide structural rigidity to non-woody plants and has been used to induce Gaussian curvature in inflatable shells. However, the route through which pressure induces this macroscopic rigidity remains unclear. We calculate the macroscopic bulk, stretching, and bending moduli of a two-dimensional single-cell thick structure with variable internal pressure in a simple model. Using these effective moduli, we then consider problems purely at a macroscale, considering the implications of our results for bryophytes (*e.g.* mosses) and soft robotics.

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Prologue

1.1 General background

Thin elastic materials are found throughout biology and engineering due to their many beneficial mechanical properties: the stiff skin of a piece of fruit protects its soft interior from bruising (Wang et al., 2017; Box et al., 2020a); plant cell walls provide both structural rigidity and protection (Falk et al., 1958; Nilsson et al., 1958); graphene (an atom-thick sheet of graphite) is the thinnest, yet one of the strongest, materials known (Lee et al., 2008; Akinwande et al., 2017). Mathematically, the *thinness* of these solids has the added attraction of simplifying the description of elastic deformation by allowing more tractable reduced models to be developed; although, a drawback is that the deformation of thin materials can become geometrically nonlinear. This thesis is concerned with the derivation of such reduced models in a number of settings motivated by applications.

The ubiquity of thin materials and the attraction of a simplified theory makes it unsurprising that the study of thin solids is a historic problem which has led to the development of a hierarchy of mathematical models as well as fundamental developments in mathematics itself. For example, Euler, Bernoulli, et al. developed the calculus of variations to be able to describe the deformations of an elastic beam back in the eighteenth century (Levien, 2008). This led to the derivation of the Euler–Bernoulli beam equation, which predates the general theory of elasticity (developed by Navier and Cauchy) by nearly a century. Since this initial theory, the study of the deformation of a beam has been extended and improved upon; for example, Timoshenko–Ehrenfest beam theory accounts for the first-order shear deformation, allowing thicker beams to be considered, and Euler’s theory of the elastica accounts for nonlinear slopes, allowing for large beam deflections — see Libai and Simmonds (2005); Howell et al. (2008); O’Reilly (2017); Elishakoff (2019) for discussions on beam theory.

Reduced models have been developed for the deformation of other structures, such as elastic foundations. For example, Winkler considered the problem of an elastic beam on a

thin elastic foundation assuming that the restoring force was proportional to the deflection (Winkler, 1867), though a more formal derivation only came from Skotheim and Mahadevan (2004). In Table 1.1, we present some common reduced models of the deflection of a thin elastic solid based on linear elasticity.

Reduced model	Deflection–load response
Euler–Bernoulli beam/Kirchhoff–Love plate ^{1–3}	$w(\mathbf{x}) = \frac{1}{B} \nabla^{-4} q(\mathbf{x})$
String/membrane ^{1–3}	$w(\mathbf{x}) = \frac{-1}{T} \nabla^{-2} q(\mathbf{x})$
Winkler foundation ^{4–6}	$w(\mathbf{x}) = \frac{1}{k_W} q(\mathbf{x})$
Elastomeric foundation ^{6,7}	$w(\mathbf{x}) = \frac{-1}{k_I} \nabla^2 q(\mathbf{x})$

Table 1.1: Common reduced models of the deflection, $w(\mathbf{x})$, of a thin isotropic elastic solid by an imposed load, $q(\mathbf{x})$. All models are based on linear elasticity but make different assumptions about the underlying deformation. (¹Timoshenko and Woinowsky-Krieger, 1959, ²Mansfield, 2005, ³Howell et al., 2008, ⁴Winkler, 1867, ⁵Skotheim and Mahadevan, 2004, ⁶Dillard et al., 2018, ⁷Dillard, 1989.)

In all these reduced mathematical models, assumptions have been made to simplify the full elastic theory; however, understanding to what extent the theory *should* be simplified and how it can affect the results is not always straightforward. In general, one wants to consider a simple representation of the problem so that mathematical progress can be made; however, oversimplifying can lead to inconsistent results and deceptive conclusions about the physical problem. For example, a model that is derived through asymptotic analysis, formally, only holds in a distinguish limit; imposing it more generally may lead to incorrect results if the underlying assumptions do not hold. In this thesis, we consider three different problems motivated by recent experiments on thin solids; each of these examples exemplifies the importance of taking care when modelling thin elastic solids and why determining an appropriate model is not always simple.

1.2 Outline of the thesis

In Chapters 2 and 3, we consider the first of these problems: the indentation of suspended two-dimensional solids. A common method for testing the mechanical properties of two-dimensional solids (*e.g.* graphene and molybdenum disulfide) is to deposit the material onto a rigid substrate that has been perforated with circular holes. Indenting the material at a point where it is suspended (for example, with an Atomic Force Microscope, AFM) yields a force–displacement response that is controlled by the mechanical parameters of the sheet. These parameters can be inferred from the response by assuming a suitable mathematical model.

However, recent experiments by different groups on ostensibly similar materials have led to reported material properties that differ by more than an order of magnitude. It has been proposed that such reported differences may arise from the inappropriate use of asymptotic results to fit data (Vella and Davidovitch, 2017) or the presence of commonly neglected effects, such as non-Hookean material behaviour (López-Polín et al., 2017). In Chapter 2, we demonstrate the first of these issues by showing the limitations of current fitting protocols, which are based on asymptotic results, by comparing them with numerical solutions of a more comprehensive model based on the Föppl–von Kármán equations. In Chapter 3, we then perform a detailed study of these indentation tests assuming linear elasticity (via the Föppl–von Kármán equations) where we derive novel (as well as unify existing) asymptotic predictions for the force–displacement relation. Furthermore, we consider the effects of non-Hookean elasticity by implementing a model which accounts for large strains and plate rotations, which is based on the work of Green and Adkins (1960).

Thin elastic solids coating a rigid substrate are encountered in a variety of scenarios and at a range of scales, from the consolidated till separating glaciers from bedrock (*e.g.* Sayag and Worster, 2011; Butler et al., 2020) to multilayer microfluidic systems (*e.g.* Thorsen et al., 2002). Unlike in the aforementioned indentation tests, here the material cannot bend out-of-plane and instead must deform internally. The simplest, and perhaps most common, model of how a coating deforms is the Winkler foundation: the material behaves like a mattress of springs. Although the Winkler foundation is known to break down for incompressible materials, it has still been used (apparently successfully) to model near-incompressible solids. In Chapter 4, we derive a model that describes the deformation of a thin foundation in response to pressure and shear loading, by applying an asymptotic expansion (akin to lubrication theory in hydrodynamics, Leal, 2007) to the elastostatic Navier–Cauchy equations. This model holds for both incompressible and compressible materials alike and can be used to explain the limitations of the Winkler foundation. We apply this model to three example problems involving an elastomeric foundation: the indentation of a soft layer with a stiff coating; the elastohydrodynamic sedimentation of a cylinder; and a Hele-Shaw flow bounded by a thin, finite, elastic layer.

The third example concerns developing a new model for the deformation of a cellular solid. Along with water being essential for basic plant functions (*e.g.* photosynthesis and nutrient transportation), the water pressure (turgor) provides structural rigidity to non-woody plants, without which they would wilt and die. However, it is unclear how the internal pressure is fundamentally linked to the macroscopic rigidity of these thin cellular structures. In Chapter 5, we calculate the macroscopic elastic properties (*i.e.* the bulk, stretching, and bending moduli) of a two-dimensional, single-cell thick structure that resembles the cellular structure found in certain species of moss (Waite and Sack, 2010) and used in inflatable

shells (Cadogan et al., 2004; Siéfert et al., 2020). Here, we utilize Kirchhoff’s rod theory for the deformation of the cell walls. Using this model, we develop new insight into the effects of internal cellular pressure, including how the extension and curvature of a leaf varies with its turgor pressure, when cells can shrink with an increase in pressure (*i.e.* the counterintuitive turgor-driven shrinking, Hofhuis et al., 2016), and discuss possible applications to soft robotics.

Finally, we note that the notation and analysis of each of these three problems is self-contained within the relevant chapters: Chapters 2 and 3 consider the indentation of suspended two-dimensional solids; Chapter 4 considers the validity of Winkler’s model for thin elastic foundations; and Chapter 5 considers the macroscopic elastic properties of thin cellular structures. Given this separation of subjects, we leave a more detailed introduction and discussion of the surrounding literature to the respective chapters. In Chapter 6, we conclude this thesis by summarizing the findings of the previous chapters and discussing a number of possible extensions.

Indentation of suspended two-dimensional solids: Simple models versus experiments

2.1 Introduction

Just as it is natural to test the inflation of a tyre and ripeness of fruit by poking with a finger, a common means of testing the mechanical properties of solids is via indentation tests. While much attention has focussed on the determination of bulk elastic constants via the indentation of a half-space (see [Harding and Sneddon, 1945](#); [Nix and Gao, 1998](#); [Perriot and Barthel, 2004](#); [Butt et al., 2005](#); [McKee et al., 2011](#), for example), a great deal of recent interest has focussed on the use of indentation to determine the mechanical properties of two-dimensional materials like graphene and molybdenum disulphide (see [Castellanos-Gomez et al., 2015](#); [Cao and Gao, 2019](#), for reviews). Flakes of such two-dimensional materials are difficult (if not impossible) to manipulate in a tensile testing machine but may be deposited on a substrate relatively easily. If the substrate on which deposition occurs is patterned with holes, the flake is then ‘suspended’ over the holes: indenting the thin material at a point where it is suspended, for example with an Atomic Force Microscope (AFM), yields a response that is largely independent of the substrate’s mechanical properties (provided that the adhesion between substrate and thin layer is sufficient to guarantee clamping at the hole edge). In particular, indentation yields a force–displacement response that is controlled by the stretching stiffness (or two-dimensional Young’s modulus), E_{2D} , of the material, together with any residual tension or bending stiffness of the suspended material.

To understand atomistic simulations ([Lu and Huang, 2009](#); [Cooper et al., 2013](#)) and experimental applications ([Akinwande et al., 2017](#)), single-atomic-layer sheets (such as graphene) have been successfully modelled as continuous two-dimensional elastic sheets. Thus, from the point of view of mechanics, the indentation of suspended two-dimensional materials has much in common with the indentation of suspended elastic membranes. This problem was studied first by [Schwerin \(1929\)](#), but has since been extended to account for

the effects of pre-tension (Norouzi et al., 2006), bending stiffness (Wan et al., 2003), and indenter geometry (Begley and Mackin, 2004; Komaragiri et al., 2005). Nevertheless, these modifications often entail approximate analyses (for example, by assuming an ansatz for the form of the solution, as in Begley and Mackin, 2004) rather than deriving asymptotic results from governing equations, such as the Föppl–von Kármán equations.

The interest in the mechanical properties of two-dimensional materials has been sparked because of their high material strength, their novel electronic properties, and the interaction between imposed elastic strain and electronic properties (Vozmediano et al., 2010; Akinwande et al., 2017; Harats et al., 2020); determining the value of the stretching stiffness, E_{2D} , as well as any pre-existing tension, is therefore the main focus of indentation experiments on suspended materials.

The first attempt to measure E_{2D} in graphene was made by Lee et al. (2008), who reported a value $E_{2D} \approx 340 \text{ N/m}$ based on AFM indentation experiments on suspended flakes. While this value is in good agreement with that predicted from first principles (Kudin and Scuseri, 2001), values as low as $E_{2D} \approx 20 \text{ N/m}$ at room temperature (Nicholl et al., 2015) and as high as $E_{2D} \approx 680 \text{ N/m}$ with imposed pre-strain (López-Polín et al., 2017) have been reported. It has also been reported that an optimal number of defects may increase the value of E_{2D} significantly (López-Polín et al., 2015).

There are many complicating factors involved in the indentation of truly two-dimensional materials like graphene, including the importance of thermal crumpling (flexural phonons) and static wrinkles that may both give rise to ‘hidden area’ (Nicholl et al., 2017), the anisotropy induced by the underlying hexagonal lattice (Kumar and Parks, 2015), as well as the possibility of slip at the boundary (Davidovitch and Guinea, 2021). In particular, excess membrane area that is hidden in static wrinkles and thermal fluctuations may lead to measurements of the stretching modulus, E_{2D} , that are small and/or load-dependent simply because out-of-plane deformations are ‘ironed out’ at low applied stress, as is common when extra material is ‘buffered-by-buckling’ (Vella, 2019). This hidden area may give rise to an apparently strain-dependent stretching modulus and, hence, be a major cause of the discrepancy in values of E_{2D} reported in the literature, while a stress-dependent stretching modulus of two-dimensional solids might also arise because of finite atomic bond lengths. Nevertheless, it has also been pointed out that inconsistencies exist in the way that the predictions of classical elasticity theory are used to interpret experimental data (Vella and Davidovitch, 2017; Jia and Ben Amar, 2020).

In this and the following chapter, the aim is to set out clearly the predictions of the standard models of mechanics that are appropriate to thin, isotropic, elastic solids as derived asymptotically from the Föppl–von Kármán equations. We will highlight the potential pitfalls that experimental attempts to characterize material properties by indentation may

fall into, focussing on the predictions of these models in the parameter regimes that are of most relevance to two-dimensional solids. A key question in this work will be how to separate the effect of geometrical nonlinearities (particularly the effect of indenter geometry) from the effect of material nonlinearities (*i.e.* non-Hookean stress-strain responses). By doing so, we hope that future experiments will be more readily reconciled, and more clearly highlight which experimental results are a consequence of the unique properties of these unusual solids.

In this chapter, we demonstrate the shortcomings of current fitting protocols and how they can be alleviated using accurate modelling of experiments. In §2.2, we present the general modelling approach, together with a simple scaling analysis that highlights the variety of possible response behaviours. In §2.3, we introduce our model for the indentation of a linearly elastic sheet by either a cylindrical or spherical-capped indenter. We then move on to discuss current fitting protocols and compare them with a more comprehensive numerical study in §2.4. In Chapter 3, we determine asymptotic solutions of the model that hold in distinguished limits, but might be used as an alternative to current fitting protocols and numerical methods — provided there is a suitable appreciation of the potential pitfalls.¹

2.2 Modelling approach

2.2.1 Physical model of indentation

Typical experimental measurements of the mechanical properties of two-dimensional solids by indentation involve a sheet being suspended over a circular hole (of radius R_{out}) on an otherwise planar substrate. While the boundary conditions at the edge of the hole are not generally well-controlled, it is usually assumed that the sheet is perfectly clamped at this boundary (*i.e.* there is no additional radial displacement at the edge as indentation progresses) by the film–substrate adhesion. We shall also assume that the clamping is perfect. When the sheet is deposited it is typically subject to a pre-existing tension, or pre-stress, T_{pre} , which may be caused by the processing or fabrication of the sheet. We shall assume that the pre-tension is uniform and isotropic and, further, that all deformations are rotationally symmetric. Although these assumptions are the intention of most experiments, rotational symmetry might be broken by, for example, a non-uniform pretension (caused by *e.g.* the initial deposition of the sheet) or not poking in the centre, which is one possible cause of experimental inconsistency. Nevertheless, we shall show that inconsistencies also exist in current fitting procedures even if this idealized setup is achieved.

A quantity of most practical interest is the two-dimensional Young’s modulus of the

¹The work presented in this and the following chapters is based on the published work by [Chandler and Vella \(2020a\)](#).

sheet, E_{2D} . For a thin, Hookean sheet of thickness t and Young's modulus E , $E_{2D} \equiv Et$, but we use E_{2D} throughout so that our theory describes equally thin elastic sheets and two-dimensional solids — for which E_{2D} actually corresponds to an *effective* stretching modulus, a result of assuming that the atomic-structure can be modelled using continuum mechanics (see [Lu and Huang, 2009](#); [Cooper et al., 2013](#), for example). We further assume that the sheet has a Poisson's ratio ν .

Indentation is typically performed by an Atomic Force Microscope (AFM) tip, which applies the force, F , required to impose a vertical displacement, δ , of the central region (measured relative to the clamped edges). We consider two tip shapes in detail: a cylindrical indenter (of radius $R_{\text{in}} < R_{\text{out}}$), which facilitates the analysis, and a spherical tip (of radius of curvature $R_s < R_{\text{out}}$), which is more representative of the tips used experimentally (see [López-Polín et al., 2017](#), for example). We shall see that the shape of the indenting tip can be important for the final interpretation of the force–displacement relationship. A sketch of the typical experimental setup is shown in Fig. 2.1.

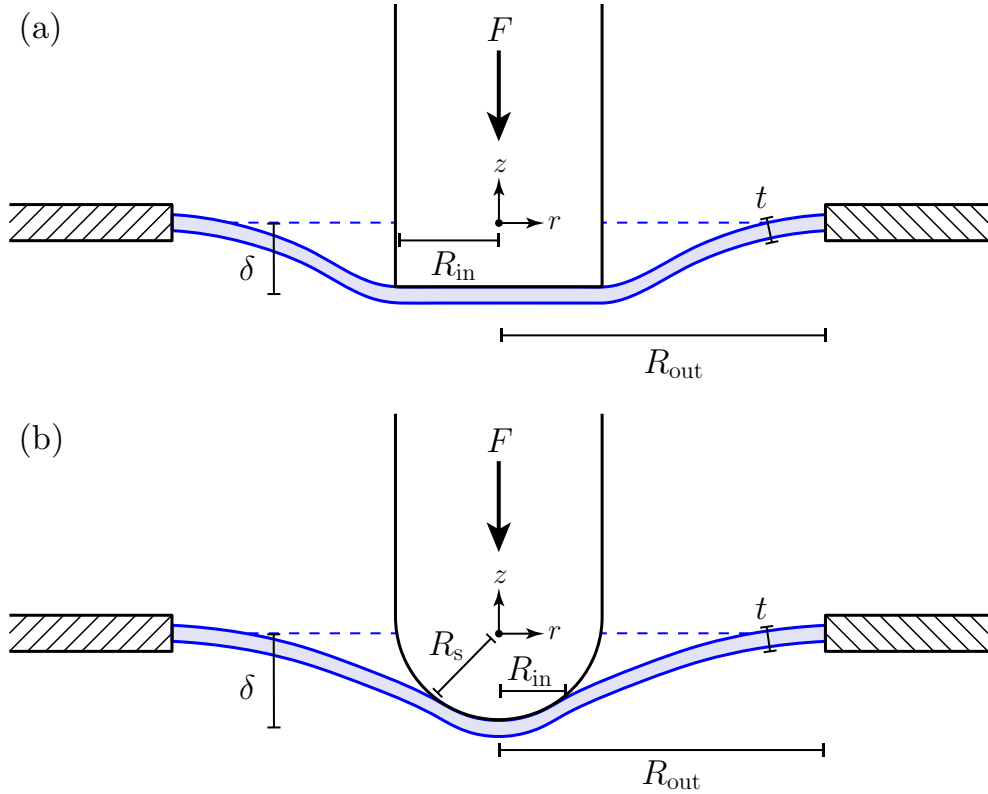


Figure 2.1: Cross-sectional sketch of the indentation of a clamped sheet by: (a) a cylindrical indenter (or punch) of known radius, R_{in} , and (b) a spherical-capped indenter of known radius of curvature, R_s , but unknown contact radius, $R_{\text{in}}(\delta)$.

2.2.2 Scalings and physical arguments

The relationship between the indentation force and depth, $F(\delta)$, holds important information for understanding the elastic properties of thin sheets. To get a first sense of the possible different behaviours of the force–indentation relation, we begin by considering the various energies in the problem at a scaling level.

The work done by the indenter, which scales as $\mathcal{U}_{\text{work}} \sim F\delta$, must be stored predominantly in either the stretching or bending energies of the sheet. To estimate the stretching energy stored within the sheet, we note that the sheet tension has two components: one caused by the pre-tension, T_{pre} , and another caused by the stretching in response to the imposed strain, $\varepsilon \sim (\delta/R_{\text{out}})^2$, which is $\Delta T \sim E_{2D} \cdot \varepsilon$. (Note that, for small displacements, $\delta \ll R_{\text{out}}$, the strain estimate $\varepsilon \sim (\delta/R_{\text{out}})^2$ can be derived from elementary geometrical considerations.) We, therefore, write $T \sim T_{\text{pre}} + E_{2D}\varepsilon$ and note that the stretching energy of the sheet must scale according to $\mathcal{U}_s \sim R_{\text{out}}^2(T_{\text{pre}} + E_{2D}\varepsilon)\varepsilon$. Finally, we note that the bending energy of the sheet scales as $\mathcal{U}_b \sim BR_{\text{out}}^2\mathcal{K}^2$, where $\mathcal{K} \sim \delta/R_{\text{out}}^2$ is the typical sheet curvature and B is the bending modulus (for an isotropic, thin, Hookean solid $B = E_{2D}t^2/[12(1 - \nu^2)]$, though we use a general bending modulus to account for truly two-dimensional solids, such as graphene, for which the effective value of B may differ significantly from this value²).

Before considering the different limits, we note that the work done, $\mathcal{U}_{\text{work}}$, must equal the sum of these different energies, *i.e.* $\mathcal{U}_{\text{work}} \sim \mathcal{U}_b + \mathcal{U}_s$, and hence

$$F \cdot \delta \sim \underbrace{R_{\text{out}}^2 T_{\text{pre}} \left(\frac{\delta}{R_{\text{out}}} \right)^2}_{\text{pre-tension}} + \underbrace{R_{\text{out}}^2 E_{2D} \left(\frac{\delta}{R_{\text{out}}} \right)^4}_{\text{stretching}} + \underbrace{R_{\text{out}}^2 B \left(\frac{\delta}{R_{\text{out}}^2} \right)^2}_{\text{bending}}. \quad (2.1)$$

There are three possible dominant balances, depending on which term dominates the right-hand side of (2.1): (i) pre-tension dominated so that $F \sim T_{\text{pre}}\delta$, corresponding to linear membrane theory with constant tension (see [Begley and Mackin, 2004](#); [Komaragiri et al., 2005](#); [Vella and Davidovitch, 2017](#), for example); (ii) stretching dominated so that $F \sim E_{2D}\delta^3/R_{\text{out}}^2$, recovering the scaling of the classical Schwerin solution (see [Schwerin, 1929](#); [Begley and Mackin, 2004](#); [Komaragiri et al., 2005](#); [Vella and Davidovitch, 2017](#)); (iii) bending stiffness dominated so that $F \sim B\delta/R_{\text{out}}^2$, and the sheet responds as a classical plate (see [Timoshenko and Woinowsky-Krieger, 1959](#); [Wan et al., 2003](#); [Komaragiri et al.,](#)

²Typically, the bending modulus of graphene is given as around 1.2 eV (see [Lu et al., 2009](#); [Akinwande et al., 2017](#), for example) or, equivalently, $B \approx 1.9 \times 10^{-19}$ J; however, based on the stretching modulus $E_{2D} = 340$ N/m and thickness $t \approx 0.335$ nm ([Akinwande et al., 2017](#)), the corresponding isotropic solid would have $B = E_{2D}t^2/12(1 - \nu^2) \approx 3.4 \times 10^{-18}$ J. As a result, graphene is 10 times more bendable than would be expected for a corresponding isotropic solid; this reflects the fact that, since it is only a single molecule thick, the usual mechanism for generating a bending stiffness (*i.e.* differential strain through the sheet thickness) is not relevant for graphene.

2005, for example).

Taking the pre-tension dominated case as a reference state, natural choices of dimensionless indentation force, depth, and bending stiffness are

$$\mathcal{F} := \frac{F E_{2D}^{1/2}}{2\pi T_{\text{pre}}^{3/2} R_{\text{out}}}, \quad d := \frac{\delta}{R_{\text{out}}} \left(\frac{E_{2D}}{T_{\text{pre}}} \right)^{1/2}, \quad \mathcal{B} := \frac{B}{T_{\text{pre}} R_{\text{out}}^2}, \quad (2.2\text{a-c})$$

respectively.

Alternatively, one could have taken the bending or stretching dominated cases as the reference state (see Komaragiri et al., 2005, for example). In the applications of current interest (especially for two-dimensional materials), the bending dominated region is of limited interest: for graphene, typical values are $B \approx 10^{-19}$ J (Lu et al., 2009) with $T_{\text{pre}} \approx 0.1$ N/m and $R_{\text{out}} \approx 1 \mu\text{m}$ (Lee et al., 2008), so that $\mathcal{B} \approx 10^{-6} \ll 1$ and the bending stiffness is expected to only be apparent in boundary layers of dimensionless width $\mathcal{B}^{1/2} \approx 10^{-3}$, corresponding to a dimensional width of ~ 1 nm. The pre-tension, therefore, dominates the bending stiffness and so we choose a non-dimensionalization that allows the limit $\mathcal{B} \rightarrow 0$ to be easily taken. Similarly, as suggested by Vella and Davidovitch (2017), many recent experiments do not always reach the stretching-dominated (or Schwerin) regime; therefore, the non-dimensionalization in (2.2) allows us to focus on the transition between the dominant balances, (i) and (ii).

In dimensionless variables the above dominant balances become: (i) $\mathcal{F} \sim d$ provided $\{\mathcal{F} \ll 1, \mathcal{B} \ll 1\}$, (ii) $\mathcal{F} \sim d^3$ provided $\{\mathcal{F} \gg \mathcal{B}^{3/2}, \mathcal{F} \gg 1\}$, and (iii) $\mathcal{F} \sim \mathcal{B}d$ provided $\{\mathcal{F} \ll \mathcal{B}^{3/2}, \mathcal{B} \gg 1\}$. These behaviours/regions were the main discussion of Komaragiri et al. (2005) who considered indentation by an idealized point-indenter (with our regions (i)–(iii) corresponding to regions 3–1 in Komaragiri et al., 2005, respectively). We emphasize that region (i) corresponds to a constant compliance, $d/\mathcal{F}$, while region (ii) corresponds to a constant ‘cubic compliance’, $d/\mathcal{F}^{1/3}$. (Discussing compliance, $d/\mathcal{F}$, rather than stiffness, $\mathcal{F}/d$, simplifies the analytical results presented later.) In this chapter and the next, we shall be focussed on understanding the dependence of each of these compliances on the material properties of the system and the behaviour of the system in-between the asymptotic regimes that correspond to regions (i) and (ii).

From these simple energy arguments, one might assume that the indenter’s geometry has little effect on the response, $d(\mathcal{F})$. This assumption has been made implicitly across a range of experimental work (Lee et al., 2008; Castellanos-Gomez et al., 2015; López-Polín et al., 2015; López-Polín et al., 2017, to name a few) — in a complex experimental setup, applying the point-indenter ‘solutions’ to experimental data allows progress to be made. In Chapter 3, we investigate the circumstances in which the radius of contact, R_{in} , and the shape of the indenter matter.

For a cylindrical indenter, we introduce the dimensionless radius

$$\mathcal{R} := \frac{R_{\text{in}}}{R_{\text{out}}}. \quad (2.3)$$

For a spherical-capped indenter, it is not immediately clear what length scale should be used to measure the sphere size; here we use the tensile length $R_{\text{out}}(E_{2D}/T_{\text{pre}})^{1/2}$, which is used to rescale vertical deflections. We, thus, introduce the dimensionless sphere radius

$$\mathcal{R}_s := \frac{R_s}{R_{\text{out}}} \left(\frac{T_{\text{pre}}}{E_{2D}} \right)^{1/2}. \quad (2.4)$$

2.3 Indentation of a linearly elastic sheet

We consider the indentation of a linearly elastic sheet by two types of indenter: a cylindrical indenter, for which the specification of the problem (especially the boundary conditions) is relatively simple; and a spherical-capped indenter, which is of more experimental relevance (see [Bhatia and Nachbar, 1968](#); [Jennings et al., 1995](#); [Begley and Mackin, 2004](#); [Lee et al., 2008](#); [López-Polín et al., 2017](#), for example). The key difference between the two types is that the radial position of the edge of contact, R_{in} , is known for the cylinder (the sheet is pinned at its rim), but is unknown for the sphere since it evolves with the indentation depth, δ , as the sheet wraps more of the indenter. Below, we formulate these two cases simultaneously.

2.3.1 Föppl–von Kármán formulation

In this chapter, we confine our attention to linear elasticity under the Kirchhoff assumptions (*i.e.* small strains and plate rotations) so that the Föppl–von Kármán (FvK) equations hold ([Mansfield, 2005](#)). The axisymmetric FvK equations link the out-of-plane displacement of the sheet, $z(r)$, to the stress via a stress potential, $\psi(r)$, which is defined such that the principal stresses are $\sigma_{rr} = \psi(r)/r$ and $\sigma_{\theta\theta} = \psi'(r)$, thereby ensuring that in-plane stress equilibrium holds automatically ([Mansfield, 2005](#)). The out-of-plane force balance and the compatibility of strains condition may both be integrated once to give

$$Br \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dz}{dr} \right) \right] = \psi \frac{dz}{dr} - \frac{F}{2\pi}, \quad (2.5a)$$

$$r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (r\psi) \right] = -\frac{E_{2D}}{2} \left(\frac{dz}{dr} \right)^2, \quad (2.5b)$$

respectively. Note that, since we are assuming linear elasticity, the deformed and undeformed configurations are interchangeable; the radial coordinate in the sheet is, thus, denoted $r \in [R_{\text{in}}, R_{\text{out}}]$. Additionally, since we are predominately interested in small bending stiffnesses, $\mathcal{B} \ll 1$, and hence small plate rotations, the Föppl–von Kármán formulation presented here is identical to the simplified Reissner theory used by *e.g.* Komaragiri et al. (2005).

At the outer rim, $r = R_{\text{out}}$, we assume a perfectly clamped boundary:

$$z(R_{\text{out}}) = 0, \quad z'(R_{\text{out}}) = 0, \quad \psi'(R_{\text{out}}) - \nu \frac{\psi(R_{\text{out}})}{R_{\text{out}}} = (1 - \nu)T_{\text{pre}}. \quad (2.6\text{a-c})$$

Here, (2.6a,b) are geometric conditions of zero vertical displacement and slope, while (2.6c) ensures the radial displacement is fixed to be that caused by the initial isotropic tension, T_{pre} .

Although steps are typically taken to ensure there is no material slippage at the outer rim (Manzanares-Negro et al., 2020), this is usually not the case at the inner rim, where intermolecular forces are often the only cause of the material binding to the indenter. Hence, *a priori*, it is not obvious what boundary condition should be imposed at the indenter. We shall assume either a perfect-slip or no-slip boundary (*i.e.* the two extreme cases) since, in reality, there may be a small amount of adhesion/friction between the indenter and sheet. The boundary conditions at the inner rim, $r = R_{\text{in}}$, are dependent on the indenter geometry:

Cylindrical indenter For the indentation by a cylindrical indenter, we impose

$$z(R_{\text{in}}) = -\delta, \quad z'(R_{\text{in}}) = 0, \quad (2.7\text{a,b})$$

with either

$$\text{perfect-slip:} \quad \psi'(R_{\text{in}}) - \frac{\psi(R_{\text{in}})}{R_{\text{in}}} = 0, \quad (2.7\text{c})$$

$$\text{or no-slip:} \quad \psi'(R_{\text{in}}) - \nu \frac{\psi(R_{\text{in}})}{R_{\text{in}}} = (1 - \nu)T_{\text{pre}}. \quad (2.7\text{d})$$

Here, (2.7a,b) are geometric conditions of continuous vertical displacement and slope; while, (2.7c) comes from a force balance with the inner (known-geometry) solution, $\psi(r) \propto r$, explaining why (2.7c) is independent of the Poisson's ratio, ν , in comparison to (2.7d), which ensures the radial displacement is fixed at its initial value.

Spherical indenter For the indentation by a spherical-capped indenter, we impose

$$z(R_{\text{in}}) = -\delta + \frac{R_{\text{in}}^2}{2R_s}, \quad z'(R_{\text{in}}) = \frac{R_{\text{in}}}{R_s}, \quad z''(R_{\text{in}}) = \frac{1}{R_s}, \quad (2.8a-c)$$

with either

$$\text{perfect-slip:} \quad \psi'(R_{\text{in}}) - \frac{\psi(R_{\text{in}})}{R_{\text{in}}} = -E_{2D} \frac{R_{\text{in}}^2}{8R_s^2}, \quad (2.8d)$$

$$\text{or no-slip:} \quad \psi'(R_{\text{in}}) - \nu \frac{\psi(R_{\text{in}})}{R_{\text{in}}} = E_{2D} \frac{u_r^*}{R_{\text{in}}}, \quad (2.8e)$$

where R_s is the sphere's radius of curvature. Here, (2.8a–c) express the geometric conditions of continuous vertical displacement, slope, and curvature; (2.8d) comes from an in-plane force balance with the contact region, in which it may easily be shown that $\psi(r) + E_{2D}r^3/16R_s^2 \propto r$ with no ν -dependence; and (2.8e) fixes the radial stretch at a value u_r^* which is, *a posteriori*, dependent on the history of contact. We again emphasize that the point of contact, R_{in} , is unknown here and must be determined as part of the solution, explaining why we require an extra boundary condition, (2.8c), compared to the cylindrical case.

Based on numerical simulations, the choice between perfect-slip or no-slip is observed to only have an effect at large deformations and is not expected to qualitatively change the asymptotic solutions presented in Chapter 3. Therefore, for the rest of this thesis, we concentrate on the perfect-slip indenter with boundary conditions (2.7c) and (2.8d), without loss of generality. We additionally assume that the sheet and indenter remain in contact for $r \leq R_{\text{in}}$.

2.3.2 Non-dimensionalization

We use the dimensionless variables suggested by the discussion of energy and scalings in §2.2.2; in particular, we define

$$\rho := \frac{r}{R_{\text{out}}}, \quad \Psi(\rho) := \frac{\psi(r)}{T_{\text{pre}} R_{\text{out}}}, \quad Z(\rho) := \frac{z(r)}{R_{\text{out}}} \left(\frac{E_{2D}}{T_{\text{pre}}} \right)^{1/2}. \quad (2.9a-c)$$

Substitution of (2.9) into (2.5)–(2.8) gives the dimensionless system

$$\mathcal{B}\rho \frac{d}{d\rho} \left[\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dZ}{d\rho} \right) \right] = \Psi \frac{dZ}{d\rho} - \mathcal{F}, \quad (2.10a)$$

$$\rho \frac{d}{d\rho} \left[\frac{1}{\rho} \frac{d}{d\rho} (\rho \Psi) \right] = -\frac{1}{2} \left(\frac{dZ}{d\rho} \right)^2, \quad (2.10b)$$

for $\mathcal{R} \leq \rho \leq 1$, subject to the boundary conditions at the outer rim,

$$Z(1) = 0, \quad Z'(1) = 0, \quad \Psi'(1) - \nu\Psi(1) = 1 - \nu. \quad (2.11a-c)$$

Those at the inner rim, however, are dependent on the indenter geometry:

Cylindrical indenter For the indentation by a perfect-slip, cylindrical indenter, we have

$$Z(\mathcal{R}) = -d, \quad Z'(\mathcal{R}) = 0, \quad \Psi'(\mathcal{R}) - \frac{\Psi(\mathcal{R})}{\mathcal{R}} = 0. \quad (2.12a-c)$$

Spherical indenter For the indentation by a perfect-slip, spherical-capped indenter, we have

$$Z(\mathcal{R}) = -d + \frac{\mathcal{R}^2}{2\mathcal{R}_s}, \quad Z'(\mathcal{R}) = \frac{\mathcal{R}}{\mathcal{R}_s}, \quad Z''(\mathcal{R}) = \frac{1}{\mathcal{R}_s}, \quad (2.13a-c)$$

$$\Psi'(\mathcal{R}) - \frac{\Psi(\mathcal{R})}{\mathcal{R}} = -\frac{\mathcal{R}^2}{8\mathcal{R}_s^2}. \quad (2.13d)$$

(Recall that $\mathcal{F}$, d , $\mathcal{B}$, $\mathcal{R}$, and $\mathcal{R}_s$ are defined in equations (2.2a-c), (2.3), and (2.4), respectively.)

Given a Poisson's ratio, ν ; dimensionless bending stiffness, $\mathcal{B}$; indentation force, $\mathcal{F}$; and indenter radius, $\mathcal{R}$ or $\mathcal{R}_s$ (for a cylinder or sphere, respectively), (2.10) subject to (2.11)–(2.13) may be solved by a standard numerical integrator (we use `bvp4c` in `MATLAB`), with the indentation depth, d , determined as part of the solution. For a spherical-capped indenter, the contact radius, $\mathcal{R}$, is also determined as part of the solution since the problem for $Z(\mathcal{R})$ is over-determined. To facilitate the computation for a spherical indenter, it is more convenient to fix the value of $\mathcal{R}$ and solve for an unknown $\mathcal{F}$ instead; by doing so, one avoids the problems associated with an unknown domain size.

Once solved numerically, this model provides a force–displacement response, $\mathcal{F}(d)$, which describes the indentation of a clamped, linearly elastic sheet by a cylindrical or spherical-capped indenter. Analytical progress can be made in the asymptotic limits of large and small displacements, which is discussed in detail in §3.1, for a cylindrical indenter, and §3.2, for a spherical-capped indenter. Here, we focus on insights from the numerical approach on current fitting protocols.

2.4 Discussion: Current fitting protocols

Even from a simple energetic scaling argument (§2.2.2), the relationship between the indentation force, F , and maximal displacement, δ , is highly dependent on the sheet's

mechanical properties — *i.e.* which term is dominant in (2.1). Consequently, fitting force–displacement curves for mechanical properties (such as the sheet’s stretching modulus, E_{2D}) is a difficult task with few analytic formulae for $F(\delta)$ existing. Instead, experiments usually rely on approximate formulae derived from different asymptotic limits of the full nonlinear problem.

2.4.1 Previous asymptotic results

One of the earliest studies of the indentation of a clamped membrane was carried out by [Schwerin \(1929\)](#), who found that the point-force, F , is cubic in the imposed deflection, δ , when the pre-tension is negligible; this corresponds to the stretching-dominated case in (2.1). Schwerin calculated the prefactor for $\nu = 1/3$ to be $\pi E_{2D}/(6R_{\text{out}}^2)$. This result can be generalized for $\nu \neq 1/3$ to give

$$F \sim \frac{q(\nu)^3 E_{2D}}{R_{\text{out}}^2} \delta^3, \quad (2.14)$$

with $q(\nu) \approx 1/(1.0491 - 0.1462\nu - 0.1158\nu^2)$, as shown numerically by [Komaragiri et al. \(2005\)](#). In this way, (2.14) describes the force–indentation response for large indentations. Subsequent studies have considered the effects of indenter geometry: [Vella and Davidovitch \(2017\)](#) showed that in the case of a no-slip cylindrical indenter the cubic coefficient, $q(\nu; \mathcal{R})$, is dependent on the indenter size, $\mathcal{R} \equiv R_{\text{in}}/R_{\text{out}}$, with $q(\nu; \mathcal{R})^{-1} \sim q(\nu)^{-1} - \{4\mathcal{R}^2/[\pi(1+\nu)]\}^{1/3}$ as $\mathcal{R} \rightarrow 0$. Whilst the case of a spherical-capped indenter has been studied by [Begley and Mackin \(2004\)](#); [Jia and Ben Amar \(2020\)](#), who found that the numerically-determined response can be approximated by a cubic coefficient, $q(\nu; R_s/R_{\text{out}})$, that is dependent on R_s/R_{out} ; however, we shall show in §3.2 that such an approximation does not truly represent the spherical response.

For small deflections and a negligible bending stiffness, the response is described by the work done against the pre-tension in the sheet, thus the force, F , is linear in the imposed displacement, δ — *i.e.* the pre-tension dominated case in (2.1). [Jennings et al. \(1995\)](#) showed that the small deflection response for a cylindrical indenter may be written as

$$F \sim \frac{2\pi T_{\text{pre}}}{\log(1/\mathcal{R})} \delta, \quad (2.15)$$

for the dimensionless radius, $\mathcal{R}$, defined in (2.3). Alternatively, the result for a spherical indenter was determined by [Bhatia and Nachbar \(1968\)](#) as

$$F \sim \frac{2\pi T_{\text{pre}}}{\log \sqrt{e/(\mathcal{R}_s \mathcal{F})}} \delta, \quad (2.16)$$

for the dimensionless force, $\mathcal{F}$, and radius, $\mathcal{R}_s$, defined in (2.2a) and (2.4), respectively. At

moderate deflections (or for small indenters), [Vella and Davidovitch \(2017\)](#) showed that the deflection is independent of the indenter size and is instead controlled by a different point-force response

$$F \sim \frac{2\pi T_{\text{pre}}}{\log(4/\mathcal{F})} \delta. \quad (2.17)$$

The stiffness for a point-force, (2.17), appears to vanish at small forces, $F/\delta \rightarrow 0$ as $\mathcal{F} \rightarrow 0$, and is *not* recovered after taking $\mathcal{R} \rightarrow 0$ in (2.15) or $\mathcal{R}_s \rightarrow 0$ in (2.16): the point limit does not make sense in this linear regime, whereas it did for the cubic regime, *i.e.* (2.14). (In Chapter 3, we show that the small, but finite, bending stiffness determines the small deflection response for a point indenter.)

Finally, for a large bending stiffness, the membrane acts according to classical plate theory (as noted by [Wan et al., 2003](#); [Komaragiri et al., 2005](#), for example) — *i.e.* the bending-dominated case in (2.1). Therefore, for a point force, the sheet responds according to

$$F \sim \frac{16\pi B}{R_{\text{out}}^2} \delta, \quad (2.18)$$

as shown by [Timoshenko and Woinowsky-Krieger \(1959\)](#). Alternatively, for a finite indenter, the stiffness is dependent on the indenter size as follows: $R_{\text{out}}^2 F/(B\delta) \sim p_c(\mathcal{R})$ for a cylinder and $R_{\text{out}}^2 F/(B\delta) \sim p_s(\mathcal{R}_s \mathcal{F}/\mathcal{B})$ for a sphere, where the functions p_c and p_s may be determined by solving the Kirchhoff–Love plate equation ([Timoshenko and Woinowsky-Krieger, 1959](#)).

These various asymptotic results show the complexity of this problem and motivate the question, when do each of these results formally hold? In Chapter 3, we perform a modelling study of the indentation test for small bending stiffnesses ($\mathcal{B} \ll 1$), where we derive some new results and unify existing asymptotic solutions. For now, we turn to discuss how these asymptotics are typically used to fit experimental data.

2.4.2 Fitting experimental data with approximate formulae

The most common fitting protocol is to use an *ad hoc* composite³ formula that interpolates between the explicit asymptotic solutions described in §2.4.1. One of the most common, and possibly simplest, expressions of this type was introduced by [Lee et al. \(2008\)](#) in their first measurement of the mechanical properties of graphene; in our notation, it takes the form

$$F = \pi T_{\text{pre}} \delta + \frac{q(\nu)^3 E_{2D}}{R_{\text{out}}^2} \delta^3. \quad (2.19)$$

This formula combines the classical Schwerin solution for a point indenter, *i.e.* (2.14), with a linear term which accounts for the work done against the pre-tension in the sheet. [Lee](#)

³Note that we use ‘composite’ here to refer to the imprecise addition of asymptotic solutions; a formal composite solution using the method of matched asymptotics has *not* been made.

et al. fixed the linear term in (2.19) using a particular indenter and sheet radius as well as $\nu = 0.16$ (relevant for graphene). The difference in the prefactor in the linear term of (2.19) compared to (2.15) suggests that such a relationship cannot hold in general (Vella and Davidovitch, 2017); nevertheless, it has been used extensively since, even in experimental setups with different dimensions (see Bertolazzi et al., 2011; Lin et al., 2013; Janshoff and Steinem, 2015; López-Polín et al., 2015; Liu and Wu, 2016; Song and Xu, 2016; Davidovikj et al., 2017; López-Polín et al., 2017; Dai et al., 2019, for example).

Two other expressions that are often used to fit material parameters are

$$F = \frac{2\pi T_{\text{pre}}}{\log(R_{\text{out}}/R_{\text{in}})}\delta + \frac{q(\nu)^3 E_{2D}}{R_{\text{out}}^2}\delta^3, \quad (2.20)$$

which uses the cylindrical indenter solution (2.15) in replace of the Lee et al. (2008) term (see Wan et al., 2003; Tanizawa and Yamamoto, 2004; Ruiz-Vargas et al., 2011; Janshoff and Steinem, 2015; Cao and Gao, 2019, for example) and

$$F = \left(\frac{16\pi B}{R_{\text{out}}^2} + \pi T_{\text{pre}} \right) \delta + \frac{q(\nu)^3 E_{2D}}{R_{\text{out}}^2}\delta^3, \quad (2.21)$$

which introduces an extra linear term to account for the effect of plate bending, *i.e.* (2.18), (see Castellanos-Gomez et al., 2012, 2015; Janshoff and Steinem, 2015; Cartamil-Bueno et al., 2017; Cao and Gao, 2019; Harbola et al., 2021; Wang et al., 2021, for example).

As highlighted by Vella and Davidovitch (2017), a simple linear combination of asymptotic results, as in (2.19)–(2.21), introduces various issues, most prominently the incorrect prefactor of $T_{\text{pre}}\delta$ in (2.19) and (2.21). In Fig. 2.2, the dimensionless versions of the composites (2.19) and (2.20) are plotted alongside the numerical solutions of the Föppl–von Kármán equations for a cylindrical indenter with (a) $\mathcal{R} = 10^{-3}$ and (b) $\mathcal{R} = 10^{-1}$. Here, the error in the linear term, $F \propto \delta$, in (2.19) and the disregard of finite indenter size in the cubic term, $F \propto \delta^3$, in (2.19) and (2.20) are apparent. Also plotted is the dimensionless version of the composite expression

$$F = \frac{2\pi T_{\text{pre}}}{\log(R_{\text{out}}/R_{\text{in}})}\delta + \frac{2\pi E_{2D}}{q_c(\nu, R_{\text{in}}/R_{\text{out}})^3 R_{\text{out}}^2}\delta^3, \quad (2.22)$$

where $q_c(\nu, \mathcal{R})$ is a numerically determined constant which satisfies $q_c(\nu, \mathcal{R}) \sim q_p(\nu) - (2\mathcal{R})^{2/3}$ as $\mathcal{R} \rightarrow 0$ with $q_p(\nu) \approx (2\pi)^{1/3}/q(\nu)$ for the $q(\nu)$ in (2.14). This expression combines the correct asymptotic solutions of the FvK equations for a cylindrical indenter, and so correctly recovers the large and small displacement behaviour observed numerically.

Directly comparing (2.19) with (2.22), we observe a 250% error in the linear term for the parameters used in Fig. 2.2(a) and an 87% error in the cubic term for the parameters used in Fig. 2.2(b); these would lead to corresponding errors in the measured values of T_{pre}

and E_{2D} if (2.19) were used to fit the numerical curve. This large error is also obvious in Fig. 2.2, which illustrates the large difference in plateaus as $\delta \rightarrow 0$.

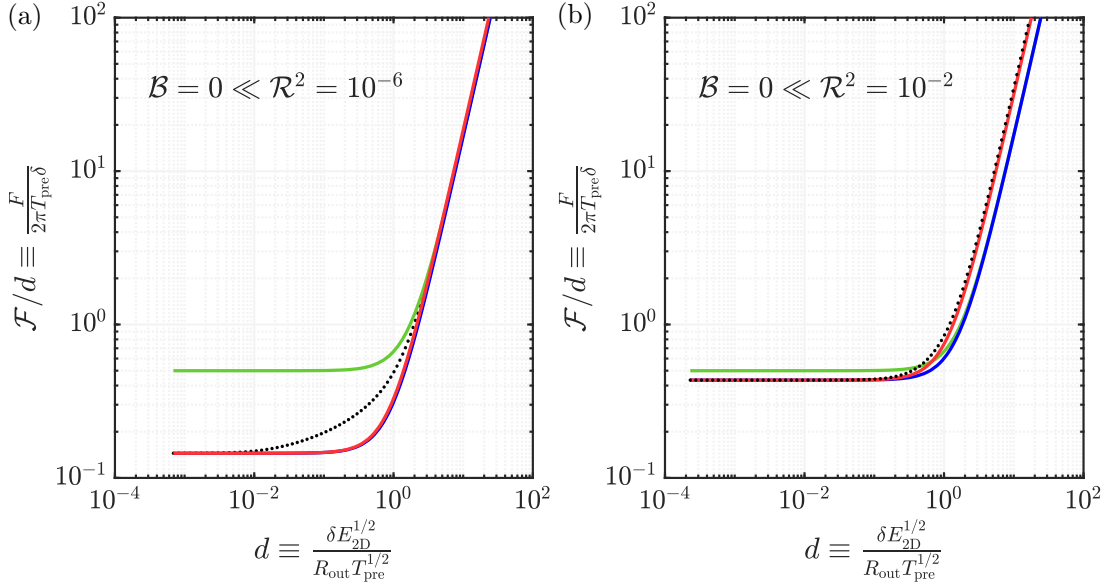


Figure 2.2: The dimensionless indentation stiffness, $\mathcal{F}/d$, obtained by numerically solving the Föppl–von Kármán equations, (2.10), for a cylindrical indenter (black dotted curve) with $\nu = 0.33$, $\mathcal{B} = 0$, and (a) $\mathcal{R} = 10^{-3}$ or (b) $\mathcal{R} = 10^{-1}$, are compared to dimensionless versions of the composite expressions: (2.19), green solid curve; (2.20), blue solid curve; and (2.22), red solid curve. The inaccuracies in the linear term, $\mathcal{F} \propto d$, in (2.19) and in the cubic term, $\mathcal{F} \propto d^3$, in (2.19) and (2.20) can be seen for small and large displacements, $d \ll 1$ and $d \gg 1$, respectively. Even if the correct asymptotic solutions are used, as is the case with (2.22), inconsistencies can still arise at moderate displacements, $d \sim 1$, solely due to the use of an *ad hoc* composite expression.

Composite expressions using the correct asymptotic solutions, such as (2.22) for the cylindrical indentation of a membrane ($\mathcal{B} \ll \mathcal{R}^2$), can only be constructed for very particular setups because of the limited number of explicit asymptotic solutions (§2.4.1). However, even ignoring the individual errors in the asymptotics, the applicability of a composite solution that combines the asymptotic results for small and large displacements is questionable: it is not clear whether (2.19)–(2.21) can be relied on to adequately describe F at intermediate values of δ . In Fig. 2.2, there is a significant error at moderate displacements for all the composite expressions; hence, even if the correct asymptotic behaviour is used, large errors may arise from using these composite formulae if the experimental data lie predominantly in the moderate indentation regime.

Vella and Davidovitch (2017) proposed that recent inconsistencies in measurements of the material properties of two-dimensional materials (*e.g.* the fact that the quoted stretching modulus of graphene varies from 20 N/m to 700 N/m, Akinwande et al., 2017) might be explained by the inappropriate application of fitting expressions, such as (2.19). In particular,

Vella and Davidovitch observed that typical experimental force–displacement data does not appear to exhibit the cubic behaviour, $F \sim \delta^3$, expected from (2.19); instead the data often appears more consistent with $F \sim \delta^2$. (This can be observed in Fig. 2.3(a), where we plot data from Castellanos-Gomez et al. 2015, fig. 3(c), as an example.) This suggests that experiments may only reach an intermediate region, in which neither of the asymptotic results in (2.19) truly hold. Alternatively, it could be that δ becomes so large that the material no longer behaves according to linear elasticity, and hence the asymptotic solutions given in §2.4.1 are not expected to be valid. We consider this possibility in Chapter 3 but focus, for now, on the difficulty of fitting experimental data by comparing the results to numerics.

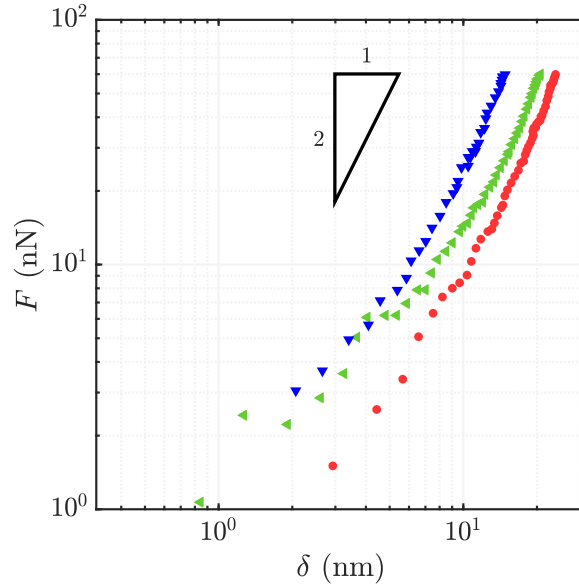


Figure 2.3: Experimental data obtained from the indentation of 2-layer (red circles, ●), 3-layer (green triangles, ▲), and 6-layer (blue triangles, ▼) mica flakes. A key feature is that there is no $F \sim \delta^3$ region, where (2.19) might be expected to hold. (The data here was captured digitally from Castellanos-Gomez et al. 2015, fig. 3(c), but has also been captured digitally and presented by Vella and Davidovitch 2017, fig. 8.)

2.4.3 Fitting experimental data with numerical results

As an alternative to using an approximate composite expression to fit experimental data, one can use a numerically determined force–displacement curve, $F(\delta)$, based on the assumption of linear elasticity — *i.e.* numerical solutions to the Föppl–von Kármán (FvK) equations, (2.10) subject to (2.11)–(2.13). This accurately models the Hookean response at moderate values of δ , where the composite expressions (2.19)–(2.21) were shown to fail.

In the case of a cylindrical indenter with negligible bending stiffness ($\mathcal{B} \ll \mathcal{R}^2$), the dimensionless force–displacement curve, $\mathcal{F}(d)$, forms a universal curve for a given indenter

size, $\mathcal{R} \equiv R_{\text{in}}/R_{\text{out}}$, and Poisson's ratio, ν . This curve can be used to fit for the unknown stretching modulus, E_{2D} , and pre-tension, T_{pre} . For a spherical-capped indenter (or if the bending stiffness is non-negligible) the situation is more complicated because both the dimensionless radius of curvature, $\mathcal{R}_s$, and bending stiffness, $\mathcal{B}$, are dependent on T_{pre} and E_{2D} , see the definitions in (2.4) and (2.2c); instead, one has to construct a family of curves with different $\mathcal{R}_s$ to properly fit the data.

In Fig. 2.4, we fit the experimental data obtained from indenting a single-layer of molybdenum disulphide (MoS_2) by a spherical-capped indenter⁴ using two procedures: one using the conventional composite expression, (2.19), and the other using the numerically-determined FvK solutions. On the one hand, the composite expression, (2.19) with $\nu = 0.25$ (the commonly accepted value for MoS_2 , Bertolazzi et al., 2011), is fitted using the nonlinear programming solver `fminsearch` in `MATLAB`, this gives the values $E_{2D}^{\text{com}} = 244.3 \text{ N/m}$ and $E_{2D}^{\text{com}} = 221.1 \text{ N/m}$ for the two indenter sizes $R_s/R_{\text{out}} = 0.076$ and $R_s/R_{\text{out}} = 0.033$, respectively. These values are significantly larger than the accepted value of 180 N/m (Bertolazzi et al., 2011), and so one might conclude that these MoS_2 flakes have an unusually large stretching modulus. On the other hand, if we instead fit the data using the numerically-determined FvK solutions, we find $E_{2D}^{\text{num}} = 199.7 \text{ N/m}$ and $E_{2D}^{\text{num}} = 189.5 \text{ N/m}$, which correspond to a difference of 10.9% and 5.27% between the fitted stretching modulus and the accepted value. (Here, the fitting of the numerical solutions is performed on the F/R_{out} versus δ/R_{out} curve using the method of least squares; that is, we compute the numerical solutions for $\nu = 0.25$, $\mathcal{B} = 0$, and various $\mathcal{R}_s \equiv R_s/R_{\text{out}} \times (T_{\text{pre}}/E_{2D})^{1/2}$, which we compare to the experimental data by calculating the sum of squared residuals, *i.e.* $\text{RSS} := \sum_i [F_i^{\text{exp}}/R_{\text{out}} - F(\delta_i^{\text{exp}}/R_{\text{out}})/R_{\text{out}}]^2$. We then find the stretching modulus, E_{2D} , and pre-tension, T_{pre} , that give the smallest value of RSS. The contour plots of the residuals are plotted in Fig. 2.5.)

Even worse are the fitted pre-tensions, which the values obtained from fitting with the composite and numerics differ by 38.9% and 35.6%, for the two indenter sizes $R_s/R_{\text{out}} = 0.076$ and $R_s/R_{\text{out}} = 0.033$, respectively. These large discrepancies are a result of inappropriately using the composite expression; however, it is unclear whether these are due to inaccuracies in the two individual asymptotic terms in (2.19) or a more general problem with using composite expressions (*e.g.* the fact that the composites do not correctly represent the solution at moderate displacements). It should also be pointed out that, even when using the numerical method, the fitted stretching moduli are still larger than the expected value: is this solely due to experimental error and within the expected error bounds, or some property of MoS_2 not predicted by linear elasticity? To address this we will need to consider the

⁴These experiments were performed by Yolanda Manzanares-Negro (Departamento de Física de la Materia Condensada, Universidad Autónoma de Madrid) and the data provided via personal communication.

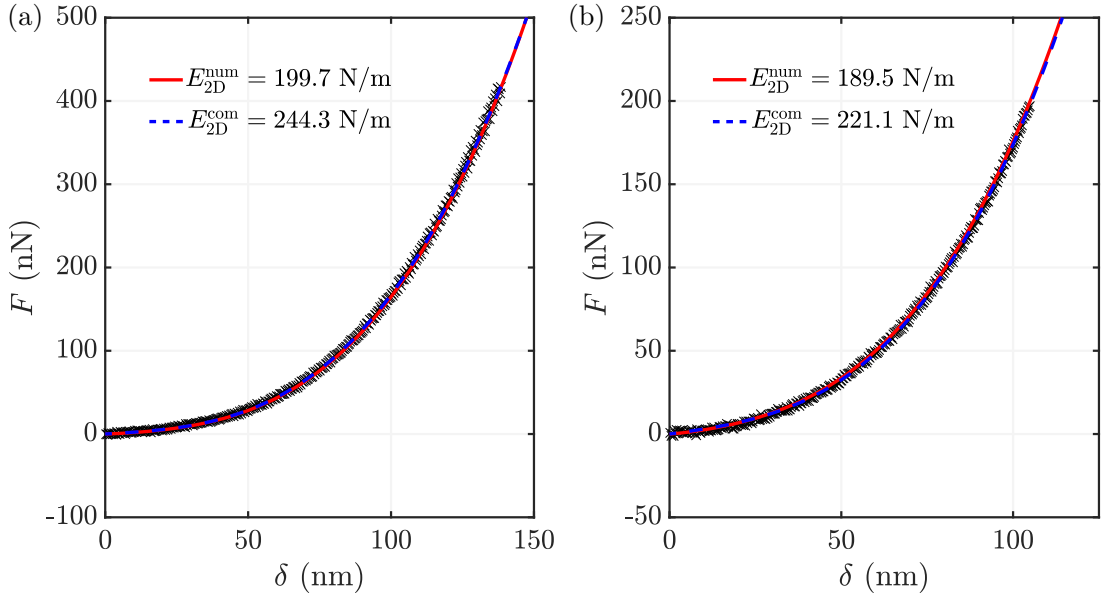


Figure 2.4: Experimental data obtained from the indentation of monolayer molybdenum disulphide (MoS_2) clamped at a radius (a) $R_{\text{out}} = 1250 \text{ nm}$ and (b) $R_{\text{out}} = 1195 \text{ nm}$ by a spherical-capped indenter of radius of curvature (a) $R_s = 95 \text{ nm}$ and (b) $R_s = 40 \text{ nm}$ are plotted as black crosses ($\times$). [This data is based on experiments performed by Yolanda Manzanares-Negro (Departamento de Física de la Materia Condensada, Universidad Autónoma de Madrid) and were provided via personal communication.] Fitting the data using the composite formula (2.19) with $\nu = 0.25$ (the commonly accepted value for MoS_2 , Bertolazzi et al., 2011) for the unknown pre-tension and stretching modulus yields the values: (a) $T_{\text{pre}}^{\text{com}} = 0.0661 \text{ N/m}$ and $E_{2D}^{\text{com}} = 244.3 \text{ N/m}$; (b) $T_{\text{pre}}^{\text{com}} = 0.0917 \text{ N/m}$ and $E_{2D}^{\text{com}} = 221.1 \text{ N/m}$ — the corresponding fitted curves are plotted as blue dashed curves. These fitted stretching moduli differ significantly from the commonly accepted value of $E_{2D} = 180 \text{ N/m}$ (Bertolazzi et al., 2011) and suggest a possible dependence of E_{2D} on the relative indenter size, R_s/R_{out} . In contrast, if the data is fitted using the numerical solutions of the Föppl–von Kármán equations, (2.11)–(2.13) with $\nu = 0.25$, $\mathcal{B} \equiv 0$, and various $\mathcal{R}_s = R_s/R_{\text{out}} \times (T_{\text{pre}}/E_{2D})^{1/2}$, the pre-tension and stretching modulus are found to be: (a) $T_{\text{pre}}^{\text{num}} = 0.1082 \text{ N/m}$ and $E_{2D}^{\text{num}} = 199.7 \text{ N/m}$; (b) $T_{\text{pre}}^{\text{num}} = 0.1423 \text{ N/m}$ and $E_{2D}^{\text{num}} = 189.5 \text{ N/m}$ — the corresponding fitted curves are plotted as red solid curves. (Here, the fitting has been performed using `fminsearch` in `MATLAB` for the composite, *i.e.* by minimizing the sum of the square residuals of the F/R_{out} against δ/R_{out} curve, *i.e.* $\text{RSS} := \sum_i [F_i^{\text{exp}}/R_{\text{out}} - F(\delta_i^{\text{exp}}/R_{\text{out}})/R_{\text{out}}]^2$, for the numerically determined force–displacement relationship $F(\delta)$. The associated residuals, RSS, are plotted in the contour plots in Fig. 2.5.)

effects of nonlinear elasticity (§3.3).

In experiments, a range of thin sheets with varying properties are probed by a variety of indenter shapes. With the errors presented here being dependent on the indenter size (and, hence, its geometry), it is important to ask what the effects of the geometry are, and how can they be distinguished from the actual material response. It is also apparent that the errors

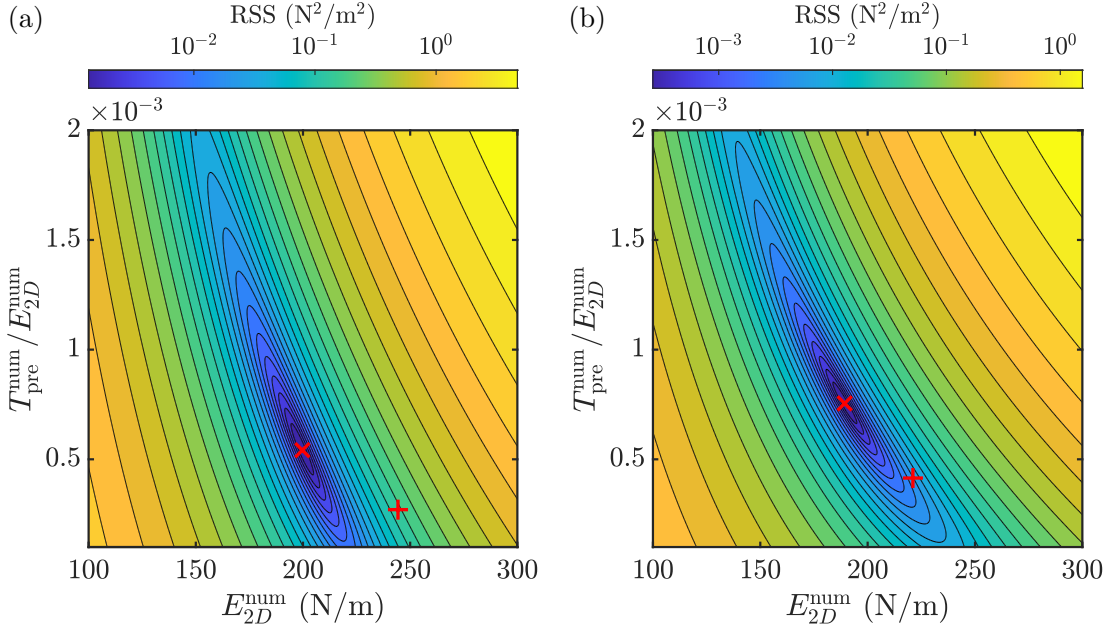


Figure 2.5: Contour plots for the residual sum of squares, $\text{RSS} := \sum_i [F_i^{\text{exp}}/R_{\text{out}} - F(\delta_i^{\text{exp}}/R_{\text{out}})/R_{\text{out}}]^2$, for the numerical fitting of the experimental data plotted in Fig. 2.4; here, plots (a) and (b) correspond to the results plotted in (a) and (b) of Fig. 2.4, respectively. The fitted stretching modulus, E_{2D}^{num} , is plotted along the x -axis while the fitted pre-strain, $T_{\text{pre}}^{\text{num}}/E_{2D}^{\text{num}}$, is plotted along the y -axis. By plotting in this way, horizontal lines correspond to a constant value of $\mathcal{R}_s \equiv R_s/R_{\text{out}} \times (T_{\text{pre}}/E_{2D})^{1/2}$ and, hence, to a single numerical solution — multiple solutions with various $\mathcal{R}_s$ (*i.e.* various T_{pre}/E_{2D}) are required to fit experiments with a spherical-capped indenter. (Note that the residuals have been interpolated in T_{pre}/Y using cubic interpolation to give a smooth contour plot, though the interpolated values were not used in the fitting.) Plotted as red crosses ($\times$) are the parameters that give the minimum residual: (a) $T_{\text{pre}}^{\text{num}} = 0.1082 \text{ N/m}$ and $E_{2D}^{\text{num}} = 199.7 \text{ N/m}$; (b) $T_{\text{pre}}^{\text{num}} = 0.1423 \text{ N/m}$ and $E_{2D}^{\text{num}} = 189.5 \text{ N/m}$. The parameters found by fitting the data to the composite expression (2.19) with $\nu = 0.25$, (a) $T_{\text{pre}}^{\text{com}} = 0.0661 \text{ N/m}$ and $E_{2D}^{\text{com}} = 244.3 \text{ N/m}$; (b) $T_{\text{pre}}^{\text{com}} = 0.0917 \text{ N/m}$ and $E_{2D}^{\text{com}} = 221.1 \text{ N/m}$, are indicted by red pluses (+) for reference, and correspond to significantly larger residuals.

associated with current fitting protocols are, at least in part, due to the inappropriate use of asymptotic expressions. Thus, we now turn to formally derive the asymptotic solutions for cylindrical and spherical-capped indenters. Our aim is to quantitatively understand the potential pitfalls of fitting protocols and address how they can be alleviated. Given the recent experimental results with graphene (*e.g.* López-Polín et al., 2015), we have a special interest in how geometrical nonlinearity can be distinguished from material nonlinearity when fitting mechanical properties of thin elastic sheets.

Indentation of suspended two-dimensional solids: Geometry versus material nonlinearity

In the previous chapter, we illustrated some of the limitations of fitting protocols that are often used to infer properties of suspended membranes from indentation experiments: experiments have been interpreted as showing anomalously large moduli, but also use asymptotic results inappropriately. We showed that using numerical solutions of a more comprehensive model based on the Föppl–von Kármán equations relieves some of these issues. In §3.1 and §3.2, we derive novel asymptotic solutions, and unify existing asymptotic solutions, for the force–displacement relation, $F(\delta)$, with a cylindrical and spherical-capped indenter, respectively. Our aim is to develop an understanding of when, and how, the effects of realistic indenter geometry is important. We are ultimately interested in how these effects can be distinguished from material nonlinearity, which has been claimed in the past (see [Nicholl et al., 2015](#); [López-Polín et al., 2017](#), for example). Thus, in §3.3, we introduce a model that describes the indentation of a nonlinear hyperelastic material (*i.e.* non-Hookean behaviour). We discuss the significance of our results for the experimental determination of elastic constants in §3.4 and then summarize our results and conclude in §3.5.¹

3.1 Cylindrical indentation of a linearly elastic sheet

In §2.3, we introduced the Föppl–von Kármán formulation for the indentation of a linearly elastic sheet under the Kirchhoff assumptions. This formulation retains the leading-order geometrical nonlinearities (for small slopes and rotations) but assumes the material behaves according to linear elasticity. In the case of a cylindrical indenter, numerical solutions to (2.10) subject to (2.11) and (2.12) describe the dimensionless force–displacement response, $\mathcal{F}(d)$, for a given dimensionless bending stiffness, $\mathcal{B}$; cylinder radius, $\mathcal{R}$; and Poisson’s ratio, ν . In this section, we make analytical progress by considering, separately, two asymptotic limits that allow for simplifications: (i) small deflections from the pre-stretched state and

¹This chapter is based on the published work by [Chandler and Vella \(2020a\)](#).

(ii) negligible bending stiffness. In §3.1.1, we consider the first of these by considering perturbations to the isotropic pre-tensed state ($\sigma_{rr} \sim \sigma_{\theta\theta} \sim T_{\text{pre}}$ or, in dimensionless variables, $\Sigma_{rr} \sim \Sigma_{\theta\theta} \sim 1$), thereby extending the work of *e.g.* Jennings et al. (1995); Wan et al. (2003); in §3.1.2, we consider the second case by focusing on the zero bending stiffness limit ($\mathcal{B} = 0$), similar to *e.g.* Bhatia and Nachbar (1968); Vella and Davidovitch (2017). This analysis allows us to reproduce some previously known results in a systematic way, whilst also uncovering new results in some regimes; we discuss the broader context of these results as they are derived.

3.1.1 Small indentation forces ($\mathcal{F} \ll \max\{\mathcal{B}^{1/2}, \mathcal{R}\}$)

For sufficiently small indentations, the stretching-induced stress of the sheet is negligible compared to the isotropic pre-tension and the effect of the (also small) bending stiffness, $\mathcal{B} \ll 1$. To investigate how bending and pre-tension interact, we follow the approach of *e.g.* Vella et al. (2012); Box et al. (2017) and linearize the governing equations, (2.10), about the initial pre-tensed state: we let $\Psi(\rho) \sim \rho + \tilde{\Psi}(\rho)$ and $Z(\rho) \sim 0 + \tilde{Z}(\rho)$ for $\tilde{\Psi}, \tilde{Z} \ll 1$. The equations describing vertical force balance and compatibility in (2.10) reduce to a single third-order differential equation for $\tilde{Z}$:

$$\mathcal{B}\rho \frac{d}{d\rho} \left[\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d\tilde{Z}}{d\rho} \right) \right] = \rho \frac{d\tilde{Z}}{d\rho} - \mathcal{F}. \quad (3.1)$$

This can be solved by a linear combination of logarithms and modified Bessel functions, see (3.A.2). Applying the boundary conditions, (2.11) and (2.12), we obtain an explicit relationship between the force and indentation depth that is linear (*i.e.* $\mathcal{F} \propto d$). We express this relationship through the (constant) compliance,

$$\frac{d}{\mathcal{F}} = \log \frac{1}{\mathcal{R}} + \frac{\hat{K}_1 \hat{I}_0^R + \hat{I}_1 \hat{K}_0^R + \mathcal{R} (\hat{K}_0 \hat{I}_1^R + \hat{I}_0 \hat{K}_1^R) - 2\mathcal{B}^{1/2} \mathcal{B}^{1/2}}{\hat{I}_1^R \hat{K}_1 - \hat{I}_1 \hat{K}_1^R} \frac{\mathcal{B}^{1/2}}{\mathcal{R}}, \quad (3.2)$$

where

$$\hat{I}_j := I_j(\mathcal{B}^{-1/2}), \quad \hat{I}_j^R := I_j(\mathcal{R}\mathcal{B}^{-1/2}), \quad (3.3a,b)$$

$$\hat{K}_j := K_j(\mathcal{B}^{-1/2}), \quad \hat{K}_j^R := K_j(\mathcal{R}\mathcal{B}^{-1/2}), \quad (3.3c,d)$$

and $I_j(x)$ and $K_j(x)$ are the j th-order modified Bessel functions of the first and second kinds, respectively (Abramowitz and Stegun, 1964).

Equation (3.2) corresponds to a constant compliance regime: the instantaneous compliance, $d/\mathcal{F}$, is a function of the indenter radius, $\mathcal{R}$, and bending stiffness, $\mathcal{B}$, only. One

could use this solution as an explicit formula to describe the small indentation compliance of a tense clamped plate. As a tool for inferring the bending stiffness from an experimental measure of the compliance, however, the complexity of this equation is daunting since the various dimensionless quantities are coupled within Bessel functions. Instead, recall that we are specifically interested in the limit of small bending stiffnesses, $\mathcal{B} \ll 1$, and note, from (3.3b,d), that $\mathcal{B}/\mathcal{R}^2$ is a key parameter; therefore, we consider separately the cases $\mathcal{B} \ll \mathcal{R}^2 < 1$ and $\mathcal{R}^2 \ll \mathcal{B} \ll 1$. We find the leading-order results

$$\frac{d}{\mathcal{F}} \sim \begin{cases} \log \frac{e^\gamma}{2\mathcal{B}^{1/2}} & \text{for } \mathcal{R}^2 \ll \mathcal{B} \ll 1, \\ \log \frac{1}{\mathcal{R}} & \text{for } \mathcal{B} \ll \mathcal{R}^2 < 1, \end{cases} \quad (3.4)$$

where $\gamma \approx 0.577$ is the Euler–Mascheroni constant (Abramowitz and Stegun, 1964).

The evolution of the compliance with indentation force is shown in Fig. 3.1 for a fixed $\mathcal{R}$ and four values of $\mathcal{B}$; for small $\mathcal{F}$, we see that the numerical results agree with our asymptotic results. In particular, note that the results for $\mathcal{B} = 10^{-8}$ and $\mathcal{B} = 10^{-9}$ are essentially indistinguishable at the scale of the plot: in both cases, $\mathcal{R}^2 = 10^{-6} \gg \mathcal{B}$ and so the small-indentation compliance is controlled by the indenter radius, rather than the bending stiffness, as predicted by (3.4). In these cases, the compliance is essentially indistinguishable from that of an ideal membrane with $\mathcal{B} = 0$. However, the results for $\mathcal{B} = 10^{-6}$ and $\mathcal{B} = 10^{-7}$ both show different compliances, indicating an effect of small, but finite, $\mathcal{B}$.

Overall, for small displacements by a cylindrical indenter ($\mathcal{F} \ll \max\{\mathcal{B}^{1/2}, \mathcal{R}\}$), the indentation compliance, $d/\mathcal{F}$, is a constant determined by the relative size of the indenter radius, $\mathcal{R}$, and the bending stiffness, $\mathcal{B}^{1/2}$. We also note that the fact that the form of the stiffness in each of the cases in (3.4) is functionally similar — both are logarithmic — is not a coincidence: in the bending-dominated case, the sheet is approximately flat over a region of horizontal scale $\mathcal{B}^{1/2}$ and it is as if the sheet were deformed by a virtual cylindrical indenter of radius $\mathcal{R} \sim 2e^{-\gamma}\mathcal{B}^{1/2}$.

