DEPARTMENT OF ECONOMICS
OxCarre (Oxford Centre for the Analysis of Resource Rich Economies)

Manor Road Building, Manor Road, Oxford OX1 3UQ
Tel: +44(0)1865 281281 Fax: +44(0)1865 281163
reception@economics.ox.ac.uk www.economics.ox.ac.uk

UNIVERSITY OF OXFORD

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Growth, Renewables and the Optimal Carbon Tax

Frederick van der Ploeg
OxCarre

&

Cees Withagen*
VU University Amsterdam

***OxCarre Research Associate**

Direct tel: +44(0) 1865 281281 E-mail: celia.kingham@economics.ox.ac.uk

GROWTH, RENEWABLES AND THE OPTIMAL CARBON TAX*

Frederick van der Ploeg, University of Oxford and VU University Amsterdam**

Cees Withagen, VU University Amsterdam***

Abstract

Optimal climate policy is investigated in a Ramsey growth model of the global economy with exhaustible oil reserves, an infinitely elastic supply of renewables, stock-dependent oil extraction costs and convex climate damages. Four regimes can occur. If the initial social cost of oil is less than that of renewables, there are two regimes starting with oil. The first one occurs if the oil stock is not too small and not too large and the initial capital stock is below its steady state in which case it is optimal to follow the oil-only phase with a renewables-only phase. The second regime occurs if the initial oil stock is large enough. It is then optimal to follow an oil-only phase with an oil-renewables phase. If it is optimal to start with renewables, a third and fourth regime emerge. The third one occurs if the initial oil stock takes on an intermediate value and the capital stock exceeds its steady-state value. It is then optimal to start with renewables and end with a phase where oil is used alongside renewables. The fourth regime occurs if the initial oil stock is low enough. Renewables are then used throughout. We also offer some policy simulations for the first and second regime, which illustrate that with a lower discount rate more oil is left in situ and renewables are phased in more quickly. In the first regime the optimal carbon tax rises during the oil-only phase, but in the second regime the optimal carbon tax can fall. Subsidizing renewables (without a carbon tax) induces more oil to be left in situ and a quicker phasing in of renewables, but oil is depleted more rapidly initially. The net effect on global warming is ambiguous.

Keywords: Green Ramsey model, carbon tax, renewables, exhaustible resources, global warming, development, growth, intergenerational inequality aversion, second best, Green Paradox

JEL codes: D90, E13, Q30, Q42, Q54

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*** Department of Economics, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands.

1. Introduction

A substantial and possibly rising carbon tax is needed not only to curb demand for fossil fuel, but also to accelerate the switch from a carbon-based to a carbon-free economy. It does this by encouraging more oil to be left in the crust of the earth and renewables to be phased in more quickly. Subsidies for carbon-free renewables are a poor substitute for a credible path of present and future carbon taxes.[1] The optimal carbon tax should be set to the social cost of carbon, which is the present value of all future marginal damages from global warming. But the social cost of carbon is highly endogenous as it depends on the level of consumption and capital in the economy as well as on whether oil[2], renewables or both are used.

Our main objective is to study these issues within a general equilibrium framework which can be used to analyze the intricate tradeoffs between economic development and fighting global warming. For this purpose we put forward a Green Ramsey model with three distinguishing features. First, the stock of oil to be left in situ is strictly positive because oil extraction costs increase without bound as less accessible reserves are exploited and because the marginal cost of global warming increases in the stock of atmospheric carbon. Second, production depends not only on capital and labour but also on energy use where oil and renewables are supposed to be perfect substitutes in production. Oil is an exhaustible reserve which will not be fully exhausted whilst renewables are infinitely elastic supplied at constant cost. Third, a substantial part of the carbon emitted from burning oil will remain in the atmosphere forever.

Our main result is that four different regimes can occur in our model depending on the sizes of the initial stock of oil and the initial stock of capital. If the initial social cost of oil (consisting of the marginal extraction cost, the Hotelling rent and the social cost of carbon) is less than that of renewables, there are two regimes starting with oil. The first one occurs if the oil stock is not too small and not too large and the initial capital stock is below its steady state in which case it is optimal to follow the oil-only phase with a final phase where only renewables are used. The second regime emerges if the initial oil stock is large enough in which case it is optimal to follow an oil-only phase with a final oil-renewables phase. These two regimes should be contrasted with the “laissez-faire” outcome, which always has an initial oil-only phase followed by a carbon-free, renewables-only phase.

If it is optimal to start with renewables, a third and fourth regime emerge. The third one occurs if the initial oil stock takes on an intermediate value and the capital stock exceeds its steady-state value. It is then optimal to start with renewables and end with a phase where oil is used alongside renewables. The fourth regime emerges if the initial oil stock is low enough. Renewables are then used throughout.

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[1] If use of renewables is associated with learning-by-doing externalities, there is a case for a renewables subsidy as well as carbon tax.
[2] We use for sake of brevity ‘oil’ rather than ‘fossil fuel’, so oil refers to natural gas, coal and the tar sands as well.

We focus in our further analysis and policy simulations on the first and second regime which start with oil, since they are currently the relevant ones. We especially analyze the optimal timing of the transition between the phases of these two regimes and the optimal amount of oil to be left in the crust of the earth. These follow from the condition that price of fossil fuel is continuous at the time of the transition and the social cost of the last extracted barrel of oil must equal the cost of renewables. For the first regime we show that the optimal carbon tax rises along its development path during the oil-only phase, but the rise flattens off as less accessible reserves are explored and the social cost of carbon increases. Furthermore, the optimal carbon tax rises as the economy develops which echoes the prescriptions of those who argue in favor of a rising ramp for the carbon tax (e.g., Nordhaus, 2007). Using a lower discount rate implies that more oil is left in situ and renewables are phased in more quickly. Subsidizing renewables (without a carbon tax) induces oil to be depleted more rapidly initially (the Green Paradox), but also more oil to be left in situ and renewables to be phased in more quickly. The net effect on global warming is ambiguous.

In the second regime consumption and capital overshoot the steady-state values of the carbon-free economy. Compared to the first regime, oil use is higher in the early part of the initial oil-only phase and is curbed back substantially in the latter part. The oil-abundant regime phases in renewables at a much later date than the oil-scarce regime, but never phases out oil. We establish that, despite zero natural decay of atmospheric carbon, the optimal carbon tax eventually falls in the second regime if the return on capital is less than the ratio of the current marginal damage of global warming to the social cost of carbon.

In both the first and second regime the switch to renewables occurs more quickly and more oil is left in situ in a socially optimum than in a market outcome which does not internalize global warming damages. We show that more oil reserves are left in situ if the social rate of discount is low, the climate challenge is acute, the initial stock of oil reserves is high and the initial state of economic development is high.

A recent paper by Golosov et al. (2011) is close to ours. It looks at backstops in a discrete-time Ramsey growth model with global warming damages in production. Given five strong assumptions (logarithmic utility, Cobb-Douglas production, 100 per cent depreciation of capital at the end of each period, zero oil extraction costs, and a negative exponential function for multiplicative output damages), the consumption-output ratio is constant and the optimal carbon tax is proportional to output.[3] We allow in our model for stock-dependent extraction costs, more general depreciation of capital and elasticities of intertemporal substitution different from one, and formally establish the existence of four different

[3] The time path of the optimal carbon tax eventually declines, which results from the assumption of natural decay of the atmospheric concentration of CO2 and full exhaustion of oil reserves.

regimes of energy use depending on the initial stocks of oil and capital. Furthermore, we show that for the first two regimes of our model the optimal carbon tax is not a constant fraction of consumption or output, especially not in the phases where only oil is used.

Our analysis extends the famous DHSS (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974) growth model with investment in manmade capital and natural exhaustible resources as factors of production to allow for renewable backstops and global warming damages. Our analysis is also related to earlier studies on pollution in the Ramsey model (e.g., van der Ploeg and Withagen, 1991), on capital accumulation, oil depletion and backstops (e.g., Tahvonen, 1997; Tsur and Zemel, 2003, 2005), on pollution and climate change in models with depletion of exhaustible resources but without investment and growth (e.g., Krautkraemer, 1985, 1998; Withagen, 1994), and on those studying regimes with simultaneous use of oil and renewables albeit in economies without capital (Hoel and Kverndokk, 1996).

Our analysis also extends results on the second-best effects of subsidizing clean backstops on oil exhaustion, speed and duration of phasing in of the backstop, and the effects on green and total welfare (Hoel, 2008; van der Ploeg and Withagen, 2010; Grafton, Kompas and Long, 2010; Gerlagh, 2011) to allow for saving, investment and capital accumulation. We thus offer a general-equilibrium analysis of the Green Paradox (Sinn, 2008ab), which says that subsidizing clean renewables (e.g., solar or wind energy) encourages oil to be pumped more quickly and accelerates global warming. However, cheaper renewables induce a bigger fraction of oil reserves to remain unexploited and global warming damages fall. The Green Paradox then need not occur. Our simulations suggest that this paradox is less likely to occur for a mature than a developing economy which has lower marginal global warming damages. This dilemma for the early stages of development reminds one of Bertolt Brecht's dictum: 'Erst kommt das Fressen: Dann kommt die Moral' (from *Die Dreigroschenoper)*.

Section 2 presents our model and the optimality conditions for the social optimum. Section 3 derives the various phases of oil use, renewables use and simultaneous use that can occur and the pasting conditions for the different phases. Section 4 characterizes the four regimes that can occur in the social optimum and the conditions necessary to pin down the optimal stock of oil to be left in situ and the time at which renewables are phased in. Section 5 establishes the two regimes of the "laissez-faire" economy, characterizes the level and time profile of the optimal carbon tax, and discusses second-best climate policy if an optimal carbon tax is infeasible and the Green Paradox. Section 6 present simulations for the social optimum and the "laissez-faire" outcome of the first regime of our model, discusses the sensitivity with respect to the state of economic development, intergenerational inequality aversion and time preference, and sheds light on the Green Paradox. Section 7 offers policy simulations for the second regime which has a final phase where oil and renewables are used simultaneously. Section 8 concludes.

2. The Green Ramsey model

Let $O(t) \geq 0$ denote oil use and $S(t) \geq 0$ the stock of remaining oil reserves at instant of time t. Then along a feasible program we have for all $t \geq 0$:

(1) $\dot{S}(t) = -O(t), \quad S(0) = S_0,$

where $S_0 > 0$ is the given initial stock of oil reserves. Hence, total oil depletion cannot exceed initial reserves: $\int_0^\infty O(t)dt \leq S_0$. We abstract from natural degradation of CO2 in the atmosphere, so the change in the stock of atmospheric carbon E is proportional to oil depletion for all $t \geq 0$:

(2) $\dot{E} = O, \quad E(0) = E_0,$

where $E_0 > 0$ is the initial stock of atmospheric carbon and we have normalized so that the CO2-emission ratio equals one. Hence, at each instant of time the stock of CO2 in the atmosphere equals the initial stock plus the accumulated sum of past CO2 emissions: $E(t) = E_0 + \int_0^t O(t)dt = E_0 + S_0 - S(t)$.[4] [5] Manmade capital K and energy are inputs in the production process, which is described by the production function F. We suppose that energy from oil, O, and energy from renewables, R, are perfect substitutes.[6] We denote by $G(S)$ the cost needed to extract one unit of oil.

Assumption 1: $G'(S) < 0$ for all $S > 0, \lim_{S \to 0} G(S) = \infty, \lim_{S \to \infty} G(S) = 0.$

We thus assume that the cost of extracting one unit of oil rises as fewer oil reserves are left, and that oil extraction costs become infinitely large as oil reserves get fully exhausted.[7] The latter ensures that oil reserves are never fully exhausted along an optimal path. The unit cost of the renewable, b, is constant.

The material balance equation of the economy and the investment dynamics are:

(3) $\dot{K} = F(K, O+R) - G(S)O - bR - C - \delta K, \quad K(0) = K_0,$

where C is consumption, δ the depreciation rate of manmade capital, and K_0 the initial capital stock.

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[4] Our system can thus be reduced to a system with only two state variables. But we prefer to work with the atmospheric carbon stock and the oil stock separately because this helps interpreting several interesting results.

[5] It is trivial to extend the result to the case where a fraction of the carbon that is emitted stays forever in the atmosphere and the remainder is returned immediately to the surface of the oceans and the earth.

[6] Wind and solar energy are in reality not perfect substitutes for oil. We assume perfect substitutability for analytical convenience; relaxing it would make introduce regimes where the two types of fuel are used alongside each other. See Smulders and van der Werf (2008) and Michielsen (2011) for partial equilibrium studies which do allow for imperfect substitution between oil and the backstop in a partial equilibrium analysis of climate policy.

[7] Alternatively, one could define the total extracted amount of oil as a state variable. Then assumption 1 implies that total extraction will never exceed the initial oil stock (cf., Farzin, 1992). In this sense, oil is not physically scarce.

Assumption 2: F has non-increasing returns to scale. It is strictly concave and increasing for positive inputs. $F(K,0)=F(0,O+R)=0$. Also, $\lim_{O+R\to 0} F_{O+R}(K,O+R)=\infty, \lim_{O+R\to\infty} F_{O+R}(K,O+R)=0,$ for all $K>0$ [8] and $\lim_{K\to 0} F_K(K,O+R)=\infty, \lim_{K\to\infty} F_K(K,O+R)=0,$ for all $O+R>0.$

The production function thus satisfies the Inada conditions. Assumption 2, together with $\delta>0$, implies that without the use of the backstop the economy will not maintain a positive constant level of consumption (Dasgupta and Heal, 1974).

Intertemporal social welfare W depends on utility of consumption and damage from accumulated CO2:

$$(4) \qquad W=\int_0^\infty e^{-\rho t}\left[U(C(t))-D(E_0+S_0-S(t))\right]dt,$$

where $\rho>0$ is the constant rate of time preference. The instantaneous utility function, U, is concave and satisfies the Inada conditions, which ensure positive consumption throughout. Global warming damages, D, and marginal damages increase in the stock of atmospheric carbon[9].

