Regularity for Almost Minimizers in Variational Problems

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Declaration of Authorship

I hereby declare that, to the best of my knowledge, the contents of this thesis are original and entirely my own work, except where otherwise indicated, cited, or commonly known. Nor has this thesis been submitted, either wholly or substantially, for another course of this Department or University, or for a course at any other institution.

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Abstract

The aim of this thesis is to contribute to the regularity theory in the calculus of variations, with a particular focus on almost minimizers. Specifically, we study (quasi-)convex variational functionals of the form

$$\mathscr{F}(u,\Omega) := \int_{\Omega} f(\nabla u)\,\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, u is an $\mathbb{R}^N$-valued map defined on Ω, and the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is of p-growth with $p \geq 1$ in the following sense: there exists a constant $L > 0$ such that

$$|f(z)| \leq L(1+|z|^p), \text{for any } z \in \mathbb{R}^{N\times n}.$$

We establish various types of regularity properties for almost minimizers, including quasiminimizers and ω-minimizers, each result being obtained under distinct natural assumptions. In particular, we emphasize that in the part concerning partial regularity (Chapter 5), no additional lower bound is posed on f, and the convexity condition assumed for f is Morrey's quasiconvexity — the natural framework in the vectorial setting.

In Chapter 4, our focus is on the p-Dirichlet energy $\mathcal{D}_p$ with $p > 1$, and we investigate the stability of quasiminimizers with respect to p. More precisely, we examine whether a quasiminimizer maintains its quasiminimality as the exponent p varies. We establish stability in the case where p increases slightly, based on a global higher integrability result. To see the behaviour of quasiminimizers as the exponent p decreases, we turn our attention to p-harmonic maps, and prove the stability of their (quasi-)minimality as p varies within a small range. Our main tools for the second result involve the Hodge decomposition on bounded regular domains and a certain type of nonlinear commutator. These tools allow us to quantify the q-Laplacian of a p-harmonic map when p and q are close to each other.

The second part (Chapter 5) concerns the partial regularity of ω-minimizers under certain quasiconvexity conditions. The first result addresses the linear growth setting ($p = 1$), where variational problems are typically relaxed to BV spaces. We establish partial $C^{1,\alpha}$ regularity for BV ω-minimizers, assuming a Dini-type condition on ω. This result is achieved through an excess decay estimate strategy, for which we utilise Ekeland's variational principle and conduct a direct harmonic approximation process as in [GK19a]. In addition, we explore the regularity of ω-minimizers without requiring any extra assumptions on ω. A partial $C^{0,\alpha}$ regularity result is obtained in the sub-quadratic context ($1 < p < 2$) by appropriately normalising the typical excess, drawing inspiration from [FM08]. By combining the arguments of the two aforementioned results, we establish partial $C^{0,\alpha}$ regularity for BV ω-minimizers of linear growth functionals.

In the final chapter (Chapter 6), the focus shifts to the Sobolev regularity of ω-minimizers, particularly in the linear growth context under the μ-ellipticity condition. This chapter is based on collaborative work with Jan Kristensen (Oxford). We examine

the regularity of BV ω-minimizers, and establish a fractional Sobolev regularity result for the absolutely continuous parts of their derivatives. This result is obtained with a Sobolev regularity result for minimizers implicitly contained in [BS13], and the argument encompasses two ingredients: an inequality characterisation of the extremality of minimizers, and a comparison of ω-minimizers and the corresponding minimizers.

Contents

Chapter 1

Introduction

The calculus of variations is a field that focuses on describing the optimal configurations of problems that arise in various areas, including differential geometry, materials science, optimal control theory and so on. In the so-called non-parametric setting, the fundamental problem can be specifically formulated as the following variational principle:

$$\text{to minimise } \mathscr{F}(u,\Omega) := \int_\Omega f(x,u,\nabla u)\,\mathrm{d}x \quad \text{over a class } \mathcal{U}, \tag{1.0.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open set. The integrand f captures the underlying physical or geometric properties of the problem and plays a crucial role in determining the optimal configurations. The class $\mathcal{U}$ is a set of admissible functions that satisfy certain constraints.

This area has seen significant developments throughout the 20th century, partially spurred by the announcement of Hilbert's 23 problems [Hil00], the last one of which is "further development of the calculus of variations". The introduction of new convexity notions provides various sufficient conditions for the existence of minimizers, which is relevant to the 20th problem. There have also been a number of regularity and non-regularity results, which are associated to the 19th problem. In spite of these advancements, there are still many problems in this field that remain open and require further understanding, especially in the context of variational problems involving vector-valued input functions.

The current thesis is focused on the regularity for almost minimizers, which encompasses both quasiminimizers and ω-minimizers. It is worth noticing that this is slightly different from the main body of the existing literature, as almost minimizers and ω-minimizers are often regarded as the same class. These relaxed minimality notions have arisen from diverse problems, such as variational problems with constraints and certain PDEs that may not have a direct connection to any variational problems.

The variational principles under consideration involve integrands of p-growth with $p \geq 1$. One of the objectives of this thesis is to explore the stability of quasiminimizers with respect to p in the super-linear setting ($p > 1$), which is the focus of Chapter 4. In addition, a significant emphasis is placed on the regularity of ω-minimizers, primarily within the linear growth context ($p = 1$). Chapter 5 and 6 are dedicated to the investigation of different types of regularity properties under a number of distinct natural conditions. There is a preliminary overview of the problem formulations and results in Section 1.3, and the subsequent chapters will provide comprehensive details on each topic.

In the following sections, we have a concise overview of the concepts of quasiminimizers and ω-minimizers, as well as the study of variation principles with linear growth functionals. A discussion of regularity theory is postponed to Chapter 3, which follows precise definitions of these minimality notions with illustrative examples.

1.1 Almost minimizers

1.1.1 Quasiminimizers

The notion of quasiminimizers was introduced in the non-parametric context by Giaquinta & Giusti in [GG82]. A quasiminimizer of a (non-negative) functional $\mathscr{F}$ is defined by the inequality

$$\mathscr{F}(u,K) \leq Q\mathscr{F}(v,K), \quad \text{for any } v \text{ with } \operatorname{supp}(u-v)=K, \tag{1.1.1}$$

where $Q \geq 1$ is a constant. See Subsection 3.1.2 for the precise definition. The purpose of introducing this concept was to unify the analysis of certain PDE solutions and minimizers in variational problems. Indeed, it is well-known that the minimizers of variational problems with regular integrands satisfy the corresponding Euler-Lagrange equations. Conversely, the solutions to some elliptic equations of divergence form are quasiminimizers of certain corresponding functionals, even if the equations are non-variational, i.e., cannot be obtained by varying any functionals. Moreover, no regularity of the coefficients is needed to conclude quasiminimality from such equations. Therefore, the study of quasiminimizers can provide a unifying approach to some regularity results for both objects.

In addition to elliptic equations, quasiminimizers also appear in other problems. For instance, quasi-regular maps and solutions to certain obstacle problems both fall within the category of quasiminimizers. In Subsection 3.1.2, the aforementioned examples are presented with more details to provide a clearer idea of this concept.

As mentioned above, minimizers of regular functionals fulfil the corresponding Euler-Lagrange equations. Thus, it is possible to investigate the regularity for minimizers by studying those equations, for which we refer to Chapter 8, 9, 10 in [Giu03] and the references therein. However, the defining inequality (1.1.1) does not allow us to derive an Euler-Lagrange equation in any obvious or canonical way as for classical minimizers, and only variational methods can be applied in the study of quasiminimizers. On the other hand, this also allows us to consider a wider range of problems as there is no need to assume any smoothness of the integrands.

Quasiminimality defined in (1.1.1) implies Caccioppoli-type inequalities, and scalar quasiminimizers are further in De Giorgi classes ([Giu03], Chapter 6 and 7). With these two properties, some regularity results for minimizers are also approachable within the framework of quasiminimizers. In particular, quasiminimizers of the Dirichlet energy $\mathcal{D}_p$ (denoted by (p,Q)-minimizers, see Definition 3.1.2) exhibit certain characteristics similar to p-harmonic maps.

The concept of quasiminimizers was briefly mentioned in [GG82], and the authors later conducted a relatively comprehensive study of it in [GG84a]. A higher integrability result was obtained with a generalised version of Gehring's lemma. For scalar quasiminimizers, they established Hölder regularity, and then a weak form of the maximum principle and Liouville's theorem with the techniques developed in [GG82]. Moreover, the authors also discussed quasiminimizers defined on 1-dimensional intervals and their stability under Γ-convergence.

Subsequent developments in the regularity of quasiminimizers have been primarily focused on the scalar case. MALÝ [Mal83] established the strong maximum principle, Liouville's theorem and Harnack's inequality for (p,Q)-minimizers with restrictions on the exponent p. In [DBT84], DI BENEDETTO and TRUDINGER obtained Harnack's inequality for general quasiminimizers by utilising a covering lemma by KRYLOV and SAFONOV [KS80], and their result was obtained near the boundary of any bounded domain by ZIEMER [Zie86]. Some properties of (sub)harmonic functions were extended to quasi(sub)minimizers in [Tol86], with which the author further explored the removability of singularities. Based on [GG84a], MARTIO and SBORDONE thoroughly discussed (p,Q)-minimizer defined on 1-dimensional intervals in [MS07], and there is an extract of the first half of their paper in Section 4.3 to provide some inspiration for the study in higher dimensions.

Quasiminimizers are also studied in the set-up of metric measure spaces equipped with a doubling measure and supporting a Poincaré inequality, with most of the regularity results mentioned above retained; see [KS01, KM03, KMM07, KKL10, KS12]. An analogue of this notion was introduced in the parabolic context in [Wie87] with further developments in [Par08, MMPP13, FHKM14, Hab15, MP15, Hab16, FH17, FHM18]. The list here is far from complete, and more relevant results can be found in the references of the above works.

1.1.2 ω-minimizers

The concept of ω-minimizers involves certain non-negative functions ω defined on $[0,\infty)$, which vanish as the argument goes to 0. In literature, the category of ω-minimizers is also known as almost minimizers. Nevertheless, in the current thesis we adopt a broader definition of almost minimizers, which encompasses quasiminimizers as well, and stick to the term "ω-minimizer" for the specific sub-class (precisely defined in Subsection 3.1.3).

ALMGREN introduced the notion of ω-minimizers in the context of geometric measure theory in [Alm76], where the term "(F,ε,δ)-minimal set" was used. He considered this category as it fits the framework of a wide class of geometric variational problems with constraints, such as obstacle problems, minimal partitions, capillarity problems and variational problems with partially free boundaries; see [Alm76], §III.2, [Lin85], [DS02] and [Mag12], §21.1 for concrete examples.

This concept was generalised to the non-parametric setting by ANZELLOTTI in [Anz83]. Roughly speaking, a map $u\colon \Omega \to \mathbb{R}^N$ is an ω-minimizer of a p-growth functional $\mathscr{F}$ if it satisfies

$$\mathscr{F}(u,B_R) \le \mathscr{F}(v,B_R) + \omega(R)\int_{B_R} (1+|\nabla v|^p)\,\mathrm{d}x \tag{1.1.2}$$

for any ball $B_R \Subset \Omega$ satisfying a prescribed size condition and any map v with $\operatorname{supp}(u-v) \Subset B_R$. Precise definitions are given in Section 3.1, and see Section 2.1 for notation. This formulation bears strong resemblance to the one in geometric measure theory, and many regularity results for ω-minimizers in the two contexts are also analogous to some extent.

Similar to the geometric context, ω-minimizers also appear in variational problems with constraints, and examples of such problems include obstacle problems and volume-constrained problems. More specifically, the solutions to certain constrained problems are ω-minimizers of corresponding functionals without constraints. Moreover, the minimizer in a non-autonomous problem can be regarded as an ω-minimizer of a family of autonomous functionals. See Section 3.1 for specific instances and illustrations, which are extracted from [Anz83] and [DGG00].

It is worth mentioning that any $C^{1,\alpha}$ map is an ω-minimizer of the Dirichlet energy $\mathcal{D}_2$ with $\omega(r) \sim r^{2\alpha}$. Indeed, given any ω satisfying a Dini-type condition, it is possible to construct an ω-minimizer of $\mathcal{D}_2$ (as elaborated in Subsection 3.1.3 and [DGG00]). This observation indicates the existence of a significant number of ω-minimizers, and also provides the optimal regularity one can expect from ω-minimizers with a prescribed ω.

Based on the definition, an ω-minimizer can be understood as a local perturbation of the corresponding minimizer. Consequently, one may expect that the regularity for minimizers also holds for ω-minimizers to some extent. The specific extent is determined by the rate at which $\omega(r)$ decreases to 0 as $r \to 0$. The reality fits such expectation well in the study of partial regularity for (ω-)minimizers of quasiconvex functionals, as evidenced by the comparison of the results in Chapter 5 (cf. [Anz83, DGG00]).

However, the situation becomes more intricate when we aim for (fractional) Sobolev regularity in the context of linear growth functionals. Even with a reasonably good ω (i.e., an ω that decreases to 0 at a reasonably fast rate), ω-minimizers can exhibit singular behaviour as demonstrated in Subsection 6.3.2. This suggests the challenges of achieving certain types of regularity for ω-minimizers, particularly in less regular set-ups such as variational problems with linear growth integrands.

Analogous to quasiminimizers, no Euler-Lagrange equations are readily available for ω-minimizers based on the minimality defined as in (1.1.2). Instead, it is typically possible to derive an Euler-Lagrange-type inequality, which allows one to utilise the corresponding (linearised) elliptic equation to establish certain regularity results as in the case of minimizers. This strategy is used in Chapter 5.

An alternative and perhaps more obvious approach to studying the regularity of ω-minimizers is to compare them directly with the corresponding minimizers. Since the deviation of the former from the latter is measured by $\omega(R)$ over a ball with radius R, and can be suitably small under appropriate assumptions on ω, the regularity for minimizers can be then partially transferred to ω-minimizers. Detailed proofs illustrating this approach can be found in Section 6.5 and [Giu03], §8.5.

On the other hand, the approximation schemes, which are commonly employed for minimizers (and for PDE solutions) with low regularity, do not readily adapt to ω-minimizers. The perturbation inherent in ω-minimizers provides more flexibility, but also makes it more difficult to construct a sequence that converges to a specific ω-minimizer. See Chapter 6 for more details.

Extensive studies have been conducted on the regularity of ω-minimizers in various contexts.

As mentioned above, the study of ω-minimizers in geometric measure theory was initiated in [Alm76], where the author established almost everywhere regularity for m-dimensional (F, ε, δ)-minimal sets with $m \geq 2$. This result was then applied to tackle the minimal partitioning problem. TAYLOR [Tay76] showed that a 2-dimensional (M, cr^{α}, δ)-minimal set exhibits precisely the singularities observed in soap bubble clusters. In [Mor94], MORGAN investigated (M, cr^{α}, δ)-minimal curves, which filled the missing regularity in the 1-dimensional setting. TAMANINI explored the regularity for almost minimal boundaries in [Tam84]. Recent developments in this field can be found in [DPM15, DPM17, MSS19] and the references therein.

In the field of non-parametric problems, ANZELLOTTI first investigated the regularity of ω-minimizers in [Anz83], and proved first-order Hölder regularity for the scalar ω-minimizers of certain quadratic functionals with $\omega(r) \sim r^{2\alpha}$ near the origin. Zero-order Hölder regularity was then obtained in [DEF96, EM99] under more general assumptions. Partial regularity results were subsequently established in the vectorial case in [DGG00, DK02, DGK05], where the functionals are assumed to be quasiconvex and of p-growth with $p > 1$. The authors of [KM05] showed certain fractional Sobolev regularity of the ω-minimizers of convex p-growth functionals with $p > 1$, and further utilised this result to estimate the Hausdorff dimension of the corresponding singular sets. Moreover, partial regularity was achieved by SCHMIDT [Sch14] for the ω-minimizers of convex functionals in the linear growth setting.

There have also been studies of ω-minimizers in the context of free-boundary problems in [DT15, DET19, DESVGT21, DESVGT23]. These studies focus on more specific functionals, which enables the authors provide clear characterisation of the regularity as well as the singularities of the associated ω-minimizers.

1.2 Linear growth functionals

Linear growth functionals comprise a class of variational functionals with integrands that grow linearly at infinity. More precisely, consider functionals of the form

$$\mathscr{F}(u, \Omega) := \int_{\Omega} f(x, \nabla u)\, \mathrm{d}x, \quad u\colon \Omega \to \mathbb{R}^N,$$

where $\Omega \subset \mathbb{R}^n$ is an open set and the integrand $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies the growth condition

$$|f(x, z)| \leq L(1 + |z|), \quad (x, z) \in \Omega \times \mathbb{R}^{N\times n}$$

for some $L > 0$.

This type of functionals arises in various variational problems. One well-known example in the scalar case ($N = 1$) is the study of minimal surfaces [Giu84], for which the area integrand $f(\cdot) = (1 + |\cdot|^2)^{\frac{1}{2}}$ is considered. In image restoration models [ROF92, CL97], the corresponding integrand often takes the form $f(\cdot) = |\cdot|$. In the past years, such variational problems have been investigated in more general settings, including both the scalar ($N = 1$) and the vectorial ($N > 1$) case.

Variational principles with p-growth functionals for $p > 1$ are typically analysed on the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$. Similarly, one can define linear growth functionals on the space $W^{1,1}(\Omega, \mathbb{R}^N)$ and seek minimizers in (certain sub-classes of) this class. However, challenges arise in the application of the direct method in this context. Given a competitor class $\mathcal{U}$ on which $\mathscr{F}$ is bounded from below, the process of establishing the existence of a minimizer in $\mathcal{U}$ of $\mathscr{F}$ with the direct method can be divided into four steps as follows:

(i) take a minimizing sequence $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{U}$ such that

$$\mathscr{F}(u_k,\Omega)\to\inf_{\mathcal{U}}\mathscr{F}(\cdot,\Omega)\quad\text{as } k\to\infty;$$

(ii) the coercivity of $\mathscr{F}$ implies the uniform boundedness of $\{u_k\}$ in $\mathcal{U}$;

(iii) the compactness of $\mathcal{U}$ and the uniform boundedness of $\{u_k\}$ imply the existence of a subsequence of $\{u_k\}$ that converges to some $u_\infty\in\mathcal{U}$ in an appropriate sense;

(iv) the lower semicontinuity of $\mathscr{F}$ on $\mathcal{U}$ implies that u_∞ minimizers $\mathscr{F}$ within $\mathcal{U}$.

The space $W^{1,1}(\Omega,\mathbb{R}^N)$ is non-reflexive, and thus we cannot expect bounded subsets of it to be compact, which results in an issue in step (iii). This suggests that the space $W^{1,1}$ is too restrictive, necessitating a broader class $\mathcal{U}$ for variational principles involving linear growth functionals. Simultaneously, to maintain a connection to the original problem, it is reasonable to impose the following requirements:

(a) the class $\mathcal{U}$ should include $W^{1,1}(\Omega,\mathbb{R}^N)$ as a dense subspace with respect to a proper topology;

(b) the functional $\mathscr{F}$ should be continuous on $\mathcal{U}$ under the chosen topology.

It turns out that $BV(\Omega,\mathbb{R}^N)$ equipped with the area-strict topology is a suitable choice. Correspondingly, given a bounded open set $\Omega\subset\mathbb{R}^n$, the functional $\mathscr{F}(\cdot,\Omega)$ is relaxed to BV maps by the formula

$$\bar{\mathscr{F}}(u,\Omega)=\int_\Omega f(x,\nabla u)\,\mathrm{d}x+\int_\Omega f^\infty\left(x,\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u|,$$

where

$$Du=\nabla u\mathscr{L}^n\llcorner\Omega+D^s u$$

is the Lebesgue-Radon-Nikodým decomposition of the weak derivative of $u\in BV(\Omega,\mathbb{R}^N)$ and f^∞ is the recession function of f. This relaxation $\bar{\mathscr{F}}$ indeed extends $\mathscr{F}$ from $W^{1,1}(\Omega,\mathbb{R}^N)$ to $BV(\Omega,\mathbb{R}^N)$ under the area-strict topology ([Rin18], Theorem 11.2), and thus (b) is fulfilled. For more details about $\bar{\mathscr{F}}$, we refer to Subsection 3.1.4, where there are another two relaxations $\mathscr{F}_*$ and $\mathscr{F}_g$ of $\mathscr{F}$ to BV maps with different restrictions.

The study of variational principles involving linear growth functionals is significantly different from that in the superlinear case due to several obstacles. One such obstacle comes from the low regularity of BV maps. The weak derivative of a BV map is only "integrable" in the L^1-sense, and thus no weak reverse Hölder inequalities are available in this setting. This renders the application of (generalised) Gehring's lemma, as in Chapter 4, ineffective for achieving higher integrability. The possible singular parts of BV maps also make the situation more intricate, which can be seen from Chapter 6. Additional difficulties arise from the non-uniform ellipticity in this context. A standard ellipticity condition imposed on a linear growth integrand f is the so-called μ-ellipticity condition:

$$\ell\frac{|\xi|^2}{(1+|z|^2)^{\frac{\mu}{2}}} \leq f''(z)[\xi,\xi] \leq L\frac{|\xi|^2}{(1+|z|^2)^{\frac{1}{2}}}, \quad \text{for any } \xi \in \mathbb{R}^{N\times n},$$

where ℓ, L and μ are positive constants with $\ell < L$ and $\mu > 1$. It is not difficult to see that f'' is non-uniformly elliptic in the following sense: the ratio

$$\Lambda_f(z) := \frac{\max_{|\xi|=1} f''(z)[\xi,\xi]}{\min_{|\xi|=1} f''(z)[\xi,\xi]},$$

a crucial number for studying regularity properties, may blow-up as $z \to \infty$. It turns out that such an imbalance is inherent to linear growth integrands (see Section 6.1), and it makes this problem degenerate. Nevertheless, when the blow-up occurs at sufficiently slow rate, it is still possible to obtain certain regularity for the gradients of BV minimizers as shown in [BF02, Bil02, BS13, Gme20].

Variational problems with linear growth functionals in the scalar case ($N = 1$), especially the area functional defined by the integrand $f(\cdot) = (1+|\cdot|^2)^{\frac{1}{2}}$, has been widely studied, for which we refer to [Mir64, JS68, BDGM69, BDGG69, Ser69, Giu70, Mir71, Sim76] and the monograph [Giu84]. In [Tau78], Tausch investigated linear growth functionals with the Uhlenbeck structure, in which case the integrand f depends on an input map through the norm of its derivative, i.e., $f(\nabla u) = g(|\nabla u|^2)$ for an appropriate function g. Based on the global gradient estimates in [Ser69] and [Tru72], the existence of Lipschitz minimizers was established. Another type of non-autonomous functionals were examined in [GMS79]. In addition to the existence of minimizers, the authors also presented examples with interior singularities and provided sufficient conditions for interior regularity.

The literature on the vectorial setting, where $N > 1$, is relatively limited.

An early piece of work in this direction is [AG88], where partial $C^{1,\alpha}$ regularity was proved for BV minimizers of convex functionals with linear growth integrands. More recently, the quasiconvex setting was investigated in [GK19a], where a similar result was obtained. The second author subsequently established a similar regularity result for BD minimizers of symmetric gradient functionals in [Gme21].

The study of full regularity under the μ-ellipticity condition was initiated by Bildhauer and Fuchs in [BF02, Bil02], based on previous work by Seregin [Ser85, Ser90, Ser96]. Their results can be divided into two parts. In the case where $\mu = 3$, an $L\log L$-estimate was established for the gradient of one particular BV minimizer. Under the Uhlenbeck structure and an L^∞-assumption, full $C^{1,\alpha}$ regularity and uniqueness of minimizers upto a constant were obtained in the case $1 < \mu < 3$. The $C^{1,\alpha}$ result was then extended to a more general setting in [BS15]. Beck and Schmidt did a thorough investigation of minimizers in the borderline setting $\mu = 3$ in [BS13]. Apart from an $L\log L$-estimate for all minimizers, they also obtained uniqueness of minimizers upto a constant without assuming the Uhlenbeck structure. Moreover, they discovered that all the minimizers of such a variational problem with an area-type integrand form a 1-parameter family, and provided a characterisation of their boundary behaviour as well. Relevant studies in the symmetric gradient setting can be found in [GK19b, Gme21].

1.3 Outline with description of the contribution

As mentioned above, the main purpose of this thesis is to study the regularity of quasiminimizers and ω-minimizers. Regularity theory aims to determine the extent to which (almost) minimizers exhibit better smoothness properties compared to generic competitor maps do. In this sense, the underlying scale of spaces that is used to measure smoothness plays a crucial role, notable examples being the Hölder space scale $\{C^{k,\alpha}\}_{k\in\mathbb{N},\alpha\in(0,1]}$ and the (fractional) Sobolev space scale $\{W^{s,p}\}_{s>0,p\in[1,\infty]}$.

In this section, a brief description of the structure of the current thesis is provided. We present the problems that will be in focus and highlight the main results and the approaches employed in each chapter.

The primary focus will be the following problems:

(**P1**) Stability of quasiminimizers: whether a quasiminimizer of the Dirichlet energy $\mathcal{D}_p$ retains its quasiminimality when the exponent p varies;

(**P2**) Partial regularity for ω-minimizers of quasiconvex functionals under different assumptions on ω;

(**P3**) Sobolev regularity for ω-minimizers of convex functionals.

The following is an outline of this thesis, and the original work is contained in Chapter 4-6.

Chapter 2: Preliminaries. In this preliminary section, we fix general notation used throughout this thesis, and then introduce certain concepts and auxiliary results that will be necessary for our study. The second part covers specific function spaces, the p-capacity, convexity notions and some other topics, as needed in the main body.

Chapter 3: Minimality notions and regularity theory. In the first half of this chapter, the precise definitions of the minimality notions involved in this thesis are presented, accompanied by illustrative examples. In addition, we provide a concise review of the regularity results in the existing literature, with a particular focus on those related to the subsequent chapters.

Chapter 4: Stability of quasiminimizers. This chapter is devoted to the investigation of Problem (**P1**). The problem is motivated by a global higher integrability result for quasiminimizers and weak solutions to elliptic PDEs, and the study of 1-dimensional quasiminimizers, which are revisited in this chapter. As a corollary of the integrability result, we establish that a (p, Q)-minimizer retains its quasiminimality as the exponent p increases slightly. To explore the behaviour of (p, Q)-minimizers when p decreases, we first focus on the minimizers in this setting, considering that their nature is relatively simpler and there are additional analytic tools available. Precisely, we study p-harmonic maps and establish the stability of their (quasi-)minimality when p varies within a small range. The main tools used for this stability result are a specific type of nonlinear commutator and the Hodge decomposition on bounded regular domains, which allow us to quantify the q-Laplacian of a p-harmonic map for any q close to p.

Chapter 5: Partial regularity for ω-minimizers of quasiconvex functionals. Problem **(P2)** is addressed in this chapter. The main result in this chapter is partial $C^{1,\alpha}$ regularity for BV ω-minimizers, which is obtained in the linear growth setting with a Dini-type condition on ω. The strategy used to attain this result involves establishing an excess decay estimate. To achieve this we employ Ekeland's variational principle and utilise a direct harmonic approximation process as in [GK19a]. Without extra assumptions on ω, the above regularity cannot be expected. However, by suitably normalising the typical excess, which is inspired by [FM08], we are able to obtain a partial $C^{0,\alpha}$ result in the sub-quadratic setting $(1 < p < 2)$. Such a technique, unfortunately, does not seem to apply in the borderline case $(p = 1)$ due to the non-uniform ellipticity. Nevertheless, near those "good" points where the average of the derivative is uniformly bounded, it is still possible to establish $C^{0,\alpha}$ regularity for a BV ω-minimizer, thereby to achieve a partial regularity result in this context. This chapter is mainly based on the paper [Li22], with the additional inclusion of partial $C^{0,\alpha}$ regularity for BV ω-minimizers.

Chapter 6: Sobolev regularity for BV ω-minimizers. In this chapter, we consider problem **(P3)** and focus on (fractional) Sobolev regularity of (ω-)minimizers, within the framework of linear growth functionals under the μ-ellipticity condition. Before delving into the investigation of ω-minimizers, we first present a Sobolev regularity result for BV minimizers, which is attained with the vanishing viscosity method. This result is not original work by the author as it is already implicitly contained in [BS13]. Based on such a regularity result for minimizers, a fractional Sobolev regularity result for BV ω-minimizers is established, which particularly involves the absolutely continuous parts of their derivatives. To achieve this, we employ an inequality characterisation of the extremality of minimizers that incorporates both the singular parts and boundary values of test maps. In addition, we conduct a comparison of a BV ω-minimizer and the corresponding minimizer. The behaviour of the singular part of a BV ω-minimizer is not yet fully understood in this context, and we cannot expect to rule it out even with a reasonably good ω as shown in a counterexample. Exploring this aspect will be the subject of our future work. This chapter is based on an ongoing project in collaboration with Jan Kristensen.

Chapter 2

Preliminaries

In this chapter, we prepare for the later ones by fixing necessary notations, introducing useful concepts, and presenting some relevant results. Section 2.1 provides a complete list of the notations that are used throughout the thesis. Each of the following sections is for a certain topic, where we present some related concepts that will appear in later discussion, and collect scattered results pertaining to the corresponding topic from the existing body of literature with references indicated when necessary.

2.1 Basic notations

Constants:

- C and c are used to denote positive constants throughout the thesis, and the values of them may change from one line to another. The dependence of them on factors $a, b, \dots$ will be indicated in the form $C(a, b, \dots)$ and $c(a, b, \dots)$. When appearing with subscripts, they represent some specific constants that are fixed.

- $\sim$ is used for two comparable quantities. Given two functions $a, b\colon X \to \mathbb{R}$ defined a topological space X, by $a(t) \sim b(t)$, we mean that

  $$c_1 b(t) \leq a(t) \leq c_2 b(t)$$

  holds true with some $c_1, c_2 > 0$ when t is in a certain subset or going to a certain limit. The subset or limit should be clear from context.

- $\vee$ is the operation of taking the maximum of two numbers, i.e., $a \vee b = \max\{a, b\}$ for any $a, b \in \mathbb{R}$.

Vectors and vector spaces:

- $\mathbb{R}^d$ is the usual d dimensional Euclidean space, equipped with the standard norm $|\cdot|$ and inner product $x \cdot y$, $x, y \in \mathbb{R}^d$. The standard basis of $\mathbb{R}^d$ is denoted by $\{e_i\}_{i=1}^d$.

- $\mathbb{R}^{N\times n}$ is the space of $N\times n$ real matrices, equipped with the inner product $z\cdot w = \operatorname{tr} z^t w$, $z, w \in \mathbb{R}^{N\times n}$, and the induced norm $|\cdot|$.

- I or I_V is either the identity map or the corresponding matrix of a certain vector space V, which will be clear from the context.
- $\bigodot^2(\mathbb{R}^{N\times n})$ is the space of symmetric and real bilinear forms on $\mathbb{R}^{N\times n}$, i.e., $\bigodot^2(\mathbb{R}^{N\times n})$ consists of the maps $\mathbb{A}\colon \mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n}\to\mathbb{R}$ such that
$$\mathbb{A}[z,w]=\mathbb{A}[w,z],\quad \mathbb{A}[az_1+z_2,w]=a\mathbb{A}[z_1,w]+\mathbb{A}[z_2,w]$$
for any $z,w,z_1,z_2\in\mathbb{R}^{N\times n}$ and $a\in\mathbb{R}$. The operator norm of $\mathbb{A}\in\bigodot^2(\mathbb{R}^{N\times n})$ is $|\mathbb{A}|=\sup\{\mathbb{A}[z,w]\colon\ |z|,|w|\le 1\}$.
- $a\otimes b$ is the tensor product of two vectors $a\in\mathbb{R}^N$ and $b\in\mathbb{R}^n$, i.e., $a\otimes b\in\mathbb{R}^{N\times n}$ and
$$(a\otimes b)_{ij}=a_ib_j,\quad i=1,\dots,N,\ j=1,\dots,n.$$
- $\simeq$ indicates an isomorphism between two Banach spaces.
- $\hookrightarrow$ indicates a continuous embedding of one Banach space into another.

Set notation:

- $\subset\subset$ indicates compact embedding in topological spaces. Given two subsets A,B of a topological space X, we say that A is compactly embedded in B ($A\subset\subset B$) if $A\subset\bar{A}\subset\mathrm{int}B$ and $\bar{A}$ is compact. $\supset\supset$ is the inverse of $\subset\subset$.
- Given a metric space (X,d) and two sets $A,B\subset X$, then
  - $\mathrm{diam}(A)$ is the diameter of A, i.e.,
  $$\mathrm{diam}(A):=\sup\{d(x,y)\colon\ x,y\in A\};$$
  - $\mathrm{dist}(A,B)$ is the distance between A and B, i.e.,
  $$\mathrm{dist}(A,B):=\inf\{d(x,y)\colon\ x\in A, y\in B\}.$$
- Fix the Euclidean space $\mathbb{R}^d$ with $d\in\mathbb{N}^+$, then:
  - $\partial\Omega$ is the boundary of the set $\Omega\subset\mathbb{R}^d$;
  - $B(x,R)$ or $B_R(x)$ is the open ball in $\mathbb{R}^d$ centred at $x\in\mathbb{R}^d$ with radius $R>0$, i.e.,
  $$B(x,R)=B_R(x):=\{y\in\mathbb{R}^d\colon\ |y-x|<R\},$$
  and we may abbreviate it to B_R when the centre is clear;
  - $Q(x,R)$ or $Q_R(x)$ is the open cube in $\mathbb{R}^d$ centred at $x\in\mathbb{R}^d$ with side-length $2R>0$ and all of the sides parallel to the axes, i.e.,
  $$Q(x,R)=Q_R(x):=\{y\in\mathbb{R}^d\colon\ |y_i-x_i|<R\},$$
  and such a cube with all of the sides parallel to the axes is also called a *d-cube*;

  - $\mathbb{B}^d$ or $\mathbb{B}$ is the unit ball $B(0,1)$;
  - $\mathbb{S}^{d-1}$ or $\mathbb{S}$ is the unit sphere, i.e., the boundary of $\mathbb{B}^d$.

- $B_{\mathbb{C}}(z,R)$ is the open ball in $\mathbb{C}$ centred at $z \in \mathbb{C}$ with radius $R > 0$, where $\mathbb{C}$ is the complex plane.

- Given an open set $\Omega \subset \mathbb{R}^d$, then

  - $\Omega(x,R)$ or $\Omega_R(x)$ is the intersection of Ω and a ball $B(x,R) \subset \mathbb{R}^d$, i.e.,

  $$\Omega(x,R) = \Omega_R(x) := \Omega \cap \mathbb{B}(x,R),$$

  and we may abbreviate it to Ω_R when the centre is clear;
  - Ω^ε is the set obtained by shrinking Ω by size $\varepsilon > 0$, i.e.,

  $$\Omega^\varepsilon := \{x \in \Omega \colon\ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

Measures and integrals:

- $\mathscr{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$.

- ω_d the Lebesgue measure of the unit ball in $\mathbb{R}^d$.

- $\mathcal{H}^k$ is the k-dimensional Hausdorff measure, and $\dim_{\mathcal{H}}$ is the Hausdorff dimension; see [EG15], Chapter 2.

- Given a measure space $(X, \mathcal{A}, \mu)$, where $\mathcal{A}$ is a σ-algebra on the set X and μ is a measure on $(X, \mathcal{A})$, then

  - $\int_X f \,\mathrm{d}\mu$ is the integral of any $\mathcal{A}$-measurable function f defined on X, and we may use the notation $\int_X f(x) \,\mathrm{d}\mu(x)$ to indicate that the integral is taken over the variable $x \in X$;
  - f_S is the average of f on $S \in \mathcal{A}$, i.e.,

  $$f_S := \fint_S f \,\mathrm{d}\mu := \frac{1}{\mu(S)} \int_S f \,\mathrm{d}\mu,$$

  where f is an $\mathcal{A}$-measurable function defined on X, and we abbreviate $f_{B(x,R)}$ to $f_{x,R}$ or f_R when $X = \mathbb{R}^d$, $\mu = \mathscr{L}^d$ and $B(x,R) \subset \mathbb{R}^d$ is a ball;
  - $\mu \llcorner A$ is the measure μ restricted to $A \in \mathcal{A}$, which is defined by

  $$(\mu \llcorner A)(C) := \mu(C \cap A), \quad \text{for any } C \in \mathcal{A}.$$

- Given a Borel set $X \subset \mathbb{R}^d$ for some positive integer d and a normed vector space V, then

  - $\mathscr{M}(X, V)$ is the set of (finite) V-valued Radon measures;
  - $\mathscr{M}^+(X)$ is the set of (finite) positive Radon measures on X;

  - $\mathscr{M}^1(X)$ is the set of probability measures defined on X, i.e., positive Radon measures on X with total mass 1.

- μ_S is the average of μ on $S \subset \mathbb{R}^d$ with respect to $\mathscr{L}^d$, i.e.,

$$\mu_S := \frac{\mu(S)}{\mathscr{L}^d(S)},$$

  where $\mu \in \mathscr{M}(X, V)$ and X, S are both Borel sets in $\mathbb{R}^d$ with $S \subset X$. We may abbreviate $\mu_{B(x,R)}$ to $\mu_{x,R}$ or μ_R, where $B(x, R) \subset \mathbb{R}^d$ is a ball.

Derivatives:

- Sobolev maps: Given a Sobolev map u defined on $\Omega \subset \mathbb{R}^d$, the distributional gradient of it is denoted by ∇u, and the distributional derivative along the direction e_i by $\partial^i u$, $i = 1, \ldots, d$. These notations also apply to classical derivatives of differentiable maps.

- BV maps: Given a BV map u defined on $\Omega \subset \mathbb{R}^d$, the distributional gradient of it is denoted by Du, which is a vector-valued Radon measure. The absolutely continuous part of Du with respect to $\mathscr{L}^d \llcorner \Omega$ is written as ∇u, and the singular part as $D^s u$, which indicates that the Lebesgue-Radon-Nikodým decomposition of Du is

$$Du = \nabla u \mathscr{L}^d \llcorner \Omega + D^s u.$$

- Given an integrand $f \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ with sufficient differentiability, we define

$$f'(z)w := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(z + tw), \quad f''(z)[w, w] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} f(z + tw).$$

  The derivative $f'(z)$ here is considered as an $N \times n$ matrix as well as a linear map, and $f''(z)$ as a symmetric bilinear form in $\bigodot^2(\mathbb{R}^{N\times n})$.

- $\Delta_h^s u$ is the s^{th}-difference quotient of size h of map u; see Section 2.3.

Function spaces:

- Continuous function spaces: Suppose that $\Omega \subset \mathbb{R}^d$ is an open set, then
  - $C(\bar{\Omega}, \mathbb{R}^m)$ ($C(\Omega, \mathbb{R}^m)$) is the space of all continuous $\mathbb{R}^m$-valued functions defined on $\bar{\Omega}$ (Ω) equipped with the supremum norm;
  - $C_c(\Omega, \mathbb{R}^m)$ is the space of all continuous $\mathbb{R}^m$-valued functions defined on Ω with compact support, where the support of a function $f \colon \Omega \to \mathbb{R}^m$ is

$$\mathrm{supp}(f) := \overline{\{x \in \Omega\colon\ |f(x)| \neq 0\}} \cap \Omega;$$

  - $C^{k,\alpha}(\Omega, \mathbb{R}^m)$ is the space of $\mathbb{R}^m$-valued functions defined on Ω that are k-times continuously differentiable with α-Hölder continuous k^{th}-derivatives, where $k \in \mathbb{N}$ and $\alpha \in [0, 1)$, and we may abbreviate it to $C^k(\Omega, \mathbb{R}^m)$ when $\alpha = 0$;

  - $C^\infty(\Omega, \mathbb{R}^m)$ is the space of all $\mathbb{R}^m$-valued smooth functions defined on Ω, which is equivalent to the characterisation $C^\infty(\Omega, \mathbb{R}^m) = \cap_{k=0}^\infty C^k(\Omega, \mathbb{R}^m)$;
  - $\mathscr{D}(\Omega, \mathbb{R}^m) = C_c^\infty(\Omega, \mathbb{R}^m)$ is the space of all $\mathbb{R}^m$-valued test functions, i.e., smooth functions with compact support, defined on Ω, and is also considered to be the space of test functions. Correspondingly, the space of $\mathbb{R}^m$-valued *distributions* is denoted by $\mathscr{D}'(\Omega, \mathbb{R}^m)$.

- Fix a bounded open set $\Omega \subset \mathbb{R}^d$, then we will consider the following function spaces:
  - $L^p(\Omega, \mathbb{R}^m)$ is the standard Lebesgue space with $p \geq 1$;
  - $L^\Phi(\Omega, \mathbb{R}^m)$ is the Orlicz space corresponding to the function $\Phi\colon [0, \infty) \to [0, \infty)$ defined by

$$L^\Phi(\Omega, \mathbb{R}^m) := \left\{ f \in L^1_{loc}(\Omega, \mathbb{R}^m)\colon \int_\Omega \Phi(\frac{|f|}{\lambda})\, \mathrm{d}x < \infty \text{ for some } \lambda > 0 \right\}$$

equipped with the norm

$$\|f\|_{L^\Phi(\Omega, \mathbb{R}^m)} := \inf \left\{ \lambda > 0\colon \int_\Omega \Phi(\frac{|f|}{\lambda})\, \mathrm{d}x \leq 1 \right\},$$

where Φ is a Young function, i.e., it is convex and satisfies

$$\lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{f(t)}{t} = \infty.$$

  - $W^{k,p}(\Omega, \mathbb{R}^m)$ $(W_0^{1,p}(\Omega, \mathbb{R}^m))$ is the standard Sobolev space with $k \in \mathbb{N}$ and $p \geq 1$;
  - $BV(\Omega, \mathbb{R}^m)$ is the space of $\mathbb{R}^m$-maps of bounded variation; see Subsection 2.2.2.
  - $W_u^{1,p}(\Omega, \mathbb{R}^m)$ with some $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ and $p > 1$ is defined by

$$W_u^{1,p}(\Omega, \mathbb{R}^m) := \{ v \in W^{1,p}(\Omega, \mathbb{R}^m)\colon\ u - v \in W_0^{1,p}(\Omega, \mathbb{R}^m) \},$$

and can be equivalently characterised as

$$W_u^{1,p}(\Omega, \mathbb{R}^m) := \{ v \in W^{1,p}(\Omega, \mathbb{R}^m)\colon\ \mathrm{tr}_\Omega\, u = \mathrm{tr}_\Omega\, v \}$$

when Ω is bounded with Lipschitz boundary.

  - $W^{s,p}(\Omega, \mathbb{R}^m)$ with some $s \in (0, 1)$ and $p \geq 1$ is the fractional Sobolev space or Sobolev-Slobodeckij space, which consists of those maps $u \in L^p(\Omega, \mathbb{R}^m)$ with the norm

$$\|u\|_{W^{s,p}(\Omega, \mathbb{R}^m)} = (\|u\|^p_{L^p(\Omega, \mathbb{R}^m)} + [u]^p_{W^{s,p}(\Omega, \mathbb{R}^m)})^{\frac{1}{p}}$$

finite, where the semi-norm is defined by

$$[u]^p_{W^{s,p}(\Omega, \mathbb{R}^m)} := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}}\, \mathrm{d}x\, \mathrm{d}y. \tag{2.1.1}$$

Such spaces can also be defined on an embedded d-dimensional submanifold X of $\mathbb{R}^n$ analogously, where the semi-norm $[u]^p_{W^{s,p}(X, \mathbb{R}^m)}$ is defined with respect to

  $\mathcal{H}^d$ on X.
  - $\mathbf{Y}^{\mathscr{M}}(\Omega,\mathbb{R}^m)$ ($\mathbf{BVY}(\Omega,\mathbb{R}^m)$) is the space of generalised Young measures (BV-Young measures); see Subsection 2.5.2.
  - $BMO(\Omega,\mathbb{R}^m)$ is the space of $\mathbb{R}^m$-valued functions of bounded mean oscillation, which consists of functions $f\in L^1_{loc}(\Omega,\mathbb{R}^m)$ with

    $$\|f\|_{BMO(\Omega,\mathbb{R}^m)} := \sup\left\{\fint_Q |f-f_Q|\,\mathrm{d}x\colon\ Q\subset\subset\Omega \text{ is a } d\text{-cube}\right\}<\infty$$

    and is a Banach space under the above norm.
  - $L^{p,\lambda}(\Omega,\mathbb{R}^m)$ ($\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^m)$) is the Morrey (Camapanato) space; see Subsection 2.2.3.

- Fix a bounded open set $\Omega\subset\mathbb{R}^d$ with Lipschitz boundary and $u\in BV(\Omega,\mathbb{R}^m)$, and define $w_{u,v}$ by

  $$w_{u,v}=\begin{cases}u-v, & \text{on } \Omega,\\ 0, & \text{on } \mathbb{R}^d\setminus\Omega\end{cases}$$

  for any $v\in BV(\Omega,\mathbb{R}^m)$, then define

  $$W^{1,1}_u(\Omega,\mathbb{R}^m) := \{v\in W^{1,1}(\Omega,\mathbb{R}^m)\colon\ w_{u,v}\in BV(\mathbb{R}^n,\mathbb{R}^m) \text{ and } |Dw_{u,v}|(\partial\Omega)=0\}; \tag{2.1.2}$$

  $$BV_u(\Omega,\mathbb{R}^m) := \{v\in BV(\Omega,\mathbb{R}^m)\colon\ w_{u,v}\in BV(\mathbb{R}^n,\mathbb{R}^m) \text{ and } |Dw_{u,v}|(\partial\Omega)=0\}. \tag{2.1.3}$$

- $V_{loc}(\Omega,\mathbb{R}^m)$ is the local version of the space $V(\Omega,\mathbb{R}^m)$ and consists of those functions that satisfy the defining property of $V(\Omega,\mathbb{R}^m)$ locally in Ω, where V indicates any of the spaces defined above (when applicable).

Functions and integrals:

- Assumptions on integrands: a function $f\colon\ \Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}$ with $\Omega\subset\mathbb{R}^n$ open is said to be
  - of *p-growth* if there exists $L>0$ such that

    $$|f(x,z)|\le L(1+|z|^p),$$

    and we also use the expression “linear growth” when $p=1$;
  - *Carathéodory* if $f(x,\cdot)$ is continuous for $\mathscr{L}^n$-a.e. $x\in\Omega$, and $f(\cdot,z)$ is measurable for every $z\in\mathbb{R}^{N\times n}$.

- *Referenced integrands*: Fix the Euclidean space $\mathbb{R}^d$, then
  - for any $p\ge 1$, define

    $$E_p(z) := \langle z\rangle^p - 1 := (1+|z|^2)^{\frac{p}{2}}-1,\quad z\in\mathbb{R}^d, \tag{2.1.4}$$

    and E_1 is written as E for convenience;

  - for any $\mu \geq 0$ and $p \geq 1$, define

$$E_p^\mu(z) := ((1+\mu)^2 + |z|^2)^{\frac{p}{2}} - (1+\mu)^p, \quad z \in \mathbb{R}^d, \tag{2.1.5}$$

    then it is obvious that $E_p^\mu(z) = (1+\mu)^p E_p(\frac{z}{1+\mu})$, and we set $E_p^A := E_p^{|A|}$ for any $A \in \mathbb{R}^d$;

  - for any $\mu > 1$, define

$$W_\mu(z) := (1+|z|^2)^{\frac{2-\mu}{4}}, \quad z \in \mathbb{R}^d. \tag{2.1.6}$$

- Maximal functions: given a function $f \in L^1_{loc}(\mathbb{R}^d)$, then
  - Mf is the Hardy-Littlewood maximal function of f, which is defined by

$$Mf(x) := \sup_{R>0} \fint_{B_R(x)} f(y)\,\mathrm{d}y; \tag{2.1.7}$$

  - $M_{R_0}f(x)$ is the local maximal function of f, which is defined by

$$M_{R_0}f(x) := \sup_{0<R<R_0} \fint_{B_R(x)} f(y)\,\mathrm{d}y. \tag{2.1.8}$$

- χ_S is the characteristic function of $S \subset X$, where X is a set, that is, the function $\chi_S \colon X \to \mathbb{R}$ is defined by

$$\chi_S(x) := \begin{cases} 1, & x \in S \\ 0, & x \notin S. \end{cases}$$

- $f|_A$ is the restriction of the map $f \colon X \to V$ to $A \subset X$, where X is a set and V is a vector space. We may denote by $f|_{\partial\Omega}$ the trace of the map $f \colon \Omega \to V$, where $\Omega \subset \mathbb{R}^d$ is open, if it exists.

- Given a Carathéodory integrand $f \colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^n$ is open, then
  - $\mathscr{F}(u,\Omega)$ is the integral defined by

$$\mathscr{F}(u,\Omega) := \int_\Omega f(x,\nabla u)\,\mathrm{d}x$$

    for any admissible map $u \colon \Omega \to \mathbb{R}^N$, and we may abbreviate it to $\mathscr{F}(u)$ when the domain Ω is clear from context;

  - $\mathscr{F}_g, \bar{\mathscr{F}}, \mathscr{F}_*$ are relaxed functionals when the integrand is of linear growth (see Subsection 3.1.4 for precise definitions).

- $\mathcal{D}_p$ is the p-Dirichlet energy, i.e.,

$$\mathcal{D}_p(u,\Omega) = \int_\Omega |\nabla u|^p\,\mathrm{d}x$$

  for some open set $\Omega \subset \mathbb{R}^d$ an d any admissible $u \colon \Omega \to \mathbb{R}^m$.

2.2 Function spaces

2.2.1 Negative Sobolev spaces $W^{-1,p}$

Sobolev spaces of order -1 will appear in the proofs in Chapter 4 and 6, so we briefly review the definition and some useful results here. The discussion about $W^{-1,1}$ is based on the relevant parts in [BS13] and [Gme20]. In this section, the domain Ω is assumed to be open in $\mathbb{R}^n$ and p' is defined as $\frac{p}{p-1}$ for any $p > 1$.

Definition 2.2.1. Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and $p > 1$. The negative Sobolev space $W^{-1,p}(\Omega, \mathbb{R}^N)$ is defined as the dual space of $W^{1,p'}(\Omega, \mathbb{R}^N)$. For any $f \in W^{-1,p}(\Omega, \mathbb{R}^N)$, the $W^{-1,p}$ norm of it is defined as the dual norm in $(W^{1,p'}(\Omega, \mathbb{R}^N))^*$, i.e.,

$$\begin{aligned}&\|f\|_{W^{-1,p}(\Omega,\mathbb{R}^N)} := \|f\|_{W^{1,p'}(\Omega,\mathbb{R}^N)^*}\\&:= \sup\{\langle f, u\rangle : \ u \in W^{1,p'}(\Omega, \mathbb{R}^N), \|u\|_{(W^{1,p'}(\Omega,\mathbb{R}^N))'} = 1\},\end{aligned}$$

where $\langle \cdot\,, \cdot\rangle$ is the duality pairing of $(W^{1,p'}(\Omega, \mathbb{R}^N))^*$ and $W^{1,p'}(\Omega, \mathbb{R}^N)$.

It is easy to see that any homogeneous partial differential operator of order 2 induces a bounded linear operator from $W^{1,p}(\Omega, \mathbb{R}^N)$ (and thus from $W_0^{1,p}(\Omega, \mathbb{R}^N)$) to $W^{-1,p}(\Omega, \mathbb{R}^N)$. Indeed, for any $j, k \in \{1, \dots , n\}$ and $v \in W^{1,p}(\Omega, \mathbb{R}^N)$, the distribution $\partial^j \partial^k v$ is defined by

$$\left\langle \partial^j \partial^k v, u \right\rangle := -\int_\Omega \partial^k v \partial^j u \, dx \left(-\int_\Omega \partial^j v \partial^k u \, dx \right)$$

for any $u \in C_c^\infty(\Omega, \mathbb{R}^N)$, which can be naturally extended to $u \in W_0^{1,p'}(\Omega, \mathbb{R}^N)$. This implies

$$\left\| \partial^j \partial^k v \right\|_{W^{-1,p}(\Omega,\mathbb{R}^N)} \leq \|\nabla v\|_{L^p(\Omega,\mathbb{R}^{N\times n})} \leq \|v\|_{W^{1,p}(\Omega,\mathbb{R}^N)} .$$

In particular, the operator induced by Δ is an isomorphism between $W_0^{1,p}(\Omega, \mathbb{R}^N)$ and $W^{-1,p}(\Omega, \mathbb{R}^N)$ when Ω is regular.

Theorem 2.2.2 ([Sim72], Theorem 4.6)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with C^1 boundary and $p > 1$. Then the Laplace operator Δ induces an isomorphism between $W_0^{1,p}(\Omega, \mathbb{R}^N)$ and $W^{-1,p}(\Omega, \mathbb{R}^N)$ in the following sense: For any $v \in W_0^{1,p}(\Omega, \mathbb{R}^N)$, define $\Delta v \in W^{-1,p}(\Omega, \mathbb{R}^N)$ by*

$$\langle \Delta v, u\rangle := -\int_\Omega \nabla v \cdot \nabla u \, dx, \quad \text{for any } u \in W_0^{1,p'}(\Omega, \mathbb{R}^N).$$

The operator $\Delta\colon W_0^{1,p}(\Omega, \mathbb{R}^N) \to W^{-1,p}(\Omega, \mathbb{R}^N)$ is a linear bijection with

$$C \|v\|_{W^{1,p}(\Omega,\mathbb{R}^N)} \leq \|\Delta v\|_{W^{-1,p}(\Omega,\mathbb{R}^N)} \leq \|v\|_{W^{1,p}(\Omega,\mathbb{R}^N)} , \tag{2.2.1}$$

where $C = C(n, N, p, \Omega) > 0$.

The duality formulation above does not work for $W^{-1,1}$, and alternatively we consider the following definition.

Definition 2.2.3. Suppose that $\Omega \subset \mathbb{R}^n$ is open. Define the space $W^{-1,1}(\Omega, \mathbb{R}^N)$ by

$$W^{-1,1}(\Omega, \mathbb{R}^N) := \{T \in \mathscr{D}'(\Omega, \mathbb{R}^N) \colon\ T = w_0 + \sum_{s=1}^{n} \partial^s w_s, w_0, w_1, \dots, w_n \in L^1(\Omega, \mathbb{R}^N)\}, \tag{2.2.2}$$

where $\mathscr{D}'(\Omega, \mathbb{R}^N)$ is the space of $\mathbb{R}^N$-valued distributions on Ω. For any $T \in W^{-1,1}(\Omega, \mathbb{R}^N)$, the $W^{-1,1}$ norm of it is defined by

$$\|T\|_{W^{-1,1}(\Omega,\mathbb{R}^N)} := \inf \sum_{s=0}^{n} \|w_s\|_{L^1(\Omega,\mathbb{R}^N)},$$

where the infimum is taken over all possible representations $T = w_0 + \sum_{s=1}^{n} \partial^s w_s$ with $w_s \in L^1(\Omega, \mathbb{R}^N)$, $s = 0, 1, \dots, n$.

It is clear that $W^{-1,1}(\Omega, \mathbb{R}^N)$ equipped with $\|\cdot\|_{W^{-1,1}(\Omega,\mathbb{R}^N)}$ is a linear normed space. It is actually complete with this norm, and thus a Banach space.

Proposition 2.2.4. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open set. Then the space $W^{-1,1}(\Omega, \mathbb{R}^N)$ is a Banach space with the norm $\|\cdot\|_{W^{-1,1}(\Omega,\mathbb{R}^N)}$.*

Proof. Define a map P as follows:

$$\begin{aligned} P \colon\ (L^1(\Omega, \mathbb{R}^N))^{n+1} &\longrightarrow W^{-1,1}(\Omega, \mathbb{R}^N) \\ (w_0, w_1, \dots, w_n) &\longmapsto w_0 + \sum_{s=1}^{n} \partial^s w_s. \end{aligned}$$

It is obvious that P is linear and bounded, and thus the kernel $\ker P$ of it is a closed subspace of $(L^1(\Omega, \mathbb{R}^N))^{n+1}$. Then the quotient space $(L^1(\Omega, \mathbb{R}^N))^{n+1}/\ker P$ with the quotient norm is a Banach space ([Yos80], I.11). Considering the definition of $W^{-1,1}$ and the corresponding norm, we know that the quotient map

$$\bar{P} \colon\ (L^1(\Omega, \mathbb{R}^N))^{n+1}/\ker P \longrightarrow W^{-1,1}(\Omega, \mathbb{R}^N)$$

is an isometric isomorphism, which indicates the completeness of $W^{-1,1}(\Omega, \mathbb{R}^N)$. □

From the definition, it is also easy the see that the following inequalities hold for any $w \in L^1(\Omega, \mathbb{R}^N)$ and any $s \in \{1, \dots, n\}$:

$$\|w\|_{W^{-1,1}(\Omega,\mathbb{R}^N)} \leq \|w\|_{L^1(\Omega,\mathbb{R}^N)}, \tag{2.2.3}$$

$$\|\partial^s w\|_{W^{-1,1}(\Omega,\mathbb{R}^N)} \leq \|w\|_{L^1(\Omega,\mathbb{R}^N)}. \tag{2.2.4}$$

Remark 2.2.5. We can also define $W^{-1,p}(\Omega, \mathbb{R}^N)$ and the corresponding norm $\|\cdot\|_{W^{-1,p}(\Omega,\mathbb{R}^N)}$ for $p > 1$ analogously to Definition 2.2.3, and the normed space obtained in this way is exactly the one in Definition 2.2.1 ([AF03], Theorem 3.12).

2.2.2 Maps of bounded variation

In the study of functionals of linear growth, the appropriate function space is BV (the space of maps of bounded variation) due to the lack of compactness in $W^{1,1}$. For BV maps and the relevant results, we refer to [AFP00]. Some definitions and results are stated here to facilitate later references. See Section 2.6 for undefined convexity notions.

Consider a bounded open set $\Omega \subset \mathbb{R}^n$. A map $u\colon \Omega \to \mathbb{R}^N$ is said to be of bounded variation if it is in $L^1(\Omega, \mathbb{R}^N)$ and its distributional derivative can be represented by a bounded $\mathbb{R}^{N\times n}$-valued Radon measure, i.e.,

$$|Du|(\Omega) := \sup\left\{\int_\Omega u \cdot \operatorname{div}(\varphi)\,\mathrm{d}x\colon\ \varphi \in C_c^1(\Omega, \mathbb{R}^{N\times n}), |\varphi| \le 1\right\} < \infty.$$

The space of maps of bounded variation equipped with the norm $\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega)$ is a Banach space.

Convergence under the BV norm is rather strong and rarely used. Actually, the space $W^{1,1}(\Omega, \mathbb{R}^N)$ is not dense with respect to the strong norm topology as it is complete itself. Instead, we consider two other forms of convergence: Suppose that $\{u_j\} \subset BV(\Omega, \mathbb{R}^N)$, $u \in BV(\Omega, \mathbb{R}^N)$ and $u_j \to u$ in $L^1(\Omega, \mathbb{R}^n)$. We say that $\{u_j\}$ converges to u in the BV (*area-*)*strict sense* if $\{Du_j\}$ converges to Du in the (area-)strict sense as in Definition 2.5.1.

The trace operator can extended to BV maps on a Lipschitz domain in a manner similar to the definition for $W^{1,p}$ maps.

Theorem 2.2.6 ([AFP00], Theorem 3.87)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is an open set with Lipschitz boundary and $u \in BV(\Omega, \mathbb{R}^N)$. Then for $\mathcal{H}^{n-1}$-almost every $x \in \partial\Omega$, there exists $u^\Omega(x) \in \mathbb{R}^N$ such that*

$$\lim_{r\to 0^+} r^{-n} \int_{\Omega\cap B_r(x)} \left|u(y) - u^\Omega(x)\right| \,\mathrm{d}y = 0. \tag{2.2.5}$$

The map u^Ω is in $L^1(\partial\Omega, \mathcal{H}^{n-1} \llcorner \partial\Omega)$ with

$$\|u^\Omega\|_{L^1(\partial\Omega)} \le C\|u\|_{BV(\Omega)}, \tag{2.2.6}$$

where $C > 0$ is a constant depending only on Ω. Extend u by $0 \in \mathbb{R}^N$ to $\mathbb{R}^n \setminus \Omega$ and denote the extension by $\tilde{u}$. Then the map $\tilde{u}$ of bounded variation on $\mathbb{R}^n$ with

$$D\tilde{u} = Du - (u^\Omega \otimes \nu_{\partial\Omega})\mathcal{H}^{n-1} \llcorner \partial\Omega,$$

where Du is considered as a measure on $\mathbb{R}^n$ with mass in Ω, and ν_Ω is the outward unit normal along $\partial\Omega$.

Thus, we have a bounded linear operator

$$\operatorname{tr}_\Omega\colon\ BV(\Omega, \mathbb{R}^N) \to L^1(\partial\Omega, \mathcal{H}^{n-1} \llcorner \partial\Omega),$$

and u^Ω in the above lemma is called the *trace* of u. Moreover, the trace operator is also continuous with respect to the strict topology ([AFP00], Theorem 3.88). As the trace operator from $W^{1,1}(\Omega, \mathbb{R}^N)$ to $L^1(\partial\Omega, \mathcal{H}^{n-1} \llcorner \partial\Omega)$ is surjective, the trace of u for any $u \in BV(\Omega, \mathbb{R}^N)$ is also the trace for some $W^{1,1}(\Omega, \mathbb{R}^N)$ map, which justifies our definition of $W^{1,1}_u(\Omega, \mathbb{R}^N)$ in 2.1.2.

Corollary 2.2.7 ([AFP00], Corollary 3.89)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $u \in BV(\Omega, \mathbb{R}^N)$ and $v \in BV(\bar{\Omega}^c, \mathbb{R}^N)$. Then the function*

$$w(x) = \begin{cases} u(x), & x \in \Omega \\ v(x), & x \in \bar{\Omega}^c \end{cases}$$

is in $BV(\mathbb{R}^n, \mathbb{R}^N)$ and

$$Dw = Du + Dv + (v^{\mathbb{R}^n \setminus \bar{\Omega}} - u^{\Omega}) \otimes \nu_\Omega) \mathcal{H}^{n-1} \llcorner \partial\Omega.$$

This can be obtained directly from Theorem 2.2.6 by considering $\tilde{u}$ and $\tilde{v}$. In particular, if $u \in BV(\mathbb{R}^n, \mathbb{R}^N)$, we have $|Du|(\partial\Omega) = 0$ if and only if $\operatorname{tr}_\Omega u = \operatorname{tr}_{\mathbb{R}^n \setminus \bar{\Omega}} u$.

It is well-known that smooth maps are dense in $BV(\Omega, \mathbb{R}^N)$ in the BV area-strict sense. Furthermore, the approximation of a BV map on Ω can be done with the boundary value preserved as stated in the following lemma. See [KR10a], Lemma 1 for a proof.

Lemma 2.2.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set without any additional regularity assumptions on $\partial\Omega$. If $u \in BV(\Omega, \mathbb{R}^N)$, there exists a sequence $\{u_j\} \subset W^{1,1}_u \cap C^\infty(\Omega, \mathbb{R}^N)$ such that $u_j \to u$ in the BV area-strict sense. If $u \in W^{1,1}(\Omega, \mathbb{R}^N)$, we can further require strong convergence in $W^{1,1}(\Omega, \mathbb{R}^N)$.*

The next result is a Fubini-type property for BV maps. It involves BV maps on submanifolds of $\mathbb{R}^n$, which are well-defined by local charts and partitions of unity. In the following, we only consider $(n-1)$-spheres, which can be covered by two local charts that correspond to the stereographic projections from two antipodal points. To be more precise, consider the stereographic projection $\pi_1 \colon \mathbb{S}^{n-1} \setminus \{N\} \to \mathbb{R}^{n-1}$ with respect to the north pole $N = (0, \dots, 0, 1)$, which is a diffeomorphism. Take $R > 0$ such that the lower half ball $\{x \in \mathbb{S}^{n-1} \colon x_n < 0\}$ is compactly contained in $\pi^{-1}(B(0,R)) =: U_1$. Assume that the local coordinates on U_1 are $y = (y_1, \dots, y_{n-1})$ with $|y| < R$, then the volume form of the induced metric (considering $\mathbb{S}^{n-1}$ as an embedded submanifold of $\mathbb{R}^n$) is

$$\mathrm{d}V = \frac{4}{(1+|y|^2)^2} \, \mathrm{d}y_1 \dots \mathrm{d}y_{n-1},$$

which is comparable to that of the standard metric on $\mathbb{R}^{n-1}$. Define another chart (U_2, π_2) with respect to the south pole S analogously. With the covering $\{U_1, U_2\}$ and the partition of unity subject to it, we can apply various results for maps defined on (open subsets of) $\mathbb{R}^{n-1}$.

For a BV map $u \colon \partial B \to \mathbb{R}^N$, we denote its *tangential approximate gradient* by $\nabla_\tau u$, which exists $\mathcal{H}^{n-1}$-almost everywhere on ∂B. Its *tangential distributional derivative* is denoted by $D_\tau u$. Indeed, the former is the absolutely continuous part of the latter with respect to $\mathcal{H}^{n-1} \llcorner \partial B$, and the two coincide when $u \in W^{1,p}(\partial B, \mathbb{R}^N)$ with $p \geq 1$.

Lemma 2.2.9. *Denote by B_R a ball $B(x_0, R) \subset \mathbb{R}^n$ and let u be a map in $BV(B_R, \mathbb{R}^N)$. Then for $\mathscr{L}^1$-almost every $\rho \in (0, R)$, the pointwise restriction $u|_{\partial B_\rho}$ coincides with the traces of u from B_ρ and $B_R \setminus \bar{B}_\rho$, and is in $BV(\partial B_\rho, \mathbb{R}^N)$. For any two radii r_1, r_2 with $0 < r_1 < r_2 < R$, we can find $\rho \in (r_1, r_2)$ such that the above holds and the total variation* *of $u|_{\partial B_\rho}$ on ∂B_ρ is bounded by that of u:*

$$\int_{\partial B_\rho} |D_\tau(u|_{\partial B_\rho})| \leq \frac{C(n,N)}{r_2 - r_1} \int_{B_{r_2}\setminus \bar{B}_{r_1}} |Du|. \tag{2.2.7}$$

This lemma is Lemma 2.3 in [GK19a] and allows us to work on those balls over the boundary of which a BV map has nice properties.

Lemma 2.2.10. *Let B be a ball $B(x_0,R) \subset \mathbb{R}^n$ and $v \in BV(\partial B, \mathbb{R}^N)$. Then we have $v \in W^{1-\frac{1}{r},r}(\partial B, \mathbb{R}^N)$ and*

$$\left(\fint_{\partial B} \int_{\partial B} \frac{|v(x)-v(y)|^r}{|x-y|^{n+r-2}} \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}\mathcal{H}^{n-1}(y) \right)^{\frac{1}{r}} \leq CR^{\frac{1}{r}} \fint_{\partial B} |D_\tau v|, \tag{2.2.8}$$

where $C = C(n,N,r) > 0$. The range of r depends on the dimension:

$$\begin{cases} r = \dfrac{n}{n-1}, & n \geq 3 \\ r \in (1,2), & n = 2. \end{cases}$$

The result in the case $n \geq 3$ is from [BBM04], Lemma D.1, which gives the continuous embedding

$$BV(\mathbb{R}^d) \hookrightarrow W^{s,r}(\mathbb{R}^d), \quad \text{with } d \geq 2, s \in (0,1) \text{ and } \frac{1}{r} = 1 - \frac{1-s}{d}. \tag{2.2.9}$$

Here we have $d = n-1, r = \frac{n}{n-1}, s = \frac{1}{n}$.

In the case $n = 2$ we need to combine the following continuous embeddings, for which see [Tar07], Lemma 38.1 and [Tri83], §3.3.1, respectively:

$$BV(\mathbb{R}) \cap L^\infty(\mathbb{R}) \hookrightarrow B_\infty^{\frac{1}{2},2}(\mathbb{R}); \tag{2.2.10}$$

$$B_\infty^{\frac{1}{2},2}(-1,1) \hookrightarrow B_r^{1-\frac{1}{r},r}(-1,1) = W^{1-\frac{1}{r},r}(-1,1). \tag{2.2.11}$$

Notice that a BV map v on ∂B can be decomposed as two BV maps v_1, v_2 which are compactly supported on the two aforementioned local charts respectively. Each $v_i, i = 1,2$, is equivalent to a BV map $\bar{v}_i$ compactly supported on $(-1,1)$, which is in $BV(\mathbb{R})$. By the first embedding we know $v_i \in B_\infty^{\frac{1}{2},2}(\mathbb{R})$ and thus in $B_\infty^{\frac{1}{2},2}(-1,1) \subset W^{1-\frac{1}{r},r}(-1,1)$, which gives the desired result.

2.2.3 Morrey and Campanato spaces

In this subsection we revisit Morrey and Campanato spaces briefly. The introduction of Campanato spaces [Cam63] extends the notion of bounded mean oscillation (BMO), which was first studied by JOHN and NIRENBERY [JN61]. These spaces are useful in the study of partial differential equations, as they offer integral characterisations of Hölder continuity.

The definitions of Morrey and Campanato spaces are recalled here with some relevant results. For a systematic discussion and more details, we refer to [Gia83], §III.1, [Giu03], §2.3 and §2.4, and [GM12], §5.1. At the end of this subsection, we show that a Campanato-type estimate implies some Hölder-type regularity for Radon measures, which will be used in Chapter 5.

In the following, we denote by Ω a bounded open set in $\mathbb{R}^n$. The discussion is for scalar-valued functions but can be extended to the vectorial context smoothly.

Definition 2.2.11. Let $p \geq 1$ and $\lambda \geq 0$. The *Morrey space* $L^{p,\lambda}(\Omega)$ is the linear space of functions $u \in L^p(\Omega)$ with

$$\|u\|_{L^{p,\lambda}(\Omega)} := \sup_{\substack{x\in\Omega,\\ 0<\rho<\operatorname{diam}(\Omega)}} \left(\rho^{-\lambda} \int_{\Omega(x,\rho)} |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty, \tag{2.2.12}$$

where $\Omega(x,\rho) := \Omega \cap B(x,\rho)$.

The condition (2.2.12) concerns more the local behaviour of u, as when $\rho > \varepsilon > 0$, the integral is controlled by

$$\left(\rho^{-\lambda} \int_{\Omega(x,\rho)} |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \left(\varepsilon^{-\lambda} \int_{\Omega} |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

It is also easy to check that $\|\cdot\|_{L^{p,\lambda}(\Omega)}$ is a norm, under which $L^{p,\lambda}(\Omega)$ is a Banach space.

Proposition 2.2.12. *Given $p, q \geq 1$ and $\lambda, \mu \geq 0$, we have the following properties of Campanato spaces:*

(i) $L^{p,0}(\Omega) \simeq L^p(\Omega)$;

(ii) $L^{p,n}(\Omega) \simeq L^\infty(\Omega)$;

(iii) $L^{p,\lambda}(\Omega) \simeq \{0\}$ *if* $\lambda > n$;

(iv) $L^{q,\mu}(\Omega) \subset L^{p,\lambda}(\Omega)$ *if* $p \leq q$, $\frac{n-\lambda}{p} \geq \frac{n-\mu}{q}$.

The above properties are easy to verify. Both (ii) and (iii) can be obtained with the Lebesgue differentiation theorem, and (iv) with Hölder's inequality.

The definition of Morrey spaces involves local integrals of an L^p function, while that of Campanato spaces is related to local oscillation.

Definition 2.2.13. Let $p \geq 1$ and $\lambda \geq 0$. The *Campanato space* $\mathcal{L}^{p,\lambda}(\Omega)$ is the linear space of functions $u \in L^p(\Omega)$ with

$$[u]_{\mathcal{L}^{p,\lambda}} := \sup_{\substack{x\in\Omega,\\ 0<\rho<\operatorname{diam}(\Omega)}} \left(\rho^{-\lambda} \int_{\Omega(x,\rho)} |u - u_{x,\rho}|^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty, \tag{2.2.13}$$

where $u_{x,\rho} := u_{\Omega(x,\rho)}$.

The quantity $[u]_{\mathcal{L}^{p,\lambda}}$ in (2.2.13), also written as $[u]_{p,\lambda}$, is a seminorm, and is finite if and only if

$$\sup_{\substack{x\in\Omega,\\ 0<\rho<\operatorname{diam}(\Omega)}} \inf_{c\in\mathbb{R}} \left(\rho^{-\lambda} \int_{\Omega(x,\rho)} |u - c|^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

The space $\mathcal{L}^{p,\lambda}(\Omega)$ is a Banach space under the norm

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} := \|u\|_{L^p(\Omega)} + [u]_{p,\lambda}.$$

It is easy to see that the analogue of (iv) holds for Campanato spaces.

To further discuss the relation between Morrey spaces and Campanato spaces, we introduce the following regularity condition on domains.

Definition 2.2.14. We say that a bounded open set $\Omega \subset \mathbb{R}^n$ has *no external cusps* if there is a constant $A > 0$ such that for any $x_0 \in \Omega$ and any $0 < \rho < \operatorname{diam}(\Omega)$, we have

$$\mathscr{L}^n(\Omega(x_0,\rho)) \geq A\mathscr{L}^n(B(x_0,\rho)). \tag{2.2.14}$$

Proposition 2.2.15. *Suppose that a bounded open set $\Omega \subset \mathbb{R}^n$ has no external cusps and $p \geq 1$. Then for any $\lambda \in [0,n)$, we have*

$$L^{p,\lambda}(\Omega) \simeq \mathcal{L}^{p,\lambda}(\Omega).$$

The inclusion $\mathcal{L}^{p,\lambda}(\Omega) \subset L^{p,\lambda}(\Omega)$ is clear, and the other direction is more involved (see Proposition 1.2 in [Gia83], §III.1 and Proposition 5.4 in [GM12]).

The above isomorphism does not hold true when $\lambda \geq n$. For $\lambda > n$, the Morrey space $L^{p,n}(\Omega)$ only contains the zero function, while the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ is non-trivial. Actually, we have the following result for $\mathcal{L}^{p,\lambda}(\Omega)$ with $\lambda > n$.

Theorem 2.2.16 ([Gia83], §III.1, Theorem 1.3)**.** *Suppose that a bounded open set $\Omega \subset \mathbb{R}^n$ has no external cusps and $p \geq 1$. Then for any $\lambda \in (n, n+p]$, we have*

$$\mathcal{L}^{p,\lambda}(\Omega) \simeq C^{0,\alpha}(\Omega)$$

with $\alpha = \frac{\lambda - n}{p}$. Moreover, if $u \in \mathcal{L}^{p,\lambda}(\Omega)$ with $\lambda > n+p$, it is constant in Ω.

Remark 2.2.17. One corollary of the above theorem is the following: if $u \in W^{1,p}(\Omega)$ with $\nabla u \in L^{p,p\alpha}_{loc}(\Omega, \mathbb{R}^n)$, where $1 \leq p \leq n$ and $\alpha \in (0,1]$, then by Poincaré's inequality we have that $u \in C^{0,\alpha}_{loc}(\Omega)$.