While the indenter-dominated solution, (3.4) with $\mathcal{B} \ll \mathcal{R}^2 < 1$, has been previously derived (see Jennings et al., 1995, for example), we believe that the bending-dominated solution, (3.4) with $\mathcal{R}^2 \ll \mathcal{B} \ll 1$, is novel; this describes scenarios when the bending stiffness is small, but cannot be neglected because the indenter size is smaller than the virtual bending-induced indenter radius, which is, therefore, the relevant length scale.

3.1.2 Small bending stiffnesses ($\mathcal{B}^{1/2} \ll \max\{\mathcal{F}, \mathcal{R}\}$)

In many experiments the dimensionless bending stiffness, $\mathcal{B}$, is small enough that the sheet can be modelled as a thin membrane (though knowing this without knowing T_{pre} is difficult).

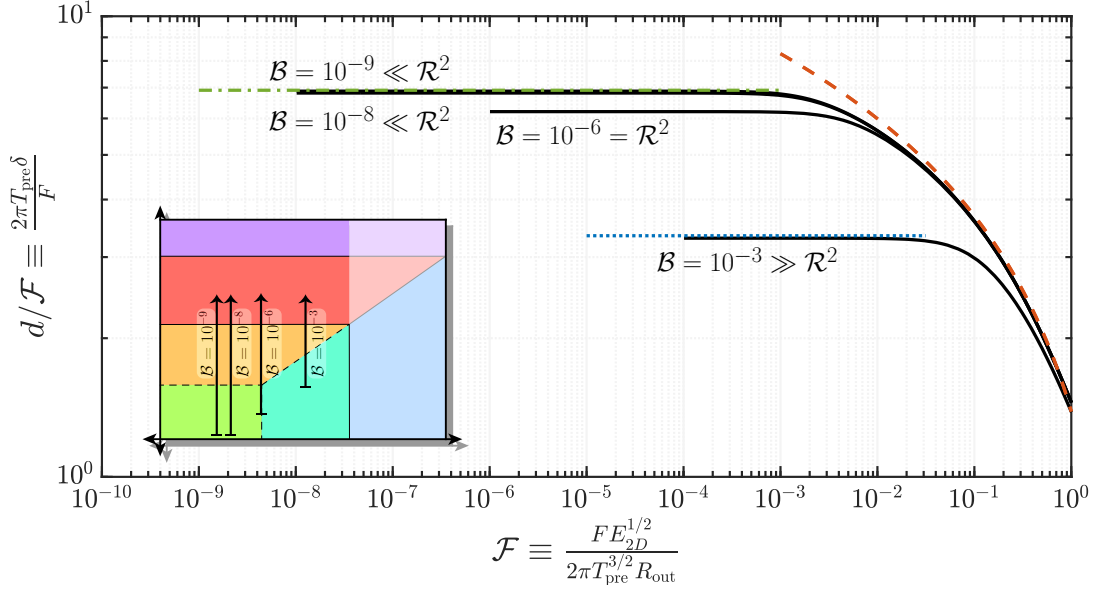


Figure 3.1: The evolution of the instantaneous linear compliance as a function of imposed force, $\mathcal{F}$, for sheets with different bending stiffnesses ($\mathcal{B} = 10^{-3}, 10^{-6}, 10^{-8}, 10^{-9}$) and $\nu = 1/3$. Results are shown for a cylindrical indenter with $\mathcal{R} = 10^{-3}$. Numerical solutions to the FvK equations are shown as black solid curves while the corresponding asymptotic solutions are shown as broken curves as follows: (3.4) with $\mathcal{R}^2 \ll \mathcal{B}$ in blue (dotted), (3.7) with $\mathcal{F} \ll \mathcal{R}$ in green (dash-dotted), and (3.7) with $\mathcal{R} \ll \mathcal{F}$ in red (dashed). The inset shows the routes through the $(\mathcal{B}, \mathcal{F})$ -parameter space taken by indentation in each case — see the regime diagram in Fig. 3.7(a) .

In this case, we can simplify the model by taking $\mathcal{B} \rightarrow 0$ in (2.10a) and dropping the highest-order boundary conditions, (2.11b) and (2.12b) — ignoring the effects of the edge boundary layers, which occur over a typical length scale $\rho \sim \mathcal{B}^{1/2}$. The resulting formulation is equivalent to that presented by Vella and Davidovitch (2017), but with the assumption of a perfect-slip, instead of no-slip, indenter. The following asymptotic analysis is, therefore, very close to that presented by Vella and Davidovitch (2017); however, presenting it here will be helpful for discussing the novel case for a spherical-capped indenter (§3.2).

Vella and Davidovitch (2017, App. B) solved the membrane problem for a point indenter ($\mathcal{R} = 0$); we extend their work by solving the finite cylinder case (the full derivation can be found in Appendix 3.A.2). Ultimately, we obtain the parametric force–displacement relation

$$d = 2 \frac{\sinh^{-1} \Phi_1^{1/2} - \sinh^{-1} \Phi_0^{1/2}}{A(\Phi_0, \Phi_1; \mathcal{R}, \nu)^{1/2}}, \quad \mathcal{F} = \frac{2}{\mathcal{R}^2} \frac{\Phi_0(1 + \Phi_0^{-1})^{-1/2}}{A(\Phi_0, \Phi_1; \mathcal{R}, \nu)^{3/2}}, \quad (3.5a,b)$$

$$A(\Phi_0, \Phi_1; \mathcal{R}, \nu) := \frac{2}{1 - \nu} \frac{\Phi_0}{\mathcal{R}^2} \sqrt{\frac{1 + \Phi_1^{-1}}{1 + \Phi_0^{-1}} - \frac{1 + \nu}{1 - \nu} \Phi_1}, \quad (3.5c)$$

which is parametrized by the boundary stresses $\Phi_1 := A\Psi(1)$ and $\Phi_0 := A\mathcal{R}\Psi(\mathcal{R})$, these ‘stresses’ must further satisfy the equation

$$\frac{\Phi_1^{3/2}}{(1 + \Phi_1)^{1/2}} - \mathcal{R}^{-2} \frac{\Phi_0^{3/2}}{(1 + \Phi_0)^{1/2}} = \left[\sinh^{-1} \Phi^{1/2} - (1 + \Phi^{-1})^{-1/2} \right]_{\Phi_0}^{\Phi_1}. \quad (3.5d)$$

For a given indentation force, $\mathcal{F}$, (3.5b) and (3.5d) form a pair of equations for the two unknowns Φ_0 and Φ_1 . Equation (3.5a), therefore, yields an implicit force–displacement relation, $d(\mathcal{F}; \mathcal{R}, \nu)$. In the limit of the inner stress (or, equivalently, the radius) being taken to zero, $\Phi_0 = \mathcal{O}(\mathcal{R}^{4/3}) \rightarrow 0$, this system yields the point-indenter result of Vella and Davidovitch (2017, eq. (12)–(14)).

To make further analytical progress with a finite indenter radius, we concentrate on the two asymptotic limits $\mathcal{F} \rightarrow 0$ (§3.1.2.1) and $\mathcal{F} \rightarrow \infty$ (§3.1.2.2). This allows us to determine explicit forms that are formally valid only in these limits, but that might be expected to apply more broadly.

3.1.2.1 Moderate indentation forces ($\mathcal{F} \ll 1$)

The small indentation limit, $\mathcal{F} \rightarrow 0$, is directly equivalent to taking $A(\Phi_0, \Phi_1) \rightarrow \infty$; consequently, we find that $A \sim \Phi_1 = 4/\mathcal{F}^2 + \mathcal{O}(1)$ as $\mathcal{F} \rightarrow 0$. Inserting this expression into (3.5) gives the implicit force–displacement relation

$$\frac{d}{\mathcal{F}} = \log \frac{4}{\mathcal{F}} - \sinh^{-1} \Phi_0^{1/2} + \mathcal{O}(\mathcal{F}^2), \quad (3.6a)$$

where Φ_0 is given by

$$\mathcal{R}^{-2} \frac{\Phi_0^{3/2}}{(1 + \Phi_0)^{1/2}} = \frac{4}{\mathcal{F}^2} + \mathcal{O}(1). \quad (3.6b)$$

The parameter Φ_0 can be eliminated from (3.6) to give an explicit equation for d in terms of $\mathcal{F}$; however, the result is complicated and does not readily reveal the limiting behaviours. Instead, we consider the sub-cases $\mathcal{F} \ll \mathcal{R}$ and $\mathcal{F} \gg \mathcal{R}$, which correspond to taking $\Phi_0 \rightarrow \infty$ and $\Phi_0 \rightarrow 0$, respectively.

At leading-order, we find that

$$\frac{d}{\mathcal{F}} \sim \begin{cases} \log \frac{1}{\mathcal{R}} & \text{for } \mathcal{F} \ll \mathcal{R} < 1, \\ \log \frac{4}{\mathcal{F}} & \text{for } \mathcal{R} \ll \mathcal{F} \ll 1. \end{cases} \quad (3.7)$$

The very small displacement solution in (3.7) (found when $\mathcal{F} \ll \mathcal{R}$) matches precisely with our earlier solution from (3.4) with $\mathcal{B} \ll \mathcal{R}^2$. Moreover, the moderate displacement solution in (3.7) (found when $\mathcal{R} \ll \mathcal{F} \ll 1$) is that presented by Vella and Davidovitch (2017) for a

point indenter, and is shown as the red-dashed curve in Fig. 3.1. In summary, for a cylinder, as the indentation force increases the indentation compliance, $d/\mathcal{F}$, evolves from being a constant (controlled by the indenter radius) to a logarithmic behaviour.

3.1.2.2 Large indentation forces ($\mathcal{F} \gg 1$)

The large indentation limit, $\mathcal{F} \rightarrow \infty$, is equivalent to taking $A(\Phi_0, \Phi_1)/\Phi_1 \rightarrow 0$; consequently, we find that $A = \mathcal{O}(\mathcal{F}^{-2/3})$ and $\Phi_1 = \mathcal{O}(1)$ as $\mathcal{F} \rightarrow \infty$. Inserting these into (3.5) gives the force–displacement relation

$$\frac{d}{\mathcal{F}^{1/3}} = \frac{2}{(1+\nu)^{1/3}} \frac{\sinh^{-1} \Phi_1^{1/2} - \sinh^{-1} \Phi_0^{1/2}}{\Phi_1^{1/2} (1+\Phi_1)^{-1/6}} + \mathcal{O}(\mathcal{F}^{-2/3}), \quad (3.8a)$$

where Φ_1 and Φ_0 are the solutions to

$$\mathcal{R}^{-2} \frac{\Phi_0^{3/2}}{(1+\Phi_0)^{1/2}} = \frac{(1+\nu)}{2} \frac{\Phi_1^{3/2}}{(1+\Phi_1)^{1/2}} + \mathcal{O}(\mathcal{F}^{-2/3}), \quad (3.8b)$$

$$\frac{1-\nu}{2} \Phi_1 = \frac{(1+\Phi_1)^{1/2}}{\Phi_1^{1/2}} \left[\sinh^{-1} \Phi_1^{1/2} - \left(1+\Phi_1^{-1}\right)^{-1/2} \right]_{\Phi_0}^{\Phi_1} + \mathcal{O}(\mathcal{F}^{-2/3}). \quad (3.8c)$$

The expression in (3.8a) corresponds to a constant cubic compliance: $d/\mathcal{F}^{1/3}$ is a function of the indenter size, $\mathcal{R}$, and Poisson’s ratio, ν , alone.

In experiments, the indenter is often orders of magnitude smaller than the sheet clamping radius, so $\mathcal{R} \ll 1$ (see Lee et al., 2008, for example); this limit corresponds to $\Phi_0 = \mathcal{O}(\mathcal{R}^{4/3}) \rightarrow 0$ and so the cylindrical result, (3.8), simplifies to

$$\frac{d}{\mathcal{F}^{1/3}} = \frac{2}{(1+\nu)^{1/3}} \frac{\sinh^{-1} \Phi_1^{1/2}}{\Phi_1^{1/2} (1+\Phi_1)^{-1/6}} - (2\mathcal{R})^{2/3} + \mathcal{O}(\mathcal{F}^{-2/3}, \mathcal{R}^2), \quad (3.9a)$$

where Φ_1 is the solution to

$$1 + \frac{1-\nu}{2} \Phi_1 = \frac{(1+\Phi_1)^{1/2}}{\Phi_1^{1/2}} \sinh^{-1} \Phi_1^{1/2} + \mathcal{O}(\mathcal{F}^{-2/3}, \mathcal{R}^2). \quad (3.9b)$$

This is the point-indenter result from Vella and Davidovitch (2017) with an additional term, the $(2\mathcal{R})^{2/3}$ term on the right-hand side of (3.9a), to account for the small (but finite) indenter size. Overall, we find that the cubic compliance

$$\frac{d}{\mathcal{F}^{1/3}} \sim q_c(\nu, \mathcal{R}) = q_p(\nu) - (2\mathcal{R})^{2/3} + \mathcal{O}(\mathcal{R}^2), \quad (3.10)$$

where $q_c(\nu, \mathcal{R})$ and $q_p(\nu)$ are given by the leading-order equations in (3.8) and (3.9),

respectively. Thus, for large indentations by a cylindrical indenter ($\mathcal{B}^{1/2} \ll 1 \ll \mathcal{F}$), the cubic compliance, $d/\mathcal{F}^{1/3}$, is a constant controlled by the sheet's Poisson's ratio, ν , and indenter radius, $\mathcal{R}$. An analogous perturbative result was found by Vella and Davidovitch (2017, eq. (48)) for the case of a no-slip indenter: $d/\mathcal{F}^{1/3} \sim [2\pi/\alpha_0(\nu)]^{1/3} - [8\mathcal{R}^2/(1+\nu)]^{1/3}$ where $[2\pi/\alpha_0(\nu)]^{1/3} \equiv q_p(\nu)$. Figure 3.2 shows the comparison between numerical simulations, the prediction for a point indenter ($\mathcal{R} = 0$), and the expression (3.10) for $\mathcal{R} = 0.1$. We see that the effect of finite indenter size is non-negligible in the large force limit.

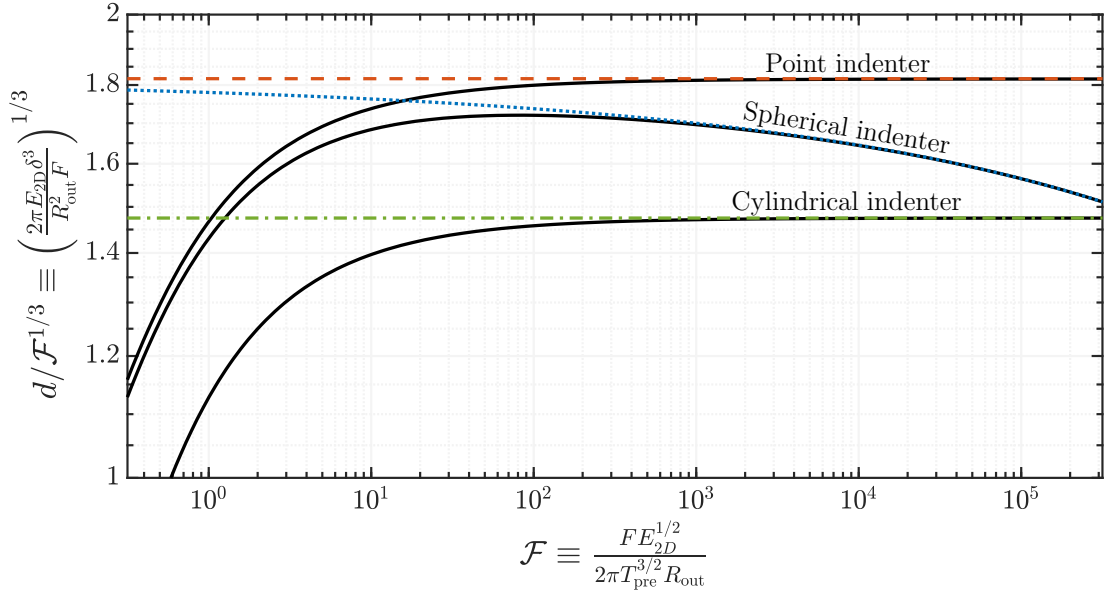


Figure 3.2: The cubic compliance associated with large indentations of a linearly elastic membrane ($\nu = 1/3$, $\mathcal{B} = 0$) for indenters of different type: results are shown for point ($\mathcal{R} \rightarrow 0$), cylindrical ($\mathcal{R} = 10^{-1}$), and spherical-capped ($\mathcal{R}_s = 10^{-3}$) indenters. Numerical solutions of the FvK equations are shown as solid black curves while the corresponding asymptotic solutions are shown as broken curves as follows: (3.10) with $\mathcal{R} = 0$ in red (dashed), (3.13a) in blue (dotted), and (3.10) in green (dash-dotted). Observe the difference made by geometry: results for a spherical indenter never reach the constant cubic compliance regime, $d/\mathcal{F}^{1/3} = \text{const}$, while even for cylindrical indenters the finite indenter size can significantly alter the constant cubic compliance.

The asymptotic predictions (3.4), (3.7), and (3.10) are summarized in §3.5 and presented in the regime diagram Fig. 3.7(a). We, now, move on from the case of a cylindrical indenter to one with a spherical tip.

3.2 Spherical indentation of a linearly elastic sheet

Having considered the simplest case of a finite cylindrical indenter in some detail, we now move on to a case of more experimental relevance: an indenter with a spherically-capped tip (see Bhatia and Nachbar, 1968; Jennings et al., 1995; Begley and Mackin, 2004; Lee

et al., 2008; López-Polín et al., 2017, for example). For such an indenter, (2.10) subject to (2.11) and (2.13) describe the dimensionless force–displacement response, $\mathcal{F}(d)$, for given dimensionless bending stiffness, $\mathcal{B}$; sphere radius, $\mathcal{R}_s$; and Poisson’s ratio, ν . To make analytical progress we apply the same asymptotic simplifications as in the cylindrical case: (i) linearize around the pre-stretched base state, *i.e.* take $\Psi(\rho) \sim \rho + \tilde{\Psi}(\rho)$ and $Z(\rho) \sim 0 + \tilde{Z}$ for $\tilde{\Psi}, \tilde{Z} \ll 1$; and (ii) consider the membrane theory limit, *i.e.* take $\mathcal{B} \rightarrow 0$ and drop the boundary conditions (2.11b) and (2.13c). The analysis of these asymptotic limits are analogous to the cylindrical indenter case presented in §3.1.1 and §3.1.2, but with the added detail that the dimensionless radius of contact, $\mathcal{R}$, is unknown; we shall only present the final results below (the full analysis can be found in Appendix 3.A.1, for small indentation forces, and Appendix 3.A.2, for small bending stiffnesses).

3.2.1 Small indentation forces ($\mathcal{F} \ll \max\{\mathcal{B}^{1/2}, \mathcal{R}_s\}$)

Linearizing around the pre-tensed state, we find that

$$\frac{d}{\mathcal{F}} \sim \begin{cases} \log \frac{e^\gamma}{2\mathcal{B}^{1/2}} & \text{for } \mathcal{R}_s\mathcal{F} \ll \mathcal{B} \ll 1, \\ \log \sqrt{\frac{e}{\mathcal{R}_s\mathcal{F}}} & \text{for } \mathcal{B} \ll \mathcal{R}_s\mathcal{F} < 1, \end{cases} \quad (3.11a)$$

with

$$\mathcal{R}^2 \sim \begin{cases} 4\mathcal{B}e^{-2\gamma} \exp\left(-\frac{4\mathcal{B}}{\mathcal{R}_s\mathcal{F}}\right) & \text{for } \mathcal{R}_s\mathcal{F} \ll \mathcal{B} \ll 1, \\ \mathcal{R}_s\mathcal{F} & \text{for } \mathcal{B} \ll \mathcal{R}_s\mathcal{F} < 1, \end{cases} \quad (3.11b)$$

where $\gamma \approx 0.577$ is, again, the Euler–Mascheroni constant. Thus, for small displacements by a spherical-capped indenter ($\mathcal{F} \ll \max\{\mathcal{B}^{1/2}, \mathcal{R}_s\}$), the instantaneous indentation compliance, $d/\mathcal{F}$, evolves from being a constant (controlled by the bending-induced radius discussed in the cylindrical indenter problem) to a logarithmic behaviour (controlled by the radius of contact, which, in turn, depends on the force). This evolution is shown in Fig. 3.3 and demonstrates excellent agreement with the numerical results in the relevant limits.

The asymptotic compliance, (3.11a), is directly equivalent to that for a cylindrical indenter, *i.e.* (3.4), and can be recovered by accounting for the extra indentation depth due to the sphere geometry — $d \mapsto d + \mathcal{R}^2/2\mathcal{R}_s$, *i.e.* compare (2.12a) and (2.13a) — and inserting the contact radius expression, (3.11b). It should be noted that the indenter-dominated solution, (3.11) with $\mathcal{B} \ll \mathcal{R}_s\mathcal{F}$, was also given by Bhatia and Nachbar (1968, eq (43b)), from analytical solutions to small-rotation Reissner theory (analogous to FvK here), and Norouzi et al. (2006, eq. (14)), by minimizing the energy of the constant-tension problem.

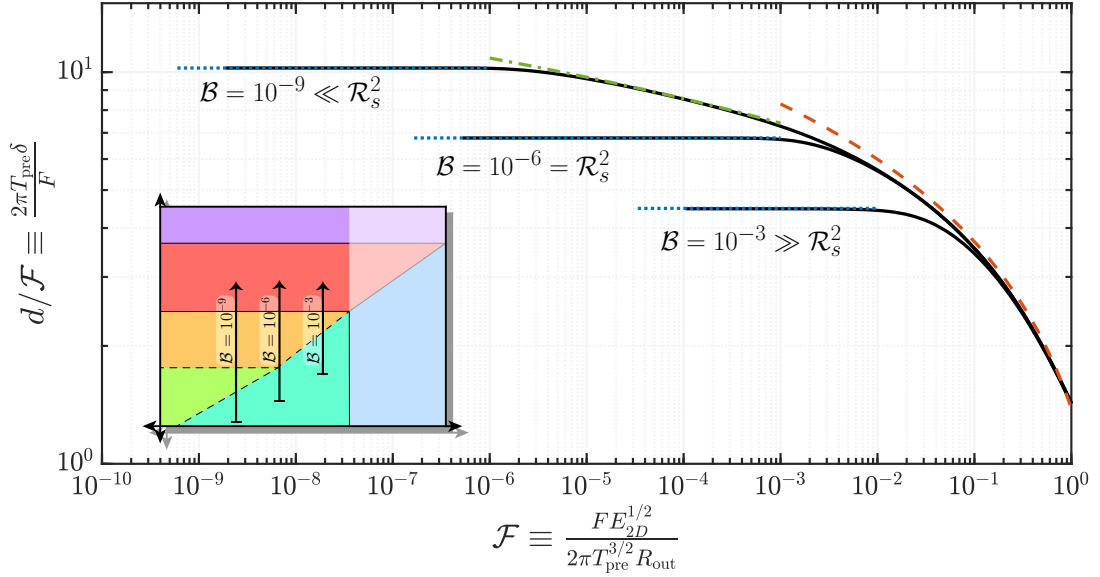


Figure 3.3: The evolution of the instantaneous linear compliance as a function of imposed force, $\mathcal{F}$, for sheets with different bending stiffnesses ($\mathcal{B} = 10^{-3}, 10^{-6}, 10^{-8}, 10^{-9}$) and Poisson's ratio $\nu = 1/3$. Results are shown for a spherical indenter with $\mathcal{R}_s = 10^{-3}$. Numerical solutions to the FvK equations are shown as black solid curves while the corresponding asymptotic solutions are shown as broken curves as follows: (3.11a) with $\mathcal{R}_s \mathcal{F} \ll \mathcal{B}$ in blue (dotted), (3.11a) with $\mathcal{B} \ll \mathcal{R}_s \mathcal{F}$ in green (dash-dotted), and (3.12a) with $\mathcal{R}_s \ll \mathcal{F}$ in red (dashed). The inset shows the routes through the $(\mathcal{B}, \mathcal{F})$ -parameter space taken by indentation in each case — see the regime diagram in Fig. 3.7(b).

3.2.2 Moderate indentation forces ($\min\{\mathcal{B}^{1/2}, \mathcal{B}/\mathcal{R}_s\} \ll \mathcal{F} \ll 1$)

Taking the membrane theory limit and assuming small indentation forces ($\mathcal{F} \ll 1$) we find that

$$\frac{d}{\mathcal{F}} \sim \begin{cases} \log \sqrt{\frac{e}{\mathcal{R}_s \mathcal{F}}} & \text{for } \mathcal{B}/\mathcal{R}_s \ll \mathcal{F} \ll \mathcal{R}_s \ll 1, \\ \log \frac{4}{\mathcal{F}} & \text{for } \max\{\mathcal{R}_s, \mathcal{B}^{1/2}\} \ll \mathcal{F} \ll 1, \end{cases} \quad (3.12a)$$

with

$$\mathcal{R}^2 \sim \begin{cases} \mathcal{R}_s \mathcal{F} & \text{for } \mathcal{B}/\mathcal{R}_s \ll \mathcal{F} \ll \mathcal{R}_s \ll 1, \\ 4(\sqrt{2} - 1)\sqrt{\mathcal{R}_s^3 \mathcal{F}} & \text{for } \max\{\mathcal{R}_s, \mathcal{B}^{1/2}\} \ll \mathcal{F} \ll 1. \end{cases} \quad (3.12b)$$

Thus, for moderate indentations by a spherical-capped indenter ($\min\{\mathcal{B}^{1/2}, \mathcal{B}/\mathcal{R}_s\} \ll \mathcal{F} \ll 1$), the instantaneous indentation compliance, $d/\mathcal{F}$, evolves from one logarithmic behaviour (controlled by the radius of contact) to another (controlled by a stretching-induced radius); these asymptotic results are also shown in Fig. 3.3.

Solution (3.12) is directly equivalent to the cylindrical result, (3.7), and can again be obtained by accounting for the extra indentation depth due to the sphere geometry —

$d \mapsto d + \mathcal{R}^2/2\mathcal{R}_s$, *i.e.* compare (2.12a) and (2.13a) — and inserting the contact radius expression, (3.12b).

3.2.3 Large indentation forces ($\mathcal{F} \gg 1$)

Taking the membrane theory limit and assuming large indentation forces ($\mathcal{F} \gg 1$), we find that

$$\frac{d}{\mathcal{F}^{1/3}} \sim q_s(\nu, \mathcal{R}_s^3 \mathcal{F}) = q_p(\nu) - 2(2 - \sqrt{2})(\mathcal{R}_s^3 \mathcal{F})^{1/6} + \mathcal{O}(\sqrt{\mathcal{R}_s^3 \mathcal{F}}), \quad (3.13a)$$

with

$$\mathcal{R}^2 \sim 4(\sqrt{2} - 1)\sqrt{\mathcal{R}_s^3 \mathcal{F}}. \quad (3.13b)$$

Note that this result suggests a different effect of the geometry of the indenter than that proposed by Jia and Ben Amar (2020) who proposed a constant cubic response, $\mathcal{F} \sim kd^3/(2\pi)$ with the cubic stiffness $k := 3.62 + 5.52R_s/R_{\text{out}}$. Jia and Ben Amar assumed an ansatz $k = c_1 + c_2R_s/R_{\text{out}}$ and determined the constants, c_1 and c_2 , by fitting this to their numerical solutions of the indentation of an incompressible, Neo-Hookean material with zero pre-tension ($\nu = 1/2$ and $T_{\text{pre}} = 0$). We also emphasize that the result (3.13) is different from the results presented by Begley and Mackin (2004) who neglect azimuthal strain and axial force balance to facilitate an approximate solution of the governing equations; in our notation, this leads them to identify a constant cubic compliance that is dependent on R_s/R_{out} as well. Instead, our asymptotic analysis shows that the cubic compliance is *not* constant, but has a correction at $\mathcal{O}(\mathcal{R}_s^{1/2} \mathcal{F}^{1/6})$ that makes it weakly dependent on both the force and the indenter radius.

Above, we have implicitly assumed that $\mathcal{R}_s^3 \mathcal{F} \ll 1$, which must be true for the assumption of small slopes to hold ($\mathcal{R} \ll 1$). Thus, for large indentations by a spherical indenter ($\mathcal{F} \gg 1$), the cubic compliance, $d/\mathcal{F}^{1/3}$, is controlled by the Poisson's ratio of the sheet, ν , and contact radius, $\mathcal{R} \propto (\mathcal{R}_s^3 \mathcal{F})^{1/4}$. This is shown in Fig. 3.2, and demonstrates that, in the spherical case, the cubic compliance never saturates at a constant value (even though it does for a cylindrical indenter). Nevertheless, the asymptotic prediction for the cubic compliance, (3.13), is in good agreement with the numerically-observed behaviour.

To our knowledge, the solution (3.13) is new. Moreover, while it is functionally similar to the corresponding cylindrical result, (3.10), it cannot be obtained by accounting for the extra indentation depth ($d \mapsto d + \mathcal{R}^2/2\mathcal{R}_s$) and inserting the contact radius expression, (3.13b), as was possible for small indentation depths. This difference is because the stress in the portion of the membrane that is in contact with the indenter is different between the two models — compare the stress boundary conditions at the edge of the contact regions, (2.12c) and (2.13d). The asymptotic results (3.11a), (3.12a), and (3.13a) are summarized in §3.5 and presented in the regime diagram Fig. 3.7(b).

3.2.4 Numerical results for a spherical indenter

Figure 3.4 shows numerical results for the force–indentation relationship and the contact radius obtained from our FvK model with a spherical indenter. A common means of plotting experimental data is to plot the instantaneous estimate of $q(\nu)^3 E_{2D}$ from the point-indenter model, defined as $q(\nu)^3 E_{2D}^{\text{est}} := FR_{\text{out}}^2/\delta^3$ with $q(\nu) := (2\pi)^{1/3}/q_p(\nu)$, as a function of the applied force. (This procedure is particularly common when fitting experimental data using composite expressions, (2.19)–(2.21), since it corresponds to the cubic coefficient expected for a point indenter, (2.14).) Figure 3.4(a) mimics this by plotting $q(\nu)^3 E_{2D}^{\text{est}}$ as a function of $F/(E_{2D}^{\text{true}} R_{\text{out}})$. Results are shown for a range of assumed pre-tensions and sphere radii, with the typical maximal applied loads used in previous AFM indentation experiments indicated (see Table 3.1 for details).

Reference	Material	$\frac{R_s}{R_{\text{out}}}$	F_{max} [nN]	$\frac{R_{\text{out}} T_{\text{pre}}^{3/2}}{E_{2D}^{1/2}}$ [nN]	$E_{2D} R_s$ [nN]
Lee et al. (2008)	graphene	0.02 – 0.06	1200 – 2900	0.5 – 25.5	5600 – 9400
Song et al. (2010)	h-BN	0.05	200	—	1100 – 25500
Bertolazzi et al. (2011)	MoS ₂	0.05	200	0.05 – 0.5	2200
Lee et al. (2013)	graphene	0.03 – 0.08	2000	0.9 – 1.3	8600 – 12500
López-Polín et al. (2017)	graphene	0.02 – 0.12	1200 – 2100	0.2 – 80	1500 – 4000
Harats et al. (2020)	WS ₂	0.02 – 0.10	300 – 900	0.7 – 4.3	8500

Table 3.1: Typical experimental values of the parameters relevant to our model as determined in previous indentation experiments on various two-dimensional materials. (h-BN is hexagonal boron nitride, MoS₂ is molybdenum disulphide, and WS₂ is tungsten disulphide.) Values of R_s/R_{out} are used to inform the parameters used in the numerical results shown in Fig. 3.4, while typical values of the maximum indentation force reached experimentally for graphene (denoted F_{max}) are indicated by the shaded region in Fig. 3.4. The values in the last two columns are pertinent to the discussion in §3.4.3, and are calculated using the reported values of E_{2D} and T_{pre} in the corresponding reference.

These numerical results show three particular features. Firstly, increasing the pre-tension may have the (undesired) effect of preventing measurements from approaching the horizontal asymptote before the maximum indentation force that can be applied is reached. Secondly, results with larger indenter radii of curvature, R_s/R_{out} , enhance the effect of the geometric nonlinearity, again pushing results further from the ideal point-indenter solution of Schwerin (1929) and making it more difficult to infer the true value of E_{2D} . Thirdly, it seems that the fraction of the indenter that is wrapped by the membrane is relatively large, with R_{in}/R_s lying in the interval $0.3 \lesssim R_{\text{in}}/R_s \lesssim 0.7$ in the regime of experimental interest.

More importantly, however, the results of Fig. 3.4 show that the cubic compliance does

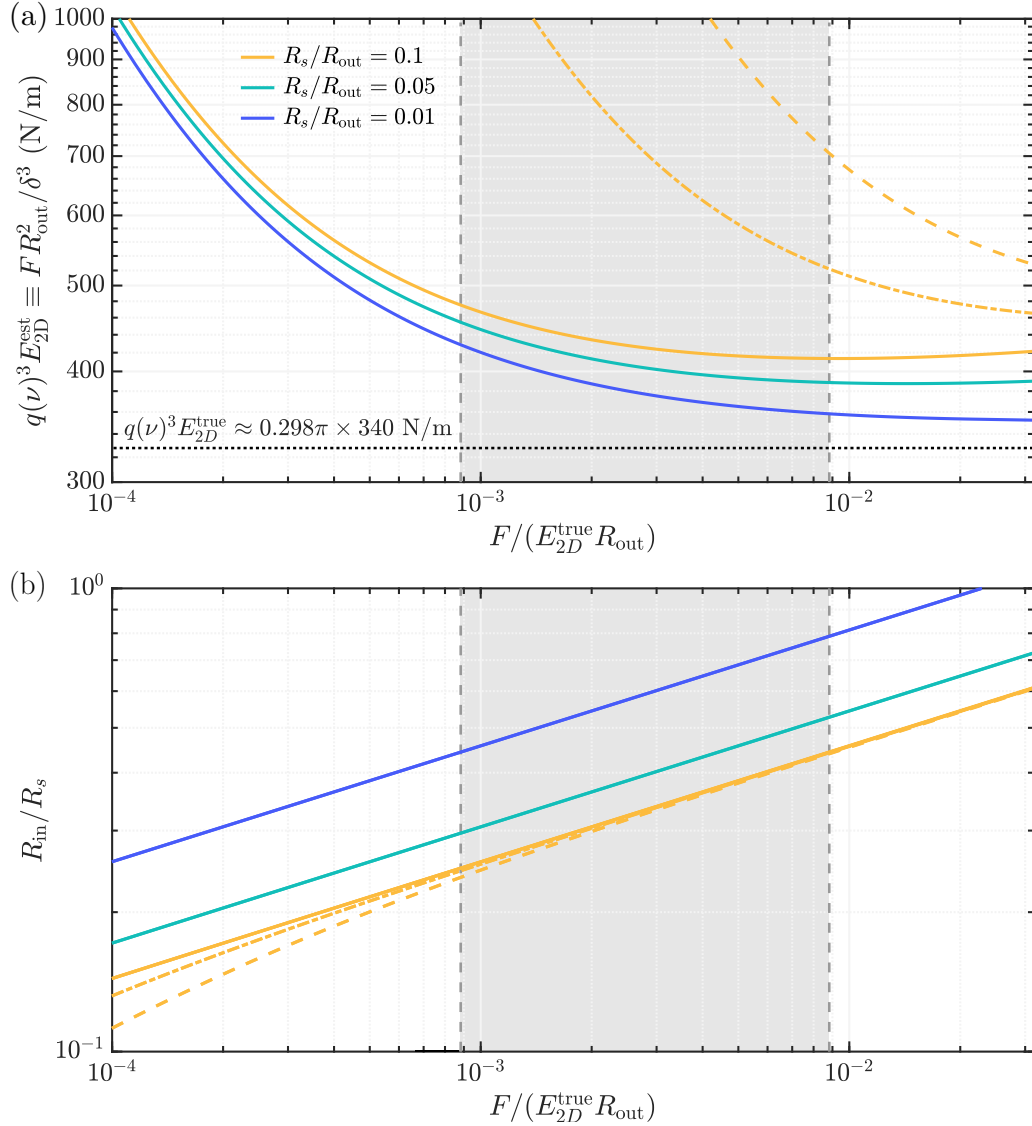


Figure 3.4: Numerical results obtained from the Föppl-von Kármán model with a spherical-capped indenter show the nonlinear effects associated with indenter geometry. (a) The instantaneous estimate of $q(\nu)^3 E_{2D}^{\text{est}} \equiv F R_{\text{out}}^2 / \delta^3$ plotted as a function of $F / (E_{2D}^{\text{true}} R_{\text{out}})$ for different values of R_s/R_{out} [indicated by colour: $R_s/R_{\text{out}} = 0.1$ (yellow), $R_s/R_{\text{out}} = 0.05$ (green) and $R_s/R_{\text{out}} = 0.01$ (blue)] and pre-tension T_{pre}/E_{2D} [indicated by line style: $T_{\text{pre}}/E_{2D} = 10^{-3}$ (solid curves), $T_{\text{pre}}/E_{2D} = 5 \times 10^{-3}$ (dash-dotted curves) and $T_{\text{pre}}/E_{2D} = 10^{-2}$ (dashed curves)]. The prediction of Schwerin (1929) for a point indenter with zero pre-tension and $E_{2D}^{\text{true}} = 340$ N/m is shown by the horizontal black dotted line. The typical range of experimental indentation forces used when fitting for the stretching modulus of graphene ($300 \text{ nN} \lesssim F \lesssim 3000 \text{ nN}$, as described in Table 3.1) is indicated by the shaded region under the assumption that $E_{2D}^{\text{true}} R_{\text{out}} = 3.4 \times 10^{-4}$ N. (b) The evolution of the radius at the edge of contact, R_{in}/R_s , is plotted as a function of force, with the same key as in (a) illustrating the varying pre-tension and sphere size. In both plots, ‘true’ values, $\nu = 0.165$ and $E_{2D}^{\text{true}} = 340$ N/m (the commonly accepted values for graphene, Akinwande et al., 2017), are assumed to present the numerical results in dimensional form. Note that, in the regime of experimental interest, the sphere is significantly wrapped (*i.e.* $R_{\text{in}}/R_s > 0.25$).

not asymptote to a constant for spherical indenters with realistic material parameters, as assumed previously (see Figure 6 of [Begley and Mackin, 2004](#), for example). Instead, the cubic compliance asymptotically decreases with increasing $\mathcal{F}$, as described by (3.13a) at large indentation forces. A decrease in cubic compliance corresponds to an increase in the instantaneous estimate, $E_{2D}^{\text{est}} \propto F/\delta^3$, as seen in Fig. 3.4(a); therefore, it seems plausible that such non-constant behaviour might be interpreted experimentally as a nonlinear material effect, either softening or stiffening. Before discussing this possibility further, we now turn to consider how such material nonlinearities should be modelled.

3.3 Nonlinear elastic materials

Thus far, through the application of the Föppl–von Kármán equations, we have retained the leading-order geometrical nonlinearities associated with deformation but have assumed that the material response remains Hookean: we have neglected any effect of a nonlinear constitutive response. This assumption even fails before indentation occurs if the pre-strain $\sim T_{\text{pre}}/E_{2D} = O(1)$ or during indentation if the stretching induced strain $\sim F/(rE_{2D}) = O(1)$ for $R_{\text{in}} \leq r \leq R_{\text{out}}$. In dimensionless variables these two conditions are equivalent to requiring

$$\mathcal{E} \ll 1 \quad \text{and} \quad \mathcal{F} \ll \mathcal{R}/\mathcal{E}^{3/2}, \quad (3.14\text{a,b})$$

for a Hookean response, where we have introduced the new dimensionless variable,

$$\mathcal{E} := \frac{T_{\text{pre}}}{E_{2D}}, \quad (3.15)$$

as a measure of the pre-strain in the sheet caused by the pre-tension, T_{pre} . In the case of a spherical indenter (for which the radius of contact, $\mathcal{R}$, is unknown), an analogous bound to (3.14b) can be formulated by noting that our detailed analysis in §3.2 showed that $\mathcal{R} \sim (\mathcal{R}_s^3 \mathcal{F})^{1/4}$, (3.13b), so the linear analysis holds provided that $\mathcal{F} \ll \mathcal{R}_s/\mathcal{E}^2$.

Many recent experiments have shown behaviour different to that expected on the basis of the FvK model of point indentation, and concluded that they are probing the nonlinear mechanical response of graphene (*e.g.* [Lee et al., 2008](#); [López-Polín et al., 2017](#)). However, we have also seen that the geometry of a spherical indenter can give behaviour that differs from the usual cubic response (or constant cubic compliance) expected from the FvK equations. The question, therefore, is what happens when the indentation advances beyond the small-strain limit, and how should one distinguish this regime from the geometrically nonlinear effects associated with indenter shape?

We introduce a model that allows for the possibility of large slopes and material strains. Specifically, we use the work of [Green and Adkins \(1960\)](#), who derived a generalized model

for the large deformations of an elastic membrane. This formulation allows the membrane to have a stress-strain relationship that is nonlinear (and, hence, the solid is non-Hookean), as well as accounting for large membrane slopes/rotations. We constrain the elastic constants introduced to recover the stretching modulus, E_{2D} , at small strains, and do not refer to an instantaneous effective stretching modulus at finite strains. The formulation presented below follows similar work by [Yang and Feng \(1970\)](#); [Long et al. \(2010\)](#); [Pearce et al. \(2011\)](#); [Laprade et al. \(2013\)](#), all of whom built upon [Green and Adkins'](#) formulation.

We concentrate on the specific case of indentation by a perfect-slip, spherical-capped wedge (with radius of curvature R_s and wedge angle $2\epsilon \ll 1$, as shown in Fig. 3.5). This type of indenter is commonly used in experiments (*e.g.* [Lee et al., 2008](#); [López-Polín et al., 2017](#)) and reduces to the spherical-capped indenter used in §3.2 provided the indentation depth is sufficiently small. (Hence, the results of this section should deviate from those of §3.2 only at large indentation depths.) Here, we also ignore the effects of the bending stiffness ($\mathcal{B} = 0$) since we are ultimately interested in the stretching dominant limit.

3.3.1 Governing ODEs

To allow for large rotations of the sheet, it is useful to introduce intrinsic coordinates: the radial arclength, ξ , and angle of rotation, α , (which is measured with respect to the radial- r -axis). We then have the geometrical conditions

$$\frac{dz}{d\xi} = \sin \alpha \quad \text{and} \quad \frac{dr}{d\xi} = \cos \alpha, \quad (3.16a,b)$$

where we recall that $0 \leq r \leq R_{\text{out}}$ and z are the radial and vertical coordinates of the sheet, respectively. These variables are sketched in Fig. 3.5.

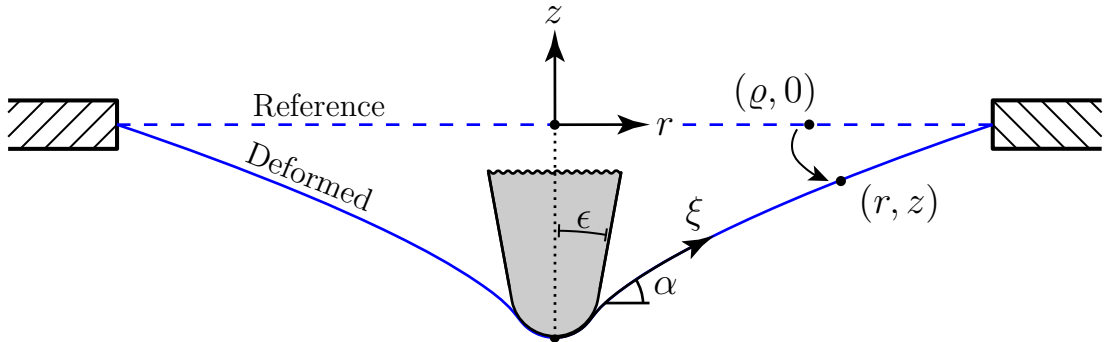


Figure 3.5: Cross-sectional sketch of the indentation of a clamped membrane showing the intrinsic and cylindrical coordinates, (ξ, α) and (r, z) , respectively. The reference configuration of the membrane is illustrated by the dashed line and the deformed configuration by the solid curve. Note the different variables used to describe the reference and deformed configurations.

The membrane is split into two regions: the region in which the membrane contacts the tip and a non-contacting region. In the contacting region (which occupies $0 \leq r \leq R_{\text{in}}$, with R_{in} yet unknown), we require

$$\sin \alpha = \begin{cases} \frac{r}{R_s} & \text{if } r \leq R_s \cos \epsilon, \\ \cos \epsilon & \text{if } r > R_s \cos \epsilon, \end{cases} \quad (3.16c)$$

for wedge angle 2ϵ . (Note that the wedge region is introduced to avoid the possibility that $r(\xi)$ becomes non-monotonic once the membrane inclination angle becomes close to $\pi/2$, but is also close to the experimentally observed tip shape. Alternative indenter geometries, *e.g.* a cylindrical punch, may be imposed here.) In the out-of-contact region (*i.e.* $R_{\text{in}} \leq r \leq R_{\text{out}}$), we impose the (integrated) vertical force balance,

$$rT_\xi \sin \alpha = \frac{F}{2\pi}, \quad (3.16d)$$

where T_ξ and T_ϕ are the thickness-averaged, in-plane, radial and azimuthal stresses, which must satisfy the in-plane force balance,

$$\frac{d}{dr} (rT_\xi) = T_\phi. \quad (3.16e)$$

Note that the out-of-plane stress is forced to be zero for thin sheets, *i.e.* $t \ll \eta_{\text{curv}}$, where t is the thickness of the sheet and η_{curv} is the minimum principal radius of curvature of the midsurface (Green and Adkins, 1960).

In modelling finite deformations of a thin membrane, one must distinguish between the reference and deformed configurations. Here, we take the reference configuration to be the planar sheet subjected to an isotropic tension T_{pre} ; this configuration is parametrized by $(\varrho, 0)$ for $0 \leq \varrho \leq R_{\text{out}}$ (where clamping is imposed at $\varrho = r = R_{\text{out}}$). The variables describing the deformed configuration are expressed as functions of ϱ , hence we define the principal stretches as

$$\lambda_\xi := \frac{d\xi}{d\varrho}, \quad \lambda_\phi := \frac{r}{\varrho}, \quad \lambda_z := \frac{t}{t_0}. \quad (3.17a-c)$$

Here, λ_ξ is the longitudinal stretch (along a cross-sectional curve in the r - z plane), λ_ϕ is the latitudinal stretch (along the direction normal to r - z plane), and λ_z is the out-of-plane stretch (a measure of membrane thickness, t , compared to its reference value, t_0). We also introduce the planar and out-of-plane pre-stretches $\Lambda := \Lambda_\xi = \Lambda_\phi$ and Λ_z which measure the initial isotropic stretching of the sheet; we will relate these pre-stretches to the pre-tension, T_{pre} , shortly.

System (3.16) is closed by imposing a constitutive relation that links the stresses with the

principal stretches, (3.17). In particular, by assuming a hyperelastic isotropic medium, we can introduce a constitutive strain energy density (per unit-volume) function, $W(\lambda_1, \lambda_2, \lambda_3)$, which directly links the stretches and stresses (see [Holzapfel, 2002](#), for example).

Using this formalism, we first calculate the pre-stretch, Λ , by solving the initial stress state,

$$\frac{\hat{t}_0}{\Lambda} \frac{\partial W}{\partial \lambda_1}(\Lambda, \Lambda, \Lambda_z) \equiv \frac{\hat{t}_0}{\Lambda} \frac{\partial W}{\partial \lambda_2}(\Lambda, \Lambda, \Lambda_z) = T_{\text{pre}}, \quad (3.18a)$$

with $\hat{t}_0$ being the undeformed sheet thickness and Λ_z chosen to satisfy the zero out-of-plane stress condition,

$$\frac{\partial W}{\partial \lambda_3}(\Lambda, \Lambda, \Lambda_z) = 0. \quad (3.18b)$$

The stresses in the deformed configuration are then computed using

$$T_\xi = \frac{\hat{t}_0}{\Lambda \lambda_\phi} \frac{\partial W}{\partial \lambda_1}(\Lambda \lambda_\xi, \Lambda \lambda_\phi, \Lambda_z \lambda_z) \quad \text{and} \quad T_\phi = \frac{\hat{t}_0}{\Lambda \lambda_\xi} \frac{\partial W}{\partial \lambda_2}(\Lambda \lambda_\xi, \Lambda \lambda_\phi, \Lambda_z \lambda_z), \quad (3.19a,b)$$

where λ_z satisfies the zero out-of-plane stress condition,

$$\frac{\partial W}{\partial \lambda_3}(\Lambda \lambda_\xi, \Lambda \lambda_\phi, \Lambda_z \lambda_z) = 0. \quad (3.19c)$$

Note that the stretches used in (3.19) are the products of the pre-stretches and indentation-induced stretches — *i.e.* $\Lambda \lambda_\xi$, $\Lambda \lambda_\phi$, and $\Lambda_z \lambda_z$ — which correspond to the total principal stretches in the sheet. The stresses are formulated in this way to account for the possibility of large pre-tensions, T_{pre} ; this contrasts with the approach of *e.g.* [Laprade et al. \(2013\)](#), who superpose the pre-tension and indentation-induced stress, and do not compute the pre-stretches.

Together, (3.16)–(3.19) form a system of three ordinary differential equations for the three unknowns — the two in-plane stretches, $\lambda_\xi(r)$ and $\lambda_\phi(r)$, and the vertical displacement, $z(r)$:

$$\frac{d\lambda_\xi}{dr} = \frac{(T_\phi - T_\xi)\lambda_\xi \cos \alpha - \lambda_\phi T_{\xi,\phi}(\lambda_\xi \cos \alpha - \lambda_\phi)}{r \lambda_\xi T_{\xi,\xi} \cos \alpha}, \quad (3.20a)$$

$$\frac{d\lambda_\phi}{dr} = \left(1 - \frac{\lambda_\phi}{\lambda_\xi \cos \alpha}\right) \frac{\lambda_\phi}{r}, \quad (3.20b)$$

$$\frac{dz}{dr} = \tan \alpha, \quad (3.20c)$$

where $\alpha(r)$ is given by (3.16c), for $0 \leq r \leq R_{\text{in}}$, and (3.16d), for $R_{\text{in}} \leq r \leq R_{\text{out}}$; and $T_\xi(\lambda_\xi, \lambda_\phi)$, $T_\phi(\lambda_\xi, \lambda_\phi)$, and $T_{\xi,j}(\lambda_\xi, \lambda_\phi) := dT_\xi/d\lambda_j$ are given by (3.18) and (3.19). To proceed further requires a particular choice of strain energy function, W , and so we turn to discuss this now.

3.3.2 Choice of strain energy density function

The choice of strain energy density function, $W(\lambda_1, \lambda_2, \lambda_3)$, is informed by the material of interest. In this chapter, we present results for two hyperelastic models to show the influence of this choice: a neo-Hookean material (the natural extension of the Hookean response that is implicit in the FvK equations) and a Gent hyperelastic material (which is a model developed for polymeric materials with finite extensibility, but is chosen here to account qualitatively for the finite bond-lengths in graphene). The Gent model contains a parameter, b , that captures the finite chain length and recovers the neo-Hookean strain energy function as $b \rightarrow 0$ (corresponding to infinite chain extensibility). The formulation of the compressible strain energy density functions is a lengthy process (the details are presented in Appendix 3.B). The final form of the Gent energy density function used is

$$W = \frac{E}{4(1+\nu)} \left[-\frac{1}{b} \log(1 - b(I_1 - 3)) + (\beta - b)(I_3 - 1) - (1 + \beta - b) \log I_3 \right], \quad (3.21)$$

where $\beta := \nu/(1 - 2\nu)$ is some known constant; $\nu \neq 1/2$ is the Poisson's ratio; $I_i(\lambda_1, \lambda_2, \lambda_3)$ are the usual tensor invariants, defined in (3.B.2); and b is an empirical parameter based on the finite extensibility of the material, defined such that $1/b \equiv I_1^\infty - 3$.

3.3.3 Boundary conditions

Having split the domain into contacting ($0 \leq r \leq R_{\text{in}}$) and non-contacting ($R_{\text{in}} \leq r \leq R_{\text{out}}$) regions, the problem is a multi-point boundary value problem, so we require boundary conditions at three positions: $r = 0$, $r = R_{\text{in}}$, and $r = R_{\text{out}}$.

At the outer edge, $r = R_{\text{out}}$, the sheet is perfectly clamped,

$$\lambda_\phi(R_{\text{out}}) = 1 \quad \text{and} \quad z(R_{\text{out}}) = 0. \quad (3.22\text{a,b})$$

At the origin, $r = 0$, we require an isotropic stretch (due to the symmetry of the problem),

$$\lambda_\phi(0) = \lambda_\xi(0) \quad \text{and} \quad z(0) = -\delta. \quad (3.22\text{c,d})$$

*At the interface between the contacting and non-contacting regions, $r = R_{\text{in}}$, a local force balance reveals that we require continuity in radial stress, $T_\xi(\lambda_\xi, \lambda_\phi)$, and membrane slope, α . Coupling this with the physical requirement of continuity of deformed variables, z and $r = \varrho\lambda_\phi$, we require continuity in all our variables, α , z , λ_ξ , and λ_ϕ . (This would not have been the case if there was a corner in the imposed indenter geometry — *e.g.* for a cylindrical punch.) Note that continuity in α gives an additional equation for the unknown point of contact, $R_{\text{in}}(F)$.*

3.3.4 Non-dimensionalization

To non-dimensionalize the problem we use the same choice of dimensionless radius and vertical coordinate, ρ and Z , as in the Föppl–von Kármán formulation, (2.9a,c), and define also

$$\wp := \frac{\varrho}{R_{\text{out}}}, \quad \hat{T}_\xi := \frac{T_\xi}{T_{\text{pre}}}, \quad \hat{T}_\phi := \frac{T_\phi}{T_{\text{pre}}}, \quad (3.23\text{a–c})$$

and

$$\hat{W}(\lambda_\xi, \lambda_\phi, \lambda_z) := \frac{W(\lambda_\xi, \lambda_\phi, \lambda_z)}{E}, \quad (3.23\text{d})$$

where we use $E_{2D} := \hat{t}_0 E$ to be the two-dimensional Young’s Modulus of the undeformed sheet for small strains — this is equivalent to the FvK choice under small strains.

Substitution of (2.9a,c) and (3.23) into equations (3.16c,d), (3.20), and (3.22) yields a dimensionless system for $\lambda_\xi(\rho)$, $\lambda_\phi(\rho)$, $Z(\rho)$, and $\alpha(\rho)$, with Cauchy stresses given by

$$\hat{T}_\xi = \frac{\mathcal{E}^{-1}}{\Lambda \lambda_\phi} \frac{\partial \hat{W}}{\partial \lambda_1}(\Lambda \lambda_\xi, \Lambda \lambda_\phi, \Lambda_Z \lambda_z), \quad \hat{T}_\phi = \frac{\mathcal{E}^{-1}}{\Lambda \lambda_\xi} \frac{\partial \hat{W}}{\partial \lambda_2}(\Lambda \lambda_\xi, \Lambda \lambda_\phi, \Lambda_Z \lambda_z), \quad (3.24\text{a,b})$$

$$\frac{\partial \hat{W}}{\partial \lambda_3}(\Lambda \lambda_\xi, \Lambda \lambda_\phi, \Lambda_Z \lambda_z) = 0, \quad (3.24\text{c})$$

and where the principal pre-stretches, Λ and Λ_z , solve

$$\frac{1}{\Lambda} \frac{\partial \hat{W}}{\partial \lambda_1}(\Lambda, \Lambda, \Lambda_z) \equiv \frac{1}{\Lambda} \frac{\partial \hat{W}}{\partial \lambda_2}(\Lambda, \Lambda, \Lambda_z) = \mathcal{E} \quad \text{and} \quad \frac{\partial \hat{W}}{\partial \lambda_3}(\Lambda, \Lambda, \Lambda_z) = 0. \quad (3.24\text{d,e})$$

For a given strain energy density function, $\hat{W}(\lambda_1, \lambda_2, \lambda_3)$ — we use (3.21) — the above system can be solved by a standard numerical integrator (we use `bvp4c` in `MATLAB`) for given parameters ν , $\mathcal{F}$, $\mathcal{E}$, and $\mathcal{R}_s$, with unknown $\mathcal{R}$. The associated indentation depth, $d(\mathcal{F})$, can then be calculated from (3.22d). In practice, however, it is more convenient to impose $\mathcal{R}$ instead of $\mathcal{F}$ and use the first derivative of (3.16d) to form a differential equation for the out-of-contact angle, α :

$$\frac{d\alpha}{d\rho} = -\frac{T_\phi \tan \alpha}{\rho T_\xi}, \quad (3.25)$$

in $\mathcal{R} \leq \rho \leq 1$, with a continuity boundary condition at $\rho = \mathcal{R}$; the indentation force, $\mathcal{F}$, can then be extracted post-computation, along with d . By doing so, one avoids the issue of an unknown domain size.

3.3.5 Results

Our aim in introducing a nonlinear elastic model is to investigate when the effects of material (as opposed to geometric) nonlinearity are observed in the key force–displacement curve. Although the quantitative results are highly dependent on the chosen constitutive strain

energy density function, (3.21), the transition from the linearized-material asymptotics of §3.2 occurs at a similar indentation force, independent of the choice of strain energy function. We are, therefore, able to investigate numerically the effect of varying the sphere radius, $\mathcal{R}_s \equiv (R_s/R_{\text{out}})\mathcal{E}^{1/2}$, and pre-tension, $\mathcal{E} \equiv T_{\text{pre}}/E_{2D}$. In Fig. 3.6, we present a table of force–displacement curves obtained as $\mathcal{E}$ and $\mathcal{R}_s$ vary; observe that the dimensionless force at which the results deviate from the Föppl–von Kármán solutions increases with R_s/R_{out} and decreases with $\mathcal{E} = T_{\text{pre}}/E_{2D}$ — reminiscent of the prediction that $\mathcal{F} \sim \mathcal{R}_s/\mathcal{E}^2$, (3.14b). It is also interesting to note that, close to the point at which the non-Hookean results deviate significantly from the FvK results, the effect of material nonlinearity is to soften the material response. This is in contrast to the geometrical nonlinearities discussed in §3.2, which acted to increase the instantaneous estimate of E_{2D} , *i.e.* to stiffen the indentation response. We discuss these results, and their significance for indentation probes of the elastic constants of thin materials, now.

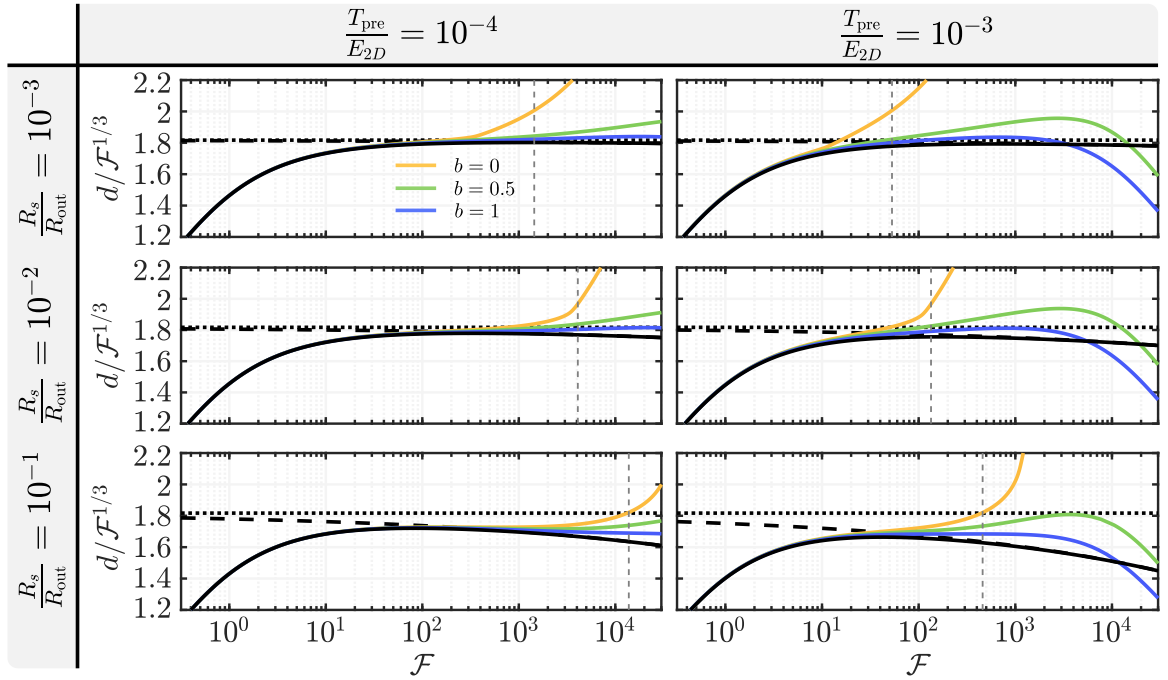


Figure 3.6: Table of plots showing how the onset of the effects of nonlinear elasticity depends on the spherical-cap curvature (rows) and the applied pre-tension (columns) of the membrane ($\nu = 1/3$, $\mathcal{B} = 0$); here, we have used an indenter wedge angle of $2\epsilon = \pi/9$ (matching the value in *e.g.* López-Polín et al., 2017). Numerical solutions are shown as solid curves: Föppl–von Kármán (linear elastic) in black, Gent ($b = 1$) in blue, Gent ($b = 0.5$) in green, and neo-Hookean ($b = 0$) in yellow. The asymptotic results from the FvK analysis for a point and spherical indenter, (3.13a) with $\mathcal{R}_s = 0$ and $\mathcal{R}_s = (R_s/R_{\text{out}})(T_{\text{pre}}/E_{2D})^{1/2}$, are shown as dotted and dashed curves, respectively. The vertical dashed lines show where the relative difference between the neo-Hookean and FvK models reaches 10%.

3.4 Discussion: Application to fitting protocols

The controlled indentation of thin sheets is a common, but delicate, experimental technique used to extract mechanical properties of thin (including two-dimensional) materials. Our detailed analysis of cylindrical and spherical indentation has led to several asymptotic results that highlight the complexity of this problem. We summarize these results in §3.5, but note that there are several potential pitfalls in this fitting procedure that must be appreciated to be avoided, and a reliable measurement of the quantity of interest obtained. Below, we discuss these pitfalls in the context of measurements of three properties: the sheet pre-tension, T_{pre} ; the two-dimensional Young's modulus, E_{2D} ; and the non-Hookean material behaviour (*i.e.* the behaviour not governed by linear elasticity). We shall concentrate on spherical-capped indenters since these are among the most used in practice.

3.4.1 Measuring sheet pre-tension

Our results showed the existence of a small-indentation regime in which the pre-tension dominates and $F \propto T_{\text{pre}}\delta$ (Fig. 3.7). Based on our analysis of this region, there are two potential pitfalls that may cause errors when attempting to infer T_{pre} from experimental measurements of $\delta(F)$:

- *Bending stiffness.* Although the bending stiffness is small at a macroscopic level, *i.e.* $B \ll T_{\text{pre}}R_{\text{out}}^2$, it is not necessarily negligible in the early stages of indentation with a spherical-capped indenter (see Fig. 3.7(b)): at small indentations, the spherical-cap is barely wrapped by the sheet, the contact radius is small, and it is the bending stiffness of the sheet that dominates through an effective, bending-induced, radius, $R_{\text{in}}^{\text{eff}} \propto (B/T_{\text{pre}})^{1/2}$, instead of the contact radius. Avoiding this requires that the applied force be sufficiently large; in particular, that $F \gg B/R_s$, for small bending stiffnesses ($B^{1/2} \ll R_s T_{\text{pre}}/E_{2D}^{1/2}$), and $F \gg B^{1/2} T_{\text{pre}}/E_{2D}^{1/2}$, for moderate bending stiffnesses ($R_s T_{\text{pre}}/E_{2D}^{1/2} \ll B^{1/2} \ll R_{\text{out}} T_{\text{pre}}^{1/2}$). For example, typical experiments for graphene have $B \approx 1.9 \times 10^{-19} \text{ N m}$, $E_{2D} \approx 340 \text{ N/m}$, $T_{\text{pre}} \approx 0.1 \text{ N/m}$, and $R_s \approx 10 \text{ nm}$ – 100 nm , thus we require $F \gg B/R_s \approx B^{1/2} T_{\text{pre}}/E_{2D}^{1/2} = \mathcal{O}(10^{-3} \text{ nN})$ to be able to neglect the bending stiffness. This is easily satisfied in all experiments of which we are aware (with $F \gtrsim 1 \text{ nN}$).
- *Sheet stretching.* If the indentation depths used are not sufficiently small, the pre-tension might be insignificant compared to the tension associated with indentation-induced stretching. This would cause a non-linear response (*i.e.* the sheet transitions from the linear, $\mathcal{F} \sim d$, towards the cubic, $\mathcal{F} \sim d^3$, behaviour). Avoiding this requires that the indentation force is not too large, in particular that $F \ll R_{\text{out}} T_{\text{pre}}^{3/2}/E_{2D}^{1/2}$.

Combining these requirements, we find that indentation tests aimed at measuring the pre-tension in a sheet should focus on indentation forces, $F_{T_{\text{pre}}}$, such that

$$\min \left\{ \frac{B}{R_s}, \frac{B^{1/2} T_{\text{pre}}}{E_{2D}^{1/2}} \right\} \ll F_{T_{\text{pre}}} \ll \frac{R_{\text{out}} T_{\text{pre}}^{3/2}}{E_{2D}^{1/2}}. \quad (3.26)$$

3.4.2 Measuring 2D Young's modulus

The measurement of the two-dimensional Young's modulus of a sheet requires experiments to be performed in the stretching-dominated regime, where $F \propto E_{2D} \delta^3 / R_{\text{out}}^2$ (Fig. 3.7). Based on our analysis of this regime, there are two immediate potential pitfalls that may cause fitting errors:

- *Sheet pre-tension.* If the sheet is not indented sufficiently, the effect of the pre-tension may still be significant, leading to a non-cubic response (*i.e.* the sheet is still transitioning between the linear, $\mathcal{F} \sim d$, and cubic, $\mathcal{F} \sim d^3$, behaviours, see Fig. 3.6). To avoid this, sufficiently large forces should be applied, in particular that $F \gg R_{\text{out}} T_{\text{pre}}^{3/2} / E_{2D}^{1/2}$.
- *Mechanical nonlinearities.* If the sheet is indented too much, the stress in the sheet exceeds that for which the linear elastic (Hookean) constitutive response is valid, and material nonlinearities become important. In this case, the expected cubic response may not be observed (see the solution divergence in Fig. 3.6), causing errors in the fitted value of E_{2D} . To avoid this possibility requires $F \ll E_{2D} R_s$.

Combining these requirements, we find that indentation tests aimed at measuring the stretching modulus of a sheet should focus on indentation forces, $F_{E_{2D}}$, such that

$$\frac{R_{\text{out}} T_{\text{pre}}^{3/2}}{E_{2D}^{1/2}} \ll F_{E_{2D}} \ll E_{2D} R_s. \quad (3.27)$$

A third potential pitfall is less obvious and independent of the indentation force applied:

- *Geometrical nonlinearities.* When fitting experimental data to obtain a value for the 2D Young's modulus, it is common practice to use the [Schwerin \(1929\)](#) point-indenter solution, which, in our notation, reads $\mathcal{F} = Q(\nu) d^3$ for $Q(\nu) := q_p(\nu)^{-3}$; for example, a point-wise estimate, $E_{2D} = F R_{\text{out}}^2 / [2\pi Q(\nu) \delta^3]$, is often used (see [López-Polín et al., 2017](#), for example). However, the analysis presented here shows that the indenter geometry is important and may lead to large errors in the fitted value of the 2D Young's modulus if not accounted for. In particular, for spherical indenters, the

point-wise estimate of E_{2D} may not converge as the force increases — the constant value (as predicted by [Schwerin](#)) is never observed (see Figs 3.2 and 3.6). To avoid a significant effect of the spherical geometry requires $R_s/R_{\text{out}} \ll 1$; however, we caution that decreasing R_s or increasing R_{out} may have the undesired effect of decreasing the desired range of indentation forces in (3.27), risking other fitting errors. In practice, $R_s/R_{\text{out}} \lesssim 10^{-2}$ appears to be sufficient (see Fig. 3.6); in many experimental setups $R_s/R_{\text{out}} \sim 0.1$ (see [López-Polín et al., 2017](#), for example), suggesting that indenter geometry may play a role in the correct interpretation of previous experimental results.

3.4.3 Measuring non-Hookean material behaviour

If interested in examining the non-Hookean material behaviour of the sheet (*i.e.* the behaviour beyond the Hookean linear stress–strain relation), there are two potential pitfalls to avoid:

- *Remaining in the linear elastic regime.* The main concern when measuring non-Hookean behaviour is whether the strains induced by indentation are large enough to be controlled by a nonlinear constitutive law. For a spherical-capped indenter, this requires

$$E_{2D}R_s \ll F. \quad (3.28)$$

- *Geometrical nonlinearities.* A less obvious trap is that deviating from the Schwerin solution, $\mathcal{F} = Q(\nu)d^3$, might be interpreted as nonlinear mechanical effects, but actually result from nonlinear geometry. For example, for sufficiently large spherical indenters, even the Föppl–von Kármán response will never achieve the plateau expected from the Schwerin result (see Figs 3.2 and 3.6). This failure to reach the Schwerin regime might be interpreted as a material nonlinearity rather than a universal geometric property that is predicted by Hookean elasticity. To avoid this possibility requires geometric effects to be negligible throughout the linear elastic regime, that is $R_s \ll R_{\text{out}}$. However, as we now discuss, our results suggest that the effect of geometrical and material nonlinearities on the cubic compliance, $d/\mathcal{F}^{1/3}$, are qualitatively different.

3.4.4 Distinguishing geometrical and material nonlinearities

Together, (3.27) and (3.28) provide limits on the validity of fitting force–indentation curves to determine the stretching modulus, E_{2D} , of a two-dimensional material, and to distinguish between geometrical and material nonlinearities in this procedure. (Of course, their validity can only be checked with some initial parameter estimate or *a posteriori*.) It should be noted that these are only based on order arguments and are not concrete cut off points. For example, the numerical results presented in Figs 3.2 and 3.6 suggest that to observe

the asymptotic results for $\mathcal{F} \gg 1$ in practice requires $\mathcal{F} \gtrsim 100$; hence, a useful guide for satisfying the condition (3.27) in practice is

$$200\pi \times \frac{R_{\text{out}} T_{\text{pre}}^{3/2}}{E_{2D}^{1/2}} \lesssim F_{E_{2D}} \ll E_{2D} R_s. \quad (3.29)$$

Comparison of the relevant experimental parameters collected in Table 3.1 with (3.29) suggests that the maximum indentation force applied experimentally, F_{max} , does not always reach the large multiple of $R_{\text{out}} T_{\text{pre}}^{3/2} / E_{2D}^{1/2}$ required for accurate measures of E_{2D} from asymptotic results — as is commonly assumed within the literature, §2.4.

When fitting, one should always ensure the desired asymptotic behaviour is observed. To fit the stretching stiffness, E_{2D} , this might be most simply done by ensuring that there is indeed a cubic plateau, $F/d^3 \sim \text{const}$. However, our results show that a true plateau is only obtained with a cylindrical indenter, while most experiments use a shape that is closer to a spherical cap. Fortunately, qualitatively similar asymptotic results may be derived from the Föppl–von Kármán equations for a spherical-capped indenter; in this regard, (3.13a) might be expected to be especially helpful and so we note that it may be rewritten in dimensional terms as

$$\frac{\delta}{R_{\text{out}}} \sim \frac{q_p(\nu)}{(2\pi)^{1/3}} \left(\frac{F}{E_{2D} R_{\text{out}}} \right)^{1/3} - \frac{2(\sqrt{2}-1)}{\pi^{1/2}} \left(\frac{R_s}{R_{\text{out}}} \right)^{1/2} \left(\frac{F}{E_{2D} R_{\text{out}}} \right)^{1/2}. \quad (3.30)$$

Note that (3.30) reduces to the classical Schwerin result as $R_s/R_{\text{out}} \rightarrow 0$, but shows that the perturbation caused by spherical geometry is *not* of the prefactor (as assumed by Begley and Mackin, 2004; Jia and Ben Amar, 2020, for example), but rather is additive. Moreover, (3.30) shows that the cubic compliance, $\delta/F^{1/3}$, is decreased by the effect of the indenter’s radius of curvature, R_s : the geometry of the indenter means that the apparent stiffness of the suspended solid is *increased* compared to a point indenter, and increases further with increasing load. As a rule of thumb, our results suggest that material nonlinearities tend to soften the response initially (at least for the strain energy functions considered here), while geometrical nonlinearities tend to stiffen the response.

3.5 Summary of results and conclusions

3.5.1 Summary

We have presented a series of asymptotic results that may be used in combination with experimental force–displacement data to fit the mechanical properties of thin materials, subject to an appreciation of the pitfalls described in §3.4. These results apply in various

asymptotic regimes determined by the relative sizes of the dimensionless force, $\mathcal{F}$; bending stiffness, $\mathcal{B}$; indenter radius, $\mathcal{R}$ or $\mathcal{R}_s$ (for a cylinder or sphere, respectively); and pre-tension, $\mathcal{E}$. Here, we summarize these results for a cylindrical and spherical-capped indenter; these are most easily expressed in terms of the vertical indentation depth, d , achieved for a fixed force, $\mathcal{F}$.

Cylindrical indenter For indentation by a cylindrical indenter, the asymptotic response is dependent on the relative size of the bending stiffness, $\mathcal{B}$, and indenter area, $\mathcal{R}^2$. We have that

$$\text{if } \mathcal{B}^{1/2} \ll \mathcal{R} < 1: \quad d \sim \begin{cases} \mathcal{F} \log \frac{1}{\mathcal{R}} & \text{for } 0 \leq \mathcal{F} \ll \mathcal{R}, \\ \mathcal{F} \log \frac{4}{\mathcal{F}} & \text{for } \mathcal{R} \ll \mathcal{F} \ll 1, \\ \mathcal{F}^{1/3} q_c(\nu, \mathcal{R}) & \text{for } 1 \ll \mathcal{F}, \end{cases} \quad (3.31a)$$

whilst

$$\text{if } \mathcal{R} \ll \mathcal{B}^{1/2} \ll 1: \quad d \sim \begin{cases} \mathcal{F} \log \frac{e^\gamma}{2\mathcal{B}^{1/2}} & \text{for } 0 \leq \mathcal{F} \ll \mathcal{B}^{1/2}, \\ \mathcal{F} \log \frac{4}{\mathcal{F}} & \text{for } \mathcal{B}^{1/2} \ll \mathcal{F} \ll 1, \\ \mathcal{F}^{1/3} q_c(\nu, \mathcal{R}) & \text{for } 1 \ll \mathcal{F}. \end{cases} \quad (3.31b)$$

Here, $\gamma \approx 0.577$ is the Euler–Mascheroni constant (Abramowitz and Stegun, 1964) and $q_c(\nu, \mathcal{R}) \sim q_p(\nu) - (2\mathcal{R})^{2/3}$ is defined in (3.10). These solutions are valid provided $\mathcal{E} \ll 1$ and $\mathcal{F} \ll \mathcal{R}\mathcal{E}^{-3/2}$; if either of these conditions fail then the material instead behaves according to a nonlinear constitutive law (non-Hookean behaviour). The regions of the regime diagram Fig. 3.7(a) are delineated by the expressions in (3.31).