Assumption 3: $U'(C)>0, U''(C)<0,$ for all $C>0,$ and $\lim_{C\to 0} U'(C)=\infty.$

$D'(E)>0, D''(E)>0,$ for all $E>0.$

The social planner maximizes (4) subject to (1)-(3) and the non-negativity constraints. The social cost of carbon is denoted by τ and equals the present discounted value of marginal global warming damages:

$$(5) \qquad \tau(t)\equiv\frac{\int_t^\infty e^{-\rho(s-t)}D'(E(s))ds}{U'(C(t))}.$$

It corresponds to the reduction in the capital stock resulting from keeping an extra unit of carbon in the soil rather than in the atmosphere evaluated along an optimum path (e.g., Nordhaus, 2011). Defining the elasticity of intertemporal substitution $\sigma(C)\equiv -U'(C)/CU''(C)>0$, the marginal product of energy as $p=F_{O+R}(K,O+R),$ and the net rate of return on capital as $r\equiv F_K(K,O+R)-\delta$, we get proposition 1.

Proposition 1: The social optimum satisfies (1)-(3) and the optimality conditions:

$$(6) \qquad p\le b,\ R\ge 0, \text{c.s.},$$

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[8] For the derivative of the production function with respect to energy, we use F_O, F_R and F_{O+R} interchangeably.

[9] Temperature is a concave function of accumulated CO2 emissions. So even if damages are a convex function of temperature, damages need not necessarily be a convex function of accumulated CO2 emissions. An alternative is to suppose that global warming damages production as in the RICE and DICE models of Nordhaus (2007) or in the Ramsey growth models of Golosov et al. (2011) and Gerlagh and Liski (2012).

(7) $\dot{p} = r[p - G(S)] - \dfrac{D'(E)}{U'(C)}$ if $O > 0,$

(8) $\dot{C} = \sigma(r - \rho)C,$

(9) $\dot{\tau} = r\tau - D'(E)/U'(C),$

(10) $\lim_{t\to\infty}\left[K(t) + \left(p(t) - G(S(t))\right)S(t) - \tau(t)(E_0 + S_0)\right]U'(C(t))e^{-\rho t} = 0.$

Proof: Let μ_K be the marginal social value of manmade capital, μ_S the marginal social value of oil reserves and μ_E the marginal social cost of carbon in the atmosphere. The current value Hamiltonian is:

$$H \equiv U(C) - D(E) + \mu_K\left[F(K, O+R) - C - G(S)O - bR - \delta K\right] - \mu_S O - \mu_E O.$$

Necessary conditions for optimality read:

(11a) $U'(C) = \mu_K,$

(11b) $F_R(K, O+R) \le b$ and $R \ge 0,$ c.s.,

(11c) $F_O(K, O+R) - G(S) - (\mu_S + \mu_E)/\mu_K \le 0$ and $O \ge 0,$ c.s.,

(11d) $\rho\mu_K - \dot{\mu}_K = \left[F_K(K, O+R) - \delta\right]\mu_K,$

(11e) $\rho\mu_S - \dot{\mu}_S = -G'(S)O\mu_K,$

(11f) $\rho\mu_E - \dot{\mu}_E = D'(E),$

(11g) $\lim_{t\to\infty}\left[\mu_K(t)K(t) + \mu_S(t)S(t) - \mu_E(t)E(t)\right]e^{-\rho t} = 0.$

Conditions (11a), (11b) and (11c), where c.s. means complementary slackness, follow from the maximization of the Hamiltonian with respect to consumption, renewables use and oil use, respectively. Assumption 3 implies that consumption is always positive and assumption 2 that total energy use is always positive. Equations (11d)-(11f) describe the optimal time paths of the co-state variables. Condition (11g) is the transversality condition. If $R > 0,$ then $p = F_R = b$ which gives (6). If $O > 0,$ then (11c) gives $p = G(S) + (\mu_S + \mu_E)/\mu_K$. Taking the derivative with respect to time and using (11c)-(11e) yields

$$\dot{p} = \frac{-D'(E) - (\mu_S + \mu_E)(\delta - F_K)}{\mu_K}.$$

Using (11a) and (11c) again gives (7). Equation (8) follows from (11a) and (11d). The shadow cost of atmospheric CO2, μ_E, is positive, because it measures the value in welfare terms of having a smaller CO2 stock. It follows from (11f) that $\mu_E(t) = \int_t^\infty e^{-\rho(s-t)} D'(E(s))ds$, so that τ defined in (5) equals μ_E / μ_K. This then yields (9). Given that $U'(C) = \mu_K > 0$ (from assumption 3), the transversality condition (11g) can be written as (10). Q.E.D.

Equation (6) states that the marginal product of the renewable backstop, if in use, equals its marginal cost b. To interpret (7) note from (11c) that if oil is used in the production process, its marginal product should equal total marginal costs consisting of the sum of the extraction costs $G(S)$, the marginal costs of extracting today rather than in the future, thereby causing all future extraction costs to be higher, $\mu_S(t) / \mu_K(t) = -\int_t^\infty e^{-\rho(s-t)} G'(S(s))O(s)\mu_K(s)ds / \mu_K(t)$, and the marginal cost of global warming $\tau(t) = \mu_E(t) / \mu_K(t) = \int_t^\infty e^{-\rho(s-t)} D'(E(s))ds / \mu_K(t)$ (where in the latter two terms we have converted the costs from *utility* units to *final goods* units). We thus have $p = G(S) + (\mu_S / \mu_K) + \tau$. Discounting in goods units, we can also write the social cost of carbon as:

$$(5') \qquad \tau(t) = \int_t^\infty e^{-\int_t^s r(s')ds'} D'(E(s)) / U'(C(s))ds > 0.$$

This gives a modified Hotelling rule: The return on leaving a marginal barrel in the earth $\dot{p}$ should equal the social rate of return *r* on the net revenues of depleting a marginal barrel ($p - G(S)$) minus the marginal global warming damages ($D'(E)/U'(C)$). Without global warming externalities ($D \equiv 0$) and extraction costs ($G(S) \equiv 0$), we get the familiar Hotelling rule which says that the capital gains on oil should equal the marginal product of capital. Note that the Hotelling rent on oil $p - G(S)$ vanishes if the "laissez-faire" economy relying only on oil approaches the moment in time where the renewable backstop is introduced (Heal, 1976). Equation (7) can be written as $\frac{d\left[p - G(S)\right]}{dt} = r\left[p - G(S)\right] + G'(S)O - \frac{D'(E)}{U'(C)}$, which says that the Hotelling rent on oil increases at a lower rate than the social rate of return on capital for two reasons. First, extracting more oil pushes up extraction costs, which makes oil depletion more conservative. Second, extracting more oil raises the stock of atmospheric carbon and this pushes up marginal climate damages, which makes oil depletion also more conservative.

Equation (8) is the Keynes-Ramsey rule which states that consumption growth is high, for given σ, if the return on capital is high and consumers are relatively patient. The coefficients of relative intergenerational inequality aversion and relative risk aversion equal $1/\sigma$. Consumption decreases in the social value of capital μ_K, since the marginal utility of consumption must equal the social value of capital, $U'(C) = \mu_K$.

Finally, along an optimal program the transversality condition (10) has to be satisfied.

3. The phases of optimal energy use for the Green Ramsey model

When it comes to energy use three possible phases of energy use arise: only oil, only renewables and simultaneous use of oil and renewables. We characterize the optimal program for each of these phases and discuss the conditions that need to be fulfilled to make a transition from one phase of energy use to another. This leads to a set of conditions pasting the phases. Section 4 sums up the regimes that can occur.

Since $p = F_{O+R}(K, O+R)$ by definition and the production function is well-behaved, we can write:

(12) $$O + R = V(K, p)$$

with $V_K(K, p) = -F_{KV} / F_{VV} > 0$ and $V_p = 1/F_{VV} < 0$. We also define output net of renewable cost as $\tilde{F}(K,b) \equiv F(K,V(K,b)) - bV(K,b),$ where $\tilde{F}_K = F_K > 0$ and $\tilde{F}_b = -V(K,b) < 0.$

3.1. Renewables-only phase

By definition the carbon-free economy relies on renewables only and uses no oil: $O = 0$. Moreover, we have from (6) that $p = b = F_R(K, R)$ so that demand for renewables is given by $R = V(K,b)$. Hence, the material balance equation and the Keynes-Ramsey rule fully characterize the renewables-only phase:

(3R) $$\dot{K} = \tilde{F}(K,b) - \delta K - C,$$

(8R) $$\dot{C}/C = \sigma(C)\left[\tilde{F}_K(K,b) - \delta - \rho\right].$$

Next suppose that this phase lasts forever, possibly after some instant of time. Assumption 2 implies that there exists a unique interior steady state of (3R) and (8R) denoted by (K^*, C^*), which also yields $R^* = V(K^*, b)$, defined by $F_K(K^*, R^*) = \rho + \delta, F_R(K^*, R^*) = b$ and $C^* = F(K^*, R^*) - bR^* - \delta K^*$. From any initial capital stock, the system (3R) and (8R) converges towards the steady state. The optimal saddlepath path leading to the steady state is denoted by superscript R to indicate that we are in the renewables-only phase: $C(t) = \Theta^R\left(K(t), b\right)$, where $\Theta_K^R\left(K, b\right) > 0$ and $\Theta_b^R\left(K, b\right) < 0$.[10] [11] The carbon-free steady state satisfies $C^* = \Theta^R(K^*, b)$.

We are now interested to determine the exact conditions under which it is optimal to use only renewables from the start. To facilitate the analysis we make assumption 4.

Assumption 4: $\dfrac{D'(E_0)}{\rho U'(\Theta^R(K^*, b))} < b.$

Given this assumption and assumption 1, the following equation yields a positive value of S^* :

(13) $$G(S^*) + \frac{D'(E_0)}{\rho U'(\Theta^R(K^*, b))} = b.$$

We define the 'pivotal' economy as the economy which starts at time zero from $K_0 = K^*$ and $S_0 = S^*$. Condition (13) ensures that at time zero the economy is indifferent between using renewables and oil. Indeed, if at time zero a marginal barrel is extracted and no extraction takes place thereafter, the shadow price of oil in situ is zero, marginal direct extraction costs are $G(S^*)$ and the social cost of carbon is $\tau(0) = D'(E_0) / \rho U'(\Theta^R(K^*, b))$. The program $\{R(t), X(t), C(t), S(t), K(t)\} = \{0, X^*, C^*, S^*, K^*\}$ for all $t \geq 0$ satisfies the necessary conditions and, given our concavity-convexity assumptions 1, 2 and 3, is thus an optimal program. The pivotal economy, defined this way, corresponds to the initial stocks of oil reserves and capital so that it is optimal to remain forever in the carbon-free steady state. If assumption 4 is not satisfied and $K_0 = K^*$, it is then even with a large initial oil stock optimal to start with the backstop and never to use oil because the initial pollution level is too high compared to the cost of renewables to warrant any oil use. Note that this does not imply that for any initial capital stock the economy should always abstain from using oil. For a very small initial capital stock, the initial rate of consumption on the stable saddlepath is small as well and its marginal utility is close to zero. Hence, oil use is still an option but it will cease quickly. The following lemma gives the conditions on the initial oil and capital stocks under which it is optimal to use only renewables from the start onwards.

Lemma 1: Suppose the initial oil stock and initial capital stock satisfy

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[10] The stable manifold $C = \Theta^R(K, b)$ with $C^* = \Theta^R(K^*, b)$ is found by eliminating time and solving the resulting first-order differential equation $dC / dK = \sigma C[\tilde{F}_K(K, b) - \delta - \rho] / [\tilde{F}(K, b) - \delta K - C]$, where the steady-state values of K and C pin down the solution.

[11] If there is an exogenous shock , say, a drop in the cost of renewables b , this induces on impact an upward jump in consumption and subsequently consumption and capital increase along the saddlepath.

(14) $\quad b < G(S_0) + \dfrac{D'(E_0)}{\rho U'(\Theta^R(K_0,b))}$ and $K_0 < K^*$ *or*

(15) $\quad b < G(S_0) + \dfrac{D'(E_0)}{\rho U'(\Theta^R(K^*,b))}$ and $K_0 > K^*$.

Then the optimal program uses only renewables.

Proof: Take $O(t) = 0, \mu_E(t) = D'(E_0)/\rho$ and $C(t) = \Theta^R\left(K(t),b\right)$ for all $t \geq 0$. With this choice we have $G(S(t)) + \dfrac{\mu_S(t)+\mu_E(t)}{\mu_K(t)} = G(S_0) + \dfrac{D'(E_0)}{\rho U'(\Theta^R(K(t),b))}$ for all $t \geq 0$. If (14) holds, then $b < G(S(t)) + \dfrac{\mu_S(t)+\mu_E(t)}{\mu_K(t)}$ for all $t \geq 0$, because $C(t) = \Theta^R\left(K(t),b\right) > \Theta^R\left(K_0,b\right)$ for all $t \geq 0$ since capital and consumption are increasing. If (15) holds then the same result is obtained because $C(t) = \Theta^R\left(K(t),b\right) > \Theta^R\left(K^*,b\right)$ for all $t \geq 0$. The proposed program satisfies all the necessary conditions and is thus the unique optimum. Q.E.D.

3.2. Simultaneous use of oil and renewables.

Along an interval of time with simultaneous use of oil and renewables (indicated by OR), we have:

(1OR) $\quad \dot{S} = -O$

(3OR) $\quad \dot{K} = \tilde{F}(K,b) + (b - G(S))O - \delta K - C,$

(6OR) $\quad p = b,$

(7OR) $\quad 0 = (\tilde{F}_K(K,b) - \delta)(G(S) - b) - \dfrac{D'(E_0 + S_0 - S)}{U'(C)},$

(8OR) $\quad \dot{C}/C = \sigma(C)(\tilde{F}_K(K,b) - \delta - \rho).$

Equation (6OR) follows from (6) with $R > 0$. We also have $O + R = V(K,p) = V(K,b)$ and $\tilde{F}(K,b) \equiv F(K,V(K,b)) - bV(K,b) + bO$. This yields (3OR). Then (7OR) follows from (7) with $p = b, \dot{p} = 0$ and $F_K(K, O+R) = \tilde{F}_K(K,b)$. Oil use O can be found as a function of K, S and C from (7OR) by taking the derivative with respect to time and using (1OR), (3OR) and (8OR).

