

Tubular configurations: equivariant scanning and splitting

Richard Manthorpe and Ulrike Tillmann

July 2014

Abstract

Replacing configurations of points by configurations of tubular neighbourhoods (or discs) in a manifold M we are able to define a natural scanning map that is equivariant under the action of the diffeomorphism group of the manifold. We also construct the so-called power set map of configuration spaces diffeomorphism equivariantly. Combining these two constructions yields stable splittings in the sense of Snaith and generalisations thereof that are equivariant. In particular one deduces stable splittings of homotopy orbit spaces. As an application the homology injectivity is proved for diffeomorphism of M that fix an increasing number of points. Throughout we work with configurations spaces with labels in a fibre bundle over M .

1 Introduction

There has been much recent interest in configuration spaces of manifolds. In one direction, the work on factorisation algebras and non-commutative Poincare duality by Lurie [Lur], see also Francis [Fra], is based on the classical work on configuration spaces of May [May72], Segal [Seg73], McDuff [McD75] and Salvatore [Sal01]. This has also ignited increased interest in the Goodwillie-Weiss embedding calculus [GW99]. From this point of view, one is more interested in configurations of embedded discs than points, and needs to understand the interaction with the diffeomorphism group of the background space M . Our approach to configurations spaces will address both these points.

In another direction, moduli spaces of manifolds and the scanning map have been central in the work on the Mumford conjecture and analogues; see [MW07], [Gal11] and also [Til12] for a survey. In this context, configuration spaces are moduli spaces of zero dimensional manifolds and have provided much intuition. The work here was motivated by some basic question of diffeomorphism equivariance that arose in this context.

Contents and results:

Let $C_k(M; X)$ denote the space of k unordered, distinct particles in a smooth, connected manifold M without boundary and with labels in a pointed space X . For a closed submanifold $M_0 \subset M$, let $C(M, M_0; X)$ denote the space of configurations of particles in M which vanish in M_0 or at the base-point of X .

The goal of this paper is two-fold. First we want to revisit the foundations of the subject and provide a natural and equivariant scanning map that relates the configuration spaces $C(M, M_0; X)$ to mapping spaces or section spaces more generally. The study of these maps goes back to May [May72] and Segal [Seg73] in the case when $M = \mathbb{R}^n$, and to McDuff [McD75] and Bökigheimer

[Böd87] for general manifolds. The diffeomorphism group of M acts naturally on both the configuration spaces and the section spaces. However, the standard scanning maps, which involve choosing a metric on M and are defined by collapsing ε -balls around the particles, do not commute with these actions.

Our approach here is to replace a configuration by its space of tubular neighbourhoods yielding the tubular configuration spaces $E_k(M; X)$ and $E(M; X)$. As the space of tubular neighbourhoods is contractible this construction does not change the homotopy type of $C_k(M; X)$ and, less obviously, also not of $C(M, M_0; X)$. This will be proved in section 2. In section 3 the scanning map on these enlarged configuration spaces is defined by simply collapsing M onto the configuration of tubular neighbourhoods. This defines a map $\gamma : E(M; X) \rightarrow \Gamma_c(M, M_0; X)$ to the space of sections of the fiberwise compactified tangent bundle TM smashed fiberwise with X which are compactly supported on M without the interior of M_0 . The map γ is a homotopy equivalence and is equivariant under the action of the diffeomorphism group. See Theorem 3.8.

Our second goal is to revisit the classical splitting theorems for function spaces going back to Snaith [Sna74] when M is a Euclidean space and generalised by Bökigheimer [Böd87] to arbitrary manifolds. Using the above results we construct these splittings equivariantly under the action of the diffeomorphism group; the main result in this direction is Theorem 4.8. In particular this gives stable splittings of the corresponding homotopy orbit spaces, something that for actions of compact Lie groups and a restricted class of manifolds was previously shown by Bökigheimer and Madsen [BM88] by different methods that do not extend to the non-compact setting.