Spherical indenter For indentation by a spherical-capped indenter, we find the asymptotic response

$$d \sim \begin{cases} \mathcal{F} \log \frac{e^\gamma}{2\mathcal{B}^{1/2}} & \text{for } 0 \leq \mathcal{F} \ll \min\{\mathcal{B}/\mathcal{R}_s, \mathcal{B}^{1/2}\}, \\ \mathcal{F} \log \sqrt{\frac{e}{\mathcal{R}_s \mathcal{F}}} & \text{for } \mathcal{B}/\mathcal{R}_s \ll \mathcal{F} \ll \mathcal{R}_s, \\ \mathcal{F} \log \frac{4}{\mathcal{F}} & \text{for } \max\{\mathcal{R}_s, \mathcal{B}^{1/2}\} \ll \mathcal{F} \ll 1, \\ \mathcal{F}^{1/3} q_s(\nu, \mathcal{R}_s^3 \mathcal{F}) & \text{for } 1 \ll \mathcal{F}, \end{cases} \quad (3.32a)$$

with the (*a priori* unknown) contact radius given by

$$\mathcal{R}^2 \sim \begin{cases} 4\mathcal{B}e^{-2\gamma} \exp(-4\mathcal{B}/\mathcal{R}_s\mathcal{F}) & \text{for } 0 \leq \mathcal{F} \ll \min\{\mathcal{B}/\mathcal{R}_s, \mathcal{B}^{1/2}\}, \\ \mathcal{R}_s\mathcal{F} & \text{for } \mathcal{B}/\mathcal{R}_s \ll \mathcal{F} \ll \mathcal{R}_s, \\ 4(\sqrt{2}-1)\mathcal{R}_s^{3/2}\mathcal{F}^{1/2} & \text{for } \max\{\mathcal{R}_s, \mathcal{B}^{1/2}\} \ll \mathcal{F}. \end{cases} \quad (3.32b)$$

Here, $q_s(\nu, \mathcal{R}_s^3\mathcal{F}) \sim q_p(\nu) - 2(2 - \sqrt{2})(\mathcal{R}_s^3\mathcal{F})^{1/6}$ is defined in (3.13) and we note that the dimensional version of (3.32a) for $\mathcal{F} \gg 1$ is given in (3.30). These solutions are valid provided $\mathcal{E} \ll 1$ and $\mathcal{F} \ll \mathcal{R}_s\mathcal{E}^{-2}$; if either of these conditions fail then the material instead behaves according to a nonlinear constitutive law (non-Hookean behaviour). The regions of the regime diagram Fig. 3.7(b) are delineated by the expressions in (3.32).

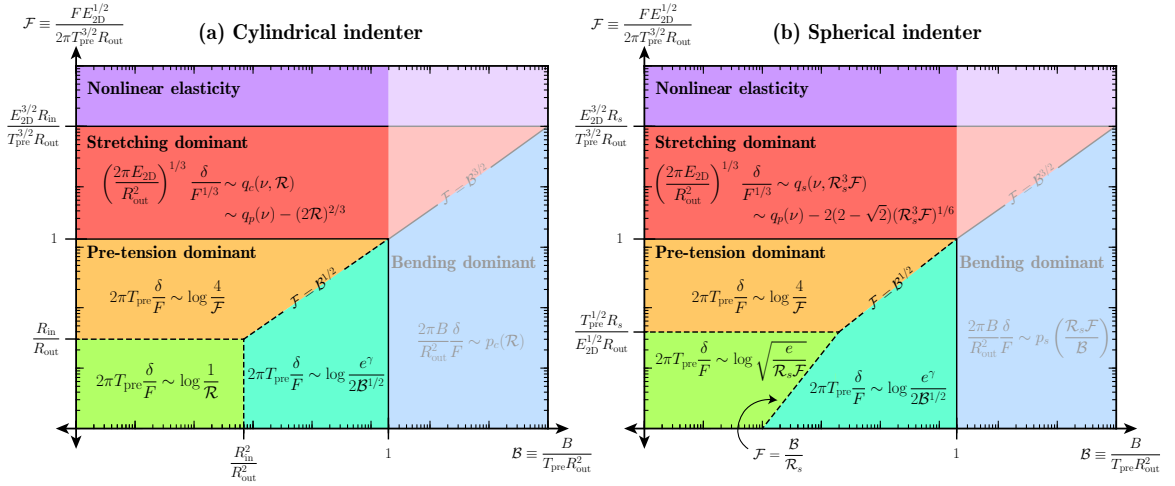


Figure 3.7: Regime diagram for the indentation of a suspended sheet: shown are the regions of the dimensionless $(\mathcal{B}, \mathcal{F})$ -space in which different force–displacement relationships are expected to be observed for (a) a cylindrical indenter of dimensionless radius $\mathcal{R} \equiv R_{\text{in}}/R_{\text{out}}$ and (b) a spherical indenter of dimensionless radius $\mathcal{R}_s \equiv T_{\text{pre}}^{1/2} R_s/(E_{2D}^{1/2} R_{\text{out}})$. Also shown are the asymptotic solutions derived in this chapter, which are summarized in (3.31) for a cylindrical indenter and (3.32) for a spherical indenter; we concentrate only on small bending stiffnesses, $\mathcal{B} \ll 1$.

3.5.2 Conclusions

Altogether, our work provides a comprehensive description of the Hookean response of a sheet subject to localized indentation accounting for geometrical nonlinearities, while additionally providing information about when a non-Hookean response can be expected. It is common in experiments to use dimensional versions of similar asymptotic solutions to extract information about the material of interest (*e.g.* its 2D Young’s modulus, E_{2D} , or pre-tension, T_{pre}). The asymptotic results presented in (3.31) and (3.32) show the number of different

regimes that exist and, hence, the difficulty of choosing the appropriate asymptotic result. Nevertheless, understanding the appropriate regime for each of these results is important since incorrect choices may lead to large errors in the fitted values obtained. For instance, if the indenter was assumed to be point-like, so that $q_c(\nu, \mathcal{R}) \approx q_p(\nu)$ or $q_s(\nu, \mathcal{R}_s^3 \mathcal{F}) \approx q_p(\nu)$ (which is implicit when using the popular composite expressions, (2.19)–(2.21), for example), we would obtain $\mathcal{O}(\mathcal{R}^{2/3})$ or $\mathcal{O}(\mathcal{R}_s^{1/2} \mathcal{F}_{\max}^{1/6})$ errors in the fitted Young’s modulus — which can be significant in such a sensitive process. We, therefore, emphasize the importance of using the correct response when fitting parameters and suggest the use of (3.30) to account for indenters with a hemispherical tip. Furthermore, when experimentally measuring material properties, we advise that multiple tests are performed (on many similar samples) using different setups (*e.g.* different indenter sizes) to ensure consistency and that there are no unaccounted-for effects within the fitting procedure.

Finally, we note that the asymptotic regimes considered here have been motivated by recent indentation experiments on ultra-thin materials, including few-layer graphene. For such materials, $B \ll T_{\text{pre}} R_{\text{out}}^2$, which corresponds to $\mathcal{B} \ll 1$, and so the different regimes described in (3.31) and (3.32) have been derived explicitly for $\mathcal{B} \ll 1$. As the dimensionless bending stiffness becomes larger, $\mathcal{B} = O(1)$, the balances that lead to these results are expected to change. In particular, we expect that for dimensionless forces $\mathcal{F} \lesssim 1$ and $\mathcal{B} \sim 1$ new results would be required (indicated by the greyed-out regions in Fig. 3.7).

Appendix 3.A Asymptotic solutions of the Föppl–von Kármán equations

In this appendix, we provide more details of the calculations that lead to the asymptotic results (3.31) and (3.32).

3.A.1 Asymptotic solutions for small indentation forces

In this section, we extend the arguments of §3.1.1 and §3.2.1 to derive the asymptotic solutions (3.4) and (3.11). We shall present the analysis for the cylindrical and spherical-capped indenters alongside one another.

We consider a small perturbation of the initial pre-tensed configuration: $\Psi(\rho) \sim \rho + \tilde{\Psi}(\rho)$ and $Z(\rho) \sim \tilde{Z}(\rho)$, anticipating that $\tilde{\Psi}, \tilde{Z} \ll 1$. At leading order (in $\tilde{\Psi}$ and $\tilde{Z}$), the Föppl–von

Kármán equations, (2.10), take the form

$$\mathcal{B}\rho \frac{d}{d\rho} \left[\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d\tilde{Z}}{d\rho} \right) \right] = \rho \frac{d\tilde{Z}}{d\rho} - \mathcal{F}, \quad (3.A.1a)$$

$$\rho \frac{d}{d\rho} \left[\frac{1}{\rho} \frac{d}{d\rho} (\rho \tilde{\Psi}) \right] = -\frac{1}{2} \left(\frac{d\tilde{Z}}{d\rho} \right)^2. \quad (3.A.1b)$$

Most importantly, (3.A.1a) no longer couples the out-of-plane deflection with the in-plane stress; the third order differential equation for the sheet profile, $Z(\rho) \sim \tilde{Z}(\rho)$, can immediately be integrated to give

$$Z \sim \tilde{Z} = c_1 I_0 \left[\frac{\rho}{\mathcal{B}^{1/2}} \right] + c_2 K_0 \left[\frac{\rho}{\mathcal{B}^{1/2}} \right] + \mathcal{F} \log \rho + c_3, \quad (3.A.2)$$

with the constants, c_1 , c_2 , and c_3 , yet to be determined. (The stress profile may be determined by substituting (3.A.2) into (3.A.1b); we omit this here since it affects the force-indentation response only at higher-order in d .)

The constants in (3.A.2) are to be determined to ensure that the sheet profile satisfies the boundary conditions

$$Z(1) = 0, \quad Z'(1) = 0, \quad (3.A.3a,b)$$

with

$$Z(\mathcal{R}) = -d, \quad Z'(\mathcal{R}) = 0, \quad (3.A.4a,b)$$

for a cylindrical indenter ($\mathcal{R}$ known), or

$$Z(\mathcal{R}) = -d + \frac{\mathcal{R}^2}{2\mathcal{R}_s}, \quad Z'(\mathcal{R}) = \frac{\mathcal{R}}{\mathcal{R}_s}, \quad Z''(\mathcal{R}) = \frac{1}{\mathcal{R}_s}, \quad (3.A.5a-c)$$

for a spherical indenter ($\mathcal{R}$ unknown). To satisfy (3.A.3) and (3.A.4b) requires the coefficients satisfy the expressions

$$\frac{c_1}{\mathcal{F}\mathcal{B}^{1/2}} = \frac{\hat{K}_1^R - \hat{K}_1/\mathcal{R}}{\hat{I}_1^R \hat{K}_1 - \hat{I}_1 \hat{K}_1^R}, \quad \frac{c_2}{\mathcal{F}\mathcal{B}^{1/2}} = \frac{\hat{I}_1^R - \hat{I}_1/\mathcal{R}}{\hat{I}_1^R \hat{K}_1 - \hat{I}_1 \hat{K}_1^R}, \quad (3.A.6a,b)$$

and

$$c_3 = -c_1 \hat{I}_0 - c_2 \hat{K}_0, \quad (3.A.6c)$$

for a cylindrical indenter. For a spherical indenter, we impose (3.A.3) and (3.A.5b,c) to find that

$$\frac{c_1}{\mathcal{F}\mathcal{B}^{1/2}} = \frac{\hat{K}_1 \mathcal{R} / \mathcal{F}\mathcal{R}_s + \hat{K}_1^R - \hat{K}_1 / \mathcal{R}}{\hat{I}_1^R \hat{K}_1 - \hat{I}_1 \hat{K}_1^R}, \quad \frac{c_2}{\mathcal{F}\mathcal{B}^{1/2}} = \frac{\hat{I}_1 \mathcal{R} / \mathcal{F}\mathcal{R}_s + \hat{I}_1^R - \hat{I}_1 / \mathcal{R}}{\hat{I}_1^R \hat{K}_1 - \hat{I}_1 \hat{K}_1^R}, \quad (3.A.7a,b)$$

and

$$c_3 = -c_1 \hat{I}_0 - c_2 \hat{K}_0, \quad (3.A.7c)$$

with $\mathcal{R}$ given by the solution of

$$\frac{1}{\mathcal{R}_s} + \frac{\mathcal{F}}{\mathcal{R}^2} = \frac{1}{2\mathcal{B}} \left[c_1 \left(\hat{I}_0^R + \hat{I}_2^R \right) + c_2 \left(\hat{K}_0^R + \hat{K}_2^R \right) \right]. \quad (3.A.7d)$$

(In each case, $\hat{I}_j$, $\hat{I}_j^R$, $\hat{K}_j$, and $\hat{K}_j^R$ are as defined in (3.3).)

The small force–displacement relations are then found by imposing the boundary conditions (3.A.4a) and (3.A.5a), to give

$$\frac{d}{\mathcal{F}} = \log \frac{1}{\mathcal{R}} + \frac{\hat{K}_1 \hat{I}_0^R + \hat{I}_1 \hat{K}_0^R + \mathcal{R} \left(\hat{K}_0 \hat{I}_1^R + \hat{I}_0 \hat{K}_1^R \right) - 2\mathcal{B}^{1/2} \mathcal{B}^{1/2}}{\hat{I}_1^R \hat{K}_1 - \hat{I}_1 \hat{K}_1^R} \frac{\mathcal{B}^{1/2}}{\mathcal{R}}, \quad (3.A.8)$$

for a cylindrical indenter; and

$$\begin{aligned} \frac{d}{\mathcal{F}} = \log \frac{1}{\mathcal{R}} + & \frac{\hat{K}_1 \hat{I}_0^R + \hat{I}_1 \hat{K}_0^R + \mathcal{R} \left(\hat{K}_0 \hat{I}_1^R + \hat{I}_0 \hat{K}_1^R \right) - 2\mathcal{B}^{1/2} \mathcal{B}^{1/2}}{\hat{I}_1^R \hat{K}_1 - \hat{I}_1 \hat{K}_1^R} \frac{\mathcal{B}^{1/2}}{\mathcal{R}} \\ & + \frac{\mathcal{B}^{1/2} - \hat{I}_1 \hat{K}_0^R - \hat{I}_0^R \hat{K}_1}{\hat{I}_1^R \hat{K}_1 - \hat{I}_1 \hat{K}_1^R} \frac{\mathcal{R} \mathcal{B}^{1/2}}{\mathcal{R}_s \mathcal{F}} + \frac{\mathcal{R}^2}{2\mathcal{R}_s \mathcal{F}}, \end{aligned} \quad (3.A.9a)$$

with $\mathcal{R}$ given by the solution of

$$\frac{\mathcal{F} \mathcal{R}_s + \mathcal{B}^{-1/2} \left(\mathcal{R}^2 - \mathcal{F} \mathcal{R}_s \right) \left(\hat{I}_0^R \hat{K}_1 + \hat{K}_0^R \hat{I}_1 \right)}{\hat{I}_1^R \hat{K}_1 - \hat{I}_1 \hat{K}_1^R} = 2\mathcal{R}, \quad (3.A.9b)$$

for a spherical indenter.

In the limit of small bending stiffnesses ($\mathcal{B} \ll 1$), which is relevant here, (3.A.8) and (3.A.9) can be simplified using the asymptotic behaviour of modified Bessel functions to give

$$\frac{d}{\mathcal{F}} = \log \frac{1}{\mathcal{R}} - \mathcal{B}^{1/2} \frac{\hat{K}_0^R}{\mathcal{R} \hat{K}_1^R} - \mathcal{B}^{1/2} \frac{\hat{I}_0}{\hat{I}_1} + \text{e.s.t.}, \quad (3.A.10)$$

for a cylindrical indenter. For a spherical indenter, the same procedure gives

$$\frac{d}{\mathcal{F}} = \log \frac{1}{\mathcal{R}} - \frac{2\mathcal{B}}{\mathcal{R}_s \mathcal{F}} + \frac{\mathcal{R}^2}{2\mathcal{R}_s \mathcal{F}} - \mathcal{B}^{1/2} \frac{\hat{I}_0}{\hat{I}_1} + \text{e.s.t.}, \quad (3.A.11a)$$

where $\mathcal{R}$ is given implicitly by the solution of

$$\left(\mathcal{R}_s \mathcal{F} - \mathcal{R}^2 \right) \frac{\hat{K}_0^R}{\mathcal{R} \hat{K}_1^R} = 2\mathcal{B}^{1/2} + \text{e.s.t.} \quad (3.A.11b)$$

Finally, considering the limits $\mathcal{B} \ll \mathcal{R}^2 < 1$ and $\mathcal{R}^2 \ll \mathcal{B} \ll 1$ leads to the leading-order solutions (3.4), for a cylindrical indenter, and (3.11), for a spherical-capped indenter.

3.A.2 Asymptotic solutions for small bending stiffnesses

In this section, we expand the arguments of §3.1.2, §3.2.2, and §3.2.3 to derive the asymptotic solutions for large indentation depths and negligible bending stiffnesses (3.7), (3.10), (3.12), and (3.13). We present the case of cylindrical and spherical-capped indenters simultaneously, and follow the analysis of a point-indenter by Vella and Davidovitch (2017, App. B).

Following the discussion in §3.1.2, we let $\eta = \rho^2$, $\Phi = \rho\Psi$ and set $\mathcal{B} = 0$ in (2.10) to give

$$2\Phi \frac{dZ}{d\eta} = \mathcal{F} \quad \text{and} \quad \frac{d^2\Phi}{d\eta^2} = -\frac{1}{2} \left(\frac{dZ}{d\eta} \right)^2 = -\frac{\mathcal{F}^2}{8\Phi^2}. \quad (3.A.12a,b)$$

Neglecting $\mathcal{B}$ reduces the order of the system, so we suppress the boundary conditions on the highest-order quantities: (2.11b), (2.12b), and (2.13c). Therefore, (3.A.12) is to be solved subject to

$$2\Phi'(1) - (1 + \nu)\Phi(1) = 1 - \nu, \quad Z(1) = 0, \quad (3.A.13a,b)$$

with

$$Z(\mathcal{R}^2) = -d, \quad \mathcal{R}\Phi'(\mathcal{R}^2) - \frac{\Phi(\mathcal{R}^2)}{\mathcal{R}} = 0, \quad (3.A.14a,b)$$

for a cylindrical indenter, or

$$Z(\mathcal{R}^2) = -d + \frac{\mathcal{R}^2}{2\mathcal{R}_s}, \quad Z'(\mathcal{R}^2) = \frac{1}{2\mathcal{R}_s}, \quad \mathcal{R}\Phi'(\mathcal{R}^2) - \frac{\Phi(\mathcal{R}^2)}{\mathcal{R}} = -\frac{\mathcal{R}^3}{16\mathcal{R}_s^2}, \quad (3.A.15a-c)$$

for a spherical indenter.

Two integrations of (3.A.12b) leads to

$$(A\Phi)^{1/2}(1 + A\Phi)^{1/2} - \sinh^{-1} \sqrt{A\Phi} = \frac{\mathcal{F}A^{3/2}}{2}\eta + B, \quad (3.A.16)$$

for integration constants A and B , while (3.A.12a) leads to

$$Z = \frac{2}{\sqrt{A}} \sinh^{-1} \sqrt{A\Phi} - C, \quad (3.A.17)$$

for some constant C .

Applying the boundary conditions (3.A.13)–(3.A.15) yields a system for the unknown constants A , B , and C . To simplify these equations, we define $\Phi_0 := A\Phi(\mathcal{R}^2)$ and $\Phi_1 :=$

$A\Phi(1)$; the system then takes the form

$$\Phi_1^{1/2}(1 + \Phi_1)^{1/2} - \sinh^{-1} \Phi_1^{1/2} = \frac{\mathcal{F}A^{3/2}}{2} + B, \quad (3.A.18a)$$

$$\Phi_0^{1/2}(1 + \Phi_0)^{1/2} - \sinh^{-1} \Phi_0^{1/2} = \frac{\mathcal{F}A^{3/2}}{2}\mathcal{R}^2 + B, \quad (3.A.18b)$$

$$\mathcal{F}A^{3/2} \left(1 + \frac{1}{\Phi_1}\right)^{1/2} - (1 + \nu)\Phi_1 = (1 - \nu)A, \quad (3.A.18c)$$

$$\frac{2}{A^{1/2}} \sinh^{-1} \Phi_1^{1/2} = C, \quad (3.A.18d)$$

with

$$\mathcal{F}A^{3/2} \left(1 + \frac{1}{\Phi_0}\right)^{1/2} = \frac{2\Phi_0}{\mathcal{R}^2}, \quad (3.A.19a)$$

$$\frac{2}{A^{1/2}} \sinh^{-1} \Phi_0^{1/2} = C - d, \quad (3.A.19b)$$

for a cylindrical indenter, or

$$\mathcal{F}A^{3/2} \left(1 + \frac{1}{\Phi_0}\right)^{1/2} = \frac{2\Phi_0}{\mathcal{R}^2} - \frac{\mathcal{R}^2 A}{8\mathcal{R}_s^2}, \quad (3.A.20a)$$

$$\frac{2}{A^{1/2}} \sinh^{-1} \Phi_0^{1/2} = C - d + \frac{\mathcal{R}^2}{2\mathcal{R}_s}, \quad (3.A.20b)$$

$$\Phi_0 = A\mathcal{F}\mathcal{R}_s, \quad (3.A.20c)$$

for a spherical indenter. Eliminating B and C from system (3.A.18) with (3.A.19) or (3.A.20) yields a force–displacement equation given implicitly by parameters Φ_0 , Φ_1 , and A (with $\mathcal{R}$ as an extra unknown in the spherical case); the resulting system for a cylindrical indenter is given in (3.5).

We, now, consider the two limits $\mathcal{F} \rightarrow 0$ and $\mathcal{F} \rightarrow \infty$ to derive explicit asymptotic force–displacement relations; in particular, equation (3.A.18c) determines how A and Φ_1 relate to the applied force, $\mathcal{F}$.

3.A.2.1 Moderate indentation forces ($\mathcal{F} \rightarrow 0$)

In the limit of small indentation forces, $\mathcal{F} \rightarrow 0$, (3.A.18c) gives $A \sim \Phi_1 = 4/\mathcal{F}^2 + \mathcal{O}(1)$, which leads to the parametric force–displacement relationship for a cylindrical indenter

$$\frac{d}{\mathcal{F}} = \log \frac{4}{\mathcal{F}} - \sinh^{-1} \Phi_0^{1/2} + \mathcal{O}(\mathcal{F}^2), \quad (3.A.21a)$$

where Φ_0 solves

$$\frac{\Phi_0^{3/2}}{(1 + \Phi_0)^{1/2}} \sim \frac{4\mathcal{R}^2}{\mathcal{F}^2}. \quad (3.A.21b)$$

For a spherical indenter, we find that

$$\frac{d}{\mathcal{F}} = \log \frac{4}{\mathcal{F}} - \sinh^{-1} \Phi_0^{1/2} + \frac{\mathcal{R}^2}{2\mathcal{R}_s \mathcal{F}} + \mathcal{O}(\mathcal{F}^2), \quad (3.A.22a)$$

where Φ_0 and $\mathcal{R}$ are given by

$$\Phi_0 \sim \frac{4\mathcal{R}_s}{\mathcal{F}} \quad \text{and} \quad \frac{\mathcal{R}^2}{2\mathcal{R}_s \mathcal{F}} \sim \Phi_0^{1/2} \left[(2 + \Phi_0)^{1/2} - (1 + \Phi_0)^{1/2} \right]. \quad (3.A.22b,c)$$

Taking the limits $\mathcal{F} \ll \mathcal{R} < 1$ and $\mathcal{R} \ll \mathcal{F} \ll 1$ yields the leading-order solutions (3.7), for a cylindrical indenter, and (3.12), for a spherical-capped indenter.

3.A.2.2 Large indentation forces ($\mathcal{F} \rightarrow \infty$)

In the limit of large indentation forces, $\mathcal{F} \rightarrow \infty$, (3.A.18c) gives $A = \mathcal{O}(\mathcal{F}^{-2/3})$ and $\Phi_1 = \mathcal{O}(1)$; inserting these into the relevant system, and rearranging, yields the parametric force–displacement relation for a cylindrical indenter

$$\frac{d}{\mathcal{F}^{1/3}} = \frac{2}{(1 + \nu)^{1/3}} \frac{\sinh^{-1} \Phi_1^{1/2} - \sinh^{-1} \Phi_0^{1/2}}{\Phi_1^{1/2} (1 + \Phi_1)^{-1/6}} + \mathcal{O}(\mathcal{F}^{-2/3}), \quad (3.A.23a)$$

where Φ_1 and Φ_0 solve

$$\frac{1 + \nu}{2} \frac{\Phi_1^{3/2}}{(1 + \Phi_1)^{1/2}} \sim \frac{\left[\Phi_1^{1/2} (1 + \Phi_1)^{1/2} - \sinh^{-1} \Phi_1^{1/2} \right]^{\Phi_1}}{1 - \mathcal{R}^2}, \quad (3.A.23b)$$

$$\frac{1 + \nu}{2} \frac{\Phi_1^{3/2}}{(1 + \Phi_1)^{1/2}} \sim \mathcal{R}^{-2} \frac{\Phi_0^{3/2}}{(1 + \Phi_0)^{1/2}}. \quad (3.A.23c)$$

For a spherical indenter, we find that

$$\frac{d}{\mathcal{F}^{1/3}} = \frac{2}{(1 + \nu)^{1/3}} \frac{\sinh^{-1} \Phi_1^{1/2} - \sinh^{-1} \Phi_0^{1/2}}{\Phi_1^{1/2} (1 + \Phi_1)^{-1/6}} + \frac{\mathcal{R}^2}{2\mathcal{R}_s \mathcal{F}^{1/3}} + \mathcal{O}(\mathcal{F}^{-2/3}), \quad (3.A.24a)$$

where Φ_1 , Φ_0 , and $\mathcal{R}$ are given by

$$\frac{1+\nu}{2} \frac{\Phi_1^{3/2}}{(1+\Phi_1)^{1/2}} \sim \frac{\left[\Phi^{1/2}(1+\Phi)^{1/2} - \sinh^{-1} \Phi^{1/2} \right]_{\Phi_0}^{\Phi_1}}{1-\mathcal{R}^2}, \quad (3.A.24b)$$

$$\frac{1+\nu}{2} \frac{\Phi_1^{3/2}}{(1+\Phi_1)^{1/2}} \sim \frac{\Phi_0^{3/2}}{(\mathcal{R}_s^3 \mathcal{F})^{1/2}}, \quad (3.A.24c)$$

$$\frac{\mathcal{R}^2}{4(\mathcal{R}_s^3 \mathcal{F})^{1/2}} \sim (2+\Phi_0)^{1/2} - (1+\Phi_0)^{1/2}. \quad (3.A.24d)$$

An asymptotic expansion in $\Phi_0 \rightarrow 0$ (equivalent to taking $\mathcal{R} \rightarrow 0$) leads to the leading-order solutions (3.10), for a cylindrical indenter, and (3.13), for a spherical-capped indenter. (Note that, although $\Phi_0 = \Phi_1 = 0$ is a solution of (3.A.23b,c), this does not correspond to the limit of large indentation depth. A local expansion around $\Phi_0 = \Phi_1 = 0$ shows that this is the correct solution only if $\mathcal{R}^2 = (3\nu - 1)/(\nu + 1)$; this was observed by Vella and Davidovitch (2017) for the case of a point indenter ($\mathcal{R} = 0$), where they noted that the trivial solution is only viable if $\nu = 1/3$. Instead, equations (3.A.23b) and (3.A.24b) must be divided through by $\Phi_1^{3/2}$.)

Appendix 3.B Choice of strain energy density function

The construction of a strain energy density function, $W(\lambda_1, \lambda_2, \lambda_3)$, is a difficult process, being dependent on both the material and experiment at hand (see discussion in *e.g.* Holzapfel, 2002, Chap. 6). Here, our focus is on a qualitative representation of hyperelasticity, so we seek suitable strain energy density functions. In this appendix, we construct the strain energy function used in this chapter.

We require our stresses to be consistent with Hookean elasticity in the small strain limit, *i.e.* $T_\xi \sim \sigma_{rr}$ and $T_\phi \sim \sigma_{\theta\theta}$ in the limit $\Lambda\lambda_\xi, \Lambda\lambda_\phi \rightarrow 1$, for T_ξ and T_ϕ defined in (3.19). Taylor expanding (3.19) around $\Lambda\lambda_\xi = \Lambda\lambda_\phi = 1$ (see Horgan and Saccomandi, 2004, for example), provides the consistency conditions

$$W_{I_1} + W_{I_2} = - (W_{I_2} + W_{I_3}) = \frac{G}{2}, \quad (3.B.1a)$$

$$W_{I_1 I_1} + 4W_{I_1 I_2} + 2W_{I_1 I_3} + 4W_{I_2 I_2} + 4W_{I_2 I_3} + W_{I_3 I_3} = \frac{K}{4} + \frac{G}{3}, \quad (3.B.1b)$$

for the bulk and shear moduli K and G , tensor invariants

$$I_1 := \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 := \lambda_1^2 \lambda_2^2 + \lambda_3^2 \lambda_1^2 + \lambda_2^2 \lambda_3^2, \quad I_3 := \lambda_1^2 \lambda_2^2 \lambda_3^2, \quad (3.B.2a-c)$$

and where we define

$$W_{I_i} := \left. \frac{\partial W}{\partial I_i} \right|_{\lambda_1=\lambda_2=\lambda_3=1} \quad \text{and} \quad W_{I_i I_j} := \left. \frac{\partial^2 W}{\partial I_i \partial I_j} \right|_{\lambda_1=\lambda_2=\lambda_3=1}. \quad (3.B.3a,b)$$

(Provided (3.B.1) holds, the membrane equations, (3.20), linearize to the zero bending-stiffness Föppl-von Kármán equations, *i.e.* (2.5) with $B = 0$, independent of the choice of strain energy density.)

As discussed in the main text, we concentrate on neo-Hookean and Gent solids. To formulate their strain density functions, it is convenient to split the strain energy into an isochoric, W_{iso} , and volumetric, W_{vol} , parts:

$$W(\lambda_1, \lambda_2, \lambda_3) \equiv W_{\text{iso}}(I_1, I_2, I_3) + W_{\text{vol}}(J), \quad (3.B.4)$$

for the tensor invariants (3.B.2) and Jacobian $J = I_3^{1/2}$. The benefit of this approach is that the isochoric part contains the full hyperelastic material model; that is,

$$\text{Neo-Hookean:} \quad W_{\text{iso}} = \frac{G}{2} \left[(I_1 - 3) - \log I_3 \right], \quad (3.B.5a)$$

$$\text{Gent:} \quad W_{\text{iso}} = \frac{G}{2} \left[-\frac{1}{b} \log \left(1 - b(I_1 - 3) \right) - \log I_3 \right], \quad (3.B.5b)$$

for some empirical parameter b based on the finite extensibility limit of the material ($1/b \equiv I_1^\infty - 3$). Since the neo-Hookean model is recovered as $b \rightarrow 0$ in the Gent model, we consider only the Gent formulation henceforth. The volumetric part is a requirement of compressible materials ($J \neq 1$) and can take many constitutive forms (see Holzapfel, 2002; Horgan and Saccomandi, 2004, for example). We take

$$W_{\text{vol}} = c_1 \log J + c_2 (\log J)^2 + c_3 (J^2 - 1), \quad (3.B.6)$$

for constants c_i ; the c_i are determined by inserting the volumetric strain energy, (3.B.6), with Gent isometric strain energy, (3.B.5b), into the consistency relations, (3.B.1), this gives

$$c_1 = -2c_3 = c_2 + \left(b + \frac{1}{3} \right) G - \frac{1}{2} K, \quad (3.B.7)$$

where we are free to choose c_2 . We choose $c_2 = 0$ to suppress the $(\log J)^2$ term, this form was first proposed by Simo and Miehe (1992) in the context of thermoplasticity. Coupling

with (3.B.5b), our strain energy density function takes the final form

$$W = \frac{G}{2} \left[-\frac{1}{b} \log(1 - b(I_1 - 3)) + (\beta - b)(I_3 - 1) - (1 + \beta - b) \log I_3 \right], \quad (3.B.8)$$

for $\beta := K/(2G) - 1/3 \equiv \nu/(1 - 2\nu)$.

Although (3.B.8) is not necessarily the simplest choice of strain energy function, it does contain some desirable properties for our work. These include: compressibility (*i.e.* a choice of Poisson's ratio, ν), which is needed to match up with the general Föppl-von Kármán equations; a variable nonlinearity (through the finite-extensibility parameter, b), giving us the ability to continuously vary our hyperelastic material model and include the common neo-Hookean hyperelasticity as a sub-case ($b = 0$); and the finite-extensibility property of Gent ($b > 0$), which could qualitatively model properties seen in complex materials, such as graphene and MoS₂ (specifically the finite length of bonds between atoms).

Validity of Winkler's mattress model for thin elastic foundations

4.1 Introduction

Thin elastic solids coating another, more rigid, substrate are encountered in a range of scenarios and at a range of scales, from the consolidated till separating a glacier from bedrock (*e.g.* Sayag and Worster, 2011; Butler et al., 2020) to multilayer microfluidic systems (*e.g.* Thorsen et al., 2002). In many of these scenarios, it is crucial to be able to determine how the coating deforms. The simplest, and perhaps most common, model of this type is the Winkler model (Dillard et al., 2018), in which the vertical displacement of the coating's interface, denoted $-\zeta(x)$, is proportional to the applied pressure, with some 'Winkler modulus' k_W , *i.e.* we may write

$$-\zeta(x) = \frac{p(x)}{k_W}. \quad (4.1)$$

On dimensional grounds, the stiffness k_W must be proportional to G/d with G the shear modulus of the layer and d its thickness (Johnson, 1985). Using an analogue of hydrodynamic lubrication theory, Skotheim and Mahadevan (2004) showed that, for a thin elastic solid, the prefactor is determined in terms of the Poisson's ratio, ν , by

$$k_W = \frac{2(1-\nu)}{1-2\nu} \frac{G}{d}. \quad (4.2)$$

Physically, the Winkler model corresponds to the elastic coating being made up of a series of disconnected springs, each of which deforms in proportion to the local pressure. For this reason, Winkler's model is often referred to as the 'mattress model' (Johnson, 1985). While this is an extremely simple, as well as natural, model for many elastic deformations, the modulus k_W in (4.2) diverges as the Poisson's ratio $\nu \rightarrow 1/2$: an incompressible spring cannot deform with any finite applied pressure. Nevertheless, the Winkler model is often

successfully used to describe experiments in which the thin coating is elastomeric and so might, ordinarily, be considered to be close to incompressible (see [Saintyves et al., 2016](#); [Rallabandi et al., 2017](#); [Saintyves et al., 2020](#), for example). Why does the Winkler model work so well in such situations?

Of course, incompressible elastic materials *are* able to deform in response to loading even if this is contrary to the suggestion of the Winkler model. For an incompressible material, deformation occurs via the shear of the material — an effect that is missed by the Winkler model. In fact, based on the work of [Gent and Meinecke \(1970\)](#), [Dillard \(1989\)](#) showed that the correct model for a one-dimensional, perfectly incompressible, coating is that the deformation of the interface is proportional to the second derivative of the pressure, *i.e.* that

$$\zeta(x) = \frac{1}{k_I} \frac{d^2 p}{dx^2}, \quad (4.3)$$

for some incompressible modulus $k_I \propto G/d^3$ (where the coefficient of proportionality depends on the precise boundary conditions assumed).

To generalize and combine these two limits, [Lai et al. \(1992\)](#) and [Bert \(1994\)](#) used an ansatz for the displacement field in which the horizontal displacement is both parabolic and up-down symmetric, while the vertical strain is assumed uniform. This assumption led to an expression for the vertical surface displacement, ζ , that combines (4.1) and (4.3), albeit with stiffness k_W that deviates from (4.2) for materials not close to incompressible. In this chapter, we use asymptotic analysis, exploiting the slenderness of the coating, to combine the two results (4.1) and (4.3) more formally. In §4.2, we derive a combined foundation model that can be used when the response is not known, *a priori*, to lie in either the Winkler or incompressible regimes. We show that in this ‘combined foundation’ model, the displacement from the Winkler and incompressible models can be added to give ζ in terms of the surface pressure, $p(x, y)$, that is

$$-\zeta_C = \frac{p}{k_W} - \frac{1}{k_I} \nabla_H^2 p, \quad (4.4)$$

where $\nabla_H^2 = (\partial_x^2 + \partial_y^2)$ is the two-dimensional Laplacian. In §4.3 and §4.4, we apply this combined foundation model to two examples with an infinite lateral domain, and discuss the signatures of the transition between the Winkler and incompressible regimes, and how this transition might be identified experimentally. In §4.5, we discuss how the combined foundation, (4.4), is corrected in the face of inconsistent lateral boundary conditions, *e.g.* a rigid vertical wall. We derive a model for the deflection in the associated boundary layers, which is then implemented for a finite domain example in §4.6. We begin, however, by considering the full three-dimensional formulation of the substrate deformation, before exploiting the coating’s slenderness to derive (4.4) as well as determining the coefficients k_W

and k_I .¹

4.2 General formulation and analysis

4.2.1 Problem setup

Consider a thin deformable layer (of thickness d , Poisson's ratio ν , and shear modulus G) bonded to a rigid substrate. The surface of the deformable layer is subject to a given pressure field $p(\mathbf{x})$ and is displaced from $z = 0$ to $z = \zeta(\mathbf{x})$; here, and throughout this chapter, $\mathbf{x} := (x, y)$ denotes the Cartesian coordinates in the plane $z = 0$. A cross-section of this setup is sketched in Fig. 4.1.

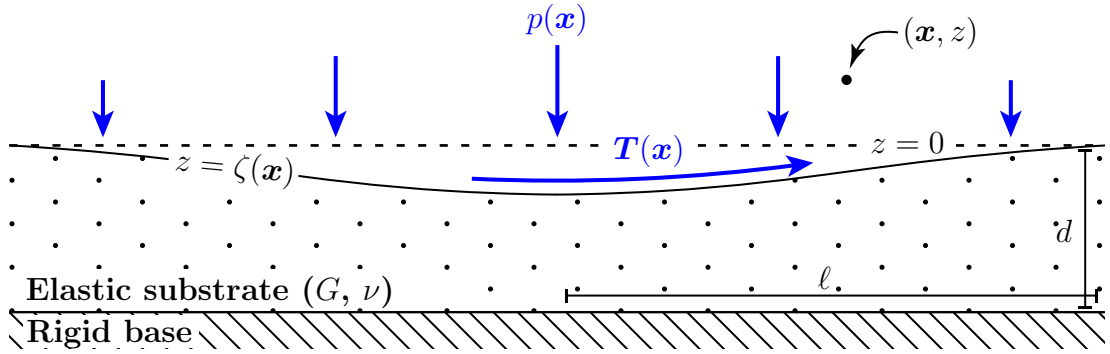


Figure 4.1: A cross-section sketch of the general setup considered in this chapter: the surface (at $z = 0$) of a three-dimensional elastic block adhered to a rigid base (at $z = -d$). A pressure field, $p(\mathbf{x})$, and shear stress, $\mathbf{T}(\mathbf{x})$, act to displace the surface of the solid over a horizontal length scale $\ell \gg d$. The position of the deformed surface is $z = \zeta(\mathbf{x})$.

Assuming small, steady displacements, the planar and vertical displacements, $\mathbf{u}(\mathbf{x}, z) = (u, v)$ and $w(\mathbf{x}, z)$, respectively, are governed by the steady Navier–Cauchy equations (see [Howell et al., 2008](#), for example),

$$\nabla_H(\nabla_H \cdot \mathbf{u} + w_z) + (1 - 2\nu) \left(\nabla_H^2 \mathbf{u} + \mathbf{u}_{zz} \right) = \mathbf{0}, \quad (4.5a)$$

$$\frac{\partial}{\partial z}(\nabla_H \cdot \mathbf{u} + w_z) + (1 - 2\nu) \left(\nabla_H^2 w + w_{zz} \right) = 0. \quad (4.5b)$$

Here, and henceforth, $\nabla_H := (\partial_x, \partial_y)$ denotes the planar del-operator and subscripts denote partial derivatives.

The deformable layer is assumed to be perfectly adhered to the base since this is the intention of most experiments. We, therefore, apply a zero-displacement boundary condition,

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad w = 0, \quad (4.6a,b)$$

¹The work presented in this chapter is based on the published work by [Chandler and Vella \(2020b\)](#).

on $z = -d$. The top surface is assumed free, so we apply normal and tangential stress balances,

$$\frac{2G}{1-2\nu} (\nu \nabla_H \cdot \mathbf{u} + (1-\nu)w_z) \equiv \tau_{zz} = -p(\mathbf{x}), \quad (4.6c)$$

$$G(\mathbf{u}_z + \nabla_H w) \equiv (\tau_{xz}, \tau_{yz}) = \mathbf{T}(\mathbf{x}), \quad (4.6d)$$

on $z = 0$, respectively. (Here, we apply the free surface boundary conditions at $z = 0$, consistent with the infinitesimal displacements assumed in linear elasticity.)

In (4.6d), $\mathbf{T}(\mathbf{x})$ denotes the two-dimensional vector of shear stresses applied to the surface. Often, $\mathbf{T}(\mathbf{x})$ may be coupled to the normal loading, namely the pressure $p(\mathbf{x})$, while in other circumstances, a particular shear stress distribution is required to ensure a particular boundary condition on the surface $z = 0$ is satisfied. We shall consider specific examples of this coupling in the two problems considered in §4.3 and §4.4. For now, however, we consider the problem for general pressure and shear stress profiles, $p(\mathbf{x})$ and $\mathbf{T}(\mathbf{x})$.

Solving (4.5) subject to the boundary conditions (4.6) is difficult and cannot, in general, be done analytically; progress can be made, however, in the case of a two-dimensional substrate or a rotationally-symmetric substrate by using Fourier or Hankel transforms, respectively — such cases have been addressed by [Sneddon \(1995\)](#), for example. In many applications of interest the deformable layer has a small aspect ratio, $\epsilon := d/\ell \ll 1$ where ℓ is the horizontal length scale over which the substrate is deformed (see [Chen et al., 2005](#); [Hyun and Jeong, 2009](#); [Kaltenbrunner et al., 2013](#); [Davies et al., 2018](#); [Saintyves et al., 2016, 2020](#); [Zhang et al., 2020](#); [Box et al., 2020b](#), for example). We, therefore, turn now to consider how this small aspect ratio can be exploited to make asymptotic progress.

4.2.2 Thin layer analysis

The observation that many applications have a small aspect ratio, together with the success of lubrication theory in viscous fluid mechanics ([Leal, 2007](#)), motivates the posing of an asymptotic expansion as $\epsilon := d/\ell \rightarrow 0$. Below, we present the formal asymptotic expansion of the steady Navier–Cauchy equations, (4.5), subject to (4.6).

We begin by noting that a geometrical argument leads to the conclusion that the stresses and strains in the plane are much smaller than their vertical counterparts ([Howell et al., 2008](#), Ch. 6). We therefore assume that $[\mathbf{u}] \sim \epsilon[w]$ and $[\mathbf{T}] \sim \epsilon[p]$ to simplify the analysis, though assuming instead that $[\mathbf{u}] \sim [w]$ and $[\mathbf{T}] \sim [p]$ ultimately gives the same solution.

Based on this rescaling, we introduce the dimensionless variables

$$\mathbf{X} := \frac{\mathbf{x}}{\ell} \equiv \frac{\epsilon \mathbf{x}}{d}, \quad Z := \frac{z}{d}, \quad U(\mathbf{X}, Z) := \frac{\mathbf{u}(\mathbf{x}, z)}{\epsilon[w]}, \quad W(\mathbf{X}, Z) := \frac{w(\mathbf{x}, z)}{[w]}, \quad (4.7a-d)$$

with dimensionless applied pressure and shear stress

$$P(\mathbf{X}) := \frac{d}{[w]} \frac{p(\mathbf{x})}{G} \quad \text{and} \quad \mathcal{T}(\mathbf{X}) := \frac{d}{\epsilon[w]} \frac{\mathbf{T}(\mathbf{x})}{G}, \quad (4.7\text{e,f})$$

respectively.

By inserting (4.7) into the system (4.5) subject to (4.6), we obtain the dimensionless elastostatic equations

$$W_{ZZ} = -\frac{\epsilon^2}{2(1-\nu)} \left(\nabla_H \cdot \mathbf{U}_Z + (1-2\nu) \nabla_H^2 W \right), \quad (4.8\text{a})$$

$$\mathbf{U}_{ZZ} + \frac{1}{1-2\nu} \nabla_H W_Z = -\frac{\epsilon^2}{1-2\nu} \left(\nabla_H (\nabla_H \cdot \mathbf{U}) + (1-2\nu) \nabla_H^2 \mathbf{U} \right), \quad (4.8\text{b})$$

in $-1 \leq Z \leq 0$, subject to the boundary conditions

$$W = 0 \quad \text{and} \quad \mathbf{U} = \mathbf{0} \quad \text{on } Z = -1, \quad (4.9\text{a,b})$$

$$W_Z + \frac{\nu\epsilon^2}{1-\nu} \nabla_H \cdot \mathbf{U} = -\frac{1-2\nu}{2(1-\nu)} P(\mathbf{X}) \quad \text{and} \quad \mathbf{U}_Z + \nabla_H W = \mathcal{T}(\mathbf{X}) \quad \text{on } Z = 0. \quad (4.9\text{c,d})$$

The dimensionless system, (4.8) subject to (4.9), contains only two dimensionless parameters: the substrate's aspect ratio, ϵ , and Poisson's ratio, ν . Assuming $(1-2\nu) = \mathcal{O}(1)$, we seek asymptotic solutions of this system by inserting the expansions

$$W(\mathbf{X}, Z) = W^{(0)}(\mathbf{X}, Z) + \epsilon^2 W^{(1)}(\mathbf{X}, Z) + \mathcal{O}(\epsilon^4), \quad (4.10\text{a})$$

$$\mathbf{U}(\mathbf{X}, Z) = \mathbf{U}^{(0)}(\mathbf{X}, Z) + \epsilon^2 \mathbf{U}^{(1)}(\mathbf{X}, Z) + \mathcal{O}(\epsilon^4), \quad (4.10\text{b})$$

and solving order-by-order as $\epsilon \rightarrow 0$.

At leading-order, $\mathcal{O}(\epsilon^0)$, the displacements $W^{(0)}(\mathbf{X}, Z)$ and $\mathbf{U}^{(0)}(\mathbf{X}, Z)$ satisfy

$$W_{ZZ}^{(0)} = 0 \quad \text{and} \quad \mathbf{U}_{ZZ}^{(0)} = -\frac{1}{1-2\nu} \nabla_H W_Z^{(0)}, \quad (4.11\text{a,b})$$

in $-1 \leq Z \leq 0$, subject to the boundary conditions

$$\mathbf{U}^{(0)}(\mathbf{X}, -1) = \mathbf{0}, \quad W^{(0)}(\mathbf{X}, -1) = 0, \quad (4.12\text{a,b})$$

$$W_Z^{(0)}(\mathbf{X}, 0) = -\frac{1-2\nu}{2(1-\nu)} P(\mathbf{X}), \quad \mathbf{U}_Z^{(0)}(\mathbf{X}, 0) + \nabla_H W^{(0)}(\mathbf{X}, 0) = \mathcal{T}(\mathbf{X}). \quad (4.12\text{c,d})$$

Integrating with respect to Z leads to the solutions

$$\mathbf{U}^{(0)}(\mathbf{X}, Z) = (1 + Z) \mathcal{T}(\mathbf{X}) - \frac{\nu - 1/4}{1 - \nu} (1 + Z) \left(1 - \frac{Z}{4\nu - 1} \right) \nabla_H P(\mathbf{X}), \quad (4.13a)$$

$$W^{(0)}(\mathbf{X}, Z) = -\frac{1 - 2\nu}{2(1 - \nu)} (1 + Z) P(\mathbf{X}). \quad (4.13b)$$

These describe the leading-order displacements provided $(1 - 2\nu) = \mathcal{O}(1)$ and, hence, reproduce the results of the Winkler limit given by [Skotheim and Mahadevan \(2004\)](#). However, as $\nu \rightarrow 1/2$ (*i.e.* in the incompressible limit) $W^{(0)}(\mathbf{X}, Z) \rightarrow 0$ and so higher-order terms will dominate; in particular, $\mathbf{U} \sim \mathbf{U}^{(0)}$ and $W \sim \epsilon^2 W^{(1)}$ for $(1 - 2\nu) \ll \epsilon^2$, where we determine $W^{(1)}$ next.

At next-order, $\mathcal{O}(\epsilon^2)$, the displacement $W^{(1)}(\mathbf{X}, Z)$ satisfies

$$W_{ZZ}^{(1)} = -\frac{1}{2(1 - \nu)} \left(\nabla_H \cdot \mathbf{U}_Z^{(0)} + (1 - 2\nu) \nabla_H^2 W^{(0)} \right), \quad (4.14)$$

in $-1 \leq Z \leq 0$, subject to the boundary conditions

$$W^{(1)}(\mathbf{X}, -1) = 0 \quad \text{and} \quad W_Z^{(1)}(\mathbf{X}, 0) = -\frac{\nu}{1 - \nu} \nabla_H \cdot \mathbf{U}^{(0)}(\mathbf{X}, 0). \quad (4.15a,b)$$

Substituting the known solutions for $\mathbf{U}^{(0)}$ and $W^{(0)}$, (4.13), and integrating with respect to Z leads to

$$\begin{aligned} W^{(1)}(\mathbf{X}, Z) = & -\frac{\nu - 1/4}{1 - \nu} (1 + Z) \left(1 + \frac{Z}{4\nu - 1} \right) \nabla_H \cdot \mathcal{T}(\mathbf{X}) \\ & + \frac{2\nu(\nu - 1/4)}{3(1 - \nu)^2} (1 + Z) \left(1 + \frac{Z}{2} - \frac{1 - \nu}{4\nu - 1} Z^2 \right) \nabla_H^2 P(\mathbf{X}). \end{aligned} \quad (4.16)$$

Overall, combining (4.13) with (4.16), we obtain asymptotic series for the displacements, $\mathbf{U}(\mathbf{X}, Z)$ and $W(\mathbf{X}, Z)$, that depend on the applied surface pressure and shear stress, $P(\mathbf{X})$ and $\mathcal{T}(\mathbf{X})$, respectively. These expressions hold for all values of the substrate's Poisson's ratio ($-1 \leq \nu \leq 1/2$); in particular, for both incompressible ($\nu = 1/2$) and compressible ($\nu \neq 1/2$) substrates alike. Substituting $z = 0$, yields the dimensionless surface displacements

$$\mathbf{U}(\mathbf{X}, 0) = \mathcal{T}(\mathbf{X}) - \beta(\nu) \nabla_H P(\mathbf{X}) + \mathcal{O}(\epsilon^2), \quad (4.17a)$$

$$W(\mathbf{X}, 0) = -\alpha(\nu)(1 - 2\nu)P(\mathbf{X}) - \left[\beta(\nu) \nabla_H \cdot \mathcal{T}(\mathbf{X}) - \gamma(\nu) \nabla_H^2 P(\mathbf{X}) \right] \epsilon^2 + \mathcal{O}(\epsilon^4), \quad (4.17b)$$

where, for convenience, we introduce the coefficients

$$\alpha(\nu) := \frac{1}{2(1 - \nu)}, \quad \beta(\nu) := \frac{\nu - 1/4}{1 - \nu}, \quad \gamma(\nu) := \frac{2\nu(\nu - 1/4)}{3(1 - \nu)^2}. \quad (4.18a-c)$$

In dimensional terms, the surface displacement expansion, (4.17), takes the form

$$\mathbf{u}(\mathbf{x}, 0) = \frac{d}{G} \mathbf{T}(\mathbf{x}) - \beta(\nu) \frac{d^2}{G} \nabla_H p(\mathbf{x}) + \mathcal{O}(\ell \epsilon^3), \quad (4.19a)$$

$$w(\mathbf{x}, 0) = -\alpha(\nu)(1 - 2\nu) \frac{d}{G} p(\mathbf{x}) - \beta(\nu) \frac{d^2}{G} \nabla_H \cdot \mathbf{T}(\mathbf{x}) + \gamma(\nu) \frac{d^3}{G} \nabla_H^2 p(\mathbf{x}) + \mathcal{O}(d \epsilon^3). \quad (4.19b)$$

Equations (4.19) are the main result of this chapter: the surface deformation of a thin deformable layer is a linear combination of the applied normal pressure, $p(\mathbf{x})$; the tangential shear stress, $\mathbf{T}(\mathbf{x})$; and their spatial derivatives. We note here that (4.19) may not hold close to lateral boundaries if the imposed boundary conditions are incompatible with the displacement field suggested by (4.19); in such a scenario, we expect a boundary layer of size $[\mathbf{x}] \sim d$ to resolve this incompatibility. The analysis of these boundary layers will be discussed further in §4.5. For now, we concentrate on problems with compatible boundary conditions (*e.g.* far-field decay conditions).

The quantity of most interest in applications is the vertical displacement of the surface, $\zeta(\mathbf{x}) := w(\mathbf{x}, 0)$. While this is given very simply by (4.19b), we consider two common cases in some detail: no (or negligible) surface-shear, $\mathbf{T}(\mathbf{x}) \equiv \mathbf{0}$, and zero surface-slip, $\mathbf{u}(\mathbf{x}, 0) \equiv \mathbf{0}$. In these cases the deformed surface profile, (4.19b), simplifies to the expression

$$\zeta(\mathbf{x}) := w(\mathbf{x}, 0) \sim -\alpha(\nu)(1 - 2\nu) \frac{d}{G} p(\mathbf{x}) + \Gamma(\nu) \frac{d^3}{G} \nabla_H^2 p(\mathbf{x}), \quad (4.20)$$

where

$$\Gamma(\nu) := \begin{cases} \gamma(\nu) = \frac{2\nu(\nu - 1/4)}{3(1 - \nu)^2} & \text{for zero surface-shear,} \\ \gamma(\nu) - \beta(\nu)^2 = \frac{(3/4 - \nu)(\nu - 1/4)}{3(1 - \nu)^2} & \text{for zero surface-slip.} \end{cases} \quad (4.21)$$

Note that the zero surface-shear and zero surface-slip results also hold for a divergence-free applied shear stress, $\nabla_H \cdot \mathbf{T}(\mathbf{x}) \equiv 0$, and a divergence-free applied surface displacement, $\nabla_H \cdot \mathbf{u}(\mathbf{x}, 0) \equiv 0$, respectively. For such solenoidal fields, the applied shear stress/displacement is only evident in the in-plane displacements, $\mathbf{u}(\mathbf{x}, z)$, with surface value given by (4.19a).

For incompressible substrates, the prefactor $\Gamma(\nu)$ for zero surface-shear, (4.21), is precisely four times that for zero surface-slip, as is familiar in hydrodynamic lubrication theory (Leal, 2007; Mukherjee and Dillard, 2018). However, for compressible substrates ($\nu \neq 1/2$), the ratio of these factors is ν -dependent because the displacement and stress profiles, (4.13) and (4.16), are not symmetric about $z = -d/2$. Also, although the incompressible response of a thin elastomer, (4.3), can be directly compared to the classical results of lubrication theory

for an incompressible fluid, specifically Reynolds' equation (Leal, 2007) — as noted by Gent and Meinecke (1970) — we are not aware of an analogue of (4.1) for a compressible fluid with spatially varying pressure (as was considered by Almqvist et al., 2019, for example).

The surface deflection described in (4.20) is precisely a linear superposition of the Winkler and incompressible foundation models, (4.1) and (4.3), respectively. While this result might seem intuitive, here we have derived it formally. Our formal asymptotic technique is more rigorous than the previous work by Lai et al. (1992) and Bert (1994), in which a parabolic displacement field and uniformly distributed out-of-plane strain were posed as an ansatz to solve the problem. Moreover, our technique allows us to consider the effect of surface shear forces and other boundary conditions on the surface of the substrate. Nonetheless, the relevant results of Lai et al. (1992, eq. (24)) and Bert (1994, eq. (10)) are recovered in the incompressible limit ($\nu \rightarrow 1/2$) of (4.20) for the zero surface-slip case.

To understand the relative importance of the Winkler and incompressible responses, we estimate the ratio between the respective terms in (4.20) as $(1 - 2\nu)/\epsilon^2$: for sufficiently compressible materials, $(1 - 2\nu) \gg \epsilon^2$, the surface deflection is dominated by the $p(\mathbf{x})$ -term and the layer is well-described by the Winkler foundation model. However, for materials that are 'close' to incompressible, $(1 - 2\nu) \ll \epsilon^2 \ll 1$, the substrate response is dominated by the $\nabla_H^2 p(\mathbf{x})$ -term, and the layer instead behaves according to the incompressible elastomeric foundation model.

Our combined model of the layer deflection shows that whether the layer behaves as an incompressible or a compressible layer depends not simply on the value of its Poisson's ratio, ν , but also on its slenderness: even layers that should be expected to be incompressible on the basis of their Poisson's ratio, *i.e.* $\nu \approx 1/2$, may behave in a compressible manner if they are sufficiently thin that $\epsilon^2 \lesssim (1 - 2\nu)$. Since quoted values of ν for elastomeric layers suggest $0.45 \lesssim \nu \lesssim 0.5^2$ (Dogru et al., 2018), we typically expect $1 - 2\nu \sim 10^{-1}$ and, hence, it is clearly feasible that in applications elastic layers will be sufficiently thin to behave in a compressible manner, *i.e.* the Winkler model may still be appropriate. Furthermore, one can readily imagine an experiment in which both compressible (Winkler's) and incompressible responses are observed with slightly different parameter values.

Of course, the above discussion depends on the value of the parameter ϵ , which in turn depends on the length scale ℓ . The challenge, however, is that ℓ is not, in general, known *a priori*; instead ℓ must be determined as part of the solution of a given problem. In the following sections, we illustrate this dependence and discuss when Winkler, rather than incompressible behaviour, can be expected in two concrete examples: the point-like indentation of a thin elastic layer with a stiff coating (see §4.3) and the elastohydrodynamic lift generated by a thin liquid layer above a thin elastic substrate (see §4.4).

²Given this range of ν , we do not consider the possibility that $\Gamma(\nu) = 0$, which occurs when $\nu = 1/4$.

4.3 Example 1: Indentation of a soft layer with a stiff coating

As a first example of the application of the combined foundation model, we consider the response of a relatively stiff elastic layer coating a much softer layer (the elastomer). Such coatings are used as a way to reinforce soft materials (Takei et al., 2011) or to combine electrical conductivity with material flexibility, as in flexible electronics (Rogers and Huang, 2009; Kaltenbrunner et al., 2013). Motivated by the increase in effective stiffness of the substrate created by using a stiff coating, we consider the response to point-indentation. This is a problem that has been considered recently by Cao et al. (2018); Niu et al. (2018); Liu et al. (2019); Box et al. (2020a) for small indentations of a coated elastic half-space (*i.e.* a substrate of infinite depth); in this limit, Box et al. (2020a) showed that the applied point-force, F , required to produce a localized indentation depth δ is $F = K_\infty \delta$, where the constant indentation stiffness

$$K_\infty = \frac{3^{3/2}}{(1-\nu)^{2/3}} B^{1/3} G^{2/3}, \quad (4.22)$$

with B the bending stiffness of the coating and G and ν denoting the moduli of the substrate.

Hertz (1884) considered the point-indentation of an elastic plate with a substrate response that is linear in the vertical displacement — while this response was envisaged to arise from the hydrostatic pressure in liquid underlying floating ice, it is mathematically identical to the indentation of a Winkler foundation coated by a plate, and hence represents the thin, compressible analogue of (4.22). In our notation, Hertz showed that the indentation stiffness $K := F/\delta = 8(Bk_W)^{1/2}$, where k_W is the Winkler modulus of the substrate, as defined in (4.1). In this section, we study how these results are modified when the substrate is slender and may be close to incompressible.

4.3.1 Mathematical modelling

We consider small, axisymmetric deformations of an infinite plate (of bending stiffness B) adhered to a thin substrate (of depth d , Poisson's ratio ν , and shear modulus G) in response to point-like indentation. Note that the coating could alternatively be parameterized by its thickness t , Poisson's ratio ν_c , and shear modulus G_c , by replacing

$$B \equiv \frac{G_c t^3}{6(1-\nu_c)}. \quad (4.23)$$

The setup we consider is sketched in Fig. 4.2.

The vertical displacement of the plate, $\zeta(r)$, in response to indentation and the restoring pressure from the underlying substrate, $q(r)$, are coupled using the zero-stretching Kirchhoff–

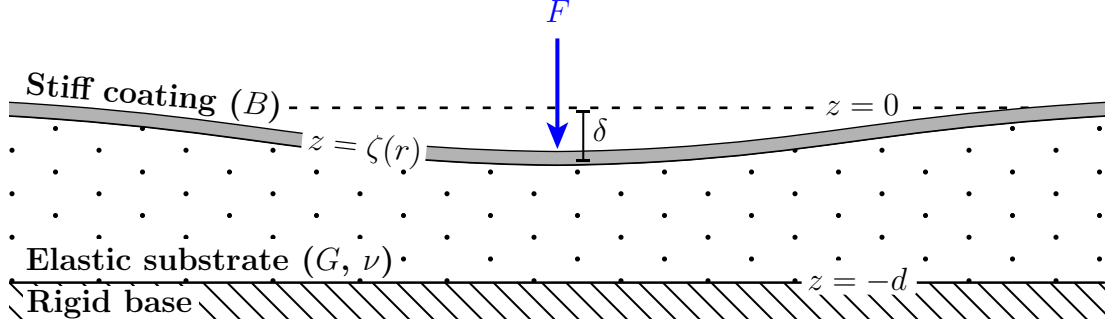


Figure 4.2: A sketch of the first example: the point-like indentation of a stiff coating of a thin, deformable substrate. The coating is loaded by a point-force, F , applied at $r = 0$, resulting in a vertical displacement δ ; the vertical deflection profile, $\zeta(r)$, decays away from the indenter over the length scale ℓ .

Love plate equation (Howell et al., 2008, Ch. 4) together with the combined foundation model — *i.e.* taking $p(\mathbf{x}) \mapsto q(r)$ in (4.20). We, therefore, have that

$$B\nabla_H^4 \zeta(r) = q(r) - \frac{F}{2\pi} \frac{\delta_{\text{Dirac}}(r)}{r}, \quad (4.24a)$$

$$\zeta(r) = -\alpha(\nu)(1-2\nu) \frac{d}{G} q(r) + \Gamma(\nu) \frac{d^3}{G} \nabla_H^2 q(r), \quad (4.24b)$$

with $r \geq 0$ the radial coordinate and F the applied force.

The plate is assumed to be perfectly adhered to the substrate and, since vertical deflections are small, we neglect horizontal displacements of the plate, which are no larger than quadratic in δ (Vella and Davidovitch, 2018; Box et al., 2020a): the boundary condition on the upper surface of the substrate is $u(r, 0) \equiv 0$. We therefore use $\Gamma(\nu)$ to denote the coefficient given in (4.21) for zero surface-slip. We seek solutions of (4.24) subject to the boundary conditions

$$\zeta(0) = -\delta, \quad \zeta'(0) = 0, \quad q'(0) = 0, \quad (4.25a-c)$$

$$\zeta(r), q(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty. \quad (4.25d,e)$$

Equation (4.25a) enforces the indentation depth δ , while (4.25b,c) ensure symmetry about the indentation point; the conditions (4.25d,e) ensure decay far from the indenter.

4.3.1.1 Scaling laws

As a first step, we seek to determine the scaling behaviour of the radial scale, ℓ . For a given indentation depth δ , the length scale ℓ is chosen to minimize the sum of two opposing elastic energies (Box et al., 2020a). The bending energy of the plate, $U_{\text{plate}} \sim B \int (\delta/\ell^2)^2 dA \sim B(\delta/\ell)^2$, penalizes the curvature of the interface, $\sim \delta/\ell^2$, over the deformed area $A \sim \ell^2$ and

so favours large ℓ . In contrast, the substrate's elastic energy, $U_{\text{subs}} \sim \int \mathcal{T} \mathcal{E} \, dV \sim \mathcal{T} \mathcal{E} (\ell^2 d)$, penalizes the strain energy density $\sim \mathcal{T} \mathcal{E}$ within the deformed volume $V \sim \ell^2 d$ and, as we shall see, favours small ℓ . There is, therefore, an optimal ℓ that minimizes the total elastic energy, $U_{\text{elast}} = U_{\text{plate}} + U_{\text{subs}}$. However, the result of this minimization depends on the typical size of the stress, $\mathcal{T}$, and strain, $\mathcal{E}$, within the substrate, which are in turn dependent on the deformation mode of the substrate (*i.e.* how compressible it is). For sufficiently compressible substrates (the Winkler response), the deformation is dominated by the vertical bulk-strain, so that $\mathcal{T} \sim G \mathcal{E}_{zz} / (1 - 2\nu)$ with $\mathcal{E} \sim \mathcal{E}_{zz} \sim \delta / d$, leading to $U_{\text{subs}} \sim G / (1 - 2\nu) \ell^2 \delta^2 / d$. For sufficiently incompressible substrates (incompressible response), the deformation is instead dominated by the shear-strain associated with a horizontal displacement u , which may be estimated as $u \sim \delta \ell / d$ from incompressibility. We, therefore, have $\mathcal{T} \sim G \mathcal{E}_{rz}$ with $\mathcal{E} \sim \mathcal{E}_{rz} \sim u / d \sim \delta \ell / d^2$ and, hence, $U_{\text{subs}} \sim G \ell^4 \delta^2 / d^3$. Minimizing the total elastic energy in each of the two cases gives:

$$\ell \propto \begin{cases} d \left[\frac{B}{G d^3} (1 - 2\nu) \right]^{1/4} & \text{(Winkler),} \\ d \left[\frac{B}{G d^3} \right]^{1/6} & \text{(incompressible).} \end{cases} \quad (4.26)$$

(Note from (4.23) that

$$\frac{B}{G d^3} \equiv \frac{1}{6(1 - \nu_c)} \frac{G_c}{G} \left(\frac{t}{d} \right)^3, \quad (4.27)$$

so that $B / G d^3$ combines the ratio of mechanical properties, G_c / G , with the geometrical ratio t / d .) Following this balance, the total elastic energy scales according to $U_{\text{elast}} \sim B \delta^2 / \ell^2$ in both cases. The work done to reach this deformed state must be done by the indenter, so $U_{\text{ind}} \sim F \delta \sim U_{\text{elast}}$ (Box et al., 2020a), which gives a constant stiffness response

$$\frac{F}{\delta} \sim \frac{B}{\ell^2} \propto \begin{cases} \left[\frac{GB}{d(1 - 2\nu)} \right]^{1/2} & \text{(Winkler),} \\ \left[\frac{GB^2}{d^3} \right]^{1/3} & \text{(incompressible).} \end{cases} \quad (4.28)$$

These results give a first understanding for how the elastic body responds in both the Winkler and incompressible regimes. However, our aim is to understand the transition between the two regimes beyond a scaling analysis; we begin by non-dimensionalizing the model problem, (4.24) subject to (4.25).

4.3.1.2 Non-dimensionalization

Although the natural choice for dimensionless pressure and surface displacement follows from the plate equation, (4.24a) and (4.25a), the choice of radial scale depends on the substrate response, as in (4.26). From the substrate equation (4.24b), we observe that there are two natural choices:

$$\ell_W := d \left[\frac{B}{Gd^3} \alpha(\nu)(1-2\nu) \right]^{1/4} \quad \text{and} \quad \ell_I := d \left[\frac{B}{Gd^3} \Gamma(\nu) \right]^{1/6}. \quad (4.29a,b)$$

We refer to the lengths ℓ_W and ℓ_I as the Winkler and incompressible lengths, respectively.

For definiteness and without loss of generality, we choose the incompressible length as the radial scale, *i.e.* we let $\ell \equiv [r] = \ell_I$, and hence define dimensionless variables

$$\rho := \frac{r}{\ell_I}, \quad Z(\rho) := \frac{\zeta(r)}{\delta}, \quad Q(\rho) := \frac{q(r)}{B\delta/\ell_I^4}. \quad (4.30a-c)$$

With the non-dimensionalization (4.30) the system, (4.24) subject to (4.25), becomes

$$\nabla_H^4 Z(\rho) = Q(\rho) - K_I \frac{\delta_{\text{Dirac}}(\rho)}{\rho}, \quad (4.31a)$$

$$Z(\rho) = -\mathcal{C}Q(\rho) + \nabla_H^2 Q(\rho), \quad (4.31b)$$

subject to

$$Z(0) = -1, \quad Z'(0) = 0, \quad Q'(0) = 0, \quad (4.32a-c)$$

$$Z(\rho), Q(\rho) \rightarrow 0 \quad \text{as } \rho \rightarrow \infty. \quad (4.32d,e)$$

Note that this non-dimensionalization introduces two key dimensionless parameters

$$\mathcal{C} := \left(\frac{\ell_W}{\ell_I} \right)^4 = (1-2\nu) \frac{\alpha(\nu)}{\Gamma(\nu)^{2/3}} \left(\frac{B}{Gd^3} \right)^{1/3}, \quad (4.33a)$$

$$\text{and} \quad K_I := \frac{F\ell_I^2}{2\pi B\delta} = \frac{\Gamma(\nu)^{1/3}}{2\pi} \left(\frac{d^3}{GB^2} \right)^{1/3} \times \frac{F}{\delta}. \quad (4.33b)$$

Here, $\mathcal{C} \propto (1-2\nu)$ measures how far from perfect incompressibility the substrate is — it is a measure of the *substrate compressibility*. The second dimensionless parameter, K_I , is a dimensionless indentation stiffness.

Since the problem is linear, only a particular value of K_I will allow the boundary condition (4.32a) to be satisfied for a given value of $\mathcal{C}$; determining $K_I(\mathcal{C})$ is the primary aim of our analysis. However, the general behaviour of K_I as the substrate compressibility, $\mathcal{C}$, varies can

be outlined: for $\mathcal{C} \ll 1$, the substrate is effectively incompressible and ℓ_I is the appropriate horizontal length scale, so we expect $K_I \sim \text{const.}$ For $\mathcal{C} \gg 1$, the substrate is highly compressible and so ℓ_W is expected to be the appropriate horizontal length scale — to cancel the factor ℓ_I^2 in the definition of K_I — we expect to find $K_I \propto \mathcal{C}^{-1/2}$. These expectations are both consistent with the scaling results (4.28); we now turn to analytical solutions of (4.31) subject to (4.32) to determine the relevant prefactors.

4.3.2 Analytical results

4.3.2.1 General solution

The axisymmetric problem, (4.31), may be solved directly using Hankel transforms (see [Sneddon, 1995](#), for example). An alternative approach (used by [Vella et al., 2012](#); [Box et al., 2017](#), for example) is to seek solutions in terms of modified Bessel functions of the second kind — *i.e.* to let $Z(\rho) \propto K_0(\lambda\rho)$ with $\text{Re } \lambda > 0$ chosen to ensure that the far-field condition, $Z(\rho \rightarrow \infty) = 0$, is satisfied. Since $\nabla_H^2 Z = \lambda^2 Z$ by construction, we obtain the auxiliary equation

$$(\lambda^2)^3 - \mathcal{C}(\lambda^2)^2 - 1 = 0. \quad (4.34)$$

An analysis of this equation shows that for $\mathcal{C} \geq 0$ (*i.e.* $\nu \leq 1/2$) there is only one real solution for λ^2 and so we may write the general solution as

$$Z(\rho) = c_1 K_0(\lambda\rho) + c_2 \text{Re} [K_0(\Lambda\rho)] + c_3 \text{Im} [K_0(\Lambda\rho)], \quad (4.35a)$$

$$Q(\rho) = \nabla_H^4 Z(\rho) = c_1 \lambda^4 K_0(\lambda\rho) + c_2 \text{Re} [\Lambda^4 K_0(\Lambda\rho)] + c_3 \text{Im} [\Lambda^4 K_0(\Lambda\rho)], \quad (4.35b)$$

for unknown real constants c_i and roots of (4.34), λ and Λ , chosen such that $\arg \lambda = 0$ and $0 < \arg \Lambda \leq \pi/2$. Furthermore, (4.34) may be used to write the complex root, Λ , in terms of λ :

$$\Lambda^2 = \frac{i\sqrt{4\lambda^6 - 1} - 1}{2\lambda^4}, \quad (4.35c)$$

where λ^2 is determined by solving the cubic, (4.34), and may be written explicitly as

$$\lambda^2 = \frac{1}{3} \left(\mathcal{C} + \frac{\mathcal{C}^2}{\Delta(\mathcal{C})} + \Delta(\mathcal{C}) \right) \quad \text{with} \quad \Delta(\mathcal{C}) := \left[\mathcal{C}^3 + \frac{3}{2} \left(9 + \sqrt{81 + 12\mathcal{C}^3} \right) \right]^{1/3}. \quad (4.35d,e)$$

The constants, c_i , are determined by imposing the boundary conditions at $\rho = 0$, (4.32a–c); we find that

$$Z(\rho) = \frac{\text{Im} \left[\frac{K_0(\lambda\rho) - K_0(\Lambda\rho)}{\lambda^4 - \Lambda^4} \right]}{\text{Im} \left[\frac{\log(\lambda/\Lambda)}{\lambda^4 - \Lambda^4} \right]} \quad \text{and} \quad Q(\rho) = \frac{\text{Im} \left[\frac{\lambda^4 K_0(\lambda\rho) - \Lambda^4 K_0(\Lambda\rho)}{\lambda^4 - \Lambda^4} \right]}{\text{Im} \left[\frac{\log(\lambda/\Lambda)}{\lambda^4 - \Lambda^4} \right]}. \quad (4.36a,b)$$

The indentation stiffness K_I is calculated from the resultant shear force at the origin, *i.e.*

$$K_I = \lim_{\rho \rightarrow 0} \left[-\rho \frac{d}{d\rho} \nabla_H^2 Z(\rho) \right] = \frac{\text{Im} \left[\frac{1}{\lambda^2 + \Lambda^2} \right]}{\text{Im} \left[\frac{\log(\lambda/\Lambda)}{\lambda^4 - \Lambda^4} \right]}. \quad (4.36c)$$

Figure 4.3 shows the surface profile predicted by (4.36a) for three different values of $\mathcal{C}$. (Since different values of $\mathcal{C}$ correspond to the incompressible or Winkler characters of the substrate dominating, this figure shows each profile plotted with respect to the different horizontal length scales ℓ_I and ℓ_W on the left and right, respectively.) The behaviour of the indentation stiffness, K_I , as a function of $\mathcal{C}$, as given in (4.36c), is plotted in Fig. 4.4(a).

