Optimal Investment, Valuation and Hedging under Model Ambiguity



Jing Ye

St John's College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Summer 2022

To my parents,
Hong Li & Yushui Ye,
for their continuous love and support.

To my grandmother,
Changying Xu,
for her love and care.

In memory of my grandfather,
Yongchang Li.

Acknowledgements

First, I would like to express my deepest gratitude to my supervisor, mentor and friend, Prof. Michael Monoyios, for his guidance, and for sharing his knowledge and expertise. Without his support and insightful advice, this thesis would not have been possible.

Moreover, I would like to thank Prof. Samuel Cohen and Prof. Hanqing Jin, for their valuable feedback and comments in my Transfer and Confirmation of Status.

I am also grateful to my college advisor, Prof. Jan Obłój, and the maths tutors in St John's, Prof. Charles Batty and Prof. Stuart White, for making St John's a welcoming place for both graduate and undergraduate mathematicians. I would like to thank St John's College as well, for their support, especially during the lockdowns in the Covid-19 pandemic. I thank Prof. Mihail Zervos, my masters academic advisor, for inspiring my interest in research in the field of Financial Mathematics.

My special thanks goes to my dear friends and colleagues, Wolfgang Stockinger, Yuhan Yao, Mingrui Zhou, Christina Zou, with whom I had the pleasure of sharing my happiness and worries throughout my study. I thank my colleagues Jiawei Li, Siyuan Li, Yixuan Wang, Zhenru Wang, Yufei Zhang, for various conversations on interesting maths and on careers in maths. I am grateful for meeting all my friends in maths and in John's. My DPhil life has been filled with joy with your company.

Finally, I am deeply grateful to my parents Hong Li and Yushui Ye, for supporting me spiritually and financially over all these years. I could not have undertaken this journey without their continuous love and support.

Abstract

In this thesis, we study several utility maximisation problems under model uncertainty, involving optimal investment, valuation and hedging.

We first derived martingale distortion representations for classical utility maximisation problems in a non-Markovian stochastic factor model, with power, logarithmic and exponential utilities.

We then study multiple priors power utility maximisation problems when the reference model is a non-Markovian stochastic factor model, and derive a BSDE representation of the value process, optimal strategy and least favourable model.

We also study a variational preferences logarithmic utility maximisation problem when the reference model is a non-Markovian stochastic factor model, and when the plausible models are penalised by an additive entropic penalty function. The robust problem is transferred to a classical utility maximisation problem with a power utility whose risk aversion is dependent on the model ambiguity aversion parameter. We then fully solve the robust problem and perform numerical tests comparing several classical and robust strategies for a Stein-Stein stochastic volatility model and for a Heston model.

Finally, we study robust exponential valuation and hedging problems in a basis risk model, using both the multiple priors approach and the variational preferences approach with a multiplicative penalty function. Specialising to a constant parameter basis risk model, the robust problems admit explicit or approximate solutions, thus we are able to conduct large-scale simulation based tests on performances of various pricing and hedging strategies.

Contents

1 Introduction **1**

1.1 Utility maximisation under model uncertainty 1

1.2 Exponential hedging of basis risk 4

1.3 Main results 5

1.4 Some standing notations 8

2 Martingale distortion method for optimal investment in a stochastic factor model **9**

2.1 A non-Markovian stochastic factor model 10

2.2 Classical power utility maximisation 11

2.2.1 A martingale distortion representation 13

2.2.2 The Markovian case 16

2.3 Classical logarithmic utility maximisation 18

2.4 Classical exponential utility maximisation with random endowment . 19

3 Multiple priors power utility maximisation in non-Markov stochastic volatility models **23**

3.1 Introduction 23

3.2 A martingale distortion representation under any possible model . . . 25

3.3 A BSDE representation of the value process 27

3.4 The Markovian case 29

4 Variational preferences logarithmic utility maximisation in stochastic factor models **32**

4.1 Introduction 32

4.2 The robust value process in terms of the value process of a classical utility maximisation problem 34

4.3 A representation of the value process 36

4.4 The Markovian case 39

4.5 Examples and numerical tests 40
4.5.1 Stein-Stein model 40
4.5.2 Heston model 60
4.5.3 Robustness to model error (or mis-specification of SV models) 62
4.5.4 Summary 71

5 Robust exponential hedging of claims on non-traded assets 72
5.1 Introduction 72
5.2 Multiple priors exponential hedging 76
5.2.1 Martingale distortion solution for each model 78
5.2.2 A BSDE representation of the robust value process 80
5.2.3 The Markovian case 83
5.3 Variational preferences exponential hedging 84
5.3.1 An HJBI PDE for the zero-sum stochastic differential game . 86
5.3.2 Verification 88
5.4 Robust valuation and hedging in the log-normal basis risk model with constant parameters 91
5.4.1 A multiple priors basis risk problem 92
5.4.2 A variational preferences problem 97
5.4.3 The residual risk process 99
5.4.4 Analytic approximations for indifference prices and hedging strategies 101
5.4.4.1 Power series expansions for an indifference pricing function and its derivative 102
5.4.4.2 Analytic approximations for robust indifference prices and hedging strategies 104
5.4.5 Numerical tests 106

6 Conclusions 120

A Classical exponential utility maximisation with random endowment 124

B Additional numerical test results for Chapter 4 127
B.1 Stein-Stein stochastic volatility model ((Section 4.5.1)) 127
B.2 Robustness to model error test (Section 4.5.3) 128

Bibliography 135

List of Figures

4.1 Stein-Stein model (TEST 1): Histograms of terminal utilities for different strategies. Top figure: $\lambda^* = 1.5$, $\beta = 0.5$. With the parameter choices, the histogram for the robust Stein-Stein strategy indeed overlaps with that for the optimal strategy. Bottom figure: $\lambda^* = 0.5$, $\beta = 0.5$. 45

4.2 Stein-Stein model (TEST 1): Parameter values are chosen as in Table 4.1 and $\lambda^* = 1.5$, $\beta = 0.5$. Top figure: example wealth processes for one realisation. Bottom figure: paths of robust strategies $(\widehat{\pi}_t^{\text{robustSS}})_{t\in[0,T]}$ for 200 realisations. In the bottom figure, $\widehat{\pi}^{\text{robustSS}}$ essentially stays at the value $\frac{\lambda^*}{1+\beta} = \lambda = 1.0$ over $t \in [0, T]$, which explains the almost identical wealth paths of the robust SS agent and the optimal agent, as well the overlapping histograms. 47

4.3 Stein-Stein model: Illustrations of average terminal utilities plotted against choices of β. Yellow lines, blue lines and green lines are average terminal utilities of the optimal agent (that is, a classical agent who knows the exact model parameters), the classical agent and the robust agent, respectively. The values of λ^* are chosen with an increment of 0.45 for illustrative purposes. 48

4.4 Stein-Stein model: The optimal choice $\widehat{\beta}$ plotted against λ^* when the parameter λ is over-estimated ($\lambda^* > \lambda$). Optimal values shown on the green line are within a range of 0.286, as a result of discrete increments in β. Purple line is the optimal choice of β if the robust strategy scales the classical strategy by a factor of $\frac{1}{1+\beta}$. 49

4.5 Stein-Stein model (TEST 4): Top figure: dynamically estimated MPRs. Middle figure: true (yellow thick solid line) and estimated volatility paths, with Y^{SV} (blue solid line) the volatility process estimated using Cuchiero–Teichmann algorithm and $\mathrm{vol}^{\mathrm{modMerton}}$ (red solid line) the updated volatility estimated by the modified Merton agent. Bottom figure: strategies (MPR/volatility processes) of the optimal agent (orange), the classical SV agent (blue) and the modified Merton agent (red). 54
4.6 Stein-Stein model (TEST 4): Histograms of terminal utilities. Top figure: $\beta = 0.5$. Bottom figure: $\beta = 1.0$. 55
4.7 Stein-Stein model (TEST 4): Path-by-path additional terminal utilities over modified Merton agent (1000 realisations). Data points are reorganised in the order of increasing robust SS terminal utilities for illustration purposes. Bottom figure fills the areas where the robust SS agent performs better than the classical SS agent with green colour, and the alternative scenario with blue colour. 57
4.8 Stein-Stein model (TEST 4): Histogram of the path-by-path mean for estimated λ for classical and robust SS agents. 58
4.9 Stein-Stein model (TEST 4): Histograms of the path-by-path mean for estimated m/α (mean-reversion level) and α (mean-reversion speed) for classical and robust SS agents. 59
4.10 Stein-Stein model (TEST 4): Histograms of the path-by-path mean for estimated b (volatility of volatility) and ρ (correlation between price and volatility) for classical and robust SS agents. 59
4.11 Stein-Stein model (TEST 4): Histograms of the path-by-path mean for estimated MPR and volatility for the modified Merton agent. 60
4.12 Heston model: Histograms of terminal utilities for different strategies. Top figure: $\lambda^* = 1.5$, $\beta = 0.5$. With the parameter choices, the histogram for the robust Heston strategy indeed overlaps with that for the optimal strategy. Bottom figure: $\lambda^* = 0.5$, $\beta = 0.5$. 63
4.13 Heston model: Parameter values are chosen as in Table 4.6 and $\lambda^* = 1.5$, $\beta = 0.5$. Top figure: example wealth processes for one realisation. Bottom figure: robust strategies $(\widehat{\pi}_t^{\mathrm{robustHes}})_{t\in[0,T]}$. In the bottom figure, $\widehat{\pi}^{\mathrm{robustHes}}$ essentially stays at the value $\frac{\lambda^*}{1+\beta} = \lambda = 1.0$ over $t \in [0, T]$, which explains the almost identical wealth paths of the robust Heston agent and the optimal agent, as well the overlapping histograms. . . 64

4.14 Robustness to model error (TEST 1): Histograms of terminal utilities. Top figure: $\beta = 0.5$. Bottom figure: $\beta = 1.0$. 66
4.15 Robustness to model error (TEST 1): Path-by-path additional terminal utilities over modified Merton agent (1000 realisations). Data points are re-organised in the order of increasing robust SS terminal utilities for illustration purposes. Bottom figure fills the areas where the robust SS agent performs better than the classical SS agent with green colour, and the alternative scenario with blue colour. 68
4.16 Robustness to model error (TEST 1): Histogram of the path-by-path mean for estimated λ for classical and robust SS agents. 69
4.17 Robustness to model error (TEST 1): Histograms of the path-by-path mean for estimated m/α (mean-reversion level) and α (mean-reversion speed) for classical and robust SS agents. 69
4.18 Robustness to model error (TEST 1): Histograms of the path-by-path mean for estimated b (volatility of volatility) and ρ (correlation between price and volatility) for classical and robust SS agents. 70
4.19 Robustness to model error (TEST 1): Histograms of the path-by-path mean for estimated MPR and volatility for the modified Merton agent. 70

5.1 Basis risk model: Asset prices, hedging strategies (number of shares of stocks to hold), hedging portfolio values for one simulation for testing scheme 1 when drift μ is over-estimated by 50% and a is under-estimated by 50%. 111
5.2 Basis risk model: Asset prices, hedging strategies (number of shares of stocks to hold), hedging portfolio values for one simulation for testing scheme 1 when drift μ is under-estimated by 50% and a is over-estimated by 50%. 114

B.1 Stein-Stein model (TEST 5): Histogram of the path-by-path mean for estimated λ for classical and robust SS agents. 128
B.2 Stein-Stein model (TEST 5): Histograms of the path-by-path mean for estimated m/α (mean-reversion level) and α (mean-reversion speed) for classical and robust SS agents. 129
B.3 Stein-Stein model (TEST 5): Histograms of the path-by-path mean for estimated b (volatility of volatility) and ρ (correlation between price and volatility) for classical and robust SS agents. 129

B.4 Stein-Stein model (TEST 5): Histograms of the path-by-path mean for estimated MPR and volatility for the modified Merton agent. 130
B.5 Robustness to model error (TEST 2): Histograms of terminal utilities. 131
B.6 Robustness to model error (TEST 2): Path-by-path additional terminal utilities over modified Merton agent (1000 realisations). Data points are re-organised in the order of increasing robust SS terminal utilities for illustration purposes. Bottom figure fills the areas where the robust SS agent performs better than the classical SS agent with green colour, and the alternative scenario with blue colour. 132
B.7 Robustness to model error (TEST 2): Histogram of the path-by-path mean for estimated λ for classical and robust SS agents. 133
B.8 Robustness to model error (TEST 2): Histograms of the path-by-path mean for estimated m/α (mean-reversion level) and α (mean-reversion speed) for classical and robust SS agents. 133
B.9 Robustness to model error (TEST 2): Histograms of the path-by-path mean for estimated b (volatility of volatility) and ρ (correlation between price and volatility) for classical and robust SS agents. 134
B.10 Robustness to model error (TEST 2): Histograms of the path-by-path mean for estimated MPR and volatility for the modified Merton agent. 134

Chapter 1

Introduction

In this thesis, we study optimal investment, pricing and hedging problems under model uncertainty, when the reference model is a non-Markovian stochastic factor model, comprising a single stock whose drift and volatility are adapted to the filtration generated by a Brownian motion correlated with that driving the stock. Underlying these so-called robust problems are classical utility maximisation problems in a continuous-time market.

1.1 Utility maximisation under model uncertainty

One version of Merton's portfolio optimisation problem involves an agent maximising expected utility from terminal wealth, X_T, for some finite horizon $T < \infty$, via trading available assets in the market in an optimal way. With admissible strategies $\pi \in \mathcal{A}$, the agent seeks to find the strategy achieving the supremum

$$\sup_{\pi \in \mathcal{A}} \mathbb{E}^0[U(X_T)]$$

where the expectation is taken under a fixed probability measure $\mathbb{P}^0$. Merton ([38, 39]) solved the problem in constant parameter Brownian models using stochastic dynamic programming techniques which require the price processes to be Markovian. A dual approach which applies to non-Markovian models was developed (among others) by Karatzas et al. [28] for a complete Itô process market and in [29] for an incomplete Itô process model, culminating in the seminal treatment of Kramkov and Schachermayer [30, 31] for an incomplete general semimartingale market.

Model uncertainty, also called *model ambiguity* or *Knightian uncertainty*, arises if the agent is uncertain about the subjective probability measure $\mathbb{P}^0$ (the *reference measure*) specified above and is averse to the risk brought about by this uncertainty.

A first approach to take model uncertainty into consideration is referred to as the *multiple priors* ("worst case") approach. The agent considers a set $\mathcal{P}$ of measures (models) instead of only the reference measure $\mathbb{P}^0$, and takes the least favourable expected utility under this set of measures before maximising this utility functional over admissible strategies. The problem can be expressed as

$$\sup_{\pi \in \mathcal{A}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}[U(X_T)], \tag{1.1.1}$$

where $\mathbb{E}[\cdot] \equiv \mathbb{E}^{\mathbb{P}}[\cdot]$ denotes expectation under each plausible measure $\mathbb{P} \in \mathcal{P}$. The axiomatic foundations of this approach were developed by the seminal work of Gilboa and Schmeidler [20]. Another way of treating the objective in (1.1.1) is to study it from a two-player zero-sum stochastic differential game point of view, where the market is seen as the minimiser and the agent seen as the maximiser (Mataramvura and Øksendal [33], Talay and Zheng [62]).

One expects that the multiple priors approach will lead to overly conservative investment strategies since it considers the worst case amongst all possible models. Unduly safe investment decisions from worst case optimisation, is one potential drawback we wish to affirm or refute here. A potentially less conservative approach, the *variational preferences* approach, weights the models according to their supposed plausibility by adding a penalty which is usually based on their "distance" to the reference measure. Hansen and Sargent [21] introduced this idea and used relative entropy with respect to the reference measure as the penalty function. The rigorous decision-theoretic foundation for this approach was provided by Maccheroni et al. [32]. See also Anderson et al. [1], Hansen et al. [22] for studying the robust utility maximisation problem from an economic point of view. In the variational preferences approach, the robust optimal investment problem is then

$$\sup_{\pi \in \mathcal{A}} \inf_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}[U(X)] + \gamma(\mathbb{P}) \right\}, \tag{1.1.2}$$

where $\gamma(\mathbb{P})$ is the penalty function assigned to each possible model.

Broadly speaking, there are two approaches to solving the robust utility maximisation problems of the form in (1.1.1) and (1.1.2). One approach is a dual approach which transforms the *sup-inf* problem of looking for a saddle point into an *inf-inf* problem, to which a stochastic control approach (dynamic programming or backward stochastic differential equations (BSDEs)) can then be applied. Building on the work of Kramkov and Schachermayer [30, 31] for the classical case without model uncertainty, Quenez [55] developed a duality theory for multiple priors utility maximisation

in a market with semi-martingale stock price processes. She proved the existence of a saddle point to the *sup-inf* problem under suitable conditions on the utility function and the set of models, the minimax property that one can interchange the maximisation over strategies and minimisation over measures, and the duality relations between the primal and dual problems. She then used BSDE techniques to analyse the dual problems for the cases of logarithmic utilities and power utilities in a market with Itô stock price processes and Brownian filtrations. Schied and Wu [59] extended this dual approach and weakened the assumptions on the set of models $\mathcal{P}$, by not requiring all plausible measures to be equivalent to the reference measure as in [55]. Schied [58] further extended the duality theories of Kramkov and Schachermayer to the variational preferences optimal investment problem, and gave a class of time-consistent penalty functions for which stochastic dynamic programming methods can be employed.

Robust investment problems are, typically, hard to solve in explicit form. One class of market models where more explicit characterisations of solutions to the robust problems might be feasible, is a form of Markovian stochastic factor model, where the drift and volatility of the stock are assumed to depend on the current value of a non-traded stochastic factor correlated with the stock price. By using dynamic programming techniques to study the dual problem associated with a multiple priors power utility maximisation problem (in [24]) and a variational preferences logarithmic utility maximisation problem (in [25]), Hernández-Hernández and Schied characterised the value functions and optimal strategies in terms of non-linear PDEs. Existence of solutions to these PDEs was obtained, but no concrete solutions.

A second approach to studying the robust utility maximisation problem is through the use of BSDEs, which applies to non-Markovian models compared with the usual dynamic programming approach. Since the seminal paper on general nonlinear BSDEs by Pardoux and Peng [52], the connections of BSDEs with stochastic control, mathematical finance and PDEs have been widely studied (see El Karoui, Peng and Quenez [15] for finance applications, and El Karoui an Mazliak [14], Pham [53, Chapter 6], Touzi [64, Chapter 10, 11], Cohen and Elliott [8, Chapter 19, A.9], Xing and Žitković [66] for more). In using BSDEs to solve classical (that is, not considering model uncertainty) utility maximisation problems, Hu et al. [26] and Cheridito and Hu [7] used a martingale argument applied to the primal maximal utility process in an Itô market with power, logarithmic and exponential utilities. The authors constructed a family of processes, each of which is a super-martingale for any strategy and a martingale (which is the value process) for the optimal strategy, and expressed

the value process as the solution to a BSDE. For the multiple priors utility maximisation problem, Müller [46] used a martingale argument applied to the primal maximal utility process and described the maximising strategy and minimising probability measure by solution to a BSDE (see also Matoussi et al. [34], Yang et al. [67] for a BSDE approach to study the multiple priors optimal investment-consumption problem with both drift and volatility uncertainties with exponential, power and logarithmic utilities). The BSDE approach also arises in optimal investment-consumption problems with recursive utilities. See Epstein and Zin [17], Duffie et al. [13], Xing [65], Matoussi and Xing [35] (a dual approach) for the classical formulation, Chen and Epstein [6], Epstein and Schneider [16] for the multiple priors formulation, and Skiadas [60], Bordigoni et al. [5], Faidi et al. [18] for the variational preferences formulation. However, this is a different utility objective thus not in the scope of this thesis.

1.2 Exponential hedging of basis risk

One typical exponential utility maximisation problem is to hedge a claim written on a non-traded asset using a correlated traded asset. The mathematical formulation of the problem can be written as

$$\sup_{\pi \in \mathcal{A}} \mathbb{E}^0[-\exp\{-\alpha(X_T + nG_T)\}], \tag{1.2.1}$$

where the agent with n claims, each with payoff G_T, is interested in pricing and hedging her position in the option. This incomplete markets problem was introduced by Davis [11], where the author solved it using dual methods. Henderson [23] looked at power and exponential utilities and obtained asymptotic results for the indifference prices. Monoyios [40] used the distortion method of Zariphopoulou [68] and expansion techniques to give analytic approximations (in powers of $\epsilon^2 = 1-\rho^2$) of optimal pricing and hedging strategies for the claim and analysed their performances. Monoyios [43] continued [40] and looked at the effect of using erroneous parameters. The results showed that parameter errors could be benign or ruinous, depending on the sign of the error. The author then suggested an approach via filtering the asset drifts to deal with uncertainty in the drift parameters. Monoyios [44] then did a full partial information approach, filtering asset drifts, deriving asymptotic expansions of the indifference price for small values of risk aversion. The author also conducted simulation-based numerical tests on the performances of the strategies, which could be dependent on the accuracy of the prior distribution for the drifts. For results on a classical exponential

utility maximisation problem of the form in (1.2.1) under a general market model, see Delbaen et al. [12], Schachermayer [57], Owen [51], Becherer [2, 3], etc.

The corresponding multiple priors exponential hedging problem can be formulated as

$$\sup_{\pi\in\mathcal{A}}\inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}[-\exp\{-\alpha(X_T+nG_T)\}].$$

Owari [50] studied the robust exponential hedging and indifference valuation problem for a general semi-martingale model, extending the duality theory in Schied and Wu [59] to the exponential utility case with a random endowment. Owari [49] then considered the problem under a Markovian stochastic factor model and obtained a PDE representation of the optimal strategy.

1.3 Main results

While most of the literature has focused on theoretical characterisations of robust utility maximisation problems in fairly abstract models, there have been few papers (except a recent paper by Sass and Westphal [56] in a robust Merton problem) on explicit characterisations of solutions under more specific market models. Moreover, no explicit results were provided on the optimal policies, beyond their characterisation in terms of derivatives of a function solving the PDE in question. Perhaps equally significant, no examination of the performance of robust investment strategies was provided, and thus one of our goals is to provide some quantitative results to this effect.

In Chapter 2, we will study classical utility maximisation problems in a non-Markovian stochastic factor model, which assumes only that the drift and volatility processes of a single stock are adapted to the filtration generated by a Brownian motion correlated with the Brownian motion driving the stock. We will derive martingale distortion representations of the value processes in the cases of power utility, logarithmic utility and exponential utility. The method and problem solved in this chapter will underpin subsequent robust results in later chapters.

The martingale distortion representation, developed by Zariphopoulou [68] in a Markovian stochastic factor model and by Tehranchi [63] in the non-Markovian case, is a reduced form representation for the value process of an optimal investment problem under a stochastic factor model. In the Markov case, if (x, y) denote the initial values (at some time $t \in [0, T]$) of wealth and the stochastic factor, the distortion representation of the value function is $u(t, x, y) = U(x)(v(t, y))^{\delta}$ where $U(\cdot)$ is the

utility function, and $v(\cdot,\cdot)$ solves a linear PDE, and the distortion power δ is a constant depending on the correlation and the risk aversion. For the non-Markovian case, the justification of the representation then makes use of the martingale property of the value function. See Monoyios [41] for discussions of the application of the distortion representation in obtaining the dual optimal measure and Fouque and Hu [19] for the application of the distortion representation to a utility maximisation problem under rough fractional stochastic volatility models.

Here, we will show how the martingale distortion representation developed in Fouque and Hu [19] can be extended to a general non-Markovian stochastic factor model, with unbounded market price of risk (MPR) (compared with the bounded MPR in [19]), and all we require is that the MPR is adapted to the filtration $\widetilde{\mathbb{F}}$ generated by a correlated Brownian motion. We will also extend the method to an exponential utility maximisation problem with a random endowment.

In Chapter 3, we study a multiple priors power utility maximisation problem of the form in (1.1.1). We first interchange the order of maximisation over strategies and minimisation over models and derive a martingale distortion representation of the value process of the optimal investment problem under each fixed plausible model, associated with probability measure $\mathbb{P} \in \mathcal{P}$ (Proposition 3.2.3). We then minimise over models, and in Proposition 3.3.1 provide a semi-explicit characterisation of the robust value process in terms of a linear BSDE, which will reduce to the PDE representation (after a change of variable) obtained by Hernández-Hernández and Schied [24] in the Markovian case.

In Chapter 4, we treat a variational preferences logarithmic utility maximisation problem of the form (1.1.2), by transforming the robust problem to a classical problem with an adjusted utility function (Proposition 4.2.1), where the ambiguity aversion parameter translates into a risk aversion parameter for the derived power utility. We give semi-explicit representations for the robust value process and the optimal strategy in Proposition 4.3.2. The main contribution in this chapter is that we are able to get fully explicit results for robust problems under Stein-Stein and Heston stochastic volatility models. This allows us to perform thorough numerical tests on the performances of the strategies.

We carry out large-scale simulation studies of the effectiveness of robust strategies versus classical strategies which assume a fixed model, in the face of parameter misestimation, with versions of the Stein-Stein and Heston stochastic volatility models as the reference model. The results indicate that the robust strategies are most beneficial when the agent is overly optimistic/pessimistic about the growth rate of the stock.

Moreover, with an appropriate level of uncertainty aversion in line with the extent of the estimation errors of the model parameters, the robust agent can have terminal utilities distributed similarly as a classical agent who has correct estimates of the model parameters.

In some of the tests, we include an ad hoc variant of the Merton strategy as one of the strategies we evaluate. In this "modified Merton" strategy, the agent uses a Merton model as the underlying model and dynamically re-calibrates the model parameters using a moving window of stock price observations. The motivation for including this variant of a classical Merton strategy is two-fold: (i) to assess its performance relative to the robust SV (stochastic volatility) strategy; (ii) to have a scheme that is easier to calibrate than the robust SV scheme. To make a fair comparison over all the strategies we compare, we also use re-calibrated model parameters for the classical and robust SV strategies. We learn from the results that even with the correct model assumed, because of the difficulty of estimating the parameters for the complicated model, the classical SV agent does not do better than the modified Merton agent. On the other hand, the robust SV strategy performs better even with the naively estimated model parameters in our tests, and is more stable when the underlying model is a Heston model but the SV agents are using Stein-Stein models.

In Chapter 5, we study optimal hedging problems (including both a multiple priors and a variational preferences approach) in a non-Markovian version of a basis risk model with a claim on a non-traded asset being hedged using a correlated stock. The innovations here are: (i) a non-Markov setting; (ii) a robust setting where, in particular, we use a penalised objective functional, with a multiplicative form of penalty function (first suggested by Mataramvura and Øksendal [33]), rather than the additive form as in (1.1.2).

Then, specialising to an uncertain (penalised) log-normal basis risk model with constant parameters, we produce explicit formulae and asymptotic expressions for robust indifference prices and option hedges. Finally, we numerically test their efficacy. It turns out that in essence, the robust approach introduces one extra degree of freedom – namely the ambiguity aversion parameter. It is an open question as to how this could be chosen – future research would be needed to address this.

Chapter 6 concludes the thesis.

1.4 Some standing notations

There will be some standing notations used throughout this thesis. We use $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}^0)$ to denote a filtered complete probability space, with $T < \infty$ some fixed finite time horizon. The filtrations used in the thesis are: $\mathbb{F}$, generated by two independent Brownian motions $(W^0, W^{0,\perp})$, and $\widetilde{\mathbb{F}}$, generated by a correlated Brownian motion $\widetilde{W}^0 := \rho W^0 + \sqrt{1-\rho^2}W^{0,\perp}$, for $\rho \in (-1, 1)$ a constant correlation. When there is model uncertainty considered in the problem in any of the chapters, we will use

- under the reference measure $\mathbb{P}^0$: expectation $\mathbb{E}^0$; independent Brownian motions $(W^0, W^{0,\perp})$;
- under any plausible measure $\mathbb{P} \in \mathcal{P}$: expectation $\mathbb{E} \equiv \mathbb{E}^{\boldsymbol{\eta}} \equiv \mathbb{E}^{\mathbb{P}}$, where $\boldsymbol{\eta} = (\eta, \eta^{\perp})$ denotes integrands characterising the measure $\mathbb{P}$; independent Brownian motions $(W, W^{\perp})$.

Chapter 2

Martingale distortion method for optimal investment in a stochastic factor model

In this chapter, we will study a non-Markovian stochastic factor model, in which the drift and volatility processes of a single stock are adapted to the filtration generated by a Brownian motion correlated with the Brownian motion driving the stock. We will derive martingale distortion representations (which we describe further below) in the manner of Tehranchi [63] and Fouque and Hu [19] for classical (that is, no model uncertainty) utility maximisation problems in this stochastic factor model. First, for a classical power utility maximisation problem, we will extend [19, Proposition 2.2] to a general non-Markovian stochastic factor model (given in (2.1.1)), with unbounded MPR (compared with the bounded MPR in [19]). We then extend the method to an exponential utility maximisation problem with a random endowment, in a general non-Markovian stochastic factor model, with unbounded MPR.

We shall describe in detail the solution of the power utility optimal investment problem. In the case of logarithmic utility, and also the problem with exponential utilities and a random endowment, the method works, but in the logarithmic utility case, the value process has an additive decomposition (as opposed to a multiplicative decomposition that is obtained in the power utility case).

It is important to note that, in this chapter, the probability measures and model parameters are those of some generic model. In subsequent chapters, the results in this chapter will be applied to stochastic factor models under various measures, when the model parameters satisfy the assumptions needed under the respective measures.

2.1 A non-Markovian stochastic factor model

In this section, we introduce a non-Markovian stochastic factor model that will be studied in various guises, in this thesis, and we introduce some standing assumptions.

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P}^0)$ be a filtered complete probability space with the filtration $\mathbb{F}$ satisfying the usual conditions of completeness and right-continuity. The measure $\mathbb{P}^0$ stands for some generic probability measure. Let $T < \infty$ be some fixed finite time horizon. Write $\widetilde{W}^0 := \rho W^0 + \overline{\rho} W^{0,\perp}$, where W^0, $W^{0,\perp}$ are independent $\mathbb{P}^0$-Brownian motions, and $\overline{\rho} := \sqrt{1-\rho^2}$ with $\rho \in (-1, 1)$ constant[1]. Introduce two filtrations

- $\mathbb{F}$, the $\mathbb{P}^0$-augmentation of the filtration generated by Brownian motions W^0 and $W^{0,\perp}$. (In some of the literature on the distortion solution, such as Monoyios [41, 42], Tehranchi [63], Fouque and Hu [19], Zariphopoulou [68], $\mathbb{F}$ is defined as the filtration generated by the correlated $\mathbb{P}$-Brownian motions W^0 and $\widetilde{W}^0$, but both definitions are equivalent.)
- $\widetilde{\mathbb{F}}$, the $\mathbb{P}^0$-augmentation of the filtration generated by the $\mathbb{P}^0$-Brownian motion $\widetilde{W}^0$.

Consider a market with zero interest rate and a single stock S, whose dynamics are given by

$$\mathrm{d}S_t = S_t\mu_t\,\mathrm{d}t + S_t\sigma_t\,\mathrm{d}W_t^0. \tag{2.1.1}$$

The key feature relevant to us is that the drift μ and volatility σ are $\widetilde{\mathbb{F}}$-adapted processes.

Remark 2.1.1 (A Markovian version). In a typical Markovian version of the model, the parameters $\mu \equiv \mu(Y)$, $\sigma \equiv \sigma(Y)$ are continuous functions of the current value of a stochastic factor process Y, that is, we have

$$\mathrm{d}S_t = S_t\mu(Y_t)\,\mathrm{d}t + S_t\sigma(Y_t)\,\mathrm{d}W_t^0, \tag{2.1.2}$$

$$\mathrm{d}Y_t = a(Y_t)\,\mathrm{d}t + b(Y_t)\,\mathrm{d}\widetilde{W}_t^0. \tag{2.1.3}$$

Tehranchi [63] and Monoyios [42] studied the non-Markovian model, while Monoyios [41] and Zariphopoulou [68] studied the Markov case.

[1]The correlation parameter ρ needs to be constant (instead of, for example, state dependent) for the distortion approach to work.

Remark 2.1.2. This model includes a lot of useful investment models. One example is to take continuous functions $\mu_t = \mu(t, Y_t)$ and $\sigma_t = \sigma(t, Y_t)$ where Y is a general stochastic process that is adapted to $\widetilde{\mathbb{F}}$, which is studied by Fouque and Hu [19]. Another example is a (possibly non-Markovian) stochastic factor model where the drift and volatility processes in (2.1.1) depend on a stochastic factor Y which is an Itô process where the underlying Brownian motion is the correlated Brownian motion $\widetilde{W}^0$ and the parameter processes are $\widetilde{\mathbb{F}}$-adapted. Other examples include stochastic volatility models, basis risk models where Y is the price process of a non-traded correlated asset in the market, and investment models where the drift process of the stock is assumed to be mean-reverting and follows, for example, some Ornstein–Uhlenbeck (OU) process. The key feature is that the parameters are $\widetilde{\mathbb{F}}$-adapted in this model.

At the outset, we assume only that the parameters μ, σ are $\widetilde{\mathbb{F}}$-adapted processes such that there exists a unique strong solution to the SDE (2.1.1), and $\sigma \geq \epsilon$ for some constant $\epsilon > 0$. Additional assumptions on the parameters will be given on a case-by-case basis where needed. The market price of risk (MPR) of the stock is

$$\lambda_t := \frac{\mu_t}{\sigma_t}, \qquad t \in [0, T],$$

assumed to satisfy $\int_0^T \lambda_s^2 \, \mathrm{d}s < \infty$, $\mathbb{P}^0$-a.s..

An agent in this market invests in the stock S and cash in a self-financing way. Denote the proportion of wealth invested in the stock by the progressively measurable process π, assumed to satisfy

$$\int_0^T (|\pi_s \mu_s| + |\pi_s \sigma_s|^2) \, \mathrm{d}s < \infty, \quad \mathbb{P}^0\text{-a.s..} \tag{2.1.4}$$

The wealth process $X \equiv X^{x,\pi}$ of the agent with initial wealth $X_0 = x > 0$ has dynamics

$$\mathrm{d}X_t = X_t \pi_t \sigma_t (\lambda_t \, \mathrm{d}t + \mathrm{d}W_t^0). \tag{2.1.5}$$

The definition of the set of admissible strategies starting at time $t \in [0, T]$, denoted by $\mathcal{A}_{t,T}$ with the convention $\mathcal{A} \equiv \mathcal{A}_{0,T}$, will be given later, with specific additional conditions augmenting (2.1.4) where necessary.

2.2 Classical power utility maximisation

We introduce a classical power utility maximisation problem (under the probability measure $\mathbb{P}^0$) and give a martingale distortion representation for the value function.

The structural result we obtain here will be used in later chapters as one of the building blocks of robust investment problems, where one introduces uncertainty in the probability measure underlying the model.

In a classical power utility maximisation problem (without considering model uncertainty), the objective of the agent is given by the value process u, defined by

$$u_t := \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{t,T}} \mathbb{E}^0 \left[U\left(X_T\right) | \mathcal{F}_t \right], \quad t \in [0,T], \tag{2.2.1}$$

where $\mathbb{E}^0$ denotes the expectation taken under the probability measure $\mathbb{P}^0$. The utility function $U : (0,\infty) \to \mathbb{R}$ is specified as the power utility

$$U(x) = \frac{x^p}{p}, \quad p < 1,\ p \neq 0.$$

For concreteness, we shall assume $p < 0$ throughout this section. In the case when $p \in (0,1)$, the approach used in this section still goes through unaltered, with different conditions imposed on the admissible strategies and modified assumptions on the parameter processes. The differences in sets of admissible strategies and assumptions will be pointed out in Remark 2.2.5.

For the strategies in (2.2.1), the following additional integrability condition is introduced:

$$\mathbb{E}^0 \left[\exp\left(cp^2 \int_0^T \pi_t^2 \sigma_t^2 \, \mathrm{d}t \right) \right] < \infty, \quad \text{for some constant } c > 1. \tag{2.2.2}$$

Thus the set of admissible strategies for any starting time $t \in [0,T]$ is defined by

$$\mathcal{A}_{t,T} := \Big\{ \pi_s \text{ is } (\mathcal{F}_s)\text{-progressively measurable, } \forall s \in [t,T] \;\Big|\; \pi \text{ satisfies (2.1.4) and (2.2.2)} \Big\},$$

where $X \geq 0$ $\mathbb{P}^0$-a.s. follows from the dynamics (2.1.5) of the wealth process with $X_0 = x > 0$.

Remark 2.2.1. The additional admissibility condition (2.2.2) will be needed later, when $p < 0$, to ensure that the exponential local martingale (in (2.2.11) in the proof of Proposition 2.2.4) is a $(\mathbb{P}^0, \mathbb{F})$-martingale. Note that this additional integrability condition, as well as the other integrability conditions to be introduced later, are only sufficient (but not necessary). In practice, weaker versions of these assumptions are possible.

2.2.1 A martingale distortion representation

In the following, we will derive a martingale distortion representation in the manner of Tehranchi [63] and Fouque and Hu [19] for the utility maximisation problem (2.2.1), extending [19, Proposition 2.2] to the general non-Markovian stochastic factor model (2.1.1), with unbounded MPR (compared with the bounded MPR in [19]).

Define the measure $\widetilde{\mathbb{P}}^0 \sim \mathbb{P}^0$ on $\widetilde{\mathcal{F}}_T$ by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}^0}{\mathrm{d}\mathbb{P}^0} = \mathcal{E}\left(-\int_0^\cdot q\rho\lambda_s \,\mathrm{d}\widetilde{W}_s^0\right)_T, \tag{2.2.3}$$

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential, and the constant q is defined by

$$q := -\frac{p}{1-p}. \tag{2.2.4}$$

The following additional assumptions 2.2.2 and 2.2.3 are made on the market price of risk, and the martingale in the martingale distortion representation (2.2.10).

Assumption 2.2.2. We assume that the market price of risk $(\lambda_t)_{t\in[0,T]}$ satisfies

$$\mathbb{E}^0\left[\exp\left(4cq^2\int_0^T \lambda_t^2 \,\mathrm{d}t\right)\right] < \infty. \tag{2.2.5}$$

where c is the same constant as in (2.2.2).

Under Assumption 2.2.2 above, the probability measure $\widetilde{\mathbb{P}}^0$ defined by the Radon-Nikodym derivative (2.2.3) is a well-defined probability measure. Also, the condition (2.2.5) is a sufficient condition that we shall use in the proof of Proposition 2.2.4 for guaranteeing that the optimal strategy satisfies the additional admissibility condition (2.2.2).

Define the positive $(\widetilde{\mathbb{P}}^0, \widetilde{\mathbb{F}})$-martingale M by

$$M_t := \widetilde{\mathbb{E}}^0\left[\exp\left(-\frac{q}{2\delta}\int_0^T \lambda_s^2 \,\mathrm{d}s\right) \,\middle|\, \widetilde{\mathcal{F}}_t\right], \quad t \in [0,T], \tag{2.2.6}$$

where $\widetilde{\mathbb{E}}^0$ denotes expectation under $\widetilde{\mathbb{P}}^0$, and the distortion power δ is given by

$$\delta := \frac{1}{1-q\rho^2}. \tag{2.2.7}$$

Then M will have a stochastic integral representation

$$M_t = M_0 + \int_0^t M_s\xi_s \,\mathrm{d}\widetilde{B}_s^0, \quad t \in [0,T], \tag{2.2.8}$$

for some $\widetilde{\mathbb{F}}$-adapted square integrable process ξ, where $\widetilde{B}^0 = \widetilde{W}^0 + \int_0^\cdot q\rho\lambda_s \,\mathrm{d}s$ is a $\widetilde{\mathbb{P}}^0$-Brownian motion. Note that M is well-defined because $q, \delta > 0$.

Assumption 2.2.3. We assume that ξ satisfies

$$\mathbb{E}^0\left[\exp\left(c_\xi \int_0^T \xi_t^2 \,\mathrm{d}t\right)\right] < \infty, \tag{2.2.9}$$

where

$$c_\xi = \max\left\{\frac{c}{c-1}(\rho^2\delta^2 + \frac{1}{2}\overline{\rho}^2\delta^2), 4cq^2\rho^2\delta^2\right\},$$

and c is the same constant as in (2.2.2).

Assumption 2.2.3 is a technical assumption used in the proof of Proposition 2.2.4 below for proving that an exponential local martingale (in (2.2.11)) is indeed a martingale and that the optimal strategy satisfies the additional admissibility condition (2.2.2).

Proposition 2.2.4. *Under Assumptions 2.2.2 and 2.2.3, the value process u of* (2.2.1) *admits the martingale distortion representation*

$$u_t = \frac{\widehat{X}_t^p}{p} V_t^\delta, \quad V_t := \widetilde{\mathbb{E}}^0\left[\exp\left(-\frac{q}{2\delta}\int_t^T \lambda_s^2 \,\mathrm{d}s\right) \,\middle|\, \widetilde{\mathcal{F}}_t\right], \quad t \in [0,T], \tag{2.2.10}$$

where $\widetilde{\mathbb{E}}^0$ denotes expectation under the probability measure $\widetilde{\mathbb{P}}^0$ defined in (2.2.3)*, the constants q and δ are as in* (2.2.4) *and* (2.2.7)*, and $\widehat{X}$ denotes the optimal wealth process. The optimal trading strategy for the optimal investment problem* (2.2.1) *is given by*

$$\widehat{\pi}_t = \frac{\lambda_t + \rho\delta\xi_t}{(1-p)\sigma_t}, \quad t \in [0,T],$$

where ξ is the integrand in the martingale representation of M in (2.2.8)*.*

Proof. Define u^π as the analogue of the value process u in (2.2.10), but with $X \equiv X^\pi$ the wealth process in (2.1.5) when using any admissible strategy $\pi \in \mathcal{A}_{t,T}$, $t \in [0,T]$. Since $u_0^\pi = u_0^{\widehat{\pi}}$, we denote them as u_0. To verify that the process $u \equiv u^{\widehat{\pi}}$ given by (2.2.10) is the value process, we need to check that u^π is a $(\mathbb{P}^0, \mathbb{F})$-super-martingale for any strategy π, and a $(\mathbb{P}^0, \mathbb{F})$-martingale for strategy $\widehat{\pi}$.

First rewrite u^π in terms of the $(\widetilde{\mathbb{P}}^0, \widetilde{\mathbb{F}})$-martingale M defined in (2.2.6):

$$u_t^\pi = \frac{X_t^p}{p}\left(\widetilde{\mathbb{E}}^0\left[\exp\left(-\frac{q}{2\delta}\int_t^T \lambda_s^2 \,\mathrm{d}s\right) \,\middle|\, \widetilde{\mathcal{F}}_t\right]\right)^\delta = \frac{X_t^p}{p}\mathrm{e}^{N_t} M_t^\delta,$$

where $N_t := \frac{q}{2}\int_0^t \lambda_s^2 \,\mathrm{d}s$, $t \in [0,T]$. Applying Itô's formula to u^π yields

$$\mathrm{d}u_t^\pi = u_t^\pi \left(p\pi_t\sigma_t \,\mathrm{d}W_t^0 + \delta\xi_t \,\mathrm{d}\widetilde{W}_t^0 \right) + u_t^\pi \left(p \cdot f(\pi_t) \right) \mathrm{d}t,$$

where

$$f(\pi_t) = \frac{p-1}{2}\sigma_t^2\pi_t^2 + \sigma_t\lambda_t\pi_t + \rho\delta\sigma_t\xi_t\pi_t + \frac{q}{2p}\lambda_t^2 + \frac{q\rho\delta}{p}\xi_t\lambda_t + \frac{\delta(\delta-1)}{2p}\xi_t^2.$$

Note that $f(\pi)$ is quadratic with respect to π, and $\frac{p-1}{2}\sigma^2 < 0$, thus it achieves its maximum value, zero, at

$$\widehat{\pi}_t = \frac{\lambda_t + \rho\delta\xi_t}{(1-p)\sigma_t}, \quad t \in [0,T].$$

Solving for u^π gives

$$u_t^\pi = u_0 \exp\left(\int_0^t p \cdot f(\pi_s) \,\mathrm{d}s \right) \mathcal{E}\left(\int_0^\cdot (p\pi_s\sigma_s + \rho\delta\xi_s) \,\mathrm{d}W_s^0 + \int_0^\cdot \overline{\rho}\delta\xi_s \,\mathrm{d}W_s^{0,\perp} \right)_t,$$

for $t \in [0,T]$. To check that u^π is a $(\mathbb{P}^0, \mathbb{F})$-super-martingale for any strategy π, and a martingale for strategy $\widehat{\pi}$, first calculate that

$$\mathbb{E}^0[u_t^\pi] = u_0\mathbb{E}^0\left[\exp\left(\int_0^t p \cdot f(\pi_s) \,\mathrm{d}s \right) \mathcal{E}\left(\int_0^\cdot (p\pi_s\sigma_s + \rho\delta\xi_s) \,\mathrm{d}W_s^0 + \int_0^\cdot \overline{\rho}\delta\xi_s \,\mathrm{d}W_s^{0,\perp} \right)_t \right]$$

$$\leq u_0\mathbb{E}^0\left[\mathcal{E}\left(\int_0^\cdot (p\pi_s\sigma_s + \rho\delta\xi_s) \,\mathrm{d}W_s^0 + \int_0^\cdot \overline{\rho}\delta\xi_s \,\mathrm{d}W_s^{0,\perp} \right)_t \right] = u_0, \tag{2.2.11}$$

where the inequality comes from the fact that $u_0 < 0$, $f(\pi_t) \leq 0$, $\forall \pi$ and $f(\widehat{\pi}_t) = 0$ for any $t \in [0,T]$, and the second equality follows from Novikov's condition, which is guaranteed by assumptions (2.2.2) and (2.2.9). By (2.2.11), u^π is a $(\mathbb{P}^0, \mathbb{F})$-super-martingale for any π. Since equality can be obtained at $\pi = \widehat{\pi}$, it is a martingale for $\widehat{\pi}$.

It remains to verify that the optimal strategy $\widehat{\pi}$ satisfies the additional admissibility condition (2.2.2). By Hölder's inequality and the exponential moment conditions (2.2.5) on λ and (2.2.9) on ξ,

$$\begin{aligned}
&\mathbb{E}^0\left[\exp\left(cp^2 \int_0^T \widehat{\pi}_t^2\sigma_t^2 \,\mathrm{d}t \right) \right] = \mathbb{E}^0\left[\exp\left(cp^2 \int_0^T \frac{(\lambda_t + \rho\delta\xi_t)^2}{(1-p)^2} \,\mathrm{d}t \right) \right] \\
&\leq \mathbb{E}^0\left[\exp\left(2cq^2 \int_0^T (\lambda_t^2 + \rho^2\delta^2\xi_t^2) \,\mathrm{d}t \right) \right] \\
&\leq \left(\mathbb{E}^0\left[\exp\left(4cq^2 \int_0^T \lambda_t^2 \,\mathrm{d}t \right) \right] \right)^{\frac{1}{2}} \left(\mathbb{E}^0\left[\exp\left(4cq^2\rho^2\delta^2 \int_0^T \xi_t^2 \,\mathrm{d}t \right) \right] \right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

□

Remark 2.2.5 (The case $p \in (0,1)$). For $p \in (0,1)$, minor amendments to the definition of admissible policies and to the conditions in Assumptions 2.2.2 and 2.2.3 are needed. First, in this case, the martingale property mentioned in Remark 2.2.1 is only needed for the optimal strategy (as $u_0 > 0$, the super-martingale property of u^π does not require the exponential local martingale in (2.2.11) to be a martingale for any admissible strategy) thus the set of admissible strategies given a starting time $t \in [0,T]$ is defined by

$$\mathcal{A}_{t,T} := \Big\{ \pi_s \text{ is } (\mathcal{F}_s)\text{-progressively measurable, } \forall s \in [t,T] \;\Big|\; \pi \text{ satisfies (2.1.4)} \Big\}.$$

Second, (2.2.5) is replaced by

$$\mathbb{E}^0 \left[\exp\left(2q^2 \int_0^T \lambda_t^2 \, \mathrm{d}t \right) \right] < \infty, \tag{2.2.12}$$

and additionally,

$$\widetilde{\mathbb{E}}^0 \left[\exp\left(-\frac{q}{2\delta} \int_0^T \lambda_t^2 \, \mathrm{d}t \right) \right] < \infty. \tag{2.2.13}$$

The condition (2.2.12) is needed for guaranteeing that $\widetilde{\mathbb{P}}^0$ is a well-defined probability measure and for verifying the optimality of the optimal strategy. Condition (2.2.13) ensures that the process M in (2.2.6) is well-defined.

Third, the constant c_ξ in (2.2.9) is replaced by

$$c_\xi = \frac{2}{(1-p)^2} \rho^2 \delta^2 + \overline{\rho}^2 \delta^2.$$

2.2.2 The Markovian case

In the Markovian case (2.1.2)-(2.1.3) in Remark 2.1.1, the value function u and optimal strategies $\widehat{\pi}$ have more explicit representations in terms of the solution to a linear PDE, as stated in the following proposition. Since the results in Proposition 2.2.6 are well-known in Zariphopoulou [68] and Monoyios [41], we only give a short proof on how the representations can be derived from Proposition 2.2.4 by specialising to the Markovian case.

Proposition 2.2.6. *Under Assumptions 2.2.2 and 2.2.3, the value process u in* (2.2.1) *admits the martingale distortion representation*

$$\begin{aligned} u_t &= u(t, \widehat{X}_t, Y_t) = \frac{\widehat{X}_t^p}{p} \left(v(t, Y_t) \right)^\delta, \\ v(t, Y_t) &:= \widetilde{\mathbb{E}}^0 \left[\exp\left(-\frac{q}{2\delta} \int_t^T \lambda^2(Y_s) \, \mathrm{d}s \right) \;\middle|\; Y_t \right], \quad t \in [0,T], \end{aligned} \tag{2.2.14}$$

where $\widetilde{\mathbb{E}}^0$ *denotes expectation under the probability measure* $\widetilde{\mathbb{P}}^0$ *defined in* (2.2.3)*, and* q *and* δ *are as in* (2.2.4) *and* (2.2.7)*. The optimal trading strategy for the optimal investment problem* (2.2.1) *is given by*

$$\widehat{\pi}_t = \frac{\lambda(Y_t) + \rho\delta\xi(t, Y_t)}{(1-p)\sigma(Y_t)}, \quad \textit{with } \xi(t, Y_t) = \frac{b(Y_t)}{v(t, Y_t)}\frac{\partial v}{\partial y}(t, Y_t), \quad t \in [0, T]. \tag{2.2.15}$$

The function $v(\cdot, \cdot)$ satisfies the linear PDE

$$\begin{aligned} &\frac{\partial v}{\partial t}(t, y) + (a(y) - q\rho\lambda(y)b(y))\frac{\partial v}{\partial y}(t, y) \\ &\qquad + \frac{1}{2}b^2(y)\frac{\partial^2 v}{\partial y^2}(t, y) - \frac{q}{2\delta}\lambda^2(y)v(t, y) = 0, \end{aligned} \tag{2.2.16}$$

with $v(T, y) \equiv 1$.

Proof. It only remains to obtain the more explicit form for the process ξ in (2.2.15). First note that by Proposition 2.2.4 and the definition of $v(\cdot, \cdot)$,

$$v(t, Y_t) = V_t = \mathrm{e}^{N_t/\delta} M_t, \qquad t \in [0, T],$$

where the process M is defined in (2.2.6) and $N_\cdot = \frac{q}{2}\int_0^\cdot \lambda^2(Y_s)\,\mathrm{d}s$. By the fact that $v(\cdot, \cdot)$ is well-defined and that there exists a unique strong solution to the SDE satisfied by the factor process Y, $v \in C^{1,2}([0, T] \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$. Applying Itô's formula to the $(\widetilde{\mathbb{P}}^0, \widetilde{\mathbb{F}})$-martingale M yields

$$\begin{aligned} \mathrm{d}M_t &= \mathrm{d}\left(\mathrm{e}^{-N_t/\delta} v(t, Y_t)\right) \\ &= \mathrm{e}^{-N_t/\delta} b(Y_t)\frac{\partial v}{\partial y}(t, Y_t)\,\mathrm{d}\widetilde{B}^0_t + \mathrm{e}^{-N_t/\delta}\left\{\frac{\partial v}{\partial t}(t, Y_t) + (a(Y_t) - q\rho\lambda(Y_t)b(Y_t))\frac{\partial v}{\partial y}(t, Y_t)\right. \\ &\quad \left. + \frac{1}{2}b^2(Y_t)\frac{\partial^2 v}{\partial y^2}(t, Y_t) - \frac{q}{2\delta}\lambda^2(Y_t)v(t, Y_t)\right\}\mathrm{d}t. \end{aligned} \tag{2.2.17}$$

The martingale property thus yields the PDE (2.2.16) satisfied by $v(\cdot, \cdot)$. Equating (2.2.8) and (2.2.17), the process ξ has the representation in (2.2.15):

$$\xi_t = \xi(t, Y_t) = \frac{b(Y_t)}{v(t, Y_t)}\frac{\partial v}{\partial y}(t, Y_t), \quad t \in [0, T].$$

Thus the optimal strategy for the classical optimal investment problem (2.2.1) in the Markovian stochastic factor model is as given in (2.2.15).