It is not easy to see what happens when $\lambda = n$. One simple example is $u(x) = \log x$, which is in $\mathcal{L}^{p,1}(0,1)$ but not in $L^{p,1}(0,1) = L^\infty(0,1)$. If Ω is a cube, then obviously $\mathcal{L}^{1,n}(\Omega) = BMO(\Omega)$. The following result shows that this also applies to $\mathcal{L}^{p,n}(\Omega)$ for any $p > 1$.

Proposition 2.2.18. *Suppose that $\Omega \subset \mathbb{R}^n$ is a cube. Then we have*

$$\mathcal{L}^{p,n}(\Omega) \simeq BMO(\Omega)$$

for any $p \geq 1$.

The above proposition is also true when Ω has no external cusps. See [Gia83], §2.4 or [GM12], §6.3.1 for a proof and a more detailed discussion.

For a summary of the relations between Morrey and Campanato spaces, see Table 2.1

	$\lambda = 0$	$\lambda \in (0,n)$	$\lambda = n$	$\lambda \in (n, n+p]$	$\lambda > n+p$
$L^{p,n}$	$\simeq L^p$	$\simeq \mathcal{L}^{p,\lambda}$	$\simeq L^\infty$	$= \{0\}$	$= \{0\}$
$\mathcal{L}^{p,\lambda}$	$\simeq L^p$	$\simeq L^{p,\lambda}$	$\simeq BMO \supsetneq L^\infty$	$\simeq C^{0,\alpha} (\alpha = \frac{\lambda - n}{p})$	$= \{\text{constants}\}$

Table 2.1: Relations between Morrey spaces $L^{p,\lambda}(\Omega)$ and Campanato spaces $\mathcal{L}^{p,\lambda}(\Omega)$, where the bounded open set $\Omega \subset \mathbb{R}^n$ has no external cusps.

Now we end this subsection with the following result, which is a generalisation of Theorem 2.2.16 to Radon measures and will be used in Chapter 5. The proof is a mollification of Theorem 5.5 in [GM12].

Lemma 2.2.19. *Assume that the function* $h\colon [0,\infty) \to [0,\infty)$ *satisfies the following*

(a) *h is non-decreasing with* $h(0) = 0 = \lim_{r\to 0} h(r)$;

(b) *For any* $\rho > 0$,

$$H(\rho) := \int_0^\rho \frac{\sqrt{h(r)}}{r}\,\mathrm{d}r < \infty;$$

(c) *There exists* $c > 0$ *such that* $H(2\rho) \le cH(\rho), h(2\rho) \le ch(\rho)$.

Suppose that $\Omega \subset \mathbb{R}^n$ *is a bounded open set, and* μ *is a bounded* $\mathbb{R}^d$*-valued Radon measure on* Ω *with some* $d \in \mathbb{N}$. *Furthermore, given any open subset* $O \subset\subset \Omega$, *there exists* $C_O > 0$ *such that*

$$\fint_{B(x_0,\rho)} E(\mu - \mu_{B(x_0,\rho)}) \le C_O h(\rho) \tag{2.2.15}$$

for any $x_0 \in O$, $\rho > 0$ *with* $B(x_0,\rho) \subset\subset \Omega$, *where* $E(z) = (1+|z|^2)^{\frac{1}{2}} - 1$ *is as defined in* (2.1.4) *and the integral* $\fint_{B(x_0,\rho)} E(\mu - \mu_{B(x_0,\rho)})$ *is defined as in Subsection* 2.5.1. *Then* μ *is absolutely continuous with respect to the Lebesgue measure* $\mathscr{L}^n$ *and* $\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}$ *is continuous in* Ω *with a local modulus of continuity* $\rho \mapsto \sqrt{h(\rho)} + H(\rho)$.

Proof. Take an open subset $O \subset\subset \Omega$, and define $[\mu]_{h,O}$ to be the smallest constant C_O with which (2.2.15) holds. Abbreviate $\mu_{B(x_0,\rho)}$ to $\mu_{x_0,\rho}$ and take $R < \operatorname{dist}(O, \partial\Omega)$. Then for any $x_0 \in O$ and $0 < r < s \le R$ we have, by the convexity of E and Jensen's inequality,

$$E(\mu_{x_0,s} - \mu_{x_0,r}) \le \fint_{B(x_0,r)} E(\mu - \mu_{x_0,s}) \le \left(\frac{s}{r}\right)^n [\mu]_{h,O} h(s). \tag{2.2.16}$$

Let $r_i = 2^{-i}R$ and insert $s = r_{i-1}, r = r_i$ into the above to obtain

$$E(\mu_{x_0,r_{i-1}} - \mu_{x_0,r_i}) \le 2^n [\mu]_{h,O} h(r_{i-1}).$$

We further require R to be so small that $2^n[\mu]_{h,O}h(R) < 1$. Notice that E is strictly increasing in the norm of the argument and quadratic near 0 (see (5.2.4)), which implies $|\mu_{x_0,r_{i-1}} - \mu_{x_0,r_i}| \le C2^{\frac{n}{2}}\sqrt{[\mu]_{h,O}h(r_{i-1})}$. With the triangle inequality we have

$$|\mu_{x_0,R} - \mu_{x_0,r_k}| \le \sum_{i=1}^{k} |\mu_{x_0,r_{i-1}} - \mu_{x_0,r_i}|$$

$$\begin{aligned}&\le C2^{\frac{n}{2}}\sqrt{[\mu]_{h,O}}\sum_{i=1}^{k}\sqrt{h(r_{i-1})}\\&\le C\sqrt{[\mu]_{h,O}}H(2R)\le C\sqrt{[\mu]_{h,O}}H(R),\end{aligned}$$

where the summability of $\sqrt{h(r_{i-1})}$ is from (cf. Lemma 5.3.5)

$$\begin{aligned}\sqrt{h(r_i)}&\le \fint_{r_i}^{r_{i-1}}\sqrt{h(r)}\,\mathrm{d}r=\frac{1}{r_{i-1}-r_i}\int_{r_i}^{r_{i-1}}\sqrt{h(r)}\,\mathrm{d}r\\&=\frac{2}{r_{i-1}}\int_{r_i}^{r_{i-1}}\sqrt{h(r)}\,\mathrm{d}r\le 2\int_{r_i}^{r_{i-1}}\frac{\sqrt{h(r)}}{r}\,\mathrm{d}r.\end{aligned}$$

For $\rho\in[r_k,r_{k-1})$ with $k\ge 1$,

$$|\mu_{x_0,R}-\mu_{x_0,\rho}|\le|\mu_{x_0,R}-\mu_{x_0,r_k}|+|\mu_{x_0,r_k}-\mu_{x_0,\rho}|\le C\sqrt{[\mu]_{h,O}}H(R)+C2^{\frac{n}{2}}\sqrt{[\mu]_{h,O}h(\rho)}.\tag{2.2.17}$$

This estimate indicates that $\{\mu_{x,R}\}_R$ converges to some $v(x)$ uniformly in $x\in O$, which is the density of μ with respect to $\mathscr{L}^n$. By a mollified version of Lemma 1.2 and Theorem 1.30 in [EG15], we know that μ is absolutely continuous with respect to $\mathscr{L}^n$ and the density $v(x)=\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}(x)$ is L^1-integrable in O. The integrability implies the continuity of $\mu_{x,R}$ in x and thus that of $v(x)$.

To show the that v has the claimed modulus of continuity, we consider $x,y\in O$ with $R=|x-y|<\frac{1}{2}\mathrm{dist}(O,\partial\Omega)$ and $2^n[\mu]_{h,O}h(R)<1$. In (2.2.17), let $\rho\to 0$ and we have

$$|\mu_{x_0,R}-v(x_0)|\le C\sqrt{[\mu]_{h,O}}H(R),$$

which holds true for x,y. Then

$$\begin{aligned}|v(x)-v(y)|&\le|v(x)-\mu_{x,2R}|+|\mu_{x,2R}-\mu_{y,2R}|+|\mu_{y,2R}-v(y)|\\&\le C\sqrt{[\mu]_{h,O}}H(2R)+|\mu_{x,2R}-\mu_{y,2R}|.\end{aligned}$$

For any $z\in B(x,2R)\cap B(y,2R)(\supset B(x,R)\cup B(y,R))$,

$$|\mu_{x,2R}-\mu_{y,2R}|\le|\mu_{x,2R}-v(z)|+|v(z)-\mu_{y,2R}|.$$

Integrating in z on $B(x,2R)\cap B(y,2R)$, we obtain

$$\begin{aligned}|\mu_{x,2R}-\mu_{y,2R}|&\le\frac{1}{|B(x,R)|}\int_{B(x,2R)}|\mu_{x,2R}-v(z)|\,\mathrm{d}z+\frac{1}{|B(y,R)|}\int_{B(y,2R)}|v(z)-\mu_{y,2R}|\,\mathrm{d}z\\&\le 2^nE^{-1}\left(\fint_{B(x,2R)}E(v(z)-\mu_{x,2R})\,\mathrm{d}z\right)+2^nE^{-1}\left(\fint_{B(y,2R)}E(v(z)-\mu_{y,2R})\,\mathrm{d}z\right)\\&\le C\sqrt{[\mu]_{h,O}h(2R)},\end{aligned}$$

where the last line is from the fact that E is quadratic near 0 (or by Lemma 5.2.3). The above estimates finally gives

$$|v(x) - v(y)| \leq C\sqrt{[\mu]_{h,O}}(\sqrt{h(|x-y|)} + H(|x-y|)).$$

□

2.3 Difference quotients

Difference quotients are widely used in the study of PDEs to show differentiability of higher orders. In the following, the definition of difference quotients is recalled with some relevant properties and results, based on [GT01], §7.11. In addition, we present a lemma in [KM05], which shows how difference quotients can help with fractional Sobolev regularity and will be useful in Section 6.5.

Throughout this section, we denote by Ω an open set in $\mathbb{R}^n$ and by $\{e_i\}_{i=1}^n$ the canonical basis of $\mathbb{R}^n$. For any $r > 0$, define the shrunk set $\Omega^r := \{x \in \Omega\colon\ \operatorname{dist}(x, \partial\Omega) > r\}$.

Definition 2.3.1. Suppose that $\Omega \subset \mathbb{R}^n$ is open and $u\colon \Omega \mapsto \mathbb{R}^N$ is a vector-valued function. For any $h \in \mathbb{R} \setminus \{0\}$, the s^{th}-difference quotient of size h of u is defined by

$$\Delta_h^s u(x) := \frac{u(x + he_s) - u(x)}{h} \tag{2.3.1}$$

for any $x \in \Omega^{|h|}$ and any $s \in \{1, \ldots, n\}$.

Several properties follow immediately from the definition.

Property 2.3.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is open, and the two maps u and v defined on Ω take values in $\mathbb{R}^N$. The constant $h \in \mathbb{R}$ is non-zero and $s \in \{1, \ldots, n\}$ is an index.*

(i) *If $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $p \geq 1$, then $\Delta_h^s u \in L^p(\Omega^{|h|}, \mathbb{R}^N)$ and we have*

$$\partial^i(\Delta_h^s u) = \Delta_h^s(\partial^i u) \tag{2.3.2}$$

in $\Omega^{|h|}$ for any $i \in \{1, \ldots, n\}$.

(ii) *If at least one of u and v is supported in $\Omega^{|h|}$, then we have*

$$\int_\Omega u \Delta_h^s v \, \mathrm{d}x = -\int_\Omega v \Delta_{-h}^s u \, \mathrm{d}x \tag{2.3.3}$$

whenever the two integrals make sense.

(iii) *We have*

$$\Delta_h^s(uv) = v(x)\Delta_h^s u(x) + u(x + he_s)\Delta_h^s v(x). \tag{2.3.4}$$

With the above properties, we can treat difference quotients as the corresponding derivatives to some extent.

For Sobolev maps in $W^{1,p}(\Omega, \mathbb{R}^N)$, the difference quotients of them can be controlled by the corresponding weak derivatives in the L^p-sense.

Lemma 2.3.3 ([GT01], Lemma 7.23)**.** *Suppose that* $\Omega \subset \mathbb{R}^n$ *and* $U \subset\subset \Omega$ *are open sets, and the constant* $h \in \mathbb{R} \setminus \{0\}$ *satisfies* $|h| < \operatorname{dist}(U, \partial\Omega)$*. Given any map* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *with* $p \geq 1$*, we have* $\Delta_h^s u \in L^p(U, \mathbb{R}^N)$ *and*

$$\|\Delta_h^s u\|_{L^p(U,\mathbb{R}^N)} \leq \|\partial^s u\|_{L^p(\Omega,\mathbb{R}^N)} \tag{2.3.5}$$

for any $s \in \{1, \ldots, n\}$*.*

The inverse of the Lemma also holds true, but only for $p > 1$.

Lemma 2.3.4 ([GT01], Lemma 7.24)**.** *Suppose that* $\Omega \subset \mathbb{R}^n$ *is an open set, and* $p > 1$ *and* $s \in \{1, \ldots, n\}$ *are fixed. Assume that there exists* $K > 0$ *such that, for any open set* $U \subset\subset \Omega$ *and any* $h \in \mathbb{R} \setminus \{0\}$ *with* $|h| < \operatorname{dist}(U, \partial\Omega)$*, we have* $\Delta_h^s u \in L^p(U, \mathbb{R}^N)$ *and*

$$\|\Delta_h^s u\|_{L^p(U,\mathbb{R}^N)} \leq K. \tag{2.3.6}$$

Then the weak derivative $\partial^s u$ *exists in* $L^p(\Omega, \mathbb{R}^N)$ *with*

$$\|\partial^s u\|_{L^p(\Omega,\mathbb{R}^N)} \leq K. \tag{2.3.7}$$

The following lemma can be found in [KM05], in which the authors used it to obtain some fractional Sobolev regularity with a certain difference quotients estimate.

Lemma 2.3.5 ([KM05], Lemma 2.5)**.** *Suppose that* $\Omega \subset \mathbb{R}^n$ *is a bounded open set and* $G \in L^p(\Omega, \mathbb{R}^{N\times n})$ *with* $p > 1$*. For some* $\theta \in (0, 1)$*,* $M \geq 0$ *and an open set* $U \subset\subset \Omega$*, there holds*

$$\int_U \sum_{s=1}^{n} |\Delta_h^s G(x)|^p \,\mathrm{d}x \leq M^p |h|^{p(\theta-1)} \tag{2.3.8}$$

for any $h \in \mathbb{R} \setminus \{0\}$ *with* $|h| < \delta := \min\{1, \operatorname{dist}(U, \partial\Omega)\}$*. Then we have* $G \in W^{b,p}_{loc}(U, \mathbb{R}^{N\times n})$ *for any* $b \in (0, \theta)$*. Moreover, given any open set* $V \subset\subset U$*, the estimate*

$$\|G\|_{W^{b,p}(V,\mathbb{R}^{N\times n})} \leq C(M + \|G\|_{L^p(\Omega,\mathbb{R}^{N\times n})}) \tag{2.3.9}$$

holds true with some $C = C(n, p, \delta, \theta, b, \mathscr{L}^n(\Omega), \operatorname{dist}(V, \partial U)) > 0$*.*

2.4 p-capacity

In this section, we recall the notion of the p-capacity, i.e., the capacity associated to the Sobolev space $W^{1,p}$. Some relevant concepts and results are also presented for later use. For the following contents and a thorough discussion about p-capacities, we refer to [HKM18]. Throughout this section, the set $\Omega \subset \mathbb{R}^n$ is open but not necessarily bounded. The exponent p is in the range $(1, \infty)$.

For any compact set $K \subset\subset \Omega$, set

$$W(K, \Omega) := \{u \in C_c^\infty(\Omega) \colon\ u \geq 1 \text{ on } K\}.$$

Definition 2.4.1. Suppose that $\Omega \subset \mathbb{R}^n$ is an open set, and $K \subset\subset \Omega$ is compact. Then the *p-capacity* of the condenser (K, Ω) is defined by

$$\operatorname{cap}_p(K;\Omega) := \inf_{u \in W(K,\Omega)} \int_\Omega |\nabla u|^p \,\mathrm{d}x. \tag{2.4.1}$$

When Ω is fixed, this quantity is also called the p-capacity of K. For any open subset U of Ω, the corresponding p-capacity is defined by

$$\operatorname{cap}_p(U;\Omega) := \sup_{\substack{K \subset U \\ \text{compact}}} \operatorname{cap}_p(K;\Omega). \tag{2.4.2}$$

Finally, we define the p-capacity of any $E \subset \Omega$ by

$$\operatorname{cap}_p(E;\Omega) := \inf_{\substack{E \subset U \subset \Omega \\ U \text{ open}}} \operatorname{cap}_p(E;\Omega). \tag{2.4.3}$$

One easy observation is that $W(K, \Omega)$ in the above definition can be replaced by

$$W_0(K,\Omega) := \{u \in W_0^{1,p}(\Omega) \cap C(\Omega) \colon\ u \geq 1 \text{ on } K\}.$$

Directly from the definition, we know that the quantity $\operatorname{cap}_p(\cdot\,;\cdot)$ is monotone in both arguments, and it is also possible to show subadditivity for the first one.

Proposition 2.4.2 ([HKM18], Theorem 2.2)**.** *The p-capacity $\operatorname{cap}_p(\cdot\,;\cdot)$ defined in Definition 2.4.1 satisfies the following properties:*

(i) *The map $E \mapsto \operatorname{cap}_p(E;\Omega)$ is increasing, that is, for any $E_1 \subset E_2 \subset \Omega$ we have*

$$\operatorname{cap}_p(E_1;\Omega) \leq \operatorname{cap}_p(E_2;\Omega); \tag{2.4.4}$$

(ii) *The map $\Omega \mapsto \operatorname{cap}_p(E;\Omega)$ is decreasing, that is, for any open sets Ω_1, Ω_2 with $E \subset \Omega_1 \subset \Omega_2$ we have*

$$\operatorname{cap}_p(E;\Omega_1) \geq \operatorname{cap}_p(E;\Omega_2); \tag{2.4.5}$$

(iii) *If $E = \cup_{i=1}^\infty E_i \subset \Omega$, we have*

$$\operatorname{cap}_p(E;\Omega) \leq \sum_{i=1}^\infty \operatorname{cap}_p(E_i;\Omega). \tag{2.4.6}$$

The fact $\operatorname{cap}_p(\emptyset;\Omega) = 0$ and (iii) imply that $\operatorname{cap}_p(\cdot\,;\Omega)$ is an outer measure on Ω.

Analogous to the almost everywhere and quasi-continuity concepts in measure theory, we have the following ones for the p-capacity.

Definition 2.4.3. (a) A property is said to hold *p-quasi-everywhere* (also abbreviated to *p-q.e.*) on Ω, if it holds on Ω except a subset with zero p-capacity.

(b) A function $u \colon \Omega \to \mathbb{R} \cup \{\infty\}$ is said to be *q-quasi-continuous* on Ω if the following holds: for any $\varepsilon > 0$, there exists an open set $G \subset \Omega$ such that $\operatorname{cap}_p(G;\Omega) < \varepsilon$ and $u|_{\Omega \setminus G}$ is finite-valued and continuous.

Furthermore, any $W^{1,p}$ function has a p-quasi-continuous representative, which is unique in the p-q.e. sense.

Theorem 2.4.4 ([HKM18], Theorem 4.14)**.** *Suppose that $u \in W^{1,p}(\Omega)$. Then there exists a p-quasi-continuous function $v \in W^{1,p}(\Omega)$ such that $u = v$ $\mathscr{L}^n$-a.e. in Ω. Moreover, if there is another p-quasi-continuous $\tilde{v} \in W^{1,p}(\Omega)$ with $u = \tilde{v}$ $\mathscr{L}^n$-a.e. in Ω, then we have $v = \tilde{v}$ q-a.e. in Ω.*

Notice that the definition of p-quasi-continuity and the above theorem also applies to vector-valued functions.

It is generally uneasy to compute capacities explicitly, but possible for some regular condensers, such as the spherical one $(B(x_0, r), B(x_0, R))$.

Example 2.4.1 ([HKM18], Example 2.12)**.** Fix a point $x_0 \in \mathbb{R}^n$, and denote the ball $B(x_0, \rho)$ by B_ρ for any $\rho > 0$. Then for any r, R with $0 < r < R < \infty$, the p-capacity of $(\bar{B}_r, B_R)$ is

$$\operatorname{cap}_p(\bar{B}_r; B_R) = \begin{cases} \omega_{n-1} \left(\dfrac{|n-p|}{p-1} \right)^{p-1} \left| R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}} \right|^{1-p}, & p \neq n, \\ \omega_{n-1} \left(\log \dfrac{R}{r} \right)^{1-n}, & p = n. \end{cases} \tag{2.4.7}$$

In particular, we have

$$\operatorname{cap}_p(\bar{B}_r; B_{2r}) = c_1(n,p) r^{n-p}, \qquad p > 1; \tag{2.4.8}$$

$$\operatorname{cap}_p(\bar{B}_r; \mathbb{R}^n) = c_2(n,p) r^{n-p}, \qquad 1 < p < n; \tag{2.4.9}$$

$$\operatorname{cap}_p(\{x_0\}; B_r) = c_1(n,p) r^{n-p}, \qquad p > n. \tag{2.4.10}$$

From the definition of the p-capacity of an open set and (2.4.7), we can see

$$\operatorname{cap}_p(B_r; B_R) = \operatorname{cap}_p(\bar{B}_r; B_R).$$

As mentioned above, the p-capacity $\operatorname{cap}_p(\cdot\,; \Omega)$ is an outer measure on Ω, and it actually provides a finer measurement of sets than the Lebesgue measure $\mathscr{L}^n \llcorner \Omega$ does.

Theorem 2.4.5 ([HKM18], Theorem 2.26)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and $1 < p \leq n$. Then it holds that*

$$\operatorname{cap}_p(E; \Omega) = 0 \quad \textit{implies} \quad \mathscr{L}^n(E) = 0$$

for any $\mathscr{L}^n$-measurable set $E \subset \Omega$. Furthermore, for any $E \subset \Omega$ with $\operatorname{cap}_p(E; \Omega) = 0$, we have

$$\dim_{\mathcal{H}}(E) \leq n - p,$$

where $\dim_{\mathcal{H}}(E)$ is the Hausdorff dimension of E.

Notice that if $p > n$, any non-empty set is of positive p-capacity by (2.4.9).

Finally, we introduce the concept of uniform p-thickness, which describes boundary regularity of domains.

Definition 2.4.6. A set $E \subset \mathbb{R}^n$ is said to be *uniformly p-thick* with constants R_0 and T_0 if for any $x \in E$ and $0 < r < R_0$, we have

$$\operatorname{cap}_p(E \cap \bar{B}(x,r); B(x,2r)) \geq T_0 \operatorname{cap}_p(\bar{B}(x,r); B(x,2r)). \tag{2.4.11}$$

Intuitively speaking, the concept of uniform p-thickness gives a control of the sharpness of any outward cusps of a domain. Notice that if $p > n$, any nonempty set is uniformly p-thick by (2.4.8) and (2.4.10). In addition, Hölder's inequality implies that any uniformly p-thick set is actually uniformly q-thick for any $q > p$. Combining Theorem 1 in [Lew88] and Theorem 2.38 in [HKM18], we have the following self-improving property of the uniform p-thickness property.

Proposition 2.4.7 ([KK94], Proposition 2.3)**.** *Suppose that $p > 1$, the two constants R_0, T_0 are positive and $E \subset \mathbb{R}^n$ is a closed set. If E is uniformly p-thickness with radius R_0 and T_0, then there exist an exponent q with $1 < q < \min\{p,n\}$ and $T = T(n,p,T_0) > 0$ such that E is uniformly q-thick with radius R_0 and T.*

2.5 Linear growth functionals

The functionals considered in Chapter 5 and 6 are of linear growth, so we now have a preliminary discussion about this type of functionals. Subsection 2.5.1 is about functionals defined on measures, with which we can relax the definition of normal functionals and extend them to BV maps. Subsection 2.5.2 is for generalised Young measures, which are effective tools in the calculus of variations, and will be used in Chapter 6 to show the strong convergence of a certain sequence.

In this section, the set $\Omega \subset \mathbb{R}^n$ is assumed to be open and bounded. To prepare for the following contents, we first define the space $\mathbf{E}(\Omega, \mathbb{R}^m)$ for some positive $m \in \mathbb{N}$. Denote by $\mathbb{B}^m$ and $\mathbb{S}^{m-1}$ the unit ball and the unit sphere in $\mathbb{R}^m$ respectively, and consider the linear transformation S on $C(\bar{\Omega} \times \mathbb{B}^m)$ given by

$$(Sf)(x, \tilde{z}) := (1 - |\tilde{z}|) f\left(x, \frac{\tilde{z}}{1 - |\tilde{z}|}\right), \quad x \in \bar{\Omega}, \tilde{z} \in \mathbb{B}^m,$$

for any $f \in C(\bar{\Omega} \times \mathbb{R}^m)$. Then set

$$\mathbf{E}(\Omega, \mathbb{R}^m) := \{f \in C(\bar{\Omega} \times \mathbb{R}^m) \colon\ Sf \in C(\bar{\Omega} \times \bar{\mathbb{B}}^m)\}.$$

Notice that for any $f \in C(\bar{\Omega} \times \mathbb{R}^m)$, by $Sf \in C(\bar{\Omega} \times \bar{\mathbb{B}}^m)$ we mean that Sf is in $C(\bar{\Omega} \times \mathbb{B}^m)$ and can be extended continuously to $\bar{\Omega} \times \bar{\mathbb{B}}^m$, and the extension is also denoted by Sf. The norm on $\mathbf{E}(\Omega, \mathbb{R}^m)$ is the natural choice

$$\|f\|_{\mathbf{E}(\Omega,\mathbb{R}^m)} := \|Sf\|_{C(\bar{\Omega} \times \bar{\mathbb{B}}^m)} = \sup_{(x,\tilde{z}) \in \bar{\Omega} \times \bar{\mathbb{B}}^m} |(Sf)(x, \tilde{z})|,$$

under which $\mathbf{E}(\Omega, \mathbb{R}^m)$ is a Banach space and $S \colon C(\bar{\Omega} \times \mathbb{R}^m) \to C(\bar{\Omega} \times \bar{\mathbb{B}}^m)$ is an isometric isomorphism ([Rin18], §12.1).

If $f \in \mathbf{E}(\Omega, \mathbb{R}^m)$, it is easy to see that f is of linear growth, i.e., there exists $L > 0$ such that

$$|f(x,z)| \leq L(1+|z|), \quad \text{for any } (x,z) \in \bar{\Omega} \times \mathbb{R}^m.$$

Moreover, the *recession function*

$$f^\infty(x,z) := \limsup_{\substack{x' \to x, z' \to z \\ t \to \infty}} \frac{f(x', tz')}{t} \tag{2.5.1}$$

exists for any $(x,z) \in \bar{\Omega} \times \mathbb{R}^m$. Indeed, in (2.5.1) the limit superior can be replaced by the limit (in which case f^∞ is called the *strong recession function*), and the convergence is locally uniformly in (x,z) by the definition of $\mathbf{E}(\Omega, \mathbb{R}^m)$. The function f^∞ is positively homogeneous in the second argument and coincides with Sf on $\bar{\Omega} \times \mathbb{R}^m$. The subclass $\mathbf{E}(\mathbb{R}^m)$ is the collection of those functions in $\mathbf{E}(\Omega, \mathbb{R}^m)$ that are independent of the first argument $x \in \bar{\Omega}$.

2.5.1 Functionals defined on measures

Let μ be an $\mathbb{R}^m$-valued Radon measure on $\Omega \subset \mathbb{R}^n$. Then the total variation $|\mu|$ of it is a real-valued Radon measure on Ω. The measure μ is said to be a bounded Radon measure if $|\mu|(\Omega) < \infty$. By the Lebesgue-Radon-Nikodým theorem, we can decompose μ as

$$\mu = \mu^{ac} + \mu^s = \frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n} \mathscr{L}^n + \frac{\mathrm{d}\mu}{\mathrm{d}|\mu^s|} |\mu^s|.$$

Let f be a function in $\mathbf{E}(\Omega, \mathbb{R}^m)$, and now we define the signed Radon measure $f(\cdot, \mu)$ as follows: for any Borel set A compactly contained in Ω, set

$$f(\cdot,\mu)(A) := \int_A f(\cdot,\mu) := \int_A f\left(\cdot, \frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\right) \mathrm{d}\mathscr{L}^n + \int_A f^\infty\left(\cdot, \frac{\mathrm{d}\mu}{\mathrm{d}|\mu^s|}\right) \mathrm{d}|\mu^s|. \tag{2.5.2}$$

For any $z \in \mathbb{R}^m$, we write $f(\mu - z)$ as a short-hand of $f(\mu - z\mathscr{L}^n)$. If μ is bounded, the definition above can be extended to all Borel subsets of Ω and $f(\cdot\,, \mu)$ is a bounded Radon measure on Ω. We now recall the convergence of Radon measures with respect to some particular integrands.

Definition 2.5.1. Suppose that $\{\mu_j\}$ and μ are Radon measures defined on Ω such that $\mu_j \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega, \mathbb{R}^m)$ and $f(\mu_j)(\Omega) \to f(\mu)(\Omega)$, where $f\colon \mathbb{R}^m \to \mathbb{R}$ is a Borel function of linear growth. Then μ_j is said to *converge to* μ *f-strictly.* In particular,

(a) μ_j is said to *converge to* μ *strictly* if $f(\cdot) = |\cdot|$;

(b) μ_j is said to *converge to* μ *area-strictly* if $f = E$ (see (2.1.4)).

Under the above topologies, smooth functions are dense in the space of Radon measures. The following approximation result is a modification of Theorem 2.2 in [AFP00].

Lemma 2.5.2. *Any Radon measure $\mu \in \mathscr{M}(\Omega, \mathbb{R}^m)$ can be locally area-strictly approximated by $\mathbb{R}^m$-valued smooth maps. If μ is bounded on Ω, the approximation is global.*

A generalisation of the continuity theorem by Reshetnyak (see [Res68], Theorem 2.39 in [AFP00] and Theorem 5 in [KR10b]) states that given any $f \in \mathbf{E}(\Omega, \mathbb{R}^m)$, the corresponding functional is continuous with respect to the area-strict topology.

Theorem 2.5.3 (Reshetnyak continuity theorem)**.** *Suppose that $f \in \mathbf{E}(\Omega, \mathbb{R}^m)$ and $\mu_j, \mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$ are Radon measures such that μ_j converges to μ in the area-strict sense. Then we have*

$$\int_\Omega f(\cdot, \mu_j) \to \int_\Omega f(\cdot, \mu), \quad \text{as } j \to \infty. \tag{2.5.3}$$

In particular, convergence in the area-strict sense implies that in the strict sense.

There is an analogue of the above continuity result for BV maps. See Section 2.6 for the concepts of rank-one convexity and quasiconvexity involved in the following statements.

Lemma 2.5.4 ([KR10b], Theorem 4)**.** *Suppose that $G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is rank-one convex and of linear growth. If $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $u_j, u \in BV(\Omega, \mathbb{R}^N)$ and $u_j \to u$ in the BV area-strict sense, then*

$$\int_\Omega G(Du_j) \to \int_\Omega G(Du) \quad \text{as } j \to \infty. \tag{2.5.4}$$

Such a continuity result together with Lemma 2.5.4 gives a direct corollary as follows, which allows us to approximate a BV map with respect to a certain energy.

Lemma 2.5.5. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. For any $u \in BV(\Omega, \mathbb{R}^N)$, there exists a sequence $\{u_j\} \subset W^{1,1}_u \cap C^\infty(\Omega, \mathbb{R}^N)$ such that $u_j \to u$ in the BV area strict sense. Furthermore, for any function $G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ that is rank-one convex and of linear growth, we have*

$$\int_\Omega G(Du_j) \to \int_\Omega G(Du) \quad \text{as } j \to \infty. \tag{2.5.5}$$

Remark 2.5.6. Notice that $E(\cdot - z_0)$ for any $z_0 \in \mathbb{R}^{N\times n}$ is convex by (5.2.3), and thus rank-one convex. Obviously it is of linear growth and then Lemma 2.5.5 holds for $E(\cdot - z_0)$. The lemma also holds for quasiconvex functions as quasiconvexity implies rank-one convexity (see 2.6.2).

2.5.2 Generalised Young measures

This subsection is about generalised Young measures and the subclass BV-Young measures. See [Rin18], Chapter 12, and [KR20] for more details.

Definition 2.5.7. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set. An $\mathbb{R}^m$-valued *generalised Young measure* defined on Ω is a triple $\nu = ((\nu_x)_{x\in\Omega}, \lambda_\nu, (\nu_x^\infty)_{x\in\bar\Omega})$, where

(i) $(\nu_x)_{x\in\Omega} \subset \mathscr{M}^1(\mathbb{R}^m)$ is the *oscillation measure*;

(ii) $\lambda_\nu \in \mathscr{M}^+(\bar\Omega)$ is the *concentration measure*;

(iii) $(\nu_x^\infty)_{x\in\bar\Omega} \subset \mathscr{M}^1(\mathbb{S}^{m-1})$ is the *concentration-direction measure*;

and it satisfies

(iv) the map $x \mapsto \nu_x$ is weakly* measurable with respect to $\mathscr{L}^n \llcorner \Omega$, i.e., the function $x \mapsto \langle \nu_x, f(x, \cdot)\rangle$ is $\mathscr{L}^n$-measurable for any bounded Borel function $f\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$;

(v) the map $x \mapsto \nu_x^\infty$ is weakly* measurable with respect to λ_ν, i.e., the function $x \mapsto \langle \nu_x^\infty, f(x,\cdot)\rangle$ is λ_ν-measurable for any bounded Borel function $f\colon \Omega \times \mathbb{S}^{m-1} \to \mathbb{R}$;

(vi) $x \mapsto \langle \nu_x, |\cdot|\rangle \in L^1(\Omega)$, i.e., the 1-moment $\int_\Omega \int_{\mathbb{R}^m} |z| \,\mathrm{d}\nu_x \,\mathrm{d}x$ is finite.

The collection of all $\mathbb{R}^m$-valued generalised Young measures on Ω is denoted by $\mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$.

In the definition above, the pairing $\langle \mu, g(\cdot)\rangle$ of a function g and a measure on its domain is the integral $\int g \,\mathrm{d}\mu$. The collection $\mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$ actually consists of equivalence classes: we identify two generalised Young measures $\mu, \nu \in \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$ if

- $\mu_x = \nu_x$ for $\mathscr{L}^n$-a.e. $x \in \Omega$;
- $\lambda_\mu = \lambda_\nu$;
- $\mu_x^\infty = \nu_x^\infty$ for λ_ν-a.e. $x \in \bar{\Omega}$.

In the following, the term "Young measures" will be used instead for convenience.

The functions in $\mathbf{E}(\Omega, \mathbb{R}^m)$ can be considered to be "test integrands" for Young measures. For $f \in \mathbf{E}(\Omega, \mathbb{R}^m)$ and $\nu \in \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$, the duality pairing is defined by

$$\begin{aligned}\langle\!\langle \nu, f \rangle\!\rangle &:= \int_\Omega \langle \nu_x, f(x,\cdot)\rangle \,\mathrm{d}x + \int_{\bar{\Omega}} \langle \nu_x^\infty, f^\infty(x,\cdot)\rangle \,\mathrm{d}\lambda_\nu(x) \\ &= \int_\Omega \int_{\mathbb{R}^m} f(x,z) \,\mathrm{d}\nu_x(z) \,\mathrm{d}x + \int_{\bar{\Omega}} \int_{\mathbb{S}^{m-1}} f^\infty(x,z) \,\mathrm{d}\nu_x^\infty(z) \,\mathrm{d}\lambda_\nu(x).\end{aligned} \tag{2.5.6}$$

Then space $\mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$ can be then considered a part of $(\mathbf{E}(\Omega, \mathbb{R}^m))^*$.

Definition 2.5.8. A sequence $\{\nu_j\}_{j\in\mathbb{N}} \subset \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$ is said to *converge weakly** to $\nu \in \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$ (written as $\nu_j \overset{*}{\rightharpoonup} \nu$) if the following holds for any $f \in \mathbf{E}(\Omega, \mathbb{R}^m)$:

$$\langle\!\langle \nu_j, f \rangle\!\rangle \to \langle\!\langle \nu, f \rangle\!\rangle \quad \text{as } j \to \infty. \tag{2.5.7}$$

Given a Young measure $\nu \in \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$, we define the *barycentre* of it by

$$\bar{\nu} := \bar{\nu}_\cdot \mathscr{L}^n \llcorner \Omega + \bar{\nu}_\cdot^\infty \lambda_\nu, \tag{2.5.8}$$

which is a measure on $\bar{\Omega}$. Notice that $\bar{\mu} := \int z \,\mathrm{d}\mu(z)$ is the barycentre of the measure μ. It is easy to see that

$$\nu_j \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m) \quad \text{implies} \quad \bar{\nu}_j \to \bar{\nu} \quad \text{in } \mathscr{M}(\bar{\Omega}, \mathbb{R}^m), \tag{2.5.9}$$

which can be proved by testing with $f(x,z) = \varphi(x) z_i$ for any $\varphi \in C(\bar{\Omega})$ and any $i \in \{1, \dots, m\}$.

Definition 2.5.9. (i) Given any finite Radon measure $\gamma \in \mathscr{M}(\bar{\Omega}, \mathbb{R}^m)$ with the Lebesgue-Radon-Nikodým decomposition

$$\gamma = \gamma^{ac} \mathscr{L}^n \llcorner \Omega + \gamma^s,$$

we associate to it an *elementary Young measure* $\varepsilon_\gamma \in \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$, which is given by

$$\varepsilon_\gamma = ((\delta_{\gamma^{ac}(x)})_{x\in\Omega}, |\gamma^s|, (\delta_{P(x)})_{x\in\bar{\Omega}}), \tag{2.5.10}$$

where δ_z is the dirac measure supported at $z \in \mathbb{R}^m$ and $P := \frac{\mathrm{d}\gamma^s}{\mathrm{d}|\gamma^s|}$.

(ii) Given a sequence $\{\gamma_j\}_{j\in\mathbb{N}} \subset \mathscr{M}(\Omega, \bar{\mathbb{R}}^m)$ with $\sup_j|\gamma_j|(\bar{\Omega}) < \infty$, we say that $\{\gamma_j\}$ *generates* $\nu \in \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$ (written as $\gamma_j \xrightarrow{\mathbf{Y}} \nu$) if $\varepsilon_{\gamma_j} \overset{*}{\rightharpoonup} \nu$ in $\mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$, i.e.,

$$\langle\!\langle \varepsilon_{\gamma_j}, f\rangle\!\rangle \to \langle\!\langle \nu, f\rangle\!\rangle \quad \text{as } j \to \infty$$

for any $f \in \mathbf{E}(\Omega, \mathbb{R}^m)$.

The definition of elementary Young measures is motivated by a more complete version of the Reschetnyak continuity theorem (Theorem 2.5.3), which says (see [Rin18], Proposition 12.4)

$$\gamma_j \longrightarrow \gamma \quad \text{area-strictly} \quad \Longleftrightarrow \quad \gamma_j \xrightarrow{\mathbf{Y}} \varepsilon_\gamma. \tag{2.5.11}$$

The following result, which is considered to be the fundamental theorem of generalised Young measures, implies the compactness of $\mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$ under the weak* topology.

Theorem 2.5.10 ([Rin18], Theorem 12.5)**.** *Suppose that $\{\gamma_j\}_{j\in\mathbb{N}} \subset \mathscr{M}(\bar{\Omega}, \mathbb{R}^m)$ is a sequence of Radon measures with $\sup_j|\gamma_j|(\bar{\Omega}) < \infty$. Then there exists a subsequence $\{\gamma_{j_k}\}_{k\in\mathbb{N}}$ and a Young measure $\nu \in \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$ such that $\gamma_{j_k} \xrightarrow{\mathbf{Y}} \nu$.*

As indicated in the terms, given a Young measure $\nu \in \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$ generated by $\{\nu_j\}_{j\in\mathbb{N}} \subset \mathscr{M}(\bar{\Omega}, \mathbb{R}^m)$, the oscillation measure $(\nu_x)_{x\in\Omega}$ describes the oscillation of the sequence $\{\nu_j\}$, and the concentration measure λ_ν with the concentration-direction measure $(\nu_x^\infty)_{x\in\bar{\Omega}}$ the concentration of $\{\nu_j\}$. If either of them vanishes, the convergence holds in a better sense. We only consider the following special case for later use.

Suppose that $\{v_j\}_{j\in\mathbb{N}} \subset L^1(\Omega, \mathbb{R}^m)$ and $v \in L^1(\Omega, \mathbb{R}^m)$, then we say that

(a) v_j *converges to* v *in* $\mathscr{L}^n$*-measure*, if for any $\varepsilon > 0$ we have

$$\mathscr{L}^n(\{x \in \Omega\colon\ |v_j(x) - v(x)| > \varepsilon\});$$

(b) $\{v_j\}$ is *uniformly integrable* if

$$\sup_j \int_{\{|v_j|>t\}} |v_j(x)|\, \mathrm{d}x \to 0 \quad \text{as } t \to \infty.$$

Theorem 2.5.11 ([KR20], Theorem 2.17)**.** *Suppose that $\{v_j\}_{j\in\mathbb{N}}$ is a sequence in $L^1(\Omega, \mathbb{R}^m)$ and $\{v_j\mathscr{L}^n \llcorner \Omega\}$ generates $\nu \in \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$. Then v_j converges to some $v \in L^1(\Omega, \mathbb{R}^m)$ in $\mathscr{L}^n$-measure if and only if*

$$v(x) = \bar{\nu}_x, \quad \nu_x = \delta_{\bar{\nu}_x} \quad \text{for } \mathscr{L}^n\text{-a.e. } x \in \Omega.$$

Theorem 2.5.12 ([KR20], Theorem 2.18)**.** *Suppose that $\{v_j\}_{j\in\mathbb{N}}$ is a sequence in $L^1(\Omega, \mathbb{R}^m)$ and $\{v_j\mathscr{L}^n \llcorner \Omega\}$ generates $\nu \in \mathbf{Y}^{\mathscr{M}}(\Omega, \mathbb{R}^m)$. Then given any $f \in \mathbf{E}(\Omega, \mathbb{R}^m)$, the sequence $\{f(\cdot, v_j)\}$ is uniformly integrable if and only if*

$$\langle \nu_x^\infty, f^\infty(x, \cdot)\rangle = 0 \quad \text{for } \mathscr{L}^n\text{-a.e. } x \in \bar{\Omega}.$$

In particular, the sequence $\{v_j\}$ is uniformly integrable if and only if $\lambda_\nu \equiv 0$.

Consider a sequnece $\{v_j\}_{j\in\mathbb{N}} \in L^1(\Omega,\mathbb{R}^m)$ such that $\{v_j\mathscr{L}^n \llcorner \Omega\}$ generates $\nu \in \mathbf{Y}^{\mathscr{M}}(\Omega,\mathbb{R}^m)$. Then from the following Vitali convergence theorem, we can see that the only factors that may prevent $\{v_j\}$ from converging strongly in L^1 to $v \in L^1(\Omega,\mathbb{R}^m)$ with $v(x) = \bar{\nu}_x$ are exactly possible oscillation and concentration.

Theorem 2.5.13 ([KR20], Theorem 1.1)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, and let $\{v_j\}_{j\in\mathbb{N}} \subset L^1(\Omega,\mathbb{R}^m)$ and $v \in L^1(\Omega,\mathbb{R}^m)$. Then $v_j \to v$ in $L^1(\Omega,\mathbb{R}^m)$ if and only if $v_j \to v$ in $\mathscr{L}^n$-measure and $\{v_j\}$ is uniformly integrable.*

The subclass of generalised Young measures, generated by gradients, is important in the study of variational problems. Here, we consider the gradients of $\mathbb{R}^N$-valued BV maps defined on Ω, and thus the corresponding Young measure space is $\mathbf{Y}^{\mathscr{M}}(\Omega,\mathbb{R}^{N\times n})$.

Given any $u \in BV(\Omega,\mathbb{R}^N)$, we associate to $Du \in \mathscr{M}(\Omega,\mathbb{R}^{N\times n})$ the elementary Young measure ε_{Du} as above. The collection of *BV-Young measures* is composed of those generated by BV-derivatives, i.e.,

$$\begin{aligned}\mathbf{BVY}(\Omega,\mathbb{R}^{N\times n}) := \{\nu \in \mathbf{Y}^{\mathscr{M}}(\Omega,\mathbb{R}^{N\times n}):\ &\text{there exists}\\ &\{u_j\} \subset BV(\Omega,\mathbb{R}^N) \text{ with } Du_j \overset{\mathbf{Y}}{\to} \nu\}.\end{aligned}$$

The fundamental theorem of generalised Young measures (Theorem 2.5.10) implies the compactness of $\mathbf{BVY}(\Omega,\mathbb{R}^{N\times n})$:

Corollary 2.5.14. *Suppose that $\{u_j\}_{j\in\mathbb{N}} \subset BV(\Omega,\mathbb{R}^N)$ is bounded, then there exists a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ and $\nu \in \mathbf{BVY}(\Omega,\mathbb{R}^{N\times n})$ such that*

$$Du_{j_k} \overset{\mathbf{Y}}{\to} \nu.$$

2.6 Convexity notions

Apart from convexity, there are a few other convexity notions introduced in the calculus of variations for different purposes. One such important notion is quasiconvexity, which plays a significant role in the study of some variational problems and was first introduced by Morrey in his fundamental work [Mor52]. Quasiconvexity is a necessary and sufficient condition for a functional being lower semi-continuous under appropriate assumptions, which is essential in the application of the direct method to establish the existence of minimizers. We give a brief introduction to various convexity notions and discuss some properties of quasiconvex functions.

2.6.1 Definitions and connections

Definition 2.6.1. A Borel measurable function $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{\infty\}$ is said to be *quasiconvex*, if for any $z \in \mathbb{R}^{N\times n}$ and any $\varphi \in W_0^{1,\infty}((0,1)^n,\mathbb{R}^N)$ we have

$$f(z) \le \int_{(0,1)^n} f(z+\nabla\varphi(x))\,\mathrm{d}x. \tag{2.6.1}$$

Remark 2.6.2. The unit cube $(0,1)^n$ in the definition can be replaced by any open and bounded Lipschitz set $D \subset \mathbb{R}^n$ with the right-hand side being the average of $f(z + \nabla\varphi)$ on D (see [Rin18], Lemma 5.2).

The focus of this thesis is on functionals depending on the gradients of their input functions. Then the above definition fits our case well as it gives Jensen's inequality (and thus convexity in average) along those "gradient directions". Under standard assumptions, an integrand f being quasiconvex is equivalent to the lower semicontinuity of the corresponding functional (Subsection 2.6.2). On the other hand, this convexity condition is also closely related to the coercivity of the functional (Subsection 2.6.3). Considering the ingredients needed for the direct method (as listed in Section 1.2), quasiconvexity is indeed the natural framework for variational problems in the vectorial setting.

For completeness, we also recall the definitions of polyconvexity and rank-one convexity. The former was introduced by Ball [Bal77] in the study of nonlinear elasticity, where the energy does not satisfy the p-growth assumption and thus Theorem 2.6.4 does not apply. It can be rather difficult to verify quasiconvexity as it is not described in a pointwise way. Alternatively, we have the concept of rank-one convexity, which is slightly weaker and equivalent to the ellipticity of the corresponding Euler-Lagrange systems.

Definition 2.6.3. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{\infty\}$ be a function. Then f is said to be

(a) *polyconvex* if there exists a convex function $F\colon \mathbb{R}^{\tau(N,n)} \to n$ such that, for any $\xi \in \mathbb{R}^{N\times n}$, we have $f(\xi) = F(T(\xi))$, where

$$\tau(N,n) = \sum_{s=1}^{N\wedge n} \binom{n}{s}\binom{N}{s}$$

and $T(\xi) \in \mathbb{R}^{\tau(N,n)}$ is the vector composed of all the $s \times s$ minors of ξ for $1 \le s \le N \wedge n := \min\{N, n\}$;

(b) *rank-one convex* if $f(z + t\xi)$ is convex in $t \in \mathbb{R}$ for any $z, \xi \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}(\xi) \le 1$.

Note that any rank one vector $\xi \in \mathbb{R}^{N\times n}$ can be written as $a \otimes b$ with some $a \in \mathbb{R}^N, b \in \mathbb{R}^n$. Then the rank-one convexity implies that if f is second-order differentiable, its second derivative is elliptic in the sense that

$$f''(z)[a \otimes b, a \otimes b] \ge 0 \quad \text{for any } a \in \mathbb{R}^N, b \in \mathbb{R}^n.$$

The relation between these notions is given in the following diagram (see [Dac08], Theorem 5.3 for a proof):

$$\text{convexity} \implies \text{polyconvexity} \implies \text{quasiconvexity} \implies \text{rank-one convexity} \tag{2.6.2}$$

None of the implication arrows can be inversed in general, and there are still some special cases that are not well-understood. For instance, whether the inverse implication between rank-one convexity and quasiconvexity holds true is unclear when $n \ge N = 2$, but the answer is otherwise negative by Šverák's example in [Šve92]. The following is an example to give some sense:

Example 2.6.1 (Alibert-Dacorogna-Marcellini, [DM88, AD92])**.** Let $n = N = 2$ and define

$$h_\gamma(A) := |A|^2(|A|^2 - 2\gamma\,\mathrm{det}A), \quad A \in \mathbb{R}^{2\times 2}.$$

The precise ranges of γ for h_γ being convex, polyconvex and rank-one convex are clear, while that for quasiconvexity is still unknown.

h_γ is	convex	polyconvex	quasiconvex	rank-one convex
$\iff$	$\lvert\gamma\rvert \leq \gamma_c := \frac{2\sqrt{2}}{3}$	$\lvert\gamma\rvert \leq \gamma_{pc} := 1$	$\lvert\gamma\rvert \leq \gamma_{qc} \in (1, \frac{2}{\sqrt{3}}]$	$\lvert\gamma\rvert \leq \gamma_{rc} := \frac{2}{\sqrt{3}}$

Table 2.2: The Alibert-Dacorogna-Marcellini example: convexity of h_γ.

2.6.2 Lower semicontinuity

Now we return to quasiconvexity and state the well-known lower-semicontinuity theorem. Results of this type were first established by Morrey [Mor52], and then generalised to functionals with Carathéodory integrands (see [AF84] and [Rin18], Theorem 5.16).

Theorem 2.6.4. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, the function $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is Carathéodory and of p-growth with $p \in (1, \infty)$. If we furthermore assume that $f(x, \cdot)$ is quasiconvex for $\mathscr{L}^n$-a.e. $x \in \Omega$, then the functional*

$$\mathscr{F}(u, \Omega) := \int_\Omega f(x, \nabla u)\,\mathrm{d}x$$

is weakly lower-semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^N)$.

A parallel version of the above theorem holds for the relaxed functional in the linear growth context ($p = 1$) with some extra assumptions. See the work by Ambrosio & Dal Maso [ADM92] and by Fonseca & Müller [FM93]. Rindler [Rin12] gave a new proof via Young measures without Alberti's rank-one theorem.

Theorem 2.6.5. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, and $f\colon \bar{\Omega} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies the following assumptions:*

(i) *f is a Carathéodory function;*

(ii) *$|f(x, z)| \leq L(1 + |z|)$ for some $L > 0$ and any $(x, z) \in \bar{\Omega} \times \mathbb{R}^{N\times n}$;*

(iii) *the strong recession function f^∞ exists and is in $C(\bar{\Omega} \times \mathbb{R}^{N\times n})$;*

(iv) *$f(x, \cdot)$ is quasiconvex for any $x \in \bar{\Omega}$.*

Then the relaxed functional

$$\mathscr{F}_*(u, \Omega) = \int_\Omega f(x, \nabla u)\,\mathrm{d}x + \int_\Omega f^\infty(x, \frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|})\,\mathrm{d}|D^s u| + \int_{\partial\Omega} f^\infty(x, u|_{\partial\Omega} \otimes \nu_\Omega^-)\,\mathrm{d}\mathcal{H}^{n-1}$$

is lower semicontinuous with respect to the weak topology in $BV(\Omega, \mathbb{R}^N)$, where $u|_{\partial\Omega} = u^\Omega$ is the trace of u and ν_Ω^- is the inward unit normal on $\partial\Omega$.*

See Subsection 3.1.4 for more details about relaxed functionals.

With respect to a stronger topology, the area-strict topology, the assumption on the integrand f can be relaxed to rank-one convexity. In this case, we actually have continuity of the relaxed functional without the boundary term as indicated in Lemma 2.5.4.

2.6.3 Lipschitz continuity and coercivity

It is well-known that convexity (of certain types) implies local Lipschitz continuity (see [Mor66], p.112, and [Fus80, Mar85]). Lemma 2.2 in [BKK00] gives a better estimate constant. As a corollary of the above results, the following lemma gives the growth of the derivative of a quasiconvex integrand.

Lemma 2.6.6. *Suppose that* $G\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ *is a real-valued function and of* p*-growth with* $p\in[1,\infty)$*, i.e.,*

$$|G(z)|\le L(1+|z|^p)$$

for some $L>0$ *and any* $z\in\mathbb{R}^{N\times n}$*. If* G *is furthermore quasiconvex, then there exists a constant* $C=C(n,N,p)>0$ *such that*

$$|G'(z)|\le CL(1+|z|^{p-1}). \tag{2.6.3}$$

In particular, G *is Lipschitz when* $p=1$*.*

Quasiconvexity is also connected to the coercivity of variational functionals. In the following we consider a p-growth integrand $f\colon\ \mathbb{R}^{N\times n}\to\mathbb{R}$ with $p\ge 1$, and the corresponding functional $\mathscr{F}(u,\Omega):=\int_\Omega f(\nabla u)\,\mathrm{d}x$ defined on some bounded open set Ω, which is well-defined on $W^{1,p}_g(\Omega,\mathbb{R}^N)$ for some $g\in W^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$.

Definition 2.6.7. Fix $q\in[1,p]$. We say that f is

(a) *pointwise* q*-coercive* if there exist $c_1>0$ and $c_2\in\mathbb{R}$ such that

$$f(z)\ge c_1|z|^q+c_2,\quad \text{for any } z\in\mathbb{R}^{N\times n}; \tag{2.6.4}$$

(b) L^q *coercive* if for any non-empty and bounded open set $\Omega\subset\mathbb{R}^n$ and any boundary value $g\in W^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$, we have that

$$\begin{matrix}\{u_j\}_{j\in\mathbb{N}}\in W^{1,p}_g(\Omega,\mathbb{R}^N)\\ \|\nabla u_j\|_{L^q(\Omega,\mathbb{R}^{N\times n})}\overset{j\to\infty}{\longrightarrow}\infty\end{matrix}\quad \text{implies}\quad \mathscr{F}(u_j,\Omega)\overset{j\to\infty}{\longrightarrow}\infty; \tag{2.6.5}$$

(c) L^q *mean coercive* if for any non-empty and bounded open set $\Omega\subset\mathbb{R}^n$ and any boundary value $g\in W^{1,p}(\mathbb{R}^n,\mathbb{R}^{N\times n})$, there exist $c_1>0$ and $c_2\in\mathbb{R}$ depending on f,Ω,g such that

$$\mathscr{F}(u,\Omega)\ge\int_\Omega(c_1|\nabla u|^q+c_2)\,\mathrm{d}x,\quad \text{for any } u\in W^{1,p}_g(\Omega,\mathbb{R}^N). \tag{2.6.6}$$

The last two forms of coercivity are defined in terms of Ω and g, but turn out to be properties of f itself and thus independent of the choice of Ω and g (see [CK17], Proposition

3.1). Pointwise coercivity is relatively strong and usually corresponds to convexity. The other two, which are defined in the sense of integral, are indeed equivalent to the convexity in the mean-value sense — quasiconvexity.

Theorem 2.6.8 ([CK17], Theorem 1.1)**.** *Suppose that the function* $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *is continuous and of* p*-growth for some* $p \geq 1$*. Then for any* $q \in [1,p]$*, the following statements are equivalent:*

(i) f *is* L^q *coercive;*

(ii) f *is* L^q*-mean coercive;*

(iii) *there exist* $c > 0$ *and* $z_0 \in \mathbb{R}^{N\times n}$ *such that* $z \mapsto f(z) - c|z|^q$ *is quasiconvex at* z_0*.*

2.7 Estimates for elliptic systems

We will need some results about Legendre-Hadamard elliptic systems. A bilinear form $\mathbb{A} \in \bigodot^2(\mathbb{R}^{N\times n})$ is said to satisfy the *strong Legendre-Hadamard condition* if there exists $\lambda > 0$ such that

$$\mathbb{A}[\eta\otimes\xi,\eta\otimes\xi] \geq \lambda|\eta|^2|\xi|^2, \text{for any } \eta\in\mathbb{R}^N, \xi\in\mathbb{R}^n. \tag{2.7.1}$$

For convenience, we usually assume that $\mathbb{A}$ is bounded by some $\Lambda > 0$, i.e.,

$$\mathbb{A}[z,z] \leq \Lambda|z|^2, \text{for any } z \in \mathbb{R}^{N\times n}. \tag{2.7.2}$$

We say that a map $u\colon \Omega \to \mathbb{R}^N$ is $\mathbb{A}$-harmonic on some open set Ω if it satisfies

$$-\operatorname{div}(\mathbb{A}\nabla u) = 0 \tag{2.7.3}$$

in the distributional sense in Ω.

Lemma 2.7.1. *Suppose that* $\mathbb{A} \in \bigodot^2(\mathbb{R}^{N\times n})$ *satisfies* (2.7.1) *and* 2.7.2*. If* $h \in W^{1,1}(B_R,\mathbb{R}^N)$ *is* $\mathbb{A}$*-harmonic on the ball* $B_R = B(x_0,R) \subset \mathbb{R}^n$*, then* h *is* C^∞ *on* B_R*, and for any* $z \in \mathbb{R}^{N\times n}$ *and some* $c_a = c_a(n,N,\frac{\Lambda}{\lambda}) > 0$ *we have*

$$\sup_{B_{\frac{R}{2}}}|\nabla h - z| + R\sup_{B_{\frac{R}{2}}}|\nabla^2 h| \leq c_a \fint_{B_R}|\nabla h - z|\,\mathrm{d}x. \tag{2.7.4}$$

Proof. It will be easier to work with $W^{1,2}$ maps when dealing with such an elliptic system. So we take the standard mollifiers $\{\phi_\varepsilon\}$ and set $h_\varepsilon := \phi_\varepsilon * h$. Then $h_\varepsilon \in C^\infty(B_{R-\varepsilon},\mathbb{R}^N)$ and thus in $W^{1,2}(B_{R-2\varepsilon},\mathbb{R}^N)$ when $\varepsilon < \frac{R}{4}$. The mollified map h_ε is also $\mathbb{A}$-harmonic in $B_{R-\varepsilon}$. Actually, take any $\varphi \in C_c^\infty(B_{R-\varepsilon},\mathbb{R}^N)$, and then by Fubini's theorem and the symmetry of ϕ_ε we have

$$\begin{aligned}\int_{B_{R-\varepsilon}} \mathbb{A}[\nabla h_\varepsilon,\nabla\varphi]\,\mathrm{d}x &= \int_{B_R} \mathbb{A}[\phi_\varepsilon * (\nabla h),\nabla\varphi]\,\mathrm{d}x \\ &= \int_{B_R} \mathbb{A}[\nabla h,\phi_\varepsilon * (\nabla\varphi)]\,\mathrm{d}x\end{aligned}$$

$$= \int_{B_R} \mathbb{A}[\nabla h, \nabla \varphi_\varepsilon]\,\mathrm{d}x = 0$$

as $\varphi_\varepsilon \in C_c^\infty(B_R, \mathbb{R}^N)$. Fix $0 < \varepsilon < \frac{R}{4}$ and let R_ε be $R - 2\varepsilon$. The discussion in §III.2 in [Gia83] shows, for any $\tau < \varepsilon$,

$$\sup_{B_{\frac{R_\varepsilon}{2}}} |\nabla h_\tau - z| \le C \left(\fint_{B_{R_\varepsilon}} |\nabla h_\tau - z|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

Apply the method of Corollary 7.1 in [Giu03] to obtain

$$\sup_{B_{tR_\varepsilon}} |\nabla h_\tau - z| \le C \left(\frac{1}{(1-t)^n} \fint_{B_{R_\varepsilon}} |\nabla h_\tau - z|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}, \quad \text{for any } t \in (0,1),$$

and Theorem 7.3 in the same book implies

$$\sup_{B_{tR_\varepsilon}} |\nabla h_\tau - z| \le C \frac{1}{(1-t)^n} \fint_{B_{R_\varepsilon}} |\nabla h_\tau - z| \,\mathrm{d}x \le C \frac{1}{(1-t)^n} \fint_{B_R} |\nabla h - z| \,\mathrm{d}x. \tag{2.7.5}$$

The sequence $\{\nabla h_\tau\}_{\tau < \varepsilon}$ converges to ∇h in L^1, and thus $\mathscr{L}^n$-a.e. in B_{R_ε} by passing to a subsequence. With (2.7.5) and Fatou's lemma we know

$$\sup_{B_{tR_\varepsilon}} |\nabla h - z| \le C \frac{1}{(1-t)^n} \fint_{B_{R_\varepsilon}} |\nabla h - z| \,\mathrm{d}x, \tag{2.7.6}$$

which implies that $|\nabla h - z| \in L^2(B_{R_{2\varepsilon}})$ and then $h \in W^{1,2}(B_{R_{2\varepsilon}}, \mathbb{R}^N)$. Hence, the map h is in $C^\infty(B_{R_{2\varepsilon}}, \mathbb{R}^N)$ by [Gia83], §III.2, and is actually in $C^\infty(B_R, \mathbb{R}^N)$ as ε can be taken arbitrarily small.

The estimates in [Gia83], §III.2 also give

$$R_\varepsilon \sup_{B_{\frac{R_\varepsilon}{2}}} |\nabla^2 h| \le C \left(\fint_{B_{\frac{3}{4}R_\varepsilon}} |\nabla h - z|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \le C \sup_{B_{\frac{3}{4}R_\varepsilon}} |\nabla h - z| \overset{(2.7.6)}{\le} C \fint_{B_{R_\varepsilon}} |\nabla h - z| \,\mathrm{d}x.$$

Now take $\varepsilon \to 0$ to get the desired result. □

The next result is also classical, and see Proposition 2.11 in [GK19a] and the references therein for a proof.

Lemma 2.7.2. *Suppose that* $\mathbb{A} \in \bigodot^2(\mathbb{R}^{N\times n})$ *satisfies* (2.7.1) *with some* $\Lambda, \lambda > 0$. *Let* $r \in (1, \infty)$, $q \in [2, \infty)$ *and* $\mathbb{B}$ *be the unit ball in* $\mathbb{R}^n$.

(a) *For any* $g \in W^{1-\frac{1}{r},r}(\partial\mathbb{B}, \mathbb{R}^N)$, *the elliptic system*

$$\begin{cases} -\operatorname{div}(\mathbb{A}\nabla h) = 0, & \text{in } \mathbb{B} \\ h|_{\partial\mathbb{B}} = g, & \text{on } \partial\mathbb{B} \end{cases} \tag{2.7.7}$$

admits a unique solution $h \in W^{1,r}(\mathbb{B}, \mathbb{R}^N)$, *and there exists* $C = C(n, N, r, \frac{\Lambda}{\lambda}) > 0$

such that

$$\|h\|_{W^{1,r}(\mathbb{B},\mathbb{R}^N)} \leq C\,\|g\|_{W^{1-\frac{1}{r},r}(\partial\mathbb{B},\mathbb{R}^N)}.$$

(b) *For any $f \in L^q(\mathbb{B},\mathbb{R}^N)$, the elliptic system*

$$\begin{cases} -\operatorname{div}(\mathbb{A}\nabla w) = f, & \text{in } \mathbb{B} \\ w|_{\partial\mathbb{B}} = 0, & \text{on } \partial\mathbb{B} \end{cases} \tag{2.7.8}$$

admits a unique solution $w \in W^{2,q} \cap W_0^{1,q}(\mathbb{B},\mathbb{R}^N)$, and there exists $C = C(n,N,q,\frac{\Lambda}{\lambda}) > 0$ such that

$$\|w\|_{W^{2,q}(\mathbb{B},\mathbb{R}^N)} \leq C\|f\|_{L^q(\mathbb{B},\mathbb{R}^N)}.$$

Remark 2.7.3. If we only consider the gradient ∇h above, it is enough to control $\|\nabla h\|_{L^r(\mathbb{B},\mathbb{R}^N)}$ with $[g]_{W^{1-\frac{1}{r},r}(\partial\mathbb{B},\mathbb{R}^N)}$ by considering $g - (g)_{\partial\mathbb{B}}$. In particular, if $g \in W^{1,r}(\mathbb{B},\mathbb{R}^N)$, its trace $\operatorname{tr}_{\mathbb{B}} g$ exists in $W^{1-\frac{1}{r},r}(\mathbb{B},\mathbb{R}^N)$ (see [Gio17], §18.4). The estimate of $\|h\|_{W^{1,r}(\mathbb{B},\mathbb{R}^N)}$ in (a) can be then replaced by $\|g\|_{W^{1,r}(\mathbb{B},\mathbb{R}^N)}$.

2.8 Ekeland variational principle

Compared with BV maps, it is usually more convenient to work with those in Sobolev spaces. Therefore, we may need proper approximation sequences in the study of BV (almost-)minimizers, for which the Ekeland variational principle is helpful. See [Eke74] and [Giu03], Theorem 5.6.

Theorem 2.8.1 (Ekeland variational principle)**.** *Suppose that (X,d) is a complete metric space and the function $\mathcal{F}\colon X \to \mathbb{R}\cup\{\infty\}$ is lower-semicontinuous with respect to the metric topology, bounded from below and not identifically ∞. If for some $u \in X$ and $\varepsilon > 0$ we have*

$$\mathcal{F}(u) \leq \inf_{v\in X} \mathcal{F}(v) + \varepsilon,$$

then there exists $w \in X$ satisfying the following:

(a) $d(u,w) \leq 1$*;*

(b) $\mathcal{F}(w) \leq \mathcal{F}(u)$*;*

(c) $\mathcal{F}(w) \leq \mathcal{F}(v) + \varepsilon d(v,w)$ *for any $v \in X$.*

Proof. We construct a sequence by induction that converges to the desired w. Take $u_1 = u$ and suppose that $u_1,\ldots,u_k$ have been chosen. Then consider

$$S_k := \{v \in X\colon\ \mathcal{F}(v) \leq \mathcal{F}(u_k) - \varepsilon d(u_k,v)\},$$

which contains u_k and thus non-empty. Take $u_{k+1} \in S_k$ such that

$$\mathcal{F}(u_{k+1}) \leq \frac{1}{2}(\mathcal{F}(u_k) + \inf_{S_k} \mathcal{F}). \tag{2.8.1}$$

The sequence $\{u_k\}_{k\in\mathbb{N}}$ we constructed is a Cauchy sequence. Indeed, from the choice of u_{k+1} we know

$$\varepsilon d(u_k, u_{k+1}) \le \mathcal{F}(u_k) - \mathcal{F}(u_{k+1}),$$

which implies implies that $\mathcal{F}(u_k)$ decreases in k and thus has a limit $\lambda \in \mathbb{R}$ as $\mathcal{F}$ is bounded from below in X. Besides, by the triangle inequality we have, for any positive integer m,

$$\varepsilon d(u_k, u_{k+m}) \le \varepsilon \sum_{i=1}^{m} d(u_{k+i-1}, u_{k+i}) \le \mathcal{F}(u_k) - \mathcal{F}(u_{k+m}), \tag{2.8.2}$$

from which we know that $\{u_k\}$ is a Cauchy sequence in X.

Suppose that $\lim_{k\to\infty} u_k = w \in X$, then the lower-semicontinuity implies

$$\mathcal{F}(w) \le \liminf_{k\to\infty} \mathcal{F}(u_k) = \lambda.$$

The inequality (2.8.2) gives, as $m \to \infty$

$$\varepsilon d(u_k, w) \le \mathcal{F}(u_k) - \mathcal{F}(w). \tag{2.8.3}$$

In particular, when $k = 1$ it becomes

$$0 \le \varepsilon d(u, w) \le \mathcal{F}(u) - \mathcal{F}(w) \le \mathcal{F}(u) - \inf_X \mathcal{F} \le \varepsilon,$$

and then (a) and (b) hold true.

To show (c), we proceed by contradiction and assume that

$$\mathcal{F}(v) < \mathcal{F}(w) - \varepsilon d(v, w) \tag{2.8.4}$$

for some $v \in X$. This inequality together with (2.8.3) implies

$$\mathcal{F}(v) < \mathcal{F}(u_k) - \varepsilon(d(u_k, w) + d(v, w)) \le \mathcal{F}(u_k) - \varepsilon d(u_k, v).$$

By definition, the set S_k contains v for any k and $\inf_{S_k} \mathcal{F} \le \mathcal{F}(v)$. From (2.8.1) and (2.8.4) we have

$$2\mathcal{F}(u_{k+1}) - \mathcal{F}(u_k) \le \mathcal{F}(v) < \mathcal{F}(w) - \varepsilon d(v, w). \tag{2.8.5}$$

Notice that $\mathcal{F}(u_{k+1}) - \mathcal{F}(u_k) \to 0$ as $k \to \infty$, and then passing to the limit in (2.8.5) gives

$$\mathcal{F}(w) < \mathcal{F}(w) - \varepsilon d(v, w),$$

which is a contradiction. □

The following version is more helpful in our discussion, and can be showed by considering the metric $\tilde{d} = \frac{1}{\sqrt{\varepsilon}} d$.

Corollary 2.8.2. *Suppose that (X, d) is a complete metric space and the function $\mathcal{F}\colon X \to \mathbb{R} \cup \{\infty\}$ is lower-semicontinuous with respect to the metric topology, bounded from below* *and not identically* ∞. *If for some* $u \in X$ *and* $\varepsilon > 0$ *we have*

$$\mathcal{F}(u) \leq \inf_{v \in X} \mathcal{F}(v) + \varepsilon,$$

then there exists $w \in X$ *satisfying the following:*

(a) $d(u, w) \leq \sqrt{\varepsilon}$*;*

(b) $\mathcal{F}(w) \leq \mathcal{F}(u)$*;*

(c) $\mathcal{F}(w) \leq \mathcal{F}(v) + \sqrt{\varepsilon} d(v, w)$ *for any* $v \in X$*.*

2.9 Two auxiliary lemmas

In this section we show two auxiliary lemmas that will be used at different points in this thesis.

The following one is an iteration-type result, which is a modification of Lemma 1.1 in [GG82], and also Lemma 6.1 in [Giu03], and is widely used to prove Caccioppoli type estimates.

Lemma 2.9.1. *Suppose that there are two positive functions* $\Phi, \Psi \colon (0, R] \to \mathbb{R}$, *where* Φ *is bounded, and* Ψ *is decreasing with* $\Psi(\sigma t) \leq \sigma^{-2}\Psi(t)$ *for any* $t \in (0, R], \sigma \in (0, 1]$*. If for any* $\frac{R}{2} \leq r < s \leq R$ *there holds*

$$\Phi(r) \leq \theta\, \Phi(s) + \Psi(s - r) + B \tag{2.9.1}$$

with given $B > 0$ *and* $\theta \in (0, 1)$*, then we have, for some* $C = C(\theta) > 0$*,*

$$\Phi\left(\frac{R}{2}\right) \leq C(\Psi(R) + B). \tag{2.9.2}$$

Proof. Take $\lambda \in (0, 1)$ to be determined later, and set

$$r_0 = \frac{R}{2}, \qquad r_{i+1} - r_i = (1 - \lambda)\lambda^i \frac{R}{2} \quad \text{for } i \in \mathbb{N}.$$

Then we apply (2.9.1) to $r_i, r_i + 1$ iteratively and obtain

$$\begin{aligned}
\Phi(r_0) &\leq \theta\Phi(r_1) + \Psi\left((1-\lambda)\frac{R}{2}\right) + B \\
&\leq \theta(\theta\Phi(r_2) + \Psi\left((1-\lambda)\lambda\frac{R}{2}\right) + B) + \Psi\left((1-\lambda)\frac{R}{2}\right) + B \\
&\leq \cdots \leq \theta^{k+1}\Phi(r_{k+1}) + \sum_{i=0}^{k} \theta^i \left(\Phi((1-\lambda)\lambda^i \frac{R}{2}) + B\right).
\end{aligned}$$

Since $\theta^i\Phi((1-\lambda)\lambda^i\frac{R}{2}) \leq 4(1-\lambda)^{-2}\lambda^{-2i}\theta^i\Phi(R)$, with $\sqrt{\theta} < \lambda < 1$ the above inequality becomes

$$\Phi(r_0) \leq \theta^{k+1}\Phi(r_{k+1}) + \frac{4}{(1-\lambda)^2}\frac{1-(\theta/\lambda^2)^{k+1}}{1-(\theta/\lambda^2)}\Psi(R) + \frac{1-\theta^{k+1}}{1-\theta}B.$$

The desired result can be obtained by taking $k \to \infty$. □

Remark 2.9.2. It is obvious that we can replace $\frac{R}{2}$ by any $0 < R' < R$ and get $\Phi(R') \leq C(\Psi(R - R') + B)$ with a similar proof.

The second one is helpful for estimating the difference between two quantities of a certain type, and will be used for several times in the main body.

Lemma 2.9.3. *Fix the Euclidean space $\mathbb{R}^d$ with some $d \in \mathbb{N}^+$, and suppose that x and y are distinct elements in $\mathbb{R}^d$ and $s, \gamma \geq 0$ are constants. Then we have the following estimates for the integral $\int_0^1 (1-t)^s (\gamma^2 + |tx + (1-t)y|^2)^{\frac{q}{2}} \,\mathrm{d}t$ for q in different ranges (assume that $\gamma > 0$ when $q \leq -1$):*

(a) $-1 < q < \infty$*: there exists* $c_1 = c_1(q, \gamma, s) \geq 1$ *such that*

$$\frac{1}{c_1}(\gamma^2 + |x|^2 + |y|^2)^{\frac{q}{2}} \leq \int_0^1 (1-t)^s (\gamma^2 + |tx + (1-t)y|^2)^{\frac{q}{2}} \,\mathrm{d}t \leq c_1 (\gamma^2 + |x|^2 + |y|^2)^{\frac{q}{2}}; \quad (2.9.3)$$

(b) $q = -1$*: there exists* $c_2, c_3 > 0$ *depending on* γ, s *such that*

$$c_2(\gamma^2 + |x|^2 + |y|^2)^{-\frac{1}{2}} \leq \int_0^1 (1-t)^s (\gamma^2 + |tx + (1-t)y|^2)^{\frac{q}{2}} \,\mathrm{d}t \leq c_3 \frac{1 + \log(\gamma^2 + |x|^2 + |y|^2)}{(\gamma^2 + |x|^2 + |y|^2)^{\frac{1}{2}}}; \quad (2.9.4)$$

(c) $q < -1$*: there exists* $c_4, c_5 > 0$ *depending on* q, γ, s *such that*

$$c_4(\gamma^2 + |x|^2 + |y|^2)^{\frac{q}{2}} \leq \int_0^1 (1-t)^s (\gamma^2 + |tx + (1-t)y|^2)^{\frac{q}{2}} \,\mathrm{d}t \quad (2.9.5)$$
$$\leq c_5 \begin{cases} (\gamma^2 + |x|^2 + |y|^2)^{\frac{q}{2}}, & \max\{|x|, |y|\} \leq 1 \\ (\gamma^2 + |x|^2 + |y|^2)^{-\frac{1}{2}}, & \max\{|x|, |y|\} > 1. \end{cases}$$

Proof. By the triangle inequality and the fact $\int_0^1 (1-t)^s \,\mathrm{d}t = \frac{1}{s+1}$ for any $s \geq 0$, we know that the following inequalities hold:

- the left inequality in (a) when $q < 0$;
- the right inequality in (a) when $q \geq 0$;
- the left inequalities in (b) and (c).