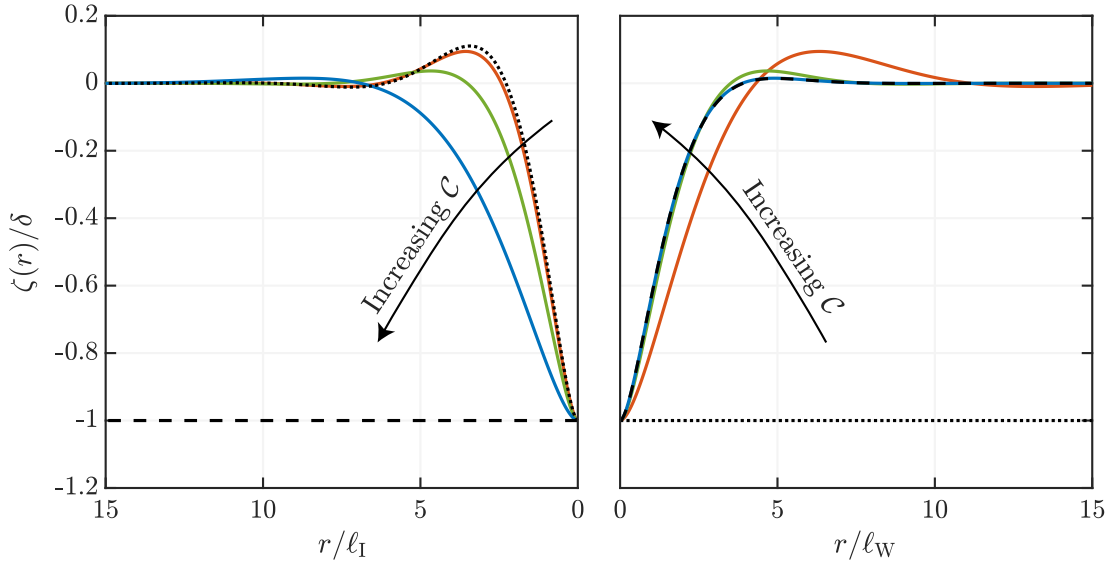


Figure 4.3: The surface deflection of a stiff coating of a thin deformable substrate as predicted by the combined foundation model for different values of the compressibility parameter, $\mathcal{C}$, defined in (4.33a). Left: deflection profiles rescaled by the incompressible length scale, $\ell_I = (\Gamma(\nu)Bd^3/G)^{1/6}$ (as appropriate for the perfectly incompressible case). Right: deflection profiles rescaled by the Winkler length scale, $\ell_W = (\alpha(\nu)(1 - 2\nu)Bd/G)^{1/4}$ (as appropriate for the compressible case). In each case, dashed curves show the interface profile predicted for a Winkler foundation, (4.38a), while dotted curves show the interface profile predicted for an incompressible foundation, (4.37a). Solid curves show the profiles predicted by the combined foundation model, (4.4), for $\mathcal{C} = 0.1$ (red), $\mathcal{C} = 1$ (green), and $\mathcal{C} = 10$ (blue).

To further understand the behaviour of the solution in (4.36), we consider the two limits of interest: the incompressible limit ($\mathcal{C} \rightarrow 0$) and the Winkler/compressible limit ($\mathcal{C} \rightarrow \infty$).

4.3.2.2 Incompressible foundation ($\mathcal{C} \rightarrow 0$)

To recover the incompressible foundation behaviour from (4.36) we let $\mathcal{C} \rightarrow 0$, leading to $\lambda \sim 1$ and $\Lambda \sim e^{i\pi/3}$. The deflection profile may, therefore, be written in the dimensional

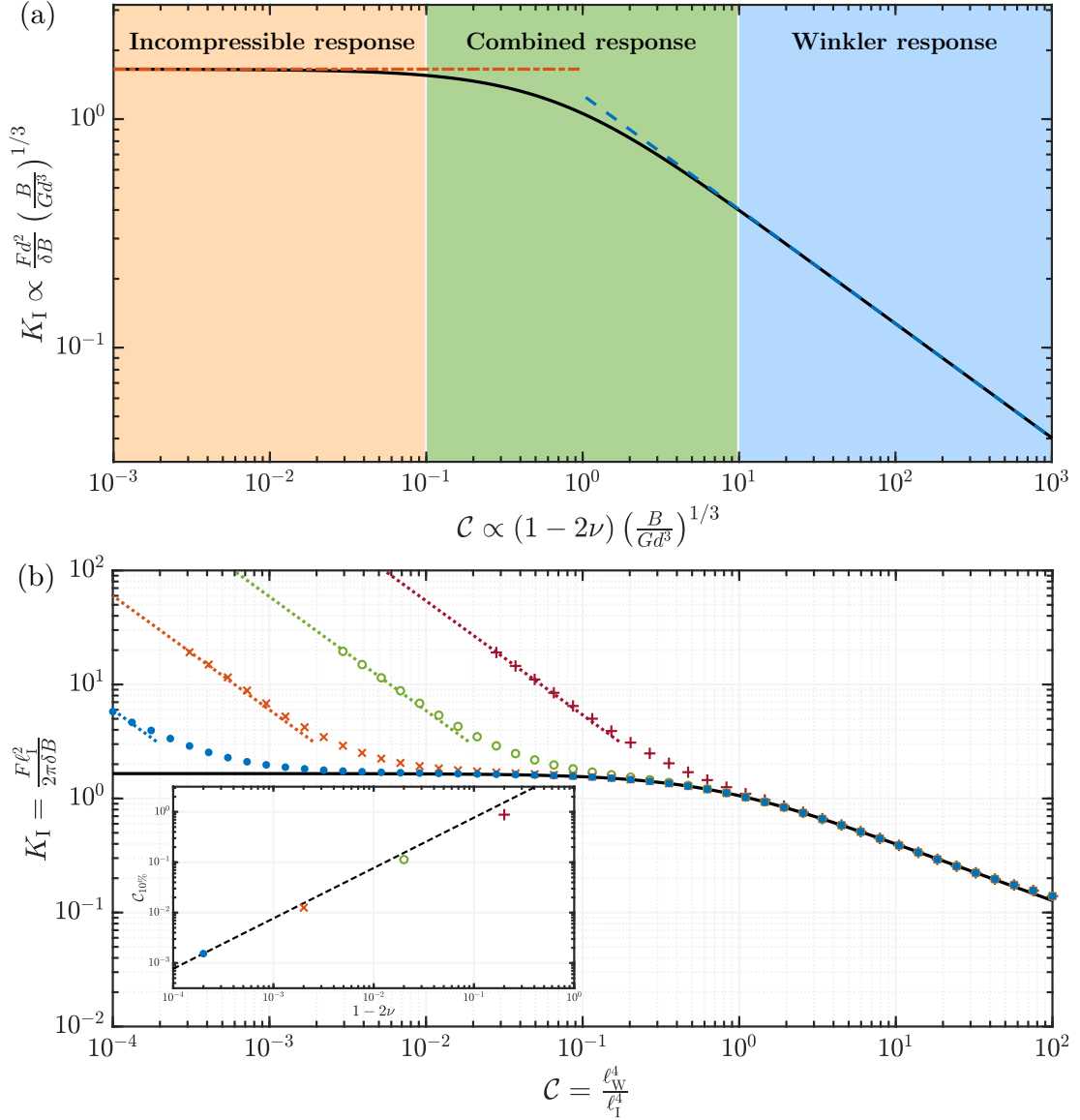


Figure 4.4: The dimensionless indentation stiffness, $K_I \propto F/\delta$, for a stiff coating of a thin elastomeric layer with varying compressibility, C . (a) The prediction of the combined foundation model, (4.36c), recovers the result for a Winkler mattress, (4.38b), (dashed line) in the compressible limit, $C \rightarrow \infty$, and that for an incompressible foundation, (4.37b), (dash-dotted line) in the incompressible limit, $C \rightarrow 0$. Shading illustrates the relevant substrate response in each range of values of C . (b) Main figure: Comparison between the prediction of the combined foundation model (solid curve) and numerical solution of the full problem, (4.A.2)–(4.A.4), (points). Numerical results are shown for different values of the Poisson's ratio, ν , as follows: $\nu = 0.4$ (+), $\nu = 0.49$ (○), $\nu = 0.499$ (×), and $\nu = 0.4999$ (●). We observe good agreement between the numerics and the combined foundation model provided that $C \gg 1 - 2\nu$. For $C \lesssim 1 - 2\nu$, the numerical results tend to the analytical result for an infinitely deep substrate, (4.22), (dotted lines). Inset: The C for which the relative error in the use of the combined foundation model is 10%, denoted $C_{10\%}$, scales linearly with $1 - 2\nu$ (dashed line).

form

$$\frac{\zeta(r)}{\delta} \sim \frac{\sqrt{3}}{\pi} \operatorname{Re} \left[K_0 \left(\frac{r}{\ell_I} \right) - 2e^{i\pi/3} K_0 \left(\frac{r}{\ell_I} e^{i\pi/3} \right) \right], \quad (4.37a)$$

where δ is related to the applied load, F , by the stiffness

$$\frac{F}{\delta} \sim 6\sqrt{3} \frac{B}{\ell_I^2} \equiv 6\sqrt{3} \left(\frac{B^2 G}{\Gamma(\nu) d^3} \right)^{1/3}. \quad (4.37b)$$

While the related case of line-indentation of a plate on top of a thin, perfectly incompressible foundation has been studied previously (Dillard, 1989; Bert, 1994; Dillard et al., 2018; Mukherjee and Dillard, 2018), to our knowledge the axisymmetric solution, (4.37), is new.

4.3.2.3 Winkler foundation ($\mathcal{C} \rightarrow \infty$)

The Winkler foundation behaviour is recovered in the limit $\mathcal{C} \rightarrow \infty$, for which $\lambda \sim \mathcal{C}^{1/2}$ and $\Lambda \sim e^{i\pi/4} \mathcal{C}^{-1/4}$. The deflection profile may, therefore, be written in the dimensional form

$$\frac{\zeta(r)}{\delta} \sim \frac{4}{\pi} \operatorname{Im} \left[K_0 \left(\frac{r}{\ell_W} e^{i\pi/4} \right) \right] \equiv \frac{4}{\pi} \operatorname{kei}_0 \left(\frac{r}{\ell_W} \right), \quad (4.38a)$$

where $\operatorname{kei}_0(x)$ is the Kelvin kei -function of zeroth-order (see Abramowitz and Stegun, 1964, for example); and δ is related to F by

$$\frac{F}{\delta} \sim 8 \frac{B}{\ell_W^2} \equiv 8 \left(\frac{BG}{\alpha(\nu)(1-2\nu)d} \right)^{1/2}, \quad (4.38b)$$

which is equivalent to the result of Hertz (1884) for the response of a plate floating on a liquid bath. (In Hertz' problem, the hydrostatic pressure within the liquid is linear in the deflection and, hence, behaves as a perfect Winkler foundation — see Dempsey et al., 1991; Wang et al., 2005; Box et al., 2017, for example.)

The results presented in Fig. 4.4(a) show that the combined foundation model interpolates between the incompressible and Winkler limits. We find that the incompressible foundation is recovered for $\mathcal{C} \lesssim 10^{-1}$, while the Winkler foundation is recovered for $\mathcal{C} \gtrsim 10^1$. For $10^{-1} \lesssim \mathcal{C} \lesssim 10^1$ the substrate does not sit cleanly in either limit and the combined foundation model, (4.4), must be used.

4.3.2.4 Model validity

The analysis presented so far rests on the assumption that the length scale ℓ is large enough to justify the thin layer analysis of the substrate deformation *and* the use of the plate

equation. Together these assumptions amount to assuming that

$$\max\{d^2, t^2\} \ll \ell^2. \quad (4.39)$$

Since the appropriate value of ℓ depends on $\mathcal{C}$, we divide both sides by d^2 and note that $(\ell_1/d)^2 \equiv C\Gamma(\nu)/[(1-2\nu)\alpha(\nu)]$ and $(\ell_W/d)^2 \equiv \mathcal{C}^{3/2}\Gamma(\nu)/[(1-2\nu)\alpha(\nu)]$ so that we can rewrite (4.39) as

$$\max\{1, \hat{t}^2\} \ll \frac{\Gamma(\nu)}{(1-2\nu)\alpha(\nu)} \max\{\mathcal{C}, \mathcal{C}^{3/2}\}, \quad (4.40)$$

where $\hat{t} := t/d$ is the ratio of sheet thickness to the substrate depth.

The limit on the size of $\hat{t}$ in (4.40) expresses when modelling the stiff coating as a Kirchhoff–Love plate is valid. Of more interest here, however, is the first inequality, which describes the validity of the small foundation aspect ratio assumption ($\epsilon^2 \equiv d^2/\ell^2 \ll 1$) that was used to derive (4.20). In particular, for the combined foundation model to be valid, we require that

$$1 \ll \frac{\Gamma(\nu)}{(1-2\nu)\alpha(\nu)} \max\{\mathcal{C}, \mathcal{C}^{3/2}\} \equiv \max \left\{ \left[\Gamma(\nu) \frac{B}{Gd^3} \right]^{1/3}, \left[(1-2\nu)\alpha(\nu) \frac{B}{Gd^3} \right]^{1/2} \right\}. \quad (4.41)$$

To understand this condition better, we now compare the predictions of the combined foundation model with numerical solutions of the full equilibrium equations for the substrate, (4.5).

4.3.3 Comparison with numerical results

When the assumption of small aspect ratios ($\epsilon \ll 1$) is no longer valid, one must instead solve the full two-dimensional equilibrium equations, (4.5), on $-d \leq z \leq 0$ and $0 \leq r < \infty$ subject to the boundary conditions (4.6) with $\mathbf{T}(\mathbf{x})$ chosen such that $u(r, 0) = 0$ and $p(\mathbf{x}) \equiv q(r)$ given by the plate equation, (4.24a) and (4.25). Non-dimensionalizing using the scalings presented in (4.30) and noting that the aspect ratio can be written as

$$\epsilon^2 = \frac{d^2}{\ell_1^2} \equiv \frac{\alpha(\nu)(1-2\nu)}{\Gamma(\nu)\mathcal{C}}, \quad (4.42)$$

we obtain a system controlled by the Poisson's ratio ν and the compressibility $\mathcal{C}$. In the full problem, the Poisson's ratio can no longer be scaled out of the problem (as was possible for $\epsilon \rightarrow 0$): ν is an additional dimensionless parameter. The resulting system of partial differential equations may be solved using standard numerical techniques; here, we use the method of finite differences. The full system and further numerical discussion is presented in Appendix 4.A.1.

For given values of the parameters ν and $\mathcal{C}$, we can compute the indentation stiffness using the first integral of (4.31a) for any foundation aspect ratio, (4.42). Figure 4.4(b) shows the dimensionless stiffness, K_I , for varying $\mathcal{C}$ and ν as computed numerically, together with the prediction of the combined foundation model, (4.36c), and the corresponding result for an infinitely deep foundation, (4.22), which was presented by Box et al. (2020a). This plot shows excellent agreement between the combined foundation model and the numerics for sufficiently large $\mathcal{C}$ (or, equivalently, sufficiently small aspect ratios, $\epsilon \ll 1$). For small $\mathcal{C}$ (or, equivalently, large aspect ratios, $\epsilon \gg 1$), however, we see that the numerical results are better predicted by (4.22), which is valid for an infinitely deep foundation. Crucially, the domain of validity of the combined foundation model, (4.4), reaches smaller $\mathcal{C}$ as $\nu \rightarrow 1/2$, as expected from (4.41).

In the example considered in this section, the horizontal length scale was set simply by the compressibility of the substrate and, hence, was a material parameter, independent of the applied load (*i.e.* F). We now consider a more involved example in which the horizontal length scale varies with the applied load.

4.4 Example 2: Elastohydrodynamic sedimentation of a cylinder

A classic problem in fluid mechanics is the ability to lubricate two solid bodies using a thin layer of a viscous liquid (Jeffrey and Onishi, 1981). Lubrication relies on tangential motion generating a vertical force that can support load; given its ubiquity, it is at first surprising that with rigid, fore-aft symmetric boundaries, viscous fluid flow cannot generate a significant vertical lift force — a consequence of Purcell’s so-called ‘Scallop Theorem’ (Purcell, 1977; Lauga, 2011). There are many means by which this symmetry can be broken (including cavitation, Ashmore et al., 2005; non-Newtonian effects; and inertia, Lauga, 2011) but one of the most common is the inclusion of a deformable boundary (Lauga, 2011). Since the coupling between elastic deformation and fluid flow can make analytical work difficult, a popular model of a deformable boundary is the Winkler model; several works have studied the coupling between a Winkler mattress and the flow induced by a horizontally translating cylinder (*e.g.* Skotheim and Mahadevan, 2004, 2005; Salez and Mahadevan, 2015) or sphere (*e.g.* Weekley et al., 2006; Davies et al., 2018).

Several recent studies have focussed on experimental realizations of the sedimentation of a rigid cylinder in a viscous liquid above an inclined plane with a soft, elastomeric coating (see Saintyves et al., 2016, 2020; Rallabandi et al., 2017, for example). Such an experiment is conducted under force-controlled conditions: the cylinder sediments until the separation between the substrate and its base generates sufficient hydrodynamic force to balance the

component of the cylinder’s weight normal to the plane; at this point, the cylinder translates at a constant height above the surface of the substrate and may also rotate (Saintyves et al., 2016). The theoretical understanding of these experiments, presented by Rallabandi et al. (2017) and Saintyves et al. (2020), generally rests on the predictions of Winkler based models, while experiments use elastomeric coatings that are (close to) incompressible — for example Saintyves et al. (2016) use polyacrylamide (PAA) and polydimethylsiloxane (PDMS), which are both known to have a Poisson’s ratio in the region $0.45 \lesssim \nu \lesssim 0.5$ (Dogru et al., 2018; Boudou et al., 2006). Given the preceding discussion, it is natural to ask whether/when such experiments operate in the compressible or incompressible regimes? In this section, we use the combined model, (4.4), to address this question.

4.4.1 Mathematical modelling

We consider a two-dimensional cylinder (of radius R and density $\rho + \Delta\rho$) translating perpendicular to its axis at a uniform speed U and rotating clockwise about its axis at an angular speed ω in a viscous fluid (of viscosity μ and density ρ) suspended above a thin elastic substrate (of depth d , Poisson’s ratio ν , and shear modulus G), which is inclined at an angle θ to the horizontal. This setup is shown schematically in Fig. 4.5. The motion of the cylinder parallel to the foundation forces liquid to flow beneath the cylinder and out again, introducing a one-dimensional pressure field $p(x)$ (measured with respect to atmospheric pressure) that is positive ahead of the cylinder and negative behind it. (Note that here, negative pressures are relative to atmospheric and so do not, in general, correspond to cavitation of the liquid.) The deformation of the substrate introduces an asymmetry in $p(x)$ that has a non-zero integral and, hence, can provide a sustained lift to support the cylinder. The magnitude of this lift will depend on the minimum separation between the cylinder and the undeflected surface of the soft substrate, which we denote by h .

The parabolic geometry near the minimum cylinder–substrate separation means that lubrication problems of this type have a characteristic horizontal length scale $\ell \propto (hR)^{1/2}$ (see Jeffrey and Onishi, 1981, for example). However, h is itself not known *a priori* — it is determined by the requirement to generate enough lift force to support the cylinder’s weight, which in turn depends on the type of substrate response (Winkler or incompressible), hence h depends on ℓ . To understand this coupling better, we study the behaviour in terms of the experimental control parameters (the density of the cylinder, *etc.*) rather than emergent physical properties (such as h). To do this, we use the combined foundation model — *i.e.* taking $p(\mathbf{x}) \mapsto p(x)$ in (4.20) — coupled with Reynolds’ equation (Leal, 2007) to describe the fluid flow in the narrow gap beneath the cylinder. In particular, in the reference

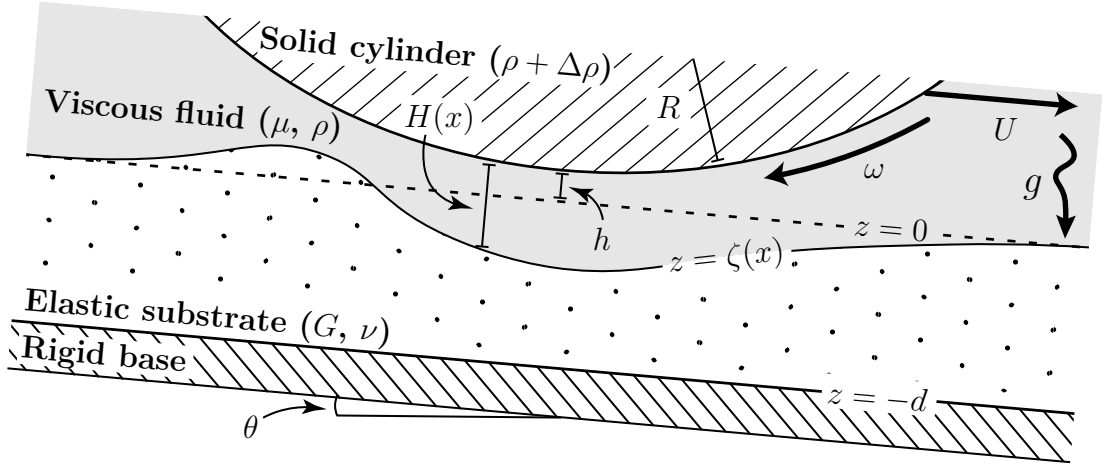


Figure 4.5: A sketch of the second example: a two-dimensional cylinder (of radius R) translates and rotates above an inclined elastic substrate. The cylinder's weight is supported by the lift force that is generated by the interaction between viscous fluid flow and substrate deformation. The resulting gap profile is denoted $H(x) = h + x^2/2R - \zeta(x)$ where $\zeta(x)$ is the profile of the deformed surface and h is the minimum gap thickness without deformation.

frame of the translating cylinder, we have

$$\frac{d}{dx} \left[H(x)^3 \frac{dp}{dx} + 6\mu (U + \omega R) H(x) \right] = 0, \quad (4.43a)$$

where x is the local coordinate parallel to the wall and $H(x)$ is the separation between the cylinder surface and that of the elastomeric coating. Since the length scale $\ell = \sqrt{2hR}$ characterizes the horizontal length over which the lubrication forces are important, we can apply a parabolic approximation, which is valid for all nonconforming contacts, for the shape of the cylindrical surface (Skotheim and Mahadevan, 2004, 2005):

$$H(x) = h + \frac{x^2}{2R} - \zeta(x). \quad (4.43b)$$

The displacement of the elastomer surface, $\zeta(x)$, is given in terms of the pressure field by (4.20), *i.e.*

$$\zeta(x) = -\alpha(\nu)(1 - 2\nu) \frac{d}{G} p(x) + \Gamma(\nu) \frac{d^3}{G} \frac{d^2 p}{dx^2}, \quad (4.43c)$$

where we neglect the effect of the viscous shear stress on the elastomer surface; this shear stress is $T(x) \sim H(x)p'(x)$, so provided $[H] \ll d$ this neglect of T is appropriate and we may use the expression for $\Gamma(\nu)$ for zero surface-shear in (4.21). Solutions of (4.43) are subject to far-field decay conditions,

$$p(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm\infty. \quad (4.44)$$

In experiments (*e.g.* Saintyves et al., 2016, 2020), the gap thickness, h ; translational speed, U ; and angular velocity, ω , evolve until they reach steady values in which the cylinder experiences no net force perpendicular or parallel to the plane and no net torque. Within our lubrication model, these conditions may be written as

$$\pi R^2 \Delta \rho g \cos \theta = \int_{-\infty}^{\infty} p(x) dx, \quad (4.45a)$$

$$\pi R^2 \Delta \rho g \sin \theta = \int_{-\infty}^{\infty} \left[\frac{x}{R} p(x) + \tau_{xz}|_{z=H} \right] dx, \quad (4.45b)$$

$$0 = \int_{-\infty}^{\infty} R \tau_{xz}|_{z=H} dx = R \int_{-\infty}^{\infty} \left[\frac{1}{2} H(x) p'(x) + \frac{\mu(U - \omega R)}{H} \right] dx, \quad (4.45c)$$

respectively (see Skotheim and Mahadevan, 2005; Rallabandi et al., 2017, for example). Note that (4.45c) immediately eliminates the second term in the integrand of (4.45b).

4.4.1.1 Scaling analysis

A key difference between the present problem and that presented in §4.3 is that the pressure scale that determines the deformation of the elastomeric foundation is not clearly related to the pressure that supplies the normal force to balance the weight of the cylinder. This is because the pressure field associated with the leading-order problem (the rigid case) is anti-symmetric and, hence, does not supply a normal load (Skotheim and Mahadevan, 2004). This leading-order problem does, however, dominate the tangential force balance (4.45b), which, when combined with a scaling analysis of Reynolds' equation, (4.43a), allows us to determine the leading-order pressure and velocity scales, p_* and $U_* \sim R\omega_*$, respectively. In particular, assuming that the steady translation of the cylinder occurs with a separation h from the undisturbed level of the substrate and, hence, that the relevant horizontal scale is $\propto (hR)^{1/2}$, Reynolds' equation, (4.43a), predicts that

$$p_* \propto \frac{\mu U_* R^{1/2}}{h^{3/2}} \sim \frac{\mu \omega_* R^{3/2}}{h^{3/2}}, \quad (4.46)$$

while (4.45b) predicts that $\Delta \rho g R^2 \sin \theta \propto h p_*$. Solving these two equations gives

$$p_* \propto \frac{\Delta \rho g R^2 \sin \theta}{h} \quad \text{and} \quad U_* \sim R\omega_* \propto \frac{\Delta \rho g R^{3/2} h^{1/2} \sin \theta}{\mu}, \quad (4.47a,b)$$

with h still undetermined. As already discussed, h must depend on the deformability of the substrate: without substrate deformation, no value of h can generate an equilibrium lift force. Nevertheless, we can use the scaling estimates (4.47) to estimate the deflection of the

foundation for each of the Winkler and incompressible limits, denoted ζ_* ; we find that

$$\zeta_* \propto \begin{cases} (1 - 2\nu) \frac{d}{G} p_* & \text{(Winkler),} \\ \frac{d^3}{G} \frac{p_*}{hR} & \text{(incompressible).} \end{cases} \quad (4.48)$$

The final piece of the scaling puzzle is to understand the effect of the deflection ζ_* on the pressure field; the perturbation to the pressure field, δp , is what supports the cylinder's load. Since the pressure depends on gap thickness algebraically, the relative change in pressure should be proportional to the relative change in gap width, *i.e.* $\delta p/p_* \propto \zeta_*/h$. The vertical force balance, (4.45a), then gives $\Delta \rho g R^2 \cos \theta \propto \delta p (hR)^{1/2} \propto p_* \zeta_* (R/h)^{1/2}$ and two possible scalings for the separation h , depending on which of the deflection scales in (4.48) is more appropriate,

$$h \propto \begin{cases} \left(\frac{(1 - 2\nu) \Delta \rho g d \sin \theta \tan \theta}{G} \right)^{2/5} R & \text{(Winkler),} \\ \left(\frac{\Delta \rho g d^3 \sin \theta \tan \theta}{G} \right)^{2/7} R^{3/7} & \text{(incompressible).} \end{cases} \quad (4.49)$$

These scalings have been given before by [Skotheim and Mahadevan \(2004, 2005\)](#); our aim here is to go beyond the scaling analysis and investigate the transition between the Winkler and incompressible regimes. We turn first to the non-dimensionalization of the problem, which will be informed by the scaling behaviour determined above.

4.4.1.2 Non-dimensionalization

From the preceding scaling analysis, there are two limiting scalings for the horizontal length scale, ℓ :

$$\ell_W := d \left(\alpha(\nu) (1 - 2\nu) \frac{2R^5 \Delta \rho g \sin \theta \tan \theta}{d^4 G} \right)^{1/5} \quad (4.50a)$$

$$\text{or } \ell_I := d \left(\Gamma(\nu) \frac{2R^5 \Delta \rho g \sin \theta \tan \theta}{d^4 G} \right)^{1/7}, \quad (4.50b)$$

which correspond to foundations whose response is of the Winkler or incompressible type, respectively. Without loss of generality, we choose the incompressible length scale, ℓ_I , as the horizontal scale and, motivated by the preceding scaling analysis, define vertical scales $[h] = \ell_I^2/(2R)$ and $[\zeta] := \ell_I^3/(2R^2 \tan \theta)$. (We emphasize that these scales are defined in terms of experimental control parameters rather than the emergent properties h , U , and ω — which were used instead in [Skotheim and Mahadevan, 2004, 2005](#), for example.)

We introduce the dimensionless variables

$$X := \frac{x}{\ell_1}, \quad Z(X) := \frac{2R^2 \tan \theta}{\ell_1^3} \zeta(x), \quad (4.51a,b)$$

$$\mathcal{H}(X) := \frac{2R}{\ell_1^2} H(x), \quad P(X) := \frac{\ell_1^2}{R^3 \Delta \rho g \sin \theta} p(x). \quad (4.51c,d)$$

Inserting (4.51) into the model, (4.43) subject to (4.44), we find the dimensionless system

$$\frac{d}{dX} [\mathcal{H}(X)^3 P'(X) + 6(\hat{U} + \hat{\omega}) \mathcal{H}(X)] = 0, \quad (4.52a)$$

$$\mathcal{H}(X) = \hat{h} + X^2 - \hat{\phi} Z(X), \quad Z(X) = -\hat{\mathcal{C}} P(X) + P''(X), \quad (4.52b,c)$$

which is to be solved subject to $P(X) \rightarrow 0$ as $X \rightarrow \pm\infty$ and the integral constraints (4.45), which after simplification read

$$\hat{\phi} = \frac{1}{\pi} \int_{-\infty}^{\infty} P(X) dX, \quad 1 = \frac{1}{\pi} \int_{-\infty}^{\infty} X P(X) dX, \quad (4.52d,e)$$

$$0 = \int_{-\infty}^{\infty} \frac{1}{2} \mathcal{H}(X) P'(X) + \frac{\hat{U} - \hat{\omega}}{\mathcal{H}(X)} dX. \quad (4.52f)$$

Note that Reynolds' equation, (4.52a), with $\mathcal{H}(X) = \mathcal{O}(X^2)$ as $|X| \rightarrow \infty$ gives $P(X) = \mathcal{O}(1/X^3)$ as $|X| \rightarrow \infty$; using this and the combined foundation (4.52c) gives

$$\int_{-\infty}^{\infty} \frac{1}{2} \mathcal{H}(X) P'(X) dX = - \int_{-\infty}^{\infty} X P(X) dX = -\pi, \quad (4.53)$$

allowing further simplification of (4.52f).

The system (4.52) is governed by the, experimentally-controlled, dimensionless parameters

$$\hat{\mathcal{C}} := \frac{\alpha(\nu)(1-2\nu)}{\Gamma(\nu)} \left(\frac{\ell_1}{d} \right)^2 \quad \text{and} \quad \hat{\phi} := \frac{\ell_1}{R \tan \theta}, \quad (4.54a,b)$$

but also involves three emergent quantities, the dimensionless gap thickness, $\hat{h}$; translation speed, $\hat{U}$; and rotation speed, $\hat{\omega}$, which are to be determined as part of the solution. The physical minimum gap thickness, h ; translation speed, U ; and rotation speed, ω , are given by

$$h = \hat{h} \times \frac{\ell_1^2}{2R}, \quad U = \hat{U} \times \frac{\ell_1 R \Delta \rho g \sin \theta}{4\mu}, \quad \omega = \hat{\omega} \times \frac{\ell_1 \Delta \rho g \sin \theta}{4\mu}. \quad (4.54c-e)$$

We also note that the dimensionless parameter $\hat{\mathcal{C}} \equiv (\ell_W/\ell_1)^5 \propto (1-2\nu)$, so $\hat{\mathcal{C}}$ is a measure of the compressibility of the substrate, analogous to the parameter $\mathcal{C}$ in the previous example, (4.33a). In addition, $\hat{\phi} \equiv [\zeta]/[H]$, so $\hat{\phi}$ is a measure of how pliable the substrate is. In experiments, $\hat{\phi}$ is often small (for example, we estimate that $\hat{\phi} = \mathcal{O}(10^{-2}-10^{-1})$ in

Saintyves et al., 2016, 2020): the elastic substrate is relatively stiff and makes only small departures from its undeformed planar state. In the limit $\hat{\phi} \rightarrow 0$, it is possible to make analytical progress, so we turn first to this limit before considering numerical solutions.

4.4.2 Analytical results for $\hat{\phi} \ll 1$

Following the perturbative approach of Skotheim and Mahadevan (2004, 2005) for a Winkler foundation, we pose the regular expansions

$$Z(X) = Z_0(X) + \hat{\phi}Z_1(X) + \mathcal{O}(\hat{\phi}^2), \quad P(X) = P_0(X) + \hat{\phi}P_1(X) + \mathcal{O}(\hat{\phi}^2), \quad (4.55a,b)$$

$$\hat{h} = \hat{h}_0 + \hat{\phi}\hat{h}_1 + \mathcal{O}(\hat{\phi}^2), \quad \hat{U} = \hat{U}_0 + \hat{\phi}\hat{U}_1 + \mathcal{O}(\hat{\phi}^2), \quad \hat{\omega} = \hat{\omega}_0 + \hat{\phi}\hat{\omega}_1 + \mathcal{O}(\hat{\phi}^2), \quad (4.55c-e)$$

in (4.52) and consider the problem order-by-order. Note that the definition of the gap separation, $\mathcal{H}(X)$, in (4.52b) included a factor of $\hat{\phi}$ multiplying $Z(X)$, hence the deflection *does* return to the rigid parabolic geometry as $\hat{\phi} \rightarrow 0$.

At leading-order, $\mathcal{O}(\hat{\phi}^0)$, we recover the rigid-substrate problem considered by Jeffrey and Onishi (1981), as expected. This determines the leading-order surface deflection, $\hat{\phi}Z_0(X)$, which is dependent on $P_0(X)$:

$$P_0(X) = \frac{2(\hat{U}_0 + \hat{\omega}_0)X}{(\hat{h}_0 + X^2)^2}, \quad (4.56a)$$

$$Z_0(X) = -\frac{2(\hat{U}_0 + \hat{\omega}_0)X}{(\hat{h}_0 + X^2)^2}\hat{\mathcal{C}} - \frac{24(\hat{U}_0 + \hat{\omega}_0)X(\hat{h}_0 - X^2)}{(\hat{h}_0 + X^2)^4}. \quad (4.56b)$$

Substituting these expressions into the integral constraints, (4.52d–f) at $\mathcal{O}(\hat{\phi}^0)$, yields

$$\hat{U}_0 = \hat{h}_0^{1/2} \quad \text{and} \quad \hat{\omega}_0 = 0. \quad (4.57a,b)$$

At the next-order, $\mathcal{O}(\hat{\phi}^1)$, MATHEMATICA can be used to obtain explicit expressions for $P_1(X)$ and $Z_1(X)$ that are omitted here for reasons of space. Substituting these expressions into the integral constraints, (4.52d–f) at $\mathcal{O}(\hat{\phi}^1)$, yields

$$\hat{h}_0^{7/2} = \frac{3}{8}\hat{\mathcal{C}}\hat{h}_0 + \frac{45}{16}, \quad \hat{U}_1 = \frac{\hat{h}_1}{2\hat{h}_0^{1/2}}, \quad \hat{\omega}_1 = 0. \quad (4.58a-c)$$

Solving (4.57a) and (4.58a) gives the leading-order gap thickness, $\hat{h} \sim \hat{h}_0$, and translational speed, $\hat{U} \sim \hat{U}_0 = \hat{h}_0^{1/2}$. (Note that for general values of $\hat{\mathcal{C}}$, (4.58a) is an algebraic equation for $\hat{h}_0$ that must be solved numerically, although asymptotic results are readily available in the limits $\hat{\mathcal{C}} \ll 1$ and $\hat{\mathcal{C}} \gg 1$.) Determining the leading-order rotational speed $\hat{\omega}$, however, requires that we continue to the next-order.

At the next-order, $\mathcal{O}(\hat{\phi}^2)$, we obtain expressions for $P_2(X)$ and $Z_2(X)$ — which are again omitted here. Substituting these into the integral constraints, (4.52d–f) at $\mathcal{O}(\hat{\phi}^2)$, gives

$$\hat{h}_1 = 0 \quad \text{and} \quad \hat{\omega}_2 = \frac{21\hat{\mathcal{C}}^2\hat{h}_0^2 + 462\hat{\mathcal{C}}\hat{h}_0 + 3042}{256\hat{h}_0^{11/2}}. \quad (4.59a,b)$$

Using the solution of (4.58a), (4.59b) gives the leading-order rotational speed $\hat{\omega} \sim \hat{\phi}^2\hat{\omega}_2$.

The full asymptotic behaviour (as $\hat{\phi} \rightarrow 0$) for varying substrate compressibility, $\hat{\mathcal{C}}$, is plotted in Fig. 4.6 together with numerical solutions for non-zero $\hat{\phi}$ (see §4.4.3 below for details). These numerical results demonstrate that (4.57)–(4.59) recover the correct behaviour for small $\hat{\phi}$.

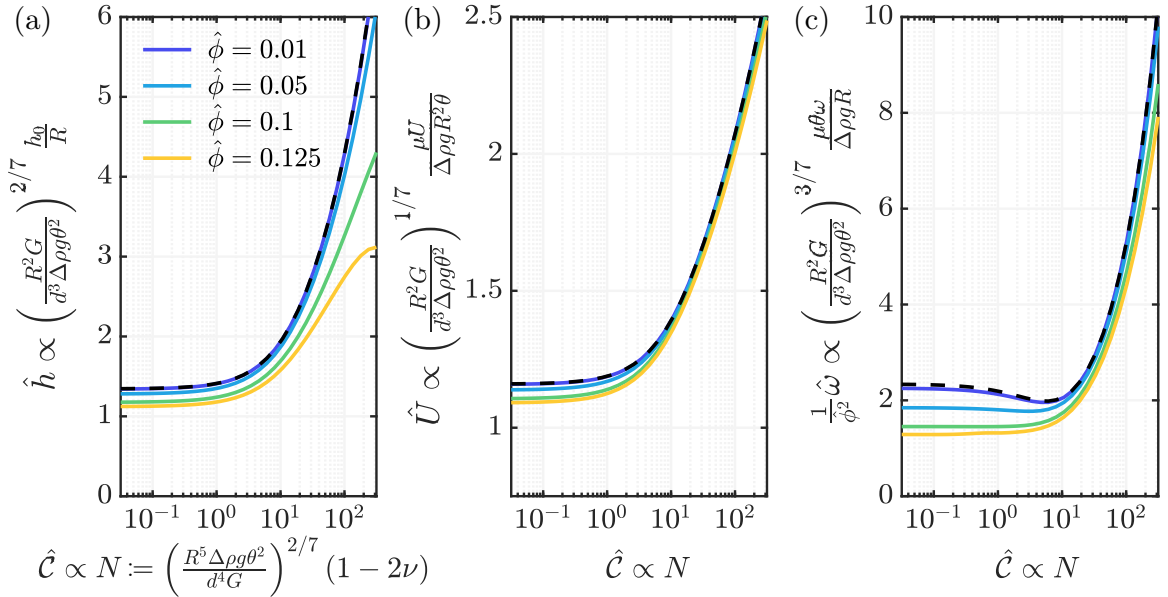


Figure 4.6: Comparison of asymptotic (valid for small pliability, $\hat{\phi} \ll 1$) and numerical results of the combined foundation model applied to a cylinder sedimenting close to a thin, elastomeric layer of dimensionless compressibility $\hat{\mathcal{C}}$. Plotted are (a) minimum gap thickness, (b) steady translation speed, and (c) steady rotation speed. In each figure, the asymptotic prediction (4.57)–(4.59), valid for $\hat{\phi} \ll 1$, is shown by dashed curves while solid curves represent the numerical solution of the combined foundation problem with different values of the pliability parameter $\hat{\phi}$, as given in the legend. Here, we have assumed $\theta \ll 1$ to simplify the axes labels.

To understand the implications of the above asymptotic results better, we move now to discuss the behaviour in the incompressible limit ($\hat{\mathcal{C}} \rightarrow 0$, incompressible foundation) and the limit of high compressibility ($\hat{\mathcal{C}} \rightarrow \infty$, Winkler foundation) in more detail.

4.4.2.1 Incompressible foundation ($\hat{\mathcal{C}} \rightarrow 0$)

For an incompressible foundation, $\hat{\mathcal{C}} \rightarrow 0$, we immediately find from (4.57)–(4.59) that $\hat{h} \sim (45/16)^{2/7}$, $\hat{U} \sim (45/16)^{1/7}$, and $\hat{\omega} \sim 338/225 \times (45/16)^{3/7} \hat{\phi}^2$, so that the dimensional gap thickness, translational speed, and rotational speed are

$$h \sim \frac{\ell_I^2}{2R} \times \left(\frac{45}{16}\right)^{2/7}, \quad U \sim \frac{R\ell_I \Delta \rho g \sin \theta}{4\mu} \times \left(\frac{45}{16}\right)^{1/7}, \quad (4.60a,b)$$

$$\omega \sim \frac{\Delta \rho g \ell_I^3 \cos \theta}{4\mu R^2 \tan \theta} \times \frac{338}{225} \left(\frac{45}{16}\right)^{3/7}, \quad (4.60c)$$

respectively. In this incompressible limit, we also have the pressure and surface-displacement profiles

$$P(X) \sim \frac{2\hat{h}_0^{1/2} X}{(\hat{h}_0 + X^2)^2} + \frac{48(11\hat{h}_0^2 - 28\hat{h}_0 X^2 + 21X^4)}{35(\hat{h}_0 + X^2)^7} \hat{h}_0 \hat{\phi}, \quad (4.61a)$$

$$Z(X) \sim -\frac{24\hat{h}_0^{1/2} X(\hat{h}_0 - X^2)}{(\hat{h}_0 + X^2)^4}. \quad (4.61b)$$

The displacement profile (4.61b) is plotted as the solid curve in Fig. 4.7(a) and illustrates that this profile has four turning points, located at $X = \pm\{\hat{h}_0(1 \pm 2/\sqrt{5})\}^{1/2} = \pm(45/16)^{1/7}(1 \pm 2/\sqrt{5})^{1/2}$. Although previous studies on the scalings of the incompressible limit have been made by Skotheim and Mahadevan (2004, 2005); Rallabandi et al. (2017), both the prefactors in (4.60) and the displacement profile (4.61b) are believed to be new.

4.4.2.2 Winkler foundation ($\hat{\mathcal{C}} \rightarrow \infty$)

The case of a Winkler foundation, $\hat{\mathcal{C}} \rightarrow \infty$, has been studied extensively previously by Skotheim and Mahadevan (2004, 2005); Salez and Mahadevan (2015); Rallabandi et al. (2017). In this limit, we find from (4.57)–(4.59) that $\hat{h} \sim (3\hat{\mathcal{C}}/8)^{2/5}$, $\hat{U} \sim (3\hat{\mathcal{C}}/8)^{1/5}$, and $\hat{\omega} \sim 7/12 \times (3\hat{\mathcal{C}}/8)^{3/5} \hat{\phi}^2$, so that the dimensional gap thickness, translational speed, and rotational speed are

$$h \sim \frac{\ell_W^2}{2R} \times \left(\frac{3}{8}\right)^{2/5}, \quad U \sim \frac{R\ell_W \Delta \rho g \sin \theta}{4\mu} \times \left(\frac{3}{8}\right)^{1/5}, \quad (4.62a,b)$$

$$\omega \sim \frac{\Delta \rho g \ell_W^3 \cos \theta}{4\mu R^2 \tan \theta} \times \frac{7}{12} \left(\frac{3}{8}\right)^{3/5}, \quad (4.62c)$$

respectively. Moreover, we obtain the pressure and surface-displacement profiles

$$P(X) \sim \frac{2\hat{h}_0^{1/2}X}{(\hat{h}_0 + X^2)^2} + \frac{3(3\hat{h}_0 - 5X^2)}{5(\hat{h}_0 + X^2)^5}\hat{\mathcal{C}}\hat{h}_0\hat{\phi}, \quad (4.63a)$$

$$Z(X) \sim -\frac{2\hat{\mathcal{C}}\hat{h}_0^{1/2}X}{(\hat{h}_0 + X^2)^2}. \quad (4.63b)$$

(Recall that $P(X)$, $Z(X)$, and X are all non-dimensionalized with respect to the incompressible response, hence there is an explicit dependence on $\hat{\mathcal{C}}$ above.) The displacement profile, (4.63b), recovers that given previously by Skotheim and Mahadevan (2004, 2005); it is plotted in Fig. 4.7(a) and, in contrast to the incompressible profile, (4.61b), has only two turning points located at $X = \pm(\hat{h}_0/3)^{1/2} = \pm(\hat{\mathcal{C}}/24\sqrt{3})^{1/5}$. The gap thickness, translation speed, and rotation speed, (4.62), agree with the previous results found by Rallabandi et al. (2017).

4.4.3 Comparison with numerical results

The analytical progress that has been made above has exploited the doubly asymptotic limit of small substrate pliability, $\hat{\phi} \ll 1$, and small aspect ratio, $\epsilon \equiv d/\ell \ll 1$. To investigate when the asymptotic results are valid, we use numerical solutions of the problem to study what happens when each of these limits fail.

4.4.3.1 Moderate substrate pliability

For moderate values of $\hat{\phi}$, but retaining small aspect ratio, there is a nonlinear feedback between Reynolds' equation and the combined foundation response, so the system, (4.52), must be solved numerically. This can be done by eliminating $Z(X)$ and $\mathcal{H}(X)$ from (4.52) to form a third-order integro-differential equation for the pressure $P(X)$ with given values of $\hat{\mathcal{C}}$ and $\hat{\phi}$, and unknown constants $\hat{h}$, $\hat{U}$, and $\hat{\omega}$. In practice, the system can be solved by coupling finite difference derivatives and quadrature with a standard nonlinear system solver (we use `fsolve` in `MATLAB`). Once the pressure, $P(X)$, has been computed, the displacement profile, $Z(X)$, can be found using (4.52c). (An alternative approach, based on similarity solutions in the long-wavelength limit, was presented recently by Essink et al. (2021) for large deformations of a Winkler foundation.)

Numerical results for the translation velocity, $\hat{U}(\hat{\phi}, \hat{\mathcal{C}})$, and gap thickness, $\hat{h}(\hat{\phi}, \hat{\mathcal{C}})$, are presented in Fig. 4.6 and show that the asymptotic predictions (4.57)–(4.59) are recovered as $\hat{\phi} \rightarrow 0$, giving a good quantitative description of the full numerical results provided that $\hat{\phi} \lesssim 0.05$. Example surface profiles are presented in Fig. 4.7(b,c) for a range of $\hat{\mathcal{C}}$; these show that the asymptotic results quantitatively describe numerical results for $\hat{\phi} = 0.01$, but only

qualitatively for $\hat{\phi} = 0.1$.

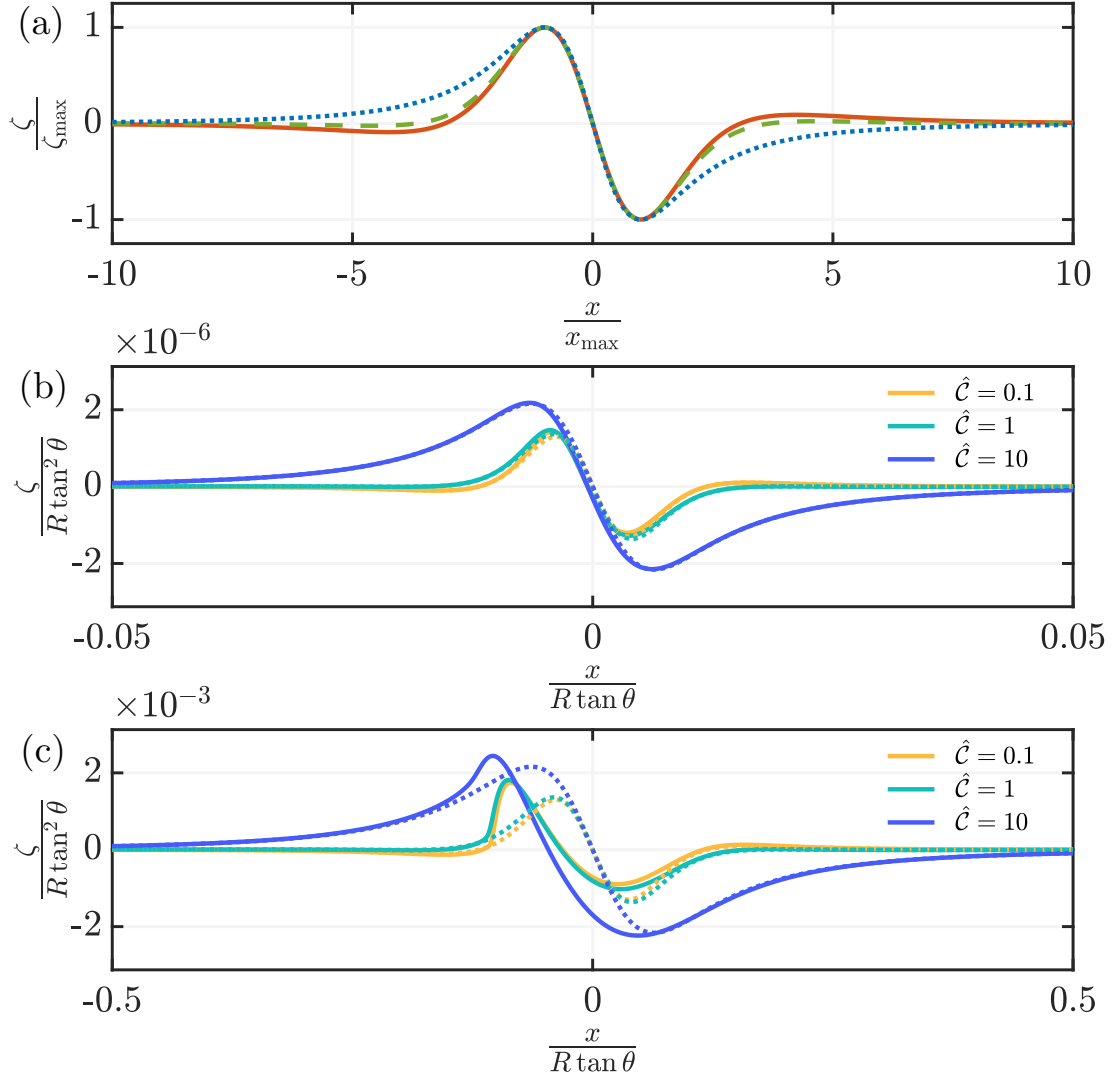


Figure 4.7: Results of the combined foundation model for moderate substrate pliability, $\hat{\phi}$. (a) Analytical predictions for the surface deformation profile with $\hat{\phi} \ll 1$ for an incompressible foundation, $\hat{C} \rightarrow 0$, (4.61b) (solid curve); a combined foundation, $\hat{C} = 1$, (4.56b) with $\hat{h}_0$ found by solving (4.58) numerically (dashed curve); and a Winkler foundation, $\hat{C} \rightarrow \infty$, (4.63b) (dotted curve). To facilitate a direct comparison between the three cases, the vertical and horizontal scales have been modified to ensure that the maximum deflection occurs at $(-1, 1)$. (b,c) Comparison between the predictions of the analytical results for $\hat{\phi} \ll 1$ with numerical results for moderate $\hat{\phi}$ from the combined foundation model for different values of $\hat{C}$ (as described in the legend). In (b) $\hat{\phi} = 0.01$ and in (c) $\hat{\phi} = 0.1$ are fixed with numerical results plotted as solid curves and the asymptotic predictions from (4.56b), plotted as dashed curves. Note that, for small $\hat{C}$, the additional turning points of the interface deflection predicted for an incompressible foundation (see panel (a)) are just visible.

4.4.3.2 Model validity

The results presented thus far provide a mathematical description of an infinite cylinder sliding and rotating in a viscous fluid above a thin elastic foundation. Although we have demonstrated that our asymptotic results for foundations with small pliability, $\hat{\phi} \ll 1$, give good intuition for the behaviour of the combined foundation model that is observed numerically for moderate $\hat{\phi}$, these numerical results rely on the description of the substrate deformation by the combined foundation model. Now, therefore, we use numerical simulations of the full elastic–substrate problem to test when this combined foundation model is appropriate. We begin by considering the scale separation required for the model to be valid in light of the detailed analysis performed.

Scaling argument In deriving the coupled system (4.43), we implicitly assumed that the horizontal length scale, $[x] \equiv \ell$, is large in comparison to both the coating thickness ($d^2 \ll \ell^2$, for the combined foundation to be valid) and to the minimum gap thickness ($h^2 \ll \ell^2$, for hydrodynamic lubrication theory to be valid). As already discussed, we also require that $h \ll d$ to be able to neglect the viscous shear force, *i.e.* $T(x) \sim H(x)p'(x) \ll p'(x)d$. Altogether, we require

$$h^2 \ll d^2 \ll \ell^2. \quad (4.64)$$

Using the two possible horizontal length scales available in (4.50), we can rewrite the separation of scales required for the combined foundation model and lubrication theory to hold, (4.64), as

$$\left(\frac{\hat{\phi} \hat{h} \tan \theta}{2} \right)^2 \ll \frac{\alpha(\nu)(1-2\nu)}{\Gamma(\nu)} \frac{1}{\hat{\mathcal{C}}} \ll \max\{1, \hat{\mathcal{C}}^{2/5}\}. \quad (4.65)$$

The first inequality in (4.65) provides a parameter bound for the fluid flow to be modelled using lubrication theory. Of more interest here, however, is the second inequality, which expresses the small foundation aspect ratio limit ($\epsilon^2 \equiv d^2/\ell^2 \ll 1$) that was used to derive the combined foundation model, (4.20). The asymptotic results in the limits $\hat{\mathcal{C}} \gg 1$ and $\hat{\mathcal{C}} \ll 1$ allow us to write the condition for the combined foundation model to be valid as

$$\begin{aligned} 1 &\ll \frac{\Gamma(\nu)}{\alpha(\nu)(1-2\nu)} \max\{\hat{\mathcal{C}}, \hat{\mathcal{C}}^{7/5}\} \\ &\equiv \max \left\{ \left[\Gamma(\nu) \frac{2R^5 \Delta \rho g \sin \theta \tan \theta}{d^4 G} \right]^{2/7}, \left[\alpha(\nu)(1-2\nu) \frac{2R^5 \Delta \rho g \sin \theta \tan \theta}{d^4 G} \right]^{2/5} \right\}. \end{aligned} \quad (4.66)$$

To see this prediction in action, we turn to solving the full equilibrium equations, (4.5), for a foundation of finite (rather than infinitesimal) depth.

Finite substrate depth If the asymptotic simplification of small aspect ratios ($\epsilon \ll 1$) is not assumed, one must solve the full two-dimensional equilibrium equations, (4.5), in the domain $-d \leq z \leq 0$ and $-\infty < x < \infty$. The appropriate boundary conditions are (4.6) with $\mathbf{T}(\mathbf{x}) \equiv \mathbf{0}$ and $p(\mathbf{x}) \equiv p(x)$ given by Reynolds' equation, (4.43a,b); (4.44); and (4.45). Non-dimensionalizing using the scalings presented in (4.51) and noting that the aspect ratio can be written as

$$\epsilon^2 = \frac{d^2}{\ell_1^2} \equiv \frac{\alpha(\nu)(1-2\nu)}{\Gamma(\nu)\hat{\mathcal{C}}}, \quad (4.67)$$

we obtain a system controlled by the Poisson's ratio ν , the compressibility $\hat{\mathcal{C}}$, and the pliability $\hat{\phi}$, where $\hat{\mathcal{C}}$ and $\hat{\phi}$ are defined in (4.54a,b). As in the indentation problem, the Poisson's ratio now acts as an extra dimensionless parameter that cannot be scaled out of the problem. The resulting system of partial differential equations may be solved using standard numerical techniques; details of these and the full problem are discussed in Appendix 4.A.2.

For given values of ν , $\hat{\mathcal{C}}$, and $\hat{\phi}$ and, hence, ϵ via (4.67), the deflection of the substrate surface together with the equilibrium gap thickness, $\hat{h}$; translational speed, $\hat{U}$; and rotational speed, $\hat{\omega}$, may be computed. Figure 4.8 shows the substrate deformation determined numerically and compares this with the asymptotic prediction, (4.56b), which accurately reproduces the solution of the combined foundation model for $\hat{\phi} \ll 1$ (see Fig. 4.7). Also note that, even with $\nu = 0.4999$ the surface profile for $\hat{\mathcal{C}} = 10$ in Fig. 4.8(a) exhibits just the two turning points characteristic of the Winkler (rather than incompressible) response.

A comparison between the numerically-determined $\hat{U}$ obtained with a finite foundation thickness and that predicted by the combined foundation model is shown in Fig. 4.9 for varying $\hat{\mathcal{C}}$ and ν , and with $\hat{\phi} = 0.01 \ll 1$ fixed. Figures 4.8 and 4.9 show excellent agreement between the prediction of the combined foundation model and the numerical solutions of the full problem for large $\hat{\mathcal{C}}$ (*i.e.*, from (4.67), small aspect ratios, $\epsilon \ll 1$). However, as $\hat{\mathcal{C}}$ decreases (*i.e.* increasing aspect ratios, ϵ) we see that the numerical results deviate from the prediction of the combined foundation model; this discrepancy becomes appreciable for $\hat{\mathcal{C}} \lesssim 1 - 2\nu$, consistent with the prediction of (4.66). Results in Fig. 4.9(a) with $\nu = 0.49$ do not show the plateau in $\hat{U}$ predicted by the incompressible foundation model for $\hat{\mathcal{C}} \ll 1$ because the small aspect ratio assumption breaks down before the incompressible limit becomes applicable. However, with $\nu = 0.4999$ the expected plateau region is observed for $10^{-2} \lesssim \hat{\mathcal{C}} \lesssim 1$.

Finally, we turn to discuss the good agreement between experiments (which use approximately incompressible substrates) and theory (which is based on a model of a compressible substrate, *i.e.* a Winkler response). The relative error in using the Winkler foundation compared to using the combined foundation or the numerical solutions of a foundation with finite depth are plotted in Fig. 4.9(b). For a Poisson's ratio of $\nu = 0.4$, the

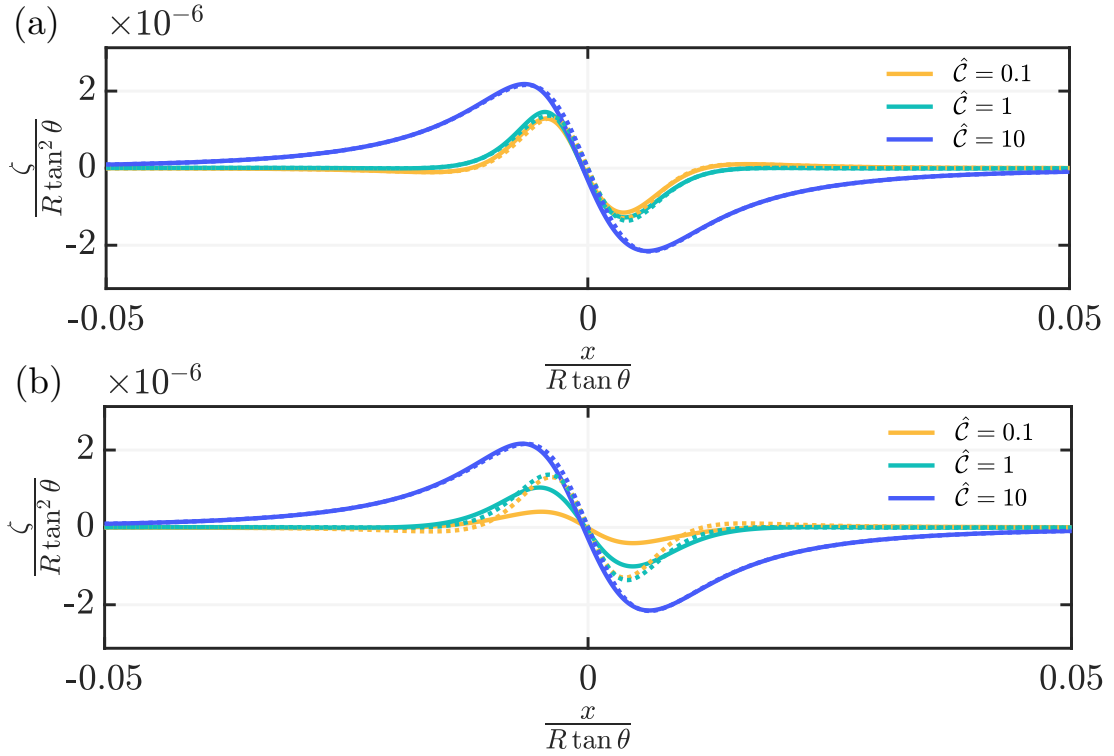


Figure 4.8: Comparison between numerically determined surface profiles, accounting for finite foundation thickness, (solid curves) and the asymptotic predictions (4.56b) from the combined foundation model (dashed curves). Results are shown with $\hat{\phi} = 0.01$ and different values of $\hat{C}$ (as described in the legend). In (a) $\nu = 0.4999$, while in (b) $\nu = 0.49$. Note that disagreement between numerics and asymptotic results is observed when $\hat{C} \lesssim (1 - 2\nu)$, as expected from (4.66).

error remains small for a considerable range of $\hat{C}$: the effect of finite substrate thickness looks more like a Winkler foundation than the combined foundation. For near incompressible materials, however, the error is appreciable at $\hat{C} \sim 1$ and is more than 50% at $\hat{C} = 0.1$. In this case, the incompressible plateau seen in Fig. 4.9(a) is apparent. Thus, if the compressibility parameter, $\hat{C}$, is well known, the errors of using Winkler, compared to combined, should be observable. We do not have sufficient details of the experimental parameters, particularly the Poisson's ratio ν , to say clearly in which regime the model would suggest such experiments lie. However, assuming that $0.45 \lesssim \nu \lesssim 0.49$, we find that the experiments of Saintyves et al. (2020) appear to have $10^{-2} \lesssim \hat{C} \lesssim 10^2$, suggesting that experiments may span the transition between incompressible and compressible (Winkler) behaviour. We hope that future experiments might include explicit estimates of the parameters $\hat{C}$ and ν allowing comparisons with the theory presented here to be made.

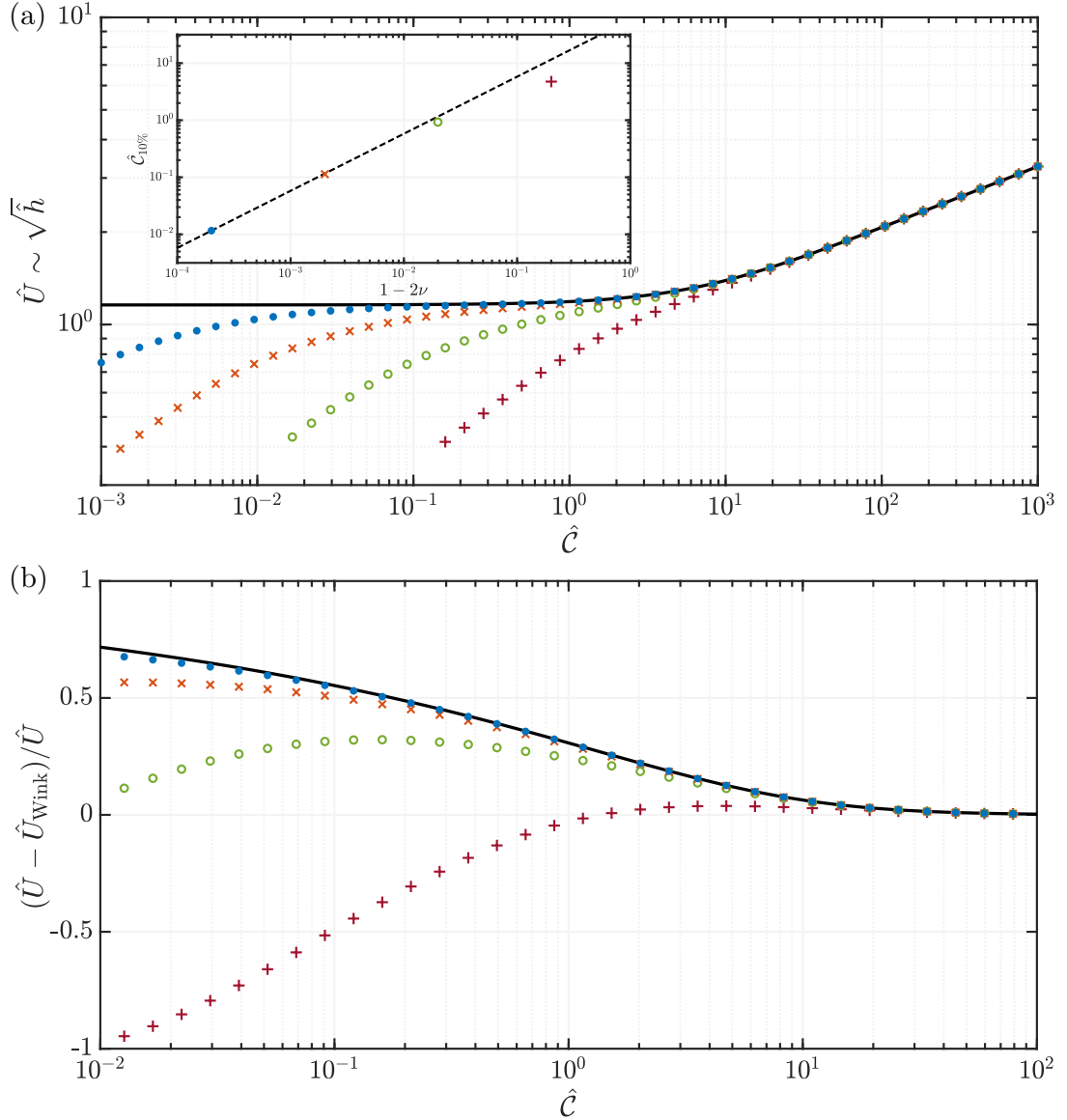


Figure 4.9: (a) Main figure: Comparison between the prediction of the combined foundation (solid curves) and numerical results of the finite-thick foundation (points) for the sedimentation of a cylinder besides an elastomeric coating. Numerical results are obtained by solving (4.A.6) with (4.A.7) and (4.52a) and are shown for different values of the Poisson's ratio, ν , as follows: $\nu = 0.4$ (+), $\nu = 0.49$ (O), $\nu = 0.499$ (x), and $\nu = 0.4999$ (●). We observe good agreement between the numerics and the combined foundation model provided that $\hat{C} \gg (1 - 2\nu)$. Note that, here, $\hat{\phi} = 0.01 \ll 1$ so $\hat{U} \sim \hat{h}^{1/2}$ and $\hat{\omega} \ll 1$. Inset: The $\hat{C}$ for which the relative error in the use of the combined foundation model is 10%, denoted $\hat{C}_{10\%}$, scales linearly with $1 - 2\nu$, as indicated by the dashed line. (b) The relative error in the cylinder's translation velocity when assuming a Winkler foundation, $\hat{U}_{\text{Wink}}$, compared to the value obtained for a combined foundation (solid curve) and numerical results of the finite-thick foundation (points). The coloured symbols represent the different Poisson's ratios, as in (a). For large $\hat{C}$, the Winkler foundation does a good job at representing the other models, whilst at small $\hat{C}$, the effects of a finite depth and an incompressible material cannot be ignored.

4.5 Finite domains and inconsistent boundary conditions

In the examples considered in §4.3 and §4.4, the elastic substrate was assumed to be infinite in the plane (neither problem contained a lateral boundary of the substrate). Lateral boundaries are, of course, necessary in experiments; in fact, in certain cases, they are known to have an important effect on the substrate deformation (see [Box et al., 2020b](#), for example). The problem is that the asymptotic solution, (4.19), cannot hold close to these boundaries if the imposed boundary conditions are incompatible with the displacements determined in the asymptotic analysis. For example, in the elasto-hydrodynamic sedimentation problem considered in §4.4, Reynolds' equation, (4.43a), gives the fluid pressure gradient

$$p'(x) = -\frac{Q + 2(U + \omega R)H(x)}{12\mu H(x)^3}, \quad (4.68)$$

for a gap thickness $H(x) \approx h + x^2/(2R)$ and fluid flux Q determined by the pressure boundary conditions. This pressure-gradient deforms the substrate according to the combined foundation, (4.19); critically, the horizontal surface displacement takes the form

$$u(x, 0) \sim \beta(\nu) \frac{d^2}{G} \frac{Q + 2(U + \omega R)H(x)}{12\mu H(x)^3}, \quad (4.69)$$

which only vanishes as $|x| \rightarrow \infty$. If the substrate was alternatively bound at some lateral boundaries we would require $u(x, 0) = 0$ at some finite value of x . In such a case, we expect a boundary layer of size $[x] \sim d$ to resolve this incompatibility.

The elastomeric materials that are commonly used as substrates in experiments (*e.g.* polydimethylsiloxane, PDMS) are cured inside of a mould with a desired shape ([Dogru et al., 2018](#)). These moulds are expected to adhere perfectly to the substrate material at the base and at some lateral boundaries. With this in mind, in this section, we consider the boundary layers associated with a fixed, zero-displacement, lateral wall. In particular, we derive the governing equations for a two-dimensional substrate in §4.5.1 and a rotationally-symmetric substrate in §4.5.2. These formulations extend the analysis of the bulk deformation in §4.2. Boundary layers associated with other incompatible conditions (*e.g.* a zero-stress free boundary) can be analysed in a similar manner and give comparable results.

4.5.1 Two-dimensional boundary layer

Consider a thin, two-dimensional, elastic layer perfectly adhered to a rigid base at $z = -d$ and a rigid wall at $x = 0$. The surface of the substrate is displaced by a surface pressure and shear stress, $p(x)$ and $T(x)$, respectively. This setup is shown in Fig. 4.10. Using the dimensionless variables defined in (4.7), the elastic displacements, $U(X, Z)$ and $W(X, Z)$,

are governed by the elastostatic equations, (4.8) in $-1 \leq Z \leq 0$ and $X \leq 0$, subject to the fixed-base condition, (4.9a,b) on $Z = -1$; given surface stress, (4.9c,d) on $Z = 0$; and fixed-lateral wall condition,

$$W = 0 \quad \text{and} \quad U = 0, \quad (4.70a,b)$$

on $X = 0$.

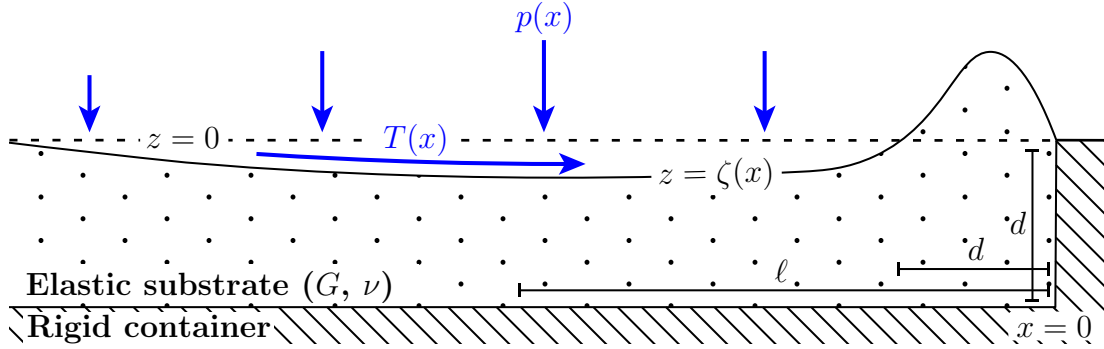


Figure 4.10: A sketch of the two-dimensional setup with a fixed lateral boundary: the surface (at $z = 0$) of an elastic block adhered to a rigid container (at $z = -d$ and $x = 0$), with aspect ratio $\epsilon := d/\ell \ll 1$, is displaced by an applied pressure, $p(x)$. The deformed surface profile, $z = \zeta(x)$, forms a boundary layer local to the lateral wall, *i.e.* for $x = \mathcal{O}(d)$.

In §4.2, we derived an asymptotic solution of the elastostatic equations subject to a fixed base and given surface stress; this will form the outer solutions, so we write

$$\begin{aligned} W_{\text{out}}(X, Z) = & -\alpha(\nu)(1-2\nu)(1+Z)P(X) - \beta(\nu)(1+Z)\left(1 + \frac{Z}{4\nu-1}\right)\mathcal{T}'(X)\epsilon^2 \\ & + \gamma(\nu)(1+Z)\left(1 + \frac{Z}{2} - \frac{1-\nu}{4\nu-1}Z^2\right)P''(X)\epsilon^2 + \mathcal{O}(\epsilon^4), \end{aligned} \quad (4.71a)$$

$$U_{\text{out}}(X, Z) = (1+Z)\mathcal{T}(X) - \beta(\nu)(1+Z)\left(1 - \frac{Z}{4\nu-1}\right)P'(X) + \mathcal{O}(\epsilon^2), \quad (4.71b)$$

as $\epsilon \rightarrow 0$. This expansion holds close to the lateral wall, $X = 0$, provided it satisfies (4.70), that is if the applied normal pressure, $P(X)$; the tangential shear stress, $\mathcal{T}(X)$; and their spatial derivatives, $P'(X)$, $P''(X)$, and $\mathcal{T}'(X)$, are zero at $X = 0$. If the pressure and shear stress do not vanish, the higher-order terms in the combined foundation, (4.17), are required in the dominant balance, *i.e.* a boundary layer of size $X = \mathcal{O}(\epsilon)$ is needed to transition from (4.71) to (4.70); we analyse this case now.

In the boundary layer, U and W must change rapidly in order to satisfy (4.70): the small aspect ratio assumption used to derive (4.71) no longer holds, instead we expect $[x] \sim [z]$

and $[u] \sim [w]$. We, hence, define an inner variable

$$\tilde{X} := \frac{X}{\epsilon}, \quad (4.72a)$$

and the displacements in the inner region

$$\tilde{W}(\tilde{X}, Z) := W(\epsilon\tilde{X}, Z) \quad \text{and} \quad \tilde{U}(\tilde{X}, Z) := \epsilon U(\epsilon\tilde{X}, Z). \quad (4.72b,c)$$

Furthermore, we shall assume that the applied normal pressure and shear stress, $P(X)$ and $\mathcal{T}(X)$, vary within the boundary layer; thus, we introduce the local applied pressure and shear stress as

$$\tilde{P}(\tilde{X}) := P(\epsilon\tilde{X}) \quad \text{and} \quad \tilde{\mathcal{T}}(\tilde{X}) := \epsilon \mathcal{T}(\epsilon\tilde{X}), \quad (4.72d,e)$$

respectively. Above, and throughout this section, we assume the substrate is bounded on the right at $X = 0$ (as in Fig. 4.10); if instead the substrate is bounded on the left at $X = 0$, then the associated boundary layer can be analysed by taking $\tilde{X} \mapsto -\tilde{X}$ henceforth.