□

Remark 2.2.7. In the Markovian case, the additional condition (2.2.2) on the set of admissible strategies is not needed, and all we need to assume for the distortion representation (2.2.14) is that $v(\cdot,\cdot)$ is well-defined and $\widetilde{\mathbb{P}}^0$ is a well-defined probability measure equivalent to $\mathbb{P}^0$ on $\widetilde{\mathcal{F}}_T$. This simplification arises because the problem in (2.2.1) is just a classical power utility maximisation problem in a Markovian stochastic factor model. Formally, the PDE (2.2.16) and optimal strategy can be obtained by a standard dynamic programming method and then seeking a separable solution $u(t,x,y) = U(x)(v(t,y))^\delta$. The Feynman-Kac theorem then gives the stochastic representation (2.2.14) of $v(\cdot,\cdot)$. The proof that $v(\cdot,\cdot)$ is a classical solution to its associated PDE then proceeds from defining v in its stochastic representation (2.2.14), where the smoothness of $v(\cdot,\cdot)$ does not require any boundedness assumption on λ. Applying Itô's formula to the $(\widetilde{\mathbb{P}}^0, \widetilde{\mathbb{F}})$-martingale $(\exp(-\frac{q}{2\delta}\int_0^t \lambda^2(Y_s)\,\mathrm{d}s)v(t,Y_t))_{t\in[0,T]}$, the PDE (2.2.16) with given terminal condition follows from the martingale property.

2.3 Classical logarithmic utility maximisation

The distortion representation for the value process of a classical logarithmic utility maximisation problem of the form (2.2.1) with $U(x) = \ln x$ is, in some sense, a degenerate case, and can be seen as a special case of the power utility problem of Section 2.2 when $q = 0$. Thus, we simply write down the results without repeating all details.

In the logarithmic utility case, the value process u admits the additive representation

$$u_t = \ln \widehat{X}_t + V_t, \quad V_t := \mathbb{E}^0\left[\frac{1}{2}\int_t^T \lambda_s^2\,\mathrm{d}s \middle| \widetilde{\mathcal{F}}_t\right], \quad t \in [0,T].$$

The optimal trading strategy for this optimal investment problem is then given by the familiar myopic form

$$\widehat{\pi}_t = \frac{\lambda_t}{\sigma_t}, \quad t \in [0,T].$$

For the Markovian case in (2.1.2)-(2.1.3), we have $V_t = v(t,Y_t)$, $t \in [0,T]$, where the function $v(\cdot,\cdot)$ satisfies the linear PDE

$$\frac{\partial v}{\partial t}(t,y) + a(y)\frac{\partial v}{\partial y}(t,y) + \frac{1}{2}b^2(y)\frac{\partial^2 v}{\partial y^2}(t,y) + \frac{1}{2}\lambda^2(y) = 0, \quad v(T,y) \equiv 0.$$

2.4 Classical exponential utility maximisation with random endowment

We now consider the scenario where, in addition to the market with a stock price process S as in Section 2.1, there is a European claim with $\widetilde{\mathcal{F}}_T$-measurable payoff G_T at time T. For example, $G_T = g(Y_T)$ for some function $g : \mathbb{R} \to \mathbb{R}$, where Y is the price process of some non-traded asset that is $\widetilde{\mathbb{F}}$-adapted and correlated with the traded asset S.

We consider an agent who has a random endowment of n ($n \in \mathbb{R}$) claims. This agent faces the problem of hedging her position using the traded stock S, which is correlated with the risk posed by the claim. Let $\theta \in L(S)$ (progressively measurable and S-integrable) denote the number of shares of stock held by the agent. Let $\pi = \theta S$ denote the process for wealth invested in the stock. The wealth process $X \equiv X^{x,\pi}$ of the agent with initial wealth $X_0 = x \in \mathbb{R}$ is given by

$$X_t = X_0 + \int_0^t \theta_u \, \mathrm{d}S_u = X_0 + \int_0^t \lambda_u \sigma_u \pi_u \, \mathrm{d}u + \int_0^t \sigma_u \pi_u \, \mathrm{d}W_u^0, \quad t \in [0, T].$$

The strategy π is assumed to satisfy the minimal conditions

$$\int_0^T (|\pi_s \mu_s| + |\pi_s \sigma_s|^2) \, \mathrm{d}s < \infty, \quad \mathbb{P}^0\text{-a.s.}. \tag{2.4.1}$$

The definition of the set of admissible strategies, denoted by $\mathcal{A}_{t,T}$ with convention $\mathcal{A} \equiv \mathcal{A}_{0,T}$, will be given when we formulate the exponential hedging problem, with specific additional conditions augmenting (2.4.1) where necessary.

In a classical exponential hedging problem, the objective of the agent is described by the value process u defined by

$$u_t := \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{t,T}} \mathbb{E}^0 \left[U \left(X_T + nG_T \right) | \mathcal{F}_t \right], \quad t \in [0, T], \tag{2.4.2}$$

where $\mathbb{E}^0$ denotes the expectation taken under the probability measure $\mathbb{P}^0$. The utility function $U : \mathbb{R} \to \mathbb{R}$ is specified as the exponential utility

$$U(x) = -\mathrm{e}^{-\alpha x}, \quad \alpha > 0.$$

For the set of admissible strategies $\mathcal{A}$ in (2.4.2), the following additional integrability condition is introduced:

$$\mathbb{E}^0 \left[\exp \left(c\alpha^2 \int_0^T \pi_t^2 \sigma_t^2 \, \mathrm{d}t \right) \right] < \infty, \quad \text{for some constant } c > 1. \tag{2.4.3}$$

Thus the set of admissible strategies is defined by

$$\mathcal{A}_{t,T} := \Big\{ \pi_s \text{ is } (\mathcal{F}_s)\text{-measurable, } \forall s \in [t,T] \;\Big|\; \pi \text{ satisfies (2.4.1), (2.4.3),} \\ \text{and is such that } \frac{\pi}{S} \cdot S \text{ is a } (\mathbb{Q},\mathbb{F})\text{-martingale for any } \mathbb{Q} \in \mathcal{M}_{e,loc}^{\mathbb{P}^0} \cap \mathcal{M}_f^{\mathbb{P}^0} \Big\},$$

where $\frac{\pi}{S} \cdot S$ denotes the stochastic integral with respect to S, and it is assumed that $\mathcal{M}_{e,loc}^{\mathbb{P}^0} \cap \mathcal{M}_f^{\mathbb{P}^0} \neq \emptyset$ is the set of equivalent local martingale measures with respect to $\mathbb{P}^0$ with finite relative entropy with $\mathbb{P}^0$.

The way to derive the martingale distortion representation for the value process for this exponential utility maximisation problem under the non-Markovian stochastic factor model is the same as for the power utility maximisation problem. Thus we simply write down the assumptions as well as the martingale distortion representation, for use in later chapters. The detailed derivations are given in Appendix A. Similar notations such as $\widetilde{\mathbb{P}}^0$ and M of earlier sections appear, but they have slightly different definitions in the exponential utility case.

Define the measure $\widetilde{\mathbb{P}}^0 \sim \mathbb{P}^0$ on $\widetilde{\mathcal{F}}_T$ by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}^0}{\mathrm{d}\mathbb{P}^0} = \mathcal{E}\left(-\int_0^{\cdot} \rho\lambda_s \,\mathrm{d}\widetilde{W}_s^0 \right)_T. \tag{2.4.4}$$

Assumption 2.4.1. Assume that the market price of risk $(\lambda_t)_{t\in[0,T]}$ satisfies

$$\mathbb{E}^0\left[\exp\left(4c \int_0^T \lambda_t^2 \,\mathrm{d}t \right) \right] < \infty. \tag{2.4.5}$$

where c is the same constant as in (2.4.3).

Introduce the positive $(\widetilde{\mathbb{P}}^0, \widetilde{\mathbb{F}})$-martingale M by

$$M_t := \widetilde{\mathbb{E}}^0\left[\exp\left(-\frac{1}{2\delta}\int_0^T \lambda_s^2 \,\mathrm{d}s - \frac{\alpha}{\delta} n G_T \right) \;\Big|\; \widetilde{\mathcal{F}}_t \right], \quad t \in [0,T], \tag{2.4.6}$$

and the distortion power δ in the exponential utility case is given by

$$\delta := \frac{1}{1-\rho^2}. \tag{2.4.7}$$

Then M has representation

$$M_t = M_0 + \int_0^t M_s \xi_s \,\mathrm{d}\widetilde{B}_s^0, \quad t \in [0,T], \tag{2.4.8}$$

for some ξ, where $\widetilde{B}_{\cdot}^0 = \widetilde{W}_{\cdot}^0 + \int_0^{\cdot} \rho\lambda_s \,\mathrm{d}s$ is standard $\widetilde{\mathbb{P}}^0$-Brownian motion. Note that M is well-defined because $\delta > 0$.

Assumption 2.4.2. It is assumed that

$$\mathbb{E}^0\left[\exp\left(c_\xi \int_0^T \xi_t^2 \,\mathrm{d}t\right)\right] < \infty, \tag{2.4.9}$$

where

$$c_\xi = \max\left\{\frac{c}{c-1}(\rho^2\delta^2 + \frac{1}{2}\overline{\rho}^2\delta^2), 4c\rho^2\delta^2\right\}.$$

Proposition 2.4.3. *Under Assumptions 2.4.1 and 2.4.2, the value process u admits the martingale distortion representation*

$$\begin{aligned} u_t &= -\mathrm{e}^{-\alpha\widehat{X}_t} V_t^\delta, \quad t \in [0,T], \\ V_t &:= \widetilde{\mathbb{E}}^0\left[\exp\left(-\frac{1}{2\delta}\int_t^T \lambda_s^2 \,\mathrm{d}s - \frac{\alpha}{\delta} n G_T\right) \,\middle|\, \widetilde{\mathcal{F}}_t\right], \end{aligned} \tag{2.4.10}$$

where $\widetilde{\mathbb{E}}^0$ denotes expectation taken under the probability measure $\widetilde{\mathbb{P}}^0$ defined in (2.4.4)*, and δ is as in* (2.4.7)*. The optimal strategy for the exponential utility maximisation problem* (2.4.2) *is given by*

$$\widehat{\pi}_t = \frac{\lambda_t + \rho\delta\xi_t}{\alpha\sigma_t}, \quad t \in [0,T], \tag{2.4.11}$$

where ξ is the integrand in the martingale representation of M in (2.4.8)*.*

A byproduct of Proposition 2.4.3 is the indifference selling price and the optimal hedging strategy of the option G_T. The **indifference selling price** p is defined such that the utility maximising agent is indifferent between either selling one option at this price and optimally investing with the initial wealth plus the proceedings, or just optimally investing with the initial wealth. The **optimal hedging strategy for shorting one option** is defined as the additional wealth invested in the traded asset as a result of selling the option, which is

$$\widehat{\pi}_t^h = \widehat{\pi}_t^{(-1)} - \widehat{\pi}_t^{(0)},$$

where $\widehat{\pi}^{(-1)}$ is the optimal trading strategy (2.4.11) when the agent sells one option ($n = -1$), and $\widehat{\pi}^{(0)}$ is the optimal trading strategy when the agent does not have the option position ($n = 0$).

Proposition 2.4.4. *The indifference selling price p at time zero of the European claim with $\widetilde{\mathcal{F}}_T$-measurable payoff G_T is given by*

$$p = \frac{\delta}{\alpha}(\ln V_0^{(-1)} - \ln V_0^{(0)}), \tag{2.4.12}$$

where $V^{(-1)}$ (respectively, $V^{(0)}$) is defined in (2.4.10) *with $n=-1$ (respectively, $n=0$). The optimal hedging strategy $\widehat{\pi}^h$ for a short position in one claim is given by*

$$\widehat{\pi}^h_t = \frac{\rho\delta\big(\xi^{(-1)}_t - \xi^{(0)}_t\big)}{\alpha\sigma_t}, \tag{2.4.13}$$

where $\xi^{(-1)}$ (respectively, $\xi^{(0)}$) is defined in (2.4.8) *with $n=-1$ (respectively, $n=0$) in* (2.4.6).

Proof. The indifference price follows from its definition and Proposition 2.4.3. The maximal utility of the agent shorting the option with initial wealth x should be equal to her maximal utility without the position in the option with initial wealth $x-p$:

$$-\mathrm{e}^{-\alpha x}(V_0^{(-1)})^\delta = -\mathrm{e}^{-\alpha(x-p)}(V_0^{(0)})^\delta,$$

which yields the indifference price (2.4.12). The optimal hedging strategy (2.4.13) follows directly from its definition. □

Remark 2.4.5 (The Markovian case). In a Markovian stochastic factor model (2.1.2)-(2.1.3), the value function u and optimal strategies π have the following explicit representations in terms of the solution to a linear PDE, as in Monoyios [41]. In this case, the additional assumption on the set of admissible strategies (2.4.3) is not needed, and all we need to assume for the distortion representation (2.4.14) in Proposition 2.4.6 is that $v(\cdot,\cdot)$ is well-defined and $\widetilde{\mathbb{P}}^0$ is a well-defined probability measure equivalent to $\mathbb{P}^0$ on $\widetilde{\mathcal{F}}_T$.

Proposition 2.4.6 (The Markovian case). *The value process u admits the martingale distortion representation*

$$\begin{aligned} u_t &= u(t,\widehat{X}_t,Y_t) = -\mathrm{e}^{-\alpha\widehat{X}_t}\left(v(t,Y_t)\right)^\delta, \\ v(t,Y_t) &:= \widetilde{\mathbb{E}}^0\left[\exp\left(-\frac{1}{2\delta}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s - \frac{\alpha}{\delta}ng(Y_T)\right)\,\middle|\, Y_t\right], \quad t\in[0,T], \end{aligned} \tag{2.4.14}$$

with δ as in (2.4.7). *The optimal strategy for the exponential utility maximisation problem* (2.4.2) *is given by*

$$\widehat{\pi}_t = \frac{\lambda(Y_t)+\rho\delta\xi(t,Y_t)}{\alpha\sigma(Y_t)}, \quad \textit{where } \xi(t,Y_t) = \frac{b(Y_t)}{v(t,Y_t)}\frac{\partial v}{\partial y}(t,Y_t),\ t\in[0,T]. \tag{2.4.15}$$

The function $v(\cdot,\cdot)$ satisfies the linear PDE

$$\begin{aligned} &\frac{\partial v}{\partial t}(t,y) + \left(a(y)-\rho\lambda(y)b(y)\right)\frac{\partial v}{\partial y}(t,y) \\ &\qquad + \frac{1}{2}b^2(y)\frac{\partial^2 v}{\partial y^2}(t,y) - \frac{1}{2\delta}\lambda^2(y)v(t,y) = 0, \end{aligned} \tag{2.4.16}$$

with $v(T,y) = \exp\left(-\frac{\alpha}{\delta}ng(y)\right)$.

Chapter 3

Multiple priors power utility maximisation in non-Markov stochastic volatility models

3.1 Introduction

In this chapter, we study multiple priors (or worst case) power utility maximisation problems of the form (1.1.1), in a non-Markovian stochastic factor model. Under the so-called reference measure $\mathbb{P}^0$, the dynamics in this model are of the same form as the generic model (2.1.1) of Chapter 2:

$$\mathrm{d}S_t = S_t\mu_t\,\mathrm{d}t + S_t\sigma_t\,\mathrm{d}W_t^0, \tag{3.1.1}$$

where the drift μ and volatility σ are $\widetilde{\mathbb{F}}$-adapted processes. A Markov version was studied by Hernández-Hernández and Schied [24], which produced a PDE representation via stochastic control methods.

Here, we show that a non-Markov version of this model can be studied and a semi-explicit representation for the value process can be obtained via the martingale distortion approach of Chapter 2. The (robust) value function, optimal strategy and "least favourable model" will be expressed via a solution to a BSDE (Proposition 3.3.1). The PDE representation in [24] can then be reproduced for the Markovian case.

We consider a model-ambiguous scenario, where the set of models $\mathcal{P}$ is defined by

$$\mathcal{P} := \Bigg\{ \mathbb{P} \sim \mathbb{P}^0 \text{ on } \mathcal{F}_T \;\Bigg|\; \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^0}\Big|_{\mathcal{F}_t} = D_t = \mathcal{E}\left(\int_0^\cdot \eta_s\,\mathrm{d}W_s^0 + \int_0^\cdot \eta_s^\perp\,\mathrm{d}W_s^{0,\perp}\right)_t, \\ t\in[0,T],\ \boldsymbol{\eta} = (\eta,\eta^\perp) \in \mathcal{N} \equiv \mathcal{N}_{0,T} \Bigg\}. \tag{3.1.2}$$

The set $\mathcal{N}_{s,T}$ for $0 \leq s \leq t \leq T$ denotes the set of all $\widetilde{\mathbb{F}}$-adapted [1] processes $(\boldsymbol{\eta}_t)_{t\in[s,T]}$ such that $\boldsymbol{\eta}_t \in \Gamma$, $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}^0$-a.e., where $\Gamma \subset \mathbb{R}^2$ is some fixed compact [2] convex set. Since there is a one-to-one correspondence between measures $\mathbb{P} \in \mathcal{P}$ and processes $\boldsymbol{\eta} \in \mathcal{N}$ (up to $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}^0$-nullsets), $\boldsymbol{\eta}$ can then be used to characterise models $\mathbb{P}$ in the optimisation problem that we define further below. The set of measures corresponding to $\boldsymbol{\eta} \in \mathcal{N}_{s,T}$ will be denoted by $\mathcal{P}_{s,T}$, with the usual convention $\mathcal{P} \equiv \mathcal{P}_{0,T}$.

The objective of the agent is described by a robust value function, which, given initial wealth $x > 0$, is defined by

$$u(x) := \sup_{\pi\in\mathcal{A}} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}\left[U\left(X_T\right)\right], \quad x > 0, \tag{3.1.3}$$

where $\mathbb{E}[\cdot]$ denotes expectation taken under $\mathbb{P} \in \mathcal{P}$, $\mathcal{A}$ denotes the set of admissible strategies defined further below, and the utility function $U : (0, \infty) \to \mathbb{R}$ is specified as the power utility

$$U(x) = \frac{x^p}{p}, \quad p < 1,\ p \neq 0.$$

For concreteness, we shall assume $p < 0$ throughout this chapter. In the case when $p \in (0, 1)$, the approach used in this chapter goes through largely unaltered, by changing the optimisations over $\boldsymbol{\eta}$ in (3.3.1) as well as (3.3.3) from supremum to infimum, and with different sets of admissible strategies and assumptions on the parameter processes. The differences in sets of admissible strategies and assumptions will be pointed out later in Remark 3.2.4.

For the set of admissible strategies $\mathcal{A}$ in (3.1.3), the following additional integrability condition is introduced:

$$\mathbb{E}\left[\exp\left(cp^2 \int_0^T \pi_t^2 \sigma_t^2 \,\mathrm{d}t\right)\right] < \infty, \quad \text{for some constant } c > 1. \tag{3.1.4}$$

Thus for the multiple priors utility maximisation problem (3.1.3), the set of admissible strategies is defined by

$$\mathcal{A}_{t,T} := \Big\{\pi_s \text{ is } (\mathcal{F}_s)\text{-progressively measurable, } \forall s \in [t, T] \;\Big|\; \pi \text{ satisfies (2.1.4) and (3.1.4)}\Big\}. \tag{3.1.5}$$

[1] We are concerned, in essence, with drift uncertainty in models of the form (3.1.1), so it is natural to consider all models which remain in this class. Indeed, in a Markovian version of (3.1.1), considered by Hernández-Hernández and Schied [24, 25], the stochastic control approach to the robust problem led to optimal models which remained in the same class as the original reference model. There may be some loss of generality, however, we are only looking at models with the same structure.

[2] The compactness condition can be lifted in some cases, but would lead to other regularity assumptions being needed.

It can be easily verified that condition (3.1.4) does not destroy the convexity of the set $\mathcal{A}$ of admissible strategies needed for the application of the minimax theorem.

In the rest of this chapter, the problem (3.1.3) will be solved by first using a minimax theorem to interchange the maximisation over strategies and minimisation over models. Under each model $\mathbb{P}$, we get a stochastic factor model of the form that was studied in Chapter 2. This allows us to obtain a distortion representation (Proposition 3.2.3) for an investment problem under one of those models. Then, we minimise over models, and get a BSDE representation for the robust value process, optimal strategy and "least favourable model" (Proposition 3.3.1). As an example, a Markovian stochastic factor model will be studied where the robust value function and optimal strategy will be expressed via solution to a PDE.

3.2 A martingale distortion representation under any possible model

Since power utility is a utility function satisfying the Inada conditions ($U'(0) = \infty$, $U'(\infty) = 0$) and has asymptotic elasticity ($AE(U) := \limsup_{x\to\infty} \frac{xU'(x)}{U(x)}$) less than 1, the set of admissible strategies $\mathcal{A}$ is convex (needed in [59, Lemma 3.4] for the application of the minimax theorem), and the set of models $\mathcal{P}$ in (3.1.2) satisfies Schied and Wu [59, Assumption 2.1] in the sense that $\mathcal{P}$ is convex, $\mathbb{P} \ll \mathbb{P}^0$, $\exists \mathbb{P} \sim \mathbb{P}^0$ on $\mathcal{F}_T$ and $\{\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^0}\}$ is closed in $L^0(\mathbb{P}^0)$, so the duality results in [59] are applicable. Particularly, the minimax theorem applies and the maximisation over strategies and minimisation over probability measures in (3.1.3) can be interchanged. Thus the robust value function (3.1.3) may be rewritten as

$$u(x) = \inf_{\mathbb{P}\in\mathcal{P}} u^{\mathbb{P}}(x), \tag{3.2.1}$$

where $u^{\mathbb{P}}(x)$ denotes the value function of an optimal investment problem under any fixed measure $\mathbb{P} \in \mathcal{P}$, so defined by

$$u^{\mathbb{P}}(x) := \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[U\left(X_T\right)\right], \quad x > 0. \tag{3.2.2}$$

The dynamic value processes corresponding to (3.2.1) and (3.2.2) are defined by

$$u_t := \operatorname*{ess\,inf}_{\mathbb{P}\in\mathcal{P}_{t,T}} u_t^{\mathbb{P}}, \tag{3.2.3}$$

$$u_t^{\mathbb{P}} := \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{t,T}} \mathbb{E}[U(X_T)|\mathcal{F}_t], \qquad t \in [0,T]. \tag{3.2.4}$$

Under $\mathbb{P}$, the stock process (2.1.1) has dynamics

$$\mathrm{d}S_t = S_t\sigma_t\left((\lambda_t+\eta_t)\,\mathrm{d}t + \mathrm{d}W_t\right),$$

and the correlated $\mathbb{P}$-Brownian motion can be written as $\widetilde{W} = \rho W + \overline{\rho}W^{\perp}$, where W and $W^{\perp}$ denote independent $\mathbb{P}$-Brownian motions given by

$$W_t = W_t^0 - \int_0^t \eta_s\,\mathrm{d}s, \qquad W_t^{\perp} = W_t^{0,\perp} - \int_0^t \eta_s^{\perp}\,\mathrm{d}s, \qquad t\in[0,T].$$

The new MPR $\lambda+\eta$ and the volatility σ of the stock under the measure $\mathbb{P}$ are both $\widetilde{\mathbb{F}}$-adapted. Thus, the martingale distortion representation given in (2.2.10) is applicable for the utility maximisation problem (3.2.4) under each fixed model $\mathbb{P}$, with some additional assumptions on the MPR process and the martingale in the martingale distortion representation. Define the measure $\widetilde{\mathbb{P}}\sim\mathbb{P}$ on $\widetilde{\mathcal{F}}_T$ by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(-\int_0 q\rho(\lambda_s+\eta_s)\,\mathrm{d}\widetilde{W}_s\right)_T, \tag{3.2.5}$$

where the constant q is as defined in (2.2.4).

Assumption 3.2.1. Assume the market price of risk $(\lambda_t)_{t\in[0,T]}$ satisfies

$$\mathbb{E}\left[\exp\left(8cq^2\int_0^T \lambda_t^2\,\mathrm{d}t\right)\right] < \infty, \tag{3.2.6}$$

where c is the same constant as in (3.1.4).

Define the positive $(\widetilde{\mathbb{P}},\widetilde{\mathbb{F}})$-martingale M by [3]

$$M_t := \widetilde{\mathbb{E}}\left[\exp\left(-\frac{q}{2\delta}\int_0^T(\lambda_s+\eta_s)^2\,\mathrm{d}s\right)\;\middle|\;\widetilde{\mathcal{F}}_t\right], \quad t\in[0,T],$$

where δ is as in (2.2.7). Then M has representation

$$M_t = M_0 + \int_0^t M_s\xi_s\,\mathrm{d}\widetilde{B}_s, \quad t\in[0,T], \tag{3.2.7}$$

for some ξ, where $\widetilde{B}_\cdot = \widetilde{W}_\cdot + \int_0^\cdot q\rho(\lambda_s+\eta_s)\,\mathrm{d}s$ is standard $\widetilde{\mathbb{P}}$-Brownian motion.

Assumption 3.2.2. It is assumed that

$$\mathbb{E}\left[\exp\left(c_\xi\int_0^T \xi_t^2\,\mathrm{d}t\right)\right] < \infty, \tag{3.2.8}$$

where

$$c_\xi = \max\left\{\frac{c}{c-1}(\rho^2\delta^2 + \frac{1}{2}\overline{\rho}^2\delta^2), 4cq^2\rho^2\delta^2\right\}.$$

[3] This is the analogue of the martingale M defined in Chapter 2 in (2.2.6). For ease of notation, we use the same symbol for this process as for the generic martingale M in Chapter 2. The same would apply to other symbols.

Proposition 3.2.3. *Under Assumption 3.2.1 and 3.2.2, the value process* $u^{\mathbb{P}}$ *in* (3.2.4) *admits the martingale distortion representation*

$$u_t^{\mathbb{P}} = \frac{\widehat{X}_t^p}{p}\left(\widetilde{\mathbb{E}}\left[\exp\left(-\frac{q}{2\delta}\int_t^T(\lambda_s+\eta_s)^2\,\mathrm{d}s\right)\,\middle|\,\widetilde{\mathcal{F}}_t\right]\right)^{\delta}, \quad t\in[0,T], \tag{3.2.9}$$

where $\widetilde{\mathbb{P}}$ *is defined in* (3.2.5)*,* q *and* δ *are as in* (2.2.4) *and* (2.2.7)*. The optimal trading strategy for the optimal investment problem* (3.2.2) *is given by*

$$\widehat{\pi}_t^{\mathbb{P}} = \frac{\lambda_t+\eta_t+\rho\delta\xi_t}{(1-p)\sigma_t}, \quad t\in[0,T], \tag{3.2.10}$$

where ξ *is the integrand in the martingale representation of* M *in* (3.2.7)*.*

Proof. It is trivial to verify that Assumption 3.2.1 and the boundedness of $\boldsymbol{\eta}$ imply Assumption 2.2.2 with MPR $\lambda+\eta$ under the measure $\mathbb{P}$. Combined with Assumption 3.2.2, we can apply Proposition 2.2.4 to the optimal investment problem (3.2.4) under each fixed measure $\mathbb{P}$. □

Remark 3.2.4 (The case $p\in(0,1)$). For $p\in(0,1)$, in line with Remark 2.2.5, the following amendments to the definition of admissible policies and to the conditions in Assumptions 3.2.1 and 3.2.2 are needed: (3.1.5) is replaced by

$$\mathcal{A}_{t,T} := \left\{\pi_s \text{ is } (\mathcal{F}_s)\text{-progressively measurable, } \forall s\in[t,T]\;\middle|\;\pi \text{ satisfies (2.1.4)}\right\},$$

(3.2.6) is replaced by

$$\mathbb{E}\left[\exp\left(4q^2\int_0^T\lambda_t^2\,\mathrm{d}t\right)\right]<\infty,$$

and additionally,

$$\widetilde{\mathbb{E}}\left[\exp\left(-\frac{q}{\delta}\int_0^T\lambda_t^2\,\mathrm{d}t\right)\right]<\infty,$$

and c_ξ in (3.2.8) is replaced by

$$c_\xi = \frac{2}{(1-p)^2}\rho^2\delta^2+\overline{\rho}^2\delta^2.$$

3.3 A BSDE representation of the value process

By Proposition 3.2.3 and (3.2.3), the robust value process may be written in the form

$$u_t = \operatorname*{ess\,inf}_{\mathbb{P}\in\mathcal{P}_{t,T}} u_t^{\mathbb{P}} = \frac{\widehat{X}_t^p}{p}V_t^\delta, \quad t\in[0,T],$$

where the $\widetilde{\mathbb{F}}$-adapted value process $(V_t)_{t\in[0,T]}$ is associated with the auxiliary problem under $\widetilde{\mathbb{P}}$:

$$V_t := \operatorname*{ess\,sup}_{\boldsymbol{\eta}\in\mathcal{N}_{t,T}} \widetilde{\mathbb{E}}\left[\exp\left(-\frac{q}{2\delta}\int_t^T(\lambda_s+\eta_s)^2\,\mathrm{d}s\right)\;\middle|\;\widetilde{\mathcal{F}}_t\right],\quad t\in[0,T]. \tag{3.3.1}$$

Denote by $\mathcal{S}^2$ the set of progressively measurable processes ψ such that

$$\mathbb{E}^0\left[\sup_{t\in[0,T]}|\psi_t|^2\right]<\infty,$$

and by $\mathbb{H}^2$ the set of progressively measurable processes ψ such that

$$\mathbb{E}^0\left[\int_0^T|\psi_t|^2\,\mathrm{d}t\right]<\infty.$$

Then, with a bounded market price of risk process λ, one can obtain a BSDE representation for V, as in the following proposition.

Proposition 3.3.1. *Under Assumption 3.2.2, and assuming additionally that λ is bounded, then $(V,\zeta)\in\mathcal{S}^2\times\mathbb{H}^2$ is the unique solution to the BSDE*

$$\mathrm{d}V_t = -f_t(V_t,\zeta_t)\,\mathrm{d}t+\zeta_t\,\mathrm{d}\widetilde{W}^0_t,\qquad V_T=1, \tag{3.3.2}$$

where the generator $(f_t(\cdot,\cdot))_{t\in[0,T]}$ is defined by

$$f_t(v,z) := \operatorname*{ess\,sup}_{\boldsymbol{\eta}\in\mathcal{N}}\left\{-\frac{q}{2\delta}(\lambda_t+\eta_t)^2v-[q\rho\lambda_t-\rho(1-q)\eta_t-\overline{\rho}\eta_t^{\perp}]z\right\}. \tag{3.3.3}$$

The optimal strategy for the robust optimal investment problem (3.1.3) *is then given by*

$$\widehat{\pi}_t=\frac{\lambda_t+\widehat{\eta}_t+\rho\delta\xi_t}{(1-p)\sigma_t},\qquad \textit{with } \xi_t=\frac{\zeta_t}{V_t},\quad t\in[0,T], \tag{3.3.4}$$

where $\widehat{\boldsymbol{\eta}}=(\widehat{\eta},\widehat{\eta}^{\perp})$ is the optimal control that realises the essential supremum in (3.3.3).

Proof. Define the process $V^{\boldsymbol{\eta}}$ by

$$V^{\boldsymbol{\eta}}_t := \widetilde{\mathbb{E}}\left[\exp\left(-\frac{q}{2\delta}\int_t^T(\lambda_s+\eta_s)^2\,\mathrm{d}s\right)\;\middle|\;\widetilde{\mathcal{F}}_t\right]=\mathrm{e}^{N_t/\delta}M_t,\quad t\in[0,T]$$

with terminal value $V^{\boldsymbol{\eta}}_T=1$, where $N_\cdot := \frac{q}{2}\int_0^\cdot(\lambda_s+\eta_s)^2\,\mathrm{d}s$. By Itô's formula, the dynamics of $V^{\boldsymbol{\eta}}_t$ under the reference measure $\mathbb{P}^0$ are

$$\mathrm{d}V^{\boldsymbol{\eta}}_t = V^{\boldsymbol{\eta}}_t\xi_t\,\mathrm{d}\widetilde{W}^0_t+\left\{\frac{q}{2\delta}(\lambda_t+\eta_t)^2V^{\boldsymbol{\eta}}_t+[q\rho\lambda_t-\rho(1-q)\eta_t-\overline{\rho}\eta_t^{\perp}]V^{\boldsymbol{\eta}}_t\xi_t\right\}\mathrm{d}t.$$

Then for any η, with the change of variable $\zeta_t^{\boldsymbol{\eta}} = V_t^{\boldsymbol{\eta}}\xi_t$, the pair $(V^{\boldsymbol{\eta}}, \zeta^{\boldsymbol{\eta}})$ can be expressed as a solution to the BSDE

$$\mathrm{d}V_t^{\boldsymbol{\eta}} = -f_t^{\boldsymbol{\eta}}(V_t^{\boldsymbol{\eta}}, \zeta_t^{\boldsymbol{\eta}})\,\mathrm{d}t + \zeta_t^{\boldsymbol{\eta}}\,\mathrm{d}\widetilde{W}_t^0, \qquad V_T^{\boldsymbol{\eta}} = 1, \tag{3.3.5}$$

where

$$f_t^{\boldsymbol{\eta}}(V_t^{\boldsymbol{\eta}}, \zeta_t^{\boldsymbol{\eta}}) := -\frac{q}{2\delta}(\lambda_t + \eta_t)^2 V_t^{\boldsymbol{\eta}} - [q\rho\lambda_t - \rho(1-q)\eta_t - \overline{\rho}\eta_t^{\perp}]\zeta_t^{\boldsymbol{\eta}}.$$

Since $f_t^{\boldsymbol{\eta}}(v, z)$ is linear with respect to v, z and $\lambda_t, \eta_t, \eta_t^{\perp}$ are bounded, $f_t^{\boldsymbol{\eta}}(v, z)$ is uniformly Lipschitz in v, z. Moreover, $\{f_t^{\boldsymbol{\eta}}(0,0), t \in [0,T]\} \in \mathbb{H}^2$ (as $f_t^{\boldsymbol{\eta}}(0,0) \equiv 0$) and the terminal condition is constant thus in L^2, so according to Touzi [64, Theorem 10.2], with standard parameters $(f^{\boldsymbol{\eta}}, 1)$, there exists a unique solution $(V_t^{\boldsymbol{\eta}}, \zeta_t^{\boldsymbol{\eta}}) \in \mathcal{S}^2 \times \mathbb{H}^2$ to the BSDE (3.3.5). Moreover, by the measurable selection results in [8, Theorem A.10.3], there exists $\widehat{\boldsymbol{\eta}} \in \mathcal{N}$ realising the essential supremum of $f_t^{\boldsymbol{\eta}}(v, z)$ over $\boldsymbol{\eta} \in \mathcal{N}$, that is,

$$f_t(v, z) := \operatorname*{ess\,sup}_{\boldsymbol{\eta} \in \mathcal{N}} f_t^{\boldsymbol{\eta}}(v, z) = f_t^{\widehat{\boldsymbol{\eta}}}(v, z).$$

Then by the comparison theorem [64, Theorem 10.4], because the terminal condition is constant, the value process V in (3.3.1) is given as

$$V_t = \operatorname*{ess\,sup}_{\boldsymbol{\eta} \in \mathcal{N}} V_t^{\boldsymbol{\eta}} = V_t^{\widehat{\boldsymbol{\eta}}},$$

and $(V, \zeta) = (V^{\widehat{\boldsymbol{\eta}}}, \zeta^{\widehat{\boldsymbol{\eta}}})$ is the unique solution to the BSDE (3.3.5) with $\boldsymbol{\eta} = \widehat{\boldsymbol{\eta}}$, which is BSDE (3.3.2).

□

In the following section, we give an example of a Markovian stochastic factor model, where the value function and optimal strategy can be characterised in a more explicit form via the solution to a non-linear PDE, either by a direct application of the dynamic programming principle in the optimisation over models problem in (3.3.1) (which does not require bounded MPR λ), or by specialising to the Markov model in the BSDE representation (3.3.2).

3.4 The Markovian case

Under the reference measure $\mathbb{P}^0$, we begin with a model of a similar form to the generic model (2.1.2)–(2.1.3) of Chapter 2:

$$\mathrm{d}S_t = S_t\mu(Y_t)\,\mathrm{d}t + S_t\sigma(Y_t)\,\mathrm{d}W_t^0, \tag{3.4.1}$$

$$\mathrm{d}Y_t = a(Y_t)\,\mathrm{d}t + b(Y_t)\,\mathrm{d}\widetilde{W}_t^0. \tag{3.4.2}$$

An existing approach to the robust optimal investment problem (3.1.3) with power utility and $p < 0$ in the Markovian stochastic factor model was studied in Hernández-Hernández and Schied [24] with very stringent conditions on the parameters. The authors first applied the duality theory of Schied and Wu [59] to transform the original problem of finding the saddle-point of a two-dimensional optimisation problem to the dual problem of finding the infimum over a two dimensional set, and then applied stochastic control techniques to the dual problem. As a result, the robust value function and optimal strategy are characterised by the solution of a non-linear PDE. Here, we show how these results can be obtained from our earlier results, with weaker conditions on the parameters.

For the Markov case, the assumption that the MPR process λ of the stock is bounded is not necessary since the optimisation problem (3.3.1) can be solved using a stochastic control approach without reference to BSDE theories.

Assuming that Assumptions 3.2.1 and 3.2.2 hold, Proposition 3.2.3 yields the value process $u^{\mathbb{P}}$ in (3.2.9) and the optimal strategy $\widehat{\pi}^{\mathbb{P}}$ in (3.2.10) for the optimal investment problem under fixed model $\mathbb{P}$ with $\lambda_t \equiv \lambda(Y_t)$, $\sigma_t \equiv \sigma(Y_t)$. Then the minimisation over models problem (3.3.1) becomes the following stochastic control problem where value process (V_t) defined in (3.3.1) is a function of time and the current state of process Y:

$$v(t,y) = \sup_{\boldsymbol{\eta}\in\mathcal{N}_{t,T}} \widetilde{\mathbb{E}}\left[\exp\left(-\frac{q}{2\delta}\int_t^T (\lambda(Y_s)+\eta_s)^2\,\mathrm{d}s\right)\;\middle|\; Y_t = y\right],$$

where the dynamics of the process Y under the measure $\widetilde{\mathbb{P}}$ are

$$\mathrm{d}Y_t = \left(a(Y_t) + \rho(1-q)b(Y_t)\eta_t + \overline{\rho}b(Y_t)\eta_t^{\perp} - q\rho b(Y_t)\lambda(Y_t)\right)\,\mathrm{d}t + b(Y_t)\,\mathrm{d}\widetilde{B}_t.$$

The HJB equation associated with $v(t,y)$ is then

$$\begin{aligned}\frac{\partial v}{\partial t}(t,y) + a(y)\frac{\partial v}{\partial y}(t,y) + \frac{1}{2}b^2(y)\frac{\partial^2 v}{\partial y^2}(t,y) + f(t,y,v,b(y)\frac{\partial v}{\partial y}(t,y)) = 0,\\ v(T,y)\equiv 1,\end{aligned} \tag{3.4.3}$$

where

$$f(t,y,v,z) = \sup_{\boldsymbol{\eta}\in\Gamma}\left\{-\frac{q}{2\delta}(\lambda(y)+\eta)^2 v - [q\rho\lambda(y) - \rho(1-q)\eta - \overline{\rho}\eta^{\perp}]z\right\}. \tag{3.4.4}$$

If $\widehat{\boldsymbol{\eta}}(t,y) = (\widehat{\eta}(t,y), \widehat{\eta}^{\perp}(t,y))$ realises the supremum in (3.4.4), $v \in C^{1,2}([0,T]\times\mathbb{R}) \times C([0,T]\times\mathbb{R})$ follows from its boundedness and the assumption that there exists a unique strong solution to the SDEs satisfied by the asset prices. Thus Itô's formula

may be used to the $\widetilde{\mathbb{P}}^{\widehat{\boldsymbol{\eta}}}$-martingale $\exp\big\{-\frac{q}{2\delta}\int_0^t(\lambda(Y_s)+\widehat{\eta}(s,Y_s))^2\,\mathrm{d}s\big\}v(t,Y_t)$. The PDE (3.4.3) then follows from the martingale property. The rest is then a classical stochastic verification argument.

From the general relation (2.2.15) between the optimal strategy and the distortion solution in the Markovian case, we have the optimal trading strategy expressed via solutions to the PDE (3.4.3):

$$\widehat{\pi}_t = \frac{\lambda(Y_t)+\widehat{\eta}(t,Y_t)+\rho\delta\xi(t,Y_t)}{(1-p)\sigma(Y_t)}, \qquad \text{with } \xi(t,y)=\frac{b(y)}{v(t,y)}\frac{\partial v}{\partial y}(t,y), \quad (3.4.5)$$

where $\widehat{\eta}(t,y)$ is the optimal feedback control function realising the supremum in (3.4.4).

Remark 3.4.1 (A Markovian BSDE approach). Assume that λ is bounded and that Assumption 3.2.2 holds, the above PDE (3.4.3) can also be obtained from a Markovian BSDE approach. The Markovian BSDE corresponds to the case when the coefficients of the BSDE (3.3.5) and (3.3.2) are such that

$$f_t^{\boldsymbol{\eta}}(v,z)=f^{\boldsymbol{\eta}}(t,Y_t,v,z), \qquad f_t(v,z)=\sup_{\boldsymbol{\eta}\in\Gamma} f^{\boldsymbol{\eta}}(t,Y_t,v,z)=f(t,Y_t,v,z),$$

where $f^{\boldsymbol{\eta}}$, f are continuous real-valued functions on $[0,T]\times\mathbb{R}^3$ and process Y is as in (3.4.2). In this case, there is a PDE associated with the BSDE (3.3.2), and the robust value function and the optimal trading strategy can also be expressed via the solution to this PDE. The associated PDE is (3.4.3) but with bounded $\lambda(\cdot)$. By El Karoui et al. [15, Theorem 4.2], if $(V_t,\zeta_t)_{t\in[0,T]}$ is the solution to the BSDE (3.3.2), then the function $v(t,y)$ defined by $v(t,y)=V_t$ with $Y_t=y$ is a viscosity solution to the PDE (3.4.3). Furthermore, if for each $R>0$ there exists a continuous function $m_R:\mathbb{R}_+\to\mathbb{R}_+$ such that $m_R(0)=0$ and

$$|f(t,y,v,z)-f(t,y',v,z)|\le m_R(|y-y'|(1+z))$$

for all $t\in[0,T]$, $|y|$, $|y'|\le R$, and $|z|\le R$ for $z\in\mathbb{R}$, then $v(\cdot,\cdot)$ is the unique viscosity solution of the PDE (3.4.3). Conversely, by [15, Proposition 4.3], supposing $v(\cdot,\cdot)\in C^{1,2}([0,T)\times\mathbb{R})$ is a classical solution to the PDE (3.4.3) with linear growth conditions on $v(\cdot,\cdot)$ and $b(\cdot)\frac{\partial v}{\partial y}(\cdot,\cdot)$, then the pair (V,ζ) defined by

$$V_t=v(t,Y_t), \qquad \zeta_t=b(Y_t)\frac{\partial v}{\partial y}(t,Y_t), \quad t\in[0,T]$$

is the unique solution to BSDE (3.3.2). Thus by (3.3.4), the optimal trading strategy can be expressed via solutions to the PDE (3.4.3) and takes the same form as in (3.4.5), with bounded λ.

Chapter 4

Variational preferences logarithmic utility maximisation in stochastic factor models

4.1 Introduction

In this chapter, we take a less conservative approach than in Chapter 3 – we penalise plausible models according to their plausibilities. This appears to be difficult to achieve (and, to the best of our knowledge, this approach has not been well-developed) with power utility. We thus use, in this chapter, a logarithmic utility. We apply the martingale optimality principles again, in a non-Markov stochastic factor model setting, and get semi-explicit results. Then, specialising to a Markov case (and specifically, robust versions of Stein-Stein (SS) and Heston (Hes) stochastic volatility (SV) models), we get fully explicit results. This allows for fast computations of strategies which can then be tested numerically over simulated trajectories of the models. We then perform numerical tests of the achievable utility from robust and classical strategies, which will be referred to as robust SV (SS/Hes), classical SV (SS/Hes), re-calibrated Merton (modified Merton). The results indicate that, unless the agent has already-conservative estimates for the return of the stock, the robust strategy would give a more favourable distribution of the terminal utilities achieved, and would yield higher terminal utilities (than the other strategies compared here) most of the time. The robust strategy also seems to perform well in the existence of model error where we assume the true model is a Heston model but the agents are using Stein-Stein strategies.

Under the so-called reference measure $\mathbb{P}^0$, the dynamics in the stochastic factor model to be studied in this chapter are as in (3.1.1), which is again of the same

form as the generic model of Chapter 2. In the variational preferences approach, since the implausible models can be excluded by associating infinite penalties, it is possible to consider all probability measures that are equivalent to the reference measure. For any measure $\mathbb{P} \sim \mathbb{P}^0$, there exists a progressively measurable process $\boldsymbol{\eta} = (\eta, \eta^\perp) \in \mathcal{N} := \left\{ \boldsymbol{\eta} \;\middle|\; \int_0^t ||\boldsymbol{\eta}_s||^2 \, \mathrm{d}s < \infty \; \mathbb{P}^0\text{-a.s. for all t} \right\}$ such that

$$\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^0}\right|_{\mathcal{F}_t} = D_t = \mathcal{E}\left(\int_0 \eta_s \, \mathrm{d}W_s^0 + \int_0 \eta_s^\perp \, \mathrm{d}W_s^{0,\perp} \right)_t, \quad t \in [0,T], \quad \mathbb{P}^0\text{-a.s.}$$

Each measure $\mathbb{P}$ will then be penalised according to its plausibility by a penalty function $\gamma_{t,T}(\mathbb{P})$, which is chosen to take the form of an entropic penalty:

$$\gamma_{t,T}(\mathbb{P}) := \frac{1}{\beta}\mathcal{H}_{t,T}(\mathbb{P}|\mathbb{P}^0) = \frac{1}{\beta}\mathbb{E}\left[\ln \frac{D_T}{D_t} \middle| \mathcal{F}_t \right] = \frac{1}{2\beta}\mathbb{E}\left[\int_t^T ||\boldsymbol{\eta}_s||^2 ds \middle| \mathcal{F}_t \right], \quad \text{with constant } \beta > 0, \; t \in [0,T].$$

Denote by $\mathcal{P}_{t,T}$ the set of probability measures $\mathbb{P} \sim \mathbb{P}^0$ with $\mathcal{H}_{t,T}(\mathbb{P}|\mathbb{P}^0) < \infty$ a.s.. Note that $\gamma_{t,T}(\mathbb{P}^0) = 0$, which means that the reference model is the most plausible model. This form of penalty function was used by Hansen and Sargent [21] as well as Bordigoni et al. [5]. Relative entropy $\mathcal{H}_{t,T}(\mathbb{P}|\mathbb{P}^0)$ characterises the "distance" of any candidate measure $\mathbb{P}$ from the reference measure $\mathbb{P}^0$, and $\beta > 0$ characterises the agent's level of ambiguity aversion (confidence in the candidate measures): when $\beta \to 0$, the agent has low ambiguity aversion (is more confident with the reference measure) so that the objective (4.1.1) becomes the classical utility maximisation problem under reference measure $\mathbb{P}^0$, and when $\beta \to +\infty$, the agent has very high ambiguity aversion and considers all candidate measures as equally possible and takes the worst case of them.

The objective of the agent at time t is to maximise the utility functional which is the infimum of the expected utility from terminal wealth over the set of models penalised by the penalty function. The robust value process with starting time $t \in [0,T]$ is defined by

$$u_t = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{t,T}} \operatorname*{ess\,inf}_{\mathbb{P} \in \mathcal{P}_{t,T}} \left\{ \mathbb{E}\left[U\left(X_T\right) | \mathcal{F}_t \right] + \gamma_{t,T}(\mathbb{P}) \right\}, \quad t \in [0,T], \tag{4.1.1}$$

where the utility function $U : (0,\infty) \to \mathbb{R}$ is the logarithmic utility

$$U(x) = \ln x.$$

To define the set of admissible strategies, first introduce the additional integrability conditions

$$\mathbb{E}^0[X_T^{-\beta}] < \infty, \tag{4.1.2}$$

$$\mathbb{E}^0\left[\exp\left(c\beta^2\int_0^T \pi_t^2\sigma_t^2\,\mathrm{d}t\right)\right]<\infty,\quad \text{for some constant } c>1, \tag{4.1.3}$$

and

$$\mathbb{E}\left[\int_0^T \left(|\pi_t\mu_t|+|\pi_t\sigma_t|^2\right)\mathrm{d}t\right]<\infty. \tag{4.1.4}$$

Then the set of admissible strategies is defined by

$$\mathcal{A}_{t,T}:=\Big\{\pi_s \text{ is } (\mathcal{F}_s)\text{-progressively measurable, } \forall s\in[t,T]\;\Big|\;\pi \text{ satisfies (4.1.2), (4.1.3) and (4.1.4)}\Big\}.$$

Remark 4.1.1. Conditions (4.1.2), (4.1.3) and (4.1.4) are technical conditions for Propositions 4.2.1 and 4.3.2. Condition (4.1.2) is equivalent to the assumption that $\mathrm{e}^{-\beta U(X_T)}\in L^1(\mathbb{P}^0)$ with logarithmic utility, and for $\beta>0$, it roughly means that the admissible strategies shall not drive the wealth process to zero. Condition (4.1.4) is a sufficient condition for guaranteeing that $\mathbb{E}[U(X_T)]<\infty$ for any probability measure $\mathbb{P}\in\mathcal{P}_{0,T}$. Condition (4.1.3) corresponds to condition (2.2.2) under the reference measure $\mathbb{P}^0$ and is needed for applying Proposition 2.2.4 under the reference measure.

In the following, the robust problem (4.1.1) will be solved by being transformed to a classical problem with an adjusted (power) utility function (Proposition 4.2.1). Then a semi-explicit representation can be obtained for the robust value process (Proposition 4.3.2) using the martingale distortion approach. Examples will be provided where explicit solutions are obtained for the Stein-Stein and Heston volatility models. Numerical tests of the effectiveness of the robust strategies compared with the classical strategies (not considering model uncertainty) and re-calibrated Merton strategies are also conducted for the examples.

4.2 The robust value process in terms of the value process of a classical utility maximisation problem

Proposition 4.2.1. *The robust value process* (4.1.1) *can be written as*

$$u_t=-\frac{1}{\beta}\ln\left(-\beta\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{t,T}}\mathbb{E}^0\Big[-\frac{1}{\beta}\mathrm{e}^{-\beta U(X_T)}\Big|\mathcal{F}_t\Big]\right),\quad t\in[0,T].$$

A static version (at time 0) of the above result can be found in Schied [58, Example 3.3], using the duality relation between entropy and free energy

$$-\ln \mathbb{E}^0[\mathrm{e}^{-R}] = \inf_{\mathbb{P}\in\mathcal{P}} \left\{ \mathbb{E}[R] + \mathcal{H}_{0,T}(\mathbb{P}|\mathbb{P}^0) \right\}, \quad \text{for random variable } R.$$

In the following, a short proof of a dynamic version of the above duality will be provided, then Proposition 4.2.1 follows directly by taking $R = \beta U(X_T) = \beta \ln X_T$ in Lemma 4.2.2 and noticing that conditions (4.1.2), (4.1.4) and $\mathcal{H}_{0,T}(\mathbb{P}|\mathbb{P}^0) < \infty$ ensure that $\mathrm{e}^{-\beta U(X_T)} \in L^1(\mathbb{P}^0)$ and $U(X_T) \in L^1(\mathbb{P})$.