The key to the splitting is the construction in section 4.3 of diffeomorphism equivariant power set maps for configuration spaces. For this we replace the orbits under the symmetric group action by their homotopy orbits, that is we replace $C(M, M_0; X)$ by $\mathcal{C}(M, M_0; X)$, and use the Barratt-Eccles [BE74a] model for the free infinite loop space functor.

Our main results (in simplified form) can be summarised by the following theorem. Counting the number of points in a configuration defines a natural filtration of $C(M, M_0; X)$. Let $D^k(M, M_0; X)$ denote the associated k -th filtration quotient.

Theorem 1.1. *Let G be a subgroup of the diffeomorphism group of M without the interior of M_0 . There is a zigzag of G -equivariant maps*

$$\Gamma_c(M, M_0; X) \longleftarrow E(M, M_0; X) \longrightarrow C(M, M_0; X) \longleftarrow \mathcal{C}(M, M_0; X) \longrightarrow \bigvee_{k \geq 1} D^k(M, M_0; X)$$

with the four left maps being weak homotopy equivalences and the right one a stable homotopy equivalence. The zigzag induces a weak homotopy equivalence

$$\Omega^\infty \Sigma^\infty(EG \times_G \Gamma_c(M, M_0; X)) \simeq \prod_{k \geq 1} \Omega^\infty \Sigma^\infty(EG \times_G D^k(M, M_0; X)).$$

In the final section, for connected M with non-empty boundary, the splitting methods are applied to show that the inclusion $b : C_k(M; X) \rightarrow C_{k+1}(M; X)$, which is well-known to be stably split injective, is indeed equivariantly so. As an immediate consequence we prove that a natural homomorphism

$$\bar{b} : \text{Diff}(M \setminus \mathbf{k}) \rightarrow \text{Diff}(M \setminus \mathbf{k} + \mathbf{1})$$

of diffeomorphisms of M fixing a set of k points to those fixing a set of $k + 1$ points induces a split injection in homology on classifying spaces; this is the content of Theorem 4.15.

Much of the literature restricts itself to configurations with labels in a constant space X . We emphasise that more generally we consider here configuration spaces with twisted coefficients, that is where X is replaced by a fibre bundle π over M and the label space may vary with the points in M . On the one hand we will need this in the application we have in mind [Til] and on the other hand it allows us to replace sections spaces with mapping spaces; see Example 4.10.

Most of the paper is written so as to treat the case when M has no boundary and we consider the full diffeomorphisms group of M . For M with boundary, we remark here that it is well-known that the configuration space of M and that of its interior $\text{int}(M)$ have the same homotopy type. We therefore *define* the configuration space of M to be that of its interior. All the theorems in the main body of the paper also apply then to manifolds with boundary. We note however, that the restriction map from $\text{Diff}(M)$ to $\text{Diff}(\text{int}(M))$ is in general not surjective and not a homotopy equivalence, and even though the configuration space is that of $\text{int}(M)$ any group action will have to factor through $\text{Diff}(M)$.

Future work and extensions:

In forthcoming work of the second author [Til], using the results established here, the map \bar{b} and generalisations thereof will be shown to also induce isomorphisms in homology in a range growing with k .

In another direction, the methods of this paper can be extended to treat configurations of submanifolds in M as considered by Palmer [Pal] and provide equivariant stable split injections for the stable homology isomorphisms in that setting.

Acknowledgements: We would like to thank the referee for the critical reading and many helpful comments.

2 Tubular configuration spaces and twisted labels

We define tubular configuration spaces and show that they are homotopic to the usual configuration spaces.