Substituting the inner variables, (4.72), into our governing system, (4.8) subject to (4.9) and (4.70), we find that the displacements, $\tilde{U}$ and $\tilde{W}$, satisfy the equilibrium equations,

$$(1 - 2\nu)\tilde{W}_{\tilde{X}\tilde{X}} + 2(1 - \nu)\tilde{W}_{ZZ} + \tilde{U}_{\tilde{X}Z} = 0, \quad (4.73a)$$

$$(1 - 2\nu)\tilde{U}_{ZZ} + 2(1 - \nu)\tilde{U}_{\tilde{X}\tilde{X}} + \tilde{W}_{\tilde{X}Z} = 0, \quad (4.73b)$$

in $-1 \leq Z \leq 0$ and $-\infty < \tilde{X} \leq 0$, subject to the boundary conditions

$$\frac{2(1 - \nu)}{1 - 2\nu}\tilde{W}_Z + \frac{2\nu}{1 - 2\nu}\tilde{U}_{\tilde{X}} = -\tilde{P}(\tilde{X}) \quad \text{and} \quad \tilde{U}_Z + \tilde{W}_{\tilde{X}} = \tilde{\mathcal{T}}(\tilde{X}) \quad \text{on } Z = 0, \quad (4.74a,b)$$

$$\tilde{W} = 0 \quad \text{and} \quad \tilde{U} = 0 \quad \text{on } Z = -1, \quad (4.74c,d)$$

$$\tilde{W} = 0 \quad \text{and} \quad \tilde{U} = 0 \quad \text{on } \tilde{X} = 0. \quad (4.74e,f)$$

Solutions to the inner problem must match with the outer solution, (4.71), that is we require

$$\tilde{W}(\tilde{X}, Z) \sim W_{\text{out}}(\epsilon\tilde{X}, Z) \quad \text{and} \quad \tilde{U}(\tilde{X}, Z) \sim \epsilon U_{\text{out}}(\epsilon\tilde{X}, Z), \quad (4.74g,h)$$

as $\tilde{X} \equiv X/\epsilon \rightarrow -\infty$.

For a general pressure and shear stress, the inner problem, (4.73) subject to (4.74), is independent of the aspect ratio, ϵ ; hence, in general, it must be solved numerically. We use the method of finite differences — this is discussed in §4.6.3 for a physical example. Analytical progress can be made, however, if the localized pressure and shear stress can

locally be approximated by a quadratic and linear polynomial,

$$\tilde{P}(\tilde{X}) = \tilde{P}_0 + \tilde{P}_1 \tilde{X} + \tilde{P}_2 \frac{\tilde{X}^2}{2} \quad \text{and} \quad \tilde{\mathcal{T}} = \tilde{\mathcal{T}}_0 + \tilde{\mathcal{T}}_1 \tilde{X}, \quad (4.75\text{a,b})$$

respectively. These localized polynomials are equivalent to requiring that the applied pressure and shear stress, $P(X)$ and $\mathcal{T}(X)$, do not vary much over the inner variable's length scale, $\tilde{X} \equiv X/\epsilon = \mathcal{O}(1)$, and can thus be approximated by Taylor's expansions truncated to $o(\epsilon^2)$ — the accuracy of the outer solution, (4.71) — with coefficients

$$P_i \equiv P^{(i)}(0) = \frac{\tilde{P}^{(i)}(0)}{\epsilon^i} \equiv \frac{\tilde{P}_i}{\epsilon^i} \quad \text{and} \quad \mathcal{T}_i \equiv \mathcal{T}^{(i)}(0) = \frac{\tilde{\mathcal{T}}^{(i)}(0)}{\epsilon^{i+1}} \equiv \frac{\tilde{\mathcal{T}}_i}{\epsilon^{i+1}}. \quad (4.76\text{a,b})$$

If the local applied stresses are given by (4.75), the outer displacements, $W_{\text{out}}(\epsilon\tilde{X}, Z)$ and $\epsilon U_{\text{out}}(\epsilon\tilde{X}, Z)$, satisfy the (inner) equilibrium equations, (4.73), by construction. Substituting

$$\tilde{W}(\tilde{X}, Z) = W_{\text{out}}(\epsilon\tilde{X}, Z) + \hat{W}(\tilde{X}, Z), \quad (4.77\text{a})$$

$$\tilde{U}(\tilde{X}, Z) = \epsilon U_{\text{out}}(\epsilon\tilde{X}, Z) + \hat{U}(\tilde{X}, Z), \quad (4.77\text{b})$$

into (4.73) and (4.74), we find that the supplementary displacements, $\hat{W}(\tilde{X}, Z)$ and $\hat{U}(\tilde{X}, Z)$, satisfy equilibrium equations analogous to (4.73) in $-1 \leq Z \leq 0$ and $-\infty < \tilde{X} \leq 0$, subject to the boundary conditions

$$\frac{2(1-\nu)}{1-2\nu} \hat{W}_Z + \frac{2\nu}{1-2\nu} \hat{U}_{\tilde{X}} = 0 \quad \text{and} \quad \hat{U}_Z + \hat{W}_{\tilde{X}} = 0 \quad \text{on } Z = 0, \quad (4.78\text{a,b})$$

$$\hat{W} = 0 \quad \text{and} \quad \hat{U} = 0 \quad \text{on } Z = -1, \quad (4.78\text{c,d})$$

$$\hat{W} \rightarrow 0 \quad \text{and} \quad \hat{U} \rightarrow 0 \quad \text{as } \tilde{X} \rightarrow -\infty, \quad (4.78\text{e,f})$$

with

$$\begin{aligned} \hat{W} = & \alpha(\nu)(1-2\nu)(1+Z)\tilde{P}_0 + \beta(\nu)(1+Z)\left(1 + \frac{Z}{4\nu-1}\right)\tilde{\mathcal{T}}_1 \\ & - \gamma(\nu)(1+Z)\left(1 + \frac{Z}{2} - \frac{1-\nu}{4\nu-1}Z^2\right)\tilde{P}_2, \end{aligned} \quad (4.78\text{g})$$

$$\hat{U} = -(1+Z)\tilde{\mathcal{T}}_0 + \beta(\nu)(1+Z)\left(1 - \frac{Z}{4\nu-1}\right)\tilde{P}_1, \quad (4.78\text{h})$$

on $\tilde{X} = 0$.

Finally, we use the system's linearity to remove the dependence on the applied pressure

and shear stress, *i.e.* $\tilde{P}_i$ and $\tilde{T}_i$, by writing

$$\hat{W}(\tilde{X}, Z) = \sum_{i=0}^2 \epsilon^i A_i \hat{W}_{P_i}(\tilde{X}, Z) + \sum_{i=0}^1 \epsilon^{i+1} B_i \hat{W}_{T_i}(\tilde{X}, Z), \quad (4.79a)$$

$$\hat{U}(\tilde{X}, Z) = \sum_{i=0}^2 \epsilon^i A_i \hat{U}_{P_i}(\tilde{X}, Z) + \sum_{i=0}^1 \epsilon^{i+1} B_i \hat{U}_{T_i}(\tilde{X}, Z), \quad (4.79b)$$

where the coefficients, A_i and B_i , are defined such that the corresponding displacements equal 1 or 0 at $Z = \tilde{X} = 0$:

$$A_0 := \alpha(\nu)(1 - 2\nu)P(0), \quad A_1 := \beta(\nu)P'(0), \quad A_2 := -\gamma(\nu)P''(0), \quad (4.80a-c)$$

$$B_0 := -\mathcal{T}(0), \quad B_1 := \beta(\nu)\mathcal{T}'(0). \quad (4.80d,e)$$

The displacements, $(\hat{W}_{P_i}, \hat{U}_{P_i})$ and $(\hat{W}_{T_i}, \hat{U}_{T_i})$, each solve a system of PDEs that depend only on the Poisson's ratio, ν — these systems are written explicitly in Appendix 4.B.1. The resulting inner displacements, (4.77) with (4.79), are valid up to $o(\epsilon^2)$ — the same order as the outer displacements, (4.79). (As previously mentioned, here we have assumed the wall, $X = 0$, bounds the substrate on the right. For a left bound, one should take $\tilde{X} \mapsto -\tilde{X}$, along with $A_1 \mapsto -A_1$ and $B_1 \mapsto -B_1$, to account for the change in orientation.)

Although there are now five PDE systems to solve numerically, (4.B.1)–(4.B.3), compared to the one we had earlier, (4.73) subject to (4.74), these new systems are independent of the applied surface pressure and shear stress; thus, the PDEs only have to be solved once for a given value of the Poisson's ratio. Furthermore, the values of $P^{(i)}(0)$ and $\mathcal{T}^{(i)}(0)$ are often known *a priori*, hence usually only one or two PDE systems have to be solved for a particular Poisson's ratio; we demonstrate this for the soft Hele-Shaw flow problem considered in §4.6, for example. The supplementary displacements associated with an applied pressure, $\hat{W}_{P_i}(\tilde{X}, 0)$ and $\hat{U}_{P_i}(\tilde{X}, 0)$, and an applied shear stress, $\hat{W}_{T_i}(\tilde{X}, 0)$ and $\hat{U}_{T_i}(\tilde{X}, 0)$, are plotted in Figs. 4.11 and 4.12, respectively, for a variety of Poisson's ratio, ν . From these plots, we can see that the supplementary surface displacements are only weakly dependent on the Poisson's ratio, ν : the majority of the Poisson's ratio's effect has been removed through the coefficients (4.80). The incompressible case, $\nu = 1/2$, can thus be used as a first-order approximation for near-incompressible substrates (which are of most interest to our work). This is useful as the incompressible elastostatic equations are, typically, easier to solve than their compressible counterparts: the two second-order PDEs for $\hat{U}$ and $\hat{W}$, (4.73), are supplemented with a Lagrange multiplier, equivalent to the elastic pressure, which enforces the incompressibility condition $\hat{U}_X + \hat{W}_Z = 0$ (Howell et al., 2008). Introducing a stream function ψ , defined such that $\hat{U} \equiv \psi_Z$ and $\hat{W} \equiv -\psi_{\tilde{X}}$, and eliminating the Lagrange multiplier reduces the system to the biharmonic equation for ψ .

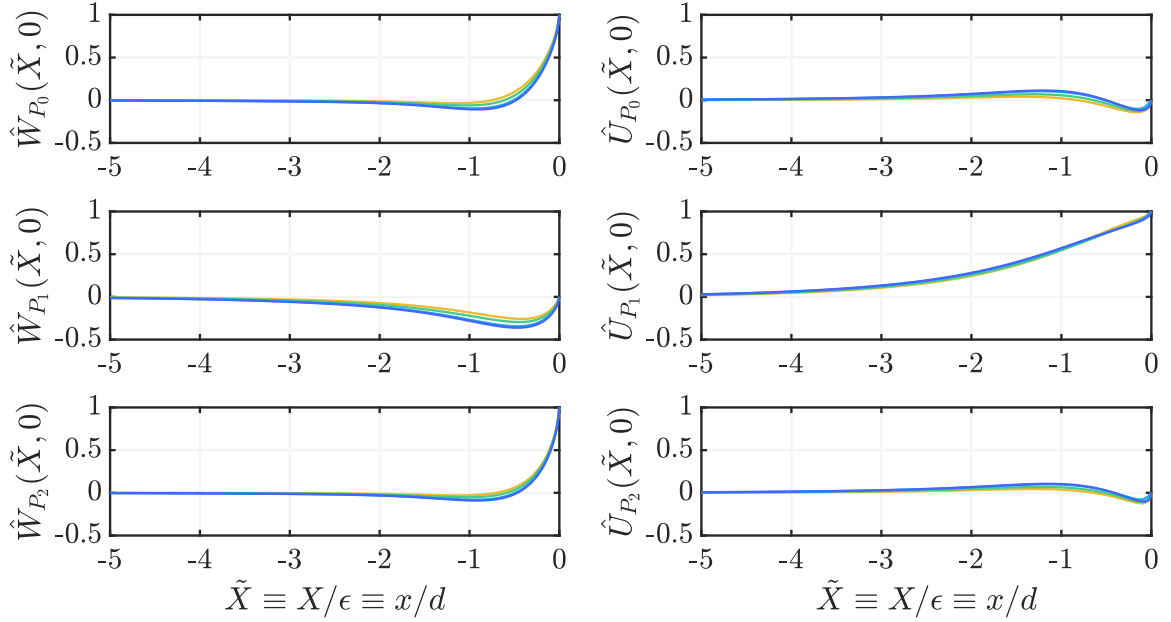


Figure 4.11: Numerical plot of the two-dimensional supplementary displacements associated with an applied pressure, $\hat{W}_{P_i}(\tilde{X}, 0)$ and $\hat{U}_{P_i}(\tilde{X}, 0)$, as a function of the inner variable, $\tilde{X}$, for Poisson's ratio $\nu = 0.4$, $\nu = 0.45$, $\nu = 0.49$, and $\nu = 0.5$ (yellow to blue). These supplementary displacements have a visibly weak dependence on the Poisson's ratio and so the incompressible case ($\nu = 1/2$) may be used as a first-order approximation.

By combining the outer, (4.71), and inner, (4.77) with (4.79), solutions, we can construct the composite surface displacements,

$$\begin{aligned}
 W_{\text{comp}}(X, 0) := & -\alpha(\nu)(1 - 2\nu) \left[P(X) - P(0)\hat{W}_{P_0}(\tilde{X}, 0) \right] - \mathcal{T}(0)\hat{W}_{\mathcal{T}_0}(\tilde{X}, 0) \epsilon \\
 & + \beta(\nu)P'(0)\hat{W}_{P_1}(\tilde{X}, 0) \epsilon - \beta(\nu) \left[\mathcal{T}'(X) - \mathcal{T}'(0)\hat{W}_{\mathcal{T}_1}(\tilde{X}, 0) \right] \epsilon^2 \\
 & + \gamma(\nu) \left[P''(X) - P''(0)\hat{W}_{P_2}(\tilde{X}, 0) \right] \epsilon^2 + \mathcal{O}(\epsilon^3),
 \end{aligned} \tag{4.81a}$$

$$\begin{aligned}
 U_{\text{comp}}(X, 0) := & \alpha(\nu)(1 - 2\nu)P(0)\hat{U}_{P_0}(\tilde{X}, 0) \frac{1}{\epsilon} + \mathcal{T}(X) - \mathcal{T}(0)\hat{U}_{\mathcal{T}_0}(\tilde{X}, 0) \\
 & - \beta(\nu) \left[P'(X) - P'(0)\hat{U}_{P_1}(\tilde{X}, 0) \right] + \beta(\nu)\mathcal{T}'(0)\hat{U}_{\mathcal{T}_1}(\tilde{X}, 0) \epsilon \\
 & - \gamma(\nu)P''(0)\hat{U}_{P_2}(\tilde{X}, 0) \epsilon + \mathcal{O}(\epsilon^2),
 \end{aligned} \tag{4.81b}$$

for inner variable $\tilde{X} \equiv X/\epsilon$. We reiterate that this solution only holds for applied pressures and shear stresses that do not vary much within the boundary layer. The applied stress expansions, (4.75), are *not* expected to hold if the applied stress is dependent on the substrate deformation (*e.g.* for a fast-flowing, lubricating fluid, §4.6.3) since the surface displacements in the boundary layer can induce large changes in the surface pressure and shear stress. For such cases, progress must be made on a case-by-case basis; however, often the inner problem,

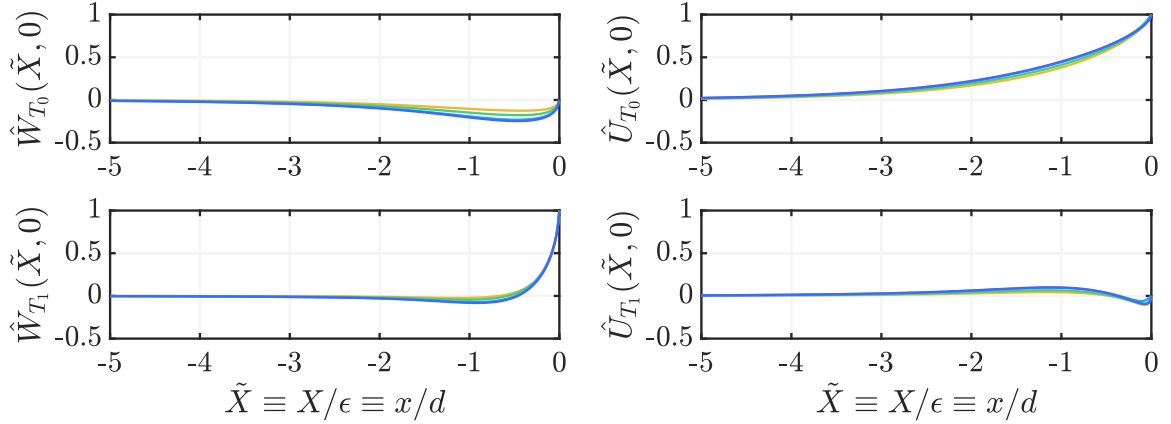


Figure 4.12: Numerical plot of the two-dimensional supplementary displacements associated with an applied shear stress, $\hat{W}_{T_i}(\tilde{X}, 0)$ and $\hat{U}_{T_i}(\tilde{X}, 0)$, as a function of the inner variable, $\tilde{X}$, for Poisson's ratio $\nu = 0.4$, $\nu = 0.45$, $\nu = 0.49$, and $\nu = 0.5$ (yellow to blue). These supplementary displacements have a visibly weak dependence on the Poisson's ratio and so the incompressible case, $\nu = 1/2$, may be used as a first-order approximation.

(4.73) subject to (4.74), may not be analytically tractable and must be solved numerically. In such cases, it can be easier to solve the full elastostatic equations over the whole domain, (4.8) subject to (4.9) and (4.70), instead of matching the numerical solutions of the inner region with the combined foundation for the outer region, (4.71).

Comparing (4.81) with the (bulk) combined foundation, (4.17), one can infer the physical meaning behind the various supplementary displacements: $\hat{W}_{P_0}$, $\hat{W}_{T_1}$, and $\hat{W}_{P_2}$ ensure there is no vertical displacement at $X = 0$, while $\hat{U}_{P_1}$ and $\hat{U}_{T_0}$ ensure there is no horizontal displacement at $X = 0$; for the elastomer to be at equilibrium, the substrate must also deform orthogonally according to $\hat{U}_{P_0}$, $\hat{U}_{T_1}$, $\hat{U}_{P_2}$, $\hat{W}_{P_1}$, and $\hat{W}_{T_0}$, respectively.

As described in §4.2.2, the vertical surface displacement, $\zeta(X) := W(X, 0)$, is likely to be of most importance in applications. In our work, we are particularly interested in two cases: zero surface-slip, $U(X, 0) = 0$, (*e.g.* for stiff coatings, §4.3) and negligible shear stress, $\mathcal{T} = 0$, (*e.g.* for lubricating fluids, §4.4). For the case of zero-surface slip, the substrate is overly restricted at $X = 0$ and one can show from the governing system, (4.73) subject to (4.74), that $P(0) = P'(0) = P''(0) = 0$ must hold to avoid singular behaviour; hence, $\mathcal{T}(X) \equiv \beta(\nu)P(X)$ and we do not expect a boundary layer (the combined foundation model, (4.20), is sufficient). Note that this is not necessarily the case for other boundary conditions at $X = 0$; for example, for a zero-stress lateral wall, the substrate has the freedom to deform vertically at $X = 0$ and so can deform under a local applied pressure. For negligible applied shear stress, $\mathcal{T} \equiv 0$ and so, provided the pressure can be locally approximated by a quadratic,

we find that the composite solution takes the dimensional form

$$\begin{aligned} \zeta(x) \sim & -\alpha(\nu)(1-2\nu)\frac{d}{G}\left[p(x)-p(0)\hat{\zeta}_{P_0}\left(\frac{x}{d};\nu\right)\right] \\ & +\beta(\nu)\frac{d^2}{G}p'(0)\hat{\zeta}_{P_1}\left(\frac{x}{d};\nu\right)+\gamma(\nu)\frac{d^3}{G}\left[p''(x)-p''(0)\hat{\zeta}_{P_2}\left(\frac{x}{d};\nu\right)\right], \end{aligned} \quad (4.82)$$

for $\hat{\zeta}_{P_i}(\hat{X};\nu) := \hat{W}_{P_i}(\hat{X},0)$. Notice that $\hat{\zeta}_{P_0}$ and $\hat{\zeta}_{P_2}$ ensure $\zeta(0) = 0$, while the $\hat{\zeta}_{P_1}$ term is due to the material gathering at the wall because of the shearing caused by a pressure gradient. This means that even if $p(0) = p''(0) = 0$ and hence $\zeta(0) = 0$ by the combined foundation, (4.20), we can still expect to observe a boundary layer if $p'(0) \neq 0$.

Before we discuss the implications of these results to a physical problem (as is done in §4.6 for a two-dimensional Hele-Shaw flow), let us consider the effects of a curved outer boundary.

4.5.2 Axisymmetric boundary layer

One might naturally expect that a boundary layer close to a curved wall will be identical to a boundary layer close to a flat (two-dimensional) wall since a curved wall can be locally approximated as flat. However, for our foundation model, this is not precisely true: the outer solution, (4.10) with (4.13) and (4.16), is accurate up to $o(\epsilon^2)$, hence higher-order terms in the inner region — including those which contain the effects of boundary curvature — must be included to correctly match with the outer solution. Using the dimensionless variables (4.7), we consider the deformation of a thin, axisymmetric substrate with a fixed wall at $R = R_0 := r_0/\ell$, where ℓ is the characteristic length scale of the outer problem. If the applied pressure and shear stress, $P(R)$ and $\mathcal{T}(R)$, do not vanish in the right way at $R = R_0$, we can expect a boundary layer of size $R - R_0 = \mathcal{O}(\epsilon)$. We shall analyse such a boundary layer, following a similar analysis to that of §4.5.1 for a two-dimensional substrate, now.

In the inner layer, we introduce the variables

$$\tilde{R} := \frac{R - R_0}{\epsilon}, \quad \tilde{W}(\tilde{R}, Z) := W(R_0 + \epsilon\tilde{R}, Z), \quad \tilde{U}(\tilde{R}, Z) := \epsilon U(R_0 + \epsilon\tilde{R}, Z), \quad (4.83a-c)$$

and the locally variable applied pressure and shear stress,

$$\tilde{P}(\tilde{R}) := P(R_0 + \epsilon\tilde{R}) \quad \text{and} \quad \tilde{\mathcal{T}}(\tilde{R}) := \epsilon \mathcal{T}(R_0 + \epsilon\tilde{R}), \quad (4.83d,e)$$

respectively. We shall assume that the wall is an exterior bound to the foundation ($R \leq R_0$), to consider an interior bound ($R \geq R_0$) one can take $\tilde{R} \mapsto -\tilde{R}$, henceforth.

Substituting (4.83) into the outer system, (4.8) subject to (4.9) and (4.70), we find that the inner displacements, $\tilde{W}$ and $\tilde{U}$, satisfy the equations

$$(1 - 2\nu)\tilde{W}_{\tilde{R}\tilde{R}} + 2(1 - \nu)\tilde{W}_{ZZ} + \tilde{U}_{\tilde{R}Z} = -\frac{\tilde{U}_Z + (1 - 2\nu)\tilde{W}_{\tilde{R}}}{R_0/\epsilon + \tilde{R}}, \quad (4.84a)$$

$$(1 - 2\nu)\tilde{U}_{ZZ} + 2(1 - \nu)\tilde{U}_{\tilde{R}\tilde{R}} + \tilde{W}_{\tilde{R}Z} = -\left(\frac{2(1 - \nu)\tilde{U}}{R_0/\epsilon + \tilde{R}}\right)_{\tilde{R}}, \quad (4.84b)$$

subject to the boundary conditions

$$\frac{2(1 - \nu)}{1 - 2\nu}\tilde{W}_Z + \frac{2\nu}{1 - 2\nu}\tilde{U}_{\tilde{R}} = -\tilde{P}(\tilde{R}) \quad \text{and} \quad \tilde{U}_Z + \tilde{W}_{\tilde{R}} = \tilde{T}(\tilde{R}) \quad \text{on } Z = 0, \quad (4.85a,b)$$

$$\tilde{W} = 0 \quad \text{and} \quad \tilde{U} = 0 \quad \text{on } Z = -1, \quad (4.85c,d)$$

$$\tilde{W} = 0 \quad \text{and} \quad \tilde{U} = 0 \quad \text{on } \tilde{R} = 0, \quad (4.85e,f)$$

$$\tilde{W} \sim W_{\text{out}}(R_0 + \epsilon\tilde{R}, Z) \quad \text{and} \quad \tilde{U} \sim \epsilon U_{\text{out}}(R_0 + \epsilon\tilde{R}, Z) \quad \text{as } \tilde{R} \rightarrow -\infty, \quad (4.85g,h)$$

for the matching displacements

$$\begin{aligned} W_{\text{out}}(R_0 + \epsilon\tilde{R}, Z) &:= -\alpha(\nu)(1 - 2\nu)(1 + Z)\tilde{P}(\tilde{R}) \\ &\quad - \beta(\nu)(1 + Z)\left(1 + \frac{Z}{4\nu - 1}\right)\left[\tilde{T}'(\tilde{R}) + \frac{\tilde{T}(\tilde{R})}{R_0/\epsilon + \tilde{R}}\right] \\ &\quad + \gamma(\nu)(1 + Z)\left(1 + \frac{Z}{2} - \frac{1 - \nu}{4\nu - 1}Z^2\right)\left[\tilde{P}''(\tilde{R}) + \frac{\tilde{P}'(\tilde{R})}{R_0/\epsilon + \tilde{R}}\right], \end{aligned} \quad (4.86a)$$

$$\epsilon U_{\text{out}}(R_0 + \epsilon\tilde{R}, Z) := (1 + Z)\tilde{T}(\tilde{R}) - \beta(\nu)(1 + Z)\left(1 - \frac{Z}{4\nu - 1}\right)\tilde{P}'(\tilde{R}). \quad (4.86b)$$

Unlike in the case of a two-dimensional boundary layer (§4.5.1), here the inner system is dependent on the aspect ratio, ϵ . This dependency is due to the curvature of the boundary: for a general applied pressure and shear stress, $\tilde{P}(\tilde{R})$ and $\tilde{T}(\tilde{R})$, the inner equations are dependent on $\epsilon/R_0 \equiv d/r_0$, a measure of the effective boundary curvature. Note that, for an exterior boundary, $\epsilon/R_0 \leq \epsilon \ll 1$; however, for an interior boundary, ϵ/R_0 may be large if the boundary is suitably close to the origin.

On the one hand, if $\epsilon/R_0 \ll 1$, the wall lies sufficiently far from the origin and an expansion in powers of ϵ/R_0 can be made:

$$\tilde{W}(\tilde{R}, Z) = \tilde{W}_0(\tilde{R}, Z) + \frac{\epsilon}{R_0}\tilde{W}_1(\tilde{R}, Z) + \left(\frac{\epsilon}{R_0}\right)^2\tilde{W}_2(\tilde{R}, Z) + \mathcal{O}\left(\frac{\epsilon}{R_0}\right)^3, \quad (4.87a)$$

$$\tilde{U}(\tilde{R}, Z) = \tilde{U}_0(\tilde{R}, Z) + \frac{\epsilon}{R_0}\tilde{U}_1(\tilde{R}, Z) + \left(\frac{\epsilon}{R_0}\right)^2\tilde{U}_2(\tilde{R}, Z) + \mathcal{O}\left(\frac{\epsilon}{R_0}\right)^3, \quad (4.87b)$$

as $\epsilon/R_0 \equiv d/r_0 \rightarrow 0$. The resulting PDE systems for $\tilde{W}_i$ and $\tilde{U}_i$ (which are omitted here

for brevity) are only dependent on the Poisson's ratio, ν . In general, these systems must be solved numerically; we use the method of finite differences, which is discussed in detail in §4.6.3. Note that the leading-order displacements, $\tilde{W}_0$ and $\tilde{U}_0$, are exactly the two-dimensional solutions of (4.73) subject to (4.74), whereas the higher-order terms account for the boundary curvature. To suitably match with the outer solution, n -terms are required in the expansion (4.87), where n is such that $(\epsilon/R_0)^{n-1} = \mathcal{O}(\epsilon^2)$.

On the other hand, if $\epsilon/R_0 \gtrsim 1$, the boundary curvature cannot be ignored at leading-order and, for a general applied pressure and shear stress, (4.85) subject to (4.84) must be solved numerically.

As was done for the two-dimensional problem, analytical progress can be made (regardless of the size of ϵ/R_0) if the applied pressure and shear stress do not vary much over the boundary layer and, hence, can be approximated by a Taylor's expansion:

$$\tilde{P}(\tilde{R}) = \tilde{P}_0 + \tilde{P}_1 \tilde{R} + \tilde{P}_2 \frac{\tilde{R}^2}{2} \quad \text{and} \quad \tilde{\mathcal{T}} = \tilde{\mathcal{T}}_0 + \tilde{\mathcal{T}}_1 \tilde{R}, \quad (4.88a,b)$$

for $\tilde{P}_i \equiv \epsilon^i P^{(i)}(R_0)$ and $\tilde{\mathcal{T}}_i \equiv \epsilon^{i+1} \mathcal{T}^{(i)}(R_0)$.

For a parabolic applied pressure and linear shear stress, (4.88), the outer displacements, (4.86), solve the inner equations (4.84) and boundary conditions (4.85a–d,g,h). Utilizing the linearity of the problem, we find that the inner displacements can be written as:

$$\tilde{W}(\tilde{R}, Z) = W_{\text{out}}(R_0 + \epsilon \tilde{R}, Z) + \sum_{i=0}^2 \epsilon^i C_i \hat{W}_{P_i}(\tilde{R}, Z) + \sum_{i=0}^1 \epsilon^{i+1} D_i \hat{W}_{\mathcal{T}_i}(\tilde{R}, Z), \quad (4.89a)$$

$$\tilde{U}(\tilde{R}, Z) = \epsilon U_{\text{out}}(R_0 + \epsilon \tilde{R}, Z) + \sum_{i=0}^2 \epsilon^i C_i \hat{U}_{P_i}(\tilde{R}, Z) + \sum_{i=0}^1 \epsilon^{i+1} D_i \hat{U}_{\mathcal{T}_i}(\tilde{R}, Z), \quad (4.89b)$$

where the coefficients,

$$C_0 := \alpha(\nu)(1 - 2\nu)P'(R_0), \quad C_1 := \beta(\nu)P'(R_0), \quad D_0 := -\mathcal{T}(R_0), \quad (4.90a-c)$$

$$C_2 := -\gamma(\nu) \left[P''(R_0) + \frac{P'(R_0)}{R_0} \right], \quad D_1 := \beta(\nu) \left[\mathcal{T}'(R_0) + \frac{\mathcal{T}(R_0)}{R_0} \right], \quad (4.90d,e)$$

are chosen such that the corresponding displacements equals 1 or 0 at $Z = \tilde{X} = 0$. The supplementary displacements, $\hat{W}_{P_i}$, $\hat{W}_{\mathcal{T}_i}$, $\hat{U}_{P_i}$, and $\hat{U}_{\mathcal{T}_i}$, solve a set of PDE systems which are dependent on the Poisson's ratio, ν , and curvature parameter, $\epsilon/R_0 \equiv d/r_0$. Notably, these equations are independent of the applied stresses, $P(R)$ and $\mathcal{T}(R)$, (see Appendix 4.B.1). These systems must be solved numerically; however, if $\epsilon/R_0 \ll 1$, one can further expand in powers of ϵ/R_0 as was done in (4.87).

Overall, provided the applied pressure and shear stress can be locally expanded as a

Taylor's expansion, (4.89) gives the composite displacements for the deformation of a thin, rotationally-symmetric, elastic foundation bounded at $R = R_0$ and $Z = -1$. A notable case of interest is that for zero surface-shear, $\mathcal{T} \equiv 0$, for which the vertical composite displacement takes the dimensional form

$$\begin{aligned} \zeta(r) \sim & -\alpha(\nu)(1-2\nu)\frac{d}{G}\left[p(r)-p(r_0)\hat{\zeta}_{P_0}\left(\frac{r-r_0}{d};\frac{d}{r_0},\nu\right)\right] \\ & +\beta(\nu)\frac{d^2}{G}p'(r_0)\hat{\zeta}_{P_1}\left(\frac{r-r_0}{d};\frac{d}{r_0},\nu\right) \\ & +\gamma(\nu)\frac{d^3}{G}\left[\nabla_H^2p(r)-\nabla_H^2p(r_0)\hat{\zeta}_{P_2}\left(\frac{r-r_0}{d};\frac{d}{r_0},\nu\right)\right], \end{aligned} \quad (4.91)$$

where $\hat{\zeta}_{P_i}(\hat{R}; d/r_0, \nu) := \hat{W}_{P_i}(\hat{R}, 0)$ and $\nabla_H^2 := d^2/d^2r + 1/r d/dr$ is the Laplace operator confined to the plane. Comparing (4.82) with (4.91), one can infer the same meanings behind the $\hat{\zeta}_{P_i}$ terms as those in the two-dimensional case, but here we have an additional dependency on the effective curvature, d/r_0 , which can be non-negligible.

In both the two-dimensional composite, (4.82), and the axisymmetric composite, (4.91), we see that the dominant term is dependent on the relative size of the substrate's aspect ratio, $\epsilon \equiv d/\ell$, and how close its Poisson's ratio is to $1/2$. However, the relevant measure of these terms is subtly different in the bulk and the boundary layer. Recall that, in the bulk, a Winkler response ($\zeta \propto dp/G$) is expected if $(1-2\nu) \gg \epsilon^2$, otherwise an elastomeric response ($\zeta \propto d^3\nabla_H^2p/G$) is expected. However, in the boundary layer, a Winkler response is expected provided $(1-2\nu) \gg \epsilon$, otherwise the pressure gradient-shear term ($\zeta \propto d^2p'(\mathbf{x}_0)/G$) is dominant. This means that the deformed surface profile will be most prominent near the substrate bounds if $(1-2\nu) \ll \epsilon \ll 1$ (assuming $p'(0) \neq 0$). Crucially, this can be true even when a Winkler response is appropriate for the bulk deflection — *i.e.* for $\epsilon^2 \ll (1-2\nu) \ll \epsilon$ we have $\zeta \sim -\alpha(\nu)(1-2\nu)dp(\mathbf{x})/G$ in the bulk, but $\zeta \sim \beta(\nu)d^2p'(\mathbf{x}_0)/G$ in the boundary layer. Thus, the successful modelling of a substrate's bulk deflection by a Winkler response does not necessarily imply that a Winkler response near the substrate bounds is appropriate, instead higher-order terms might be needed.

The challenge of the above discussion is even more difficult when the horizontal length scale, ℓ , and hence the aspect ratio, ϵ , is not known *a priori*, which was the case for the indentation and elastohydrodynamic sedimentation examples considered in §4.3 and §4.4, respectively. In §4.6, we demonstrate how our boundary layer corrections are implemented for a finite, two-dimensional, Hele-Shaw flow bounded by a thin elastic layer.

4.6 Example 3: Hele-Shaw flow bounded by a thin elastic layer

Stokes flow between two rigid plates separated by a small gap, commonly known as a Hele-Shaw flow, is an old problem in the theory of viscous fluids (Saffman, 1986). The confinement of the fluid gives rise to a Poiseuille flow, with a flux proportional to the product of the pressure-gradient, $\nabla_H p(\mathbf{x})$, and the cube of the gap thickness, H^3 :

$$Q = -\frac{H^3}{12\mu} \nabla_H p(\mathbf{x}), \quad (4.92)$$

where μ is the fluid viscosity. The relative simplicity of this lubrication-like flow, as well as the intriguing properties of confined viscous flows (for example, the Saffman–Taylor instability and the analogy to flow in porous media, see *e.g.* Leal, 2007), have made these flows a popular area of research with many applications. Recently, with applications such as soft robotics and microfluidics (Thorsen et al., 2002) in mind, there has been an interest in the compliant Hele-Shaw problem: one or both of the bounding walls is replaced by a deformable material. These elastic boundaries can take the form of an elastic beam/plate (*e.g.* Alvarado et al., 2017; Pihler-Puzović et al., 2018; Inamdar et al., 2020), a semi-infinite substrate (*e.g.* Wang and Christov, 2019), or an elastic substrate of finite depth (*e.g.* Box et al., 2020b). In each case, the common feature is that large normal forces may deform the boundary, increasing the gap thickness, H , and, hence, decreasing the normal fluid force.

As a result of this negative feedback, the channel thickness and elastic deformation of the bounding wall are closely coupled through the fluid pressure. For example, in a recent study, Box et al. (2020b) shared that a fast-moving, axisymmetric (though still Stokes) flow can deform an elastic wall sufficiently that it interrupts the flow. This so-called ‘choking’ gives rise to a hydraulic fuse (by analogy with a fuse in electronics, which cuts current above a threshold value). Crucially, choking is observed to occur close to the substrate boundary, within a boundary layer of the same characteristic width as the substrate’s depth. Box et al. concluded that this was a consequence of (i) the radial pressure deforming the elastomer through shear, (ii) the incompressibility of the soft solid, and (iii) the elastomer’s spatial constraints imposed by its rigid enclosure. In this section, we use the combined foundation model for a finite, two-dimensional, elastic substrate (as introduced in §4.5.1) to study the localized bulging of a Hele-Shaw flow bounded by a thin elastic layer. Here, we concentrate on the two-dimensional flow, instead of the axisymmetric case considered by *e.g.* Box et al. (2020b), so that we can generalize the conclusions (i)–(iii) in the simplest case, and so we do not have to worry about the pressure singularity at the point of fluid injection.

4.6.1 Mathematical modelling

We consider a two-dimensional Hele-Shaw flow with a compliant boundary: a viscous fluid (of viscosity μ) is injected between a rigid wall and a thin elastic substrate (of depth d , width x_0 , Poisson's ratio ν , and shear modulus G). The elastic substrate is assumed to be perfectly adhered to an underlying rigid base and two lateral walls. This setup is shown in Fig. 4.13.

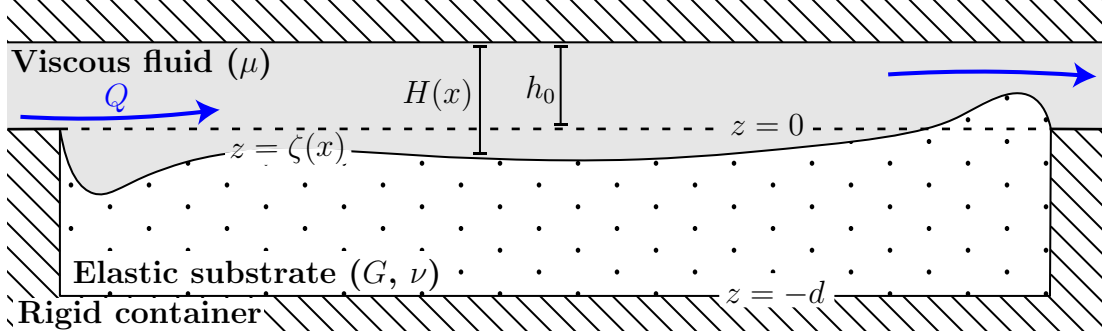


Figure 4.13: A sketch of the third example: a viscous fluid is injected between a rigid upper wall and a finite elastic substrate at a flux Q . The lubricating fluid induces a large pressure profile which deforms the substrate. The resulting gap width is denoted $H(x) = h_0 - \zeta(x)$, where $\zeta(x)$ is the profile of the deformed surface and h_0 is the gap thickness in the absence of deformation.

Unlike in the (unbounded) elastohydrodynamic sedimentation example considered in §4.4, here the horizontal length scale is set by the substrate's width, $\ell = x_0$. Hence, the assumption of a small aspect ratio, both in the viscous fluid and elastic substrate, may be ensured *a priori*. We denote the gap between the rigid wall and the undeflected elastic substrate as h_0 . The typical aspect ratio of the fluid flow, h_0/x_0 , and substrate, $\epsilon \equiv d/x_0$, are assumed to be small so that the hydrodynamic interaction can be modelled using Reynolds' equation (see Leal, 2007, for example) coupled with the combined foundation — *i.e.* taking $p(\mathbf{x}) \mapsto p(x)$ in (4.20). We further assume that $h_0 \ll d$ so that Reynolds' equation also holds within the elastic boundary layers that we anticipate at $x = 0$ and $x = x_0$. Together, we require

$$h_0 \ll d \ll x_0, \quad (4.93)$$

which is equivalent to (4.64) in the elastohydrodynamic sedimentation example.

The fluid pressure is governed by (4.92), *i.e.*

$$H(x)^3 \frac{dp}{dx} = -12\mu Q, \quad (4.94a)$$

where $H(x)$ is the total thickness of the fluid layer, given by

$$H(x) = h_0 - \zeta(x), \quad (4.94b)$$

for $0 \leq x \leq x_0$. The Reynolds' equation in (4.94a) is equivalent to (4.43a) in the earlier example for zero boundary velocity, $U = \omega R = 0$; here, however, the fluid flux, Q , and undeformed gap width, h_0 , are known and are set by the behaviour at the inlet.

The deformed elastic substrate surface, $\zeta(x)$, is given in terms of the pressure profile, $p(x)$, as

$$\zeta(x) = -\alpha(\nu)(1 - 2\nu)\frac{d}{G}p(x) + \gamma(\nu)\frac{d^3}{G}p''(x) + \hat{\zeta}_L\left(\frac{-x}{d}\right) + \hat{\zeta}_R\left(\frac{x - x_0}{d}\right), \quad (4.94c)$$

where $\hat{\zeta}_L$ and $\hat{\zeta}_R$ are the left and right boundary layer corrections, needed to ensure the substrate displacements vanish at the side walls, $x = 0$ and $x = x_0$, respectively: the material is assumed to be perfectly adhered to the side walls, *i.e.* $u(0, z) = w(0, z) = 0$ and $u(x_0, z) = w(x_0, z) = 0$. In general, $\hat{\zeta}_L$ and $\hat{\zeta}_R$ are determined by solving the local elastostatic equations, which involve the fluid pressure (this was the principal discussion of §4.5). A tractable case, however, is when the fluid pressure profile can be approximated as locally parabolic (in x) close to the confining side walls; the surface displacement then takes the form (4.82), for a substrate bound on the right at $x = 0$, or, in this case,

$$\begin{aligned} \zeta(x) = & -\alpha(\nu)(1 - 2\nu)\frac{d}{G}\left[p(x) - p(0)\hat{\zeta}_{P_0}\left(\frac{-x}{d}; \nu\right) - p(x_0)\hat{\zeta}_{P_0}\left(\frac{x - x_0}{d}; \nu\right)\right] \\ & + \beta(\nu)\frac{d^2}{G}\left[-p'(0)\hat{\zeta}_{P_1}\left(\frac{-x}{d}; \nu\right) + p'(x_0)\hat{\zeta}_{P_1}\left(\frac{x - x_0}{d}; \nu\right)\right] \\ & + \gamma(\nu)\frac{d^3}{G}\left[p''(x) - p''(0)\hat{\zeta}_{P_2}\left(\frac{-x}{d}; \nu\right) - p''(x_0)\hat{\zeta}_{P_2}\left(\frac{x - x_0}{d}; \nu\right)\right], \end{aligned} \quad (4.94d)$$

where $\hat{\zeta}_{P_i}(\hat{X}; \nu) := \hat{W}_{P_i}(\hat{X}, 0)$ solve the system of PDEs given in Appendix 4.B.1, and depend only on the Poisson's ratio, ν . (Note that the functions $\hat{\zeta}_{P_i}$ are independent of the applied pressure by the Taylor's expansion performed in §4.5.1.) Here, we have used the fact that the surface shear stress is negligible for a thin fluid flow provided $h_0 \ll d$, as already assumed in (4.93).

Solutions to (4.94), are subject to the boundary conditions,

$$p(x_0) = 0, \quad p'(x_0) = p'(0) = -\frac{12\mu Q}{h_0^3}, \quad p''(x_0) = p''(0) = 0. \quad (4.95a-e)$$

Condition (4.95a) results from setting the reference pressure at $x = x_0$, while (4.95b-e)

come directly from Reynolds' equation, (4.94a), with $\zeta(0) = \zeta(x_0) = 0$ (zero-displacement walls) and $\zeta'(0) = \zeta'(x_0) = 0$ (zero-displacement walls with zero-surface shear stress). Note that the conditions (4.95) greatly simplify the surface displacement for the case of a locally parabolic pressure, (4.94d).

4.6.1.1 Non-dimensionalization

Unlike in the examples considered in §4.3 and §4.4, here the horizontal length scale is set by the finite size of the substrate, $\ell = x_0$, and is independent of the substrate deformation. Similarly, the pressure scale is set directly from the inlet fluid flux, (4.95c). The choice of substrate deflection scale is dependent on which term dominates in the response (4.94d); there are three natural choices,

$$\zeta_W^* := \frac{\alpha(\nu)(1-2\nu)d}{x_0} \frac{12\mu Q x_0^2}{G h_0^3}, \quad \zeta_S^* := \frac{\beta(\nu)d^2}{x_0^2} \frac{12\mu Q x_0^2}{G h_0^3}, \quad \zeta_I^* := \frac{\gamma(\nu)d^3}{x_0^3} \frac{12\mu Q x_0^2}{G h_0^3}. \quad (4.96a-c)$$

We refer to the lengths ζ_W^* , ζ_S^* , and ζ_I^* as the Winkler, shear, and incompressible scales, respectively. On the one hand, away from the boundaries, the substrate deforms according to the combined foundation model, and so the correct deflection scale is ζ_W^* or ζ_I^* , depending on the relative size of $(1-2\nu)$ and the square of the aspect ratio, $\epsilon^2 := (d/x_0)^2$ — a similar choice had to be made for the previous two examples, (4.29) and (4.50). On the other hand, close to the boundaries, the substrate deforms either due to a Winkler response or due to a build-up of material caused by the accumulated shearing induced by pressure. The correct choice is, thus, either ζ_W^* or ζ_S^* depending on the relative size of $(1-2\nu)$ and the aspect ratio, $\epsilon := d/x_0$. It should be noted that $\zeta_I^* \ll \zeta_S^*$ regardless of the value of ν since $\epsilon \ll 1$. Without loss of generality, we choose the shear length scale as the deflection scale, $[\zeta] = \zeta_S^*$. Note that this contrasts the choice made in the first two examples, §4.3 and §4.4, where we chose the incompressible length scale; here, we choose the shear length scale to achieve the correct scalings inside the boundary layer and to simplify the small deflection analysis (which we present in §4.6.2).

We introduce the dimensionless variables

$$X := \frac{x}{x_0}, \quad \mathcal{H}(X) := \frac{H(x)}{h_0}, \quad (4.97a,b)$$

$$P(X) := \frac{h_0^3}{12\mu Q x_0} p(x), \quad Z(X) := \frac{h_0^3 G}{12\beta(\nu)d^2\mu Q} \zeta(x). \quad (4.97c,d)$$

Inserting the non-dimensionalization, (4.97), into the system, (4.94), gives

$$\mathcal{H}(X)^3 P'(X) = -1 \quad \text{for} \quad \mathcal{H}(X) = 1 - \Phi Z(X), \quad (4.98a,b)$$

in $0 \leq Z \leq 1$, with the substrate deflection given by

$$Z(X) = -\mathcal{C}_W P(X) + \mathcal{C}_I P''(X) + \hat{Z}_L \left(\frac{-X}{\epsilon} \right) + \hat{Z}_R \left(\frac{X-1}{\epsilon} \right), \quad (4.98c)$$

where

$$\Phi := \frac{12\beta(\nu)d^2\mu Q}{h_0^4 G}, \quad \mathcal{C}_W := \frac{\alpha(\nu)(1-2\nu)}{\beta(\nu)} \frac{1}{\epsilon}, \quad \mathcal{C}_I := \frac{\gamma(\nu)}{\beta(\nu)} \epsilon, \quad (4.99a-c)$$

for aspect ratio $\epsilon := d/x_0 \ll 1$. If the fluid pressure can be approximated as a parabola close to the side walls, then we may also write

$$\begin{aligned} Z(X) = & -\mathcal{C}_W \left[P(X) - P(0) \hat{Z}_{P_0} \left(\frac{-X}{\epsilon}; \nu \right) \right] + \mathcal{C}_I P''(X) \\ & + \hat{Z}_{P_1} \left(\frac{-X}{\epsilon}; \nu \right) - \hat{Z}_{P_1} \left(\frac{X-1}{\epsilon}; \nu \right). \end{aligned} \quad (4.100)$$

Solutions to (4.98) are subject to the boundary conditions

$$P(1) = 0, \quad P'(1) = P'(0) = -1, \quad P''(1) = P''(0) = 0. \quad (4.101a-e)$$

(Note that, by imposing the combined foundation here, we have implicitly assumed that the substrate responds according to linear elasticity; thus, in addition to the constraints (4.93), we also require that $|\zeta(x)| \ll d$ or, equivalently, $\Phi|Z(X)| \ll d/h_0$, which gives the parameter restraints $\Phi \ll d/h_0$ and $\Phi\mathcal{C}_W \ll d/h_0$.)

The three dimensionless parameters, Φ , $\mathcal{C}_W$, and $\mathcal{C}_I$, can be interpreted as follows. The parameter Φ measures how much the substrate deforms under the flux-induced pressure, we refer to Φ as the *pliability*, it is analogous to $\hat{\phi}$ in the elastohydrodynamic sedimentation problem, (4.54b). We also note that $\mathcal{C}_W \equiv \zeta_W^*/\zeta_S^*$ and $\mathcal{C}_I \equiv \zeta_I^*/\zeta_S^*$, these measure the relative size of the Winkler, shear, and incompressible terms in (4.100). The ratio $\mathcal{C}_W/\mathcal{C}_I$ measures the *compressibility* of the elastic bulk, it is analogous to the parameters $\mathcal{C}$ and $\hat{\mathcal{C}}$ in the previous two examples, (4.33a) and (4.54a), respectively; similarly, $\mathcal{C}_W$ measures the *compressibility* inside of the boundary layer. Since $\mathcal{C}_W$ and $\mathcal{C}_I$ are functions of ν and ϵ , one can equivalently give the Poisson's ratio and aspect ratio instead of the compressibility parameters.

Overall, given a substrate pliability/input flux, Φ ; aspect ratio, ϵ ; and Poisson's ratio, ν , (4.98) and (4.101) form a differential equation for the fluid pressure, $P(X)$, and substrate deflection, $Z(X)$. In general, the pressure is nonlinearly dependent on the substrate deflection due to Reynolds' equation, (4.98a,b), and so the expression (4.98c) must be used for the

surface deflection. This means that the full equilibrium equations, (4.73) with (4.74) and $\mathcal{T} = 0$, must be solved in the inner layer (this case is discussed further in §4.6.3). Progress can be made in the limit of small substrate deflections, $\Phi|Z(X)| \ll 1 \ll d/h_0$. In this limit, the pressure and substrate deflection decouple at leading-order, meaning the locally quadratic expression for $Z(X)$, (4.100), can be used. We consider this case now.

4.6.2 Asymptotic results for $\Phi \ll 1$

For $\Phi|Z(X)| \ll 1 \ll d/h_0$ (or, equivalently, $\Phi \ll 1$ and $\mathcal{C}_W\Phi \ll 1$), the leading-order fluid pressure is independent of the surface displacement. We, thus, expect the pressure to only be a function of the outer variable, X , and so it can be approximated by a Taylor's expansion inside of the boundary layer; thus, (4.100) describes the surface displacement. To consider this limit, we pose the expansions

$$Z(X) = Z_0(X) + \Phi Z_1(X) + \mathcal{O}(\Phi^2) \quad \text{and} \quad P(X) = P_0(X) + \Phi P_1(X) + \mathcal{O}(\Phi^2), \quad (4.102a,b)$$

as $\Phi \rightarrow 0$. (Note that a similar expansion was considered for the elastohydrodynamic sedimentation problem in §4.4.2.) Inserting (4.102) into (4.98), (4.100), and (4.101), we consider the problem order-by-order:

At leading-order, $\mathcal{O}(\Phi^0)$, the fluid pressure is equivalent to the pressure in a rigid lubricating channel,

$$P_0(X) = 1 - X. \quad (4.103)$$

Since this pressure can be written as a quadratic local to the substrate walls (it is a linear function over the whole domain), the expression (4.100) can be used to determine the leading-order surface displacement. We obtain,

$$Z_0(X) = -\mathcal{C}_W \left[1 - X - \hat{Z}_{P_0} \left(-\frac{X}{\epsilon}; \nu \right) \right] - \hat{Z}_{P_1} \left(\frac{X-1}{\epsilon}; \nu \right) + \hat{Z}_{P_1} \left(-\frac{X}{\epsilon}; \nu \right). \quad (4.104)$$

Hence, at leading-order, we find that the bulk elastomer is displaced according to Winkler response, $Z_0(X) \propto -\mathcal{C}_W P_0(X)$, and since $P_0''(X) = 0$, the substrate bulk is not expected to deform for perfectly incompressible materials ($\mathcal{C}_W = 0$). Close to the boundaries, there is an additional displacement, $\hat{Z}_{P_1}$, which is due to the pressure-gradient shearing material away from $X = 0$ to $X = 1$. The leading-order surface displacement, (4.104), is plotted in Fig. 4.14, for $\epsilon = 0.01$ and various ν , and Fig. 4.15, for $\nu = 0.5$ and various ϵ . In these plots, we clearly observe that the boundary layers at $X = 0$ and $X = 1$ are of width $\mathcal{O}(\epsilon)$, and the bulk deforms for $\mathcal{C}_W \neq 0$ (*i.e.* compressible substrates, $\nu \neq 1/2$).

At next-order, $\mathcal{O}(\Phi^1)$, the leading-order surface displacement, (4.104), induces a fluid

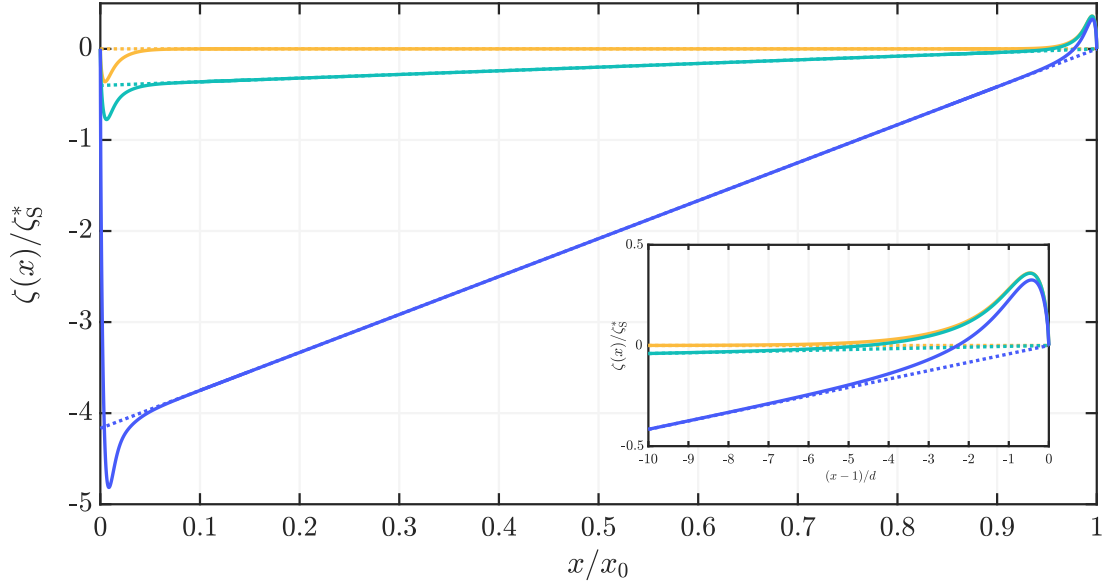


Figure 4.14: Main figure: The small pliability ($\Phi \ll 1$) surface displacement, (4.104), (solid curves) and bulk solution (dotted lines) as a function of the outer variable, $X \equiv x/x_0$. Plotted are the profiles for aspect ratio $\epsilon = 0.01$ and Poisson's ratio $\nu = 0.5$ (yellow), $\nu = 0.499$ (green), and $\nu = 0.49$ (blue). For compressible substrates ($\nu \neq 1/2$), the elastomer bulk deforms according to the Winkler foundation, whilst the boundary layer, which is due to the pressure-gradient shearing, occurs for all values of ν . Inset: The surface displacements as a function of the inner variable near to the right-hand boundary, $\hat{X} \equiv (x-1)/d$.

pressure, which takes the form

$$P_1(X) = 3 \int_X^1 Z_0(\xi) d\xi = -\mathcal{C}_W \frac{3(1-X)^2}{2} - 3\epsilon \int_{-\infty}^{-\frac{X}{\epsilon}} \mathcal{C}_W \hat{Z}_{P_0}(\hat{X}; \nu) + \hat{Z}_{P_1}(\hat{X}; \nu) d\hat{X} + 3\epsilon \int_{\frac{X-1}{\epsilon}}^0 \hat{Z}_{P_1}(\hat{X}; \nu) d\hat{X}. \quad (4.105)$$

where we have replaced the lower limit of the second integral, $-1/\epsilon$, with $-\infty$, consistent with the bounds of the inner functions $\hat{Z}_{P_i}$. The three terms in (4.105) correspond to the pressure due to the Winkler response, the pressure across the boundary layer at $X = \mathcal{O}(\epsilon)$, and the pressure across the boundary layer at $1-X = \mathcal{O}(\epsilon)$, respectively. Since the pressure, (4.105), is a function of the inner variables, X/ϵ and $(X-1)/\epsilon$, it cannot be approximated as a quadratic in the two boundary layers. Hence, the corresponding displacement, Z_1 , must be numerically computed by solving the full elastostatic equations, (4.73) subject to (4.74), with $\mathcal{T} = 0$ and $P = P_1(X)$. We can, however, use the combined foundation, (4.20), in the bulk to conclude that

$$Z_1(X) = -\mathcal{C}_W P_1(X) + \mathcal{C}_1 P_1''(X) = 3\mathcal{C}_W \left(\mathcal{C}_W \frac{(1-X)^2}{2} - \mathcal{C}_1 \right), \quad (4.106)$$

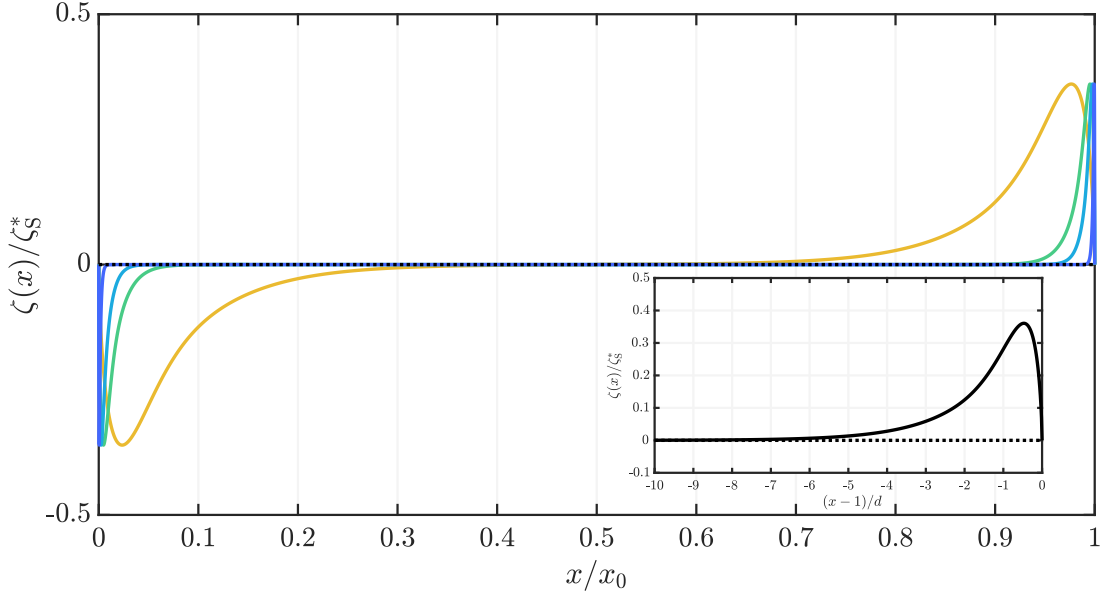


Figure 4.15: Main figure: The small pliability ($\Phi \ll 1$) surface displacement, (4.104), (solid curves) and bulk solution (dotted lines) as a function of the outer variable, $X \equiv x/x_0$, for a perfectly incompressible foundation ($\nu = 1/2$). Plotted are the profiles for substrate aspect ratio $\epsilon = 0.001$, $\epsilon = 0.005$, $\epsilon = 0.01$, and $\epsilon = 0.05$ (blue to yellow). Observe that the pressure-gradient induced boundary layers have width $\mathcal{O}(\epsilon)$. Inset: The surface displacement near to the substrate's bounds collapse as a function of the inner variable, $\hat{X} \equiv (x-1)/d$.

for $\epsilon \ll X \ll 1 - \epsilon$ (*i.e.* away from the bounding walls).

Based on Fig. 4.14, we expect the maximum substrate height to occur close to the fluid outlet, $X = 1$, and to be caused by the pressure-gradient shearing material from left to right. The Winkler response is expected to only compress the substrate (since $P(X) \geq 0$). If the pliability (or, equivalently, the input fluid flux Φ) is large enough, this surface buckling will impede the fluid flow. Based on the asymptotic solution above, this occurs when

$$\min_X \mathcal{H}(X) = 1 - \Phi \max_X Z_0(X) + \mathcal{O}(\Phi^2) \approx 0, \quad (4.107)$$

i.e. $\Phi \approx 1/0.36 = 2.778$, by Figs. 4.14 and 4.15. For such values of Φ , the asymptotic results of this section fail to be valid. Instead, the full nonlinear problem must be considered. Before discussing this, it is worth isolating the specific case of an incompressible foundation ($\mathcal{C}_W \rightarrow 0$).

4.6.2.1 Incompressible foundation ($\mathcal{C}_W \rightarrow 0$)

For an incompressible foundation, $\mathcal{C}_W \rightarrow 0$, the pressure and surface displacement are asymptotically given by

$$P(X) \sim 1 - X + 3\Phi\epsilon \left[\int_{\frac{X-1}{\epsilon}}^0 \hat{Z}_{P_1}(\hat{X}; \nu) d\hat{X} - \int_{-\infty}^{-\frac{X}{\epsilon}} \hat{Z}_{P_1}(\hat{X}; \nu) d\hat{X} \right], \quad (4.108a)$$

$$Z(X) \sim \hat{Z}_{P_1}\left(-\frac{X}{\epsilon}; \nu\right) - \hat{Z}_{P_1}\left(\frac{X-1}{\epsilon}; \nu\right), \quad (4.108b)$$

as $\Phi \rightarrow 0$. Here, the deformation is localized around the bounding walls such that the solid volume is conserved ($\nu = 1/2$): $\int_0^1 Z(X) dX = 0$. This volume conservation means the inlet pressure is $P(0) = 1 + \mathcal{O}(\Phi^2)$: the effects of the substrate deformation are not felt by the inlet fluid until $\mathcal{O}(\Phi^2)$, in contrast to a compressible substrate, for which $P(0) = 1 - 3\mathcal{C}_W\Phi \left(1/2 - \epsilon \int_{-\infty}^0 \hat{Z}_{P_0}(\hat{X}) d\hat{X}\right) + \mathcal{O}(\Phi^2)$. The surface profile, (4.108b), is plotted in Fig. 4.15 for various ϵ . An incompressible substrate only deforming near the lateral boundaries, as seen here, is actually true for all values of Φ (not just infinitesimal ones): in the bulk, Reynolds' equation, (4.98a,b), is coupled with the incompressible response, $Z(X) = \mathcal{C}_I P''(X)$, which is solved by a linear pressure profile, $P(X) = P_0 - X$ where P_0 is unknown and set by the pressure jumps across the boundary layers.

4.6.3 Numerical results for moderate Φ

For $\mathcal{O}(1)$ values of Φ , the substrate deflection and fluid pressure are nonlinearly dependent on each other through Reynolds' equation, (4.98a,b). In this case, the parabolic approximations of the pressure profile inside of the boundary layers are no longer valid, and the substrate's displacement is given by (4.98c). We approach this problem numerically by solving the inner and outer asymptotic regions simultaneously, subject to matching conditions at fixed interfaces: $X = X_{\text{match}}$ and $X = 1 - X_{\text{match}}$ for the left and right boundary layers, respectively. In our numerics, we usually use $X_{\text{match}} = \epsilon^{1/2}$, although any power of ϵ satisfying $\epsilon \ll \epsilon^\alpha \ll 1$ should suffice.

In the outer region, $X_{\text{match}} \leq X \leq 1 - X_{\text{match}}$, Reynolds' equation, (4.98a,b), is coupled with the combined foundation, (4.20), which takes the dimensionless form

$$Z(X) = -\mathcal{C}_W P(X) + \mathcal{C}_I P''(X). \quad (4.109)$$

This system is solved using the method of finite differences in X .

In the left and right inner regions, $-X_{\text{match}}/\epsilon \leq \hat{X} \equiv -X/\epsilon \leq 0$ and $-X_{\text{match}}/\epsilon \leq \hat{X} \equiv (X-1)/\epsilon \leq 0$, respectively, Reynolds' equation, (4.98a,b), is coupled with the localized

elastostatic equations, (4.73) subject to (4.74) and (4.101). Here, we have assumed that Reynolds' equation holds in the inner region (*i.e.* the lubrication approximations still hold), which is true provided the parameter limits (4.93) hold. These inner systems are solved using the method of finite differences in $\hat{X}$ and Z .

Overall, for given parameters Φ , ν , and ϵ , the inner and outer systems, subject to continuity in the pressure and displacements at $X = X_{\text{match}}$ and $X = 1 - X_{\text{match}}$, can be solved simultaneously. We use the nonlinear system solver `fsolve` in `MATLAB`.

In the above formulation, using the combined foundation, (4.109), in the outer region meant that we could avoid solving the elastostatic equations in the bulk numerically. For moderate–large values of ϵ , the asymptotic combined foundation fails to hold, and instead the elastostatic equations must be solved over the whole domain, $0 \leq X \leq 1$. The formulation of these equations is presented in Appendix 4.A.3; numerical solutions to these equations have been used to validate the above asymptotic model. Since we are predominately interested in the localized bulging of the substrates (*i.e.* boundary layers), however, we do not analyse $\epsilon \gtrsim \mathcal{O}(1)$ further; although, we note here that progress can be made for very thick substrates ($\epsilon \gg 1$).

In Fig. 4.16, the fluid pressure at the configuration's inlet ($X = 0$) and midpoint ($X = 1/2$) — found by solving the above problem numerically — are plotted as functions of the fluid flux, $\Phi/[\epsilon\beta(\nu)] \propto Q$. At small fluxes, the Poiseuille solution, $p(x) = 12\mu Q(x_0 - x)/h_0^3$, is recovered, as is expected from our asymptotic analysis, (4.103). For large fluxes, however, the substrate's bulk deforms proportionally to the compressibility of the substrate, $\mathcal{C}_W \propto (1 - 2\nu)$, this deformation increases the flux gap thickness, $\mathcal{H}(X)$, and, hence, decreases the corresponding pressure. Based on the asymptotic pressure, (4.105), this deviation is expect to occur for $\Phi\mathcal{C}_W = \mathcal{O}(1)$ or, correspondingly, $\Phi/[\epsilon\beta(\nu)] = \mathcal{O}(1/(1 - 2\nu))$. For a perfectly incompressible substrate, $\mathcal{C}_W = 0$, the bulk does not deform (as discussed in §4.6.2.1), so the deviation from the Poiseuille pressure is solely due to the boundary layer bulging, which occurs at $\Phi\epsilon = \mathcal{O}(1)$ — *i.e.* $\Phi/[\epsilon\beta(\nu)] = \mathcal{O}(1/\epsilon^2)$ — see (4.108a). Furthermore, since the pressure is increased over the outlet bulge (where $Z(X) \geq 0$) and decreased over the inlet bulge (where $Z(X) \leq 0$), the pressure deviation at the inlet, $X = 0$, is only seen if there is an asymmetry in substrate deflection, which only occurs at even larger fluxes — see (4.108a) with $X = 0$, for example. This crudely explains why the deviation from Poiseuille flow for $\nu = 1/2$ is more apparent in Fig. 4.16(b) than in Fig. 4.16(a).

Although the effects of the substrate bulging are discernible from comparing the pressures at $X = 0$ and $X = 1/2$, *i.e.* comparing Figs. 4.16(a) and 4.16(b), the associated maximum deflection is $\max_X \Phi Z(X) \equiv \max_x \zeta(x)/h_0 \approx 0.264$, which occurs at $\nu = 1/2$ and $\Phi/[\epsilon\beta(\nu)] = 100$. Larger deflections are expected for larger fluid fluxes, $\Phi \gg 1$; however, the complete choking of the flow (as seen in Box et al., 2020b, for example) cannot occur

for steady flows as it contradicts the imposed flux, Q , in Reynolds' equation, (4.98a,b). Moreover, larger values of Φ require more elaborate numerical methods to account for large changes in the local pressure. That being said, the combined foundation can still provide insight into the choking mechanism; for example, according to the competition of terms in (4.100), the substrate compressibility ($\mathcal{C}_W > 0$) reduces the maximum deflection of the substrate within the right boundary layer, $1 - X = \mathcal{O}(\epsilon)$, thus compressibility could provide a physical mechanism for preventing choking. Analysing the combined foundation in such limits is left for future work.

4.7 Conclusions

In this chapter, we have presented an asymptotic analysis of the deformation of a thin elastic substrate in response to pressure and shear loading at its surface. This analysis led to a combined foundation model, (4.19), that is able to describe the deformation of a slender foundation irrespective of how close it is to being perfectly incompressible ($\nu = 1/2$). Supplementary displacements were found to be required near to lateral boundaries that impose displacements inconsistent with the combined foundation model; these are apparent in boundary layers of width comparable to the substrate thickness (§4.5). We used this combined foundation model with the assumption of either zero surface-shear or zero surface-slip, (4.20), to demonstrate that whether a given material behaves as an incompressible or compressible substrate is not determined solely by its Poisson's ratio: even with a Poisson's ratio that is arbitrarily close to $\nu = 1/2$, other properties of the system, particularly its slenderness, may lead to behaviour that is effectively compressible and, hence, may be well-described by the classical Winkler model.

Generally speaking, which of these behaviours (compressible or incompressible) is observed in the substrate's bulk is determined by a compressibility parameter,

$$\mathcal{C} \propto (1 - 2\nu) \frac{\ell^2}{d^2}, \quad (4.110)$$

where d is the depth of the foundation and ℓ is the characteristic horizontal length scale. Local to lateral boundaries, a modified compressibility parameter,

$$\mathcal{C}_{\text{BL}} \propto (1 - 2\nu) \frac{\ell}{d}, \quad (4.111)$$

may be appropriate, depending on the imposed boundary conditions. Unfortunately, it is usually the case that the length scale ℓ can only be determined by the solution of the problem at hand; in this regard, the combined foundation model can be used as a quantitative alternative to scaling analysis to determine the relevant behaviour.

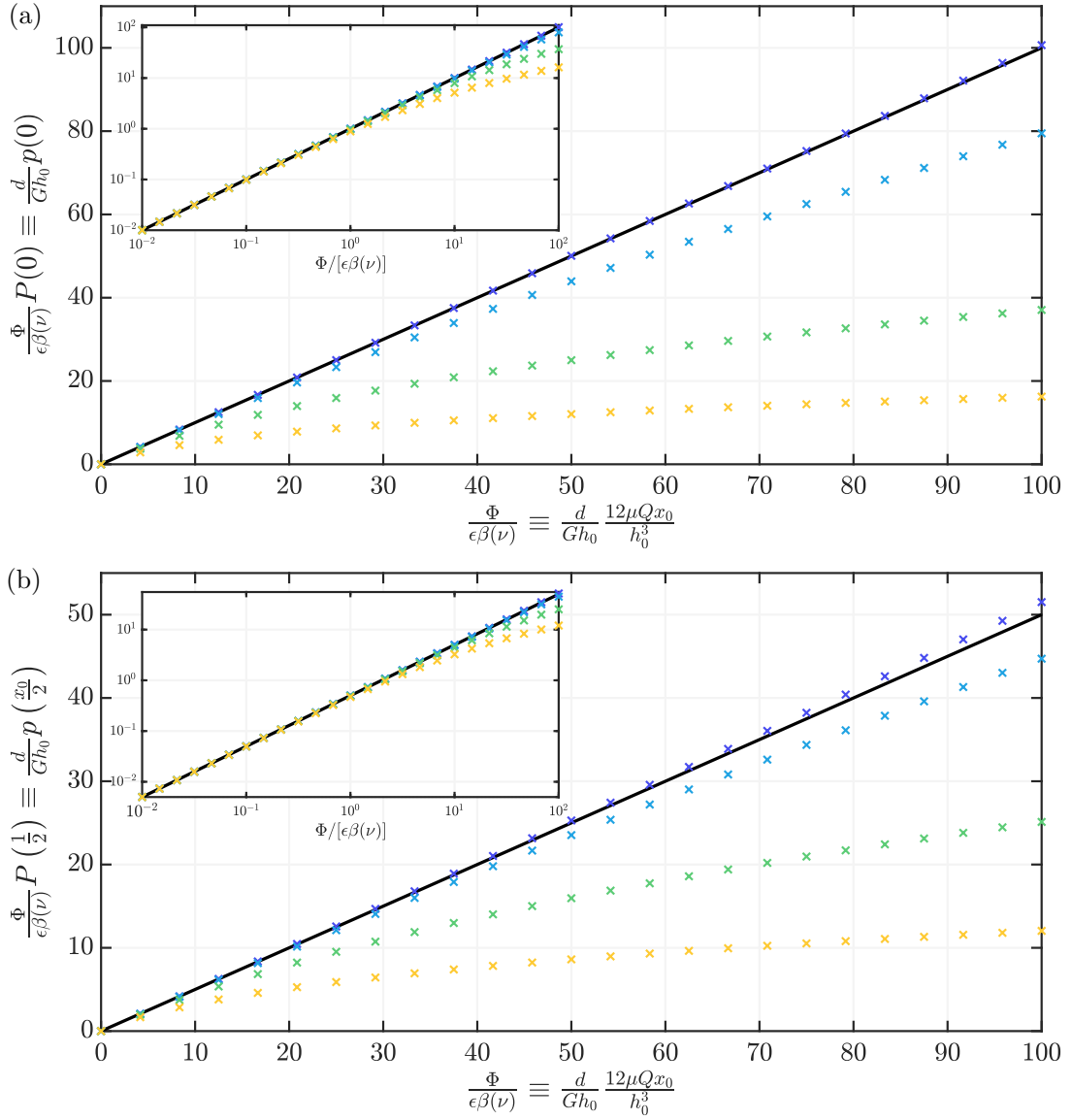


Figure 4.16: Main figures: Dimensionless fluid pressure at the (a) inlet ($X = 0$) and (b) midpoint ($X = 1/2$) as a function of the dimensionless fluid flux, $\Phi/[\epsilon\beta(\nu)] \propto Q$. Plotted are the numerical solutions of the two-dimensional Hele-Shaw flow problem using the combined foundation (crosses, $\times$) with aspect ratio $\epsilon = 0.01$ and Poisson's ratio $\nu = 0.5$, $\nu = 0.499$, $\nu = 0.49$, and $\nu = 0.45$ (blue to yellow). For small fluid fluxes, the results for the compliant Hele-Shaw cell are comparable to the linear Poiseuille flow, (4.103), (black solid line). For large fluxes, the substrate's bulk deforms proportional to the substrate compressibility, $\mathcal{C}_W \propto (1 - 2\nu)$, which increases the fluid gap, decreasing the required pressure. Insets: The corresponding logarithmic plots show that the Poiseuille flow, (4.103), is recovered in the asymptotic limit $\max\{\Phi, \Phi\mathcal{C}_W\} \ll 1$.