Lemma 4.2.2. *The following duality relation holds for random variable R assuming $R \in L^1(\mathbb{P})$ for any $\mathbb{P} \in \mathcal{P}_{t,T}$ and $\mathrm{e}^{-R} \in L^1(\mathbb{P}^0)$:*

$$-\ln \mathbb{E}^0[\mathrm{e}^{-R}|\mathcal{F}_t] = \operatorname*{ess\,inf}_{\mathbb{P}\in\mathcal{P}_{t,T}} \left\{ \mathbb{E}[R|\mathcal{F}_t] + \mathcal{H}_{t,T}(\mathbb{P}|\mathbb{P}^0) \right\}, \quad t \in [0,T].$$

Proof. The proof goes by first applying a change of measure and Jensen's inequality for conditional expectations to obtain that the left-hand-side is smaller than or equal to the right-hand-side. Then equality is attained by choosing some $\mathbb{P} = \mathbb{P}^*$ on the right-hand-side. In fact, by a change of measure and Jensen's inequality,

$$\begin{aligned}
-\ln \mathbb{E}^0[\mathrm{e}^{-R}|\mathcal{F}_t] &= -\ln \mathbb{E}\left[\frac{D_t}{D_T} \mathrm{e}^{-R} \Big| \mathcal{F}_t \right] \\
&= -\ln D_t - \ln \mathbb{E}[\mathrm{e}^{-R-\ln D_T}|\mathcal{F}_t] \\
&\le -\ln D_t + \mathbb{E}[-\ln \mathrm{e}^{-R-\ln D_T}|\mathcal{F}_t] \\
&= \mathbb{E}[R|\mathcal{F}_t] + \mathcal{H}_{t,T}(\mathbb{P}|\mathbb{P}^0).
\end{aligned}$$

Thus

$$-\ln \mathbb{E}^0[\mathrm{e}^{-R}|\mathcal{F}_t] \le \operatorname*{ess\,inf}_{\mathbb{P}\in\mathcal{P}_{t,T}} \left\{ \mathbb{E}[R|\mathcal{F}_t] + \mathcal{H}_{t,T}(\mathbb{P}|\mathbb{P}^0) \right\}.$$

By taking $\mathbb{P}^*$ defined by the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}^0} = D_T^* := \frac{\mathrm{e}^{-R}}{\mathbb{E}^0[\mathrm{e}^{-R}]},$$

and denoting the expectation taken under the measure $\mathbb{P}^*$ by $\mathbb{E}^*$, the right-hand-side is equal to

$$\begin{aligned}
\mathbb{E}^*[R|\mathcal{F}_t] + \mathcal{H}_{t,T}(\mathbb{P}^*|\mathbb{P}^0) &= -\ln D_t^* + \mathbb{E}^*[R + \ln D_T^*|\mathcal{F}_t] \\
&= -\mathbb{E}^*[\ln D_t^* + \ln \mathbb{E}^0[\mathrm{e}^{-R}]|\mathcal{F}_t] \\
&= -\mathbb{E}^*\left[\ln \frac{\mathbb{E}^0[\mathrm{e}^{-R}|\mathcal{F}_t]}{\mathbb{E}^0[\mathrm{e}^{-R}]} + \ln \mathbb{E}^0[\mathrm{e}^{-R}] \;\Big|\; \mathcal{F}_t \right] \\
&= -\ln \mathbb{E}^0[\mathrm{e}^{-R}|\mathcal{F}_t],
\end{aligned}$$

which proves the required dual relation. □

By Proposition 4.2.1, with logarithmic utility, the robust utility maximisation problem (4.1.1) can be transformed to a classical problem with a different utility function

$$U_A(x) = \frac{x^{-\beta}}{-\beta},$$

which is a power utility function with parameter $-\beta < 0$. The robust value process can be written as

$$u_t = -\frac{1}{\beta} \ln \left(-\beta \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{t,T}} \mathbb{E}^0[U_A(X_T)|\mathcal{F}_t] \right), \quad t \in [0,T]. \tag{4.2.1}$$

Define the value process for the classical optimal investment problem with the adjusted utility as

$$u_t^A := \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{t,T}} \mathbb{E}^0[U_A(X_T)|\mathcal{F}_t], \quad t \in [0,T]. \tag{4.2.2}$$

The robust problem in (4.2.1) can then be solved by finding the distortion representation for the above classical problem in (4.2.2) with adjusted utility $U_A : (0,\infty) \to \mathbb{R}$. Although the "*sup-inf*" structure that defines the variational preferences approach by considering a set of measures has been lost after the transformation, the agent's aversion towards model uncertainty has been reflected by the parameter β in the adjusted utility function of the resulting classical problem (4.2.2). As can be observed in the two examples in Section 4.5, for the agent with logarithmic utility who is concerned about model uncertainty, the optimal strategy $\widehat{\pi}$ may depend on the time horizon T and so may no longer be myopic.

4.3 A representation of the value process

Define the measure $\widetilde{\mathbb{P}}^0 \sim \mathbb{P}^0$ on $\widetilde{\mathcal{F}}_T$ by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}^0}{\mathrm{d}\mathbb{P}^0} = \mathcal{E}\left(-\int_0^{\cdot} q\rho\lambda_s \,\mathrm{d}\widetilde{W}_s^0 \right)_T, \tag{4.3.1}$$

where the constant q is defined by

$$q := \frac{\beta}{\beta+1} \in (0,1). \tag{4.3.2}$$

Under Assumption 4.3.1, Proposition 4.3.2 below follows from Proposition 2.2.4. The only difference is the additional integrability conditions (4.1.2) and (4.1.4) on the admissible strategies, but it is straightforward to verify under Assumption 4.3.1 that the optimal strategy indeed satisfies these conditions.

Assumption 4.3.1. (i) Assume the market price of risk λ satisfies

$$\mathbb{E}^0\left[\exp\left(c_\lambda \int_0^T \lambda_t^2 \,\mathrm{d}t\right)\right] < \infty, \tag{4.3.3}$$

$$\mathbb{E}\left[\int_0^T \lambda_t^2 \,\mathrm{d}t\right] < \infty. \tag{4.3.4}$$

where $c_\lambda = \max\{4cq^2, 4q^2+2\}$, and c is the same constant as in (4.1.3).

(ii) Define the positive $(\widetilde{\mathbb{P}}^0, \widetilde{\mathbb{F}})$-martingale M by

$$M_t := \widetilde{\mathbb{E}}^0\left[\exp\left(-\frac{q}{2\delta}\int_0^T \lambda_s^2 \,\mathrm{d}s\right) \,\middle|\, \widetilde{\mathcal{F}}_t\right] \in (0,1], \quad t \in [0,T],$$

where $\widetilde{\mathbb{E}}^0$ denotes the expectation taken under the probability measure $\widetilde{\mathbb{P}}^0$, and

$$\delta := \frac{1}{1-q\rho^2} \in (1,\infty). \tag{4.3.5}$$

Then M has representation

$$M_t = M_0 + \int_0^t M_s \xi_s \,\mathrm{d}\widetilde{B}_s^0, \tag{4.3.6}$$

for some ξ, where $\widetilde{B}_\cdot^0 = \widetilde{W}_\cdot^0 + \int_0^\cdot q\rho\lambda_s \,\mathrm{d}s$ is standard $\widetilde{\mathbb{P}}^0$-Brownian motion. It is assumed that

$$\mathbb{E}^0\left[\exp\left(c_\xi \int_0^T \xi_t^2 \,\mathrm{d}t\right)\right] < \infty, \tag{4.3.7}$$

$$\mathbb{E}\left[\int_0^T \xi_t^2 \,\mathrm{d}t\right] < \infty, \tag{4.3.8}$$

where

$$c_\xi = \max\left\{\frac{c}{c-1}(\rho^2\delta^2 + \frac{1}{2}\overline{\rho}^2\delta^2), \frac{10\beta^2+4\beta}{(1+\beta)^2}\rho^2\delta^2, 4cq^2\rho^2\delta^2, \frac{4c}{c-1}q^2\rho^2\delta^2\right\}.$$

Proposition 4.3.2. *Under Assumption 4.3.1, the value process u^A has the martingale distortion representation*

$$u_t^A = \frac{\widehat{X}_t^{-\beta}}{-\beta}\left(\widetilde{\mathbb{E}}^0\left[\exp\left(-\frac{q}{2\delta}\int_t^T \lambda_s^2 \,\mathrm{d}s\right) \,\middle|\, \widetilde{\mathcal{F}}_t\right]\right)^\delta, \quad t \in [0,T],$$

where $\widetilde{\mathbb{P}}^0$ is defined in (4.3.1)*, q and δ are as in* (4.3.2) *and* (4.3.5)*. The robust value process u in* (4.1.1) *then has the following representation:*

$$u_t = -\frac{1}{\beta}\ln\left(-\beta u_t^A\right) = \ln\widehat{X}_t - \frac{\delta}{\beta}\ln\widetilde{\mathbb{E}}^0\left[\exp\left(-\frac{q}{2\delta}\int_t^T \lambda_s^2 \,\mathrm{d}s\right) \,\middle|\, \widetilde{\mathcal{F}}_t\right], \ t \in [0,T].$$

The optimal trading strategy is given by

$$\widehat{\pi}_t = \frac{\lambda_t + \rho\delta\xi_t}{(1+\beta)\sigma_t}, \quad t \in [0,T], \tag{4.3.9}$$

where ξ is the integrand in the martingale representation of M in (4.3.6).

Proof. This proposition is a direct result of Proposition 2.2.4. The remaining is to check that the optimal strategy (4.3.9) satisfies the additional integrability conditions (4.1.2) and (4.1.4).

First, by the dynamics of the wealth process (2.1.5) and substituting in the optimal strategy, it can be calculated that

$$\begin{aligned}
&\mathbb{E}^0[\widehat{X}_T^{-\beta}] \\
&= X_0^{-\beta}\mathbb{E}^0\bigg[\exp\bigg(-\int_0^T \beta\Big(\frac{\lambda_t}{1+\beta}+\frac{\rho\delta\xi_t}{1+\beta}\Big)\,\mathrm{d}W_t^0 - \int_0^T \beta^2\Big(\frac{\lambda_t}{1+\beta}+\frac{\rho\delta\xi_t}{1+\beta}\Big)^2\,\mathrm{d}t\bigg) \\
&\qquad \exp\bigg(-\beta\int_0^T\Big(\frac{\lambda_t^2}{1+\beta}+\frac{\rho\delta\lambda_t\xi_t}{1+\beta}\Big)\,\mathrm{d}t + \Big(\frac{\beta}{2}+\beta^2\Big)\int_0^T\Big(\frac{\lambda_t}{1+\beta}+\frac{\rho\delta\xi_t}{1+\beta}\Big)^2\,\mathrm{d}t\bigg)\bigg] \\
&\le X_0^{-\beta}\bigg(\mathbb{E}^0\bigg[\exp\bigg(-\int_0^T 2q(\lambda_t+\rho\delta\xi_t)\,\mathrm{d}W_t^0 - \int_0^T 2q^2(\lambda_t+\rho\delta\xi_t)^2\,\mathrm{d}t\bigg)\bigg]\bigg)^{\frac{1}{2}} \\
&\qquad \bigg(\mathbb{E}^0\bigg[\exp\bigg(-2q\int_0^T(\lambda_t^2+\rho\delta\lambda_t\xi_t)\,\mathrm{d}t + \frac{\beta+2\beta^2}{(1+\beta)^2}\int_0^T(\lambda_t+\rho\delta\xi_t)^2\,\mathrm{d}t\bigg)\bigg]\bigg)^{\frac{1}{2}}.
\end{aligned}$$

The first expectation is equal to one by (4.3.3) and (4.3.7) which guarantee Novikov's condition, and the second expectation is finite by Hölder's inequality and assumptions (4.3.3) and (4.3.7).

For condition (4.1.4),

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^T\big(|\widehat{\pi}_t\mu_t| + |\widehat{\pi}_t\sigma_t|^2\big)\,\mathrm{d}t\bigg] \\
&= \mathbb{E}\bigg[\int_0^T\Big|\frac{\lambda_t^2}{1+\beta}+\frac{\rho\delta\lambda_t\xi_t}{1+\beta}\Big|\,\mathrm{d}t\bigg] + \frac{1}{(1+\beta)^2}\mathbb{E}\bigg[\int_0^T(\lambda_t+\rho\delta\xi_t)^2\,\mathrm{d}t\bigg] \\
&\le \frac{1}{1+\beta}\mathbb{E}\bigg[\int_0^T\lambda_t^2\,\mathrm{d}t\bigg] + \frac{1}{2(1+\beta)}\bigg\{\mathbb{E}\bigg[\int_0^T\lambda_t^2\,\mathrm{d}t\bigg] + \mathbb{E}\bigg[\int_0^T\rho^2\delta^2\xi_t^2\,\mathrm{d}t\bigg]\bigg\} \\
&\quad + \frac{2}{(1+\beta)^2}\bigg\{\mathbb{E}\bigg[\int_0^T\lambda_t^2\,\mathrm{d}t\bigg] + \mathbb{E}\bigg[\int_0^T\rho^2\delta^2\xi_t^2\,\mathrm{d}t\bigg]\bigg\} < \infty,
\end{aligned}$$

by assumptions (4.3.4) and (4.3.8).

□

Define the process

$$V_t := \widetilde{\mathbb{E}}^0 \left[\exp\left(-\frac{q}{2\delta} \int_t^T \lambda_s^2 \, \mathrm{d}s \right) \,\middle|\, \widetilde{\mathcal{F}}_t \right], \quad t \in [0,T] \tag{4.3.10}$$

such that $u_t^A = \frac{X_t^{-\beta}}{-\beta}(V_t)^\delta$ and $u_t = \ln X_t - \frac{\delta}{\beta} \ln V_t$. By definition of M_t, V_t has the representation

$$V_t = \mathrm{e}^{N_t/\delta} M_t, \quad \text{where } N_t := \frac{q}{2} \int_0^t \lambda_s^2 \, \mathrm{d}s, \quad t \in [0,T].$$

Thus similarly as the BSDE representation for the multiple priors utility maximisation problem in (3.3.2), $(V, \zeta) = (V, V\xi)$ also admits a BSDE representation

$$\mathrm{d}V_t = -\left(-\frac{q}{2\delta} \lambda_t^2 V_t - q\rho\lambda_t \zeta_t \right) \mathrm{d}t + \zeta_t \, \mathrm{d}\widetilde{W}_t^0, \qquad V_T = 1.$$

4.4 The Markovian case

In this section, we study the robust optimal investment problem (4.1.1) in a Markovian stochastic factor model (3.4.1)-(3.4.2) as an example where the optimal strategy may be obtained in a more explicit form via solution to a linear PDE. An existing approach which involves applying stochastic control techniques to the dual problem has been used in Hernández-Hernández and Schied [25] where the authors focused on using a very general penalty function and deriving conditions on when the value function may be expressed as a solution to the PDE. By studying the specific case of an entropic penalty function here, the assumptions in [25] on the parameter functions in (3.4.1)-(3.4.2), which exclude some well-known stochastic volatility models, are weakened.

For the Markovian case, Proposition 2.2.6 is applicable to the classical optimal investment problem (4.2.2). The process V defined by (4.3.10) is a function of time and the current state of the stochastic factor Y:

$$V_t := \widetilde{\mathbb{E}}^0 \left[\exp\left(-\frac{q}{2\delta} \int_t^T \lambda^2(Y_s) \, \mathrm{d}s \right) \,\middle|\, \widetilde{\mathcal{F}}_t \right] = v(t, Y_t), \quad t \in [0,T].$$

The function $v(\cdot,\cdot)$ satisfies the PDE

$$\begin{aligned} &\frac{\partial v}{\partial t}(t,y) + (a(y) - q\rho\lambda(y)b(y)) \frac{\partial v}{\partial y}(t,y) \\ &\qquad + \frac{1}{2} b^2(y) \frac{\partial^2 v}{\partial y^2}(t,y) - \frac{q}{2\delta} \lambda^2(y) v(t,y) = 0, \end{aligned} \tag{4.4.1}$$

with terminal condition $v(T, y) \equiv 1$. The process ξ in (4.3.9) has the representation

$$\xi_t = \xi(t, Y_t) = \frac{b(Y_t)}{v(t, Y_t)} \frac{\partial v}{\partial y}(t, Y_t), \quad t \in [0, T].$$

The robust value function and optimal strategy are given by

$$\begin{aligned} u_t &= u(t, \widehat{X}_t, Y_t) = \ln \widehat{X}_t - \frac{\delta}{\beta} \ln v(t, Y_t), \\ \widehat{\pi}_t &= \widehat{\pi}(t, Y_t) = \frac{1}{(1+\beta)\sigma(Y_t)} \left(\lambda(Y_t) + \frac{\rho \delta b(Y_t)}{v(t, Y_t)} \frac{\partial v}{\partial y}(t, Y_t) \right), \quad t \in [0, T]. \end{aligned} \tag{4.4.2}$$

Remark 4.4.1. In the Markovian case, by Remark 2.2.7, the additional assumption on the set of admissible strategies (4.1.3) is not needed. Assumptions (4.3.3) and (4.3.4) are still needed for $\lambda_t \equiv \lambda(Y_t)$ in order to guarantee that the optimal strategy satisfies the integrability conditions (4.1.2) and (4.1.4). The constant c_ξ in assumption (4.3.7), with $\xi_t = \xi(t, Y_t) = \frac{b(Y_t)}{v(t, Y_t)} \frac{\partial v}{\partial y}(t, Y_t)$, may be replaced by a (potentially) slightly smaller constant $c_\xi = \max \left\{ \frac{10\beta^2 + 4\beta}{(1+\beta)^2} \rho^2 \delta^2, \frac{4c}{c-1} q^2 \rho^2 \delta^2 \right\}$.

4.5 Examples and numerical tests

The following two examples in Sections 4.5.1 and 4.5.2 illustrate the case when explicit solutions can be obtained for the PDE (4.4.1). We then numerically compare the performances of the robust strategies with that of the classical strategies when the agent has wrong estimates for the model parameters. We also did an initial exploration in Section 4.5.3 when there is a mis-specification of stochastic volatility model – the true model is a Heston model but both the robust and classical agents are estimating and implementing a Stein-Stein model.

4.5.1 Stein-Stein model

Assume that $\lambda(y) = \lambda y$ for some constant $\lambda \neq 0$, and $a(y) = m - \alpha y$, $b(y) = b$ for some positive constants m, α, b. The dynamics of the stock and the volatility processes under the reference model are

$$\begin{aligned} \mathrm{d}S_t &= Y_t S_t (\lambda Y_t \, \mathrm{d}t + \mathrm{d}W_t^0), \\ \mathrm{d}Y_t &= \big(m - \alpha Y_t\big) \, \mathrm{d}t + b \, \mathrm{d}\widetilde{W}_t^0. \end{aligned}$$

Substituting in the parameter functions in (4.4.1) yields the PDE satisfied by v

$$\begin{aligned} \frac{\partial v}{\partial t}(t, y) + \big(m - (\alpha + q\rho b\lambda) y\big) \frac{\partial v}{\partial y}(t, y) + \frac{1}{2} b^2 \frac{\partial^2 v}{\partial y^2}(t, y) - \frac{q\lambda^2}{2\delta} y^2 v(t, y) = 0, \\ v(T, y) = 1. \end{aligned}$$

The solution is given by (see Monoyios [45], obtained by making the following ansatz and solving the ODEs satisfied by deterministic functions $c(\cdot)$, $d(\cdot)$, $e(\cdot)$ [1])

$$v(t,y) = \exp\left(-c(t)y^2 - d(t)y - e(t)\right), \tag{4.5.1}$$

where

$$\begin{aligned}
c(t) &= \frac{q\lambda^2 \sinh(a_1(T-t))}{2\delta\psi(a_1(T-t))}, \qquad d(t) = \frac{qm\lambda^2}{a_1\delta}\left(\frac{\cosh(a_1(T-t))-1}{\psi(a_1(T-t))}\right), \\
e(t) &= \frac{1}{2}\left(\ln\left(\frac{\psi(a_1(T-t))}{a_1}\right) - a_2(T-t)\right) - \frac{1}{2}\left(\frac{m}{a_1 b}\right)^2\left((a_1^2+a_2^2-2a_1)(T-t)\right. \\
&\quad \left. - \frac{2a_2 q b^2\lambda^2}{a_1\delta}\left(\frac{1}{\psi(a_1(T-t))} - \frac{1}{a_1}\right) + \frac{q\lambda^2 b^2}{a_1^2\delta}\left(\frac{q\lambda^2 b^2}{\delta} - a_2^2\right)\frac{\sinh(a_1(T-t))}{\psi(a_1(T-t))}\right),
\end{aligned}$$

and

$$\begin{aligned}
&a_1 = \sqrt{\alpha^2 + 2\alpha q\rho b\lambda + qb^2\lambda^2}, \qquad a_2 = \alpha + q\rho b\lambda, \\
&\psi(\tau) := a_1\cosh(\tau) + a_2\sinh(\tau).
\end{aligned} \tag{4.5.2}$$

The robust value function and robust optimal strategy can be obtained from (4.4.2) along with (4.5.1), which yields

$$\begin{aligned}
u_t &= \ln X_t + \frac{\delta}{\beta}\left(c(t)Y_t^2 + d(t)Y_t + e(t)\right), \\
\widehat{\pi}_t^{\text{robustSS}} &= \frac{1}{\beta+1}\left(\lambda - 2\rho\delta bc(t) - \rho\delta b\frac{d(t)}{Y_t}\right), \quad t\in[0,T].
\end{aligned} \tag{4.5.3}$$

Note that since $c(T) = d(T) = 0$, the proportion of wealth invested in the stock will be closer to the constant $\lambda/(1+\beta)$ when time approaches T. The larger the ambiguity aversion parameter β is, the smaller the proportion of wealth invested in the stock tends to be. Compared with the classical optimal strategy

$$\widehat{\pi}_t^{\text{classicalSS}} = \lambda, \quad t\in[0,T],$$

which is only dependent on the parameter λ, the robust strategy in (4.5.3) introduces dependence on the current value of the volatility process, the ambiguity aversion of the agent β, the investment horizon T, as well as other model parameters ρ, m, α and b. This on one hand has made the robust agent take more factors into consideration while making investment decisions, on the other hand, it has potentially made the agent exposed to more parameter estimation errors, especially considering that in reality the volatility process Y cannot be observed.

[1] A similar (but less explicit) solution can be found in Benth and Karlsen [4].

With the above explicit expressions for the value function and optimal strategies, we can numerically compare the performances of various strategies. The true model parameter values are chosen as in Table 4.1[1]. In the following tests, we will denote the set of model parameters as $\Theta := (\lambda, m, \alpha, b, \rho, Y)$, and the set of "estimated"[2] model parameters as Θ^*.

S_0	Y_0	λ	m	α	b	ρ	X_0	T
100	0.3	1.0	4.8	16	0.4	-0.2	1000	1 year

Table 4.1: True parameter values for Stein-Stein model

In the following tests, we will always use the "optimal strategy" as a reference, which refers to the ideal case when the agent knows everything – the true model and the true model parameters. We compare the robust strategy with: the classical strategy where no model uncertainty is taken into account, and the "optimal strategy" which refers to the classical strategy when the agent has correct estimates for the model parameters. The following four schemes are then used for testing the performances of these strategies:

- TEST 1: $\Theta^* = (\lambda^*, m, \alpha, b, \rho, Y)$.
  Only λ unknown, all other elements of Θ known.
  No actual estimation of λ – assumed values of optimistic or pessimistic estimates (for illustrative purposes).

- TEST 2: $\Theta^* = (\lambda^*, m^*, \alpha^*, b^*, \rho^*, Y)$, 1 estimation + N tests.
  All elements of Θ unknown, except Y.
  One estimation path over $[-T, 0]$ (estimation procedure as described in detail in the testing scheme descriptions below), N testing paths over $[0, T]$.

- TEST 3: $\Theta^* = (\lambda^*, m^*, \alpha^*, b^*, \rho^*, Y)$, N (estimation + test)s.
  All elements of Θ unknown, except Y.
  N paths over $[-T, T]$.

- TEST 4: $\Theta^* = (\lambda^*, m^*, \alpha^*, b^*, \rho^*, Y^*)$, 1 estimation + N tests.
  All elements of Θ unknown.
  One estimation path over $[-T, 0]$ (estimation procedure as described in detail in the testing scheme descriptions below), N testing paths over $[0, T]$.
  Include a modified Merton strategy (Merton strategy with re-calibrated model parameters) into comparison. All strategies compared have dynamically updated model parameters with a moving estimation window.

TEST 1: $\Theta^* = (\lambda^*, m, \alpha, b, \rho, Y)$**.**

In this simple test, we assume everything is known to the agents, except for λ. We compare two types of agents in this scenario: (i) A classical agent who will implement the classical Stein-Stein strategies, usually with an erroneous estimate of the parameter λ. (ii) A robust agent who will implement the robust Stein-Stein strategies with (as for the classical agent) the same erroneous estimate of λ. We will also consider this agent with different magnitudes of ambiguity aversion β.

The agents' "estimate" of λ (which we are in fact choosing in these experiments) is denoted by λ^*. We will consider cases in which

$$\lambda^* \begin{cases} > \lambda, & \text{an (optimistic) over-estimate,} \\ = \lambda, & \text{a precise-estimate,} \\ < \lambda, & \text{an under-estimate.} \end{cases}$$

A classical agent who over-estimates stock returns will (we conjecture) take an overly aggressive stock position, and, in the face of such parameter error (and, in particular, in a highly volatile environment) will, we suppose, suffer the consequences, in forms of achievable average utility (over the many simulated paths in our experiments). The robust agent will, presumably, be protected to some extent (dependent on her magnitude of ambiguity aversion) from these dangers, because her objective is more conservative, accounting for such over-optimistic errors. However, when the parameter λ is mis-estimated in the other direction, the classical agent would suffer from taking an overly conservative stock position. The robust agent will, we suppose, suffer more, as a result of being even more conservative than the classical agent.

The procedure of the test is as follows. We simulate the true model dynamics, implement the above alternative strategies using the estimate λ^* in the optimal trading strategy associated with each objective, repeat over many ($N = 10^5$) sample paths, and compute the average terminal utilities $u_{T,\text{avg}} = \frac{1}{N}\sum_{j=1}^{N} \ln(X_T^{(j)})$, where $X_T^{(j)}$ denotes the terminal wealth for the j-th sample path.

Although the above tests are highly artificial, we are therefore able to mimic the basic scenario when an agent (robust or non-robust in their objective functional) over or under-estimates stock return. The results are shown in Table 4.2, which, to some extent, are in line with our intuition. With standard errors that are smaller than 0.0015 (see the standard deviations in the brackets), we can confirm that the robust agent does achieve greater average terminal utility when the stock return is an

[1]We refer to Stein and Stein [61] for the choices of m, α, b.

[2]In some tests, we have assumed the estimated parameter value(s) in order to study the different scenarios and their impacts on the performances of the various strategies.

optimistic over-estimate. But, with the parameter values we used, the improvement in average utility is of the order of $0.1\% - 0.3\%$. Instead, the main impact of β is on the standard deviation, rather than on the mean of the terminal utilities achieved. So we conclude that robust strategies do appear to have some benefit in the case that the stock return is over-estimated, but perhaps the severity of the model error in this experiment limited the extent of this benefit.

	λ^*	β	$u_{T,\text{avg}}^{\text{robustSS}}$	$u_{T,\text{avg}}^{\text{classicalSS}}$	$u_{T,\text{avg}}^{\text{optimal}}$
over-estimate	1.5	0.5	6.96 (0.31)	6.94 (0.46)	6.96 (0.31)
		1.0	6.95 (0.23)		
precise-estimate	1.0	0.5	6.95 (0.20)	6.96 (0.31)	
		1.0	6.94 (0.15)		
under-estimate	0.5	0.5	6.93 (0.10)	6.94 (0.15)	
		1.0	6.93 (0.08)		

Table 4.2: Stein-Stein model (TEST 1): Average terminal utilities of different strategies when the agent over/precisely/under estimates λ (with values of λ^* chosen for illustrative purposes). Numbers in the brackets are standard deviations of the terminal utilities. The "optimal" utility $u_{T,\text{avg}}^{\text{optimal}}$ denotes the average terminal utility achieved by a classical agent who knows the exact model parameters.

Similar conclusions can be made from the histograms in Figure 4.1. As shown by the top figure of Figure 4.1 when the agent over-estimates λ by 50%, compared with the classical strategy, with $\beta = 0.5$, the robust strategy has achieved a fairly similar distribution of terminal utilities as the optimal strategy. While in cases where the agent under-estimates the return of the stock (bottom figure in Figure 4.1), since the robust strategies are already more conservative than the classical strategies with the same estimates, this conservativeness will not be as beneficial for the agent: the average terminal utilities have a slightly lower mean and is even more concentrated around the mean than the already-conservative classical strategies.

Indeed, when the agent over-estimates the parameter λ, with a suitable ambiguity aversion parameter β, the robust strategies can essentially coincide with the optimal strategies. Figure 4.2 shows an example simulation path of the wealth processes of the strategies when the agent has over-estimated the growth of the stock by 50%. While the (wrong) optimistic estimate has made the classical agent's investment strategies quite aggressive, the robust agent's wealth behaves in a very similar way as the classical agent knowing the exact model parameters. The bottom figure in Figure 4.2 shows the trajectories of robust strategies over 200 realisations of price paths. For our parameter choices in Table 4.1, the robust strategies basically stay at the value

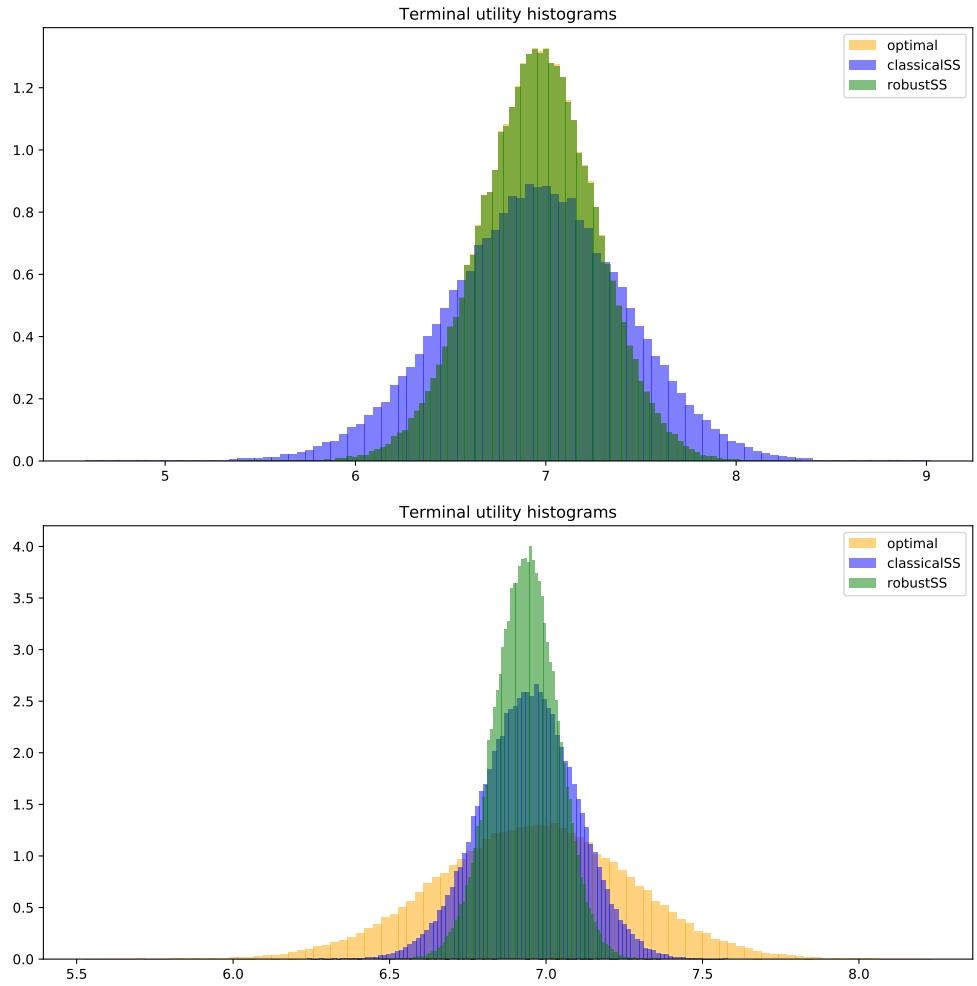


Figure 4.1: Stein-Stein model (TEST 1): Histograms of terminal utilities for different strategies. Top figure: $\lambda^* = 1.5$, $\beta = 0.5$. With the parameter choices, the histogram for the robust Stein-Stein strategy indeed overlaps with that for the optimal strategy. Bottom figure: $\lambda^* = 0.5$, $\beta = 0.5$.

$\frac{\lambda^*}{1+\beta}$ – which happens to be the same as the true λ with $\lambda^* = 1.5\lambda$ and $\beta = 0.5$, with the small spikes possibly due to low volatilities in (4.5.3).

Remark 4.5.1 (More comments on the choice of ambiguity aversion parameter β). Dependent on the errors in estimated λ, the average terminal utilities achieved by the robust agent can vary differently when we increase the values of the ambiguity aversion parameter β. Figure 4.3 illustrates some example average terminal utilities against choices of β when, from left to right, top to bottom, we set $\lambda^* < 0 < \lambda$, $0 < \lambda^* < \lambda$, and $\lambda^* > \lambda$. The procedure for obtaining the figures are the same as TEST 1, where we vary the values of λ^* and β and assume the agents have precise estimates for the other parameters.

When $\beta = 0$, the robust strategy coincides with the classical strategy thus the terminal utilities coincide. For values of $\lambda^* < 0 < \lambda$ (graphs in the first row and first two graphs in the second row), with increasing β, the average terminal utility obtained by the robust agent increases, but will not reach that of the optimal agent – since the agent has the wrong sign of λ estimated thus the wrong side of position in the stock. While for $0 < \lambda^* < \lambda$ (third graph in the second row and first graph in the third row), the robust agent indeed becomes worse off when we increase β, as a result of being too conservative with an already-conservative estimate for the return of the stock. Finally, when λ is over-estimated ($\lambda^* > \lambda$ in the rest of the graphs), there could be an "optimal" choice $\widehat{\beta}$ of the ambiguity aversion parameter β that could help the robust agent reach the expected utility of an optimal agent: The robust average terminal utility increases before β gets to this value $\widehat{\beta}$, reaches the optimal terminal average utility, and decreases (eventually to a lower value than the classical agent) when β is increasing beyond $\widehat{\beta}$.

Figure 4.4 shows how the optimal choice $\widehat{\beta}$ changes as λ^* varies as in Figure 4.3, in the case when λ is over-estimated ($\lambda^* > \lambda$). We can see that $\widehat{\beta}$ changes linearly with λ^*, and lies close to (or on) the line $\frac{\lambda^*}{\lambda} - 1$. This again confirms our previous observation (also see bottom figure in Figure 4.2) that in many cases, the robust strategy is approximately $\frac{\lambda^*}{1+\beta}$, which simply scales the classical strategy by the factor $\frac{1}{1+\beta}$ and would equal the optimal strategy if the robust agent could choose $\widehat{\beta} = \frac{\lambda^*}{\lambda} - 1$ (though infeasible in practice).

TEST 2: $\Theta^* = (\lambda^*, m^*, \alpha^*, b^*, \rho^*, Y)$, 1 estimation + N tests.

The results in TEST 1 might have been affected by the limited form of the model error we used (only λ is not known), thus in TEST 2, we expand the list of mis-estimated parameters from λ to $(\lambda, m, \alpha, b, \rho)$, with the volatility process Y itself,

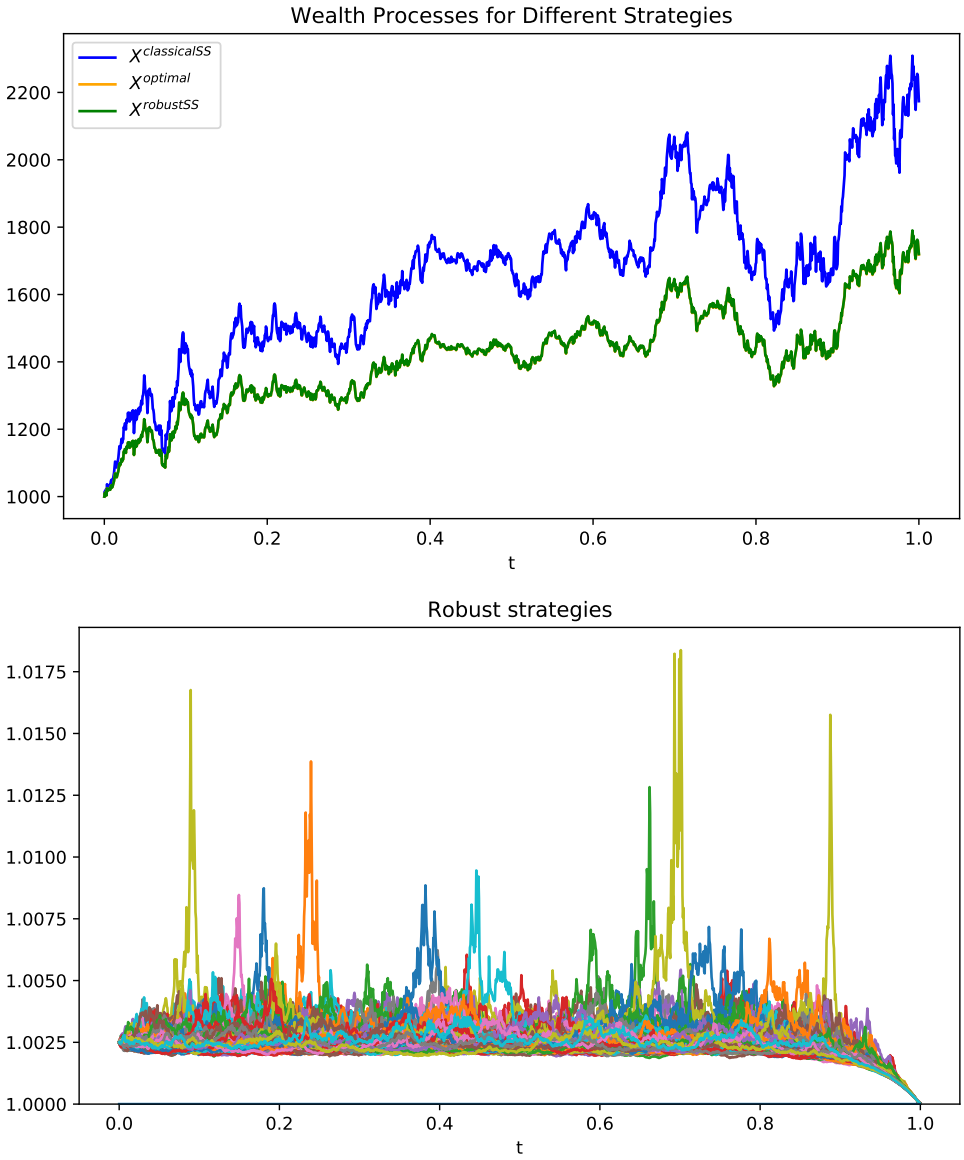


Figure 4.2: Stein-Stein model (TEST 1): Parameter values are chosen as in Table 4.1 and $\lambda^* = 1.5$, $\beta = 0.5$. Top figure: example wealth processes for one realisation. Bottom figure: paths of robust strategies $(\widehat{\pi}_t^{\text{robustSS}})_{t\in[0,T]}$ for 200 realisations. In the bottom figure, $\widehat{\pi}^{\text{robustSS}}$ essentially stays at the value $\frac{\lambda^*}{1+\beta} = \lambda = 1.0$ over $t \in [0, T]$, which explains the almost identical wealth paths of the robust SS agent and the optimal agent, as well the overlapping histograms.

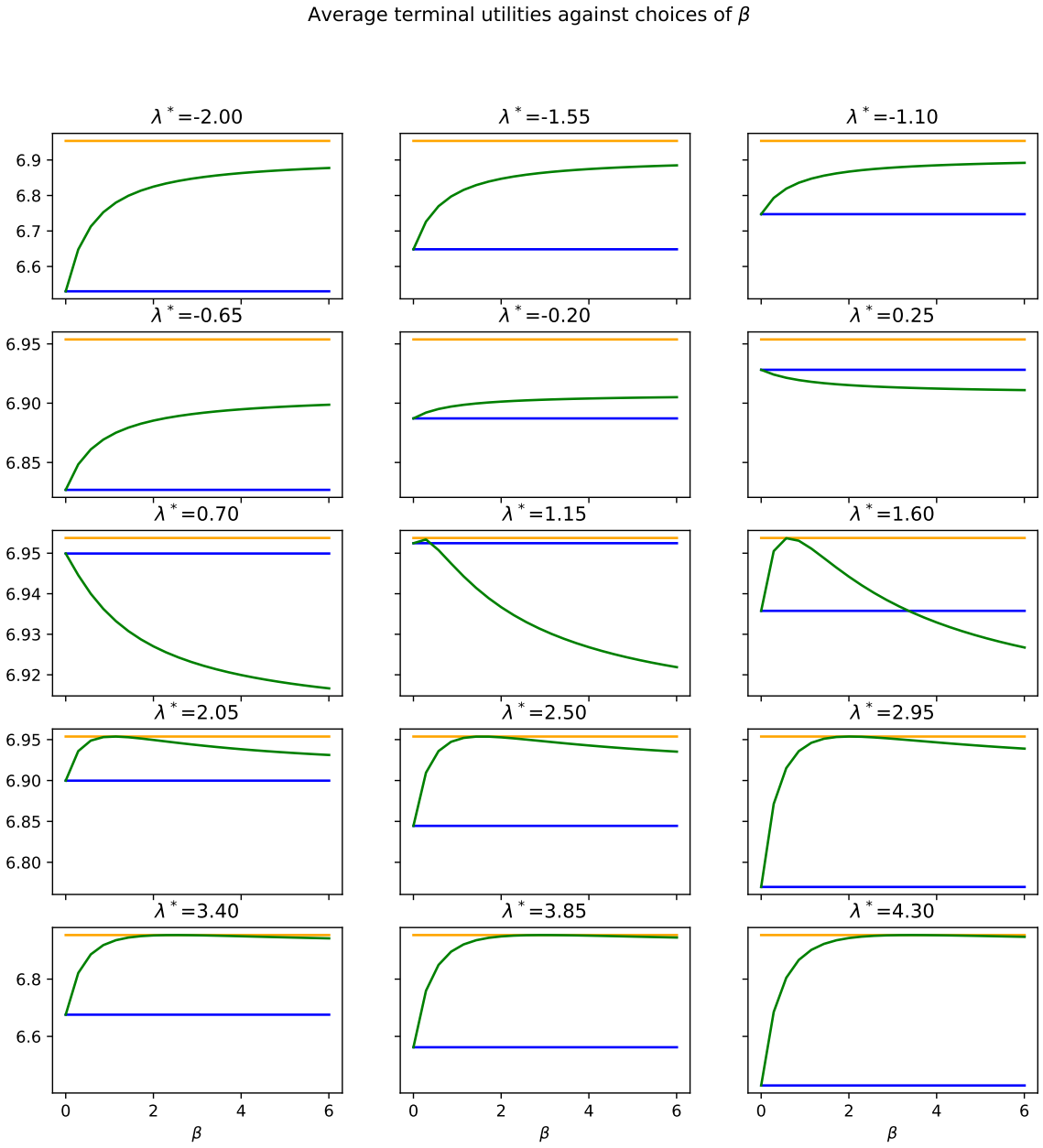


Figure 4.3: Stein-Stein model: Illustrations of average terminal utilities plotted against choices of β. Yellow lines, blue lines and green lines are average terminal utilities of the optimal agent (that is, a classical agent who knows the exact model parameters), the classical agent and the robust agent, respectively. The values of λ^* are chosen with an increment of 0.45 for illustrative purposes.

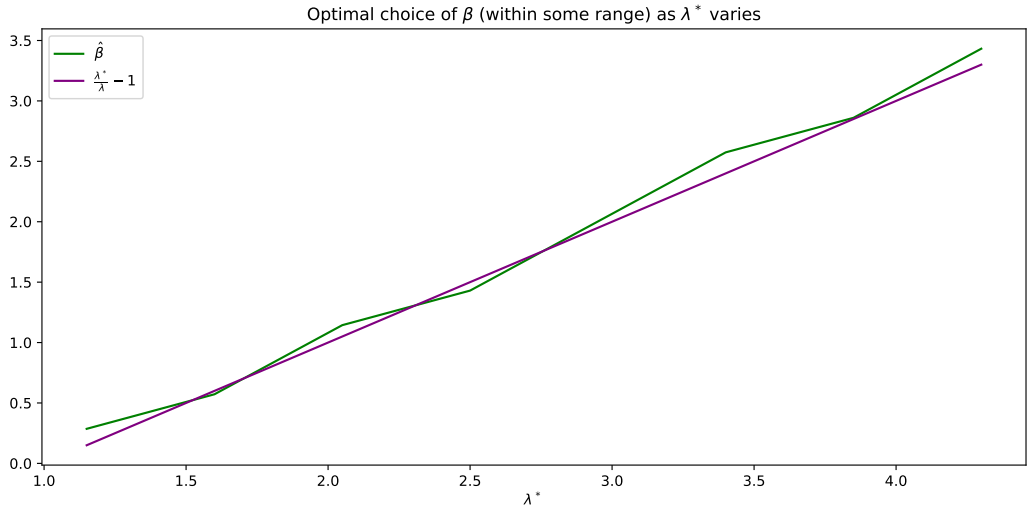


Figure 4.4: Stein-Stein model: The optimal choice $\widehat{\beta}$ plotted against λ^* when the parameter λ is over-estimated ($\lambda^* > \lambda$). Optimal values shown on the green line are within a range of 0.286, as a result of discrete increments in β. Purple line is the optimal choice of β if the robust strategy scales the classical strategy by a factor of $\frac{1}{1+\beta}$.

known. In this experiment, we actually carried out a somewhat ad-hoc estimation procedure for the parameters, using one pair of simulated sample paths under the true model in the estimation period which is of the same time horizon T as the test period. Supposing we have $M+1$ equally-spaced observations in some interval $[0, T]$, that is, S_i, Y_i at $t_i = iT/M$, $i = 0, 1, ..., M$, we use the quadratic variation of volatilities to estimate the volatility of volatility, the co-variation of logarithmic prices and volatilities to estimate the correlation, the sample mean of the logarithm prices to estimate the drifts, and the invariant distribution $\mathcal{N}(\frac{m}{\alpha}, \frac{b^2}{2\alpha})$ of the Ornstein–Uhlenbeck process to estimate the long-run mean and rate of mean reversion of the stochastic volatility, which yields the following naive estimates for the model parameters:

$$
\begin{aligned}
b^* &= \sqrt{\frac{1}{T}\sum_{i=1}^{M}(Y_i - Y_{i-1})^2}, \qquad \rho^* = \frac{1}{Tb^*}\sum_{i=1}^{M}(\ln S_i - \ln S_{i-1})(\ln Y_i - \ln Y_{i-1}), \\
\lambda^* &= \frac{1}{2} + \frac{\sum_{i=1}^{M}(\ln S_i - \ln S_{i-1})/Y_{i-1}}{T\sum_{i=1}^{M} Y_{i-1}/M}, \\
\alpha^* &= \frac{(b^*)^2}{2 \cdot \mathrm{var}(Y)}, \qquad m^* = \alpha^* \cdot \mathrm{mean}(Y).
\end{aligned}
\tag{4.5.4}
$$

Here, some parameter estimates (e.g. b^* and ρ^*) were fairly accurate (since we assume Y was known and they came from the quadratic variation of Y and co-variation of

$\log S$ and $\log Y$). The performances of the robust strategies, the optimal strategies and the classical strategies are then tested over a large number of sample paths ($N = 10^5$) in the test period.

There might be some drawbacks to this experiment (e.g. we are not in control of the "estimation" errors), but we can see some more evidence of the effect of adopting a robust approach. For example, when $\lambda^* = 3.21, m^* = 3.93, \alpha^* = 13.1$ (Table 4.3, first row), which is in line with an optimistic estimate (since the agent over-estimates the return of the stock), the expected utility benefit to the robust agent is of the order of 2.5%. As the model error is greater than in TEST 1, the robust approach has shown a greater benefit in this experiment.

With $\lambda^* = 2.00, m^* = 9.18, \alpha^* = 26.36$ (Table 4.3, second row) where the estimates for m and α were far away from their true values (though the mean reversion level m^*/α^* is only 16% higher than its true value $m/\alpha = 0.3$), the robust agent still benefited from adopting a robust strategy. This indicates that for our true model, the robust strategy might not be sensitive to estimation errors in these two parameters though it introduced dependence on these parameters (thus is exposed to estimation errors in these parameters).

In the case when the agents under-estimate λ but still think λ is positive where $\lambda^* = 0.34, m^* = 3.28, \alpha^* = 10.62$ (Table 4.3, third row), the classical agent invests less in the stock than she should have done, and the robust agent is even more conservative. Although they cannot be differentiated by their average utilities, the investment results are less interesting and vary very little.

Finally, it also happened in our experiments that the parameter λ can be severely mis-estimated in the sense that $\lambda^* < 0$ while the true value $\lambda = 1$ (Table 4.3, last two rows). When $\lambda^* = -0.73$, the classical agent indeed suffered less than the scenario when she is overly aggressive ($\lambda^* = 3.21$), and the robust agent's gain in expected utility is only of the order 0.6%. While when the model error is more severe ($\lambda^* = -2.03$), the expected utility benefit to the robust agent is of the order 2.4%, as the robust agent is less aggressive in mistakenly shorting the stock than the classical agent.

TEST 3: $\Theta^* = (\lambda^*, m^*, \alpha^*, b^*, \rho^*, Y)$, N (estimation + test)s.

As we can see from TEST 1 and TEST 2, for a fixed model, depending on the different scenarios, the estimation errors can be beneficial or destructive for the robust strategies. Thus, in this scheme, we would like to test the average performances of these investment strategies, where a greater variety of estimation errors are considered

Parameter estimates					Average terminal utilities		
λ^*	m^*	α^*	b^*	ρ^*	$u_{T,\text{avg}}^{\text{robustSS}}$	$u_{T,\text{avg}}^{\text{classicalSS}}$	$u_{T,\text{avg}}^{\text{optimal}}$
3.21	3.93	13.10	0.39	−0.18	6.89 (0.66)	6.72 (0.99)	
2.00	9.18	26.36	0.40	−0.21	6.95 (0.41)	6.91 (0.61)	
0.34	3.28	10.62	0.40	−0.20	6.93 (0.07)	6.93 (0.10)	6.96 (0.31)
−0.73	3.56	11.91	0.40	−0.20	6.85 (0.15)	6.81 (0.22)	
−2.03	9.17	24.00	0.40	−0.19	6.69 (0.41)	6.52 (0.62)	

Table 4.3: Stein-Stein model (TEST 2): Average terminal utilities of different strategies for five independent samples of the model parameter estimates. Numbers in the brackets are standard deviations of the terminal utilities. True parameter values are as in Table 4.1, and ambiguity aversion parameter $\beta = 0.5$.

and the average performance of these strategies is tested over a large number of realisations. The process is similar as in TEST 2, except that the estimation-testing procedure is treated as a whole: the agent first observes the stock and volatility paths in the estimation period, estimates the parameter values, and then applies the robust/classical/optimal strategy with the corresponding estimated parameters (or true parameters for the optimal strategy) in the test period. The whole process is repeated $N = 10^5$ times to get the distributions of terminal utilities.

As we can see in Table 4.4, the average performance of the robust agent is better than the classical agent: she has achieved higher average terminal utility (of the order 3.9% than the classical agent) with smaller standard deviation. Although the robust strategy is not so favourable in cases when the agent only slightly under-estimates the growth of the stock, in general, the robust strategy still helps to achieve better terminal investment results than the classical ones.