2.1 The definition of tubular configuration spaces

Let M be a smooth, connected manifold without boundary. The configurations space of k ordered particles in M is the subspace of the k -fold Cartesian product of M

$$\tilde{C}_k(M) := \{(m_1, \dots, m_k) \in M^k : m_i \neq m_j \text{ if } i \neq j\}.$$

Equivalently, $\tilde{C}_k(M)$ is the embedding space $\text{Emb}(\mathbf{k}, M)$, where \mathbf{k} denotes the 0-manifold with k points. The symmetric group Σ_k acts freely and the configuration space of k unordered particles in M is the orbit space $C_k(M)$. When $k = 0$ there is only one configuration, the empty configuration, and $\tilde{C}_0(M) = C_0(M) = *$.

Let $M_0 \subset M$ be a (possibly empty) submanifold which is closed in M . The configuration space

of particles in M modulo M_0 is then defined as

$$C(M, M_0) := \left(\prod_{k=0}^{\infty} C_k(M) \right) / \sim$$

where $(m_1, \dots, m_k) \sim (m_1, \dots, m_{k-1})$ if $m_k \in M_0$. We think of this relation as particles vanishing in M_0 .

To define the tubular configuration spaces of particles in M we replace a configuration $\mathbf{m} = (m_1, \dots, m_k)$ by the space of tubular neighbourhoods of its particles considered as a 0-dimensional submanifold of M .

Let $P \subset M$ be a neat submanifold, let ν be its normal bundle and identify P with the image of the zero section in ν . By a tubular neighbourhood of P in M we mean an embedding $f : \nu \rightarrow M$ which restricts to the identity on the zero-section and for which the composition

$$\nu \hookrightarrow \nu \oplus TP = T\nu|_P \xrightarrow{df} TM|_P \rightarrow \nu \quad (\text{id})$$

is the identity on ν . We call this last property (id). Furthermore, the image of each fibre of ν must have compact closure in M . Denote the space of tubular neighbourhoods of P by $\text{Tub}(P)$ and topologise it as a subspace of the embedding space $\text{Emb}(\nu, M)$ with the C^∞ topology.

For an ordered or unordered configuration \mathbf{m} of M let $\text{Tub}(\mathbf{m})$ denote the space of tubular neighbourhoods of \mathbf{m} , where \mathbf{m} is considered as a 0-dimensional submanifold of M . Define the tubular configuration space of k ordered particles in M as the disjoint union

$$\tilde{E}_k(M) := \coprod_{\mathbf{m} \in \tilde{C}_k(M)} \text{Tub}(\mathbf{m})$$

equipped with an appropriate topology which restricts to the ordinary C^∞ topology on the fibres. We defer a description of the topology to Section 2.2. We refer to a tubular neighbourhood $f \in \tilde{E}_k(M)$ as a tubular configuration and we often write it as a collection of its components (f_1, \dots, f_k) , with $f_i : T_{m_i}M \rightarrow M$. The symmetric group on k points acts freely on $\tilde{E}_k(M)$ and we define the tubular configuration space of k unordered particles as the orbit space $E_k(M)$.

Analogous to the ordinary configuration spaces, we define the tubular configuration space of particles in M modulo M_0 as

$$E(M, M_0) := \left(\prod_{k=0}^{\infty} E_k(M) \right) / \sim$$

where $(f_1, \dots, f_k) \sim (f_1, \dots, f_{k-1})$ if $f_k(0) \in M_0$. Components of a tubular configuration vanish if their midpoint is in M_0 .

Remark 2.1. As an alternative approach, consider configuration spaces with neighbourhoods of each particle without an identification with the normal bundle. We think of this as configurations of submanifolds diffeomorphic to a finite union of open disks with one marked point. Note that for an ordered configuration \mathbf{m} of k particles in M , there are isomorphisms

$$\begin{aligned} \text{Tub}(\mathbf{m}) &\cong \text{WeakTub}(\mathbf{m}) / \bigoplus_i \text{GL}_n(\mathbb{R}) \\ &\text{and} \\ \text{Tub}(\mathbf{m}) / \bigoplus_i \text{Diff}(T_{m_i}M, \text{id}) &\cong \text{WeakTub}(\mathbf{m}) / \bigoplus_i \text{Diff}(T_{m_i}M, 0) \end{aligned}$$