When $q \geq 0$, we first assume $|x| \leq |y|$. Then for $t \in [0, \frac{1}{3}]$, it is easy to see

$$|y + t(x-y)|^2 \geq ((1-t)|y| - t|x|)^2 \geq \left(\frac{2}{3}|y| - \frac{1}{3}|x|\right)^2 \geq \frac{1}{18}(|x|^2 + |y|^2).$$

The left inequality for $q \geq 0$ then follows by considering the integral on $[0, \frac{1}{3}]$. In the case $|x| > |y|$, we replace the interval $[0, \frac{1}{3}]$ by $[\frac{2}{3}, 1]$.

Now we show the right inequalities in (a)-(c) with $q < 0$ together. Without loss of generality, we assume $|x| \le |y|$, and consider the function $\varphi_z(t) := (\gamma^2 + |tz + (1-t)y|^2)^{\frac{q}{2}}$ for $z \in \mathbb{R}^d$ and $q < 0$. Let x_0 be the point on the line $\ell_{x,y} := \{y + t(x-y) : t \in \mathbb{R}\}$ with the least norm, and then we have $t_0 := \frac{|x_0 - y|}{|x-y|} \ge \frac{1}{2}$. Now set

$$z^t := tx + (1-t)y, \quad z_0^t := tx_0 + (1-t)y, \quad y^t := (1-t)y.$$

When $t_0 \ge 1$, it is clear that $|z_0^t| \le |z^t|$ and thus $\varphi_x(t) \le \varphi_{x_0}(t)$ for any $t \in (0,1)$. When $t_0 \in [\frac{1}{2}, 1)$, we have $2t_0 - t \in (0, t_0)$ for any $t \in (t_0, 1)$, and then z^t and $z^{2t_0 - t}$ are symmetric with respect to x_0. Thus, the integral of $\varphi_x(t)$ on $(t_0, 1)$ can be estimated by

$$\int_{t_0}^1 \varphi_x(t)\,\mathrm{d}t = \int_{t_0}^1 \varphi_x(2t_0 - t)\,\mathrm{d}t = \int_{2t_0-1}^{t_0} \varphi_x(t)\,\mathrm{d}t \le \int_0^{t_0} \varphi_x(t)\,\mathrm{d}t,$$

where the inequality holds as $2t_0 - 1 \ge 0$. Furthermore, we have

$$\int_0^1 \varphi_x(t)\,\mathrm{d}t \le 2\int_0^{t_0} \varphi_x(t)\,\mathrm{d}t = 2t_0 \int_0^1 \varphi_{x_0}(t)\,\mathrm{d}t \le 2\int_0^1 \varphi_{x_0}(t)\,\mathrm{d}t.$$

From the above discussion and the fact $|z_0^t| \ge |y^t|$, we know that in both cases there holds

$$\int_0^1 \varphi_x(t)\,\mathrm{d}t \le 2\int_0^1 \varphi_{x_0}(t)\,\mathrm{d}t \le 2\int_0^1 \varphi_0(t)\,\mathrm{d}t \le 2\int_0^1 (\gamma + t|y|)^q\,\mathrm{d}t.$$

The last integral above can be estimated as follows:

(i) $0 < |y| \le \gamma$:

$$\int_0^1 (\gamma + t\,|y|)^q\,\mathrm{d}t \le \gamma^q \le 3^{-\frac{q}{2}}(\gamma^2 + |x|^2 + |y|^2)^{\frac{q}{2}}.$$

(ii) $|y| > \gamma$: In this case, the integral can be calculated directly.

- $-1 < q < 0$:

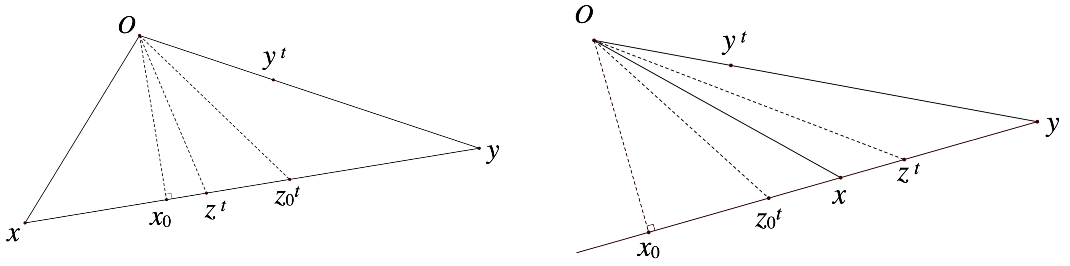
$$\begin{aligned}\int_0^1 (\gamma + t\,|y|)^q\,\mathrm{d}t &= \frac{1}{(q+1)\,|y|}((\gamma + |y|)^{q+1} - 1) \\ &\le \frac{2}{q+1}(\gamma + |y|)^q \le c(q)(\gamma^2 + |x|^2 + |y|^2)^{\frac{q}{2}}.\end{aligned}$$

- $q = -1$:

$$\int_0^1 (\gamma + t\,|y|)^{-1}\,\mathrm{d}t = \frac{1}{|y|}(\log(\gamma + |y|) - \log\gamma) \le c(\gamma)\frac{1 + \log(\gamma^2 + |x|^2 + |y|^2)}{(\gamma^2 + |x|^2 + |y|^2)^{\frac{1}{2}}}.$$

- $q < -1$:

$$\begin{aligned}\int_0^1 (\gamma + t\,|y|)^q\,\mathrm{d}t &= -\frac{1}{(q+1)\,|y|}(\gamma^{q+1} - (\gamma + |y|)^{q+1}) \\ &\le -\frac{3^{\frac{1}{2}}}{q+1}\frac{\gamma^{q+1}}{(\gamma^2 + |x|^2 + |y|^2)^{\frac{1}{2}}}.\end{aligned}$$



(a) $t_0 \in [\frac{1}{2}, 1)$ (b) $t_0 \geq 1$

Figure 2.9.1: Illustration of Lemma 2.9.3.

It is not necessary to consider the factor $(1-t)^s$ in these cases as it is controlled by 1. Combining the above estimates together, we can obtain the desired result.

□

Remark 2.9.4. The right inequalities in (b) and (c) are optimal to some extent. Indeed, we can take x, y such that $|x| = 1$ and $y = rx$ with $r > 1$. Then let $\gamma = 1$, $s = 0$ and $t' = t + (1-t)r$, and we have

$$\int_0^1 \varphi_x(t)\,\mathrm{d}t = \int_0^1 (1 + |t + (1-t)r|^2)^{\frac{q}{2}}\,\mathrm{d}t$$
$$= \frac{1}{r-1}\int_1^r (1 + (t')^2)^{\frac{q}{2}}\,\mathrm{d}t' \geq \frac{1}{2^{\frac{q}{2}}(r-1)}\int_1^r (1+t)^q\,\mathrm{d}t.$$

Again, we consider the following two cases:

(i) $q = -1$:

$$\frac{1}{r-1}\int_1^r (1+t)^q\,\mathrm{d}t = \frac{\log(1+r) - \log 2}{r-1},$$

then for large r the quantity is comparable to $\log(1 + |x|^2 + |y|^2)(1 + |x|^2 + |y|^2)^{-\frac{1}{2}}$.

(ii) $q < -1$:

$$\frac{1}{r-1}\int_1^r (1+t)^q\,\mathrm{d}t = -\frac{1}{(q+1)(r-1)}(2^{q+1} - (1+r)^{q+1}),$$

and for large r this quantity is comparable to $(1 + |x|^2 + |y|^2)^{-\frac{1}{2}}$.

For the case where $q < -1$ and $\max\{|x|, |y|\} < 1$, we take $\gamma = 0, s = 0, y = 2x$ and $|x| = r < \frac{1}{2}$. Then the integral becomes

$$\int_0^1 \varphi_x(t)\,\mathrm{d}t = \int_0^1 (2-t)^q r^q\,\mathrm{d}t = -\frac{1-2^{q+1}}{q+1} r^q,$$

which is comparable to $(\gamma^2 + |x|^2 + |y|^2)^{\frac{q}{2}}$ as $r \to 0^+$.

Chapter 3

Minimality notions and regularity theory

In this chapter, we introduce three minimality notions involved in this thesis, including minimizers, quasiminimizers and ω-minimizers. The first one, as the most natural concept in variational problems, has been widely studied in various set-ups. The other two are together considered to be "almost minimizers" in this thesis and minimise the corresponding functionals in a more general sense. In addition to the definition, more background information and some examples are also presented for the two relaxed minimality notions.

The rest part is denoted to a brief review of regularity theory, with an emphasis on those regularity results connected to the subsequent chapters. We first consider higher integrability results in both the scalar ($N = 1$) and the vectorial ($N > 1$) contexts in Section 3.2. Section 3.3 contains a review of Hölder regularity results of zero-order as well as first-order in the scalar setting. In Section 3.4, we provide a concise summary of partial regularity results in the vectorial setting.

3.1 Definitions and examples

This thesis is focuses on variational principles involving p-growth functionals. In the following, we specify the context and provide the relevant definitions.

Consider a bounded open set $\Omega \subset \mathbb{R}^n$ and the functional defined as follows:

$$\mathscr{F}(u, \Omega) := \int_\Omega f(x, u, \nabla u)\,\mathrm{d}x, \tag{3.1.1}$$

where the map u defined on Ω takes values in $\mathbb{R}^N$ and the integrand $f\colon\ \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies the following properties:

(**G1**) f is a Carathéodory function;

(**G2**) for any $(x, u, z) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{N\times n}$, we have

$$|f(x, u, z)| \le L(1 + |z|^p), \tag{3.1.2}$$

where $p \ge 1$ and $L > 0$ is a constant.

With the above assumptions, it is easy to see that $\mathscr{F}(u,\Omega)$ is well-defined for $u \in W^{1,p}(\Omega)$, which is the appropriate function class for studying the corresponding variational problem when $p > 1$. For the linear growth case ($p = 1$), the suitable functional space is BV and the functional in (3.1.1) needs to be relaxed, which will be discussed in Subsection 3.1.4.

The prototype of such functionals is the p-Dirichlet integral $\mathcal{D}_p := \int_\Omega |\nabla u|^p \,\mathrm{d}x$, which will be the focus in Chapter 4. The discussion in this chapter is for functionals of the general form (3.1.1).

3.1.1 Minimizers

In the study of a variational problem with a functional $\mathscr{F}$ as described above, the objective is typically to minimise $\mathscr{F}$ within a specific subclass of $W^{1,p}(\Omega,\mathbb{R}^N)$. Here, we restrict ourselves to the commonly studied Dirichlet problem, and consider minimizers within certain Dirichlet classes $W^{1,p}_g(\Omega,\mathbb{R}^N)$, where $g \in W^{\Omega,\mathbb{R}^N}$ is the Dirichlet data.

Definition 3.1.1. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, the integrand $f\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies **(G1)** and **(G2)** with $p > 1$, and $\mathscr{F}$ is defined as in (3.1.1). A map $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ is said to be a *minimizer* of $\mathscr{F}$, if for any $\varphi \in W^{1,p}_0(\Omega,\mathbb{R}^N)$ we have

$$\mathscr{F}(u,\Omega) \le \mathscr{F}(u+\varphi,\Omega). \tag{3.1.3}$$

When the test map φ coincides with u outside of a compact subset of Ω, the above inequality (3.1.3) also holds locally. Consequently, we can also define *local minimizers* in $W^{1,p}_{loc}(\Omega,\mathbb{R}^N)$, which minimise the functional $\mathscr{F}$ in a local sense. It is worth mentioning that local minimality is an interior property and does not guarantee global integrability, which can be illustrated by the following example.

Example 3.1.1 ([Giu03], §6.1)**.** Consider the 2-Dirichlet energy

$$\mathcal{D}_2(u,\mathbb{B}) := \int_{\mathbb{B}} |\nabla u|^2 \,\mathrm{d}x$$

defined on the unit ball $\mathbb{B} \subset \mathbb{R}^2$. The function

$$u(x,y) = \frac{y}{(x+1)^2 + y^2}$$

is harmonic in $\mathbb{B}$ and thus a local minimizer of $\mathcal{D}_2$. However, by calculation we know that

$$\mathcal{D}_2(u,\mathbb{B}) = \int_{\mathbb{B}} |\nabla u|^2 \,\mathrm{d}x = \int_{\mathbb{B}} ((x+1)^2 + y^2)^{-2} \,\mathrm{d}x\,\mathrm{d}y = \infty$$

due to the singularity at $(-1,0)$.

The minimizers of a specific functional, which are the solutions of the corresponding variational principle, are closely connected to (elliptic) PDEs. Assume that the integrand f defined in Definition 3.1.1 is C^1, and $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ is a minimizer of $\mathscr{F}$. Set $g(t) = \mathscr{F}(u+t\varphi,\Omega)$ with some $\varphi \in C^\infty_c(\Omega,\mathbb{R}^N)$, then g achieves its minimum at $t = 0$ and satisfies

$g'(0) = 0$, which gives

$$\int_\Omega \left(\frac{\partial f}{\partial z_i^\alpha}(x, u(x), \nabla u(x)) \frac{\partial \varphi^\alpha}{\partial x_i} + \frac{\partial f}{\partial u^\alpha}(x, u(x), \nabla u(x)) \varphi^\alpha \right) \mathrm{d}x = 0, \tag{3.1.4}$$

where we used the Einstein summation convention. The integral equation (3.1.4) is the weak formulation of the system

$$-\frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial z_i^\alpha}(x, u(x), \nabla u(x) \right) + \frac{\partial f}{\partial u^\alpha}(x, u(x), \nabla u(x)) = 0, \tag{3.1.5}$$

which is known as the *Euler-Lagrange equation (system)* of the variational problem associated to $\mathscr{F}$. The system (3.1.5) is often expressed in a more concise form as follows:

$$-\operatorname{div}(\partial_z f(x, u, \nabla u)) + \partial_u f(x, u, \nabla u) = 0, \tag{3.1.6}$$

This system is elliptic under certain convexity assumptions on f.

The inverse statement, i.e., that a weak solution to the Euler-Lagrange system (3.1.5) minimises $\mathscr{F}$, also holds true when the functional is convex. Again, we take the p-Dirichlet energy

$$\mathcal{D}_p(u, \Omega) = \int_\Omega |\nabla u|^p \,\mathrm{d}x$$

as an example. It is well-known that, when $p > 1$, there is a one-to-one corresponding between local minimizers of D_p and p-harmonic functions, which has already been used in Example 3.1.1.

3.1.2 Quasiminimizers

From the above, we know that the minimizers of a variational functional correspond to the solutions to its Euler-Lagrange system. More generally, the solutions to a specific system may also satisfy certain variational properties, though the system does not arise from the variation of any specific functional. To unify the analysis of PDE solutions and minimizers, Giaquinta & Giusti introduced the concept of quasiminimizers in [GG84a].

Before presenting the definition of quasiminimzers, we strengthen **(G2)** to the following condition:

(G2′) for any $(x, u, z) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{N\times n}$, we have

$$\ell |z|^p \le |f(x, u, z)| \le L(1 + |z|^p), \tag{3.1.7}$$

where $L \ge \ell > 0$ are constants.

Definition 3.1.2. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, the integrand $f\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies **(G1)** and **(G2′)** with $p > 1$, and $\mathscr{F}$ is defined as in (3.1.1). A map $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ is said to be a *quasiminimizer* of $\mathscr{F}$ with constant $Q \ge 1$ (a *Q-minimizer* of $\mathscr{F}$), if for any $\varphi \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $\operatorname{supp}(\varphi) \subset\subset \Omega$ we have

$$\mathscr{F}(u, \operatorname{supp}(\varphi)) \le Q \mathscr{F}(u + \varphi, \operatorname{supp}(\varphi)). \tag{3.1.8}$$

In particular, a Q-minimizer of the Dirichlet integral $\mathcal{D}_p$ is called a *(p, Q)-minimizer.*

Analogously, we can define *cubic* (or *spherical*) *quasiminimizers*: for any n-cube Q_R (or ball B_R) compactly contained in Ω and any $\varphi \in W_0^{1,p}(Q_R, \mathbb{R}^N)$ (or $W_0^{1,p}(B_R, \mathbb{R}^N)$), the inequality (3.1.8) holds with $\operatorname{supp}(\varphi)$ replaced by Q_R (B_R).

It is obvious that the notion of cubic (spherical) quasiminimizers is more general than that of quasiminimizers. In general, the class of cubic (spherical) quasiminimizers contains that of quasiminimizers as a proper subset as shown in the following example.

Example 3.1.2 ([GG84a], Example d)**.** Assume that $n > 2$, and $u(x)$ is homogeneous of degree β with $0 > \beta > 1 - \frac{n}{2}$ and continuously differentiable in $\mathbb{R}^n \setminus \{0\}$ without any stationary points. In addition, we require u to be non-constant on the boundary of any n-cube in $\mathbb{R}^n$. For convenience, take $u(x) = |x|^\beta$.

The function u is a cubic quasiminimizer of the 2-Dirichlet energy $\mathcal{D}_2$ on $\mathbb{R}^n$ but not a quasiminimizer. The way to show the former is to prove that, for any n-cube $Q(x_0, R) = Q_R \subset \mathbb{R}^n$, there holds

$$R \int_{Q_R} |\nabla u|^2 \, \mathrm{d}x \le C \int_{\partial Q_R} |u - u_{\partial Q_R}|^2 \, \mathrm{d}\mathcal{H}^{n-1},$$

where $u_{\partial Q_R}$ is the mean value of u on ∂Q_R with respect to the $(n-1)$-Hausdorff measure $\mathcal{H}^{n-1}$. We refer to [GG84a] for details. If $u - v \in W_0^{1,2}(Q_R)$, then $u = v$ on ∂Q_R in the sense of trace. By Theorem 3.20 in [Giu03], we have

$$\int_{\partial Q_R} |u - u_{\partial Q_R}|^2 \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial Q_R} |v - v_{\partial Q_R}|^2 \, \mathrm{d}\mathcal{H}^{n-1} \le CR \int_{Q_R} |\nabla v|^2 \, \mathrm{d}x,$$

which establishes the first claim.

On the other hand, $v = \min\{u, 1\}$ is different from u on the unit ball $\mathbb{B}$ with $u - v \in W_0^{1,2}(\mathbb{B})$. It is easy to see that $\int_B |\nabla v|^2 \, \mathrm{d}x = 0$ while $\int_B |\nabla u|^2 \, \mathrm{d}x \neq 0$, which incidates that u is not a quasiminimizer as claimed. $\square$

The class of quasiminimizers includes a wide range of functions, particularly the solutions to some elliptic equations and systems in divergence form. We illustrate this with the following examples.

Example 3.1.3. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set. Consider the following system

$$\operatorname{div}(A(x)\nabla u(x)) = 0, \tag{3.1.9}$$

where $A\colon \Omega \to \bigodot^2(\mathbb{R}^{N\times n})$. Furthermore, we require that the coefficients satisfy the following uniform ellipticity condition

$$A(x)[\xi, \xi] \ge \nu |\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^{N\times n} \tag{3.1.10}$$

with some $\nu > 0$, and the uniform bound

$$\sup_{|\xi|=1} |A(x)\xi| \le M \tag{3.1.11}$$

with some $M > 0$.

Assume that $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ is a weak solution to (3.1.9). Now take a map $v \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ that coincides with u outside of a compact set $K \subset\subset \Omega$, and set $\varphi := v - u$. The by (3.1.9) we have

$$\int_K A(x)[\nabla u, \nabla u]\,\mathrm{d}x = \int_K A(x)[\nabla u, \nabla v]\,\mathrm{d}x, \tag{3.1.12}$$

and (3.1.10) and (3.1.11) together imply

$$\nu \int_K |\nabla u|^2\,\mathrm{d}x \leq M \int_K |\nabla u||\nabla v|\,\mathrm{d}x \leq M \left(\int_K |\nabla u|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_K |\nabla v|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Then it follows that u is a quasiminimizer of $\mathcal{D}_2$ with constant $\nu^{-2}M^2$. □

Another class of quasiminimizers is composed of quasi-regular mappings. A differentiable map $u\colon \Omega \to \mathbb{R}^n$ is said to be *quasi-regular* if there exists $A > 0$ such that

$$|\nabla u(x)|^n \leq A \det(\nabla u(x)), \quad \text{for } \mathscr{L}^n\text{-a.e. } x \in \Omega. \tag{3.1.13}$$

This concept is a generalisation of *quasi-conformal mappings*, which are assumed to be homeomorphisms.

Example 3.1.4. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set and $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ is a quasi-regular mapping satisfying (3.1.13). Then it is a quasiminimizer of the n-Dirichlet functional $\mathcal{D}_n$ on Ω.

For any $\varphi \in W^{1,n}(\Omega, \mathbb{R}^n)$ with $K := \operatorname{supp}(\varphi) \subset\subset \Omega$, we have

$$\int_K \det(\nabla u)\,\mathrm{d}x = \int_K \det(\nabla u + \nabla\varphi)\,\mathrm{d}x \leq c(n) \int_K |\nabla u + \nabla \varphi|^n\,\mathrm{d}x,$$

where the first equality holds as the determinant is a null-Lagrangian (see [Rin18], §5.2). Then the conclusion follows from (3.1.13), and the quasiminimising constant is $Q = c(n)A$. □

Apart from solutions to elliptic systems and quasi-regular mappings, the solutions of some variational problems with constraints are also fall in the class of quasiminimizers. As an example, we consider the following obstacle problem:

Example 3.1.5 ([Giu03], Example 6.4)**.** Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, the integrand $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies **(G1)** and **(G2′)**, and $\psi \in W^{1,p}(\Omega)$ is a given function. Moreover, we assume that $u \in W^{1,p}_{loc}(\Omega)$ minimises the functional

$$\mathscr{F}(v, \Omega) = \int_\Omega f(x, v, \nabla v)\,\mathrm{d}x$$

in the class

$$\mathcal{O}_{\psi,\Omega} := \{v \in W^{1,p}_{loc}(\Omega)\colon\ v \geq \psi\ \mathscr{L}^n\text{-a.e. in } \Omega\},$$

i.e., among all the $W^{1,p}_{loc}$ functions with graphs above the obstacle function $\psi \in W^{1,p}(\Omega)$. This actually means that for any $v \in W^{1,p}_{loc}(\Omega)$ with $v \geq \psi$ and $K = \operatorname{supp}(u - v) \subset\subset \Omega$, we have

$$\mathscr{F}(u,K) \leq \mathscr{F}(v,K).$$

Given any $v \in W^{1,p}_{loc}(\Omega)$ with $K = \operatorname{supp}(u-v) \subset\subset \Omega$, set $S := \{x \in \Omega \colon v(x) \geq \psi(x)\}$ and the function $w = v \vee \psi$. Then by the minimality of u within $\mathcal{O}_{\psi,\Omega}$ we obtain

$$\begin{aligned}\mathscr{F}(u,K) &\leq \mathscr{F}(w,K) = \mathscr{F}(v,K \cap S) + \mathscr{F}(\psi, K \setminus S)\\ &\leq \mathscr{F}(v,K) + \mathscr{F}(\psi,K).\end{aligned}$$

Then adding the extra term $\mathscr{F}(\psi,K)$ to both sides implies

$$\mathcal{G}(u,K) \leq 2\mathcal{G}(v,K),$$

where

$$\mathcal{G}(u,\Omega) = \int_\Omega (f(x,u,\nabla u) + \gamma(x))\,\mathrm{d}x, \quad \gamma(x) = f(x,\psi(x),\nabla\psi(x)).$$

Then we know that a minimizer of $\mathscr{F}$ above the obstacle ψ is a free quasiminimizer of the functional $\mathcal{G}$[1]. This actually also holds for quasiminimizers. □

In order to limit the textual length, the examples presented above are in relatively simple forms. For more examples with detailed analysis, see [GG84a] and [Giu03], Chapter 6.

Remark 3.1.3. The lower bound in (3.1.7) is relatively strict and not practical in many cases (see [BD95], [Iwa02], [IL93] and [CK17]). In Section 3.2, we will replace it by a strong quasi-convexity condition. An alternative option is to assume the coercivity of the functional as in [CK17] in the autonomous setting (where the integrand f only depends on ∇u). The authors of [CK17] showed the equivalence between a strong quasi-convexity condition and the coercivity of the corresponding functional as stated in Theorem 2.6.8.

3.1.3 ω-minimizers

The concept of ω-minimizers was introduced in the context of geometric measure theory by Almgren in [Alm76], with the motivation that they arise from various geometric variational problems with constraints. The study of ω-minimizers in the non-parametric setting was initiated by Anzellotti in [Anz83], and for more subsequent regularity results we refer to Subsection 1.1.2.

The function ω quantifies the extent to which ω-minimizers deviate from being true minimizers, and is required to satisfy

($\boldsymbol{\omega}$) $\omega\colon [0,\infty) \to [0,\infty)$ is non-decreasing with $\lim_{r\to 0^+} \omega = \omega(0) = 0$.

Definition 3.1.4. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, the integrand $f\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies **(G1)** and **(G2)**, and the functional $\mathscr{F}$ is defined as in (3.1.1). Given a function $\omega\colon [0,\infty) \to [0,\infty)$ satisfying ($\boldsymbol{\omega}$), a map $u \in W^{1,p}_{loc}(\Omega,\mathbb{R}^N)$ with $p > 1$ is said

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[1] Notice that the growth control of the integrand of $\mathcal{G}$ is

$$\ell|z|^p + \gamma(x) \leq f(x,u,z) + \gamma(x) \leq L(1+|z|^p) + \gamma(x),$$

which contains an extra function in x compared to **(G2′)**. We consider the latter, the simplified version for convenience, and the former may appear in certain problems (see [Giu03], Chapter 6).

to be an *ω-minimizer* of $\mathscr{F}$ with $R_0 > 0$, if for any ball $B_R \subset\subset \Omega$ with $R < R_0$ and any $\varphi \in W_0^{1,p}(B_R, \mathbb{R}^N)$ we have

$$\mathscr{F}(u, B_R) \leq \mathscr{F}(u + \varphi, B_R) + \omega(R) \int_{B_R} (1 + |\nabla(u + \varphi)|^p) \,\mathrm{d}x. \tag{3.1.14}$$

Notice that the boundary value g in the definition of generalised minimizers may not be specified when we only focus on interior regularity.

Remark 3.1.5. (a) We may replace the extra term with ω in the above definition by $\omega(R) \int_{B_R} (1 + |\nabla u|^p + |\nabla\varphi|^p) \,\mathrm{d}x$, or $\omega(R)\mathscr{F}(u + \varphi, B_R)$ under the condition (**G2′**), which makes no significant difference.

(b) In certain contexts, n-cubes instead of balls are used to define ω-minimizers, but this distinction is not essential.

(c) The term “almost minimizer” is also used for ω-minimizers in the predominant body of existing literature. However, our class of almost minimizers is broader and includes quasiminimizers as well.

From the definition, we know that ω-minimizers can be considered to be local perturbation of the corresponding minimizers, and thus allow more flexibility. As a consequence, the class of ω-minimizers includes the solutions of diverse variational problems, such as non-autonomous problems and problems with constraints. We present the following examples for illustration.

One example arises from non-autonomous variational principles, where the integrands also depend on x in addition to ∇u.

Example 3.1.6. Suppose that the integrand $g\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is of p-growth and satisfies

$$|g(x_1, z) - g(x_2, z)| \leq \omega(|x_1 - x_2|)(1 + |z|^p)$$

for any $x_1, x_2 \in \Omega$ and $z \in \mathbb{R}^{N\times n}$, where condition ($\boldsymbol{\omega}$) holds true for the function ω. Then any local minimizer $u \in W_{loc}^{1,p}(\Omega, \mathbb{R}^N)$ of the functional

$$\mathcal{G}(v, \Omega) := \int_\Omega g(x, \nabla v) \,\mathrm{d}x$$

is a (2ω)-minimizer of the family of functionals G_{x_0}, where $\mathcal{G}_{x_0}$ is the frozen functional defined by $g(x_0, \cdot)$ for any $x_0 \in \Omega$. Notice that we only have the analogue of (3.1.14) hold for $\mathcal{G}_{x_0}$ on small balls containing x_0.

As mentioned above, ω-minimizers also arise in certain constrained problems, such as variational problems with obstacles or volume-constraints as demonstrated below.

Example 3.1.7 (Obstacle problem, [Anz83], §3)**.** Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set and the map ψ is in the space $C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^N)$ with some $\alpha \in (0, 1)$ with each entry $\psi^i, i = 1, \ldots, N$, being non-positive on the boundary. We consider to minimise the 2-Dirichlet functional $\mathcal{D}_2$ in the set

$$\mathcal{O}_{\psi,\Omega} := W_0^{1,2}(\Omega, \mathbb{R}^N) \cap \{v\colon\ \Omega \to \mathbb{R}^N\colon\ v^i \geq \psi^i, i = 1, \ldots, N\}.$$

Any minimizer of this problem is an ω-minimizer of $\mathcal{D}_2$ with $\omega(R) = cR^{2\alpha}$, where the constant $c > 0$ is related to the $C^{1,\alpha}$-norm of ψ. We prove this claim in the following.

Fix a ball $B_R = B(x_0, R) \subset\subset \Omega$ and take the minimizer h of $\mathcal{D}_2$ in the Dirichlet class $W^{1,2}_u(B_R, \mathbb{R}^N)$, which is harmonic in B_R. Notice that the harmonicity of h implies

$$\int_{B_R} \nabla h \cdot \nabla \varphi \, \mathrm{d}x = 0, \quad \text{for any } \varphi \in W^{1,2}_0(B_R, \mathbb{R}^N). \tag{3.1.15}$$

Then we have, for any φ as above,

$$\begin{aligned}\int_{B_R} |\nabla u|^2 \, \mathrm{d}x &= \int_{B_R} |\nabla h|^2 \, \mathrm{d}x + \int_{B_R} |\nabla (u-h)|^2 \, \mathrm{d}x \\ &\leq \int_{B_R} |\nabla (u+\varphi)|^2 \, \mathrm{d}x + \int_{B_R} |\nabla (u-h)|^2 \, \mathrm{d}x.\end{aligned}$$

Given any $v \in \mathcal{O}_{\psi,\Omega}$, set $\ell(t) := \int_{B_R} |\nabla(u + t(v-u))|^2 \, \mathrm{d}t$. Then by the minimality of u within $\mathcal{O}_{\psi,\Omega}$, we know $\ell'(0) \geq 0$, which is

$$\int_{B_R} \nabla u \cdot \nabla (u - v) \, \mathrm{d}x \leq 0. \tag{3.1.16}$$

Take a map v defined on Ω with $v^i := h^i \vee \psi^i$, $i = 1, \dots, N$, then (3.1.15), (3.1.16) and Young's inequality together imply

$$\begin{aligned}\int_{B_R} |\nabla (u-h)|^2 \, \mathrm{d}x &= \int_{B_R} \nabla (u-h) \cdot \nabla (u-v) \, \mathrm{d}x + \int_{B_R} \nabla (u-h) \cdot \nabla (v-h) \, \mathrm{d}x \qquad (3.1.17)\\ &\leq \frac{1}{2} \int_{B_R} |\nabla (u-h)|^2 \, \mathrm{d}x + \frac{1}{2} \int_{B_R} |\nabla (v-h)|^2 \, \mathrm{d}x.\end{aligned}$$

Notice that we have $v \in \mathcal{O}_{\psi,\Omega}$ and again the harmonicity of h implies

$$\begin{aligned}\int_{B_R} |\nabla (v^i - h^i)|^2 \, \mathrm{d}x &= \int_{B_R} (\nabla (h^i \vee \psi^i) - (\nabla \psi^i)_{B_R}) \cdot \nabla (h^i \vee \psi^i - h^i) \, \mathrm{d}x \\ &= \int_{B_R \cap \{h^i \leq \psi^i\}} (\nabla \psi^i - (\nabla \psi^i)_{B_R}) \cdot \nabla (\psi^i - h^i) \, \mathrm{d}x \qquad (3.1.18)\\ &\leq \frac{1}{2} \int_{B_R} |\nabla \psi^i - (\nabla \psi^i)_{B_R}|^2 \, \mathrm{d}x + \frac{1}{2} \int_{B_R} |\nabla (v^i - h^i)|^2 \, \mathrm{d}x.\end{aligned}$$

From the regularity of ψ, we know that $|\nabla \psi^i - (\nabla \psi^i)_{B_R}| \leq cR^\alpha$. Thus, the two estimates (3.1.17) and (3.1.18) gives

$$\int_{B_R} |\nabla (u-h)|^2 \, \mathrm{d}x \leq \int_{B_R} |\nabla (v-h)|^2 \, \mathrm{d}x \leq cR^{2\alpha} \leq cR^{2\alpha} \int_B (1 + |\nabla (u+\varphi)|^2) \, \mathrm{d}x,$$

which indicates that u is an ω-minimizer of $\mathcal{D}_2$ with $\omega(R) = cR^{2\alpha}$. There are no restrictions on the size of the ball B_R.

If instead, we assume that the modulus of continuity of $\nabla \psi$ is μ, i.e.,

$$|\nabla \psi(x_1) - \nabla \psi(x_2)| \leq \mu(|x_1 - x_2|), \quad \text{for any } x_1, x_2 \in \Omega,$$

the above argument can still be carried out with $\omega = \mu^2$. □

Example 3.1.8 (Volume-constrained problem, [DGG00], §2)**.** This example is also about minimising the 2-Dirichlet functional $\mathcal{D}_2$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, and $v_0 \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a fixed map. We aim to minimise $\mathcal{D}_2$ within the following class:

$$\mathcal{V}_{v_0,\Omega} := W^{1,2}_{v_0} \cap \left\{ v\colon \Omega \to \mathbb{R}^N\colon \int_\Omega v \, \mathrm{d}x = \int_\Omega v_0 \, \mathrm{d}x \right\}.$$

Assume that $u \in \mathcal{V}_{v_0,\Omega}$ is the minimizer in this problem, then we claim that it is an ω-minimizer of $\mathcal{D}_2$ with $\omega(R) = cR$ with some constant $c > 0$.

Take

$$R_1 := \sup_{x\in\Omega}\{\sup\{r\colon B_r(x) \subset\subset \Omega\}\},$$

and set $R_0 = \min\{\frac{R_1}{4}, 1\}$. For any $x_0 \in \Omega$, we show the ω-minimality of u on any ball $B_R = B_R(x_0) \subset\subset \Omega$ with $0 < R \le R_0$. By the selection of R_1, we know that there exists $y_0 \in \Omega$ such that $\tilde{B} := B_{R_1/4}(y_0) \subset\subset \Omega$ and $\tilde{B} \cap B_R = \emptyset$. Fix $\psi \in W^{1,2}_0(B_{R_1/4}(0), \mathbb{R}^N)$ with $\int_{B_{R_1/4}(0)} \psi^i \, \mathrm{d}x \neq 0$ for any $i = 1, \dots, N$, and define $\eta(x) := \psi(x - y_0)$.

Now for any $\varphi \in W^{1,2}_0(B_R, \mathbb{R}^N)$, take the numbers

$$t_i := \frac{-\int_{B_R} \varphi^i \, \mathrm{d}x}{\int_{\tilde{B}} \eta^i \, \mathrm{d}x}, \quad i = 1, \dots, N.$$

By Poincaré's inequality, the numbers are controlled as follows:

$$|t_i| \le cR^{1+\frac{n}{2}} \left(\int_{B_R} |\nabla \varphi^i|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}, \tag{3.1.19}$$

where the constant c depends on n and ψ. Define a map w on Ω by

$$w^i(x) = \begin{cases} u^i(x) + \varphi^i(x), & x \in B_R \\ u^i(x) - t_i\eta^i(x), & x \in \tilde{B} \\ u^i(x), & x \in \Omega \setminus (B_R \cup \tilde{B}) \end{cases} \qquad i = 1, \dots, N,$$

and it is obvious that $w \in \mathcal{V}_{v_0,\Omega}$. The minimality of u within $\mathcal{V}_{v_0,\Omega}$ implies

$$\int_{B_R} |\nabla u|^2 \, \mathrm{d}x \le \int_{B_R} |\nabla (u + \varphi)|^2 \, \mathrm{d}x + \int_{\tilde{B}} |\nabla (u + t\eta)|^2 \, \mathrm{d}x - \int_{\tilde{B}} |\nabla u|^2 \, \mathrm{d}x,$$

where $t\eta = (t_i \eta^i)_{i=1}^N$. We expand the second term on the right-hand side and estimate it with the third one:

$$\int_{\tilde{B}} |\nabla (u + t\eta)|^2 \, \mathrm{d}x - \int_{\tilde{B}} |\nabla u|^2 \, \mathrm{d}x \le 2 \left| \sum_{i=1}^N \int_{\tilde{B}} t_i \nabla u^i \cdot \nabla \eta^i \, \mathrm{d}x \right| + \sum_{i=1}^N t_i^2 \int_{\tilde{B}} |\nabla \eta^i|^2 \, \mathrm{d}x. \tag{3.1.20}$$

By the Cauchy-Schwarz inequality, (3.1.19) and Young's inequality we have

$$\begin{aligned}\left|\sum_{i=1}^{N}\int_{\tilde{B}} t_i\nabla u^i\cdot\nabla\eta^i\,\mathrm{d}x\right| &\le \left(\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\sum_{i=1}^{N} t_i^2\int_{\tilde{B}}|\nabla\eta^i|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\ &\le \left(\int_\Omega |\nabla v_0|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(cR^{n+2}\int_{B_R}|\nabla\varphi|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\ &\le c\left(R^{n+1}+R\int_{B_R}|\nabla\varphi|^2\,\mathrm{d}x\right),\end{aligned}$$

where the last constant c depends on n, ψ, v_0. The estimate for last term in (3.1.20) is the same as above and thus we have

$$\int_{B_R}|\nabla u|^2\,\mathrm{d}x \le \int_{B_R}|\nabla(u+\varphi)|^2\,\mathrm{d}x + cR\int_{B_R}(1+|\nabla\varphi|^2)\,\mathrm{d}x,$$

which indicates the desired ω-minimality of u. □

Another interesting fact is that any map u with certain regularity is an ω-minimizer of D_2, where ω is determined by the continuity of ∇u.

Example 3.1.9. Suppose that $u\colon \bar{\Omega}\to\mathbb{R}^N$ is in the space $C^{1,\alpha}(\bar{\Omega},\mathbb{R}^N)$ with some $\alpha\in(0,1)$, then it is an ω-minimizer of $\mathcal{D}_2$ with $\omega(R)=cR^{2\alpha}$, where the constant $c>0$ is related to the $C^{1,\alpha}$-norm of u. Fix a ball $B_R = B_R(x_0)\subset\subset\Omega$. By an argument similar to that in Example 3.1.7, we have

$$\int_{B_R}|\nabla(u-h)|^2\,\mathrm{d}x \le \int_{B_R}|\nabla u-(\nabla u)_{B_R}|^2\,\mathrm{d}x \le cR^{n+2\alpha},$$

where h is the harmonic map with the same boundary value as u. Then the claim can be proved as in Example 3.1.7.

Furthermore, the assumption on the regularity of u can be generalised, and we still have ω-minimality with an appropriate prescribed ω. Suppose that ω satisfies ($\boldsymbol{\omega}$**1**), ($\boldsymbol{\omega}$**2**) and ($\boldsymbol{\omega}$**3**) in Section 3.4, which are basically the smallness condition ($\boldsymbol{\omega}$) and a Dini-type condition. Then it is possible to construct an ω-minimizer u of $\mathcal{D}_2$ such that the modulus of continuity of ∇u is given by $\Xi_{\frac{1}{2}}$, where

$$\Xi_\sigma(r) := \int_0^r \frac{\omega^\alpha(\rho)}{\rho}\,\mathrm{d}\rho.$$

With the same process we can show that u is an ω-minimizer of $\mathcal{D}_2$. See [DGG00], Example 3 for details. □

3.1.4 Linear-growth functionals

In the current subsection, we discuss the linear growth setting and present the corresponding definitions analogous to those in the previous subsections.

Suppose that $\Omega\subset\mathbb{R}^n$ is a bounded open set with Lipschitz boundary. Consider the following functional

$$\mathscr{F}(u,\Omega) := \int_\Omega f(\nabla u)\,\mathrm{d}x, \quad u\colon \Omega \to \mathbb{R}^N, \tag{3.1.21}$$

where the integrand f satisfies the following growth condition

(**LG1**) $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is of linear growth, that is, for some $L > 0$ and any $z \in \mathbb{R}^{N\times n}$, we have

$$|f(z)| \leq L(1+|z|). \tag{3.1.22}$$

For the discussion in this subsection, we also assume quasiconvexity:

(**LG2**) f is quasiconvex, that is, for any $z \in \mathbb{R}^{N\times n}$ and any $\varphi \in C_c^\infty((0,1)^n,\mathbb{R}^N)$, we have

$$\int_{(0,1)^n} f(z+\varphi)\,\mathrm{d}x \geq f(z). \tag{3.1.23}$$

As in the super-linear setting, the functional $\mathscr{F}$ is well-defined in $W^{1,1}(\Omega,\mathbb{R}^N)$. However, the study of the corresponding variational principle cannot be conducted in $W^{1,1}(\Omega,\mathbb{R}^N)$ due to a compactness issue as explained in the following.

A commonly used method to achieve the existence of minimizers is the direct method in the calculus of variations. Fix $g \in W^{1,1}(\Omega,\mathbb{R}^N)$ as the boundary value, and take a minimising sequence $\{u_k\} \subset W_g^{1,1}(\Omega,\mathbb{R}^N)$ such that

$$\mathscr{F}(u_k,\Omega) \to \inf_{W_g^{1,1}(\Omega,\mathbb{R}^N)} \mathscr{F} \qquad \text{as } k\to\infty.$$

With a certain coercivity assumption, we can obtain the boundedness of such a sequence in the Sobolev space $W^{1,1}(\Omega,\mathbb{R}^N)$. In the super-linear case ($p > 1$), there exists a weakly converging subsequence of $\{u_k\}$ by the weak compactness of $W^{1,p}(\Omega,\mathbb{R}^N)$. Under the quasiconvexity assumption as (**LG2**),the functional $\mathscr{F}$ is weakly lower semicontinuous (see Subsection 2.6.2), which then implies the existence of a minimizer.

However, this argument fails in the linear growth context as the space $W^{1,1}(\Omega,\mathbb{R}^N)$ is not weakly compact. To proceed with the direct method, we need to consider a larger class, the one consisting of maps of bounded variation, in which a bounded sequence is weakly* precompact. Correspondingly, the definition of the functional $\mathscr{F}$ needs to be relaxed for BV maps in such a way that it is lower semicontinuous with respect to the weak* topology.

Following Lebesgue [Leb02], Serrin [Ser61] and Marcellini [Mar86], we define

$$\mathscr{F}_g(u,\Omega) := \inf\left\{\liminf_{k\to\infty}\int_\Omega f(\nabla u_k)\,\mathrm{d}x\colon\ \{u_k\}\subset W_g^{1,1}(\Omega,\mathbb{R}^N), u_k\to u \text{ in } L^1(\Omega,\mathbb{R}^N)\right\}. \tag{3.1.24}$$

This relaxation of $\mathscr{F}$ is sequentially lower semicontinuous with respect to the strong convergence in L^1, and thus to the weakly* convergence in BV. An integral expression of $\mathscr{F}_g(u,\Omega)$ was found in [KR10b], which is based on the work by Ambrosio & Dal Maso [ADM92] and Fonseca & Müller [FM93]. When f is quasiconvex, of linear growth and L^1 mean coercive, we have

$$\mathscr{F}_g(u,\Omega) = \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u| + \int_{\partial\Omega} f^\infty((g-u)\otimes\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.1.25}$$

where ν_Ω is the outward unit normal on $\partial\Omega$. The third term is present as the trace operator is not continuous with respect to the weak* topology in BV.

Therefore, any minimizing sequence $\{u_k\} \subset W^{1,1}_g(\Omega, \mathbb{R}^N)$ has a subsequence weakly* converging to some $u \in BV(\Omega, \mathbb{R}^N)$, and the map u is indeed a *minimizer* of $\mathscr{F}_g$.

Under the area-strict topology of BV, which is stronger than the weak* one, the extension of $\mathscr{F}$ to BV maps is the following

$$\bar{\mathscr{F}}(u, \Omega) := \left(\int_\Omega f(Du) := \right) \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty \left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|} \right) \mathrm{d}|D^s u|, \tag{3.1.26}$$

for any $u \in BV(\Omega, \mathbb{R}^N)$. This relaxation is lower semicontinuous with respect to weak* convergence in BV (see [ADM92, FM93]). More generally, we can consider the functional

$$\int_\Omega f(\nabla u)\,\mathrm{d}x + \int_{\partial\Omega} f^\infty(u|_{\partial\Omega} \otimes \nu_\Omega^-)\,\mathrm{d}\mathcal{H}^{n-1}$$

with the boundary value of a map u in $W^{1,1}(\Omega, \mathbb{R}^N)$ incorporated, where ν_Ω^- is the inward unit normal on $\partial\Omega$. The area-strict extension of this functional to BV maps, by Proposition 12.24 in [Rin18], is

$$\mathscr{F}_*(u, \Omega) := \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty \left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|} \right) \mathrm{d}|D^s u| + \int_{\partial\Omega} f^\infty(u|_{\partial\Omega} \otimes \nu_\Omega^-)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.1.27}$$

which is also lower semicontinuous with respect to weak* convergence in BV ([Rin18], Theorem 12.25).

Similar to the minimality notions in the super-linear setting, we have the following:

Definition 3.1.6. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz boundary, and $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies (**LG1**) and (**LG2**). The functional $\mathscr{F}$ and its relaxations $\bar{\mathscr{F}}$ and $\mathscr{F}_g$ are defined as in (3.1.21), (3.1.26) and (3.1.25).

(a) A map $u \in BV(\Omega, \mathbb{R}^N)$ is said to be a *generalised minimizer* of $\mathscr{F}_g$ (of $\mathscr{F}$ in the Dirichlet class $g + W^{1,1}_0(\Omega, \mathbb{R}^N)$) for some $g \in BV(\Omega, \mathbb{R}^N)$, if for any $w \in BV(\Omega, \mathbb{R}^N)$, we have

$$\mathscr{F}_g(u, \Omega) \le \mathscr{F}_g(w, \Omega); \tag{3.1.28}$$

(b) A map $u \in BV_{loc}(\Omega, \mathbb{R}^N)$ is said to be an *ω-minimizer* of $\bar{\mathscr{F}}$ with consant $R_0 > 0$, if for any ball $B_R = B(x_0, R) \subset\subset \Omega$ with $0 < R < R_0$ and any $w \in BV_u(B_R, \mathbb{R}^N)$, we have

$$\bar{\mathscr{F}}(u, B_R) \le \bar{\mathscr{F}}(w, B_R) + \omega(R) \int_{B_R} (1 + |Dw|). \tag{3.1.29}$$

Remark 3.1.7. Suppose that the integrand f is assumed to be of linear-growth in the following sense

(**LG1**)′ $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies

$$\ell|z| \le f(z) \le L(1 + |z|) \tag{3.1.30}$$

for any $z \in \mathbb{R}^{N\times n}$, where the constants L and ℓ are positive with $\ell \le L$.

Then we can use the following defining inequality

$$\bar{\mathscr{F}}(u, B_R) \leq (1 + \omega(R))\bar{\mathscr{F}}(w, B_R) \tag{3.1.31}$$

for ω-minimizers instead of (3.1.29).

3.2 Higher integrability for the first derivatives

This section provides a review of higher integrability results in the study of variational principles (or elliptic PDEs). Such results indicate that (almost) minimizers in certain variational problems exhibit higher integrability compared to general competitor maps.

In Subsection 3.2.1, we discuss higher L^p-integrability in the super-linear setting. Subsection 3.2.2 focuses on an $L \log L$ regularity result for BV minimizers in the linear growth context.

3.2.1 Higher L^p-integrability

Autonomous functionals are considered in this subsection for illustration. For more general cases, we refer to [Giu03], Chapter 6, and the references in the following.

Fix a bounded open set $\Omega \subset \mathbb{R}^n$ and consider a functional defined by

$$\mathscr{F}(u, \Omega) := \int_\Omega f(\nabla u)\, \mathrm{d}x, \quad u\colon\ \Omega \to \mathbb{R}^N, \tag{3.2.1}$$

where the integrand $f\colon\ \mathbb{R}^{N\times n} \to \mathbb{R}$ is required to satisfy the following properties with $p > 1$:

(**HI1**) There exists $L > 0$ such that for any $z \in \mathbb{R}^{N\times n}$ we have

$$|f(z)| \leq L(1 + |z|^p);$$

(**HI2**) The function $z \mapsto f(z) - \nu|z|^p$ is quasiconvex, i.e.,

$$\int_{(0,1)^n} (f(z + \nabla\varphi) - \nu|z + \nabla\varphi|^p)\, \mathrm{d}x \geq f(z) - \nu|z|^p$$

for any $z \in \mathbb{R}^{N\times n}$ and any $\varphi \in C_c^\infty((0,1)^n, \mathbb{R}^N)$.

With the above assumptions, it is possible to establish the following higher integrability result for spherical quasiminimizers. This result obviously extends to quasiminimizers, ω-minimizers and certainly minimizers.

Theorem 3.2.1 ([GG84a], Theorem 3.1)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, the integrand $f\colon\ \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* (**HI1**) *and* (**HI2**) *with $p > 1$, and the map $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ is a spherical minimizer of $\mathscr{F}$ as defined in* (3.2.1) *with constant $Q \geq 1$. Then there exists $\tau = \tau(n, N, p, Q, L, \nu) > 0$ such that $u \in W^{1,q}_{loc}(\Omega, \mathbb{R}^N)$ for any $q \in (p, p + \tau)$. Moreover, given any $q \in (p, p + \tau)$ and any ball $B_R = B(x_0, R) \subset\subset \Omega$, we have*

$$\left(\fint_{B_{\frac{R}{2}}} |\nabla u|^q\, \mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(\fint_{B_R} (1 + |\nabla u|^p)\, \mathrm{d}x \right)^{\frac{1}{p}}, \tag{3.2.2}$$

where the constant $C = C(n, N, p, q, Q, L, \nu) > 0$.

In the proof of the above theorem, a Caccioppoli-type inequality and a generalised version of Gehring's lemma are employed. With the quasiminimality of u and the quasiconvexity of the integrand f, one can achieve the following Caccioppoli-type inequality:

$$\int_{B_{\frac{R}{2}}} |\nabla u|^p \, \mathrm{d}x \leq C \int_{B_R} \left(\frac{|u - u_R|^p}{R^p} + 1 \right) \mathrm{d}x, \tag{3.2.3}$$

where B_R is an arbitrary ball compactly contained in Ω. A weak reverse Hölder is furthermore available from (3.2.3) with the Sobolev-Poincaré inequality:

$$\fint_{B_{\frac{R}{2}}} |\nabla u|^p \, \mathrm{d}x \leq C \left(\left(\fint_{B_R} |\nabla u|^{p_*} \, \mathrm{d}x \right)^{\frac{p}{p_*}} + 1 \right), \tag{3.2.4}$$

where $p_* = \min\{\frac{np}{n+p}, 1\}$. Finally, a generalised version of Gehring's lemma (see Theorem 4.2.2) implies the higher integrability as in Theorem 3.2.1.

This type of higher integrability results has a long history. It was initiated by Bojarski in [Boj57, Boj55], where the author established higher integrability for solutions to first order elliptic systems of Beltrami's type in two dimensions. Meyers [Mey63] later proved a similar result for solutions of second order linear elliptic equations with L^∞ coefficients. Gehring [Geh73] obtained a higher integrability result for quasi-conformal mappings with a different method, which was subsequently improved in [EM75, GM79, Str84] to establish similar results for nonlinear elliptic systems.

Under appropriate boundary conditions, a global version of Theorem 3.2.1 is also approachable (see [Giu03], Theorem 6.8, and [KK94]). Such a result is also contained in the proof of Theorem 4.1.1, and is the motivation of the problem we will address in Chapter 4. See Section 4.1 for a more detailed discussion.

3.2.2 $L \log L$ regularity for generalised minimizers

The higher integrability result above only holds true for $p > 1$, and the argument does not extend to the linear-growth setting. The main issue is that we cannot obtain a suitable weak reverse Hölder inequality from the Caccioppoli inequality (3.2.3) by using the Sobolev-Poincaré inequality, and thus cannot further apply the generalised Gehring's lemma to get such a result.

However, higher integrability results are still available, at least for (generalised) minimizers in the linear-growth context, with a different approach under a certain convexity condition.

Consider the following functional defined for maps in $W^{1,1}(\Omega, \mathbb{R}^N)$:

$$\mathscr{F}(u, \Omega) := \int_\Omega f(\nabla u) \, \mathrm{d}x. \tag{3.2.5}$$

The integrand $f \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ is assumed to be of linear growth and convex with some μ-ellipticity condition. More precisely, we assume

(**LD1**) $\ell\,|z| \le f(z) \le L(1+|z|)$ for any $z \in \mathbb{R}^{N\times n}$;

(**LD2**) $f \in C^2(\mathbb{R}^{N\times n})$, and for some $\mu \in (1,3]$ we have

$$\ell \frac{|\xi|^2}{(1+|z|^2)^{\frac{\mu}{2}}} \le f''(z)[\xi,\xi] \le L \frac{|\xi|^2}{(1+|z|^2)^{\frac{1}{2}}}, \quad \text{for any } \xi \in \mathbb{R}^{N\times n}. \tag{E$_\mu$}$$

Bildhauer and Fuchs did a systematic study in [Bil02, Bil03a, Bil03b, BF02] of the corresponding variational problem, and their work contains various regularity results for generalised minimizers in this context. In [Bil02], an $L\log L$-regularity was established for one specific generalised minimizer of $\mathscr{F}$ with a vanishing viscosity argument. This result was later extended to all minimizers in [BS13], where the authors used a similar method and applied Ekeland's variational principle to construct an approximating sequence for any generalised minimizer.

Theorem 3.2.2 ([BS13], Theorem 1.10)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, and the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is C^2 satisfies* (**LD1**), (**LD2**) *with $\mu = 3$, and $f(0) \le \lambda$. For any generalised minimizer $u \in BV \cap L^\infty(\Omega,\mathbb{R}^N)$ of $\mathscr{F}$ defined in* (3.2.5)*, we have*

$$u \in W^{1,1}(\Omega,\mathbb{R}^N), \quad |\nabla u|\log(1+|\nabla u|^2) \in L^1_{loc}(\Omega), \quad \textit{and} \tag{3.2.6}$$

$$\int_{B_r} |\nabla u|\log(1+|\nabla u|^2)\,\mathrm{d}x \le C\left(\lambda + \frac{1}{r}\sup_{B_{2r}}|u|\right)\int_{B_{2r}} (1+|\nabla u|)\,\mathrm{d}x \tag{3.2.7}$$

for any ball $B_{2r} = B(x_0,2r) \subset\subset \Omega$ with $C = C(n,N,\ell,L) > 0$.

Certain boundedness conditions are assumed in both [Bil02] and [BS13], and are helpful in the proof of the $L\log L$-regularity. Indeed, the local boundedness of a given generalised minimizer can be obtained under an extra structure condition

(**LD3**) $(a^\top f'(z))\cdot(a^\top z) \ge -\ell|a|^2$ for any $z \in \mathbb{R}^{N\times n}$ and any $a \in \mathbb{R}^N$.

This is also proved in [BS13]:

Theorem 3.2.3 ([BS13], Theorem 1.11)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, and the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* (**LD1**) *and* (**LD3**)*, and is C^1 and convex with $f(0) \le \lambda$. Then any generalised minimizer $u \in BV(\Omega,\mathbb{R}^N)$ of $\mathscr{F}$ defined in* (3.2.5) *is locally bounded (i.e., $u \in L^\infty_{loc}(\Omega,\mathbb{R}^N)$), and we have, for any pair of concentric balls $B(x_0,r) \subset\subset B(x_0,R) \subset\subset \Omega$,*

$$\sup_{B_r}|u| \le \frac{C}{(R-r)^n}\int_{B_R} ((R-r)\lambda + |u|)\,\mathrm{d}x \tag{3.2.8}$$

with $C = C(n,N,\ell,L,\lambda) > 0$.

Combining the two results above, we can remove the boundedness assumption in Theorem 3.2.2 and get the following corollary.

Corollary 3.2.4 ([BS13], Corollary 1.13)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set, and the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* (**LD1**)-(**LD3**) *with $\mu = 3$ and $f(0) \le \lambda$. Then for any generalised minimizer $u \in BV(\Omega,\mathbb{R}^N)$ of $\mathscr{F}$ defined in* (3.2.5)*, we have*

$$u \in W^{1,1}(\Omega,\mathbb{R}^N), \quad |\nabla u|\log(1+|\nabla u|^2) \in L^1_{loc}(\Omega). \tag{3.2.9}$$

Furthermore, there exists a constant $C = C(n, N, \ell, L, \lambda) > 0$ *such that for any ball* $B_{3r} = B(x_0, 3r) \subset\subset \Omega$ *there holds*

$$\frac{1}{r^n} \int_{B_r} |\nabla u| \log(1 + |\nabla u|^2) \, \mathrm{d}x \leq C \left(1 + \frac{1}{r^n} \int_{B_{3r}} |\nabla u| \, \mathrm{d}x \right)^2 . \tag{3.2.10}$$

To obtain Theorem 3.2.2, the authors regularised the integrand f by adding a viscosity term related to k, which is quadratic and vanishes as $k \to \infty$. Then a sequence $\{u_k\}$ that approximates a given generalised minimizer u is constructed with Ekeland's variational principle. Each u_k satisfies an Euler-Lagrange type inequality, with which the second differentiability can be further obtained with an estimate uniform in k. The $L \log L$-regularity for u_k is then proved with such an estimate, and is indeed retained by the minimizer u. We remark that the argument also applies to the case $1 < \mu < 3$.

The proof in [BS13] actually implies a Sobolev type regularity of $W_\mu(\nabla u)$ and thus a higher integrability of ∇u when μ is close to 1, where u is any given generalised minimizer of $\mathscr{F}$ and $W_\mu(z) = (1 + |z|^2)^{\frac{2-\mu}{4}}$. This is not stated in their paper, and we present the proof for completeness in Section 6.4. Based on this result, we can obtain a fractional Sobolev regularity for BV ω-minimizers in Section 6.5.

It is worth mentioning that some variational problems originated from, for example, plasticity, are posed in the space of maps of bounded deformation, where symmetric gradients $Eu := \frac{1}{2}(Du + Du^t)$ are considered instead of gradients Du. Higher integrability results are also obtained for generalised minimizers in the BD context under the μ-ellipticity condition (E_μ) in [GK19b, Gme20]. Their argument is similar to that in [BS13], but more involved due to the presence of bounded deformation and Ornstein's non-inequality.

3.3 Hölder regularity in the scalar case ($N = 1$)

In this section, we give a brief review of Hölder regularity results for (almost) minimizers in the scalar case ($N = 1$). The review includes both zero-order and first-order Hölder regularity, and is far from being complete. More detailed discussions can be found in [Mor66], [Gia83], [Giu03] and [Min06].

3.3.1 Zero-order Hölder regularity

In the fundamental work [DG57], De Giorgi established Hölder regularity for solutions to a certain type of linear elliptic equations. This method has since been generalised to cover more general solutions of nonlinear equations of divergence form in, for instance, [Sta58, Sta60, Sta65] and [LU68].

De Giorgi's result was obtained by Nash [Nas58] independently for parabolic equations. Shortly afterwards, Moser [Mos60] provided a different proof with Harnack's inequality.

De Giorgi's method was later extended to minimizers in variational problems without turning to Euler-Lagrange equations. Frehse first studied a particular case with relatively restrictive assumptions in [Fre75]. In [GG82], Giaquinta and Giusti applied De Giorgi's method directly to minimizers of variational problems, and then quasiminimizers [GG84a]. Subsequently, Di Benedetto and Trudinger [DBT84] proved Harnack's inequality for functions in the so-called De Giorgi classes (see Definition 3.3.2), and thus for quasiminimizers. The Hölder regularity result was further extended to ω-minimizers in [DEF96] and [EM99].

We present those (simplified) results related to quasiminimizers and ω-minimizers below for illustration. Consider the following functional

$$\mathscr{F}(u,\Omega) = \int_\Omega f(x,u,\nabla u)\,\mathrm{d}x, \tag{3.3.1}$$

where the integrand f satisfies

(**HC1**) $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^n$ is a Carathéodory function;

(**HC2**) for any $(x,u,z)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$, we have

$$\ell|z|^p \le |f(x,u,z)| \le L(1+|z|^p), \tag{3.3.2}$$

where L and ℓ are positive constants with $0<\ell\le L$, and $p>1$.

Theorem 3.3.1 ([GG84a], Theorem 4.2)**.** *Suppose that $\Omega\subset\mathbb{R}^n$ is a bounded open set, and the integrand $f\colon\Omega\times\mathbb{R}\times\mathbb{R}^n$ satisfies* (**HC1**) *and* (**HC2**)*. If $u\in W^{1,p}_{loc}(\Omega)$ is a quasiminimizer of $\mathscr{F}$ defined in* (3.3.1) *with constant $Q\ge 1$, then there exists $\alpha=\alpha(n,p,\ell,L,Q)\in(0,1)$ such that $u\in C^{0,\alpha}_{loc}(\Omega)$.*

The proof of this result is based on Caccioppoli-type inequalities on the level sets of u. Take an arbitrary ball $B_R = B(x_0,R)\subset\subset\Omega$ and a number $k\in\mathbb{R}$, and define

$$\begin{aligned} A(k,R) &= \{x\in B_R\colon\ u(x)>k\},\\ B(k,R) &= \{x\in B_R\colon\ u(x)<k\}. \end{aligned}$$

Then with the quasiminimality of u, it is possible to derive the following inequalities

$$\int_{A(k,\rho)} |\nabla u|^p\,\mathrm{d}x \le C\left(\frac{1}{(R-\rho)^p}\int_{A(k,R)}(u-k)^p\,\mathrm{d}x + \mathscr{L}^n(A(k,R))\right), \tag{3.3.3}$$

$$\int_{B(k,\rho)} |\nabla u|^p\,\mathrm{d}x \le C\left(\frac{1}{(R-\rho)^p}\int_{B(k,R)}(k-u)^p\,\mathrm{d}x + \mathscr{L}^n(B(k,R))\right) \tag{3.3.4}$$

for any $\rho\in(0,R)$ and any $k\in\mathbb{R}$, where $C=C(p,\ell,L,Q)>0$. These inequalities actually provide the characterisation of the so-called De Giorgi classes.

Definition 3.3.2. Suppose that $\Omega\subset\mathbb{R}^n$ is a bounded open set. A function $u\in W^{1,p}(\Omega)$ with $p>1$ is said to be in the *De Giorgi class* $DG^+_p(\Omega)$ (of $DG^-_p(\Omega)$), if there exists a constant $c_d>0$, a number $k_0\in\mathbb{R}$ and a radius $R_0>0$ such that, for any pair of concentric balls $B(x_0,r)\subset\subset B(x_0,R)\subset\Omega$ with $R\le R_0$ and any $k\ge k_0$ (or $k\le k_0$), we have (3.3.3) (or (3.3.4)) hold true with C replaced by c_d. The class $DG_p(\Omega)$ is defined as $DG^+_p(\Omega)\cap DG^-_p(\Omega)$.

In [DG57], De Giorgi proved boundedness and Hölder regularity for the functions in the De Giorgi class $DG_p(\Omega)$, where the Hölder exponent $\alpha\in(0,1)$ for each $u\in DG_p(\Omega)$ is determined by n,p,c_d.

In addition to Hölder continuity, another regularity result concerning the De Giorgi classes and thus quasiminimizers is the Harnack inequality by Di Benedetto & Trudinger [DBT84]. They employed a covering theorem by Krylov & Safonov [KS80] to establish the following result:

Theorem 3.3.3 ([DBT84], Theorem 3)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, and u is a positive function in $DG_p(\Omega)$ with constants c_d, k_0, R_0. Then there exists $C = C(n,p,c_d) > 0$ such that for any $x_0 \in \Omega$ and $\rho < \min\{\frac{R_0}{2}, \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega)\}$, we have*

$$\operatorname{ess\,sup}_{B_\rho} u(x) \leq C(\operatorname{ess\,inf}_{B_\rho} u(x) + \rho). \tag{3.3.5}$$

Moreover, it is also possible to show Hölder continuity for ω-minimizers with a comparison argument, for which we refer to [EM99].

Theorem 3.3.4 ([EM99], Theorem 6)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, the integrand $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n$ satisfies* (**HC1**) *and* (**HC2**) *with $1 < p < n$, and the function $\omega\colon [0,\infty) \to [0,\infty)$ satisfies* ($\boldsymbol{\omega}$)*. Then there exists $\alpha = \alpha(n,p,\ell,L) \in (0,1)$ depending on such that any ω-minimizer $u \in W^{1,p}_{loc}(\Omega)$ of the functional $\mathscr{F}$ defined in* (3.3.1) *is in $C^{0,\alpha}_{loc}(\Omega)$.*

3.3.2 First order Hölder continuity

For quasiminimizers, the best regularity we can expect is zero-order Hölder continuity. This can be illustrated by following example.

Example 3.3.1 ([Giu03], Example 8.1)**.** Assume that $n \geq 2$, and consider $u = \frac{x_1}{|x|^\alpha}$ on the unit ball $\mathbb{B} \subset \mathbb{R}^n$ with some $\alpha \in (0,1)$. Then u is a weak solution to the equation

$$\operatorname{div}(a^{ij}(x)\partial_j u) = 0 \tag{3.3.6}$$

on $\mathbb{B}$, where we used the Einstein summation convention and the coefficients $\{a^{ij}\}_{i,j=1}^n$ are defined by

$$a^{ij}(x) = \delta^{ij} + \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)}\frac{x_i x_j}{|x|^2}.$$

From the equation, it can be derived that u is a minimizer of the functional

$$\int_B \left(|\nabla u|^2 + \sigma \left(\frac{x \cdot \nabla u}{|x|} \right)^2 \right) \mathrm{d}x$$

with $\sigma = \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)}$. Notice that the integrand $F(x,z)$ satisfies

$$|z|^2 \leq F(x,z) \leq (1+\sigma)|z|^2,$$

which implies that u is a quasiminimizer of the 2-Dirichlet integral. However, it is easily seen that ∇u does not even have a limit when $x \to 0$. □

Back to minimizers (and solutions to elliptic equations), first-order Hölder regularity is achievable under appropriate assumptions. Moreover, ω-minimizers retain the first-order regularity of the corresponding minimizers to certain extent, if the function $\omega(r)$ decreases to 0 as $r \to 0$ at a suitable rate.

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set. Consider the functional $\mathscr{F}$ defined by

$$\mathscr{F}(u, \Omega) = \int_\Omega f(x, u, \nabla u)\,\mathrm{d}x, \tag{3.3.7}$$

where the integrand $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n$ is is Carathéodory and C^2 in $z \in \mathbb{R}^n$, and satisfies the following properties for any $(x, u, z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$:

(HD1) $\ell V(z)^p \le f(x, u, z) \le L V(z)^p$ for some positive constants ℓ, L with $0 < \ell \le L$;

(HD2) $|f_{zz}(x, u, z)| \le L V(z)^{p-2}$;

(HD3) $f_{z_i z_j}(x, u, z)\xi_i \xi_j \ge V(z)^{p-2}|\xi|^2$;

(HD4) $|f(x, u, z) - f(y, v, z)| \le \vartheta(|x - y| + |u - v|)V^p(z)$.

The function $V(z)$ is defined as $\sqrt{1 + |z|^2}$, and we assume that $p > 1$.

Theorem 3.3.5 ([Giu03], Theorem 8.5 & 8.8)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, and the integrand $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n$ is Carathéodory and C^2 in $z \in \mathbb{R}^n$, and satisfies* (**HD1**)-(**HD4**) *with $p > 1$. In addition, assume that the function $\omega\colon [0, \infty) \to [0, \infty)$ satisfies* ($\boldsymbol{\omega}$)*, and $\vartheta(t) + \omega(t) \le \theta t^\tau$ for some $\tau \in (0, 1)$ and $\theta > 0$. If $u \in W^{1,p}_{loc}(\Omega)$ is an ω-minimizer of the functional $\mathscr{F}$ defined in (3.3.7), then there exists $\alpha = \alpha(n, p, \ell, L, \tau, \theta) \in (0, 1)$ such that $\nabla u \in C^{0,\alpha}_{loc}(\Omega, \mathbb{R}^n)$.*

It is worth mentioning that the integrand f is not required to be differentiable in u, and thus the Euler-Lagrange equation of the variational problem associated to $\mathscr{F}$ does not exist. The methods used to prove Theorem 3.3.5 involves frozen functionals. Given a ball $B(x_0, R) \subset\subset \Omega$, the corresponding frozen functional $\mathscr{F}_0$ is defined by

$$\mathscr{F}_0(v, \Omega) := \int_\Omega f(x_0, u(x_0), \nabla v)\,\mathrm{d}x.$$

The basic idea is to compare a minimizer of $\mathscr{F}$ and a minimizer of $\mathscr{F}_0$ with the same boundary value on $B(x_0, R)$ for sufficiently small R. This strategy can be extended to ω-minimizers as well.

The methods employed in proving results similar Theorem 3.3.5 are mostly taken from the regularity theory for quasi-linear elliptic equations of divergence form, for which there have also been analogous first-order results. Relevant references for the latter context include, for instance, [Ura68], [Uhl77] and [Eva82].

The extension of those methods to variational problems was first conducted by Giaquinta and Giusti in [GG83, GG84b] in the quadratic setting ($p = 2$). The super-quadratic case ($p > 2$) was addressed in [GM86a] under relatively strong assumptions, and Manfredi [Man88] established the corresponding result in a more general context with $p \neq 2$. Notice that the degenerate case, where $V(z)$ is replaced by $|z|$, is also included in [GM86a, Man88]. In addition, we refer to [Lew83, DiB83, Tol83, Man86] for first-order Hölder regularity concerning solutions of degenerate elliptic equations.

With the first-order Hölder regularity for a minimizer (or a solution to a specific elliptic equation) as in Theorem 3.3.5, higher order regularity up to analyticity can be obtained via a boot-strap argument, with sufficient regularity of the integrand (or the coefficients). Such results nowadays are known as Schauder estimates, and a relevant discussion can be found in [Giu03], Chapter 10.

3.4 Partial regularity in the vectorial case

In the vectorial setting ($N > 1$), full regularity is no longer expected for minimizers, not to mention almost minimizers. Minimizers of convex functionals may exhibit singularities, even if with sufficiently smooth integrands.

Such counterexamples were firstly given by De Giorgi [DG68] and Maz'ya [Maz68] independently. In each example, a minimizer with singularities was constructed for a certain quadratic functional of the form $f(x,z) = \mathbb{A}(x)[z,z]$, where $\mathbb{A}(x)$ is discontinuous. Shortly afterwards, Giusti and Miranda [GM69] constructed an example with a quadratic integrand $f(u,z) = \mathbb{A}(u)[z,z]$, where $\mathbb{A}(u)$ is analytic. Counterexamples with autonomous integrands can be found in [Neč77, HLN96, ŠY00, ŠY02]. More recently, Mooney and Savin gave an example in low dimensions ($n = 3, N = 2$).

We present the example by Giusti and Miranda for illustration.

Example 3.4.1 ([GM69])**.** Let $N = n$ be large enough, and consider the functional

$$\mathcal{A}(u,\mathbb{B}) = \int_{\mathbb{B}} A^{ij}_{\alpha\beta}(u)\partial_i u^\alpha \partial_j u^\beta \, \mathrm{d}x,$$

where the coefficients are

$$A^{ij}_{\alpha\beta}(u) = \delta_{\alpha\beta}\delta_{ij} + \left(\delta_{i\alpha} + \frac{4}{n-2}\cdot\frac{u_i u_\alpha}{1+|u|^2}\right)\left(\delta_{j\beta} + \frac{4}{n-2}\cdot\frac{u_j u_\beta}{1+|u|^2}\right).$$

It is easy to see that $A^{ij}_{\alpha\beta}(u)$ is bounded for any i, j, α, β, and the matrix satisfies

$$A^{ij}_{\alpha\beta}(u)\xi^\alpha_i \xi^\beta_j \geq |\xi|^2.$$

When the dimension n is large enough, the map $u = \frac{x}{|x|}$ is the unique minimizer of $\mathcal{A}$ in $W^{1,2}_u(\mathbb{B},\mathbb{R}^n)$ but has a singularity at $x = 0$. □

Despite of possible singularities of minimizers, regularity in a partial sense (the so-called "partial regularity") is still available. More precisely, under appropriate assumptions on the variational problem defined on $\Omega \subset \mathbb{R}^n$, one can show that a minimizer is regular on an open subset Ω_0 of Ω with $\mathscr{L}^n(\Omega \setminus \Omega_0) = 0$.

The strategy used for most partial regularity results dates back to De Giorgi [DG61] and Almgren [Alm68], who worked on minimal surfaces and minimising varifolds, respectively. This method was later adapted by Giusti and Miranda [GM69] to prove the partial regularity for minimizers in certain variational problems in the non-parametric setting, and by Morrey [Mor68] for the solutions to certain elliptic systems. It was Evans [Eva86] who showed the first partial regularity result for quasiconvex functionals, which involves quadratic-growth integrands. Shortly afterwards, Fusco and Hutchinson

[FH85], and Giaquinta and Modica [GM86a] extended this result to functionals with general integrands of the form $f(x,u,\nabla u)$, and Acerbi and Fusco [AF87] dealt with integrands of p-growth with $p \geq 2$. Carozza, Fusco and Mingione [CFM98] first studied the subquadratic case $(1 < p < 2)$, and there are various results afterwards, including [AF89, CPdN96, DLSV12, DM04, DGG00, DK02, DGK05]. As to the linear growth context $(p = 1)$, there are only limited references. Anzellotti and Giaquinta [AG88] showed a partial regularity result for convex functionals, which was extended to the degenerate setting in [Sch14]. Some recent progress in the quasiconvex case is given by Gmeineder and Kristensen [GK19a].