Lemma 2: Simultaneous use of oil and renewables can only occur if $K > K^*$.

Proof: Along any interval of time with simultaneous use we have from (11b) and (11c) that $b = G(S(t)) + \frac{\mu_S(t) + \mu_E(t)}{\mu_K(t)}$. Using (11e) and (11f), and (11b) again, we find that along the interval

$0 = \frac{\rho\mu_E(t) - D'(E(t)) + \rho\mu_S(t)}{\mu_K(t)} + \left[F_K(K(t), V(K(t),b)) - \delta - \rho\right]\left[b - G(S(t))\right]$, because total energy use if $R > 0$ is given by $O(t) + R(t) = V(K(t),b)$. We have from (11f) $\ddot{\mu}_E = \rho\dot{\mu}_E - D''(E)\dot{E}$. Since $D''(E) > 0$ and $\dot{E} > 0$ along an interval of time with simultaneous use, it follows that $\rho\mu_E(t) - D'(E(t)) \geq 0$ for all $t \geq 0$. Indeed if μ_E would ever fall along an interval with simultaneous use, it will become negative, which is ruled out, because μ_E is the value in welfare terms of a lower CO2 stock. So, simultaneous use only occurs if $F_K(K(t), V(K(t),b)) < \delta + \rho$ which demands $K(t) > K^*$. Q.E.D.

It has already been demonstrated that oil use can be written as a function of the oil stock, the capital stock and the rate of consumption. It also follows from (7OR) that the oil stock is a function of capital and the rate of consumption, so the system of equations (1OR), (3OR) and (6OR)-(8OR) can be reduced to a system of two differential equations in C and K only. The saddlepath solution of this system is denoted by $C = \Theta^{OR}(K,b)$ (see appendix 2).

If, starting from an initial capital stock $K_0 > K^*$, we have simultaneous use forever, we must converge to (K^*, S^*, C^*) as oil will be phased out eventually given its limited supply and its extraction cost becoming prohibitively high. This requires a specific initial rate of consumption, and hence a specific initial oil stock. This defines a locus in (K_0, S_0)-space (see fig. 1 below).

Lemma 3: The locus of initial stocks in (K_0, S_0)-space, for which it is optimal to have simultaneous use throughout, starts at (K^*, S^*) and slopes upwards. Furthermore, capital stocks on the locus are smaller than some finite upper bound.

Proof: If $K_0 = K^*$ and $S_0 > S^*$, it is optimal to use only oil initially. The reason is that extraction costs are smaller than in the pivotal economy and that the shadow price of oil in the ground relative to capital is smaller as well. If $K_0 > K^*$ and $S_0 = S^*$, the shadow price of oil relative to capital is higher than in the pivotal economy, and it is optimal to use only renewables initially. Hence, simultaneous use in the neighborhood of the carbon-free steady state can only occur to the north-east of it. In fact, the locus is increasing globally. If this were not so, there would exists an initial $K_0 > K^*$ and initial

$S_{03} > S_{02} > S_{01} > S^*$ such that with $S_{02} > S_0 > S_{01}$ it is optimal to use only oil initially, with $S_0 = S_{02}$ to start with simultaneous use of oil and renewables, and with $S_{03} > S_0 > S_{02}$ to start with only renewables. But this contradicts the fact that with increasing abundance of oil, ceteris paribus, it becomes more attractive to use oil only. Since $D'(E_0)/\left[F_K(K_0,V(K_0,b))-\delta\right]U'(\Theta^{OR}(K_0,b))$ goes to infinity if K_0 goes to infinity, there is an upper bound on the initial capital stock for which it is optimal to start with simultaneous use. Q.E.D.

3.3. Only oil use

Along intervals of time where the economy uses only oil, we have $R = 0$, $O = V(K, p)$ and:

(1O) $\dot{S} = -V(K,p), \quad S(0) = S_0,$

(3O) $\dot{K} = F(K,V(K,p)) - G(S)V(K,p) - C - \delta K, \ \ K(0) = K_0,$

(7O) $\dot{p} = \left[F_K(K,V(K,p)) - \delta\right]\left[p - G(S)\right] - \dfrac{D'(E_0 + S_0 - S)}{U'(C)},$

(8O) $\dot{C} = \sigma(C)\left[F_K(K,V(K,p)) - \delta - \rho\right]C.$

Using only oil throughout or from some instant of time on is never optimal. If the economy would only use oil forever, consumption would go to zero eventually and marginal utility to infinity which can be avoided by using renewables.

3.4 The threshold economy

In the sequel a key role is played by the so-called threshold economy, which we now characterize. Let the initial capital stock $K_0 \le K^*$ be given. The threshold economy is defined as the economy that is endowed with a particular initial oil stock that we denote by S_0^* (which depends on K_0). The endowments (K_0, S_0^*) are such that it is optimal to start using only oil until a critical time T^* when the economy exactly reaches the steady state of the carbon-free economy (K^*, C^*) from below and stays there forever after: $(K(t), O(t), R(t), C(t), S(t)) = (K^*, 0, R^*, C^*, S(T^*))$ for all $t \ge T^*$. At time T^*, the economy must be indifferent between oil and renewables:

(16) $G(S(T^*)) + \dfrac{D'(E_0 + S_0^* - S(T^*))}{\rho U'(C^*)} = b.$

Lemma 4: For any $K_0 \leq K^*$, there is at most one $S_0 = S_0^*$ for which the threshold economy described here is optimal. S_0^* is a decreasing function of K_0. For $S_0 > S_0^*$, the time path for *K* will overshoot K^*.

Proof: Let (K_0, S_0) and *T* be given. We take S_0 such that $G(S_0) + \dfrac{D'(E_0)}{\rho U'(\Theta^R(K_0, b))} < b$ because else it is not optimal to start with oil only (lemma 1). We also set $C(T) = C^*, p(T) = b$. Then the solution of the system (1O), (3O)-(8O) yields $K(T)$ and $S(T)$. We then solve for S_0^* and T^* so that $K(T^*) = K^*$ and (16) are satisfied. We restrict attention to those values of K_0 for which a solution satisfying $S_0^* \geq 0$ and $T^* \geq 0$ exists. This gives the unique initial oil stock and the associated switch time:

(17) $\quad S_0^* = S_0^*(K_0; b), \quad T^* = T^*(K_0; b).$

If $K_0 = K^*$ and $S_0 = S^*$, so that $b = G(S_0) + D'(E_0) / \rho U'(C^*)$, then $T^* = 0$ and we have the pivotal economy. For $S_0 < S^*$ and still $K_0 = K^*$, oil will never be used (because of lemma 1). For a larger S_0, and still $K_0 = K^*$, the economy will overshoot $K_0 = K^*$ because it can be shown that for any regime the capital stock is monotonically increasing as long as it is below its steady state value (see lemma A1, appendix 1). So, the threshold value of S_0^* depends negatively on the initial stock of capital as a lower K_0 requires a higher S_0^* and is associated with a higher switch time T^*. Q.E.D.

A lower K_0 thus increases the threshold value S_0^*, but also depresses initial consumption $C(0) = C_0^*$ and the initial social price of oil $p(0) = p_0^*$ in the threshold economy. With more capital, the economy uses more oil, produces more and reaches the carbon-free steady state more quickly. However, the economy will also consume more and save less, so that capital will grow less quickly and this tends to postpone the date at which K^* is reached. Computations with the parameter values used in sections 6 and 7 suggest that the former effect dominates the latter effect so that $\partial T^* / \partial K_0 > 0$.

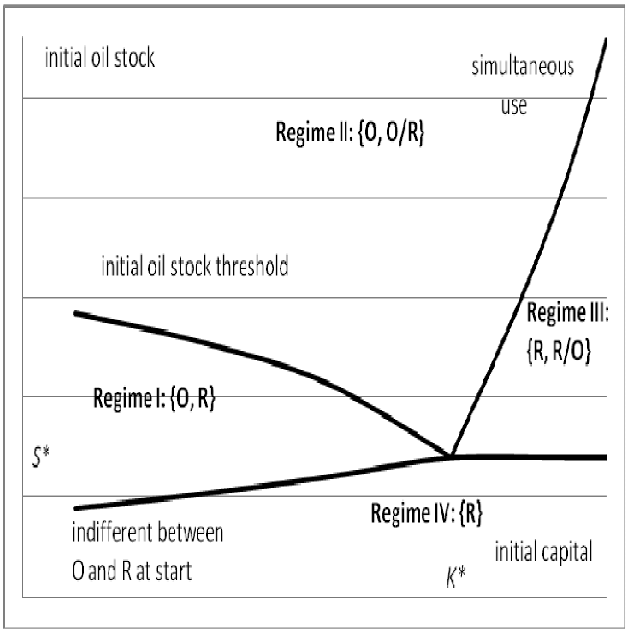
4. Characterization of the four regimes of the socially optimal outcome

Fig. 1(a) depicts the four socially optimal regimes that can occur in our Green Ramsey model, where each regime depends on a particular combination of the initial capital stock given on the horizontal axis and the initial stock of oil reserves given on the vertical axis. In the design of figure 1 we have kept the initial CO2 stock fixed. There are four dividing curves meeting in the point (S^*, K^*), which is the pivotal economy discussed in section 3.1. The four curves have been derived in the previous analysis. The horizontal curve starting from the pivotal economy and extending to the east (i.e., $S_0 = S^*, K_0 > K^*$) is given by the first part of equation (15) holding with equality. The curve starting in the pivotal economy extending to the south-west is given by the first part of equation (14) holding with equality. The curve leading to the north-east is the locus of initial values of the capital and the oil stocks for which simultaneous use throughout is optimal (see lemma 3 of section 3.2).[12] Finally, the locus leading to the north-west describes the initial stocks where the threshold economy finds itself (see lemma 4 of section 3), meaning that if the initial stocks are on this curve, it is optimal to use only oil until the pivotal economy is reached, within finite time, after which the economy will remain carbon-free.

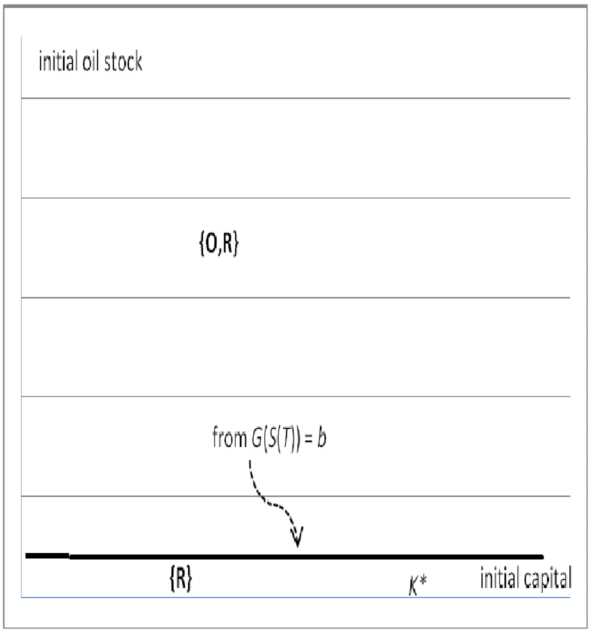
Figure 1 should then be read as follows. Suppose the initial oil stock and the initial capital stock are located in the region denoted by I. It is optimal to have only oil use initially, but after some instant of time only renewables are used. For the other regions a similar reasoning applies. Note that the figure should not be interpreted as a phase diagram, because it is drawn for a given initial value of the CO2 stock. But this stock obviously changes over time if oil use takes place. We now demonstrate that the four regimes depicted in fig. 1(a) are the optimal regimes.

Figure 1: Characterization of regimes of the Green Ramsey model

(a) Social optimum



(b) "Laissez-faire"



[12] Since it can be shown that $\Theta_K^{OR} > \Theta_K^R > 0$ (see appendix 2) and $F_{KK} < 0,$ we find that this locus separating regimes II and III is steeper than the locus (14) with equality separating regimes I and IV in fig. 1(a).

Regime I: The economy transitions from using only oil to only renewables

We start with the region indicated by I which occurs for intermediate values of the initial oil stock and an initial capital stock smaller than K^*, so that

(18) $$b > G(S_0) + \frac{D'(E_0)}{\rho U'(\Theta^R(K_0,b))}.$$

Clearly, it is not optimal to start with simultaneous use, because $K_0 < K^*$. Neither is it optimal to start with only renewables. To see this note first that it cannot be optimal to have a renewables-only regime forever. If that were the case, then $\mu_S(t) = 0$ and $\mu_E(t) = D'(E_0)/\rho$ for all $t \geq 0$. Moreover, $C(0) = U'(\Theta^R(K_0,b))$. But then it follows from (11b) and (11c) that (18) does not hold. Hence, if we start with only renewables, a transition should take place at some instant of time to the use of oil. The transition obviously takes place before K^* is reached. This requires from (11b) and (11c) that $(\mu_E + \mu_S)/\mu_K$ is decreasing over time just before the transition takes place. It follows from (11d)-(11f) that $\frac{D'(E_0)}{(F_K - \delta)U'(C)} > \frac{\mu_E + \mu_S}{\mu_K}$ just before the transition. But $F_K - \delta > \rho$ and $C > \Theta^R(K_0,b)$. This contradicts (18) again. Therefore, it is optimal to start with only oil. Appendix 1 (lemma A1) shows that along the optimum the capital stock and the rate of consumption are both monotonically increasing. In appendix 1 (lemma A2) we also show that a period of time during which oil is used cannot be interrupted by a period of time where only renewables are used. Then it follows from the construction of the locus corresponding with the threshold economy that for $S_0 < S^*(K,b)$ there will be a transition to the carbon-free economy before K^* is reached.