$u_{T,\text{avg}}^{\text{robustSS}}$	$u_{T,\text{avg}}^{\text{classicalSS}}$	$u_{T,\text{avg}}^{\text{optimal}}$
6.73 (0.78)	6.46 (1.29)	6.96 (0.31)

Table 4.4: Stein-Stein model (TEST 3): Average terminal utilities of different strategies. Numbers in the brackets are standard deviations of the terminal utilities. True parameter values are as in Table 4.1, and ambiguity aversion parameter $\beta = 0.5$.

TEST 4: $\Theta^* = (\lambda^*, m^*, \alpha^*, b^*, \rho^*, Y^*)$, 1 estimation + N tests.

We are aware that the volatility process is non-observable in reality, thus the agents using a stochastic volatility model as the "believed" model will have to estimate the volatility process, especially the robust agent whose strategies depend explicitly on the volatility process. This might impair the benefits of the robust agent as well as

the classical agent who also needs to estimate the volatility process in order to get an estimation for λ using our naive estimation method. Thus we would wonder whether an agent using a simple strategy, such as a Merton agent with dynamically calibrated model parameters (referred to as modified Merton agent later), would indeed perform better in terms of average utilities, as a result of having fewer parameters to estimate.

Motivated by the above, we compare the following three types of agents: (i) A classical agent who will implement the classical Stein-Stein strategies, but with dynamically updated parameter values using a moving estimation window of length T. (ii) A robust agent who will implement the robust Stein-Stein strategies, but also with dynamically updated parameter values using the moving estimation window. (iii) A modified Merton agent who will implement the Merton strategies $\widehat{\pi}^{\text{modMerton}}$ with dynamically calibrated model parameters, using the same moving estimation window.

For the modified Merton agent, she believes the model is a Merton model, with constant MPR θ and volatility ν, and employs the Merton strategies $\widehat{\pi}^{\text{modMerton}} = \theta/\nu$ with the following method to estimate the parameters

$$\nu^* = \sqrt{\frac{1}{T}\sum_{i=1}^{M}(\ln S_i - \ln S_{i-1})^2}, \qquad \theta^* = \frac{\nu^*}{2} + \frac{1}{T\nu^*}(\ln S_M - \ln S_0). \quad (4.5.5)$$

While for the classical and robust agents using the Stein-Stein stochastic volatility model, we suppose they are using the Cuchiero-Teichmann volatility estimator in Cuchiero and Teichmann [9]. The code used was from Merkle et. al. [37], which was part of the code implemented in Merkle et. al. [36]. Then the estimated volatility process is used as a data input to estimate other parameters, in the same way as described in TEST 2 (see (4.5.4)).

The procedure of the test is as follows. We simulate one pair of stock price and volatility paths in the period $[-T, 0]$, and $N = 10^5$ pairs of sample paths in the test period $[0, T]$. The agents will start with the estimation window $[-T, 0]$ to obtain the initial estimates of their corresponding model parameters for use at time 0. As time goes from 0 to T in the test period, along each sample path in our simulations, they will move the estimation window (which is of a fixed length T) forward, update their estimates, and implement their investment strategies. Then we will compute and compare their average terminal utilities.

Figure 4.5 (middle figure) shows example volatility processes estimated for the Stein-Stein (SS) agents and the modified Merton agent (updated constant volatilities in this case). It shows that the Cuchiero-Teichmann volatility estimator yields a

reasonably good estimation of the volatility process (by the fact that the blue thin line lies on the black thick line most of the time), but the volatility of volatility seems to be under-estimated. On the other hand, the updated volatilities used by the modified Merton agent seem to be able to catch the average volatility level and trend. The corresponding estimated MPR paths are also shown in Figure 4.5 (top figure). For the SS agents, the MPR is $Y^{\mathrm{CT}}\lambda^*$, while for the modified Merton agent, it is the updated estimates θ^*. Both estimates are more volatile than the true MPR process λY, and they seem to move in line with each other for most of the times. The bottom figure of Figure 4.5 shows the estimated MPR/volatility paths for the SS agents and the modified Merton agent. For the classical SS agent and the modified Merton agent, these paths coincide with their strategy paths (proportion of wealth invested in stock), and for the robust SS agent, this will be essentially scaled by a factor $1/(1+\beta)$ to give the strategy path. Then we can expect that the robust strategy would yield investment results that are closer to the optimal strategy.

The results are shown in Table 4.5, with the histograms of terminal utilities in Figure 4.6 (top figure). We can learn from both that although the classical agent was right to assume that the underlying model is a Stein-Stein stochastic volatility model, however, because of the difficulty of estimating the model parameters for this complicated model, she ended up with lower/similar average terminal utilities as the modified Merton agent who used the simple Merton model with dynamically calibrated model parameters. The robust agent, on the other hand, has gained in expected utility (of the order 3.5% compared with the classical agent), and achieved terminal utilities distributed in a closer way to those of the agent knowing the true model as well as the true model parameters and the volatility paths. Figure 4.6 (bottom figure) also shows the histograms if the robust agent is more conservative and chooses $\beta = 1.0$. The terminal utilities of the robust SS agent are distributed closer to the optimal agent, however, this seems to be due to a greater improvement in terminal utility when the estimation error is more severe. In fact, the frequency of the robust SS agent performing better than the classical SS and the modified Merton agent has not increased (it indeed decreases slightly), as there would be more times when the robust SS agent becomes too conservative. Thus we will only focus on the less conservative choice of ambiguity aversion parameter.

Similar conclusions can also be made from Figure 4.7 which shows the path-by-path respective additional terminal utilities of the classical SS agent and robust SS agent over the modified Merton agent. For the randomly selected 1000 realisations in the figures, we can see that, for more than half (about 60%, according to our

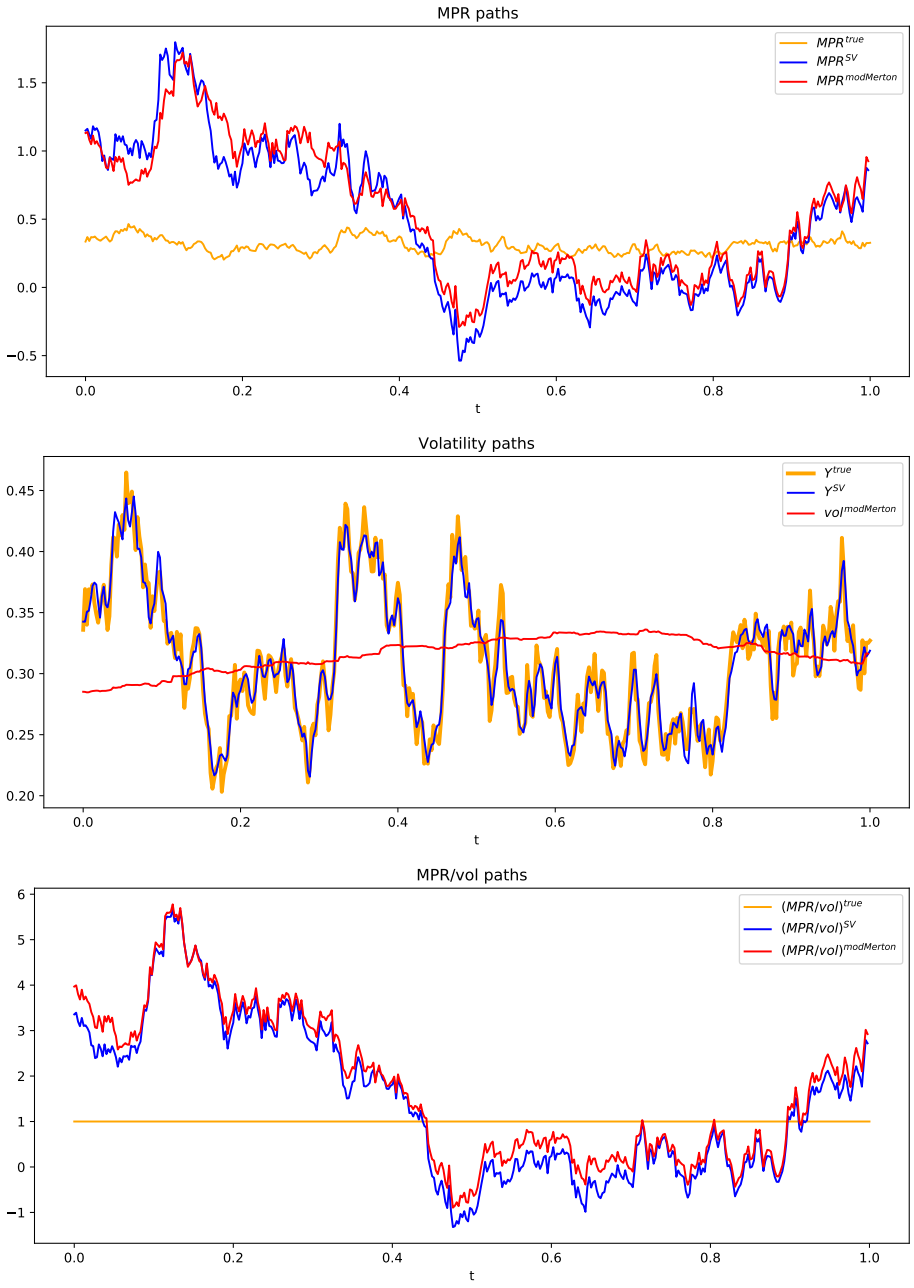


Figure 4.5: Stein-Stein model (TEST 4): Top figure: dynamically estimated MPRs. Middle figure: true (yellow thick solid line) and estimated volatility paths, with Y^{SV} (blue solid line) the volatility process estimated using Cuchiero–Teichmann algorithm and $\mathrm{vol}^{\mathrm{modMerton}}$ (red solid line) the updated volatility estimated by the modified Merton agent. Bottom figure: strategies (MPR/volatility processes) of the optimal agent (orange), the classical SV agent (blue) and the modified Merton agent (red).

$u^{\text{robustSS}}_{T,\text{avg}}$	$u^{\text{classicalSS}}_{T,\text{avg}}$	$u^{\text{modMerton}}_{T,\text{avg}}$	$u^{\text{optimal}}_{T,\text{avg}}$
6.79 (0.64)	6.55 (0.86)	6.58 (0.84)	6.96 (0.31)

Table 4.5: Stein-Stein model (TEST 4): Average terminal utilities of different strategies. Numbers in the brackets are standard deviations of the terminal utilities. True parameter values are as in Table 4.1, and ambiguity aversion parameter $\beta = 0.5$.

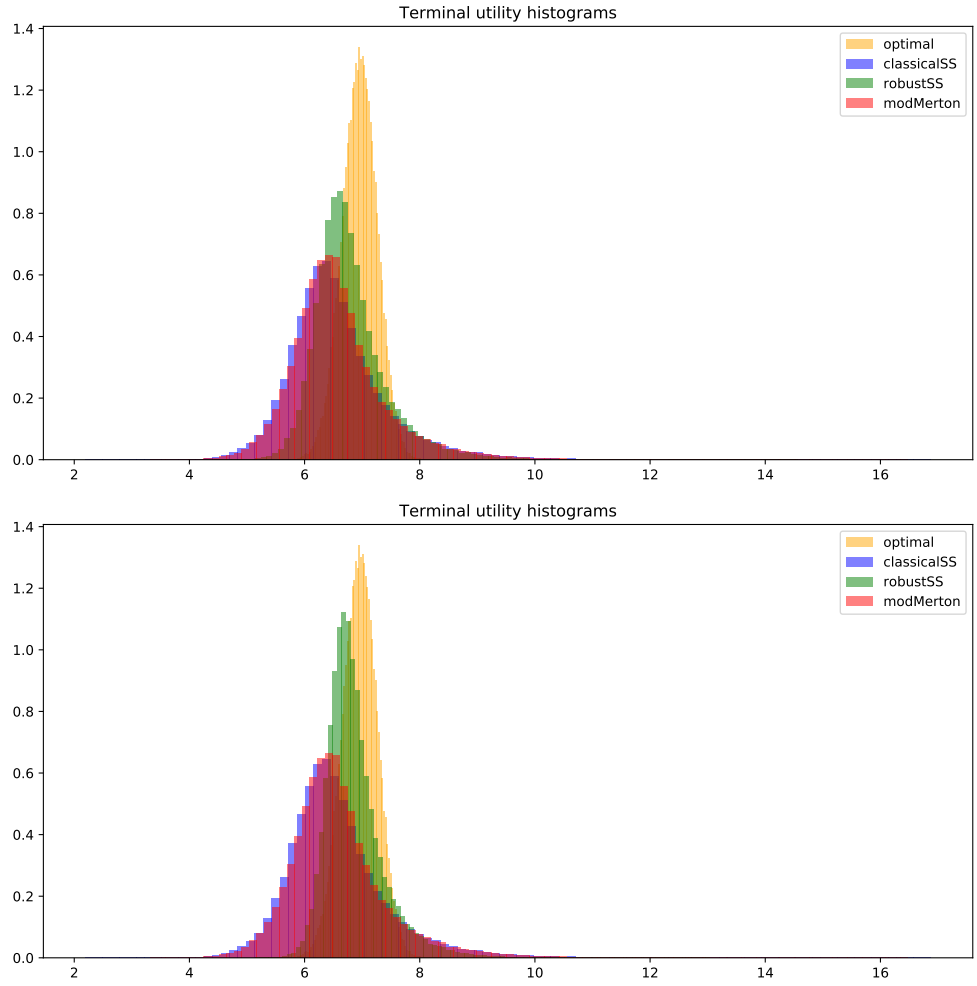


Figure 4.6: Stein-Stein model (TEST 4): Histograms of terminal utilities. Top figure: $\beta = 0.5$. Bottom figure: $\beta = 1.0$.

data) of them, the classical SS agent has terminal utilities lower than or similar as the modified Merton agent, while the robust SS agent has higher terminal utilities for about 86% of the realisations shown. If we compare the robust SS agent with the classical SS agent, in the bottom figure, for most of the times (about 87%) in the 1000 paths shown, the robust SS agent performs better than the classical SS agent (paths coloured green).

To better understand the results, we would like to see how the estimated parameters are distributed. Since all agents have been updating their parameter estimates using the moving window, we show the distributions of the path-by-path mean of the estimated parameters for each realisation. Taking parameter λ as an example, for realisation j ($j = 1, ..., N$), the agent has estimated values $\lambda_i^{*,(j)}$ at time $t_{i-1} = (i-1)T/M$ for $i = 1, ..., M$. The path-by-path mean is defined as below:

$$\text{Mean}^{(j)} := \frac{1}{M} \sum_{i=0}^{M-1} \lambda_i^{*,(j)}.$$

The histograms are then shown in Figures 4.8 - 4.10 for the classical and robust SS agents and Figure 4.11 for the modified Merton agent.

From Figure 4.8, we can see that for the SS agents, the parameter λ is more frequently over-estimated. Indeed, it happens less often that λ is mildly under-estimated ($0 < \lambda^* < \lambda$), which is when the robust SS agent is doing not so well due to being too conservative. Combined with our conclusion from the previous tests that the robust agent achieves greater terminal utility when the stock return is an optimistic over-estimate ($\lambda^* > \lambda > 0$) or an overly-pessimistic under-estimate ($\lambda^* < 0 < \lambda$), this explains why the robust SS agent, on average, has higher terminal utilities than the classical SS agent.

Figure 4.9 shows the histograms of the path-by-path mean for the estimated parameters m/α (mean-reversion level) and α (mean-reversion speed) for the SS agents. The estimated values of the mean-reversion level mostly lie in the interval $[(1-8\%)m/\alpha, (1+17\%)m/\alpha]$, and are more frequently over-estimated. While the estimates for the mean-reversion speed are much worse (within $[(1-69\%)\alpha, (1+25\%)\alpha]$), and are more frequently lower than the true value. For the parameters b (volatility of volatility) and ρ (correlation between price and volatility processes) for the SS agents, from Figure 4.10, we can see that the volatility of volatility is always under-estimated (about 35% lower), while the extent of correlation between the price and its volatility ($|\rho|$) is almost always under-estimated (worst could be 75% lower). It is difficult to say whether this would yield constantly higher or lower values of estimated volatilities,

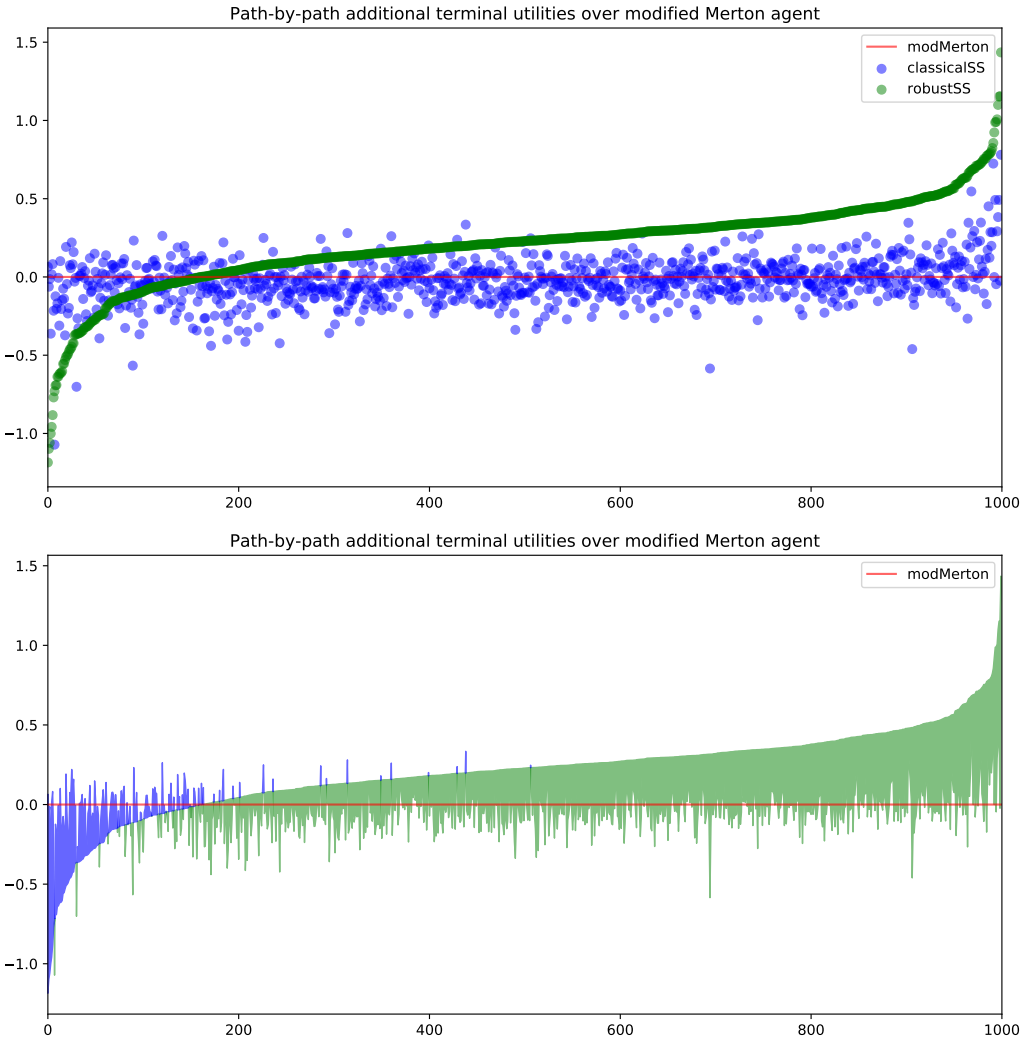


Figure 4.7: Stein-Stein model (TEST 4): Path-by-path additional terminal utilities over modified Merton agent (1000 realisations). Data points are re-organised in the order of increasing robust SS terminal utilities for illustration purposes. Bottom figure fills the areas where the robust SS agent performs better than the classical SS agent with green colour, and the alternative scenario with blue colour.

as the agents have more frequently over-estimated mean-reversion level but under-estimated mean-reversion speed, at the same time of under-estimating volatility of volatility and extent of correlation between stock price and its volatilities.

Despite the wrong estimates for these model parameters that the classical SS agent's strategy does not depend on explicitly, the robust SS agent still manages to achieve a higher expected utility than the classical SS agent. This again confirms our observation in TEST 2 that the robust strategy $\widehat{\pi}^{\text{robustSS}}$ is not very sensitive to model parameters other than λ.

Finally, the modified Merton agent only estimates the MPR (the term λY in the Stein-Stein model), and the believed constant volatility (process Y in the Stein-Stein model). Her MPR estimates are also quite volatile and can vary from around -1.5 to 2.5 when the true value is normally in-between 0 and 1 in our case. However, her volatility estimates, as illustrated by the example paths in Figure 4.5 (middle figure), can capture the average volatility and the trend of the volatility process as the estimation window moves forward. Moreover, as illustrated in Figure 4.5 (bottom figure), the strategy path (updated MPR/volatility) taken by the modified Merton agent follows similar trends as the classical SS agent. This explains why the modified Merton agent, in this test, has similar (or slightly better, of order around 0.4%) performances as the classical SS agent, when both are using our naive approach of estimating the model parameters.

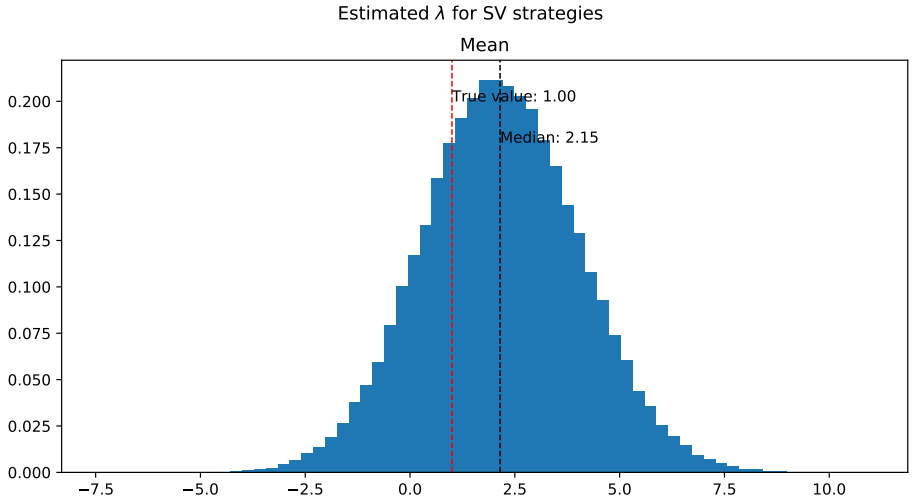


Figure 4.8: Stein-Stein model (TEST 4): Histogram of the path-by-path mean for estimated λ for classical and robust SS agents.

We have also performed the numerical test (TEST 5 in Appendix B) which is an analogy of TEST 4, but with N integrated estimation and test paths. The results

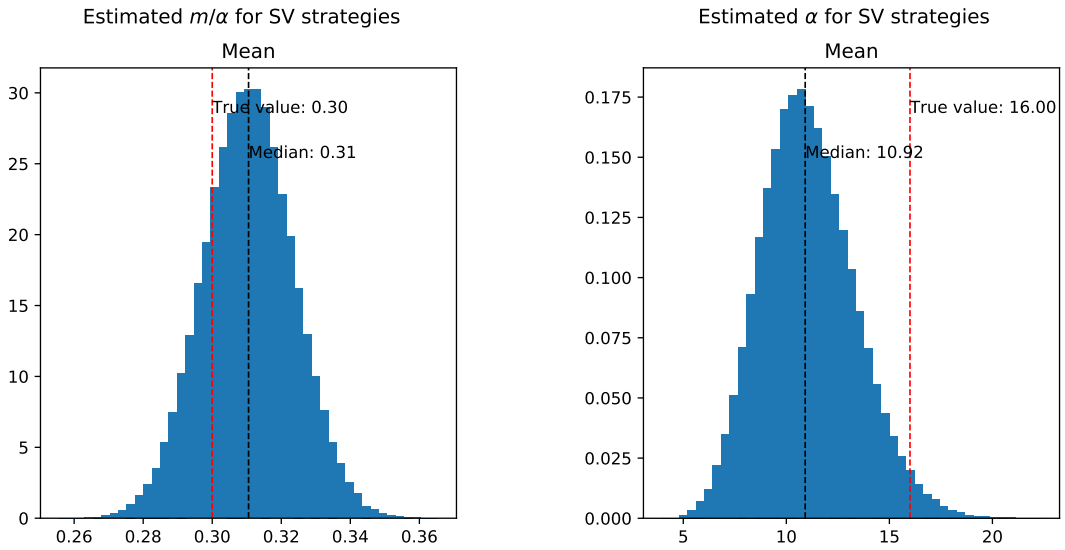


Figure 4.9: Stein-Stein model (TEST 4): Histograms of the path-by-path mean for estimated m/α (mean-reversion level) and α (mean-reversion speed) for classical and robust SS agents.

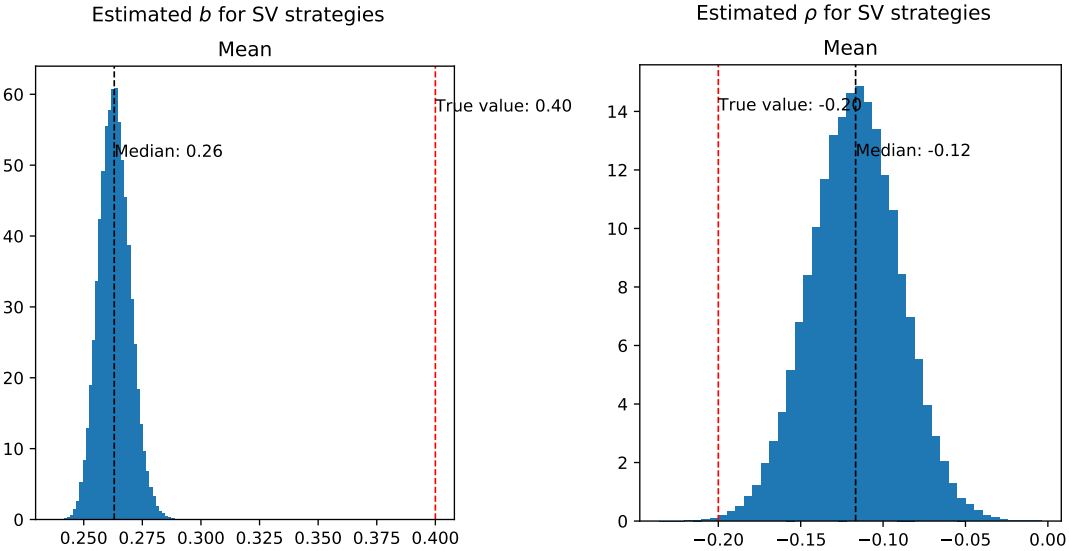


Figure 4.10: Stein-Stein model (TEST 4): Histograms of the path-by-path mean for estimated b (volatility of volatility) and ρ (correlation between price and volatility) for classical and robust SS agents.

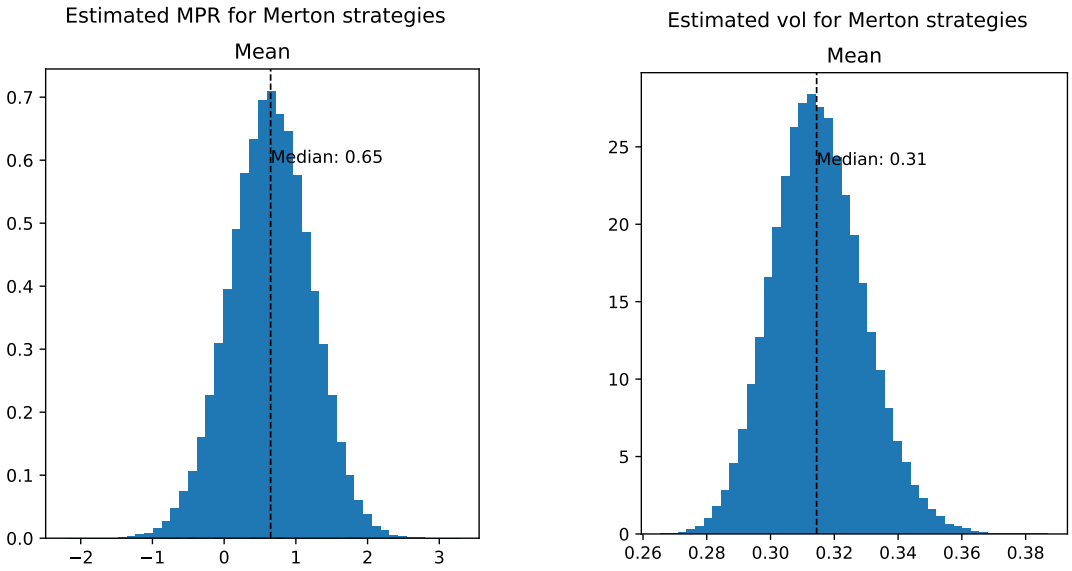


Figure 4.11: Stein-Stein model (TEST 4): Histograms of the path-by-path mean for estimated MPR and volatility for the modified Merton agent.

are in line with those in TEST 4, except that the parameter estimation errors are more severe as a result of this test scheme allowing for more possibilities in the estimation period. One could then expect that all agents perform worse compared with themselves in TEST 4, but the relative performances remain the same. Thus we put the test results in Appendix B.

4.5.2 Heston model

In the Heston stochastic volatility model, the stock S and its volatility Y are assumed to follow

$$\begin{aligned} \mathrm{d}S_t &= Y_t S_t(\lambda Y_t \,\mathrm{d}t + \,\mathrm{d}W_t^0), \\ \mathrm{d}Y_t &= \left(\frac{m}{Y_t} - \alpha Y_t\right) \mathrm{d}t + b\,\mathrm{d}\widetilde{W}_t^0, \end{aligned} \tag{4.5.6}$$

where parameter functions in the general model (3.4.1)-(3.4.2) are specified by $\lambda(y) = \lambda y$, $a(y) = \frac{m}{y} - \alpha y$, $b(y) = b$ for some constants $\lambda \neq 0, m > 0$, $\alpha > 0$ and $b > 0$.

Substituting in the above parameter functions in (4.4.1), the PDE satisfied by v is

$$\frac{\partial v}{\partial t}(t,y) + \left(\frac{m}{y} - (\alpha + q\rho b\lambda)y\right)\frac{\partial v}{\partial y}(t,y) + \frac{1}{2}b^2\frac{\partial^2 v}{\partial y^2}(t,y) - \frac{q\lambda^2}{2\delta}y^2 v(t,y) = 0,$$
$$v(T,y) = 1.$$

Using the same method as for the Stein-Stein model, the solution to the above PDE is given by

$$v(t,y) = \exp\left(-c(t)y^2 - e(t)\right),$$

where

$$c(t) = \frac{q\lambda^2 \sinh(a_1(T-t))}{2\delta\psi(a_1(T-t))},$$
$$e(t) = \frac{2m+b^2}{2b^2}\left(\ln\left(\frac{\psi(a_1(T-t))}{a_1}\right) - a_2(T-t)\right),$$

and the constants a_1, a_2 and function $\psi(\cdot)$ are defined in (4.5.2). Thus the robust value function and robust optimal strategy are

$$u_t = \ln X_t + \frac{\delta}{\beta}\left(c(t)Y_t^2 + e(t)\right),$$
$$\widehat{\pi}_t^{\text{robustHes}} = \frac{1}{\beta+1}\left(\lambda - 2\rho\delta b c(t)\right), \quad t \in [0,T]. \tag{4.5.7}$$

In comparison with the classical optimal strategy given by

$$\widehat{\pi}_t^{\text{classicalHes}} = \lambda, \quad t \in [0,T],$$

which keeps a constant proportion of wealth in the stock, the robust strategy is, instead, a deterministic function of time t, the ambiguity aversion of the agent β, the investment horizon T, as well as other model parameters ρ, α and b. Similarly to the Stein-Stein model, the robust strategy also has constant terminal value $\lambda/(1+\beta)$.

Assuming the agent has wrong estimates λ^* for parameter λ, we then numerically compare the performances of the robust strategies with those of the classical strategies and optimal strategies, using the above explicit expressions of the optimal strategies. The true parameters are chosen as in Table 4.6 and the results for average utilities of different strategies are shown in Table 4.7.

S_0	Y_0	λ	m	α	b	ρ	X_0	T
100	0.3	1.0	0.1	1.5	0.25	-0.30	1000	1 year

Table 4.6: Parameter values for Heston model (4.5.6).

From Table 4.7 and Figure 4.12, coherent with the observations from the Stein-Stein example, the robust strategies are more useful when the agent over-estimates λ and is moderately ambiguity averse (top figure in Figure 4.12). In cases when the agent is strongly ambiguity averse, she will barely invest in the risky asset. Although

it seems better to be more conservative if the agent over-estimates λ with some large error, in most cases when the agent under/precisely-estimates λ, being too conservative will not give the agent favourable distributions of terminal utilities (bottom figure in Figure 4.12).

	λ^*	β	$u_{T,\text{avg}}^{\text{robustHes}}$	$u_{T,\text{avg}}^{\text{classicalHes}}$	$u_{T,\text{avg}}^{\text{optimal}}$
over-estimate	1.5	0.5	6.95 (0.29)	6.94 (0.44)	6.95 (0.29)
		1.0	6.95 (0.22)		
precise-estimate	1.0	0.5	6.95 (0.19)	6.95 (0.29)	
		1.0	6.94 (0.15)		
under-estimate	0.5	0.5	6.93 (0.10)	6.94 (0.15)	
		1.0	6.93 (0.07)		

Table 4.7: Heston model: Average terminal utilities of different strategies when the agent over/under/precisely estimates λ. Numbers in the brackets are standard deviations of the average terminal utilities.

Another similar observation as in the Stein-Stein example is that, for our parameter choices (Table 4.6), the robust strategy essentially scales the classical strategy by a factor $\frac{1}{1+\beta} < 1$ and the second term in the robust strategy (4.5.7) is very small. As shown by the bottom figure in Figure 4.13, the robust strategy process essentially stays at the value $\frac{\lambda^*}{1+\beta}$ throughout $t \in [0, T]$. In our case when the agent over-estimates λ by 50% and is moderately ambiguity averse with $\beta = 0.5$, the robust strategy indeed coincides with the optimal strategy thus has the same wealth process (top figure in Figure 4.13) and terminal utility distribution (top figure in Figure 4.12).

4.5.3 Robustness to model error (or mis-specification of SV models)

As we can see from the previous results, the robust strategies, being more conservative, can achieve better terminal utilities than the classical strategies and the modified Merton strategies (Merton strategy with dynamically calibrated model parameters), in the cases when the robust agent has the correct model assumed. Then we would like to test how "robust" the robust strategies are in terms of model error – when the robust agent assumes a different model than the true underlying model.

In the following, we will conduct some initial attempt towards testing the model robustness of the strategies. We will assume the true underlying model is a Heston stochastic volatility model, with parameters chosen as in Table 4.6. Then we compare the average terminal utilities achieved by: (i) A classical Stein-Stein agent who takes

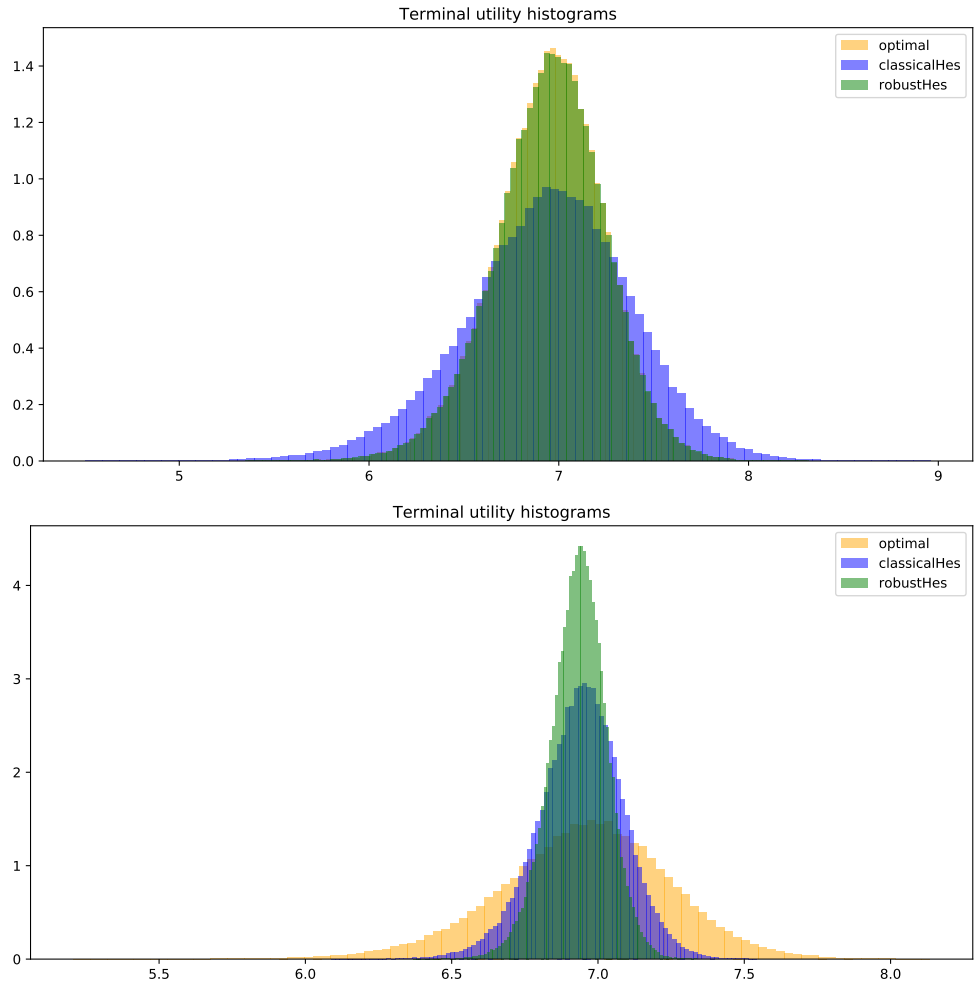


Figure 4.12: Heston model: Histograms of terminal utilities for different strategies. Top figure: $\lambda^* = 1.5$, $\beta = 0.5$. With the parameter choices, the histogram for the robust Heston strategy indeed overlaps with that for the optimal strategy. Bottom figure: $\lambda^* = 0.5$, $\beta = 0.5$.

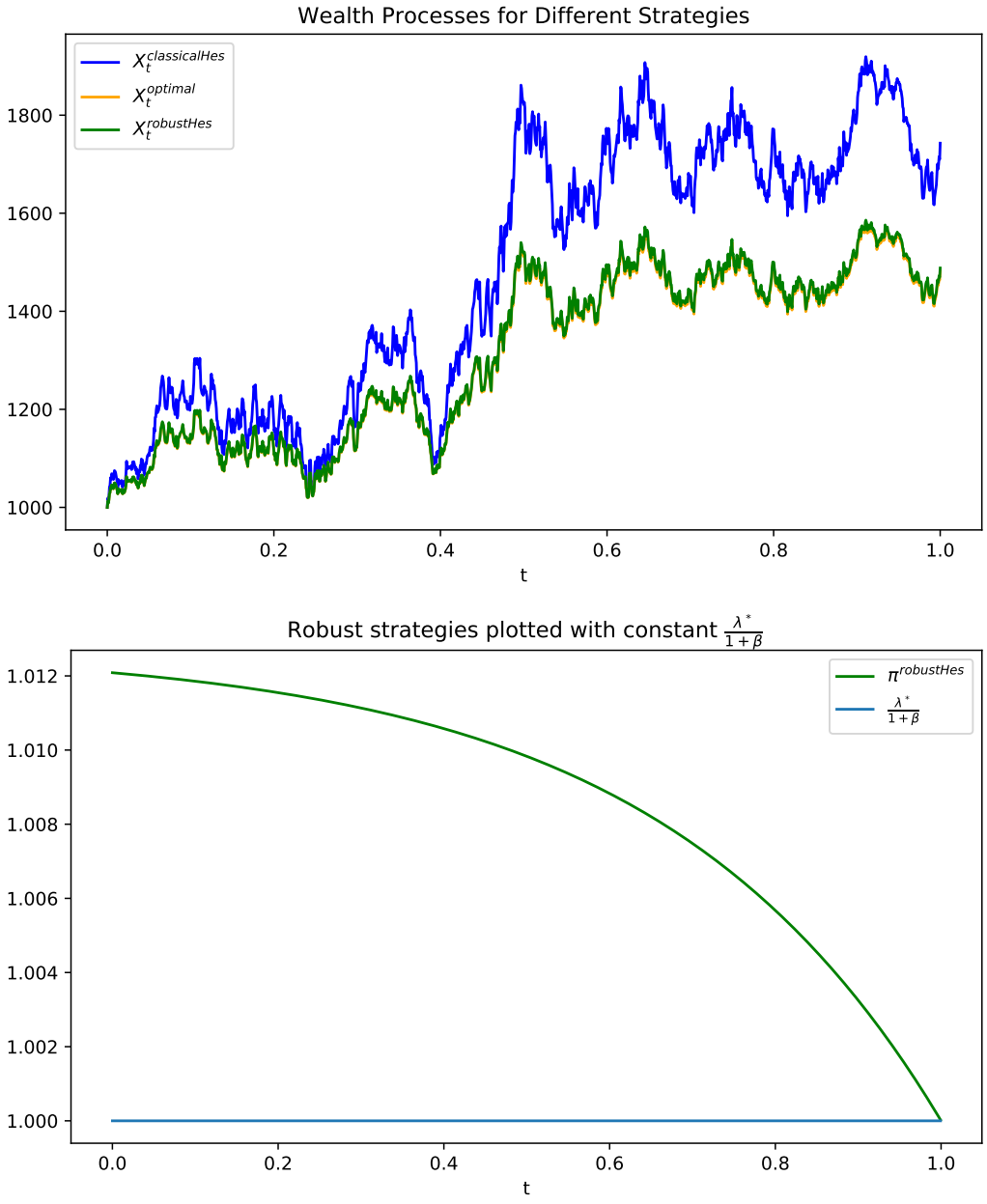


Figure 4.13: Heston model: Parameter values are chosen as in Table 4.6 and $\lambda^* = 1.5$, $\beta = 0.5$. Top figure: example wealth processes for one realisation. Bottom figure: robust strategies $(\widehat{\pi}_t^{\text{robustHes}})_{t\in[0,T]}$. In the bottom figure, $\widehat{\pi}^{\text{robustHes}}$ essentially stays at the value $\frac{\lambda^*}{1+\beta} = \lambda = 1.0$ over $t \in [0,T]$, which explains the almost identical wealth paths of the robust Heston agent and the optimal agent, as well the overlapping histograms.

a Stein-Stein volatility model as the "true" model and uses the strategy $\widehat{\pi}^{\text{classicalSS}}$ with dynamically updated model parameters; (ii) A robust Stein-Stein agent who takes a Stein-Stein volatility model as the "true" model but uses the robust strategy $\widehat{\pi}^{\text{robustSS}}$ with dynamically updated model parameters; (iii) A modified Merton agent who believes in the Merton model and uses the Merton strategy with dynamically updated model parameters; (iv) An optimal agent who knows everything including the true model as well as the model parameters (thus uses the true constant model parameters).

In the following tests, the methods used by the agents to estimate the model parameters are as they have used in the Stein-Stein example in Section 4.5.1. The Stein-Stein (SS) agents will use the Cuchiero-Teichmann algorithm for estimating the volatility process and take that as an input for volatility for our naive estimation scheme in (4.5.4) for their believed Stein-Stein model. The modified Merton agent will use the simple estimation method in (4.5.5) to estimate the MPR and the volatility for her believed Merton model. All agents will use the same moving window for dynamically updating the estimated model parameters.

TEST 1: True model = Heston, 1 estimation + N tests.

We first use the scheme where there is one path in the estimation period for the agents to get their initial estimates for the parameters, and N paths in the test period for the agents to implement their strategies with updated parameter values with the moving window.

As we can see from both Table 4.8 and Figure 4.14, in this case when all agents have the wrong model assumed, the results are still consistent with our previous observations. The robust SS agent achieves the highest average terminal utilities, of the order of 3.2% compared with the classical SS agent and 2.9% compared with the modified Merton agent. The distribution of the terminal utilities of the robust SS agent is also closer to the optimal agent, with a higher mean and lower standard deviation. While choosing a more conservative ambiguity aversion parameter β (bottom figure in Figure 4.14) for the robust SS agent improves the average terminal utilities and reduces the standard deviation (as a result of higher terminal utilities in face of severe parameter estimation errors), it does not increase the frequency of the robust SS agent outperforming the classical SS agent and the modified Merton agent. Hence we only show the results when $\beta = 0.5$.

By the path-by-path diagrams in Figure 4.15 for additional terminal utilities over modified Merton agent, we can also confirm that the robust SS agent has, more often,

$u_{T,\text{avg}}^{\text{robustSS}}$	$u_{T,\text{avg}}^{\text{classicalSS}}$	$u_{T,\text{avg}}^{\text{modMerton}}$	$u_{T,\text{avg}}^{\text{optimal}}$
6.79 (0.63)	6.57 (0.86)	6.59 (0.85)	6.95 (0.29)

Table 4.8: Robustness to model error (TEST 1): Average terminal utilities of different strategies. Numbers in the brackets are standard deviations of the terminal utilities. True parameter values are as in Table 4.6, and ambiguity aversion parameter $\beta = 0.5$.

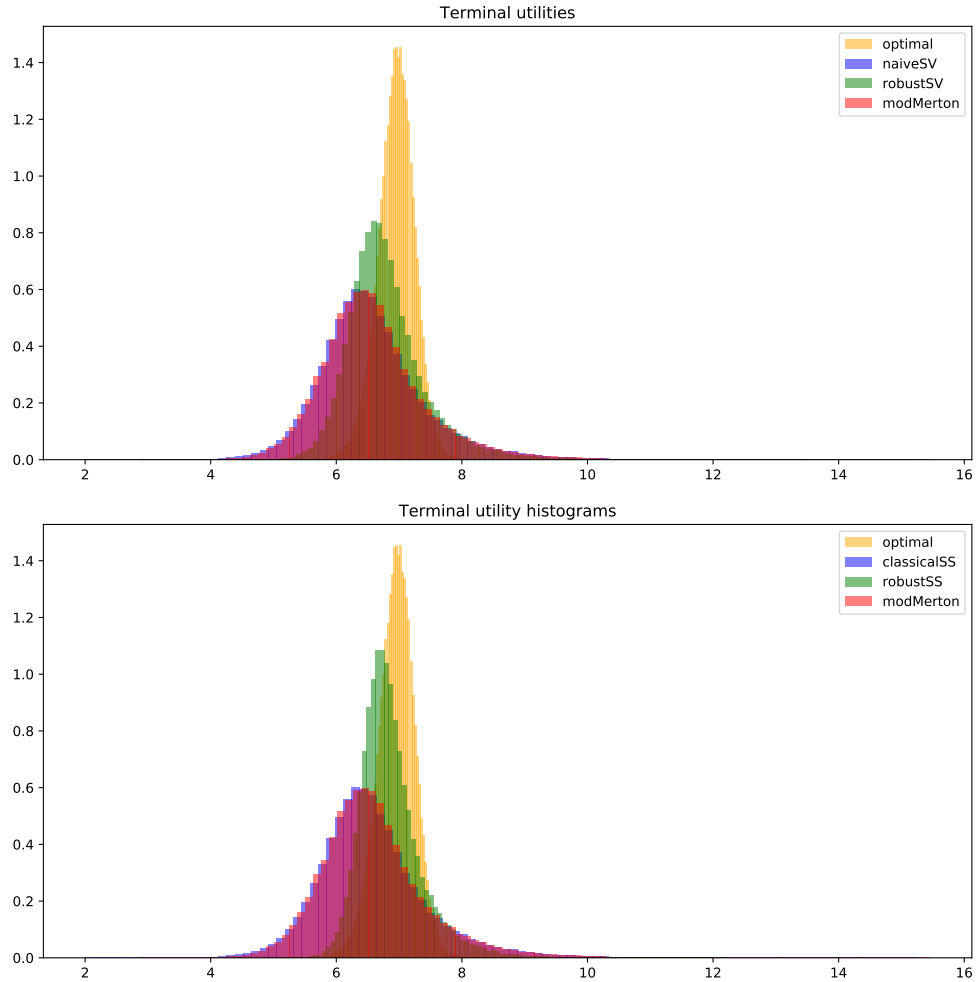


Figure 4.14: Robustness to model error (TEST 1): Histograms of terminal utilities. Top figure: $\beta = 0.5$. Bottom figure: $\beta = 1.0$.

achieved higher terminal utilities than the classical SS agent (85% of all times in the N realisations and 86% of the 1000 randomly chosen realisations in the figure) as well as the modified Merton agent (81% of all times in the N realisations and of the 1000 randomly chosen realisations in the figure). The classical SS agent only did better than the modified Merton agent for 43% of all times in the N realisations and 44% of the 1000 randomly chosen realisations in the figure.

The estimated parameter distributions are shown in Figures 4.16 – 4.18 for the classical SS and robust SS agents and in Figure 4.19 for the modified Merton agent. For the classical and robust SS agents, from the structure of the Stein-Stein and the Heston model, we can see that the stock return parameter λ, the stochastic volatility process, the volatility of volatility parameter, and the correlation between the Brownian motions driving the price and the volatility, coincide for both models. Thus we are still able to compare their estimated values against the true values.

We could see from Figure 4.16 that the estimates for the stock return parameter are essentially as diverse as before (for example, see Figure 4.8). Figure 4.17 gives us an idea of the mean estimated mean-reversion level and speed of the volatility process used by the SS agents. The estimated mean-reversion level used by the SS agents mostly lies within 33% around 0.3, which is close to the long-run average volatility 0.296 implied by the parameter choices in Table 4.6 for the Heston model. The mean-reversion speed used by the SS agents for the volatility process is higher, for most of the times, than the mean-reversion speed (which is $2\alpha^{\text{Heston}} = 3$) of the variance process of the Heston model. Then, for the volatility of volatility and correlation parameters, the tendency of our estimation method to under-estimate volatility of volatility and the extent of correlation are coherent with our previous findings when the SS agents have the correct underlying model assumed for the volatility process.

For the modified Merton agent, she also has estimated MPRs and volatilities distributed as diverse as in our previous TEST 4 when the true underlying model is a Stein-Stein model. With the wrong model and the naively estimated parameters, she eventually managed to achieve higher terminal utilities for about 57% of all times than the classical SS agent. This is, again, consistent with the case when the underlying model is a Stein-Stein stochastic volatility model (when the classical SS agent has the correct model assumed).

Then we also test the strategies using the alternative test scheme where we use N joint estimation and test paths. The observations are, again, in agreement with those from TEST 1, thus we have put the results in Appendix B. All agents perform worse than themselves in TEST 1, but the loss of average terminal utilities of the robust SS

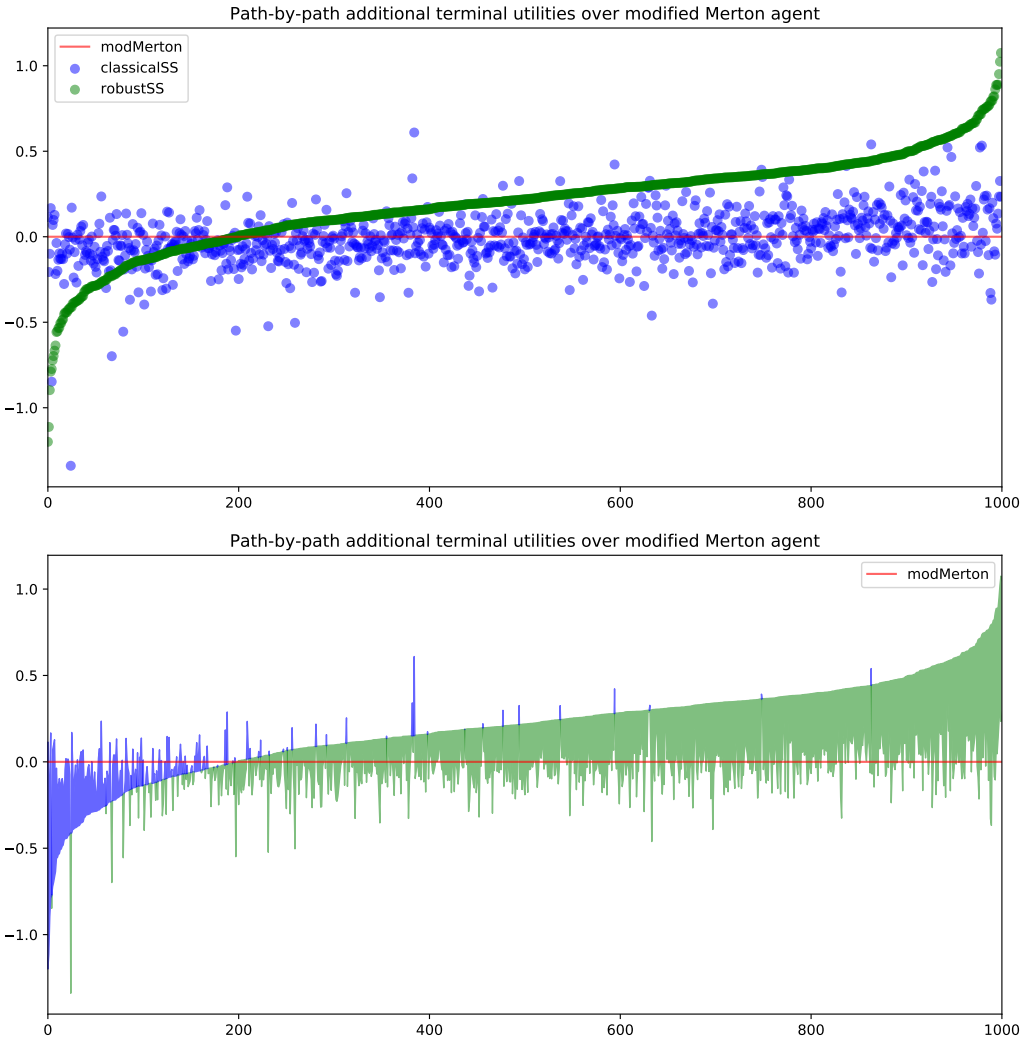


Figure 4.15: Robustness to model error (TEST 1): Path-by-path additional terminal utilities over modified Merton agent (1000 realisations). Data points are re-organised in the order of increasing robust SS terminal utilities for illustration purposes. Bottom figure fills the areas where the robust SS agent performs better than the classical SS agent with green colour, and the alternative scenario with blue colour.