The literature on regularity for quasiconvex functionals is extensive, and the list above is far from complete. We refer to [GK19a] and the monograph by Giusti [Giu03] for a thorough review.

The question about the size of singular sets in partial regularity results remains open, but see [KM05, KM06, KM07] for some estimates of the Hausdorff dimensions of singular sets in various set-ups.

In the following, we present the results in [DGG00, DK02, DGK05] for ω-minimizers, which are related to Chapter 5.

Consider a bounded open set $\Omega \subset \mathbb{R}^n$, and the functional $\mathscr{F}$ defined by

$$\mathscr{F}(u,\Omega) = \int_\Omega f(\nabla u)\,\mathrm{d}x.$$

The assumptions for the super-quadratic setting $(p \geq 2)$ and the sub-quadratic one $(1 < p < 2)$ are slightly different. For the former, we require the integrand f to satisfy the following properties:

(P1) $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is C^2, and for some $L \geq 0$ we have

$$|f''(z)| \leq L(1+|z|^{p-2}), \quad \text{for any } z \in \mathbb{R}^{N\times n};$$

(P2) f is strictly quasiconvex in the following sense: there exists $\ell > 0$ such that for any $z \in \mathbb{R}^{N\times n}$ and any $\varphi \in W_0^{1,p}((0,1)^n, \mathbb{R}^N)$ we have

$$\int_{(0,1)^n} (f(z+\nabla\varphi) - f(z))\,\mathrm{d}x \geq \ell \int_{(0,1)^n} (|\nabla\varphi|^2 + |\nabla\varphi|^p)\,\mathrm{d}x.$$

It is worth mentioning that **(P1)** implies the p-growth of f

(P1′) $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is C^2, and for some $L \geq 0$ we have

$$|f(z)| \leq L(1+|z|^p), \quad \text{for any } z \in \mathbb{R}^{N\times n}.$$

Notice that if we utilise **(P1)** to obtain **(P1′)**, the constant L in the latter should be different from the one in the former.

The growth condition **(P1′)** is needed for the case $1 < p < 2$. In addition, we assume a quasiconvexity condition of the following form:

(P2′) there exists $\ell > 0$ such that for any $z \in \mathbb{R}^{N\times n}$ and any $\varphi \in W_0^{1,p}((0,1)^n, \mathbb{R}^N)$ we have

$$\int_{(0,1)^n} (f(z+\nabla\varphi) - f(z))\,\mathrm{d}x \geq \ell \int_{(0,1)^n} (1+|z|^2+|\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2\,\mathrm{d}x.$$

For the function ω, the smallness around the origin as in Subsection 3.1.3 is assumed

($\boldsymbol{\omega}$1) $\omega\colon [0,\infty) \to [0,\infty)$ is non-decreasing with $\omega(r) \leq 1$ and $\lim_{r\to 0^+}\omega = \omega(0) = 0$.

Instead of giving a precise growth rate as in Subsection 3.3.2, we require the following:

($\boldsymbol{\omega}$2) there exists $\beta \in (0,1)$ such that $r \mapsto \frac{\omega(r)}{r^{2\beta}}$ is non-increasing in $(0,\infty)$;

($\boldsymbol{\omega}$3) the Dini-type condition: for any $\rho > 0$, we have $\Xi_{\frac{1}{2}}(\rho) < \infty$, where

$$\Xi_\alpha(\rho) := \int_0^\rho \frac{\omega^\alpha(r)}{r}\,\mathrm{d}r.$$

Under these conditions, it is possible to get partial regularity for the first derivatives of ω-minimizers:

Theorem 3.4.1 ([DGG00, DK02, DGK05])**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open sets, and the function ω satisfies ($\boldsymbol{\omega}$1)-($\boldsymbol{\omega}$3). Furthermore, we assume that the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies (P1) and (P2) if $p \geq 2$, and (P1′) and (P2′) if $1 < p < 2$. If $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of $\mathscr{F}$, then there exists a relatively closed subset $S_u \subset \Omega$ such that $\mathscr{L}^n(S_u) = 0$ and $u \in C^1(\Omega \setminus S_u)$. Moreover, in $\Omega \setminus S_u$ the gradient ∇u has a local modulus of continuity $\rho \mapsto \rho^\alpha + \Xi_{\frac{1}{2}}(\rho)$ for any $\alpha \in (0,1)$. The singular set S_u is contained in $\Sigma_1 \cup \Sigma_2$, where the two sets Σ_1 and Σ_2 are defined by*

$$\Sigma_1 = \left\{x_0 \in \Omega\colon \liminf_{r\to 0} \fint_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^p\,\mathrm{d}x > 0\right\}, \tag{3.4.1}$$

$$\Sigma_2 = \left\{x_0 \in \Omega\colon \limsup_{r\to 0} |(\nabla u)_{x_0,r}| = \infty\right\}. \tag{3.4.2}$$

It is worth mentioning that zero-order partial regularity for ω-minimizers can be obtained only with ($\boldsymbol{\omega}$1). The result for $p = 2$ is presented in [DK02], and in Section 5.4 we establish such results for $1 < p < 2$ and the linear growth context.

In Chapter 5, we focus on ω-minimizers of quasiconvex functionals, primarily within the linear growth context, and establish three partial regularity results. In particular, the one gives first-order partial regularity for BV ω-minimizers extends the result for BV minimizers in [GK19a], and also bridges the gap at the borderline $p = 1$ for ω-minimizers (as shown in Theorem 3.4.1).

Chapter 4

Stability of quasiminimizers

The focus of this chapter is to study the stability of quasiminimizers. To be more precise, we will study whether the quasi-minimality of a map with respect to the Dirichlet integrals $\mathcal{D}_p$ is stable with respect to the exponent p. When the exponent p increases within a certain range, the quasi-minimality of a map is retained (Theorem 4.1.1). However, for exponents smaller than p, we only have a stability result for minimizers, i.e., p-harmonic maps (Theorem 4.1.2).

The problem is contextualised first in Section 4.1, where there are also the statements of the two results mentioned above. In Section 4.2 we have a brief look at Gehring's lemma, and present the proof of a generalised version, which is used in the proof of Theorem 4.1.1. Section 4.3 is devoted to a discussion of quasiminimizers on 1-dimensional intervals, which is based on [MS07]. Theorem 4.1.1, the result concerning quasiminimality of a higher order, is proved in Section 4.4. The proof of the stability result Theorem 4.1.2 is then presented in Section 4.5.

4.1 Contextualisation and main results

In this chapter, we consider some specific functionals–the Dirichlet energies $\mathcal{D}_p$ with $p > 1$, and the corresponding (p, Q)-minimizers as defined in Definition 3.1.2. The focus is on the stability of (p, Q)-minimizers with respect to p, i.e., whether the quasi-minimality of a (p, Q)-minimizer remains valid for $\mathcal{D}_q$ with a $q > 1$ close to p.

This problem is inspired by a higher integrability result for quasiminimizers. Quasiminimizers of a wide class of functionals have locally improved integrability as presented in Subsection 3.2.1, and a global version of this result on a domain Ω is also achievable under regularity conditions on $\partial\Omega$ and the boundary value of the quasiminimizer (see [Giu03], Theorem 6.8, and [KK94]). If we consider this result for (p, Q)-minimizers, a corollary is the following quasiminimality result of a higher order.

Theorem 4.1.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, and $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ is a (p, Q)-minimizer with $p > 1, Q \geq 1$. Given two positive constants T_0 and R_0, there exists $q_0 > p$ depending on n, N, p, Q, R_0, T_0, and for any $q \in (p, q_0)$ a constant $Q_q = Q_q(n, N, p, Q, R_0, T_0, q) > 0$, such that the following holds: For any open set $O \subset\subset \Omega$ such that O^c is uniformly p-thick with radius $R_0 > 0$ and constant $T_0 > 0$, the map u quasiminimises $\mathcal{D}_q$ on O with constant Q_q. That is, for any $v \in W^{1,q}(O, \mathbb{R}^N)$ and $u - v \in$*

$W_0^{1,q}(O,\mathbb{R}^N)$ *we have*

$$\int_O |\nabla u|^q \,\mathrm{d}x \le Q_q \int_O |\nabla v|^q \,\mathrm{d}x. \tag{4.1.1}$$

Notice that we only have the above conclusion for $p > 1$. In the linear-growth case $p = 1$, there is no weak reverse Hölder inequality since the Sobolev embeddings only hold for exponents no less than one and it is impossible to obtain a decrease in the exponent as in Proposition 4.4.4.

The above theorem is inspired by [KK94], in which the authors showed that, if the domain satisfies some thickness assumption as above and the boundary values are also of a higher integrability, solutions to certain elliptic PDEs are globally integrable of a higher order. It is possible to modify this result so that it applies to the case of quasiminimizers of polynomial-growth functionals, in particular, the Dirichlet energy $\mathcal{D}_p$. By choosing a proper comparison map, we are able to further conclude quasiminimality of a higher order as stated in Theorem 4.1.1.

The main tool used in the proof of Theorem 4.1.1 is a generalisation of Gehring's lemma (Theorem 4.2.3), which is actually the self-improving property of weak reverse Hölder inequalities. To obtain a global higher integrability, such inequalities are needed not only interiorly but also near the boundary of the domain in consideration. Thus, apart from the generalised Gehring's lemma, the proof is mainly focused on showing a weak reverse Hölder inequality near the boundary, which includes a Caccioppoli-type inequality with the boundary value involved (Subsection 4.4.2) and a Sobolev-Poincaré inequality with the p-capacity involved (Subsection 4.4.1).

It is worth mentioning that (strong) reverse Hölder inequalities similar to (4.2.1) are not expected in general for PDE solutions or quasiminimizers. Indeed, in [IM01], §14.5 the authors showed that the unique continuation principle applies to functions that satisfy reverse Hölder inequalities, while there are non-trivial solutions to certain elliptic PDEs which fulfil weak reverse Hölder inequalities but vanish on an open non-empty subset of their domain (see [Pli63, Mar88]).

Theorem 4.1.1 indicates a certain type of stability of (p,Q)-minimality. Precisely, the (p,Q)-minimality of a map remains valid when the exponent p slightly increases. Then it is natural to ask what will happen when the exponent decreases.

In the one-dimensional case where $n = 1$, the answer is clear. For maps defined on a 1-d interval, a (p,Q)-minimizer satisfies a certain reverse Hölder inequality (see Section 4.3), and a (p,Q)-minimizer actually quasiminimises $\mathcal{D}_q$ for any $q \in [1,p)$ (Theorem 4.3.7). The discussion in this simple case indicates the possibility of a similar result in the higher-dimensional case ($n > 1$), though the latter is more complicated and we cannot expect a neat conclusion as in the former.

To see whether a (p,Q)-minimizer retains its minimality when the exponent decreases, we first consider the relatively simple case of minimizers, i.e., p-harmonic maps. It turns out that p-harmonic maps quasiminimises $\mathcal{D}_q$ for q close to p.

Theorem 4.1.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set and $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ is a p-harmonic map. Then there exists $\tau > 0$ depending on p, n, N such that for any $q \in (p-\tau, p+\tau)$, the map u is a spherical (q, Q_q)-minimizer on Ω with Q_q depending on p, q, n, N. More* *precisely, for any ball* $B_R \subset\subset \Omega$ *and any* $w \in W^{1,q}(B_R, \mathbb{R}^N)$ *with* $u - w \in W_0^{1,q}(B_R, \mathbb{R}^N)$*, we have*

$$\int_{B_R} |\nabla u|^q \, \mathrm{d}x \leq Q_q \int_{B_R} |\nabla w|^q \, \mathrm{d}x. \tag{4.1.2}$$

Furthermore, the constant Q_q *can be chosen such that* $Q_q \to 1$ *as* $q \to p$*.*

However, the situation becomes more complicated when we consider (p, Q)-minimizers with $Q > 1$, and it is still unclear whether an analogue of the above theorem holds true in this case.

The strategy we used to investigate the minimality of a p-harmonic map with respect to $\mathcal{D}_q$ is to quantify the q-Laplacian of u and then compare it with a q-harmonic map with the same boundary value. This argument was carried out with the help of the Hodge decomposition and a certain type of nonlinear commutators as in Subsection 4.5.2, the latter of which has seen its application to different equations and systems (see [IS94, CGI02, IM01] and the references therein). A similar idea also appears in [IS94], where the authors made use of nonlienar commutators in a slightly different way to deal with weak p-harmonic maps.

4.2 Gehring's lemma and its generalisation

In this section, we have a short discussion of Gehring's lemma and some of its variants.

An inequality of the form

$$\fint_Q f^q \, \mathrm{d}x \leq b \left(\fint_Q f \, \mathrm{d}x \right)^q \tag{4.2.1}$$

on some n-cube $Q \subset \mathbb{R}^n$ with some $q > 1, b \geq 1$ is called a *reverse Hölder inequality.* Notice that the inverse inequality holds with $b = 1$ by Hölder's inequality. A *weak reverse Hölder inequality* is an inequality of the same form with Q on the right-hand side replaced by $2Q$ (or more generally by μQ with some $\mu > 1$). In certain contexts, such inequalities are presented with respect to concentric balls instead of cubes, and may contain inhomogeneous terms on the right-hand side.

Reverse Hölder inequalities first appeared in the study of weighted maximal functions [Muc72] and quasiconformal mappings [Geh73]. See [CUN95, IMS15, IM01] and the references therein for a systematic discussion about (weak) reverse Hölder inequalities.

We first give a brief introduction to Gehring's lemma and the corresponding generalisation in Subsection 4.2.1. In Subsection 4.2.2 there are two auxiliary lemmas related to Stieltjes integrals. The two lemmas are useful in the proof of Theorem 4.2.3 in Subsection 4.2.3, which is a generalised version of Gehring's lemma on the whole space. The contents of this section are not claimed to be new, and are just presented in a form suitable for our use.

4.2.1 A short introduction

One celebrated result related to reverse Hölder inequalities is Gehring's lemma, proved by Gehring in his work [Geh73] on quasiconformal mappings.

Theorem 4.2.1 ([Geh73], Lemma 3)**.** *Suppose that $Q_0 \subset \mathbb{R}^n$ is an n-cube, and the function $f\colon Q_0 \to [0,\infty)$ is in $L^q(Q_0)$ for some $q > 1$. In addition, the following inequality*

$$\fint_Q f^q \,\mathrm{d}x \le b \left(\fint_Q f \,\mathrm{d}x \right)^q \tag{4.2.2}$$

is fulfilled for any n-cube $Q \subset Q_0$ with some $b \ge 1$. Then there exists $\tau = \tau(n,q,b) > 0$ such that $f \in L^p(Q_0)$ with

$$\fint_{Q_0} f^p \,\mathrm{d}x \le \frac{\tau}{q+\tau-p} \left(\fint_{Q_0} f^q \,\mathrm{d}x \right)^{\frac{p}{q}} \le \frac{\tau b^{\frac{p}{q}}}{q+\tau-p} \left(\fint_{Q_0} f \,\mathrm{d}x \right)^p \tag{4.2.3}$$

for any $p \in [q, q+\tau)$.

Notice that (4.2.3) conversely implies (4.2.2) up to a constant, and thus the above result is considered the self-improving preperty of reverse Hölder inequalities.

From the inequality (4.2.2) it is easy to obtain the following

$$M(f^q)(x) \le b M^q(f)(x), \quad \text{for any } x \in \mathbb{R}^n \tag{4.2.4}$$

if Q_0 is replaced by $\mathbb{R}^n$, and the self-improving property in Theorem 4.2.1 remains true for reverse Hölder inequalities in the form of maximal functions as (4.2.4), which is Lemma 2 in [Geh73].

Gehring's lemma was later extended to weak reverse Hölder inequalities by ELCRAT & MEYERS [EM75], GIAQUINTA & MODICA [GM79], STREDULINSKY [Str84], where the generalised versions found their use in the regularity theory of elliptic PDEs. There are also Orlicz and non-local versions given in [IM01] (Theorem 14.3.1) and [KMS14, KMS15], respectively.

We present the following version for illustration. It is in the form of maximal functions, and helpful in the regularity theory of elliptic equations or systems (see [Gia83]).

Theorem 4.2.2 ([Gia83], §V.1)**.** *Suppose that Q_0 is an n-cube in $\mathbb{R}^n$ and $f \in L^q(Q_0), g \in L^r(Q_0)$ with $1 < q < r$. For any $x_0 \in Q_0$ and $R < \min\{\frac{1}{2}\operatorname{dist}(x_0, \partial Q_0), R_0\}$, there holds*

$$\fint_{Q_R(x_0)} f^q \,\mathrm{d}x \le b \left(\fint_{Q_{2R}(x_0)} f \,\mathrm{d}x \right)^q + \fint_{Q_{2R}(x_0)} g^q \,\mathrm{d}x + \beta \fint_{Q_{2R}(x_0)} f^q \,\mathrm{d}x, \tag{4.2.5}$$

where $R_0 > 0$, $b \ge 1$ and $\beta \in [0,1)$ are constants. Then there exists $\varepsilon = \varepsilon(n,\beta,b,q,r) > 0$ such that $f \in L^p_{loc}(Q_0)$ for any $p \in [q, q+\varepsilon)$, and we have

$$\left(\fint_{Q_R} f^p(x) \,\mathrm{d}x \right)^{\frac{1}{p}} \le C \left(\left(\fint_{Q_{2R}} f^q(x) \,\mathrm{d}x \right)^{\frac{1}{q}} + \left(\fint_{Q_{2R}} g^p(x) \,\mathrm{d}x \right)^{\frac{1}{p}} \right) \tag{4.2.6}$$

for any n-cube $Q_{2R} \subset\subset Q$ with $R < R_0$ and $C(n,\beta,b,q,p) \ge 1$.

In the version given by IWANIEC & MARTIN in [IM01], the corresponding estimate is on the whole space and can be used to show a global higher integrability on regular domains. We state it with some modification and give the proof in Subsection 4.2.3.

Theorem 4.2.3. *Suppose that* $\mu > 1$*,* $R_0 > 0$*, and* $f \in L^q(\mathbb{R}^n)$ *and* $g \in L^r(\mathbb{R}^n)$ *are non-negative functions with* $1 < q < r < \infty$*. Furthermore, the inequality*

$$\left(\fint_{B_R} f^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le b_1 \left(\fint_{B_{\mu R}} f \,\mathrm{d}x\right) + b_2 \left(\fint_{B_{\mu R}} g^q \,\mathrm{d}x\right)^{\frac{1}{q}} \tag{4.2.7}$$

holds for any ball $B_R \subset \mathbb{R}^n$ *with* $R < R_0$*. Then there exists* $\tau = \tau(n, q, b_1, b_2, R_0) > 0$ *such that for any* $\theta \in [0, \min(\{\tau, r - q\})$*, we have* $f \in L^{q+\theta}(\mathbb{R}^n)$ *and the following estimate*

$$\int_{\mathbb{R}^n} f^{q+\theta} \,\mathrm{d}x \le C_1 \left(\int_{\mathbb{R}^n} f^q \,\mathrm{d}x\right)^{\frac{q+\theta}{q}} + C_2 \int_{\mathbb{R}^n} g^{q+\theta} \,\mathrm{d}x \tag{4.2.8}$$

with some $C_1, C_2 > 0$ *depending on* $n, q, b_1, b_2, R_0, \theta$*.*

We finish this part with a surprising result about weak reverse Hölder inequalities (see [IN85], Theorem 2 or [BKL95], Lemma 1.4). Given a function satisfying a weak reverse Hölder inequality without any inhomogeneous terms, it is possible to obtain arbitrary "improvement" in one direction in the following sense: Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and $f \in L^p_{loc}(\Omega)$ with $p > 0$ satisfies

$$\left(\fint_{B} f^p \,\mathrm{d}x\right)^{\frac{1}{p}} \le C \left(\fint_{2B} f^s \,\mathrm{d}x\right)^{\frac{1}{s}}, \quad \text{for any } B \subset 2B \subset\subset \Omega \tag{4.2.9}$$

with some $0 < s < p$. Then for any $0 < r < s$ and any $\sigma > 1$, we have

$$\left(\fint_{B} f^p \,\mathrm{d}x\right)^{\frac{1}{p}} \le C(r,\sigma) \left(\fint_{\sigma B} f^r \,\mathrm{d}x\right)^{\frac{1}{r}}, \quad \text{for any } B \subset \sigma B \subset\subset \Omega. \tag{4.2.10}$$

4.2.2 Auxiliary lemmas about Riemann-Stieltjes integrals

Before showing Theorem 4.2.3, we present two auxiliary lemmas related to Riemann-Stieltjes integrals that will be used in the proof. See [Rud76], Chapter 6 for the definition of and a discussion about the Riemann-Stieltjes integral, and in the following we will simply refer to them as Stieltjes integrals.

The first one gives the connection between normal Lebesgue integrals and Stieltjes integrals of a certain type, that is, an L^q-Lebesgue integral can be written as some Stieltjes integral.

Lemma 4.2.4. *Suppose that* f *is non-negative and in* $L^q(\mathbb{R}^n)$ *with some* $q > 1$*, and set*

$$E(f,t) := \{x \in \mathbb{R}^n : f(x) > t\}, \quad h(t) := -\int_{E(f,t)} f \,\mathrm{d}x.$$

Then the L^q*-integral of* f *on its level set* $E(f,t)$ *for any* $t \ge 0$ *can be written as a corresponding Stieltjes integral:*

$$\int_{E(f,t)} f^q \,\mathrm{d}x = -\int_{(t,\infty)} s^{q-1} \,\mathrm{d}h(s). \tag{4.2.11}$$

Proof. We denote $E(f,t) \setminus E(f,s)$ by $E(f,t,s)$ for any $0 \leq t \leq s < \infty$. Fix a $t \geq 0$ and take a finite $T > t$. Given a partition $P = \{t = t_0 < t_1 < \cdots < t_k = T\}$ of the interval $[t,T]$, define the upper and lower Riemann sum

$$\underline{S}(P,h,q) := -\sum_{i=0}^{k-1} t_i^{q-1}(h(t_{i+1}) - h(t_i)) = \sum_{i=0}^{k-1} t_i^{q-1} \int_{E(f,t_i,t_{i+1})} f \,\mathrm{d}x,$$
$$\overline{S}(P,h,q) := -\sum_{i=0}^{k-1} t_{i+1}^{q-1}(h(t_{i+1}) - h(t_i)) = \sum_{i=0}^{k-1} t_{i+1}^{q-1} \int_{E(f,t_i,t_{i+1})} f \,\mathrm{d}x.$$

It is easy to see that

$$\underline{S}(P,h,q) \leq \sum_{i=0}^{k-1} \int_{E(f,t_i,t_{i+1})} f^q \,\mathrm{d}x = \int_{E(f,t,T)} f^q \,\mathrm{d}x = \sum_{i=0}^{k-1} \int_{E(f,t_i,t_{i+1})} f^q \,\mathrm{d}x \leq \overline{S}(P,h,q),$$

which implies

$$\begin{aligned} 0 \leq \overline{S}(P,h,q) - \underline{S}(P,h,q) &= \sum_{i=0}^{k-1} (t_{i+1}^{q-1} - t_i^{q-1}) \int_{E(f,t_i,t_{i+1})} f \,\mathrm{d}x \\ &\leq \sum_{i=0}^{k-1} C(t_{i+1} - t_i) \int_{E(f,t_i,t_{i+1})} f \,\mathrm{d}x \leq C\mathrm{norm}(P) \int_{E(f,t,T)} f \,\mathrm{d}x, \end{aligned}$$

where $C = C(q,t,T) > 0$ and $\mathrm{norm}(P) := \max\{|t_{i+1} - t_i| : i = 0, \ldots, k-1\}$. When $\mathrm{norm}(P) \to 0^+$, we have that both $\overline{S}(P,h,q)$ and $\underline{S}(P,h,q)$ converge to $\int_{E(f,t,T)} f^q \,\mathrm{d}x$, and thus the corresponding Stieltjes integral exists with

$$\int_{E(f,t,T)} f^q \,\mathrm{d}x = -\int_{(t,T]} s^{q-1} \,\mathrm{d}h(s). \tag{4.2.12}$$

Since we have $\int_{E(f,T)} f^q \,\mathrm{d}x \to \infty$ as $T \to \infty$, take $T \to \infty$ in (4.2.12) and then the equality (4.2.11) follows. □

The second one is a higher-integrability result for certain Stieltjes integrals, which will be used in Subsection 4.2.3 to obtain the claimed higher integrability.

Lemma 4.2.5. *Suppose that $q \in (1,\infty)$, $a_1 > 1, a_2 > 0$, and $h, H \colon [1,\infty) \to [0,\infty)$ are non-increasing and right-continuous functions satisfying*

(a) $\lim_{t\to\infty} h(t) = \lim_{t\to\infty} H(t) = 0$*;*

(b) $-\int_{(t,\infty)} s^{q-1} \,\mathrm{d}h(s) \leq a_1 t^{q-1} h(t) + a_2(-\int_{(t,\infty)} s^{q-1} \,\mathrm{d}H(s))$ *for any $t \in [1,\infty)$, where the integrals are finite in the Stieltjes sense.*

Then for any $\theta \in [0, \frac{a_1 q-1}{a_1-1})$, we have the following estimate

$$-\int_{(1,\infty)} t^{q+\theta-1} \,\mathrm{d}h(t) \leq \frac{q-1}{q-1-(a_1-1)\theta} \left(-\int_{(1,\infty)} t^{q-1} \,\mathrm{d}h(t) \right)$$

$$
\begin{aligned}
&+\frac{a_2(q+\theta-1)}{q-1-(a_1-1)\theta}\left(-\int_{(1,\infty)} t^{q+\theta-1}\,\mathrm{d}H(t)\right) \qquad (4.2.13)\\
&-\frac{a_2(q-1)}{q-1-(a_1-1)\theta}\left(-\int_{(1,\infty)} t^{q-1}\,\mathrm{d}H(t)\right)
\end{aligned}
$$

where the second integral on the right-hand side may take the value ∞.

Proof. For any $r > 1$, define

$$I(r) = -\int_{(1,\infty)} t^{r-1}\,\mathrm{d}h(t), \quad J(r) = -\int_{(1,\infty)} t^{r-1}\,\mathrm{d}H(t),$$

where the two integrals may take the value ∞.

We first show this lemma for truncated h's, and the general case can be obtained by taking the limit.

Step 1. Assume that $h(t) = 0$ on $[j,\infty)$ for some $j > 1$. Take $q > 1$ as in the assumption, and another exponent $p \geq q$. Notice that in this case

$$I(p) = -\int_{(1,j]} t^{p-1}\,\mathrm{d}h(t) < \infty$$

for any p, then we have

$$
\begin{aligned}
I(p) - I(q) &= -\int_{(1,\infty)} (t^{p-1} - t^{q-1})\,\mathrm{d}h(t) = -\int_{(1,\infty)} t^{q-1}(t^{p-q}-1)\,\mathrm{d}h(t)\\
&= -\int_{(1,\infty)} (t^{p-q}-1)\,\mathrm{d}\left(-\int_{(t,\infty)} s^{q-1}\,\mathrm{d}h(s)\right) \qquad (4.2.14)\\
&= \int_{(1,\infty)} (p-q)t^{p-q-1}\left(-\int_{(t,\infty)} s^{q-1}\,\mathrm{d}h(s)\right)\mathrm{d}t,
\end{aligned}
$$

where the last line is obtained by integration by parts with the fact

$$-\int_{(t,\infty)} s^{q-1}\,\mathrm{d}h(s) = 0 \quad \text{for any } t \geq j.$$

Assumption (b) implies

$$
\begin{aligned}
I(p) - I(q) &\leq \int_{(1,\infty)} (p-q)t^{p-q-1}\left(a_1 t^{q-1}h(t) + a_2\left(-\int_{(t,\infty)} s^{q-1}\,\mathrm{d}H(s)\right)\right)\mathrm{d}t \qquad (4.2.15)\\
&= a_1(p-q)\int_{(1,\infty)} t^{p-2}h(t)\,\mathrm{d}t + a_2(p-q)\int_{(1,\infty)} t^{p-q-1}\left(-\int_{(t,\infty)} s^{q-1}\,\mathrm{d}H(s)\right)\mathrm{d}t\\
&= a_1\frac{p-q}{p-1}\int_{(1,\infty)} h(t)\,\mathrm{d}t^{p-1} + a_2\int_{(1,j]}\left(-\int_{(t,\infty)} s^{q-1}\,\mathrm{d}H(s)\right)\mathrm{d}t^{p-q}.
\end{aligned}
$$

Without loss of generality, we can assume that

$$J(p) = \int_{(1,\infty)} t^{p-1}\, \mathrm{d}H(t) < \infty,$$

since otherwise the inequality (4.2.13) holds trivially with $\theta = p - q$.

Step 2. Now we consider the general case. For $1 \le j < \sigma < \infty$, the following is easy to see

$$h(j)j^{q-1} - h(\sigma)j^{q-1} = -j^{q-1}\int_{(j,\sigma]} \mathrm{d}h(s) \le -\int_{(j,\sigma]} s^{q-1}\,\mathrm{d}h(s),$$

which becomes

$$h(j)j^{q-1} \le \int_{(j,\infty)} s^{q-1}\,\mathrm{d}h(s) \tag{4.2.16}$$

as $\sigma \to \infty$. Set

$$h_j(t) := \begin{cases} h(t), & t \in [1,j) \\ 0, & t \in [j,\infty). \end{cases}$$

Then the function h_j is non-increasing and right-continuous. For any $t \in [1,j)$, we have

$$\begin{aligned} -\int_{(t,\infty)} s^{q-1}\,\mathrm{d}h_j(s) &= -\int_{(t,j]} s^{q-1}\,\mathrm{d}h(s) + h(j)j^{q-1} \overset{(4.2.16)}{\le} -\int_{(t,\infty)} s^{q-1}\,\mathrm{d}h(s) \\ &\overset{(b)}{\le} a_1 t^{q-1}h(t) + a_2\left(-\int_{(t,\infty)} s^{q-1}\,\mathrm{d}H(s)\right) \qquad (4.2.17) \\ &= a_1 t^{q-1}h_j(t) + a_2\left(-\int_{(t,\infty)} s^{q-1}\,\mathrm{d}H(s)\right), \end{aligned}$$

which is exactly the condition (b) for h_j and H. For $t \ge j$, the condition holds trivially. Notice that the first line in (4.2.17) holds for any $r > 1$, and thus we have

$$-\int_{(t,j]} s^{r-1}\,\mathrm{d}h(s) \le -\int_{(t,j]} s^{r-1}\,\mathrm{d}h(s) + h(j)j^{r-1} = -\int_{(t,\infty)} s^{r-1}\,\mathrm{d}h_j(s), \tag{4.2.18}$$

$$\text{and } -\int_{(t,\infty)} s^{r-1}\,\mathrm{d}h_j(s) = -\int_{(t,j]} s^{r-1}\,\mathrm{d}h(s) + h(j)j^{r-1} \le -\int_{(t,\infty)} s^{r-1}\,\mathrm{d}h(s). \tag{4.2.19}$$

Now set $I_j(r) := \int_{(1,\infty)} t^{r-1}\,\mathrm{d}h_j(t)$, and apply the conclusion in 4.2.2 to h_j and H to obtain

$$\begin{aligned} -\int_{(t,j]} s^{p-1}\,\mathrm{d}h(s) &\overset{(4.2.18)}{\le} I_j(p) &(4.2.20)\\ &\le \frac{1}{p-1-a_1(p-q)}((q-1)I_j(q) - a_2(q-1)J(q) + a_2(p-1)J(p)) &(4.2.21)\\ &\overset{(4.2.19)}{\le} \frac{1}{p-1-a_1(p-q)}((q-1)I(q) - a_2(q-1)J(q) + a_2(p-1)J(p)). &(4.2.22) \end{aligned}$$

The desired estimate (4.2.13) then follows as $j \to \infty$ with $\theta = p - q$. □

4.2.3 The proof of a generalised version

Now we are in the place to prove the self-improving result Theorem 4.2.3, which is a modification of Lemma 14.3.1 in [IM01]. Lemma 4.2.5 is where we actually obtain a higher integrability, so the main purpose of the following is to get an inequality like (b) from the assumed weak reverse Hölder inequality (4.2.7).

Our proof follows the main idea of Gehring [Geh73]. The strategy is to estimate the L^q-integral of a normalised version of f on its level sets with the Calderón-Zygmund decomposition and the assumed weak reverse Hölder inequality (4.2.7). The estimate obtained can be transferred to Stieltjes integrals by Lemma 4.2.4, and then with Lemma 4.2.5 we will have a higher integrability as claimed.

Proof of Theorem 4.2.3. Without loss of generality, we can assume $f \not\equiv 0$, and set

$$F = \frac{f}{\|f\|_{L^q(\mathbb{R}^n)}}, \quad G = \frac{g}{\|f\|_{L^q(\mathbb{R}^n)}}.$$

For any given $t \geq 1$, set $s = \lambda t$ with $\lambda > 1$ to be determined.

By applying the Calderón-Zygmund decomposition ([Ste70], §I.3, Theorem 4), we obtain a collection of disjoint n-cubes $\{Q_j\}_{j=1}^{\infty}$ such that

$$s^q < \fint_{Q_j} F^q \,\mathrm{d}x \leq 2^n s^q \text{ for all } j, \tag{4.2.23}$$

$$\text{and} \quad F(x) \leq s \text{ a.e. on } \mathbb{R}^n \setminus \cup_j Q_j. \tag{4.2.24}$$

Define $E(F,s) = \{x \in \mathbb{R}^n \colon F(x) > s\}$, then we have $E(F,s) \subset \cup_j Q_j$ up to an $\mathscr{L}^n$-null set, and thus

$$\int_{E(F,s)} F^q \,\mathrm{d}x \leq \sum_j \int_{Q_j} F^q \,\mathrm{d}x \overset{(4.2.23)}{\leq} 2^n s^q \sum_j |Q_j|. \tag{4.2.25}$$

Now the aim is to estimate the integral of F^q on the level set $E(F,s)$, for which we need to control $\sum_j |Q_j|$.

Considering (4.2.23) and the facts $\int_{Q_j} F^q \,\mathrm{d}x \leq 1$ and $t \geq 1$, we know that

$$|Q_j| < \frac{\int_{Q_j} F^q \,\mathrm{d}x}{\lambda^q t^q} \leq \lambda^{-q},$$

and then $d_j := \operatorname{diam}(Q_j) < \sqrt{n}\lambda^{-\frac{q}{n}}$. Take $\lambda \geq (\frac{\sqrt{n}}{R_0})^{\frac{n}{q}}$, with which we have $d_j < R_0$ and then the following holds for any $x \in Q_j$:

$$\begin{aligned} s &\leq \left(\fint_{Q_j} F^q \,\mathrm{d}y\right)^{\frac{1}{q}} \leq \left(\frac{|B(x,d_j)|}{|Q_j|} \fint_{B(x,d_j)} F^q \,\mathrm{d}y\right)^{\frac{1}{q}} \\ &\leq (\omega_n n^{\frac{n}{2}})^{\frac{1}{q}} \left(b_1 \left(\fint_{B(x,\mu d_j)} F \,\mathrm{d}y\right) + b_2 \left(\fint_{B(x,\mu d_j)} G^q \,\mathrm{d}y\right)^{\frac{1}{q}} \right). \end{aligned} \tag{4.2.26}$$

The first integral on the right hand side is controlled by

$$\fint_{B(x,\mu d_j)} F\,\mathrm{d}y \leq \frac{1}{|B(x,\mu d_j)|}\int_{B(x,\mu d_j)\cap E(F,t)} F\,\mathrm{d}y + t. \tag{4.2.27}$$

To estimate the second integral in (4.2.26), notice that

$$z^{\frac{1}{q}} \leq t + \frac{z}{t^{q-1}} \text{ for any } z > 0, t > 0,$$

which implies

$$\begin{aligned}\left(\fint_{B(x,\mu d_j)} G^q\,\mathrm{d}y\right)^{\frac{1}{q}} &\leq t + \frac{1}{t^{q-1}}\fint_{B(x,\mu d_j)} G^q\,\mathrm{d}y \\ &\leq 2t + \frac{1}{t^{q-1}|B(x,\mu d_j)|}\int_{B(x,\mu d_j)\cap E(G,t)} G^q\,\mathrm{d}y.\end{aligned} \tag{4.2.28}$$

Thus, the previous inequalities (4.2.26)-(4.2.28) give

$$\begin{aligned} s \leq (\omega_n n^{\frac{n}{2}})^{\frac{1}{q}}\Bigg(b_1\left(\frac{1}{|B(x,\mu d_j)|}\int_{B(x,\mu d_j)\cap E(F,t)} F\,\mathrm{d}y + t\right) \\ + b_2\left(2t + \frac{1}{t^{q-1}|B(x,\mu d_j)|}\int_{B(x,\mu d_j)\cap E(G,t)} G^q\,\mathrm{d}y\right)\Bigg),\end{aligned}$$

which is indeed

$$\begin{aligned}\lambda t|B(x,\mu d_j)| \leq (\omega_n n^{\frac{n}{2}})^{\frac{1}{q}}\Bigg((b_1+2b_2)t|B(x,\mu d_j)| \\ + b_1\int_{B(x,\mu d_j)\cap E(F,t)} F\,\mathrm{d}y + \frac{b_2}{t^{q-1}}\int_{B(x,\mu d_j)\cap E(G,t)} G^q\,\mathrm{d}y\Bigg).\end{aligned} \tag{4.2.29}$$

Choose $\lambda > (\omega_n n^{\frac{n}{2}})^{\frac{1}{q}}(b_1+2b_2) =: c$, then we obtain from (4.2.29)

$$|B(x,\mu d_j)| \leq \frac{(\omega_n n^{\frac{n}{2}})^{\frac{1}{q}}}{(\lambda-c)t}\left(b_1\int_{B(x,\mu d_j)\cap E(F,t)} F\,\mathrm{d}y + \frac{b_2}{t^{q-1}}\int_{B(x,\mu d_j)\cap E(G,t)} G^q\,\mathrm{d}y\right).$$

The collection of balls $\{B(x,\mu d_j)\colon x \in Q_j, j \in \mathbb{N}\}$ is an open cover of $\cup_j Q_j$. The Vitali covering lemma ([EG15], Theorem 1.24) indicates the existence of a countable subset $\{B_i\}_{i=1}^{\infty}$ of $\{B(x,\mu d_j)\}$ such that

$$B_i \cap B_k = \emptyset,\ i \neq k, \quad \text{and} \quad \cup_j Q_j \subset \cup_i 5B_i,$$

and thus there holds

$$\int_{E(F,s)} F^q\,\mathrm{d}x \leq 2^n s^q \sum_j |Q_j| \leq 2^n s^q \sum_i 5^n|B_i| \tag{4.2.30}$$

$$\begin{aligned}
&\le 10^n(\omega_n n^{\frac{n}{2}})^{\frac{1}{q}} \frac{\lambda^q t^q}{(\lambda - c)t} \left(b_1 \int_{E(F,t)} F \,\mathrm{d}x + \frac{b_2}{t^{q-1}} \int_{E(G,t)} G^q \,\mathrm{d}x \right) \\
&= 10^n(\omega_n n^{\frac{n}{2}})^{\frac{1}{q}} \frac{\lambda^q}{\lambda - c} \left(b_1 t^{q-1} \int_{E(F,t)} F \,\mathrm{d}x + b_2 \int_{E(G,t)} G^q \,\mathrm{d}x \right).
\end{aligned}$$

Since $\frac{\lambda^q}{\lambda-c}$ is minimised by $\lambda_0 = \frac{cq}{q-1}$ and is increasing on (λ_0, ∞), we take $\lambda = \max\{\frac{cq}{q-1}, (\frac{\sqrt{n}}{R_0})^{\frac{n}{q}}\}$. Then the L^q-integral of F on its level set $E(F,t)$ for any $t \ge 1$ is estimated by

$$\begin{aligned}
\int_{E(F,t)} F^q \,\mathrm{d}x &= \int_{E(F,t)\setminus E(F,s)} F^q \,\mathrm{d}x + \int_{E(F,s)} F^q \,\mathrm{d}x \\
&\le s^{q-1} \int_{E(F,t)} F \,\mathrm{d}x + \int_{E(F,s)} F^q \,\mathrm{d}x \qquad (4.2.31) \\
&\overset{(4.2.30)}{\le} a_1 t^{q-1} \int_{E(F,t)} F \,\mathrm{d}x + a_2 \int_{E(G,t)} G^q \,\mathrm{d}x,
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= 10^n(\omega_n n^{\frac{n}{2}})^{\frac{1}{q}} b_1 \frac{\lambda^q}{\lambda - c} + \lambda^{q-1}, \\
a_2 &= 10^n(\omega_n n^{\frac{n}{2}})^{\frac{1}{q}} b_2 \frac{\lambda^q}{\lambda - c}.
\end{aligned}$$

Set $h(t) := \int_{E(F,t)} F \,\mathrm{d}x$ and $H(t) := \int_{E(G,t)} G \,\mathrm{d}x$, and it is easy to see that h and H are non-increasing and right-continuous. Then the inequality (4.2.31) is actually, by Lemma 4.2.4,

$$-\int_{(t,\infty)} s^{q-1} \,\mathrm{d}h(s) \le a_1 t^{q-1} h(t) + a_2 \left(-\int_{(t,\infty)} s^{q-1} \,\mathrm{d}H(s) \right). \qquad (4.2.32)$$

Applying Lemma 4.2.5, we obtain the following estimate for any $\theta \in [0, \frac{a_1 q-1}{a_1-1})$

$$\begin{aligned}
-\int_{(1,\infty)} t^{q+\theta-1} \,\mathrm{d}h(t) \le{} & \frac{q-1}{q-1-(a_1-1)\theta} \left(-\int_{(1,\infty)} t^{q-1} \,\mathrm{d}h(t) \right) \qquad (4.2.33) \\
& + \frac{a_2(q+\theta-1)}{q-1-(a_1-1)\theta} \left(-\int_{(1,\infty)} t^{q+\theta-1} \,\mathrm{d}H(t) \right) \\
& - \frac{a_2(q-1)}{q-1-(a_1-1)\theta} \left(-\int_{(1,\infty)} t^{q-1} \,\mathrm{d}H(t) \right).
\end{aligned}$$

Set $\tau = \frac{a_1 q-1}{a_1-1}$, and the above inequality implies, whenever $\theta \le \min\{\tau, r-1\}$,

$$\int_{E(F,1)} F^{q+\theta} \,\mathrm{d}x \le C_1 \int_{E(F,1)} F^q \,\mathrm{d}x + C_2 \int_{E(G,1)} G^{q+\theta} \,\mathrm{d}x - C_3 \int_{E(G,1)} G^q \,\mathrm{d}x, \qquad (4.2.34)$$

where $C_i, i = 1,2,3$ are the three constants in the former inequality (4.2.33), respectively. It is clear that $C_1 \ge 1$, and then we have, considering $\int_{\{F<1\}} F^{q+\theta} \,\mathrm{d}x \le \int_{\{F<1\}} F^q \,\mathrm{d}x$ and

$\int_{\mathbb{R}^n} F^q \,\mathrm{d}x = 1$,

$$\begin{aligned}\int_{\mathbb{R}^n} F^{q+\theta} \,\mathrm{d}x &\le C_1 \int_{\mathbb{R}^n} F^q \,\mathrm{d}x + C_2 \int_{\mathbb{R}^n} G^{q+\theta} \,\mathrm{d}x \\ &= C_1 \left(\int_{\mathbb{R}^n} F^q \,\mathrm{d}x \right)^{\frac{q+\theta}{q}} + C_2 \int_{\mathbb{R}^n} G^{q+\theta} \,\mathrm{d}x.\end{aligned}$$

Going back to f and g, we finally obtain the desired inequality (4.2.8). □

Remark 4.2.6. If we remove the restriction $R < R_0$ on the radii of the balls for which (4.2.7) holds, the t in the proof could be taken from the range $(0, \infty)$. Then the inequalities in Lemma 4.2.5 hold for any $t \ge 0$, which will give the estimate

$$-\int_{(0,\infty)} t^{q+\theta_2-1} \,\mathrm{d}h(t) \le \frac{a_2(q+\theta_2-1)}{q-1-(a_1-1)\theta_2} \left(-\int_{(0,\infty)} t^{q+\theta_2-1} \,\mathrm{d}H(t) \right)$$

with $\theta_2 \in [0, \frac{a_1 q - 1}{a_1 - 1})$, and the final estimate we obtain would be

$$\int_{\mathbb{R}^n} f^{q+\theta} \,\mathrm{d}x \le C_2 \int_{\mathbb{R}^n} g^{q+\theta} \,\mathrm{d}x,$$

where the C_2 is the same as defined in the above proof. See [IM01], Theorem 14.3.1.

4.3 Quasiminimizers in the 1-dimensional case

Compared to the high-dimensional case ($n \ge 2$), the study of quasiminimizers in the 1-dimensional case is much simpler. With more tools in 1-D analysis, it is possible to obtain a clearer description of and neat results about quasiminimizers. We revisit the discussion about quasiminimizers in the 1-D case given in [MS07], which provides some inspiration for the study in higher dimensions.

Let $(a, b) \subset \mathbb{R}$ be an open interval, where a, b are not necessarily finite. Suppose $p \ge 1$ and $u \in W^{1,p}_{loc}((a,b), \mathbb{R}^N)$. By definition, the map u is said to be a (p, Q)-minimizer with some $Q \ge 1$, if it satisfies

$$\int_c^d |u'|^p \,\mathrm{d}x \le Q \int_c^d |v'|^p \,\mathrm{d}x \tag{4.3.1}$$

whenever $[c, d] \subset (a, b)$ and $u - v \subset W^{1,p}_0((c,d), \mathbb{R}^N)$. Now we show that p-harmonic maps are exactly all affine maps on (a, b).

Proposition 4.3.1. *Suppose that $(a, b) \subset \mathbb{R}$ is an open interval and $p > 1$. Then the map $u \in W^{1,p}_{loc}((a,b), \mathbb{R}^N)$ is p-harmonic if and only if it is affine.*

Proof. First assume that $u \in W^{1,p}_{loc}((a,b), \mathbb{R}^N)$ is affine. Fix a subinterval $[c, d] \subset (a, b)$, and take $w \in W^{1,p}_0((c,d), \mathbb{R}^N)$. Then it is easy to see that the map

$$v(x) := \int_c^x w'(y) \,\mathrm{d}y + u(c)$$

is absolutely continuous on $[c,d]$, and we have

$$v(x) = w(x), \quad \text{a.e. } x \in (c,d) \quad \text{and} \quad v'(x) = w'(x), \quad \text{a.e. } x \in (c,d).$$

Extending v continuously to c and d, we obtain

$$\begin{aligned}\int_c^d |u'|^p \,\mathrm{d}x = \frac{|u(d)-u(c)|^p}{|d-c|^{p-1}} &= \frac{|v(d)-v(c)|^p}{|d-c|^{p-1}} \\ &= |d-c|^{1-p}\left|\int_c^d u' \,\mathrm{d}x\right|^p \le \int_c^d |u'|^p \,\mathrm{d}x = \int_c^d |w'|^p \,\mathrm{d}x,\end{aligned}$$

which gives the minimality of u.

Conversely, assume that $u \in W^{1,p}_{loc}((a,b),\mathbb{R}^N)$ is p-harmonic. Take a sequence of intervals $\{(c_k,d_k)\}_{k\in\mathbb{N}}$ such that $c_k \searrow a$, $d_k \nearrow b$ and c_k, d_k are Lebesgue points of u. Then by the above argument, we know that the affine map $v_k(x) := u(c_k) + \frac{u(d_k)-u(c_k)}{d_k-c_k}(x-c_k)$ is p-harmonic on (c_k,d_k). The strong convexity of the functional $\mathcal{D}_p$ and the fact $u - v_k \in W^{1,p}_0((c_k,d_k),\mathbb{R}^N)$ imply $u = v_k$ $\mathscr{L}^1$-a.e. on (c_k,d_k). Then we know that u is affine on (a,b). ☐

Remark 4.3.2. Notice that the "if" part of the above proposition applies to the case $p = 1$, while the "only if" part does not as $|\cdot|$ is not strongly convex. Indeed, when $p = 1$ the minimizer of a given Dirichlet class on some $[c,d] \subset (a,b)$ is not unique. See Proposition 4.3.4, (b).

Proposition 4.3.1 together with Remark 4.3.2 gives an equivalent characterisation of quasiminimality in 1-D:

Corollary 4.3.3. *Suppose that $(a,b) \subset \mathbb{R}$ is an open interval and $p \ge 1$. The map $u \in W^{1,p}_{loc}((a,b),\mathbb{R}^N)$ is a (p,Q)-minimizer if and only if it satisfies*

$$\int_c^d |u'|^p \,\mathrm{d}x \le Q\frac{|u(d)-u(c)|^p}{|d-c|^{p-1}}, \qquad \textit{for all } [c,d] \subset (a,b). \tag{4.3.2}$$

As any map in $W^{1,p}_{loc}((a,b),\mathbb{R}^N)$ with $p \ge 1$ has a representative that is locally absolutely continuous, we will only consider locally absolutely continuous maps in the following.

Two basic observations about scalar quasiminimizers are as follows, and the first one can also be found in [GG82].

Proposition 4.3.4. *In the case of scalar maps ($N = 1$), we have the following:*

(a) *When $p > 1$, any (p,Q)-minimizer on a certain interval (a,b) is either constant or strictly monotone.*

(b) *When $p = 1$, any $(1,Q)$-minimizer on (a,b) is monotone, and thus a minimizer.*

Proof. First, suppose that $u \in W^{1,p}_{loc}(a,b)$ is a (p,Q)-minimizer for $p \ge 1$ but not monotone. Since u is continuous, there exist $c_1, c_2, c_3 \in (a,b)$ with $c_1 < c_2 < c_3$ such that $u(c_1) = u(c_3) \ne u(c_2)$. Thus,

$$\int_{c_1}^{c_3} |u'|^p \,\mathrm{d}x \ge (c_3-c_1)^{1-p}\left(\int_{c_1}^{c_3} |u'| \,\mathrm{d}x\right)^p \ge (c_3-c_1)^{1-p}\left|\int_{c_1}^{c_2} u' \,\mathrm{d}x\right|^p$$

$$= \frac{|u(c_2)-u(c_1)|}{|c_3-c_1|^{p-1}} > 0 = Q\frac{|u(c_3)-u(c_1)|^p}{|c_3-c_1|^{p-1}},$$

which contradicts (4.3.2). So u must be monotone in both cases.

When $p > 1$ and u is neither strictly monotone nor constant, it is possible to find $[c', d] \subset (a,b)$ and $e \in (a,b) \setminus [c',d]$ such that $c' < d$ and $u(c') = u(d) \neq u(e)$. Without loss of generality, we can assume $e < c'$. Set $c = \inf\{x \le d\colon\ u(x) = u(d)\}$, and it is easy to see that $c \le c' < d$. By monotonicity we have $e < c$ and

$$\begin{aligned}\int_e^c |u'|^p \,\mathrm{d}x = \int_e^d |u'|^p \,\mathrm{d}x &\overset{(4.3.2)}{\le} Q\frac{|u(d)-u(e)|^p}{|d-e|^{p-1}} = Q\frac{|u(c)-u(e)|^p}{|d-e|^{p-1}} \\ &= Q|d-e|^{1-p}\left|\int_e^c u' \,\mathrm{d}x\right|^p \le Q\left|\frac{c-e}{d-e}\right|^{p-1}\int_e^c |u'|^p \,\mathrm{d}x.\end{aligned}$$

As $\int_e^c |u'|^p \,\mathrm{d}x \neq 0$, the above inequality implies $Q|\frac{c-e}{d-e}|^{p-1} \ge 1$, which fails when $e \to c^-$. Therefore, u must be either constant or strictly monotone when $p > 1$.

On the other hand, u is monotone when $p = 1$, and thus

$$\int_c^d |u'| \,\mathrm{d}x = \left|\int_c^d u' \,\mathrm{d}x\right| = |u(d) - u(c)|.$$

Since the affine function $v(x) = u(c) + \frac{u(d)-u(c)}{d-c}(x-c)$ is a minimizer of $\mathcal{D}_1$ on (c,d), we know that u is a 1-minimizer (i.e., minimizer) on (c,d), and thus on any subinterval of (a,b). ☐

Notice that the argument for (a) also applies to the vectorial case ($N > 1$) with a slight modification, and the statement is that each (p,Q)-minimizer on (a,b) is either constant or injective.

Another result is that a (p,Q)-minimizer for $p > 1$ cannot blow-up on any finite interval. This can also be observed from (4.3.2), where the left-hand side would increase much faster than the right-hand side if b is finite and $u(d) \to \infty$ as $d \to b^-$.

Lemma 4.3.5 ([MS07], Lemma 1)**.** *Let $p > 1$ and $u \in W^{1,p}_{loc}((a,b),\mathbb{R}^N)$ be a (p,Q)-minimizer on (a,b). If $b < \infty$, the map u can be continuously extended to b and (4.3.2) holds for any $[c,d] \subset (a,b]$.*

Proof. Suppose that u is non-constant. For convenience, we assume $b = 1$, $a < 0$ and $u(0) = 0$.

For $0 \le c < t < 1$, by definition there holds

$$\begin{aligned}\int_c^t |u'| \,\mathrm{d}x &\le (t-c)^{\frac{p-1}{p}}\left(\int_c^t |u'|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \le (t-c)^{\frac{p-1}{p}}\left(\int_0^t |u'|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\overset{(4.3.2)}{\le} Q^{\frac{1}{p}}\left(\frac{t-c}{t}\right)^{\frac{p-1}{p}}\left|\int_0^t u' \,\mathrm{d}x\right| \le Q^{\frac{1}{p}}\left(1-\frac{c}{t}\right)^{\frac{p-1}{p}}\int_0^t |u'| \,\mathrm{d}x.\end{aligned}$$

Take $c = 1 - (2^p Q)^{\frac{1}{1-p}}$, with which we have

$$\left(1 - \frac{c}{t}\right)^{\frac{p-1}{p}} < (1-c)^{\frac{p-1}{p}} = \frac{1}{2Q^{\frac{1}{p}}},$$

and the above estimate of $\int_c^t |u'| \,\mathrm{d}x$ becomes

$$\int_c^t |u'| \,\mathrm{d}x \le \frac{1}{2} \int_0^t |u'| \,\mathrm{d}x.$$

This further implies

$$\int_0^t |u'| \,\mathrm{d}x = \int_0^c |u'| \,\mathrm{d}x + \int_c^t |u'| \,\mathrm{d}x \le \int_0^c |u'| \,\mathrm{d}x + \frac{1}{2} \int_0^t |u'| \,\mathrm{d}x,$$

and thus $$\int_0^t |u'| \,\mathrm{d}x \le 2 \int_0^c |u'| \,\mathrm{d}x.$$

Since the integral $\int_0^t |u'| \,\mathrm{d}x$ is increasing in t, it has a limit as $t \to 1^-$. Then the limit

$$\lim_{t \to 1^-} u(t) = \lim_{t \to 1^-} \int_0^t u' \,\mathrm{d}x$$

exists and we can extend u continuously to 1.

By the continuity of u, the inequality (4.3.2) remains true as $d \to 1^-$. This implies that $u \in W^{1,p}((c,b), \mathbb{R}^N)$ for any $c > a$, and that (4.3.2) holds true for any $[c,d] \subset (a,b]$. □

A similar argument also applies to the left end point a if it is finite. This result shows that a natural domain of a quasiminimizer in 1-D is a closed interval. In the following, by $u \in W^{1,p}([a,b], \mathbb{R}^N)$ we mean that $u \in W^{1,p}((a,b), \mathbb{R}^N)$ and is defined on the closed interval $[a,b]$.

The discussion about the high-dimensional case obviously applies to the one of 1-D, and from Subsection 3.2.1 and Section 4.4 we know that quasiminimizers satisfy certain weak reverse Hölder inequalities. The special characterisation of quasiminimizers (4.3.2) in 1-D further leads us to a stronger result, a reverse Hölder inequality.

Now we assume that $u \in W^{1,p}([a,b], \mathbb{R}^N)$ and is absolutely continuous. If u is a (p,Q)-minimizer on $[a,b]$, for any $[c,d] \subset [a,b]$ we have

$$\int_c^d |u'|^p \,\mathrm{d}x \le Q \frac{|u(d) - u(c)|^p}{|d-c|^{p-1}} \le Q(d-c)^{1-p} \left(\int_c^d |u'| \,\mathrm{d}x \right)^p,$$

which is actually the following reverse Hölder inequality

$$⨍_c^d |u'|^p \,\mathrm{d}x \le Q \left(⨍_c^d |u'| \,\mathrm{d}x \right)^p. \tag{4.3.3}$$

If u is a monotone scalar function, from (4.3.3) we can conversely get (4.3.2) and conclude that u is a (p,Q)-minimizer on $[a,b]$. Therefore, a monotone scalar function u defined on $[a,b]$ is a (p,Q)-minimizer if and only if it satisfies the reverse Hölder inequality (4.3.3) for any $[c, d] \subset [a, b]$.

The well-known Gehring's lemma [Geh73] (also see Theorem 4.2.1) and (4.3.3) indicate that u' has a higher integrability than p. In particular, the sharp improvement of integrability has been obtained in the 1-dimensional case in [DS90] and [Kor92].

For $p > 1$ and $t > 1$, set $\gamma_{p,t} \colon [p, \infty) \to \mathbb{R}$ to be

$$\gamma_{p,t}(x) = 1 + t\frac{p-x}{x}\left(\frac{x}{x-1}\right)^p. \tag{4.3.4}$$

See Proposition 2.1 in [DS90] for an analysis of the function $\gamma_{p,t}$, and from the discussion there we know that $\gamma_{p,t}$ has exactly one zero (denoted by $q(p, t)$) on $[p, \infty)$ for fixed p, t.

Theorem 4.3.6 (Theorem 2, [Kor92])**.** *Fix an interval $[a, b] \subset \mathbb{R}$ and suppose that $f \in L^p([a, b])$ with $p > 1$ satisfies the reverse Hölder inequality*

$$\fint_c^d |f|^p \,\mathrm{d}x \le Q\left(\fint_c^d |f| \,\mathrm{d}x\right)^p \tag{4.3.5}$$

for any $[c, d] \subset [a, b]$. Then we have $f \in L^s([a, b])$ for any $p \le s < q(p, Q)$ with

$$\fint_c^d |f|^s \,\mathrm{d}x \le Q_s\left(\fint_c^d |f| \,\mathrm{d}x\right)^s, \quad \textit{for any } [c, d] \subset [a, b], \tag{4.3.6}$$

where $q(p, Q)$ is the unique zero of $\gamma_{p,Q}$ defined in (4.3.4) on $[p, \infty)$ and Q_s depends on p, s, Q.

Then the following result of quasiminimality of a higher order is a corollary of the above theorem.

Theorem 4.3.7 ([MS07], Theorem 4)**.** *Suppose that $u \in W^{1,p}([a, b])$ is a (p, Q)-minimizer with $p > 1, Q \ge 1$. Then u is also a (s, Q_s)-minimizer on $[a, b]$ for any $1 \le s < q(p, Q)$, where $q(p, Q)$ is the unique zero of $\gamma_{p,Q}$ defined in (4.3.4) on $[p, \infty)$, and*

$$Q_s = \begin{cases} Q_s(p, s, Q), & p < s < q(p, Q) \\ Q^{\frac{s}{p}}, & 1 \le s \le p. \end{cases} \tag{4.3.7}$$

Proof. As shown above, the function $|u'|$ satisfies the reverse Hölder inequality (4.3.5). By Theorem 4.3.6, we know that $|u'| \in L^s([a, b])$ for $p < s < q(p, Q)$ and

$$\fint_c^d |u'|^s \,\mathrm{d}x \le Q_s\left(\fint_c^d |u'| \,\mathrm{d}x\right)^s$$

for any $[c, d] \subset [a, b]$. The function u is a (p, Q)-minimizer, and thus monotone by Proposition 4.3.4, (a), with which we further have

$$\fint_c^d |u'|^s \,\mathrm{d}x \le Q_s\left|\fint_c^d u' \,\mathrm{d}x\right|^s = Q_s\frac{|u(d) - u(c)|^s}{|d - c|^s}.$$

Corollary 4.3.3 then implies that u is a (s, Q_s)-minimizer on $[a, b]$ with some $Q_s = Q_s(p, s, Q) \ge 1$.

The case $s \in [1, p)$ is simpler. By Hölder's inequality, we have

$$\int_c^d |u'|^s \, \mathrm{d}x \le (d-c)^{\frac{p-s}{p}} \left(\int_c^d |u'|^p \, \mathrm{d}x \right)^{\frac{s}{p}}$$

$$\overset{(4.3.2)}{\le} Q^{\frac{s}{p}} (d-c)^{\frac{p-s}{p}} \frac{|u(d)-u(c)|^s}{|d-c|^{\frac{(p-1)s}{p}}} = Q^{\frac{s}{p}} \frac{|u(d)-u(c)|^s}{|d-c|^{s-1}}.$$

The (s, Q_s)-minimality of u then follows from Corollary 4.3.3. ☐

Remark 4.3.8. (i) In [DS90], a precise value of the constant Q_s in (4.3.6) is given for f with $|f|$ non-increasing:

$$Q_s = \left(\frac{pQ}{\gamma_{p,Q}(s)s} \right)^{\frac{s}{p}}.$$

Correspondingly, if the (p, Q)-minimizer u in Theorem 4.3.7 is such that $|u'|$ is non-increasing, then it is also a (s, Q_s)-minimizer with a specific Q_s as above.

(ii) The upper bound $q(p, Q)$ in Theorem 4.3.7 is sharp. To see this, consider the function

$$u(x) = \frac{p_0}{p_0 - 1} x^{1-\frac{1}{p_0}}, \ x \in [0,1], \ p_0 = q(p, Q).$$

Then $u'(x) = x^{-\frac{1}{p_0}}$ and calculation implies that u is a (p, Q)-minimizer on $[0,1]$ (see Proposition 2.3 in [DS90]). However, it is also easy to see $u' \notin L^{p_0}(0,1)$.

(iii) In the above discussion, we actually showed that p-quasiminimality for $p > 1$ is a self-improving property. To be more precise, p-quasiminimality implies q-quasiminimality for $1 \le q < p$, and any p-quasiminimizer is also a q-quasiminimizer for some $q > p$.

4.4 Quasiminimality of a higher order

In this section, we prove Theorem 4.1.1, a higher-order quasiminimality result. The key step is to obtain weak reverse Hölder inequalities near the boundary (Subsection 4.4.2) of a regular domain, for which the corresponding Caccioppoli-type inequality (Subsection 4.4.2) and Sobolev-Poincaré inequality (Subsection 4.4.1) are required. With these ingredients and the self-improving property Theorem 4.2.3, we can conclude the result in Subsection 4.4.4.

In the following, we consider a bounded open set $\Omega \subset \mathbb{R}^n$, and suppose that $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ is a (p, Q)-minimizer.

4.4.1 A Sobolev-Poincaré inequality

We first show a Sobolev-Poincaré inequality with the p-capacity involved. Notice that the analogue of this inequality with the Lebesgue measure (Theorem 3.16 in [Giu03]) is more well-known and commonly used.

Lemma 4.4.1 ([KK94], Lemma 3.1)**.** *Suppose that $1 < q < \infty$, $B_r = B(x_0, r) \subset \mathbb{R}^n$ is a ball and $v \in W^{1,q}(B_r)$ is q-quasi-continuous. Set $N(v) = \{x \in B_r \colon v(x) = 0\}$, then we* *have*

$$\left(\fint_{B_r} |v|^{\kappa q}\, \mathrm{d}x\right)^{\frac{1}{\kappa q}} \leq C(n,q)\left(\frac{1}{\mathrm{cap}_q(N(v);B_{2r})}\int_{B_r} |\nabla v|^q\, \mathrm{d}x\right)^{\frac{1}{q}}, \tag{4.4.1}$$

where the constant κ *is defined by*

$$\kappa = \begin{cases} \dfrac{n}{n-q}, & 1 < q < n, \\ 2, & q \geq n. \end{cases}$$

Proof. If the average $v_{B_r} = 0$, the inequality above follows easily from the normal Sobolev-Poincaré inequality and the fact (see 2.4.8)

$$\mathrm{cap}_q(N(v);B_{2r}) \leq \mathrm{cap}_q(B_r;B_{2r}) = C(n,q)r^{n-q}.$$

Thus, we can assume that $v_{B_r} = 1$ considering the homogeneity of (4.4.1).

By the extension theorem of $W^{1,p}$ maps ([EG15], §5.4, Theorem 1) and Theorem 2.4.4, there exists a q-quasi-continuous function $w \in W^{1,q}(B_{2r})$ such that $\mathrm{supp}(w) \subset B_{2r}$, $w = v - v_{B_r}$ a.e. in B_r, and

$$\int_{B_{2r}} |\nabla w|^q\, \mathrm{d}x \leq C(n,q)\int_{B_r} |\nabla v|^q\, \mathrm{d}x. \tag{4.4.2}$$

As v is also q-quasi-continuous, we actually have $w = v - v_{B_r}$ q-q.e. in B_r (Theorem 2.4.4). Set $\widetilde{N}(w) := \{x \in \bar{B}_r \colon w(x) = 0\}$. For any $\sigma > 0$, by the definition of quasi-continuity (Definition 2.4.3) there exists an open set $O_\sigma \subset B_{2r}$ such that w is continuous on $B_{2r} \setminus O_\sigma$ and $\mathrm{cap}_q(O_\sigma;B_{2r}) < \sigma$. Thus, the null set of $w+1$ in $\bar{B}_r \setminus O_\sigma$, which is $\widetilde{N}(w+1)\setminus O_\sigma =: E_\sigma$, is compact. Denote $\widetilde{N}(w+1) \cap O_\sigma$ by $\widetilde{E}_\sigma$, and then we have, with the fact $w = -1$ q-q.e. on E_σ and an analogue of Corollary 4.13 in [HKM18],

$$\begin{aligned} \int_{B_{2r}} |\nabla w|^q\, \mathrm{d}x &\geq \mathrm{cap}_q(E_\sigma;B_{2r}) \\ &\geq \mathrm{cap}_q(\widetilde{N}(w+1);B_{2r}) - \mathrm{cap}_q(\widetilde{E}_\sigma;B_{2r}) \\ &\geq \mathrm{cap}_q(\widetilde{N}(w+1);B_{2r}) - \mathrm{cap}_q(O_\sigma;B_{2r}) \\ &\geq \mathrm{cap}_q(N(w+1);B_{2r}) - \sigma, \end{aligned} \tag{4.4.3}$$

where the second and third inequalities follow from the subadditivity and monotonicity of the p-capacity (Proposition 2.4.2). Since $w = v - v_{B_r} = v - 1$ q-q.e. in B_r and σ can be arbitrarily small, the inequality (4.4.3) implies

$$\int_{B_{2r}} |\nabla w|^q\, \mathrm{d}x \geq \mathrm{cap}_q(N(v);B_{2r}). \tag{4.4.4}$$

Then the desired inequality can be obtained as follows:

$$\left(\fint_{B_r} |v|^{\kappa q}\, \mathrm{d}x\right)^{\frac{1}{\kappa q}} \leq \left(\fint_{B_r} |v - v_{B_r}|^{\kappa q}\, \mathrm{d}x\right)^{\frac{1}{\kappa q}} + v_{B_r}$$

$$
\begin{aligned}
&\leq \left(2^n \fint_{B_{2r}} |w|^{\kappa q}\,\mathrm{d}x\right)^{\frac{1}{\kappa q}} + 1\\
&\leq C(n,q)r\left(\fint_{B_{2r}} |\nabla w|^q\,\mathrm{d}x\right)^{\frac{1}{q}} + \left(\frac{1}{\mathrm{cap}_q(N(v);B_{2r})}\int_{B_{2r}} |\nabla w|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&\leq C(n,q)\left(\frac{1}{\mathrm{cap}_q(N(v);B_{2r})}\int_{B_r} |\nabla v|^q\,\mathrm{d}x\right)^{\frac{1}{q}},
\end{aligned}
$$

where the third line follows from the normal Sobolev-Poincaré inequality and (4.4.4), and the fourth from (4.4.2) and

$$\mathrm{cap}_q(N(v);B_{2r}) \leq \mathrm{cap}_q(B_r;B_{2r}) \overset{(2.4.8)}{=} C(n,q)r^{n-q}$$

□

Remark 4.4.2. The above lemma also holds for any vector-valued function $v \in W^{1,q}(B_r,\mathbb{R}^N)$ with $N(v) = \{x \in B_r\colon\, |v(x)| = 0\}$, to show which we apply Lemma 4.4.1 to each component v_i of v $(i = 1,\dots,N)$, and the fact that

$$\mathrm{cap}_q(N(v_i);B_{2r}) \geq \mathrm{cap}_q(N(v);B_{2r}).$$

The constant in the corresponding inequality depends on n, N and q.

4.4.2 Caccioppoli-type inequality near the boundary

To show weak reverse Hölder inequalities near the boundary of a domain, we need another ingredient–a Caccioppoli-type inequality, which is presented in this subsection. This is similar to Lemma 2.2 in [KK94].

Proposition 4.4.3. *Suppose that $u \in W^{1,p}_{loc}(\Omega,\mathbb{R}^N)$ is a (p,Q)-minimizer with $p > 1, Q \geq 1$, and $O \subset\subset \Omega$ is an open set. The map $h \in W^{1,q}(O,\mathbb{R}^N)$ for some $q > p$ is such that $u - h \in W^{1,p}_0(O,\mathbb{R}^N)$. Then for any ball $B_R = B(x_0,R) \subset \mathbb{R}^n$ with $B_{2R} \cap O^c \neq \emptyset$, we have*

$$\int_{O_R} |\nabla u|^p\,\mathrm{d}x \leq \frac{C}{R^p}\int_{O_{2R}} |u-h|^p\,\mathrm{d}x + C\int_{O_{2R}} |\nabla h|^p\,\mathrm{d}x, \tag{4.4.5}$$

where $O_r := O \cap B_r$ and $C = C(p,Q) > 0$.

Proof. Fix a ball $B(x_0,R) \subset \mathbb{R}^n$ with $B_{2R} \cap O^c \neq \emptyset$ and denote the open set $O \cap B_r$ by O_r. For $R \leq t < s \leq 2R$, take a cut-off function η between B_t and B_s with $|\nabla\eta| \leq \frac{2}{s-t}$, and set $\varphi = \eta(u-h) \in W^{1,p}_0(O_s,\mathbb{R}^N)$. Then $u - \varphi = u - \eta(u-h) = (1-\eta)(u-h) + h$. Since u is a (p,Q)-minimizer, by the triangle inequality we have

$$
\begin{aligned}
\int_{O_t} |\nabla u|^p\,\mathrm{d}x \leq \int_{O_s} |\nabla u|^p\,\mathrm{d}x &\leq Q\int_{O_s} |\nabla(u-\varphi)|^p\,\mathrm{d}x\\
&\leq C(p)Q\left(\int_{O_s\setminus O_t} |\nabla((1-\eta)(u-h))|^p\,\mathrm{d}x + \int_{O_s} |\nabla h|^p\,\mathrm{d}x\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le C(p)Q\left(\frac{2^p}{(s-t)^p}\int_{O_{2R}}|u-h|^p\,\mathrm{d}x+\int_{O_s\setminus O_t}|\nabla(u-h)|^p\,\mathrm{d}x+\int_{O_s}|\nabla h|^p\,\mathrm{d}x\right)\\
&\le C(p)Q\left(\frac{1}{(s-t)^p}\int_{O_{2R}}|u-h|^p\,\mathrm{d}x+\int_{O_s\setminus O_t}|\nabla u|^p\,\mathrm{d}x+\int_{O_{2R}}|\nabla h|^p\,\mathrm{d}x\right).
\end{aligned}
$$

Apply the hole-filling trick, that is, add $C(p)Q\int_{O_t}|\nabla u|^p\,\mathrm{d}x$ to both sides and divide the inequality by $C(p)Q+1$, to get

$$
\int_{O_t}|\nabla u|^p\,\mathrm{d}x\le\frac{C(p)Q}{C(p)Q+1}\left(\frac{2^p}{(s-t)^p}\int_{O_{2R}}|u-h|^p\,\mathrm{d}x+\int_{O_s}|\nabla u|^p\,\mathrm{d}x+\int_{O_{2R}}|\nabla h|^p\,\mathrm{d}x\right).
$$

Then the desired inequality (4.4.5) follows from Lemma 2.9.1 with

$$
\Phi(r)=\int_{O_r}|\nabla u|^p\,\mathrm{d}x,\quad \theta=\frac{C(p)Q}{C(p)Q+1},\quad \Psi(t)=\frac{1}{t^p}\int_{O_{2R}}|u-h|^p\,\mathrm{d}x,\quad B=\int_{O_{2R}}|\nabla h|^p\,\mathrm{d}x.
$$

□

4.4.3 Weak reverse Hölder inequalities near the boundary

With the conclusions in the above two subsections, we are able to show the following weak reverse Hölder inequality.

Proposition 4.4.4. *Suppose that $u\in W^{1,p}_{loc}(\Omega,\mathbb{R}^N)$ with $p>1$ is a (p,Q)-minimizer, the set $O\subset\subset\Omega$ is open, and O^c is uniformly p-thick with constants $R_0>0,T_0>0$. The map $h\in W^{1,q}(O,\mathbb{R}^N)$ with $q\ge p$ satisfies $u-h\in W^{1,p}_0(O,\mathbb{R}^N)$. Then there exists $p_0\in(1,p)$ depending on n,p,T_0 such that for any $p_1\in(p_0,p)$ and any $B_R=B(x_0,R)\subset\mathbb{R}^n$ with $B_{2R}\cap O^c\neq\emptyset$ and $R<\frac{R_0}{2}$, we have*

$$
\frac{1}{R^n}\int_{O_R}|\nabla u|^p\,\mathrm{d}x\le C\left(\frac{1}{R^n}\int_{O_{4R}}|\nabla u|^{p_1}\,\mathrm{d}x\right)^{\frac{p}{p_1}}+\frac{C}{R^n}\int_{O_{4R}}|\nabla h|^p\,\mathrm{d}x,\tag{4.4.6}
$$

where $C=C(n,N,p,p_1,Q,T_0)>0$.

Proof. Without loss of generality, we may assume that u and h are both p-quasi-continuous (and thus q-quasi-continuous for $1<q<p$ by definition and Hölder's inequality), since every function in $W^{1,p}(\Omega,\mathbb{R}^N)$ has a quasi-continuous representative by Theorem 2.4.4.