To actually calculate the optimum we proceed as follows. Consider the system (1O)-(8O). Let the switch time T be given. Let the initial values for S_0 and K_0 be given as well. We have $p(T) = b$. Moreover, $C(T) = \Theta^R\left(K(T),b\right)$. Hence, we can solve the resulting two-point-boundary-value problem for a given T. This gives $K(T)$, which is used as initial condition in the subsequent phase renewables-only phase, and $S(T)$, both as functions of the unknown switch time T. To paste the two phases of the optimal program, we use the continuity at T of p, K, C and S where $S(T)$ as a function of $K(T)$ follows from

(19) $$G(S(T)) + \frac{D'(E_0 + S_0 - S(T))}{\rho U'(\Theta^R\left(K(T),b\right))} = b.$$

The stock of oil to be left in situ at time T is endogenous as it depends on the capital stock that the carbon-free economy starts off with. However, if the elasticity of intertemporal substitution is infinite, i.e., utility is linear $U'(C) = \varphi$, $S(T)$ can be found directly from (19). In general, the stock of oil that is left in situ is high if the capital stock is high, because then consumption is high, the marginal utility of consumption is low and thus the social cost of carbon of the last extracted barrel of oil is high. Keeping more oil in situ lowers extraction costs and marginal damages and keeps the social cost of the last barrel of oil equal to the cost of renewables. Also, given $K(T)$, condition (19) implies that more oil is kept in situ if the social discount rate is small (cf., Stern, 2007) and the cost of renewables is low.

Regime II: The economy transitions from using only oil to using oil alongside renewables

Let us now move on to regime II. Consider first the case of $K_0 < K^*$. By construction of the threshold economy the optimal path starts with using only oil. Hence, the oil stock decreases and the capital stock increases, as demonstrated in appendix 1. But a transition to renewables will not take place for capital stocks below the carbon-free steady state. Hence, overshooting of capital (and consumption) will occur. For initial stocks to the north-west of the locus of simultaneous use in fig. 1(a) (and with $K > K^*$), the economy is relatively oil abundant, and the economy will only use oil. Oil use continues until the moment in time where it becomes optimal to switch to simultaneous use. This is also the final phase of the optimal program. It is shown in appendix 1 that we cannot have renewables-only use before the locus of simultaneous use is reached. It is also easily seen that in that case we cannot have a transition to renewables-only use indefinitely. The reason is that above the locus of simultaneous use, oil is preferred over renewables. If we would start with $K_0 > K^*$ the same optimum prevails: first only oil, then simultaneous use forever. No switch back to only oil can occur, because this would immediately lead to simultaneous use again. In the final phase it is optimal to approach the steady-state capital stock from above with oil and renewables used forever alongside each other and oil use vanishing eventually.

The actual calculation of the optimum is, as before, a matter of pasting two sets of differential equations. Initially we have to deal with (1O)-(8O) and after some instant of time the system is governed by (1OR) and (3OR)-(8OR). Pasting utilizes the continuity of capital, the oil stock, total energy use and consumption, where for consumption $C(T) = \Theta^{OR}(K(T), b)$. Moreover, it has to be taken into account that convergence to the steady state of the carbon-free economy must take place.

Regime III: The economy transitions from using only renewables to using oil alongside renewables

Now consider the region denoted by III, which is relevant if the initial capital stock is above its carbon-free steady state and the initial oil stock is neither too large nor too small. Here we are to the right of the locus for simultaneous use meaning that capital is relatively abundant and oil is relatively scarce. Hence it is optimal to start with only renewables and after some time to switch to simultaneous use.

Regime IV: The economy uses renewables from the outset and forever afterwards

Finally, consider the region denoted by IV which occurs if the initially oil is scarce and dear enough for using oil to be optimal. According to lemma 1 the economy then never uses oil. Hence, only renewables are used forever and escape from this regime is suboptimal. If (14) holds, consumption and capital rise monotonically. If (15) holds, consumption and capital decline monotonically.

5. Climate policy, the market and the Green Paradox

5.1. Realizing the socially optimal outcome in the market economy

The "laissez-faire" economy does not internalize global warming externalities, which implies that it acts as if $D \equiv 0$. The following proposition establishes that under "laissez faire" only two regimes can occur depending on whether oil is abundant or scarce – see fig. 1(b).

Proposition 2: The "laissez-faire" economy never has simultaneous use of oil and renewables. If oil is abundant initially, $G(S_0) < b$, there is first an oil-only phase followed at some instant of time T by a final phase of using only renewables. The amount of oil left in situ follows from $b = G(S(T))$, and is less than in the socially optimal outcome. If oil is scarce initially, $G(S_0) > b$, it is optimal to start with renewables and keep using them forever. In both cases, the economy converges asymptotically to the carbon-free steady state.

Proof: From (17) simultaneous use requires $G(S) = b$ if $D \equiv 0$ which gives a constant stock of oil which is inconsistent with positive oil use. Instead of (14), we now have $p(T) = G\big(S(T)\big) = b.$ Comparing this with (14) shows that now less oil is left in situ than in the social optimum. The rest is obvious Q.E.D.

The critical oil stock is less under "laissez faire" than in the social optimum, $G^{-1}(b) < S^*$, because the market does not internalize global warming externalities whereas the social optimum does. If the initial oil stock is high and it is cheap to deplete oil, the economy starts with using only oil. As oil reserves get depleted and oil extraction costs rise, the economy switches to using renewables. If the initial oil stock of oil is low enough, oil is never used.

Since there are no other distortions apart from the climate externality and lump-sum taxes/subsidies are available, the optimal carbon tax must equal the social cost of carbon (5) which was defined as the present discounted value of all future marginal global warming damages.

Proposition 3: The government can reproduce the socially optimal outcome by levying a carbon tax τ equal to the social cost of carbon (5) or (5′), evaluated in the first-best. The optimal carbon tax rises over time if and only if $r > [D'(E)/U'(C)]/\tau$.

Proof: Compare the first-order conditions of the "laissez-faire" outcomes with those of proposition 1. See necessary condition (9) for the second part. Q.E.D.

The numerator in $\tau = \mu_E / \mu_K$ increases over time, so the optimal carbon tax can only decline with time if the denominator increases over time. This requires that consumption declines over time and the capital stock exceeds its steady-state value. The condition of proposition 3 states that the carbon tax declines over time if the return on capital is less than the ratio of the current marginal damage, $D'(E)/U'(C)$, to the social cost of carbon, τ. The carbon tax must be maintained after the only-oil era has finished and only renewables are used. If this were not the case, the economy would switch back to using oil.

If the economy starts off with a low capital stock, consumption is low, the interest rate is high and the marginal value of consumption is high so that the social cost of carbon and the optimal carbon tax are low. As the economy develops and consumption and capital rise, the social cost of carbon and optimal carbon tax rise. Once renewables kick in, the optimal carbon tax stays constant at the level that prevails at the end of the oil-only phase. Hence, the magnitude of the optimal carbon tax depends on the state of economic development. If the economy has high enough degree of economic development, it may be optimal for the carbon tax to start off high and then diminish with time. Indeed, proposition 3 indicates that, if consumption and capital overshoot their steady-state values, the interest rate is relatively small and the marginal utility of consumption is small, there is a real possibility that the optimal carbon tax falls with time, especially if the optimal carbon tax is low and the atmospheric stock of CO2 is high.

5.2. Second-best outcome if a carbon tax is infeasible

What happens if for political or other reasons it is infeasible to levy a carbon tax?[13] The Green Paradox states that subsidizing the backstop fuel with the aim of curbing oil demand and carbon emissions encourages private well owners to pump up their oil more rapidly, thereby aggravating global warming damages. If oil extraction costs rise rapidly as reserves diminish, the market leaves less oil in the crust of the earth than the social optimum. In that case, the Green Paradox need not occur as reducing the cost of renewables then leaves more oil in situ (van der Ploeg and Withagen, 201).

[13] In practice, politicians do charge a price for carbon (e.g., the European Union Emission Trading Scheme) but lower than the social optimum. For simplicity, we assume that the government levies no carbon tax at all.

Within our context the Green Paradox highlights the second-best effects of introducing a constant backstop subsidy υ, financed by a lump-sum tax, to phase out oil more quickly and mitigate global warming given that a carbon tax is ruled out, $\tau = 0$. We are interested in the effects of a renewables subsidy on consumption, accumulation of capital, growth and welfare. There is no case anymore for a subsidy (or tax) on renewables once the extraction cost of oil is larger than the production cost of renewables. We suppose that the government can commit and keeps the renewables subsidy in place once oil is no longer used.

The economy without a subsidy on renewables leaves some oil reserves unexploited, but less than in the social optimum. A subsidy has the effect of leaving more oil unexploited[14], because $b - \upsilon = G(S(T))$, which reduces the amount of carbon in the atmosphere and brings the economy closer to the first-best. As we will see in the simulations of section 6 and 7 and is known from the literature on the Green Paradox, initially more oil is extracted. Hence, the total effect on green welfare (the negative of climate damages) is ambiguous. With a subsidy the steady-state stock of capital is higher (from $\tilde{F}_K(K^*, b-\upsilon) = \delta + \rho$). The steady-state rate of consumption is lower now, because the subsidy is financed by lump-sum taxes which curbs consumption. The effect on total welfare depends on the state of the economy. With a low level of development (low initial capital stock), the subsidy may be counterproductive. The loss of consumption implies a large loss in utility, whereas the possible gain in terms of reduced damages may be negligible compared to the utility loss. The optimal second-best renewables subsidy will thus be lower if the economy is in the early stages of development. As we have seen already, the optimal first-best carbon tax is then lower as well.

6. Policy simulations: oil-scarce regime (undershooting)

To start we offer some policy simulations for the case where the initial conditions are such that the social optimum has an initial period where only oil is used and a final period where only renewables are used. This is regime I of proposition 2 which arises for a given initial capital stock smaller than K^* and an intermediate level of the initial oil stock or for a given initial oil stock and a sufficiently low initial capital stock. We simulate both the social optimum and the decentralized market outcome (described in sections

[14] If full exhaustion of oil reserves is feasible (i.e., if oil extraction costs as oil reserves vanish tend to a small enough finite number rather than infinity), a backstop subsidy only leads to more rapid pumping of oil, faster exhaustion of oil reserves and thus to a Green Paradox (van der Ploeg and Withagen, 2010).

4 and 5). We set $S_0 = 20$ and $E_0 = 24$.[15] We use $\rho = 0.014$ as the rate of time preference. We use the iso-elastic utility function $U(C) = C^{1-1/\sigma} / (1-1/\sigma)$ with a ballpark estimate of the elasticity of intertemporal substitution equal to $\sigma = 0.5$ [16] and explore the sensitivity with respect to σ to gain insight into the effect of intergenerational inequality aversion on global warming and economic growth. The production function is Cobb Douglas $F(K, O+R)) = K^{\alpha}(O+R)^{\beta}$. The shares of capital and of oil/gas in GDP have been set at $\alpha = 0.2$ and $\beta = 0.1$. The average lifetime of manmade capital has been set at twenty years, so $\delta = 0.05$. The initial stock of capital is set at half the value that prevails in the steady state of the carbon-free economy, i.e., $K_0 = K^* / 2$. We set $b = 0.5$ [17] and the initial cost of extracting one unit of oil at $G(S_0) = 0.2$. We suppose that unit extraction costs become infinitely large as more and more oil is extracted and capture this with the specification $G(S) = \gamma S_0 / S$ with $\gamma = 0.25$. This implies that in the market outcome where global warming externalities are not internalized, half of the initial stock of oil is left in situ at the time of the switch to the renewable backstop, i.e., $S(T) = S_0 / 2 = 10$. If global warming externalities are internalized, a bigger stock is left in situ. The cost of extracting the last drop of oil thus exactly equals the cost of renewables in the market outcome, but will be less than in the socially optimal outcome. For global warming damages we use the specification $D(E) = \kappa E^2 / 2$ with $\kappa = 0.00012$.[18]

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[15] In 2000 there were oil and gas reserves in the crust of the earth corresponding to 469 and 1,121 Giga tons of carbon, respectively, whereas there had been emitted 224 plus 111 Giga tons of carbon into the atmosphere resulting from burning, respectively, oil and gas (Edenhofer and Kalkuhl, 2009). Normalizing so that $S_0 = 20$, we set $E_0 = 24$ (rather than 335x20/1,590 = 4.2) to allow for the substantial CO2 concentration that was already in the atmosphere for non-anthropogenic reasons.

[16] Some argue that the elasticity of intertemporal substitution σ is very low with an implied coefficient of relative risk aversion of about 10 (e.g., Mehra and Prescott, 1985; Campbell and Mankiw, 1989; Obstfeld, 1994); others argue that σ is one or greater than one with a much smaller and more realistic implied coefficient of relative risk aversion (e.g., Hansen and Singleton, 1982). $\sigma = 0.5$ implies a coefficient of relative risk aversion 2 which seems a little high. Rather than breaking the link between risk aversion and intertemporal substitution to allow for an elasticity of intertemporal substitution in the range 0.05 to 1 and a coefficient of relative risk aversion in the range 0.4 to 1.4 (Epstein and Zin, 1991; Crost and Traeger, 2012), we explore the sensitivity of our results with respect to different values of σ.

[17] Solar and wind energy are more expensive than fossil fuel, especially if one looks beyond marginal production costs once capacity is installed and considers the costs needed to increase capacity, deal with intermittence and repair offshore wind mills. Wind energy can be at least three times as expensive as 'grey' electricity. As far as the electricity industry is concerned, costs of renewables have fallen substantially: solar energy is currently 50% more expensive than conventional electricity; wind energy has the same cost and is (apart from the problem of intermittence) competitive; and biomass, CCS coal/gas and advanced natural gas combined cycle have mark-ups of 10%, 60% and 20%, respectively (Paltsev et al., 2009). These mark-ups are measured from a very low base and may not be so impressive when they account for a much larger market share. Hence, we use a 100% mark-up.