where $\text{WeakTub}(\mathbf{m})$ is the space of weak tubular neighbourhoods, that is the space of embeddings of the normal bundle of \mathbf{m} into M which restrict to the identity on the zero section but do not satisfy property (id), $\text{Diff}(T_{m_i}M, 0)$ is the group of diffeomorphisms which fix 0 and $\text{Diff}(T_{m_i}M, \text{id}) \subset \text{Diff}(T_{m_i}M, 0)$ are the diffeomorphisms whose derivatives at zero are the identity. The second space is the space of embedded discs with given midpoints. We note that

$$\text{Diff}(T_{m_i}M, 0) \simeq \text{GL}_n(\mathbb{R}) \text{ and } \text{Diff}(T_{m_i}M, \text{id}) \simeq *.$$

Hence, the space of embedded discs (with fixed midpoint) is homotopic to the space of tubular neighbourhoods, in other words the two spaces above are homotopic.

2.2 The topology of tubular configuration spaces

We will show that $\tilde{E}_k(M)$ can be given the structure of a fibre bundle over $\tilde{C}_k(M)$ with fibre $\text{Tub}(\mathbf{m})$ over \mathbf{m} . This will give $\tilde{E}_k(M)$ and $E_k(M)$ their natural topology.

Let M be a smooth, connected manifold without boundary. The tubular configuration space of k ordered particles in M is

$$\tilde{E}_k(M) := \{(\mathbf{m}, f) : \mathbf{m} \in \tilde{C}_k(M) \text{ and } f \in \text{Tub}(\mathbf{m})\}.$$

Consider the evaluation map

$$\tilde{p} : \tilde{E}_k(M) \rightarrow \tilde{C}_k(M)$$

mapping f to $(f_1(0), \dots, f_k(0))$, and the induced map of quotient spaces $p : E_k(M) \rightarrow C_k(M)$.

Proposition 2.2. $\tilde{p} : \tilde{E}_k(M) \rightarrow \tilde{C}_k(M)$ and $p : E_k(M) \rightarrow C_k(M)$ are fibre bundles.

Proof. Let $\mathbf{m} = (m_1, \dots, m_k)$ be a configuration in $\tilde{C}_k(M)$. We construct a local trivialisation for p around \mathbf{m} . Let $h : TM|_{\mathbf{m}} \rightarrow M$ be a tubular neighbourhood of \mathbf{m} and let DM and \mathring{DM} be the closed and open disk subbundles of TM for some fixed metric. Then the restriction of h to $DM|_{\mathbf{m}}$ is a closed tubular neighbourhood. Choose a continuous family of diffeomorphisms of $h(DM|_{\mathbf{m}})$ fixing the boundary

$$\tau : \Gamma(\mathring{DM}|_{\mathbf{m}}) \longrightarrow \text{Diff}(h(DM|_{\mathbf{m}}), \partial)$$

parameterised by the space of sections of $\mathring{DM}|_{\mathbf{m}}$ and such that for each section $s \in \Gamma(\mathring{DM}|_{\mathbf{m}})$, $\tau_s \circ h \circ s = \text{id}_{\mathbf{m}}$ where $\tau_s := \tau(s)$.

There is a homeomorphism

$$\begin{aligned} \phi : \Gamma(\mathring{DM}|_{\mathbf{m}}) &\longrightarrow \prod_i h(D_{m_i}M) =: V \subseteq \tilde{C}_k(M) \\ s &\longmapsto (h \circ s(m_1), \dots, h \circ s(m_k)) \end{aligned}$$

whose image is an open neighbourhood of \mathbf{m} in $\tilde{C}_k(M)$. Intuitively, h defines an open ball around each particle in \mathbf{m} and V is the set of all configurations with precisely one particle in each open ball. For each configuration $\mathbf{n} \in V$, $\sigma_{\mathbf{n}} := \tau \circ \phi^{-1}(\mathbf{n})$ is the diffeomorphism of M which moves the particles in \mathbf{n} onto the particles in \mathbf{m} .