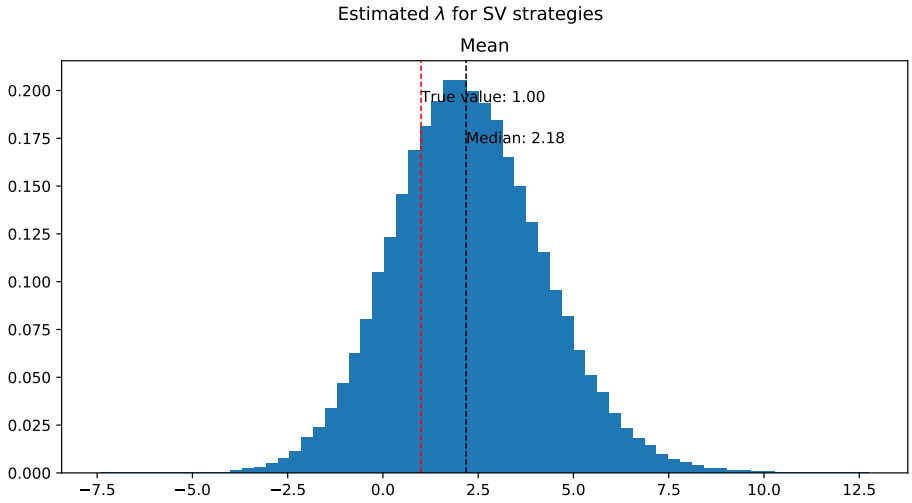


Figure 4.16: Robustness to model error (TEST 1): Histogram of the path-by-path mean for estimated λ for classical and robust SS agents.

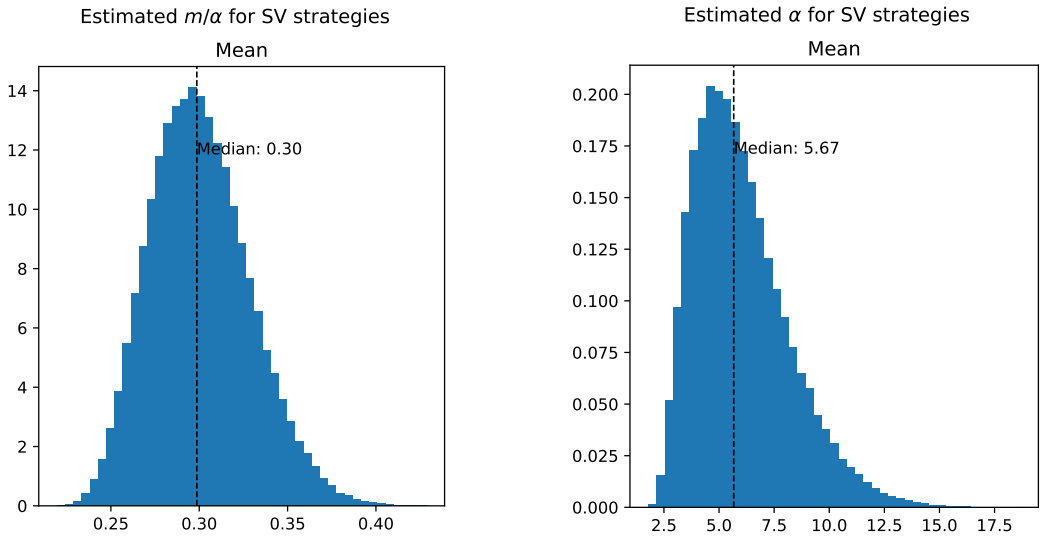


Figure 4.17: Robustness to model error (TEST 1): Histograms of the path-by-path mean for estimated m/α (mean-reversion level) and α (mean-reversion speed) for classical and robust SS agents.

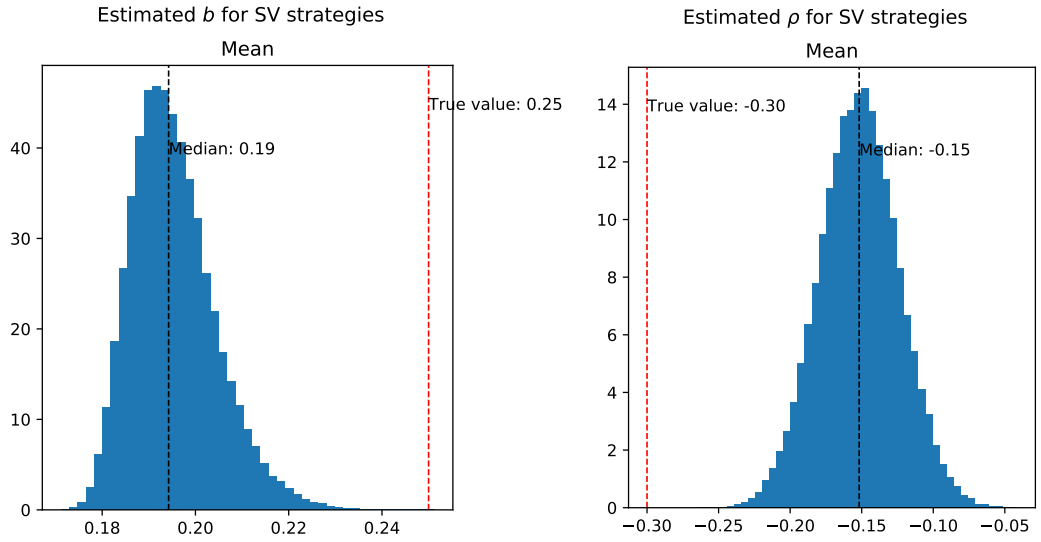


Figure 4.18: Robustness to model error (TEST 1): Histograms of the path-by-path mean for estimated b (volatility of volatility) and ρ (correlation between price and volatility) for classical and robust SS agents.

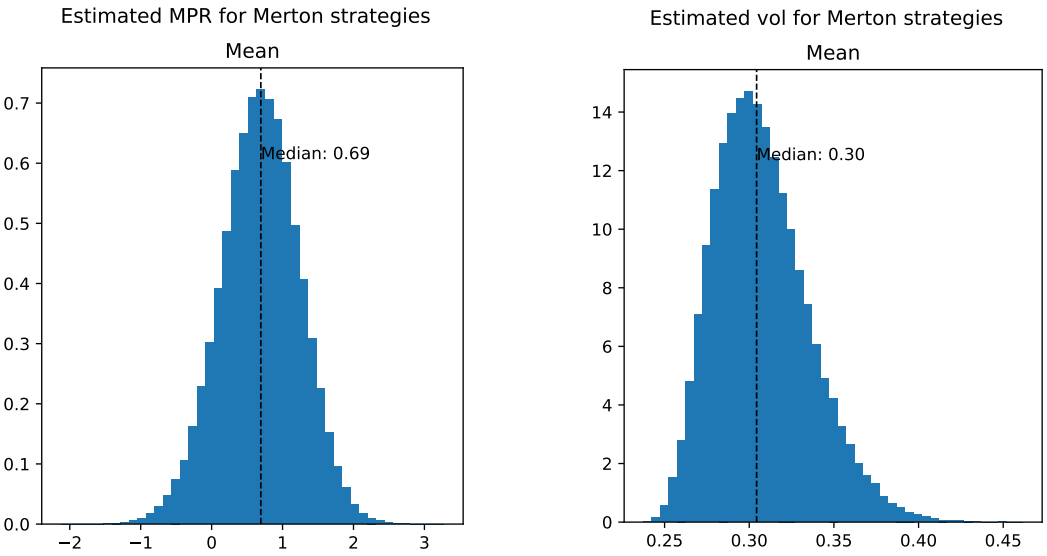


Figure 4.19: Robustness to model error (TEST 1): Histograms of the path-by-path mean for estimated MPR and volatility for the modified Merton agent.

agent has a lower order compared with other agents, which indicates that the robust strategy is more stable with respect to the variations in the procedures of our tests.

4.5.4 Summary

From the numerical examples of the Stein-Stein and Heston stochastic volatility models, we can see that the main benefit for using the robust strategies is that the robust agent does not have to worry much about being slightly aggressive (in whichever direction) about the growth of the stock she invests in: the robust strategies will partially cancel the aggressiveness of the estimate and reduce the variations in the terminal investment results (utilities).

It is worth noticing that, if we understand β as the level of uncertainty the robust agent has with the optimistic estimations, although not easy to achieve in practice, by adjusting this level of uncertainty wisely according to the estimates (or having a good sense of how reliable the estimates are), the robust agent can better approximate the distributions of terminal utilities and wealth processes of some classical agent knowing the exact value of λ, which has been illustrated in Figures 4.1, 4.2, 4.3, 4.12 and 4.13.

Another observation is that although the robust strategy has introduced direct dependence on model parameters other than the return of the stock λ, the strategy is not very sensitive to estimation errors in these parameters. According to our tests, with the true parameter values we have chosen, even with the naively estimated model parameters, the robust strategy has still performed better than the classical strategy.

Moreover, according to our tests using different test schemes and in the tests where there is model error (a mis-specification of stochastic volatility models), the robust strategies also seem to perform better and are more stable with respect to the changes in our test procedures as well as true data-generating stochastic volatility models.

One last note from the tests is that when implementing strategies under specific models, the difficulty of estimating the parameters in these models can be an important factor affecting the actual performances of these strategies. It could happen that even with the right model assumed (which is not even likely to be the case in reality), because of the difficulty of estimating parameters for a complicated model, the strategy still performs worse than a strategy with a wrong but simple model.

Chapter 5

Robust exponential hedging of claims on non-traded assets

5.1 Introduction

The ultimate goal of this chapter is to analyse how the robust approach to optimal investment impinges upon a classical incomplete market hedging problem in which a claim on a non-traded asset is hedged using a correlated traded stock, under exponential utility preferences. The non-robust version of this problem has received huge attention, with many explicit results, asymptotic analyses, and examination of the efficacy of the strategies, beginning with Davis [11] (comes from [10]) and many other papers since: Henderson [23], Monoyios [40, 43], Monoyios [44] (partial information), Musiela and Zariphopoulou [47]. To this end, we will formulate a market model containing a traded stock S, and a correlated non-traded asset with price process Y. Ultimately, both S and Y will be taken to follow correlated geometric Brownian motions (GBMs) with constant parameters under the reference model. However, to start with, we shall take S and Y to be general Itô processes (so a non-Markov set-up) with parameters that are constrained to be adapted to the filtration generated by the correlated Brownian motion driving Y.[1]

Let us denote the model and the associated optimisation problems somewhat formally for the purposes of this introduction. Precise details and model assumptions will follow in subsequent sections.

Under some reference model, on some stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}^0)$, with $\mathbb{P}^0$ some reference measure, supporting a 2-dimensional $(\mathbb{P}^0, \mathbb{F})$-Brownian motion

[1] In the multiple priors exponential hedging problem in Section 5.2, for ease of illustration, we will not involve Y explicitly and the agent faces the problem of hedging a European claim using a traded stock S correlated with the risk posed by the claim.

$(W^0, W^{0,\perp})$, a traded stock with price process, in a zero interest rate market, is taken to follow an Itô process, with dynamics of the form

$$\mathrm{d}S_t = \sigma_t S_t(\lambda_t \,\mathrm{d}t + \,\mathrm{d}W_t^0), \tag{5.1.1}$$

for some strictly positive volatility process σ and real-valued MPR process λ. Here, both σ and λ are $\widetilde{\mathbb{F}}$-adapted (as in Chapter 2), where $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$ is the filtration generated by the Brownian motion $\widetilde{W}^0$ defined by

$$\widetilde{W}^0 = \rho W^0 + \sqrt{1-\rho^2}W^{0,\perp},$$

where $\rho \in (-1, 1)$ is a constant correlation parameter. Ultimately, when studying a log-normal basis risk model, we shall take σ, λ as constants.

The unhedgeable risk in so-called basis risk models stems from the introduction of claims on a non-traded asset Y, whose main feature is that it is an $\widetilde{\mathbb{F}}$-adapted process. We shall take Y to have Itô dynamics (under $\mathbb{P}^0$) of the form

$$\mathrm{d}Y_t = a_t \,\mathrm{d}t + b_t \,\mathrm{d}\widetilde{W}_t^0,$$

for some positive volatility process b and some drift process a, with both a and b being $\widetilde{\mathbb{F}}$-adapted. Later in a numerical study, we shall take a, b as constant multiples of Y.

A European claim on Y has bounded $\widetilde{\mathcal{F}}_T$-measurable payoff $g(Y_T)$ for some function $g : \mathbb{R} \to \mathbb{R}$. More generally, one could take any $\widetilde{\mathcal{F}}_T$-measurable claim payoff G_T.

In this scenario, one can consider a variety of exponential hedging problems (with or without model uncertainty), as follows. Let us consider an agent whose risk preferences are described by an exponential utility function with constant risk aversion parameter $\alpha > 0$:

$$U(x) = -\mathrm{e}^{-\alpha x}, \quad x \in \mathbb{R}, \quad \alpha > 0.$$

The agent trading the stock plus cash is assumed to generate self-financing wealth process X, associated with some trading strategy π (the wealth held in S). Here are some utility maximisation problems that we shall compare in this chapter.

1: No model uncertainty. (Chapter 2, Section 2.4)

This is the classical problem of maximising expected utility of terminal wealth with a random endowment of nG_T, that is, a number $n \in \mathbb{R}$ of units of the claim. The objective here is

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}^0[U(X_T + nG_T)],$$

maximised over some class of admissible strategies.

2: Multiple priors (worst case) utility maximisation with random endowment.
Now, model ambiguity is introduced, in a similar spirit to earlier chapters, and one considers models $\mathbb{P} \in \mathcal{P}$, with each possible measure $\mathbb{P} \sim \mathbb{P}^0$ on $\mathcal{F}_T$, and with

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^0} = \mathcal{E}\left(\int_0^{\cdot} \eta_s \,\mathrm{d}W_s^0 + \int_0^{\cdot} \eta_s^{\perp} \,\mathrm{d}W_s^{0,\perp}\right)_T,$$

for some process $\boldsymbol{\eta} := (\eta, \eta^{\perp}) \in \mathcal{N}$ containing $\widetilde{\mathbb{F}}$-adapted processes with values in some compact convex set $\Gamma \subset \mathbb{R}^2$. As in earlier chapters, each component in $\boldsymbol{\eta}$ simply adds a drift to each of the driving Brownian motions W^0, $W^{0,\perp}$. Naturally, the model dynamics under $\mathbb{P} \in \mathcal{P}$ are of the form

$$\begin{aligned} \mathrm{d}S_t &= \sigma_t S_t((\lambda_t + \eta_t)\,\mathrm{d}t + \,\mathrm{d}W_t), \\ \mathrm{d}Y_t &= \big(a_t + b_t(\rho\eta_t + \sqrt{1-\rho^2}\eta_t^{\perp})\big)\,\mathrm{d}t + b_t\,\mathrm{d}\widetilde{W}_t, \end{aligned} \tag{5.1.2}$$

where $W, \widetilde{W}$ are now $\mathbb{P}$-Brownian motions. Here, a worst case problem has objective

$$\sup_{\pi\in\mathcal{A}} \inf_{\boldsymbol{\eta}\in\mathcal{N}} \mathbb{E}[U(X_T + nG_T)], \tag{5.1.3}$$

maximised over some (adjusted) set $\mathcal{A}$ of admissible strategies and minimised over the plausible set of models.

3: Variational preferences robust hedging.
Now we consider all possible models (so $\mathcal{N}$ contains all $\mathbb{F}$-adapted processes, and Γ can be an unbounded set), but penalised according to their plausibilities. Here, given the exponential utility function, it turns out that it is advantageous to use a multiplicative penalty so that, with some constant ambiguity aversion parameter $\beta > 0$, the objective becomes

$$\sup_{\pi\in\mathcal{A}} \inf_{\boldsymbol{\eta}\in\mathcal{N}} \mathbb{E}\left[\exp\Big(-\frac{1}{2\beta}\int_t^T ||\boldsymbol{\eta}_s||^2 ds\Big) U\left(X_T + nG_T\right)\right], \tag{5.1.4}$$

This type of penalty was suggested by Mataramvura and Øksendal [33], (and in essence works because the penalty simply modifies the terminal random endowment, given the form of the utility functional).

In this chapter, we first obtain explicit results for the second problem above, with objective as in (5.1.3), using similar techniques to Chapter 3. Specifically, in the non-Markov stochastic factor setting of (5.1.2), we first employ a martingale distortion technique to solve the optimal investment problem under $\mathbb{P} \in \mathcal{P}$ in (5.1.3),

and then minimise over models. The value process, optimal trading strategy and least favourable model are expressed in terms of the solution to a BSDE. We also obtain PDE representations for these quantities when the model is specialised to be Markovian.

The main focus in this chapter, however, is the solution of the variational preferences problem (5.1.4). We proceed directly to the Markov version of the model, with parameters λ, σ, a, b and claim payoff g all functions of Y. In this case, the max-min problem in (5.1.4) is attacked via a stochastic differential game approach, which leads to a Hamilton-Jacobi-Bellman Isaacs (HJBI) equation for the value function. Then the optimal strategy and least favourable model (so optimisers $\widehat{\boldsymbol{\eta}} = (\widehat{\eta}, \widehat{\eta}^{\perp})$) are all characterised in terms of the solution to the HJBI equation. Moreover, the HJBI equation actually leads to the value function represented in terms of a linear PDE.

The aforementioned PDE is not readily solvable, so, in the final part of this chapter, we specialise the basis risk model to one where both $\ln S$ and $\ln Y$ have constant parameters under $\mathbb{P}^0$. This is the well-studied log-normal non-traded asset model which has featured in many papers (Davis [11], Henderson [23], Monoyios [40]). Now, the full power of analytic solutions to the problems comes to the fore. We are able to write down (almost immediately, given earlier research) analytic solutions or approximations (if an analytic solution is not available) for the (robust) indifference prices and optimal hedging strategies associated with:

- classical exponential hedging,
- multiple priors (worst case) exponential hedging,
- variational preferences (penalised) exponential hedging,
- Black-Scholes (and correlation-weighted Black-Scholes) hedging, which takes S as a perfect proxy for Y (and multiplies the Black-Scholes-style hedging strategy by the correlation).

The point is that we have fast computations of a number of alternative hedging schemes, where we proceed to test over simulated paths, computing distributions of the terminal hedging errors so produced.

Broadly speaking, the variational preferences (respectively, multiple priors) approach simply adds one extra degree of freedom to the classical problem – namely, it allows for a choice of ambiguity aversion. What is more, the results (for indifference prices and hedging strategies) have the same structural form as the classical

(no model uncertainty) scenario, with simply a modified risk aversion parameter α^{vp}, given by $\alpha^{\text{vp}} = \alpha(1+\beta)$ (respectively, for the multiple priors approach, a less desirable distribution of the non-traded asset Y with a lower mean of $\ln Y$ used in the pricing formula).

The upshot here is that a robust approach, with the form of model penalisation as in (5.1.4) (or a worst case approach considering some independent constant intervals of $\eta, \eta^{\perp}$, as in (5.1.3)), reproduces the indifference prices and hedging strategies of a classical problem with modified risk aversion parameter (or the drift of the process Y to a less favourable case) that now also depends on the ambiguity aversion parameter (or the boundaries of the intervals of $\eta^{\perp}$). However, on reflection, this is an important observation and leads to interesting suggestions for future data-led calibration of ambiguity aversion in such hedging schemes. Indeed, one can envisage algorithms based on machine learning to select a suitable model ambiguity aversion parameter based on bounding crucial statistics such as variance of hedging errors. This is clearly a vast topic for future research.

5.2 Multiple priors exponential hedging

In this section, we consider a worst case scenario, in a non-Markovian stochastic factor model. Under the so-called reference measure $\mathbb{P}^0$, the dynamics in this model are as in (5.1.1) (or (3.1.1)), which is of the same form as the generic model of Chapter 2. The set of models $\mathcal{P}$ is defined by

$$\mathcal{P} := \Bigg\{ \mathbb{P} \sim \mathbb{P}^0 \text{ on } \mathcal{F}_T \;\Big|\; \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^0}\Big|_{\mathcal{F}_t} = D_t = \mathcal{E}\left(\int_0^{\cdot} \eta_s \,\mathrm{d}W_s^0 + \int_0^{\cdot} \eta_s^{\perp} \,\mathrm{d}W_s^{0,\perp} \right)_t, \\ t \in [0,T],\ \boldsymbol{\eta} = (\eta, \eta^{\perp}) \in \mathcal{N} \equiv \mathcal{N}_{0,T} \Bigg\}.$$

The set $\mathcal{N}_{s,T}$ for $0 \le s \le t \le T$ denotes the set of all $\widetilde{\mathbb{F}}$-adapted [2] processes $(\boldsymbol{\eta}_t)_{t\in[s,T]}$ such that $\boldsymbol{\eta}_t \in \Gamma$, $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}^0$-a.e., where $\Gamma \subset \mathbb{R}^2$ is some fixed compact [3] convex set. Since there is a one-to-one correspondence between measures $\mathbb{P} \in \mathcal{P}$ and processes $\boldsymbol{\eta} \in \mathcal{N}$ (up to $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}^0$-nullsets), $\boldsymbol{\eta}$ can then be used to characterise models $\mathbb{P}$ in

[2] We are concerned, in essence, with drift uncertainty in models of the form (3.1.1), so it is natural to consider all models which remain in this class. Indeed, in a Markovian version of (3.1.1), considered by Owari [49], the stochastic control approach to the robust problem led to optimal models which remained in the same class as the original reference model. There may be some loss of generality, however, we are only looking at models with the same structure.

[3] Similar as in Chapter 3, the compactness condition here can be lifted in some cases, but would lead to other regularity assumptions being needed.

the optimisation problem that we define below. The set of measures corresponding to $\boldsymbol{\eta} \in \mathcal{N}_{s,T}$ will be denoted by $\mathcal{P}_{s,T}$, with the usual convention $\mathcal{P} \equiv \mathcal{P}_{0,T}$.

The objective of the agent is described by the robust value function, given initial wealth x,

$$u(x) := \sup_{\pi \in \mathcal{A}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}\left[U\left(X_T + nG_T\right)\right], \tag{5.2.1}$$

where the utility function $U : \mathbb{R} \to \mathbb{R}$ is specified as the exponential utility.

For the set of admissible strategies $\mathcal{A}$ in (5.2.1), the following additional integrability condition is introduced:

$$\mathbb{E}\left[\exp\left\{c\alpha^2 \int_0^T \pi_t^2 \sigma_t^2 \, \mathrm{d}t\right\}\right] < \infty, \quad \text{for some constant } c > 1. \tag{5.2.2}$$

Thus $\mathcal{A}$ is defined by

$$\begin{aligned} \mathcal{A} := \Big\{ & \pi_t \text{ is } (\mathcal{F}_t)\text{-progressively measurable, } \forall t \in [0,T] \;\Big|\; \pi \text{ satisfies (2.4.1), (5.2.2),} \\ & \text{and } \frac{\pi}{S} \cdot S \text{ is a } \mathbb{Q}\text{-martingale for any } \mathbb{Q} \in \mathcal{M}_{a,loc}^{\mathbb{P}^0} \cap \mathcal{M}_f^{\mathcal{P}} \text{ with } \mathcal{H}(\mathbb{Q}|\widehat{\mathbb{P}}) < \infty \Big\}, \end{aligned} \tag{5.2.3}$$

where $\widehat{\mathbb{P}}$ is the least favourable model that achieves the "inf" in (5.2.1) for an agent with a terminal endowment of n ($n \in \mathbb{R}$) claims, and $\mathcal{M}_{a,loc}^{\mathbb{P}^0} \cap \mathcal{M}_f^{\mathcal{P}}$ is the set of absolutely continuous local martingale measures with respect to $\mathbb{P}^0$ with $\mathcal{H}(\mathbb{Q}|\mathcal{P}) := \inf_{\mathbb{P} \in \mathcal{P}} \mathcal{H}(\mathbb{Q}|\mathbb{P}) < \infty$. It can be easily verified that condition (5.2.2) does not destroy the convexity of the set $\mathcal{A}$ of admissible strategies needed for the application of the minimax theorem.

In the rest of this section, the problem in (5.2.1) will be solved, in a similar way as in Chapter 3, by first using a minimax theorem to interchange the maximisation over strategies and minimisation over models, deriving a martingale distortion representation for the utility maximisation problem (5.2.5) under each fixed model (Proposition 5.2.4), and then minimising over models using the comparison theorem in classical BSDE theory. The value process, optimal strategy and "least favourable model" will be expressed via the solution of a BSDE (Proposition 5.2.5). As an example, a Markovian stochastic factor model will be studied where the robust value function and optimal strategy will be expressed via the solution of a PDE. Similar results were obtained in a slightly different context by Owari [49].

5.2.1 Martingale distortion solution for each model

Assumption 5.2.1. (i) The market price of risk $(\lambda_t)_{t\in[0,T]}$ in (5.1.1) satisfies

$$\mathbb{E}^0\left[\exp\left\{\frac{1}{2}\int_0^T \lambda_t^2\,\mathrm{d}t\right\}\right] < \infty.$$

(ii) The payoff G_T of the European option is bounded. Examples include put options, covered call options and spreads such as butterfly and bull spread.

Under Assumption 5.2.1 and our choice of the set of plausible models $\mathcal{P}$ and admissible strategies $\mathcal{A}$, it can be verified that the assumptions (A1)-(A4) in Owari [50] are satisfied, thus the duality results in [50, Theorem 2.4] hold. Particularly, the primal problem in (5.2.1) admits a saddle point, the optimal strategy satisfies the conditions in set $\mathcal{A}$ defined by (5.2.3) (except for the additional integrability condition (5.2.2), which is verified in the derivation of the martingale distortion representation), and the minimax theorem holds - the maximisation over strategies and minimisation over probability measures can be interchanged. Indeed, (A1), (A2) and (A4) are already verified in Owari [49, Lemma 2] since our set of density processes for candidate models $\{\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^0} \mid \mathbb{P}\in\mathcal{P}\}$ is essentially the same as that chosen in [49] except that we only look at $\widetilde{\mathbb{F}}$-adapted processes $\boldsymbol{\eta}$. For (A3), it only remains to check $\{\mathrm{e}^{\alpha n G_T}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^0} \mid \mathbb{P}\in\mathcal{P}\}$ is $\mathbb{P}^0$-uniformly integrable, since the rest are trivial by the boundedness of G_T. This is done by choosing the test function in de la Vallée-Poussin criterion to be $\phi(t)=t^2$ and proving that $\sup_{\boldsymbol{\eta}\in\mathcal{N}}\mathbb{E}^0\big[\big(\mathrm{e}^{\alpha n G_T}\mathcal{E}(\int_0\boldsymbol{\eta}_t\,\mathrm{d}\boldsymbol{W}_t^0)_T\big)^2\big]<\infty$, where $\boldsymbol{W}^0:=(W^0,W^{0,\perp})'$. In fact, since G_T is bounded and $\boldsymbol{\eta}$ is in some fixed compact convex set in $\mathbb{R}^2$, we have

$$\begin{aligned}\mathbb{E}^0\left[\left(\mathrm{e}^{\alpha n G_T}\mathcal{E}\Big(\int_0\boldsymbol{\eta}_t\,\mathrm{d}\boldsymbol{W}_t^0\Big)_T\right)^2\right] &\le C\mathbb{E}^0\big[\mathrm{e}^{2\int_0^T\boldsymbol{\eta}_t\,\mathrm{d}\boldsymbol{W}_t^0-\int_0^T||\boldsymbol{\eta}_t||^2\,\mathrm{d}t}\big]\\ &\le C\mathbb{E}^0[\mathrm{e}^{2\int_0^T\boldsymbol{\eta}_t\,\mathrm{d}\boldsymbol{W}_t^0}]\\ &\le C\big(\mathbb{E}^0[\mathrm{e}^{8\int_0^T||\boldsymbol{\eta}_t||^2\,\mathrm{d}t}]\big)^{\frac{1}{2}}\le C,\end{aligned}$$

where constant C might change from line to line. The third inequality is due to Protter [54, Theorem III 39].

Thus the robust value function defined in (5.2.1) may be rewritten as

$$u(x)=\inf_{\mathbb{P}\in\mathcal{P}}u^{\mathbb{P}}(x), \tag{5.2.4}$$

where $u^{\mathbb{P}}(x)$ denotes the value function of an exponential hedging problem under any fixed measure $\mathbb{P}\in\mathcal{P}$, so defined by

$$u^{\mathbb{P}}(x):=\sup_{\pi\in\mathcal{A}}\mathbb{E}\left[U\left(X_T+nG_T\right)\right]. \tag{5.2.5}$$

The dynamic value processes corresponding to (5.2.4) and (5.2.5) are defined by

$$u_t := \operatorname*{ess\,inf}_{\mathbb{P}\in\mathcal{P}_{t,T}} u_t^{\mathbb{P}}, \tag{5.2.6}$$

$$u_t^{\mathbb{P}} := \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{t,T}} \mathbb{E}[U(X_T + nG_T)|\mathcal{F}_t], \qquad t \in [0,T]. \tag{5.2.7}$$

Under $\mathbb{P}$, the stock process is as in (5.1.2), repeated below,

$$\mathrm{d}S_t = S_t\sigma_t\left((\lambda_t + \eta_t)\,\mathrm{d}t + \mathrm{d}W_t\right),$$

and the correlated $\mathbb{P}$-Brownian motion can be written as $\widetilde{W}_t = \rho W_t + \overline{\rho}W_t^{\perp}$, where W and $W^{\perp}$ denote independent $\mathbb{P}$-Brownian motions given by

$$W_t = W_t^0 - \int_0^t \eta_s\,\mathrm{d}s, \qquad W_t^{\perp} = W_t^{0,\perp} - \int_0^t \eta_s^{\perp}\,\mathrm{d}s, \qquad t \in [0,T].$$

Both the MPR $\lambda + \eta$ and the volatility processes σ of the stock under the measure $\mathbb{P}$ are $\widetilde{\mathbb{F}}$-adapted. Thus, the martingale distortion representation given by (2.4.10) is applicable for the utility maximisation problem (5.2.7) under each fixed model $\mathbb{P}$, with some additional assumptions on the MPR process and the martingale in the martingale distortion representation. Define the measure $\widetilde{\mathbb{P}} \sim \mathbb{P}$ on $\widetilde{\mathcal{F}}_T$ by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(-\int_0 \rho(\lambda_s + \eta_s)\,\mathrm{d}\widetilde{W}_s\right)_T. \tag{5.2.8}$$

Assumption 5.2.2. Assume the market price of risk $(\lambda_t)_{t\in[0,T]}$ satisfies

$$\mathbb{E}\left[\exp\left(8c\int_0^T \lambda_t^2\,\mathrm{d}t\right)\right] < \infty,$$

where c is the same constant as in (5.2.2).

Define the positive $(\widetilde{\mathbb{P}}, \widetilde{\mathbb{F}})$-martingale M by

$$M_t := \widetilde{\mathbb{E}}\left[\exp\left(-\frac{1}{2\delta}\int_0^T (\lambda_s + \eta_s)^2\,\mathrm{d}s - \frac{\alpha}{\delta}nG_T\right)\,\middle|\,\widetilde{\mathcal{F}}_t\right],$$

where δ is as in (2.4.7). Then M has representation

$$M_t = M_0 + \int_0^t M_s\xi_s\,\mathrm{d}\widetilde{B}_s, \quad t \in [0,T], \tag{5.2.9}$$

for some ξ, where $\widetilde{B}_\cdot = \widetilde{W}_\cdot + \int_0^\cdot \rho(\lambda_s + \eta_s)\,\mathrm{d}s$ is standard $\widetilde{\mathbb{P}}$-Brownian motion.

Assumption 5.2.3. It is assumed that

$$\mathbb{E}\left[\exp\left(c_\xi \int_0^T \xi_t^2 \,\mathrm{d}t\right)\right] < \infty,$$

where

$$c_\xi = \max\left\{\frac{c}{c-1}(\rho^2\delta^2 + \frac{1}{2}\overline{\rho}^2\delta^2), 4c\rho^2\delta^2\right\}.$$

Proposition 5.2.4. *Under Assumptions 5.2.1–5.2.3, the value process $u^{\mathbb{P}}$ admits the martingale distortion representation*

$$u_t^{\mathbb{P}} = -\mathrm{e}^{-\alpha X_t}\left(\widetilde{\mathbb{E}}\left[\exp\left(-\frac{1}{2\delta}\int_t^T (\lambda_s+\eta_s)^2\,\mathrm{d}s - \frac{\alpha}{\delta}nG_T\right)\,\middle|\,\widetilde{\mathcal{F}}_t\right]\right)^\delta, \quad (5.2.10)$$

where $\widetilde{\mathbb{P}}$ is defined in (5.2.8)*, δ is as in* (2.4.7)*. The optimal trading strategy for the problem in* (5.2.5) *is*

$$\widehat{\pi}_t^{\mathbb{P}} = \frac{\lambda_t + \eta_t + \rho\delta\xi_t}{\alpha\sigma_t}, \quad t \in [0,T], \quad (5.2.11)$$

where ξ is the integrand in the martingale representation of M in (5.2.9)*.*

Proof. It is trivial to verify that Assumption 5.2.2 and the boundedness of $\boldsymbol{\eta}$ imply Assumption 2.4.1 with MPR $\lambda + \eta$ under the measure $\mathbb{P}$. Combined with Assumption 5.2.3, we can apply Proposition 2.4.3 to the exponential hedging problem (5.2.7) under each fixed measure $\mathbb{P}$. □

5.2.2 A BSDE representation of the robust value process

By Proposition 5.2.4 and (5.2.6), the robust value process may be written in the form

$$u_t = \operatorname*{ess\,inf}_{\mathbb{P}\in\mathcal{P}_{t,T}} u_t^{\mathbb{P}} = -\mathrm{e}^{-\alpha X_t} V_t^\delta, \quad t \in [0,T], \quad (5.2.12)$$

where the $\widetilde{\mathbb{F}}$-adapted value process $(V_t)_{t\in[0,T]}$ is associated with an auxiliary problem under $\widetilde{\mathbb{P}}$, given by:

$$V_t := \operatorname*{ess\,sup}_{\boldsymbol{\eta}\in\mathcal{N}_{t,T}} \widetilde{\mathbb{E}}\left[\exp\left(-\frac{1}{2\delta}\int_t^T (\lambda_s+\eta_s)^2\,\mathrm{d}s - \frac{\alpha}{\delta}nG_T\right)\,\middle|\,\widetilde{\mathcal{F}}_t\right]. \quad (5.2.13)$$

We will write $V_t^{(n)} \equiv V_t$ when we wish to emphasise dependence on the number of claims. Denote by $\mathcal{S}^2$ the set of progressively measurable processes ψ such that

$$\mathbb{E}^0\big[\sup_{t\in[0,T]} |\psi_t|^2\big] < \infty,$$

and by $\mathbb{H}^2$ the set of progressively measurable processes ψ such that

$$\mathbb{E}^0\Big[\int_0^T |\psi_t|^2 \,\mathrm{d}t\Big] < \infty.$$

Then, with a bounded market price of risk process λ, one can obtain a BSDE representation for V, as in the following proposition.

Proposition 5.2.5. *Under Assumption 5.2.1(ii) and Assumption 5.2.3, and assuming additionally that λ is bounded, then $(V,\zeta) \in \mathcal{S}^2 \times \mathbb{H}^2$ is the unique solution to the BSDE*

$$\mathrm{d}V_t = -f_t(V_t,\zeta_t)\,\mathrm{d}t + \zeta_t\,\mathrm{d}\widetilde{W}^0_t, \qquad V_T = \exp\Big(-\frac{\alpha}{\delta}nG_T\Big), \tag{5.2.14}$$

where the generator $(f_t(\cdot,\cdot))_{t\in[0,T]}$ is defined by

$$f_t(v,z) = \operatorname*{ess\,sup}_{\boldsymbol{\eta}\in\mathcal{N}} \left\{ -\frac{1}{2\delta}(\lambda_t+\eta_t)^2 v - (\rho\lambda_t - \overline{\rho}\eta_t^{\perp})z \right\}. \tag{5.2.15}$$

The optimal strategy for the robust exponential utility maximisation problem in (5.2.1) *is then given by*

$$\widehat{\pi}_t = \frac{\lambda_t + \widehat{\eta}_t + \rho\delta\xi_t}{\alpha\sigma_t}, \qquad \text{with } \xi_t = \frac{\zeta_t}{V_t}, \quad t\in[0,T],$$

where $\widehat{\boldsymbol{\eta}} = (\widehat{\eta},\widehat{\eta}^{\perp})$ is the optimal control that realises the essential supremum in (5.2.15).

Proof. Define the process $V^{\boldsymbol{\eta}}$ by

$$V^{\boldsymbol{\eta}}_t := \widetilde{\mathbb{E}}\left[\exp\left(-\frac{1}{2\delta}\int_t^T(\lambda_s+\eta_s)^2\,\mathrm{d}s - \frac{\alpha}{\delta}nG_T\right)\,\middle|\,\widetilde{\mathcal{F}}_t\right] = \mathrm{e}^{\frac{1}{\delta}N_t}M_t, \quad t\in[0,T],$$

with terminal value $V^{\boldsymbol{\eta}}_T = \mathrm{e}^{-(\alpha/\delta)nG_T}$, where $N_\cdot := \frac{1}{2}\int_0^\cdot(\lambda_s+\eta_s)^2\,\mathrm{d}s$. By Itô's formula, the dynamics of $V^{\boldsymbol{\eta}}_t$ under the reference measure $\mathbb{P}^0$ are

$$\mathrm{d}V^{\boldsymbol{\eta}}_t = V^{\boldsymbol{\eta}}_t\xi_t\,\mathrm{d}\widetilde{W}^0_t + \left\{\frac{1}{2\delta}(\lambda_t+\eta_t)^2V^{\boldsymbol{\eta}}_t + (\rho\lambda_t - \overline{\rho}\eta_t^{\perp})V^{\boldsymbol{\eta}}_t\xi_t\right\}\mathrm{d}t.$$

Then for any η, with the change of variable $\zeta^{\boldsymbol{\eta}}_t = V^{\boldsymbol{\eta}}_t\xi_t$, the pair $(V^{\boldsymbol{\eta}},\zeta^{\boldsymbol{\eta}})$ can be expressed as a solution to the BSDE

$$\mathrm{d}V^{\boldsymbol{\eta}}_t = -f^{\boldsymbol{\eta}}_t(V^{\boldsymbol{\eta}}_t,\zeta^{\boldsymbol{\eta}}_t)\,\mathrm{d}t + \zeta^{\boldsymbol{\eta}}_t\,\mathrm{d}\widetilde{W}^0_t, \qquad V^{\boldsymbol{\eta}}_T = \mathrm{e}^{-(\alpha/\delta)nG_T}, \tag{5.2.16}$$

where

$$f^{\boldsymbol{\eta}}_t(V^{\boldsymbol{\eta}}_t,\zeta^{\boldsymbol{\eta}}_t) := -\frac{1}{2\delta}(\lambda_t+\eta_t)^2V^{\boldsymbol{\eta}}_t - (\rho\lambda_t - \overline{\rho}\eta_t^{\perp})\zeta^{\boldsymbol{\eta}}_t.$$

Since $f_t^{\boldsymbol{\eta}}(v,z)$ is linear with respect to v,z and $\lambda_t,\eta_t,\eta_t^{\perp}$ are bounded, $f_t^{\boldsymbol{\eta}}(v,z)$ is uniformly Lipschitz in v,z. Moreover, $\{f_t^{\boldsymbol{\eta}}(0,0), t\in[0,T]\}\in\mathbb{H}^2$ (as $f_t^{\boldsymbol{\eta}}(0,0)\equiv 0$) and terminal condition $\mathrm{e}^{-(\alpha/\delta)nG_T}\in L^2$ (by Assumption 5.2.1(ii)), so according to Touzi [64, Theorem 10.2], with standard parameters $(f^{\boldsymbol{\eta}},\mathrm{e}^{-(\alpha/\delta)nG_T})$, there exists a unique solution $(V^{\boldsymbol{\eta}},\zeta^{\boldsymbol{\eta}})\in\mathcal{S}^2\times\mathbb{H}^2$ to the above BSDE (5.2.16). Moreover, by the measurable selection results in [8, Theorem A.10.3], there exists $\widehat{\boldsymbol{\eta}}\in\mathcal{N}$ realising the essential supremum of $f_t^{\boldsymbol{\eta}}(v,z)$ over $\boldsymbol{\eta}\in\mathcal{N}$, that is,

$$f_t(v,z):=\operatorname*{ess\,sup}_{\boldsymbol{\eta}\in\mathcal{N}} f_t^{\boldsymbol{\eta}}(v,z)=f_t^{\widehat{\boldsymbol{\eta}}}(v,z).$$

Then by the comparison theorem [64, Theorem 10.4], because the terminal conditions are the same, the value process V in (5.2.13) is given by

$$V_t:=\operatorname*{ess\,sup}_{\boldsymbol{\eta}\in\mathcal{N}} V_t^{\boldsymbol{\eta}}=V_t^{\widehat{\boldsymbol{\eta}}},$$

and $(V,\zeta)=(V^{\widehat{\boldsymbol{\eta}}},\zeta^{\widehat{\boldsymbol{\eta}}})$ is the unique solution to the BSDE (5.2.16) with $\boldsymbol{\eta}=\widehat{\boldsymbol{\eta}}$, which is BSDE (5.2.14).

□

We then have the following results for the multiple priors indifference selling price and the optimal hedging strategy, which follow from (5.2.12), the definition of indifference prices and hedging strategies (in Chapter 2), and Proposition 5.2.5.

Proposition 5.2.6. *The* ***multiple priors indifference selling price*** p *at time zero of the European claim with* $\widetilde{\mathcal{F}}_T$*-measurable payoff* G_T *is given by*

$$p=\frac{\delta}{\alpha}(\ln V_0^{(-1)}-\ln V_0^{(0)}).$$

The ***multiple priors optimal hedging strategy*** $\widehat{\pi}^h$ *for shorting one option is*

$$\widehat{\pi}_t^h=\frac{\widehat{\eta}_t^{(-1)}-\widehat{\eta}_t^{(0)}+\rho\delta\big(\xi_t^{(-1)}-\xi_t^{(0)}\big)}{\alpha\sigma_t},$$
$$\xi_t^{(-1)}=\zeta_t^{(-1)}/V_t^{(-1)},\quad \xi_t^{(0)}=\zeta_t^{(0)}/V_t^{(0)},\quad t\in[0,T].$$

The process $(V^{(-1)},\zeta^{(-1)})$ *is the solution to BSDE* (5.2.14) *with* $n=-1$ *and* $(V^{(0)},\zeta^{(0)})$ *is the solution to BSDE* (5.2.14) *with* $n=0$. *Processes* $\widehat{\eta}^{(-1)}$ *and* $\widehat{\eta}^{(0)}$ *are the respective optimisers in the BSDEs.*

In a similar spirit as the multiple priors power utility maximisation problem of Chapter 3, we have the following PDE representations under a Markovian stochastic factor model, which can be obtained by applying the dynamic programming principle in (5.2.13) or by specialising to the Markov model in the BSDE representation (similar as Remark 3.4.1 thus will not be repeated).

5.2.3 The Markovian case

An existing approach to the robust exponential hedging problem (5.2.1) under the Markovian stochastic factor model (3.4.1)–(3.4.2) was studied in Owari [49] with very stringent conditions on the option and the parameters, where the author first applied the duality theory in Owari [50] to transform the original problem of finding out the saddle-point of a two-dimensional optimisation problem to the dual problem of finding the infimum over a two dimensional set, and then applied stochastic control techniques to the dual problem. As a result, the robust value function and optimal strategy are characterised by solution to a non-linear PDE. Here, we show how these results can be obtained from our earlier results, with weaker conditions on the parameters.

For the Markovian case, the assumption that the MPR process λ of the stock is bounded is not necessary since the optimisation problem defined in (5.2.13) can be solved using a stochastic control approach without reference to BSDE theories.

Assuming that Assumptions 5.2.1–5.2.3 hold, Proposition 5.2.4 yields the value process $u^{\mathbb{P}}$ in (5.2.10) and the optimal strategy $\widehat{\pi}^{\mathbb{P}}$ in (5.2.11) for the optimal investment problem under any fixed model $\mathbb{P}$ with $\lambda_t \equiv \lambda(Y_t)$, $\sigma_t \equiv \sigma(Y_t)$. Then the minimisation over models problem (5.2.13) becomes the following stochastic control problem where value process V defined in (5.2.13) is a function of time and the current state of process Y. That is, $V_t \equiv v(t, Y_t), t \in [0, T]$, with function v given by

$$v(t,y) = \sup_{\boldsymbol{\eta} \in \mathcal{N}_{t,T}} \widetilde{\mathbb{E}}_{t,y} \left[\exp\left(-\frac{1}{2\delta} \int_t^T (\lambda(Y_s) + \eta_s)^2 \, \mathrm{d}s - \frac{\alpha}{\delta} n g(Y_T) \right) \right], \quad (5.2.17)$$

where the dynamics of the process Y under the measure $\widetilde{\mathbb{P}}$ are

$$\mathrm{d}Y_t = \left(a(Y_t) + \overline{\rho} b(Y_t) \eta_t^{\perp} - \rho b(Y_t) \lambda(Y_t) \right) \mathrm{d}t + b(Y_t) \, \mathrm{d}\widetilde{B}_t,$$

and where we recall that $\widetilde{B} = \widetilde{W} + \int_0^{\cdot} \rho(\lambda_s + \eta_s) \, \mathrm{d}s$ is standard $\widetilde{\mathbb{P}}$-Brownian motion. The HJB equation associated with $v(\cdot, \cdot)$ is then

$$\begin{aligned} \frac{\partial v}{\partial t}(t,y) + a(y) \frac{\partial v}{\partial y}(t,y) + \frac{1}{2} b^2(y) \frac{\partial^2 v}{\partial y^2}(t,y) \\ + f\left(t, y, v, b(y) \frac{\partial v}{\partial y}(t,y)\right) = 0, \end{aligned} \quad (5.2.18)$$

with terminal condition $v(T, y) = \mathrm{e}^{-(\alpha/\delta) n g(y)}$, where

$$f(t,y,v,z) = \sup_{\boldsymbol{\eta} \in \Gamma} \left\{ -\frac{1}{2\delta} (\lambda(y) + \eta)^2 v - (\rho \lambda(y) - \overline{\rho} \eta^{\perp}) z \right\}. \quad (5.2.19)$$

If $\widehat{\boldsymbol{\eta}}(t,y) = (\widehat{\eta}(t,y), \widehat{\eta}^{\perp}(t,y))$ realises the supremum in (5.2.19), $v \in C^{1,2}([0,T]\times\mathbb{R}) \times C([0,T]\times\mathbb{R})$ follows from its boundedness and the assumptions that there exists a unique strong solution to the SDEs satisfied by the asset prices. Thus Itô's formula may be applied to the $\widetilde{\mathbb{P}}^{\widehat{\boldsymbol{\eta}}}$-martingale $\exp\big\{-\frac{1}{2\delta}\int_0^t(\lambda(Y_s)+\widehat{\eta}(s,Y_s))^2\,\mathrm{d}s\big\}v(t,Y_t)$. The PDE (5.2.18) then follows from the martingale property. The rest is then a classical stochastic verification argument.

The general relation (2.4.15) between the optimal strategy and the Markovian distortion solution holds here as well, so we have the optimal trading strategy expressed via the solution to the PDE (5.2.18):

$$\widehat{\pi}_t = \frac{\lambda(Y_t)+\widehat{\eta}(t,Y_t)+\rho\delta\xi(t,Y_t)}{\alpha\sigma(Y_t)}, \quad \text{with } \xi(t,Y_t) = \frac{b(Y_t)}{v(t,Y_t)}\frac{\partial v}{\partial y}(t,Y_t), \tag{5.2.20}$$

where $\widehat{\eta}(t,y)$ is the optimal feedback control function realising the supremum in (5.2.19). Note that in cases when the optimisations over η and $\eta^{\perp}$ can be carried out independently, we have

$$\widehat{\eta}(t,y) = \arg\min_{\eta}|\lambda(y)+\eta|, \quad \widehat{\eta}^{\perp}(t,y) = \arg\max_{\eta^{\perp}} b(y)\frac{\partial v}{\partial y}(t,y)\eta^{\perp}.$$

5.3 Variational preferences exponential hedging

In this section, we will only focus on the Markovian stochastic factor model with traded asset S and non-traded asset Y under the reference measure $\mathbb{P}^0$. The model dynamics under $\mathbb{P}^0$ are as in Chapter 3 ((3.4.1)-(3.4.2)), repeated below:

$$\begin{aligned}\mathrm{d}S_t &= S_t\mu(Y_t)\,\mathrm{d}t + S_t\sigma(Y_t)\,\mathrm{d}W_t^0,\\ \mathrm{d}Y_t &= a(Y_t)\,\mathrm{d}t + b(Y_t)\,\mathrm{d}\widetilde{W}_t^0.\end{aligned}$$

In the variational preferences approach, since the implausible models can be excluded by associating infinite penalties, it is possible to consider all probability measures that are equivalent to the reference measure. For any measure $\mathbb{P}\sim\mathbb{P}^0$, there exists a progressively measurable process $\boldsymbol{\eta} = (\eta,\eta^{\perp}) \in \mathcal{N} := \big\{\boldsymbol{\eta} \;\big|\; \int_0^t ||\boldsymbol{\eta}_s||^2\,\mathrm{d}s < \infty\ \mathbb{P}^0\text{-a.s. for all t}\big\}$ such that

$$\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^0}\right|_{\mathcal{F}_t} = D_t = \mathcal{E}\left(\int_0^{\cdot}\eta_s\,\mathrm{d}W_s^0 + \int_0^{\cdot}\eta_s^{\perp}\,\mathrm{d}W_s^{0,\perp}\right)_t, \quad t\in[0,T], \quad \mathbb{P}^0\text{-a.s.} \tag{5.3.1}$$

Each measure $\mathbb{P}$ will then be penalised according to its plausibility by a penalty function $\gamma_{t,T}(\boldsymbol{\eta})$, which is chosen to take the following form

$$\gamma_{t,T}(\boldsymbol{\eta}) := \exp\left\{-\frac{1}{2\beta}\int_t^T ||\boldsymbol{\eta}_s||^2 ds\right\}, \quad \text{with constant } \beta>0,\ t\in[0,T].$$

The objective of the agent at time t is to maximise the utility functional which is the infimum of the expected utility from terminal wealth over the set of models penalised by the penalty function. The robust value function with starting time $t \in [0, T]$ is defined by

$$u(t,x,y) = \sup_{\pi \in \mathcal{A}_{t,T}} \inf_{\mathbb{P} \in \mathcal{P}_{t,T}} \mathbb{E}\left[\gamma_{t,T}(\boldsymbol{\eta}) U(X_T + ng(Y_T)) \mid (X_t, Y_t) = (x,y)\right], \tag{5.3.2}$$

where the utility function $U : \mathbb{R} \to \mathbb{R}$ is the exponential utility. Note that the penalisation is multiplicative rather than additive. This penalty functional was suggested by Øksendal and Sulem [48] and has some desirable features. The integral $\frac{1}{2}\int_t^T ||\boldsymbol{\eta}_s||^2 ds$ characterises the "distance" of the candidate measure $\mathbb{P}$ from the reference measure $\mathbb{P}^0$: as the candidate measure approaches the reference measure, $\gamma_{t,T}(\boldsymbol{\eta}) \to 1$, thus the objective functional is closer to the expected utility under the reference model. The further away the candidate measure is from the reference measure, the smaller the penalty is, and the larger the objective functional becomes – thus the candidate measures that are too far away from the reference measure are eliminated. The constant $\beta > 0$ characterises the agent's level of ambiguity aversion: when $\beta \to 0$, the agent has low ambiguity aversion so that the objective (5.3.2) becomes the classical utility maximisation problem under reference measure $\mathbb{P}^0$; and when $\beta \to +\infty$, the agent has very high ambiguity aversion and considers all candidate measures to be equally possible and takes the worst case of them. We shall call β the ambiguity aversion parameter.