[18] Peer-reviewed estimates of the social cost of carbon for 2005 have an average of \$43 per ton of carbon and a standard deviation of \$83 dollar per ton of carbon, and these estimates are likely to increase by 2 to 4 percent per year (Yohe et al., 2007). A ballpark estimate of the social cost of carbon is \$30 dollar per ton (Nordhaus, 2007).

With these parameters, we calculate $S_0^* = 20.8$ and $T^* = 24.3.$ Since we start with $S_0 = 20 < S_0^*,$ it is optimal to start with an initial oil-only phase and end with a final renewables-only phase (see section 4). Appendix 3 described the algorithm we have used to solve this problem.

6.1. The optimal carbon tax needed to attain the first-best outcome in the market economy

Fig. 2 plots the time paths of the key variables for these two outcomes (solid lines represent the optimum, long-dashes the market and also gives the time path of the optimal carbon tax that ensures that the market properly internalizes the global warming externality.[19] Both manmade capital and consumption rise over the entire optimal path, confirming the theory. In this benchmark simulation there is no overshooting of capital in the optimum, but the market does overshoot. Under "laissez-faire", manmade capital and consumption decline during the carbon-free phase and even before. Optimally internalizing global warming externalities implies that renewables get phased in more quickly than under the "laissez-faire" outcome, namely at time 22.0 rather than 42.3[20], and that 14.9 rather than 10.0 units of oil are left in situ at the time of the switch to the carbon-free economy. Switching more quickly to the carbon-free economy and leaving more oil in situ is an effective way to curb CO2 emissions and global warming. Consumption and manmade capital are smaller for the optimal than for the "laissez-faire" outcome, since oil use is cut back more quickly to limit global warming. We also note a steeper time path for the social price of oil in the socially optimal outcome due to the rising time profile of the optimal carbon tax from 0.24 to 0.31.[21]

The Green Solow model put forward by Brock and Taylor (2010) has CO2 emissions as an inevitable by-product of production and abstracts from renewables. Social welfare is maximized by choosing a constant savings rate and a constant share of abatement in output. They find an Environmental Kuznets Curve: emissions initially increase and later decrease with economic development. Within our Green Ramsey framework CO2 emissions per unit of output are initially high, since initially the marginal utility of consumption is large compared to the marginal damages of accumulated CO2. During the development growth path, the rapid accumulation of manmade capital compensates for the falling use of oil resulting from the rising price of oil and growth tapering off. Consequently, CO2 emissions are initially high and then fall rapidly over time. Once the economy has switched to a clean backstop, CO2 emissions are reduced to zero and the accumulated pollution in the atmosphere is stabilized. So, we also get an Environmental Kuznets Curve, but in a different framework with a benevolent policy maker.

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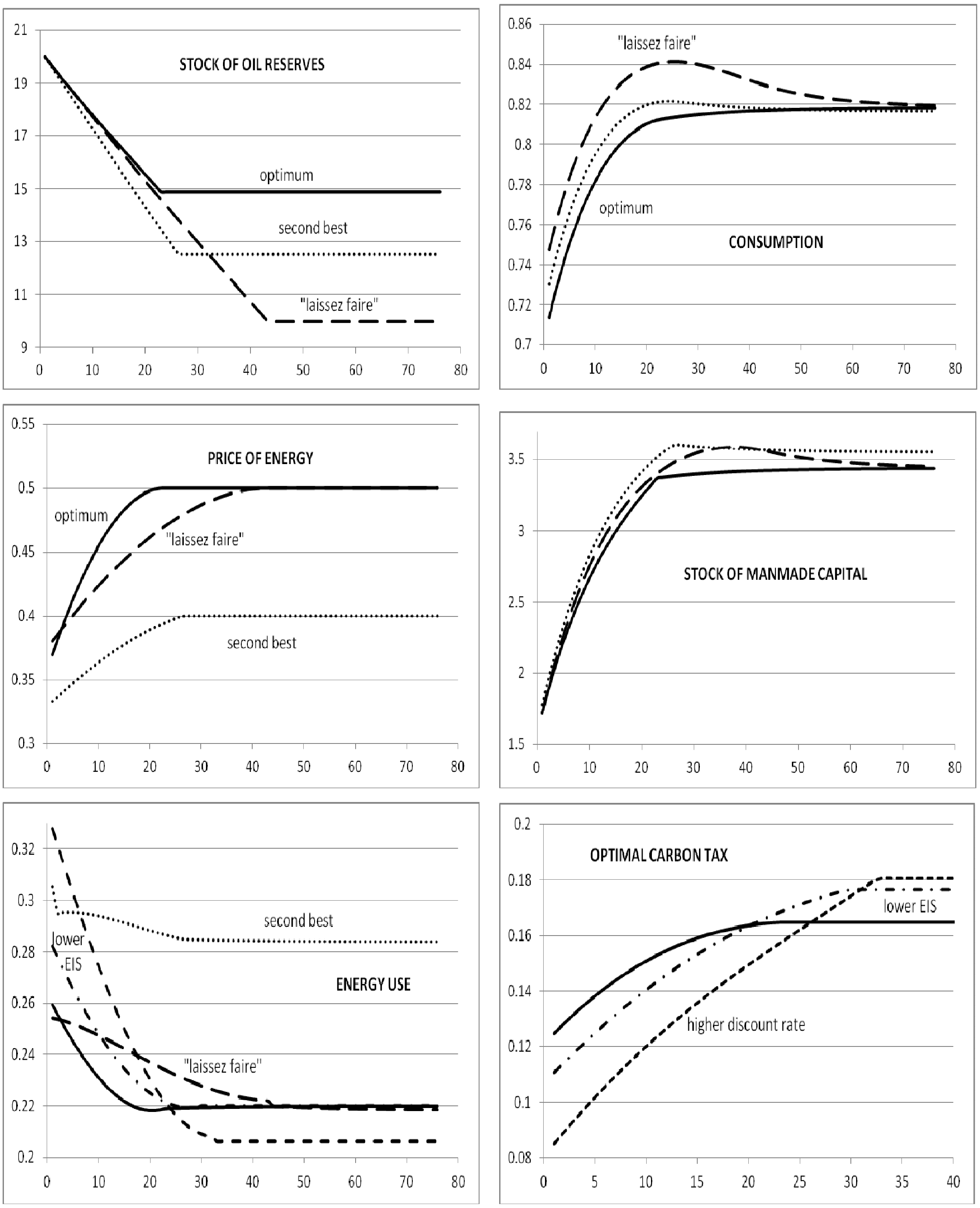
[19] Fig. 2 suggests that the time path of oil stocks is linear which is incompatible with non-constant oil use, but this is an optical illusion as the time paths of oil stocks is not linear but approximated by a linear combination of four exponential functions (see appendix 3).

[20] The switch occurs earlier than in the threshold economy, $T = 22.0 < T^* = 24.3$ as oil is scarcer.

[21] This contrasts with the inverse U-shape for the time profile of the optimal carbon tax in Golosov et al. (2011), where the eventual decline of the carbon tax results from their assumption of natural decay of atmospheric CO2.

Figure 2: Simulation trajectories for oil-scarce economy



Key: solid lines – optimum (benchmark): $T = 22.0,\ S(T) = 14.9$;
long dashes – "laissez-faire": $T = 42.3,\ S(T) = 10.0$;
dots – no carbon tax and backstop subsidy (second best): $T = 25.3,\ S(T) = 12.5$;
dots and dashes – optimum with higher inequality aversion: $T = 28.9,\ S(T) = 13.1$;
short dashes – optimum with higher rate of time preference: $T = 31.8,\ S(T) = 12.0$.

6.2. Effects of intergenerational inequality aversion and time preference

If the elasticity of intergenerational inequality aversion ($1/\sigma$) is increased from 2 in the benchmark to 4 (corresponding to halving the elasticity of intertemporal substitution ($EIS = \sigma$), we find that the time to phase out oil and switch to renewables in the social optimum is postponed from instant 22.0 to 28.9, and that, as a result of more aggressive oil use, the stock of oil that is left in situ at the end of the oil phase is decreased from 14.9 to 13.1. Both these factors tend to increase CO2 emissions and global warming, as may be expected if intergenerational inequality aversion is higher and thus more priority is given to current, relatively poor generations who have to shoulder most of the burden of combating climate change rather than to distant, relatively rich generations. This way the economy develops faster initially at the expense of global warming, albeit that the steady-state levels of the capital stock and consumption are not affected by more intergenerational inequality aversion. Fig. 2 indicates that, with a lower *EIS*, the optimal carbon tax rate for this case (dots and dashes) is lower in the initial part of the oil-only phase but higher in the latter part of this phase. Furthermore, as the oil-only phase lasts longer, the oil stock left in situ is smaller and the eventual stock of atmospheric carbon larger than if intergenerational inequality aversion is not so high. With a higher intergenerational equality aversion (lower *EIS*), current generations are better off in terms of consumption than future generations but the climate is worse off.

One might argue that the private sector employs a higher rate of time preference than the government, say a rate of time preference of 0.03 rather than 0.014 for the "laissez-faire" economy. The time of the switch towards renewables is then reduced by a tiny amount from 42.34 to 42.30 whilst the stock of oil that remains in situ remains 10.0. However, as the economy is impatient and consumes more upfront and thus invests less, it ends up in the long run with much less manmade capital (2.58 rather than 3.57) and somewhat lower consumption (0.80 rather than 0.83).[22] If the government also employs the higher rate of discount of 0.03, it initially pursues a less aggressive climate change policy resulting in much more oil use. In the latter part of the oil-only phase climate policy becomes less aggressive and oil use is below than the case where the government employs the prudent discount rate. Still, renewables are phased in more quickly than under "laissez-faire" at instant 31.8, but a lot more slowly than if the government employs a precautionary discount rate of 0.014 (at time instant 22.0). Furthermore, oil left in situ, 12.0, is less than with a prudent discount rate of 0.014, but more than in the "laissez-faire" outcome. Fig. 3 indicates that the optimal carbon tax for the case of a low discount rate (solid line) is for the most part lower than that for a high discount rate (short dashes) but in the final part of the oil-only phase is higher.

[22] The reason that C^* changes only a little compared with K^* is that the share of capital is much smaller than the combined share of all the non-energy factors (capital and labor) in value added.

6.3. Second-best outcome: subsidizing renewables does not lead to the Green Paradox

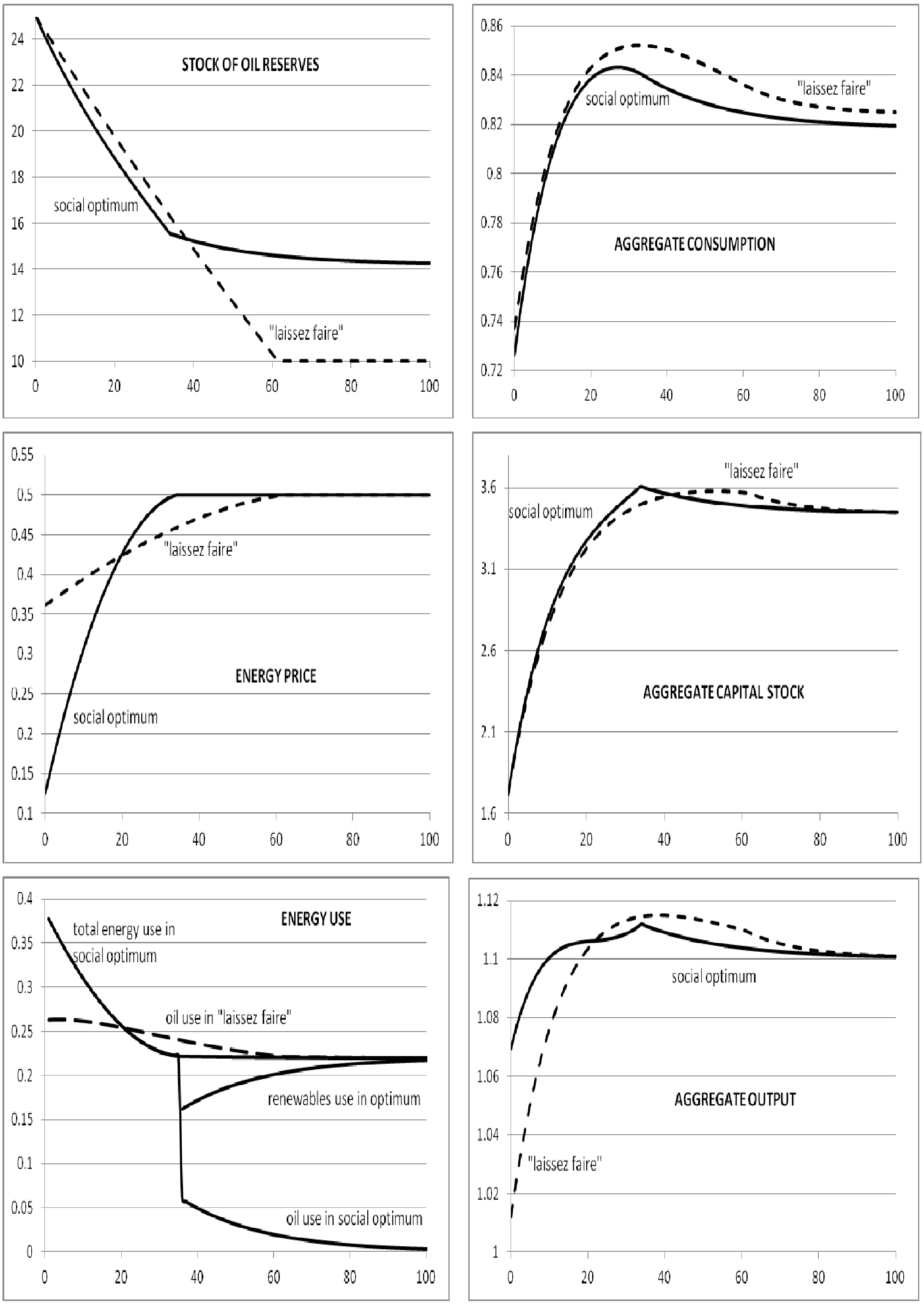
Fig. 2 also plots the time paths of the key macroeconomic and resource variables under the renewables subsidy (dotted lines) amounting to $\upsilon = 0.1$ and financed with lump-sum taxes. The date of switching from oil to renewables becomes 25.3, later than in the socially optimal outcome and earlier than in the market without the subsidy. The amount of oil left in situ increases from 10 in the "laissez-faire" economy to 12.5, which is less than in the socially optimal outcome. The time path for consumption is higher than in the social optimum but lower than in the "laissez-faire" market economy. As a result of renewables subsidy, the price of energy is much lower both during the oil-only and the carbon-free phase in this simulation. Initially private agents are encouraged to use much more oil in production than even in the "laissez-faire" outcome, because the loss of higher extraction now on future extraction cost is lower due to the fact that renewables will be phased in earlier and more oil is left in situ. This is what underpins the inexorable logic of the Green Paradox: despite renewables being phased in more quickly and more oil being left in situ, private agents pump up oil much more vigorously. For our numerical example the present value of global warming damages is reduced from 2.19 in the "laissez-faire" market outcome to 1.98 (more than in the social optimum, 1.73). Hence, despite that more oil is pumped up initially, global warming damages need not increase under the backstop subsidy as renewables are phased in more quickly and more oil is left in situ. As the renewables subsidy distorts private decisions, private welfare falls from -67.8 in the "laissez-faire" to -69.0 in the market outcome with the subsidy. The renewables subsidy thus boosts green welfare, but curbs social welfare from -70.0 to -70.9. Clearly, such a subsidy also performs worse than an optimal carbon tax.