From the fact that O^c is uniformly p-thick, it can be shown (from [Lew88], Theorem 1 and [HKM18], 2.38) that there is a $p_0\in(1,p)$ such that O^c is uniformly p_1-thick for any $p_0<p_1<p$ with constants $R_0,\tilde{T}_0$, where p_0 and $\tilde{T}_0$ depends on n,p,T_0. We can take p_0 such that $\kappa p_0>p$, where

$$
\kappa=\begin{cases}\dfrac{n}{n-p_0}, & 1<p_0<n,\\ 2, & p_0\ge n.\end{cases}
$$

Fix $p_1\in(p_0,p)$, then O^c is uniformly p_1-thick with constants R_0 and $\tilde{T}_0$. Extend $u-h$ by

0 on O^c, and thus for any $R < \frac{R_0}{2}$ we have, by Lemma 4.4.1 and Remark 4.4.2,

$$\begin{aligned}\left(\frac{1}{R^p}\int_{O_{2R}}|u-h|^p\,\mathrm{d}x\right)^{\frac{1}{p}} &\le 4^{\frac{n}{p}}R^{\frac{n}{p}-1}\omega_n^{\frac{1}{p}}\left(\fint_{B_{4R}}|u-h|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\le 4^{\frac{n}{p}}R^{\frac{n}{p}-1}\omega_n^{\frac{1}{p}}\left(\fint_{B_{4R}}|u-h|^{\kappa p_1}\,\mathrm{d}x\right)^{\frac{1}{\kappa p_1}} \qquad (4.4.7)\\ &\le 4^{\frac{n}{p}}R^{\frac{n}{p}-1}\omega_n^{\frac{1}{p}}C(n,p_1)\left(\frac{1}{\operatorname{cap}_{p_1}(N(u-h);B_{8R})}\int_{B_{4R}}|\nabla(u-h)|^{p_1}\,\mathrm{d}x\right)^{\frac{1}{p_1}},\end{aligned}$$

where $N(u-h)=\{x\in B_{4R}\colon\ |u(x)-h(x)|=0\}$. Take $y_0\in B_{2R}\setminus O$ such that $B(y_0,2R)\subset B_{4R}$ and $B_{8R}\subset B(y_0,10R)$. Notice that $N(u-h)\supset B_{4R}\setminus O$, then it is possible to get a lower bound of $\operatorname{cap}_{p_1}(N(u-h);B_{8R})$ as follows:

$$\begin{aligned}\operatorname{cap}_{p_1}(N(u-h);B_{8R})) &\ge \operatorname{cap}_{p_1}(B_{4R}\setminus O;B_{8R})\\ &\ge \operatorname{cap}_{p_1}(B(y_0;2R)\setminus O;B_{8R}) \qquad (4.4.8)\\ &\ge \operatorname{cap}_{p_1}(B(y_0;2R)\setminus O;B(y_0,10R))\\ &\ge \frac{1}{C(n,p_1)}\operatorname{cap}_{p_1}(B(y_0;2R)\setminus O;B(y_0,4R)) \ge \frac{\widetilde{T}_0}{C(n,p_1)}(2R)^{n-p_1},\end{aligned}$$

where the second and third lines are obtained by the monotonicity of the p-capacity $\operatorname{cap}_p(\cdot\,;\cdot)$ in both arguments as shown in Proposition 2.4.2, and the last inequality is from the p_1-thickness of O^c.

Combining (4.4.7) and (4.4.8), we obtain

$$\left(\frac{1}{R^p}\int_{O_{2R}}|u-h|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \le C\left(R^{(p/p_1-1)n}\int_{B_{4R}}|\nabla(u-h)|^{p_1}\,\mathrm{d}x\right)^{\frac{1}{p_1}}, \qquad (4.4.9)$$

which further implies, with Proposition 4.4.3 and Hölder's inequality,

$$\begin{aligned}\frac{1}{R^n}\int_{O_R}|\nabla u|^p\,\mathrm{d}x &\le C\left(\frac{1}{R^n}\int_{O_{4R}}|\nabla(u-h)|^{p_1}\,\mathrm{d}x\right)^{\frac{p}{p_1}}+\frac{C}{R^n}\int_{O_{2R}}|\nabla h|^p\,\mathrm{d}x\\ &\le C\left(\frac{1}{R^n}\int_{O_{4R}}|\nabla u|^{p_1}\,\mathrm{d}x\right)^{\frac{p}{p_1}}+C\left(\frac{1}{R^n}\int_{O_{4R}}|\nabla h|^{p_1}\,\mathrm{d}x\right)^{\frac{p}{p_1}}+\frac{C}{R^n}\int_{O_{4R}}|\nabla h|^p\,\mathrm{d}x\\ &\le C\left(\frac{1}{R^n}\int_{O_{4R}}|\nabla u|^{p_1}\,\mathrm{d}x\right)^{\frac{p}{p_1}}+\frac{C}{R^n}\int_{O_{4R}}|\nabla h|^p\,\mathrm{d}x,\end{aligned}$$

where $C=C(n,N,p,p_1,Q,T_0)>0$. □

4.4.4 Quasiminimality of a higher order

Now we give the proof of Theorem 4.1.1 with the weak reverse Hölder inequality obtained above and the self-improving property Theorem 4.2.3.

Proof of Theorem 4.1.1. Fix an open set $O\subset\Omega$ such that O^c is uniformly p-thick with radius $R_0>0$ and constant $T_0>0$. We have obtained weak reverse Hölder inequalities

(4.4.6) near ∂O. For an interior ball $B_R = B(x_0, R)$ with $B_R \subset B_{2R} \subset O$, the corresponding weak reverse Hölder inequality

$$\left(\fint_{B_R} |\nabla u|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \le C \left(\fint_{B_{2R}} |\nabla u|^{p_*} \, \mathrm{d}x\right)^{\frac{1}{p_*}} \tag{4.4.10}$$

also holds true, where $p_* = \max\{1, \frac{np}{n+p}\}$ and $C = C(n, N, p, Q) > 0$. By considering $\varphi = \eta(u - u_{2R})$ and proceeding as in Proposition 4.4.3, we have

$$\int_{B_R} |\nabla u|^p \, \mathrm{d}x \le \frac{C}{R^p} \int_{B_{2R}} |u - u_{2R}|^p \, \mathrm{d}x \tag{4.4.11}$$

with $C = C(p, Q) > 0$. Then (4.4.10) follows from the normal Sobolev-Poincaré inequality.

Set

$$g(x) = \begin{cases} |\nabla u(x)|, & x \in O \\ 0, & x \in O^c \end{cases} \quad \text{and} \quad f(x) = \begin{cases} |\nabla h(x)|, & x \in O \\ 0, & x \in O^c, \end{cases}$$

where h is an arbitrary map in $W^{1,q}(O, \mathbb{R}^N)$ with $u - h \in W_0^{1,p}(O, \mathbb{R}^N)$ and $q > p$ to be determined. Fix $p_1 = \frac{p+p_0}{2}$ with p_0 as in the proof of Proposition 4.4.4 and let $\sigma = \min\{1 - p_*/p, 1 - p_1/p\}$. From (4.4.6) and (4.4.10) we know that the following weak reverse Hölder inequality

$$\left(\fint_{B_R} g^p \, \mathrm{d}x\right)^{\frac{1}{p}} \le b_1 \left(\fint_{B_{4R}} g^{p(1-\sigma)} \, \mathrm{d}x\right)^{\frac{1}{p(1-\sigma)}} + b_2 \left(\fint_{B_{4R}} f^p \, \mathrm{d}x\right)^{\frac{1}{p}} \tag{4.4.12}$$

holds for any ball $B(x_0, R)$ with $R < \frac{R_0}{2}$, where $b_1, b_2 > 0$ depend on n, N, p, Q, T_0, and R_0 is the radius limit associated to the uniform p-thickness of O^c. Notice that there are no restrictions on the location of the ball $B(x_0, R)$ and the above inequality actually holds in the whole $\mathbb{R}^n$. By Theorem 4.2.3, there is a $\theta > 0$ depending on n, N, p, Q, R_0, T_0 such that the following holds: For any $\bar{p} \in (p, p + \theta)$ with $\bar{p} \le q$, we have $g \in L^{\bar{p}}(O)$ and

$$\int_O |\nabla u|^{\bar{p}} \, \mathrm{d}x \le C_1 \left(\int_O |\nabla u|^p \, \mathrm{d}x\right)^{\frac{\bar{p}}{p}} + C_2 \int_O |\nabla h|^{\bar{p}} \, \mathrm{d}x, \tag{4.4.13}$$

where $C_1, C_2 > 0$ depend on $n, N, p, Q, R_0, T_0, \bar{p}$.

The inequality (4.4.10) actually holds for any ball B_R with $B_{2R} \subset\subset \Omega$, and then Theorem 4.2.2 gives a locally higher regularity of the gradient of the quasiminimizer u. Then we know that there exists $q_1 > p$ such that $u \in W^{1,\bar{q}}_{loc}(\Omega, \mathbb{R}^N)$ for any $\bar{q} < q_1$. Take q such that $p < q < \min\{p + \theta, (q_1 - p)\}$ and then (4.4.13) holds with $\bar{p} = q$ and any $h \in W^{1,q}(O, \mathbb{R}^N)$ with $h - u \in W_0^{1,p}(O, \mathbb{R}^N)$. Notice that θ is independent of h and q, which allows us to do take h with more freedom without affecting the result. To proceed, we choose h to be the q-harmonic function on O with $u - h \in W_0^{1,q}(O, \mathbb{R}^N)$. Then from (4.4.13) and the quasiminimality of u, we have

$$\int_O |\nabla u|^q \, \mathrm{d}x \le C_1 \left(\int_O |\nabla u|^p \, \mathrm{d}x\right)^{\frac{q}{p}} + C_2 \int_O |\nabla h|^q \, \mathrm{d}x$$

$$\leq C_1 \left(Q \int_O |\nabla h|^p \, \mathrm{d}x \right)^{\frac{q}{p}} + C_2 \int_O |\nabla h|^q \, \mathrm{d}x$$
$$\leq (C_1 Q^{\frac{q}{p}} |O|^{\frac{q}{p}-1} + C_2) \int_O |\nabla h|^q \, \mathrm{d}x.$$

Since h is the minimizer of the D_q on O, the above inequality actually indicates that u quasiminimises $\mathcal{D}_q$ on O with a constant $Q_q \geq 1$ depending on n, N, p, Q, R_0, T_0, q. $\square$

Remark 4.4.5. (i) If we replace the subdomain O by Ω in the above proof, it is possible to obtain a global higher integrability of ∇u on Ω (cf. [KK94], Theorem 1.1). To achieve this, several assumptions are required additionally:

(a) Ω^c is uniformly p-thick.

(b) The boundary value φ of $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is in $W^{1,r}(\Omega)$ for some $r > p$. The boundary value is taken in the sense that $u - \varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N)$.

(c) u quasiminimizes the p-Dirichlet integral not only interiorly but also near the boundary.

(ii) Considering the values of C_1 and C_2 obtained in Theorem 4.2.3, which are from Lemma 4.2.5, we will find that as $q \to p^+$ the quasiminimizing constant Q_q above tends to $Q + C(n, N, p, Q, R_0, T_0) > Q$. The extra constant C comes from Theorem 4.2.3, where we omitted the third term in (4.2.34). This is due to the fact that we started from weak reverse Hölder inequalities of integral form (4.2.7) and cannot deal with the functions f, g on the right-hand side together. Then the consequent inequality would be of the form (b) in Lemma 4.2.5, from which a negative term like the one in (4.2.34) will appear. This issue can be avoided if we consider weak reverse Hölder inequalities in the form of maximal functions, where the second function g on the right-hand side is replaced by the maximal function $M^{\frac{1}{q}}(g^q)$. However, to get back to the original g, we have to use the boundedness of M on $L^{\frac{\bar{p}}{p}}$. The bound blows up when $\bar{p}$ tends to p, which again gives a quasiminimizing constant strictly larger than Q. Therefore, the quasiminimizing constant obtained by this method is not stable when p varies in this small range.

4.5 Stability of minimality

This section is devoted to the stability of $\mathcal{D}_p$-minimality. We start with a qualitative discussion about the stability in Subsection 4.5.1, and then present the proof of Theorem 4.1.2 in the subsequent two subsections. One type of nonlinear commutator is introduced in Subsection 4.5.2 based on [IM01], Chapter 13, and that will be the main tool in the proof. The stability result is given in Subsection 4.5.3 with a quantitative characterisation.

4.5.1 Qualitative stability

The stability of $\mathcal{D}_p$-minimality with respect to p holds true qualitatively in the scalar case. More precisely, given a p-harmonic function u on a regular domain, the q-harmonic function with the same boundary value converges to u in a proper sense as $q \to p$. See [Lin87] for a similar result on the solutions to non-homogeneous p-Laplace equations.

Proposition 4.5.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and $B_R = B(x_0, R) \subset\subset \Omega$ is a ball. The function $u \in W^{1,p}_{loc}(\Omega)$ is p-harmonic, and $\{q_m\}_{m\in\mathbb{N}}$ is a sequence in $(1, p)$ and increases to p as $m \to \infty$. Then there exists $\beta = \beta(n, p, \alpha) \in (0, \alpha]$ such that the sequence $\{h_m\}_{m\in\mathbb{N}}$ converges to u in $C^{1,\gamma}(\bar{B}_R)$ for any $\gamma \in (0, \beta)$, where h_m is the q_m-harmonic function with $u - h_m \in W^{1,q_m}_0(B_R)$.*

Proof. Since u is p-harmonic, it is in $C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1)$ ([Tol83, Uhl77, AF89, GM86a]), and thus in $C^{1,\alpha}(\bar{B}_R)$. By the global regularity result for p-harmonic functions ([Lie88], Theorem 1), we know that $h_m \in C^{1,\beta}(\bar{B}_R)$ for some $\beta = \beta(n, p, \alpha) \in (0, \alpha]$, which is independent of m. Furthermore, we have a uniform bound $M = M(n, p, \alpha, u) > 0$ of $\|h_m\|_{C^{1,\beta}(\bar{B}_R)}$.

The Arzelà-Ascoli theorem ([DS58], IV.6, Theorem 7) indicates the existence of a subsequence (still denoted by $\{h_m\}$) such that

$$(h_m, \nabla h_m) \longrightarrow (h, \nabla h) \quad \text{uniformly in } B_R \tag{4.5.1}$$

for some $h \in C^{1,\beta}(\bar{B}_R)$. Given any $\gamma \in (0, \beta)$, take any $m, \ell \in \mathbb{N}$ and any $x, y \in \bar{B}_R$. The β-Hölder continuity of h_m and h_ℓ implies

$$\begin{aligned}
&\frac{|\nabla(h_m - h_\ell)(x) - \nabla(h_m - h_\ell)(y)|}{|x - y|^\gamma} \\
&= \left(\frac{|\nabla(h_m - h_\ell)(x) - \nabla(h_m - h_\ell)(y)|}{|x - y|^\gamma}\right)^{\frac{\gamma}{\beta}} |\nabla(h_m - h_\ell)(x) - \nabla(h_m - h_\ell)(y)|^{\frac{\beta-\gamma}{\beta}} \\
&\leq (2M)^{\frac{\gamma}{\beta}} (2 \sup_{\bar{B}_R} |\nabla h_m - \nabla h_\ell|)^{\frac{\beta-\gamma}{\beta}}.
\end{aligned}$$

Considering the uniform convergence $\nabla h_m \to \nabla h$, we know that the last line vanishes as $m, \ell \to \infty$, and thus $h_m \to h$ in $C^{1,\gamma}(\bar{B}_R)$.

The uniform convergence (4.5.1) also implies $h \in W^{1,p}_u(B_R)$. Fix $k \in \mathbb{N}$, and it is easy to see the following

$$\begin{aligned}
\left(\fint_{B_R} |\nabla h|^{q_k} \,\mathrm{d}x\right)^{\frac{1}{q_k}} &= \lim_{m\to\infty} \left(\fint_{B_R} |\nabla h_m|^{q_k} \,\mathrm{d}x\right)^{\frac{1}{q_k}} \\
&\leq \lim_{m\to\infty} \left(\fint_{B_R} |\nabla h_m|^{q_m} \,\mathrm{d}x\right)^{\frac{1}{q_m}} \\
&\leq \lim_{m\to\infty} \left(\fint_{B_R} |\nabla u|^{q_m} \,\mathrm{d}x\right)^{\frac{1}{q_m}} \\
&= \left(\fint_{B_R} |\nabla u|^{p} \,\mathrm{d}x\right)^{\frac{1}{p}},
\end{aligned} \tag{4.5.2}$$

where the second inequality is due to $q_k < q_m$ as $m \to \infty$, and the third to the fact that h_m minimises $\mathcal{D}_{q_m}$ in $W^{1,q_m}_u(B_R)$. As $k \to \infty$, the above inequality (4.5.2) gives

$$\left(\fint_{B_R} |\nabla h|^{p} \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \left(\fint_{B_R} |\nabla u|^{p} \,\mathrm{d}x\right)^{\frac{1}{p}},$$

i.e., the function h also minimises $\mathcal{D}_p$ in $W^{1,p}_u(B_R)$. Considering the strict convexity of $\mathcal{D}_p$, we know that the minimizer of it in the Dirichlet class $W^{1,p}_u(B_R)$ is unique, and thus $u = h$.

From the above discussion, we can see that for any subsequence of $\{h_m\}$, there is a further subsequence that converges to u in $C^{1,\gamma}(\bar{B}_R)$. Therefore, the desired convergence holds for $\{h_m\}$ itself. $\square$

Remark 4.5.2. (a) We have the same result when $\{q_m\}$ decreases to p with a similar argument. The way to show that the limit $h \in C^{1,\beta}(\bar{B}_R)$ is actually u in this case is as follows:

$$\begin{aligned}
\left(\fint_{B_R} |\nabla h|^p \,\mathrm{d}x\right)^{\frac{1}{p}} &= \lim_{m\to\infty} \left(\fint_{B_R} |\nabla h_m|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\le \lim_{m\to\infty} \left(\fint_{B_R} |\nabla h_m|^{q_m} \,\mathrm{d}x\right)^{\frac{1}{q_m}} \\
&\le \lim_{m\to\infty} \left(\fint_{B_R} |\nabla u|^{q_m} \,\mathrm{d}x\right)^{\frac{1}{q_m}} \\
&= \left(\fint_{B_R} |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned} \tag{4.5.3}$$

(b) The above discussion also applies to any domain $O \subset\subset \Omega$ with $C^{1,\alpha}$ boundary, where α is the Hölder exponent of ∇u. See Theorem 1 in [Lie88].

(c) Notice that the proposition is only for the scalar case, and for vector-valued p-harmonic maps we lack a global regularity result as Theorem 1 in [Lie88]. In [CD89], such a result is proved for $p > \frac{2n}{n+1}$ with a higher regularity assumption on the boundary value, with which the above discussion remains true when $p > n$. Alternatively, we can employ the global Lipschitz regularity result in [Bög15], which says that, when $p \ge 2$, any vector-valued p-harmonic map on a $C^{1,\beta}$ domain with a $C^{1,\beta}$ boundary value is globally Lipschitz continuous. Then with the compact embedding $W^{1,\infty}(B_R, \mathbb{R}^N) \hookrightarrow C^{0,\gamma}(\bar{B}_R, \mathbb{R}^N)$ for any $\gamma \in (0,1)$ and the weak compactness of any bounded set in $W^{1,p}(B_R, \mathbb{R}^N)$, it is possible to show that h_m (as defined in Proposition 4.5.1) converges to u in $C^{0,\gamma}(\bar{B}_R, \mathbb{R}^N)$ for any $\gamma \in (0,1)$ when $p > 2$. Furthermore, strong convergence in $W^{1,p}(B_R, \mathbb{R}^N)$ can be proved by showing $\int_{B_R} |\nabla h_m|^p \,\mathrm{d}x \to \int_{B_R} |\nabla u|^p \,\mathrm{d}x$ and an argument similar to the one in Subsection 6.4.4.

4.5.2 Nonlinear commutators

This subsection is for a certain type of nonlinear commutators and the corresponding estimate. The discussion is based on [IM01], Chapter 13, where there is a thorough discussion about a wide range of nonlinear commutators. Such commutators have been used to deal with some p-Laplace type equations and systems (see [IS94, CGI02] and the references therein). With the help of the tool of nonlinear commutators, we will estimate the difference between a p-harmonic and a q-harmonic map with the same boundary value for two exponents p, q close to each other in Subsection 4.5.3.

We first consider a bounded linear operator $T\colon L^r(X, V) \to L^r(X, V)$ for s in some range in $(1, \infty)$ (say, a non-trivial interval $[p_1, p_2] \subset (1, \infty)$), and denote the corresponding operator norm by $\|T\|_r$ for $r \in [p_1, p_2]$. Here, the domain X is a topological space equipped with a σ-finite measure μ and V is an inner-product space of finite dimension. Since the result below depends on complex interpolation, we complexify V and T if they are originally defined over $\mathbb{R}$. After complexification, the induced inner-product of $V_{\mathbb{C}}$ is Hermitian and $T_{\mathbb{C}}$ is linear over $\mathbb{C}$. Notice that the complexified operator maintains its norm. In the following, we stick to the notations V and T for convenience.

Take $\rho > 0$ with $2\rho \le \min\{\frac{s}{p_1} - 1, 1 - \frac{s}{p_2}\}$ for some fixed $s \in (p_1, p_2)$. The discussion will be focused on the commutator of T and $f \mapsto |f|^z f$ with $f \in L^s(X, V)$ and z being a complex number in $B_{\mathbb{C}}(0, \rho)$. Define

$$T^z f := T(|f|^z f) - |Tf|^z Tf, \tag{4.5.4}$$

which is the commutator applied to $f \in L^s(X, V)$. We first give a rough control of the difference $T^z f - T^z g$.

Proposition 4.5.3. *Suppose that X is a topological space equipped with a σ-finite measure μ, and V is an inner-product space over $\mathbb{C}$ of finite dimension. The operator $T\colon : L^r(X, V) \to L^r(X, V)$ is linear over $\mathbb{C}$ and bounded for any r in a given interval $[p_1, p_2]$ with $1 < p_1 < p_2 < \infty$. For an arbitrary $s \in (p_1, p_2)$, take ρ with $2\rho \le \min\{\frac{s}{p_1} - 1, 1 - \frac{s}{p_2}\}$ and a complex number $z = x + iy \in B_{\mathbb{C}}(0, \rho)$. Then we have*

$$\|T^z f - T^z g\|_{L^{\frac{s}{1+x}}} \le C(\|f - g\|_{L^s}^{1-\rho}(\|f\|_{L^s} + \|g\|_{L^s})^{\rho})^{1+x} \tag{4.5.5}$$

hold true for any $f, g \in L^s(X, V)$, where $C = C(\|T\|_{p_1}, \|T\|_{p_2}) > 0$.

Proof. Set $z = x + iy$, then we have

$$||f|^z f| = |f|^{1+x} \quad \text{and} \quad ||Tf|^z Tf| = |Tf|^{1+x},$$

and both of them are in $L^{\frac{s}{1+x}}(X, V)$. The choice of ρ indicates $\frac{s}{1+x} \in [p_1, p_2]$, and thus the operator T^z is bounded from $L^s(X, V)$ to $L^{\frac{s}{1+x}}(X, V)$.

For any $f, g \in L^s(X, V)$, we know that $T^z f - T^z g \in L^{\frac{s}{1+x}}(X, V)$ and

$$\begin{aligned} \|T^z f - T^z g\|_{L^{\frac{s}{1+x}}} &= \|T(|f|^z f - |g|^z g) - (|Tf|^z Tf - |Tg|^z Tg)\|_{L^{\frac{s}{1+x}}} \\ &\le \|T\|_{\frac{s}{1+x}} \||f|^z f - |g|^z g\|_{L^{\frac{s}{1+x}}} + \|(|Tf|^z Tf - |Tg|^z Tg)\|_{L^{\frac{s}{1+x}}}. \end{aligned} \tag{4.5.6}$$

Define $a(\xi) := |\xi|^z \xi$ for $\xi \in V$. The derivative of this map at $\xi \ne 0$ is

$$a'(\xi) = z|\xi|^{z-2} \xi \otimes \xi + |\xi|^z I_V.$$

Then for any $\xi, \zeta \in V$ with 0 not on the segment connecting ξ and ζ, we have

$$\begin{aligned} |a(\xi) - a(\zeta)| &= \left| \int_0^1 a'(\zeta + t(\xi - \zeta)) \cdot (\xi - \zeta) \, \mathrm{d}t \right| \\ &\le |\xi - \zeta| \int_0^1 (|z| + 1)|\zeta + t(\xi - \zeta)|^x \, \mathrm{d}t \\ &\le C|\xi - \zeta|(|\xi| + |\zeta|)^x, \end{aligned} \tag{4.5.7}$$

where the last inequality follows from Lemma 2.9.3 as $|x| \leq |z| < \rho \leq \frac{1}{2}$. It is possible to take a constant C in (4.5.7) that is independent of x and ρ.

With (4.5.6) and (4.5.7), the difference $T^z f - T^z g$ can be further estimated with Hölder's inequality:

$$\begin{aligned}\|T^z f - T^z g\|_{L^{\frac{s}{1+x}}} &\leq C(\|T\|_{\frac{s}{1+x}} \|(|f|+|g|)^x (f-g)\|_{L^{\frac{s}{1+x}}} + \|(|Tf|+|Tg|)^x (Tf - Tg)\|_{L^{\frac{s}{1+x}}}) \\ &\leq C(\|T\|_{\frac{s}{1+x}} \|f-g\|_{L^s} (\|f\|_{L^s} + \|g\|_{L^s})^x + \|T(f-g)\|_{L^s} (\|Tf\|_{L^s} + \|Tg\|_{L^s})^x) \\ &\leq C(\|T\|_{\frac{s}{1+x}} + \|T\|_s^{1+x}) \|f-g\|_{L^s} (\|f\|_{L^s} + \|g\|_{L^s})^x. \qquad (4.5.8)\end{aligned}$$

For any $\theta \in (0,1)$, take $p_\theta \in (p_1, p_2)$ such that

$$\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

Then with the Riesz-Thorin interpolation theorem ([SS11], Theorem 2.1), we can control the norm of $T\colon L^{p_\theta}(X,V) \to L^{p_\theta}(X,V)$ by

$$\|T\|_{p_\theta} \leq \|T\|_{p_1}^{\theta} \|T\|_{p_2}^{1-\theta} \leq \max\{\|T\|_{p_1}, \|T\|_{p_2}\}. \qquad (4.5.9)$$

Since $|x| < \rho \leq \frac{1}{2}$, it is easy to see that the constant $C(\|T\|_{\frac{s}{1+x}} + \|T\|_s^{1+x})$ in (4.5.8) is dominated by some $C(\|T\|_{p_1}, \|T\|_{p_2}) > 0$. It is also easy the see that the following estimate

$$\|f-g\|_{L^s} (\|f\|_{L^s} + \|g\|_{L^s})^x \leq (\|f-g\|_{L^s}^{1-\rho} (\|f\|_{L^s} + \|g\|_{L^s})^\rho)^{1+x}$$

holds true, which finally implies the desired result (4.5.5). □

It is possible to improve the estimate (4.5.5) by showing that T^z depends on z analytically and vanishes at $z = 0$.

Theorem 4.5.4 ([IM01], Theorem 13.2.1)**.** *Suppose that X is a topological space equipped with a σ-finite measure μ, and V is an inner-product space over $\mathbb{C}$ of finite dimension. The operator $T\colon L^r(X,V) \to L^r(X,V)$ is linear over $\mathbb{C}$ and bounded for any r in a given interval $[p_1, p_2]$ with $1 < p_1 < p_2 < \infty$. For an arbitrary $s \in (p_1, p_2)$, take ρ with $2\rho \leq \min\{\frac{s}{p_1} - 1, 1 - \frac{s}{p_2}\}$ and a complex number $z \in B_{\mathbb{C}}(0,\rho)$. Then we have*

$$\|T^z f - T^z g\|_{L^{\frac{s}{1+x}}} \leq C\frac{|z|}{\rho} (\|f-g\|_{L^s}^{1-\rho} (\|f\|_{L^s} + \|g\|_{L^s})^\rho)^{1+x} \qquad (4.5.10)$$

for any $f, g \in L^s(X,V)$ with $C = C(\|T\|_{p_1}, \|T\|_{p_2}) > 0$. In particular, for any $f \in L^s(X,V)$ there holds

$$\|T^z f\|_{L^{\frac{s}{1+x}}} = \|T(|f|^z f) - |Tf|^z Tf\|_{L^{\frac{s}{1+x}}} \leq C\frac{|z|}{\rho} \|f\|_{L^s}^{1+x}. \qquad (4.5.11)$$

Proof. If $f = g$, the inequality holds trivially. Then without loss of generality we can assume

$$\|f-g\|_{L^s}^{1-\rho} (\|f\|_{L^s} + \|g\|_{L^s})^\rho = 1.$$

Then the estimate (4.5.5) becomes

$$\|T^z f - T^z g\|_{L^{\frac{s}{1+x}}} \leq C. \qquad (4.5.12)$$

Now consider the following function defined on $B_{\mathbb{C}}(0,\rho)$

$$F(z) := \int_X \langle T^z f - T^z g, |\phi|^{s-2-\bar{z}}\phi\rangle \,\mathrm{d}\mu,$$

where $\langle\cdot\,,\cdot\rangle$ is the inner-product on V and ϕ is an arbitrary map in $L^s(X,V)$ with $\|\phi\|_{L^s} = 1$. The map $z \mapsto |\zeta|^z\zeta$ is holomorphic for any fixed $\zeta \in V$. Since T is linear and bounded, and the inner-product is Hermitian, the function F is holomorphic on $B_{\mathbb{C}}(0,\rho)$. Furthermore, we have $F(0) = 0$ and, for any $z \in B_{\mathbb{C}}(0,\rho)$,

$$|F(z)| \leq \|T^z f - T^z g\|_{L^{\frac{s}{1+x}}} \, \||\phi|^{s-2-\bar{z}}\phi\|_{L^{\frac{s}{s-1-x}}} \leq C\|\phi\|_{L^s}^{s-1-x} = C.$$

Set

$$\omega := \frac{z}{\rho} \quad \text{and} \quad G(\omega) := \frac{F(\rho\,\omega)}{C} = \frac{F(z)}{C}.$$

Then the function G is holomorphic on the unit disk $B_{\mathbb{C}}(0,1)$ with $G(0) = 0$ and $|G(\omega)| \leq 1$ for any $\omega \in B_{\mathbb{C}}(0,1)$. The Schwarz lemma ([SS03], Lemma 2.1) implies $|G(\omega)| \leq |\omega|$ on $B(0,1)$, which is actually $F(z) \leq C\frac{|z|}{\rho}$ on $B_{\mathbb{C}}(0,\rho)$. The desired inequality then follows from

$$\begin{aligned}\|T^z f - T^z g\|_{L^{\frac{s}{1+x}}} &= \sup\{|F(z)| \colon \; \||\phi|^{s-2-\bar{z}}\phi\|_{L^{\frac{s}{s-1-x}}} = 1\} \\ &= \sup\{|F(z)| \colon \; \|\phi\|_{L^s} = 1\} \leq C\frac{|z|}{\rho}.\end{aligned}$$

For general $f, g \in L^s(X,V)$, divide them by the factor $\|f-g\|_{L^s}^{1-\rho}(\|f\|_{L^s} + \|g\|_{L^s})^{\rho}$, and then the result follows from the above discussion. ☐

Remark 4.5.5. Notice that the original estimate obtained in Proposition 4.5.3 is

$$\|T^z f - T^z g\|_{L^{\frac{s}{1+x}}} \leq C\|f-g\|_{L^s}(\|f\|_{L^s} + \|g\|_{L^s})^x.$$

We further control it with $C(\|f-g\|_{L^s}^{1-\rho}(\|f\|_{L^s} + \|g\|_{L^s})^{\rho})^{1+x}$ so that the normalisation in Theorem 4.5.4 can be done.

4.5.3 Difference estimate and the stability result

The strengthened result Theorem 4.5.4 enables us to estimate the difference between a p-harmonic map u and a q-harmonic map v with the same boundary value. With such an estimate, it is possible to conclude the stability result Theorem 4.1.2.

To apply the tool of nonlinear commutators as in Subsection 4.5.2, we first define the operator T with the help of the L^p-Hodge decomposition. The decomposition is proved by solving a Poisson's equation in the following, and can also be done with Green's functions (see [IS94], §2). For such a decomposition on the whole space, one can use Riesz transforms (see [Iwa83] and [IS94], §2).

Proposition 4.5.6 (Hodge decomposition)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with C^1 boundary. Any map $f \in L^r(\Omega,\mathbb{R}^n)$ with some $r > 1$ can be uniquely decomposed in the following form*

$$f = \nabla\Phi + \sigma, \tag{4.5.13}$$

where $\Phi \in W_0^{1,r}(\Omega)$ *and* $\sigma \in L^r(\Omega, \mathbb{R}^n)$ *with* $\operatorname{div} \sigma = 0$ *in the distributional sense. Moreover, there exists a constant* $C = C(\Omega, r, n) > 0$ *such that*

$$\|\nabla\Phi\|_{L^r(\Omega)} + \|\sigma\|_{L^r(\Omega)} \leq C\|f\|_{L^r(\Omega)}. \tag{4.5.14}$$

Proof. It is easy to see that $\operatorname{div} f$ is in $W^{-1,r}(\Omega)$, and then we only need to find a weak solution to the equation

$$\begin{cases} \Delta\Phi = \operatorname{div} f, & \text{in } \Omega, \\ \Phi = 0, & \text{on } \partial\Omega \end{cases} \tag{4.5.15}$$

in $W_0^{1,r}(\Omega)$. By Theorem 2.2.2, the operator Δ induces an isomorphism between $W_0^{1,r}(\Omega)$ and $(W_0^{1,r}(\Omega))^* = W^{-1,r}(\Omega)$. In other words, given any $G \in W^{-1,r}(\Omega)$, there exists a unique $u \in W_0^{1,r}(\Omega)$ such that

$$\langle G, v\rangle = -\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$$

for any $v \in W_0^{1,r}(O)$, where $\langle \cdot\,, \cdot\rangle$ is the duality pairing of $W^{-1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$. In addition, the estimates

$$\|u\|_{W_0^{1,r}(\Omega)} \leq C\|G\|_{W^{-1,r}(\Omega)}$$

holds true with some $C = C(\Omega, r, n) > 0$. Back to equation (4.5.15), since $\operatorname{div} f \in W^{-1,r}(\Omega)$, there exists a unique $\Phi \in W_0^{1,r}(\Omega)$ satisfying (4.5.15) and we have

$$\|\nabla\Phi\|_{L^r(\Omega)} \leq \|\Phi\|_{W_0^{1,r}(\Omega)} \leq C\|\operatorname{div} f\|_{W^{-1,r}(\Omega)} \leq C\|f\|_{L^r(\Omega)} \tag{4.5.16}$$

with the above constant $C = C(\Omega, r, n)$. Define $\sigma := f - \nabla\Phi$, and it is obvious that $\sigma \in L^r(\Omega, \mathbb{R}^n)$ with

$$\begin{aligned} &\operatorname{div} \sigma = 0 \quad \text{in the distributional sense,} \\ \text{and} \quad &\|\sigma\|_{L^r(\Omega)} \leq C(\Omega, r, n)\|f\|_{L^r(\Omega)}. \end{aligned}$$

The uniqueness follows from the maximum principle of harmonic functions ([GT01], Corollary 8.2). ☐

Remark 4.5.7. (a) In the assumptions of Proposition 4.5.6, the restriction on the boundary of Ω can be relaxed. By Theorem 14.5.3 in [MS09], the above proposition holds for Ω with

$$\partial\Omega \in \begin{cases} M_p^{l+1-h-\frac{1}{p}}, & p(l-h) \leq n, \\ W_p^{l+1-h-\frac{1}{p}}, & p(l-h) > n. \end{cases}$$

The precise definitions of these boundary conditions are introduced in [MS09].

(b) The decomposition also applies to maps in $L^r(\Omega, \mathbb{R}^{N\times n})$ with any $N \in \mathbb{N}$, for which we apply the above proposition row-wise.

(c) Rescaling shows that the constant $C(\Omega, r, n)$ does not depend on the size of Ω.

With the Hodge decomposition, we can define a bounded linear operator $T\colon L^r(\Omega, \mathbb{R}^{N\times n}) \to L^r(\Omega, \mathbb{R}^{N\times n})$ for any exponent $r > 1$ as follows: For any $f \in$

$L^r(\Omega, \mathbb{R}^{N\times n})$, define

$$Tf = \nabla\Phi, \tag{4.5.17}$$

where Φ is the unique map in $L^r(\Omega, \mathbb{R}^N)$ such that the decomposition

$$f = \nabla\Phi + \sigma$$

holds with some divergence-free $\sigma \in L^r(\Omega, \mathbb{R}^{N\times n})$. It is easy to see from Proposition 4.5.6 that T is linear and bounded on $L^r(\Omega, \mathbb{R}^{N\times n})$.

Now we present the proof of Theorem 4.1.2.

Proof of Theorem 4.1.2. Fix an exponent $p \in (1, \infty)$, a ball $B_R = B(x_0, R) \subset\subset \Omega$ and a p-harmonic map $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. By the regularity result of p-harmonic maps (see [Tol83], also [Uhl77, GM86a, AF89]), we know that $u \in C^{1,\beta}_{loc}(\Omega, \mathbb{R}^N)$ with some $\beta \in (0, 1)$ and is thus in $C^{1,\beta}(\bar{B}_R, \mathbb{R}^N)$. Then it is possible to take the q-harmonic map $v \in W^{1,q}(B_R, \mathbb{R}^N)$ with $u - v \in W^{1,p}_0(B_R, \mathbb{R}^N)$.

In the following, we carefully determine an interval $[p_1, p_2]$ and the range of q, with which we can apply Theorem 4.5.4 to estimate the difference between ∇u and ∇v.

Let q be in the interval $(p-\sigma_1, p+\sigma_2)$ with $\sigma_1, \sigma_2 > 0$ to be determined. We particularly require $\sigma_1 \le p - 1$ so that $q > 1$. Set $s := \frac{q}{p-1}$. To make sure $s \in [p_1, p_2]$, we need

$$\frac{p - \sigma_1}{p - 1} \ge p_1, \quad \text{and} \quad \frac{p + \sigma_2}{p - 1} \le p_2, \tag{4.5.18}$$

which implies $p' = \frac{p}{p-1} \in (p_1, p_2)$ and

$$0 < \sigma_1 \le (p - 1)(p' - p_1), \quad 0 < \sigma_2 \le (p - 1)(p_2 - p'). \tag{4.5.19}$$

Now we take p_1, p_2 as follows:

$$p_1 = \frac{1}{2}\left(1 + \frac{p'}{p}\right) = \frac{2p - 1}{2(p - 1)}, \quad \text{and} \quad p_2 = 2p' = \frac{2p}{p - 1}. \tag{4.5.20}$$

Then the ranges for σ_1 and σ_2 become

$$0 < \sigma_1 \le \min\left\{\frac{1}{2}, p - 1\right\}, \quad \text{and} \quad 0 < \sigma_2 \le p. \tag{4.5.21}$$

To get some space for manipulation, we set

$$\sigma_1 = \min\left\{\frac{1}{4}, \frac{p - 1}{2}\right\}, \quad \text{and} \quad \sigma_2 = \frac{p}{2}, \tag{4.5.22}$$

with which the exponent q should satisfy

$$\max\left\{p - \frac{1}{4}, \frac{p + 1}{2}\right\} < q < \frac{3p}{2}. \tag{4.5.23}$$

As in Subsection 4.5.2, we need $2\rho < \min\{\frac{s}{p_1} - 1, 1 - \frac{s}{p_2}\}$, where ρ is the radius of the range of the exponent z in the commutator T^z. The choice of p_1, p_2 (4.5.20) and the range of q (4.5.23) together imply

$$\frac{s}{p_1}-1=\frac{2q}{2p-1}-1>\max\left\{\frac{1}{2(2p-1)},\frac{2-p}{2p-1}\right\}\geq\frac{1}{2(2p-1)},$$
$$1-\frac{s}{p_2}=1-\frac{q}{2p}=1-\frac{3}{4}>\frac{1}{4}.$$

Then take $\rho=\min\{\frac{1}{4(2p-1)},\frac{1}{8}\}$, which is independent of the precise values of s and q as long as they are in the required ranges. For any complex number $z=x+iy\in B_{\mathbb{C}}(0,\rho)$ and $f\in L^s(B_R,\mathbb{R}^{N\times n})$, the estimate

$$\|T^z f\|_{L^{\frac{s}{1+x}}}\leq C(p_1,p_2,n,N)\frac{|z|}{\rho}\|f\|_{L^s}^{1+x} \tag{4.5.24}$$

holds true by Theorem 4.5.4, where T is defined as in (4.5.17) and T^z as in (4.5.4).

Take $f=|\nabla u|^{p-2}\nabla u$, then we have $f\in L^s(B_R,\mathbb{R}^{N\times n})$ and $Tf=0$ as u solves the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0$$

in the distributional sense. Set $z=\frac{q-p}{p-1}$, then there holds

$$|f|^z f=||\nabla u|^{p-2}\nabla u|^{\frac{q-p}{p-1}}|\nabla u|^{p-2}\nabla u=|\nabla u|^{q-2}\nabla u\in L^{q'}(B_R,\mathbb{R}^{N\times n}),$$

where $q'=\frac{q}{q-1}$. According to (b), the vector-valued function $|f|^z f$ can be decomposed as follows

$$|f|^z f=\nabla\Phi+\sigma, \tag{4.5.25}$$

where $\Phi\in W_0^{1,q'}(B_R,\mathbb{R}^N)$ and $\sigma\in L^{q'}(B_R,\mathbb{R}^{N\times n})$ with $\operatorname{div}\sigma=0$. When the exponent q satisfies (4.5.23) and is close enough to p such that $z\in B_{\mathbb{C}}(0,\rho)$, the estimate (4.5.24) implies

$$\|\nabla\Phi\|_{L^{q'}}=\|T^z f\|_{L^{\frac{s}{1+x}}}\leq C(p,n,N)\frac{|q-p|}{p-1}\|f\|_{L^s}^{1+x}=C(p,n,N)\frac{|q-p|}{p-1}\|\nabla u\|_{L^q}^{q-1}. \tag{4.5.26}$$

We are now at the place to compare the L^q-integrals of ∇u and ∇v. Lemma 4.5.9 and the fact that $v\in W_u^{1,q}(B_R,\mathbb{R}^{N\times n})$ imply the following estimate

$$\begin{aligned}
&\int_{B_R}(|\nabla u|^q-|\nabla v|^q)\,\mathrm{d}x\\
=&\int_{B_R}(|\nabla u|^q-|\nabla v|^q-q|\nabla v|^{q-2}\nabla v\cdot(\nabla u-\nabla v))\,\mathrm{d}x\\
\leq& C(q)\int_{B_R}(|\nabla u|^2+|\nabla v|^2)^{\frac{q-2}{2}}|\nabla u-\nabla v|^2\,\mathrm{d}x\\
\leq& C(q)\int_{B_R}(|\nabla v|^q-|\nabla u|^q-q|\nabla u|^{q-2}\nabla u\cdot(\nabla v-\nabla u))\,\mathrm{d}x.
\end{aligned} \tag{4.5.27}$$

It is obvious that the integral $\int_{B_R}(|\nabla v|^q-|\nabla u|^q)\,\mathrm{d}x$ is non-positive. Considering the decomposition (4.5.25) and the corresponding estimate (4.5.26), we can further proceed with (4.5.27) as follows

$$\begin{aligned}
&\int_{B_R} (|\nabla u|^q - |\nabla v|^q)\,\mathrm{d}x \le C(q) \int_{B_R} (\nabla\Phi + \sigma)\cdot(\nabla u - \nabla v)\,\mathrm{d}x \\
&= C(q)\int_{B_R} \nabla\Phi\cdot(\nabla u - \nabla v)\,\mathrm{d}x \le C(q)\|\nabla\Phi\|_{L^{q'}}\|\nabla u - \nabla v\|_{L^q} \qquad (4.5.28)\\
&\le C(p,q,n,N)\varepsilon^{-\frac{q'}{q}}\left(\frac{|q-p|}{p-1}\right)^{q'}\int_{B_R}|\nabla u|^q\,\mathrm{d}x + \varepsilon\int_{B_R}(|\nabla u|^q + |\nabla v|^q)\,\mathrm{d}x,
\end{aligned}$$

where the last inequality is obtained with Young's inequality and some $\varepsilon > 0$. If ε and $|q-p|$ are taken so small that we have

$$\varepsilon \le \frac{1}{4} \quad \text{and} \quad C(p,q,n,N)\varepsilon^{-q'/q}\left(\frac{|q-p|}{p-1}\right)^{q'} \le \varepsilon, \qquad (4.5.29)$$

the inequality (4.5.28) implies

$$\begin{aligned}
\int_{B_R}|\nabla u|^q\,\mathrm{d}x &\le \frac{1+\varepsilon}{1-\varepsilon-C(p,q,n,N)\varepsilon^{-q'/q}\left(\frac{|q-p|}{p-1}\right)^{q'}}\int_{B_R}|\nabla v|^q\,\mathrm{d}x \qquad (4.5.30)\\
&\le \frac{1+\varepsilon}{1-2\varepsilon}\int_{B_R}|\nabla v|^q\,\mathrm{d}x.
\end{aligned}$$

We now specify the range of q. Since q is taken under the condition (4.5.23), the constant $C(p,q,n,N)$ can be controlled by some $c = c(p,n,N) > 0$. Considering (4.5.29), we first take $\varepsilon = \frac{1}{4}$, and then the second inequality gives

$$\frac{|q-p|}{p-1} \le \left(\frac{\varepsilon^{q'}}{c}\right)^{\frac{1}{q'}} = \frac{1}{4}c^{-\frac{1}{q'}}.$$

The constant $c(p,n,N)$ can be taken in $[1,\infty)$, and then the range (4.5.23) of q implies

$$\frac{1}{4}c^{-\frac{1}{q'}} \ge \frac{1}{4}c^{-e(p)},$$

where $e(p) = (\frac{3p}{2})' = \frac{3p}{3p-2}$. Set

$$\tau := \min\left\{\rho, \frac{1}{4}c^{-e(p)}\right\} = \min\left\{\frac{1}{4(2p-1)}, \frac{1}{8}, \frac{1}{4}c^{-e(p)}\right\},$$

then for any $q \in (p-\tau, p+\tau)$ the condition (4.5.29) is satisfied with $\varepsilon = \frac{1}{4}$, and (4.5.30) holds true with constant $\frac{5}{2}$ on the right-hand side. When $q \to p$, we can take the constant $\varepsilon = \frac{|q-p|}{p-1}c^{e(p)}$, and thus the constant $Q_q := \frac{1+\varepsilon(q)}{1-2\varepsilon(q)}$ converges to 1 as $q \to p$.

Notice that the size and location of the ball B_R does not affect the argument above or the constants involved, and (4.5.30) holds for any ball $B_R \Subset \Omega$ with the same constant. This indicates that u is a spherical (Q_q, q)-minimizer on Ω. □

Remark 4.5.8. The above proof applies to not only balls but also subdomains of Ω for which the corresponding constants in (4.5.16) are bounded uniformly. Thus, apart from the C^1-boundary condition, we also need some restrictions on the “shape” of the subdomains as the constant C in (4.5.16) comes from a covering argument.

At the end, we show the lemma used above to compare ∇u and ∇v.

Lemma 4.5.9. *Given $q \in (1, \infty)$, ther exists a positive constant $c(q)$ such that the following estimate*

$$\begin{aligned}\frac{1}{c(q)}(|x|^2 + |y|^2)^{\frac{q-2}{2}}|x-y|^2 \leq |x|^q - |y|^q - q|y|^{q-2}y \cdot (x-y)& \\ \leq c(q)(|x|^2 + |y|^2)^{\frac{q-2}{2}}|x-y|^2& \qquad (4.5.31)\end{aligned}$$

holds for any $x, y \in \mathbb{R}^d$ for some $d \in \mathbb{N}^+$.

Notice that when $y = 0$, the quantity $|y|^{q-2}y \cdot (x-y)$ is considered as the limit

$$\lim_{y\to 0}|y|^{q-2}y \cdot (x-y).$$

It is the same for $(|x|^2 + |y|^2)^{\frac{q-2}{2}}|x-y|^2$ when $x = y = 0$.

Proof. The inequalities hold true trivially when $x = y$ or $y = 0$. Then it is sufficient to show the case where $x \neq y$ and $y \neq 0$. When 0 is on the segment connecting x and y, set $t = \frac{|x|}{|y|} \geq 0$ and thus $x = -ty$. Then we have

$$\frac{|x|^q - |y|^q - q|y|^{q-2}y \cdot (x-y)}{(|x|^2 + |y|^2)^{\frac{q-2}{2}}|x-y|^2} = \frac{t^q + qt + q - 1}{(t^2+1)^{\frac{q-2}{2}}(t+1)^2},$$

which is bounded from above and away from zero. Therefore, in this case the inequalities in (4.5.31) hold.

When 0 is not on the segment connecting x and y, consider $g(x) := |x|^q$, $x \in \mathbb{R}^d$. The first and second order derivatives of g satisfy

$$\begin{gathered}g'(x) = q|x|^{q-2}x, \\ g''(x)[w,w] = q(q-2)|x|^{q-4}|w\cdot x|^2 + q|x|^{q-2}|w|^2\end{gathered}$$

for any $x, w \in \mathbb{R}^d$ with $x \neq 0$. It is easy to see

$$q(q-1)|z|^{q-2}|w|^2 \leq g''(z)[w,w] \leq q|z|^{q-2}|w|^2$$

when $q \in (1, 2)$, and that the inverse inequalities are valid when $q \geq 2$. With Taylor's theorem we have

$$|x|^q - |y|^q - q|y|^{q-2}y \cdot (x-y) = \int_0^1 (1-t)g''(y + t(x-y))[x-y, x-y]\,\mathrm{d}t,$$

and then (4.5.31) follows from the fact that $\int_0^1 |y + t(x-y)|^{q-2}(1-t)\,\mathrm{d}t$ is comparable to $(|x|^2+|y|^2)^{\frac{q-2}{2}}$, which can be obtained from Lemma 2.9.3. □

Chapter 5

Partial regularity for ω-minimizers of quasiconvex functionals

The focus of this chapter is partial regularity for ω-minimizers of quasiconvex functionals. We establish a first-order and another two zero-order results corresponding to different assumptions on ω separately, which are stated in Section 5.1 below.

In the first section, we state the assumptions and the results with a summary of the corresponding strategies for the proofs and the difficulties in each setting. Section 5.2 is devoted for some preliminary results for the reference integrand E_p and the shifted integrand f_w, which will be useful in the proof. Theorem 5.1.1 is a first-order regularity result in the linear growth setting under the Dini-type condition ($\boldsymbol{\omega}$**3**), and the proof is in Section 5.3. In Section 5.4, Theorem 5.1.2, a zero-order regularity result for sub-quadratic growth functionals, is proved with a similar strategy, and thus some details are omitted to avoid repetition. At the end of Section 5.4, we sketch the proof of Theorem 5.1.3, an analogue of Theorem 5.1.2 in the linear growth case, based on the proofs of the previous two results.

5.1 Assumptions and main results

We first contextualise the results. The functional considered in this chapter is of p-growth and strongly quasiconvex. Precisely, we focus on the following functional

$$\mathscr{F}(u,\Omega) := \int_\Omega f(\nabla u)\,\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies:

(**LP1**) $|f(z)| \le L(1+|z|^p)$ for any $z \in \mathbb{R}^{N\times n}$ with $p \ge 1$ and $L > 0$;

(**LP2**) f is strongly quasiconvex in the sense that $f - \ell E_p$ is quasiconvex for some $\ell > 0$;

(**LP3**) f is in $C^{2,1}_{loc}(\mathbb{R}^{N\times n})$.

We emphasize that an extra lower bound for f is not posed here. Moreover, the convexity assumption in this chapter is Morrey's quasiconvexity (in a strong sense), which is the natural condition for vectorial variational problems.

In this chapter and the last one, we consider ω-minimizers. The ω here is a non-negative function defined on $[0,\infty)$, which measures the deviation of an ω-minimizers from being a minimizer, and is required to be small around the origin:

($\boldsymbol{\omega}$**1**) ω is bounded by 1 and nondecreasing with $\omega(0) = \lim_{r\to 0}\omega(r) = 0$.

The first result, partial regularity for the derivatives, requires a finer description of the smallness of ω:

($\boldsymbol{\omega}$**2**) There exists $\beta \in (0,1)$ such that $r \mapsto \frac{\omega(r)}{r^{2\beta}}$ is non-increasing in $(0,\infty)$;

($\boldsymbol{\omega}$**3**) The Dini-type condition: for any $\rho > 0$, we have $\Xi_{\frac{1}{4}}(\rho) < \infty$, where

$$\Xi_\alpha(\rho) := \int_0^\rho \frac{\omega^\alpha(r)}{r}\,\mathrm{d}r.$$

Sometimes, a more specific control of ω is assumed:

($\boldsymbol{\omega}$**4**) $\omega(r) \le Ar^{2\sigma}$ for some $\sigma \in (0,1)$.

With such an assumption, the discussion becomes more straightforward and simpler, and sometimes stronger results are approachable. (See [Giu03, KM05] and [DT15, DET19], for example.) In this case, condition ($\boldsymbol{\omega}$**3**) is satisfied, while ($\boldsymbol{\omega}$**2**) might not hold any more.

The following is the first result of this chapter, partial regularity for the first order derivatives of BV ω-minimizers under the above Dini-type condition. Since the integrand considered in this result and Theorem 5.1.3 below is of linear growth, we consider the relaxed functional $\bar{\mathscr{F}}$, which is defined in Subsection 3.1.4 as follows

$$\bar{\mathscr{F}}(u,\Omega) = \int_\Omega f(Du) = \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u|$$

for any $u \in BV(\Omega,\mathbb{R}^N)$. The corresponding definition of ω-minimizers can be found in Definition 3.1.6, and we take the definition (3.1.29) in this chapter.

Theorem 5.1.1. *Suppose that the function f satisfies* (**LP1**)-(**LP3**) *with $p = 1$, and ω satisfies* ($\boldsymbol{\omega}$**1**)-($\boldsymbol{\omega}$**3**). *If $u \in BV_{loc}(\Omega,\mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$, then it is partially regular in the following sense: there exists a relatively closed $\mathscr{L}^n$-null set $S_u \subset \Omega$ such that u is C^1 on $\Omega\setminus S_u$. Furthermore, we have that*

(a) *$Du \llcorner (\Omega\setminus S_u) = \nabla u \mathscr{L}^n$, and ∇u has a local modulus of continuity $\rho \mapsto \rho^\alpha + \Xi_{\frac{1}{2}}(\rho)$ on $\Omega\setminus S_u$ for any $\alpha \in (0,1)$;*

(b) *the singular set S_u is contained in $\Sigma_1 \cup \Sigma_2$, where*

$$\Sigma_1 := \left\{x \in \Omega\colon \liminf_{\rho\to 0^+} \fint_{B_\rho(x)} E(Du - (Du)_{x,\rho}) > 0\right\}, \tag{5.1.1}$$

$$\Sigma_2 := \left\{x \in \Omega\colon \limsup_{\rho\to 0^+} |(Du)_{x,\rho}| = \infty\right\}. \tag{5.1.2}$$

In particular, if (**ω4**) holds, we have $u \in C^{1,\sigma}_{loc}(\Omega \setminus S_u, \mathbb{R}^N)$. Example 3.1.9 shows that the modulus of continuity $\rho \mapsto \rho^\alpha + \Xi_{\frac{1}{2}}(\rho)$ obtained here is sharp.

Partial regularity for ω-minimizers under the Dini-type condition (like (**ω3**)) has been proved in the super-linear setting (see [DGG00, DK02, DGK05]). The result above gives the counterpart in the end point setting ($p = 1$), where there are difficulties from the low integrability of BV maps as stated in the discussion after Theorem 5.1.3.

In the proof of Theorem 5.1.1, the conditions (**ω2**) and (**ω3**) appear in the iteration of the excess decay estimate (Subsection 5.3.5), and the deduction from such an estimate to the final Hölder regularity (Lemma 2.2.19). Then it is natural to ask what happens if we only assume (**ω1**), i.e., the smallness of ω near the origin. In this case, the strategy used for Theorem 5.1.1 no longer works and the regularity of Du as above is no longer expected. Even if it is possible to obtain a similar excess decay estimate

$$\mathscr{E}(\rho) \le C\left(\frac{\rho}{R}\right)^{2\alpha} \mathscr{E}(R) + C\sqrt{\omega(\rho)}, \tag{5.1.3}$$

only the smallness of $\omega(\rho)$ near the origin does not give any further information of Du or $\mathscr{E}(\rho)$, since $\mathscr{E}(\rho) \to 0$ as $\rho \to 0^+$ near $\mathscr{L}^n$-almost every point in Ω. However, it is still possible to get partial Hölder continuity of u in both the subquadratic (cf. [DGK05]) and the linear settings, which is stated in Theorem 5.1.2 and 5.1.3.

For the second result, a more precise characterisation of the second derivatives of f is required and we replace (**LP3**) by the following with $L > 0$:

(**LP3**$_1$) f is C^2 with $|f''(z)| \le L(1+|z|)^{p-2}$ for any $z \in \mathbb{R}^{N\times n}$;

(**LP3**$_2$) f'' is Lipschitz and satisfies

$$|f''(z_1) - f''(z_2)| \le \frac{L|z_1 - z_2|}{(1+|z_1|+|z_2|)^{3-p}}, \quad \text{for any } z_1, z_2 \in \mathbb{R}^{N\times n}.$$

Theorem 5.1.2. *Suppose that the function f satisfies* (**LP1**), (**LP2**), (**LP3**$_1$) *and* (**LP3**$_2$) *with $1 < p < 2$, and ω satisfies* (**ω1**). *If $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of $\mathscr{F}$, then it is partially regular in the following sense: there exists a relatively closed $\mathscr{L}^n$-null set $S'_u \subset \Omega$ such that $u \in C^{0,\alpha}_{loc}(\Omega \setminus S'_u, \mathbb{R}^N)$ for any $\alpha \in (0,1)$. The singular set S'_u is contained in Σ_3, where*

$$\Sigma_3 := \left\{ x \in \Omega \colon \liminf_{\rho \to 0^+} \fint_{B_\rho(x)} E\left(\frac{Du - (Du)_{x,\rho}}{1+|(Du)_{x,\rho}|}\right) > 0 \right\}. \tag{5.1.4}$$

Notice that the condition on ω in the above theorem is much weaker than that in Theorem 5.1.1, as there is no control on the smallness of ω near the origin. In the work by Gmeineder & Kristensen [GK19a], it is shown that BV generalised minimizers of quasiconvex functionals are partially $C^{1,\alpha}$ for any $\alpha \in (0,1)$, which also holds for $p > 1$ (see [Eva86, GM86b, CFM98], and the references in Section 3.4). Thus, comparing the two results above, one can see that the behaviour of ω around the origin determines how much regularity an ω-minimizer can “inherit” from the corresponding minimizer(s).

The key step in our proof is to establish an excess decay estimate, which describes the oscillation of the first-order derivatives of an ω-minimizer, or how much an ω-minimizer locally differs from a certain affine function. With a harmonic approximation process and a Caccioppoli's inequality, one can transfer such an estimate to an ω-minimizer from a corresponding harmonic map, which solves a relevant linear elliptic system.

The proofs of the two results above are in similar spirit and there are several difficulties specially in our situation. One appears in the harmonic approximation, where it is impossible to work in the natural space $W^{1,2}$ for a linear elliptic system. This is due to the low integrability in the sub-quadratic setting ($1 \leq p < 2$). We also emphasize that in the linear setting, a weak reverse Hölder inequality is unavailable as in Subsection 3.2.1 and 4.4.4. Thus, one cannot apply Gehring's lemma to obtain a higher integrability, which is usually helpful in showing an excess decay estimate in some similar cases (cf. Subsection 5.3.7). The approximation process in Subsection 5.3.3 is adapted from an approach by Gmeineder & Kristensen ([GK19a], Section 4.3), and an explicit test map is constructed with a Fubini type property of BV maps and truncation. The difference between minimizers and ω-minimizers also leads to an issue, as for the latter there is no Euler-Lagrange equation holding. Thanks to the almost-minimality, we are able to establish an Euler-Lagrange type inequality with the help of Ekeland's variational principle.

The study without control on the smallness of ω is more subtle. The excess decay estimate cannot be carried out as in the proof of Theorem 5.1.1, and the same regularity of Du is not expected any more. Instead of estimating the typical excess, we normalise it by $1 + |(Du)_{x_0,R}|$ and then try to control the oscillation of Du on that scale. This method is inspired by [FM08], where the authors studied elliptic systems (variational functionals) with coefficients $a(x, u, Du)$ (integrands $F(x, u, Du)$) only continuous in (x, u) in the context $p \geq 2$. The solutions (minimizers) in this case may be considered as almost minimisers of a family of functionals (see [DGG00], Section 2). Similar results for scalar minimizers can be found in [CFP99, Man88, Man86, Min06].

Notice that Theorem 5.1.2 is only for the setting $1 < p < 2$. In the linear growth setting, the second derivative matrix of the integrand f is degenerately elliptic (see Section 1.2 and 6.1), and the normalisation does not work due to such an imbalance. Nevertheless, near a "good" point where the average of Du is uniformly bounded, it is possible to combine the arguments of the previous two theorems to obtain a partial zero-order regularity.

Theorem 5.1.3. *Suppose that the function f satisfies* (**LP1**)-(**LP3**) *with $p = 1$, and ω satisfies* ($\boldsymbol{\omega}$**1**). *If $u \in BV_{loc}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$, then it is partially regular in the following sense: there exists a relatively closed $\mathscr{L}^n$-null set $S'_u \subset \Omega$ such that $u \in C^{0,\alpha}_{loc}(\Omega \setminus S'_u, \mathbb{R}^N)$ with any $\alpha \in (0, 1)$. The singular set is contained in $\Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 are defined as in Theorem 5.1.1.*

We believe that the approach used to prove Theorem 5.1.1 also applies to ω-minimizers in the super-linear setting ($p > 1$), which are studied in [DGG00, DK02, DGK05]. In the sub-quadratic setting, an Euler-Lagrange type inequality is also obtained in [DGK05] (Lemma 5), with which the harmonic approximation is done indirectly. This method can also be adapted into our case (see Subsection 5.3.8 and Remark 5.4.6).

5.2 Preliminary results for integrands

To prepare for the proofs, we first give some estimates and properties for the reference integrand E_p (as defined in (2.1.4)) and the shifted version of f.

5.2.1 Reference integrand

Obviously, we have that $E_p(z) = \langle z\rangle^p - 1$ is C^2 in z, and an elementary calculation gives

$$E_p'(w) \cdot z = p\langle w\rangle^{p-2} w \cdot z, \tag{5.2.1}$$

$$E_p''(w)[z,z] = p\langle w\rangle^{p-4}(\langle w\rangle^2 |z|^2 + (p-2)|w\cdot z|^2). \tag{5.2.2}$$

Considering the two cases $p \in (1,2)$ and $p = 1$ separately, we have

$$E_p''(w)[z,z] \geq \begin{cases} p(p-1)\langle w\rangle^{p-2} |z|^2, & p \in (1,2) \\ \langle w\rangle^{-3} |z|^2, & p = 1. \end{cases} \tag{5.2.3}$$

Thus, the function E_p is a convex function. In the following, we only consider E_p with $1 \leq p < 2$. By the definition and convexity of E_p, it is easy to get the following:

Lemma 5.2.1. *Suppose that* $1 \leq p < 2$ *and set* $a_1 = \sqrt{2} - 1, a_2 = 1$. *Then the following holds*

$$a_1 \min\{|z|^p, |z|^2\} \leq E_p(z) \leq a_2 \min\{|z|^p, |z|^2\}, \tag{5.2.4}$$

$$E_p(az) \leq \max\{a, a^2\} E_p(z) \quad \text{and} \quad E_p(z+w) \leq 2(E_p(z) + E_p(w)) \tag{5.2.5}$$

for any $a > 0$ *and any* $z, w \in \mathbb{R}^d$ *with an arbitrary* $d \in \mathbb{N}^+$.

A corollary of (5.2.4) is

$$|z|^p \leq 1 + \frac{1}{a_1} E_p(z), \quad \text{for any } z \in \mathbb{R}^d,\ p \in [1,2). \tag{5.2.6}$$

Proof. The Taylor expansion of $E_p(z)$ in terms of $|z|^2$ at the point $|z|^2 = 0$ is

$$E_p(z) = 1 + \frac{p}{2}|z|^2 + O(|z|^4) - 1 = \frac{p}{2}|z|^2 + O(|z|^4),$$

which gives $E_p(z) \sim |z|^2$ when $|z| \ll 1$. It is obvious that $E_p(z) \sim |z|^p$ when $|z| \gg 1$.

To get more precise estimates, we consider the two cases $|z| > 1$ and $|z| \leq 1$ separately.

Case $t > 1$. We compare $E_p(t)$ and t^p.

$$E_p(t) = (1+t^2)^{\frac{p}{2}} - 1 \leq 1 + t^p - 1 = t^p,$$

then we have the right inequality. Set $g(t) := \frac{E_p(t)}{t^p}$, then

$$g'(t) = \frac{1}{t^{2p}}(p(1+t^2)^{\frac{p}{2}-1} t^{p+1} - p\, t^{p-1}((1+t^2)^{\frac{p}{2}} - 1)) = \frac{p}{t^{p+1}}(1 - (1+t^2)^{\frac{p}{2}-1}) > 0.$$

Thus, the function $g(t)$ is increasing on $[1, \infty)$ and $g(t) \geq g(1) = 2^{\frac{p}{2}} - 1 \geq \sqrt{2} - 1$.

Case $t \leq 1$. Consider $E_p(t) - t^2$ and $E_p(t) - (\sqrt{2}-1)t^2$ and differentiate them to get the desired result.

To show the first part of (5.2.5), consider $\ell(t) = (1+a^2t)^{\frac{p}{2}} - a^2(1+t)^{\frac{p}{2}} + a^2 - 1, t \geq 0$. When $a \geq 1$, differentiating ℓ gives

$$\ell'(t) = a^2\frac{p}{2}((1+a^2t)^{\frac{p}{2}-1} - (1+t)^{\frac{p}{2}-1}) \leq 0,$$

and thus $\ell(t) \leq \ell(0) = 0$ for any $t \geq 0$. In particular, we have $E_p(az) - a^2E_p(z) = \ell(|z|^2) \leq 0$ for $a \geq 1$. For $0 < a < 1$, the convexity of E_p implies $E_p(az) \leq aE_p(z)$.

The second estimate in (5.2.5) is a corollary of the convexity of E_p and the first one with $a = 2$:

$$E_p(z+w) = E_p\left(\frac{1}{2}(2z+2w)\right) \leq \frac{1}{2}(E_p(2z) + E_p(2w)) \leq 2(E_p(z) + E_p(w)).$$

□

Lemma 5.2.2. *Let $1 \leq p < 2$, $B \subset \mathbb{R}^n$ be an open ball and $u \in L^p(B, \mathbb{R}^m)$. Then for any $z \in \mathbb{R}^m$ we have*

$$\int_B E_p(u - u_B)\,\mathrm{d}x \leq 4\int_B E_p(u-z)\,\mathrm{d}x. \tag{5.2.7}$$

When $p = 1$, the function u can be replaced by a bounded $\mathbb{R}^m$-valued Radon measure, and the inequality holds in the relaxed sense as in (2.5.2).

Proof. For $u \in L^p(B, \mathbb{R}^m)$, the claimed inequality can be easily obtained with (5.2.5) and Jensen's inequality.

Assume $p = 1$ and fix a bounded $\mathbb{R}^m$-valued Radon measure μ. We extend it to $\mathbb{R}^n \setminus B$ by 0, i.e., for any Borel set $A \subset \mathbb{R}^n$ define $\mu(A) := \mu(A \cap B)$. Let $\{\phi_\varepsilon\}$ be the standard mollifiers and set $\mu_\varepsilon = \phi_\varepsilon * \mu$. Then $\mu_\varepsilon \in L^1(B, \mathbb{R}^m)$. Again by (5.2.5) and Jensen's inequality, we have

$$\int_B E_p(\mu_\varepsilon - (\mu_\varepsilon)_B)\,\mathrm{d}x \leq 2\int_B E_p(\mu_\varepsilon - z)\,\mathrm{d}x + 2\mathscr{L}^n(B)E_p((\mu_\varepsilon)_B - z) \tag{5.2.8}$$

$$\leq 4\int_B E_p(\mu_\varepsilon - z)\,\mathrm{d}x. \tag{5.2.9}$$

Lemma 2.5.2 implies that μ_ε converges to μ area-strictly. With Lemma 2.5.4 we can take $\varepsilon \to 0$ to obtain (5.2.7). □

The next lemma gives the estimate of $⨍_B |\mu|$ in terms of $⨍_B E_p(\mu)$.

Lemma 5.2.3. *Let $1 \leq p < 2$, $B \subset \mathbb{R}^n$ be an open ball and $f \in L^p(B, \mathbb{R}^m)$. Set $\mathcal{E} := ⨍_B E_p(f)\,\mathrm{d}x$, then we have*

$$⨍_B |f|^p\,\mathrm{d}x \leq \sqrt{\mathcal{E}^2 + 2\mathcal{E}}. \tag{5.2.10}$$

When $\mathcal{E} \leq a$, it is obvious that the right-hand side can be replaced by $\sqrt{(2+a)\mathcal{E}}$. When $p = 1$, we have the analogue holds for any bounded $\mathbb{R}^m$-valued Radon measure.

Proof. Suppose that $f \in L^p(B, \mathbb{R}^m)$, then Jensen's inequality implies

$$E\left(\fint_B |f|^p \, \mathrm{d}x\right) \leq \fint_B E(|f|^p) \, \mathrm{d}x \leq \fint_B E_p(f) \, \mathrm{d}x,$$

where we used the fact $E(t^p) \leq E_p(t)$. Notice that $E(t) = \sqrt{1+t^2} - 1$, which can be written as $t = \sqrt{E^2(t) + 2E(t)}$, and thus we have the desired result. By a mollification argument as in the proof of Lemma 5.2.2, we can also show the result for any bounded $\mathbb{R}^m$-valued Radon measure. □

By definition, we know that for $E_p^A, A \in \mathbb{R}^{N\times n}$, the analogues of Lemma 5.2.1 and 5.2.2 hold. Moreover, there exists $c > 0$ such that

$$\frac{1}{c} E_p^A(z) \leq \frac{|z|^2}{(1 + |A| + |z|)^{2-p}} \leq c E_p^A(z). \tag{5.2.11}$$

The following is a combination of Lemma 4.5.9 and Lemma 4.1 in [GK19a], and gives an estimate from below of the first order expansion of E_p.

Lemma 5.2.4. *There exist constants $c(p), c > 0$ such that the following holds for any $w, z \in \mathbb{R}^d$ with some $d \in \mathbb{N}^+$:*

$$E_p(z + w) - E_p(w) - E_p'(w) \cdot z \geq \begin{cases} c(p) E_p^w(z), & p \in (1,2), \\ cE(z) \langle w \rangle^{-3}, & p = 1. \end{cases} \tag{5.2.12}$$

Proof. When $p \in (1,2)$, by Taylor's theorem, (5.2.3) and Lemma 4.5.9 we have

$$\begin{aligned} E_p(z + w) - E_p(w) - E_p'(w) \cdot z &= \int_0^1 E''(w + tz)[z,z](1-t) \, \mathrm{d}t \\ &\geq c(p) \int_0^1 \frac{|z|^2}{(1 + |w + tz|^2)^{\frac{2-p}{2}}} \, \mathrm{d}t \\ &\geq c(p) \frac{|z|^2}{(1 + |w|^2 + |z|^2)^{\frac{2-p}{2}}}, \end{aligned}$$

and the desired result comes from (5.2.11).

For the other case $p = 1$, notice that $\langle w + tz \rangle \leq \langle w \rangle + t|z|$. Then with (5.2.3) we have

$$\begin{aligned} \int_0^1 E''(w + tz)[z,z](1-t) \, \mathrm{d}t &\geq c \int_0^1 \frac{(1-t)|z|^2}{\langle w + tz \rangle^3} \, \mathrm{d}t \\ &\geq c \int_0^1 \frac{(1-t)|z|^2}{(\langle w \rangle + t|z|)^3} \, \mathrm{d}t \\ &= \frac{c|z|^2}{\langle w \rangle^2 (\langle w \rangle + |z|)}. \end{aligned}$$

The final estimate can be obtained by considering the two cases $|z| > 1$ and $|z| \leq 1$ separately. □

5.2.2 Estimates for the shifted integrand

Given any C^2 function $f\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ and any $w\in\mathbb{R}^{N\times n}$, we define shifted integrand

$$\begin{aligned} f_w(z) &:= f(z+w)-f(w)-f'(w)\cdot z \\ &= \int_0^1 (1-t)f''(w+tz)[z,z]\,\mathrm{d}t. \end{aligned} \tag{5.2.13}$$

Lemma 5.2.5. *Suppose that $f\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ is C^2 and satisfies (**LP2**). When $p\in(1,2)$, there holds, with some constant $c=c(p)>0$,*

$$\int_B f_w(\nabla\varphi)\,\mathrm{d}x \ge c\,\ell\int_B E_p^w(\nabla\varphi)\,\mathrm{d}x, \tag{5.2.14}$$

$$f''(w)[\eta\otimes\xi,\eta\otimes\xi] \ge c\,\ell\,\frac{|\eta|^2|\xi|^2}{\langle w\rangle^{2-p}} \tag{5.2.15}$$

for any ball $B\subset\mathbb{R}^n$, $w\in\mathbb{R}^{N\times n}$, $\varphi\in W_0^{1,p}(B,\mathbb{R}^N)$, $\eta\in\mathbb{R}^N$ and $\xi\in\mathbb{R}^n$. For $p=1$, the corresponding estimates are, with some $c'>0$,

$$\int_B f_w(\nabla\varphi)\,\mathrm{d}x \ge c'\ell\int_B \langle w\rangle^{-3}E(\nabla\varphi)\,\mathrm{d}x, \tag{5.2.16}$$

$$f''(w)[\eta\otimes\xi,\eta\otimes\xi] \ge c'\ell\,\frac{|\eta|^2|\xi|^2}{\langle w\rangle^{3}}. \tag{5.2.17}$$

Proof. With the quasiconvexity condition and Remark 2.6.2, we have

$$\begin{aligned} \int_B f_w(\nabla\varphi)\,\mathrm{d}x &= \int_B (f(w+\nabla\varphi)-f(w))\,\mathrm{d}x \\ &\ge \ell\int_B (E_p(w+\nabla\varphi)-E_p(w))\,\mathrm{d}x \\ &= \ell\int_B (E_p(w+\nabla\varphi)-E_p(w)-E_p'(w)\cdot\nabla\varphi)\,\mathrm{d}x, \end{aligned}$$

where we used $\int_B z\cdot\nabla\varphi\,\mathrm{d}x=0$ for any $z\in\mathbb{R}^{N\times n}$ and any $\varphi\in W_0^{1,p}(B,\mathbb{R}^N)$. Lemma 5.2.4 then implies (5.2.14) and (5.2.16).

Since quasiconvexity implies rank-one convexity (see (2.6.2) or [Dac08], Theorem 5.4), there holds

$$f''(w)[\eta\otimes\xi,\eta\otimes\xi]-\ell E_p''(w)[\eta\otimes\xi,\eta\otimes\xi]\ge 0.$$

Then (5.2.3) gives the claimed estimates (5.2.15) and (5.2.17). □

The next two lemmas allow us to control some f_w related quantities with certain variants of E_p.