The set of admissible strategies is defined by

$$\begin{aligned}\mathcal{A}_{t,T} := \Big\{ &\pi_s \text{ is } (\mathcal{F}_s)\text{-progressively measurable, } \forall s \in [t,T] \;\Big|\; \pi \text{ satisfies} \\ &\text{(2.4.1) and is such that } (\mathrm{e}^{-\alpha X_t})_{t\in[0,T]} \text{ is } \mathbb{P}\text{-uniformly integrable}, \forall \mathbb{P} \sim \mathbb{P}^0 \Big\}.\end{aligned} \tag{5.3.3}$$

The problem defined in (5.3.2) can be seen as a zero-sum stochastic differential game, where the market is minimising the objective function and the agent is maximising the objective function. In the following, we solve the problem by first formally solving the HJB-Isaacs (HJBI) equation satisfied by the value function assuming that the game value exists, and then applying a verification theorem to verify that the game value does exist and the solution to the HJBI equation is the value function to the problem (5.3.2).

5.3.1 An HJBI PDE for the zero-sum stochastic differential game

The value function at time $t \in [0,T]$ can be written as

$$\begin{aligned} u(t,x,y) &= \sup_{\pi\in\mathcal{A}} \inf_{\boldsymbol{\eta}\in\mathcal{N}} \mathbb{E}_{t,x,y}\left[\mathrm{e}^{-\frac{1}{2\beta}\int_t^T ||\boldsymbol{\eta}_s||^2 ds} U\left(X_T + ng(Y_T)\right)\right] \\ &= \sup_{\pi\in\mathcal{A}} \inf_{\boldsymbol{\eta}\in\mathcal{N}} \mathbb{E}^0_{t,x,y}\Big[U(\widetilde{X}_T + ng(Y_T))\Big], \quad t\in[0,T]. \end{aligned} \tag{5.3.4}$$

where $\mathbb{E}^0_{t,x,y}$ denotes expectation taken under the reference measure $\mathbb{P}^0$ conditional on $\widetilde{X}_t = x$, $Y_t = y$. The dynamics of the process $\widetilde{X}$ and the non-traded asset price Y under the reference measure $\mathbb{P}^0$ are

$$\begin{aligned} \mathrm{d}\widetilde{X}_t &= \left(\lambda(Y_t)\sigma(Y_t)\pi_t + \frac{1+\beta}{2\alpha\beta}|\eta_t|^2 + \frac{1+\beta}{2\alpha\beta}|\eta_t^\perp|^2\right)\mathrm{d}t \\ &\quad + \left(\pi_t\sigma(Y_t) - \frac{1}{\alpha}\eta_t\right)\mathrm{d}W_t^0 - \frac{1}{\alpha}\eta_t^\perp\,\mathrm{d}W_t^{0,\perp}, \\ \mathrm{d}Y_t &= a(Y_t)\,\mathrm{d}t + \rho b(Y_t)\,\mathrm{d}W_t^0 + \overline{\rho} b(Y_t)\,\mathrm{d}W_t^{0,\perp}. \end{aligned}$$

Assuming that the game value exists and $u(\cdot,\cdot,\cdot) \in C^{1,2,2}([0,T],\mathbb{R},\mathbb{R})$, the common PDE satisfied by the lower value and upper value of the game is

$$\frac{\partial u}{\partial t}(t,x,y) + \sup_{\pi\in\mathbb{R}} \inf_{\boldsymbol{\eta}\in\mathbb{R}^2} \mathcal{L}^{\pi,\boldsymbol{\eta}} u(t,x,y) = 0,$$

with terminal condition $u(T,x,y) = U(x+ng(y))$, and where the generator $\mathcal{L}^{\pi,\boldsymbol{\eta}}$ is defined by

$$\begin{aligned} \mathcal{L}^{\pi,\boldsymbol{\eta}} u(t,x,y) = &\left(\pi\sigma(y)\lambda(y) + \frac{1+\beta}{2\alpha\beta}\eta^2 + \frac{1+\beta}{2\alpha\beta}(\eta^\perp)^2\right)\frac{\partial u}{\partial x}(t,x,y) \\ &+ \frac{1}{2}\left(\left(\pi\sigma(y) - \frac{1}{\alpha}\eta\right)^2 + \frac{1}{\alpha^2}(\eta^\perp)^2\right)\frac{\partial^2 u}{\partial x^2}(t,x,y) \\ &+ a(y)\frac{\partial u}{\partial y}(t,x,y) + \frac{1}{2}b^2(y)\frac{\partial^2 u}{\partial y^2}(t,x,y) \\ &+ \left(\rho b(y)\sigma(y)\pi - \frac{1}{\alpha}\rho b(y)\eta - \frac{1}{\alpha}\overline{\rho} b(y)\eta^\perp\right)\frac{\partial^2 u}{\partial x\partial y}(t,x,y). \end{aligned} \tag{5.3.5}$$

For brevity, we omit the variables (t,x,y) for the function $u(\cdot,\cdot,\cdot)$ and its derivatives in the following derivations. Using the first order condition, we obtain the optimal controls $\widehat{\eta}$ and $\widehat{\eta}^\perp$:

$$\widehat{\eta}(\pi) = \frac{\frac{1}{\alpha}\pi\sigma(y)\frac{\partial^2 u}{\partial x^2} + \frac{1}{\alpha}\rho b(y)\frac{\partial^2 u}{\partial x\partial y}}{\frac{1+\beta}{\alpha\beta}\frac{\partial u}{\partial x} + \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial x^2}}, \qquad \widehat{\eta}^\perp = \frac{\frac{1}{\alpha}\overline{\rho} b(y)\frac{\partial^2 u}{\partial x\partial y}}{\frac{1+\beta}{\alpha\beta}\frac{\partial u}{\partial x} + \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial x^2}}. \tag{5.3.6}$$

Substituting in the HJBI equation yields that

$$\begin{aligned}
&\frac{\partial u}{\partial t} + a(y)\frac{\partial u}{\partial y} + \frac{1}{2}b^2(y)\frac{\partial^2 u}{\partial y^2} \\
&+ \sup_{\pi\in\mathbb{R}}\Bigg\{\pi\sigma(y)\lambda(y)\frac{\partial u}{\partial x} + \frac{1}{2}\pi^2\sigma^2(y)\frac{\partial^2 u}{\partial x^2} + \rho b(y)\sigma(y)\pi\frac{\partial^2 u}{\partial x\partial y} \\
&\qquad - \frac{1}{2}\frac{\frac{1}{\alpha^2}\pi^2\sigma^2(y)\big(\frac{\partial^2 u}{\partial x^2}\big)^2 + \frac{2}{\alpha^2}\pi\sigma(y)\rho b(y)\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial x\partial y} + \frac{1}{\alpha^2}b^2(y)\big(\frac{\partial^2 u}{\partial x\partial y}\big)^2}{\frac{1+\beta}{\alpha\beta}\frac{\partial u}{\partial x} + \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial x^2}}\Bigg\} = 0.
\end{aligned}$$

Optimising over π, we obtain that

$$\widehat{\pi} = -\frac{\lambda(y)\frac{\partial u}{\partial x}}{\sigma(y)\frac{\partial^2 u}{\partial x^2}} - \frac{\beta}{\alpha(1+\beta)}\frac{\lambda(y)}{\sigma(y)} - \frac{\rho b(y)\frac{\partial^2 u}{\partial x\partial y}}{\sigma(y)\frac{\partial^2 u}{\partial x^2}}. \tag{5.3.7}$$

The resulting PDE is thus

$$\begin{aligned}
&\frac{\partial u}{\partial t} + a(y)\frac{\partial u}{\partial y} + \frac{1}{2}b^2(y)\frac{\partial^2 u}{\partial y^2} - \frac{1}{2}\lambda^2(y)\frac{\big(\frac{\partial u}{\partial x}\big)^2}{\frac{\partial^2 u}{\partial x^2}} - \rho b(y)\lambda(y)\frac{\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}}{\frac{\partial^2 u}{\partial x^2}} \\
&\quad - \frac{1}{2}\rho^2 b^2(y)\frac{\big(\frac{\partial^2 u}{\partial x\partial y}\big)^2}{\frac{\partial^2 u}{\partial x^2}} - \frac{\beta}{2\alpha(1+\beta)}\lambda^2(y)\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\frac{1}{\alpha^2}\overline{\rho}^2 b^2(y)\big(\frac{\partial^2 u}{\partial x\partial y}\big)^2}{\frac{1+\beta}{\alpha\beta}\frac{\partial u}{\partial x} + \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial x^2}} = 0,
\end{aligned}$$

with terminal condition $u(T,x,y) = -\mathrm{e}^{-\alpha(x+ng(y))}$. Looking for a solution of the form

$$u(t,x,y) = -\mathrm{e}^{-\alpha x}(v(t,y))^{\delta}, \quad \text{where } \delta = \frac{1}{(1+\beta)(1-\rho^2)}, \tag{5.3.8}$$

we obtain a linear PDE satisfied by $v(\cdot,\cdot)$:

$$\frac{\partial v}{\partial t} + \big(a(y) - \rho b(y)\lambda(y)\big)\frac{\partial v}{\partial y} + \frac{1}{2}b^2(y)\frac{\partial^2 v}{\partial y^2} - \frac{1}{2(1+\beta)\delta}\lambda^2(y)v = 0, \tag{5.3.9}$$

with terminal condition $v(T,x,y) = \mathrm{e}^{-(\alpha/\delta)ng(y)}$. By the Feynman-Kac formula, the function $v(\cdot,\cdot)$ has the stochastic representation

$$v(t,y) = \widetilde{\mathbb{E}}^0_{t,y}\left[\exp\left\{-\frac{1}{2(1+\beta)\delta}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s - \frac{\alpha}{\delta}ng(Y_T)\right\}\right], \tag{5.3.10}$$

where $\widetilde{\mathbb{E}}^0_{t,y}$ denotes the conditional expectation taken under the distortion measure $\widetilde{\mathbb{P}}^0$ defined by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}^0}{\mathrm{d}\mathbb{P}^0} = \mathcal{E}\left(-\int_0^{\cdot}\rho\lambda(Y_s)\,\mathrm{d}\widetilde{W}^0_s\right)_T.$$

The dynamics of the process Y under the measure $\widetilde{\mathbb{P}}^0$ is

$$\mathrm{d}Y_t = \big(a(Y_t) - \rho b(Y_t)\lambda(Y_t)\big)\,\mathrm{d}t + b(Y_t)\,\mathrm{d}\widetilde{B}^0_t, \quad t\in[0,T],$$

where $\widetilde{B}^0_\cdot = \widetilde{W}^0_\cdot + \int_0^\cdot \rho\lambda(Y_s)\,\mathrm{d}s$ is a standard $\widetilde{\mathbb{P}}^0$-Brownian motion.

Substituting the value function (5.3.8) into (5.3.7) and (5.3.6), we obtain the optimal Markov controls

$$\widehat{\pi}(t,y) = \frac{1}{\alpha\sigma(y)}\left(\frac{\lambda(y)}{1+\beta} + \frac{\rho\delta b(y)}{v(t,y)}\frac{\partial v}{\partial y}(t,y)\right),$$

and

$$\widehat{\eta}(t,y) = -\frac{\beta}{1+\beta}\lambda(y), \qquad \widehat{\eta}^{\perp}(t,y) = \frac{\overline{\rho}\beta\delta b(y)}{v(t,y)}\frac{\partial v}{\partial y}(t,y).$$

5.3.2 Verification

We first restate a verification theorem developed in Mataramvura and Øksendal [33] for a zero-sum stochastic game of the form in (5.3.4), for the case when the state processes are continuous stochastic processes without jumps.

Theorem 5.3.1 (Mataramvura and Øksendal [33], Theorem 3.2 restated for continuous state processes without jumps)**.** *Suppose there exists a function* $u(\cdot,\cdot,\cdot) \in C^{1,2}([0,T]\times\mathbb{R}^2)$ *and Markov controls* $(\widehat{\pi}(t,x,y), \widehat{\boldsymbol{\eta}}(t,x,y)) \in \mathbb{R}\times\mathbb{R}^2$ *such that*

(i) $\frac{\partial u}{\partial t}(t,x,y) + \mathcal{L}^{\pi,\widehat{\boldsymbol{\eta}}}u(t,x,y) \le 0$, $\forall \pi \in \mathbb{R}$, $t\in[0,T], x,y\in\mathbb{R}$,

(ii) $\frac{\partial u}{\partial t}(t,x,y) + \mathcal{L}^{\widehat{\pi},\boldsymbol{\eta}}u(t,x,y) \ge 0$, $\forall \boldsymbol{\eta} \in \mathbb{R}^2$, $t\in[0,T], x,y\in\mathbb{R}$,

(iii) $\frac{\partial u}{\partial t}(t,x,y) + \mathcal{L}^{\widehat{\pi},\widehat{\boldsymbol{\eta}}}u(t,x,y) = 0$, $\forall t\in[0,T], x,y\in\mathbb{R}$,

(iv) $u(T,x,y) = -\mathrm{e}^{-\alpha(x+ng(y))}$,

(v) the family $\{u(t,X_t,Y_t)\}_{t\in[0,T]}$ *is uniformly integrable,* $\forall x,y\in\mathbb{R}$, $\pi\in\mathcal{A}$, $\boldsymbol{\eta}\in\mathcal{N}$.

Then

$$\begin{aligned} u(t,x,y) &= \sup_{\pi\in\mathcal{A}}\inf_{\boldsymbol{\eta}\in\mathcal{N}} J^{\pi,\boldsymbol{\eta}}(t,x,y) = \inf_{\boldsymbol{\eta}\in\mathcal{N}}\sup_{\pi\in\mathcal{A}} J^{\pi,\boldsymbol{\eta}}(t,x,y) \\ &= \sup_{\pi\in\mathcal{A}} J^{\pi,\widehat{\boldsymbol{\eta}}}(t,x,y) = \inf_{\boldsymbol{\eta}\in\mathcal{N}} J^{\widehat{\pi},\boldsymbol{\eta}}(t,x,y) = J^{\widehat{\pi},\widehat{\boldsymbol{\eta}}}(t,x,y), \quad t\in[0,T], x,y\in\mathbb{R}, \end{aligned}$$

and $(\widehat{\pi}(t,x,y), \widehat{\boldsymbol{\eta}}(t,x,y))$ *is an optimal Markov control.*

This leads to the following results (Proposition 5.3.2 below) for the robust value function, the optimal strategies and the least favourable measure. Notice that the integrands $(\widehat{\eta}, \widehat{\eta}^{\perp})$ characterising the least favourable measure are functions of time and the current price of the non-traded asset Y, thus $\widetilde{\mathbb{F}}$-adapted processes. Although

we have considered all probability measures that are equivalent to the reference measure in the variational preferences exponential hedging problem defined in (5.3.2), the least favourable measure turns out to be a measure that preserves the original structure of the stochastic factor model. There is thus no loss of generality if we had only considered integrands $\eta, \eta^{\perp}$ in (5.3.1) that are $\widetilde{\mathbb{F}}$-adapted.

Proposition 5.3.2. *The value function $u(\cdot,\cdot,\cdot)$ of the variational preferences exponential utility maximisation problem* (5.3.2) *is*

$$u(t,x,y) = -\mathrm{e}^{-\alpha x}(v(t,y))^{\delta}, \tag{5.3.11}$$

where the constant δ is given by

$$\delta = \frac{1}{(1+\beta)(1-\rho^2)},$$

and function $v(\cdot,\cdot)$ is as defined in (5.3.10) *and is the solution to the linear PDE* (5.3.9).

The optimal strategies, assuming they are admissible as defined in (5.3.3)*, are*

$$\widehat{\pi}_t = \widehat{\pi}(t,Y_t) = \frac{1}{\alpha\sigma(Y_t)}\left(\frac{\lambda(Y_t)}{1+\beta} + \frac{\rho\delta b(Y_t)}{v(t,Y_t)}\frac{\partial v}{\partial y}(t,Y_t)\right), \quad t\in[0,T]. \tag{5.3.12}$$

The integrands characterising the least favourable measure are

$$\begin{aligned} \widehat{\eta}_t &= \widehat{\eta}(t,Y_t) = -\frac{\beta}{1+\beta}\lambda(Y_t), \\ \widehat{\eta}^{\perp}_t &= \widehat{\eta}^{\perp}(t,Y_t) = \frac{\overline{\rho}\beta\delta b(Y_t)}{v(t,Y_t)}\frac{\partial v}{\partial y}(t,Y_t), \quad t\in[0,T]. \end{aligned} \tag{5.3.13}$$

Proof. The formal derivations in Section 5.3.1 give us the stochastic definition (5.3.10) of function $v(\cdot,\cdot)$. Then $v\in C^{1,2}([0,T]\times\mathbb{R})$ follows from the fact that v is bounded and the assumption that there exists a unique strong solution to the SDE satisfied by the process Y. It can then be verified that u as defined by the separable form in (5.3.11) is $C^{1,2}([0,T]\times\mathbb{R}^2)$. Applying Itô's formula to the $\widetilde{\mathbb{P}}^0$-martingale $-\exp\big\{-\frac{1}{2(1+\beta)\delta}\int_0^t\lambda^2(Y_s)\,\mathrm{d}s\big\}v(t,Y_t)$, the PDE (5.3.9) follows from the martingale property.

We then verify the conditions (i)-(v) in Theorem 5.3.1. Using the first order condition and keeping π fixed, we obtain $\widehat{\eta}$ and $\widehat{\eta}^{\perp}$:

$$\widehat{\eta}(\pi) = \frac{\frac{1}{\alpha}\pi\sigma(y)\frac{\partial^2 u}{\partial x^2} + \frac{1}{\alpha}\rho b(y)\frac{\partial^2 u}{\partial x\partial y}}{\frac{1+\beta}{\alpha\beta}\frac{\partial u}{\partial x} + \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial x^2}}, \qquad \widehat{\eta}^{\perp} = \frac{\frac{1}{\alpha}\overline{\rho}b(y)\frac{\partial^2 u}{\partial x\partial y}}{\frac{1+\beta}{\alpha\beta}\frac{\partial u}{\partial x} + \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial x^2}}. \tag{5.3.14}$$

Similarly, for any fixed η and $\eta^{\perp}$, we have

$$\widehat{\pi}(\eta) = -\frac{\lambda(y)\frac{\partial u}{\partial x} - \frac{1}{\alpha}\frac{\partial^2 u}{\partial x^2}\eta + \rho b(y)\frac{\partial^2 u}{\partial x \partial y}}{\sigma(y)\frac{\partial^2 u}{\partial x^2}}. \tag{5.3.15}$$

By definition (5.3.5), $\mathcal{L}^{\pi,\boldsymbol{\eta}}$ is quadratic in π and η, $\eta^{\perp}$ respectively, with coefficients of the quadratic terms given by $\frac{1}{2}\sigma^2(y)\frac{\partial^2 u}{\partial x^2}$ and $\frac{1+\beta}{2\alpha\beta}\frac{\partial u}{\partial x} + \frac{1}{2\alpha^2}\frac{\partial^2 u}{\partial x^2}$. From (5.3.11), it is easy to see that $\frac{\partial^2 u}{\partial x^2} < 0$ and that $\frac{1+\beta}{2\alpha\beta}\frac{\partial u}{\partial x} + \frac{1}{2\alpha^2}\frac{\partial^2 u}{\partial x^2} = \frac{1}{2\beta}\mathrm{e}^{-\alpha x}(v(t,y))^{\delta} > 0$ for $t \in [0,T]$, $x, y \in \mathbb{R}$. Thus the above optimisers $\widehat{\pi}$ and $\widehat{\eta}$, $\widehat{\eta}^{\perp}$ satisfy conditions (i)-(ii) in Theorem 5.3.1. It is then easy to verify that (5.3.12) and (5.3.13) satisfy the above equations (5.3.14) and (5.3.15). Condition (iii) is straightforward to verify by substituting the value function (5.3.11) and optimal controls (5.3.12) and (5.3.13) into the HJBI equation, and condition (iv) follows from (5.3.11) and (5.3.10). Condition (v) is a direct result from the assumption that the optimal strategy exists and is admissible.

□

Using Proposition 5.3.2, we can also obtain the robust indifference selling price and the robust hedging strategy of the option, in a similar way as for the multiple priors problem.

Proposition 5.3.3. *The **variational preferences indifference selling price**, at time $t \in [0,T]$, of the European claim on the non-traded asset with payoff $g(Y_T)$, is given by $p(t,Y_t)$, where the function $p(\cdot,\cdot)$ of time and the current price of the non-traded asset Y is given by:*

$$\begin{aligned} p(t,y) &= \frac{\delta}{\alpha}\ln\left(\frac{v^{(-1)}(t,y)}{v^{(0)}(t,y)}\right) \\ &= \frac{\delta}{\alpha}\ln\left(\frac{\widetilde{\mathbb{E}}^0_{t,y}\left[\exp\left\{-\frac{1}{2(1+\beta)\delta}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s - \frac{\alpha}{\delta}ng(Y_T)\right\}\right]}{\widetilde{\mathbb{E}}^0_{t,y}\left[\exp\left\{-\frac{1}{2(1+\beta)\delta}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s\right\}\right]}\right). \end{aligned}$$

*The **variational preferences optimal hedging strategy** $\widehat{\pi}^h$ for a short position in one option is*

$$\widehat{\pi}^h_t = \frac{\rho\delta b(Y_t)}{\alpha\sigma(Y_t)}\left(\frac{\partial}{\partial y}\ln v^{(-1)}(t,Y_t) - \frac{\partial}{\partial y}\ln v^{(0)}(t,Y_t)\right), \quad t \in [0,T],$$

where $v^{(-1)}(\cdot,\cdot)$ (respectively, $v^{(0)}(\cdot,\cdot)$) is as in (5.3.10) with $n=-1$ (respectively, $n=0$).

5.4 Robust valuation and hedging in the log-normal basis risk model with constant parameters

In this section, we specialise to constant parameters where more explicit results can be obtained for the value function, optimal strategies and indifference prices when further assumptions on the market model and the extent of model uncertainty are made.

Assume zero risk-free rate and consider a basis risk model which has a traded asset S and a non-traded asset Y following correlated geometric Brownian motions. There is a European put option on the non-traded asset with payoff $g(Y_T) = (K - Y_T)^+$ at terminal time T where $K > 0$ is the strike. The asset price dynamics under the reference measure $\mathbb{P}^0$ are

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t^0, \\ \mathrm{d}Y_t &= aY_t \,\mathrm{d}t + bY_t\big(\rho \,\mathrm{d}W_t^0 + \sqrt{1-\rho^2}\,\mathrm{d}W_t^{0,\perp}\big), \end{aligned}$$

where W^0, $W^{0,\perp}$ are independent Brownian motions, coefficients μ, σ, a, b are assumed to be constants, with the volatilities b, σ positive, and constant correlation $\rho \in (-1, 1)$. Denote by $\lambda = \mu/\sigma$ the market price of risk of the traded stock.

An agent with exponential utility holds $n \in \mathbb{R}$ options on the non-traded asset, and faces the problem of hedging her position in the option using the correlated traded asset. Let $\theta \in L(S)$ (progressively measurable and S-integrable) denote the number of shares of the traded asset held by the agent, and let $\pi = \theta S$ denote the wealth invested in the stock. The wealth process $(X_t)_{0\le t\le T}$ of her portfolio is

$$X_t = X_0 + \int_0^t \theta_u \,\mathrm{d}S_u = X_0 + \int_0^t \lambda\sigma\pi_u \,\mathrm{d}u + \int_0^t \sigma\pi_u \,\mathrm{d}W_u^0,$$

The strategy π is assumed to satisfy the minimal condition

$$\int_0^T |\pi_s|^2 \,\mathrm{d}s < \infty, \quad \mathbb{P}^0\text{-a.s.}. \tag{5.4.1}$$

As the agent is concerned about model uncertainty, instead of considering only the reference model $\mathbb{P}^0$, she considers the range of probability measures defined by

$$\mathcal{P} := \Bigg\{ \mathbb{P} \sim \mathbb{P}^0 \text{ on } \mathcal{F}_T \;\Bigg|\; \left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^0}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^\cdot \eta_s \,\mathrm{d}W_s^0 + \int_0^\cdot \eta_s^\perp \,\mathrm{d}W_s^{0,\perp}\right)_t, \\ t \in [0,T],\ \boldsymbol{\eta} = (\eta, \eta^\perp) \in \mathcal{N} \equiv \mathcal{N}_{0,T} \Bigg\}.$$

The set $\mathcal{N}_{s,T}$ above denotes the set of all $\widetilde{\mathbb{F}}$-adapted[4] processes $(\boldsymbol{\eta}_t)_{t\in[s,T]}$ such that $\int_0^T ||\boldsymbol{\eta}_t||^2 \,\mathrm{d}t < \infty$ and $\boldsymbol{\eta}_t \in \Gamma$, $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}^0$-a.e., where $\Gamma \subset \mathbb{R}^2$ will be specified separately for the multiple priors and the variational preferences exponential hedging problems.

In the following, we will apply the results in Sections 5.2 and 5.3 to the constant parameter log-normal basis risk valuation and hedging problem. We then obtain explicit results for the indifference prices and hedging strategies (Proposition 5.4.3 and 5.4.6), and discuss properties of the residual risk processes (defined later) for the robust hedging portfolios (Proposition 5.4.7). Using analytic approximations of the indifference prices and hedging strategies (Proposition 5.4.10), we are able to compute these quantities efficiently and perform thorough numerical tests, over simulated paths, of the performance of various strategies such as the naive Black-Scholes strategy, the modified Black-Scholes strategy (given later), the classical indifference pricing and hedging strategy, and the robust strategies (with both multiple priors and variational preferences).

In order to distinguish between the various strategies, a number of superscripts will appear as labels (BS, mBS, classical, mp, vp). For example, $u^{(n,\mathrm{mp})}$ will denote the (robust) value function of the multiple priors exponential utility maximisation problem when the agent holds n options. We will omit n when not specifically referring to the number of options.

5.4.1 A multiple priors basis risk problem

In the case when the agent considers the worst case amongst all possible models in some compact set Γ, we set $\Gamma = [\underline{\eta}, \overline{\eta}] \times [\underline{\eta}^{\perp}, \overline{\eta}^{\perp}]$, a subset of $\mathbb{R}^2$ containing the origin. The set of admissible strategies for the agent is defined as the set $\mathcal{A}$ of $(\mathcal{F}_t)$-progressively measurable processes (π_t) satisfying (5.4.1) as well as

(i) $\mathbb{E}\left[\exp\left(c\alpha^2\sigma^2 \int_0^T \pi_t^2 \,\mathrm{d}t\right)\right] < \infty$, for some constant $c > 1$,

(ii) $\frac{\pi}{S} \cdot S$ is a $\mathbb{Q}$-martingale for any $\mathbb{Q} \in \mathcal{M}_{a,loc}^{\mathbb{P}^0} \cap \mathcal{M}_f^{\mathcal{P}}$ with $\mathcal{H}(\mathbb{Q}|\widehat{\mathbb{P}}) < \infty$,

where $\frac{\pi}{S} \cdot S$ denotes the stochastic integral with respect to S, $\widehat{\mathbb{P}}$ is the least favourable model that achieves the infimum in (5.4.2) for an agent holding n options, and $\mathcal{M}_{a,loc}^{\mathbb{P}^0} \cap \mathcal{M}_f^{\mathcal{P}}$ is the set of absolutely continuous local martingale measures with respect to $\mathbb{P}^0$ with $\mathcal{H}(\mathbb{Q}|\mathcal{P}) := \inf_{P\in\mathcal{P}} \mathcal{H}(\mathbb{Q}|\mathbb{P}) < \infty$.

[4]Recall that as defined in Chapter 2, $\widetilde{\mathbb{F}}$ is the filtration generated by the standard $\mathbb{P}^0$-Brownian motion $\widetilde{W}^0 := \rho W^0 + \sqrt{1-\rho^2} W^{0,\perp}$. Note that there is no loss of generality here to consider only $\widetilde{\mathbb{F}}$-adapted processes.

The goal of the agent given a starting time t with a random endowment of n claims, is described by the value function

$$u^{(\mathrm{n,mp})}(t,x,y) := \sup_{\pi\in\mathcal{A}_{t,T}} \inf_{\mathbb{P}\in\mathcal{P}_{t,T}} \mathbb{E}_{t,x,y}[U(X_T + ng(Y_T))], \tag{5.4.2}$$

where $\mathbb{E}_{t,x,y}$ denotes conditional expectation under $\mathbb{P}$ with initial wealth $X_t = x$ and non-traded asset price $Y_t = y$. In the following, we will drop n in the superscript when it does not cause confusion.

Assumption 5.4.1. Define the probability measure $\widetilde{\mathbb{P}} \sim \mathbb{P}$ on $\widetilde{\mathcal{F}}_T$ by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(-\int_0 \rho(\lambda+\eta_s)\,\mathrm{d}\widetilde{W}_s\right)_T, \tag{5.4.3}$$

and the positive bounded $\widetilde{\mathbb{P}}$-martingale M by

$$M_t := \widetilde{\mathbb{E}}_{t,y}\left[\exp\left(-\frac{1-\rho^2}{2}\int_0^T (\lambda+\eta_s)^2\,\mathrm{d}s - \alpha(1-\rho^2)ng(Y_T)\right)\right], \quad t\in[0,T].$$

Then M has representation

$$M_t = M_0 + \int_0^t M_s\xi_s\,\mathrm{d}\widetilde{B}_s, \quad t\in[0,T],$$

for some ξ, where $\widetilde{B}_\cdot = \widetilde{W}_\cdot + \int_0^\cdot \rho(\lambda+\eta_s)\,\mathrm{d}s$ is standard $\widetilde{\mathbb{P}}$-Brownian motion. We assume that

$$\mathbb{E}\left[\exp\left(c_\xi\int_0^T \xi_t^2\,\mathrm{d}t\right)\right] < \infty, \text{ where } c_\xi = \max\left\{\frac{1}{2}\frac{c}{c-1}\frac{1+\rho^2}{(1-\rho^2)^2}, 4c\frac{\rho^2}{(1-\rho^2)^2}\right\}.$$

Under Assumption 5.4.1, and with the given market model and the European option, Assumptions 5.2.1-5.2.3 are satisfied. Thus Propositions 5.2.4, 5.2.5 and the results for the Markovian case in Section 5.2.3 apply and we have the following results.

Proposition 5.4.2 (Multiple priors value function and optimal trading strategy)**.** *Under Assumption 5.4.1, the robust value function in* (5.4.2) *of the multiple priors exponential utility maximisation problem for an agent holding n European put options is given by*

$$u^{(\mathrm{mp})}(t,x,y) = -\mathrm{e}^{-\alpha x}(v^{(\mathrm{mp})}(t,y))^{\frac{1}{1-\rho^2}}, \tag{5.4.4}$$

where the function $v^{(\mathrm{mp})}$ *is*

$$v^{(\mathrm{mp})}(t,y) = \widetilde{\mathbb{E}}^{\widehat{\boldsymbol{\eta}}}_{t,y}\left[\exp\left(-\frac{1-\rho^2}{2}(\lambda+\widehat{\eta})^2(T-t) - \alpha(1-\rho^2)ng(Y_T)\right)\right], \tag{5.4.5}$$

and where $\widetilde{\mathbb{E}}^{\widehat{\boldsymbol{\eta}}}$ *denotes the expectation taken under the probability measure* $\widetilde{\mathbb{P}}^{\widehat{\boldsymbol{\eta}}}$ *defined in* (5.4.3) *with respect to the least favourable measure* $\mathbb{P}^{\widehat{\boldsymbol{\eta}}} \equiv \widehat{\mathbb{P}}$*. The function* $v^{(\mathrm{mp})}$ *is the solution to the linear PDE*

$$\frac{\partial v^{(\mathrm{mp})}}{\partial t}(t,y)+\Big(ay-\big(\rho\lambda-\overline{\rho}\widehat{\eta}^{\perp}\big)by\Big)\frac{\partial v^{(\mathrm{mp})}}{\partial y}(t,y) + \frac{1}{2}b^2y^2\frac{\partial^2 v^{(\mathrm{mp})}}{\partial y^2}(t,y)-\frac{1-\rho^2}{2}(\lambda+\widehat{\eta})^2 v^{(\mathrm{mp})}(t,y)=0, \quad (5.4.6)$$

with terminal condition $v^{(\mathrm{mp})}(T,y)=\mathrm{e}^{-\alpha(1-\rho^2)nC(y)}$.

The optimal trading strategy is

$$\widehat{\pi}_t^{(\mathrm{mp})}=\frac{1}{\alpha\sigma}\left(\lambda+\widehat{\eta}+\frac{\rho bY_t}{(1-\rho^2)v^{(\mathrm{mp})}(t,Y_t)}\frac{\partial v^{(\mathrm{mp})}}{\partial y}(t,Y_t)\right), \quad t\in[0,T]. \quad (5.4.7)$$

The integrands characterising the least favourable measure are constants given by

$$\widehat{\eta}^{\mathrm{mp}}=\operatorname*{arg\,min}_{\eta\in[\underline{\eta},\overline{\eta}]}|\lambda+\eta|, \qquad \widehat{\eta}^{\perp,\mathrm{mp}}=\begin{cases}\underline{\eta}^{\perp}, & \text{if } n<0,\\ \overline{\eta}^{\perp}, & \text{if } n>0.\end{cases} \quad (5.4.8)$$

Proof. The distortion representation (5.4.4) of the value function, the stochastic representation (5.4.5) of $v^{(\mathrm{mp})}$, the PDE (5.4.6) satisfied by $v^{(\mathrm{mp})}$, and the optimal trading strategy (5.4.7) follow directly from (5.2.12), (5.2.17), (5.2.18) and (5.2.20). Then it only remains to obtain the optimal controls $\widehat{\eta}^{\mathrm{mp}}$ and $\widehat{\eta}^{\perp,\mathrm{mp}}$. By (5.2.17), the function $v^{(\mathrm{mp})}$ is defined by

$$v^{(\mathrm{mp})}(t,y)=\sup_{\boldsymbol{\eta}\in\mathcal{N}_{t,T}}\widetilde{\mathbb{E}}_{t,y}\left[\exp\left(-\frac{1-\rho^2}{2}\int_t^T(\lambda+\eta_s)^2\,\mathrm{d}s-\alpha(1-\rho^2)ng(Y_T)\right)\right],$$

with the dynamics of the non-traded asset Y under $\widetilde{\mathbb{P}}$ given by

$$\mathrm{d}Y_t=(a+\overline{\rho}b\eta_t^{\perp}-\rho b\lambda)Y_t\,\mathrm{d}t+bY_t\,\mathrm{d}\widetilde{B}_t,$$

and where $\widetilde{B}$ is standard Brownian motion under $\widetilde{\mathbb{P}}$. Then optimal control $\widehat{\eta}^{\mathrm{mp}}=\inf_\eta|\lambda+\eta|$, and $\eta^{\perp}$ controls the distribution of the process Y. For $n>0$, we are essentially minimising the payoff of the put option, thus maximising the terminal non-traded asset price Y_T. This is equivalent to maximising the drift of $\ln Y$ (volatility is not controlled), which can only be achieved by taking $\widehat{\eta}^{\perp,\mathrm{mp}}\equiv\overline{\eta}^{\perp}$. A similar reasoning yields that $\widehat{\eta}^{\perp,\mathrm{mp}}\equiv\underline{\eta}^{\perp}$ for $n<0$. □

From Proposition 5.4.2, we can see that the least favourable model only depends on the market price of risk of the traded asset and on whether the agent is taking a

long or short position in the claim. Under one of the possible measures $\mathbb{P} \in \mathcal{P}$, the dynamics of the asset prices are

$$\begin{aligned}
\mathrm{d}S_t &= \sigma S_t\big((\lambda + \eta_t)\,\mathrm{d}t + \,\mathrm{d}W_t\big),\\
\mathrm{d}Y_t &= \big(a + (\rho\eta_t + \overline{\rho}\eta_t^{\perp})b\big)Y_t\,\mathrm{d}t + bY_t\,\mathrm{d}\widetilde{W}_t.
\end{aligned}$$

The result of considering the worst scenario amongst the above set of models is that the agent always believes the traded asset does not have a clear upwards/downwards trend, and the drift of the non-traded asset (the underlying asset for the put option) is always against her position in the option: when she is selling the option ($n < 0$), the drift of the underlying non-traded asset is the smallest it can get; and when the agent takes a long position in the option ($n > 0$), the drift of the underlying non-traded asset is the largest it can get.

We then obtain the multiple priors indifference selling price and the multiple priors hedging strategies for the put option. Recall that the time $t \in [0, T]$ indifference selling price p^{mp} of the option is the price such that

$$u^{(-1,\mathrm{mp})}(t, x, y) = u^{(0,\mathrm{mp})}(t, x - p^{\mathrm{mp}}, y),$$

and optimal hedging strategy $\widehat{\pi}^{h,\mathrm{mp}}$ is defined as

$$\widehat{\pi}_t^{h,\mathrm{mp}} := \widehat{\pi}_t^{(-1,\mathrm{mp})} - \widehat{\pi}_t^{(0,\mathrm{mp})},$$

where $\widehat{\pi}^{(\mathrm{n,mp})}$ is as in (5.4.7).

Proposition 5.4.3 (Multiple priors indifference selling price and hedging strategy)**.** *The multiple priors indifference selling price at time $t \in [0, T]$ of a European put option on the non-traded asset with payoff $g(Y_T)$ is $p(t, Y_t)$, with function $p(\cdot, \cdot)$ given by*

$$p^{\mathrm{mp}}(t, y) = \frac{1}{\alpha(1-\rho^2)} \ln \widetilde{\mathbb{E}}_{t,y}^{\widehat{\boldsymbol{\eta}}^{(-1,\mathrm{mp})}} \left[\exp\left(\alpha(1-\rho^2)g(Y_T)\right)\right]. \tag{5.4.9}$$

The optimal number of traded asset held at time $t \in [0, T]$ for hedging the position of selling one put option is

$$\widehat{\theta}_t^{h,\mathrm{mp}} = \frac{\rho b Y_t}{\sigma S_t} \frac{\partial p^{\mathrm{mp}}}{\partial y}(t, Y_t). \tag{5.4.10}$$

Proof. By the definition of indifference selling price and Proposition 5.4.2, the price function is independent of the initial wealth of the agent, and is given by

$$\begin{aligned}
p^{\mathrm{mp}}(t, y) &= \frac{1}{\alpha(1-\rho^2)} \ln \frac{v^{(-1,\mathrm{mp})}(t, y)}{v^{(0,\mathrm{mp})}(t, y)}\\
&= \frac{1}{\alpha(1-\rho^2)} \ln \widetilde{\mathbb{E}}_{t,y}^{\widehat{\boldsymbol{\eta}}^{(-1,\mathrm{mp})}} \left[\exp\left(\alpha(1-\rho^2)g(Y_T)\right)\right].
\end{aligned}$$

The optimal hedging strategy for the option is

$$\begin{aligned}\widehat{\pi}_t^{h,\mathrm{mp}} &:= \widehat{\pi}_t^{(-1,\mathrm{mp})} - \widehat{\pi}_t^{(0,\mathrm{mp})} \\ &= \frac{1}{\alpha\sigma}\left(\lambda + \widehat{\eta} + \frac{\rho b Y_t}{(1-\rho^2)v^{(-1,\mathrm{mp})}(t,Y_t)}\frac{\partial v^{(-1,\mathrm{mp})}}{\partial y}(t,Y_t)\right) - \frac{1}{\alpha\sigma}(\lambda+\widehat{\eta}) \\ &= \frac{1}{\alpha\sigma(1-\rho^2)}\frac{\rho b Y_t}{v^{(-1,\mathrm{mp})}(t,Y_t)}\frac{\partial v^{(-1,\mathrm{mp})}}{\partial y}(t,Y_t) = \frac{\rho b Y_t}{\sigma}\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,Y_t).\end{aligned}$$

Thus the number of shares of the traded asset held by the seller of the European option on the non-traded asset is given by (5.4.10). □

Following the pricing formula (5.4.9), applying the Feynman-Kac theorem to the conditional expectation which is equal to $\exp(\alpha(1-\rho^2)p^{\mathrm{mp}})$ and a change of variable yields that function p^{mp} solves the PDE

$$\begin{aligned}&\frac{\partial p^{\mathrm{mp}}}{\partial t}(t,y) + (a - \rho b\lambda + \overline{\rho} b\widehat{\eta}^{\perp})y\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,y) \\ &\qquad + \frac{1}{2}b^2y^2\frac{\partial^2 p^{\mathrm{mp}}}{\partial y^2}(t,y) + \frac{1}{2}\alpha(1-\rho^2)b^2y^2\Big(\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,y)\Big)^2 = 0,\end{aligned}$$

with terminal condition $p^{\mathrm{mp}}(T,y) = g(y)$. It then follows that the multiple priors indifference price process $(p^{\mathrm{mp}}(t,Y_t))_{t\in[0,T]}$ has the following dynamics under the reference measure $\mathbb{P}^0$:

$$\begin{aligned}\mathrm{d}p^{\mathrm{mp}}(t,Y_t) &= (\rho b\lambda - \overline{\rho} b\widehat{\eta}^{\perp})Y_t\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,Y_t)\,\mathrm{d}t - \frac{1}{2}\alpha(1-\rho^2)b^2Y_t^2\Big(\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,Y_t)\Big)^2\,\mathrm{d}t \\ &\quad + bY_t\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,Y_t)\,\mathrm{d}\widetilde{W}_t^0. \qquad (5.4.11)\end{aligned}$$

For the multiple priors indifference selling price under the basis risk model, we have the following additional result, which says that the robust indifference selling price of one option is the highest price amongst all indifference prices under the set of plausible models.

Proposition 5.4.4. *The multiple priors indifference selling price* (5.4.9) *is the maximum price amongst the utility indifference prices under all possible models:*

$$p^{\mathrm{mp}}(t,y) = \sup_{\mathbb{P}\in\mathcal{P}} p^{\mathbb{P}}(t,y), \quad t\in[0,T],$$

where $p^{\mathbb{P}}(\cdot,\cdot)$ is the utility indifference price under each fixed model

$$p^{\mathbb{P}}(t,y) = \frac{1}{\alpha(1-\rho^2)}\ln\widetilde{\mathbb{E}}_{t,y}\left[\exp\left(\alpha(1-\rho^2)g(Y_T)\right)\right].$$

Proof. We need to prove that

$$p^{\mathrm{mp}}(t,y) = \frac{1}{\alpha(1-\rho^2)} \ln\left(\sup_{\boldsymbol{\eta}\in\mathcal{N}_{t,T}} \widetilde{\mathbb{E}}_{t,y}\left[\exp\left(\alpha(1-\rho^2)g(Y_T)\right)\right]\right).$$

Define

$$w(t,y) := \sup_{\boldsymbol{\eta}\in\mathcal{N}_{t,T}} \widetilde{\mathbb{E}}_{t,y}\left[\exp\left(\alpha(1-\rho^2)g(Y_T)\right)\right], \tag{5.4.12}$$

where the dynamics of the process Y under $\widetilde{\mathbb{P}}$ is

$$\mathrm{d}Y_t = (a + \overline{\rho}b\eta_t^{\perp} - \rho b\lambda)Y_t\,\mathrm{d}t + bY_t\,\mathrm{d}\widetilde{B}_t.$$

Using the same reasoning as in the proof of Proposition 5.4.2, we have $\widehat{\eta}^{\perp} = \underline{\eta}^{\perp}$, which means that the process Y has the same initial value, drift and volatility (thus the same distribution) under the measure $\widetilde{\mathbb{P}}^{\widehat{\boldsymbol{\eta}}}$ and under the measure $\widetilde{\mathbb{P}}^{\widehat{\eta}^{\perp}}$ (which realises the supremum in (5.4.12)). Thus we conclude that the maximal conditional expectation of $\exp(\alpha(1-\rho^2)g(Y_T))$ in (5.4.12) is the same as its conditional expectation in (5.4.9). □

5.4.2 A variational preferences problem

In this section, we consider the case when the agent considers all equivalent probability measures with respect to the reference measure ($\Gamma = \mathbb{R}^2$) but penalises each measure according to its plausibility by a multiplicative penalty function. The set of admissible strategies for the agent is defined as the set $\mathcal{A}$ of $(\mathcal{F}_t)$-progressively measurable processes (π_t) satisfying (5.4.1) and is such that $(e^{-\alpha X_t})$ is $\mathbb{P}$-uniformly integrable for $\mathbb{P} \sim \mathbb{P}^0$.

The goal of the agent starting from time t with n options is described by the robust value function

$$u^{(n,\mathrm{vp})}(t,x,y) := \sup_{\pi\in\mathcal{A}_{t,T}} \inf_{\boldsymbol{\eta}\in\mathcal{N}_{t,T}} \mathbb{E}_{t,x,y}\left[\mathrm{e}^{-\frac{1}{2\beta}\int_t^T \|\boldsymbol{\eta}_s\|^2 ds} U\left(X_T + ng(Y_T)\right)\right]. \tag{5.4.13}$$

Again, in the following, we will drop n in the superscript when it does not cause any confusion. By Propositions 5.3.2 and 5.3.3, the following results hold.

Proposition 5.4.5. *The value function $u^{(n,\mathrm{vp})}$ defined in* (5.4.13) *of the variational preferences exponential utility maximisation problem for an agent holding n European put options is*

$$u^{(\mathrm{vp})}(t,x,y) = -\mathrm{e}^{-\alpha x}(v^{(\mathrm{vp})}(t,y))^{\frac{1}{(1+\beta)(1-\rho^2)}},$$

where the function $v^{(\mathrm{vp})}$ *is*

$$v^{(\mathrm{vp})}(t,y) = \widetilde{\mathbb{E}}^0_{t,y}\left[\exp\left\{-\frac{1}{2}\lambda^2(1-\rho^2)(T-t) - \alpha(1+\beta)(1-\rho^2)ng(Y_T)\right\}\right],$$

and where $\widetilde{\mathbb{E}}^0_{t,y}$ *denotes the conditional expectation taken under the distortion measure* $\widetilde{\mathbb{P}}^0$ *defined by*

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}^0}{\mathrm{d}\mathbb{P}^0} = \mathcal{E}\left(-\int_0^\cdot \rho\lambda\,\mathrm{d}\widetilde{W}^0_s\right)_T.$$

The function $v^{(\mathrm{vp})}$ *is the solution to the linear PDE*

$$\begin{aligned}\frac{\partial v^{(\mathrm{vp})}}{\partial t}(t,y) + \big(a-\rho b\lambda\big)y\frac{\partial v^{(\mathrm{vp})}}{\partial y}(t,y)\\ + \frac{1}{2}b^2y^2\frac{\partial^2 v^{(\mathrm{vp})}}{\partial y^2}(t,y) - \frac{1}{2}(1-\rho^2)\lambda^2 v^{(\mathrm{vp})}(t,y) = 0,\end{aligned}$$

with terminal condition $v^{(\mathrm{vp})}(T,x,y) = \mathrm{e}^{-\alpha(1+\beta)(1-\rho^2)nC(y)}$.

The optimal trading strategies are

$$\widehat{\pi}^{(\mathrm{vp})}_t = \frac{1}{\alpha\sigma}\left(\frac{\lambda}{1+\beta} + \frac{\rho bY_t}{(1+\beta)(1-\rho^2)}\frac{\partial}{\partial y}\ln v^{(\mathrm{vp})}(t,Y_t)\right), \quad t\in[0,T].$$

The integrands characterising the least favourable measure are given by

$$\begin{aligned}\widehat{\eta}^{\mathrm{vp}}_t &\equiv \widehat{\eta}^{\mathrm{vp}} = -\frac{\beta}{1+\beta}\lambda,\\ \widehat{\eta}^{\perp,\mathrm{vp}}_t &= \widehat{\eta}^{\perp,\mathrm{vp}}(t,Y_t) = \frac{\beta}{1+\beta}\frac{bY_t}{\sqrt{1-\rho^2}}\frac{\partial}{\partial y}\ln v^{(\mathrm{vp})}(t,Y_t), \quad t\in[0,T].\end{aligned}$$

From Proposition 5.4.5, under the least favourable measure $\widehat{\mathbb{P}} \equiv \mathbb{P}^{\widehat{\boldsymbol{\eta}}}$, the processes have the dynamics as below

$$\begin{aligned}\mathrm{d}S_t &= \sigma S_t\left(\frac{1}{1+\beta}\lambda\,\mathrm{d}t + \mathrm{d}W^{\widehat{\boldsymbol{\eta}}}_t\right),\\ \mathrm{d}Y_t &= \left(aY_t + bY_t\Big(-\frac{\rho\beta}{1+\beta}\lambda + \frac{\beta}{1+\beta}\frac{bY_t}{v^{(\mathrm{vp})}(t,Y_t)}\frac{\partial v^{(\mathrm{vp})}}{\partial y}(t,Y_t)\Big)\right)\mathrm{d}t + bY_t\,\mathrm{d}\widetilde{W}^{\widehat{\boldsymbol{\eta}}}_t.\end{aligned}$$

We can see that the least favourable model in the variational preferences case indeed scales the MPR of the traded asset by a factor $\frac{1}{1+\beta} < 1$. While for the non-traded asset, by examining the stochastic representation of $v^{(\mathrm{vp})}$, it is easy to see that $\frac{\partial v^{(\mathrm{vp})}}{\partial y}(t,y) > 0$ for $n > 0$ and $\frac{\partial v^{(\mathrm{vp})}}{\partial y}(t,y) < 0$ for $n < 0$. Thus the least favourable model also stands against the position of the agent in the put option: compared with the reference model, it decreases the drift of the underlying non-traded asset when the agent takes a short position in the put option and increases the drift of the underlying non-traded asset when the agent takes a long position in the put option.

Proposition 5.4.6 (Variational preferences indifference selling price and hedging strategy)**.** *The variational preferences indifference selling price at time $t \in [0,T]$ of a European put option on the non-traded asset with payoff $g(Y_T)$ is $p(t,Y_t)$, with function $p(\cdot,\cdot)$ given by*

$$p^{\mathrm{vp}}(t,y) = \frac{1}{\alpha(1+\beta)(1-\rho^2)} \ln \widetilde{\mathbb{E}}^0_{t,y} \left[\exp\left(\alpha(1+\beta)(1-\rho^2) g(Y_T) \right) \right]. \quad (5.4.14)$$

The optimal number of traded asset held at time $t \in [0,T]$ for hedging the position of selling one put option is

$$\widehat{\theta}^{h,\mathrm{vp}}_t = \frac{\rho b Y_t}{\sigma S_t} \frac{\partial p^{\mathrm{vp}}}{\partial y}(t,Y_t). \quad (5.4.15)$$

From (5.4.14) and (5.4.15) above, we can see that the variational preferences approach in essence yields the same non-linear expectation representation of the indifference price and hedging strategy that one gets classically (see, for example, Section 5.4.4.1 below) with modified risk aversion parameter $\alpha^{\mathrm{vp}} = \alpha(1+\beta)$.