7. Policy simulations: oil-abundant regime (overshooting)

We now offer policy simulations for regime II of proposition 2 in which the oil-only phase is followed by a final phase where oil and renewables are used simultaneously. In contrast, in the final phase of the "laissez-faire" outcome only renewables are used. We use the same parameters as in section 6, including $K_0 = K^* / 2,$ except that we set the initial oil stock equal to $S_0 = 25 > S_0^* = 20.8$ instead of 20. The policy simulations of this oil-abundant regime are depicted in fig. 3.

It is optimal to extract more oil initially than in the market economy. The aim is not to increase consumption but to build up capital more rapidly than in the market economy. The social price of energy in the social optimum starts below that under "laissez-faire" and then during the latter part of the oil-only phase is above it. As a result, oil use is only lower than the "laissez-faire" oil use in the latter part of the only-oil phase. As soon as renewables take over under "laissez-faire", the market price has caught up with

Figure 3: Simulation trajectories for the oil-abundant economy



Note: Social optimum $T = 34.0; S(T) = 15.6; S(\infty) = 14.2$; "Laissez-faire" $T = 61.4; S(T) = S(\infty) = 10$.

the social price of oil again. Still, renewables fall ever so slightly from then on as capital falls during this final phase under "laissez-faire". We also observe that *clean* renewables are phased in much later under "laissez-faire" (i.e., at time 61.4 instead of 34.0), albeit that the social optimum never phases out oil completely and only gradually ramps up the use of renewables. Output overshoots in both outcomes. The swinging time profile of output in the social optimum reflects, on the one hand, the rapid growth of the capital stock during the early parts of the initial oil-only phase, and, on the other hand, the substantial curbing of oil use during the oil-only phase. The reason for the kink in net investments and output is that at the transition, capital, consumption and energy use are continuous, but oil and renewables are not.[23] Oil use jumps down at the moment renewables are phased in.

The social optimum achieves an improvement in green welfare at the expense of private welfare by substantially curbing oil use and carbon emissions and thus flattening output growth during the latter part of the oil-only phase, despite the increase in oil use during the early part of this phase. Consumption under "laissez-faire" is higher throughout, especially during the latter part of the oil-only phase. The social optimum also leaves much more oil in situ and thus puts less carbon in the atmosphere.

The time profile of the optimal carbon tax

The optimal carbon tax for the oil-abundant economy is depicted in fig. 4(a). It continues to rise after renewables have been phased in, since in the case at hand the decrease in consumption is not too fast. This combined with the increase in marginal climate damages as more carbon is emitted into the atmosphere leads to an upward time profile of the optimal carbon tax in the final phase. During this final phase use of oil in the production process is gradually winded down whilst that of renewables is gradually ramped up.

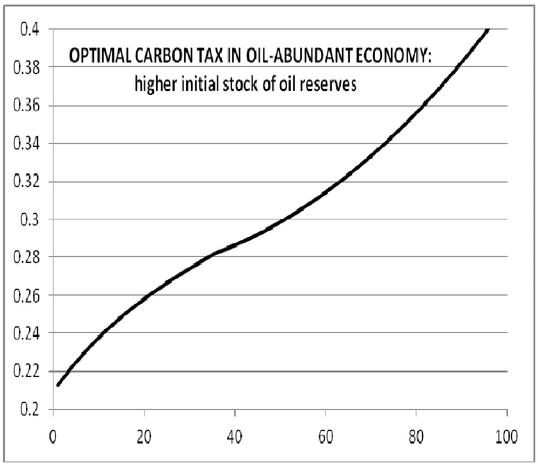
If we would start with a larger initial capital stock, say $K_0 = 0.75K^*$ and keep $S_0 = 20$, the threshold for the initial oil stock will be lower, i.e., $S_0^* = 18.7 < 20.8$. This is an alternative way to let the economy move from an oil-scarce to an oil-abundant regime, so that oil is never phased out. At instant of time 20.1 renewables are phased in alongside oil (see appendix 4 for the corresponding figures). Figure 4(b) indicates that the time profile of the optimal carbon tax for the oil-abundant economy starting with a higher initial capital stock (rather than a higher initial stock of oil) has an inverted-U shape. It starts off higher but ends up lower than in the oil-scarce, developing economy discussed in section 6. The social cost of carbon for the mature economy is initially higher due to the lower interest rate and the lower marginal utility of consumption. In spite of the fact that there is no decay of the CO2 stock, the carbon tax

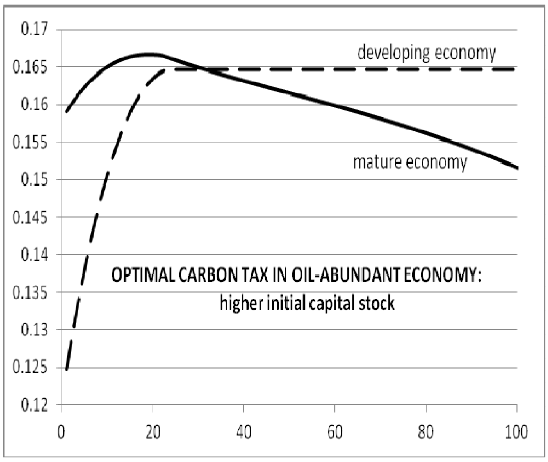
[23] $bR + G(S)O = b(O+R) + (G(S)-b)O$ is discontinuous at the transition because $G(S) - b = (\mu_S + \mu_E)/\mu_K \neq 0$.

eventually decreases. Note that the optimal carbon tax is a lot higher if we move into an oil-abundant economy by having more initial oil reserves and thus potentially more carbon to emit into the atmosphere (see fig. 4(a)) than by having a higher initial capital stock (see fig. 4(b)).

Figure 4: The optimal carbon tax in the oil-abundant economy

(a) $K_0 = 0.5K^*, S_0 = 25 > S_0^* = 20.8$ (b) $K_0 = 0.75K^*, S_0 = 20 > S_0^* = 18.7$





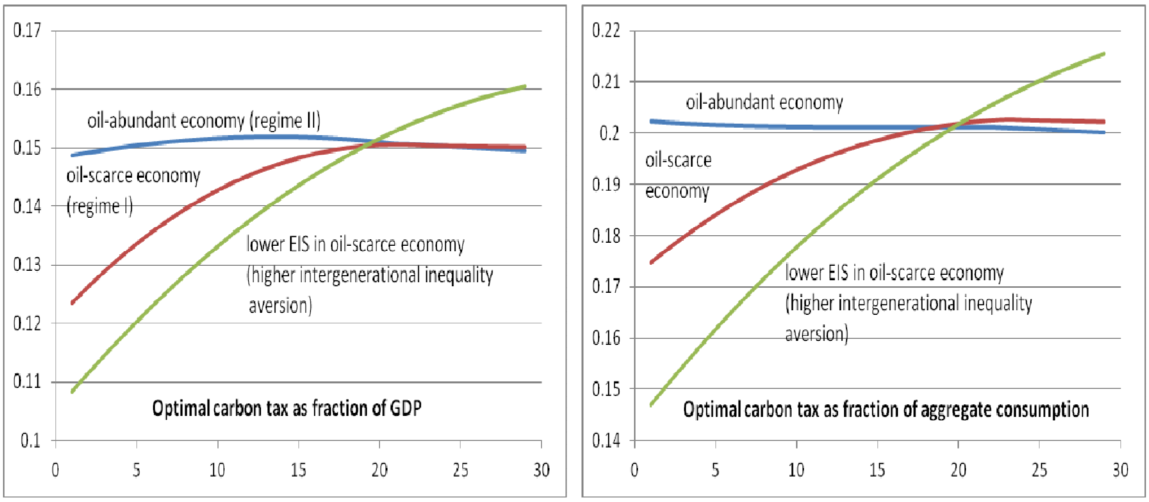
Finally, Golosov et al. (2011) have argued that setting the carbon tax equal to a constant fraction of national income or equivalently of aggregate consumption might not be bad approximation of the optimum. To check this, fig. 5 plots the optimal carbon tax in the oil-scarce (the red lines) and oil-abundant economies (the blue lines) already displayed in fig. 2 and 3 for an *EIS* of 0.5, respectively, as ratios of GDP and aggregate consumption. Although the time path of the optimal carbon tax is fairly flat in the oil-abundant regime II, this is not the case in the oil-scarce regime I, especially during the oil-only phase. In the early periods of development the optimal carbon tax is low relative to GDP or aggregate consumption compared with the later periods of the development process. The reason is that during the early periods of development the marginal utility of consumption is high and thus it is optimal to pay relatively more attention to that than to fighting climate change.

From (5) the social cost of carbon with iso-elastic utility can be written as:

$$(5'') \qquad \tau(t) = \left[\int_t^{\infty} e^{-\rho(s-t)} D'(E(s)) ds \right] C(t)^{\sigma}.$$

If utility is logarithmic($\sigma = 1$), (5″) indicates that the social cost of carbon is proportional to consumption in any phase where only renewables are used. But with $\sigma < 1$, as in our simulations, we see that in this phase of regime I the optimal carbon tax is an increasing and concave function of consumption. The red line in the right panel of fig. 5 suggests that the social cost of carbon rises much more during the oil-only phase. The green line suggests that a higher degree of intergenerational inequality aversion (i.e., *EIS* =0.25) implies that the social planner is less prepared to hurt welfare of current generations to benefit future generations, and therefore the optimal carbon tax starts off being much less ambitious but then rises more as the economy becomes wealthier. As already noted, in regime II the social cost of carbon falls, albeit not very fast. The reason is that, even though the stock of atmospheric carbon rises, consumption falls over time in the later phases and this depresses the social cost of carbon.

Figure 5: Optimal carbon tax as fraction of GDP and of aggregate consumption



If, as in Golosov et al. (2011), we also allow for production damages from global warming by replacing output $Y = F(K, O+R)$ by $Y = A_0 e^{-\xi E} F(K, O+R)$, with A_0 and ξ positive constants, we can show that the social cost of carbon as fraction of GDP equals:

$$(20) \qquad \frac{\tau(t)}{Y(t)} = \left[\int_t^\infty e^{-\rho(s-t)} \left[D'(E(s)) + \xi Y(s) C(s)^{-\sigma} \right] ds \right] \frac{C(t)^\sigma}{Y(t)}.$$

Supposing that utility is logarithmic (i.e., the *EIS* =1) and defining $c \equiv C / Y$, we get:

$$(20') \qquad \frac{\tau(t)}{Y(t)} = \left[\int_t^\infty e^{-\rho(s-t)} \left[D'(E(s)) + \frac{\xi}{c(s)} \right] ds \right] c(t).$$

In the discrete-time model of Golosov et al. (2011) with 100% depreciation of capital, a Cobb-Douglas production function and no damages in welfare, the consumption-output ratio c is constant and we see from (20′) that the social cost of carbon is proportional to output, $\tau(t) / Y(t) = \xi / \rho$. In general, $EIS < 1$ and the consumption-output ratio is not constant, especially during the oil-only phases and thus the ratios of the optimal carbon tax to GDP or aggregate consumption are not constant either.

8. Conclusion

We have analyzed optimal climate policy in a Ramsey growth model with exhaustible oil reserves, an infinitely elastic supply of renewables, stock-dependent oil extraction costs and convex climate damages. In a market economy there are only two regimes: either the initial stock of oil reserves is so low that there is an initial oil-only phase followed by a final renewables-only phase or only renewables are used forever from the outset. Under "laissez faire" oil and renewables are never simultaneous used. We have also characterized the four regimes that are socially optimal. The two most likely regimes I and II occur if it is optimal for the economy to start with oil rather than renewables, which requires relatively high initial oil stocks and low initial capital stocks so that the initial oil extraction cost and the social cost of carbon are high enough for the social cost of oil to be below the cost of renewables.