Lemma 5.2.6. *Suppose that $f\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ satisfies (**LP1**)-(**LP3**) with $p\in[1,2)$. Then for any $m>0$ and any $w\in\mathbb{R}^{N\times n}$ satisfying $|w|\le m$, we have*

$$|f_w(z)|\le CE_p(z) \tag{5.2.18}$$

hold for $C=C(m,n,N,L,p)>0$.

Alternatively, if we only assume (**LP3$_1$**) *with* $p \in (1,2)$*, the estimate becomes*

$$|f_w(z)| \le C E_p^w(z) \tag{5.2.19}$$

for any $w \in \mathbb{R}^{N\times n}$ *with a fixed constant* $C = C(L,p) > 0$.

Proof. The first estimate (5.2.18) can be shown directly from the definition by considering the two cases $|z| \le 1$ and $|z| > 1$ separately. For the former case, we have

$$|f_w(z)| = \left| \int_0^1 (1-t) f''(w+tz)[z,z]\,\mathrm{d}t \right| \le \frac{1}{2} \sup_{|w|\le m+1} |f''(w)||z|^2 \le C(m,p)E_p(z).$$

In the latter, each term in the expression of $f_w(z)$ can be estimated with (**LP1**) and Lemma 2.6.6:

$$\begin{aligned} |f_w(z)| &= |f(z+w) - f(w) - f'(w)z| \\ &\le |f(z+w)| + |f(w)| + |f'(w)||z| \\ &\le L(2 + |z+w|^p + |w|^p) + CL(1+|w|^{p-1})|z|. \end{aligned}$$

From the facts $|w| \le m$ and $|z| > 1$, it is obvious that $|f_w(z)|$ can be controlled by $E_p(z)$ up to a constant.

For (5.2.19), by the definition of f_w, Taylor's theorem and (**LP3$_1$**), we have the estimate

$$\begin{aligned} |f_w(z)| &= |f(z+w) - f(w) - f'(w)z| \\ &= \left| \int_0^1 f''(w+tz)(1-t)\,\mathrm{d}t\,[z,z] \right| \\ &\le \int_0^1 \frac{(1-t)L}{(1+|w+tz|)^{2-p}}\,\mathrm{d}t|z|^2. \end{aligned}$$

Lemma 2.9.3 implies that the integral in the last line is controlled by $CL(1+|w|+|z|)^{p-2}$. The estimate (5.2.11) then gives the desired result. ☐

Lemma 5.2.7. *Suppose that* $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *satisfies* (**LP1**)-(**LP3**) *with* $p \in [1,2)$. *Then for any* $m > 0$ *and any* $w \in \mathbb{R}^{N\times n}$ *with* $|w| \le m$*, there exists a constant* $C = C(m,n,N,L,p) > 0$ *such that*

$$|f_w''(0)z - f_w'(z)| \le CE(z). \tag{5.2.20}$$

Alternatively, with (**LP3$_1$**) and (**LP3$_2$**) *instead of* (**LP3**)*, and no bound for* w*, we have*

$$|f_w''(0)z - f_w'(z)| \le C(1+|w|)^{p-2}E^w(z) \tag{5.2.21}$$

with $C = C(n,N,L,p) > 0$.

Proof. The estimate (5.2.20) can be easily obtained by considering the two cases $|z| \le 1$ and $|z| > 1$ separately. When $|z| \le 1$, there holds

$$\begin{aligned} |f_w''(0)z - f_w'(z)| &= |f''(w)z - (f'(w+z) - f'(w))| \\ = \left| \int_0^1 (f''(w) - f''(w+tz)) \cdot z\,\mathrm{d}t \right| &\le C\int_0^1 t|z|^2\,\mathrm{d}t \le CE(z), \end{aligned}$$

where the last line is obtained with **(LP3)** and the fact $w+tz, w \in B(0,m+1)$. In the other case, we estimate the three terms directly with Lemma 2.6.6:

$$\begin{aligned}
&|f''_w(0)z - f'_w(z)| = |f''(w)z - (f'(w+z) - f'(w))| \qquad (5.2.22)\\
\le\ & C(m)|z| + CL(2 + |w+z|^{p-1} + |z|^{p-1}) \overset{|z|>1}{\le} C(m,L)|z| \le CE(z).
\end{aligned}$$

The proof of (5.2.21) is in a similar manner. When $|z| \le 1 + |w|$, the condition **(LP3**$_2$**)** implies

$$\begin{aligned}
\left|f''_w(0)z - f'_w(z)\right| &= \left|\int_0^1 (f''(w) - f''(w+tz)) \cdot z \,\mathrm{d}t\right|\\
&\le L\int_0^1 \frac{t\,|z|^2}{(1+|w|+|w+tz|)^{3-p}}\,\mathrm{d}t\\
&\overset{p<2}{\le} L(1+|w|)^{p-1}\frac{|z|^2}{(1+|w|)^2}\\
&\le C(1+|w|)^{p-2}E^w(z).
\end{aligned}$$

When $|z| > 1 + |w|$, the estimate can be obtained in a way similar to (5.2.22) with Lemma 2.6.6 and **(LP3**$_1$**)**. ☐

5.2.3 Mean coercivity

To close this part, we present the L^p mean coercivity of f, which helps us control the L^p-integral of $|\nabla v|$ for $v \in W^{1,p}$ by $\int f(\nabla v)\,\mathrm{d}x$. This is a variant of (iii) $\implies$ (ii) in Theorem 2.6.8.

Lemma 5.2.8. *Suppose that* $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *satisfies* **(LP1)** *and* **(LP2)** *with* $p \in [1,2)$*. Fix a ball* $B_R = B(x_0,R) \subset \mathbb{R}^n$ *and a map* $u \in W^{1,p}(B_R,\mathbb{R}^N)$*, then there exist* $a_3 = a_3(p,\ell) > 0, a_4 = a_4(n,N,L,\ell,p,f) \in \mathbb{R}$ *and* $a_5 = a_5(n,N,L,\ell,p) > 0$ *such that*

$$a_3 \fint_{B_R} |\nabla v|^p \,\mathrm{d}x + a_4 \le \fint_{B_R} f(\nabla v)\,\mathrm{d}x + a_5 \fint_{B_R} |\nabla u|^p \,\mathrm{d}x \qquad (5.2.23)$$

holds true for any $v \in W^{1,p}_u(B_R,\mathbb{R}^N)$.

Proof. First, with the triangle inequality and (5.2.6) we have

$$\begin{aligned}
\fint_{B_R} |\nabla v|^p \,\mathrm{d}x &\le C(p)\fint_{B_R} (|\nabla v - \nabla u|^p + |\nabla u|^p)\,\mathrm{d}x \qquad (5.2.24)\\
&\le C(p)\fint_{B_R} (1 + a_1^{-1}E_p(\nabla v - \nabla u) + |\nabla u|^p)\,\mathrm{d}x.
\end{aligned}$$

Notice that $v - u \in W^{1,p}_0(B_R,\mathbb{R}^N)$, then **(LP2)** implies

$$\ell \fint_{B_R} E_p(\nabla v - \nabla u)\,\mathrm{d}x \le \fint_{B_R} f(\nabla v - \nabla u)\,\mathrm{d}x - f(0). \qquad (5.2.25)$$

To estimate the integral on the right-hand side, we apply Lemma 2.6.6 to get

$$\begin{aligned}
&\left|\fint_{B_R} (f(\nabla v - \nabla u) - f(\nabla v))\,\mathrm{d}x\right| \leq \fint_{B_R}\int_0^1 \left|f'(\nabla v - t\nabla u)\right| |\nabla u| \,\mathrm{d}t\,\mathrm{d}x \qquad (5.2.26)\\
&\leq CL\fint_{B_R}\int_0^1 (1+|\nabla v - t\nabla u|^{p-1})\,|\nabla u| \,\mathrm{d}t\,\mathrm{d}x\\
&\leq CL\fint_{B_R} (|\nabla u| + |\nabla u|^p + |\nabla u|\,|\nabla v|^{p-1})\,\mathrm{d}x\\
&\leq CL\fint_{B_R} (1 + (1+\sigma^{1-p})\,|\nabla u|^p + \sigma\,|\nabla v|^p)\,\mathrm{d}x,
\end{aligned}$$

where the σ is to be determined. Combining (5.2.24)-(5.2.26), we know that

$$\begin{aligned}
\fint_{B_R} |\nabla v|^p \,\mathrm{d}x \leq c_1 &\fint_{B_R} (1 + (1+\sigma^{1-p})\,|\nabla u|^p + \sigma\,|\nabla v|^p)\,\mathrm{d}x\\
&+ c_2\fint_{B_R} (f(\nabla v) - f(0))\,\mathrm{d}x + C(p),
\end{aligned}$$

where $c_1 = c_1(n,N,p,L,\ell) > 0, c_2 = c_2(p,\ell) > 0$. Take $\sigma = \frac{1}{2c_1}$, then (5.2.23) follows. ☐

Remark 5.2.9. The convexity of $|\cdot|$ together with Remark 2.5.6 tells us that (5.2.23) can be extended to maps in $BV_u(B_R,\mathbb{R}^N)$ if $p = 1$.

5.3 First-order partial regularity

This section is devoted to the proof of Theorem 5.1.1, the partial regularity for the first derivatives of BV ω-minimizers when ω satisfies **($\boldsymbol{\omega}$2)** and the Dini-type condition **($\boldsymbol{\omega}$3)**. The process is composed of six steps, each corresponding to one subsection below. An indirect argument is given at the end, in which the harmonic approximation is proved by contradiction as in [DGK05]. Throughout this section, we fix a BV ω-minimizer u of $\bar{\mathscr{F}}$ as defined in Definition 3.1.6.

5.3.1 Caccioppoli-type inequality

We now give a Caccioppoli-type inequality of the second kind for u. Such an inequality allows us to control Du with u in the integral sense.

Proposition 5.3.1. *Suppose that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* **(LP1)**-**(LP3)** *with $p = 1$, and $u \in BV_{loc}(\Omega,\mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$ with constant $R_0 > 0$, where ω satisfies* **($\boldsymbol{\omega}$1)**. *Then for any $m > 0$, there exists $c = c(m,n,N,L,\ell) \geq 1$ such that the following holds: for any $B(x_0,R) \subset\subset \Omega$ with $R < R_0$, and any affine map $a\colon \mathbb{R}^n \to \mathbb{R}^N$ satisfying $|\nabla a| \leq m$, we have*

$$\int_{B_{\frac{R}{2}}} E(D(u-a)) \leq c\left(\int_{B_R} E\left(\frac{u-a}{R}\right)\mathrm{d}x + \omega(R)R^n\right). \qquad (5.3.1)$$

Proof. Let $\tilde{f} = f_{\nabla a}$ as defined in (5.2.13), $\tilde{u} = u - a$. Then it is obvious that $\tilde{f}$ is quasiconvex and $\tilde{u}$ is an ω-minimizer of the functional corresponding to $\tilde{f}$. Fix r, s such that $\frac{R}{2} < r < s <$

R. Take a smooth cut-off function ρ between B_r and B_s with $\rho \in C_c^\infty(B_s)$ and $|\nabla \rho| \le \frac{2}{s-r}$, and set $\varphi = \rho\tilde{u}, \psi = (1-\rho)\tilde{u}$. Let $\{\phi_\varepsilon\}$ be the standard mollifiers and $\varphi_\varepsilon = \varphi * \phi_\varepsilon$, then $\varphi_\varepsilon \in W_0^{1,1}(B_s, \mathbb{R}^N)$ when $\varepsilon < \operatorname{dist}(\operatorname{supp}(\rho), \partial B_s)$.

The strong quasiconvexity of f gives, by (5.2.16),

$$C(m,\ell)\int_{B_s} E(\nabla\varphi_\varepsilon)\,\mathrm{d}x \le \int_{B_s} \tilde{f}(\nabla\varphi_\varepsilon)\,\mathrm{d}x.$$

It is easy to see that φ_ε converges to φ in the area-strict sense in $BV(B_s, \mathbb{R}^N)$. Take $\varepsilon \to 0$, then by the quasiconvexity of $\tilde{f}$ and E (E is actually convex), and Lemma 2.5.5, there holds

$$C\int_{B_s} E(D\varphi) \le \int_{B_s} \tilde{f}(D\varphi).$$

We can further proceed as follows:

$$\begin{aligned}
&C\int_{B_r} E(D\tilde{u}) \le C\int_{B_s} E(D\varphi) \le \int_{B_s} \tilde{f}(D\varphi)\\
=\ &\int_{B_s} \tilde{f}(D\tilde{u}) + \int_{B_s} \tilde{f}(D\varphi) - \int_{B_s} \tilde{f}(D\tilde{u})\\
\le\ &\int_{B_s} \tilde{f}(D\psi) + \omega(s)\int_{B_s}(1+|D\psi|) + \int_{B_s} \tilde{f}(D\varphi) - \int_{B_s} \tilde{f}(D\tilde{u})\\
\overset{(5.2.18)}{\le}\ &C\int_{B_s} E(D\psi) + \omega(s)\int_{B_s}(1+|D\psi|) + C\int_{B_s\setminus B_r}(E(D\varphi)+E(D\tilde{u})).
\end{aligned}$$

The second term can be estimated by the fact $|z| \le \frac{1}{a_1}E(z) + 1$:

$$\begin{aligned}
\omega(s)\int_{B_s}(1+|D\psi|) &\le \omega(s)\left(\mathscr{L}^n(B_s) + \int_{B_s}|D\psi|\right)\\
&\le 2\mathscr{L}^n(B_s)\omega(s) + C\int_{B_s} E(D\psi).
\end{aligned}$$

Inserting this into the estimate of $C\int_{B_r} E(D\tilde{u})$, we obtain

$$\begin{aligned}
&\int_{B_r} E(D\tilde{u}) \le C\left(\int_{B_s} E(D\psi) + \int_{B_s\setminus B_r}(E(D\varphi)+E(D\tilde{u})) + \omega(s)s^n\right)\\
&= C\left(\int_{B_s} E((1-\rho)D\tilde{u} - \tilde{u}\otimes\nabla\rho) + \int_{B_s\setminus B_r}(E(D\tilde{u}) + E(\rho D\tilde{u} + \tilde{u}\otimes\nabla\rho)) + \omega(s)s^n\right)\\
&\le C\left(\int_{B_s\setminus B_r} E(D\tilde{u}) + \int_{B_s} E\left(\frac{\tilde{u}}{s-r}\right)\mathrm{d}x + \omega(R)R^n\right).
\end{aligned}$$

Now we can apply the hole-filling trick, adding $C\int_{B_r} E(D\tilde{u})$ to both sides and dividing the inequality by $C+1$. Finally, the desired inequality can be obtained by applying Lemma

2.9.1 with

$$\Phi(r) = \int_{B_r} E(D\tilde{u}), \quad \theta = \frac{C}{C+1}, \quad \Psi(t) = \frac{C}{C+1}\int_{B_R} E\left(\frac{\tilde{u}}{t}\right), \quad B = \frac{C}{C+1}\omega(R)R^n.$$

□

5.3.2 Euler-Lagrange inequality

The minimizers of a functional with a regular integrand satisfy the corresponding Euler-Lagrange equation. In the case of ω-minimizers, the equation does not hold anymore, while we are able to obtain a corresponding inequality instead with the help of Ekeland's variational principle.

Proposition 5.3.2. *Suppose that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* (**LP1**)-(**LP3**) *with $p = 1$, ω satisfies* (**ω1**) *and $u \in BV_{loc}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$ with constant $R_0 > 0$. Take $B_R = B(x_0, R) \subset\subset \Omega$ with $R < R_0$. Given any $\delta > 0$ and any ball $B_R = B(x_0, R) \subset\subset \Omega$ with $R < R_0$, there exists a map $w \in W^{1,1}_u(B_R, \mathbb{R}^N)$ satisfying*

$$\fint_{B_R} E\left(\frac{u-w}{R}\right) \mathrm{d}x \le \delta + C\sqrt{\varepsilon+\delta}, \quad \textit{and} \tag{5.3.2}$$

$$\left|\fint_{B_R} f'(\nabla w)\nabla\varphi \,\mathrm{d}x\right| \le \sqrt{\varepsilon+\delta}\fint_{B_R} |\nabla\varphi| \,\mathrm{d}x, \quad \textit{for any } \varphi \in W^{1,1}_0(B_R, \mathbb{R}^N), \tag{5.3.3}$$

where $C = C(n, N) > 0$ and

$$\varepsilon = \omega(R)\fint_{B_R}(a_6 + a_7|Du|), \quad a_6 = 1 + \frac{L-a_4}{a_3}, \quad a_7 = \frac{a_5+L}{a_3}, \tag{5.3.4}$$

with the constants a_3, a_4, a_5 in Lemma 5.2.8. Furthermore, given any affine map $a\colon \mathbb{R}^n \to \mathbb{R}^N$, we have

$$\fint_{B_R} E(\nabla w - \nabla a) \,\mathrm{d}x \le 2\fint_{B_R} E(Du - \nabla a) + 2\sqrt{\varepsilon+\delta} + C(\delta, \nabla a), \tag{5.3.5}$$

where the constant $C(\delta, \nabla a)$ is positive and vanishes as $\delta \to 0^+$.

Proof. For a given $\delta > 0$, Remark 2.5.6 implies that there exists $u_\delta \in W^{1,1}_u(B_R, \mathbb{R}^N)$ such that

$$\fint_{B_R} \frac{|u-u_\delta|}{R} \,\mathrm{d}x < \delta, \quad \left|\fint_{B_R} E(Du) - \fint_{B_R} E(\nabla u_\delta) \,\mathrm{d}x\right| < \delta, \tag{5.3.6}$$

$$\left|\fint_{B_R} f(\nabla u_\delta) \,\mathrm{d}x - \fint_{B_R} f(Du)\right| < \delta. \tag{5.3.7}$$

By the ω-minimality of u we have

$$\bar{\mathscr{F}}(u, B_R) \le \bar{\mathscr{F}}(v, B_R) + \omega(R)\int_{B_R}(1 + |Dv|) \tag{5.3.8}$$

for any $v \in BV_u(B_R, \mathbb{R}^N)$.

Again from Remark 2.5.6, we know that

$$\inf_{v\in W^{1,1}_u(B_R,\mathbb{R}^N)} \mathscr{F}(v,B_R) = \inf_{v\in BV_u(B_R,\mathbb{R}^N)} \bar{\mathscr{F}}(v,B_R) =: I$$

and there exists $\{v_j\}\subset W^{1,1}_u(B_R,\mathbb{R}^N)$ such that $\mathscr{F}(v_j,B_R)\to I$. The mean coercivity of f (Lemma 5.2.8) implies

$$\begin{aligned}
⨍_{B_R}(1+|Dv_j|)\,\mathrm{d}x &\le 1+\frac{1}{a_3}\left(⨍_{B_R} f(\nabla v_j)\,\mathrm{d}x + a_5 ⨍_{B_R}|Du| - a_4\right)\\
&\le 1-\frac{a_4}{a_3}+\frac{a_5}{a_3}⨍_{B_R}|Du| + \frac{1}{a_3}\left(⨍_{B_R} f(Du)+\delta_j\right)\\
&\le ⨍_{B_R}(a_6+a_7\,|Du|)+\frac{\delta_j}{a_3},
\end{aligned} \tag{5.3.9}$$

where $\delta_j := \frac{1}{|B_R|}(\mathscr{F}(v_j,B_R)-I)\to 0$ as $j\to\infty$, and $a_6 = 1+\frac{L-a_4}{a_3}$, $a_7=\frac{a_5+L}{a_3}$ as defined in (5.3.4). Take v to be v_j in (5.3.8), and the above estimate gives, when $j\to\infty$,

$$\bar{\mathscr{F}}(u,B_R)\le \inf_{v\in BV_u(B_R,\mathbb{R}^N)}\bar{\mathscr{F}}(v,B_R)+\omega(R)\int_{B_R}(a_6+a_7|Du|)\,. \tag{5.3.10}$$

Set $\varepsilon=\omega(R)⨍_{B_R}(a_6+a_7|Du|)$ as in (5.3.4), and it is easy to see $\varepsilon>0$ from the above estimate (5.3.9). Then by (5.3.7) we have

$$\begin{aligned}
\mathscr{F}(u_\delta,B_R) &\le \inf_{v\in BV_u(B_R,\mathbb{R}^N)}\bar{\mathscr{F}}(v,B_R)+\omega_n R^n(\varepsilon+\delta)\\
&= \inf_{v\in W^{1,1}_u(B_R,\mathbb{R}^N)}\mathscr{F}(v,B_R)+\omega_n R^n(\varepsilon+\delta).
\end{aligned}$$

Consider the complete metric space $X = W^{1,1}_u(B_R,\mathbb{R}^N)$ with $d(w_1,w_2) = ⨍_{B_R}|\nabla(w_1-w_2)|\,\mathrm{d}x$. Take $\mathcal{F}(w) = ⨍_{B_R} f(\nabla w)\,\mathrm{d}x$ and apply Corollary 2.8.2 with ε replaced by $\varepsilon+\delta$, then there exists $w\in W^{1,1}_u(B_R,\mathbb{R}^N)$ such that

(a) $d(u_\delta,w)\le\sqrt{\varepsilon+\delta}$;

(b) $\mathcal{F}(w)\le\mathcal{F}(u_\delta)$;

(c) $\mathcal{F}(w)\le\mathcal{F}(v)+\sqrt{\varepsilon+\delta}\,d(w,v)$, for any $v\in X=W^{1,1}_u(B_R,\mathbb{R}^N)$.

From the choice of u_δ and Lemma 5.2.1, we have

$$\begin{aligned}
&⨍_{B_R} E\left(\frac{u-w}{R}\right)\mathrm{d}x \le ⨍_{B_R}\frac{|u-w|}{R}\,\mathrm{d}x\\
&\le ⨍_{B_R}\frac{|u-u_\delta|}{R}\,\mathrm{d}x + ⨍_{B_R}\frac{|u_\delta-w|}{R}\,\mathrm{d}x\\
&\le \delta + C(n,N)⨍_{B_R}|\nabla u_\delta-\nabla w|\,\mathrm{d}x\\
&\le \delta + C\sqrt{\varepsilon+\delta},
\end{aligned}$$

where the third line comes from (5.3.6) and Poincaré's inequality, and the fourth from (a). This gives (5.3.2).

For any $\varphi \in W_0^{1,1}(B_R, \mathbb{R}^N)$, we take $v = w + t\varphi$ and insert it into (c) to obtain

$$\fint_{B_R} f(\nabla w)\,\mathrm{d}x \le \fint_{B_R} f(\nabla(w + t\varphi))\,\mathrm{d}x + \sqrt{\varepsilon + \delta}|t| \fint_{B_R} |\nabla\varphi|\,\mathrm{d}x.$$

Move the first term of the right-hand side to the left, divide both sides by $|t|$ and take $t \to 0$, then we have the Euler-Lagrange inequality (5.3.3) as claimed.

Finally, we set $\tilde{u} := u - a$ and $\tilde{u}_\delta = u_\delta - a$. Lemma 5.2.1 then implies

$$\begin{aligned}
&\fint_{B_R} E(\nabla w - \nabla a)\,\mathrm{d}x \le 2\fint_{B_R} E(\nabla w - \nabla u_\delta)\,\mathrm{d}x + 2\fint_{B_R} E(\nabla\tilde{u}_\delta)\,\mathrm{d}x \\
&\le 2\fint_{B_R} |\nabla w - \nabla u_\delta|\,\mathrm{d}x + 2\left(\fint_{B_R} E(\nabla\tilde{u}_\delta)\,\mathrm{d}x - \fint_{B_R} E(D\tilde{u})\right) + 2\fint_{B_R} E(D\tilde{u}) \\
&\overset{(a)}{\le} 2\sqrt{\varepsilon + \delta} + 2\left(\fint_{B_R} E(\nabla\tilde{u}_\delta)\,\mathrm{d}x - \fint_{B_R} E(D\tilde{u})\right) + 2\fint_{B_R} E(D\tilde{u}).
\end{aligned}$$

By Remark 2.5.6 we know that the second term in the last line can be controlled by some constant $C(\delta, \nabla a) > 0$, which goes to 0 as $\delta \to 0^+$. ☐

5.3.3 Harmonic approximation

Again, we take an affine map $a\colon \mathbb{R}^n \to \mathbb{R}^N$ with $|\nabla a| \le m$ for some $m > 0$, and $\tilde{f} = f_{\nabla a}$. Now we compare u with a harmonic map h which coincides with u on the boundary of a certain ball.

Proposition 5.3.3. *Suppose that the function $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* **(LP1)**-**(LP3)** *with $p = 1$, ω satisfies ($\boldsymbol{\omega}$**1**) and $u \in BV_{loc}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$ with constant $R_0 > 0$. Let $m > 0$ be a fixed constant, $a\colon \mathbb{R}^n \to \mathbb{R}^N$ be affine with $|\nabla a| \le m$ and $\tilde{f} := f_{\nabla a}$. Assume that $B_R = B(x_0, R) \subset\subset \Omega$ is a ball such that $|Du|(\partial B_R) = 0$ and $u|_{\partial B_R} \in BV(\partial B_R, \mathbb{R}^N)$. Then the system*

$$\begin{cases} -\operatorname{div}(\tilde{f}''(0)\nabla h) = 0, & \text{in } B_R \\ h|_{\partial B_R} = u|_{\partial B_R}, & \text{on } \partial B_R \end{cases} \tag{5.3.11}$$

admits a unique solution $h \in W^{1,r}(B_R, \mathbb{R}^N)$ such that

$$\left(\fint_{B_R} |\nabla h - \nabla a|^r\,\mathrm{d}x\right)^{\frac{1}{r}} \le C\fint_{\partial B_R} |D_\tau(u - a)|. \tag{5.3.12}$$

The exponent r is as in Lemma 2.2.10 and $C = C(m, n, N, L, \ell, r) > 0$. Furthermore, for any $q \in (1, \frac{n}{n-1})$, there exists a constant $C = C(m, n, N, L, \ell, q) > 0$ such that

$$\fint_{B_R} E\left(\frac{u - h}{R}\right)\mathrm{d}x \le C\left(\fint_{B_R} E(D(u - a))\right)^q + C(\sqrt{\varepsilon} + \sqrt{\varepsilon}^q), \tag{5.3.13}$$

where $\varepsilon = \omega(R) \fint_{B_R}(a_6 + a_7|Du|)$ is as in Proposition 5.3.2.

Proof. From (**LP3**) and (5.2.17) we have that $|\tilde{f}''(0)| \le C(m)$ and satisfies the Legendre-Hadamard condition (2.7.1) with $\lambda = \lambda(\ell, m)$. By Lemma 2.2.10 we know that $u|_{\partial B_R} \in W^{1-\frac{1}{r},r}(\partial B_R, \mathbb{R}^N)$ for a proper $r > 1$ and

$$\left(\fint_{\partial B_R}\int_{\partial B_R}\frac{|u(x)-u(y)|^r}{|x-y|^{n+r-2}}\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}\mathcal{H}^{n-1}(y)\right)^{\frac{1}{r}} \le CR^{\frac{1}{r}}\fint_{\partial B_R}|D_\tau u|. \tag{5.3.14}$$

Lemma 2.7.2 implies the existence of a unique solution $h \in W^{1,r}(B_R, \mathbb{R}^N)$ to (5.3.11). Then with (5.3.14) and Remark 2.7.3, we have the estimate (5.3.12).

Fix $\delta > 0$, and let w be as in Proposition 5.3.2. Take an arbitrary $\varphi \in C_c^\infty(B_R, \mathbb{R}^N)$, then by (5.3.11) we have

$$\begin{aligned}&\fint_{B_R}\tilde{f}''(0)[\nabla(w-h), \nabla\varphi]\,\mathrm{d}x = \fint_{B_R}\tilde{f}''(0)[\nabla\tilde{w}, \nabla\varphi]\,\mathrm{d}x\\ &= \fint_{B_R}(\tilde{f}''(0)[\nabla\tilde{w}, \nabla\varphi] - \tilde{f}'(\nabla\tilde{w})\nabla\varphi)\,\mathrm{d}x + \fint_{B_R}f'(\nabla w)\nabla\varphi\,\mathrm{d}x\\ &\le C\fint_{B_R}E(\nabla\tilde{w})|\nabla\varphi|\,\mathrm{d}x + \sqrt{\varepsilon+\delta}\fint_{B_R}|\nabla\varphi|\,\mathrm{d}x,\end{aligned} \tag{5.3.15}$$

where $\tilde{w} = w - a$ and the last line is obtained by Lemma 5.2.7 and (5.3.3). By approximation, φ can be taken in $W_0^{1,\infty} \cap C^1(B_R, \mathbb{R}^N)$. To obtain the desired estimate, we need to find a proper test map φ, before which we scale to the unit ball $B(0,1)(=: B)$.

Define $\psi := w - h$, and set

$$\Psi(y) := \frac{\psi(x_0+Ry)}{R}, \quad \Phi(y) := \frac{\varphi(x_0+Ry)}{R}, \quad \tilde{W}(y) := \frac{\tilde{w}(x_0+Ry)}{R}.$$

Consider, with $\mathbb{A} := \tilde{f}''(0)$,

$$\begin{cases} -\operatorname{div}(\mathbb{A}\nabla\Phi) = T(\Psi), & \text{in } B\\ \Phi|_{\partial B} = 0, & \text{on } \partial B,\end{cases} \tag{5.3.16}$$

where

$$T(\Psi) := \begin{cases}\Psi, & |\Psi| \le 1\\ \dfrac{\Psi}{|\Psi|}, & |\Psi| > 1.\end{cases} \tag{5.3.17}$$

As $T(\Psi) \in L^\infty(B_R, \mathbb{R}^N)$, the solution Φ exists and lies in $W_0^{1,s} \cap W^{2,s}(B_R, \mathbb{R}^N)$ for any $s > 1$. We take $s > n$ so that by Morrey's inequality

$$\|\nabla\Phi\|_{L^\infty} \le C\|\Phi\|_{W^{2,s}} \le C\|T(\Psi)\|_{L^s} \le C\left(\fint_B E(\Psi)\,\mathrm{d}x\right)^{\frac{1}{s}}. \tag{5.3.18}$$

Thus, we can test (5.3.16) with Ψ to get

$$\fint_B E(\Psi)\,\mathrm{d}x \le a_2\fint_B \min\{|\Psi|, |\Psi|^2\}\,\mathrm{d}x = a_2\fint_B [T(\Psi), \Psi]\,\mathrm{d}x$$

$$
\begin{aligned}
&= a_2 \fint_B \mathbb{A}[\nabla\Phi, \nabla\Psi]\,\mathrm{d}x = a_2 \fint_B \mathbb{A}[\nabla\Psi, \nabla\Phi]\,\mathrm{d}x \\
&\overset{(5.3.15)}{\leq} C \fint_B E(\nabla\tilde{W})|\nabla\Phi|\,\mathrm{d}x + a_2\sqrt{\varepsilon+\delta}\fint_B |\nabla\Phi|\,\mathrm{d}x \\
&\overset{(5.3.18)}{\leq} C\left(\fint_B E(\nabla\tilde{W})\,\mathrm{d}x + \sqrt{\varepsilon+\delta}\right)\left(\fint_B E(\Psi)\,\mathrm{d}x\right)^{\frac{1}{s}}.
\end{aligned}
$$

Setting $q = s' = \frac{s}{s-1}$, we obtain

$$
\fint_B E(\Psi)\,\mathrm{d}x \leq C\left(\fint_B E(\nabla\tilde{W})\,\mathrm{d}x\right)^q + C(\varepsilon+\delta)^{\frac{q}{2}}. \tag{5.3.19}
$$

Back to B_R, the above inequality becomes

$$
\fint_{B_R} E\left(\frac{w-h}{R}\right)\mathrm{d}x \leq C\left(\fint_{B_R} E(\nabla(w-a))\,\mathrm{d}x\right)^q + C(\varepsilon+\delta)^{\frac{q}{2}}. \tag{5.3.20}
$$

To compare u and h, we decompose $u-h$ as $(u-w)+(w-h)$, and use (5.3.2) and (5.3.20):

$$
\begin{aligned}
\fint_{B_R} E\left(\frac{u-h}{R}\right)\mathrm{d}x &\leq C\fint_{B_R}\left(E\left(\frac{u-w}{R}\right) + E\left(\frac{w-h}{R}\right)\right)\mathrm{d}x \qquad (5.3.21)\\
\leq 2\delta + C\sqrt{\varepsilon+\delta} &+ C\left(\fint_{B_R} E(\nabla(w-a))\,\mathrm{d}x\right)^q + C(\varepsilon+\delta)^{\frac{q}{2}}.
\end{aligned}
$$

The term concerning $\nabla(w-a)$ can be controlled in terms of $D(u-a)$ with (5.3.5). Combining the estimates above and taking $\delta \to 0$, we have (5.3.13) hold. ☐

5.3.4 Excess decay estimate

Suppose that u is a BV ω-minimizer of $\tilde{F}$ as in the last subsection. For any ball $B(x_0,R) \subset\subset \Omega$, define the excess of u as

$$
\mathscr{E}(x_0,R) := \fint_{B_R} E(Du - (Du)_{B_R}). \tag{5.3.22}
$$

This quantity measures the oscillation of Du on B_R (or the deviation of u from being an affine map with gradient $(Du)_R$, as it vanishes when u is affine). We will abbreviate $\mathscr{E}(x_0,R)$ as $\mathscr{E}(R)$ when the centre x_0 is clear in the context.

Proposition 5.3.4. *Suppose that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* **(LP1)**-**(LP3)** *with $p = 1$, $\omega\colon [0,\infty) \to [0,\infty)$ satisfies* **($\boldsymbol{\omega}$1)** *and $u \in BV_{loc}(\Omega,\mathbb{R}^N)$ is an ω-minimizer of $\bar{\tilde{\mathscr{F}}}$ with constant $R_0 > 0$. If a ball $B_R = B(x_0,R) \subset\subset \Omega$ is such that $R < R_0$ and*

$$
|(Du)_{B_R}| < m, \quad \fint_{B_R} |Du - (Du)_{B_R}| \leq 1 \tag{5.3.23}
$$

for some $m > 0$, then we have

$$
\mathscr{E}(\sigma R) \leq c(\sigma^2 + \sigma^{-(n+2)}\mathscr{E}(R)^{q-1})\mathscr{E}(R) + c\sigma^{-(n+2)}\sqrt{\omega(R)} \tag{5.3.24}
$$

holds for any $\sigma \in (0,1)$ *and any* $q \in (1, \frac{n}{n-1})$ *with some* $c = c(m,n,N,L,\ell,q) > 0$.

Proof. When $\sigma \geq \frac{1}{5}$, (5.3.24) is easy to show, and thus we only consider the case $\sigma \in (0, \frac{1}{5})$. Set $a(x) = u_{B_R} + (Du)_{B_R}(x - x_0)$ and $\tilde{u} = u - a$, $\tilde{f} = f_{\nabla a}$. Take $\rho \in (\frac{9}{10}R, R)$ such that $|D\tilde{u}|(\partial B_\rho) = 0$ and $\tilde{u}|_{\partial B_\rho} \in BV(\partial B_\rho, \mathbb{R}^N)$, then by Lemma 2.2.10 and 2.2.9, we have $\tilde{u}|_{\partial B_\rho} \in W^{1-\frac{1}{r},r}(\partial B_\rho, \mathbb{R}^N)$, where $r = \frac{n}{n-1}$ if $n \geq 3$, and r can be taken arbitrarily in $(1,2)$ if $n = 2$. In addition, the corresponding estimate is as follows:

$$[\tilde{u}|_{\partial B_\rho}]_{W^{1-\frac{1}{r},r}} \leq C(n,N,r) \int_{\partial B_\rho} |D_\tau(\tilde{u}|_{\partial B_\rho})| \leq \frac{C(n,N,r)}{R} \int_{B_R} |D\tilde{u}|. \tag{5.3.25}$$

Let h be the harmonic map determined by (5.3.11) with R replaced by ρ. We moreover define

$$\tilde{h} = h - a, \quad A(x) = \tilde{h}(x_0) + \nabla\tilde{h}(x_0)(x - x_0), \quad a_0 = a + A.$$

Then Lemma 2.7.1, 2.7.2 and (5.3.25) imply

$$\begin{aligned}|\nabla\tilde{h}(x_0)| &\leq \sup_{B_{\frac{\rho}{2}}} |\nabla\tilde{h}| \leq C \fint_{B_\rho} |\nabla\tilde{h}|\,\mathrm{d}x \leq C \left(\fint_{B_\rho} |\nabla\tilde{h}|^r\,\mathrm{d}x \right)^{\frac{1}{r}} \\ &\leq \frac{C}{\rho^{n-1}} [\tilde{u}|_{\partial B_\rho}]_{W^{1-\frac{1}{r},r}} \leq \frac{C}{R\rho^{n-1}} \int_{B_R} |D\tilde{u}| \leq C \fint_{B_R} |D\tilde{u}|.\end{aligned}$$

Then by assumption, we have

$$|\nabla a_0| \leq |\nabla a| + |\nabla A| \leq |(Du)_{B_R}| + C \fint_{B_R} |D\tilde{u}| \leq m + C =: C_m.$$

For any $\sigma \in (0, \frac{1}{5})$, we have $2\sigma R < \frac{\rho}{2}$. Lemma 5.2.2 gives

$$\fint_{B_{\sigma R}} E(Du - (Du)_{B_{\sigma R}}) \leq 4 \fint_{B_{\sigma R}} E(D(u - a_0)), \tag{5.3.26}$$

and the Caccioppoli inequality (5.3.1) implies

$$\begin{aligned}\fint_{B_{\sigma R}} E(D(u - a_0)) &\leq C \left(\fint_{B_{2\sigma R}} E\left(\frac{u - a_0}{2\sigma R} \right) \mathrm{d}x + \omega(2\sigma R) \right) \\ &\leq 2C \fint_{B_{2\sigma R}} \left(E\left(\frac{\tilde{u} - \tilde{h}}{2\sigma R} \right) + E\left(\frac{\tilde{h} - A}{2\sigma R} \right) \right) \mathrm{d}x + C\omega(2\sigma R).\end{aligned} \tag{5.3.27}$$

By Lemma 2.7.1 we have, for $x \in B_{2\sigma R}$,

$$\begin{aligned}\frac{|\tilde{h}(x) - A(x)|}{2\sigma R} &\leq C \sup_{B_{2\sigma R}} |\nabla^2 h| \frac{|x - x_0|^2}{2\sigma R} \leq C\sigma R \sup_{B_{\frac{\rho}{2}}} |\nabla^2 \tilde{h}| \\ &\leq C\sigma \fint_{B_\rho} |\nabla\tilde{h}|\,\mathrm{d}x \leq C\sigma \fint_{\partial B_\rho} |D_\tau(\tilde{u}|_{\partial B_\rho})| \\ &\overset{(5.3.25)}{\leq} C\sigma \fint_{B_R} |Du - (Du)_{B_R}|\end{aligned}$$

$$\leq C\sigma \left(\fint_{B_R} E(Du - (Du)_{B_R}) \right)^{\frac{1}{2}},$$

where the last inequality is by Lemma 5.2.3, (5.2.4) and the bound we assume in (5.3.23). Thus, again by (5.2.4) we have

$$\begin{aligned} \fint_{B_{2\sigma R}} E\left(\frac{\tilde{h} - A}{2\sigma R} \right) \mathrm{d}x &\leq E\left(C\sigma \left(\fint_{B_R} E(Du - (Du)_{B_R}) \right)^{\frac{1}{2}} \right) \\ &\leq a_2 C\sigma^2 \fint_{B_R} E(Du - (Du)_{B_R}). \end{aligned} \tag{5.3.28}$$

The term concerning $\tilde{u} - \tilde{h}$ can be estimated with (5.3.13):

$$\begin{aligned} \fint_{B_{2\sigma R}} E\left(\frac{\tilde{u} - \tilde{h}}{2\sigma R} \right) \mathrm{d}x &\leq \frac{C}{\sigma^{n+2}} \fint_{B_\rho} E\left(\frac{\tilde{u} - \tilde{h}}{\rho} \right) \mathrm{d}x \\ \leq \frac{C}{\sigma^{n+2}} &\left(\left(\fint_{B_\rho} E(D(u - a)) \right)^q + \sqrt{\varepsilon} + \sqrt{\varepsilon}^q \right), \end{aligned} \tag{5.3.29}$$

where $\varepsilon = \omega(\rho) \fint_{B_\rho} (a_6 + a_7 |Du|)$ with a_6, a_7 as in (5.3.10). Considering $|Du| \leq |Du - (Du)_{B_R}| + |(Du)_{B_R}|$, we obtain by assumption that $\varepsilon \leq C\omega(\rho) \leq C$. The above estimates (5.3.26)-(5.3.29) and the estimate for ε together give

$$\begin{aligned} \fint_{B_{\sigma R}} E(Du - (Du)_{B_{\sigma R}}) \leq \frac{C}{\sigma^{n+2}} &\left(\left(\fint_{B_R} E(Du - (Du)_{B_R}) \right)^q + \sqrt{\omega(R)} \right) \\ &+ C\sigma^2 \fint_{B_R} E(Du - (Du)_{B_R}) + C\omega(2\sigma R), \end{aligned} \tag{5.3.30}$$

from which (5.3.24) can be easily derived. □

5.3.5 Iteration

Now it is time to do iteration with (5.3.24) and to get some regularity result of u. Before that we present a lemma concerning the summability of ω to a certain power.

Lemma 5.3.5. *For any fixed $r > 0$, $\alpha \in [\frac{1}{4}, 1)$ and $\tau \in (0, 1)$, we have*

$$\sum_{j=0}^{\infty} \omega^\alpha(\tau^j r) \leq \frac{2\alpha\beta}{1 - \tau^{2\alpha\beta}} \Xi_\alpha(r), \tag{5.3.31}$$

where β is as in ($\boldsymbol{\omega 2}$). In particular, with $C(\alpha, \beta) = 2\alpha\beta \leq 2$ we have

$$\omega^\alpha(r) \leq C(\alpha, \beta) \Xi_\alpha(r). \tag{5.3.32}$$

Proof. The idea is to transform the terms in the sum on the left-hand side into integrals on disjoint subintervals of $[0,r]$. Indeed, by **(ω2)**

$$\begin{aligned}\int_{\tau^j r}^{\tau^{j-1}r}\frac{\omega^\alpha(\rho)}{\rho}\,\mathrm{d}\rho &\geq \frac{\omega^\alpha(\tau^{j-1}r)}{(\tau^{j-1}r)^{2\alpha\beta}}\int_{\tau^j r}^{\tau^{j-1}r}\rho^{2\alpha\beta-1}\,\mathrm{d}\rho\\ &= \frac{\omega^\alpha(\tau^{j-1}r)}{(\tau^{j-1}r)^{2\alpha\beta}}\frac{1}{2\alpha\beta}((\tau^{j-1}r)^{2\alpha\beta}-(\tau^j r)^{2\alpha\beta})\\ &= \frac{1}{2\alpha\beta}(1-\tau^{2\alpha\beta})\,\omega^\alpha(\tau^{j-1}r).\end{aligned}$$

Summing over j we obtain

$$\sum_{j=0}^{\infty}\omega^\alpha(\tau^j r)\leq\frac{2\alpha\beta}{1-\tau^{2\alpha\beta}}\int_0^r\frac{\omega^\alpha(\rho)}{\rho}\,\mathrm{d}\rho=\frac{2\alpha\beta}{1-\tau^{2\alpha\beta}}\Xi_\alpha(r).$$

□

With the above lemma, we are able to iterate the process in Proposition 5.3.4.

Proposition 5.3.6. *Suppose that $f\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ satisfies* **(LP1)**-**(LP3)** *with $p=1$, $\omega\colon[0,\infty)\to[0,\infty)$ satisfies* **(ω1)**-**(ω3)** *and $u\in BV_{loc}(\Omega,\mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$ with constant $R_0>0$. For any $\alpha\in(\frac{\beta}{2},1)$ and $m>0$, there exist $C=C(m,n,N,L,\ell,\alpha,\beta)>0,\varepsilon_m>0$ and $0<R_1<R_0$ such that the following holds: if $B_R=B(x_0,R)\subset\subset\Omega$ is such that*

$$|(Du)_{B_R}|<m,\quad \mathscr{E}(x_0,R)<\frac{\varepsilon_m}{2},\quad R<R_1, \tag{5.3.33}$$

then for any $0<\rho<R$, there holds

$$\mathscr{E}(\rho)\leq C\left(\frac{\rho}{R}\right)^{2\alpha}\mathscr{E}(R)+C\sqrt{\omega(\rho)}. \tag{5.3.34}$$

Proof. By Lemma 5.2.3, we have

$$\fint_{B_R}|Du-(Du)_{B_R}|\leq\sqrt{3\mathscr{E}(R)}$$

if $\mathscr{E}(R)\leq 1$. Then set $\varepsilon_m<\frac{1}{3}$ so that $\fint_{B_R}|Du-(Du)_{B_R}|<1$. Meanwhile, we take R_1 such that $\omega(R_1)<1$. The assumptions of Proposition 5.3.4 are satisfied and then, with a fixed $q\in(1,\frac{n}{n-1})$ we have

$$\mathscr{E}(\sigma R)\leq c(\sigma^2+\sigma^{-(n+2)}\mathscr{E}(R)^{q-1})\mathscr{E}(R)+c\sigma^{-(n+2)}\sqrt{\omega(R)}, \tag{5.3.35}$$

where $c=c(m,n,N,L,\ell,q)$. Set $C_{m+1}=c(m+1,n,N,L,\ell,q))$, and it is obvious that (5.3.35) holds with C_{m+1} instead of the constant c above. Take $\sigma\in(0,\frac{1}{5})$ and then $\varepsilon_m\in(0,\frac{1}{3})$ such that

$$C_{m+1}\sigma^2<\frac{1}{2}\sigma^{2\alpha},\quad C_{m+1}\sigma^{-(n+2)}\varepsilon_m^{q-1}<\frac{1}{2}\sigma^{2\alpha}.$$

In this case, with $c_1:=C_{m+1}\sigma^{-(n+2)}$, the estimate (5.3.35) becomes

$$\mathscr{E}(\sigma R)\leq\sigma^{2\alpha}\mathscr{E}(R)+c_1\sqrt{\omega(R)}.$$

To do the iteration, we consider the following

$$(\mathbf{I}_j) \qquad |(Du)_{B_{\sigma^j R}}| \le m+1,$$

$$(\mathbf{II}_j) \qquad \mathscr{E}(\sigma^j R) \le \sigma^{2j\alpha}\mathscr{E}(R) + c_2\sqrt{\omega(\sigma^j R)},$$

$$(\mathbf{III}_j) \qquad \mathscr{E}(\sigma^j R) \le \varepsilon_m,$$

where $c_2 = \frac{c_1}{\sigma^\beta - \sigma^{2\alpha}}$. The three conditions above hold for $j = 0$. We assume that they hold for $j = 0, 1, \dots, k-1$ with $k \ge 1$ and argue inductively in the following. For any $j \in \{0, 1, \dots, k-1\}$, condition ($\mathbf{III}_j$) together with $\varepsilon_m < \frac{1}{3}$ implies $⨍_{B_{\sigma^j R}} |Du - (Du)_{B_{\sigma^j R}}| < 1$. Combining this with ($\mathbf{I}_j$) we have, by applying Proposition 5.3.4 for k times, the choice of σ, ε_m and condition ($\boldsymbol{\omega}\mathbf{2}$),

$$\begin{aligned}\mathscr{E}(\sigma^k R) &\le \sigma^{2k\alpha}\mathscr{E}(R) + c_1\sum_{j=0}^{k-1}\sigma^{2(k-j-1)\alpha}\sqrt{\omega(\sigma^j R)}\\ &\le \sigma^{2k\alpha}\mathscr{E}(R) + c_1\sum_{j=0}^{k-1}\sigma^{(j-k)\beta+2(k-j-1)\alpha}\sqrt{\omega(\sigma^k R)}\\ &\le \sigma^{2k\alpha}\mathscr{E}(R) + \frac{c_1}{\sigma^\beta - \sigma^{2\alpha}}\sqrt{\omega(\sigma^k R)},\end{aligned}$$

which actually gives ($\mathbf{II}_k$). Take σ and R_1 small enough such that $\sigma^{2\alpha} < \frac{1}{2}, c_2\sqrt{\omega(R_1)} < \frac{\varepsilon_m}{2}$, and we furthermore have ($\mathbf{III}_k$). Finally, to get ($\mathbf{I}_k$) we use the triangle inequality

$$|(Du)_{B_{\sigma^k R}}| \le |(Du)_{B_R}| + \sum_{j=0}^{k-1}|(Du)_{B_{\sigma^{j+1}R}} - (Du)_{B_{\sigma^j R}}|.$$

For any $j \in \{0, 1, \dots, k-1\}$, by Lemma 5.2.3, ($\mathbf{III}_j$) and ($\mathbf{II}_j$) we have

$$\begin{aligned}&|(Du)_{B_{\sigma^{j+1}R}} - (Du)_{B_{\sigma^j R}}| \le \sigma^{-n}⨍_{B_{\sigma^j R}}|Du - (Du)_{B_{\sigma^j R}}|\\ &\le \sigma^{-n}\sqrt{3\mathscr{E}(\sigma^j R)} \le \sigma^{-n}(3\sigma^{2j\alpha}\mathscr{E}(R) + 3c_2\sqrt{\omega(\sigma^j R)})^{\frac{1}{2}}\\ &\le \sigma^{-n}(\sqrt{3}\sigma^{j\alpha}\sqrt{\mathscr{E}(R)} + \sqrt{3c_2}\omega^{\frac{1}{4}}(\sigma^j R)).\end{aligned}$$

Sum up the above from 0 to $k-1$ with the help of Lemma 5.3.5 to obtain

$$\begin{aligned}|(Du)_{B_{\sigma^k R}}| &\le m + \sqrt{3}\sigma^{-n}\sum_{j=0}^{k-1}(\sigma^{j\alpha}\sqrt{\mathscr{E}(R)} + \sqrt{c_2}\omega^{\frac{1}{4}}(\sigma^j R))\\ &\le m + \sqrt{3}\sigma^{-n}\left(\frac{\sqrt{\mathscr{E}(R)}}{1-\sigma^\alpha} + \frac{\sqrt{c_2}\beta}{2(1-\sigma^{\frac{\beta}{2}})}\Xi_{\frac{1}{4}}(R)\right).\end{aligned}$$

We require

$$\frac{\sqrt{3}\sigma^{-n}}{1-\sigma^\alpha}\sqrt{\varepsilon_m} < \frac{1}{2}, \quad \frac{\sqrt{3c_2}\beta\sigma^{-n}}{2(1-\sigma^{\frac{\beta}{2}})}\Xi_{\frac{1}{4}}(R_1) < \frac{1}{2},$$

which can be satisfied when $\varepsilon_m \ll 1, R_1 \ll 1$. Then $(\mathbf{I}_k)$ also holds true. Notice that in the above we have chosen $\sigma, \varepsilon_m, R_1$ in order such that $|(Du)_{B_R}| < m, \mathscr{E}(R) < \frac{\varepsilon_m}{2}$ and $R < R_1$ imply $(\mathbf{I}_j)$-$(\mathbf{III}_j)$ for any $j \in \mathbb{N}$. Given any $\rho \in (0, R)$, we can take $\sigma^{k+1}R < \rho \le \sigma^k R$ and get the desired estimate for $\mathscr{E}(\rho)$ by controlling it with $\mathscr{E}(\sigma^k R)$. □

5.3.6 Regularity near a "good" point

Assume that $f \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies **(LP1)**-**(LP3)** with $p = 1$, $\omega \colon [0,\infty) \to [0,\infty)$ satisfies $(\boldsymbol{\omega}\mathbf{1})$-$(\boldsymbol{\omega}\mathbf{3})$ and $u \in BV_{loc}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$ with constant $R_0 > 0$. From the results we obtained above, it is possible to deduce a first-step partial regularity result. We claim that there exists a relatively closed $\mathscr{L}^n$-null set $S_u \subset \Omega$ such that $u \in C^1_{loc}(\Omega \setminus S_u, \mathbb{R}^N)$, $Du \llcorner (\Omega \setminus S_n) = \nabla u \mathscr{L}^n$, and ∇u has a local modulus of continuity $\rho \mapsto \rho^\alpha + \Xi_{\frac{1}{4}}(\rho)$ on $\Omega \setminus S_u$ with any $\alpha \in (0,1)$.

Actually, for any $x_0 \in \Omega$ such that

$$\limsup_{R\to 0^+} |(Du)_{x_0,R}| < \infty \quad \text{and} \quad \liminf_{R\to 0^+} \fint_{B(x_0,R)} E(Du - (Du)_{x_0,R}) \,\mathrm{d}x = 0, \tag{5.3.36}$$

one can show that there is a small neighbourhood $\mathcal{N}$ of x_0 such that $Du \llcorner \mathcal{N} = \nabla u \mathscr{L}^n$ and ∇u has the desired modulus of continuity in $\mathcal{N}$ (cf. Proposition 4.9 in [GK19a]).

Denote by $\mathcal{R}_1$ and $\mathcal{R}_2$ the two subsets of Ω where the two conditions in (5.3.36) are satisfied, respectively. Fix an arbitrary $\alpha \in (\frac{\beta}{2}, 1)$, and take $x_0 \in \mathcal{R}_1 \cap \mathcal{R}_2$. Suppose $\limsup_{R\to 0^+} |(Du)_{x_0,R}| = m < \infty$, and select $R_2 \le R_1$ such that

$$|(Du)_{B_{R_2}}| < m + \frac{1}{2}, \quad \mathscr{E}(x_0, R_2) < \tau < 1,$$

where the constant R_1 is as in Proposition 5.3.6 for $m+1$ instead of m and τ is to be determined. For any $y \in B(x_0, \frac{R_2}{2})$, it is easy to see $B(y, \frac{R_2}{2}) \subset B(x_0, R_2)$ and

$$\begin{aligned}
\mathscr{E}(y, R_2/2) &\le 2 \fint_{B(y,R_2/2)} E(Du - (Du)_{x_0,R_2}) + 2E((Du)_{y,R_2/2} - (Du)_{x_0,R_2}) \\
&\le 4 \fint_{B(y,R_2/2)} E(Du - (Du)_{x_0,R_2}) \\
&\le 4 \cdot 2^n \mathscr{E}(x_0, R_2) < 4 \cdot 2^n \tau
\end{aligned}$$

by (5.2.5) and the convexity of E. The average of Du on $B(y, \frac{R_2}{2})$ can also be controlled as follows with Lemma 5.2.3

$$\begin{aligned}
|(Du)_{y,R_2/2}| &\le |(Du)_{y,R_2/2} - (Du)_{x_0,R_2}| + |(Du)_{x_0,R_2}| \\
&\le \fint_{B(y,R_2/2)} |Du - (Du)_{x_0,R_2}| + m + \frac{1}{2} \\
&\le \sqrt{3\mathscr{E}(x_0, R_2)} + m + \frac{1}{2} \le \sqrt{3\tau} + m + \frac{1}{2}.
\end{aligned}$$

Then take τ such that

$$4 \cdot 2^n \tau < \varepsilon_{m+1}, \quad \text{and} \quad \sqrt{3\tau} < \frac{1}{2},$$

where ε_{m+1} is the ε in Proposition 5.3.6 with $m+1$ instead of m. From the above discussion, we know that for any $y \in B(x_0, \frac{R_2}{2})$ the conditions of Proposition 5.3.6 are satisfied with $m+1$ on $B(y, \frac{R_2}{2})$, and thus the estimate

$$\mathscr{E}(y,\rho) \le C\left(\frac{2\rho}{R_2}\right)^{\alpha} \mathscr{E}(y, R_2/2) + C\sqrt{\omega(\rho)} \le 4\cdot 2^n C\left(\frac{2\rho}{R_2}\right)^{\alpha} \mathscr{E}(x_0, R_2) + C\sqrt{\omega(\rho)} \quad (5.3.37)$$

holds for any $\rho < \frac{R_2}{2}$.

From Lemma 2.2.19 with $h(\rho) = \rho^\alpha + \sqrt{\omega(\rho)}$, we can see that $Du \llcorner B(x_0, \frac{R_2}{2}) = \nabla u \mathscr{L}^n$. Moreover, the gradient ∇u is continuous with a modulus of continuity $\rho \mapsto \rho^\alpha + \Xi_{\frac{1}{4}}(\rho)$ in $B(x_0, \frac{R_2}{2})$.

It is easy to see that $\mathcal{R}_1 \cap \mathcal{R}_2$ is of full measure in Ω. Indeed, by (a modification of) Theorem 1.31 in [EG15], we know that $\mathscr{L}^n(\Omega \setminus \mathcal{R}_u) = 0$, where

$$\mathcal{R}_u := \left\{x \in \Omega : \frac{\mathrm{d}|D^s u|}{\mathrm{d}\mathscr{L}^n}(x) = 0\right\} \cup \{x \in \Omega : x \text{ is a Lebesgue point of } \nabla u\}.$$

For any $x_0 \in \mathcal{R}_u$, we have

$$\limsup_{R\to 0} |(Du)_{x_0,R}| = |\nabla u(x_0)| < \infty$$

and that, with $B_R = B(x_0, R)$

$$\begin{aligned}
\mathscr{E}(x_0, R) &= \fint_{B_R} E(Du - (Du)_R) \\
&\le 2\fint_{B_R} E(\nabla u - (\nabla u)_R)\,\mathrm{d}x + 2E\left(\frac{D^s u(B_R)}{\mathscr{L}^n(B_R)}\right) + \frac{|D^s u|(B_R)}{\mathscr{L}^n(B_R)} \\
&\longrightarrow 0 \quad \text{as } R \to 0.
\end{aligned}$$

Then one can see $\mathcal{R}_u \subset \mathcal{R}_1 \cap \mathcal{R}_2$ and $\mathscr{L}^n(\Omega \setminus (\mathcal{R}_1 \cap \mathcal{R}_2)) = 0$. Thus, our claim at the beginning of this subsection holds true, and there is a singular set S_u of u with $S_u \subset \Omega \setminus (\mathcal{R}_1 \cap \mathcal{R}_2) = \Sigma_1 \cup \Sigma_2$.

Remark 5.3.7. The modulus of continuity of ∇u obtained here is $\rho \mapsto \rho^\alpha + \Xi_{\frac{1}{4}}(\rho)$, not the claimed one $\rho \mapsto \rho^\alpha + \Xi_{\frac{1}{2}}(\rho)$ in Theorem 5.1.1. This modulus comes from the excess decay estimate (5.3.34), where the exponent of $\omega(\rho)$ on the right-hand side is $\frac{1}{2}$, while the ideal one to obtain the claimed modulus is 1 as in (5.3.46). There are two places where we lost some power of ω:

- In (5.3.18), the order of $\nabla\Phi$ on the left-hand side is 1, while that of Ψ on the right-hand side is $\frac{2}{s}$ (if we assume Ψ to be small).
- In (5.3.21), we controlled $E\left(\frac{u_\delta - w}{R}\right)$ by $|\frac{u_\delta - w}{R}|$, where the former is possibly of power 2 if it is small, while the latter is of 1.

5.3.7 Improvement of regularity

With the regularity proved above, it is possible to further show that the local modulus of continuity of Du is $\rho \mapsto \rho^\alpha + \Xi_{\frac{1}{2}}(\rho)$ for any $\alpha \in (0,1)$. On any open set $\Omega' \subset\subset \Omega \setminus S_u$, the C^1-norm of u is finite, and then it is sufficient to perform the proof of regularity in the quadratic case as in [Giu03], §9.4. For completeness, we sketch the process here.

Proposition 5.3.8. *Suppose that* $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *satisfies* **(LP1)**-**(LP3)** *with* $p = 1$, $\omega\colon [0,\infty) \to [0,\infty)$ *satisfies* **(ω1)**-**(ω3)** *and* $u \in BV_{loc}(\Omega, \mathbb{R}^N)$ *is an* ω*-minimizer of* $\bar{\mathscr{F}}$ *with constant* $R_0 > 0$*. Let* S_u *be the relatively closed singular set as in last subsection. Take* $\Omega' \subset\subset \Omega \setminus S_u$ *and* $M = M(\Omega') = \|u\|_{C^1(\Omega')} > 0$*. For any ball* $B(x_0, R) \subset\subset \Omega'$ *with* $R < R_0$*, if* $a\colon \mathbb{R}^n \to \mathbb{R}^N$ *is an affine map with* $|\nabla a| \le m$ *for some* $m > 0$*, then there exists* $C = C(m, n, N, L, \ell, M) > 0$ *such that*

$$\fint_{B_{\frac{R}{2}}} |\nabla(u-a)|^2 \,\mathrm{d}x \le C\left(\fint_{B_R} \frac{|u-a|^2}{R^2}\,\mathrm{d}x + \omega(R)\right). \tag{5.3.38}$$

Proof. In Proposition 5.3.1 we have already obtained a Caccioppoli's inequality in the form of E. By the basic properties of E, we have $E(\frac{u-a}{R}) \le a_2\frac{|u-a|^2}{R^2}$. To deal with the left-hand side, notice that $|\nabla(u-a)| \le |\nabla u| + |\nabla a| \le M + m$ and then $E(\nabla(u-a)) \ge C(M+m)|\nabla(u-a)|^2$. □

In this case, we also have a higher integrability of ∇u as in Section 3.2.

Proposition 5.3.9. *Suppose that* $f, \omega, u, S_u, \Omega'$ *and* $M(\Omega')$ *are as in Proposition 5.3.8 and* $z_0 \in \mathbb{R}^{N\times n}$ *satisfies* $|z_0| \le m$*. There exist* $q = q(m, n, N, L, \ell, M) > 1$ *and a constant* $C = C(m, n, N, L, \ell, M, q) \ge 1$*, such that* $|\nabla u - z_0| \in L^{2q}_{loc}(\Omega')$*, and for any ball* $B(y_0, R) \subset\subset \Omega'$ *we have*

$$\left(\fint_{B(y_0,\frac{R}{2})} |\nabla u - z_0|^{2q}\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\fint_{B(y_0,R)} |\nabla u - z_0|^2\,\mathrm{d}x + \omega(R)\right). \tag{5.3.39}$$

Proof. Pick $B_\rho = B(x_0,\rho) \subset B(y_0,R)$ with $\rho < R_0$ and $a(x) = u_{B_\rho} + z_0(x - x_0)$. The average of $u - a$ on B_ρ vanishes by the definition of a, and then the Sobolev-Poincaré inequality implies

$$\fint_{B_\rho} \frac{|u-a|^2}{\rho^2}\,\mathrm{d}x \le C\left(\fint_{B_\rho} |\nabla(u-a)|^{2_*}\,\mathrm{d}x\right)^{\frac{2}{2_*}} = C\left(\fint_{B_\rho} |\nabla u - z_0|^{2_*}\,\mathrm{d}x\right)^{\frac{2}{2_*}},$$

where $2_* = \frac{2n}{n+2} < 2$. Combining Proposition 5.3.8 we have the weak reverse Hölder inequality

$$\fint_{B_{\frac{\rho}{2}}} |\nabla u - z_0|^2\,\mathrm{d}x \le C\left(\fint_{B_\rho} |\nabla u - z_0|^{2_*}\,\mathrm{d}x\right)^{\frac{2}{2_*}} + C\omega(\rho). \tag{5.3.40}$$

The above estimate holds for any ball $B_\rho \subset B(y_0,R)$ with $\rho < R_0$ and we can replace the $\omega(\rho)$ on the right-hand side by $\omega(R)$. By the generalised Gehring lemma (see [Gia83], Chap.V, [Str84], §2.3 and also Theorem 4.2.3), we know that there is an $q_0 > 1$ such that $|\nabla u - z_0| \in L^{2q}(B(y_0,R))$ for any $q \in (1, q_0)$ with (5.3.39) holding true. □

To get the regularity of u, we compare it with a harmonic map again, which is now taken to be a minimizer of a quadratic functional. Take a ball $B(x_0, 2R) \subset\subset \Omega'$ and consider

$$\begin{cases} -\operatorname{div}(\mathbb{A}\nabla h) = 0, & \text{in } B_R \\ h|_{\partial B_R} = u|_{\partial B_R}, & \text{on } \partial B_R, \end{cases} \tag{5.3.41}$$

where $\mathbb{A} = \tilde{f}''(0)$ with $\tilde{f} = f_{\nabla a}$ and $a(x) = u_{B_R} + (\nabla u)_{B_{2R}}(x - x_0)$. It is obvious that h is the minimizer of

$$\mathscr{G}(v, B_R) := \int_{B_R} \left(f(\nabla a) + f'(\nabla a)\nabla(v - a) + \frac{1}{2} f''(\nabla a)[\nabla(v - a), \nabla(v - a)]\right) \mathrm{d}x.$$

Lemma 5.3.10. *Let $f, \omega, u, S_u, \Omega', M(\Omega')$ be as in Proposition 5.3.8, q be the exponent obtained in Proposition 5.3.9 and $\tilde{f}, a, B_R$ be as above. Then there exists $h \in C^\infty \cap W^{1,r}_u(B_R, \mathbb{R}^N)$ which solves (5.3.41) for any $r > 1$. In addition, with some $C = C(n, N, L, \ell, M, q) > 0$ we have the following estimate*

$$\fint_{B_R} |\nabla(u - h)|^2 \,\mathrm{d}x \le C \left(\fint_{B_{2R}} |\nabla(u - a)|^2 \,\mathrm{d}x\right)^{1+\frac{1}{q'}} + C\omega(2R). \tag{5.3.42}$$

Proof. The existence of h is clear by Lemma 2.7.2 and the fact $u \in C^1(\bar{B}_R, \mathbb{R}^N)$. Furthermore, we have $|\nabla(u - h)| \in L^2(B_R)$ and by (5.2.17)

$$\begin{aligned} \int_{B_R} |\nabla(u - h)|^2 \,\mathrm{d}x &\le \frac{1}{2} C \int_{B_R} \tilde{f}''(0)[\nabla(u - h), \nabla(u - h)] \,\mathrm{d}x \\ &= C(\mathscr{G}(u) - \mathscr{G}(h)) \\ &= C(\mathscr{G}(u) - \mathscr{F}(u) + \mathscr{F}(u) - \mathscr{F}(h) + \mathscr{F}(h) - \mathscr{G}(h)) \\ &=: C(I + II + III). \end{aligned}$$

The ω-minimality of u, Hölder's inequality and the L^2-estimate of (5.3.41) (see [Giu03], §10.4) give

$$\begin{aligned} II &\le \omega(R) \int_{B_R} (1 + |\nabla h|) \,\mathrm{d}x \\ &\le \omega(R)(\mathscr{L}^n(B_R) + C R^{\frac{n}{2}} \|\nabla u\|_{L^2(B_R)}) \\ &\le \omega(R)\omega_n R^n (1 + CM). \end{aligned}$$

By the C^1 boundedness of u, it is clear that $|\nabla(u - a)| \le 2M$ and then we can estimate I with (**LP3**$_1$) as follows

$$\begin{aligned} I &= -\int_{B_R} \int_0^1 (1 - t)(f''(\nabla a + t\nabla(u - a)) - f''(\nabla a))[\nabla(u - a), \nabla(u - a)] \,\mathrm{d}x \\ &\le C(M) \int_{B_R} |\nabla(u - a)|^3 \,\mathrm{d}x \\ &\le C(M)\omega_n R^n \left(\fint_{B_R} |\nabla(u - a)|^{q'} \,\mathrm{d}x\right)^{\frac{1}{q'}} \left(\fint_{B_R} |\nabla(u - a)|^{2q} \,\mathrm{d}x\right)^{\frac{1}{q}} \end{aligned}$$

$$\overset{(5.3.39)}{\leq} C\omega_n R^n \left(\fint_{B_R} |\nabla(u-a)|^2 \,\mathrm{d}x\right)^{\frac{1}{q'}} \left(\fint_{B_{2R}} |\nabla(u-a)|^2 \,\mathrm{d}x + \omega(2R)\right),$$

where the q can be taken smaller than 2 so that there holds the inequality $|\nabla(u-a)|^{q'} \leq (2M)^{q'-2}|\nabla(u-a)|^2$. The estimate of III is similar with the help of the L^p-estimate of (5.3.41) (see [Giu03], Section 10.4). Summing up the estimates for I, II and III gives the desired inequality. □

With the estimate between u and the harmonic function h, we can obtain the following excess decay estimate from the regularity of h.

Proposition 5.3.11. *Suppose that $f, \omega, u, S_u, \Omega', M(\Omega')$ are as in Proposition 5.3.8 and q as in Proposition 5.3.9. Take a ball $B(x_0, R)$ such that $R < R_0$ and $B(x_0, 2R) \subset\subset \Omega'$. For any $\sigma, \gamma \in (0,1)$ we have*

$$\mathscr{E}_1(\sigma R) \leq C(\sigma^{-n} + \sigma^{2\gamma}) \left(\mathscr{E}_1(2R)^{1+\frac{1}{q'}} + \omega(2R)\right) + C\sigma^{2\gamma}\mathscr{E}_1(2R) \tag{5.3.43}$$

with $C = C(n, N, L, \ell, M, q, \gamma) > 0$, where $\mathscr{E}_1$ is the L^2-excess of u

$$\mathscr{E}_1(x_0, \rho) := \fint_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}|^2 \,\mathrm{d}x.$$

Proof. Suppose that h is as in Lemma 5.3.10. For any $\rho < R$ and any fixed $\gamma \in (0,1)$, the map h satisfies, by §III.2 in [Gia83],

$$\fint_{B_\rho} |\nabla h - (\nabla h)_{B_\rho}|^2 \,\mathrm{d}x \leq C\left(\frac{\rho}{R}\right)^{\gamma} \fint_{B_R} |\nabla h - (\nabla h)_{B_R}|^2 \,\mathrm{d}x. \tag{5.3.44}$$

Then the excess of ∇u can be estimated by comparing ∇u and ∇h, which gives

$$\begin{aligned}\fint_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}| &\leq 6\fint_{B_\rho} |\nabla u - \nabla h|^2 \,\mathrm{d}x + 3\fint_{B_\rho} |\nabla h - (\nabla h)_{B_\rho}|^2 \,\mathrm{d}x \\ &\leq 6\left(\frac{R}{\rho}\right)^n \fint_{B_R} |\nabla u - \nabla h|^2 \,\mathrm{d}x + 3C\left(\frac{\rho}{R}\right)^{2\gamma} \fint_{B_R} |\nabla h - (\nabla h)_{B_R}|^2 \,\mathrm{d}x.\end{aligned}$$

The last term of the right-hand side can be estimated by

$$\fint_{B_R} |\nabla h - (\nabla h)_{B_R}|^2 \,\mathrm{d}x \leq 6\fint_{B_R} |\nabla u - \nabla h|^2 \,\mathrm{d}x + 3\fint_{B_R} |\nabla u - (\nabla u)_{B_R}|^2 \,\mathrm{d}x.$$

Taking Lemma 5.3.10 into consideration we have the following desired estimate

$$\begin{aligned}&\fint_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}|^2 \,\mathrm{d}x \\ &\leq C\left(\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^{2\gamma}\right) \fint_{B_R} |\nabla(u-h)|^2 \,\mathrm{d}x + C\left(\frac{\rho}{R}\right)^{2\gamma} \fint_{B_R} |\nabla u - (\nabla u)_{B_R}|^2 \,\mathrm{d}x \\ &\leq C\left(\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^{2\gamma}\right) \left(\left(\fint_{B_{2R}} |\nabla u - (\nabla u)_{B_{2R}}|^2 \,\mathrm{d}x\right)^{1+\frac{1}{q'}} + \omega(2R)\right)\end{aligned}$$

$$+ C\left(\frac{\rho}{R}\right)^{2\gamma} \fint_{B_{2R}} |\nabla u - (\nabla u)_{B_{2R}}|^2 \,\mathrm{d}x.$$

□

Replace $2R$ by R and then the excess estimate is

$$\mathscr{E}_1(\sigma R) \le C(\sigma^{-n} + \sigma^{2\gamma}) \left(\mathscr{E}_1(R)^{1+\frac{1}{q'}} + \omega(R)\right) + C\sigma^{2\gamma}\mathscr{E}_1(R). \tag{5.3.45}$$

It indeed holds for $\sigma \in (0,1)$ as the case $\sigma \in (\frac{1}{2}, 1)$ is obvious. Given $\alpha \in (0,1)$, we take $\gamma > \alpha$ and do iteration as in Subsection 5.3.5. The final statement is as follows: There exist $\varepsilon_0 > 0$ and $R_2 > 0$ such that if $B(x_0, R) \subset\subset \Omega'$ satisfies

$$\mathscr{E}_1(R) < \frac{\varepsilon_0}{2}, \quad R < R_2,$$

we have

$$\mathscr{E}_1(\rho) \le C\left(\frac{\rho}{R}\right)^{2\alpha} \mathscr{E}_1(R) + C\omega(\rho) \tag{5.3.46}$$

for any $\rho \in (0,R)$ with some $C = C(n,N,L,\ell,M,\alpha) > 0$. With an argument similar with that in Subsection 5.3.6, it is easy to see that ∇u has a local modulus of continuity $\rho \mapsto \rho^\alpha + \Xi_{\frac{1}{2}}(\rho)$ in Ω', and thus in $\Omega \setminus S_u$. The proof of Theorem 5.1.1 is now complete.

5.3.8 Indirect argument

In [DGK05] the authors showed partial regularity for ω-minimizers in the sub-quadratic case ($1 < p < 2$), where the harmonic approximation was done via an indirect argument. That method can be adapted to the linear growth context to obtain out partial regularity result Theorem 5.1.1. A sketch of the proof is presented in the following, and we mainly show the part different from that in [DGK05]. For some skipped details we refer the above reference. Notice that with this method, only C^2 regularity of f is needed. In other words, we replace **(LP3)** by

(LP3)$'$ F is C^2 and

$$\left|f''(z_1) - f''(z_2)\right| \le \nu_M(|z_1 - z_2|)$$

for any $z_1, z_2 \in B(0, M+1)$, where ν_M is concave and non-decreasing on $[0,\infty)$ with $\nu_M(0) = \lim_{t\to 0} \nu_M(t) = 0$.

The Dini type condition of ω can also be relaxed to $\Xi_{\frac{1}{2}}(\rho) < \infty$ for any $\rho > 0$, as in the excess decay estimate $\omega(R)$ with the desired exponent will be obtained with one attempt.

For this argument, most of the steps in [DGK05] remain the same. The difference is twofold: the Sobolev-Poincaré inequality and the harmonic approximation.

Define two maps V, W on any Euclidean space $\mathbb{R}^d$ with $d \in \mathbb{N}^+$:

$$V(\xi) := \frac{\xi}{(1+|\xi|^2)^{\frac{1}{4}}}, \quad W(\xi) := \frac{\xi}{\sqrt{1+|\xi|}}, \quad \xi \in \mathbb{R}^d.$$

It is easy to see the following properties:

(i) $|W(\xi)| \le |V(\xi)| \le 2^{-\frac{1}{4}}|W(\xi)|$;

(ii) $|W(\xi)|^2$ is convex in ξ;

(iii) $|W(\xi)|^2 \sim |\xi|^2$ near the origin and $|W(\xi)|^2 \sim |\xi|$ near the infinity, which is similar with $E(\xi)$ (see (5.2.4)).