We have illustrated the importance of the compressibility parameter, $\mathcal{C}$, in three concrete examples and shown how the combined foundation model may be used to derive results valid for intermediate values of the compressibility, $\mathcal{C}$, where scaling results are not available.

We have also shown that the combined foundation model breaks down when the aspect ratio of the foundation, d/ℓ , becomes $\mathcal{O}(1)$, at which point the foundation behaves more akin to an elastic half-space.

A feature of our results that is, at first sight, surprising, is that materials with Poisson's ratio very close to $1/2$ may be well-described by the Winkler model. For example, the results in Fig. 4.9 with $\nu = 0.4999$ indicate that the (compressible) Winkler model is valid for extremely slender foundations provided the aspect ratio $\epsilon \lesssim 5 \times 10^{-3}$.

The transition from incompressible to Winkler behaviour might also give a sensitive test of the Poisson's ratio, ν , for materials that are close to incompressible, $\nu \approx 1/2$. For example, in the first example considered here, the indentation of a stiff thin sheet coating a thin soft layer, $\mathcal{C} = \ell_W^4/\ell_1^4 \propto (1 - 2\nu)(B/Gd^3)^{1/3}$. If the position of the 'kink' in the dimensionless stiffness, Fig. 4.4, can be accurately measured and identified with $\mathcal{C} = \mathcal{O}(1)$, then it would be possible to obtain an accurate measure of $1 - 2\nu$ and, hence, the deviation of ν from $1/2$. For example, in Fig. 4.4, a 10% discrepancy between the incompressible plateau and the combined foundation stiffness occurs at $\mathcal{C} \approx 0.1547$. Hence, if one is able to plot an experiment's indentation stiffness, K_I , for varying compressibility, $(B/Gd^3)^{1/3}$, and measure a 10% discrepancy at $(B/Gd^3)^{1/3} = X$ say, then our results suggest that the substrate's Poisson's ratio is $\nu \approx 0.5 - 0.4054X$.

Alternatively, for the third example, the Hele-Shaw flow on a thin, compliant foundation, the inlet pressure required to achieve a specific fluid flux, Fig. 4.16(a), could also provide a sensitive test for the Poisson's ratio. For example, for a fluid flux $12\mu Qd^2/(Gh_0^4) = 1$ and substrate aspect ratio $d/x_0 = 0.01$, the dimensionless inlet pressure is $dp(0)/(Gh_0) \approx 100.7$ for $\nu = 1/2$ while $dp(0)/(Gh_0) \approx 37.1$ for $\nu = 0.49$ — see Fig. 4.16(a). So, provided the inlet pressure and associated fluid flux can be measure accurately, this sensitive dependence on the Poisson's ratio could be taken advantage of. We leave these and other applications of the combined foundation model to future work.

Appendix 4.A Numerical solutions of the finite depth foundation

In this appendix, we present further details of the numerical simulations to describe an elastic foundation with finite thickness. We present the governing equations for the examples of §4.3, §4.4, and §4.6, and discuss the numerical solutions of these systems.

4.A.1 Example 1: Indentation of a soft layer with a stiff coating

In §4.3, we considered the point-indentation of a soft substrate with a stiff coating. Using the scalings (4.30) and (4.7), we introduce the dimensionless coordinates, displacements, and pressure:

$$\rho := \frac{r}{\ell_I}, \quad Z := \frac{z}{d}, \quad (4.A.1a,b)$$

$$U(\rho, Z) := \frac{u(r, z)}{\epsilon \delta}, \quad W(\rho, Z) := \frac{w(r, z)}{\delta}, \quad Q(\rho) := \frac{\ell_I^4 q(r)}{B \delta}, \quad (4.A.1c-e)$$

where ℓ_I is the horizontal length scale for an incompressible foundation, (4.29b), and $\epsilon = d/\ell_I$ is the aspect ratio, (4.42). With this non-dimensionalization, the displacements $U(\rho, Z)$ and $W(\rho, z)$ satisfy

$$W_{ZZ} = -\frac{\epsilon^2}{2(1-\nu)} \left(\frac{1}{\rho} \frac{\partial(\rho U_Z)}{\partial \rho} + (1-2\nu) \frac{1}{\rho} \frac{\partial(\rho W_\rho)}{\partial \rho} \right), \quad (4.A.2a)$$

$$U_{ZZ} + \frac{W_{\rho Z}}{1-2\nu} = -\frac{2(1-\nu)}{1-2\nu} \epsilon^2 \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial(\rho U)}{\partial \rho} \right), \quad (4.A.2b)$$

in $-1 \leq Z \leq 0$ and $0 \leq \rho < \infty$, subject to the boundary conditions

$$W, U \rightarrow 0 \quad \text{as } \rho \rightarrow \infty, \quad (4.A.3a,b)$$

$$W_\rho = 0 \quad \text{and} \quad U = 0 \quad \text{on } \rho = 0, \quad (4.A.3c,d)$$

$$W = 0 \quad \text{and} \quad U = 0 \quad \text{on } Z = -1, \quad (4.A.3e,f)$$

$$W_Z + \frac{\nu \epsilon^2}{1-\nu} \frac{1}{\rho} \frac{\partial(\rho U)}{\partial \rho} = -\mathcal{C} Q(\rho) \quad \text{and} \quad U = 0 \quad \text{on } Z = 0, \quad (4.A.3g,h)$$

$$W = -1 \quad \text{at } Z = \rho = 0. \quad (4.A.3i)$$

where $Q(\rho)$ is given by the Kirchhoff–Love plate equation,

$$Q(\rho) = \nabla_H^4 W(\rho, 0) + K_I \frac{\delta_{\text{Dirac}}(\rho)}{\rho}. \quad (4.A.4)$$

For given values of ν and $\mathcal{C}$, (4.A.2)–(4.A.4) describe a system for the substrate displacements, $U(\rho, Z)$ and $W(\rho, Z)$, with the problem for $W(\rho, 0)$ over-determined, determining the stiffness, K_I .

The system (4.A.2)–(4.A.4) can be solved by standard numerical techniques; we use the method of finite differences after first eliminating $U(\rho, Z)$ by cross-differentiating to obtain the biharmonic formulation of the elastostatic equations. This gives a fourth-order PDE for $W(\rho, z)$, instead of a system of second-order PDEs for $W(\rho, z)$ and $U(\rho, Z)$. The

results presented in this chapter were obtained by applying a second-order central difference approximation for the derivatives on a uniform mesh of $[0, \rho_{\text{out}}] \times [-1, 0]$, where $\rho_{\text{out}} \gg 1$ is a given truncation of the radial domain. The resulting linear system was solved using `mldivide` (or `\`) in `MATLAB`. Numerical convergence was ensured by varying the density of the mesh as well as the truncation value ρ_{out} .

4.A.2 Example 2: Elastohydrodynamic sedimentation of a cylinder

In §4.4, we considered the equilibrium state of an infinite cylinder sliding and rotating above an inclined soft substrate within a viscous liquid. Using the scalings (4.7) and (4.51), we non-dimensionalize by letting

$$X := \frac{x}{\ell_1}, \quad U(X, Z) := \frac{2R^2 \tan \theta}{\ell_1^2 d} u(x, z), \quad \mathcal{H}(X) := \frac{2RH(x)}{\ell_1^2}, \quad (4.A.5a-c)$$

$$Z := \frac{z}{d}, \quad W(X, Z) := \frac{2R^2 \tan \theta}{\ell_1^3} w(x, z), \quad P(X) := \frac{\ell_1^2 p(x)}{R^3 \Delta \rho g \sin \theta}, \quad (4.A.5d-f)$$

where ℓ_1 is the horizontal length scale associated with an incompressible response, defined in (4.50b). The dimensionless foundation displacements, $U(X, Z)$ and $W(X, Z)$, then satisfy

$$W_{ZZ} = -\frac{\epsilon^2}{2(1-\nu)} [U_{XZ} + (1-2\nu)W_{XX}], \quad (4.A.6a)$$

$$U_{ZZ} + \frac{W_{XZ}}{1-2\nu} = -\frac{2(1-\nu)}{1-2\nu} \epsilon^2 U_{XX}, \quad (4.A.6b)$$

in $-1 \leq Z \leq 0$ and $-\infty < X < \infty$, subject to the boundary conditions

$$W, U \rightarrow 0 \quad \text{as } X \rightarrow \pm\infty, \quad (4.A.7a,b)$$

$$W = 0 \quad \text{and} \quad U = 0 \quad \text{on } Z = -1, \quad (4.A.7c,d)$$

$$W_Z + \frac{\nu \epsilon^2}{1-\nu} U_X = -\hat{\mathcal{C}} P(X) \quad \text{and} \quad U_Z + W_X = 0 \quad \text{on } Z = 0, \quad (4.A.7e,f)$$

where $\epsilon \equiv d/\ell_1$ is the aspect ratio, (4.67), and $P(X)$ satisfies Reynolds' equation, (4.52a) with $\mathcal{H}(X) = \hat{h} + X^2 - \hat{\phi} W(X, 0)$; this is to be solved subject to $P(X) \rightarrow 0$ as $X \rightarrow \pm\infty$ and the integral constraints (4.52d-f).

Overall, given values for the dimensionless parameters ν , $\hat{\mathcal{C}}$, and $\hat{\phi}$, (4.A.6) and (4.A.7) with (4.52a,d-f) describe a system for the substrate displacements, $U(X, Z)$ and $W(X, Z)$, and the fluid pressure, $P(X)$; the gap thickness, $\hat{h}$, translational speed, $\hat{U}$, and rotational speed, $\hat{\omega}$, are determined as part of the solution since the problem for the pressure field, P , is over-determined. This system can be solved using standard numerical techniques. The results presented in this chapter were obtained by applying a second-order central difference

approximation for the derivatives and trapezoidal rule for the integrals on a uniform mesh of $[-x_{\text{out}}, x_{\text{out}}] \times [-1, 0]$, where $x_{\text{out}} \gg 1$ is a given truncation of the lateral domain. The resulting nonlinear system was solved using `fsolve` in `MATLAB`. Numerical convergence was ensured by varying the density of the mesh, the truncation value x_{out} , and the termination tolerances associated with `fsolve`.

4.A.3 Example 3: Hele-Shaw flow bounded by a thin elastic layer

In §4.6, we considered a viscous fluid injected between a rigid wall and finite elastic layer which is deformed by the fluid pressure. Using the scalings, (4.97), we non-dimensionalize the full elastostatic problem using

$$X := \frac{x}{x_0}, \quad U(X, Z) := \frac{h_0^3 x_0 G}{12\beta(\nu)d^3\mu Q} u(x, z), \quad \mathcal{H}(X) := \frac{H(x)}{h_0}, \quad (4.A.8a-c)$$

$$Z := \frac{z}{d}, \quad W(X, Z) := \frac{h_0^3 G}{12\beta(\nu)d^2\mu Q} w(x, z), \quad P(X) := \frac{h_0^3}{12\mu Q x_0} p(x). \quad (4.A.8d-f)$$

The dimensionless substrate displacements, $U(X, Z)$ and $W(X, Z)$, satisfy the equations

$$W_{ZZ} = -\frac{\epsilon^2}{2(1-\nu)} [U_{XZ} + (1-2\nu)W_{XX}], \quad (4.A.9a)$$

$$U_{ZZ} + \frac{W_{XZ}}{1-2\nu} = -\frac{2(1-\nu)}{1-2\nu} \epsilon^2 U_{XX}, \quad (4.A.9b)$$

in $-1 \leq Z \leq 0$ and $0 \leq X \leq 1$, subject to the boundary conditions

$$W = 0 \quad \text{and} \quad U = 0 \quad \text{on} \quad X = 0, \quad (4.A.10a,b)$$

$$W = 0 \quad \text{and} \quad U = 0 \quad \text{on} \quad X = 1, \quad (4.A.10c,d)$$

$$W = 0 \quad \text{and} \quad U = 0 \quad \text{on} \quad Z = -1, \quad (4.A.10e,f)$$

$$W_Z + \frac{\nu\epsilon^2}{1-\nu} U_X = -\mathcal{C}_W P(X) \quad \text{and} \quad U_Z + W_X = 0 \quad \text{on} \quad Z = 0, \quad (4.A.10g,h)$$

where $\epsilon \equiv d/x_0$ is the aspect ratio of the substrate. The fluid pressure, $P(X)$, satisfies Reynold's equation, (4.98a) with gap thickness $\mathcal{H} = 1 - \Phi W(X, 0)$, which is further subject to boundary conditions (4.101). Here, $\mathcal{C}_W$ and Φ are dimensionless constants dependent on the foundation's aspect ratio, ϵ , and Poisson's ratio, ν , as described by (4.99a,b).

Overall, given the dimensionless parameters, Φ , ν , and ϵ , (4.A.9), (4.A.10), (4.101a), and (4.98) describe a system for the substrate displacements, $U(X, Z)$ and $W(X, Z)$, and the fluid pressure, $P(X)$. This system can be solved using standard numerical techniques; we used a second-order central difference approximation for the derivatives on a uniform mesh of $[0, 1] \times [-1, 0]$. The resulting nonlinear system was solved using `fsolve` in `MATLAB`.

and numerical convergence was ensured by varying the density of the mesh as well as the termination tolerances associated with `fsolve`.

Note that, for $\epsilon \ll 1$, we expect the boundary conditions (4.A.10a–d) to give rise to boundary layers close to $X = 0$ and $X = 1$. To obtain an accurate solution in this case, one should take a discretization that is densely populated inside the boundary layers, or even split the domain into inner and outer regions and solve each simultaneously (as we did in §4.6.3).

Appendix 4.B Governing systems for the inner displacements

In this appendix, we present the governing systems for the supplementary displacements used in §4.5.1 for a two-dimensional substrate and in §4.5.2 for a rotationally symmetric substrate.

4.B.1 Two-dimensional boundary layer

In §4.5.1, we considered the boundary layer associated with a two-dimensional, thin substrate with a zero-displacement bounding wall at $x = x_0$. We showed that the dimensionless inner displacements, $\tilde{W}(\tilde{X}, Z)$ and $\tilde{U}(\tilde{X}, Z)$, solve the governing equations (4.73) subject to (4.74). These equations were made tractable by assuming the applied surface pressure and shear stress, $P(X)$ and $\mathcal{T}(X)$, do not vary with the inner variable, $\tilde{X}$, and hence can be approximated as parabolas close to the bounding wall, (4.75). For such applied stresses, the inner displacement's dependence on $P(X)$ and $\mathcal{T}(X)$ may be ‘removed’ by writing the displacements in the form (4.77) with (4.79). Here, we write out the governing systems for the supplementary displacements, $\hat{W}_{P_i}$, $\hat{U}_{P_i}$, $\hat{W}_{T_i}$, and $\hat{U}_{T_i}$, used in (4.79).

The supplementary displacements, $\hat{W}_{P_i}$, $\hat{U}_{P_i}$, $\hat{W}_{T_i}$, and $\hat{U}_{T_i}$, originate from applying the series (4.79) to the system (4.73) subject to (4.78). Using the linearity of the system, we find that the displacement pairs, $(\hat{U}_{P_i}, \hat{W}_{P_i})$ and $(\hat{U}_{T_i}, \hat{W}_{T_i})$, satisfy the following equations:

$$(1 - 2\nu)\hat{W}_{\tilde{X}\tilde{X}} + 2(1 - \nu)\hat{W}_{ZZ} + \hat{U}_{\tilde{X}Z} = 0, \quad (4.B.1a)$$

$$(1 - 2\nu)\hat{U}_{ZZ} + 2(1 - \nu)\hat{U}_{\tilde{X}\tilde{X}} + \hat{W}_{\tilde{X}Z} = 0, \quad (4.B.1b)$$

in $-1 \leq Z \leq 0$ and $-\infty < \tilde{X} \leq 0$, subject to the boundary conditions

$$\frac{2(1-\nu)}{1-2\nu}\hat{W}_Z + \frac{2\nu}{1-2\nu}\hat{U}_{\tilde{X}} = 0 \quad \text{and} \quad \hat{U}_Z + \hat{W}_{\tilde{X}} = 0 \quad \text{on } Z = 0, \quad (4.B.2a,b)$$

$$\hat{W} = 0 \quad \text{and} \quad \hat{U} = 0 \quad \text{on } Z = -1, \quad (4.B.2c,d)$$

$$\hat{W} \rightarrow 0 \quad \text{and} \quad \hat{U} \rightarrow 0 \quad \text{as } \tilde{X} \rightarrow -\infty, \quad (4.B.2e,f)$$

with wall displacement boundary conditions on $\tilde{X} = 0$ that depend on the problem at hand. In particular, for

$$P_0 \text{ displacements:} \quad \hat{W} = 1 + Z \quad \text{and} \quad \hat{U} = 0, \quad (4.B.3a)$$

$$P_1 \text{ displacements:} \quad \hat{W} = 0 \quad \text{and} \quad \hat{U} = (1 + Z) \left(1 - \frac{Z}{4\nu - 1} \right), \quad (4.B.3b)$$

$$P_2 \text{ displacements:} \quad \hat{W} = (1 + Z) \left(1 + \frac{Z}{2} - \frac{1-\nu}{4\nu-1} Z^2 \right) \quad \text{and} \quad \hat{U} = 0, \quad (4.B.3c)$$

$$\mathcal{T}_0 \text{ displacements:} \quad \hat{W} = 0 \quad \text{and} \quad \hat{U} = 1 + Z, \quad (4.B.3d)$$

$$\mathcal{T}_1 \text{ displacements:} \quad \hat{W} = (1 + Z) \left(1 + \frac{Z}{4\nu - 1} \right) \quad \text{and} \quad \hat{U} = 0. \quad (4.B.3e)$$

A case of particular interest is that for incompressible materials ($\nu = 1/2$), in such a case the above system is supplemented with a Lagrange multiplier, equivalent to the elastic pressure, which enforces the incompressibility condition $\hat{U}_{\tilde{X}} + \hat{W}_Z = 0$ (Howell et al., 2008). We can, thus, introduce a stream function ψ , defined such that $\hat{U} = \psi_Z$ and $\hat{W} = -\psi_{\tilde{X}}$. Eliminating the Lagrange multiplier, the governing equations, (4.B.1), are then equivalent to solving the biharmonic equation for ψ .

Overall, given the Poisson's ratio, ν , the various supplementary displacements from (4.79) can be determined by solving (4.B.1) subject to (4.B.2) with the appropriate condition from (4.B.3). These systems must be solved numerically; we use the method of finite differences to form a linear system that is solved using `mldivide` (or `\`) in **MATLAB** — analogous to the methods described in Appendix 4.A. Solutions to (4.B.1)–(4.B.3) are plotted in Figs. 4.11 and 4.12.

4.B.2 Axisymmetric boundary layer

In §4.5.2, we considered the boundary layer associated with an axisymmetric, thin substrate with a zero-displacement bounding wall at $r = r_0$. We showed that the dimensionless displacements, $\tilde{W}(\tilde{R}, Z)$ and $\tilde{U}(\tilde{R}, Z)$, solve the governing equations (4.84) subject to (4.85). These equations are dependent on the effective wall curvature $\epsilon/R_0 \equiv d/r_0$ and Poisson's ratio, ν . To make analytical progress, we again assumed the applied surface pressure and

shear stress, $P(R)$ and $\mathcal{T}(R)$, do not vary much with the inner variable, $\tilde{R}$, and hence can be approximated as parabolas close to the bounding wall, (4.88). We were then able to ‘remove’ the inner displacement’s dependence on $P(R)$ and $\mathcal{T}(R)$ through the expansion (4.89). Here we present the governing systems for the supplementary displacements, $\hat{W}_{P_i}$, $\hat{U}_{P_i}$, $\hat{W}_{T_i}$, and $\hat{U}_{T_i}$, used in (4.89).

For the axisymmetric case, the supplementary displacements satisfy the local equilibrium equations

$$(1 - 2\nu)\tilde{W}_{\tilde{R}\tilde{R}} + 2(1 - \nu)\tilde{W}_{ZZ} + \tilde{U}_{\tilde{R}Z} = -\frac{\tilde{U}_Z + (1 - 2\nu)\tilde{W}_{\tilde{R}}}{R_0/\epsilon + \tilde{R}}, \quad (4.B.4a)$$

$$(1 - 2\nu)\tilde{U}_{ZZ} + 2(1 - \nu)\tilde{U}_{\tilde{R}\tilde{R}} + \tilde{W}_{\tilde{R}Z} = -\left(\frac{2(1 - \nu)\tilde{U}}{R_0/\epsilon + \tilde{R}}\right)_{\tilde{R}}, \quad (4.B.4b)$$

in $-1 \leq Z \leq 0$ and $-\infty < \tilde{R} \leq 0$, subject to the boundary conditions (4.B.2) and (4.B.3) after taking $\tilde{X} \mapsto \tilde{R}$.

In the case of an incompressible material ($\nu = 1/2$), the system is again supplemented with an incompressibility condition, $\hat{U}_{\tilde{R}} + \hat{U}/(R_0/\epsilon + \tilde{R}) + \hat{W}_Z = 0$. We can, thus, introduce a stream function ψ , defined such that $\hat{U} = \psi_Z$ and $\hat{W} = -\psi_{\tilde{R}} - \psi/(R_0/\epsilon + \tilde{R})$. The equations (4.B.4) are then equivalent to the biharmonic equation for ψ .

Unlike in the two dimensional case described in Appendix 4.B.1, here the governing system, (4.B.4), is dependent on the aspect ratio, ϵ , through the effective wall curvature $\epsilon/R_0 \equiv d/r_0$. For the case of small curvatures, $\epsilon/R_0 \ll 1$, (*i.e.* for a wall away from the origin) an expansion in powers of ϵ/R_0 may be made, (4.87). At leading-order, the displacements, $\hat{W}_0$ and $\hat{U}_0$, satisfy the two-dimensional system (4.B.1)–(4.B.3). At higher-orders, the governing equations account for the curvature of the boundary and are dependent on the lower-order solutions. For example, at order $\mathcal{O}(\epsilon/R_0)$, the displacements, $\hat{W}_1$ and $\hat{U}_1$, satisfy the system

$$(1 - 2\nu)\tilde{W}_{1,\tilde{R}\tilde{R}} + 2(1 - \nu)\tilde{W}_{1,ZZ} + \tilde{U}_{1,\tilde{R}Z} = -\tilde{U}_{0,Z} - (1 - 2\nu)\tilde{W}_{0,\tilde{R}}, \quad (4.B.5a)$$

$$(1 - 2\nu)\tilde{U}_{1,ZZ} + 2(1 - \nu)\tilde{U}_{1,\tilde{R}\tilde{R}} + \tilde{W}_{1,\tilde{R}Z} = -2(1 - \nu)\tilde{U}_{0,\tilde{R}}, \quad (4.B.5b)$$

where we use commas to separate the subscripts from the partial derivatives. Solutions to (4.B.5) are subject to the boundary conditions (4.B.2) with a zero-displacement condition

$$\hat{W}_1 = 0 \quad \text{and} \quad \hat{U}_1 = 0, \quad (4.B.6a,b)$$

on $\tilde{X} = 0$. Further terms in the asymptotic expansion may be formulated in a similar way. Three terms are sufficient for matching with the outer solution when $R_0 = \mathcal{O}(1)$, *i.e.* $\ell \sim r_0$.

Overall, given the effective wall curvature, ϵ/R_0 , and Poisson's ratio, ν , the various supplementary displacements from (4.89) can be determined by solving (4.B.4) subject to (4.B.2) with the corresponding condition from (4.B.3). These systems must be solved numerically; we use the method of finite differences to form a linear system which is solved using `mldivide` (or `\`) in `MATLAB` — analogous to the methods described in Appendix 4.A.

Macroscopic elastic properties of thin cellular structures

5.1 Introduction

Along with water being essential for basic plant functions (*e.g.* photosynthesis and nutrient transportation), water pressure (turgor) provides structural rigidity to non-woody structures, such as herbaceous plants and leaves, without which the plants would wilt and die. Typically, a dehydrated plant leaf is unable to support its own weight and so it sags under gravity, which could reduce its exposure to light; rehydrating the plant may restore the leaf provided it is not too desiccated.

Given the ubiquity of turgor as a means of supporting non-woody plants, it is important to understand what the mechanical effects of turgor pressure are: How does a structure's stiffness depend on turgidity? How much turgor is needed to prevent bending under gravity? Why do cellular structures sometimes shrink instead of expand with pressure (as observed by [Hofhuis et al., 2016](#), for example)? Does the curvature observed in dehydrated leaves predominantly minimize exposed tissue, or does it also make a significant difference to the rigidity of such leaves (see [Charron and Quatrano, 2009](#); [Hu et al., 2016](#), for examples in moss)?

In addition to biological tissue, cellular materials have recently been used in inflatable three-dimensional structures (see [Cadogan et al., 2004](#); [Siéfert et al., 2020](#), for example). For example, [Siéfert et al. \(2019, 2020\)](#) showed how to design a network of parallel channels that when inflated create an array of cylinders and induce a desired Gaussian curvature via the anisotropy of the pattern design (Fig. 5.1). A cross-section of this array (as seen in Fig. 5.1(b)) resembles a chain of cells connected by parallel seam lines, which is comparable to the single-cell thick structures found in some plant species, particularly mosses (see the photos in Fig. 5.2 from [Waite and Sack, 2010](#)). The effective bending modulus of this pressurized structure was found to be $B_{\text{eff}}^{\parallel} \propto Eh^3$, when bending parallel to the seams, and

$B_{\text{eff}}^{\perp} \propto Ehw^2$, when bending perpendicular to the seams, where w is the channel width, h is the thickness of the sheet, and E is its Young's modulus. These results are pressure independent, and hence do not explain the turgor dependence seen in nature as well as the clear observation that a turgid structure is more rigid than the flaccid one.

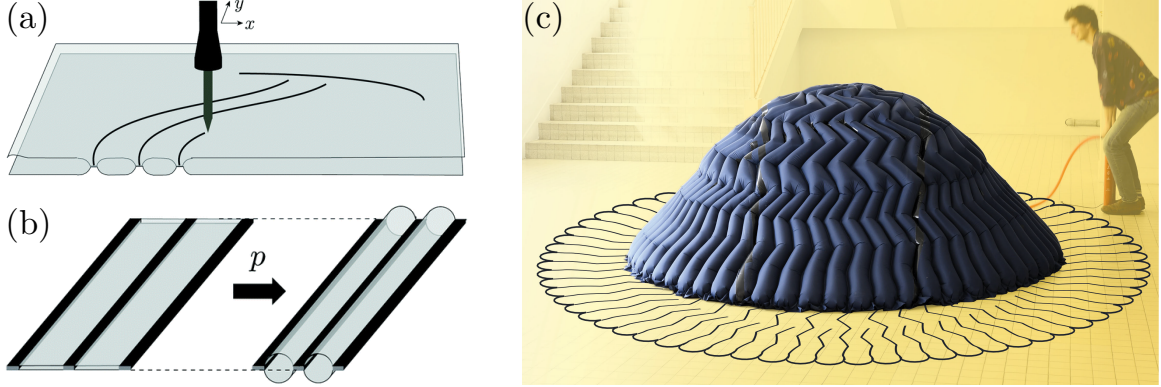


Figure 5.1: The design and inflation of programmable shells by Siéfert et al. (2019, 2020). In (a), two flat superimposed fabric sheets are heat-sealed along a desired path to create a pattern of parallel channels. In (b), the channels are inflated to some pressure, p , forming an array of inflated cylinders. The cross-section of this array is comparable to the single-cell thick structure found in some plant species (*e.g.* mosses, Fig 5.2). In (c), the sheets are sealed with a Miura-ori pattern that, when inflated, forms a 3 m-wide and 1.2 m-high paraboloid structure. (These images are taken from Siéfert et al., 2020.)

In this chapter, we construct a simple model of the single-cell thick structure found in many moss leaves (Waite and Sack, 2010) and used in inflatable shells (Cadogan et al., 2004; Siéfert et al., 2020). We shall use this model to study the mechanical consequences of varying the internal pressure, with the hope that it will help explain phenomena seen in more complex structures. Instead of the thin seam lines considered by Siéfert et al. (2020), we shall assume that neighbouring cells share a deformable wall, which is more in line with the structure seen in plant tissue (Fig. 5.2). Furthermore, in the absence of these seam lines, which act as soft hinges when bending parallel to the seams (Siéfert et al., 2020), the cells must deform against the internal pressure when bending; thus, we expect the resulting bending stiffness to be pressure-dependent. However, this comes with the difficult challenge of modelling the mechanics of the individual cell walls. We begin, in §5.2, by introducing the effective elastic moduli of the macroscale problem, discussing models found in the literature, and formally introducing the microscale structure considered in the rest of this chapter. In §5.3, we mathematically model the structure's response to an applied tension and moment using Kirchhoff's rod theory and assuming up-down cell symmetry; the implications for plant tissue and other applications are then discussed in §5.4. Following this, in §5.5, we consider the possible macroscale curvatures induced by cellular asymmetry; the consequences

of which are discussed in §5.6. We then conclude and summarize our results in §5.7.

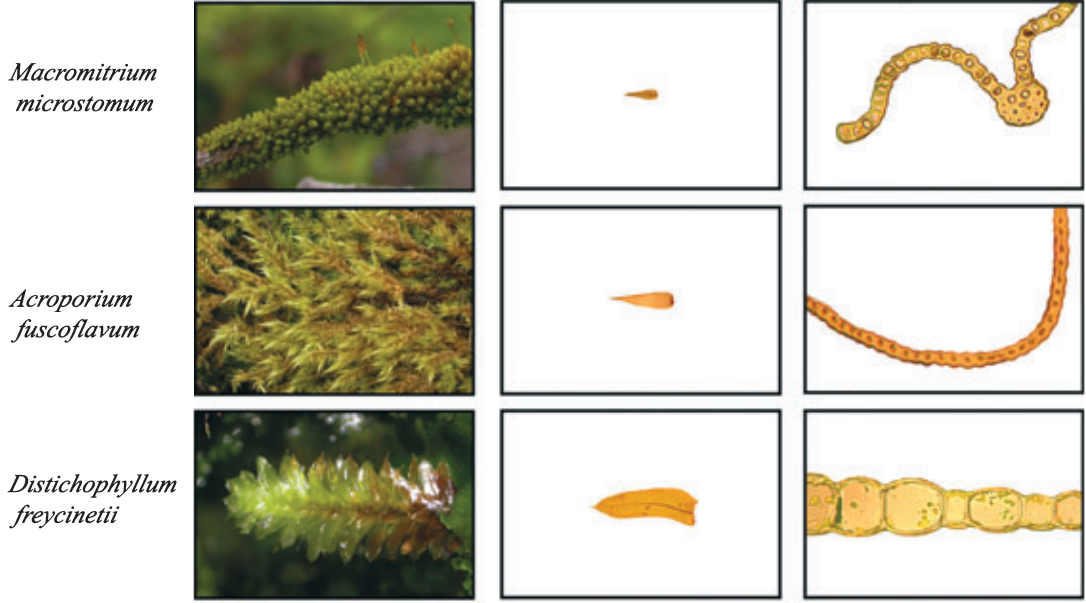


Figure 5.2: Photographs of three Hawaiian species of moss: the branch-dwelling *Macromitrium microstomum* (first row), the trunk-dwelling *Acroporium fuscoflavum* (second row), and the ground-dwelling *Distichophyllum freycinetii* (third row). The first, second, and third columns correspond to photographs of the canopy, leaf, and mid-leaf cross-sections, respectively. (This image is taken from [Waite and Sack, 2010](#).)

5.2 Previous models of the monolayer tissue

A thin, inhomogeneous material, such as the cellular structure found in plant tissue, may be identified with its homogeneous counterpart by introducing its effective stretching and bending moduli, Y_{eff} and B_{eff} , respectively. These moduli measure the material's response to an applied tension and moment, and are analogous to the homogeneous values: $Y \equiv E_{2D} = Eh$ and $B = EI$, where E is the Young's modulus, h is the thickness, and I is the second moment of inertia of the material. Accordingly, we define Y_{eff} to be the slope of the stress-strain relation — *i.e.* an in-plane tension, τ , stretches the material by an in-plane strain, ϵ , that satisfies Hooke's law,

$$\tau = Y_{\text{eff}} \epsilon. \quad (5.1)$$

Whereas, we define B_{eff} to be the slope of the moment-curvature relation — *i.e.* under a moment, μ , the curvature of the midline, κ , satisfies the expression

$$\mu = B_{\text{eff}} (\kappa - \kappa_0), \quad (5.2)$$

where κ_0 is the intrinsic curvature of the material (Howell et al., 2008; O’Reilly, 2017). At a macroscale, we can then approximate the inhomogeneous material as a homogeneous beam parametrized by these effective moduli and, in theory, not have to worry about the complex microscale structure — a rudimentary asymptotic homogenization.

It should be noted that one could alternatively define the elastic moduli as the coefficients of the quadratic terms in the energy associated with an in-plane strain, ϵ , and change in curvature, $\kappa - \kappa_0$, *i.e.*

$$U_{\text{beam}} = L \left[\frac{B_{\text{eff}}}{2} (\kappa - \kappa_0)^2 + \frac{Y_{\text{eff}}}{2} \epsilon^2 \right], \quad (5.3)$$

where L is the length of the beam (Howell et al., 2008). (For example, the effective bending stiffness of a chain of magnetic particles was calculated by Vella et al. (2014) using such an energetic formulation.) These two methods are equivalent at infinitesimal strains/curvatures (as shown in Appendix 5.A for the cellular model introduced in §5.2.2); however, they may deviate at finite strains and curvatures, where the inhomogeneity becomes more apparent. In general, we are most interested in calculating the infinitesimal values; thus, without loss of generality, we concentrate on the force/torque definitions of Y_{eff} and B_{eff} , (5.1) and (5.2), throughout this chapter.

The aim of this work is to understand the rigidity of thin plant tissue (*e.g.* monolayer leaves) and other cellular materials for a varying level of hydration/pressure. Accordingly, we want to determine the effective elastic moduli, Y_{eff} and B_{eff} , of a simple representation of the complex pressurized structure found in *e.g.* non-woody plants. We begin by showing what goes wrong if the tissue model is oversimplified, by considering a two-dimensional inflated channel. We then discuss more complex existing models in §5.2.2, and introduce the configuration considered in the rest of this chapter.

5.2.1 A simplistic tissue model

Perhaps the simplest way to incorporate the pressurized space found inside thin cellular materials is to model the tissue as a pressurized channel, resembling *e.g.* a bicycle inner-tube (Comer and Levy, 1963). Here, the two-dimensional strip consists of two beams (of thickness h and Young’s modulus E) held parallel to each other at a distance t ; these beams correspond to the exterior walls of a leaf, for example. To ensure the two beams are held at a distance t , requires either some lateral supporting struts (analogous to the interior cell walls found in plant tissue, Karttunen and Reddy, 2020) or some confinement in the third dimension (a bicycle inner tube has an annular cross-section, for example). Pressurizing the interior to a pressure p increases the interior volume and stretches the two walls, inducing a tension, T_p . The resulting structure is akin to ‘sandwich beams’, which have been used to model plant leaves as a lightweight elastic core with a stiff elastic coating (for example, Gibson

et al., 1988 used a two-dimensional sandwich beam to model the bending of iris leaves, whilst Niklas, 1991 considered the axisymmetric case to model *Juncus effusus* leaves). Here, we replace the elastic core by a pressurized interior to better mimic the effect of turgor pressure. We now calculate the effective stretching and bending moduli, Y_{eff} and B_{eff} , of this configuration by imposing an additional tension, τ , and moment, μ , respectively. We assume that the spatial confinement (*e.g.* the supporting struts or curvature in the third dimension) does not affect the elastic response of the channel, thus can be neglected here. The considered configuration is sketched in Fig. 5.3.

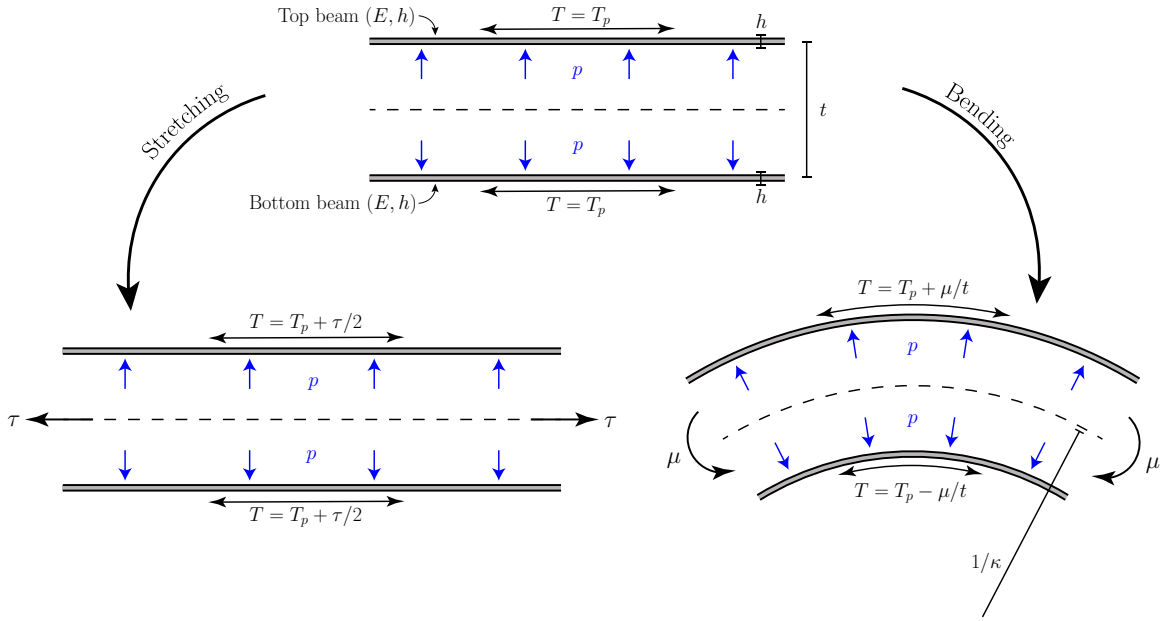


Figure 5.3: Sketch of the stretching and bending of the pressurized two-dimensional channel. In the top sketch, an interior pressure, p , induces a tension, T_p , in the two bounding beams. In the left and right sketches, the pressurized beam is deformed by an imposed tension, τ , and moment, μ , respectively, which induce an additional strain in the two beams. Relating the additional tensions with the midline strains, using (5.1) and (5.2), results in an effective stretching modulus $Y_{\text{eff}} = 2Eh$ and bending modulus $B_{\text{eff}} = Eht^2/2$.

An applied in-plane tension, τ , is split evenly between the top and bottom beams, providing an additional tension, $\tau/2$, in both (see the left sketch in Fig. 5.3). Assuming the resulting tension in the beams, $T = T_p + \tau/2$, is small enough that they deform according to linear elasticity — *i.e.* Hooke’s law, (5.1) — the additional strain, ϵ , is given by

$$\frac{\tau}{2} = Y\epsilon, \quad (5.4)$$

where $Y \equiv Eh$ is the stretching modulus of the beams. Comparing (5.4) with (5.1), we

deduce the effective stretching modulus

$$Y_{\text{eff}} = 2Y \equiv 2Eh. \quad (5.5)$$

Bending the strip downwards to form a midline curvature, κ , induces a differential strain which stretches the top beam and compresses the bottom beam by a strain $\epsilon \equiv \pm t\kappa/2$, respectively. Again assuming Hooke's law (*i.e.* the pressure- and curvature-induced strains are sufficiently small), the top and bottom beams are at a tension $T = T_p \pm Yt\kappa/2$, respectively (see the right sketch in Fig. 5.3). The moment (about the midline) required to impose such tensions is

$$\mu = \frac{t}{2} \left(T_p + Y \frac{t}{2} \kappa \right) - \frac{t}{2} \left(T_p - Y \frac{t}{2} \kappa \right) = Y \frac{t^2}{2} \kappa. \quad (5.6)$$

Comparing this with (5.2) for $\kappa_0 = 0$ (the configuration is intrinsically flat), gives an effective bending modulus

$$B_{\text{eff}} = Y \frac{t^2}{2} \equiv Eh \frac{t^2}{2}. \quad (5.7)$$

(Note that we could have obtained this effective bending modulus by using the homogeneity of the strip in the longitudinal direction, and so $B = EI$ where I is the second moment of inertia given by

$$I = \int_{t/2-h/2}^{t/2+h/2} z^2 dz + \int_{-t/2-h/2}^{-t/2+h/2} z^2 dz = \frac{t^2 h}{2} + \frac{h^3}{6} \sim \frac{t^2 h}{2}, \quad (5.8)$$

for $h^2 \ll t^2$, [Speck et al., 1990](#). The extra term here, $I \propto h^3$, can be identified as the bending stiffness of the two individual beams, $B = Eh^3/12$, which did not appear in (5.7) since we ignored the differential strain across the individual beams. However, for more complex models, where a linear differential strain cannot be assumed — such as the cellular model presented in §5.2.2 — this second moment of inertia argument cannot be used, and the moment–curvature relation must be considered in detail.)

Overall, we have found that the effective stretching modulus, Y_{eff} , and bending modulus, B_{eff} , of this simplistic model are independent of the internal pressure, and only depend on the wall stretching modulus, $Y = Eh$, and the configuration thickness, t . These moduli do not help to explain the increase in rigidity of plant tissue with the increase in turgor pressure and are smaller than the values needed to support an extended leaf. This problem stems from the symmetry and homogeneity of the model: here, the pressure acts perpendicular to the midline and so an in-plane or differential strain, associated with stretching or bending respectively, will not depend on the pressure; furthermore, the up–down symmetry means the second moment of inertia, I , and hence bending stiffness, $B = EI$, are small.

[Haseganu and Steigmann \(1994\)](#) showed that wrinkling on the underside of a pressurized

cylindrical membrane (analogous to the bottom beam) occurs when the stress along the wall vanishes, creating an asymmetry in the configuration and, hence, inducing a pressure-dependent stiffness. For example, taking the membrane limit, $h \rightarrow 0$, in the two-dimensional structure presented here, wrinkling occurs in the lower wall when $T = T_p - Yt\kappa/2 \leq 0$ since membranes cannot support negative tensions. In this case, the lower wall tension is instead set to zero and the required moment, (5.6), takes the alternative form

$$\mu = \frac{t}{2} \left(T_p + Y \frac{t}{2} \kappa \right). \quad (5.9)$$

This moment is pressure-dependent through the T_p -term; however, since $T_p \leq Yt\kappa/2$, it is smaller than the non-wrinkling solution, (5.6). Furthermore, in the resulting effective bending modulus, $B_{\text{eff}} \equiv \mu/\kappa = T_p t/(2\kappa) + Yt^2/4$, the pressure term vanishes at large curvatures, and yields a modulus which is smaller than the non-wrinkling case, (5.7) — *i.e.* $B_{\text{eff}} \sim Yt^2/4$ as $\kappa \rightarrow \infty$. Additionally, such wrinkles have not been reported in plant tissue. To improve upon this first model, then, we need to consider a more realistic configuration which incorporates the cellular structure.

5.2.2 More realistic tissue models

Modelling the cellular structure found in *e.g.* plant tissue is a difficult process, and has previously been approached in a number of different ways, including

Composite materials Here, the cellular structure is separated into regions of known elastic moduli (from either experimental measurements or from first principle calculations), Fig 5.4(a). These regions correspond to the various anatomical components of the tissue — for example, the mesophyll, epidermis, and midrib found in leaves are known to have varying elastic properties and turgidity (Zimmermann et al., 1980; Frensch and Schulze, 1988; Nonami and Schulze, 1989; Onoda et al., 2015). Typically, an upper or lower bound on the rule of mixtures (*e.g.* Voigt and Reuss models, as discussed by Niklas, 1992) is then used to combine the elastic parameters of these regions to provide effective moduli (see Vincent, 1982, 1989; Niklas, 1992; Faisal et al., 2013, for example).

Honeycomb/foam-like structure Here, a lattice structure consisting of an array of elastic beams/shells with a fluidic, elastic, or pressurized interior are used to model the cellular structure, Fig 5.4(b). The resulting effective Young’s modulus is found to be dependent on the ratio of the density of the cell structure to that of the cell walls (Warner et al., 2000; Georget et al., 2003). Such a model is effective for modelling bulk material, where a homogenization of the cellular structure is appropriate (see Kimmel,

1992; Warner et al., 2000; Georget et al., 2003; Ongaro et al., 2016, for example); but does not, typically, take the mechanics of the cell walls into consideration.

Thin shell models Here, individual cell walls are modelled as thin membranes or beams (for three- or two-dimensional models, respectively) with an elastic, fluidic, or pressurized interior, Fig 5.4(c). In contrast to foam-like structures, these models consider the mechanics of the cell walls (see Falk et al., 1958; Nilsson et al., 1958; Hofhuis et al., 2016; Myhan et al., 2020, for example). However, this often results in a formulation that is too detailed, *e.g.* using finite element methods (Hofhuis et al., 2016), or an *a priori* assumption about the shape of the deformed shell (Falk et al., 1958; Nilsson et al., 1958).

Thick shell models For certain plant species the cell walls are far from thin, hence thin shell theory cannot be utilized, Fig 5.4(d). Instead, the walls and interiors must be modelled individually. Such a case is often approached as a composite material (Niklas, 1989).

Each of these methods has its advantages and disadvantages; however, typically, they are unable to model thin (a few cells thick) structures, where the cell mechanics is important, or, if they do, are so detailed that they can only be approached via numerical simulations.

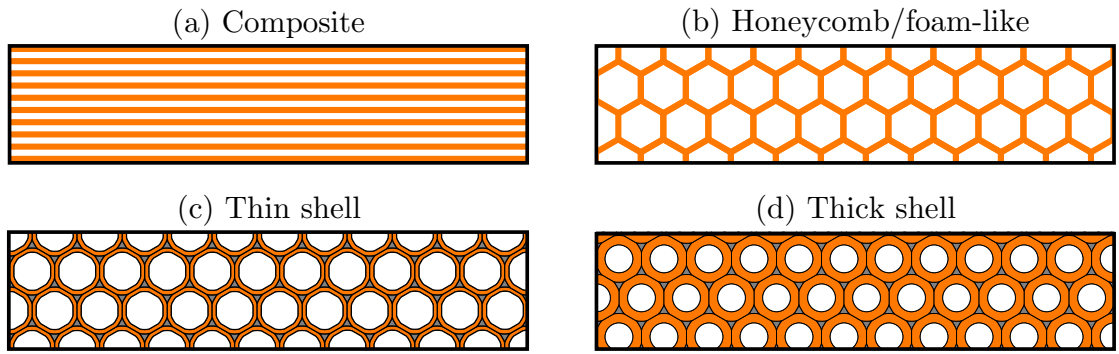


Figure 5.4: Diagrams of four tissue models used to model the deformation of cellular structures: (a) composite material (the structure is separated into regions of known material parameters), (b) honeycomb/foam-like structure (the structure is modelled as an array of elastic beams/shells forming a honeycomb-like pattern), (c) thin-shell model (the cells are modelled using thin shell or beam theory), (d) thick-shell model (thin shell theory cannot be utilized and, instead, the walls and interior must be modelled individually).

Our niche In this chapter, we study the deformation of structures that are a single cell thick, accounting for wall deformation; such structures can be found in certain species of moss (see Fig, 5.2 and Waite and Sack, 2010, for example). We use a model based on Kirchhoff's

rod theory (O'Reilly, 2017) to model a chain of two-dimensional cells each consisting of four intersecting beams: two side (in-contact) walls and two lateral (out-of-contact) walls. The cells' interior is pressure-controlled with positive pressure stretching the side walls and deforming the lateral walls. This configuration is sketched in Fig. 5.5. Note that the pressure-controlled system is relevant for the long timescale problems (where the turgor is effectively constant throughout the tissue) that we are interested in, for example the gravitational deflection of a leaf; for problems with a shorter timescale (where the turgor does not have sufficient time to adjust to a volume change), a system controlled by the interior volume or Boyle's Law may be more appropriate. We aim to understand the pressure-dependent properties of this tissue model (such as the macroscale rigidity and the pressure-induced curvature), in the hope that it will help explain phenomena seen in more complex structures — *e.g.* the quick recovery of desiccated sunflower leaves (Fellows and Boyer, 1978).

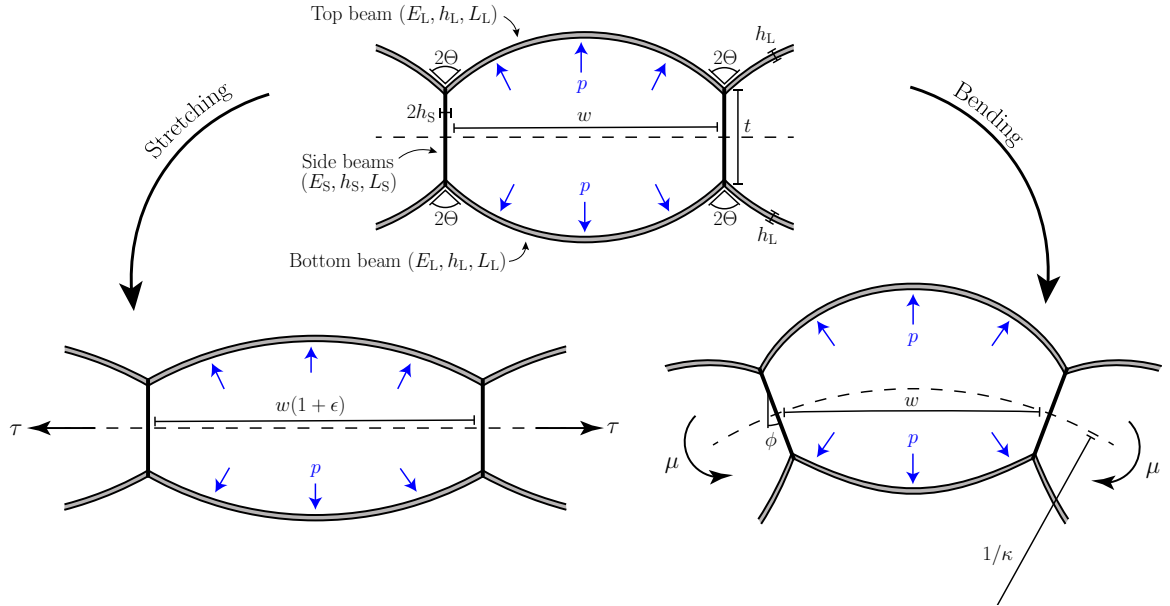


Figure 5.5: Sketch of the two-dimensional, pressurized cell model considered in this chapter. Each ‘cell’ consists of four distinct beams, corresponding to the top, bottom, and two side (or in-contact) walls. In the top sketch, the internal pressure, p , stretches the side beams and deforms the top and bottom beams. Imposing an additional tension, τ , (left sketch) or moment, μ , (right sketch) stretches or bends the cell, respectively. This deformation is modelled by the Kirchhoff equations in §5.3.1, and the associated effective stretching and bending moduli are computed in §5.3.3 and §5.3.4, respectively.

Compared to the oversimplified model presented in §5.2.1, in this model the cell walls are able to bend out-of-plane when the structure is stretched and bent, so that the symmetry and homogeneity are broken and larger, pressure-dependent, effective moduli are expected. However, note that, here, bending is not expected to produce a linear differential strain over the cross-section (since the lateral walls can individually bend out-of-plane in response to an

applied moment), and so the second moment of inertia is irrelevant for the calculation of an effective bending stiffness. Instead, the four walls must be modelled as elastic beams that are able to individually deform.

5.3 Symmetric cellular structure

In this section, we formulate the problem of the cellular structure subject to a general imposed tension, τ , and bending moment, μ , with the assumption that the cells are intrinsically up–down and left–right symmetric.

5.3.1 Basic model setup

We model a two-dimensional cell using four distinct beams: the two side walls (of thickness h_S , Young’s modulus E_S , relaxed length L_S , and deformed length t) are held flat by the adjacent cells at an angle ϕ to the vertical and at an average distance w away from each other (see Fig. 5.5, lower right). The two lateral walls (of thickness h_L , Young’s modulus E_L , and relaxed length L_L)¹ are clamped to the ends of the two side walls at an exterior angle Θ . Together these beams form a trapezoidal shape (Fig. 5.5), which replicate the structure seen in *e.g.* moss leaves, Fig. 5.2. The cell interior is pressurized to a constant pressure, p , and subject to a horizontal tension, τ , and moment, μ , which stretch and rotate the side walls and deform the lateral walls. (Note that τ and μ correspond to the macroscopic applied tension and bending moment, respectively; at a cellular scale, τ and μ are imposed at infinity and, thus, correspond to the total tension and torque acting on a cell by its neighbour.) The horizontal cell width, w ; side wall length, t ; and angle, ϕ , are set by requiring that the cell be at equilibrium. We expect $\phi \equiv 0$ for zero applied moment, $\mu = 0$, due to the up–down symmetry; we assume right–left symmetry throughout. The first step in understanding the cell’s equilibrium state is to consider how the walls interact, and so we do this now.

Equilibrium of walls For the left (and correspondingly right) side wall to be at equilibrium, the end forces which originate from the top and bottom beams, (T_T, N_T) and (T_B, N_B) , respectively, must balance the force from the interior pressure acting over the walls length, $-pt(\cos \phi, \sin \phi)$, and the tension imposed by a neighbouring cell, τ , that is

$$T_T + T_B = \tau + pt \cos \phi \quad \text{and} \quad N_T + N_B = pt \sin \phi, \quad (5.10a,b)$$

¹We assume that the walls are isotropic beams, so that their bending and stretching stiffnesses can be written as $B = Eh^3/12$ and $Y = Eh$, respectively. This assumption may not be true in general, and it may, instead, be preferable to model the beams according to their own effective moduli, B and Y . We assume the isotropic beam forms for clarity only.

where the side walls length, t , and angle, ϕ , are still to be determined. (Note that τ is assumed horizontal since the symmetric cell cannot support a vertical force applied to the side walls.) Likewise, the moment of the end forces around the side wall centre must balance the moment imposed by a neighbouring cell, μ , that is

$$(T_T \cos \phi + N_T \sin \phi) \frac{t}{2} - (T_B \cos \phi + N_B \sin \phi) \frac{t}{2} = \mu. \quad (5.10c)$$

These three equations give conditions for determining the end forces, T_T , T_B , N_T , and N_B , as sketched in Fig. 5.6(c).

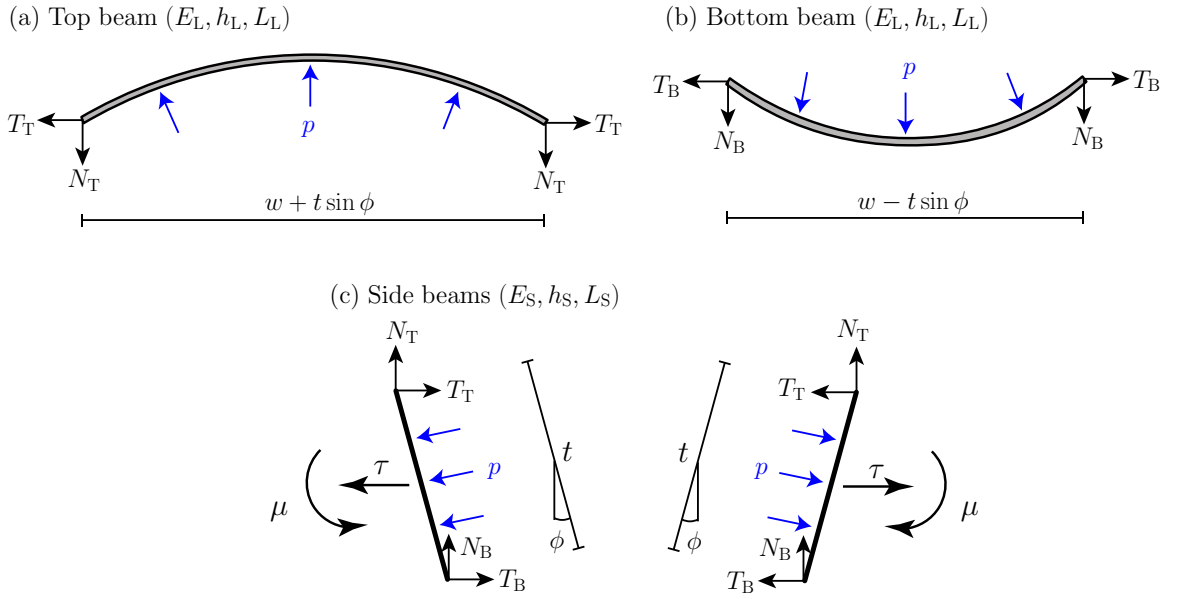


Figure 5.6: A force diagram of the four distinct beams which make up the cell sketched in Fig. 5.5: (a) top beam, (b) bottom beam, and (c) the two side beams. For the individual beams to be at equilibrium, the forces and moments must be in balance, this gives the expressions for the end forces (5.11) and (5.12).

The final condition comes from applying a vertical force balance on the lateral beams: making use of the left–right symmetry of the cell, the vertical end forces must balance the pressure acting over the projected length, $w + t \sin \phi$ and $w - t \sin \phi$ for the top and bottom beams, respectively — as sketched in Fig. 5.6(a,b) — that is

$$N_T = \frac{p}{2} (w + t \sin \phi) \quad \text{and} \quad N_B = -\frac{p}{2} (w - t \sin \phi), \quad (5.11a,b)$$

where the cell width, w , is also to be determined. Inserting (5.11) into (5.10), gives the

horizontal end forces

$$T_T = \frac{\tau}{2} + \frac{\mu}{t \cos \phi} - \frac{p}{2} (w \tan \phi - t \cos \phi), \quad (5.12a)$$

$$T_B = \frac{\tau}{2} - \frac{\mu}{t \cos \phi} + \frac{p}{2} (w \tan \phi + t \cos \phi). \quad (5.12b)$$

With the expressions for the end forces, (5.11) and (5.12), we are now able to consider the deformation of each beam individually.

5.3.1.1 Deformation of the side walls

By construction, the side walls remain flat for all interior pressures, p , and only deform longitudinally due to an in-plane tension, $T(S)$, which incorporates the end forces and applied tension:

$$T(S) := -N_B \cos \phi + T_B \sin \phi - \tau \sin \phi \frac{S}{L_S} \equiv \frac{pw}{2 \cos \phi} - \frac{\mu \tan \phi}{t} + \tau \sin \phi \left(\frac{1}{2} - \frac{S}{L_S} \right), \quad (5.13)$$

where $0 \leq S \leq L_S$ is the arclength of the undeformed side wall. Here, we have assumed that the tension imposed by a neighbouring cell, τ , is uniformly distributed over the reference arclength, $0 \leq S \leq L_S$. We could have alternatively assumed that τ was applied as a point force at the beam centre by taking $\tau \sin \phi S / L_S \mapsto \tau \sin \phi H(S - L_S/2)$ in (5.13), where $H(S)$ is the Heaviside step function; this ultimately gives the same result.

Assuming the side walls deform according to Hooke's law (*i.e.* according to linear elasticity), the in-plane tension can be written as

$$T(S) = Y_S (\alpha(S) - 1), \quad (5.14)$$

where $Y_S := E_S h_S$ is the two-dimensional Young's modulus of the side walls and α is the axial stretch,

$$\alpha(S) := \frac{ds}{dS}, \quad (5.15)$$

which measures the deformation of the reference (stress-free) configuration, with arclength S , to the deformed configuration, with arclength s . Note that the introduction of a reference and deformed configuration is required for Kirchhoff's rod theory, which shall be used to model the deformation of the lateral walls in §5.3.1.2.

Integrating the axial stretch, (5.15), with (5.13) and (5.14) over the reference length, $0 \leq S \leq L_S$, results in a quadratic equation for the deformed length, t :

$$\frac{t}{L_S} = 1 + \frac{1}{Y_S} \left(\frac{pw}{2 \cos \phi} - \frac{\mu \tan \phi}{t} \right). \quad (5.16)$$

To determine the side wall angle, ϕ , and cell width, w , requires us to consider the deformation of the lateral walls, which we do now.

5.3.1.2 Deformation of the lateral walls

Since we have a particular interest in turgid cells and, hence, large interior pressures (in a sense to be defined later), we cannot, in general, use a small slope approximation for the lateral walls. Since such a small angle assumption is implicit in many popular linear elastic beam models — *e.g.* Euler–Bernoulli beam theory and the Föppl–von Kármán equations — we instead choose to model beam deflections using the Kirchhoff beam equations (Howell et al., 2008; O'Reilly, 2017), which account for large (nonlinear) slopes but still assumes infinitesimal midline strains (linear elasticity).

For a Kirchhoff rod, one must distinguish between the reference (stress-free) configuration, with arclength $0 \leq S \leq L_L$, and the deformed configuration, with arclength s . To simplify the following formulation, we work only in the reference coordinate, S , and present the formulation for the top and bottom beams simultaneously. A similar formulation was presented by Pandey et al. (2014) for the dynamics of the elastic snap-through of arches.

The deformation of the beam is measured by the axial stretch, $\alpha(S)$, which is defined in (5.15), and the beam position, $(x(S), z(S))$, which is given by the geometric conditions

$$\frac{dx}{dS} = \alpha \cos \theta \quad \text{and} \quad \frac{dz}{dS} = \alpha \sin \theta, \quad (5.17a,b)$$

where $\theta(S)$ is the angle of the midline of the beam with respect to the horizontal.

The tangential and perpendicular resultant forces, $T(S)$ and $N(S)$, and resultant moment around the y -axis, $M(S)$, satisfy a force and moment balance,

$$\frac{1}{\alpha} \frac{d}{dS} \left[T(S) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} - N(S) \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} \right] = \pm p \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}, \quad (5.18a)$$

$$\frac{1}{\alpha(S)} \frac{dM}{dS} = -N(S), \quad (5.18b)$$

where $\pm p$ refers to the pressure being applied from above or below, *i.e.* for the top or bottom beam, respectively. Integrating (5.18a) using (5.17) and setting $x(0) = z(0) = 0$ yields the resultant forces

$$T(S) = T_0 \cos \theta + N_0 \sin \theta \pm p(z \cos \theta - x \sin \theta), \quad (5.19a)$$

$$-N(S) = T_0 \sin \theta - N_0 \cos \theta \pm p(z \sin \theta + x \cos \theta), \quad (5.19b)$$

where (T_0, N_0) is the imposed end force, which equals (T_T, N_T) or (T_B, N_B) — from (5.11)

and (5.12) — for the top or bottom beam, respectively. Integrating (5.19b) using (5.17) and (5.18b) yields the resultant moment

$$M(S) = M_0 + T_0 z - N_0 x \pm p \frac{x^2 + z^2}{2}, \quad (5.20)$$

where $M_0 := M(0)$ is the unknown moment applied to the end.

Overall, (5.19) and (5.20) describe the resultant forces acting on the lateral walls. To close this system, we introduce two constitutive laws: Hooke's law (linear elasticity),

$$T(S) = Y_L(\alpha(S) - 1), \quad (5.21)$$

and a quadratic strain energy (the moment is linearly related to the curvature),

$$M(S) = B_L \left(\frac{d\theta}{dS} - \kappa_0 \right), \quad (5.22)$$

where κ_0 is the intrinsic beam curvature, and $B_L \equiv E_L h_L^3/12$ and $Y_L \equiv E_L h_L$ are the bending and stretching moduli of the lateral beams, respectively. (Note that these constitutive laws are equivalent to assuming a strain energy density function $\psi = B_L(\theta' - \kappa_0)^2/2 + Y_L(\alpha - 1)^2/2$. One could assume alternative strain energy density functions, here, to model *e.g.* nonlinear elastic effects, O'Reilly, 2017.)

Combining (5.20) and (5.22) leaves a first-order differential system for the beam angle, $\theta(S)$, and position, $(x(S), z(S))$:

$$B_L \left(\frac{d\theta}{dS} - \kappa_0 \right) = M_0 + T_0 z - N_0 x \pm p \frac{x^2 + z^2}{2}, \quad (5.23)$$

with (5.17) in $0 \leq S \leq L_L$, where the axial stretch, $\alpha(S)$, is given by

$$Y_L(\alpha - 1) = T_0 \cos \theta + N_0 \sin \theta \pm p(z \cos \theta - x \sin \theta), \quad (5.24)$$

and the edge forces, (T_0, N_0) , are given by (5.11) and (5.12).

5.3.1.3 Boundary conditions

It is not obvious what mechanism should be imposed at the side-lateral wall interfaces to correctly model the complex structure found in *e.g.* plant cells. Based on photographs of moss leaves (see Fig 5.2 and Waite and Sack, 2010), it appears that the exterior angle between the two walls is typically in the range $0 \leq \Theta \leq \pi/4$ (depending on the species), whilst in inflatable structures (Siéfert et al., 2020) and detached exocarp cells during explosive seed

dispersal (Hofhuis et al., 2016), an exterior angle of $\Theta = \pi/2$ is more appropriate. Thus, for simplicity and versatility, we shall concentrate on clamped boundary conditions with an imposed exterior angle $0 \leq \Theta \leq \pi/2$. Note that Θ is independent of any natural curvature the walls may have, *i.e.* κ_0 .

At $S = 0$, we impose clamped boundary conditions,

$$x(0) = z(0) = 0, \quad \theta(0) = \pm \left(\frac{\pi}{2} - \Theta \right) + \phi, \quad (5.25a-c)$$

while at $S = L_L/2$, we impose

$$x(L_L/2) = \frac{w}{2} \pm \frac{t}{2} \sin \phi \quad \text{and} \quad \theta(L_L/2) = 0, \quad (5.25d,e)$$

where the $\pm$ refers to the top and bottom beams, respectively. Here, (5.25a,b,d) are conditions based on the geometry of the cell (see Fig. 5.5), (5.25c) is a clamping condition, and (5.25e) comes from the symmetry of the cell.

Note that the extra boundary condition for θ , (5.25e), determines the unknown end moment, $M_0 \equiv B_L[\theta'(0) - \kappa_0]$; whilst those for x , (5.25d) for the top and bottom beam, determine the unknown cell width, w , and side wall angle, ϕ . Thus, solving (5.17), (5.23), and (5.24) subject to (5.25) for both the top and bottom beams determines w , ϕ , and, hence, t by (5.16).

5.3.1.4 Non-dimensionalization

To non-dimensionalize the problem, we take the relaxed length of the lateral beams, L_L , as the reference length scale. We, thus, define the dimensionless parameters

$$\xi := \frac{S}{L_L}, \quad X(\xi) := \frac{x(S)}{L_L}, \quad Z(\xi) := \frac{z(S)}{L_L}, \quad (5.26a-c)$$

and consider the axial stretch and beam angle, α and θ , as functions of the dimensionless arclength, ξ , instead of its dimensional counterpart, S .

Inserting (5.26) into (5.17), (5.23), and (5.24) yields a system of first-order ODEs for the lateral beam slope, $\theta(\xi)$, and dimensionless position of a point on the beam, $(X(\xi), Z(\xi))$, which read

$$\frac{dX}{d\xi} = \alpha \cos \theta, \quad \frac{dZ}{d\xi} = \alpha \sin \theta, \quad (5.27a,b)$$

$$\hat{\beta} \frac{d\theta}{d\xi} = \hat{M}_0 + \hat{T}_0 Z - \hat{N}_0 X \pm \hat{p} \frac{X^2 + Z^2}{2}, \quad (5.27c)$$

in $0 \leq \xi \leq 1$, with the axial stretch

$$\alpha(\xi) = 1 + \hat{T}_0 \cos \theta + \hat{N}_0 \sin \theta \pm \hat{p} (Z \cos \theta - X \sin \theta), \quad (5.28)$$

end forces

$$\hat{T}_0 := \frac{T_0}{Y_L} = \frac{\hat{p}}{2} \left(\hat{L} \hat{t} \cos \phi \mp \hat{w} \tan \phi \right) + \frac{\hat{\tau}}{2} \pm \frac{\hat{\mu}}{\hat{t} \cos \phi}, \quad (5.29a)$$

$$\hat{N}_0 := \frac{N_0}{Y_L} = \frac{\hat{p}}{2} \left(\hat{L} \hat{t} \sin \phi \pm \hat{w} \right), \quad (5.29b)$$

and side wall stretch, $\hat{t} := t/L_S$, solves

$$\hat{t} = 1 + \frac{1}{\hat{Y}} \left(\frac{\hat{p} \hat{w}}{2 \cos \phi} - \frac{\hat{\mu} \tan \phi}{\hat{t}} \right), \quad (5.30)$$

where the $\pm$ refers to the top and bottom walls, respectively.

Solutions to (5.27) are subject to

$$X(0) = Z(0) = \theta(1/2) = 0, \quad (5.31a-c)$$

$$\theta(0) = \pm \left(\frac{\pi}{2} - \Theta \right) + \phi, \quad X(1/2) = \frac{\hat{w}}{2} \pm \frac{\hat{L} \hat{t}}{2} \sin \phi, \quad (5.31d,e)$$

where the extra boundary conditions for θ and X determine the unknown side wall angle, ϕ , and end moments and cell width,

$$\hat{M}_0 := \hat{\beta} \frac{d\theta}{d\xi}(0) \quad \text{and} \quad \hat{w} := \frac{w}{L_L}, \quad (5.32a,b)$$

respectively. Note that the top and bottom beams must be solved simultaneously to determine the cell width, $\hat{w}$, and side wall angle, ϕ .

Here

$$\hat{p} := \frac{p L_L}{Y_L}, \quad \hat{\tau} := \frac{\tau}{Y_L}, \quad \hat{\mu} := \frac{\mu}{Y_L L_S}, \quad (5.33a-c)$$

$$\hat{Y} := \frac{Y_S}{Y_L} \equiv \frac{E_S h_S}{E_L h_L}, \quad \hat{L} := \frac{L_S}{L_L}, \quad \hat{\beta} := \frac{B_L}{Y_L L_L^2} \equiv \frac{h_L^2}{12 L_L^2}. \quad (5.33d-f)$$

These dimensionless parameters can be interpreted as follows: $\hat{p}$, $\hat{\tau}$, and $\hat{\mu}$ are measures of the internal pressure, applied tension, and applied moment, respectively, compared to the stretching of the lateral walls; $\hat{\beta} \ll 1$ compares the lateral beams' bending and stretching moduli, *i.e.* $B_L \equiv E_L h_L^3/12$ and $Y_L \equiv E_L h_L$, and is a Föppl–von Kármán number; $\hat{Y}$ and $\hat{L}$ are dimensionless stretching modulus and relaxed length ratios, respectively, comparing the value for the side and lateral walls.

A common simplification in cellular models is that the shell is homogeneous, *i.e.* all the walls have the same elastic parameters and thicknesses. In our notation, this is equivalent to imposing $\hat{Y} \equiv 1$. However, in single-cell thick moss leaves, the ‘interior wall thickness’ and ‘surface wall thickness’, or equivalently h_S and h_L , are often quite different so that $0.3 \lesssim h_S/h_L \lesssim 1.2$ depending on the species (Waite and Sack, 2010). That said, the Young’s modulus of the cell walls are difficult to measure in practice, with values ranging from ~ 100 MPa to ~ 500 MPa (Cosgrove, 2016), so the ratio E_S/E_L is not well constrained. For simplicity, we shall usually assume a homogeneous shell, $\hat{Y} \equiv 1$, as well. Furthermore, we shall predominantly restrict our study to $\hat{p} \ll 1$, which ensures small axial strains (*i.e.* linear elasticity); for $\hat{p} \gtrsim \mathcal{O}(1)$, a nonlinear elastic model should be implemented in place of the constitutive laws, (5.14) and (5.22). Note that the large pressure required for a turgid cell — *i.e.* such that the pressure dominates the bending of the lateral walls ($\hat{p} \gtrsim \hat{\beta}$) — is still expected to be modelled by linear elasticity since $\hat{\beta} \ll 1$. Moreover, for $\hat{p} \lesssim \mathcal{O}(\hat{\beta}) \ll 1$, the stretching of the cell walls is negligible and (5.27) reduces to the equations for the elastica (Howell et al., 2008; O’Reilly, 2017).

We have now setup the problem for a symmetric (up–down and left–right) cellular structure. In general, the system of ODEs (5.27) subject to (5.31) must be solved numerically; we use the BVP solver `bvp4c` in `MATLAB`. To understand the behaviour of this model, we now study the results in three cases: no additional forcing ($\hat{\tau} = \hat{\mu} = 0$), a pure applied tension ($\hat{\tau} \neq 0$ and $\hat{\mu} = 0$), and a pure applied bending moment ($\hat{\tau} = 0$ and $\hat{\mu} \neq 0$).

5.3.2 The effects of pressure on cell geometry ($\mu = \tau = 0$)

For no additional forcing, $\hat{\mu} = \hat{\tau} = 0$, the cell walls are in equilibrium with the internal pressure, $\hat{p}$. The problem is up–down symmetric, and we expect the side walls to be vertical, $\phi = 0$, by symmetry. We denote the (*a priori* unknown) dimensionless cell thickness and width, $\hat{t}$ and $\hat{w}$, as $\hat{t}_0$ and $\hat{w}_0$, respectively. The formulation presented in §5.3.1 reduces to solving the system of ODEs (5.27) in $0 \leq \xi \leq 1$, with the axial stretch (5.28), end forces

$$\hat{T}_0 = \hat{p} \frac{\hat{L}\hat{t}_0}{2} \quad \text{and} \quad \hat{N}_0 = \pm \hat{p} \frac{\hat{w}_0}{2}, \quad (5.34a,b)$$

and side wall stretch

$$\hat{t}_0 = 1 + \hat{p} \frac{\hat{w}_0}{2\hat{Y}}, \quad (5.35)$$

subject to the boundary conditions (5.31a–c),

$$\theta(0) = \pm \left(\frac{\pi}{2} - \Theta \right), \quad \text{and} \quad X(1/2) = \frac{\hat{w}_0}{2}, \quad (5.36a,b)$$

where the $\pm$ corresponds to the top and bottom beams, respectively; however, by symmetry, we only need to solve for one of these beams in practice. This system describes the effect of the internal pressure, $\hat{p}$, on the cell's geometry, for given cell parameters $\hat{Y}$, $\hat{L}$, $\hat{\beta}$, and Θ , as defined in (5.33). For example, for relatively inextensible lateral walls ($\hat{\beta} \ll 1$), we might expect the cell to shrink as $\hat{p}$ increases: increasing the pressure, typically, causes a large wall curvature, but $\hat{\beta} \ll 1$ makes stretching expensive. For general parameters, the problem must be solved numerically (we used `bvp4c` in `MATLAB`); however, analytical progress can be made in the case of large or vanishing pressures (*i.e.* a turgid or flaccid cell), which we consider now.