Local trivialisations over V are then given by

$$\begin{aligned} \tilde{E}_k(M)|_V &\longmapsto V \times \text{Tub}(\mathbf{m}) \\ \text{Tub}(\mathbf{n}) \ni f &\longmapsto (\mathbf{n}, \sigma_{\mathbf{n}} \circ f \circ (d\sigma_{\mathbf{n}})^{-1}). \end{aligned}$$

Here $(d\sigma_{\mathbf{n}})^{-1}$ maps the normal bundle of \mathbf{m} to the normal bundle of \mathbf{n} , f maps this to a neighbourhood of \mathbf{n} and $\sigma_{\mathbf{n}}$ maps this neighbourhood to a neighbourhood of \mathbf{m} , thus the composition is indeed a tubular neighbourhood of \mathbf{m} .

As \tilde{p} is a Σ_k -equivariant map of free Σ_k -spaces, the induced map on quotients remains a fibre bundle. \square

For each $k \geq 0$ there is a covering map $q : \tilde{C}_k(M) \rightarrow C_k(M)$. The tubular configuration space of k ordered particles in M is the pullback $\tilde{E}_k(M) := q^*E_k(M)$.

The projection \tilde{p} has a right inverse $\tilde{\sigma}$. To define σ choose a Riemannian metric on M . For a configuration \mathbf{m} let ε_1 be the smallest distance in M between any two of its particles and let ε_2 be the greatest value such that for each i , $\exp_{m_i} : B_{\varepsilon_2}(T_{m_i}M, 0) \rightarrow B_{\varepsilon_2}(M, m_i)$ is a diffeomorphism. Let $\varepsilon_{\mathbf{m}} = \min\{\varepsilon_1, \varepsilon_2\}$, then $\mathbf{m} \mapsto \varepsilon_{\mathbf{m}}$ is a continuous map $\tilde{C}_k(M) \rightarrow \mathbb{R}_{>0}$. For each i define $f_i : T_{m_i}M \rightarrow M$ by $v \mapsto \exp_{m_i}\left(\frac{2\varepsilon_{\mathbf{m}}v}{\pi|v|} \arctan|v|\right)$, then $(m_1, \dots, m_k) \mapsto (f_1, \dots, f_k)$ is a section $\tilde{\sigma} : \tilde{C}_k(M) \rightarrow \tilde{E}_k(M)$. Moreover, this section is Σ_k -equivariant and descends to a section $\sigma : C_k(M) \rightarrow E_k(M)$.

Corollary 2.3. *The projections p and \tilde{p} are homotopy equivalences with homotopy inverses given by the global sections σ and $\tilde{\sigma}$.*

Proof. We have seen that the projections p and \tilde{p} give tubular configuration spaces the structure of fibre bundles. The fibre over any configuration \mathbf{m} is the space of tubular neighbourhoods $\text{Tub}(\mathbf{m})$. The space of tubular neighbourhoods of any compact submanifold is contractible, see for example [God08]. The contractions of each fibre determine homotopies $\sigma \circ p \sim \text{id}_{E_k(M)}$ and $\tilde{\sigma} \circ \tilde{p} \sim \text{id}_{\tilde{E}_k(M)}$. \square

2.3 Twisted labels and homotopy equivalences

It is common to add local data to configurations in the form of labels in a parameter space. In this paper we consider an extension of this notion which allows the parameter space to vary as the fibre of a fibre bundle over the underlying manifold. We say the configurations have *twisted labels*.