The following is a Sobolev-Poincaré type inequality with respect to W, which is analogous to Theorem 2 in [DGK05] but more involved.

Theorem 5.3.12. *Let $B_R(x_0) \subset \mathbb{R}^n$ be a ball with $n \geq 2$. Then for any $u \in BV(B_R(x_0), \mathbb{R}^N)$ there holds*

$$\left(\fint_{B_R(x_0)} \left| W\left(\frac{u(x) - u_{x_0,R}}{R} \right) \right|^{\frac{2n}{n-1}} \mathrm{d}x \right)^{\frac{n-1}{2n}} \leq c_s \left(\fint_{B_R(x_0)} |W(Du)|^2 \right)^{\frac{1}{2}}, \tag{5.3.47}$$

where the constant c_s depends on n, N. It also holds with a different constant if W is replaced by V.

Proof. We only show for maps in $W^{1,1} \cap C^\infty(B_R(x_0), \mathbb{R}^N)$. As $|W|^2$ is convex and thus $\int |W|^2$ is continuous with respect to convergence in the area-strict sense in BV, the general case can be done by approximation. For any two distinct points $x, y \in B_R(x_0)$, it is easy to see $|x-y| < 2R$. Then fix an $x \in B_R(x_0)$, we have

$$\frac{|u(x) - u_{x_0,R}|}{2R} \leq \fint_{B_R(x_0)} \frac{|u(x) - u(y)|}{2R} \, \mathrm{d}y \leq \fint_{B_R(x_0)} \fint_0^{|x-y|} |Du(x + r\omega)\omega| \, \mathrm{d}r \, \mathrm{d}y,$$

where $\omega = \frac{y-x}{|y-x|}$. The function $W^2(t)$ is non-decreasing in $t \in [0, \infty)$ and convex, then we can apply it to both sides and use Jensen's inequality to obtain

$$W^2 \left(\frac{|u(x) - u_{x_0,R}|}{2R} \right) \leq \fint_{B_R(x_0)} \fint_0^{|x-y|} W^2(|Du(x + r\omega)\omega|) \, \mathrm{d}r \, \mathrm{d}y.$$

Define $\widetilde{W}(|Du(z)|) = W(|Du(z)|)\chi_{B_R(x_0)}(z)$ and consider the balls centered at x. Then the right-hand side can be estimated with Tonelli's theorem as follows:

$$\begin{aligned}
\text{RHS} &\leq \frac{1}{\mathscr{L}^n(B_R(x_0))} \int_{B_{2R}(x)} \fint_0^{|x-y|} \widetilde{W}^2(|Du(x + r\omega)|) \, \mathrm{d}r \, \mathrm{d}y \\
&\leq \frac{1}{\mathscr{L}^n(B_R(x_0))} \int_{\mathbb{S}^{n-1}} \int_0^{2R} s^{n-2} \int_0^s \widetilde{W}^2(|Du(x + r\omega)|) \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}\omega \\
&\leq \frac{1}{\mathscr{L}^n(B_R(x_0))} \int_{\mathbb{S}^{n-1}} \int_0^{2R} \widetilde{W}^2(|Du(x + r\omega)|) \int_s^{2R} s^{n-2} \, \mathrm{d}s \, \mathrm{d}r \, \mathrm{d}\omega \\
&\leq \frac{(2R)^{n-1}}{(n-1)\mathscr{L}^n(B_R(x_0))} \int_{B_R(x_0)} \frac{W^2(|Du(y)|)}{|y-x|^{n-1}} \, \mathrm{d}y.
\end{aligned}$$

Integrating with respect to x in $B_R(x_0)$, we get

$$\fint_{B_R(x_0)} \left| W\left(\frac{u(x) - u_{x_0,R}}{R} \right) \right|^2 \mathrm{d}x$$

$$
\begin{aligned}
&\leq \frac{c}{R} \fint_{B_R(x_0)} \int_{B_R(x_0)} \frac{W^2(|Du(y)|)}{|x-y|^{n-1}} \,\mathrm{d}y \,\mathrm{d}x \\
&\leq \frac{c}{R} \fint_{B_R(x_0)} W^2(|Du(y)|) \int_{B_{2R}(y)} |x-y|^{1-n} \,\mathrm{d}x \,\mathrm{d}y \qquad (5.3.48)\\
&\leq c \fint_{B_R(x_0)} W^2(|Du(y)|) \,\mathrm{d}y.
\end{aligned}
$$

For the estimate of a higher order integral of $W(|u-u_{x_0,R}|/R)$, we need the classical Sobolev inequality. Consider $g = (u-u_{x_0,R})/R$ and $U = W^2(|g|)$. Notice that $W^2(|\cdot|)$ is Lipschitz, and then $U \in W^{1,1}(B_R(x_0))$ with

$$
DU(x) = \frac{|g(x)|(2+|g(x)|)}{(1+|g(x)|)^2} Dg(x) \frac{g(x)}{|g(x)|} \quad \text{in } \{x\colon\ g(x) \neq 0\}.
$$

The Sobolev embedding for $W^{1,1}$ gives

$$
\left(\fint_{B_R(x_0)} |U(x)|^{\frac{n}{n-1}} \,\mathrm{d}x \right)^{\frac{n-1}{n}} \leq C \left(R \fint_{B_R(x_0)} |DU(x)| \,\mathrm{d}x + \fint_{B_R(x_0)} |U(x)| \,\mathrm{d}x \right). \qquad (5.3.49)
$$

To further control the integral of $|DU|$, property (iii) is needed in the following. When $|Du(x)| \geq 1$, from the expression of DU we have

$$
R|DU(x)| \leq 2|Du(x)| \leq cW^2(|Du(x)|).
$$

When $0 < |Du(x)| < 1$, apply Young's inequality and then

$$
\begin{aligned}
R|DU(x)| &\leq \frac{1}{2} \frac{|g|^2(2+|g|)^2}{(1+|g|)^4} + \frac{1}{2}|Du|^2 \\
&\leq 2\min\{|g|, |g|^2\} + cW^2(|Du|) \\
&\leq c(W^2(|g|) + W^2(|Du|)).
\end{aligned}
$$

Thus, the first term on the right-hand side of (5.3.49) is controlled by

$$
R \fint_{B_R(x_0)} |DU(x)| \,\mathrm{d}x \leq C \fint_{B_R(x_0)} \left(W^2\left(\frac{u(x)-u_{x_0,R}}{R} \right) + W^2(|Du(x)|) \right) \mathrm{d}x.
$$

Combining (5.3.48) we have the desired inequality. □

We have obtained the Caccioppoli inequality in Proposition 5.3.1. It is easy to see from (i), (iii) that $V^2(t) \sim E(t)$, so we have Caccioppoli's inequality with respect to V^2.

Lemma 5.3.13. *Suppose that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* **(LP1)**,**(LP2)** *and* **(LP3)**$'$ *with $p=1$, ω satisfies* **($\boldsymbol{\omega}$1)**, *and $u \in BV_{loc}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$ with constant $R_0 > 0$. Fix $m > 0$, then there exists $c_c = c_c(m,n,N,L,\ell)$ such that for any $B_R = B(x_0,R) \subset\subset \Omega$ with $R < R_0$ and any affine map $a : \mathbb{R}^n \to \mathbb{R}^N$ with $|\nabla a| \leq m$, there holds*

$$
\fint_{B_{\frac{R}{2}}} |V(D(u-a))|^2 \leq c_c \left(\fint_{B_R} \left| V\left(\frac{u-a}{R} \right) \right|^2 \mathrm{d}x + \omega(R) \right). \qquad (5.3.50)
$$

The ω-minimality of u implies that it is almost an $\mathbb{A}$-harmonic map with a proper $\mathbb{A}$, to present which we define the excess for u with $A \in \mathbb{R}^{N\times n}$:

$$\mathscr{E}_2(x_0, R, A) := \left(\fint_{B_R(x_0)} |V(Du) - V(A)|^2 \right)^{\frac{1}{2}}.$$

When x_0 (and R) and A are fixed, we abbreviate the quantity as $\mathscr{E}_2(R)(\mathscr{E}_2)$. The following can be showed with the proof of Lemma 4 in [DGK05] by considering ∇u and $D^s u$ separately.

Lemma 5.3.14 (Approximate harmonicity)**.** *Suppose that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* **(LP1)**,**(LP2)** *and* **(LP3)**$'$ *with $p = 1$, ω satisfies* **($\boldsymbol{\omega}$1)** *with constant $R_0 > 0$, and $u \in BV_{loc}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$. For any $m > 0$, there exists $c_e = c_e(m, n, N, L) > 0$ such that for any ball $B_R = B(x_0, R) \subset\subset \Omega$ with $R < R_0$ and any $A \in \mathbb{R}^{N\times n}$ with $|A| \le m$, we have*

$$\left| \fint_{B_R} f''(A)[Du - A, D\varphi] \right| \le c_e \left(\sqrt{\nu_M(\mathscr{E}_2)}\mathscr{E}_2 + \mathscr{E}_2^2 + \sqrt{\omega(R)} \right) \sup_{B_R} |D\varphi| \tag{5.3.51}$$

for any $\varphi \in C_0^1(B_R, \mathbb{R}^N)$.

The inequality (5.3.51) obtained above can be considered to be a linearised Euler-Lagrange inequality, with which we are able to approximate u by an $f''(A)$-harmonic map by the following lemma:

Lemma 5.3.15. *Fix an arbitrary $\gamma \in (0, 1]$ and a bilinear form $\mathbb{A} \in \bigodot^2(\mathbb{R}^{N\times n})$ that satisfies*

$$\begin{cases} \mathbb{A}[\eta\otimes\xi, \eta\otimes\xi] \ge \lambda|\eta|^2|\xi|^2, & \textit{for any } \eta \in \mathbb{R}^N, \xi \in \mathbb{R}^n, \\ \mathbb{A}[z, z] \le \Lambda|z|^2, & \textit{for any } z \in \mathbb{R}^{N\times n} \end{cases} \tag{5.3.52}$$

for some constants λ, Λ with $0 < \lambda \le \Lambda$. For any $\varepsilon > 0$, there exists $\delta = \delta(n, N, \lambda, \Lambda, \varepsilon) \in (0, 1]$ such that the following holds: for any $v \in BV(B_R(x_0), \mathbb{R}^N)$ satisfying

$$\fint_{B_R} |W(Dv)|^2 \le \gamma^2 \le 1, \tag{5.3.53}$$

$$\fint_{B_R} \mathbb{A}[Dv, D\varphi] \le \gamma\delta \sup_{B_R}|D\varphi|, \quad \textit{for any } \varphi \in C_0^1(B_R, \mathbb{R}^N), \tag{5.3.54}$$

there exists an $\mathbb{A}$-harmonic map $h \in C^\infty(B_R, \mathbb{R}^N)$ such that

$$\fint_{B_R} |W(Dh)|^2 \le 1, \quad \fint_{B_R} \left| W\left(\frac{v - \gamma h}{R} \right) \right|^2 \mathrm{d}x \le \gamma^2 \varepsilon. \tag{5.3.55}$$

The proof of this lemma is by contradiction, for which see [DGK05], Lemma 6. It is mainly a blow-up argument, in which Theorem 5.3.12 is applied. The integrals concerning ∇u and $D^s u$ need to be considered separately when necessary. Notice that the scaling between $B_R(x_0)$ and $B_1(0)$ for BV maps does not hold straightforward but can proved by approximation with $W^{1,1} \cap C^\infty$ maps.

Consider $x_0 \in \Omega$ such that $|(Du)_{x_0,R}| \le m$. Let $A = (Du)_{x_0,R}$ and $\mathbb{A} = f''(A)$, then the bilinear form $\mathbb{A}$ satisfies (5.3.52) with λ, Λ depending on ℓ, L, m. We take $\varepsilon = \sigma^{n+4}$, where

$\sigma \in (0, \frac{1}{4}]$ is to be determined. Then by Lemma 5.3.15 there is a $\delta = \delta(n, N, \ell, L, \sigma) \in (0, 1]$ such that the statement holds. The constants $c_e = c_e(m, n, N, L), c_a = c_a(n, N, \frac{\Lambda}{\lambda})$ are as in Lemma 5.3.14 and Lemma 2.7.1. Set $\mathscr{E}_2(\rho) = \mathscr{E}_2(x_0, \rho, (Du)_{x_0,\rho})$ and

$$\Gamma(R) = \sqrt{\mathscr{E}_2^2(R) + 4\frac{\omega(R)}{\delta^2}}, \quad v = u - (Du)_{x_0,R}(x - x_0), \quad \gamma = c_1 c_e \Gamma(R), \tag{5.3.56}$$

where c_1 is such that (see Lemma 1 (vi) in [DK02])

$$|V(\xi) - V(\eta)| \leq c_1 |V(\xi - \eta)|, \quad \text{for any } |\eta| \leq m.$$

Then we can apply Lemma 5.3.14 to u and Lemma 5.3.15 to v to get an $\mathbb{A}$-harmonic map $h \in C^\infty(B_R, \mathbb{R}^N)$ such that (5.3.55) holds true. By comparing v and γh, we can transfer some regularity of h to v and then obtain an excess decay estimate (see the argument in Subsection 5.3.4 and Lemma 7 in [DGK05]).

Lemma 5.3.16. *Suppose that* $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *satisfies* **(LP1)**,**(LP2)** *and* **(LP3)**$'$ *with* $p = 1$, ω *satisfies* **($\boldsymbol{\omega}$1)**-**($\boldsymbol{\omega}$3)** *with* $\Xi_{\frac{1}{4}}$ *replaced by* $\Xi_{\frac{1}{2}}$, *and* $u \in BV_{loc}(\Omega, \mathbb{R}^N)$ *is an* ω*-minimizer of* $\bar{\mathscr{F}}$ *with constant* $R_0 > 0$. *Fix a* $\sigma \in (0, \frac{1}{4}]$ *and assume that for some* $R \in (0, R_0]$ *the following holds:*

$$|(Du)_{x_0,R}| \leq m, \; 2\sqrt{2} c_a \gamma \leq 1, \; \sqrt{\nu_M(\mathscr{E}_2(R))} + \mathscr{E}_2(R) \leq \frac{\delta}{2}, \; c_e c_a \mathscr{E}_2(R) \leq 1, \tag{5.3.57}$$

where γ, c_e, c_a *are taken as in* (5.3.56) *and* δ *is obtained in Lemma* 5.3.15 *with* γ *and* $\varepsilon = \sigma^{n+4}$. *Then there exist constants* $\tilde{c} = \tilde{c}(n, N, L, \ell, m)$ *and* $\hat{c} = \hat{c}(n, N, L, \ell, m, \sigma)$ *such that the following holds:*

$$\mathscr{E}_2^2(\sigma R) \leq \tilde{c}\sigma^2 \mathscr{E}_2^2(R) + \hat{c}\omega(R). \tag{5.3.58}$$

The next step is to do iteration as in Subsection 5.3.5 (see also §5 in [DGG00]). More precisely, we argue inductively on $B(x_0, \sigma^k R), k = 0, 1, \ldots,$ where σ (and subsequently ε and the upper bound of R) is chosen in such a way that (5.3.57) holds with $m+1$. Then if a ball $B_R = B(x_0, R) \subset\subset \Omega$ is such that $|(Du)_{x_0,R}| \leq m$, and $\mathscr{E}_2(R)$ and R are small enough, we will have

$$\mathscr{E}_2^2(\rho) \leq C\left(\left(\frac{\rho}{R}\right)^{2\alpha} \mathscr{E}_2(R) + \omega(R)\right).$$

The desired partial regularity hence follows with an argument similar with that in Subsection 5.3.6.

5.4 Zero-order partial regularity

The main strategy in the previous section also applies to Theorem 5.1.2 and 5.1.3, partial regularity for ω-minimizers with weaker assumptions on ω. In the following, we will avoid repetition and only present the difference. Theorem 5.1.2 is shown first with more details, and in Subsection 5.4.5 we briefly sketch the proof of Theorem 5.1.3 based on the former and Section 5.3.

5.4.1 Caccioppoli-type inequality

To show Theorem 5.1.2, we need to consider a normalised excess as mentioned Section 5.1. Correspondingly, the Caccioppoli inequality in this case is also presented with a normalising factor $(1+|A|)$.

Proposition 5.4.1. *Suppose that $f\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ satisfies* **(LP1)**, **(LP2)**, **(LP3$_1$)** *and* **(LP3$_2$)** *with $p\in(1,2)$, and ω satisfies* **($\boldsymbol{\omega}$1)**. *The map $u\in BV_{loc}(\Omega,\mathbb{R}^N)$ is an ω-minimizer of $\mathscr{F}$ with constant $R_0>0$. Then for any ball $B_R=B(x_0,R)\subset\subset\Omega$ with $R<R_0$ and any affine map $a\colon\mathbb{R}^n\to\mathbb{R}^N$ with $\nabla a=A\in\mathbb{R}^{N\times n}$, there exists a constant $C=C(n,N,L,\ell,p)$ independent of map a such that*

$$\int_{B_{\frac{R}{2}}} E_p\left(\frac{\nabla u-A}{1+|A|}\right)\mathrm{d}x\le C\left(\int_{B_R} E_p\left(\frac{u-a}{R(1+|A|)}\right)\mathrm{d}x+\omega(R)R^n\right). \tag{5.4.1}$$

Proof. Set $\tilde f:=f_A$, $\tilde u=u-a$, and fix $\frac{R}{2}<t<s<R$. Take a smooth cut-off function ρ between B_t and B_s with $\rho\in C_c^\infty(B_s)$ and $|\nabla\rho|\le\frac{2}{s-t}$, and set $\varphi=\rho\tilde u,\psi=(1-\rho)\tilde u$. Then $\varphi\in W_0^{1,p}(B_s,\mathbb{R}^N)$, and the quasiconvex condition **(LP2)** with (5.2.14) implies

$$\int_{B_s}\tilde f(\nabla\varphi)\,\mathrm{d}x\ge c\,\ell\int_{B_s}E_p^A(\nabla\varphi)\,\mathrm{d}x.$$

The rest part can be carried out as in Proposition 5.3.1 with E replaced by E_p^A defined in Section 2.1. We estimate the term with $\omega(s)$ as follows

$$\begin{aligned}\omega(s)\int_{B_s}(1+|\nabla\psi|^p)\,\mathrm{d}x&=\omega(s)(1+|A|)^p\int_{B_s}\frac{1+|\nabla\psi|^p}{(1+|A|)^p}\,\mathrm{d}x\\&\overset{(5.2.6)}{\le}\omega(s)(1+|A|)^p\int_{B_s}\left(2+\frac{1}{a_1}E_p\left(\frac{\nabla\psi}{1+|A|}\right)\right)\mathrm{d}x\\&\le 2\omega(R)\omega_nR^n(1+|A|)^p+\frac{1}{a_1}\int_{B_s}E_p^A(\nabla\psi)\,\mathrm{d}x.\end{aligned}$$

Then proceed further with Lemma 2.9.1 to obtain

$$\int_{B_{\frac{R}{2}}}E_p^A(\nabla u-A)\le C\int_{B_R}E_p^A\left(\frac{u-a}{R}\right)\mathrm{d}x+C\omega(R)R^n(1+|A|)^p, \tag{5.4.2}$$

and then (5.4.1) follows. □

5.4.2 Harmonic approximation

The result in this subsection can be obtained by modifying the process in Subsections 5.3.2 and 5.3.3, so we will omit the repetitive part and only present the difference.

Suppose that $f\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ satisfies **(LP1)**, **(LP2)**, **(LP3$_1$)** and **(LP3$_2$)** with $p\in(1,2)$, and ω satisfies **($\boldsymbol{\omega}$1)**. The map $u\in BV_{loc}(\Omega,\mathbb{R}^N)$ is an ω-minimizer of $\mathscr{F}$ with constant $R_0>0$. Take $B_R=B(x_0,R)\subset\subset\Omega$ with $R<R_0$, and fix $A\in\mathbb{R}^{N\times n}$. Similar with

Subsection 5.3.2, we have

$$\mathscr{F}(u, B_R) \leq \inf_{v \in W_u^{1,1}(B_R, \mathbb{R}^N)} \mathscr{F}(v, B_R) + \omega_n R^n \varepsilon, \tag{5.4.3}$$

where $\varepsilon = \omega(R) \fint_{B_R} (a_6 + a_7 |\nabla u|^p) \, \mathrm{d}x$ with a_6, a_7 as in (5.3.10). Consider the complete metric space $X = W_u^{1,p}(B_R, \mathbb{R}^N)$ with

$$d(w_1, w_2) = (1 + |A|)^{\frac{p}{2}-1} \left(\fint_{B_R} |\nabla(w_1 - w_2)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

The Ekeland variational principle (Theorem 2.8.1) then implies the existence of $w \in W_u^{1,p}(B_R, \mathbb{R}^N)$ such that, with $\mathcal{F}(u) = \fint_{B_R} F(\nabla u) \, \mathrm{d}x$,

(a) $d(u, w) \leq \sqrt{\varepsilon}$;

(b) $\mathcal{F}(w) \leq \mathcal{F}(u)$;

(c) $\mathcal{F}(w) \leq \mathcal{F}(v) + \sqrt{\varepsilon} \, d(w, v)$, for any $v \in X = W_u^{1,p}(B_R, \mathbb{R}^N)$.

Subsequently, we have the Euler-Lagrange inequality: for any $\varphi \in W_0^{1,p}(B_R, \mathbb{R}^N)$ there holds

$$\left| \fint_{B_R} f'(\nabla w) \cdot \nabla \varphi \, \mathrm{d}x \right| \leq \sqrt{\varepsilon} (1 + |A|)^{\frac{p}{2}-1} \left(\fint_{B_R} |\nabla \varphi|^p \, \mathrm{d}x \right)^{\frac{1}{p}}. \tag{5.4.4}$$

Proposition 5.4.2. *Suppose that $f \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ satisfies* (**LP1**), (**LP2**), (**LP3**$_1$) *and* (**LP3**$_2$) *with $p \in (1, 2)$, and ω satisfies* ($\boldsymbol{\omega}$**1**). *The map $u \in W_{loc}^{1,p}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of $\mathscr{F}$ with constant $R_0 > 0$. For any ball $B_R = B(x_0, R) \subset\subset \Omega$ with $R < R_0$ and any affine map $a \colon \mathbb{R}^n \to \mathbb{R}^N$ with $\nabla a = A \in \mathbb{R}^{N \times n}$, the system*

$$\begin{cases} -\operatorname{div}(f''(A) \nabla h) = 0, & \text{in } B_R \\ h|_{\partial B_R} = u|_{\partial B_R}, & \text{on } \partial B_R \end{cases} \tag{5.4.5}$$

admits a unique solution $h \in W_u^{1,p}(B_R, \mathbb{R}^N)$ such that

$$\left(\fint_{B_R} |\nabla h - A|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \leq C \left(\fint_{B_R} |\nabla u - A|^p \, \mathrm{d}x \right)^{\frac{1}{p}}, \tag{5.4.6}$$

where $C = C(n, N, \frac{L}{\ell}, p) > 0$. Furthermore, set

$$\varepsilon = \fint_{B_R} (a_6 + a_7 |\nabla u|^p) \, \mathrm{d}x, \ \varepsilon_{A,p} = \frac{\varepsilon}{(1 + |A|)^p}, \ r = \max\left\{ 2, \frac{np'}{n + p'} \right\},$$

and denote $\frac{r'}{p} = \min\{\frac{2}{p}, \frac{n}{n-p}\}$ by s, then there exists a constant $C = C(n, N, L, \ell, p) > 0$ such that

$$\fint_{B_R} E_p \left(\frac{u - h}{R(1 + |A|)} \right) \mathrm{d}x \leq C \left(\fint_{B_R} E_p \left(\frac{\nabla u - A}{1 + |A|} \right) \mathrm{d}x \right)^s + C(\varepsilon_{A,p}^{\frac{p}{2}} + \varepsilon_{A,p}^{\frac{r'}{2}}). \tag{5.4.7}$$

Proof. Define $\mathbb{A} := f''(A)(1 + |A|)^{2-p}$. Then from (**LP3**$_1$) and Lemma 5.2.5 we know that $|\mathbb{A}| \leq L$ and the operator satisfies the Legendre-Hadamard condition. Lemma 2.7.1 and the comment after it indicate that there exists a unique solution $h \in W^{1,p}_u(B_R, \mathbb{R}^N)$ to (5.4.5) satisfying (5.4.6).

Set $\tilde{f} = f_A, \tilde{u} = u - a$ and $\tilde{w} = w - a$, where the map w is obtained by Ekeland's variational principle as above and thus satisfies (5.4.4). As in (5.3.15), we have, by Lemma 5.2.7, (5.4.4), Höler's inequality and the fact $E(z)^p \leq c(p)E_p(z)$,

$$
\begin{aligned}
&\fint_{B_R} \tilde{f}''(0)[\nabla(w-h), \nabla\varphi]\, \mathrm{d}x \qquad (5.4.8)\\
&\leq (1+|A|)^{p-1}\left(C \fint_{B_R} E\left(\frac{\nabla\tilde{w}}{1+|A|}\right)|\nabla\varphi|\ \mathrm{d}x + \varepsilon^{\frac{1}{2}}_{A,p} \fint_{B_R} |\nabla\varphi|\, \mathrm{d}x\right)\\
&\leq (1+|A|)^{p-1}\left(\fint_{B_R} |\nabla\varphi|^{p'}\ \mathrm{d}x\right)^{\frac{1}{p'}}\left(C\left(\fint_{B_R} E_p\left(\frac{\nabla\tilde{w}}{1+|A|}\right)\ \mathrm{d}x\right)^{\frac{1}{p}} + \varepsilon^{\frac{1}{2}}_{A,p}\right)
\end{aligned}
$$

for any $\varphi \in W^{1,\infty}_0 \cap C^1(B_R, \mathbb{R}^n)$. To find a proper test map φ, we again scale to the unit ball $B = B(0,1)$, define Φ, Ψ and $\tilde{W}$ as Proposition 5.3.3 and consider

$$
\begin{cases}
-\operatorname{div}(\mathbb{A}\nabla\Phi) = T_p\left(\dfrac{\Psi}{1+|A|}\right), & \text{in } B\\
\Phi|_{\partial B} = 0, & \text{on } \partial B,
\end{cases}
\tag{5.4.9}
$$

where for any $y \in \mathbb{R}^N$

$$
T_p(y) = \begin{cases} y, & |y| \leq 1\\ |y|^{p-2}\, y, & |y| > 1. \end{cases}
$$

Then we have $T_p(\frac{\Psi}{1+|A|}) \in L^{p'}(B, \mathbb{R}^N)$ and that (5.4.9) has a unique solution $\Psi \in W^{1,p'}_0 \cap W^{2,p'}(B, \mathbb{R}^N)$ satisfying

$$
\|\Phi\|_{W^{2,r}} \leq C(n,N,r)\left\|T_p\left(\frac{\Psi}{1+|A|}\right)\right\|_{L^r}, \quad \text{for any } r \in [2, p']. \tag{5.4.10}
$$

Take $r = \max\{2, \frac{np'}{n+p'}\}$, which is smaller than p', then $\|\nabla\Phi\|_{L^{p'}}$ can be controlled in the following way with the Sobolev embedding

$$
\|\nabla\Phi\|_{L^{p'}} \leq C(p,n,N)\,\|\Phi\|_{W^{2,r}} \leq C(p,n,N)\left\|T_p\left(\frac{\Psi}{1+|A|}\right)\right\|_{L^r}. \tag{5.4.11}
$$

When $|y| \leq 1$, it is easy to see that $|T_p(y)|^r \leq |y|^2 \leq \frac{1}{a_1}E_p(y)$. If $|y| > 1$, we consider two cases:

- $\frac{2n}{n+2} \leq p < 2$, i.e., $\frac{np'}{n+p'} \leq 2$ and $r = 2$: $(p-1)r = 2(p-1) \leq p$ as $p < 2$;
- $1 < p < \frac{2n}{n+2}$, i.e., $\frac{np'}{n+p'} > 2$ and $r = \frac{np'}{n+p'}$: $(p-1)r = \frac{np(p-1)}{n(p-1)+p} < p$.

In both cases, we have $|T_p(y)|^r = |y|^{(p-1)r} \leq |y|^p \leq \frac{1}{a_1}E_p(y)$. Thus, with (5.4.8), (5.4.11) and the difference between u and w (see (a)), the estimate (5.4.7) can be obtained as in Proposition 5.3.3. □

5.4.3 Excess decay estimate

For a ball $B_R = B(x_0, R) \subset\subset \Omega$, we define the excess

$$\mathcal{E}(x_0, R) := \fint_{B_R} E_p\left(\frac{\nabla u - (\nabla u)_R}{1 + |(\nabla u)_R|}\right) \mathrm{d}x. \tag{5.4.12}$$

When the centre x_0 is fixed, we will abbreviate the excess as $\mathcal{E}(R)$.

Proposition 5.4.3. *Suppose that* $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *satisfies* **(LP1)**, **(LP2)**, **(LP3$_1$)** *and* **(LP3$_2$)** *with* $p \in (1, 2)$, *and* ω *satisfies* **($\boldsymbol{\omega}$1)**. *The map* $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ *is an* ω*-minimizer of* $\mathscr{F}$ *with* $R_0 > 0$. *For any* $\sigma \in (0, 1)$, *there exists* $\varepsilon_1 > 0$ *such that if*

$$R < R_0, \quad \mathcal{E}(x_0, R) < \varepsilon_1 \tag{5.4.13}$$

for some ball $B_R = B(x_0, R) \subset\subset \Omega$, *then we have*

$$\mathcal{E}(\sigma R) \le c_1\sigma^{-(n+2)}(\mathcal{E}(R)^s + \omega(R)^{\frac{p}{2}}) + c_2\sigma\mathcal{E}(R) + c_3\omega(2\sigma R), \tag{5.4.14}$$

where s *is as in Proposition* 5.4.2 *and* $c_i = c_i(n, N, L, \ell, p) > 0$, $i = 1, 2, 3$.

Proof. We only consider $\sigma \in (0, \frac{1}{4})$ as it is obvious when $\sigma \in [\frac{1}{4}, 1)$. As in Proposition 5.3.4, we define $a(x) = u_{B_R} + (\nabla u)_{B_R}(x - x_0)$, $\tilde{u} = u - a$ and $\tilde{f} = f_{\nabla a}$. Let h be the harmonic map determined by (5.4.5) and set

$$\tilde{h} = h - a, \quad a_1(x) = \tilde{h}(x_0) + \nabla\tilde{h}(x_0)(x - x_0), \quad a_0 = a + a_1.$$

With Lemma 2.7.1 and 2.7.2 we have

$$|\nabla h(x_0) - (\nabla u)_R| = |\nabla\tilde{h}(x_0)| \le C\fint_{B_R} |\nabla\tilde{h}|\,\mathrm{d}x \tag{5.4.15}$$

$$\le C\left(\fint_{B_R} |\nabla\tilde{h}|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \le c_4\left(\fint_{B_R} |\nabla\tilde{u}|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{5.4.16}$$

The normalising factor needed in each step varies, and we now give the comparison of them. The first one is as follows:

$$\begin{aligned}
1 + |(\nabla u)_R| &\le 1 + |(\nabla u)_{\sigma R}| + \sigma^{-n}\fint_{B_R} |\nabla u - (\nabla u)_R|\,\mathrm{d}x \\
&\le 1 + |(\nabla u)_{\sigma R}| + \frac{1 + |(\nabla u)_R|}{\sigma^n}\left(\fint_{B_R}\left(\frac{|\nabla u - (\nabla u)_R|}{1 + |(\nabla u)_R|}\right)^p \mathrm{d}x\right)^{\frac{1}{p}} \qquad (5.4.17)\\
&\le 1 + |(\nabla u)_{\sigma R}| + \frac{1 + |(\nabla u)_R|}{\sigma^n}(3\mathcal{E}(R))^{\frac{1}{2p}},
\end{aligned}$$

where the last line is from Lemma 5.2.3 if we take $\varepsilon_1 < 1$. We further require $\sigma^{-n}(3\varepsilon_1)^{\frac{1}{2p}} < \frac{1}{2}$, i.e., $\varepsilon_1 < \frac{\sigma^{2np}}{3\cdot 4^p}$, then the above estimate gives

$$1 + |(\nabla u)_R| \le 2(1 + |(\nabla u)_{\sigma R}|). \tag{5.4.18}$$

For $1+|\nabla h(x_0)|$ and $1+|(\nabla u)_{\sigma R}|$, we have

$$\begin{aligned}\frac{1+|\nabla h(x_0)|}{1+|(\nabla u)_{\sigma R}|} &\le 1+\frac{1}{1+|(\nabla u)_{\sigma R}|}(|\nabla h(x_0)-(\nabla u)_R|+|(\nabla u)_R-(\nabla u)_{\sigma R}|)\\ &\le 1+(c_4+\sigma^{-n})\left(\fint_{B_R}\left(\frac{|\nabla u-(\nabla u)_R|}{1+|(\nabla u)_{\sigma R}|}\right)^p \mathrm{d}x\right)^{\frac{1}{p}}\\ &\le 1+2(c_4+\sigma^{-n})\left(\fint_{B_R}\left(\frac{|\nabla u-(\nabla u)_R|}{1+|(\nabla u)_{R}|}\right)^p \mathrm{d}x\right)^{\frac{1}{p}}\\ &\le 1+2(c_4+\sigma^{-n})(3\mathcal{E}(R))^{\frac{1}{2p}},\end{aligned}$$

where (5.4.16), Hölder's inequality, (5.4.18) and Lemma 5.2.3 are applied. Taking $2(c_4+\sigma^{-n})(3\varepsilon_1)^{\frac{1}{2p}}<1$, i.e., $\varepsilon_1<\frac{1}{3\cdot 4^p}(c_4+\sigma^{-n})^{-2p}$, we have

$$\frac{1+|\nabla h(x_0)|}{1+|(\nabla u)_{\sigma R}|}\le 2. \tag{5.4.19}$$

The comparison between $1+|(\nabla u)_R|$ and $1+|\nabla h(x_0)|$ is similar:

$$\begin{aligned}\frac{1+|(\nabla u)_R|}{1+|\nabla h(x_0)|} &\le 1+\frac{|(\nabla u)_R-\nabla h(x_0)|}{1+|(\nabla u)_R|}\cdot\frac{1+|(\nabla u)_R|}{1+|\nabla h(x_0)|}\\ &\le 1+c_4(\mathcal{E}(R))^{\frac{1}{2p}}\frac{1+|(\nabla u)_R|}{1+|\nabla h(x_0)|}\\ &\le 1+\frac{1}{2}\cdot\frac{1+|(\nabla u)_R|}{1+|\nabla h(x_0)|},\end{aligned}$$

which implies

$$\frac{1+|(\nabla u)_R|}{1+|\nabla h(x_0)|}\le 2. \tag{5.4.20}$$

Now we estimate $\mathcal{E}(\sigma R)$: by (5.2.5) and (5.4.19) there holds

$$\mathcal{E}(\sigma R)=\fint_{B_{\sigma R}}E_p\left(\frac{\nabla u-(\nabla u)_{\sigma R}}{1+|(\nabla u)_{\sigma R}|}\right)\mathrm{d}x\le 16\fint_{B_{\sigma R}}E_p\left(\frac{\nabla u-\nabla h(x_0)}{1+|\nabla h(x_0)|}\right)\mathrm{d}x. \tag{5.4.21}$$

The right-hand side can be estimated by the Caccioppoli-type inequality (5.4.1)

$$\fint_{B_{\sigma R}}E_p\left(\frac{\nabla u-\nabla h(x_0)}{1+|\nabla h(x_0)|}\right)\mathrm{d}x\le C\fint_{B_{2\sigma R}}E_p\left(\frac{u-a_0}{2\sigma R(1+|\nabla h(x_0)|)}\right)\mathrm{d}x+C\omega(2\sigma R). \tag{5.4.22}$$

The term involving $u-a_0$ can be estimated, like in Proposition 5.3.4, by decomposing $u-a_0$ into $\tilde{u}-\tilde{h}$ and $\tilde{h}-a_1$. Applying (5.4.20) and (5.4.7), we have

$$\begin{aligned}\fint_{B_{2\sigma R}}E_p\left(\frac{\tilde{u}-\tilde{h}}{2\sigma R(1+|\nabla h(x_0)|)}\right)\mathrm{d}x &\le C\sigma^{-(n+2)}\fint_{B_R}E_p\left(\frac{u-h}{R(1+|(\nabla u)_R|)}\right)\\ &\le C\sigma^{-(n+2)}\left(\mathcal{E}(R)^s+\varepsilon_{A,p}^{\frac{p}{2}}+\varepsilon_{A,p}^{\frac{r'}{2}}\right).\end{aligned}$$

The estimate of $|\nabla^2 h(x_0)|$ in Lemma 2.7.1 and (5.4.20) implies

$$\fint_{B_{2\sigma R}} E_p\left(\frac{\tilde{h}-a_1}{2\sigma R(1+|\nabla h(x_0)|)}\right)\mathrm{d}x \le E_p\left(C\sigma \fint_{B_R}\frac{|\nabla u-(\nabla u)_R|}{1+|(\nabla u)_R|}\,\mathrm{d}x\right) \le C\sigma\mathcal{E}(R), \tag{5.4.23}$$

where we used (5.2.5) and Jensen's inequality. Notice that the term $\varepsilon_{A,p}$ can be estimated with the triangle inequality and Lemma 5.2.3, and we obtain

$$\varepsilon_{A,p} = \omega(R)\fint_{B_R}\frac{a_6+a_7|\nabla u|^p}{(1+|(\nabla u)_R|)^p}\,\mathrm{d}x \le C\omega(R).$$

Thus, combining (5.4.21)-(5.4.23) we have the desired estimate (5.4.14) of $\mathcal{E}(\sigma R)$ under the condition $\varepsilon_1 < \frac{1}{3}\min\{\frac{\sigma^{2np}}{4^p}, \frac{1}{4^p}(c_4+\sigma^{-n})^{-2p}\}$. □

5.4.4 Final conclusion

In this subsection, we use the excess decay estimate above to further obtain a Morrey-type estimate for ∇u, which then implies the Hölder regularity of u.

For any $\alpha \in (0,1)$, we take $\gamma = p(\alpha-1)+n \in (n-p,n)$.

Proposition 5.4.4. *Suppose that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* (**LP1**), (**LP2**), (**LP3**$_1$) *and* (**LP3**$_2$) *with $p \in (1,2)$, and ω satisfies* ($\boldsymbol{\omega}$**1**). *The map $u \in W^{1,p}_{loc}(\Omega,\mathbb{R}^N)$ is an ω-minimizer of $\mathscr{F}$ with $R_0 > 0$. There exist $R_1 \in (0,R_0), \varepsilon_2 \in (0,1)$ such that for any ball $B_R = B(x_0,R) \subset\subset \Omega$ with*

$$0 < R < R_1, \quad \mathcal{E}(x_0,R) < \varepsilon_2,$$

we have

$$\int_{B_\rho}|\nabla u|^p\,\mathrm{d}x \le c_5\left(\left(\frac{\rho}{R}\right)^\gamma\int_{B_R}|\nabla u|^p\,\mathrm{d}x + \rho^\gamma\right) \tag{5.4.24}$$

for some $c_5 = c_5(n,N,L,\ell,p,\gamma) > 0$, where $\gamma \in (n-p,n)$ is defined as above.

Proof. Fix the constants σ, ε_2 and R_1 in order:

$$\sigma = \min\left\{\frac{1}{2}, \frac{1}{4c_2}, 2^{-\frac{p}{n-\gamma}}\right\}, \tag{5.4.25}$$

$$\varepsilon_2 = \min\left\{\varepsilon_1, \left(\frac{\sigma^{n+2}}{4c_1}\right)^{\frac{1}{s-1}}, \frac{1}{3\cdot 16^{p-1}}, \frac{\sigma^{2n}}{3\cdot 4^{p-1}}\right\}, \tag{5.4.26}$$

$$R_1 \in (0,R_0) \text{ such that } \omega(R_1) \le \min\left\{\left(\frac{\sigma^{n+2}\varepsilon_2}{4c_1}\right)^{\frac{2}{p}}, \frac{\varepsilon_2}{4c_3}\right\}, \tag{5.4.27}$$

where s is as in Proposition 5.4.2, and ε_1 and $c_i, i = 1,2,3$, are as in Proposition 5.4.3.

Suppose that for the ball $B_R = B(x_0,R) \subset\subset \Omega$ with some $R \in (0,R_1)$ there holds

$$\mathcal{E}(x_0,R) < \varepsilon_2. \tag{5.4.28}$$

We will show that

$$(\mathbf{I}_k) \qquad\qquad \mathcal{E}(\sigma^k R) < \varepsilon_2$$

holds for any $k \geq 0$ by induction. Obviously, it holds for $k=0$, and we assume that ($\mathbf{I}_k$) holds for some $k \geq 0$. With our choice of ε_2, Proposition 5.4.3 implies

$$\mathcal{E}(\sigma^{k+1}R) \leq c_1\sigma^{-(n+2)}(\mathcal{E}(\sigma^k R)^s + \omega(R)^{\frac{p}{2}}) + c_2\sigma\mathcal{E}(\sigma^k R) + c_3\omega(2\sigma^{k+1}R).$$

By (5.4.25)-(5.4.27), we know

$$c_1\sigma^{-(n+2)}\varepsilon_2^{s-1} \leq \frac{1}{4},\ c_1\sigma^{-(n+2)}\omega(R)^{\frac{p}{2}} \leq \frac{\varepsilon_2}{4},\ c_2\sigma \leq \frac{1}{4},\ c_3\omega(R) \leq \frac{\varepsilon_2}{4},$$

which thus gives ($\mathbf{I}_{k+1}$). Therefore, we have ($\mathbf{I}_k$) holds for any $k \in \mathbb{N}$.

With ($\mathbf{I}_k$) and Lemma 5.2.3 we have

$$\begin{aligned}
&\int_{B_{\sigma^{k+1}R}} |\nabla u|^p \leq 2^{p-1}\left(\int_{B_{\sigma^{k+1}R}} |\nabla u - (\nabla u)_{\sigma^k R}|^p \,\mathrm{d}x + \omega_n(\sigma^{k+1}R)^n |(\nabla u)_{\sigma^k R}|^p\right)\\
&\leq 2^{p-1}(1+|(\nabla u)_{\sigma^k R}|)^p \int_{B_{\sigma^{k+1}R}} \frac{|\nabla u - (\nabla u)_{\sigma^k R}|^p}{(1+|(\nabla u)_{\sigma^k R}|)^p}\,\mathrm{d}x + 2^{p-1}\sigma^n \int_{B_{\sigma^k R}} |\nabla u|^p \,\mathrm{d}x\\
&\leq 2^{p-1}(1+|(\nabla u)_{\sigma^k R}|)^p \omega_n(\sigma^k R)^n \sqrt{3\mathcal{E}(\sigma^k R)} + 2^{p-1}\sigma^n \int_{B_{\sigma^k R}} |\nabla u|^p \,\mathrm{d}x\\
&\leq 2^{p-1}(2^{p-1}\sqrt{3\varepsilon_2} + \sigma^n) \int_{B_{\sigma^k R}} |\nabla u|^p \,\mathrm{d}x + 2^{2(p-1)}\omega_n(\sigma^k R)^n \sqrt{3\varepsilon_2}
\end{aligned}$$

From the choice of ε_2, σ, it is easy to see

$$2^{2(p-1)}\sqrt{3\varepsilon_2} \leq 1, \quad 2^{p-1}(2^{p-1}\sqrt{3\varepsilon_2} + \sigma^n) \leq 2^p\sigma^n \leq \sigma^\gamma.$$

Set $\lambda_p(\rho) := \int_{B_\rho} |\nabla u|^p \,\mathrm{d}x$, then the above gives

$$\lambda_p(\sigma^{k+1}R) \leq \sigma^\gamma \lambda_p(\sigma^k R) + \omega_n(\sigma^k R)^n \tag{5.4.29}$$

for any integer $k \geq 0$. With the following lemma we can further obtain

$$\lambda_p(t) \leq c_5\left(\left(\frac{t}{R}\right)^\gamma \lambda(R) + t^\gamma\right), \tag{5.4.30}$$

where $c_5 = c_5(n,\gamma,\sigma) = c_5(n,N,L,\ell,p,\gamma) = c_5(n,N,L,\ell,p,\alpha)$. □

To obtain (5.4.30), the following lemma is needed. One special case of it can also be found in [Giu03] (Lemma 7.3).

Lemma 5.4.5. *Suppose that* $\lambda\colon \mathbb{R}_+ \to \mathbb{R}$ *is positive, and there exists* $\tau \in (0,1)$*, positive exponents* δ, β*,* $R_0 > 0$ *and* $A, B > 0$ *such that*

$$\lambda(\tau R) \leq \tau^\delta \lambda(R) + AR^\beta, \qquad \text{for any } R < R_0, \tag{5.4.31}$$

$$\lambda(t) \leq B\lambda(\tau^k R), \qquad \text{for any } t \in (\tau^{k+1}R, \tau^k R). \tag{5.4.32}$$

Then for any $\rho < R \leq R_0$, we have

$$\lambda(\rho) \leq C\left(\left(\frac{\rho}{R}\right)^{\alpha}\lambda(R) + A\rho^{\alpha}\right), \tag{5.4.33}$$

where

- *if $\delta \neq \beta$: $\alpha = \min\{\delta, \beta\}, C = C(\tau, \delta, \beta, B) > 0$;*
- *if $\delta = \beta$: $\alpha \in (0, \delta)$ is arbitrary, and $C = C(\tau, \delta, B, \alpha) > 0$.*

Proof. Applying (5.4.31) inductively, we obtain

$$\lambda(\tau^{k+1}R) \leq \tau^{(k+1)\delta}\lambda(R) + AR^{\beta}\tau^{k\beta}\sum_{j=0}^{k}\tau^{j(\delta-\beta)}.$$

It is obvious that the series on the right-hand side converges if $\delta > \beta$, and thus we have

$$\lambda(\tau^{k+1}R) \leq \tau^{(k+1)\delta}\lambda(R) + CA(\tau^k R)^{\beta}.$$

The value of $\lambda(t)$ for a general t can be controlled with (5.4.32). With the fact $R^{\beta} \leq R_0^{\beta-\delta}R^{\delta}$, the proof of the case $\delta < \beta$ is similar.

When $\delta = \beta$, the control we obtain from (5.4.31) is

$$\lambda(\tau^{k+1}R) \leq \tau^{(k+1)\delta}\lambda(R) + AR^{\delta}\tau^{k\delta}(k+1).$$

Since $\tau < 0$, the function $x \mapsto \tau^{(\delta-\alpha)x}(x+1) \to 0$ as $x \in \infty$, and is thus bounded on $(0, \infty)$ by $c = c(\tau, \delta, \alpha) > 0$. The desired inequality then follows. □

Then by a discussion similar to that at the end of Subsection 5.3.5, there exists a relatively closed $\mathscr{L}^n$-null set $S_u' \subset \Omega$ such that $|\nabla u|$ is in the Morrey space $L^{p,\gamma}_{loc}(\Omega \setminus S_u')$. The Sobolev embedding implies that u lies in the Campanato space $\mathcal{L}^{p,\gamma+p}_{loc}(\Omega \setminus S_u', \mathbb{R}^N)$, which is actually $C^{0,\alpha}_{loc}(\Omega \setminus S_u', \mathbb{R}^N)$ as $\gamma = p(\alpha - 1) + n$. The proof of Theorem 5.1.2 is then complete.

Remark 5.4.6. Theorem 5.1.2 can also be approached by an indirect argument, similar to that in [DGK05] or [FM08], by choosing normalising factors carefully. For such an argument, the Lipschitz continuity of f'' can be relaxed to

(LP3$_2$**)**$'$ For any $z_1, z_2 \in \mathbb{R}^{N\times n}$ we have

$$|f''(z_1) - f''(z_2)| \leq \nu\left(\frac{|z_1 - z_2|}{1 + |z_1| + |z_2|}\right)\frac{1}{(1 + |z_1| + |z_2|)^{2-p}},$$

where ν is a concave, non-decreasing function on $[0, \infty)$ with $\nu(0) = \lim_{t\to 0}\nu(t) = 0$ and $\nu \leq 1$.

5.4.5 Linear growth setting

The above argument does not apply to the linear growth setting ($p = 1$) due to the degenerate ellipticity of f'', which also appears in Chapter 6 in the form of convexity.

Suppose that the integrand $f\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ is of linear growth, then it is reasonable to assume that $|f''(z)|\le L(1+|z|)^{-1}$ as in (**LP3**$_1$). However, the Legendre-Hadamard condition of $f''(z)$ given by the quasiconvexity (**LP2**) comes with a coefficient comparable to $(1+|z|)^{-3}$ (see (5.2.17)). Thus, the normalisation of $f''(A)$ as in Proposition 5.4.2 does not work here. Nevertheless, it is still possible to work near those points in Ω where the average of Du remains bounded.

In the following, we suppose that $f\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ satisfies (**LP1**)-(**LP3**) with $p=1$, and ω satisfies (**ω1**). The map $u\in BV_{loc}(\Omega,\mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$ with $R_0>0$. Then what we did in Subsection 5.3.1-5.3.4 remains true as (**ω2**) and (**ω3**) were not used in this process. In particular, Proposition 5.3.4 holds for the ω-minimizer u. The iteration in Subsection 5.3.5 was carried on under (**ω2**) and (**ω3**), and thus is not expected here. Arguing as in Subsection 5.4.4, we are still able to obtain a Morrey-type estimate for Du near those regular points.

Proposition 5.4.7. *Suppose that $f\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ satisfies* (**LP1**)-(**LP3**) *with $p=1$, and ω satisfies* (**ω1**). *The map $u\in BV_{loc}(\Omega,\mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$ with $R_0>0$. For any $\gamma\in(n-1,1)$ and any $m>0$, there exists $C=C(m,n,N,L,\ell,\gamma)>0, \varepsilon'_m>0, R_1>0$ such that the following holds: if $B_R=B(x_0,R)\subset\subset\Omega$ is such that*

$$\sup_{0<r<R}|(Du)_{x_0,R}|<m,\quad \mathscr{E}(x_0,R)<\varepsilon'_m,\quad R<R_1, \tag{5.4.34}$$

then for any $\rho\in(0,R)$ there holds

$$\int_{B_\rho}|Du|\le C\left(\left(\frac{\rho}{R}\right)^\gamma\int_{B_R}|Du|+\rho^\gamma\right). \tag{5.4.35}$$

The excess $\mathscr{E}$ here is as defined in (5.3.22).

Proof. Fix an exponent $q\in(1,\frac{n}{n-1})$ (for example, the middle point $\frac{2n-1}{2(n-1)}$), and take

$$\sigma=\min\left\{1,\frac{1}{2\sqrt{c_m}}\right\}, \tag{5.4.36}$$

$$\varepsilon'_m=\min\left\{1,\left(\frac{\sigma^{n+2}}{4c_m}\right)^{\frac{1}{q-1}}\right\}, \tag{5.4.37}$$

$$R_1\in(0,R_0)\text{ such that }\omega(R_1)\le\min\left\{1,\left(\frac{\sigma^{n+2}\varepsilon'_m}{2c_m}\right)^2\right\}, \tag{5.4.38}$$

where c_m is the constant obtained in Proposition 5.3.4 with the m and q taken above.

Suppose that (5.4.34) holds for $B_R=B(x_0,R)\subset\subset\Omega$. Then we show by induction that the condition

$$(\mathbf{I}_k)\qquad\qquad \mathscr{E}(\sigma^kR)<\varepsilon'_m$$

holds for any integer $k\ge 0$.

It is obvious that $(\mathbf{I}_0)$ holds true. Now assume that we have $(\mathbf{I}_k)$ for some $k\ge0$, and

Proposition 5.3.4 implies

$$\mathscr{E}(\sigma^{k+1}R) \le c_m(\sigma^2 + \sigma^{-(n+2)}\mathscr{E}(\sigma^k R)^{q-1})\mathscr{E}(\sigma^k R) + c_m\sigma^{n+2}\sqrt{\omega(\sigma^k R)}.$$

From the choice of σ, ε'_m and R_1 we have

$$c_m\sigma^2 \le \frac{1}{4}, \quad c_m\sigma^{-(n+2)}(\varepsilon'_m)^{q-1} \le \frac{1}{4}, \quad c_m\sigma^{-(n+2)}\sqrt{\omega(R_1)} \le \frac{\varepsilon'_m}{2},$$

and thus $\mathscr{E}(\sigma^{k+1}R) < \varepsilon'_m$, which is exactly $(\mathbf{I}_{k+1})$.

The condition $(\mathbf{I}_k)$ implies an estimate for $\int|Du|$. By the triangle inequality, the choice of ε'_m and Lemma 5.2.3, we get

$$\begin{aligned}\int_{B_{\sigma^{k+1}R}}|Du| &\le \int_{B_{\sigma^k R}}|Du-(Du)_{x_0,\sigma^k R}| + \omega_n(\sigma^{k+1}R)^n|(Du)_{x_0,\sigma^k R}| \\ &\le \omega_n(\sigma^k R)^n\sqrt{3\mathscr{E}(\sigma^k R)} + \sigma^n\int_{B_{\sigma^k R}}|Du| \\ &\le c_0(\sigma^k R)^n + \sigma^n\int_{B_{\sigma^k R}}|Du|\end{aligned}$$

for any integer $k \ge 0$, where $c_0 = \omega_n\sqrt{3\varepsilon'_m}$. Set $\lambda(\rho) = \int_{B_\rho}|Du|$, then the above estimate becomes

$$\lambda(\sigma^{k+1}R) \le \sigma^n\lambda(\sigma^k R) + c_0(\sigma^k R)^n. \tag{5.4.39}$$

Lemma 5.4.5 implies that the following Morrey-type estimate holds

$$\lambda(\rho) \le c_1\left(\left(\frac{\rho}{R}\right)^\gamma\lambda(R) + \rho^\gamma\right), \tag{5.4.40}$$

where the coefficient c_1 depends on $n, \varepsilon'_m, \sigma, \gamma$, and thus on m, n, N, L, ℓ, γ. $\square$

An argument as that in Subsection 5.3.6 together with the Sobolev inequality for BV maps shows that $u \in \mathcal{L}^{1,\gamma+1}(\mathcal{N}, \mathbb{R}^N)$, where $\mathcal{N}$ is a small neighbourhood of the point x_0 in Proposition 5.4.7. Notice that for any point $x \in \Omega$ satisfying

$$\liminf_{\rho\to 0^+}\fint_{B_\rho(x)}E(Du-(Du_{x,\rho})) = 0, \quad \limsup_{\rho\to 0^+}|(Du)_{x,\rho}| < \infty,$$

the conditions in (5.4.34) holds with some $R > 0$ and $m > 0$. Therefore, we know that there exists a set $S'_u \subset \Sigma_1 \cup \Sigma_2$ with Σ_1, Σ_2 defined in (5.1.1) and (5.1.2), and that $u \in \mathcal{L}^{1,\gamma+1}_{loc}(\Omega\setminus S'_u, \mathbb{R}^N) \hookrightarrow C^{0,\alpha}_{loc}(\Omega\setminus S'_u, \mathbb{R}^N)$, where $\alpha = \gamma+1-n \in (0,1)$. The proof of Theorem 5.1.2 is then complete.

Chapter 6

Sobolev regularity for BV (ω-)minimizers

The current chapter focuses on Sobolev regularity of BV minimizers and ω-minimizers, and the functionals we study are defined by integrands of linear growth and satisfying the so-called μ-ellipticity condition (**LD2**). A certain Sobolev regularity for BV minimizers, as specified in Theorem 6.1.1, is first established. Then there follows a result on the fractional Sobolev regularity of the absolutely continuous part of ω-minimizers, as presented in Theorem 6.1.2. At the time of writing the thesis, it remains unclear whether the singular part of a BV ω-minimizer can be ruled out, and we plan to further explore this in future work. Notice that the Sobolev regularity result for generalised minimizers is already implicitly contained in [BS13].

The problem is contextualised in Section 6.1 together with the statement of our main results. There are examples of integrands in Section 6.2, and counterexamples in which the minimizers have singularities in Section 6.3. Theorem 6.1.1, which is about Sobolev regularity for minimizers, is then proved in Section 6.4. Based on the first result, we are able to discuss regularity for ω-minimizers and show Theorem 6.1.2 in Section 6.5.

6.1 Set-up and main results

The functional we study in this chapter is also of linear growth, and assumed to be strongly convex. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and consider the following functional

$$\mathscr{F}(u,\Omega) := \int_\Omega f(\nabla u)\,\mathrm{d}x. \tag{6.1.1}$$

When the domain Ω is clear from the context, we may abbreviate $\mathscr{F}(u,\Omega)$ as $\mathscr{F}(u)$. For the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$, we consider the following conditions with constants L, ℓ satisfying $0 < \ell \le L$:

(**LD1**) $\ell\,|z| \le f(z) \le L(1+|z|)$ for any $z \in \mathbb{R}^{N\times n}$;

(**LD2**) $f \in C^2(\mathbb{R}^{N\times n})$, and for some $\mu \in (1,3]$ we have

$$\ell\frac{|\xi|^2}{(1+|z|^2)^{\frac{\mu}{2}}} \le f''(z)[\xi,\xi] \le L\frac{|\xi|^2}{(1+|z|^2)^{\frac{1}{2}}}, \quad \text{for any } \xi \in \mathbb{R}^{N\times n}. \tag{E_μ}$$

The left inequality in (E_μ) is the so-called μ-ellipticity. This is considered to be a non-uniform ellipticity condition since the ratio

$$\Lambda_f(z) := \frac{\max_{|\xi|=1} f''(z)[\xi,\xi]}{\min_{|\xi|=1} f''(z)[\xi,\xi]}$$

may blow up as $z \to \infty$. This imbalance is inevitable for linear growth integrands in the following sense: The upper bound in (**LD2**) is the natural controlled growth condition associated to linear growth integrands, and attained by some examples (for instance, $e_{\lambda,p}$ in Example 6.2.1). However, the μ in the lower bound must be larger than 1, since the anti-derivative of $(1+t^2)^{-\frac{1}{2}}$ grows as $\log(1+|t|^2)$ near infinity, and thus an integrand satisfying (**LD2**) with $\mu = 1$ grows super-linearly and the condition (**LD1**) cannot hold at the same time.

In addition to the above two conditions, the following one is needed for the boundedness result in [BS13]:

(**LD3**) $(a^\top f'(z)) \cdot (a^\top z) \geq -\ell |a|^2$ for any $z \in \mathbb{R}^{N\times n}$ and any $a \in \mathbb{R}^N$.

The following statement concerns Sobolev regularity for generalised minimizers. Notice that we do not claim the novelty of this result, as it can be derived from the proof of Theorem 1.10 and B.2 in [BS13]. We present the statement as well as the proof for completeness as this result is needed for Theorem 6.1.2.

Theorem 6.1.1. *Suppose that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies (**LD1**)-(**LD3**). If $u \in BV(\Omega, \mathbb{R}^N)$ is a generalised minimizer of $\bar{\mathscr{F}}$, then we have*

$$u \in W^{1,1}(\Omega, \mathbb{R}^N), \quad \textit{and} \quad W_\mu(\nabla u) \in W^{1,2}_{loc}(\Omega), \tag{6.1.2}$$

where $W_\mu(z) = (1+|z|^2)^{\frac{2-\mu}{4}}$ is defined on $\mathbb{R}^{N\times n}$. For any ball $B_R = B(x_0, R) \subset\subset \Omega$ and any $r \in (0, R)$, the following estimate

$$\int_{B_r} |\nabla W_\mu(\nabla u)|^2 \, \mathrm{d}x \leq \frac{C}{(R-r)^2} \int_{B_R} (1 + |\nabla u|) \, \mathrm{d}x \tag{6.1.3}$$

holds with a constant $C = C(n, N, L, \ell)$. In addition, when $1 < \mu < 1 + \frac{2}{n}$, we furthermore have the following higher integrability of the gradient of u:

$$\begin{cases} \nabla u \in L^q_{loc}(\Omega, \mathbb{R}^{N\times n}), & n \geq 3, \\ W_\mu(\nabla u) \in L^\Phi_{loc}(\Omega), & n = 2, \end{cases} \tag{6.1.4}$$

where $q = \frac{(2-\mu)n}{n-2} > 1$ and L^Φ_{loc} is the local Orlicz space with $\Phi(t) = e^{t^2} - 1$.

The condition (**LD3**) is useful in the proof of the boundedness of u (see Theorem 3.2.2). Instead, we can replace this condition by the a priori assumption that $u \in L^\infty(\Omega, \mathbb{R}^N)$ to obtain the above result. When $1 < \mu < 1 + \frac{2}{n}$, the same result actually holds true without (**LD3**) (see Remark 6.4.7).

The $W^{1,1}$ regularity was already obtained in [BS13] for $\mu = 3$, and their argument also works for $1 < \mu < 3$. With such regularity, the Euler-Lagrange equation can be written down normally as in the super-linear case. Nevertheless, the low integrability of ∇u makes it more difficult to work with such an equation directly (see Remark 6.4.8), and we applied the so-called vanishing viscosity method to tackle this issue. Our argument is a modification of that in [BS13], and a similar approach also appears in [Bil02].

To show the above regularity for a generalised minimizer u, we construct a new series of integrands $\{f_k\}$ by adding to f a viscosity term related to k, which is quadratic and vanishes as $k \to \infty$. The regularity result in [Bil02] is only shown for one generalised minimizer. However, an approximating sequence $\{u_k\}$ of any given generalised minimizer u with nice properties can be determined by applying Ekeland's variational principle (as in [BS13]). Each u_k satisfies an Euler-Largrange type inequality, with which the second differentiability of u_k is further obtained with an estimate uniform in k. Instead of showing an $L \log L$ regularity as in [Bil02] and [BS13], we prove that u_k converges to u strongly in $W^{1,1}$ with the tool of Young measures, and thus the corresponding estimate remains to be true for u, which gives the desired regularity.

Based on the Theorem 6.1.1, we investigated regularity for BV ω-minimizers in this context. The ω considered here is as in Chapter 5 but satisfies the more specific condition ($\boldsymbol{\omega}$**4**). We re-state the conditions here for convenience:

($\boldsymbol{\omega}$**1**) ω is bounded by 1 and nondecreasing with $\omega(0) = \lim_{r\to 0} \omega(r) = 0$.

($\boldsymbol{\omega}$**4**) $\omega(r) \le A r^{2\sigma}$ for some $\sigma \in (0,1)$.

With the Sobolev regularity for generalised minimizers above, we managed to show the following fractional Sobolev regularity for ω-minimizers. Here we take the definition (3.1.31).

Theorem 6.1.2. *Suppose that* $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *satisfies* (**LD1**)-(**LD3**) *with* $1 < \mu < 2$. *If* $u \in BV(\Omega, \mathbb{R}^N)$ *is an* ω*-minimizer of* $\bar{\mathscr{F}}$ *with* $R_0 > 0$ *and* $\omega(R)$ *satisfying* ($\boldsymbol{\omega}$**1**), ($\boldsymbol{\omega}$**4**), *then* $W_\mu(\nabla u) \in W^{t,2}_{loc}(\Omega)$ *for any* $t \in (0, \frac{\sigma}{1+\sigma})$, *where* $W_\mu(z) = (1+|z|^2)^{\frac{2-\mu}{4}}$ *and* $\nabla u \mathscr{L}^n \llcorner \Omega$ *is the absolutely continuous part of* Du. *In addition, for any* $\Omega' \subset\subset \Omega$, *there holds*

$$[W_\mu(\nabla u)]^2_{W^{t,2}(\Omega')} \le C \int_\Omega (1 + f(Du)) \tag{6.1.5}$$

with $C = C(n, N, L, \ell, t, \sigma, R_0, \Omega', \Omega) > 0$. *In particular, when* $\mu < 1 + \frac{2\sigma}{n(1+\sigma)}$, *we have* $|\nabla u| \in L^q_{loc}(\Omega)$ *for any* $q \in (1, \frac{(1+\sigma)(2-\mu)n}{n+(n-2)\sigma})$.

This result is obtained by comparing a BV ω-minimizer u with a corresponding generalised minimizer v on any small ball. The process can be carried out smoothly in the super-linear setting ($p > 1$) as in [Giu03], Chapter 8[1], and [KM05], while difficulties appear in the linear growth one. One issue is the possible singular part of Du. Another one, which is closely related, is that the two maps u and v may not coincide on the boundary. Thus, we cannot test the Euler-Lagrange equation with $u - v$ as is usually done. Alternatively, it is possible to describe the extremality of v by an inequality of the Euler-Lagrange type with $D^s u$ and the incoincidence of the boundary values of u and v incorporated. With such an inequality, we are able to compare ∇v and ∇u (the absolutely continuous parts) in terms of

---

[1] The discussion in [Giu03], Chapter 8 focuses on the scalar case, where $N = 1$. However, the argument related to ω-minimizers can be smoothly adapted to the vectorial setting.

the auxiliary function W_μ, and then conclude the fractional Sobolev regularity in Theorem 6.1.2.

Notice that the vanishing viscosity approach in [BS13] does not directly apply to ω-minimizers. The corresponding approximating sequence $\{u_k\}$ can only be constructed on small balls with radius $R < R_0$, since the almost minimality is only defined locally. The distance between u_k and u will then be related to $\omega(R)$, and we have to take $R \to 0^+$ to make the approximation work, which unfortunately does not give any regularity in the end. More refined covering arguments may appear promising, but all seem to require additional regularity of the BV map u on the boundary of each small tile. Such regularity is possible to achieve on each individual tile (see, for example, Lemma 2.2.9 and 2.2.10), but the corresponding estimate may blow up as the tile diameter goes to 0, which makes such an argument impractical.

Theorem 6.1.2 is only for the absolutely continuous part of Du, and it is unclear at the time of writing this thesis how the singular part $D^s u$ behaves. The example in Subsection 6.3.2 indicates that singularities may arise even with a reasonably "good" ω. Therefore, to rule out the singular part, we need to impose better assumptions on ω (such as $\omega(r) \sim r^{2\alpha}$ with $\alpha > \frac{1}{2}$). We will further investigate this aspect in our future work.

6.2 Examples of μ-elliptic integrands

We give some concrete integrands that satisfy the assumptions (**LD1**)-(**LD3**) in this section.

One example, and also a model integrand, is the following e_2 and its generalisation. In the scalar case $N = 1$, the integral $\int_\Omega e_2(\nabla u)\,\mathrm{d}x$ is the n-dimensional area of the graph of the function $u\colon \Omega \to \mathbb{R}$. In the vectorial setting, where $N > 1$, the area integral is different from e_2.

Example 6.2.1. Consider the following maps defined on the Euclidean space $\mathbb{R}^d$ for any $d \in \mathbb{N}^+$:

$$
\begin{aligned}
e_p(z) &:= (1 + |z|^p)^{\frac{1}{p}}, \\
e_{\lambda,p}(z) &:= (1 + (\lambda^2 + |z|^2)^{\frac{p}{2}})^{\frac{1}{p}}.
\end{aligned}
$$

The condition (**LD1**) is obviously satisfied. Direct calculation implies, when $1 < p \leq 2$ and $z \neq 0$,

$$
\begin{aligned}
e_p'(z) &= \frac{|z|^{p-2}\, z}{(1 + |z|^p)^{1-\frac{1}{p}}}, \\
e_p''(z)[\xi, \xi] &= \frac{|z|^{p-2}\,(1 + |z|^p)\,|\xi|^2 + (p - 2 - |z|^p)\,|z|^{p-4}\,|z \cdot \xi|^2)}{(1 + |z|^p)^{2-\frac{1}{p}}} \\
&\geq \frac{(p-1)\,|z|^{p-2}\,|\xi|^2}{(1 + |z|^p)^{2-\frac{1}{p}}} \geq (p-1)2^{\frac{(p-2)(2p-1)}{2p}} \frac{|\xi|^2}{(1 + |z|^2)^{\frac{p+1}{2}}}.
\end{aligned}
$$

In this case $\mu = p + 1$. Notice that when $1 < p < 2$, the integrand e_p is not differentiable at $z = 0$ and does not satisfy the right inequality of (**LD2**). However, there are no such issues for $e_{\lambda,p}$ with $\lambda > 0$. It is easy to see that $e_{\lambda,p}$ is twice differentiable everywhere, and that with $\langle z\rangle_\lambda := (\lambda^2 + |z|^2)^{\frac{1}{2}}$

$$\begin{aligned} e'_{\lambda,p}(z) &= \frac{\langle z\rangle_\lambda^{p-2}\, z}{(1+\langle z\rangle_\lambda^p)^{1-\frac{1}{p}}}, \\ e''_{\lambda,p}(z)[\xi,\xi] &= \frac{\langle z\rangle_\lambda^{p-2}\,(1+\langle z\rangle_\lambda^p)\,|\xi|^2 + (p-2-\langle z\rangle_\lambda^p)\,\langle z\rangle_\lambda^{p-4}\,|z\cdot\xi|^2)}{(1+\langle z\rangle_\lambda^p)^{2-\frac{1}{p}}} \\ &\geq \frac{(p-1)\,\langle z\rangle_\lambda^{p-2}\,|\xi|^2}{(1+\langle z\rangle_\lambda^p)^{2-\frac{1}{p}}} \geq C(p,\lambda)\frac{|\xi|^2}{(1+|z|^2)^{\frac{p+1}{2}}}. \end{aligned}$$

To obtain the upper bound of $e''_{\lambda,p}(z)[\xi,\xi]$, we just omit the second term in the numerator as $p-2-\langle z\rangle_\lambda^p < 0$, and thus it is easy to see

$$e''_{\lambda,p}(z)[\xi,\xi] \leq C(p,\lambda)\frac{|\xi|^2}{(1+|z|^2)^{\frac{1}{2}}}.$$

Actually, it is possible to construct an example satisfying (**LD1**) and (**LD2**) for any $\mu > 1$ as follows (cf. [BF02], §1 and [BFM01], §3).

Example 6.2.2. Given any $\mu > 1$, define

$$\varphi(r) := \int_0^r \int_0^s (1+t^2)^{-\frac{\mu}{2}}\,\mathrm{d}t\,\mathrm{d}s$$

for any $r \geq 0$, and $f(z) := \varphi(|z|)$ on the Euclidean space $\mathbb{R}^d$ for any $d \in \mathbb{N}^+$. Then we have, for $z \neq 0$,

$$\begin{aligned} f'(z) &= \int_0^{|z|} (1+t^2)^{-\frac{\mu}{2}}\,\mathrm{d}t\frac{z}{|z|}, \quad \text{and} \\ f''(z)[\xi,\xi] &= (1+|z|^2)^{-\frac{\mu}{2}}\frac{|z\cdot\xi|^2}{|z|^2} + \int_0^{|z|}(1+t^2)^{-\frac{\mu}{2}}\,\mathrm{d}t\frac{|z|^2|\xi|^2 - |z\cdot\xi|^2}{|z|^3} \geq 0. \end{aligned}$$

By definition, it is easy to see that f, f' are also differentiable at $z = 0$ with $f'(0) = 0 \in \mathbb{R}^d$ and $f''(z) = I_{\mathbb{R}^d} \otimes I_{\mathbb{R}^d}$. Thus, we know that f (and φ) is convex.

Notice that

$$|f'(z)| \leq \int_0^\infty (1+t^2)^{-\frac{\mu}{2}}\,\mathrm{d}t = C(\mu) < \infty,$$

which implies $|f(z)| \leq C(\mu)|z|$. As φ is convex, we have

$$f(z) = \varphi(|z|) \geq \varphi(1) + \varphi'(1)(|z|-1),$$

where $\varphi'(1) = \int_0^1 (1+t^2)^{-\frac{\mu}{2}} > 0$. Then (**LD1**) is satisfied by f up to an additive constant.

For (**LD2**), we first look at the left-hand side. Since

$$|z|^2|\xi|^2 - |z\cdot\xi|^2 \geq 0 \quad \text{and} \quad \int_0^{|z|}(1+t^2)^{-\frac{\mu}{2}}\,\mathrm{d}t \geq |z|(1+|z|^2)^{-\frac{\mu}{2}}, \tag{6.2.1}$$

we have the following lower bound

$$f''(z)[\xi,\xi] \geq (1+|z|^2)^{-\frac{\mu}{2}}\frac{|z\cdot\xi|^2}{|z|^2} + (1+|z|^2)^{-\frac{\mu}{2}}\frac{|z|^2|\xi|^2-|z\cdot\xi|^2}{|z|^2} = \frac{|\xi|^2}{(1+|z|^2)^{\frac{\mu}{2}}},$$

which is exactly the one in (**LD2**).

Again with the second inequality in (6.2.1), the second derivatives of f satisfies

$$f''(z)[\xi,\xi] \leq \int_0^{|z|}(1+t^2)^{-\frac{\mu}{2}}\,\mathrm{d}t\frac{|\xi|^2}{|z|}.$$

When $|z|\geq 1$, it can be seen that

$$\begin{aligned}&\int_0^{|z|}(1+t^2)^{-\frac{\mu}{2}}\,\mathrm{d}t \leq \int_0^{\infty}(1+t^2)^{-\frac{\mu}{2}}\,\mathrm{d}t = C(\mu)<\infty,\\ \text{and}\quad &\frac{|\xi|^2}{|z|}\leq\frac{\sqrt{2}|\xi|^2}{(1+|z|^2)^{\frac{1}{2}}}.\end{aligned}$$

When $|z|<1$, we have

$$\int_0^{|z|}(1+t^2)^{-\frac{\mu}{2}}\,\mathrm{d}t \leq |z| \quad\text{and}\quad |\xi|^2\leq\frac{\sqrt{2}|\xi|^2}{(1+|z|^2)^{\frac{1}{2}}}.$$

The right-hand side of (**LD2**) thus holds true for the function f.

6.3 Counterexamples

6.3.1 μ-elliptic functionals with $\mu\geq 3$

In our study, we only consider functionals satisfying (E_μ) with $\mu\in(1,3]$. When $\mu>3$, the problem seems to be more singular and the hope of obtaining regularity is small. There are two examples given in [GMS79], II, showing that there might exist singularities when $\mu>3$, and even in the borderline case $\mu=3$. This section is devoted to the discussion of those counterexamples, where the integrand is of the form $f(x,z)$ and the corresponding minimizers have interior jumps. The two examples are defined on the $1-d$ interval $(-1,1)$, and there is another one defined on a higher-dimensional annulus of similar spirit in [BF03]. Notice that it is not yet clear how to construct counterexamples with integrands that have no x-dependence.