Regime I occurs for intermediate values of the initial stock of oil and an initial capital stock lower than the carbon-free steady state. It is then optimal to have an initial oil-only phase followed by a renewables-only, carbon-free phase. Our simulations suggest that in this regime a lower discount rate or a lower degree of intergenerational inequality aversion command a higher long-run carbon tax for the market economy, which ensures that more oil is left in situ and renewables are phased in more quickly. The optimal carbon tax rises as the economy moves along its development path during the oil-only phase. The carbon tax rises over time for two reasons. First, as oil reserves diminish and the stock of atmospheric carbon rises, the marginal cost of global warming rises. Second, as consumption increases, the marginal utility of consumption falls. The rise in the carbon tax flattens off as less accessible reserves have to be explored, because oil well owners deplete more conservatively then which is good for the environment. If a carbon tax is infeasible and renewables are subsidized (a second-best policy), renewables are phased in more quickly and more oil is left in situ. However, oil is also pumped up more vigorously initially (a manifestation of the Green Paradox), so the effect on global warming is ambiguous. The renewables subsidy hurts consumption.

Regime II occurs if the initial oil stock is large enough. The social optimum has an initial oil-only phase followed with a final phase where both oil and renewables are used; in the "laissez- faire" outcome oil is phased out and renewables phased in much later. The energy price starts below that in "laissez-faire", but during the latter part of the oil-only phase rises above it. Oil use is thus initially higher than "laissez-faire" and only lower in the latter part of the only-oil phase. In the final phase use of renewables is gradually ramped up. Consumption and capital overshoot their steady-state values in regime II, but not in regime I. The social optimum boosts green welfare at the expense of private welfare by substantially curbing oil use and carbon emissions and thus flattening output growth during the latter part of the oil-only phase, despite the increase in oil use during the early part of this phase. Consumption under "laissez-faire" is higher throughout, especially during the latter part of the oil-only phase. The social optimum also leaves more oil in situ and thus less carbon in the atmosphere. Regime II may also apply for an economy with an initial capital stock that exceeds the carbon-free steady state, but not by too much. Even though there is no decay of atmospheric carbon, the optimal carbon tax may decline in regime II. Although the optimal carbon tax appears to be roughly a constant proportion of aggregate consumption or GDP in regime II where consumption is falling anyway (cf., Golosov et al., 2011), we find it is a strongly rising proportion over time for the oil-only phase of regime I where consumption is rising along the development path.

If it is optimal to start with renewables rather than oil, two further regimes emerge. For intermediate values of the initial oil stock and the initial capital stock above its steady state, regime III prevails with an initial renewables-only phase followed by a final oil-renewables phase. If the initial stock of oil is low enough, regime IV prevails where renewables are used forever. Until there is a breakthrough in renewables technology, these latter two regimes seem unlikely to occur.

Our analysis has abstracted from various features to highlight the importance of recognizing different regimes for the design of the optimal carbon tax. First, an upward-sloping supply schedule of renewables (e.g., Sinn, 2008ab) introduces regimes where more and more renewables are phased in alongside oil as the energy price rises (van der Ploeg and Withagen, 2010). Second, technical progress in renewables (e.g., Bovenberg and Smulders, 1996; Popp, 2002; Bosetti et al., 2009; Acemoglu et al., 2012; van der Meijden and Smulders, 2012; Daubanes et al., 2012) leads to a gradual decline in the price of renewables, thus bringing forward the date of the switch from fossil fuel to renewables and kick-starting green innovation. Third, technical progress and population growth will affect the optimal carbon tax. Imperfect substitution between energy and other production factors is weak in the short run, but strong in the long run due to directed energy-saving technical change (Hassler et al., 2011). Fourth, imperfect substitution between different sources of energy matters (Smulders and van der Werf, 2008; Michielsen, 2011; Pelli, 2012). Fifth, natural decay of the atmospheric stock of CO2 makes the optimal carbon tax eventually fall over time (cf, Golosov et al., 2011). Sixth, if coal is the relevant backstop, it is optimal to have a more conservative oil depletion strategy and delay the switch to using coal (van der Ploeg and Withagen, 2011). Seventh, it matters a lot whether global warming damages affect production multiplicatively or additively (Rezai et al., 2012). Finally, fast-growing China and India have less appetite for an aggressive climate policy than the OECD economies. OPEC has even less interest. A multi-country analysis will shed more light on the different tradeoffs between climate and development facing different parts of the world.

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Appendix 1: Details of proofs

Lemma A1: Consumption and capital increase monotonically if $K < K^*$

Proof: Define total energy use $V \equiv O + R$. Recall that (K^*, V^*) is defined by $F_V(K^*, V^*) = b$ and $F_K(K^*, V^*) = \rho + \delta$. It follows from concavity of F that $\left[\rho + \delta - F_K(K,V)\right](K - K^*) + \left[b - F_V(K,V)\right](V - V^*) \geq 0$. We have $K < K^*$ by assumption, and $b \geq F_V(K,V)$. Suppose $F_K(K,V) - \rho - \delta \leq 0$. Then $V \geq V^*$. But with $K < K^*$ and $V \geq V^*$ we have $F_K(K,V) > F_K(K^*, V^*) = \rho + \delta$, which is a contradiction. It follows that *C* is rising. Regarding the proof that *K* is rising, we do not have to consider the case of simultaneous use, because simultaneous use only occurs for $K \geq K^*$. The statement is true if we are in a renewables-only phase, because once such a phase starts, it will last forever and the economy approaches its steady state with *K* monotonically increasing. So, we assume that along some interval of time with only fossil fuel use, *K* is decreasing and establish a contradiction. Since the economy will eventually approach the steady-state capital stock, the decrease will not be permanent. Hence, there exist instants of time $t_1 < t_2$ with $K(t_1) = K(t_2) = K < K^*$ and $S(t_1) > S(t_2)$ such that $\dot{K}(t_1) = F(K, O(t_1)) - G(S(t_1))O(t_1) - \delta K - C(t_1) < 0$ and either

Case 1: $\dot{K}(t_2) = F(K, O(t_2)) - G(S(t_2))O(t_2) - \delta K - C(t_2) > 0$ or

Case 2: $\dot{K}(t_2) = F(K, R(t_2)) - bR(t_2) - \delta K - C(t_2) > 0.$

Consumption is increasing over time, so that $C(t_1) < C(t_2)$. In case 1 we therefore have $F(K,O(t_1)) - G(S(t_1))O(t_1) < F(K,O(t_2)) - G(S(t_2))O(t_2)$. Moreover, we have $F_O(K,O(t_i)) = G(S(t_i)) + \mu(t_i)/\mu_K(t_i), i = 1,2$. From the fact that we have the same capital stock at both instants of time but higher social cost of oil in t_2 we have $\mu(t_2)/\mu_K(t_2) > \mu(t_1)/\mu_K(t_1)$. It then follows that $F_O(K,O(t_1)) < F_O(K,O(t_2))$ so that $O(t_2) < O(t_1)$. Reduce $O(t_1)$ to $O(t_2)$. Then we have $F(K,O(t_2)) - G(S(t_1))O(t_2) = F(K,O(t_2)) - G(S(t_2))O(t_2) + (G(S(t_2)) - G(S(t_1))O(t_2)$ $> F(K,O(t_1)) - G(S(t_1))O(t_1)$. So, by decreasing $O(t_1)$ we get more net production and less pollution. Welfare can be improved and we were not on an optimal path. In case 2 we have $F(K,R(t_2)) - bR(t_2) > F(K,O(t_1)) - G(S(t_1)O(t_1)$. Indeed, we now have $F_R(K,R(t_2)) = b > G(S(t_1))$ and $F_O(K,O(t_1)) < b$ from the necessary conditions ($b \geq G(S(t_1)) + \mu(t_1)/\mu_K(t_1)$). Hence, $R(t_2) < O(t_1)$ and the same contradiction is reached. Q.E.D.

Lemma A2: A period of time with oil use cannot be interrupted by a phase with only use of the backstop.
Proof: Suppose that this does not hold. Assume that at $t_1 > 0$ a transition takes place from oil use to use of the backstop only, and that the reverse takes place at $t_2 > t_1$. Then, according to (11b) and (11c) $b = G(S(t_1)) + (\mu_S(t_1) + \mu_E(t_1))/\mu_K(t_1) = G(S(t_2)) + (\mu_S(t_2) + \mu_E(t_2))/\mu_K(t_2)$ implying from $S(t_1) = S(t_2)$ that $(\mu_S(t_1) + \mu_E(t_1))/\mu_K(t_1) = (\mu_S(t_2) + \mu_E(t_2))/\mu_K(t_2)$. Note that the initial stocks only differ with regard to capital. Suppose $K(t_1) > K(t_2)$. Then $(\mu_S(t_1) + \mu_E(t_1))/\mu_K(t_1) > (\mu_S(t_2) + \mu_E(t_2))/\mu_K(t_2)$ since capital is relatively scarce at t_2. Hence we obtain a contradiction. This same argument can be used to show that $K(t_1) < K(t_2)$ is ruled out. Hence $K(t_1) = K(t_2)$. Now, with equal stocks at t_1 and t_2 the programs to be pursued should be equal as well. This is a contradiction.

Appendix 2: Details of solving for the optimal time paths of regime II

The dynamics of consumption and the stock of manmade capital during the final oil- renewables phase ($t \geq T$) are given by (3OR) and (8OR). The indifference condition (7OR) gives the stock of oil in situ:

(A1) $\quad S = S(C,K).$

Differentiating (A1) and using (3OR) and (7OR), we obtain the oil-depletion dynamics:

$$\text{(A2)} \qquad O = -\dot{S} = \frac{-S_C \sigma C\left[\tilde{F}_K(K,b) - \delta - \rho\right] - S_K\left[\tilde{F}(K,b) - \delta K - C\right]}{1 + S_K\left[b - G(S)\right]}.$$

Equations (3OR) and (7OR) with *S* given by (A1) and *O* given by (A2) can be solved as a two-dimensional, two-point-boundary value problem for a given switch time *T* and *K*(*T*). The saddlepath corresponding to the stable manifold of this system is denoted by $C(t) = \Theta^{OR}(K(t);b)$. Denoting deviations from the carbon-free steady state by Δ, the saddlepath can be written as $\Delta C = \theta \Delta K$ with the guess $\Theta^{OR} = \theta.$ We get $\Delta\dot{C} = \sigma F_{KK} C^* \Delta K = \theta(\alpha_1 \Delta K - \alpha_2 \Delta C),$ where the linearization around steady state gives $\alpha_1 \equiv \left\{\rho - \sigma F_{KK} C^* S_C [b - G(S^*)]\right\} \alpha_2 > 0$ and $\alpha_2 \equiv \left\{1 + [b - G(S^*)] S_K\right\}^{-1} > 0.$ The method of undetermined coefficients requires solving the quadratic $\alpha_2 \theta^2 - \alpha_1 \theta + \sigma F_{KK} C^* = 0,$ which yields the solution $\theta = \left(0.5\alpha_1 \pm 0.5\sqrt{\alpha_1^{\ 2} - 4\alpha_2 \sigma F_{KK} C^*}\right) / \alpha_2.$ Picking the positive value of θ corresponds to the saddlepath associated with the stable manifold and gives $\Theta_K^{OR} = 0.5\sqrt{r^{*2} - 4r^* \sigma F_{KK} C^*} + 0.5r^* > 0,$ where $r^* \equiv \rho - \sigma F_{KK} C^* S_C \left[b - G(S)\right] > \rho > 0.$ Note that as $r^* > \rho,$ we have $\Theta_K^{OR} > \Theta_K^R > 0.$

Pasting the oil-only and oil-renewables phases

The solution of the final oil-renewables phase also gives consumption at the beginning of that phase, $C(T) = \Theta^{OR}\left(K(T), b\right),$ where $\Theta^{OR}(.)$ is the stable manifold of the system. This serves as terminal condition for the oil-only phase. The switch time *T* is chosen such that the *S*(*T*) at the end of the oil-only phase matches the oil stock at the beginning of the oil-renewables phase, so from (A1) we require $S(T) = S\left(\Theta^{OR}\left(K(T), b\right), K(T)\right).$ Similarly, we require that capital at the end of the oil-only phase must equal capital at the beginning of the oil-renewables phase. Since energy prices and energy use must be continuous at time *T*, renewables use starts with a positive amount and thus oil use must fall at time *T* by a corresponding amount. Oil and renewables use are thus not continuous at time *T*.

Appendix 3: Spectral decomposition algorithm for solving the oil-scarce economy

We have used a fourth-order Runge-Kutta algorithm to solve (1O), (3O), (7O) and (8O) given $K(0) = K_0, S(0) = S_0$ and guesses for $C(0)$ and $p(0)$; and Gauss-Newton iterations to adjust $C(0)$, $p(0)$ and T to satisfy $p(T) = b$, $b - G(S(T)) = \dfrac{D'(E_0 + S_0 - S(T))}{\rho U'\left(C(T)\right)}$, and $C(T) = \Theta^R\left(K(T), b\right)$. This was numerically sensitive, so we report the results from our linearized model. We now describe the algorithm that we used for this purpose.