5.3.2.1 Flaccid cell ($\hat{p} = 0$)

For a flaccid cell with no internal pressure, $\hat{p} = 0$, integrating (5.27c) with $\hat{T}_0 = \hat{N}_0 = 0$ and applying (5.31c) and (5.36a) yields the intrinsic equations of the top and bottom beams,

$$\theta(\xi) = \pm \left(\frac{\pi}{2} - \Theta \right) (1 - 2\xi), \quad (5.37)$$

respectively. Note that, here, the beams form circular arcs because of the imposed clamping and, since the beam ends are free to move horizontally, they do not stretch and have unit arclength ($\alpha(\xi) \equiv 1$). The corresponding beam positions, $(X(\xi), Z(\xi))$, are found by integrating (5.27a,b) and imposing (5.31a,b). The resulting unpressurized cell has thickness and projected length

$$\hat{t}_0 = 1 \quad \text{and} \quad \hat{w}_0 = \frac{2 \cos \Theta}{\pi - 2\Theta}, \quad (5.38a,b)$$

respectively. The enclosed area, A_0 , consists of two circular segments and a rectangular section, computing these gives the dimensionless area

$$\hat{A}_0 := \frac{A_0}{L_S L_L} = \frac{2 \cos \Theta}{\pi - 2\Theta} + \frac{1}{\hat{L}} \frac{\pi - 2\Theta - \sin(2\Theta)}{(\pi - 2\Theta)^2}. \quad (5.39)$$

(Note that we non-dimensionalize the area with respect to the case $\Theta = \pi/2$, where $A_0 = L_S L_L$.) Although the cell width increases monotonically from $\hat{w}_0 = 2/\pi$, for $\Theta = 0$, to $\hat{w}_0 = 1$, for $\Theta = \pi/2$, the cell area is maximized at a particular clamping angle, $\Theta_{\max}$, satisfying

$$\sin \Theta_{\max} = \hat{L} \left(\frac{\pi}{2} - \Theta_{\max} \right). \quad (5.40)$$

The cell of maximum area (*i.e.* for $\Theta = \Theta_{\max}$) is a circle (of radius $2/(\pi - 2\Theta_{\max})$) with two removed segments (of chord length $\hat{t}_0 = \hat{L}$) corresponding to the two side walls; this case is pertinent to our discussion below.

5.3.2.2 Turgid cell ($\hat{\beta} \ll \hat{p} \ll 1$)

In the other extreme, for a turgid cell with a large internal pressure, but still governed by linear elasticity (*i.e.* $\hat{\beta} \ll \hat{p} \ll 1$), the right-hand side of (5.27c) must vanish away from the ends of the lateral beams ($\xi = 0$ and $\xi = 1$), and hence we find that

$$\left(X(\xi) - \frac{\hat{w}_0}{2}\right)^2 + \left(Z(\xi) \pm \frac{\hat{t}_0 \hat{L}}{2}\right)^2 = \hat{R}_0^2 := \frac{\hat{w}_0^2 + (\hat{t}_0 \hat{L})^2}{4}, \quad (5.41)$$

for the top and bottom beams. The corresponding axial stretch, (5.28), is given by

$$\alpha(\xi) = 1 + \hat{p} \hat{R}_0. \quad (5.42)$$

Here, the clamping conditions, (5.36a), are only apparent in boundary layers of width $\xi = \mathcal{O}(\hat{\beta}/\hat{p})$ at the ends of the lateral beams. The pressure maximizes the internal area and the cell is a circle (of radius $R_0 \equiv L_L \hat{R}_0$) with two removed segments (of chord length $t_0 \equiv L_S \hat{t}_0$) corresponding to the two side walls — this is analogous to the circular cell observed for $\hat{p} = 0$ and $\Theta = \Theta_{\max}$, (5.40). From elementary geometry, the total deformed length of the lateral beams, *i.e.* $\int_0^1 \alpha(\xi) d\xi$, must equal $\hat{R}_0(\pi - \gamma_0)$ where $\gamma_0 := 2 \arctan(\hat{t}_0 \hat{L}/\hat{w}_0)$ is the angle of the removed segments. Thus, by integrating (5.42), we require

$$\int_0^1 \alpha(\xi) d\xi = 1 + \hat{p} \hat{R}_0 = \hat{R}_0 (\pi - \gamma_0), \quad (5.43)$$

which, when coupled with (5.35), forms a system for the unknown cell width, $\hat{w}_0$, and thickness, $\hat{t}_0$. The enclosed area, $\hat{A}_0$, is the area of the circle minus the two segments, giving

$$\hat{A}_0 = \frac{\hat{w}_0 \hat{t}_0}{2} + \frac{\hat{R}_0^2}{\hat{L}} (\pi - \gamma_0), \quad (5.44)$$

where $\hat{t}_0$ and $\hat{w}_0$ solve (5.35) and (5.43).

5.3.2.3 Moderate internal pressures

As the cell pressure increases from $\hat{p} = 0$ (§5.3.2.1) to $\hat{\beta} \ll \hat{p} \ll 1$ (§5.3.2.2), the cell area is expected to increase: a hydrated plant is swollen compared to its dehydrated counterpart. In our model, an increase in cell area,

$$\hat{A}_0 := \frac{A_0}{L_S L_L} = \hat{w}_0 \hat{t}_0 + \frac{1}{\hat{L}} \int_0^1 Z_T \frac{dX_T}{d\xi} - Z_B \frac{dX_B}{d\xi} d\xi, \quad (5.45)$$

where (X_T, Z_T) and (X_B, Z_B) denote the dimensionless positions of the top and bottom walls, respectively, is caused by an increase in cell thickness, $\hat{t}_0$; cell width, $\hat{w}_0$; and the

deformation of the top and bottom beams as the internal pressure increases.

In Fig. 5.7, we plot $\hat{w}_0$ and $\hat{A}_0$ as functions of the internal pressure, $\hat{p}$, for $\hat{\beta} = 10^{-4}$, $\hat{L} = 1/2$, $\hat{Y} = 1$, and various clamping angles, Θ . Note that $\hat{t}_0$ increases (approximately) linearly with $\hat{p}$, (5.35), and is not plotted. For $\Theta = \pi/2$, $3\pi/8$, and $\pi/4$ the cell width, $\hat{w}_0$, does not increase monotonically with the pressure: initially the increase in pressure bends the top and bottom cell walls out-of-plane, which pulls in the side walls and shrinks the cell. This is an example of turgor-driven shrinkage (as reported by Scoffoni et al., 2014; Hofhuis et al., 2016; Siéfert et al., 2020, for example) and is discussed further in §5.4.1. In the inset of Fig. 5.7(b), we observe the non-monotonic dependence of the flaccid cell area on the clamping angle, Θ , (5.39); for these geometrical parameters, $\hat{A}_0$ is maximized at $p = 0$ when $\Theta = \Theta_{\max} \approx 0.54$, for which $\hat{A}_0 \approx 1.39$. In fact, this observation is inextricably linked to the values of Θ for which $\hat{w}_0$ does not increase monotonically: for $\Theta = \Theta_{\max}$, the flaccid-cell's area is maximized and the cell is circular in shape (see ② in Fig. 5.7); thus, an increase in pressure can only increase the internal area by stretching the lateral walls ($\hat{w}_0$ is initially constant). Consequently, we can observe two regimes: for $\Theta > \Theta_{\max}$, the flaccid cell resembles a horizontal ellipse (see ① in Fig. 5.7), and the initial increase in pressure increases the internal area by pulling in the two side walls ($\hat{w}_0$ decreases); whilst for $\Theta < \Theta_{\max}$, the flaccid cell resembles a vertical ellipse (see ③ in Fig. 5.7), and the initial increase in pressure increases the internal area by pushing out the two side walls ($\hat{w}_0$ increases).

Varying the other cell parameters has the following effects, $\hat{L}$ alters the critical angle, $\Theta_{\max}$, according to (5.40) (with $\Theta_{\max} \rightarrow 0$ as $\hat{L} \rightarrow 0$ and $\Theta_{\max} \rightarrow \pi/2$ as $\hat{L} \rightarrow \infty$); $\hat{\beta}$ changes the onset of the turgid regime ($\hat{\beta} \ll \hat{p} \ll 1$); and $\hat{Y}$ controls the size of the side walls, which only quantitatively changes the results at moderate pressures. To conclude our analysis of the effects of internal pressure on the cell geometry, we now introduce a measure of the cell's resistance to an increase in pressure/volume.

For an isotropic elastic body, a common measure of the substance's resistance to compression is the bulk modulus, $K = -V \mathrm{d}P / \mathrm{d}V$, where P is the imposed external pressure and V is the volume of the substance. An analogous *effective bulk modulus* can be introduced for the two-dimensional cell, which measures the resistance to a change in internal area as the internal pressure increases, that is

$$K_{\mathrm{eff}} = A_0 \frac{\mathrm{d}p}{\mathrm{d}A_0}, \quad (5.46)$$

where p is the internal pressure and A_0 is the cell area. (Note that the sign change between K and K_{eff} occurs because of the switch from an external to internal pressure.) For example, Proctor et al. (1998) measured the effective bulk modulus of turgid bryophyte cells, obtaining values $K_{\mathrm{eff}} = 1 \text{ MPa} - 20 \text{ MPa}$ (depending on the species).

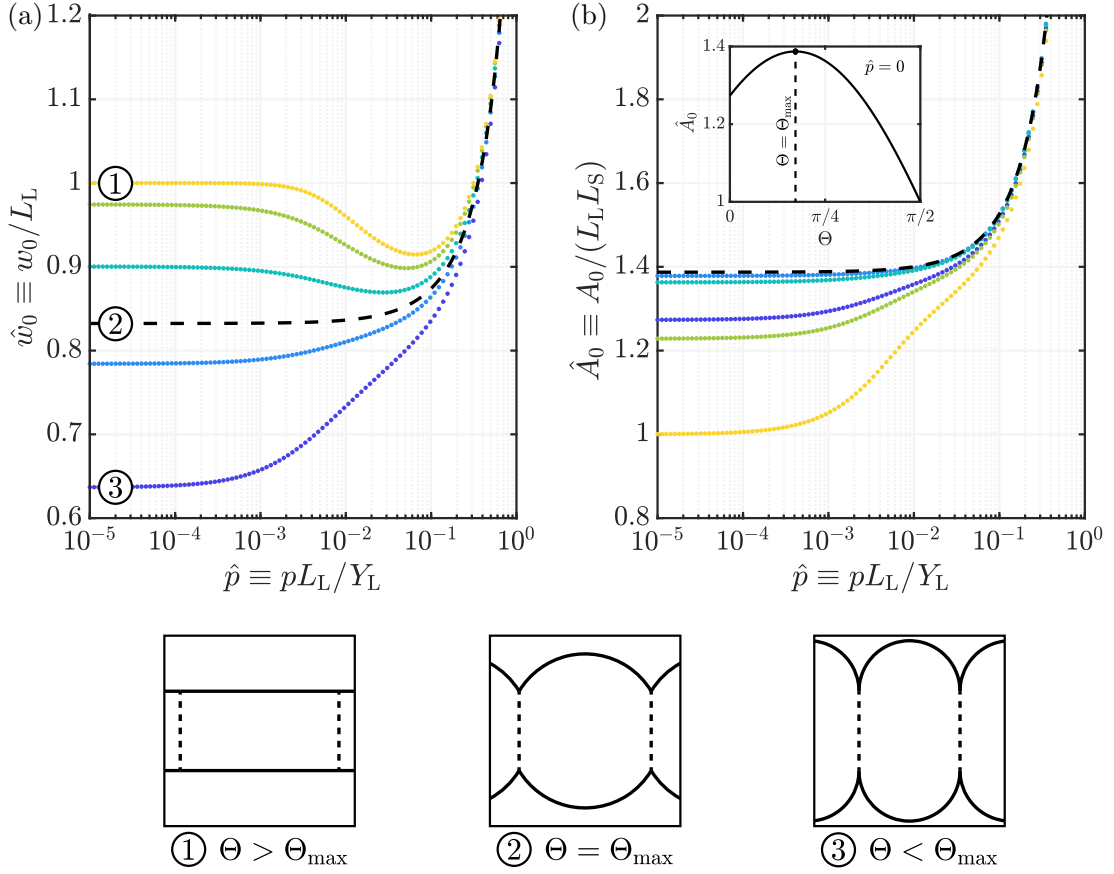


Figure 5.7: Plots of (a) the horizontal width, $\hat{w}_0$, and (b) the enclosed area, $\hat{A}_0$, of the cell as functions of the internal pressure, $\hat{p}$. These plots were numerically-determined by solving the Kirchhoff equations with parameters: $\hat{Y} = 1$, $\hat{\beta} = 10^{-4}$, $\hat{L} = 1/2$, and $\Theta = \pi/2$, $3\pi/8$, $\pi/4$, $\pi/8$, and 0 (yellow to blue dots, ●). In the inset of (b), the area of the flaccid cell (*i.e.* $\hat{A}_0$ at $\hat{p} = 0$) is plotted as a function of the clamping angle, Θ ; this is maximized at $\Theta = \Theta_{\max} \approx 0.54$. In the main figures, the solution for $\Theta = \Theta_{\max}$ is plotted as a black dashed curve. In this critical case, the cell is circular in shape and qualitatively separates the different behaviours: for $\Theta > \Theta_{\max}$, the cell resembles a horizontal ellipse and an increase in pressure initially decreases the cell width, $\hat{w}_0$; whilst, for $\Theta < \Theta_{\max}$, the cell resembles a vertical ellipse and an increase in pressure initially increases the cell width, $\hat{w}_0$. These regimes are delineated by ①–③.

To non-dimensionalize the cell area, $\hat{A}_0$; width, $\hat{w}_0$; and thickness, $\hat{t}_0$, we use the flaccid cell ($\hat{p} = 0$) solution for $\Theta = \pi/2$ as a reference configuration — *i.e.* the rectangular cell plotted in Fig. 5.7①. We choose to non-dimensionalize the effective bulk modulus, K_{eff} , using this case, also². Thus, we need the first two terms in the expansion of $\hat{A}_0$ as $\hat{p} \rightarrow 0$ so

²Note that there are many different ways to non-dimensionalize the effective bulk modulus, K_{eff} ; a more natural choice of reference state might be the flaccid cell ($\hat{p} = 0$) with the relevant clamping angle, Θ . However, as we show shortly, our choice leads to a solution independent of Θ at large pressures ($\hat{p} \gg \hat{\beta}$) — a case which we have a considerable interest in — but it comes with the caveat that the effects of Θ will be more obvious at small pressures. An identical choice in reference state shall be made for the effective stretching and bending moduli, Y_{eff} and B_{eff} , in §5.3.3 and §5.3.4, respectively.

that we can take the derivative with respect to $\hat{p}$, (5.46). For $\Theta = \pi/2$, the leading-order beam angle, (5.37), vanishes, it is easy to show that corresponding enclosed area, (5.45), is

$$\hat{A}_0 = 1 + \hat{p} \left(\frac{\hat{L}}{2} + \frac{1}{2\hat{Y}} + \frac{1}{360\hat{\beta}\hat{L}} \right) + \mathcal{O}(\hat{p}^2), \quad (5.47)$$

for $0 \leq \hat{p} \ll \hat{\beta} \ll 1$. At leading-order, $\hat{A}_0 \sim 1$ which is by the choice of dimensionless area, (5.39). The three expressions in the $\mathcal{O}(\hat{p})$ -term describe the initial increase in area (shown numerically in Fig. 5.7) caused by the stretching of the lateral walls, stretching of the side walls, and (most prominently since $\hat{\beta} \ll 1$) bending of the lateral walls, respectively. Inserting (5.47) into (5.46) gives the bulk modulus

$$K_{\text{eff}}^0 := \frac{E_L h_L}{L_L} \times \left(\frac{\hat{L}}{2} + \frac{1}{2\hat{Y}} + \frac{1}{360\hat{\beta}\hat{L}} \right)^{-1}, \quad (5.48)$$

for $\hat{p} = 0$ and $\Theta = \pi/2$. We use K_{eff}^0 as the reference value for the general effective bulk modulus, K_{eff} , and so define its dimensionless form as

$$\hat{K}_{\text{eff}} := \frac{K_{\text{eff}}}{K_{\text{eff}}^0} \equiv \left(\frac{\hat{L}}{2} + \frac{1}{2\hat{Y}} + \frac{1}{360\hat{\beta}\hat{L}} \right) \hat{A}_0 \frac{d\hat{p}}{d\hat{A}_0}. \quad (5.49)$$

In Fig. 5.8, $\hat{K}_{\text{eff}}$ is plotted as a function of $\hat{p}$ for the variables used in Fig. 5.7. This plot shows that, for a fixed $\Theta \neq \Theta_{\text{max}}$ and $\hat{p} \ll 1$, an increase in pressure is progressively more resisted (*i.e.* $\hat{K}_{\text{eff}}$ increases with $\hat{p}$). For $\hat{p} \sim 1$, the pressure is large enough to stretch the cell walls and increase the area further; as a result, $\hat{K}_{\text{eff}}$ decreases. It should be noted that the assumption of small beam strains (linear elasticity) only holds for $\hat{p} \ll 1$; hence, for $\hat{p} \gtrsim \mathcal{O}(1)$ the response is governed by nonlinear elasticity. In the critical case, $\Theta = \Theta_{\text{max}}$, the action of the effective bulk modulus is significantly different from that in the other cases: the cell is circular for all internal pressures, so the enclosed area is sufficiently maximized (see Fig. 5.7) and $\hat{K}_{\text{eff}}$ is approximately constant for $\hat{p} \ll 1$.

Having studied the basic pressurized state, we now consider the response of the pressurized unit cell to an additional stretch (§5.3.3) and pure bend (§5.3.4).

5.3.3 Effective stretching modulus ($\mu = 0$)

For an additional applied tension, $\hat{\tau}$, but no applied moment, $\hat{\mu} = 0$, the pressurized cellular configuration of §5.3.2 is expected to stretch longitudinally, leading to some strain ϵ (Fig. 5.5, lower left). The cell width takes the form $\hat{w} = \hat{w}_0(1 + \epsilon)$, where $\hat{w}_0$ is the width of the cell with no additional strain, $\hat{\tau} = 0$, (*i.e.* the system presented in §5.3.2). The up-down symmetry is not broken by pure stretching and the angle of the side walls must still vanish,

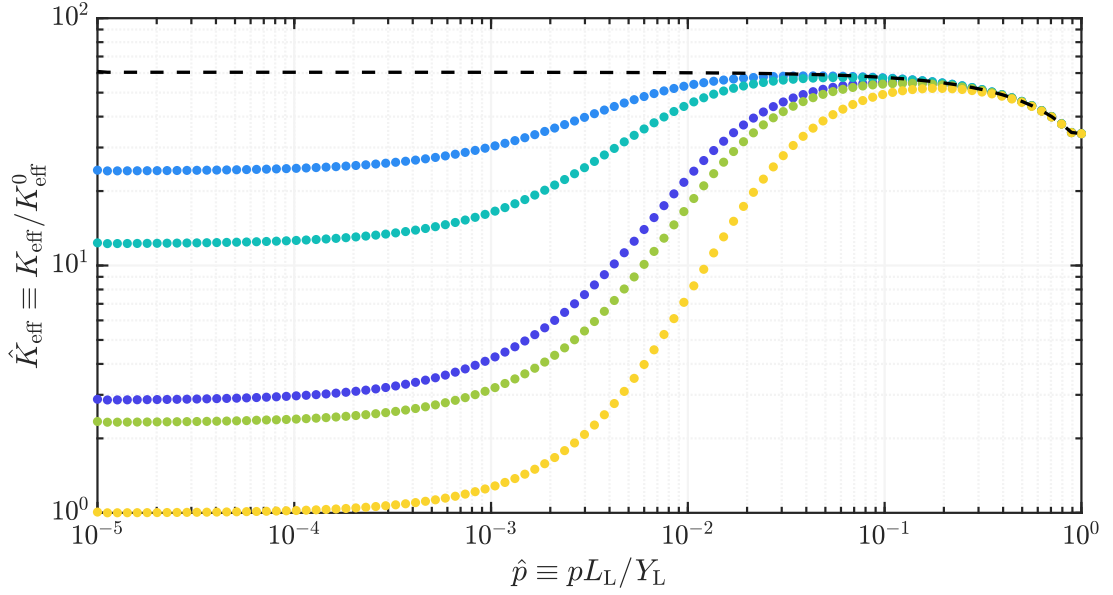


Figure 5.8: Plot of the numerically-determined bulk modulus, $\hat{K}_{\text{eff}}$, as a function of the internal pressure, $\hat{p}$, for the configuration parameters: $\hat{Y} = 1$, $\hat{\beta} = 10^{-4}$, $\hat{L} = 1/2$, and $\Theta = \pi/2, 3\pi/8, \pi/4, \pi/8$, and 0 (yellow to blue dots, $\bullet$) — which are equivalent to the values used in Fig. 5.7. Plotted as a black dashed curve is the circular cell solution with $\Theta = \Theta_{\text{max}} \approx 0.54$, (5.40). For small beam strains ($\hat{p} \ll 1$), the increase in pressure gets progressively more resisted as the cell tends towards the circular configuration (which is achieved for the $\Theta = \Theta_{\text{max}}$ case at all $\hat{p}$).

$\phi = 0$. The formulation presented in §5.3.1 reduces to solving the system of ODEs (5.27) in $0 \leq \xi \leq 1$, with the axial stretch (5.28), end forces

$$\hat{T}_0 = \hat{p} \frac{\hat{L}\hat{t}}{2} + \frac{\hat{\tau}}{2} \quad \text{and} \quad \hat{N}_0 = \pm \hat{p} \frac{\hat{w}_0}{2} (1 + \epsilon), \quad (5.50\text{a,b})$$

and side wall stretch

$$\hat{t} = 1 + \hat{p} \frac{\hat{w}_0}{2\hat{Y}} (1 + \epsilon) \equiv \hat{t}_0 + \hat{p} \frac{\hat{w}_0}{2\hat{Y}} \epsilon, \quad (5.51)$$

subject to the boundary conditions (5.31a–c),

$$\theta(0) = \pm \left(\frac{\pi}{2} - \Theta \right), \quad \text{and} \quad X(1/2) = \frac{\hat{w}_0}{2} (1 + \epsilon), \quad (5.52\text{a,b})$$

where the $\pm$ corresponds to the top and bottom beams, respectively. The solution of this overdetermined system yields a relationship between the applied tension, $\hat{\tau}$, and the midline strain, ϵ . For a given $\hat{\tau}$, we can compute ϵ (or *vice versa*) and infer an *effective stretching modulus* using Hooke's law, (5.1):

$$Y_{\text{eff}} := E_L h_L \times \frac{\hat{\tau}}{\epsilon}. \quad (5.53)$$

To non-dimensionalize Y_{eff} , we use the stiffness of a flaccid cell ($\hat{p} = 0$) with $\Theta = \pi/2$, which was also used to non-dimensionalize the effective bulk modulus, $\hat{K}_{\text{eff}}$, and cell area, $\hat{A}_0$, and is sketched in Fig. 5.7④. As might be expected, the longitudinal stretching of this rectangular case is identical to the simplistic model discussed in §5.2.1, and results in the same effective modulus,

$$Y_{\text{eff}}^0 := 2E_L h_L. \quad (5.54)$$

We use Y_{eff}^0 as the reference value for the general effective stretching modulus, Y_{eff} , and so define the dimensionless stretching modulus as

$$\hat{Y}_{\text{eff}} := \frac{Y_{\text{eff}}}{Y_{\text{eff}}^0} \equiv \frac{1}{2} \times \frac{\hat{\tau}}{\epsilon}. \quad (5.55)$$

Note that, here, one could write the applied tension, $\hat{\tau}$, as a sum of those applied to the top and bottom walls, *i.e.* $\hat{\tau} \mapsto \hat{\tau}_T + \hat{\tau}_B$; thus, $\hat{\tau}/2$ may be thought of as the thickness-averaged tension.

Although $\hat{Y}_{\text{eff}}$ can be computed for any imposed tension, $\hat{\tau}$, it can only be interpreted as a macroscale stiffness at infinitesimal values of $\hat{\tau} \sim \epsilon$. For larger values, (5.55) deviates from the corresponding energetic definition of the stretching modulus, (5.3), and thus loses its physical meaning (Appendix 5.A). We, therefore, concentrate on infinitesimal values, $\hat{\tau} \sim \epsilon \ll 1$.

As $\hat{\tau} \sim \epsilon \rightarrow 0$, the above system, (5.27) subject to (5.31a–c) and (5.52), can be solved by posing an asymptotic expansion in powers of ϵ as the solution of the force balance equations, (5.27) with (5.28), (5.50), and (5.51). At leading-order, $\mathcal{O}(\epsilon^0)$, we recover the zero-forcing equations of §5.3.2 and, hence, recover the cell width and thickness, $\hat{w}_0$ and $\hat{t}_0$. At next-order, $\mathcal{O}(\epsilon^1)$, we obtain a system for the infinitesimal effective stretching modulus, $\hat{Y}_{\text{eff}}$, which is dependent on the previous-order. The resulting asymptotic systems are presented in Appendix 5.A.2. In general, these equations must be solved numerically (we use `bvp4c` in `MATLAB`), however analytical progress can be made in the asymptotic limits of large and small pressures (*i.e.* of turgid and flaccid cells), which we consider now.

5.3.3.1 Flaccid cell ($\hat{p} = 0$)

In §5.3.2.1, we showed that the behaviour of the ‘undeformed’ flaccid cell ($\hat{p} = 0$) is governed by the clamping angle, Θ ; the cell resembles two circular segments attached to a rectangular centre. An additional tension, $\hat{\tau}$, acts to stretch the cell and flatten the lateral beams. Solving (5.27) subject to (5.31a–c) and (5.52) with (5.28), $\hat{T}_0 = \hat{\tau}/2$, $\hat{N}_0 = 0$, and $\hat{t} = 1$, using an asymptotic expansion as $\hat{\tau} \sim \epsilon \rightarrow 0$, we find the intrinsic equation of the deformed

beams to be

$$\theta(\xi) = \theta_0(\xi) \pm \frac{\hat{\tau}}{2\hat{\beta}(\pi - 2\Theta)^2} \left\{ (1 - 2\xi) \cos \Theta - \sin \left[\left(\frac{\pi}{2} - \Theta \right) (1 - 2\xi) \right] \right\} + \mathcal{O}(\hat{\tau}^2), \quad (5.56a)$$

and axial stretch

$$\alpha(\xi) = 1 + \frac{\hat{\tau}}{2} \cos \left[\left(\frac{\pi}{2} - \Theta \right) (1 - 2\xi) \right] + \mathcal{O}(\hat{\tau}^2), \quad (5.56b)$$

where $\theta_0(\xi) = \pm(\pi/2 - \Theta)(1 - 2\xi)$ is the leading-order solution, (5.37). Inserting these expressions into (5.27b) and integrating over the domain, $0 \leq \xi \leq 1/2$, yields an expression for the stretched cell length, $2X(1/2) = \hat{w}_0(1 + \epsilon)$. At $\mathcal{O}(\epsilon)$, this results in a linear relation between the applied tension, $\hat{\tau}$, and the longitudinal strain, ϵ , giving the effective stretching modulus

$$\hat{Y}_{\text{eff}} \equiv \frac{\hat{\tau}}{2\epsilon} \sim \frac{4\hat{\beta}(\pi - 2\Theta)^3 \cos \Theta}{\left[1 + \hat{\beta}(\pi - 2\Theta)^2 \right] (\pi - 2\Theta) (\pi - 2\Theta + \sin 2\Theta) - 4(1 + \cos 2\Theta)}. \quad (5.57)$$

Here, the dependence on $\hat{\beta} \ll 1$ determines whether the lateral beams respond by bending or stretching. For $\Theta = \pi/2$, the cell is rectangular in shape and the walls can only stretch with $\hat{Y}_{\text{eff}} \sim 1$ (by choice of reference value). In the other extreme, $\Theta = 0$, the cell resembles two semicircles attached to a rectangular centre, and we find that $\hat{Y}_{\text{eff}} \sim 4\pi^3\hat{\beta}/(\pi^4\hat{\beta} + \pi^2 - 8) \ll 1$. Note that $\hat{Y}_{\text{eff}}$ is small for $\Theta = 0$ because the excess length is ‘buffered-by-buckling’ (Vella, 2019), this is not possible for the rectangular cell (*i.e.* for $\pi/2 - \Theta \lesssim \mathcal{O}(\hat{\beta}^{1/2}) \ll 1$), which can only stretch under an imposed tension and $\hat{Y}_{\text{eff}} \sim [1 + (\pi/2 - \Theta)^2/(180\hat{\beta})]^{-1}$.

5.3.3.2 Turgid cell ($\hat{\beta} \ll \hat{p} \ll 1$)

For large internal pressures, but still under the assumption of linear elasticity (*i.e.* $\hat{\beta} \ll \hat{p} \ll 1$), the right hand side of (5.27c) must vanish: the upper and lower beams are circular arcs with radius $R \equiv L_L \hat{R}$ and sector angle $\pi - \gamma$, such that

$$\hat{R}^2 = \frac{\hat{w}_0^2(1 + \epsilon)^2 + \left(\hat{L}\hat{t} + \hat{\tau}/\hat{p} \right)^2}{4} \quad \text{and} \quad \gamma = 2 \arctan \left(\frac{\hat{L}\hat{t} + \hat{\tau}/\hat{p}}{\hat{w}_0(1 + \epsilon)} \right), \quad (5.58a,b)$$

where $\hat{t}$ is given by (5.51). The stretched turgid cell is in the shape of a symmetric lens with truncated tips (corresponding to the two side walls); the circular profile from §5.3.2.2 is recovered by taking $\hat{\tau} = \epsilon = 0$. The clamped boundary conditions, (5.52a), are only apparent in boundary layers of width $\mathcal{O}(\hat{\beta}/\hat{p})$ at the ends of the lateral beams, we assume that these boundary layers are negligible when calculating $\hat{Y}_{\text{eff}}$ at large pressures. Inserting (5.58) into the arclength requirement, (5.43), and expanding in powers of $\epsilon \sim \hat{\tau} \rightarrow 0$, yields a linear

tension–strain relationship at $\mathcal{O}(\epsilon)$, which gives the effective stretching modulus

$$\hat{Y}_{\text{eff}} \equiv \frac{\hat{\tau}}{2\epsilon} \sim \frac{\hat{p}\hat{w}_0}{2} \left(\frac{\hat{w}_0 + 2\hat{R}_0\hat{L}\hat{t}_0}{2\hat{R}_0\hat{w}_0 - \hat{L}\hat{t}_0} - \frac{\hat{L}\hat{p}}{2\hat{Y}} \right), \quad (5.59)$$

where $\hat{R}_0^2 := (\hat{w}_0^2 + \hat{L}^2\hat{t}_0^2)/4$, $\hat{w}_0$, and $\hat{t}_0$ are the initial cell radius, width, and thickness, respectively (which were determined in §5.3.2.2). For small axial strains, $\hat{p} \ll 1$, (5.59) increases linearly with the pressure, whilst for large axial strains, $\hat{p} \sim 1$, the pressure has a nonlinear effect; however, at these values, the assumption of linear elasticity is not expected to hold.

5.3.3.3 Moderate internal pressure

In order to determine the effective stretching modulus, Y_{eff} , at moderate interior pressures, the governing system, (5.27) subject to (5.31a–c) and (5.52), must be solved numerically. In Fig. 5.9, the numerically-determined $\hat{Y}_{\text{eff}}$ (at infinitesimal values of $\hat{\tau} \sim \epsilon$) is plotted as a function of $\hat{p}$, for $\hat{\beta} = 10^{-4}$, $\hat{Y} = 1$, $\hat{L} = 1/2$, and various Θ . Also plotted are the asymptotic results at small and large pressures, (5.57) and (5.59), respectively. As well as showing that the expected asymptotic behaviour is recovered by these numerics, Fig. 5.9 shows that $\hat{Y}_{\text{eff}}$ may initially decrease with pressure if the exterior clamping angle, Θ , is sufficiently large, but ultimately increases with $\hat{p}$. Also apparent is that at large pressures, $\hat{p} = \mathcal{O}(1)$, the nonlinear dependence on $\hat{p}$ in (5.59) is sublinear (though a nonlinear elastic model may alter this observation). Altering the cell parameters, $\hat{Y}$, $\hat{\beta}$, and $\hat{L}$, appears to only quantitatively affect the results presented here.

5.3.4 Effective bending modulus ($\tau = 0$)

For a moment, μ , applied about the midline, the cellular structure is expected to bend forming some midline curvature, κ , (see Fig. 5.5, lower right). This induced curvature deforms the lateral beams and breaks the up–down symmetry; thus, the side-wall angle is not expected to vanish, $\phi \neq 0$. As we are ultimately interested in computing the effective bending modulus at infinitesimal values of $\hat{\mu}$, we shall assume $\phi \ll 1$. As $\phi \rightarrow 0$, the symmetry of bending the cell upwards or downwards (negative or positive ϕ , respectively) means the width of the cell remains unchanged, $\hat{w} = \hat{w}_0 + \mathcal{O}(\phi^2)$, where $\hat{w}_0$ was found in §5.3.2. The midline curvature can be written as

$$\hat{\kappa} = \frac{2 \sin \phi}{\hat{w}} = \frac{2\phi}{\hat{w}_0} + \mathcal{O}(\phi^3), \quad (5.60)$$

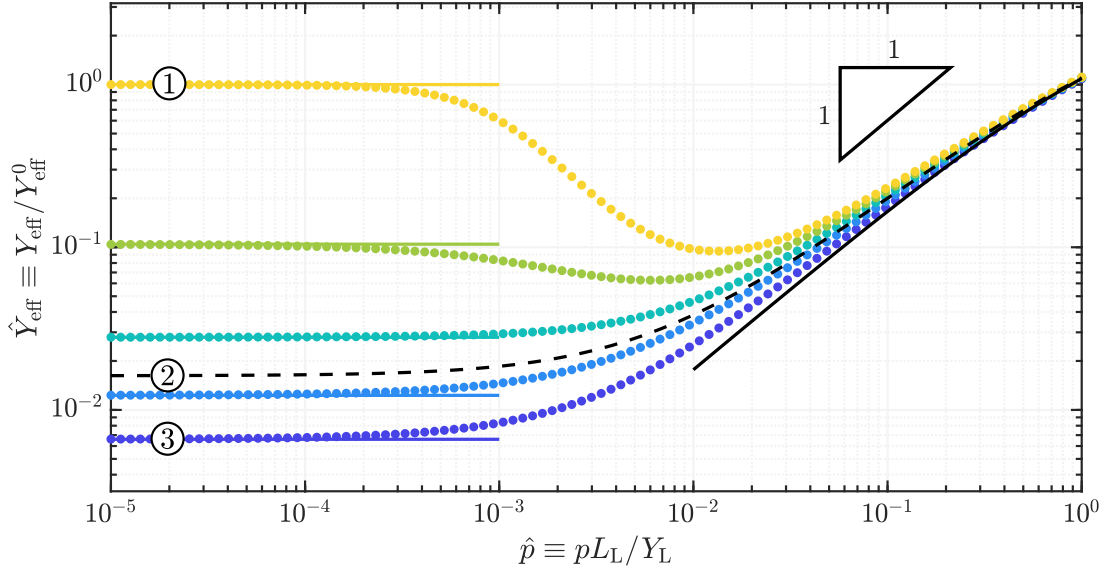


Figure 5.9: Plot of the effective stretching modulus, $\hat{Y}_{\text{eff}}$, as a function of the internal pressure, $\hat{p}$, for the configuration parameters: $\hat{Y} = 1$, $\hat{\beta} = 10^{-4}$, $\hat{L} = 1/2$, and $\Theta = \pi/2, 3\pi/8, \pi/4, \pi/8$, and 0 (yellow to blue). The $\hat{Y}_{\text{eff}}$ found by numerically solving the Kirchhoff equations, (5.27) subject to (5.31a–c) and (5.52), for infinitesimal $\tau \sim \epsilon \ll 1$, are plotted as coloured dots (●) whilst asymptotic results are plotted as solid curves: (5.57), valid for small pressures, is shown in colour and (5.59), valid for large pressures, is in black. Plotted as a black dashed curve is the numerical solution with maximum area, $\Theta = \Theta_{\text{max}} \approx 0.54$, (5.40). ①–③ correspond to the three regimes sketched in Fig. 5.7. Although $\hat{Y}_{\text{eff}}$ may initially decrease with $\hat{p}$ (*i.e.* for large values of Θ), it ultimately increases with $\hat{p}$ according to (5.59), transitioning from a linear to sublinear dependence on $\hat{p}$.

where $\phi \ll 1$ is still to be determined, for a given μ . The general formulation of §5.3.1 simplifies to solving the system of ODEs (5.27) in $0 \leq \xi \leq 1$, with the axial stretch (5.28), end forces

$$\hat{T}_0 = \hat{p} \left(\frac{\hat{L}\hat{t}_0}{2} \mp \frac{\hat{w}_0}{2}\phi \right) \pm \frac{\hat{\mu}}{\hat{t}_0} + \mathcal{O}(\phi^2) \quad \text{and} \quad \hat{N}_0 = \pm \hat{p} \left(\frac{\hat{w}_0}{2} \pm \frac{\hat{L}\hat{t}_0}{2}\phi \right) + \mathcal{O}(\phi^2), \quad (5.61\text{a,b})$$

and side wall stretch $\hat{t} = \hat{t}_0 + \mathcal{O}(\phi^2)$, for $\hat{t}_0$ given by (5.35), subject to the boundary conditions (5.31a–c),

$$\theta(0) = \pm \left(\frac{\pi}{2} - \Theta \right) + \phi, \quad \text{and} \quad X(1/2) = \frac{\hat{w}_0}{2} \pm \frac{\hat{L}\hat{t}_0}{2}\phi + \mathcal{O}(\phi^2), \quad (5.62\text{a,b})$$

where the $\pm$ corresponds to the top and bottom beams, respectively. This system describes the relationship between the applied moment, $\hat{\mu}$, and the side wall angle, ϕ . For a given $\hat{\mu}$ we can compute ϕ (or *vice versa*) and implement the bending modulus definition, (5.2), to

deduce an *effective bending modulus*,

$$B_{\text{eff}} := E_L h_L L_S L_L \times \frac{\hat{\mu}}{\hat{\kappa}} \equiv E_L h_L L_S L_L \times \frac{\hat{w}_0}{2} \frac{\hat{\mu}}{\phi} + \mathcal{O}(\phi^2). \quad (5.63)$$

To non-dimensionalize B_{eff} , we again use the flaccid case ($\hat{p} = 0$) with $\Theta = \pi/2$ — *i.e.* the rectangular cell sketched in Fig. 5.7①. Bending this configuration by an infinitesimal amount is identical to bending the simplistic model discussed in §5.2.1, hence it has an effective bending modulus

$$B_{\text{eff}}^0 := E_L h_L \frac{L_S^2}{2}. \quad (5.64)$$

(The differential strain associated with bending stretches the upper wall and compresses the bottom wall by a strain $L_S \kappa/2$.) Using B_{eff}^0 as the reference value for the general bending modulus, B_{eff} , we define its dimensionless form as

$$\hat{B}_{\text{eff}} := \frac{B_{\text{eff}}}{B_{\text{eff}}^0} \equiv \frac{\hat{w}_0}{\hat{L}} \frac{\hat{\mu}}{\phi} + \mathcal{O}(\phi^2). \quad (5.65)$$

As was the case for the effective stretching modulus (§5.3.3), $\hat{B}_{\text{eff}}$ can only be interpreted as a macroscale stiffness for infinitesimal $\hat{\mu} \sim \phi$ (Appendix 5.A). In the limit of $\hat{\mu} \sim \phi \rightarrow 0$, the above system can be solved using an asymptotic expansion in powers of ϕ . The leading-order problem, $\mathcal{O}(\phi^0)$, corresponding to the zero-forcing equations of §5.3.2. The next-order, $\mathcal{O}(\phi^1)$, yields a system for the infinitesimal effective bending modulus, $\hat{B}_{\text{eff}}$, which is dependent on $\mathcal{O}(\phi^0)$ problem. These asymptotic systems are presented in Appendix 5.A.3 and, in general, must be solved numerically (we use `bvp4c` in `MATLAB`). Analytical progress can be made, however, in the limit of small and large pressures (*i.e.* a flaccid and turgid cell, respectively), as we show now.

5.3.4.1 Flaccid cell ($\hat{p} = 0$)

Bending the flaccid cell ($\hat{p} = 0$) from §5.3.2.1, stretches and flattens the circular top wall, whilst compressing and bulging the circular bottom wall — this response generates a stiffness. Solving (5.27) subject to (5.31a–c) and (5.62), with (5.28), $\hat{T}_0 = \pm \hat{\mu}$, $\hat{N}_0 = 0$, and $\hat{t} = 1$, using an asymptotic expansion as $\hat{\tau} \sim \epsilon \rightarrow 0$, we find the intrinsic equation of the deformed beam

$$\theta(\xi) \sim \theta_0(\xi) + (1 - 2\zeta)\phi + \frac{\mu}{\hat{\beta}(\pi - 2\Theta)^2} \left\{ (1 - 2\zeta) \cos \Theta - \sin \left[\left(\frac{\pi}{2} - \Theta \right) (1 - 2\zeta) \right] \right\}, \quad (5.66a)$$

and axial stretch

$$\alpha(\xi) \sim 1 \pm \mu \cos \left[\left(\frac{\pi}{2} - \Theta \right) (1 - 2\zeta) \right], \quad (5.66b)$$

where $\theta_0(\xi) = \pm(\pi/2 - \Theta)(1 - 2\xi)$ is the leading-order solution from (5.37). Inserting these expressions into (5.27a) and integrating over the domain, $0 \leq \xi \leq 1/2$, yields an expression for the stretched cell length, $2X(1/2) = \hat{w}_0 \pm \hat{L}\phi/2$. At $\mathcal{O}(\phi)$, this results in a linear relation between the imposed moment, $\hat{\mu}$, and the side-wall angle, ϕ , which gives the effective bending modulus

$$\hat{B}_{\text{eff}} \equiv \frac{\hat{w}_0}{\hat{L}} \frac{\hat{\mu}}{\phi} \sim \frac{\frac{4\hat{\beta}}{\hat{L}}(\pi - 2\Theta) \left[\hat{L}(\pi - 2\Theta)^2 \cos \Theta + 2(1 + \cos 2\Theta) - (\pi - 2\Theta) \sin 2\Theta \right]}{\left[1 + \hat{\beta}(\pi - 2\Theta)^2 \right] (\pi - 2\Theta) (\pi - 2\Theta + \sin 2\Theta) - 4(1 + \cos 2\Theta)}. \quad (5.67)$$

Unlike the corresponding effective stretching modulus, (5.57), $\hat{B}_{\text{eff}}$ depends on the initial height of the cell through the parameter $\hat{L}$: the differential strain is proportional to the distance from the midline. For $\Theta = \pi/2$, $\hat{B}_{\text{eff}} \sim 1$ (by choice of reference value), whilst for $\Theta = 0$, $\hat{B}_{\text{eff}} \sim 4\pi(\pi^2 + 4/\hat{L})\hat{\beta}/(\pi^4\hat{\beta} + \pi^2 - 8) \ll 1$, suggesting that, at least initially, $\hat{B}_{\text{eff}}$ increases with the exterior clamping angle, Θ . Furthermore, for a rectangular cell with $\pi/2 - \Theta \lesssim \mathcal{O}(\hat{\beta}^{1/2}) \ll 1$, we find that $\hat{B}_{\text{eff}} \sim [1 + (\pi/2 - \Theta)^2/(180\hat{\beta})]^{-1}$, which is the same result found for Y_{eff} in §5.3.3.2.

5.3.4.2 Turgid cell ($\hat{\beta} \ll \hat{p} \ll 1$)

As described in §5.3.2.1, when subject to large internal pressures (*i.e.* $\hat{\beta} \ll \hat{p}$), the right hand side of (5.27c) must vanish. We, hence, find that the upper and lower beams are circular arcs with radius $R \equiv L_L \hat{R}$ and sector angle $\pi - \gamma$, such that

$$\hat{R}^2 \sim \frac{\hat{w}_0^2 + (\hat{L}\hat{t}_0)^2}{4} \pm \frac{\hat{L}\hat{\mu}}{\hat{p}} \quad \text{and} \quad \gamma \sim 2 \arctan \left(\frac{\hat{L}\hat{t}_0}{\hat{w}_0} \right) \mp 2\phi \pm \frac{4\hat{w}_0}{\hat{w}_0^2 + (\hat{L}\hat{t}_0)^2} \frac{\hat{\mu}}{\hat{t}_0\hat{p}}, \quad (5.68\text{a,b})$$

where $\hat{w}_0$ and $\hat{t}_0$ are found by solving (5.35) and (5.43). The circular cell from §5.3.2.2 is recovered for zero imposed moment, $\hat{\mu} = \phi = 0$. Here, the clamped boundary conditions, (5.52a), are assumed to be negligible when calculating $\hat{B}_{\text{eff}}$ at large pressures since they are only apparent in boundary layers of width $\mathcal{O}(\hat{\beta}/\hat{p})$ at the ends of the lateral beams. Inserting (5.68) into the arclength requirement (5.43) yields a linear moment–angle relationship at $\mathcal{O}(\phi)$, which gives the effective bending modulus

$$\hat{B}_{\text{eff}} \equiv \frac{\hat{w}_0}{\hat{L}} \frac{\hat{\mu}}{\phi} \sim \hat{p} \frac{4\hat{R}_0^3 \hat{t}_0}{2\hat{R}_0 \hat{w}_0 - \hat{L}\hat{t}_0}, \quad (5.69)$$

where $\hat{R}_0 := (\hat{w}_0^2 + \hat{L}^2 \hat{t}_0^2)/4$ is the ‘undeformed’ cell radius found in §5.3.2.2. For small axial strains, $\hat{p} \ll 1$, (5.69) suggests that the effective bending modulus, $\hat{B}_{\text{eff}}$, linearly increases with the pressure, $\hat{p}$. For moderate–large axial strains, $\hat{p} \sim 1$, the pressure terms in the definitions of $\hat{w}_0$ and $\hat{t}_0$, (5.35) and (5.43), are important, thus $\hat{B}_{\text{eff}}$ is nonlinearly dependent

on $\hat{p}$, though at such large strains the assumption of linear elasticity is expected to break down.

5.3.4.3 Moderate internal pressures

At moderate pressures, the system (5.27) subject to (5.31a–c) and (5.62) must be solved numerically to determine the effective bending modulus, $\hat{B}_{\text{eff}}$. In Fig. 5.10, the numerically-determined $\hat{B}_{\text{eff}}$ is plotted as a function of $\hat{p}$, for $\hat{\beta} = 10^{-4}$, $\hat{Y} = 1$, $\hat{L} = 1/2$, and various Θ . Also plotted are the asymptotic results for small and large pressures, (5.67) and (5.69), respectively, which show that the correct behaviours are recovered as $\hat{p} \rightarrow 0$ and $\hat{p} \rightarrow \infty$. Although initially the effective bending modulus may decrease with pressure, $\hat{B}_{\text{eff}}$ ultimately increases with $\hat{p}$ according to (5.69). Furthermore, for $\hat{p} \sim 1$, the nonlinear dependence of (5.69) on the pressure, $\hat{p}$, appears to be superlinear — which is in contrast to the effective stretching stiffness, $\hat{Y}_{\text{eff}}$, which had a sublinear dependence (Fig. 5.9) — with the usual reminder that the beam strains are large and a nonlinear elastic model is required. Changing the configuration parameters, $\hat{Y}$, $\hat{\beta}$, and $\hat{L}$, appears to only quantitatively change the results presented here.

5.4 Discussion: Implications for plant leaves and other applications

5.4.1 Turgor-driven shape change

In the cellular structure we have studied, an increase in internal pressure gives rise to an increase in the area encapsulated by the cell (Fig. 5.7(b)). This is in accordance with what is seen in *ex vivo* experiments on plant tissue (see Proctor et al., 1998; Scoffoni et al., 2014, for example): turgor pressure drives plant cell expansion. It is then, at least at first, counterintuitive that the cells may initially shrink laterally with increasing internal pressure (as seen for $\Theta > \Theta_{\text{max}}$ in Fig. 5.7(a), for example). However, a similar occurrence has also been observed in plant tissue, for example in detached exocarp cells during explosive seed dispersal (Hofhuis et al., 2016). We, therefore, begin our discussion of plant cells by considering the relevance of our results to their observations.

Cardamine exocarp cells Hofhuis et al. (2016) found that exocarp cells contract and curl with pressure when detached from the fruit (suggesting that the cells are held flat in a state of tension when attached to the fruit). This ‘turgor-driven shrinkage’ was found to be highly dependent on the cell geometry and wall anisotropy — which were simulated using finite element modelling (FEM) of three-dimensional cells. Crucially, the wall anisotropy is required

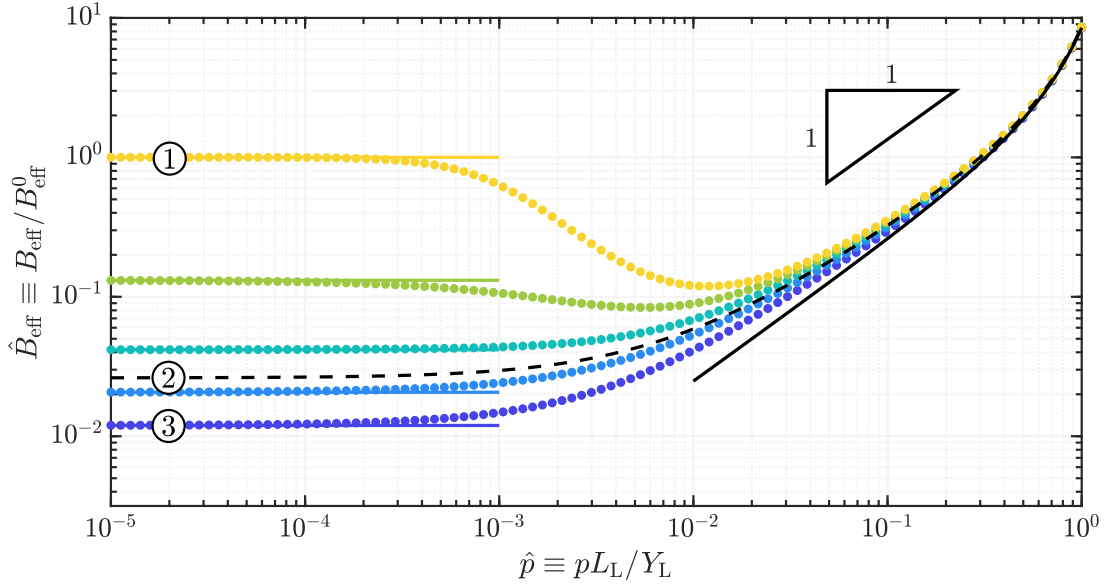


Figure 5.10: Plot of the effective bending modulus, $\hat{B}_{\text{eff}}$, as a function of the internal pressure, $\hat{p}$, for the configuration parameters: $\hat{Y} = 1$, $\hat{\beta} = 10^{-4}$, $\hat{L} = 1/2$, and $\Theta = \pi/2, 3\pi/8, \pi/4, \pi/8$, and 0 (yellow to blue). The $\hat{B}_{\text{eff}}$ found by numerically solving the Kirchhoff equations, (5.27) subject to (5.31a–c) and (5.62), for infinitesimal $\mu \sim \phi \ll 1$, are plotted as coloured dots (●) whilst asymptotic results are plotted as solid curves: (5.67), valid for small pressures, is shown in colour and (5.69), valid for large pressures, is in black. Plotted as a black dashed curve is the solution with maximum area, $\Theta = \Theta_{\text{max}} \approx 0.54$, (5.40). ①–③ correspond to the three regimes sketched in Fig. 5.7. Although B_{eff} initially decreases with $\hat{p}$ for large values of Θ , it ultimately increases with $\hat{p}$ according to (5.69), transitioning from a linear to superlinear dependence on $\hat{p}$.

to ensure that the third dimension (which is not included in our model) is soft in comparison to the longitudinal direction. More quantitatively, Hofhuis et al. measured the wall elastic moduli of a *Cardamine hirsuta* exocarp cell as $E_{\parallel} \approx 9000$ MPa and $E_{\perp} \approx 150$ MPa (parallel and perpendicular to the shrinking axis, respectively), which were found by fitting cellular force microscopy (CFM) data. With these values and a turgor pressure $p \approx 0.65$ MPa, they found a relative cell length $R_{\text{len}} \approx 0.91$ and volume $R_{\text{vol}} \approx 1.53$ from their FEM model, which can be compared to the values $R_{\text{len}} \approx 0.88$ and $R_{\text{vol}} \approx 1.53$ observed in experiments. In Fig. 5.11, the relative cell length, $\hat{w}_0$, and area, $\hat{A}_0$, from numerical solutions of our model are plotted as functions of the internal pressure using the typical cell parameters for a *Cardamine hirsuta* exocarp cell from Hofhuis et al. (2016). At an imposed pressure of $p/E_L = 0.65/9000 \approx 7.2 \times 10^{-5}$ (the same value as that used in the FEM modelling of Hofhuis et al., 2016), we obtain $R_{\text{len}} = \hat{w}_0 \approx 0.83$ and $R_{\text{vol}} = \hat{A}_0 \approx 1.53$; which are remarkably similar to the values obtained by the more sophisticated FEM simulations³. It

³Note that $\hat{A}_0$ is actually the relative internal area in our two-dimensional model, not the relative cell volume, found in Hofhuis et al. (2016). We speculate that the fact that $R_{\text{vol}} = 1.53$ (to three significant

should be noted that a slightly pressurized cell was used as a reference state in Hofhuis et al. (2016), to reflect the observations of actual plant cells. This is in comparison to the truly unpressurized state ($p = 0$), which we have used here. Using a pressurized cell has the effect of slightly increasing R_{len} and slightly decreasing R_{vol} (see Fig. 5.11), but without more details of the slight pressurization used by Hofhuis et al., we cannot express this in our results.

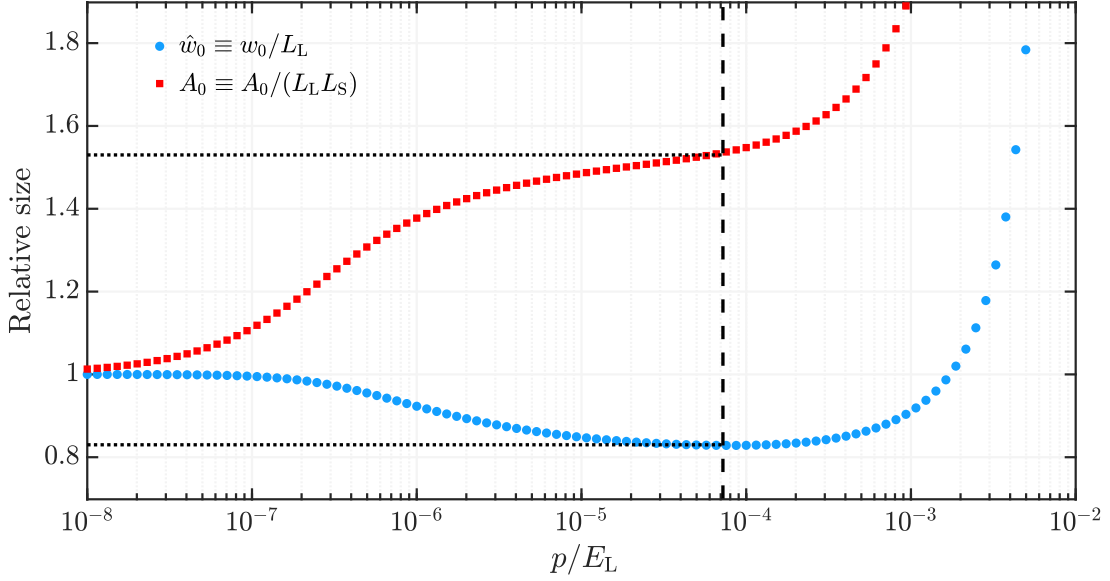


Figure 5.11: Plots of the relative cell length, $\hat{w}_0 \equiv w_0/L_L$, (blue dots, ●) and cell area, $\hat{A}_0 \equiv A_0/(L_L L_S)$, (red squares, ■) as functions of the internal pressure, p/E_L , computed numerically using the two-dimensional cell model presented here, with typical values for a *Cardamine hirsuta* exocarp cell (Hofhuis et al., 2016): $h_L = h_S = 0.2 \mu\text{m}$, $L_L = 50 \mu\text{m}$, $L_S = 20 \mu\text{m}$, $E_S = E_L$, and $\Theta = \pi/2$ (where the clamping angle is chosen to reflect the unpressurized cuboidal cells assumed in the FEM simulations of Hofhuis et al., 2016). Plotted as a dashed vertical line is $p/E_L = 0.65/9000 \approx 7.2 \times 10^{-5}$, which is the value for the turgid cell measured by Hofhuis et al. (2016), and gives the relative sizes $\hat{w}_0 \approx 0.83$ and $\hat{A}_0 \approx 1.53$, comparable to the numerical and experimental values reported by Hofhuis et al. (2016).

Inflatable structures A similar, pressure-driven, cell shrinkage has also been used by Siéfert et al. (2020) to create bio-inspired, programmable shells. The structures created by Siéfert et al. are made by heat-sealing superimposed fabric sheets to create a network of channels. (These channels are comparable to our cellular structures when $\Theta = \pi/2$ — *i.e.* the rectangular cells sketched in Fig. 5.7① — but with side walls of length $L_S = 2h_L \ll 2L_L$.) The channels are then inflated, which shrink the cells (since $\Theta = \pi/2 > \Theta_{\text{max}}$, Fig. 5.7(a)) and, by the anisotropy of the patterns, induces a Gaussian curvature, thus creating a three-

figures) in all three cases is coincidental, and it is *not* the case that $\hat{A}_0$ is very robust to changes in parameters, *e.g.* pressure (as shown in Fig. 5.11).

dimensional structure (Fig. 5.1). Unlike our cells, the seam lines of these programmable shells are thin and wide, so they act as soft hinges when bending; thus, the bending rigidity of the cellular array is determined by the side walls instead of the pressurized cells, that is $B_{\text{eff}} \approx E_S L_S^3/12 \equiv E_S (2h_L)^3/12$, which is significantly smaller than the values found in §5.3.4.

In theory, if one could impose clamping angles other than $\Theta = \pi/2$, one could impose an anisotropy in the programmable shells via the clamping angles, instead of the channel patterns. The cell-expansion or shrinkage (for $\Theta < \Theta_{\text{max}}$ and $\Theta > \Theta_{\text{max}}$, respectively) could then be used to control the induced Gaussian curvature. In fact, one could also introduce an up–down asymmetry in *e.g.* the clamping angles to impose an initial curvature other than flat — this is the subject of the discussion in §5.5.

5.4.2 Length limits on monolayer leaves

The work in this chapter was partially motivated by the leaves of mosses, which are often a single-cell thick (see Fig. 5.2). Since such leaves have no vasculature to stiffen them, a natural question is whether the length of such a leaf is limited by the ability of turgor pressure to stiffen it against gravity. A horizontally clamped beam/plate (of bending modulus B and linear/area density ϱ) bends under the influence of its weight when its length is comparable to $\ell_{\text{bend}} := (B/\varrho g)^{1/3}$, with g the gravitational acceleration (Wang, 1986). If the length of the beam is much larger than ℓ_{bend} , then it is unable to support its own weight and will sag significantly. This sagging can have two detrimental consequences for a plant leaf: (i) it decreases the incident sunlight that may be captured and (ii) it may lead to large internal strains, which may cause structural failure and damage. Previously, similar results for a *heavy elastica* have been used to compute the branch shape with maximum reach (Wei et al., 2012) and the leaf design to maximize the light intercepted for photosynthesis (Tadrist and Darbois-Textier, 2016). By taking $B \equiv B_{\text{eff}}$ and $\varrho \equiv \rho \ell_{\text{thick}}$, where ρ is the bulk density of the tissue and ℓ_{thick} is the macroscopic thickness, we can compute the typical length scales over which our cellular structures bend and strain, ℓ_{bend} and ℓ_{strain} , and compare these to values seen in nature — *e.g.* the leaf length and width, ℓ_{width} and ℓ_{length} , as sketched in Fig. 5.12. (Note that the length scale over which stretching is observed, $\ell_{\text{stretch}} := Y_{\text{eff}}/(\rho g \ell_{\text{thick}})$, is typically orders of magnitude larger than the bending length, so is not considered here.)

The change in projected area is, for small deflections, expected to be quadratic in deflection, while the differential strain caused by bending has size $\mathcal{E} \sim \ell_{\text{thick}} \kappa$, where κ is the curvature of the beam. To estimate the typical deflection, $\mathcal{D}$, and curvature, κ , we note that the small deflection solution of the heavy elastica (*i.e.* the Euler–Bernoulli beam, Howell

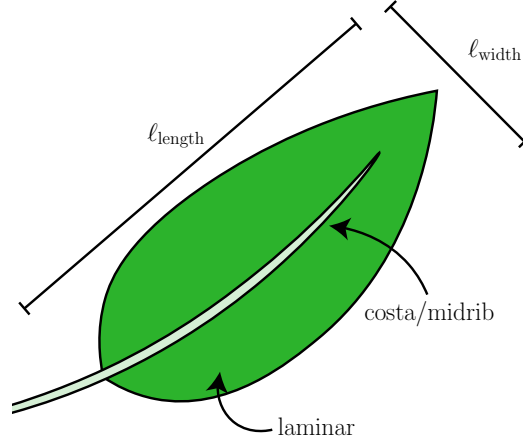


Figure 5.12: Sketch of a leaf showing the macroscale lengths ℓ_{width} and ℓ_{length} . The lamina thickness is denoted ℓ_{thick} .

et al., 2008) is

$$w(x) = \frac{1}{24\ell_{\text{bend}}^3} x^2 (6L^2 - 4Lx + x^2), \quad (5.70)$$

for beam arclength $0 \leq x \leq L$. We, therefore, estimate the deflection

$$\mathcal{D} := \frac{w(L)}{L} = \frac{L^3}{8\ell_{\text{bend}}^3}, \quad (5.71a)$$

and strain

$$\mathcal{E} := \frac{\ell_{\text{thick}}}{2} w''(L) = \frac{\ell_{\text{thick}} L^2}{4\ell_{\text{bend}}^3}. \quad (5.71b)$$

In conclusion, sufficiently small leaves (L small in comparison to both the bending length, ℓ_{bend} , and the strain length, $\ell_{\text{strain}} := \ell_{\text{bend}}^{3/2}/\ell_{\text{thick}}^{1/2}$) should be able to extend without detrimental sagging.

In Fig. 5.13, the leaf widths, ℓ_{width} , (as sketched in Fig. 5.12) of nine different species of single-cell thick moss are plotted against the bending and strain lengths, $\ell_{\text{bend}} \equiv [B_{\text{eff}}/(\rho g \ell_{\text{thick}})]^{1/3}$ and $\ell_{\text{strain}} \equiv [B_{\text{eff}}/(\rho g \ell_{\text{thick}}^2)]^{1/2}$, computed for our model. Here, we use the geometric data for the moss cells and leaves from Waite and Sack (2010), with ℓ_{thick} the macroscopic thickness of the leaf, turgor pressures in the range $0.8 \text{ MPa} \leq p \leq 2.1 \text{ MPa}$ (to reflect the typical values found in moss, Proctor et al., 2007), and Young's modulus $E_S = E_L = 100 \text{ MPa}$ (a lower bound on the typically accepted values for plant cells, Cosgrove, 2016). As is expected, the length of lamina extended horizontally from the leaf midrib is less than the bending length, $\ell_{\text{width}}/2 < \ell_{\text{bend}}$, and the strain length, $\ell_{\text{width}}/2 < \ell_{\text{strain}}$: leaves are not expected to sag under their own weight. In fact, using the small deflection solution, (5.71) with $L = \ell_{\text{width}}/2$, the expected deflection of the moss leaf is less than 1%, i.e. $[(\ell_{\text{width}}/2)/\ell_{\text{bend}}]^3 < 8 \times 10^{-2}$, and the maximal strain caused by bending is less than

0.1%, *i.e.* $[(\ell_{\text{width}}/2)/\ell_{\text{strain}}]^2 < 4 \times 10^{-3}$.

In Fig. 5.14, the leaf widths, ℓ_{width} , of the same nine different species of moss as plotted in Fig. 5.13, are plotted against the bending and strain lengths, ℓ_{bend} and ℓ_{strain} , computed for our model with zero internal pressure ($p = 0$). Here, all the bending and straining lengths do not lie below the $\mathcal{D} = 1\%$ and $\mathcal{E} = 0.1\%$ lines, respectively, suggesting that an unpressurized leaf may sag under gravity and, thus, turgor maybe required for the leaf to support its own weight.

For the moss species plotted in Figs. 5.13 and 5.14, the leaf length, ℓ_{length} , (as sketched in Fig. 5.12) is typically 3–17 times larger than its width, ℓ_{width} , (Waite and Sack, 2010); seemingly, the leaf would be expected to sag under its own weight since $\ell_{\text{length}} > \ell_{\text{bend}}$. However, the leaf costa/midrib (a multi-cell thick structure which lies along the centre of the leaf, Waite and Sack, 2010, see sketch in Fig. 5.12) provides additional longitudinal support.

Further support might also come from the lateral curvature of a leaf: dehydrated leaves often curl laterally to reduce the total exposed tissue and, hence, reduce the rate of evaporation, preventing further water loss (see Charron and Quatrano, 2009; Hu et al., 2016, for example). Such a curvature can be seen in the cut moss leaves in Fig. 5.2 (with additional photographs available in Waite and Sack, 2010). (Since these cells have been dissected, they lack turgidity; the curvature visible must be intrinsic to the cellular geometry and not an artefact of the pressure.) This intrinsic curvature can only be changed by planar stretching (which, in general, acts over a larger length scale than bending, $\ell_{\text{stretch}} > \ell_{\text{bend}}$), or via unfurling the leaves (which often comes with an energetic cost) — ‘Gaussian curvature of a surface is invariant under local isometry’, Gauss’s *Theorema Egregium* (see Taffetani et al., 2019, for example). This is implicitly used in curved tape measures to provide large standout values (see Seffen and Pellegrino, 1999; Pini et al., 2016; Taffetani et al., 2019, for example). Does curvature also provide rigidity to thin plant tissue in the absence of turgor? We now turn to consider *how* an intrinsic curvature can appear in the thin cellular structure modelled in §5.3.

5.5 Asymmetric cellular structure

A dehydrated plant leaf often curls in-order to reduce the effects of evaporation and to possibly become more rigid in the absence of turgor (Charron and Quatrano, 2009; Hu et al., 2016). Hydrating the plant unfurls these leaves, exposing more surface to the sun, and increasing the light absorbed in photosynthesis. For the leaf to curl as the pressure varies, there must be an up–down asymmetry in the plant tissue with effects that vary with the turgor pressure. To reproduce such an effect in the one-cell thick model presented in §5.3, an asymmetry between the upper and lower beams is required — *i.e.* an effective macroscale

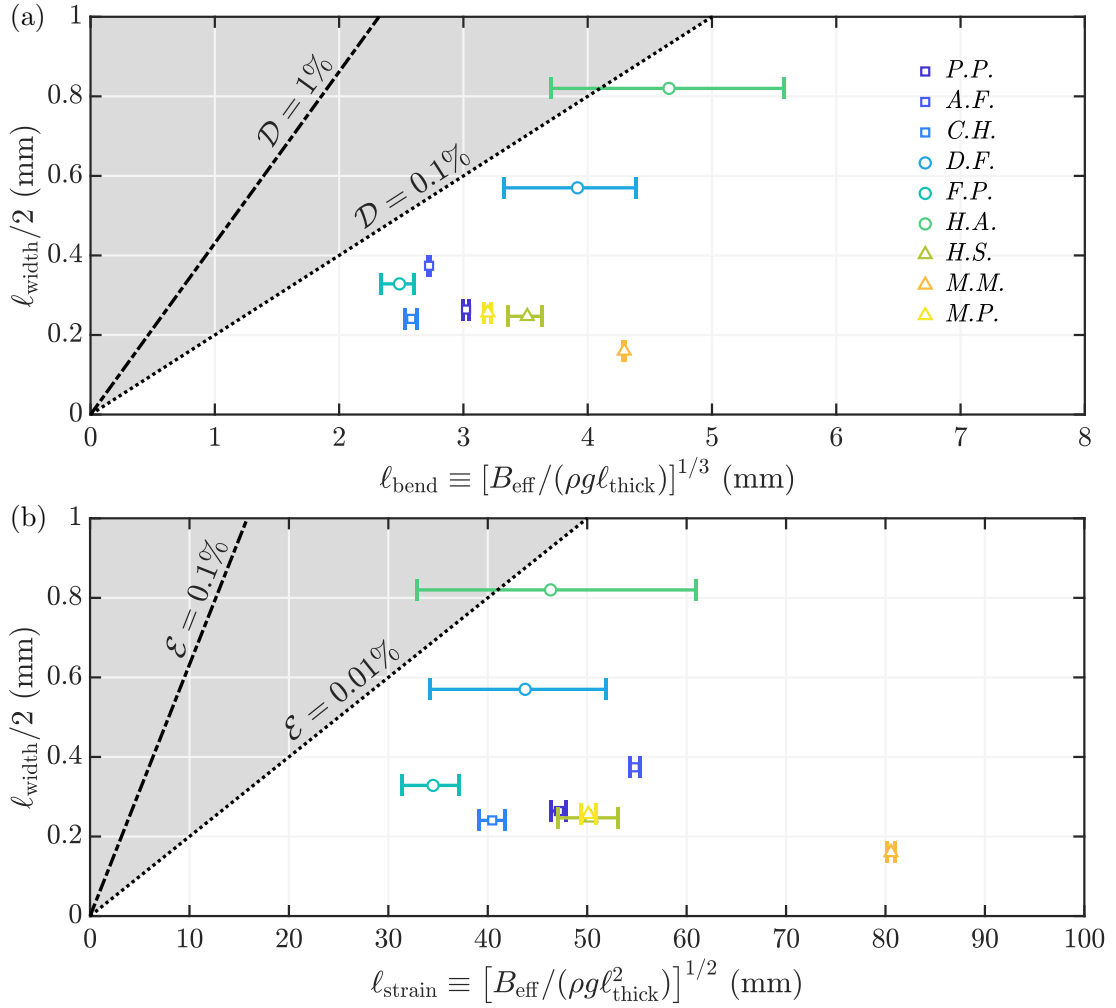


Figure 5.13: Plot of the extended lamina widths, $\ell_{\text{width}}/2$, against estimated (a) bending length, $\ell_{\text{bend}} = [B_{\text{eff}}/(\rho g \ell_{\text{thick}})]^{1/3}$, and (b) strain length, $\ell_{\text{strain}} = [B_{\text{eff}}/(\rho g \ell_{\text{thick}}^2)]^{1/2}$, for nine species of single-cell thick moss, defined by colour (see key) and with point shape denoting the type: trunk-dwelling mosses, $\square$, (*Pyrrhobryum pungens*, *Acroporium fuscoflavum*, *Campylopus hawaiiicus*); ground-dwelling mosses, $\circ$, (*Distichophyllum freycinetii*, *Fissidens pacificus*, *Hookeria acutifolia*); branch-dwelling mosses, $\triangle$, (*Holomitrium seticalycinum*, *Macromitrium microstomum*, *Macromitrium piliferum*). To compute the length scales, we have used the cell width, cell height, interior and surface cell wall thicknesses, lamina thickness, and maximum leaf width from [Waite and Sack \(2010\)](#) to estimate L_L , L_S , h_S , h_L , ℓ_{thick} , and ℓ_{width} , respectively. We further assume a density $\rho = 1000 \text{ kg/m}^3$; clamping angle $\Theta = \pi/4$ (an approximation based on photographs of the cells in [Waite and Sack, 2010](#)); turgor pressures $0.8 \text{ MPa} \leq p \leq 2.1 \text{ MPa}$ (a typical range found in moss, [Proctor et al., 2007](#)), giving the range in ℓ_{bend} and ℓ_{strain} ; and Young's moduli $E_S = E_L = 100 \text{ MPa}$, which are chosen as a lower bound to the typically accepted range in plant cells ([Cosgrove, 2016](#)). In (a), the lines $\ell_{\text{width}}/2 = 2\mathcal{D}^{1/3}\ell_{\text{bend}}$ for $\mathcal{D} = 10^{-2}$ (dash-dotted) and $\mathcal{D} = 10^{-3}$ (dotted), correspond to the deflection, $\mathcal{D}$, of a Euler–Bernoulli beam, (5.71a). Similarly, in (b), the lines $\ell_{\text{width}}/2 = 2\mathcal{E}^{1/2}\ell_{\text{strain}}$ for $\mathcal{E} = 10^{-3}$ (dash-dotted) and $\mathcal{E} = 10^{-4}$ (dotted), correspond to the maximal differential strain, $\mathcal{E}$, caused by bending a Euler–Bernoulli beam, (5.71b). The shaded regions correspond to where appreciable deformation under gravity is expected.