Example 6.3.1. Consider the following minimisation problem

$$\begin{cases}\displaystyle\int_{-1}^{1} f(t,u'(t))\,\mathrm{d}t\longrightarrow\min\\ u(-1)=-a,\quad u(1)=a,\end{cases}\tag{6.3.1}$$

where $a>0$ and the integrand f is defined by

$$f(t,z):=(1+\alpha(t)|z|^2)^{\frac{1}{2}}.$$

The function $\alpha : (-1,1) \to \mathbb{R}^+$ is given by $\alpha(t) = 1 + t^2(\log \frac{2}{|t|})^4$. Direct calculation gives that $\alpha \in C^1(-1,1) \cap C^2((-1,1) \setminus \{0\})$ with

$$\begin{aligned}
\alpha'(t) &= 2t\left(\log \frac{2}{|t|}\right)^4 - 4t\left(\log \frac{2}{|t|}\right)^3 \\
\alpha''(t) &= 2\left(\log \frac{2}{|t|}\right)^4 - 12\left(\log \frac{2}{|t|}\right)^3 + 12\left(\log \frac{2}{|t|}\right)^2 \xrightarrow{t\to 0} \infty.
\end{aligned}$$

Moreover, the function α attains its strict minimum on $(-1,1)$ at $t = 0$.

Set $u_0(t) := at$ and $p = 1$, then the problem (6.3.1) is actually to minimise the functional

$$\mathscr{F}_{u_0}(u) = \int_{-1}^{1} (1 + \alpha(t)|u'(t)|^2)^{\frac{1}{2}} + \sqrt{\alpha(-1)}|u(-1) + a| + \sqrt{\alpha(1)}|u(1) - a|$$

in $BV(-1,1)$.

Take $a > a_0$, where a_0 is given by

$$\begin{aligned}
a_0 &= \int_0^1 (\alpha(t) - \alpha(0))^{-\frac{1}{2}}\,\mathrm{d}t = \int_0^1 \frac{1}{t(\log \frac{2}{t})^2}\,\mathrm{d}t \qquad (6.3.2)\\
&= -\int_0^1 \left(\log \frac{2}{t}\right)^{-2} \mathrm{d}\left(\log \frac{2}{t}\right) = \left(\log \frac{2}{t}\right)^{-1}\Bigg|_0^1 = \frac{1}{\log 2}.
\end{aligned}$$

However, if $\alpha \in C^{1,1}(-1,1)$ is even and has a strict minimum at $t = 0$, we have

$$\alpha(t) - \alpha(0) = \int_0^t \alpha'(s)\,\mathrm{d}s = \int_0^t (\alpha'(s) - \alpha'(0))\,\mathrm{d}t \le c\frac{t^2}{2},$$

where c is the Lipschitz constant of $\alpha'(t)$. In this case, the integral $\int_0^1 (\alpha(t) - \alpha(0))^{-\frac{1}{2}}\,\mathrm{d}t$ is not finite, and we will not be able to define a_0 as above.

We claim that the unique minimizer $u \in BV(-1,1)$ which solves problem (6.3.1) satisfies the following properties:

(a) $u_+(-1)(= \lim_{t\to -1^+} \alpha(t)) = -a, u_-(1)(= \lim_{t\to 1^-} \alpha(t)) = a$;

(b) $u \notin W^{1,1}(-1,1)$;

(c) the singular part of u' is supported at $t = 0$.

Proof. (a). Since $u \in BV(-1,1)$, it can be easily shown that the limits $u_+(-1)$ and $u_-(1)$ exist. Suppose that the claim does not hold true. Then define another function

$$v(t) := \begin{cases} u(t) - u(-1) - a, & t \in [-1,0) \\ u(t) - u(1) + a, & t \in (0,1], \end{cases}$$

on which the functional takes the value

$$\mathscr{F}_{u_0}(v) = \int_{(-1,1)\setminus\{0\}} (1 + \alpha(t)|v'(t)|^2)^{\frac{1}{2}} + \sqrt{\alpha(0)}|u_+(0) - u_-(1) - u_-(0) + u_+(-1) + 2a|$$

$$
\begin{aligned}
< \int_{(-1,1)\setminus\{0\}} (1+\alpha(t)|v'(t)|^2)^{\frac{1}{2}} + \sqrt{\alpha(0)}|u_+(0)-u_-(0)| \\
+ \sqrt{\alpha(-1)}|u_+(-1)+a| + \sqrt{\alpha(1)}|u_-(1)-a|,
\end{aligned}
$$

where the inequality is strict since $|u_+(-1)+a|+|u_-(1)-a|>0$. The right-hand side in the above inequality is exactly $\mathscr{F}_{u_0}(u)$ and this gives a contradiction.

(b). Again, we show by contradiction, assuming that $u \in W^{1,1}(-1,1)$. Then u' satisfies the Euler-Lagrange equation

$$
\operatorname{div} \frac{\alpha(t)u'(t)}{(1+\alpha(t)|u'(t)|^2)^{\frac{1}{2}}} = 0 \tag{6.3.3}
$$

in the weak sense, which implies that

$$
\frac{\alpha(t)u'(t)}{(1+\alpha(t)|u'(t)|^2)^{\frac{1}{2}}} = \lambda \quad \text{a.e. on } (-1,1)
$$

for some constant λ. Then we have

$$
\begin{aligned}
\lambda^2 &\overset{\text{a.e.}}{=} \alpha(t)\frac{\alpha(t)|u'(t)|^2}{1+\alpha(t)|u'(t)|^2} \le \min_{t\in[0,1]} \alpha(t) = \alpha(0), \\
|u'(t)| &= \frac{|\lambda|}{(\alpha(t)(\alpha(t)-\lambda^2))^{\frac{1}{2}}} \le \frac{1}{(\alpha(t)-\alpha(0))^{\frac{1}{2}}} \quad \text{a.e. on } (-1,1).
\end{aligned}
$$

As $u \in W^{1,1}(-1,1)$, there holds

$$
2a = (u_-(1)-u_+(1)) \le \int_{-1}^{1} |u'(t)|\,\mathrm{d}t \le \int_{-1}^{1} \frac{1}{(\alpha(t)-\alpha(0))^{\frac{1}{2}}}\,\mathrm{d}t = 2a_0 < 2a,
$$

which is impossible.

(c). Now we show that the singular part of u' is exactly a jump at $t=0$. Decompose u' with respect to the Lebesgue measure

$$
u' = u'_{ac} + u'_s,
$$

and define another function $v \in BV(-1,1)$ with

$$
v'_{ac} = u'_{ac}, \quad v_+(-1) = u_+(-1), \quad v'_s = \int_{-1}^{1} u'_s \cdot \delta_0,
$$

and we can see that $v_-(1) = u_-(1)$. The definition of v implies

$$
\mathscr{F}_{u_0}(v) = \int_{-1}^{1} (1+\alpha(t)|v'_{ac}(t)|^2)^{\frac{1}{2}}\,\mathrm{d}t + \sqrt{\alpha(0)}\left|\int_{-1}^{1} u'_s\right| \le \int_{-1}^{1} (1+\alpha(t)|v'_{ac}(t)|^2)^{\frac{1}{2}}\,\mathrm{d}t + \sqrt{\alpha(0)}\int_{-1}^{1} |u'_s|\,.
$$

The from the property of $\alpha(t)$ and the minimality of u, we have

$$
\sqrt{\alpha(0)}\int_{-1}^{1} |u'_s| \le \int_{-1}^{1} \sqrt{\alpha(t)}|u'_s| \le \int_{-1}^{1} \sqrt{\alpha(t)}|v'_s| = \sqrt{\alpha(0)}\int_{-1}^{1} |u'_s|,
$$

and thus the support of u_s' should be in $\{0\}$, which means that u has exactly a jump at $t = 0$. ☐

The previous example is for the borderline case $\mu = 3$. The following one is of the same type and for $\mu > 3$.

Example 6.3.2. We consider the minimisation problem (6.3.1) with some $a > 0$ and an integrand f defined by

$$f(t, z) := (1 + \alpha(t)|z|^p)^{\frac{1}{p}}$$

with some $p > 2$. The function α now can be taken to be a reasonable even $C^{1,1}(-1, 1)$ function, for instance, $\alpha(t) = 1 + t^2$.

Similar to the above example, we take

$$a > a_0 := \int_0^1 \left(\alpha(t)^{\frac{1}{p-1}} - \alpha(0)^{\frac{1}{p-1}}\right)^{-\frac{1}{p}} \mathrm{d}t.$$

The claims (a)-(c) also hold in this example, which can be proven with a similar argument. ☐

6.3.2 ω-minimizers with $\omega(r) \sim r$

Consider the integrand $f(z) = (1 + |z|^p)^{\frac{1}{p}}$, and the minimisation problem of

$$\mathscr{F}(u) = \int f(\nabla u)\,\mathrm{d}x$$

on the 1-dimensional interval $(-1, 1)$. Set

$$u(x) = \begin{cases} 1, & x \in (0, 1), \\ 0, & x \in (-1, 0). \end{cases}$$

We will show that $u \in BV(-1, 1)$ is an ω-minimizer of $\mathscr{F}$ on $(-1, 1)$ with $\omega(R) = 2R$.

It is obvious that u is a minimizer away from the origin $x = 0$, then it is sufficient to consider small intervals containing the origin. Given any $r \in (0, 1)$ and any interval $I_r = (-a_r, b_r)$ with $a_r, b_r > 0$ and $a_r + b_r = 2r$, one minimizer of $\mathscr{F}_u(\cdot)$ is the affine function $v(x) = \frac{1}{2r}(x + a_r)$. Indeed, for any $w \in W^{1,1}_u(I_r)$, by Jensen's inequality and the convexity of f we have

$$\begin{aligned} \mathscr{F}_u(v, I_r) &= 2r\left(1 + \left(\frac{1}{2r}\right)^p\right)^{\frac{1}{p}} = 2r\left(1 + \left(\fint_{-a_r}^{b_r} w'(x)\,\mathrm{d}x\right)^p\right)^{\frac{1}{p}} \\ &\leq \int_{-a_r}^{b_r} \left(1 + |w'(x)|^p\right)^{\frac{1}{p}} \mathrm{d}x = \mathscr{F}_u(w, I_r). \end{aligned}$$

Then the minimality in $BV_u(I_r)$ can be obtained by approximation. Alternatively, the derivative v' of v satisfies the corresponding Euler-Lagrange equation

$$\int_{-a_r}^{b_r} f'(v'(x))\varphi'(x)\,\mathrm{d}x = \int_{-a_r}^{b_r} f'\left(\frac{1}{2r}\right)\varphi'(x)\,\mathrm{d}x = 0 \tag{6.3.4}$$

for any $\varphi \in C_c^\infty(I_r)$. The strict convexity of f then implies the minimality of v.

Notice that

$$\mathscr{F}_u(u, I_r) = 2r + 1, \quad \mathscr{F}_u(v, I_r) = 2r \left(1 + \left(\frac{1}{2r}\right)^p\right)^{\frac{1}{p}},$$

which directly implies

$$\mathscr{F}_u(u, I_r) \leq (1 + 2r)\mathscr{F}_u(v, I_r).$$

Therefore, we have

$$\mathscr{F}_u(u, I_r) \leq (1 + 2r)\mathscr{F}_u(w, I_r)$$

for any $w \in BV(-1, 1)$, and thus u is an ω-minimizer of $\mathscr{F}$ on $(-1, 1)$ with $\omega(r) \sim r$. However, it has a singularity at $x = 0$.

Remark 6.3.1. As mentioned in Example 6.2.1, the integrand f $(= e_p = (1 + |\cdot|^p)^{\frac{1}{p}})$ does not satisfy (**LD2**). We can replace e_p by $e_{\lambda,p}$ in the above example so that it provides a counterexample that precisely aligns with the assumptions in Theorem 6.1.2. With this substitution, the function u remains an ω-minimizer with $\omega(r) \sim r$, and the above argument still applies.

6.4 Sobolev regularity for generalised minimizers

In this section, we show the Sobolev regularity for generalised minimizers of $\mathscr{F}$ as in Theorem 6.1.1. The argument is modified from the vanishing viscosity method in [BS13], and is simplified as we already have the $W^{1,1}$ regularity for and do not need to incorporate the boundedness of a generalised minimizer u in our proof. In this process, a sequence $\{u_k\}$ is constructed with Ekeland's variational principle. Each u_k minimises a regularised functional, and it is possible to show $W_\mu(\nabla u_k) \in W^{1,2}_{loc}(\Omega)$. The sequence converges to u in the f-strict sense (see Definition 2.5.1 and (6.4.33)), and indeed strongly in $W^{1,1}$ as shown in Subsection 6.4.4. Then it is possible to pass the regularity of $W_\mu(\nabla u_k)$ to $W_\mu(\nabla u)$ with the difference quotient estimate for the former and Fatou's lemma.

Fix a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. The boundedness and Lipschitz property here are not necessary and assumed just for convenience, since we can restrict the discussion to a compact Lipschitz sub-domain of Ω. Suppose that the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies (**LD1**)-(**LD3**), and $u \in BV(\Omega, \mathbb{R}^N)$ is a generalised minimizer of the functional $\mathscr{F}$ (see Definition 3.1.6). We do not specify the boundary value, because Lemma 2.9 in [BS13] shows that a generalised minimizer in a certain Dirichlet class also minimises the functional with respect to its own boundary value. By Corollary 3.2.4 we know that $u \in W^{1,1} \cap L^\infty(\Omega, \mathbb{R}^N)$.

Since Ω is bounded and has a Lipschitz boundary, there exists $u_0 \in W^{1,1}(\Omega, \mathbb{R}^N)$ with $\operatorname{tr}_\Omega u_0 = \operatorname{tr}_\Omega u$. Define the Dirichlet class $\mathscr{D} := u_0 + W_0^{1,1}(\Omega, \mathbb{R}^N)$.

The proof is composed of four steps. We first regularise the functional $\mathscr{F}$ in Subsection 6.4.1 and construct the sequence $\{u_k\}$. The existence of $\nabla^2 u_k$ and a uniform estimate for the L^2-integrals of $\nabla W_\mu(\nabla u_k)$ are given in 6.4.2 and 6.4.3, respectively. The final step in Subsection 6.4.4 is to show that $u_k \to u$ in $W^{1,1}(\Omega, \mathbb{R}^N)$, which further implies the estimate for $\nabla W_\mu(\nabla u)$.

6.4.1 Regularisation and approximating sequence

In the following, we take two sequences $\{u_{0,k}\}$ and $\{v_k\}$ in turn. The first one is a suitable approximation of u_0, while the second one of u. The final approximating sequence $\{u_k\}$ will be chosen based on $\{v_k\}$ with Ekeland's variational principle. In the following, we abbreviate $\mathscr{F}(w,\Omega)$ as $\mathscr{F}(w)$ for any admissible map w.

To work with a $W^{1,2}$ regularisation, we choose a $W^{1,2}$ minimising sequence. First, approximate the boundary value u_0 by $\{u_{0,k}\} \subset W^{1,2}(\Omega,\mathbb{R}^N)$ such that

$$\|u_{0,k}-u_0\|_{W^{1,1}(\Omega,\mathbb{R}^N)} \le \frac{1}{6L_f k^2}, \tag{6.4.1}$$

where $L_f = L_f(n,N,L) > 0$ is the Lipschitz constant of f (Lemma 2.6.6). Define $\mathscr{D}_k := u_{0,k} + W_0^{1,2}(\Omega,\mathbb{R}^N)$, and choose $\{v_k\} \subset \mathscr{D}_k$ with

$$\|(v_k-u_{0,k})-(u-u_0)\|_{W^{1,1}(\Omega,\mathbb{R}^N)} \le \frac{1}{6L_f k^2}, \tag{6.4.2}$$

which implies

$$\|v_k-u\|_{W^{1,1}(\Omega,\mathbb{R}^N)} \le \frac{1}{3L_f k^2}. \tag{6.4.3}$$

With (6.4.1) and the Lipschitz continuity of f, we have

$$\inf_{\mathscr{D}} \mathscr{F} = \inf_{u_0+W_0^{1,2}(\Omega,\mathbb{R}^N)} \mathscr{F} \le \inf_{\mathscr{D}_k} \mathscr{F} + \frac{1}{6k^2}. \tag{6.4.4}$$

The inequality (6.4.3) further implies

$$\mathscr{F}(v_k) \le \mathscr{F}(u) + \frac{1}{3k^2} = \inf_{\mathscr{D}} \mathscr{F} + \frac{1}{3k^2} \le \inf_{\mathscr{D}_k} \mathscr{F} + \frac{1}{2k^2}. \tag{6.4.5}$$

Define the following regularised integrand

$$f_k(z) := f(z) + \frac{1}{2V_k k^2}|z|^2, \tag{6.4.6}$$
$$\text{where} \quad V_k := 1 + \int_\Omega |\nabla v_k|^2 \,\mathrm{d}x.$$

Then f_k is of quadratic growth, and from **(LD2)** we know that its second derivatives satisfy

$$\left(\frac{\ell}{(1+|z|^2)^{\frac{\mu}{2}}} + \frac{1}{V_k k^2}\right) \le f_k''(z)[\xi,\xi] \le \left(\frac{L}{(1+|z|^2)^{\frac{1}{2}}} + \frac{1}{V_k k^2}\right)|\xi|^2. \tag{6.4.7}$$

The regularised functional is defined with f_k on $W^{-1,1}(\Omega,\mathbb{R}^N)$:

$$\mathscr{F}_k(w) := \begin{cases} \displaystyle\int_\Omega f_k(\nabla w)\,\mathrm{d}x, & w \in \mathscr{D}_k, \\ \infty, & w \in W^{-1,1}(\Omega,\mathbb{R}^N) \setminus \mathscr{D}_k, \end{cases} \tag{6.4.8}$$

and see Subsection 2.2.1 for a short discussion about the negative Sobolev space $W^{-1,1}$. It is easy to see with (6.4.5) that

$$\begin{aligned} \mathscr{F}_k(v_k) &= \mathscr{F}(v_k) + \frac{1}{2V_k k^2}\int_\Omega |\nabla v_k|^2 \,\mathrm{d}x \\ &\le \inf_{\mathscr{D}_k} \mathscr{F} + \frac{1}{2k^2} + \frac{1}{2k^2} \\ &\le \inf_{W^{-1,1}(\Omega,\mathbb{R}^N)} \mathscr{F}_k + \frac{1}{k^2}. \end{aligned} \tag{6.4.9}$$

Notice that $W^{-1,1}(\Omega,\mathbb{R}^N)$ is a complete normed space (Proposition 2.2.4), and $\mathscr{F}_k$ as a function defined on this metric space is bounded from below and not identically ∞. The proof of its lower semi-continuity is postponed to the end of this subsection. Then applying Ekeland's variational principle (Theorem 2.8.1) with (6.4.9), we know there exists $u_k \in W^{-1,1}(\Omega,\mathbb{R}^N)$ satisfying

(a) $\|u_k - v_k\|_{W^{-1,1}(\Omega,\mathbb{R}^N)} \le \frac{1}{k}$;

(b) $\mathscr{F}_k(u_k) \le \mathscr{F}_k(v_k)$;

(c) $\mathscr{F}_k(u_k) \le \mathscr{F}_k(w) + \frac{1}{k}\|w - u_k\|_{W^{1,1}(\Omega,\mathbb{R}^N)}$.

The inequality (b) implies that $u_k \in \mathscr{D}_k$ since $\mathscr{F}_k(v_k) < \infty$. Actually, the integrals $\mathscr{F}_k(v_k)$ and thus $\mathscr{F}_k(u_k)$ are uniformly bounded

$$\mathscr{F}_k(u_k) \le \mathscr{F}_k(v_k) \le \mathscr{F}(v_k) + \frac{1}{2k^2} \overset{(6.4.5)}{\le} \inf_{\mathscr{D}} \mathscr{F} + \frac{5}{6k^2} = \mathscr{F}(u) + \frac{5}{6k^2}. \tag{6.4.10}$$

Considering the lower bound of f, we furthermore have

$$\int_\Omega \left(\ell|\nabla u_k| + \frac{1}{2V_k k^2}|\nabla u_k|^2\right) \mathrm{d}x \le \mathscr{F}_k(u_k) \le \mathscr{F}(u) + \frac{5}{6k^2}. \tag{6.4.11}$$

It is routine to obtain the following Euler-Lagrange inequality from (c) with the fact that $f \in C^2(\mathbb{R}^{N\times n})$:

$$\left|\int_\Omega f_k'(\nabla u_k)\cdot\nabla\varphi \,\mathrm{d}x\right| \le \frac{1}{k}\|\varphi\|_{W^{-1,1}(\Omega,\mathbb{R}^N)}. \tag{6.4.12}$$

Now we show that the functional $\mathscr{F}_k$ is lower semi-continuous in $W^{-1,1}(\Omega,\mathbb{R}^N)$.

Lemma 6.4.1. *Suppose that the functional $\mathscr{F}_k$ is defined on $W^{-1,1}(\Omega,\mathbb{R}^N)$ as in (6.4.8). Then $\mathscr{F}_k$ is lower semi-continuous with respect to norm convergence in $W^{-1,1}(\Omega,\mathbb{R}^N)$.*

Proof. Suppose that the sequence $\{w_l\} \subset W^{-1,1}(\Omega,\mathbb{R}^N)$ converges to w in $W^{-1,1}$ norm. Without loss of generality, we can assume

$$\liminf_{l\to\infty} \mathscr{F}_k(w_l) =: m < \infty$$

as the result is obvious when $m = \infty$. Then there exists $T \in \mathbb{N}$ such that for any $l > T$, we have $\mathscr{F}_k(w_k) \le m + 1$. This gives a uniform bound of $\|w_k\|_{W^{1,2}(\Omega,\mathbb{R}^N)}$ for $l > T$, since the definition of $\mathscr{F}_k$ implies

$$\frac{1}{2V_k k^2}\int_\Omega |\nabla w_l|^2 \,\mathrm{d}x \le \mathscr{F}_k(w_l) \le m+1, \tag{6.4.13}$$

and $\|w_k\|_{W^{1,2}(\Omega,\mathbb{R}^N)}$ is thus uniformly bounded by the fact $w_l \in \mathscr{D}_k$ and Poincaré's inequality.

Then there exists a subsequence $\{w_{l_t}\}$ of $\{w_l\}$ which weakly converges to some $\bar{w}$ in $W^{1,2}(\Omega,\mathbb{R}^N)$. From (2.2.3) we know

$$\|w_{l_t}-\bar{w}\|_{W^{-1,1}(\Omega,\mathbb{R}^N)} \le \|w_{l_t}-\bar{w}\|_{L^1(\Omega,\mathbb{R}^N)} \overset{t\to\infty}{\to} 0,$$

and thus $w=\bar{w}$ is in $\mathscr{D}_k$ since the Dirichlet class is weakly closed. The functional $\mathscr{F}_k$ is lower semi-continuous in $W^{1,2}$ since it is convex (see Theorem 8.4 in [Dac08]), and thus there holds

$$\mathscr{F}_k(w) \le \liminf_{t\to\infty} \mathscr{F}_k(w_{l_t}) = m = \liminf_{l\to\infty} \mathscr{F}_k(w_l).$$

□

6.4.2 Second differentiability of u_k

In this step, the second differentiability of u_k is shown with a difference quotient argument based on the Euler-Lagrange inequality (6.4.12).

Proposition 6.4.2. *Suppose that the integrand $f\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ satisfies* (**LD1**), (**LD2**), *and f_k is defined as in* (6.4.6) *with some constant $V_k>0$. If $u_k \in W^{1,2}(\Omega,\mathbb{R}^N)$ satisfies* (6.4.12), *then we have $u_k \in W^{2,2}_{loc}(\Omega,\mathbb{R}^N)$.*

Proof. The estimates in this proof are not uniform in k, so the constants are allowed to depend on k.

Suppose that $\{e_s\}_{s=1}^n$ is the standard basis in $\mathbb{R}^n$, and for some $h\in\mathbb{R}\setminus\{0\}$ we define the following difference quotient

$$\Delta_h^s v(x) := \frac{v(x+he_s)-v(x)}{h}$$

as in Section 2.3. Now take $\eta\in C_c^\infty(\Omega)$ with $0\le\eta\le 1$ and $h\in\mathbb{R}$ with $0<|h|<\operatorname{dist}(\operatorname{supp}(\eta),\partial\Omega)$. Test (6.4.12) with the function $\varphi := \Delta_{-h}^s(\eta^2\Delta_h^s u_k)$, which implies

$$\begin{aligned}\left|\int_\Omega f_k'(\nabla u_k)\cdot\nabla\varphi\,\mathrm{d}x\right| &\le \frac{1}{k}\|\varphi\|_{W^{-1,1}(\Omega,\mathbb{R}^N)} \overset{(2.2.4)}{\le} \frac{1}{k}\left\|\Delta_{-h}^s(\eta^2\Delta_h^s u_k)\right\|_{L^1(\Omega,\mathbb{R}^N)}\\ &\overset{\text{Lemma 2.3.3}}{\le} \frac{1}{k}\left\|\partial^s(\eta^2\Delta_h^s u_k)\right\|_{L^1(\Omega,\mathbb{R}^N)} \\ &\le \frac{1}{k}\left(\|2\eta\partial^s\eta\Delta_h^s u_k\|_{L^1(\Omega,\mathbb{R}^N)} + \left\|\eta^2\Delta_h^s\partial^s u_k\right\|_{L^1(\Omega,\mathbb{R}^N)}\right).\end{aligned} \tag{6.4.14}$$

The left-hand side above can be expanded as follows

$$\begin{aligned}\int_\Omega f_k'(\nabla u_k)\cdot\nabla\varphi\,\mathrm{d}x &= -\int_\Omega \Delta_h^s f_k'(\nabla u_k)\cdot\nabla(\eta^2\Delta_h^s u_k)\,\mathrm{d}x \\ &= -\int_\Omega \Delta_h^s f_k'(\nabla u_k)\cdot(\eta^2\Delta_h^s\nabla u_k + 2\eta\Delta_h^s u_k\otimes\nabla\eta)\,\mathrm{d}x.\end{aligned} \tag{6.4.15}$$

By the fundamental theorem of calculus, the difference quotient of $f_k'(\nabla u_k)$ can be writted as

$$\Delta_h^s f_k'(\nabla u_k(x)) = \int_0^1 f_k''(\nabla u_k(x) + th\Delta_h^s \nabla u_k(x))\,\mathrm{d}t\,[\Delta_h^s \nabla u_k(x), \cdot],$$

and we denote the symmetric bilinear form by $\mathscr{B}_{k,x}$, i.e.,

$$\mathscr{B}_{k,x} := \int_0^1 f_k''(\nabla u_k(x) + th\Delta_h^s \nabla u_k(x))\,\mathrm{d}t.$$

Then with the Cauchy-Schwarz inequality, the second term on the right-hand side on (6.4.15) is controlled as follows

$$\begin{aligned} &-2\int_\Omega \mathscr{B}_{k,x}[\eta\Delta_h^s\nabla u_k, \Delta_h^s u_k \otimes \nabla\eta]\,\mathrm{d}x \\ &\quad\le \frac{1}{2}\int_\Omega \mathscr{B}_{k,x}[\eta\Delta_h^s\nabla u_k, \eta\Delta_h^s\nabla u_k]\,\mathrm{d}x + 2\int_\Omega \mathscr{B}_{k,x}[\Delta_h^s u_k\otimes\nabla\eta, \Delta_h^s u_k\otimes\nabla\eta]\,\mathrm{d}x, \end{aligned} \tag{6.4.16}$$

and combining (6.4.14) we obtain

$$\begin{aligned} \int_\Omega \mathscr{B}_{k,x}[\eta\Delta_h^s\nabla u_k, \eta\Delta_h^s\nabla u_k]\,\mathrm{d}x \le \frac{2}{k}\left(2\int_\Omega \eta|\partial^s\eta\Delta_h^s u_k|\,\mathrm{d}x + \int_\Omega \eta^2|\Delta_h^s\partial^s u_k|\,\mathrm{d}x\right) \\ + 4\int_\Omega \mathscr{B}_{k,x}[\Delta_h^s u_k\otimes\nabla\eta, \Delta_h^s u_k\otimes\nabla\eta]\,\mathrm{d}x. \end{aligned} \tag{6.4.17}$$

The inequality (6.4.7) implies that the bilinear form $\mathscr{B}_{k,x}$ satisfies the following estimate

$$\frac{1}{V_k k^2}|\xi|^2 \le \mathscr{B}_{k,x}[\xi,\xi] \le \left(L + \frac{1}{V_k k^2}\right)|\xi|^2, \tag{6.4.18}$$

with which we can further rewrite (6.4.17) as

$$\begin{aligned} \frac{1}{V_k k^2}\int_\Omega \eta^2|\Delta_h^s\nabla u_k|^2\,\mathrm{d}x \le \frac{2}{k}\left(2\int_\Omega \eta|\partial^s\eta\Delta_h^s u_k|\,\mathrm{d}x + \int_\Omega \eta^2|\Delta_h^s\nabla u_k|\,\mathrm{d}x\right) \\ + 4\int_\Omega \left(L + \frac{1}{V_k k^2}|\Delta_h^s u_k\otimes\nabla\eta|^2\right)\mathrm{d}x. \end{aligned} \tag{6.4.19}$$

Estimating the second term on the right-hand side with Young's inequality

$$\frac{2}{k}\int_\Omega \eta^2|\Delta_h^s\nabla u_k|\,\mathrm{d}x \le \frac{1}{2V_k k^2}\int_\Omega \eta^2|\Delta_h^s\nabla u_k|^2\,\mathrm{d}x + 2V_k\int_\Omega \eta^2\,\mathrm{d}x,$$

we finally get, with Lemma 2.3.3

$$\frac{1}{V_k k^2}\int_\Omega \eta^2|\Delta_h^s\nabla u_k|^2\,\mathrm{d}x \tag{6.4.20}$$

$$\le \frac{8}{k}\int_\Omega \eta|\partial^s\eta\Delta_h^s u_k|\,\mathrm{d}x + 4V_k\int_\Omega \eta^2\,\mathrm{d}x + 8\int_\Omega\left(L + \frac{1}{V_k k^2}|\Delta_h^s u_k\otimes\nabla\eta|^2\right)\mathrm{d}x \tag{6.4.21}$$

$$\le C\left(\sup_\Omega|\nabla\eta|\int_\Omega|\nabla u_k|\,\mathrm{d}x + \sup_\Omega|\nabla\eta|^2\int_\Omega|\nabla u_k|^2\,\mathrm{d}x + \mathscr{L}^n(\Omega)\right) \tag{6.4.22}$$

Lemma 2.3.4 implies that $u_k \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$. □

6.4.3 A uniform estimate for the second derivatives

Define the following function on any finite dimensional Hilbert space

$$W_\mu(z) := (1 + |z|^2)^{\frac{2-\mu}{4}},$$

then we know that $W_\mu(\nabla u_k) \in L^2(\Omega)$. Notice that W_μ is differentiable, and

$$W'_\mu(z) = \frac{2-\mu}{2} \frac{z}{(1+|z|^2)^{\frac{2+\mu}{2}}}$$

is bounded. Since $u_k \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$, the function $W_\mu(\nabla u_k)$ is actually in the space $W^{1,2}_{loc}(\Omega)$ with

$$\partial^s W_\mu(\nabla u_k) = \frac{2-\mu}{2} \frac{\nabla u_k \cdot \partial^s \nabla u_k}{(1+|\nabla u_k|^2)^{\frac{2+\mu}{2}}}, \quad \text{for any } s \in \{1, \dots, n\}. \tag{6.4.23}$$

Indeed, it is possible to show a uniform estimate for the L^2 integrals of $\nabla W_\mu(\nabla u_k)$ on any fixed subdomain $\Omega' \subset\subset \Omega$.

We first prove the following lemma, with which we can transfer one derivative on the test map in (6.4.12) to $f'_k(\nabla u_k)$.

Lemma 6.4.3. *Suppose that the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* **(LD1)**, **(LD2)**, *and f_k is defined as in* (6.4.6) *with some constant V_k. The map $u_k \in W^{1,2}(\Omega, \mathbb{R}^N)$ satisfies* (6.4.12). *Then for any $s \in \{1, \dots, n\}$ and $\varphi \in W^{1,2}_0(\Omega, \mathbb{R}^N)$, the following inequality holds true:*

$$\left| \int_\Omega \partial^s f'_k(\nabla u_k) \cdot \nabla \varphi \, \mathrm{d}x \right| \le \frac{1}{k} \|\varphi\|_{L^1(\Omega, \mathbb{R}^N)}. \tag{6.4.24}$$

Proof. From Proposition 6.4.2 we know $u_k \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$. By definition, the map f'_k is differentiable with bounded derivatives, and thus $f'_k(\nabla u_k)$ is weakly differentiable with

$$\partial^s f'_k(\nabla u_k) = f''_k(\nabla u_k) \partial^s \nabla u_k \in L^2_{loc}(\Omega, \mathbb{R}^{N\times n}).$$

Furthermore, we have, by Lemma 2.6.6,

$$|f'_k(\nabla u_k)| \le CL + \frac{1}{V_k k^2} |\nabla u_k| \in L^2(\Omega),$$

which implies $f'_k(\nabla u_k) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^{N\times n})$. For any $\varphi \in C^\infty_c(\Omega, \mathbb{R}^N)$, there holds

$$\left| \int_\Omega \partial^s f'_k(\nabla u_k) \cdot \nabla \varphi \, \mathrm{d}x \right| = \left| \int_\Omega f'_k(\nabla u_k) \cdot \nabla (\partial^s \varphi) \, \mathrm{d}x \right| \overset{(6.4.12)}{\le} \frac{1}{k} \|\partial^s \varphi\|_{W^{-1,1}(\Omega, \mathbb{R}^N)} \overset{(2.2.4)}{\le} \frac{1}{k} \|\varphi\|_{L^1(\Omega, \mathbb{R}^N)}.$$

The inequality for a general $\varphi \in W^{1,2}_0(\Omega, \mathbb{R}^N)$ also holds by approximation. □

Proposition 6.4.4. *Suppose that the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* **(LD1)**, **(LD2)**, *and f_k is defined as in* (6.4.6) *with some constant $V_k > 0$. If $u_k \in W^{1,2}(\Omega, \mathbb{R}^N)$ satisfies*

(6.4.12), *then* $u_k \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$, *and the following estimate*

$$\ell \int_{B_r} \frac{|\nabla^2 u_k|^2}{(1+|\nabla u_k|^2)^{\frac{\mu}{2}}} \,\mathrm{d}x + \frac{1}{V_k k^2} \int_{B_r} |\nabla^2 u_k|^2 \,\mathrm{d}x \\ \leq \left(\frac{16L}{(R-r)^2} + \frac{2}{k\sqrt{n}} \right) \int_{B_R} |\nabla u_k| \,\mathrm{d}x + \frac{16}{V_k k^2 (R-r)^2} \int_{B_R} |\nabla u_k|^2 \,\mathrm{d}x \tag{6.4.25}$$

holds for any ball $B_R = B(x_0, R) \subset\subset \Omega$ *and any* $r \in (0, R)$.

Proof. Fix a ball $B_R = B(x_0, R) \subset\subset \Omega$ and $0 < r < R$, and take a cut-off function $\eta \in C^\infty_c(B_R)$ between B_r and B_R with

$$0 \leq \eta \leq 1, \quad \eta(x) \equiv 1 \text{ in } B_r, \quad |\nabla \eta| \leq \frac{2}{R-r}.$$

Then insert the test function $\varphi := \eta^2 \partial^s u_k$ into (6.4.24), which gives

$$\left| \int_{B_R} \partial^s f'_k(\nabla u_k) \cdot \nabla(\eta^2 \partial^s u_k) \,\mathrm{d}x \right| \leq \frac{1}{k} \left\| \eta^2 \partial^s u_k \right\|_{L^1(\Omega, \mathbb{R}^N)}. \tag{6.4.26}$$

The integral on the left-hand side can be written as

$$\int_{B_R} f''_k(\nabla u_k)[\eta \partial^s \nabla u_k, \eta \partial^s \nabla u_k] \,\mathrm{d}x + \int_{B_R} f''_k(\nabla u_k)[\eta \partial^s \nabla u_k, 2\partial^s u_k \otimes \nabla \eta] \,\mathrm{d}x.$$

The second term can be controlled with the Cauchy-Schwarz inequality as in (6.4.16). Then (6.4.26) and (6.4.7) implies

$$\begin{aligned} &\int_{B_r} \left(\ell \frac{|\partial^s \nabla u_k|^2}{(1+|\nabla u_k|^2)^{\frac{\mu}{2}}} + \frac{1}{V_k k^2} |\partial^s \nabla u_k|^2 \right) \mathrm{d}x \\ &\leq \int_{B_R} f''_k(\nabla u_k)[\eta \partial^s \nabla u_k, \eta \partial^s \nabla u_k] \,\mathrm{d}x \\ &\leq \frac{2}{k} \int_{B_R} \eta^2 |\partial^s u_k| \,\mathrm{d}x + 4 \int_{B_R} f''_k(\nabla u_k)[\partial^s u_k \otimes \nabla \eta, \partial^s u_k \otimes \nabla \eta] \,\mathrm{d}x \\ &\leq \frac{2}{k} \int_{B_R} \eta^2 |\partial^s u_k| \,\mathrm{d}x + \frac{16}{(R-r)^2} \int_{B_R} \left(\frac{L |\partial^s u_k|^2}{(1+|\nabla u_k|^2)^{\frac{1}{2}}} + \frac{1}{V_k k^2} |\partial^s u_k|^2 \right) \mathrm{d}x. \end{aligned} \tag{6.4.27}$$

Summing (6.4.27) over s, we have the desired estimate (6.4.25). □

Now we know that the map u_k selected in Subsection 6.4.1 satisfies (6.4.25), which can be further controlled with (6.4.11):

$$\ell \int_{B_r} \frac{|\nabla^2 u_k|^2}{(1+|\nabla u_k|^2)^{\frac{\mu}{2}}} \,\mathrm{d}x + \frac{1}{V_k k^2} \int_{B_r} |\nabla^2 u_k|^2 \,\mathrm{d}x \\ \leq C \left(\frac{1}{(R-r)^2} + \frac{1}{k} \right) \left(\int_\Omega f(\nabla u) \,\mathrm{d}x + \frac{1}{k^2} \right), \tag{6.4.28}$$

where $C = C(n, L, \ell) > 0$. Then from (6.4.23) we have the following uniform bound with

$C = C(n, L, \ell, \mu) > 0$:

$$\begin{aligned}\int_{B_r} |\nabla W_\mu(\nabla u_k)|^2 \,\mathrm{d}x &\leq C \int_{B_r} \frac{|\nabla^2 u_k|^2}{(1+|\nabla u_k|^2)^{\frac{\mu}{2}}} \,\mathrm{d}x \\ &\leq C\left(\frac{1}{(R-r)^2} + \frac{1}{k}\right)\left(\int_\Omega f(\nabla u)\,\mathrm{d}x + \frac{1}{k^2}\right). \quad (6.4.29)\end{aligned}$$

6.4.4 Strong convergence

The uniform estimate obtained in the last subsection can be passed to the generalised minimizer u, which then gives Theorem 6.1.1. To achieve this, we first show that $u_k \to u$ strongly in $W^{1,1}$ up to a subsequence, and then apply Fatou's lemma in the difference quotient estimates for u_k.

We make two claims about the convergence of $\{u_k\}$:

(i) u_k converges to u in the weak* sense up to a subsequence;

(ii) u_k converges to u in the f-strict sense up to a subsequence, i.e., there exists a subsequence $\{u_{k_l}\}$ such that

$$u_{k_l} \overset{*}{\rightharpoonup} u \quad \text{in } BV(\Omega, \mathbb{R}^N), \quad \text{and} \quad \lim_{l\to\infty} \mathscr{F}(u_{k_l}) = \mathscr{F}(u).$$

Proof of (i). Combining (6.4.3) and (a) we know that

$$\|u_k - u\|_{W^{-1,1}(\Omega,\mathbb{R}^N)} \leq \frac{1}{3L_f k^2} + \frac{1}{k} \to 0, \quad \text{as } k \to \infty,$$

and thus $u_k \to u$ in $W^{-1,1}(\Omega, \mathbb{R}^N)$.

The estimate (6.4.11) with Poincaré's inequality implies that $\{u_k\}$ is bounded in $W^{1,1}(\Omega, \mathbb{R}^N)$. Therefore, there exists a subsequence $\{u_{k_l}\}$ and a map $w \in BV(\Omega, \mathbb{R}^N)$ such that u_{k_l} converges to w in the weak* sense. In particular, we have

$$\|u_{k_l} - w\|_{L^1(\Omega,\mathbb{R}^N)} \to 0, \quad \text{as } l \to \infty.$$

By (2.2.3), the limit w must coincide with u, and thus u_{k_l} converges to u in the weak* sense. □

Proof of (ii). Consider the subsequence $\{u_{k_l}\}$ selected in the proof of (i). First, by definition and the selection of u_k we have

$$\mathscr{F}(u_{k_l}) \leq \mathscr{F}_{k_l}(u_{k_l}) \overset{(b)}{\leq} \mathscr{F}_{k_l}(v_{k_l}) \overset{(6.4.5)}{\leq} \inf_{\mathcal{D}} \mathscr{F} + \frac{5}{6k_l^2} = \mathscr{F}(u) + \frac{5}{6k_l^2}. \quad (6.4.30)$$

Taking the limit superior of both sides gives

$$\limsup_{l\to\infty} \mathscr{F}(u_{k_l}) \leq \mathscr{F}(u). \quad (6.4.31)$$

Notice that f is of linear growth and strongly convex, and thus $\bar{\mathscr{F}}$ is lower semicontinuous with respect to the weak* topology in $BV(\Omega, \mathbb{R}^N)$. This result can be found in

[ADM92] and [FM93], where the authors showed that the lower semi-continuous relaxation with respect to the $L^1_{loc}(\Omega,\mathbb{R}^N)$ topology of $\mathscr{F}$, which is originally defined on $W^{1,1}$, is $\bar{\mathscr{F}}$. See also Theorem 11.7 in [Rin18]. Thus, we have

$$\mathscr{F}(u) \leq \liminf_{l\to\infty} \mathscr{F}(u_{k_l}), \tag{6.4.32}$$

which finally implies

$$\lim_{l\to\infty} \mathscr{F}(u_{k_l}) = \mathscr{F}(u).$$

□

The next step is to show that $\{u_{k_l}\}$ converges to u strongly in $W^{1,1}(\Omega,\mathbb{R}^N)$, which is the following lemma. The statement is similar to Theorem B.2 in [BS13], but we use a different argument based on Young measures.

Lemma 6.4.5. *Suppose that the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* **(LD1)**, **(LD2)**, *and that the sequence $\{w_j\} \subset W^{1,1}(\Omega,\mathbb{R}^N)$ converges to $w \in W^{1,1}(\Omega,\mathbb{R}^N)$ in the f-strict sense, i.e.,*

$$w_j \overset{*}{\rightharpoonup} w \quad \text{in } BV(\Omega,\mathbb{R}^N), \quad \text{and} \quad \int_\Omega f(\nabla w_j)\,\mathrm{d}x \to \int_\Omega f(\nabla w)\,\mathrm{d}x \tag{6.4.33}$$

as $j\to\infty$. Then the convergence also holds true in the strong sense in $W^{1,1}(\Omega,\mathbb{R}^N)$.

Proof. From **(LD1)** and the assumption (6.4.33), we know that $\{w_j\}$ is bounded in $W^{1,1}(\Omega,\mathbb{R}^N)$. Then Corollary 2.5.14 implies that for any subsequence of it (not relabelled), there exists a further subsequence $\{w_{j_t}\}$ such that $\{\nabla w_{j_t}\}$ generates a BV-Young measure $\nu \in BVY(\Omega,\mathbb{R}^N) \subset Y^{\mathscr{M}}(\Omega,\mathbb{R}^N)$ with $\nu = ((\nu_x)_{x\in\Omega}, \lambda_\nu, (\nu_x^\infty)_{x\in\bar\Omega})$, where

$$(\nu_x)_{x\in\Omega} \subset \mathscr{M}^1(\mathbb{R}^N), \quad \lambda_\nu \in \mathscr{M}^+(\bar\Omega), \quad (\nu_x^\infty)_{x\in\bar\Omega} \subset \mathscr{M}^1(\mathbb{S}^{Nn-1}).$$

Notice that $\mathbb{S}^{Nn-1}$ is the unit sphere in $\mathbb{R}^{N\times n}$.

Since $\nabla w_{j_t} \overset{*}{\rightharpoonup} \nabla w$, by (2.5.9) we know that

$$\nabla w(\cdot)\mathscr{L}^n \llcorner \Omega = \bar\nu = \bar\nu_. \mathscr{L}^n \llcorner \Omega + \bar\nu_.^\infty \lambda_\nu \llcorner \Omega \quad \text{in } \mathscr{M}(\Omega,\mathbb{R}^{N\times n}),$$

where $\bar\nu$ is the barycentre of ν. The measure on the left-hand side is absolutely continuous with respect to $\mathscr{L}^n \llcorner \Omega$, and thus so is the one on the right, which implies

$$\lambda_\nu \llcorner \Omega \ll \mathscr{L}^n \llcorner \Omega \quad \text{and} \quad \nabla w(\cdot)\mathscr{L}^n \llcorner \Omega = (\bar\nu_. + \bar\nu_.^\infty \frac{\mathrm{d}\lambda_\nu}{\mathrm{d}\mathscr{L}^n})\mathscr{L}^n \llcorner \Omega. \tag{6.4.34}$$

Considering that ν is generated by $\{\nabla w_{j_t}\}$ and w_j converges to w in the f-strict sense, we have

$$\begin{aligned} \int_\Omega f(\nabla w)\,\mathrm{d}x &= \lim_{t\to\infty} \int_\Omega f(\nabla w_{j_t})\,\mathrm{d}x = \int_\Omega \langle \nu_x, f\rangle \,\mathrm{d}x + \int_{\bar\Omega} \langle \nu_x^\infty, f^\infty\rangle \,\mathrm{d}\lambda_\nu(x) \\ &\geq \int_\Omega \left(f(\bar\nu_x) + f^\infty\left(\bar\nu_x^\infty \frac{\mathrm{d}\lambda_\nu}{\mathrm{d}\mathscr{L}^n}\right)\right)\mathrm{d}x + \int_{\partial\Omega} \langle \nu_x^\infty, f^\infty\rangle \,\mathrm{d}\lambda_\nu(x) \\ &\geq \int_\Omega f(\nabla w)\,\mathrm{d}x + \int_{\partial\Omega} \langle \nu_x^\infty, f^\infty\rangle \,\mathrm{d}\lambda_\nu(x), \end{aligned} \tag{6.4.35}$$

where we applied Jensen's inequality, Lemma 6.4.6 in the following and (6.4.34). The above estimate indicates that $\langle \nu_x^\infty, f^\infty\rangle = 0$ λ_ν-a.e. on $\partial\Omega$. However, the assumption **(LD1)** and the fact that $\nu_x^\infty \in \mathscr{M}^1(\mathbb{S}^{Nn-1})$ implies

$$\langle \nu_x^\infty, f^\infty\rangle = \int_{\mathbb{S}^{Nn-1}} f^\infty(z)\,\mathrm{d}\nu_x^\infty(z) \geq \int_{\mathbb{S}^{Nn-1}} \ell|z|\,\mathrm{d}\nu_x^\infty(z) \geq \ell > 0, \quad \text{for any } x \in \partial\Omega.$$

Thus, we have $\lambda_\nu \llcorner \partial\Omega = 0$. Besides, both inequalities in (6.4.35) hold with equality. The function f is strictly convex on $\mathbb{R}^{N\times n}$, and thus so is f^∞ on $\mathbb{S}^{Nn-1}$. With Lemma 6.4.6 we have, for $\mathscr{L}^n$-a.e. $x \in \Omega$ and thus λ_ν-a.e. $x \in \Omega$,

$$\langle \nu_x, f\rangle = f(\bar{\nu}_x) \implies \nu_x = \delta_{\bar{\nu}_x}; \tag{6.4.36}$$

$$\langle \nu_x^\infty, f^\infty\rangle = f(\bar{\nu}_x^\infty) \implies \nu_x^\infty = \delta_{\bar{\nu}_x^\infty}; \tag{6.4.37}$$

$$f(\bar{\nu}_x) + f^\infty\left(\bar{\nu}_x^\infty \frac{\mathrm{d}\lambda_\nu}{\mathrm{d}\mathscr{L}^n}\right) = f\left(\bar{\nu}_x + \bar{\nu}_x^\infty \frac{\mathrm{d}\lambda_\nu}{\mathrm{d}\mathscr{L}^n}\right) \implies \bar{\nu}_x^\infty \frac{\mathrm{d}\lambda_\nu}{\mathrm{d}\mathscr{L}^n} = 0. \tag{6.4.38}$$

Notice that $\nu_x^\infty = \delta_{\bar{\nu}_x^\infty}$ and thus $\bar{\nu}_x^\infty \in \mathbb{S}^{Nn-1}$ for any $x \in \Omega$, then it is easy to see $\lambda_\nu \equiv 0$. Therefore Theorem 2.5.12 implies that ∇w_{j_t} is uniformly integrable on Ω, and Theorem 2.5.11 with (6.4.36) implies the convergence of $\{\nabla w_{j_t}\}$ in $\mathscr{L}^n$-measure to ∇w. Finally, we apply Vitali's convergence theorem (Theorem 2.5.13) to conclude that $\nabla w_{j_t} \to \nabla w$ strongly in $L^1(\Omega, \mathbb{R}^{N\times n})$.

Since each subsequence of $\{w_j\}$ has a further subsequence converging to w strongly in $W^{1,1}(\Omega, \mathbb{R}^N)$, the strong convergence must hold for $\{w_j\}$ itself. □

At the end of this part, we show an auxiliary lemma about f and f^∞ used in the above proof.

Lemma 6.4.6. *Suppose that the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* **(LD1)**, **(LD2)**. *Then for any $z_1, z_2 \in \mathbb{R}^{N\times n}$, we have*

$$f(z_1) + f^\infty(z_2) \geq f(z_1 + z_2) \tag{6.4.39}$$

hold true, and the equality is attained if and only if $z_2 = 0$.

Proof. Consider the function

$$g(t) := \frac{1}{t}(f(z_1 + tz_2) - f(z_1)), \quad t > 0.$$

Then with the assumptions **(LD1)** and **(LD2)** on f, we can do calculation as follows

$$g(t) = \int_0^1 f'(z_1 + stz_2)\cdot z_2\,\mathrm{d}s,$$

$$g'(t) = \int_0^1 f''(z_1 + stz_2)[sz_2, z_2]\,\mathrm{d}x \geq \int_0^1 \frac{s\ell|z_2|^2}{(1 + |z_1 + stz_2|^2)^{\frac{\mu}{2}}}\,\mathrm{d}s.$$

Notice that when $z_2 \neq 0$, the derivative $g'(t)$ is strictly positive for any $t > 0$, and thus $g(t)$

is strictly increasing in t and

$$\begin{aligned}\lim_{t\to\infty} g(t) &= \frac{1}{t}(f(z_1+tz_2)-f(tz_2)+f(tz_2)-f(z_1))\\ &\overset{\text{Lemma 2.6.6}}{\leq} \frac{1}{t}(CL|z_1|+f(tz_2)-f(z_1)) = f^\infty(z_2).\end{aligned}$$

In particular, the inequality (6.4.39) is obtained by comparing $g(1)$ and $g(\infty)$:

$$f^\infty(z_2) = \lim_{t\to\infty} g(t) > g(1) = f(z_1+z_2)-f(z_1).$$

It is obvious that (6.4.39) holds with equality when $z_2 = 0$. □

6.4.5 Differentiability of $W_\mu(\nabla u)$

Finally, we are at the place to prove the regularity of the generalised minimizer u as in Theorem 6.1.1. The sequence $\{u_{k_l}\}$ selected in the proof of (i) and (ii) converges to u in the f-strict sense, and thus strongly in $W^{1,1}(\Omega,\mathbb{R}^N)$ by Lemma 6.4.5. Then there is a further subsequence $\{u_{k_t}\}$ such that $\nabla u_{k_t} \to \nabla u$ $\mathscr{L}^n$-a.e. in Ω. Fix a ball $B_R = B(x_0,R) \subset\subset \Omega$, and take $r \in (0,R)$ and $h \in \mathbb{R}$ with $0 < |h| < r$.

The estimate (6.4.29) holds for u_{k_t}, and thus for any $s \in \{1,\dots,n\}$ we have the following difference quotient estimate by Lemma 2.3.3

$$\begin{aligned}\int_{B_{r-|h|}} |\Delta_h^s W_\mu(\nabla u_{k_t})|^2 \,\mathrm{d}x &\leq \int_{B_r} |\nabla W_\mu(\nabla u_{k_t})|^2 \,\mathrm{d}x\\ &\leq C\left(\frac{1}{(R-r)^2}+\frac{1}{k_t}\right)\left(\int_\Omega f(\nabla u)\,\mathrm{d}x + \frac{1}{k_t^2}\right).\end{aligned} \tag{6.4.40}$$

Applying Fatou's lemma to obtain

$$\int_{B_{r-|h|}} |\Delta_h^s W_\mu(\nabla u)|^2 \,\mathrm{d}x \leq \liminf_{t\to\infty} \int_{B_{r-|h|}} |\Delta_h^s W_\mu(\nabla u_{k_t})|^2 \,\mathrm{d}x \leq \frac{C}{(R-r)^2}\int_\Omega f(\nabla u)\,\mathrm{d}x. \tag{6.4.41}$$

Thus, Lemma 2.3.4 indicates that $W_\mu(\nabla u) \in W^{1,2}_{loc}(\Omega)$ with the estimate (6.1.3).

In particular, for $1 < \mu < 1+\frac{2}{n}$, a higher integrability for ∇u as in Theorem 6.1.1 follows from

$$|\nabla u|^{\frac{2-\mu}{2}} \leq W_\mu(\nabla u), \quad W_\mu(\nabla u) \in W^{1,2}_{loc}(\Omega) \hookrightarrow \begin{cases} L^{\frac{2n}{n-2}}_{loc}(\Omega), & n \geq 3,\\ L^\Phi_{loc}(\Omega), & n = 2,\end{cases} \tag{6.4.42}$$

where $L^\Phi_{loc}(\Omega)$ is the local Orlicz space with $\Phi(t) = e^{t^2}-1$. The second embedding is by Trudinger [Tru67].

Remark 6.4.7. In the case where $1 < \mu < 1+\frac{2}{n}$, the condition **(LD3)** is indeed unnecessary. This condition is used in [BS13] to show the local boundedness of any generalised minimizer, which is helpful in the proof of the $L\log L$ regularity for the gradient. When $1 < \mu < 1+\frac{2}{n}$, one can follow the above argument to construct an approximating sequence $\{u_k\}$ of a generalised minimizer $u \in BV(\Omega,\mathbb{R}^N)$ and obtain the fact that $W_\mu(\nabla u_k) \in W^{1,2}_{loc}(\Omega)$ with a local estimate uniform in k. Since the $W^{1,1}$ regularity for u is not known *a priori*, it is necessary to first approximate u with a sequence in $W^{1,1}_u(\Omega, \mathbb{R}^N)$ in the area-strict sense (see §5 in [BS13]). Then as in (6.4.42) we have $|\nabla u_k| \in L^q_{loc}(\Omega)$ with some $q = q(n, \mu) > 1$, and the local L^q-integrability of Du follows from the lower semi-continuity of the integrand $z \mapsto |z|^q$.

Remark 6.4.8. Since we already know that a generalised minimizer u is in $W^{1,1}(\Omega, \mathbb{R}^N)$ under conditions (**LD1**)-(**LD3**) from Theorem 3.2.2, the corresponding Euler-Lagrange equation holds in the weak sense

$$\int_\Omega f'(\nabla u) \cdot \nabla \varphi \, \mathrm{d}x, \quad \text{for any } \varphi \in C^\infty_c(\Omega, \mathbb{R}^N). \tag{6.4.43}$$

It is possible to do difference quotient directly with (6.4.43). However, there will be issues due to the low integrability of ∇u. Test (6.4.43) with $\varphi = \Delta^s_{-h}(\eta^2 \Delta^s_h u)$ for some cut-off function $\eta \in C^\infty_c(\Omega)$, then we will have to deal with the term

$$\int_\Omega 2\eta \Delta^s_h f'(\nabla u) \cdot \Delta^s_h u \otimes \nabla \eta \, \mathrm{d}x. \tag{6.4.44}$$

It can be controlled with the Cauchy-Schwartz inequality as in (6.4.16), and one term obtained with $\Delta^s_h \nabla u$ can be absorbed, while the other is

$$\int_\Omega \int_0^1 f''(\nabla u(x) + th\Delta^s_h \nabla u(x)) \, \mathrm{d}t [\Delta^s_h u \nabla \eta, \Delta^s_h u \nabla \eta] \, \mathrm{d}x.$$

We can further control it with (**LD2**) by the following

$$\sup_\Omega |\nabla \eta|^2 \int_{\mathrm{supp}(\eta)} \frac{|\Delta^s_h u|^2 (1 + \log(1 + |\nabla u(x)|^2 + |\nabla u(x + he_s)|^2))}{(1 + +|\nabla u(x)|^2 + |\nabla u(x + he_s)|^2)^{\frac{1}{2}}} \, \mathrm{d}x.$$

The pointwise limit of the integrand is integrable by Theorem 3.2.2, but it is not clear how to deal with the integral itself. If the integrand is bounded by an integrable function, Fatou's lemma will help to control this term.

Alternatively, we can control the term (6.4.44) by

$$\sup_\Omega |\nabla \eta| \frac{2L_f}{|h|} \int_{\mathrm{supp}(\eta)} |\Delta^s_h u| \, \mathrm{d}x \le \sup_\Omega |\nabla \eta| \frac{2L_f}{|h|} \int_{\mathrm{supp}(\eta)} |\nabla u| \, \mathrm{d}x,$$

with which we lose one $|h|$ and can only obtain $W^{r,2}_{loc}$ regularity of $W_\mu(\nabla u)$ with $r \in (0, \frac{1}{2})$.

6.5 Sobolev regularity for ω-minimizers

An ω-minimizer partially retains the Sobolev regularity of the corresponding generalised minimizers, which is specified in Theorem 6.1.2 and proved in this section. To show this result, we first consider the extremality of a generalised minimizer, and obtain an inequality similar to the corresponding Euler-Lagrange equation but with the singular part and boundary value of the test map incorporated (see Subsection 6.5.1). With such an inequality, the comparison between an ω-minimizer and a corresponding generalised minimizer on any small ball can be carried out, which is presented in Subsection 6.5.2. Then the Sobolev regularity proved in Theorem 6.1.1 can be transferred to the ω-minimizer (the absolutely continuous part) with such a process.

We remark that it is not clear to us how to rule out the singular part of the derivative of a BV ω-minimizer. In addition, the argument only works for $1 < \mu < 2$, since the estimate in Lemma 2.9.3 for $\mu \geq 2$ is not good enough to make the proof work.

6.5.1 Extremality of generalised minimizers

Suppose that $v \in BV(\Omega, \mathbb{R}^N)$ is a generalised minimizer of $\mathscr{F}_g$ with some $g \in BV(\Omega, \mathbb{R}^N)$. By Theorem 3.2.2 we know that $v \in W^{1,1}(\Omega, \mathbb{R}^N)$, then the Euler-Lagrange equation holds in $W^{1,1}$

$$\int_\Omega f'(\nabla v) \cdot \nabla \varphi \, \mathrm{d}x = 0, \quad \text{for any } \varphi \in W_0^{1,1}(\Omega, \mathbb{R}^N). \tag{6.5.1}$$

However, in BV the situation is more subtle due to the possible singular part of the test map and possible incoincidence of the boundary values. Taking the two factors into consideration, we are still able to obtain a description of the extremality of v.

Lemma 6.5.1. *Suppose that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* (**LD1**)-(**LD3**), *and $v \in BV(\Omega, \mathbb{R}^N)$ is a generalised minimizer of $\mathscr{F}_g$ with some $g \in BV(\Omega, \mathbb{R}^N)$. Then for any $w \in BV_g(\Omega, \mathbb{R}^N)$, the following inequality holds*

$$\int_\Omega f'(\nabla v) \cdot \nabla(w - v) \, \mathrm{d}x + \int_\Omega f^\infty \left(\frac{\mathrm{d}D^s w}{\mathrm{d}|D^s w|} \right) \mathrm{d}|D^s w| \\ - \int_{\partial\Omega} f^\infty(\mathrm{tr}_\Omega(w - v) \otimes \nu_\Omega) \, \mathrm{d}\mathcal{H}^{n-1} \geq 0. \tag{6.5.2}$$

Proof. Take any $w \in BV(\Omega, \mathbb{R}^N)$ and let $\varphi = w - v$. For any $t \in [0,1]$, there holds $\mathscr{F}_g(v, \Omega) \leq \mathscr{F}_g(v + t\varphi, \Omega)$, which is

$$\begin{aligned} &\int_\Omega f(\nabla v) \, \mathrm{d}x + \int_{\partial\Omega} f^\infty(\mathrm{tr}_\Omega(g - v) \otimes \nu_\Omega) \, \mathrm{d}\mathcal{H}^{n-1} \\ &\leq \int_\Omega f(\nabla v + t\nabla\varphi) \, \mathrm{d}x + t \int_\Omega f^\infty \left(\frac{\mathrm{d}D^s w}{\mathrm{d}|D^s w|} \right) \mathrm{d}|D^s w| \\ &\qquad + \int_{\partial\Omega} f^\infty(\mathrm{tr}_\Omega(g - (1-t)v - tw) \otimes \nu_\Omega) \, \mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

where ν_Ω is the unit outward normal of $\partial\Omega$, and we used the positive homogeneity of f^∞. If $w \in BV_g(\Omega, \mathbb{R}^N)$, the above inequality becomes

$$\begin{aligned} &\int_\Omega f(\nabla v) \, \mathrm{d}x + \int_{\partial\Omega} f^\infty(\mathrm{tr}_\Omega(g - v) \otimes \nu_\Omega) \, \mathrm{d}\mathcal{H}^{n-1} \\ &\leq \int_\Omega f(\nabla v + t\nabla\varphi) \, \mathrm{d}x + t \int_\Omega f^\infty \left(\frac{\mathrm{d}D^s w}{\mathrm{d}|D^s w|} \right) \mathrm{d}|D^s w| \\ &\qquad + (1-t) \int_{\partial\Omega} f^\infty(\mathrm{tr}_\Omega(w - v) \otimes \nu_\Omega) \, \mathrm{d}\mathcal{H}^{n-1} \end{aligned}$$

Denote the function in t on the right-hand side by $h(t)$, then it is easy to see that $h(t) \geq h(0)$ for any $t > 0$ and thus $h'(0) \geq 0$, which gives exactly (6.5.2). ☐

6.5.2 Comparison and fractional Sobolev regularity

Assume that $u \in BV(\Omega, \mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$ with some proper ω. In this subsection, we compare u with a corresponding generalised minimizer on any small ball, via which it is possible to establish an estimate for a fractional difference quotient of $W_\mu(\nabla u)$. Then by an argument similar to that in [KM05], one can obtain the fractional Sobolev regularity in Theorem 6.1.2.

Proposition 6.5.2. *Suppose that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* **(LD1)**-**(LD3)** *with $1 < \mu < 2$, and $u \in BV_{loc}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of $\bar{\mathscr{F}}$ with some $R_0 > 0$ and ω satisfying* **($\boldsymbol{\omega}$1)**. *Then for any ball $B_R = B(x_0, R) \subset\subset \Omega$ with $R < R_0$, we have*

$$\int_{B_r} |W_\mu(\nabla u(x)) - W_\mu(\nabla u(x + he_s))|^2 \,\mathrm{d}x \leq C\left(\frac{|h|^2}{R^2} + \omega(R)\right)\int_{B_R}(1 + f(Du)), \tag{6.5.3}$$

where $0 < r \leq \frac{R}{4}$ and $C = C(n, N, L, \ell, \mu) > 0$.

Proof. Take a generalised minimizer $v \in BV(B_R, \mathbb{R}^N)$ of $\mathscr{F}_u$ on B_R, and the existence is guaranteed by Corollary 1.9 in [BS13]. Then by Theorem 6.1.1 we know that $v \in W^{1,1}(B_R, \mathbb{R}^N)$, and $W_\mu(\nabla v) \in W^{1,2}_{loc}(B_R)$ with

$$\int_{B_{2r}} |\nabla W_\mu(\nabla v)|^2 \,\mathrm{d}x \leq \frac{C}{(R-r)^2}\int_{B_{3r}}(1 + |\nabla v|)\,\mathrm{d}x \leq \frac{C}{(R-r)^2}\int_{B_R}(1 + f(Du)) \tag{6.5.4}$$

for any $0 < r \leq \frac{R}{4}$, where we used **(LD1)** and the minimality of v to get the second inequality. Thus, for any $s \in \{1, \ldots, n\}$ and any $h \in \mathbb{R}$ with $0 < |h| < r$, the following estimate holds

$$\begin{aligned}\int_{B_r} |\Delta_h^s W_\mu(\nabla v)|^2 \,\mathrm{d}x &\leq \int_{B_{2r}} |\partial^s W_\mu(\nabla v)|^2 \,\mathrm{d}x \\ &\leq \frac{C}{(R-r)^2}\int_{B_{3r}}(1 + |\nabla v|)\,\mathrm{d}x \leq \frac{C}{(R-r)^2}\int_{B_R}(1 + f(Du)).\end{aligned} \tag{6.5.5}$$

On the other hand, Lemma 6.5.1 implies

$$\begin{aligned}\int_\Omega f'(\nabla v)\cdot\nabla(u - v)\,\mathrm{d}x + \int_\Omega f^\infty\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u| \\ - \int_{\partial\Omega} f^\infty(\mathrm{tr}_\Omega(u - v)\otimes\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} \geq 0.\end{aligned} \tag{6.5.6}$$

Then we have the following estimate

$$\begin{aligned}&\int_{B_R}(f(\nabla u) - f(\nabla v) - f'(\nabla v)\cdot\nabla(u - v))\,\mathrm{d}x \\ &= \mathscr{F}_u(u, B_R) - \mathscr{F}_u(v, B_R) - \int_\Omega f'(\nabla v)\cdot\nabla(u - v)\,\mathrm{d}x\end{aligned} \tag{6.5.7}$$

$$
\begin{aligned}
&-\int_\Omega f^\infty\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u| + \int_{\partial\Omega} f^\infty(\mathrm{tr}_\Omega(u-v)\otimes\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}\\
&\le \mathscr{F}_u(u,B_R)-\mathscr{F}_u(v,B_R).
\end{aligned}
$$

Considering the definition of $\mathscr{F}_u$ in (3.1.24), we may take $\{w_k\}\subset W^{1,1}_u(B_R,\mathbb{R}^N)$ such that

$$
w_k\to v \quad \text{in } L^1(B_R,\mathbb{R}^N) \quad \text{and} \quad \mathscr{F}(w_k,B_R)\to\mathscr{F}_u(v,B_R)
$$

as $k\to\infty$. The ω-minimality gives

$$
\mathscr{F}(u,B_R)\le \lim_{k\to\infty}(1+\omega(R))\mathscr{F}(w_k,B_R)=(1+\omega(R))\mathscr{F}_u(v,B_R),
$$

and thus there holds

$$
\int_{B_R}(f(\nabla u)-f(\nabla v)-f'(\nabla v)\cdot\nabla(u-v))\,\mathrm{d}x\le\omega(R)\mathscr{F}_u(v,B_R)\le\omega(R)\mathscr{F}_u(u,B_R). \tag{6.5.8}
$$

In addition, the left-hand side of the above estimate is controlled from below:

$$
\begin{aligned}
&\int_{B_R}(f(\nabla u)-f(\nabla v)-f'(\nabla v)\cdot\nabla(u-v))\,\mathrm{d}x\\
&=\int_{B_R}\int_0^1(1-t)f''(\nabla v+t\nabla(u-v))\,\mathrm{d}t[\nabla u-\nabla v,\nabla u-\nabla v]\,\mathrm{d}x \qquad (6.5.9)\\
&\overset{\text{(LD2)}}{\ge}\int_{B_R}\int_0^1\frac{1-t}{(1+|\nabla v+t\nabla(u-v)|^2)^{\frac{\mu}{2}}}\,\mathrm{d}t|\nabla u-\nabla v|^2\,\mathrm{d}x\\
&\overset{\text{Lemma 2.9.3}}{\ge} C\int_{B_R}\frac{|\nabla u-\nabla v|^2}{(1+|\nabla u|^2+|\nabla v|^2)^{\frac{\mu}{2}}}\,\mathrm{d}x.
\end{aligned}
$$

Lemma 2.9.3 also helps to compare $W_\mu(\nabla u)$ and $W_\mu(\nabla v)$ in the following way as $1<\mu<2$:

$$
\begin{aligned}
\int_{B_R}|W_\mu(\nabla u)-W_\mu(\nabla v)|^2\,\mathrm{d}x&\le\int_{B_R}\left(\int_0^1\frac{|\nabla u-\nabla v|}{(1+|\nabla v+t\nabla(u-v)|^2)^{\frac{\mu}{4}}}\right)^2\mathrm{d}x\\
&\overset{\text{Lemma 2.9.3}}{\le} C\int_{B_R}\frac{|\nabla u-\nabla v|^2}{(1+|\nabla u|^2+|\nabla v|^2)^{\frac{\mu}{2}}}\,\mathrm{d}x \qquad (6.5.10)\\
&\overset{(6.5.9)}{\le} C\int_{B_R}(f(\nabla u)-f(\nabla v)-f'(\nabla v)\cdot\nabla(u-v))\,\mathrm{d}x\\
&\overset{(6.5.8)}{\le} C\omega(R)\mathscr{F}_u(u,B_R).
\end{aligned}
$$

To get the estimate (6.5.3), we take $h\in\mathbb{R}$ with $0<|h|<r$ and $s\in\{1,\dots,n\}$, and use (6.5.5) and (6.5.10) to obtain

$$
\begin{aligned}
&\int_{B_r}|W_\mu(\nabla u(x+he_s))-W_\mu(\nabla u(x))|^2\,\mathrm{d}x\\
&\le 3\Bigg(\int_{B_r}|W_\mu(\nabla u)-W_\mu(\nabla v)|^2\,\mathrm{d}x+\int_{B_r}|W_\mu(\nabla v(x+he_s))-W_\mu(\nabla v(x))|^2\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{B_r}|W_\mu(\nabla u(x+he_s))-W_\mu(\nabla v(x+he_s))|^2\,\mathrm{d}x\Big) &(6.5.11)\\
&\le 6\int_{B_R}|W_\mu(\nabla u)-W_\mu(\nabla v)|^2\,\mathrm{d}x+3C\frac{|h|^2}{(R-r)^2}\int_{B_R}(1+f(Du))\\
&\le C(\omega(R)+\frac{|h|^2}{(R-r)^2})\int_{B_R}(1+f(Du)).
\end{aligned}
$$

Thus, the estimate (6.5.3) holds because of $r\le\frac{R}{4}$. ☐

Now we are ready to prove Theorem 6.1.2. For any subdomain $\Omega'\subset\subset\Omega$ of Ω, take $\Omega''\subset\subset\Omega$ with $\Omega'\subset\subset\Omega''$, $\beta\in(0,1)$ to be fixed and $h\in\mathbb{R}$ such that

$$
0<|h|^\beta<\min\left\{1,\frac{1}{5\sqrt{n}}\operatorname{dist}(\Omega'',\partial\Omega),\frac{1}{4}R_0\right\}. \tag{6.5.12}
$$

For any $x_0\in\Omega$ with $\operatorname{dist}(x_0,\Omega'')<r:=|h|^\beta$, we have

$$
Q_{\frac{r}{\sqrt{n}}}\subset B_r\subset B_{4r}\subset Q_{4r}\subset\subset\Omega,
$$

where the cubes and balls are all centred at x_0. Then (6.5.3) implies

$$
\begin{aligned}
&\int_{Q_{\frac{4}{\sqrt{n}}}}|W_\mu(\nabla u(x+he_s))-W_\mu(\nabla u(x))|^2\,\mathrm{d}x\\
&\le C\left(\frac{|h|^2}{r^2}+\omega(4r)\right)\int_{Q_{4r}}(1+f(Du)) &(6.5.13)\\
&\le C(|h|^{2-2\beta}+|h|^{2\sigma\beta})\int_{Q_{4r}}(1+f(Du)).
\end{aligned}
$$

It is possible to select a finite family of disjoint cubes $\{Q(x_i,\frac{r}{\sqrt{n}})=:Q_i\}_{i=1}^K$ such that $\Omega''\subset\cup_{i=1}^K Q_i$, where $K\in\mathbb{N}$ depends on Ω',r,n. For any $i\in\{1,\dots,K\}$, the cube $4\sqrt{n}Q_i=Q(x_i,4r)$ intersects at most $(16\sqrt{n})^n$ cubes in the family $\{4\sqrt{n}Q_i\}_{i=1}^K$. Obviously, the inequality (6.5.13) holds for each Q_i, and we sum over i and s to get

$$
\int_{\Omega''}\sum_{s=1}^n|W_\mu(\nabla u(x+he_s))-W_\mu(\nabla u(x))|^2\,\mathrm{d}x\le C(|h|^{2-2\beta}+|h|^{2\sigma\beta})\int_\Omega(1+f(Du)), \tag{6.5.14}
$$

where $C=C(n,N,L,\ell)>0$. Notice that the above estimate holds for any h satisfies (6.5.12).

Set $\alpha:=\min\{2-2\beta,2\sigma\beta\}$, which takes its maximal value $\frac{2\sigma}{1+\sigma}$ when $\beta=\frac{1}{1+\sigma}$. Then by Lemma 2.3.5 $W_\mu(\nabla u)\in W^{t,2}_{loc}(\Omega'')$ for any $t\in(0,\frac{\alpha}{2})=(0,\frac{\sigma}{1+\sigma})$. In addition, the $W^{t,2}(\Omega')$ norm of $W_\mu(\nabla u)$ is controlled as follows:

$$
\begin{aligned}
[W_\mu(\nabla u)]^2_{W^{t,2}(\Omega')}&\le C\left(\int_\Omega(1+f(Du))+\int_\Omega W_\mu(\nabla u)^2\,\mathrm{d}x\right) &(6.5.15)\\
&\le C\left(\int_\Omega(1+f(Du))+\left(\int_\Omega f(Du)\right)^{2-\mu}\mathscr{L}^n(\Omega)^{\mu-1}\right)
\end{aligned}
$$

with $C > 0$ depending on $n, N, L, \ell, t, \sigma, R_0, \operatorname{dist}(\Omega'', \partial\Omega), \operatorname{dist}(\Omega', \partial\Omega''), \mathscr{L}^n(\Omega)$. We can further apply the embedding for fractional Sobolev spaces

$$W^{t,2}(\Omega') \hookrightarrow L^{\frac{2n}{n-2t}}(\Omega')$$

when Ω' is Lipschitz, which implies, with $W_\mu(z) \geq |z|^{\frac{2-\mu}{2}}$,

$$|\nabla u| \in L^{\frac{2-\mu}{2} \cdot \frac{2n}{n-2t}}(\Omega').$$

Then for $\mu < 1 + \frac{\alpha}{n} = 1 + \frac{2\sigma}{n(1+\sigma)}$, the absolutely continuous part ∇u of Du is locally L^q-integrable with any $q \in (1, \frac{(1+\sigma)(2-\mu)n}{n+(n-2)\sigma})$. The proof of Theorem 6.1.2 is now complete.

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