Similarly, we also have the PDE satisfied by the variational preferences indifference pricing function p^{vp}

$$\begin{aligned} &\frac{\partial p^{\mathrm{vp}}}{\partial t}(t,y) + (a - \rho b\lambda) y \frac{\partial p^{\mathrm{vp}}}{\partial y}(t,y) + \frac{1}{2} b^2 y^2 \frac{\partial^2 p^{\mathrm{vp}}}{\partial y^2}(t,y) \\ &\qquad + \frac{1}{2}\alpha(1+\beta)(1-\rho^2) b^2 y^2 \Big(\frac{\partial p^{\mathrm{vp}}}{\partial y}(t,y) \Big)^2 = 0, \end{aligned}$$

with $p^{\mathrm{vp}}(T,y) = g(y)$, as well as the dynamics of the variational preferences indifference price process $(p^{\mathrm{vp}}(t,Y_t))_{t\in[0,T]}$ under the reference measure $\mathbb{P}^0$:

$$\begin{aligned} \mathrm{d}p^{\mathrm{vp}}(t,Y_t) = {} & \rho b\lambda Y_t \frac{\partial p^{\mathrm{vp}}}{\partial y}(t,Y_t)\,\mathrm{d}t - \frac{1}{2}\alpha(1+\beta)(1-\rho^2) b^2 Y_t^2 \Big(\frac{\partial p^{\mathrm{vp}}}{\partial y}(t,Y_t) \Big)^2 \mathrm{d}t \\ & + b Y_t \frac{\partial p^{\mathrm{vp}}}{\partial y}(t,Y_t)\,\mathrm{d}\widetilde{W}^0_t. \end{aligned} \quad (5.4.16)$$

5.4.3 The residual risk process

Suppose the agent sells the option at price $p^i(0,Y_0)$, $i = \mathrm{mp}, \mathrm{vp}$, at time 0, and hedges the option over $[0,T]$ using strategy $\widehat{\theta}^{h,i}$. Let $\widehat{X}^{h,i}$ denote her hedging portfolio value, the **residual risk process** R^i is defined as the difference between her hedging portfolio value and the price process of the option, that is, the value process of her overall position:

$$R^i_t := \widehat{X}^{h,i}_t - p^i(t,Y_t), \quad t \in [0,T].$$

Using

$$\mathrm{d}\widehat{X}_t^{h,i} = \widehat{\theta}_t^{h,i}\,\mathrm{d}S_t, \qquad \widehat{X}_0^{h,i} = p^i(0, Y_0),$$

and the dynamics (5.4.11) and (5.4.16) of the robust indifference price processes, we have

$$\begin{aligned}\mathrm{d}R_t^i &= \mathrm{d}\widehat{X}_t^{h,i} - \mathrm{d}p^i(t, Y_t) \qquad (5.4.17)\\ &= \begin{cases} \Big(\frac{1}{2}\alpha(1-\rho^2)b^2Y_t^2\big(\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,Y_t)\big)^2 + \sqrt{1-\rho^2}b\widehat{\eta}^{\perp}Y_t\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,Y_t)\Big)\,\mathrm{d}t \\ \qquad -\sqrt{1-\rho^2}bY_t\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,Y_t)\,\mathrm{d}W_t^{0,\perp}, & i = \mathrm{mp}, \\ \frac{1}{2}\alpha(1+\beta)(1-\rho^2)b^2Y_t^2\big(\frac{\partial p^{\mathrm{vp}}}{\partial y}(t,Y_t)\big)^2\,\mathrm{d}t \\ \qquad -\sqrt{1-\rho^2}bY_t\frac{\partial p^{\mathrm{vp}}}{\partial y}(t,Y_t)\,\mathrm{d}W_t^{0,\perp}, & i = \mathrm{vp}. \end{cases}\end{aligned}$$

Similar as the classical case (Monoyios [43] and Musiela and Zariphopoulou [47]), for the robust residual risk process, the $\mathrm{d}W_t^{0,\perp}$-term can be interpreted as the unhedgeable risk. In the case when $\rho = 1$, the robust residual risk process also becomes riskless, which indicates that the market incompleteness disappears.

It has been studied in Musiela and Zariphopoulou [47, Proposition 6] that the preference-adjusted exponential of the classical residual risk process $(-\mathrm{e}^{(-\alpha R_t^{\mathrm{classical}})})$ is a local martingale (and a martingale under the appropriate integrability conditions) under the reference measure $\mathbb{P}^0$. The following result shows that the $\mathbb{P}^0$-(local) martingale property does not hold for the preference-adjusted exponential of the robust residual risk processes, though it still holds for the multiple priors case under the corresponding least favourable measure.

Proposition 5.4.7. *The preference-adjusted exponential of the robust residual risk process*

$$U_t^i = -\mathrm{e}^{-\alpha R_t^i}, \quad i = \mathrm{mp}, \mathrm{vp},\ t \in [0, T] \qquad (5.4.18)$$

is a sub-martingale under the reference measure $\mathbb{P}^0$. The process U^{mp} is a martingale under the least favourable measure $\widehat{\mathbb{P}}^{\mathrm{mp}}$.

Proof. Applying Itô's formula to (5.4.18) and using the dynamics (5.4.17) of the robust residual risk processes, we have

$$\begin{aligned}\mathrm{d}U_t^{\mathrm{mp}} =& -U_t^{\mathrm{mp}}\alpha\sqrt{1-\rho^2}b\widehat{\eta}^{\perp,\mathrm{mp}}Y_t\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,Y_t)\,\mathrm{d}t \\ &+ U_t^{\mathrm{mp}}\alpha\sqrt{1-\rho^2}bY_t\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,Y_t)\,\mathrm{d}W_t^{0,\perp},\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}U_t^{\mathrm{vp}} = &-\frac{1}{2}U_t^{\mathrm{vp}}\beta\alpha^2(1-\rho^2)b^2Y_t^2\Big(\frac{\partial p^{\mathrm{vp}}}{\partial y}(t,Y_t)\Big)^2\,\mathrm{d}t\\ &+U_t^{\mathrm{vp}}\alpha\sqrt{1-\rho^2}bY_t\frac{\partial p^{\mathrm{vp}}}{\partial y}(t,Y_t)\,\mathrm{d}W_t^{0,\perp}.\end{aligned}$$

It then follows that

$$\begin{aligned}U_t^{\mathrm{vp}} = U_0^{\mathrm{vp}}\mathcal{E}\Big(\alpha\sqrt{1-\rho^2}b\int_0^{\cdot}Y_s\frac{\partial p^{\mathrm{vp}}}{\partial y}(s,Y_s)\,\mathrm{d}W_s^{0,\perp}\Big)_t\\ \exp\left\{-\frac{1}{2}\beta\alpha^2(1-\rho^2)b^2\int_0^t Y_s^2\Big(\frac{\partial p^{\mathrm{vp}}}{\partial y}(s,Y_s)\Big)^2\,\mathrm{d}s\right\},\quad t\in[0,T]\end{aligned}$$

is a sub-martingale under $\mathbb{P}^0$: since $U_0^{\mathrm{vp}} < 0$, the Doléans-Dade exponential is a super-martingale (under the integrability condition that $\int_0^T Y_s^2\big(\frac{\partial p^{\mathrm{vp}}}{\partial y}(s,Y_s)\big)^2\,\mathrm{d}s<\infty$, $\mathbb{P}^0$-a.s.), and the integrand in the exponential is always negative or zero.

For the integrability condition, in fact, we have $\mathbb{E}^0[\int_0^T Y_s^2\big(\frac{\partial p^{\mathrm{vp}}}{\partial y}(s,Y_s)\big)^2\,\mathrm{d}s]<\infty$: since the indifference price and its derivative with respect to the initial asset price are bounded, we only need $\mathbb{E}^0[\int_0^T Y_s^2\,\mathrm{d}s]<\infty$, which holds for the geometric Brownian motion Y according to Touzi [64, Theorem 2.2].

Using (5.4.9) and the fact that the process Y is a geometric Brownian motion and the optimal controls $\widehat{\eta}^{\mathrm{mp}},\widehat{\eta}^{\perp,\mathrm{mp}}$ (thus the least favourable measure) do not depend on y, we have that $\frac{\partial p^{\mathrm{mp}}}{\partial y}(t,y)\le 0$. Then a similar reasoning yields that

$$\begin{aligned}U_t^{\mathrm{mp}} = U_0^{\mathrm{mp}}\mathcal{E}\Big(\alpha\sqrt{1-\rho^2}b\int_0^{\cdot}Y_s\frac{\partial p^{\mathrm{mp}}}{\partial y}(s,Y_s)\,\mathrm{d}W_s^{0,\perp}\Big)_t\\ \exp\left\{-\alpha\sqrt{1-\rho^2}b\widehat{\eta}^{\perp,\mathrm{mp}}\int_0^t Y_s\frac{\partial p^{\mathrm{mp}}}{\partial y}(s,Y_s)\,\mathrm{d}s\right\},\quad t\in[0,T]\end{aligned}$$

is a sub-martingale under $\mathbb{P}^0$. The martingale property under $\widehat{\mathbb{P}}^{\mathrm{mp}}$ follows from a change of measure in the above representation to get

$$U_t^{\mathrm{mp}} = U_0^{\mathrm{mp}}\mathcal{E}\Big(\alpha\sqrt{1-\rho^2}b\int_0^{\cdot}Y_s\frac{\partial p^{\mathrm{mp}}}{\partial y}(s,Y_s)\,\mathrm{d}W_s^{\perp}\Big)_t,\quad t\in[0,T].$$

□

5.4.4 Analytic approximations for indifference prices and hedging strategies

We would like to carry out a simulation-based test of the robust hedging strategies versus the Black-Scholes strategies and the classical strategies (without taking model

uncertainty into consideration), computing the distributions of the terminal hedging errors over many asset price paths. This requires the calculation of the derivative of the option prices each time the agent re-balances the hedging portfolio. Since classical Monte-Carlo methods require a large number of simulations to evaluate the derivative, this is computationally expensive. In this section, we will use the analytic approximations developed in Monoyios [43, Proposition 3, Corollary 1 and Proposition 4] for the option prices and their derivatives, which enables us to compute the hedging strategies fast and accurately in the simulation study.

For purposes of future numerical experiments on real market data, we will assume non-zero risk-free rate $r \geq 0$ and make the following adjustments to the formulae: the utility function admits the separable form $u(t,x,y) = U(\mathrm{e}^{r(T-t)}x)(v(t,y))^{\delta}$, the optimal trading strategies $\widehat{\pi}$ (wealth invested in traded asset) and the indifference price p are discounted by the factor $e^{-r(T-t)}$. For example, for the multiple priors case, when $r \neq 0$, in Proposition 5.4.2 and 5.4.3, the value function $u^{(\mathrm{mp})}$ and optimal trading strategies $\widehat{\pi}^{(\mathrm{mp})}$ for the agent holding n options are

$$
\begin{aligned}
&u^{(\mathrm{mp})}(t,x,y) = -\exp\big(-\alpha \mathrm{e}^{r(T-t)}x\big)(v^{(\mathrm{mp})}(t,y))^{\frac{1}{1-\rho^2}},\\
&\widehat{\pi}_t^{(\mathrm{mp})} = \frac{\mathrm{e}^{-r(T-t)}}{\alpha\sigma}\left(\lambda + \widehat{\eta}^{\mathrm{mp}} + \frac{\rho b Y_t}{(1-\rho^2)v^{(\mathrm{mp})}(t,Y_t)}\frac{\partial v^{(\mathrm{mp})}}{\partial y}(t,Y_t)\right), \quad t \in [0,T],
\end{aligned}
$$

and the indifference price for selling one option is

$$
p^{\mathrm{mp}}(t,y) = \frac{\mathrm{e}^{-r(T-t)}}{\alpha(1-\rho^2)} \ln \widetilde{\mathbb{E}}_{t,y}^{\widehat{\boldsymbol{\eta}}^{(-1,\mathrm{mp})}}\left[\exp\big(\alpha(1-\rho^2)g(Y_T)\big)\right],
$$

where the rest formulae (5.4.8) for the integrands characterising the least favourable measure, (5.4.5) for $v^{(\mathrm{mp})}$, and (5.4.10) for the optimal number of shares of traded asset to hold to hedge the position of selling one option are the same as for $r = 0$. The assumption of zero risk-free rate only affects the monetary quantities while keeping the choices of the agent the same.

5.4.4.1 Power series expansions for an indifference pricing function and its derivative

We first state the analytic approximations in Monoyios [43] for an indifference pricing function of the form

$$
p(t,y) = \frac{\mathrm{e}^{-r(T-t)}}{c} \ln E_{t,y}[\exp(cg(Y_T))],
$$

where c is some constant (equal to $\alpha(1-\rho^2)$ for the classical case), $r \geq 0$ is the risk-free interest rate, $g(\cdot)$ is the payoff function of the option, and $E_{t,y}$ is the expectation taken under some probability measure[5] P conditional on $Y_t = y$.

Proposition 5.4.8 (Proposition 3 and Corollary 1 in Monoyios [43])**.** *The indifference pricing function $p(t,y)$ has the power series expansion*

$$p(t,y) = \mathrm{e}^{-r(T-t)} \sum_{j=1}^{5} \frac{1}{j!} k_j(g(Y_T)) c^{j-1} + O(c^5),$$

and the partial derivative of $p(t,y)$ with respect to y has the power series expansion

$$\frac{\partial p}{\partial y}(t,y) = \mathrm{e}^{-r(T-t)} \sum_{j=1}^{5} \frac{1}{j!} (\partial k_j) c^{j-1} + O(c^5),$$

where k_j and ∂k_j will be given below. The above expansions are valid for model parameters satisfying

$$E_{t,y}[\exp(cg(Y_T))] \leq 2.$$

For a random variable X, write the j-th moment

$$m_j \equiv m_j(X) := E[X^j | \mathcal{F}_t],$$

then $k_j \equiv k_j(X)$ are given by

$$\begin{aligned}
k_1 &= m_1, \\
k_2 &= m_2 - m_1^2, \\
k_3 &= m_3 - 3m_1m_2 + 2m_1^3, \\
k_4 &= m_4 - 4m_1m_3 - 3m_2^2 + 12m_1^2m_2 - 6m_1^4, \\
k_5 &= m_5 - 5m_1m_4 - 10m_2m_3 + 20m_1^2m_3 + 30m_1m_2^2 - 60m_1^3m_2 + 24m_1^5,
\end{aligned} \tag{5.4.19}$$

[5]Recall that for the classical case, this is the measure $\widetilde{\mathbb{P}}^0$ defined by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}^0}{\mathrm{d}\mathbb{P}^0} = \mathcal{E}\left(-\int_0^{\cdot} \rho\lambda \,\mathrm{d}\widetilde{W}_s^0\right)_T.$$

and $\partial k_j \equiv \frac{\partial k_j}{\partial y}$ *(with* $\partial m_j \equiv \frac{\partial m_j}{\partial y}$*) are given by*

$$\begin{aligned}
\partial k_1 &= \partial m_1, \\
\partial k_2 &= \partial m_2 - 2m_1\partial m_1, \\
\partial k_3 &= \partial m_3 - 3(m_1\partial m_2 + m_2\partial m_1 + 2m_1^2\partial m_1), \\
\partial k_4 &= \partial m_4 - 4(m_1\partial m_3 + m_3\partial m_1) - 6m_2\partial m_2 \\
&\quad + 12m_1(m_1\partial m_2 + 2m_2\partial m_1 - 2m_1^2\partial m_1), \\
\partial k_5 &= \partial m_5 - 5(m_1\partial m_4 + m_4\partial m_1) - 10(m_2\partial m_3 + m_3\partial m_2) \\
&\quad + 20m_1(m_1\partial m_3 + 2m_3\partial m_1) + 30m_2(2m_1\partial m_2 + m_2\partial m_1) \\
&\quad - 60m_1^2(m_1\partial m_2 + 3m_2\partial m_1 - 2m_1^2\partial m_1).
\end{aligned} \tag{5.4.20}$$

When the terminal asset price Y_T is log-normally distributed under measure P (conditional on $Y_t = y$) with

$$\ln Y_T \sim \mathcal{N}\Big(\ln y + q - \frac{1}{2}\nu^2, \nu^2\Big),$$

where q, ν are constants ($q = (a - \rho b\lambda)(T-t)$, $\nu^2 = b^2(T-t)$ for the classical case), we have the following formulae for the terms in the power series expansions.

Proposition 5.4.9 (Proposition 4 and 5 in Monoyios [43])**.** *For a put option with payoff function* $g(y) = (K-y)^+$*, where* $K > 0$ *is the strike price, the* j*th moment* $m_j := E_{t,y}[g^j(Y_T)]$ *and its partial derivative* $\partial m_j \equiv \frac{\partial m_j}{\partial y}$*,* $j \in \mathbb{N}$*, are given by*

$$m_j = \sum_{l=0}^{j} \binom{j}{l} (-y)^l K^{j-l} \mathrm{e}^{l\left(q + \frac{1}{2}(l-1)\nu^2\right)} N(-d_1 - (l-1)\nu), \tag{5.4.21}$$

$$\partial m_j = -\sum_{l=1}^{j} \binom{j}{l} (-y)^{l-1} K^{j-l} \mathrm{e}^{l\left(q + \frac{1}{2}(l-1)\nu^2\right)} l N(-d_1 - (l-1)\nu), \tag{5.4.22}$$

where $N(\cdot)$ *denotes the standard cumulative normal distribution function, and where*

$$d_1 = \frac{1}{\nu}\left[\ln\left(\frac{y}{K}\right) + q + \frac{1}{2}\nu^2\right].$$

5.4.4.2 Analytic approximations for robust indifference prices and hedging strategies

Using the power series expansions for the indifference prices and its partial derivative with respect to the underlying asset price in Proposition 5.4.8, as well as Proposition 5.4.9 for the put option with log-normally distributed terminal underlying prices, we have the following analytic approximations for the robust indifference prices and

hedging strategies. The classical indifference prices and hedging strategies, which can be found in [40, Theorem 1 and 2], are also included here for comparison with the robust ones.

Proposition 5.4.10 (Analytic approximations for robust indifference prices and hedging strategies)**.** *The robust indifference price of the put option on the non-traded asset Y with payoff function $g(y) = (K-y)^+$ has the following power series expansion*

$$p^i(t,y) = \mathrm{e}^{-r(T-t)} \sum_{j=1}^{5} \frac{1}{j!} k_j(g(Y_T)) c^{j-1} + O(c^5).$$

The optimal number of traded asset S held at time $t \in [0,T]$ for hedging a short position in one put option is

$$\widehat{\theta}_t^{h,i} = \frac{\rho b Y_t}{\sigma S_t} \frac{\partial p^i}{\partial y}(t, Y_t),$$

where

$$\frac{\partial p^i}{\partial y}(t,y) = \mathrm{e}^{-r(T-t)} \sum_{j=1}^{5} \frac{1}{j!} (\partial k_j) c^{j-1} + O(c^5).$$

The terms k_j, ∂k_j are as in (5.4.19)-(5.4.20) *with m_j, ∂m_j as in* (5.4.21)-(5.4.22) *with different choices for the constants c, q and ν for the multiple priors case and the variational preferences case*

$$\begin{aligned}
&i = \text{mp}: \quad c = \alpha(1-\rho^2),\ q = (a - \rho b\lambda + \overline{\rho} b \underline{\eta}^{\perp})(T-t),\ \nu = b\sqrt{T-t},\\
&i = \text{vp}: \quad c = \alpha(1+\beta)(1-\rho^2),\ q = (a - \rho b\lambda)(T-t),\ \nu = b\sqrt{T-t},\\
&i = \text{classical}: \quad c = \alpha(1-\rho^2),\ q = (a - \rho b\lambda)(T-t),\ \nu = b\sqrt{T-t}.
\end{aligned}$$

Note that when the ambiguity aversion parameter β in the variational preferences case approaches zero (low ambiguity aversion), the variational preferences indifference prices and hedging strategies approach those for the classical case. This is consistent with our observations from the definition of the variational preferences value function in (5.3.2).

It can also be observed from the power series expansions in Proposition 5.4.10 that the robust prices and strategies only add one extra degree of freedom ($\widehat{\eta}^{\perp,\text{mp}} = \underline{\eta}^{\perp}$ for the multiple priors case, and β for the variational preferences case) to the classical ones. Essentially, the multiple priors strategy reduces the mean in the distribution used for the logarithm price of the non-traded asset in the pricing and hedging functions, while the variational preferences strategy uses a more conservative risk aversion

parameter $\alpha^{\text{vp}} = \alpha(1+\beta)$ in the exponential utility function. In the following, we conduct simulation-based numerical tests on whether this extra degree of freedom can be chosen to compensate for model parameter estimation errors and yield better performances than the classical strategies.

With the above analytic approximations, we are then able to compute the prices and the hedging strategies of the put option fast and accurately for the numerical tests, for valid model parameters. It was found in Monoyios [40] that the approximations are accurate for all values of correlation when the risk aversion parameter α is below about 0.05, regardless of other parameter choices, which is, as mentioned in [40], a very mild restriction as typical risk aversion parameters for market participants are around 10^{-6}.

5.4.5 Numerical tests

To test the performances of the robust hedging strategies, we compare the robust strategies (both multiple priors and variational preferences) with: (i) the classical strategies where no model uncertainty is taken into account; (ii) the naive Black-Scholes strategies where the traded asset S is used as a perfect substitute for the non-traded asset Y (perfect correlation); (iii) the modified Black-Scholes strategies (proposed by Hulley and McWalter [27]) where the agent modifies the naive Black-Scholes strategies by simply multiplying correlation ρ to the number of shares of traded assets to hold. For the sake of conciseness, we will omit the super-scripts indicating the different strategies in describing the testing schemes. The selling prices, hedging strategies, hedging portfolio values and terminal hedging errors all correspond to the following strategies: Black-Scholes, modified Black-Scholes, classical basis risk, multiple priors basis risk, and variational preferences basis risk.

Scheme 1: A hypothetical test with assumed parameter estimates.

We first conduct a simple test of the strategies, assuming the agent has some wrong estimates of the model parameters. We assume that volatilities and correlation are known, and only focus on the cases when the agent has wrong estimates for the drifts of the traded and non-traded assets. This scheme is convenient for studying the effects of different scenarios of mis-estimations of these model parameters.

Suppose the agent sells the put option at time 0 on the non-traded asset Y at price $p(0, Y_0)$, which will be the initial wealth of her hedging portfolio. She will hedge her position using strategy $\widehat{\theta}^h$ as described in Proposition 5.4.10. Over the simulated price paths for S, Y under the true model, the agent will re-balance her hedge daily

(assuming 252 trading days per year). Denoting $\widehat{X}^h$ as the hedging portfolio value, the terminal hedging error is then the difference between the hedging portfolio value and the payoff of the put option

$$\mathcal{E}_T = \widehat{X}_T^h - (K - Y_T)^+.$$

Repeating the hedging process over a large number ($N = 10^5$) of sample paths, we then compare the distributions of the terminal hedging errors of different strategies.

The true model parameters are chosen as in Table 5.1. The parameters characterising the risk and ambiguity aversion of the agent are chosen as in Table 5.2. Denote the estimated parameters by $\mu^*, \sigma^*, a^*, b^*, \rho^*$. For the multiple priors agent, we assume she considers MPR for the stock in the range $[\lambda^* - 0.1, \lambda^* + 0.1]$, and drift for the logarithm of the non-traded asset in the range $[a^* - 0.1(\rho^* + \sqrt{1-(\rho^*)^2})b^*, a^* + 0.1(\rho^* + \sqrt{1-(\rho^*)^2})b^*]$, which is approximately $a^* \pm 0.14b^*$ if the agent can estimate ρ and b relatively accurately. For the variational preferences agent, we assume she chooses $\beta = 0.5$, that is, she essentially uses risk aversion parameter $\alpha^{\text{vp}} = 1.5\alpha = 0.015$ in the exponential utility functional.

S_0	Y_0	K	r	μ	σ	a	b	ρ	T
100	100	110	0.05	0.13	0.3	0.10	0.25	0.75	1 year

Table 5.1: Basis risk model: True model parameters

α	$[\underline{\eta}, \overline{\eta}]$	$[\underline{\eta}^\perp, \overline{\eta}^\perp]$	β
0.01	$[-0.1, 0.1]$	$[-0.1, 0.1]$	0.5

Table 5.2: Basis risk model: Risk and ambiguity aversion parameters

Table 5.3 shows the statistics (minimum, maximum, mean, median and standard deviation) of the terminal hedging errors of the five different strategies, together with the prices the put option are sold at, when both the drifts of the traded and non-traded assets are over-estimated by 50%. While the classical strategy sells the option at a slightly lower price than the Black-Scholes price, the more conservative multiple priors and variational preferences strategies sell the option at higher prices and perform better in terms of mean and median hedging errors. We also notice that in this case, the modified Black-Scholes strategy has greatly improved the naive Black-Scholes strategy and performs better on average than the classical strategies. In fact, as shown in Table 5.4, if we eliminate the effect of the robust strategies selling the option at higher prices and assume all strategies sell the option at the

Black-Scholes price, the modified Black-Scholes strategy indeed performs better than the classical strategy as well as the robust strategies.

	Black-Scholes	modified BS	classical	multiple priors	variational preferences
price	12.66	12.66	12.64	13.52	12.88
$\min(\mathcal{E}_T)$	-52.16	-45.97	-46.08	-45.44	-45.96
$\max(\mathcal{E}_T)$	45.94	35.90	36.01	37.43	36.47
$\text{mean}(\mathcal{E}_T)$	-0.84	0.02	-0.01	0.87	0.22
$\text{med}(\mathcal{E}_T)$	0.18	1.84	1.79	2.58	1.99
$\text{sd}(\mathcal{E}_T)$	10.29	9.54	9.53	9.53	9.53

Table 5.3: Basis risk model: Statistics for terminal hedging error for testing scheme 1 when both drifts μ and a are over-estimated by 50%. The option is assumed to be traded at the corresponding price for each strategy.

	Black-Scholes	modified BS	classical	multiple priors	variational preferences
price	12.66	12.66	12.66	12.66	12.66
$\min(\mathcal{E}_T)$	-52.81	-46.66	-46.65	-46.61	-46.62
$\max(\mathcal{E}_T)$	48.51	37.04	37.16	37.54	37.33
$\text{mean}(\mathcal{E}_T)$	-0.90	-0.04	-0.05	-0.10	-0.07
$\text{med}(\mathcal{E}_T)$	0.11	1.77	1.74	1.59	1.69
$\text{sd}(\mathcal{E}_T)$	10.26	9.53	9.53	9.52	9.53

Table 5.4: Basis risk model: Statistics for terminal hedging error for testing scheme 1 when both drifts μ and a are over-estimated by 50%. The option is assumed to be traded at the Black-Scholes price for all strategies.

Table 5.5 shows the results when both the drifts of the traded and non-traded assets are under-estimated by 50%. The classical indifference price is higher than the Black-Scholes price as the agent thinks the put option has a larger possibility of ending in the money than it actually is. Both robust strategies perform even better, at the cost of selling the option at even higher prices than the classical indifference price. However, if the option is sold at the Black-Scholes price for all strategies (Table 5.6), the modified Black-Scholes and the classical basis risk hedging strategies both perform better than the variational preferences and the multiple priors hedging strategies.

Table 5.7 shows the results when the drift of the traded asset μ is over-estimated by 50% and the drift of the non-traded asset a is under-estimated by 50%. In this case, the classical indifference price is significantly higher than the Black-Scholes price,

	Black-Scholes	modified BS	classical	multiple priors	variational preferences
price	12.66	12.66	13.64	14.56	13.90
$\min(\mathcal{E}_T)$	-53.22	-44.91	-43.94	-43.02	-43.69
$\max(\mathcal{E}_T)$	48.18	37.53	38.98	40.21	39.36
$\text{mean}(\mathcal{E}_T)$	-0.89	-0.03	0.95	1.86	1.19
$\text{med}(\mathcal{E}_T)$	0.11	1.80	2.61	3.42	2.82
$\text{sd}(\mathcal{E}_T)$	10.32	9.57	9.57	9.57	9.57

Table 5.5: Basis risk model: Statistics for terminal hedging error for testing scheme 1 when both drifts μ and a are under-estimated by 50%. The option is assumed to be traded at the corresponding price for each strategy.

	Black-Scholes	modified BS	classical	multiple priors	variational preferences
price	12.66	12.66	12.66	12.66	12.66
$\min(\mathcal{E}_T)$	-69.40	-60.04	-59.64	-59.40	-59.50
$\max(\mathcal{E}_T)$	42.67	35.07	35.76	36.23	35.96
$\text{mean}(\mathcal{E}_T)$	-0.90	-0.04	-0.10	-0.14	-0.12
$\text{med}(\mathcal{E}_T)$	0.12	1.81	1.61	1.46	1.55
$\text{sd}(\mathcal{E}_T)$	10.33	9.60	9.60	9.60	9.60

Table 5.6: Basis risk model: Statistics for terminal hedging error for testing scheme 1 when both drifts μ and a are under-estimated by 50%. The option is assumed to be traded at the Black-Scholes price for all strategies.

due to the agent thinking that the put option is more likely to end in the money and that she is hedging with a less effective traded asset than it actually is (as analysed in Monoyios [43]). The more conservative robust strategies have exaggerated the effect of the model estimation error in this case, making the selling prices higher and the agent further less competitive in the market, though with better hedging results. From Table 5.8, again, when the option is sold at the same Black-Scholes price at time 0, the simple modified Black-Scholes strategy performs better in terms of terminal hedging error distributions, than the classical strategies and the robust ones.

Figure 5.1 shows example price paths for traded and non-traded assets, the resulting hedging strategy (number of shares in stock) paths of different strategies as well as their hedging portfolio values (both with the corresponding selling price and with the same Black-Scholes price for the strategies) when μ is over-estimated by 50% and a is under-estimated by 50%. The observations are inline with those from Tables 5.7 and 5.8. In this simulation, the put option on Y expires out of the money, and all five strategies aim to hold 0 shares of stock close to expiry. Although the classical indifference strategy has priced the option very conservatively (third figure), the corresponding hedging involves shorting more stocks than the modified BS strategy (second figure). The robust strategies sell the option at more conservative prices and hold slightly larger short positions in the stock to hedge the option. Then with the same selling price for the option (fourth figure), the indifference hedging strategies (both classical and robust ones) end up with a slightly lower terminal wealth than the modified BS strategy.

	Black-Scholes	modified BS	classical	multiple priors	variational preferences
price	12.66	12.66	18.38	19.40	18.67
$\min(\mathcal{E}_T)$	-50.62	-45.16	-38.39	-37.20	-38.02
$\max(\mathcal{E}_T)$	51.18	37.70	46.00	47.32	46.42
$\text{mean}(\mathcal{E}_T)$	-0.86	-0.01	5.73	6.77	6.03
$\text{med}(\mathcal{E}_T)$	0.14	1.80	6.88	7.83	7.14
$\text{sd}(\mathcal{E}_T)$	10.31	9.56	9.61	9.63	9.62

Table 5.7: Basis risk model: Statistics for terminal hedging error for testing scheme 1 when drift μ is over-estimated by 50% and a is under-estimated by 50%. The option is assumed to be traded at the corresponding price for each strategy.

Table 5.9 shows the results when the drift of the traded asset μ is under-estimated by 50% and the drift of the non-traded asset a is over-estimated by 50%. In this case when model estimation errors are destructive for the classical strategy as the agent

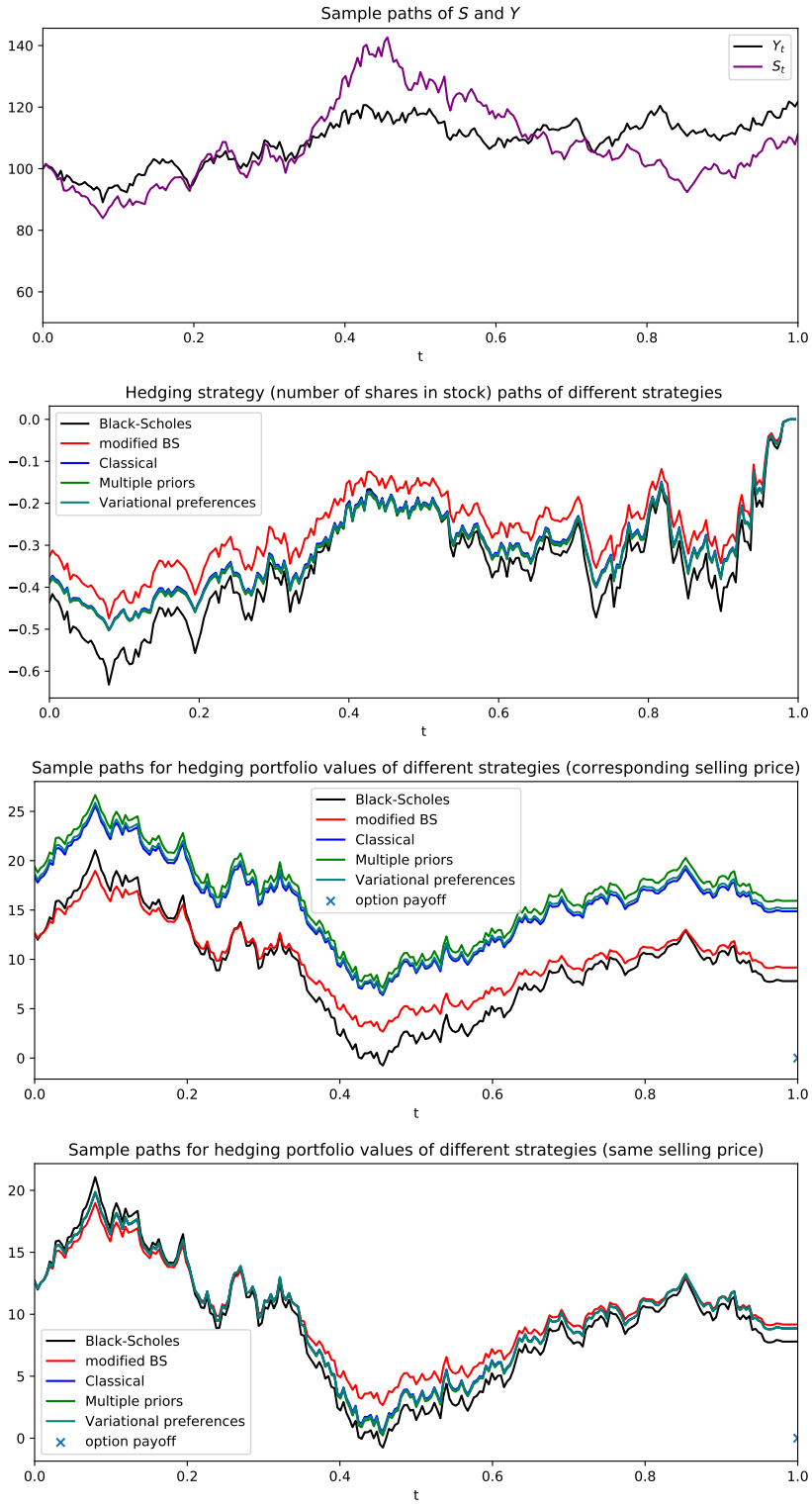


Figure 5.1: Basis risk model: Asset prices, hedging strategies (number of shares of stocks to hold), hedging portfolio values for one simulation for testing scheme 1 when drift μ is over-estimated by 50% and a is under-estimated by 50%.

	Black-Scholes	modified BS	classical	multiple priors	variational preferences
price	12.66	12.66	12.66	12.66	12.66
$\min(\mathcal{E}_T)$	-61.39	-49.47	-48.60	-48.43	-48.50
$\max(\mathcal{E}_T)$	48.53	39.72	42.35	42.65	42.48
$\text{mean}(\mathcal{E}_T)$	-0.86	-0.01	-0.28	-0.32	-0.29
$\text{med}(\mathcal{E}_T)$	0.18	1.80	0.90	0.78	0.85
$\text{sd}(\mathcal{E}_T)$	10.30	9.56	9.60	9.62	9.61

Table 5.8: Basis risk model: Statistics for terminal hedging error for testing scheme 1 when drift μ is over-estimated by 50% and a is under-estimated by 50%. The option is assumed to be traded at the Black-Scholes price for all strategies.

believes the put option is less likely to end in the money thus sells the option at a much lower price than the Black-Scholes price, the robust strategies are helpful by being more conservative and selling the option at a higher price. However, if the option is sold at the Black-Scholes price for all strategies (Table 5.10), the classical strategy and the robust strategies indeed give more favourable terminal hedging error distributions than the modified Black-Scholes and naive Black-Scholes strategies. The completely reverse order of the performances of these strategies when the option is sold at different or the same prices reflects how much the selling price is affecting the terminal hedging results.

Figure 5.2 shows example asset price paths, hedging strategy (number of shares in stock) paths as well as hedging portfolio values (both with the corresponding selling price and with the same Black-Scholes price) of different strategies, corresponding to the scenarios in Tables 5.9 and 5.10. The classical strategy prices the option very aggressively because of the ruinous drift estimation errors in this case. This is corrected (to some extent) by the more conservative robust strategies, and they all short fewer shares of stocks (than the modified BS strategy) to hedge the short position in the option – which indeed generates the highest terminal hedging portfolio values when the pricing effect is eliminated.

In summary, we can see that for the true parameter values as in Table 5.1, for the parameter estimation errors studied in this section, when the option is sold at the corresponding indifference price for each strategy, the strategy with the highest selling price always gives the most favourable distributions of terminal hedging errors. This is usually the multiple priors strategy, except when the model estimation error has made the agent under-estimating the actual risk and lowering the indifference prices significantly, in which case the modified Black-Scholes strategy gives the best

	Black-Scholes	modified BS	classical	multiple priors	variational preferences
price	12.66	12.66	8.74	9.47	8.92
$\min(\mathcal{E}_T)$	-55.31	-46.26	-50.77	-49.94	-50.56
$\max(\mathcal{E}_T)$	44.17	34.73	28.23	29.52	28.62
$\text{mean}(\mathcal{E}_T)$	-0.86	0.02	-3.85	-3.14	-3.68
$\text{med}(\mathcal{E}_T)$	0.16	1.85	-1.50	-0.90	-1.37
$\text{sd}(\mathcal{E}_T)$	10.28	9.53	9.63	9.60	9.62

Table 5.9: Basis risk model: Statistics for terminal hedging error for testing scheme 1 when drift μ is under-estimated by 50% and a is over-estimated by 50%. The option is assumed to be traded at the corresponding price for each strategy.

	Black-Scholes	modified BS	classical	multiple priors	variational preferences
price	12.66	12.66	12.66	12.66	12.66
$\min(\mathcal{E}_T)$	-59.26	-50.55	-49.79	-49.81	-49.78
$\max(\mathcal{E}_T)$	51.07	37.88	34.86	35.56	35.14
$\text{mean}(\mathcal{E}_T)$	-0.88	-0.01	0.24	0.18	0.22
$\text{med}(\mathcal{E}_T)$	0.13	1.81	2.60	2.42	2.54
$\text{sd}(\mathcal{E}_T)$	10.31	9.58	9.69	9.65	9.67

Table 5.10: Basis risk model: Statistics for terminal hedging error for testing scheme 1 when drift μ is under-estimated by 50% and a is over-estimated by 50%. The option is assumed to be traded at the Black-Scholes price for all strategies.

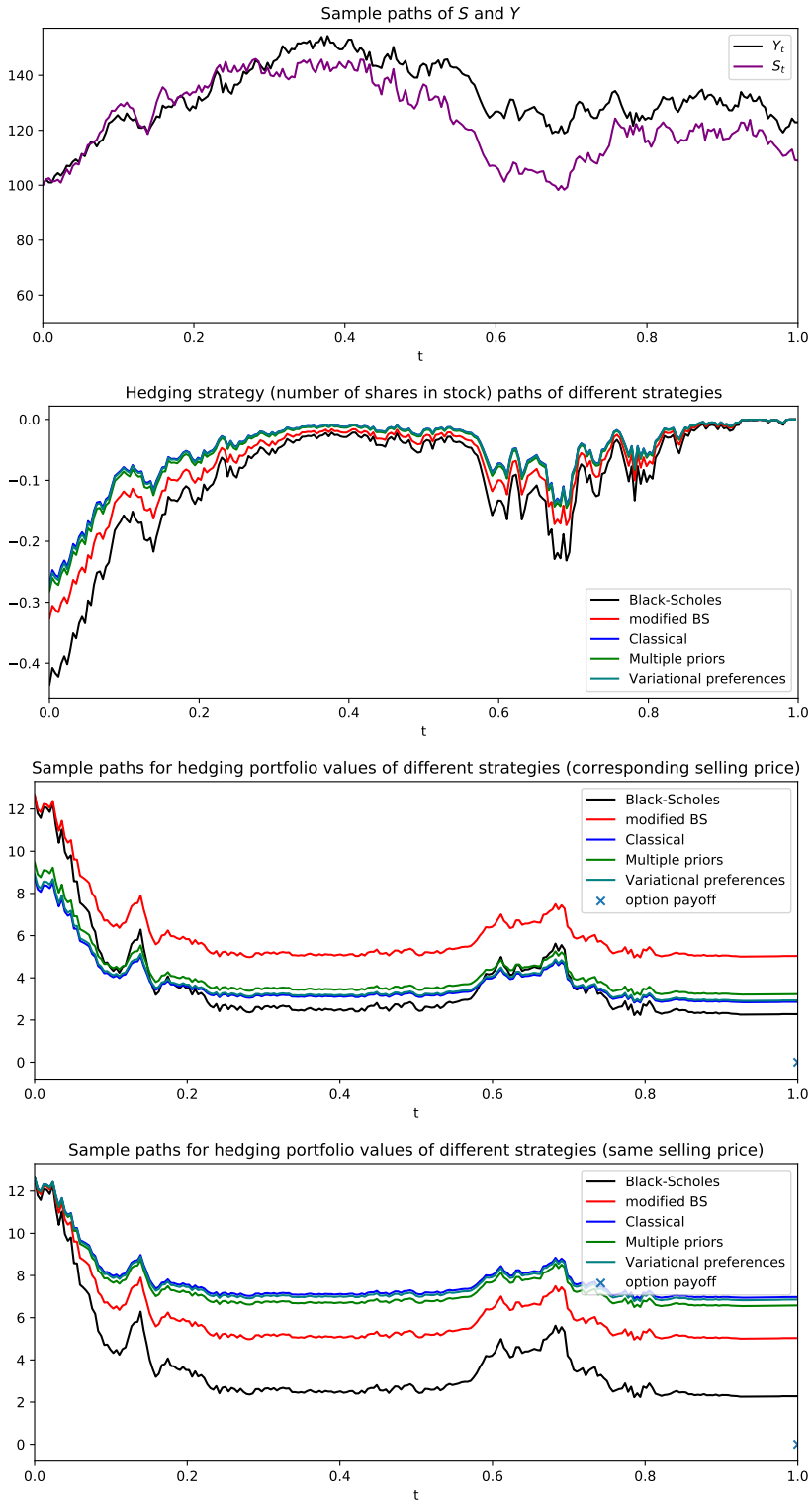


Figure 5.2: Basis risk model: Asset prices, hedging strategies (number of shares of stocks to hold), hedging portfolio values for one simulation for testing scheme 1 when drift μ is under-estimated by 50% and a is over-estimated by 50%.

hedging results. It seems that the selling price is the dominating factor affecting the terminal hedging errors, hence here we always have the order of mean/median of the terminal hedging errors of different strategies in line with the order of their selling prices from high to low.

Then, when we eliminate the effect of different selling prices, in terms of terminal hedging error distributions, in most cases of model estimations errors (except when traded asset drift μ is under-estimated by 50% and non-traded asset drift a is over-estimated by 50%), the modified Black-Scholes strategy gives the best hedging results, then the classical strategy and the variational preferences strategy. This makes sense since when it is impossible to estimate the drift parameters with any certainty with a few years' data, the modified Black-Scholes strategy, which is the naive Black-Scholes strategy adjusted by the imperfect correlation, provides the hedging that is independent of the drift parameters, thus independent of the drift estimation errors.

Scheme 2: 1 estimation + N tests.

While scheme 1 is convenient for controlling the direction and extent of model estimation errors and observing the performances of different strategies, in reality, the agent will not be able to "control" how wrong the estimates are. We now conduct a more realistic test of the strategies, assuming the agent observes some sample paths of S, Y in the estimation period and estimates the model parameters from the sample paths. The performances of different strategies are then tested over a large number of sample paths in the test period.

In this scheme, we will generate one pair of sample paths for S and Y in the estimation period of the same length T as the test period, using initial prices that are chosen randomly. Supposing we have $M+1$ equally-spaced observations in interval $[0,T]$, that is, S_i, Y_i at $t_i = iT/M$, $i = 0, 1, ..., M$, we use the quadratic variation and co-variation of logarithmic prices to estimate the volatilities and the correlation, and the sample mean of the logarithmic prices to estimate the drifts, which yields the following estimates for the model parameters:

$$\sigma^* = \sqrt{\frac{1}{T}\sum_{i=1}^{M}(\ln S_i - \ln S_{i-1})^2}, \qquad b^* = \sqrt{\frac{1}{T}\sum_{i=1}^{M}(\ln Y_i - \ln Y_{i-1})^2},$$

$$\rho^* = \frac{1}{T\sigma^* b^*}\sum_{i=1}^{M}(\ln S_i - \ln S_{i-1})(\ln Y_i - \ln Y_{i-1}),$$

$$\mu^* = \frac{1}{2}(\sigma^*)^2 + \frac{1}{T}(\ln S_T - \ln S_0), \qquad a^* = \frac{1}{2}(b^*)^2 + \frac{1}{T}(\ln Y_T - \ln Y_0).$$

The true model parameters and risk/ambiguity aversion parameters are chosen as in Tables 5.1 and 5.2. In the following, we show several different test cases when the extent of the parameter estimations errors is uncontrolled, where we can see how wrong the estimates for the drifts can be. These cases illustrate that depending on the extent of estimation errors, even with the same direction of estimation errors, different results can occur from those in scheme 1.

Table 5.11 shows one possibility of the things that can happen when both drifts of traded and non-traded assets are over-estimated apart from the controlled case in scheme 1 (Table 5.3). In this case, μ is over-estimated by 437%, and a is over-estimated by 595%. The classical and robust strategies all suffer because of their significant under-pricing of the option. Although the robust strategies are more conservative and price the option at a higher value, it is still difficult to offset the significant under-pricing caused by the severe drift estimation error.

	μ	σ	a	b	ρ
estimate	0.698	0.30	0.695	0.25	0.75
	Black-Scholes	modified BS	classical	multiple priors	variational preferences
price	12.60	12.60	3.61	4.02	3.69
$\min(\mathcal{E}_T)$	-58.25	-49.22	-61.55	-60.89	-61.37
$\max(\mathcal{E}_T)$	46.35	36.15	18.74	19.85	19.07
$\text{mean}(\mathcal{E}_T)$	-0.90	-0.01	-8.70	-8.33	-8.64
$\text{med}(\mathcal{E}_T)$	0.14	1.84	-5.54	-5.24	-5.51
$\text{sd}(\mathcal{E}_T)$	10.28	9.52	10.21	10.12	10.18

Table 5.11: Basis risk model: Statistics for terminal hedging error for testing scheme 2 when both drifts μ and a are over-estimated. The option is assumed to be traded at the corresponding price for each strategy.

In order to be able to eliminate the effect of different selling prices (thus different initial wealth of the hedging portfolios) from testing the performances of the hedging, in Table 5.12 and 5.13, we fix the seed for the random number generator and show one scenario when both drifts are under-estimated. When the option is sold at different prices, the selling price is still the dominating factor in determining the relative terminal hedging results. However, note that under this scenario, although with a lower initial wealth in the hedging portfolio, the multiple priors strategy has a better hedging result than the naive Black-Scholes strategy if we use the median instead of the mean of the terminal hedging error as the standard for comparison. When we eliminate the effect of different selling prices and assume the option is sold at the

Black-Scholes price for all strategies, the drift estimation error in this scenario is beneficial to the classical and the robust hedging strategies, which give better hedging results than the drift-error-independent modified Black-Scholes strategy.

	μ	σ	a	b	ρ
estimate	-0.429	0.30	-0.194	0.25	0.75
	Black-Scholes	modified BS	classical	multiple priors	variational preferences
price	12.66	12.66	10.29	11.09	10.49
$\min(\mathcal{E}_T)$	-62.72	-50.98	-53.23	-52.49	-53.05
$\max(\mathcal{E}_T)$	44.08	33.99	29.85	31.34	30.32
$\text{mean}(\mathcal{E}_T)$	-0.92	-0.04	-2.39	-1.61	-2.20
$\text{med}(\mathcal{E}_T)$	0.12	1.77	-0.29	0.39	-0.14
$\text{sd}(\mathcal{E}_T)$	10.29	9.53	9.58	9.55	9.57

Table 5.12: Basis risk model: Statistics for terminal hedging error for testing scheme 2 when both drifts μ and a are under-estimated. The option is assumed to be traded at the corresponding price for each strategy.

	μ	σ	a	b	ρ
estimate	-0.429	0.30	-0.194	0.25	0.75
	Black-Scholes	modified BS	classical	multiple priors	variational preferences
price	12.66	12.66	12.66	12.66	12.66
$\min(\mathcal{E}_T)$	-62.72	-50.98	-50.74	-50.84	-50.78
$\max(\mathcal{E}_T)$	44.08	33.99	32.34	32.99	32.59
$\text{mean}(\mathcal{E}_T)$	-0.92	-0.04	0.10	0.05	0.08
$\text{med}(\mathcal{E}_T)$	0.12	1.77	2.20	2.05	2.14
$\text{sd}(\mathcal{E}_T)$	10.29	9.53	9.58	9.55	9.57

Table 5.13: Basis risk model: Statistics for terminal hedging error for testing scheme 2 when both drifts μ and a are under-estimated. The option is assumed to be traded at the Black-Scholes price for all strategies.

Scheme 3: N (estimation + test)s.

As testing schemes 1 and 2 have showed, for a fixed model, depending on the different scenarios (for example, the direction and extent of the model estimation errors, and whether the option is sold at the corresponding prices), the estimation error can be beneficial or destructive for the classical and the robust strategies. In reality, it is difficult for the agent to know beforehand what scenario she is in. Thus we develop another possible approach of testing the average performances of the hedging

strategies, where we take into account that the agent can have all kinds of estimation errors and explore the average performances of different strategies over a large number of realisations.

The process is similar to scheme 2, except that the estimation-testing procedure is integrated: we generate one pair of sample path in the estimation period, estimate the model parameters, and then test the strategies over one pair of sample paths in the test period. The whole estimation-testing process is repeated N ($N = 10^5$) times to get the distribution of terminal hedging errors. This scheme is capable of evaluating how the agent does with each strategy on average.

Tables 5.14 and 5.15 show the results for testing scheme 3 when the option is sold at the corresponding prices or at the same (Black-Scholes) price for different strategies. The true model parameters are chosen as in Table 5.1, while the risk/ambiguity aversion parameters are chosen as in Table 5.2. We can see that in our case, when the option is sold at different prices, the most conservative multiple priors pricing and hedging strategies have the best performances, then the variational preferences strategies and the classical strategies. While when the option is sold at the same price for all strategies, the drift-error-independent modified Black-Scholes strategy and the less conservative classical strategy indeed give the most favourable terminal hedging error distributions, then the less ambiguity averse variational preferences strategy. The naive Black-Scholes strategy always gives the worst hedging results amongst the five compared strategies, since it completely ignores the fact that the two assets are not perfectly correlated.

	Black-Scholes	modified BS	classical	multiple priors	variational preferences
$\min(\mathcal{E}_T)$	-53.79	-48.25	-53.91	-53.35	-53.79
$\max(\mathcal{E}_T)$	49.43	40.07	60.08	61.21	60.40
$\text{mean}(\mathcal{E}_T)$	-0.84	0.05	1.87	2.71	2.10
$\text{med}(\mathcal{E}_T)$	0.14	1.82	2.07	2.78	2.25
$\text{sd}(\mathcal{E}_T)$	10.24	9.49	12.85	12.99	12.90

Table 5.14: Basis risk model: Statistics for terminal hedging error for testing scheme 3. The option is assumed to be traded at the corresponding price for each strategy.