The carbon-free phase starts at time *T* and is given by (3R) and (8R) starting with the initial condition *K*(*T*). Linearizing this system around the steady state gives:

(A3) $\dot{K} = \rho(K - K^*) - (C - C^*), \qquad \dot{C} = \sigma C^* \tilde{F}_{KK}(K - K^*).$

Suppose the stable manifold of the solution to this system is $C - C^* = \theta(K - K^*),$ then substitution yields:

(A4) $\dot{C} = \sigma C^* \tilde{F}_{KK}(K - K^*) = \theta \dot{K} = \theta\left[\rho(K - K^*) - \theta(K - K^*)\right].$

Equating the coefficients on $K - K^*$ gives the following quadratic equation in θ:

(A5) $\theta^2 - \rho\theta + \sigma C^* \tilde{F}_{KK} = 0.$

Choosing the positive solution to θ corresponding to the stable manifold gives:

(A6) $\theta = 0.5\rho + 0.5\sqrt{\rho^2 - 4\sigma C^* \tilde{F}_{KK}} > 0.$

Since with Cobb-Douglas production $F(K,R) = K^{\alpha} R^{\beta}$ we have $R = \beta F/b$, $F = [K^{\alpha}(\beta/b)^{\beta}]^{1/(1-\beta)}$, $\tilde{F} = (1-\beta)[K^{\alpha}(\beta/b)^{\beta}]^{1/(1-\beta)}$, $\tilde{F}_K = \alpha F/K$ and $\tilde{F}_{KK} = -(1-\alpha-\beta)\tilde{F}_K/(1-\beta)K,$ we get:

(A7) $\theta = 0.5\rho + 0.5\sqrt{\rho^2 + 4\sigma\left(\frac{1-\alpha-\beta}{1-\beta}\right)(\rho+\delta)C^*/K^*} > 0.$

$C(T)$ must be on the saddlepath of the carbon-free economy, hence we must have:

(A8) $C(T) = \Theta^R\left(K(T),b\right) \cong C^* + \theta\left(K(T) - K^*\right),\ \Theta^{R}{}' = \theta > 0.$

We have linearized around (S^*, K^*, C^*, b) with S^* from $b = G(S^*) + \frac{D'(E_0 + S_0 - S^*)}{\rho U'(C^*)}$. Defining $R^* = V(K^*, b),$ we get the state-space system for the oil-scarce economy (1O), (3O), (7O) and (8O):
$\dot{\underset{\sim}{x}} = \mathrm{A}\underset{\sim}{x} + \underset{\sim}{a},$ where $\underset{\sim}{x} \equiv (S - S^*, K - K^*, C - C^*, p - b)'$ and $\underset{\sim}{a} \equiv (-R^*, K^{*\alpha}R^{*\beta} - G(S^*) - \delta K^* - C^*, 0, 0)'$.
Spectral decomposition gives $\mathrm{A} = \mathrm{M}\Lambda\mathrm{M}^{-1} = \mathrm{N}^{-1}\Lambda\mathrm{N}$ where the diagonal matrix Λ contains the eigenvalues in descending order and the matrix M contains the eigenvectors. From the saddlepoint property the system has two eigenvalues with positive real part (provided the discount rate is not too large), collected in the vector $\underset{\sim}{\lambda}_u$, and two with negative real part, collected in $\underset{\sim}{\lambda}_s$, and thus satisfies the saddlepoint property. We thus have $\Lambda = \mathrm{diag}(\lambda_{u1}\lambda_{u2}, \lambda_{s1}, \lambda_{s2})$. Defining the canonical variables $\underset{\sim}{y} = \mathrm{N}\underset{\sim}{x}$ yields $\dot{\underset{\sim}{y}} = \Lambda\underset{\sim}{y} + \underset{\sim}{n},\ \ \underset{\sim}{n} \equiv \mathrm{N}\underset{\sim}{a}$. We can solve this canonical system as follows:

(A9)
$$y_{ui}(t) = e^{\lambda_{ui}(t-T)}\left[y_{ui}(T) + \bar{n}_{ui}\right] - \bar{n}_{ui},\quad \bar{n}_{ui} \equiv n_{ui}/\lambda_{ui},\quad i = 1,2,$$
$$y_{si}(t) = e^{\lambda_{si}t}\left[y_{si}(0) + \bar{n}_{si}\right] - \bar{n}_{si},\quad \bar{n}_{si} \equiv n_{si}/\lambda_{si},\quad i = 1,2,\quad \forall t \in [0,T].$$

Decomposing so that $\mathrm{M} = \begin{pmatrix} \mathrm{M}_{su} & \mathrm{M}_{ss} \\ \mathrm{M}_{uu} & \mathrm{M}_{us} \end{pmatrix}$ and $\underset{\sim}{x} = (\underset{\sim}{x}_s{}', \underset{\sim}{x}_u{}')'$, we write the initial conditions as:

(A10)
$$\underset{\sim}{x}_s(0) = \mathrm{M}_{ss}\underset{\sim}{y}_s(0) + \mathrm{M}_{su}\underset{\sim}{y}_u(0) = \mathrm{M}_{ss}\underset{\sim}{y}_s(0) + \mathrm{M}_{su}\left\{\begin{pmatrix} e^{-\lambda_{u1}T} & 0 \\ 0 & e^{-\lambda_{u2}T} \end{pmatrix}\left[\underset{\sim}{y}_u(T) + \underset{\sim}{\bar{n}}_u\right] - \underset{\sim}{\bar{n}}_u\right\}$$
$$= \underset{\sim}{x}_{so} \equiv (S_0 - S^*, K_0 - K^*)'.$$

The terminal conditions $C(T) = \Theta\left(K(T)\right)$ given above and $p(T) = b$ become:

$$\mathrm{E}\underset{\sim}{x}_s(T)+\underset{\sim}{x}_u(T)=\mathrm{E}\left[\mathrm{M}_{su}\underset{\sim}{y}_u(T)+\mathrm{M}_{ss}\underset{\sim}{y}_s(T)\right]+\left[\mathrm{M}_{uu}\underset{\sim}{y}_u(T)+\mathrm{M}_{us}\underset{\sim}{y}_s(T)\right]=$$

$$\text{(A11)}\quad \mathrm{E}\left(\mathrm{M}_{su}\underset{\sim}{y}_u(T)+\mathrm{M}_{ss}\left\{\begin{pmatrix} e^{\lambda_{s1}T} & 0 \\ 0 & e^{\lambda_{s2}T}\end{pmatrix}\left[\underset{\sim}{y}_s(0)+\underset{\sim}{\overline{n}}_s\right]-\underset{\sim}{\overline{n}}_s\right\}\right)+$$

$$\mathrm{M}_{uu}\underset{\sim}{y}_u(T)+\mathrm{M}_{us}\left\{\begin{pmatrix} e^{\lambda_{s1}T} & 0 \\ 0 & e^{\lambda_{s2}T}\end{pmatrix}\left[\underset{\sim}{y}_s(0)+\underset{\sim}{\overline{n}}_s\right]-\underset{\sim}{\overline{n}}_s\right\}=\underset{\sim}{0} \text{ where } \mathrm{E}=\begin{pmatrix} 0 & -\theta \\ 0 & 0\end{pmatrix}.$$

The initial and terminal conditions (A10) and (A11) can be solved as follows:

$$\begin{pmatrix}\underset{\sim}{y}_s(0) \\ \underset{\sim}{y}_u(T)\end{pmatrix}=\begin{pmatrix} \mathrm{M}_{ss} & \mathrm{M}_{su}\begin{pmatrix} e^{-\lambda_{u1}T} & 0 \\ 0 & e^{-\lambda_{u2}T}\end{pmatrix} \\ (\mathrm{M}_{us}+\mathrm{EM}_{ss})\begin{pmatrix} e^{\lambda_{s1}T} & 0 \\ 0 & e^{\lambda_{s2}T}\end{pmatrix} & \mathrm{M}_{uu}+\mathrm{EM}_{su}\end{pmatrix}^{-1}\begin{pmatrix} \underset{\sim}{x}_{s0}+\mathrm{M}_{su}\begin{pmatrix} 1-e^{-\lambda_{u1}T} & 0 \\ 0 & 1-e^{-\lambda_{u2}T}\end{pmatrix}\underset{\sim}{n}_u \\ \left(\mathrm{M}_{us}+\mathrm{EM}_{ss}\right)\begin{pmatrix} 1-e^{\lambda_{s1}T} & 0 \\ 0 & 1-e^{\lambda_{s2}T}\end{pmatrix}\underset{\sim}{n}_s\end{pmatrix}.$$

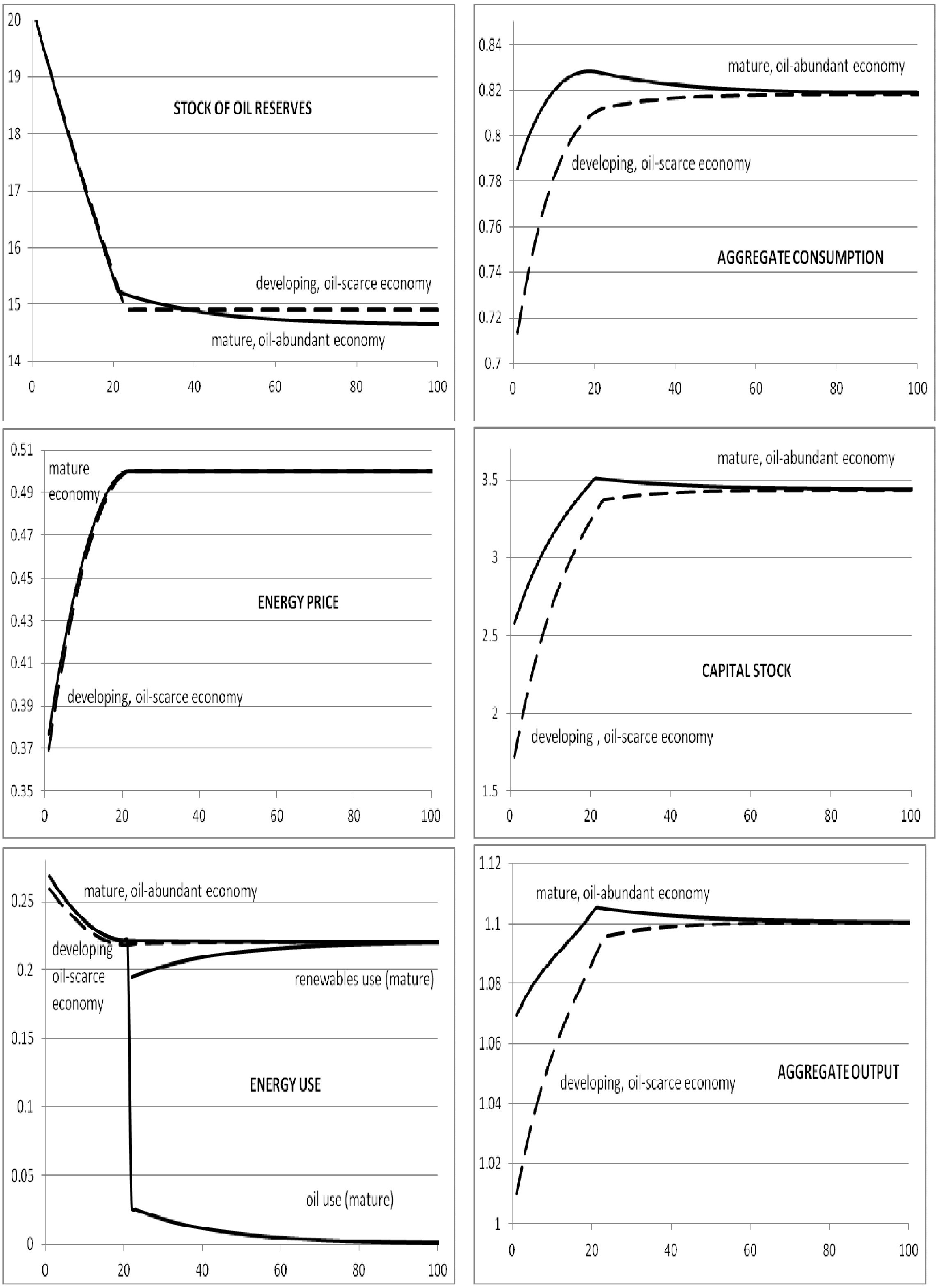
Given this we can calculate $\underset{\sim}{y}$ from (A9) and thus finally obtain $\underset{\sim}{x}=\mathrm{M}\underset{\sim}{y}$ for $\forall t\in[0,T]$. The resulting solution trajectories satisfy the necessary initial and terminal conditions, (A10) and (A11). To obtain the switch time, we solve for time *T* from $b=G(x_{s1}(T)+S^*)+\dfrac{D'(E_0+S_0-x_{s1}(T)-S^*)}{\rho U'(x_{u1}(T)+C^*)}$. This procedure implies that *S*(*T*) depends on *C*(*T*). Given *K*(*T*) obtained from the fossil-fuel economy, the carbon-free economy can be found from multiple shooting or directly from linearization:

$$\text{(A12)}\quad \begin{aligned}&\dot{K}(t)=K^*+(\rho-\theta)\left[K(t)-K^*\right]\ \Rightarrow\ K(t)=K^*+e^{(\rho-\theta)(t-T)}\left[K(T)-K^*\right]\ \text{ and}\\ &C(t)=C^*+\theta e^{(\rho-\theta)(t-T)}\left[K(T)-K^*\right],\quad \forall t\geq T,\ \text{where } \theta>\rho.\end{aligned}$$

Appendix 4: Overshooting in a mature, oil-abundant economy

If we start off with $S_0=20$ and $K_0=0.75K^*$, we get $S_0^*=18.7<20.8$ and $T^*=16.4<24.3$. The paths for the social optimum starting with $K_0=0.5K^*$ and starting with $K_0=0.75K^*$ are shown in fig. A1. They are qualitatively very similar to the paths of fig. 4 for an oil-abundant economy. This oil-abundant economy starts with a sufficiently high initial capital stock to be in regime II. It thus starts off with a higher rate of consumption, but ends up with the same capital stock and rate of consumption in steady state. The rate of consumption overshoots first and sometime later capital overshoots its steady-state value; this does not occur in a developing, oil-scarce economy. In the more mature economy $T=20.1>T^*=16.4$, but in the developing economy $T=22>T^*=24.3$. The more mature economy leaves less oil in the crust of the earth than the developing economy, which results from oil use throughout the only-oil phase being less than in the developing economy and no oil being use alongside renewables in the final phase. The more negative impact on the climate in a developing economy is less as, despite advancing more rapidly along its development path, oil use is somewhat less in the oil-only phase and zero in the final renewables-only phase. This is why the mature economy has higher carbon tax than the developing economy for the oil-only phase.

Figure A1: Comparing social optimum for a mature economy with a developing economy



Note: Mature economy $T = 20.1$, $S(T) = 15.2$; developing economy $T = 22$, $S(T) = 14.9$.