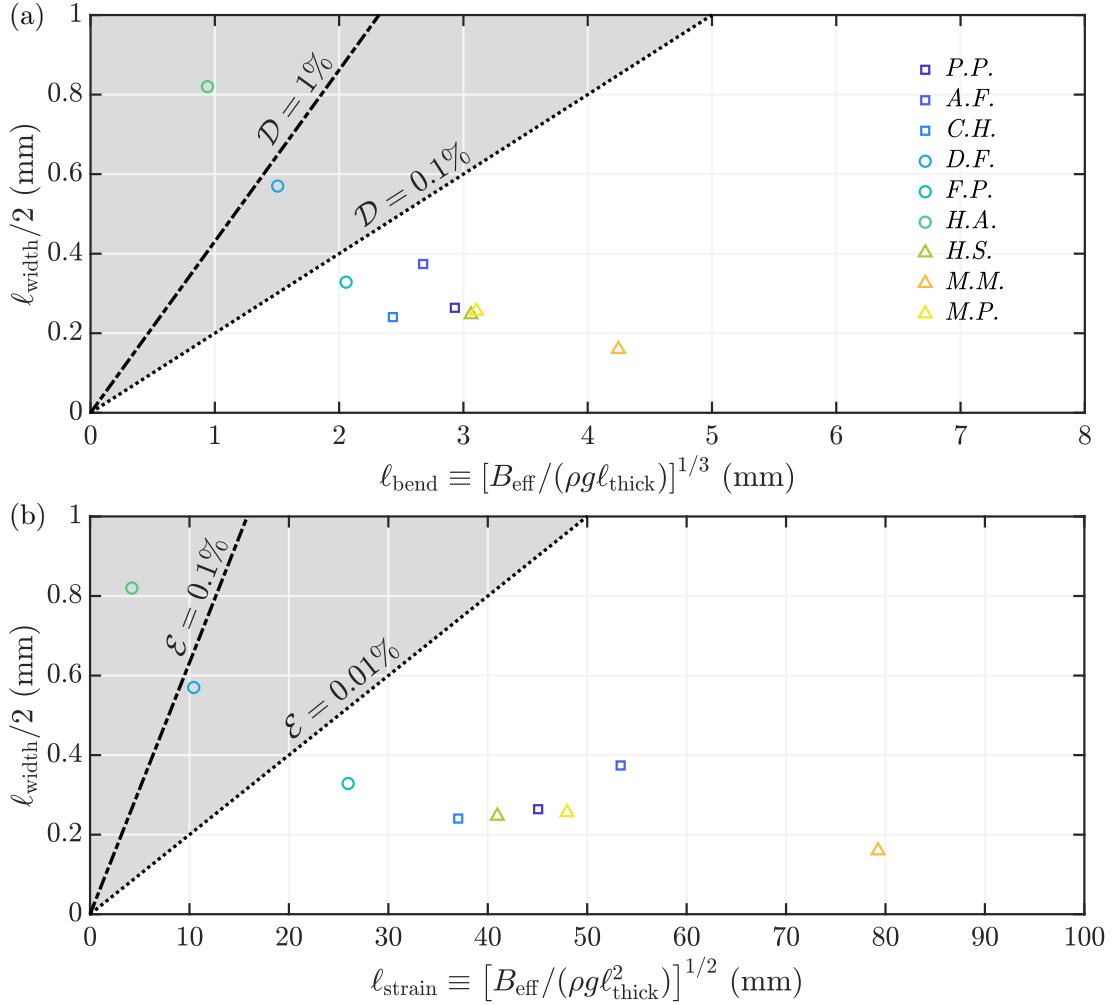


Figure 5.14: Plot of the extended lamina widths, $\ell_{\text{width}}/2$, against estimated (a) bending length, $\ell_{\text{bend}} = [B_{\text{eff}}/(\rho g \ell_{\text{thick}})]^{1/3}$, and (b) strain length, $\ell_{\text{strain}} = [B_{\text{eff}}/(\rho g \ell_{\text{thick}}^2)]^{1/2}$, with zero turgor pressure ($p = 0$), for nine species of single-cell thick moss, defined by colour (see key) and with point shape denoting the type: trunk-dwelling mosses, $\square$, (*Pyrrhobryum pungens*, *Acroporium fuscoflavum*, *Campylopus hawaiiicus*); ground-dwelling mosses, $\circ$, (*Distichophyllum freycinetii*, *Fissidens pacificus*, *Hookeria acutifolia*); branch-dwelling mosses, $\triangle$, (*Holomitrium seticalycinum*, *Macromitrium microstomum*, *Macromitrium piliferum*). To compute the length scales, we have used the same values as in Fig. 5.13 (*i.e.* the data from Waite and Sack, 2010), but with zero internal pressure ($p = 0$). In (a), the lines $\ell_{\text{width}}/2 = 2\mathcal{D}^{1/3}\ell_{\text{bend}}$ for $\mathcal{D} = 10^{-2}$ (dash-dotted) and $\mathcal{D} = 10^{-3}$ (dotted), correspond to the deflection, $\mathcal{D}$, of a Euler–Bernoulli beam, (5.71a). Similarly, in (b), the lines $\ell_{\text{width}}/2 = 2\mathcal{E}^{1/2}\ell_{\text{strain}}$ for $\mathcal{E} = 10^{-3}$ (dash-dotted) and $\mathcal{E} = 10^{-4}$ (dotted), correspond to the maximal differential strain, $\mathcal{E}$, caused by bending a Euler–Bernoulli beam, (5.71b). Notably, the length scales for a flaccid leaf ($p = 0$), plotted here, are much smaller than their turgid counterparts, plotted in Fig. 5.13, suggesting that turgor pressure stiffens the leaf against gravity.

curvature, κ_{eff} , must exist with no imposed tension, τ , or moment, μ . However, it is not clear what mechanisms or asymmetries are needed to obtain a curvature at low pressures but not at high pressures.

5.5.1 Equilibrium equations of the asymmetric cell

To introduce an asymmetry into our cell model, we assume that the top and bottom beams differ in one or more of the following properties: relaxed length, L_L ; thickness, h_L ; Young's modulus, E_L ; clamping angle, Θ . The mathematical formulation of such an asymmetric cell is identical to the symmetric case considered in §5.3, but with $\mu = \tau = 0$ (here we can neglect the additional forcing) and $\{E_L, h_L, L_L, \Theta\}$ replaced with $\{E_T, h_T, L_T, \Theta_T\}$ and $\{E_B, h_B, L_B, \Theta_B\}$ for the top and bottom beams, respectively (*i.e.* one has to distinguish between the lateral beams). This setup is sketched in Fig. 5.15.

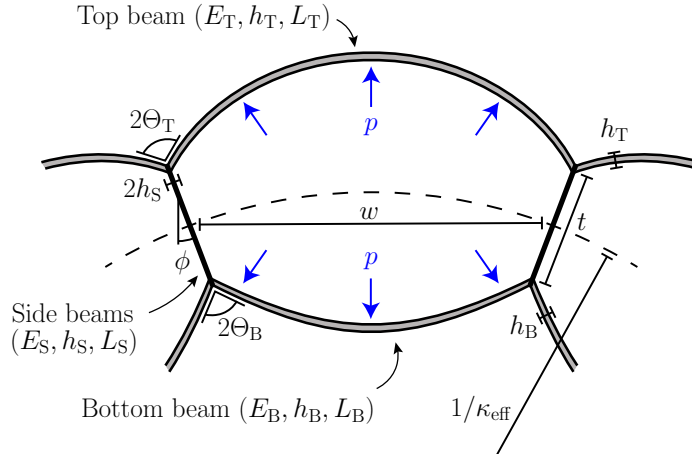


Figure 5.15: Sketch of the two-dimensional, asymmetric, pressurized cell model considered in this section. Each ‘cell’ consists of four distinct beams corresponding to the top, bottom, and two side (or in-contact) walls. The top and bottom beams have differing values of Young’s moduli, E_T and E_B ; thicknesses, h_T and h_B ; relaxed lengths, L_T and L_B ; and clamping angles, Θ_T and Θ_B . These asymmetries induce an effective curvature, κ_{eff} , which varies with the internal pressure, p .

In the dimensionless variables (5.26) (albeit with L_L suitably replaced by L_T or L_B), the intrinsic equation, $\theta(\xi)$, and positions, $(X(\xi), Z(\xi))$, satisfy the system of ODEs (5.27) in $0 \leq \xi \leq 1$, with the axial stretch (5.28), end forces

$$\hat{T}_0 = \frac{\hat{p}}{2} (\hat{L}\hat{t} \cos \phi \mp \hat{w} \tan \phi) \quad \text{and} \quad \hat{N}_0 = \frac{\hat{p}}{2} (\hat{L}\hat{t} \sin \phi \pm \hat{w}), \quad (5.72a,b)$$

and side wall stretch

$$\hat{t} := \frac{t}{L_S} = 1 + \frac{\hat{p}\hat{w}}{2 \cos \phi \hat{Y}}, \quad (5.73)$$

subject to the boundary conditions (5.31). Here, the $\pm$ refers to the top and bottom beams, respectively, and the dimensionless parameters are understood to be either

$$\hat{p} = \hat{p}_T := \frac{pL_T}{E_T h_T}, \quad \hat{Y} = \hat{Y}_T := \frac{E_S h_S}{E_T h_T}, \quad \hat{\beta} = \hat{\beta}_T := \frac{h_T^2}{12L_T^2}, \quad (5.74a-c)$$

$$\hat{L} = \hat{L}_T := \frac{L_S}{L_T}, \quad \hat{w} = \hat{w}_T := \frac{w}{L_T}, \quad \Theta = \Theta_T, \quad (5.74d-f)$$

for the top beam, or

$$\hat{p} = \hat{p}_B := \frac{pL_B}{E_B h_B}, \quad \hat{Y} = \hat{Y}_B := \frac{E_S h_S}{E_B h_B}, \quad \hat{\beta} = \hat{\beta}_B := \frac{h_B^2}{12L_B^2}, \quad (5.75a-c)$$

$$\hat{L} = \hat{L}_B := \frac{L_S}{L_B}, \quad \hat{w} = \hat{w}_B := \frac{w}{L_B}, \quad \Theta = \Theta_B, \quad (5.75d-f)$$

for the bottom beam.

Note that by introducing the parameter ratios,

$$\mathcal{R}_E := \frac{E_B}{E_T}, \quad \mathcal{R}_h := \frac{h_B}{h_T}, \quad \mathcal{R}_L := \frac{L_B}{L_T}, \quad \mathcal{R}_\Theta := \frac{\Theta_B}{\Theta_T}, \quad (5.76a-d)$$

the parameters for the bottom beam, (5.75), can be written in terms of those for the top beam, (5.74). The difference of the ratios, (5.76), from unity measures the asymmetry of the cell. Overall, given values for the top beam parameters, $\hat{p}_T$, $\hat{Y}_T$, $\hat{\beta}_T$, $\hat{L}_T$, and Θ_T , and the asymmetric ratios, $\mathcal{R}_E$, $\mathcal{R}_L$, $\mathcal{R}_h$, and $\mathcal{R}_\Theta$, the system for the top and bottom beams can be solved simultaneously to determine the cell width, $\hat{w}_T \equiv \mathcal{R}_L \hat{w}_B$; cell thickness, $\hat{t}$; and side wall angle, ϕ . We reiterate that the choice of dimensionless variables for the top and bottom beams are different if $L_T \neq L_B$ (*i.e.* $\mathcal{R}_L \neq 1$); thus, extra care must be taken when interpreting the results. In general, the system must be solved numerically; we use the BVP solver `bvp4c` in `MATLAB`. We now turn to consider the macroscopic measure of the asymmetric cells, *i.e.* the effective curvature.

5.5.2 Effective curvature

At a macroscale, individual cells cannot be identified, instead the asymmetry in *e.g.* plant tissue can be observed through an intrinsic curvature. In the case of the asymmetric cell model introduced in §5.5.1, this *effective curvature* is the curvature of the cell midline which, from simple geometrical arguments, can be written as

$$\kappa_{\text{eff}} := \frac{1}{L_T} \times \frac{2 \sin \phi}{\hat{w}_T} \equiv \frac{1}{L_B} \times \frac{2 \sin \phi}{\hat{w}_B}, \quad (5.77)$$

where we associate a positive curvature with a positive angle ϕ for the left side wall (*i.e.* the cell curves downwards, Fig. 5.15). As we show below, the scale of κ_{eff} , and how it varies with $\hat{p}$, is dependent on the type and magnitude of asymmetry chosen; thus, there is no obvious choice of a dimensionless scale for κ_{eff} . Instead, we choose L_S (the relaxed length of the side walls) as a reference length to aid the comparison between the different asymmetries, *i.e.* we define

$$\hat{\kappa}_{\text{eff}} := L_S \kappa_{\text{eff}} \equiv \frac{2\hat{L}_T \sin \phi}{\hat{w}_T} \equiv \frac{2\hat{L}_B \sin \phi}{\hat{w}_B}. \quad (5.78)$$

In general, the effective curvature, $\hat{\kappa}_{\text{eff}}$, must be found using numerical solutions (§5.5.1); however, as was true for the symmetric cell (§5.3.2), progress can be made in the case of large and vanishing pressures, *i.e.* turgid and flaccid cells. We consider these limiting cases now.

5.5.2.1 Flaccid cell ($\hat{p}_T = \hat{p}_B = 0$)

For a flaccid cell with no internal pressure, $\hat{p}_T = \hat{p}_B = 0$, integrating (5.27c) with $\hat{T}_0 = \hat{N}_0 = 0$ and applying the boundary conditions (5.31c,d) results in the intrinsic angles for the top and bottom beams,

$$\theta_T(\xi) = \left(\frac{\pi}{2} - \Theta_T + \phi \right) (1 - 2\xi) \quad \text{and} \quad \theta_B(\xi) = \left(\Theta_B - \frac{\pi}{2} + \phi \right) (1 - 2\xi), \quad (5.79\text{a,b})$$

respectively: the lateral beams form circular arcs with no axial stretching ($\alpha \equiv 1$). A similar result was found for the symmetric cell (§5.3.2.1), however here ϕ does not necessarily vanish and, instead, needs to be determined. The corresponding beam positions may be found by integrating (5.27a,b) and imposing (5.31a,b). From (5.25e), we conclude that the unpressurized, asymmetric cell has width

$$\hat{w}_T \equiv \mathcal{R}_L \hat{w}_B = \frac{\sin(\pi/2 - \Theta_T + \phi)}{\pi - 2\Theta_T + 2\phi} + \mathcal{R}_L \frac{\sin(\Theta_B - \pi/2 + \phi)}{2\Theta_B - \pi + 2\phi}, \quad (5.80)$$

where the side wall angle, ϕ , satisfies

$$\hat{L}_T \sin \phi \equiv \mathcal{R}_L \hat{L}_B \sin \phi = \frac{\sin(\pi/2 - \Theta_T + \phi)}{\pi - 2\Theta_T + 2\phi} - \mathcal{R}_L \frac{\sin(\Theta_B - \pi/2 + \phi)}{2\Theta_B - \pi + 2\phi}. \quad (5.81)$$

These relations can be used to calculate effective curvature, $\hat{\kappa}_{\text{eff}}$, via (5.78). In general, ϕ must be determined numerically from (5.81); however, we note that $2\phi = (1 - \mathcal{R}_L)/\hat{L}_T$ for $\Theta_T = \Theta_B = \pi/2$, in which case the cell resembles an annular sector with midline curvature $\hat{\kappa}_{\text{eff}} = 2(1 - \mathcal{R}_L)/(1 + \mathcal{R}_L)$. It is also interesting to note that $\hat{\kappa}_{\text{eff}}$ is independent of $\mathcal{R}_E$ and $\mathcal{R}_h$: an asymmetry in the Young's modulus or thickness of the lateral beams does *not* induce a curvature in the flaccid cell, which is to be expected because the lateral beams are not

strained for $\hat{p} = 0$, and are only restricted by the clamping angles, Θ_T and Θ_B .

5.5.2.2 Turgid cell ($\hat{\beta}_B \ll \hat{p}_B \ll 1$ and $\hat{\beta}_T \ll \hat{p}_T \ll 1$)

As was the case for the symmetric cell, for large internal pressures but still under the assumption of small axial strains ($\hat{\beta}_B \ll \hat{p}_B \ll 1$ and $\hat{\beta}_T \ll \hat{p}_T \ll 1$), the right hand side of (5.27c) must vanish, and the lateral beams form circular arcs. The top beam has dimensionless radius, $\hat{R}_T := R/L_T$, and sector angle, $\pi - \gamma_T$, given by

$$\hat{R}_T^2 = \frac{\hat{w}_T^2 + (\hat{t}\hat{L}_T \cos \phi)^2}{(2 \cos \phi)^2} \quad \text{and} \quad \gamma_T = 2 \arctan \left(\frac{\hat{L}_T \hat{t} \cos \phi - \hat{w}_T \tan \phi}{\hat{w}_T + \hat{L}_T \hat{t} \sin \phi} \right), \quad (5.82a,b)$$

whilst the bottom beam has dimensionless radius, $\hat{R}_B := R/L_B$, and sector angle, $\pi - \gamma_B$, given by

$$\hat{R}_B^2 = \frac{\hat{w}_B^2 + (\hat{t}\hat{L}_B \cos \phi)^2}{(2 \cos \phi)^2} \quad \text{and} \quad \gamma_B = 2 \arctan \left(\frac{\hat{L}_B \hat{t} \cos \phi + \hat{w}_B \tan \phi}{\hat{w}_B - \hat{L}_B \hat{t} \sin \phi} \right). \quad (5.83a,b)$$

In terms of dimensional variables, the cell is a circle of radius $R := (w^2 \sec^2 \phi + t^2)^{1/2}/2$ with two removed segments (of chord length t) that are oriented to form an angle ϕ with the vertical. To compute ϕ , R , and t , we use the arclength condition (5.43) for the top and bottom beams: we impose

$$1 + \hat{p}_T \hat{R}_T = \hat{R}_T (\pi - \gamma_T) \quad \text{and} \quad 1 + \hat{p}_B \hat{R}_B = \hat{R}_B (\pi - \gamma_B), \quad (5.84a,b)$$

which, when coupled with (5.73), gives a system for ϕ , $\hat{w}_T = \mathcal{R}_L \hat{w}_B$, and $\hat{t}$ that can be solved numerically. The cell curvature, $\hat{\kappa}_{\text{eff}}$, can then be computed by using (5.78).

Overall, the macroscopic curvature of a turgid cell is predominately set by the relaxed length ratio, $\mathcal{R}_L$. Although, at large axial strains ($\hat{p}_T \sim 1$ and $\hat{p}_B \sim 1$), the pressure in (5.73) and (5.84) is non-negligible — the beams stretch — so $\hat{\kappa}_{\text{eff}}$ also depends on the stretching modulus ratio $\mathcal{R}_E \mathcal{R}_h \equiv E_B h_B / (E_T h_T)$. Note that an asymmetry in the clamping angles ($\mathcal{R}_\Theta \neq 1$) does *not* induce a curvature in the turgid cells since the clamping conditions, (5.31d), are only apparent in narrow boundary layers close to the ends of the lateral beams.

5.5.2.3 Moderate internal pressures

At moderate internal pressures, an asymmetry in any of the beam parameters may induce an effective curvature, $\hat{\kappa}_{\text{eff}} \neq 0$. Here, the governing system (as presented in §5.5.1) must be solved numerically. In Fig. 5.16, we plot the curvature of the cell, $\hat{\kappa}_{\text{eff}}$, as a function of the internal pressure for various parameters. It is apparent that the different asymmetries

induce curvatures that differ qualitatively. Below, we explain the results for each type of asymmetry individually.

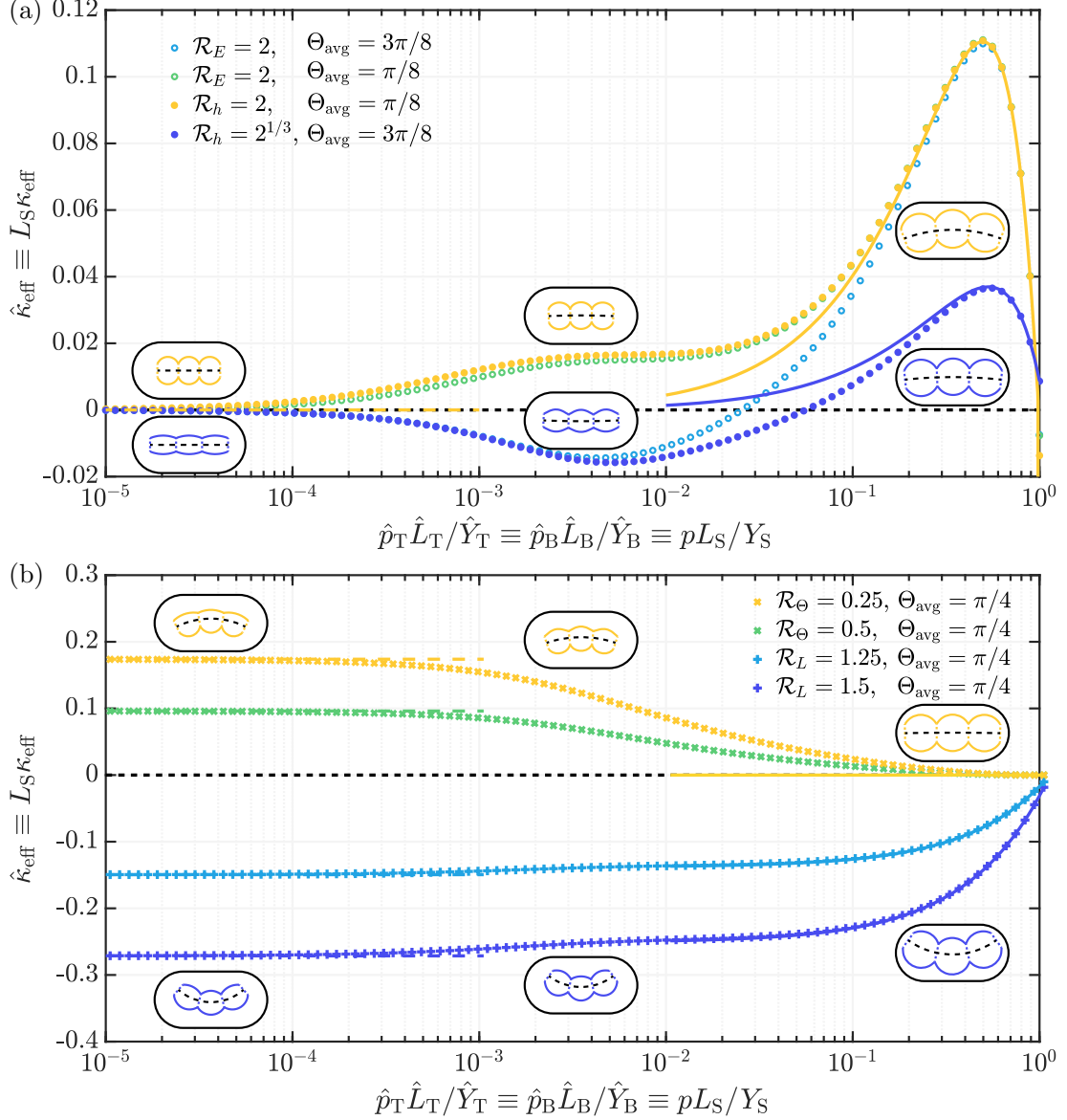


Figure 5.16: Plots of the effective cell curvature, $\hat{\kappa}_{\text{eff}}$, as a function of the internal pressure for an asymmetry in either the Young's modulus ($\circ$), thickness ($\bullet$), relaxed length ($+$), or clamping angle ($\times$) of the lateral beams, where the values of $\mathcal{R}_E$, $\mathcal{R}_h$, $\mathcal{R}_L$, or $\mathcal{R}_\Theta$ are all equal to 1 unless stated in the keys. The parameters are chosen such that their averages recover those used in Fig. 5.7, that is $(1/\hat{Y}_T + 1/\hat{Y}_B)/2 = 1$; $(\hat{\beta}_T + \hat{\beta}_B)/2 = 10^{-4}$; $(1/\hat{L}_T + 1/\hat{L}_B)/2 = 1/2$; and $(\Theta_T + \Theta_B)/2 = \Theta_{\text{avg}}$, where Θ_{avg} is given in the keys. Also plotted are the asymptotic results derived in §5.5.2.1, for small pressures, (as dashed lines) and in §5.5.2.2, for large pressures, (as solid curves), as well as the cell profiles for the asymmetry with the corresponding colour at $\hat{p} = 0$, $10^{-5/2}$, and $10^{-1/2}$ (left to right).

Asymmetry in the Young's modulus ($\mathcal{R}_E \neq 1$) At small internal pressures, the lateral beams are unstrained since the beam ends are free to move horizontally; thus, an asymmetry in the Young's modulus of the lateral beams does not induce a curvature at small pressures (§5.5.2.1). As the pressure increases, however, the beams deform in response to the additional loading. The more compliant beam can bend at lower pressures than the less compliant beam. This results in either the cell curving towards or away from the more compliant beam, depending on whether the bending decreases or increases the width of the cell, respectively — *i.e.* $\Theta < \Theta_{\max}$ or $\Theta > \Theta_{\max}$, for $\Theta_{\max}$ given by (5.40), see discussion in §5.3.2. (In the case of Fig. 5.16(a), $\mathcal{R}_E = 2 > 1$, so the top beam is more compliant; thus, for $\Theta = 3\pi/8 > \Theta_{\max} \approx 0.54$, the cell initially bends upwards and $\hat{\kappa}_{\text{eff}}$ is negative, whilst for $\Theta = \pi/8 < \Theta_{\max} \approx 0.54$, it bends downwards and $\hat{\kappa}_{\text{eff}}$ is positive.) As the pressure increases further, *i.e.* for $\hat{p}_T \gg \hat{\beta}_T$ and $\hat{p}_B \gg \hat{\beta}_B$, the cell resembles a circle (§5.5.2.2), thus the walls cannot bend any more and, instead, stretch under an increase in pressure. Again, the more compliant beam deforms more at lower pressures; this increases its length and bends the cell away from it, regardless of the clamping angle, Θ . (In Fig. 5.16(a), the cell curves downwards at large pressures for both $\Theta = 3\pi/8$ and $\pi/8$.) Finally, for $\hat{p}_T \gtrsim 1$ and $\hat{p}_B \gtrsim 1$, the pressure is sufficiently large that both beams are able to stretch freely, resulting in the cell flattening; however, here, the small axial strain assumption does not hold, and a nonlinear elastic model should be used, which might qualitatively change this behaviour.

Asymmetry in the beam thicknesses ($\mathcal{R}_h \neq 1$) An asymmetry in the lateral beam thickness is analogous to the asymmetry in the Young's modulus, with the induced curvature depending on the pressure in a qualitatively identical way. In fact, taking $\mathcal{R}_E = \mathcal{R}$ and $\mathcal{R}_h = 1$ is the same as taking $\mathcal{R}_E = 1$ and $\mathcal{R}_h = \mathcal{R}^{1/3}$ at small pressures, and as taking $\mathcal{R}_E = 1$ and $\mathcal{R}_h = \mathcal{R}$ at large pressures — as shown in Fig. 5.16(a) for $\mathcal{R} = 2$. (This mapping results from the fact that the beams only bend at small pressures, which is governed by the bending modulus $B \propto Eh^3$, and only stretch at large pressures, which is governed by the stretching modulus $Y \propto Eh$.)

Asymmetry in the beam lengths ($\mathcal{R}_L \neq 1$) If one of the lateral beams is naturally longer than the other, then the cell has a curvature which reflects this, *i.e.* the cell is curved away from the longer beam. (In the case of Fig. 5.16(b), $\mathcal{R}_L > 1$ so $L_B > L_T$ — the bottom beam is longer than the top — and the cell is curved upwards, thus $\hat{\kappa}_{\text{eff}}$ is negative.) As the pressure increases, the curvature smoothly transitions from the results for a flaccid cell, §5.5.2.1, to those for a turgid cell, §5.5.2.2. At very large pressures ($\hat{p}_T \gtrsim 1$ and $\hat{p}_B \gtrsim 1$), the walls are free to stretch, and the cell flattens (though at these pressures the axial strains are large, and a nonlinear elastic model should be used).

Asymmetry in the clamping angles ($\mathcal{R}_\Theta \neq 1$) For small internal pressures, an asymmetry in the clamping angles causes the cell to be curved away from the shallower angled beam — *i.e.* the larger value between Θ_T and Θ_B (§5.5.2.1). (In Fig. 5.16(b), $\mathcal{R}_\Theta < 1$ and so $\Theta_B < \Theta_T$, thus the cell is curved downwards with a positive $\hat{\kappa}_{\text{eff}}$.) As the pressure increases, the effects of clamping lessen. For $\hat{p}_T \equiv \hat{p}_B \gg \hat{\beta}_T \equiv \hat{\beta}_B$ the clamping angles are only apparent in boundary layers of width $\mathcal{O}(\hat{\beta}_T/\hat{p}_T \equiv \hat{\beta}_B/\hat{p}_B)$ close to the beam ends, and the cell is effectively symmetric with zero curvature (§5.5.2.2). Unlike the other asymmetries, this trend towards zero-curvature can occur at small axial strains (where the assumption of linear elasticity still holds), see Fig. 5.16(b).

In reality, multiple asymmetric properties might be combined. These properties might have conflicting effects and so the curvature, $\hat{\kappa}_{\text{eff}}$, might vary with the pressure differently to that discussed. However, the asymptotic results of §5.5.2.1 and §5.5.2.2, for small and large pressures, still hold true and can be used as a first step towards understanding how $\hat{\kappa}_{\text{eff}}$ varies with pressure.

5.6 Discussion: Curvature in plants and other applications

In plant leaves, there is a trade-off between maximizing the tissue exposed to sunlight (to increase the light absorbed for photosynthesis) and minimizing surface area (to decrease the rate of evaporation and prevent water loss). The typical observation is that a dehydrated plant (small turgor pressure) is tightly curved since the preservation of water is paramount in such situations. However, a hydrated plant (large turgor pressure) often has flat leaves in the absence of any forcing (see Charron and Quatrano, 2009; Hu et al., 2016, for examples in moss). In fact, at least in moss, this dehydrated–hydrated response occurs rapidly, with full recovery occurring after only 1–3 days (Hu et al., 2016). Given the pressure dependence of the curvatures induced by the various asymmetries in Fig. 5.16, an asymmetry in the clamping angles ($\mathcal{R}_\Theta \neq 1$) appears to qualitatively reproduce the observed behaviour: a dehydrated plant is curved, whilst a hydrated plant is flat. This suggests that the initial cell geometry (*i.e.* the clamping angles) solely determines the dehydrated curvature of moss, and is independent of the mechanical parameters of the cell walls.

In contrast to a curvature induced by dehydration, turgid plant tissue has also been observed to curl; for example, longitudinal strips removed from *Tulipa* stems spontaneously recurve (Niklas and Paolillo Jr., 1997); detached exocarp cells after explosive seed dispersal in *Cardamine hirsuta* curl away from the fruit (Hofhuis et al., 2016); and epithelial monolayers curl downwards when devoid of a supporting substrate (Fouchard et al., 2020). In all three of these examples it was proposed that the spontaneous curvature was due to an anisotropy

between the outer and inner layers of the cellular structure: [Niklas and Paolillo Jr. \(1997\)](#) suggested that the spontaneous curvature was due to differing mechanical properties across the cell walls (as first suggested by [Hofmeister, 1859](#)); [Hofhuis et al. \(2016\)](#) modelled the curling of the detached exocarp as the release of tension inside a structure made of three different materials: an active soft outer layer, a passive middle, and a stiff inner layer; and [Fouchard et al. \(2020\)](#) proposed that the asymmetry was due to an enrichment of Myosin II in the substrate-supported side. Parallels can, thus, be drawn between these observations and the asymmetric Young's modulus (or equivalently thickness) induced curvature, $\mathcal{R}_E \neq 1$ (or $\mathcal{R}_h \neq 1$), as discussed in §5.5.2. Accordingly, our model provides a tool for the qualitative understanding of the curvature of simple cellular structures; for example, it helps explain why the observed spontaneous curvature decreases with turgor pressure — *i.e.* $\kappa_{\text{eff}} \rightarrow 0$ as $p \rightarrow 0$, Fig. 5.16(a).

In addition to these biological examples, the curving cells may have implications for the design of possible mechanical devices; for example, in the development of inflatable wings ([Cadogan et al., 2004](#)) or the design of programmable inflatable shells ([Siéfert et al., 2020](#)). Asymmetries can be designed within these structures to induce a curvature at large or small pressures, as required; thus, Gaussian curvature can be generated without a spatial anisotropy (*e.g.* the channel design considered by [Siéfert et al., 2020](#)) — the anisotropy instead comes from the microscopic asymmetry of individual cells. Along similar lines, the pressure-dependent curvature induced by asymmetric clamping angles ($\mathcal{R}_\Theta \neq 1$) may be used to create a passive gripper in soft robotics, where the unpressurized device is clenched (*i.e.* curved), whilst the pressurized device is open (*i.e.* flat) — see [Ilievski et al., 2011](#); [Rus and Tolley, 2015](#), for example.

Photographs of the inflation of a 3D printed device based on our cellular structure, with an asymmetry in the clamping angles, are presented in Fig. 5.17. As we expected for an asymmetry in the clamping angles (§5.5), the device flattens as the pressure increases. Quantifying the pressure–curvature relationship seen experimentally is ongoing work.

5.7 Conclusions

In this chapter, we have constructed a two-dimensional model of the single-cell thick structure found in the leaves of certain species of moss and used in inflatable shells. In the case of an up–down symmetric cell, we computed the three primary elastic moduli of the macroscopic structure — the effective stretching, bending, and bulk moduli (Y_{eff} , B_{eff} , and K_{eff} , respectively) — and discussed how they vary with the cell geometry, mechanics, and internal pressure. Moreover, by introducing asymmetries in the Young's moduli, thicknesses, relaxed lengths, and clamping angles of the lateral walls (which have parameter ratios

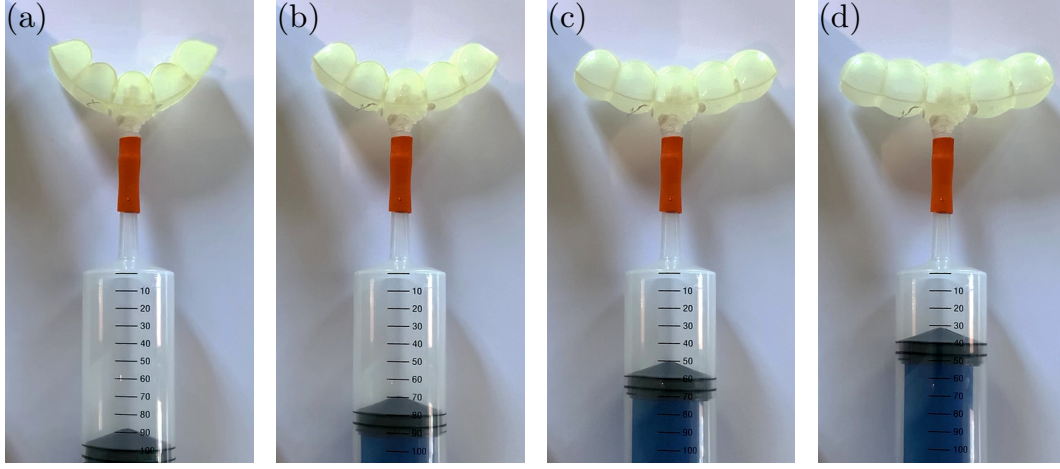


Figure 5.17: The inflation of a 3D printed device based on our asymmetric cellular structure, with the following geometrical parameters: $L_S = 10$ mm, $L_T = L_B = 20$ mm, $h_S = h_T = h_B = 1$ mm, $\Theta_T = 0$, and $\Theta_B = \pi/2$. Here, the structure has been 3D printed using the PolyJet elastomer Agilus30 (Abayazid and Ghajari, 2020) and, to enable inflation, has been printed with a ‘lid’ and ‘base’ in the third dimension (separated by 20 mm) and elliptical holes between neighbouring cells (of width 6 mm and length 12 mm). In (a), the flaccid cell has an intrinsic curvature because of the designed asymmetry in the clamping angles. In (b)–(d), the device is injected with air at increments of 20 ml, which increases the internal pressure and, hence, decreases the effective curvature, κ_{eff} . [I designed this device, Perla Maiolino (Department of Engineering Science, University of Oxford) 3D printed it, and Dominic Vella (Mathematical Institute, University of Oxford) took the photographs.]

$\mathcal{R}_E$, $\mathcal{R}_h$, $\mathcal{R}_L$, and $\mathcal{R}_\Theta$, respectively) we were able to compute the macroscopic intrinsic curvature, κ_{eff} , as a function of the internal pressure. These calculations give us enough information about the behaviour of the microstructure to concentrate on problems involving the homogenized macrostructure, which is suitably parametrized by the computed effective parameters.

In a macro-framework our model provides a tool for understanding the qualitative behaviour of thin cellular structures. We have illustrated this through computing the expected bending and strain lengths of typical moss leaves (which were shown to be related to the observed leaf widths, Fig. 5.13), determining when our structure shrinks with internal pressure (and how it compares to measured values in *Cardamine* exocarp cells in explosive seed dispersal, Fig. 5.11), and inferring the cellular asymmetries required to obtain the qualitative pressure-driven curvatures seen in dehydrated plants and restrained growth (*i.e.* a clamping angle and elastic moduli asymmetry, respectively).

This basic structure may also provide inspiration for possible mechanical devices. For example, the pressure-dependent curvature induced by asymmetric clamping angles ($\mathcal{R}_\Theta \neq 1$) may be used to create a passive gripper in soft robotics, where the unpressurized device is

clenched (*i.e.* curved), whilst the pressurized device is open (*i.e.* flat) — an example device, based on our work, was presented in Fig. 5.17. Alternatively, the asymmetries could be used inside inflatable shells (akin to the programmable shells created by Siéfert et al., 2020) to provide an anisotropy, which induces Gaussian curvature when inflated. We leave such applications to future work.

One possible avenue for extending the model is the addition of a second layer of cells to create a bi-layer or, further, multi-layer structure. It is not obvious what the quantitative consequences of this layering will be; although, a multi-layered structure is expected to be stiffer than a single-layered structure of the same thickness, as there are extra supporting walls. Additionally, one would be able to introduce an asymmetry between the layers, which might be more characteristic of what is seen in practice (*e.g.* Hofhuis et al., 2016, considered a tri-layer when modelling explosive seed dispersal). Another possible extension is to model the midrib/costa of a leaf: the midrib is believed to provide the greatest structural support for the longitudinal extension of a leaf and can appear as a cusp to the attached leaf laminae (Waite and Sack, 2010). Modelling the midrib is, thus, crucial for understanding the reach of a leaf.

Appendix 5.A Energy formulation of the effective moduli

In this appendix, we present the energy formulation of the problem to determine the effective moduli of the cellular model considered in this chapter. We show that the resulting values are identical to those obtained via the force/torque formulation (§5.3.3 and §5.3.4) at infinitesimal strains.

5.A.1 Energy formulation

For a given configuration (*i.e.* a given cell width, w ; thickness, t ; and side wall angle, ϕ), the total energy of the cell, U_{tot} , is found by combining the stored elastic energy of the top, bottom, and two side beams, U_{top} , U_{bot} , and U_{side} , with the work done by the interior pressure, $-pA$. That is,

$$U_{\text{tot}} := U_{\text{top}} + U_{\text{bot}} + 2U_{\text{side}} - pA, \quad (5.A.1)$$

where the elastic energies of the beams satisfy

$$U := \int \psi(\theta' - \kappa_0, \alpha) \, dS, \quad (5.A.2)$$

for the strain energy density function

$$\psi(\theta'(S) - \kappa_0, \alpha(S)) := \frac{B}{2} (\theta'(S) - \kappa_0)^2 + \frac{Y}{2} (\alpha(S) - 1)^2, \quad (5.A.3)$$

with B and Y the bending and stretching moduli of the beam, respectively, and the enclosed area

$$A := wt \cos \phi + \int_0^{L_L} z_T \frac{dx_T}{dS} - z_B \frac{dx_B}{dS} dS, \quad (5.A.4)$$

where (x_T, z_T) and (x_B, z_B) refer to the shapes of the top and bottom beams, respectively.

The side walls are held flat at a tension $T = Y_S(\alpha - 1)$ given by (5.13), so that the elastic energy of a side wall is

$$U_{\text{side}} = \frac{L_S}{Y_S} \frac{T^2}{2} \equiv \frac{L_S}{2E_S h_S} \left[\left(\frac{pw}{2 \cos \phi} - \frac{\mu \tan \phi}{t} \right)^2 + \frac{\tau^2 \sin \phi}{12} \right]. \quad (5.A.5)$$

The energies of the top and bottom walls are found by solving the equilibrium equations (5.17), (5.23), and (5.24) for the beam profile, $(x(S), z(S))$; angle, $\theta(S)$; and axial stretch $\alpha(S)$, and substituting into (5.A.2).

In this energy formulation, the effective bending and stretching moduli are defined as the coefficients of the quadratic terms (ϵ^2 and κ^2) in the change in stored energy associated with a midline strain, ϵ , and midline curvature, κ — *i.e.* (5.3). That is, we compute the difference in U_{tot} between the equilibrium setup of §5.3.2 and the strained setups of §5.3.3 (for $\epsilon \neq 0$ and $\kappa = 0$) or §5.3.4 (for $\epsilon = 0$ and $\kappa \neq 0$) and set it equal to the effective energy,

$$U_{\text{eff}} = w \left(B_{\text{eff}} \frac{\kappa^2}{2} + Y_{\text{eff}} \frac{\epsilon^2}{2} \right), \quad (5.A.6)$$

where w is the width of the cell. In §5.A.2 and §5.A.3, we show that this formulation of the effective elastic moduli is identical to the force/torque method used in the main text as $\epsilon, \kappa \rightarrow 0$. For moderate ϵ and κ , the effective elastic moduli obtained using the energetic and force/torque methods differ because the inhomogeneity of the cellular structure is more apparent, and it cannot be identified with a homogeneous beam with energy (5.A.6). We now turn to non-dimensionalize the above equations.

5.A.1.1 Non-dimensionalization

Using the dimensionless variables (5.26), we define the dimensionless total energy as

$$\hat{U} := \frac{U_{\text{tot}}}{L_L E_L h_L} = \int_0^1 \Psi_T d\xi + \int_0^1 \Psi_B d\xi + \frac{\hat{L}}{\hat{Y}} \left(\frac{\hat{p}\hat{w}}{2 \cos \phi} - \frac{\hat{\mu} \tan \phi}{\hat{t}} \right)^2 - \hat{p} \hat{L} \hat{w} \hat{t} \cos \phi, \quad (5.A.7)$$

where the dimensionless energy density for the lateral beams is

$$\Psi(\xi) := \frac{\hat{\beta}}{2} \left(\theta'(\xi) - \hat{\kappa}_0 \right)^2 + \frac{1}{2} (\alpha(\xi) - 1)^2 \mp \hat{p} Z(\xi) X'(\xi), \quad (5.A.8)$$

$\hat{\kappa}_0 := L_L \kappa_0$ is the dimensionless intrinsic curvature of the beams, and $\mp$ refers to the top and bottom cases, Ψ_T and Ψ_B , respectively. Here, we have dropped the term proportional to $\tau^2 \sin \phi$ in (5.A.5), without loss of generality, as we consider either the case $\hat{\mu} = \phi = 0$ or $\hat{\tau} = \epsilon = 0$. The effective moduli are determined by setting $\hat{U}$ equal to the effective energy,

$$\hat{U}_{\text{eff}} := \frac{U_{\text{eff}}}{L_L E_L h_L} = \hat{w} \left(\hat{B}_{\text{eff}} \frac{(\hat{L} \sin \phi)^2}{\hat{w}^2} + \hat{Y}_{\text{eff}} \epsilon^2 \right), \quad (5.A.9)$$

where we have used the identity $\kappa \equiv 2 \sin \phi / w$, (5.60), and the definitions (5.55) and (5.65).

5.A.2 Effective stretching modulus at infinitesimal strains

For the effective stretching modulus, we consider the setup considered in §5.3.3 and set $\hat{\mu} = \phi = 0$. The intrinsic equation, $\theta(\xi)$; beam shape, $(X(\xi), Z(\xi))$; and axial stretch, $\alpha(\xi)$, of the lateral beams are found by solving the system of ODEs (5.27) in $0 \leq \xi \leq 1$, with the axial stretch (5.28), end forces (5.50), and side wall stretch (5.51), subject to (5.31a–c) and (5.52). As $\epsilon \rightarrow 0$, we pose an expansion in powers of ϵ :

$$\theta = \theta_0 + \theta_1 \epsilon + \theta_2 \epsilon^2 + \mathcal{O}(\epsilon^3), \quad \alpha = \alpha_0 + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + \mathcal{O}(\epsilon^3), \quad (5.A.10a,b)$$

$$X = X_0 + X_1 \epsilon + X_2 \epsilon^2 + \mathcal{O}(\epsilon^3), \quad Z = Z_0 + Z_1 \epsilon + Z_2 \epsilon^2 + \mathcal{O}(\epsilon^3), \quad (5.A.10c,d)$$

$$\hat{T}_0 = \hat{T}_{00} + \hat{T}_{01} \epsilon + \hat{T}_{02} \epsilon^2 + \mathcal{O}(\epsilon^3), \quad \hat{N}_0 = \hat{N}_{00} + \hat{N}_{01} \epsilon + \hat{N}_{02} \epsilon^2 + \mathcal{O}(\epsilon^3), \quad (5.A.10e,f)$$

$$\hat{t} = \hat{t}_0 + \hat{t}_1 \epsilon + \hat{t}_2 \epsilon^2 + \mathcal{O}(\epsilon^3), \quad \hat{\tau} = \hat{\tau}_1 \epsilon + \hat{\tau}_2 \epsilon^2 + \mathcal{O}(\epsilon^3), \quad (5.A.10g,h)$$

where we assume that $\hat{\tau} = \mathcal{O}(\epsilon)$. We insert these expansions into the governing system and consider the problem order-by-order.

At *leading-order*, $\mathcal{O}(\epsilon^0)$, we recover the zero-forcing equations presented in §5.3.2: θ_0 , X_0 , and Z_0 satisfy the system of ODEs (5.27) in $0 \leq \xi \leq 1$, with the axial stretch, α_0 , given by (5.51); end forces, $\hat{T}_{00}$ and $\hat{N}_{00}$, given by (5.34); and side wall stretch, $\hat{t}_0$, given by (5.35), subject to (5.31a–c) and (5.36). Solving this system also yields the unknown cell width, $\hat{w}_0$.

At *next-order*, $\mathcal{O}(\epsilon^1)$, we obtain a system of ODEs dependent on the previous-order: θ_1 , X_1 , and Z_1 satisfy

$$\hat{\beta} \frac{d\theta_1}{d\xi} = \hat{M}_{01} + \hat{T}_{00} Z_1 + \hat{T}_{01} Z_0 - \hat{N}_{00} X_1 - \hat{N}_{01} X_0 \pm \hat{p} (X_0 X_1 + Z_0 Z_1), \quad (5.A.11a)$$

$$\frac{dX_1}{d\xi} = \alpha_1 \cos \theta_0 - \alpha_0 \theta_1 \sin \theta_0, \quad \text{and} \quad \frac{dZ_1}{d\xi} = \alpha_1 \sin \theta_0 + \alpha_0 \theta_1 \cos \theta_0, \quad (5.A.11b,c)$$

in $0 \leq \xi \leq 1$, with the axial stretch

$$\begin{aligned} \alpha_1(\xi) = & \left(\hat{T}_{01} + \hat{N}_{00}\theta_1 \right) \cos \theta_0 + \left(\hat{N}_{01} - \hat{T}_{00}\theta_1 \right) \sin \theta_1 \\ & \pm \hat{p} \left[(Z_1 - \theta_1 X_0) \cos \theta_0 - (X_1 + \theta_1 Z_0) \sin \theta_0 \right], \end{aligned} \quad (5.A.12)$$

end forces

$$\hat{T}_{01} := \hat{p} \frac{\hat{L}\hat{t}_1}{2} + \frac{\hat{\tau}_1}{2} \quad \text{and} \quad \hat{N}_{01} := \pm \hat{p} \frac{\hat{w}_0}{2}, \quad (5.A.13a,b)$$

and side wall stretch

$$\hat{t}_1 = \frac{\hat{p}\hat{w}_0}{2\hat{Y}}, \quad (5.A.14)$$

subject to (5.31a–c),

$$\theta_1(0) = 0, \quad \text{and} \quad X_1(1/2) = \frac{\hat{w}_0}{2}. \quad (5.A.15a,b)$$

In §5.3.3, this system was used to determine the effective stretching modulus, $\hat{Y}_{\text{eff}}$, at infinitesimal strains ($\hat{\tau} \sim \epsilon \ll 1$), *i.e.* (5.55) or, equivalently,

$$\hat{Y}_{\text{eff}} := \frac{\hat{\tau}_1}{2} + \mathcal{O}(\epsilon). \quad (5.A.16)$$

For the energetic formulation of $\hat{Y}_{\text{eff}}$, however, the next-order is needed to balance the effective energy, (5.A.9). The resulting system at $\mathcal{O}(\epsilon^2)$ has been omitted here for reasons of space and only the final results are shown, which were verified via symbolic computation (*i.e.* MATHEMATICA).

Expanding the energy density, Ψ , and total energy, $\hat{U}$, in powers of ϵ ,

$$\Psi = \Psi_0 + \Psi_1\epsilon + \Psi_2\epsilon^2 + \mathcal{O}(\epsilon^3), \quad \hat{U} = \hat{U}_0 + \hat{U}_1\epsilon + \hat{U}_2\epsilon^2 + \mathcal{O}(\epsilon^3), \quad (5.A.17a,b)$$

we may consider the energy of the system order-by-order, as well.

At *leading-order*, $\mathcal{O}(\epsilon^0)$, solutions to the above system yield the energy density,

$$\Psi_0 = \frac{\hat{\beta}}{2}(\theta'_0 - \hat{\kappa}_0)^2 + \frac{1}{2}(\alpha_0 - 1)^2 \mp \hat{p}Z_0X'_0, \quad (5.A.18)$$

which can be integrated to give the leading-order total energy

$$\hat{U}_0 = \int_0^1 \Psi_{T0} d\xi + \int_0^1 \Psi_{B0} d\xi + \frac{\hat{L}}{\hat{Y}} \left(\frac{\hat{p}\hat{w}_0}{2} \right)^2 - \hat{p}\hat{L}\hat{w}_0\hat{t}_0, \quad (5.A.19)$$

where Ψ_{T0} and Ψ_{B0} refer to Ψ_0 for the top and bottom beams, respectively.

At *next-order*, $\mathcal{O}(\epsilon^1)$, the governing equations, (5.A.11)–(5.A.14), simplify the expression

for the energy density and give

$$\Psi_1 = \frac{d}{d\xi} \left[\hat{\beta}(\theta'_0 - \hat{\kappa}_0)\theta_1 + \hat{T}_{00}X_1 + \hat{N}_{00}Z_1 \mp \hat{p}X_0Z_1 \right]. \quad (5.A.20)$$

Integrating this over the length of the lateral beams and applying the boundary conditions, (5.31a–c) and (5.A.15), yields the lateral beam energy,

$$\int_0^1 \Psi_1 d\xi = \hat{p} \frac{\hat{L}\hat{t}_0\hat{w}_0}{2}, \quad (5.A.21)$$

thus the total energy at $\mathcal{O}(\epsilon)$ vanishes:

$$\hat{U}_1 = \hat{p}\hat{L}\hat{t}_0\hat{w}_0 + \hat{p}^2 \frac{\hat{L}\hat{w}_0^2}{2\hat{Y}} - \hat{p}\hat{L}(\hat{w}_0\hat{t}_0 + \hat{w}_0\hat{t}_1) \equiv 0. \quad (5.A.22)$$

Note that the vanishing of $\hat{U}_1$ was to be expected since the configuration is at equilibrium and there are no net forces.

At next-order, $\mathcal{O}(\epsilon^2)$, we can similarly simplify the expression for the energy density:

$$\begin{aligned} \Psi_2 = \frac{d}{d\xi} \left[\hat{\beta} \left((\theta'_0 - \hat{\kappa}_0)\theta_2 + \frac{1}{2}\theta'_1\theta_1 \right) + \hat{T}_{00}X_2 + \frac{\hat{T}_{01}}{2}X_1 + \hat{N}_{00}Z_2 + \frac{\hat{N}_{01}}{2}Z_1 \right. \\ \left. \mp \hat{p} \left(Z_2X_0 + \frac{1}{2}X_1Z_1 \right) \right], \end{aligned} \quad (5.A.23)$$

which yields the lateral beam energy

$$\int_0^1 \Psi_2 d\xi = \left(\hat{p} \frac{\hat{L}\hat{t}_1}{2} + \frac{\hat{\tau}_1}{2} \right) \frac{\hat{w}_0}{2}, \quad (5.A.24)$$

and total energy

$$\hat{U}_2 = \left(\hat{p} \frac{\hat{L}\hat{t}_1}{2} + \frac{\hat{\tau}_1}{2} \right) \hat{w}_0 + \hat{p}^2 \frac{\hat{L}\hat{w}_0^2}{4\hat{Y}} - \hat{p}\hat{L}\hat{w}_0\hat{t}_1 \equiv \frac{\hat{\tau}_1\hat{w}_0}{2}. \quad (5.A.25)$$

Overall, balancing the change in elastic energy, $\hat{U} - \hat{U}_0 \sim \hat{U}_2\epsilon^2 = \hat{\tau}_1\hat{w}_0\epsilon^2/2$, with the effective energy from (5.A.9), $\hat{U}_{\text{eff}} \sim \hat{w}_0\hat{Y}_{\text{eff}}\epsilon^2$, recovers the effective stretching modulus (5.A.16) — the energy and force formulations of $\hat{Y}_{\text{eff}}$ are equivalent as $\hat{\tau} \sim \epsilon \rightarrow 0$.

5.A.3 Effective bending modulus at infinitesimal strains

For the effective bending modulus, we consider the setup considered in §5.3.3 and set $\hat{\tau} = \epsilon = 0$. The intrinsic equation, $\theta(\xi)$; beam shape, $(X(\xi), Z(\xi))$; and axial stretch, $\alpha(\xi)$,

of the lateral beams are found by solving the system of ODEs (5.27) in $0 \leq \xi \leq 1$, with the axial stretch (5.28), end forces (5.61), and side wall stretch $\hat{t} \sim \hat{t}_0 + \mathcal{O}(\epsilon^2)$, subject to (5.31a–c) and (5.62). As $\phi \rightarrow 0$, we pose an asymptotic expansion in powers of ϕ ; that is, we take (5.A.10a–f) after mapping $\epsilon \mapsto \phi$, with

$$\hat{t} = \hat{t}_0 + \hat{t}_2\phi^2 + \mathcal{O}(\phi^4) \quad \text{and} \quad \hat{\mu} = \hat{\mu}_1\phi + \hat{\mu}_2\phi^2 + \mathcal{O}(\phi^3), \quad (5.A.26a,b)$$

where we have used that $\hat{t}_1 = \hat{w}_1 = 0$ (by symmetry) and $\hat{\mu} = \mathcal{O}(\phi)$. We insert these expansions into the governing system and consider the problem order-by-order.

At *leading-order*, $\mathcal{O}(\phi^0)$, we recover the zero-forcing equations presented in §5.A.2.

At *next-order*, $\mathcal{O}(\phi^1)$, we obtain the system of ODEs (5.A.11) in $0 \leq \xi \leq 1$ and axial stretch (5.A.12), but with the new end forces

$$\hat{T}_{01} := \mp \hat{p} \frac{\hat{w}_0}{2} + \frac{\hat{\mu}_1}{\hat{t}_0} \quad \text{and} \quad \hat{N}_{01} := \hat{p} \frac{\hat{L}\hat{t}_0}{2}, \quad (5.A.27a,b)$$

subject to (5.31a–c),

$$\theta_1(0) = 1, \quad \text{and} \quad X_1(1/2) = \pm \frac{\hat{L}\hat{t}_0}{2}. \quad (5.A.28a,b)$$

Solutions to this system were used in §5.3.4 to compute the effective bending modulus, $\hat{B}_{\text{eff}}$, at infinitesimal curvatures ($\hat{\mu} \sim \phi \ll 1$), and yielded (5.65) or, equivalently,

$$\hat{B}_{\text{eff}} := \frac{\hat{w}_0}{\hat{L}} \hat{\mu}_1 + \mathcal{O}(\phi). \quad (5.A.29)$$

To determine $\hat{B}_{\text{eff}}$ using the energetic formulation, however, the next-order is needed to balance the effective energy, (5.A.9). The resulting system at $\mathcal{O}(\phi^2)$ has been omitted here for reasons of space and only the final results are shown, which were verified via symbolic computations (*i.e.* MATHEMATICA).

Expanding the energy density, Ψ , and total energy, $\hat{U}$, in powers of ϕ — *i.e.* (5.A.17) after mapping $\epsilon \mapsto \phi$ — we may consider the energy order-by-order, as well.

At *leading-order*, $\mathcal{O}(\phi^0)$, we obtain the energy (5.A.19).

At *next-order*, $\mathcal{O}(\phi^1)$, we obtain the energy density (5.A.20). Integrating this over the length of the lateral beams and applying the boundary conditions, (5.31a–c) and (5.A.28), gives the lateral beam energy

$$\int_0^1 \Psi_1 \, d\xi = \pm \hat{p} \frac{\hat{L}^2 \hat{t}_0^2}{2}, \quad (5.A.30)$$

thus the total energy at $\mathcal{O}(\phi)$ vanishes, $\hat{U}_1 = 0$. Again, this was to be expected, as there are no net forces acting on the system.

At *next-order*, $\mathcal{O}(\phi^2)$, we obtain the energy density (5.A.23). Integrating this over the

length of the lateral beams yields

$$\int_0^1 \Psi_2 \, d\xi = \hat{p} \frac{\hat{L}\hat{t}_0}{2} \left(\hat{w}_2 - \frac{\hat{w}_0}{2} \right) + \hat{\mu}_1 \frac{\hat{L}}{2}, \quad (5.A.31)$$

and so the total energy at $\mathcal{O}(\phi^2)$ is $\hat{U}_2 = \hat{L}\hat{\mu}_1$.

Overall, balancing the change in elastic energy, $\hat{U} - \hat{U}_0 \sim \hat{U}_2\phi^2 = \hat{L}\hat{\mu}_1\phi^2$, with the effective energy from (5.A.9), $\hat{U}_{\text{eff}} \sim \hat{L}^2 \hat{B}_{\text{eff}} \phi^2 / \hat{w}_0$, recovers the effective bending modulus (5.A.29) — the energy and torque formulations of $\hat{B}_{\text{eff}}$ are equivalent as $\hat{\mu} \sim \phi \rightarrow 0$.

Epilogue

In this thesis, we considered three distinct problems involving the deformation of a thin elastic material, each motivated by recent experiments: the indentation of suspended two-dimensional solids (Chapters 2 and 3), an alternative to the Winkler model for thin elastic foundations (Chapter 4), and the macroscopic elastic properties of thin, pressurized, cellular structures (Chapter 5). Here, we provide a summary of the main results and discuss possible extensions, which could be considered in future work.

6.1 Summary of the thesis

In Chapters 2 and 3, we considered indentation tests that are commonly performed on truly two-dimensional solids (*e.g.* graphene and molybdenum disulfide) to determine their material properties (such as their stretching modulus, E_{2D} , and pre-tension, T_{pre}). We began in Chapter 2 by demonstrating the possibility for making errors in current fitting protocols, which are based on asymptotic solutions to the force–displacement response. By comparing such fits with a more comprehensive model based on the Föppl–von Kármán equations, we showed that there are three possible sources of error: (i) application of incorrect asymptotic results (for example, Lee et al., 2008 fixed the linear term in (2.19) for a particular experimental setup, but their particular result has been applied more generally); (ii) application of asymptotic results outside their domain of validity (for example, the composite expressions, (2.19)–(2.21), do not recover the correct behaviour at moderate displacements, $d \sim 1$); (iii) unaccounted for effects — certain aspects of the experiments are often neglected to facilitate analytical progress (for example, non-Hookean material behaviour or complex indenter geometry), however these aspects may affect the material response. To address these points, in Chapter 3, we performed a modelling study of indentation tests for cylindrical and spherical-capped indenters assuming linear elasticity (through the Föppl–von Kármán equations). We also presented a model for the indentation of a non-Hookean, hyperelastic material. Through this modelling study, we derived several

asymptotic representations of the force–displacement response, which are controlled by the relative sizes of the dimensionless indentation force, $\mathcal{F}$; bending stiffness, $\mathcal{B}$; indenter radius, $\mathcal{R}$ or $\mathcal{R}_s$ (for a cylinder or sphere, respectively); and pre-tension, $\mathcal{E}$ — as summarized in §3.5. This analysis informed a discussion (§3.4) of several potential pitfalls in fitting procedures that should be appreciated to be avoided.

In Chapter 4, we considered the deformation of a thin elastic layer coating a rigid base by a spatially varying surface pressure, $p(\mathbf{x})$, and shear stress, $\mathbf{T}(\mathbf{x})$. By applying an asymptotic expansion to the elastostatic equations in the small aspect ratio limit, $\epsilon := d/\ell \rightarrow 0$, (analogous to lubrication theory in hydrodynamics, Leal, 2007) we derived an expression for the elastic deformation in terms of $p(\mathbf{x})$, $\mathbf{T}(\mathbf{x})$, and their spatial derivatives, *i.e.* (4.19). In the case of zero surface-shear or zero surface-slip, this expression recovers the Winkler response (Winkler, 1867; Dillard et al., 2018) for $\epsilon^2 \ll (1 - 2\nu)$ and the elastomeric response (Gent and Meinecke, 1970; Dillard, 1989) for $(1 - 2\nu) \ll \epsilon^2$, where ν is the Poisson’s ratio of the elastic layer. This was referred to as the combined foundation since it combines both previously studied limits (and, notably, holds in-between). Thus, whether the foundation response is akin to that of a compressible (Winkler) or of an incompressible (elastomeric) material is not determined by the Poisson’s ratio, ν , alone, but on its closeness to $1/2$ relative to the square of the aspect ratio, ϵ^2 — *i.e.* the foundation is characterized by a parameter $\mathcal{C} \propto (1 - 2\nu)/\epsilon^2$, which we refer to as its *compressibility*. Crucially, the appropriate definition of ϵ (and, hence, $\mathcal{C}$) depends on the problem at hand, which we demonstrated by applying this model to two examples involving a thin elastic layer: the indentation of a soft layer with a stiff coating (§4.3) and the sedimentation of a cylinder in a viscous fluid above a thin elastic layer (§4.4). We went on to consider the effects of a confining lateral boundary, by determining a correction to the combined foundation in the form a boundary layer of width $\mathbf{x} = \mathcal{O}(d)$. Inside of this boundary layer, a modified compressibility parameter, $\mathcal{C}_{\text{BL}} \propto (1 - 2\nu)/\epsilon$, may be more appropriate, depending on the boundary conditions. This corrected model was then applied to a third example: the two-dimensional Hele-Shaw flow bounded by a thin elastic layer (§4.6), for which the solution was shown to deviate from the typically expected linear Poiseuille flow at large fluid fluxes because of the substrate bulging close to the boundary wall.

In Chapter 5, we introduced a two-dimensional single-cell thick structure subject to a variable internal pressure, which was inspired by certain species of moss (Waite and Sack, 2010) and inflatable structures (Siéfert et al., 2020). By introducing an applied tension and torque, we calculated the macroscopic bulk, stretching, and bending moduli of this structure using Kirchhoff’s rod theory for the deformation of the cell walls. These moduli were found to vary with the internal pressure and the cell wall properties, and were used to consider problems at a macroscale. For example, we examined the (counterintuitive)

pressure-driven shrinking of cells, showing that our model leads to shrinkage comparable to detached *Cardamine hirsuta* exocarp cells during explosive seed dispersal (Hofhuis et al., 2016). We also considered whether the width of a moss leaf is limited by its weight. We found that the length scale over which our pressurized cellular structure bends and strains due to gravity is much greater than the extended width of the moss leaves considered by Waite and Sack (2010), suggesting that turgor pressure stiffens the leaf against gravity. Following this, we considered the observation that moss leaves curve when drying (losing turgor) — Charron and Quatrano (2009); Hu et al. (2016). To understand what asymmetries lead to this effect, we introduced an asymmetry in the thickness, relaxed length, clamping angle, and Young’s modulus of the lateral cell walls. We found that, to explain the typical pressure dependence of curling leaves (*i.e.* curled when flaccid and flat when turgid), an asymmetry in the clamping angles of the cell walls is required. This is in contrast to the curling observed in constrained tissue growth (see Niklas and Paolillo Jr., 1997; Hofhuis et al., 2016; Fouchard et al., 2020, for example) for which an asymmetry in the Young’s modulus or wall thickness is more appropriate. Finally, we concluded Chapter 5 by discussing the possible applications of our structure to soft robotics and programmable inflatable devices, presenting a 3D printed model of this structure.

6.2 Future work

As we showed in Chapters 2 and 3, the mechanics of even apparently simple indentation tests on two-dimensional solids is complicated. Consequently, there were certain aspects of our study that were ignored to make mathematical progress and simplify the presentation of possible pitfalls. The most pressing of these is the slipping of material at the sheet boundaries. We assumed that the material was able to perfectly slip over the indenter; in reality, there may be adhesion/friction between the indenter and sheet caused by intermolecular forces. Although numerical simulations suggest that a no-slip indenter (the diametric case to perfect-slip) behaves qualitatively similar to a perfect-slip indenter, the constants in the asymptotic solutions may differ quantitatively at large indentation depths (for example, Vella and Davidovitch, 2017 found that $d/\mathcal{F}^{1/3} \sim q_p(\nu) - [8\mathcal{R}^2/(1+\nu)]^{1/3}$ at large displacements by a no-slip cylindrical indenter, which is different from our result, $d/\mathcal{F}^{1/3} \sim q_p(\nu) - (2\mathcal{R})^{2/3}$). Furthermore, material may be hidden in wrinkles, caused by *e.g.* the initial thermal crumpling (Nicholl et al., 2017), in the in-contact region of the sheet that cannot be ‘ironed-out’ with indentation if the indenter adheres to the material, thus causing a stiffer response. Material slippage might also occur at the outer edge of the sheet (Manzanares-Negro et al., 2020), which can alter the asymptotic results further. In this case, the sheet has access to extra material, causing a softer response and the possible appearance of azimuthal compression,

which can create material wrinkles and dramatically affect the response (Davidovitch and Guinea, 2021). To understand the implications of these aspects fully, a mathematical study, like that presented here, may be required. It may be useful to note that, in general, adhesion on the indenter causes a stiffer response to what we have presented here, while slipping at the rim causes a softer response.

An alternative approach to fitting force–displacement curves with approximate models, is to use machine learning for the inverse problem (Klos et al., 2020). By training a neural network on highly accurate numerical solutions, one could create an efficient tool which is easier to implement than the current fitting protocols and avoids issues such as not being in the appropriate asymptotic regime or the inability to account for certain features. For example, Yang and Yao (2020) used machine learning to characterize the thickness, size, and strength of mechanically exfoliated two-dimensional materials (such as graphene). However, the automatic characterization of the mechanical properties of two-dimensional materials, to our knowledge, remains outstanding. Machine learning is not an easy technique to implement and provides many technical challenges; however, if achieved, it could revolutionize how the properties of materials are determined in the future.

Thermal effects often introduce a natural out-of-plane rippling or crumpling in the membrane, causing our initial assumption of planarity to fail. This is especially the case for graphene flakes (Nicholl et al., 2015). An experimental technique used to ‘iron-out’ these wrinkles, is to pre-pressurize the membrane, stretching out the buckled sheet. Our previous models can easily be adapted to include this pressurization by adding a $p(R_{\text{in}}^2 - r^2)/2$ term to the integrated out-of-plane force balances — (2.5a) for the FvK formulation and (3.16d) for the hyperelastic material formulation — where p is the pressure jump across the membrane. For small displacements, Vella and Davidovitch (2017) showed that the pressurization can be modelled effectively through a modified pressure-dependent pre-tension — for example, for $\nu = 1/3$, one may impose the effective pre-tension

$$T_{\text{eff}} \approx T_{\text{pre}} + 0.44E_{2D}^{1/3}(pR_{\text{out}})^{2/3}. \quad (6.1)$$

For large displacements, however, one cannot simply ignore the geometric effects caused by pressurization. A comprehensive understanding of its effects are still required, especially since it is often the case that the planar results are adapted for pressurized membranes in an *ad hoc* manner (see López-Polín et al., 2017, for example).

The generality of the combined foundation model derived in Chapter 4, makes it a powerful model which can be applied to a wide range of problems. The difficulty, however, is finding an example in which the transition between the compressible and incompressible behaviours — *i.e.* $(1 - 2\nu) \gg \epsilon^2$ and $(1 - 2\nu) \ll \epsilon^2$, respectively — can be observed

experimentally. If this transition could be measured accurately, then the Poisson's ratio of the underlying elastomeric material might be inferred, which is important given the typical uncertainty in their values. For example, polyacrylamide (PAA) and polydimethylsiloxane (PDMS) are commonly used experimentally, but are quoted to have Poisson's ratio anywhere in the region $0.45 \lesssim \nu \lesssim 0.5$ (Boudou et al., 2006; Dogru et al., 2018). Unless the values of $(1 - 2\nu)$ can be determined at a precision comparable to ϵ^2 , quantitative models are likely to be difficult.

One aspect we have not explored when applying the combined foundation, (4.19), is its dependence on a spatially variable surface shear, $\mathbf{T}(\mathbf{x})$. For zero applied pressure, $p(\mathbf{x}) = 0$, the surface displacements, (4.19), may be written as

$$\mathbf{u}(\mathbf{x}, 0) = \frac{d}{G} \mathbf{T}(\mathbf{x}) + \mathcal{O}(\ell \epsilon^3) \quad \text{and} \quad w(\mathbf{x}, 0) = -\beta(\nu) \frac{d^2}{G} \nabla_H \cdot \mathbf{T}(\mathbf{x}) + \mathcal{O}(d \epsilon^3), \quad (6.2a,b)$$

thus it is the gradient of the shear, $\mathbf{T}(\mathbf{x})$, that leads to a vertical surface deflection. However, for a non-zero applied pressure, the dominant behaviour could be any of the three terms in (4.19b) depending on the setup. We leave such applications to future work.

In Chapter 5, we computed the effective macroscale properties of a simple two-dimensional cellular model. These properties allow us to approximate the cellular structure as homogeneous at a macroscale and, in theory, not to worry about the complex microscale structure. It is not obvious, however, how this can be used in problems involving the third dimension. For example, the curvature of a leaf may provide structural rigidity in the absence of turgor, akin to why tape measures/springs are typically curved (see Seffen and Pellegrino, 1999; Pini et al., 2016; Taffetani et al., 2019, for example). One can approximate the bending modulus of a curved tape by calculating its second moment of inertia, I , about its neutral axis (Soykasap, 2007), that is

$$B = EI \equiv Et \frac{R^3}{2} \left(\sin \theta + \theta + 4 \frac{\cos \theta - 1}{\theta} \right), \quad (6.3)$$

where E is the Young's modulus, t is the thickness, R is the radius of curvature, and θ is the sector angle of the tape. (This is in comparison to a flat strip, $\theta \rightarrow 0$ with $w = R\theta$ fixed, for which $B = EI \equiv Ewt^3/12$, Soykasap, 2007.) For the curved cellular structure, replacing Et and R in (6.3) with the effective stretching modulus and intrinsic curvature found in Chapter 5, *i.e.* Y_{eff} and $1/\kappa_{\text{eff}}$, may provide a first approximation of its longitudinal stiffness. Comparing this with the effective bending modulus of the flat structure, *i.e.* B_{eff} , could provide insight into whether leaf curvature brings any structural benefits. However, the accuracy of (6.3) as an approximation for the bending modulus of a curved tape is not clear: how is (6.3) modified for a shallow curved strip?

One could also apply our cellular model to dynamic problems. For example, *Mimosa pudica* leaves are known to curl due to a change in thickness across the leaf hinge (pulvinus), which is caused by the motion of liquid between neighbouring cells (Guo et al., 2015), Fig. 6.1. However, it is not clear how many pairs of cells are required for an effective hinge: with one pair it will take a long time to move enough fluid to induce a big angle change, while many pairs might achieve a given curvature with less water transportation, but requires the ‘message’ to pump water to travel further. Is there an optimum length of hinge?

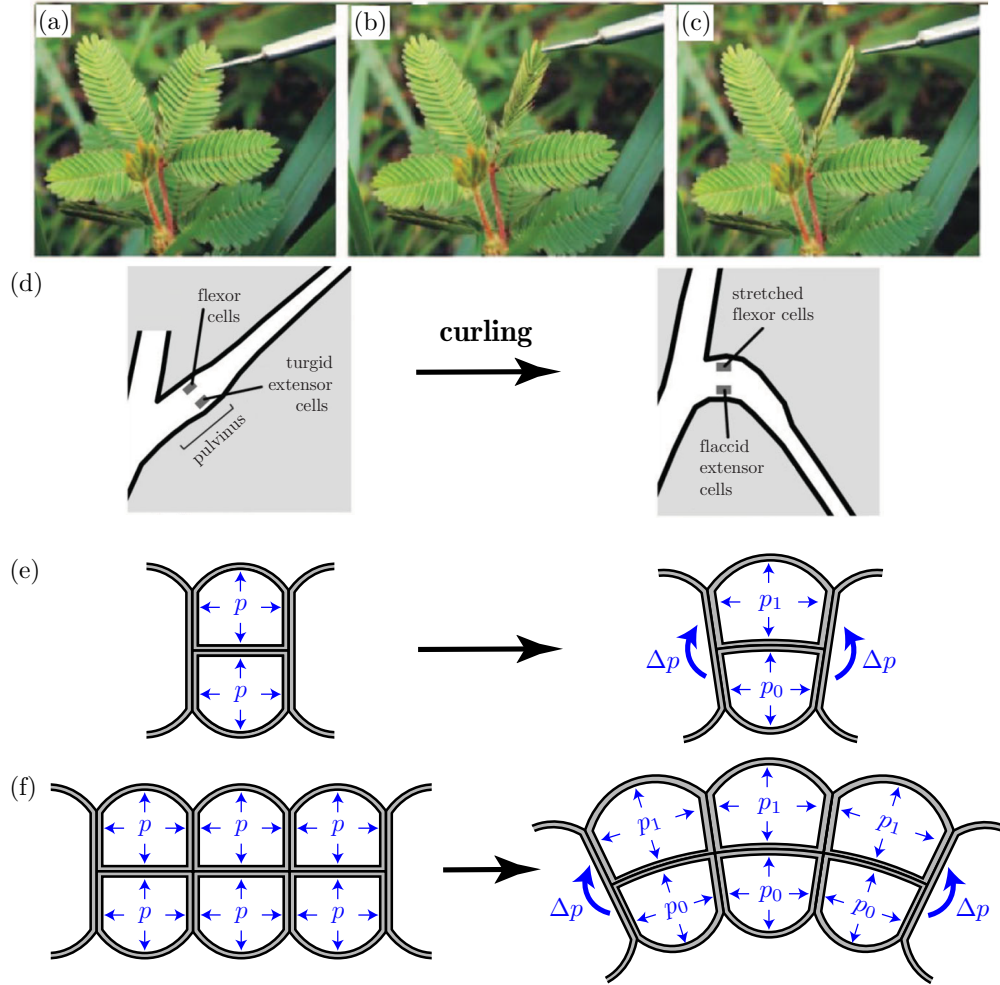


Figure 6.1: Photographs and schematics of the thigmonastic movement of leaflets in *Mimosa pudica*. (a–c) Photographs, at ~ 1 s intervals, of the touch-induced curling of the leaflets. (d) Cartoon of before (left panel) and after (right panel) of the curling within the pulvinus (hinge). (e, f) Sketch of a simple cellular model of the leaflet hinge before (left panels) and after (right panels) curling. The motion of liquid across the hinge inflates the top flexural cells, to a pressure $p_1 = p + \Delta p$, while deflating the bottom extensor cells, to a pressure $p_0 = p - \Delta p$, hence inducing a curvature. In (e) the hinge consists of one pair of cells, whilst in (f) the hinge consists of three pairs. For more pairs, less water transportation is required to achieve a given curvature, but the ‘message’ to pump water is required to travel further. (The images (a–d) are taken from Guo et al., 2015.)

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