As a final note for scheme 3, we mention that how the strategies perform on average also depends on the true model. According to our tests using scheme 3 on various models, it might occur that the drift-error-independent modified Black-Scholes strategy does not do so well when the option is sold at the same price and is the one that benefits from higher selling prices.

	Black-Scholes	modified BS	classical	multiple priors	variational preferences
$\min(\mathcal{E}_T)$	-53.81	-47.95	-50.56	-50.58	-50.57
$\max(\mathcal{E}_T)$	41.60	34.51	38.02	38.59	38.28
$\text{mean}(\mathcal{E}_T)$	-0.87	0.01	0.03	-0.02	0.01
$\text{med}(\mathcal{E}_T)$	0.14	1.80	1.67	1.55	1.63
$\text{sd}(\mathcal{E}_T)$	10.26	9.53	9.75	9.74	9.75

Table 5.15: Basis risk model: Statistics for terminal hedging error for testing scheme 3. The option is assumed to be traded at the Black-Scholes price for all strategies.

From our numerical tests using schemes 1-3, in summary, we can see that in this incomplete market, the safest and most effective protection against unhedgeable risk is selling the option at a higher price, which will certainly put the seller at an uncompetitive position. With the same selling price, in many cases (Table 5.4, 5.6, 5.8, 5.15), the modified Black-Scholes strategy, which is independent of the drift estimation error, indeed yields the most favourable terminal hedging error distributions. The naive Black-Scholes strategy is not as ideal because of its negligence of the imperfect correlation between the traded and non-traded assets. The classical strategy also yields preferable hedging results when the pricing effect is excluded. When the agent does sell the option at the indifference price, the pricing effect caused by model estimation errors can sometimes be ruinous for the classical strategy (Table 5.9 and 5.11), in which case the robust strategies are helpful in correcting the aggressive pricing to an extent dependent on how ambiguity averse the agent is. Finally, the robust strategies need to be treated with great caution. The variational preferences strategy is helpful in the sense that in all cases we've tested, no matter whether the option is sold at the corresponding indifference price or at the same price for each strategy, it always guarantees intermediate hedging results. It also provides limited protection when the classical strategy is being too aggressive and under-prices the option (Table 5.9, 5.11 and 5.12).

Chapter 6

Conclusions

This thesis studied several optimal investment, valuation and hedging problems under model uncertainty, and performed simulation-based numerical tests comparing the performances of the resulting robust strategies with some classical strategies.

First, in Chapter 2, under a generic probability measure, we have derived martingale distortion representations for utility maximisation problems in a non-Markovian stochastic factor model which comprises a single stock whose drift and volatility processes are adapted to the filtration generated by a Brownian motion correlated with the Brownian motion driving the stock. The value process has a multiplicative decomposition for the power utility case and the exponential utility case with a random endowment, and an additive decomposition for the logarithmic utility case.

Chapter 3 then considered robust power utility maximisation problems, when the agent considers the worst case within a range of models with the same feature as the reference model which is a non-Markovian stochastic factor model. We showed that semi-explicit representations in terms of the solution to a BSDE can be obtained for the value process and optimal strategy, using the martingale distortion approach of Chapter 2. This then reduced to PDE representations in the Markovian case.

In Chapter 4, we studied logarithmic utility maximisation problems, when the agent considers all probability measures equivalent to the reference measure and penalises each model with an additive entropic penalty function. We obtained semi-explicit results for the non-Markovian stochastic factor model, and fully explicit results when specialised to the Stein-Stein and Heston stochastic volatility models.

We also performed numerical tests based on simulated paths, comparing the average achievable terminal utilities of the robust strategy, the classical strategy as well as the re-calibrated Merton strategy. In the cases when the agent has over-estimated the return of the stock or under-estimated it in an unfavourable direction, the robust strategy performed better amongst the strategies compared, giving a more favourable

distribution of terminal utilities achieved. From the numerical results, we also learnt that for the (reasonable) parameter choices in the tests, the robust strategy essentially scales the classical strategy with a factor $1/(1+\beta)$ dependent on the agent's ambiguity aversion parameter β. This parameter β could be "chosen" wisely so that the robust strategy indeed coincides with the optimal strategy where the agent knows the exact values of the model parameters. Future research involving machine learning could be possible to figure out the best way for choosing this ambiguity aversion parameter based on how trustworthy the estimated return parameter (or/and other key parameters) has been for this agent.

In the final part of Chapter 4, we conducted a simple experiment when the agents' implemented model differed from the true data-generating model (although they were both stochastic volatility models). The robust SS strategy also performed better than the classical SS strategy and the Merton strategy when we assumed the true model was a Heston model while the robust SS and classical SS agents were using a Stein-Stein model with naively estimated (but also dynamically updated) model parameters.

Finally, in Chapter 5, we studied the effect of the robust approach on the classical exponential hedging problem (as described earlier in Chapter 2) for a basis risk model, in which a claim on a non-traded asset is hedged using a correlated traded stock. We first studied the case when the agent considers the worst case within a range of non-Markovian stochastic factor models, and obtained semi-explicit results for the value process and optimal strategy in terms of the solution of a BSDE. The multiple priors indifference selling price and hedging strategy for the short position in one option were also expressed via solutions to BSDEs.

We then considered the case of a Markovian version of the basis risk model, when the agent considers all probability measures equivalent to the reference measure, and penalises each measure using a multiplicative penalty function dependent on the distance of the candidate measure to the reference measure. The robust value function, optimal strategies, least favourable model as well as the variational preferences indifference price and hedging strategies were represented via solutions to linear PDEs.

In the last part of the chapter, we specialised to the constant parameter basis risk model, where the log-price processes have constant drifts and volatilities. We were therefore able to solve the robust problems explicitly, write down analytic solutions or approximations of the robust indifference prices and hedging strategies, and perform numerical tests over simulated paths to compare distributions of terminal hedging errors of various strategies.

It turned out that for the constant parameter basis risk model, by considering the worst case amongst models characterised by integrands in a fixed rectangular set, the agent would always believe that the traded stock she uses to hedge her position in the put option does not have a clear upwards/downwards trend, and the drift of the non-traded underlying asset of the option is against her position in the option – when she has a long (short) position in the option, the drift would be as large (small) as it can get. The multiple priors selling price of one option could then be proved to be the highest price amidst the indifference prices under all possible models considered by the agent. In fact, from the power series expansions of the indifference pricing functions, we could see that the multiple priors indifference price and hedging strategy of the put option only add one extra degree of freedom to the classical indifference price and hedging strategy. Essentially, the multiple priors approach reduces the mean of the log-price of the non-traded asset used in the pricing and hedging functions. The preference-adjusted exponential (that is, utility process) of the resulting value process of the overall position for the agent hedging her short position in one put option using the multiple priors approach, compared with the classical result under no model uncertainty, turned out to be a sub-martingale under the reference measure, but remained a martingale under the least favourable measure.

With a similar spirit, the result of considering the least favourable model when the agent penalises all equivalent probability measures with a multiplicative penalty function under the constant parameter basis risk model is that the agent would scale the MPR of the traded stock by a factor $1/(1+\beta)$, where $\beta > 0$ is the ambiguity aversion parameter. The effect on the drift of the non-traded asset is also always against the agent's position in the option. The variational preferences indifference price and hedging strategy also only add one extra degree of freedom to the classical strategy with no model uncertainty, modifying the risk aversion parameter α of the exponential utility to a more conservative value $\alpha^{\text{vp}} = (1+\beta)\alpha$. The residual risk process, in this case, is a sub-martingale under the reference measure.

Finally, in the numerical tests, we compared the distributions of terminal hedging errors for the Black-Scholes strategy, the modified Black-Scholes strategy (which multiplies the correlation between the traded and non-traded assets to the number of stocks to hold to hedge the option), the classical indifference pricing and hedging strategy, the multiple priors strategy, and the variational preferences strategy. The results indicated that in the case when the option is sold at the corresponding indifference prices for each strategy, the robust strategies are helpful when the parameter estimation errors are ruinous for the classical strategy, by selling the option

at higher prices. Indeed, according to our experiments, selling the option at a higher price is the most effective way to protect the agent against the unhedgeable risk in this incomplete market, which would make the agent uncompetitive in the market. In the end, when the option is sold at the Black-Scholes price for all strategies, in many cases, the modified Black-Scholes strategy, which does not require estimation of the drift parameter, performs the best amongst all strategies compared. This again showed how much impact the mis-estimation of the drift parameter, which is unlikely to be any close to accurate with the available data, has on the performances of the strategies dependent on it.

Appendix A

Classical exponential utility maximisation with random endowment

Derivation of Proposition 2.4.3. To verify the process u given by (2.4.10) is the value process, it is only needed to check that it is a $(\mathbb{P}^0, \mathbb{F})$-super-martingale for any strategy π, and a $(\mathbb{P}^0, \mathbb{F})$-martingale for strategy $\widehat{\pi}$.

First rewrite u in terms of the $(\widetilde{\mathbb{P}}^0, \widetilde{\mathbb{F}})$-martingale M defined in (2.4.6):

$$u_t = -\mathrm{e}^{-\alpha X_t}\left(\widetilde{\mathbb{E}}^0\left[\exp\left(-\frac{1}{2\delta}\int_t^T \lambda_s^2\,\mathrm{d}s + \frac{\alpha}{\delta}G_T\right)\,\middle|\,\widetilde{\mathcal{F}}_t\right]\right)^{\delta} = -\mathrm{e}^{-\alpha X_t}\mathrm{e}^{N_t}M_t^{\delta},$$

where $N_t := \frac{1}{2}\int_0^t \lambda_s^2\,\mathrm{d}s$, $t \in [0,T]$. Applying Itô's formula to u_t yields

$$\mathrm{d}u_t = u_t\left(-\alpha\pi_t\sigma_t\,\mathrm{d}W_t^0 + \delta\xi_t\,\mathrm{d}\widetilde{W}_t^0\right) + u_t f(\pi_t)\,\mathrm{d}t,$$

where

$$f(\pi_t) = \frac{\alpha^2}{2}\sigma_t^2\pi_t^2 - \alpha\lambda_t\sigma_t\pi_t - \alpha\rho\delta\xi_t\sigma_t\pi_t + \frac{1}{2}\lambda_t^2 + \rho\delta\xi_t\lambda_t + \frac{\delta(\delta-1)}{2}\xi_t^2.$$

Note that $f(\pi)$ is quadratic with respect to π, and $\frac{\alpha^2}{2}\sigma_t^2 > 0$, thus it achieves its minimum value, zero, at

$$\widehat{\pi}_t = \frac{\lambda_t + \rho\delta\xi_t}{\alpha\sigma_t}.$$

Solving for u_t gives

$$u_t = u_0\exp\left(\int_0^t f(\pi_s)\,\mathrm{d}s\right)\mathcal{E}\left(\int_0^{\cdot}(\rho\delta\xi_s - \alpha\pi_s\sigma_s)\,\mathrm{d}W_s^0 + \int_0^{\cdot}\overline{\rho}\delta\xi_s\,\mathrm{d}W_s^{0,\perp}\right)_t.$$

To check that u is a $(\mathbb{P}^0, \mathbb{F})$-super-martingale for any strategy π, and a martingale for strategy $\widehat{\pi}$, first calculate that

$$\begin{aligned}\mathbb{E}^0[u_t] &= u_0 \mathbb{E}^0\left[\exp\left(\int_0^t f(\pi_s)\,\mathrm{d}s\right)\mathcal{E}\left(\int_0^{\cdot}(\rho\delta\xi_s - \alpha\pi_s\sigma_s)\,\mathrm{d}W_s^0 + \int_0^{\cdot}\overline{\rho}\delta\xi_s\,\mathrm{d}W_s^{0,\perp}\right)_t\right] \\ &\leq u_0\mathbb{E}^0\left[\mathcal{E}\left(\int_0^{\cdot}(\rho\delta\xi_s - \alpha\pi_s\sigma_s)\,\mathrm{d}W_s^0 + \int_0^{\cdot}\overline{\rho}\delta\xi_s\,\mathrm{d}W_s^{0,\perp}\right)_t\right] = u_0, \qquad \text{(A.0.1)}\end{aligned}$$

where the first inequality comes from the fact that $u_0 < 0$, $f(\pi_t) \geq 0$, $\forall\pi$ and $f(\widehat{\pi}_t) = 0$, and the second equality comes from Novikov's condition, which is guaranteed by assumptions (2.4.3) and (2.4.9). By (A.0.1), u is a $(\mathbb{P}^0, \mathbb{F})$-super-martingale for any π. Since equality can be obtained at $\pi = \widehat{\pi}$, it is a martingale for $\widehat{\pi}$.

Then it can be verified that the optimal strategy $\widehat{\pi}$ is admissible. In fact, by Hölder's inequality and the exponential moment conditions (2.4.5) on λ and (2.4.9) on ξ,

$$\begin{aligned}&\mathbb{E}^0\left[\exp\left(c\alpha^2\int_0^T\widehat{\pi}_t^2\sigma_t^2\,\mathrm{d}t\right)\right] = \mathbb{E}^0\left[\exp\left(c\alpha^2\int_0^T\frac{(\lambda_t+\rho\delta\xi_t)^2}{\alpha^2}\,\mathrm{d}t\right)\right] \\ &\leq \mathbb{E}^0\left[\exp\left(2c\int_0^T(\lambda_t^2+\rho^2\delta^2\xi_t^2)\,\mathrm{d}t\right)\right] \\ &\leq \left(\mathbb{E}^0\left[\exp\left(4c\int_0^T\lambda_t^2\,\mathrm{d}t\right)\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^0\left[\exp\left(4c\rho^2\delta^2\int_0^T\xi_t^2\,\mathrm{d}t\right)\right]\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

□

Derivation of Proposition 2.4.6. It only remains to obtain the more explicit form for process ξ in (2.4.15). First note that by Proposition 2.4.3 and definition of $v(\cdot,\cdot)$,

$$v(t, Y_t) = V_t = \mathrm{e}^{N_t/\delta}M_t, \qquad t \in [0, T],$$

where the process M is defined in (2.4.6) and $N_\cdot = \frac{1}{2}\int_0^{\cdot}\lambda^2(Y_s)\,\mathrm{d}s$. By the fact that v is well-defined and that there exists a unique strong solution to the SDE satisfied by the factor process Y, $v \in C^{1,2}([0,T]\times\mathbb{R}) \cap C([0,T]\times\mathbb{R})$. Applying Itô's formula to the $(\widetilde{\mathbb{P}}^0, \widetilde{\mathbb{F}})$-martingale M yields

$$\begin{aligned}\mathrm{d}M_t =&\ \mathrm{d}\left(\mathrm{e}^{-N_t/\delta}v(t,Y_t)\right) \\ =&\ \mathrm{e}^{-N_t/\delta}b(Y_t)\frac{\partial v}{\partial y}(t,Y_t)\,\mathrm{d}\widetilde{B}_t^0 + \mathrm{e}^{-N_t/\delta}\left\{\frac{\partial v}{\partial t}(t,Y_t) + (a(Y_t) - \rho\lambda(Y_t)b(Y_t))\frac{\partial v}{\partial y}(t,Y_t)\right. \\ &\left. + \frac{1}{2}b^2(Y_t)\frac{\partial^2 v}{\partial y^2}(t,Y_t) - \frac{1}{2\delta}\lambda^2(Y_t)v(t,Y_t)\right\}\mathrm{d}t. \qquad \text{(A.0.2)}\end{aligned}$$

The martingale property thus yields the PDE (2.4.16) satisfied by v. Equating (2.4.8) and (A.0.2), the process ξ has the representation

$$\xi_t = \xi(t, Y_t) = \frac{b(Y_t)}{v(t, Y_t)} \frac{\partial v}{\partial y}(t, Y_t), \quad t \in [0, T].$$

Thus the optimal strategy for the classical optimal investment problem (2.4.2) in the Markovian stochastic factor model (2.1.2)-(2.1.3) is (2.4.15).

□

Appendix B

Additional numerical test results for Chapter 4

B.1 Stein-Stein stochastic volatility model ((Section 4.5.1))

TEST 5: $\Theta^* = (\lambda^*, m^*, \alpha^*, b^*, \rho^*, Y^*)$**,** N **(estimation + test)s.**

Similar as for TEST 2 and TEST 3, the procedure for TEST 5 is as for TEST 4, but the estimation period and the test period are treated as integrated and repeated N times.

The results are shown in Table B.1. Compared with the results for TEST 4 in Table 4.5, although (apart from the optimal strategy) all agents did $1 \sim 2\%$ worse in terms of average terminal utilities using the scheme in TEST 5, the relative performances of the agents remain the same. The classical SS agent achieved lower (of the order 0.3%) average terminal utilities than the modified Merton agent, possibly because of the difficulty of estimating the parameters for the correct but complicated model. The robust SS agent, being more conservative, did better (4.3% higher) in average terminal utilities.

$u_{T,\text{avg}}^{\text{robustSS}}$	$u_{T,\text{avg}}^{\text{classicalSS}}$	$u_{T,\text{avg}}^{\text{modMerton}}$	$u_{T,\text{avg}}^{\text{optimal}}$
6.72 (0.62)	6.42 (0.87)	6.44 (0.84)	6.96 (0.31)

Table B.1: Stein-Stein model (TEST 5): Average terminal utilities of different strategies. Numbers in the brackets are standard deviations of the terminal utilities. True parameter values are as in Table 4.1, and ambiguity aversion parameter $\beta = 0.5$.

The reason why the agents achieve worse average terminal utilities becomes clearer when we observe the histograms for the path-by-path mean of the estimated parameters in Figures B.1 – B.4. The estimation errors for the parameter λ, which is the

key parameter determining the performances of the SS agents, become more severe. Similar results hold for the mean-reversion level m/α, the mean-reversion speed α and the volatility of volatility b. The modified Merton agent also has more diverse estimates for the MPR compared with TEST 4. The tendency to be over or under estimated for parameters α, b and ρ for the SS agents are the same, which indicates that there might be some consistent tendencies of the estimation errors in our naive estimation method.

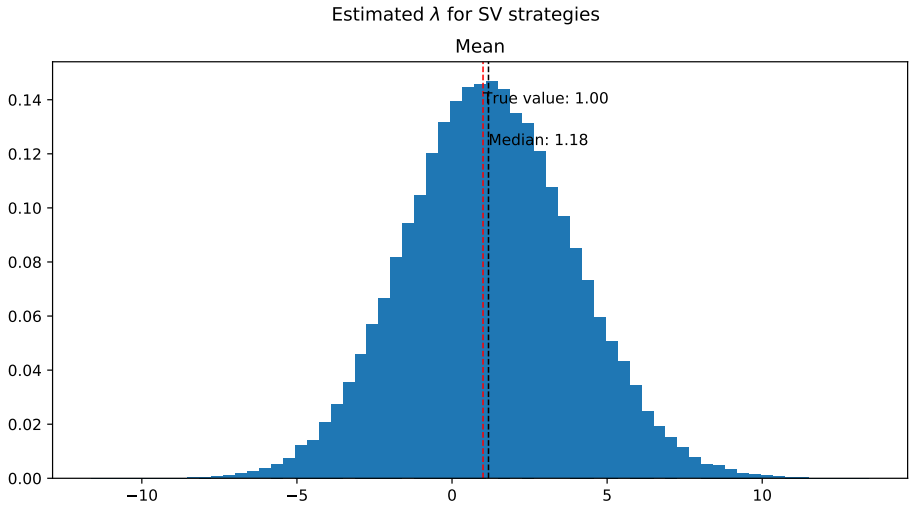


Figure B.1: Stein-Stein model (TEST 5): Histogram of the path-by-path mean for estimated λ for classical and robust SS agents.

B.2 Robustness to model error test (Section 4.5.3)

TEST 2: True model = Heston, N (estimation + test)s.

Then we use N joint estimation and test paths to allow more possibilities in the test. The procedure is similar as described in TEST 4 of the Stein-Stein example, except that now the true underlying model is a Heston stochastic volatility model.

The average terminal utilities and standard deviations are shown in Table B.2, with the histograms of terminal utilities in Figure B.5 and path-by-path additional terminal utilities over modified Merton agent in Figure B.6. From Table B.2, we can see that the relative performances of the various strategies are consistent with that in TEST 1, and this can also be confirmed by Figure B.5. Another observation is that, compared with Table 4.8, the loss of average terminal utilities caused by the different test scheme is of a lower order for the robust SS agent (1.4%) than for the classical SS agent (2.9%) and modified Merton agent (2.6%). In conjunction with the differences

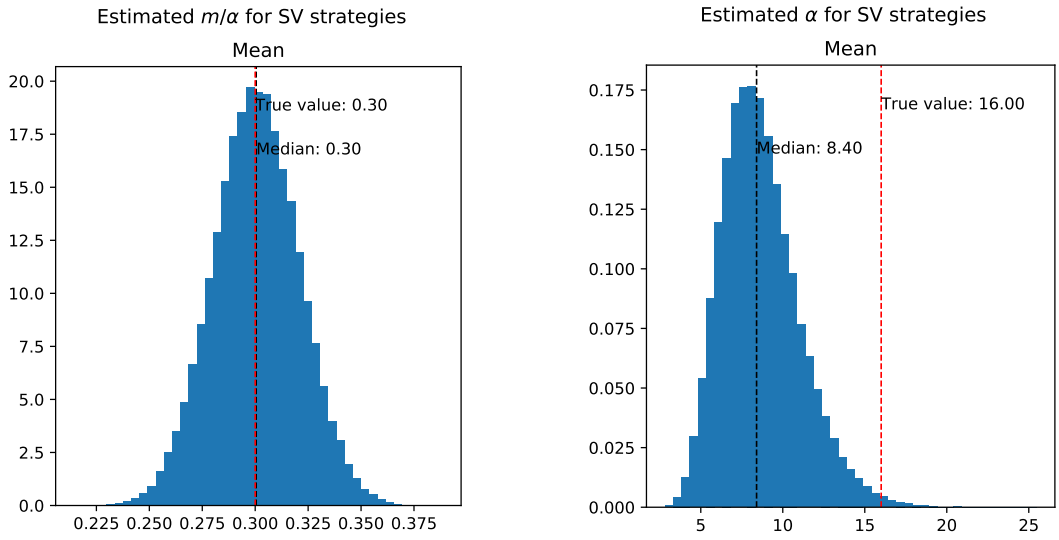


Figure B.2: Stein-Stein model (TEST 5): Histograms of the path-by-path mean for estimated m/α (mean-reversion level) and α (mean-reversion speed) for classical and robust SS agents.

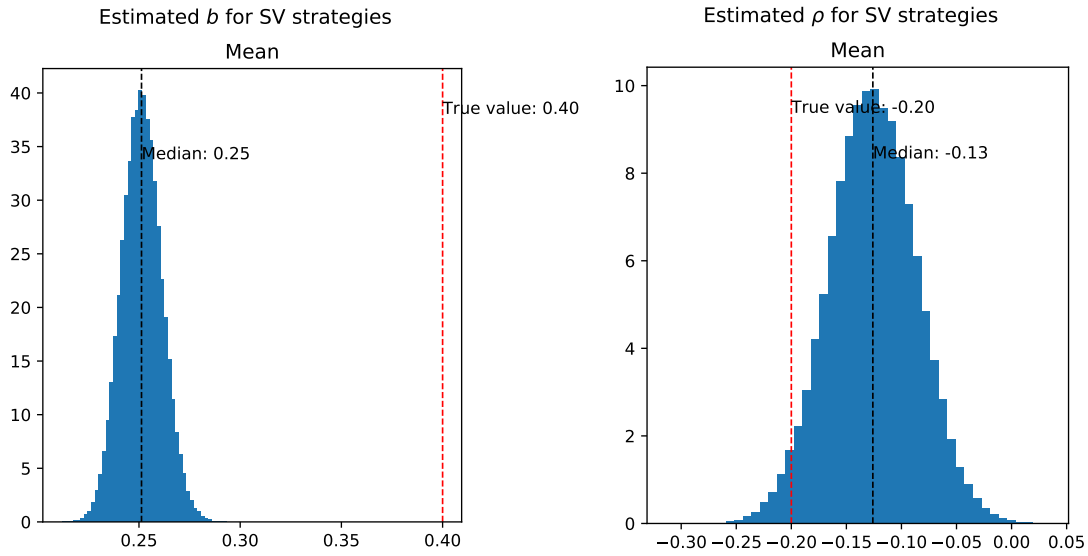


Figure B.3: Stein-Stein model (TEST 5): Histograms of the path-by-path mean for estimated b (volatility of volatility) and ρ (correlation between price and volatility) for classical and robust SS agents.

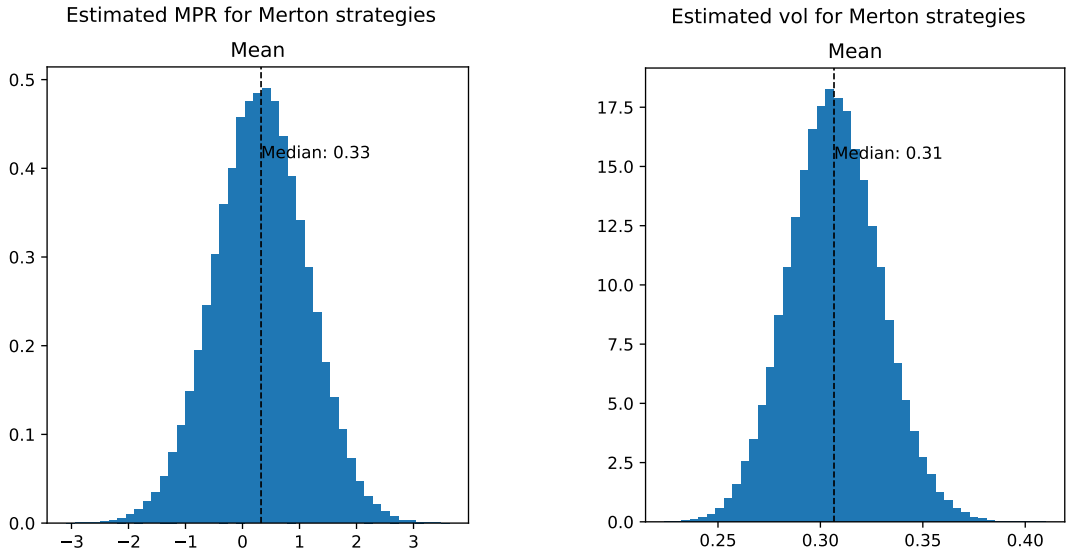


Figure B.4: Stein-Stein model (TEST 5): Histograms of the path-by-path mean for estimated MPR and volatility for the modified Merton agent.

in standard deviations between Tables 4.8 and B.2, this indicates that the robust SS strategy is most stable with respect to the variations in the procedures of our tests, while the classical SS strategy is the most unstable in these tests we've performed.

It is worth noticing that in the histograms for terminal utilities in Figure B.5, there can be extreme low values for the classical SS agent (about -19) and the modified Merton agent (about -14). This is because there are more possibilities in the repeatedly generated price and volatility paths for the estimation period, which leads to the wealth processes for the classical SS agent and the modified Merton agent touching zero in some of our experiments.

$u_{T,\text{avg}}^{\text{robustSS}}$	$u_{T,\text{avg}}^{\text{classicalSS}}$	$u_{T,\text{avg}}^{\text{modMerton}}$	$u_{T,\text{avg}}^{\text{optimal}}$
6.69 (0.64)	6.37 (0.95)	6.41 (0.87)	6.95 (0.29)

Table B.2: Robustness to model error (TEST 2): Average terminal utilities of different strategies. Numbers in the brackets are standard deviations of the terminal utilities. True parameter values are as in Table 4.6, and ambiguity aversion parameter $\beta = 0.5$.

Figure B.7 shows the histograms of the path-by-path mean of the dynamically estimated values of the return parameter λ of the stock in the experiments. We can see again that this test scheme allows for more extreme estimation errors in this key parameter, thus the poor performances of the classical SS agent. The other parameters m/α, α, b and ρ that appear in the robust SS agent's strategies are, according to Figures B.8 and B.9, still far from being accurate. However, with these

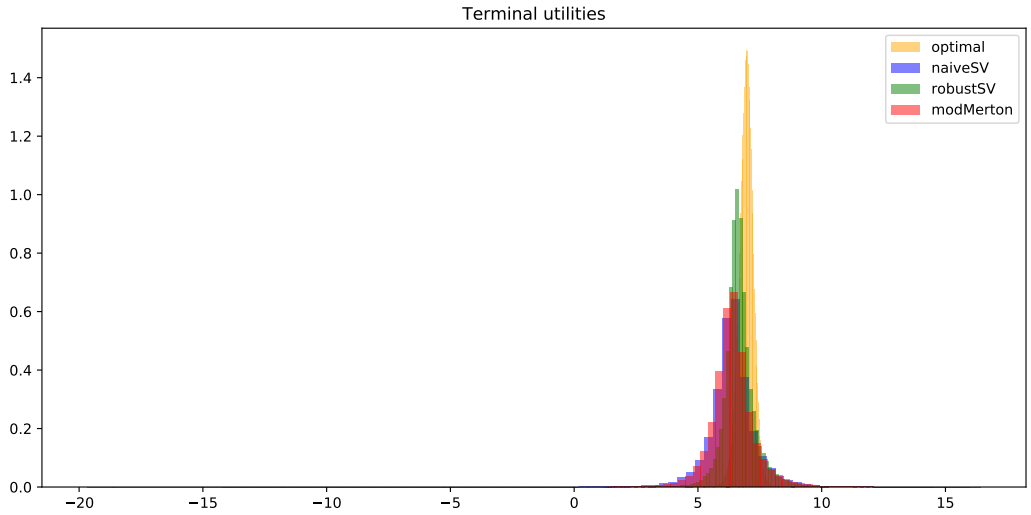


Figure B.5: Robustness to model error (TEST 2): Histograms of terminal utilities.

naive estimates for the wrong underlying model, the robust SS agent still manages to perform best amongst the strategies compared in our experiments. The modified Merton agent, as noticed in Figure B.10, also has more diverse estimates for the MPR and volatility, with a lower median for the MPR. She has also obtained better investment results, both in terms of average terminal utilities and standard deviations of the terminal utilities achieved, than the classical SS agent.

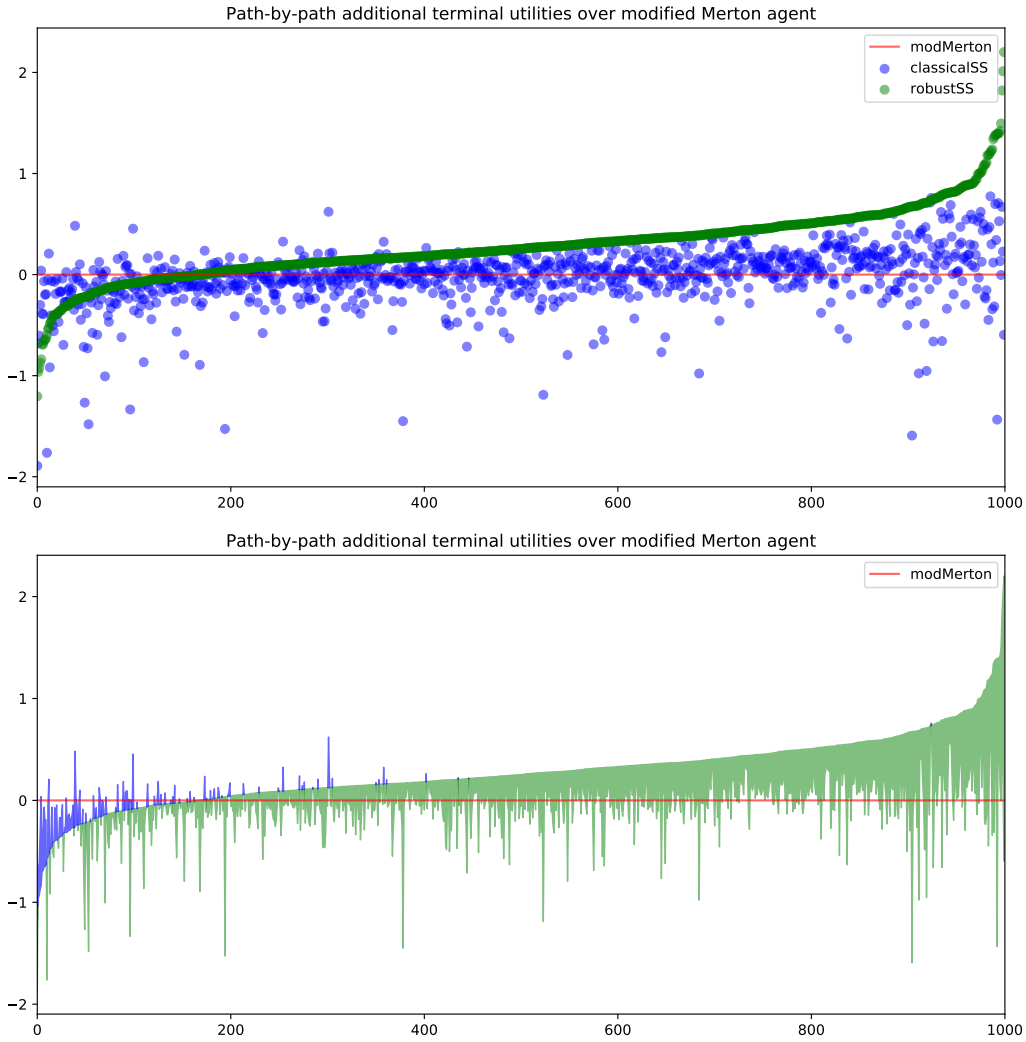


Figure B.6: Robustness to model error (TEST 2): Path-by-path additional terminal utilities over modified Merton agent (1000 realisations). Data points are re-organised in the order of increasing robust SS terminal utilities for illustration purposes. Bottom figure fills the areas where the robust SS agent performs better than the classical SS agent with green colour, and the alternative scenario with blue colour.

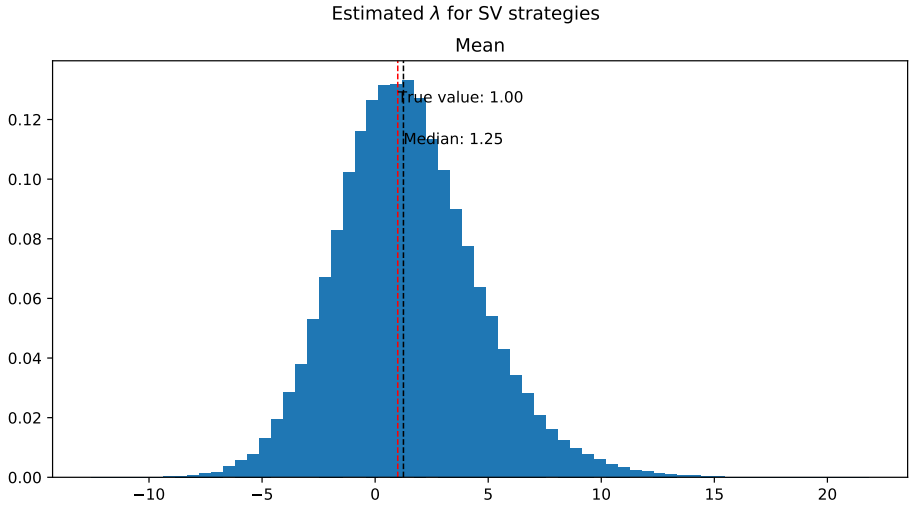


Figure B.7: Robustness to model error (TEST 2): Histogram of the path-by-path mean for estimated λ for classical and robust SS agents.

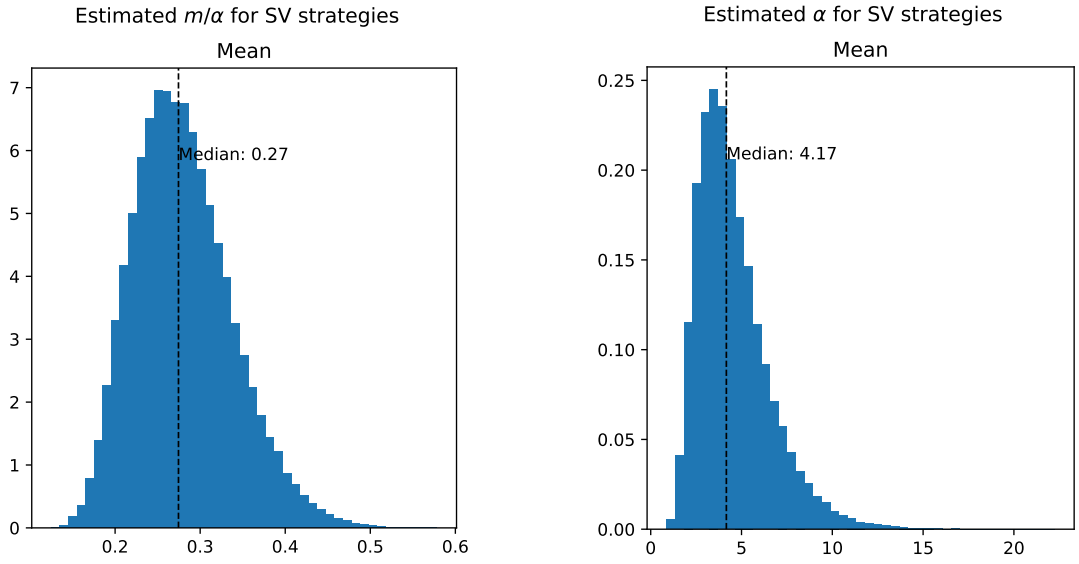


Figure B.8: Robustness to model error (TEST 2): Histograms of the path-by-path mean for estimated m/α (mean-reversion level) and α (mean-reversion speed) for classical and robust SS agents.

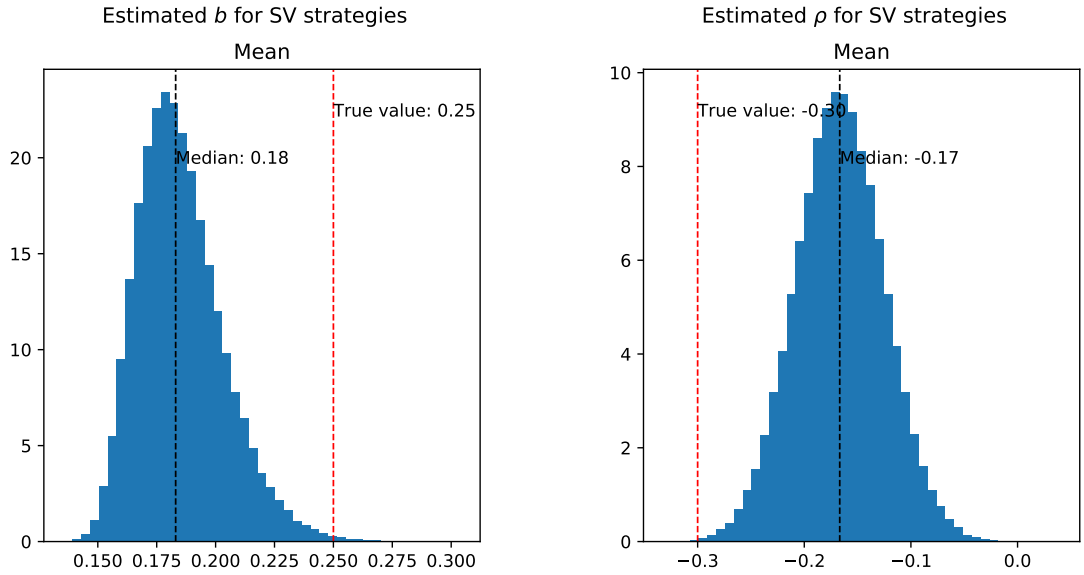


Figure B.9: Robustness to model error (TEST 2): Histograms of the path-by-path mean for estimated b (volatility of volatility) and ρ (correlation between price and volatility) for classical and robust SS agents.

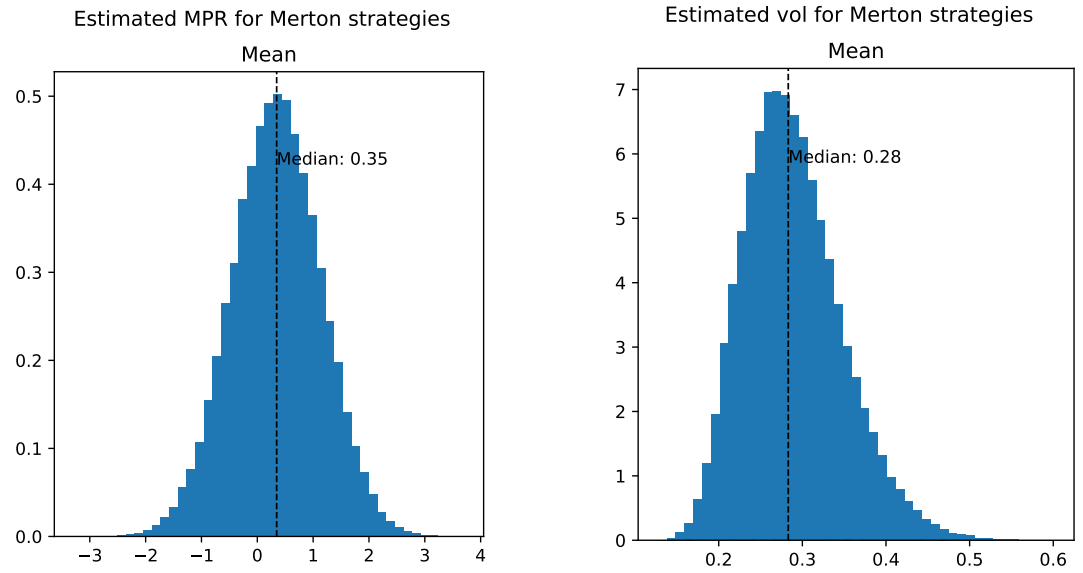


Figure B.10: Robustness to model error (TEST 2): Histograms of the path-by-path mean for estimated MPR and volatility for the modified Merton agent.

Bibliography

[1] Evan W Anderson, Lars Peter Hansen, and Thomas J Sargent. A quartet of semigroups for model specification, robustness, prices of risk, and model detection. *Journal of the European Economic Association*, 1(1):68–123, 2003.

[2] Dirk Becherer. Rational hedging and valuation of integrated risks under constant absolute risk aversion. *Insurance: Mathematics and economics*, 33(1):1–28, 2003.

[3] Dirk Becherer. Utility–indifference hedging and valuation via reaction–diffusion systems. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 460(2041):27–51, 2004.

[4] Fred Espen Benth and Kenneth Hvistendahl Karlsen. A PDE representation of the density of the minimal entropy martingale measure in stochastic volatility markets. *Stochastics An International Journal of probability and Stochastic Processes*, 77(2):109–137, 2005.

[5] Giuliana Bordigoni, Anis Matoussi, and Martin Schweizer. A stochastic control approach to a robust utility maximization problem. In *Stochastic Analysis and Applications*, pages 125–151. Springer, 2007.

[6] Zengjing Chen and Larry G Epstein. Ambiguity, risk, and asset returns in continuous time. *Econometrica*, 70(4):1403–1443, 2002.

[7] Patrick Cheridito and Ying Hu. Optimal consumption and investment in incomplete markets with general constraints. *Stochastics and Dynamics*, 11(02n03):283–299, 2011.

[8] Samuel N Cohen and Robert James Elliott. *Stochastic calculus and applications*, volume 2. Springer, 2015.

[9] Christa Cuchiero and Josef Teichmann. Fourier transform methods for pathwise covariance estimation in the presence of jumps. *Stochastic processes and their applications*, 125(1):116–160, 2015.

[10] Mark HA Davis. Optimal hedging with basis risk. 2000.

[11] Mark HA Davis. Optimal hedging with basis risk. In *From stochastic calculus to mathematical finance*, pages 169–187. Springer, 2006.

[12] Freddy Delbaen, Peter Grandits, Thorsten Rheinländer, Dominick Samperi, Martin Schweizer, and Christophe Stricker. Exponential hedging and entropic penalties. *Mathematical Finance*, 12(2):99–123, 2002.

[13] Darrell Duffie and Larry G Epstein. Stochastic differential utility. *Econometrica: Journal of the Econometric Society*, pages 353–394, 1992.

[14] Nicole El Karoui and Laurent Mazliak. *Backward stochastic differential equations*, volume 364. CRC Press, 1997.

[15] Nicole El Karoui, Shige Peng, and Marie Claire Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.

[16] Larry G Epstein and Martin Schneider. Recursive multiple-priors. *Journal of Economic Theory*, 113(1):1–31, 2003.

[17] Larry G. Epstein and Stanley E. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, 57(4):937–969, 1989.

[18] Wahid Faidi, Anis Matoussi, and Mohamed Mnif. Maximization of recursive utilities: a dynamic maximum principle approach. *SIAM Journal on Financial Mathematics*, 2(1):1014–1041, 2011.

[19] Jean-Pierre Fouque and Ruimeng Hu. Optimal portfolio under fractional stochastic environment. *Mathematical Finance*, 29(3):697–734, 2019.

[20] Itzhak Gilboa and David Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.

[21] Lars Peter Hansen and Thomas J Sargent. Robust control and model uncertainty. *American Economic Review*, 91(2):60–66, 2001.

[22] Lars Peter Hansen, Thomas J Sargent, Gauhar Turmuhambetova, and Noah Williams. Robust control and model misspecification. *Journal of Economic Theory*, 128(1):45–90, 2006.

[23] Vicky Henderson. Valuation of claims on nontraded assets using utility maximization. *Mathematical Finance*, 12(4):351–373, 2002.

[24] Daniel Hernández-Hernández and Alexander Schied. Robust utility maximization in a stochastic factor model. *Statistics & Decisions*, 24(1/2006):109–125, 2006.

[25] Daniel Hernández-Hernández and Alexander Schied. A control approach to robust utility maximization with logarithmic utility and time-consistent penalties. *Stochastic Processes and their Applications*, 117(8):980–1000, 2007.

[26] Ying Hu, Peter Imkeller, and Matthias Müller. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15(3):1691–1712, 2005.

[27] Hardy Hulley and Thomas A McWalter. Quadratic hedging of basis risk. *Journal of Risk and Financial Management*, 8(1):83–102, 2015.

[28] Ioannis Karatzas, John P Lehoczky, and Steven E Shreve. Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM Journal on Control and Optimization*, 25(6):1557–1586, 1987.

[29] Ioannis Karatzas, John P Lehoczky, Steven E Shreve, and Gan-Lin Xu. Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization*, 29(3):702–730, 1991.

[30] Dimitri Kramkov and Walter Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, pages 904–950, 1999.

[31] Dmitry Kramkov and Walter Schachermayer. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Annals of Applied Probability*, pages 1504–1516, 2003.

[32] Fabio Maccheroni, Massimo Marinacci, and Aldo Rustichini. Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74(6):1447–1498, 2006.

[33] Sure Mataramvura and Bernt Øksendal. Risk minimizing portfolios and HJBI equations for stochastic differential games. *Stochastics An International Journal of Probability and Stochastic Processes*, 80(4):317–337, 2008.

[34] Anis Matoussi, Dylan Possamaï, and Chao Zhou. Robust utility maximization in nondominated models with 2BSDE: the uncertain volatility model. *Mathematical Finance*, 25(2):258–287, 2015.

[35] Anis Matoussi and Hao Xing. Convex duality for Epstein–Zin stochastic differential utility. *Mathematical Finance*, 28(4):991–1019, 2018.

[36] Milan Merkle, Yuri F Saporito, and Rodrigo S Targino. Bayesian approach for parameter estimation of continuous-time stochastic volatility models using Fourier transform methods. *Statistics & Probability Letters*, 156:108600, 2020.

[37] Milan Merkle, Yuri F Saporito, and Rodrigo S Targino. Fourier-Bayesian estimation of stochastic volatility models. https://github.com/ysaporito/fourier-bayesian-sv-estimation, 2021.

[38] Robert C Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, pages 247–257, 1969.

[39] Robert C Merton. Optimum consumption and portfolio rules in a continuous-time model. In *Stochastic Optimization Models in Finance*, pages 621–661. Elsevier, 1975.

[40] Michael Monoyios. Performance of utility based strategies for hedging basis risk. *Quantitative Finance*, 4:245–255, 2004.

[41] Michael Monoyios. Characterisation of optimal dual measures via distortion. *Decisions in Economics and Finance*, 29(2):95–119, 2006.

[42] Michael Monoyios. The minimal entropy measure and an Esscher transform in an incomplete market model. *Statistics & Probability Letters*, 77(11):1070–1076, 2007.

[43] Michael Monoyios. Optimal hedging and parameter uncertainty. *IMA Journal of Management Mathematics*, 18(4):331–351, 2007.

[44] Michael Monoyios. Utility-based valuation and hedging of basis risk with partial information. *Applied Mathematical Finance*, 17(6):519–551, 2010.

[45] Michael Monoyios. Computing the distortion function, technical note (a short didactic note on how to compute a function which appears in the distortion solution to some investment problems). Available at https://people.maths.ox.ac.uk/monoyios/, 2020.

[46] Matthias Müller. *Market completion and robust utility maximization*. PhD thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät II, 2005.

[47] Marek Musiela and Thaleia Zariphopoulou. An example of indifference prices under exponential preferences. *Finance and Stochastics*, 8(2):229–239, 2004.

[48] Bernt Øksendal and Agnes Sulem. Portfolio optimization under model uncertainty and BSDE games. *Quantitative Finance*, 11(11):1665–1674, 2011.

[49] Keita Owari. Robust exponential hedging in a Brownian setting. *JSIAM Letters*, 1:64–67, 2009.

[50] Keita Owari. Robust exponential hedging and indifference valuation. *International Journal of Theoretical and Applied Finance*, 13(07):1075–1101, 2010.

[51] Mark P Owen. Utility based optimal hedging in incomplete markets. *The Annals of Applied Probability*, 12(2):691–709, 2002.

[52] Etienne Pardoux and Shige Peng. Adapted solution of a backward stochastic differential equation. *Systems & Control Letters*, 14(1):55–61, 1990.

[53] Huyên Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 61. Springer Science & Business Media, 2009.

[54] Philip E. Protter. *Stochastic Integration and Differential Equations*. Springer-Verlag, second edition, 2004.

[55] Marie Claire Quenez. Optimal portfolio in a multiple-priors model. In *Seminar on Stochastic Analysis, Random Fields and Applications IV*, pages 291–321. Springer, 2004.

[56] Jörn Sass and Dorothee Westphal. Robust utility maximizing strategies under model uncertainty and their convergence. *Mathematics and Financial Economics*, 16(2):367–397, 2022.

[57] Walter Schachermayer. Optimal investment in incomplete markets when wealth may become negative. *Annals of Applied Probability*, pages 694–734, 2001.

[58] Alexander Schied. Optimal investments for risk-and ambiguity-averse preferences: a duality approach. *Finance and Stochastics*, 11(1):107–129, 2007.

[59] Alexander Schied and Ching-Tang Wu. Duality theory for optimal investments under model uncertainty. *Statistics & Decisions*, 23(3/2005):199–217, 2005.

[60] Costis Skiadas. Robust control and recursive utility. *Finance and Stochastics*, 7(4):475–489, 2003.

[61] Elias M Stein and Jeremy C Stein. Stock price distributions with stochastic volatility: an analytic approach. *The review of financial studies*, 4(4):727–752, 1991.

[62] Denis Talay and Ziyu Zheng. Worst case model risk management. *Finance and Stochastics*, 6(4):517–537, Oct 2002.

[63] Michael Tehranchi. Explicit solutions of some utility maximization problems in incomplete markets. *Stochastic Processes and their Applications*, 114(1):109–125, 2004.

[64] Nizar Touzi. *Optimal stochastic control, stochastic target problems, and backward SDE*, volume 29. Springer Science & Business Media, 2012.

[65] Hao Xing. Consumption–investment optimization with Epstein–Zin utility in incomplete markets. *Finance and Stochastics*, 21(1):227–262, 2017.

[66] Hao Xing and Gordan Žitković. A class of globally solvable Markovian quadratic BSDE systems and applications. *The Annals of Probability*, 46(1):491–550, 2018.

[67] Zhou Yang, Gechun Liang, and Chao Zhou. Constrained portfolio-consumption strategies with uncertain parameters and borrowing costs. *Mathematics and Financial Economics*, Dec 2018.

[68] Thaleia Zariphopoulou. A solution approach to valuation with unhedgeable risks. *Finance and Stochastics*, 5(1):61–82, 2001.