Let M be a smooth, connected manifold without boundary and let $\pi : Y \rightarrow M$ be a fibre bundle and a zero section $o : M \rightarrow Y$. Furthermore, assume that for each $m \in M$, the fibre Y_m over m is well-pointed with base-point $o(m)$. We define the configuration space of k ordered particles in M with twisted labels in π as

$$\tilde{C}_k(M; \pi) := \{(\mathbf{m}; \mathbf{x}) \in \tilde{C}(M) \times Y^k : \pi(x_k) = m_k\}.$$

It will be convenient to think of \mathbf{x} as a partially defined section.

Example 2.4. An important example is when Y is the trivial bundle $M \times X$ where X is well-pointed with base-point $*$ and the zero section is $o(m) = (m, *)$. In this case we write $\tilde{C}_k(M; X)$ for the configuration space with labels in X .

As in the unlabelled case, we define $C_k(M; \pi)$, the configuration space of k unordered particles with twisted labels in π , as the orbit space under the obvious Σ_k -action and $C_0(M; \pi) = *$. Now

let $M_0 \subset M$ be a compact submanifold, then the configuration space of particles in M modulo M_0 with twisted labels in π is

$$C(M, M_0; \pi) := \left(\prod_{k=0}^{\infty} C_k(M; \pi) \right) / \sim$$

where $(m_1, \dots, m_k; x_1, \dots, x_k) \sim (m_1, \dots, m_{k-1}; x_1, \dots, x_{k-1})$ if $m_k \in M_0$ or $x_k = o|_{m_k}$. Here x_i denotes the i th component of x , i.e. $x_i = x|_{m_i}$.

In order to define twisted coefficients for the tubular configurations we use the space of partially defined maps $P_o(M, Y)$ with open domains and the paro compact-open topology as introduced in [BB78], [AAB80] and which we will now recall.

Let X and Y be topological spaces. A partial map $f : X \rightarrow Y$ is a map $A \rightarrow Y$ for some subspace $A \subseteq X$. We call A the domain of f and denote it by $\mathcal{D}(f)$. A paro map or partial map with open domain is a partial map f such that $\mathcal{D}(f)$ is open in X . Let $P_o(X, Y)$ denote the set of paro maps $X \rightarrow Y$ with open domains. The paro mapping space is $P_o(X, Y)$ equipped with the paro compact-open topology which has sub-base the sets

$$(K, U) := \{f \in P_o(X, Y) : K \subseteq \mathcal{D}(f), f(K) \subseteq U\}$$

for all compact sets $K \subseteq X$ and open sets $U \subseteq Y$.

Remark 2.5. Given a fixed open set $A \in X$, the topology on the subset of paro maps $X \rightarrow Y$ which are defined precisely on A coincides with the ordinary compact-open topology on the mapping space $\text{Map}(A, Y)$.

We now define the tubular configuration space of k ordered particles in M with twisted labels in π as

$$\tilde{E}_k(M; \pi) := \{(f, s) \in \tilde{E}_k(M) \times P_o(M, Y) : \mathcal{D}(s) = \text{im}(f), \pi s = \text{id}_{\text{im}(f)}\}$$

and the unordered tubular configuration space as the orbit space $E_k(M; \pi)$ under the Σ_n -action. Thus labels in π for tubular configurations are sections of π defined over the tubular neighbourhoods.

The following propositions are proved using similar techniques as in the proof of Proposition 2.2 and Corollary 2.3.

Proposition 2.6. *The projection $q : E_k(M; \pi) \rightarrow E_k(M)$, $(f; s) \mapsto f$ is a fibre bundle with fibre $q^{-1}(f) = \Gamma(Y|_{\text{im}(f)})$, the space of sections of Y over the image of f . \square*

Proposition 2.7. *The projection $p : E_k(M; \pi) \rightarrow C_k(M; \pi)$, $(f; s) \mapsto (p(f); s|_{p(f)})$ is a fibre bundle with contractible fibres. \square*

Analogous to the ordinary configuration spaces, we define the tubular configurations space of particles in M modulo M_0 with twisted labels in Y to be

$$E(M, M_0; \pi) := \left(\prod_{k=0}^{\infty} E_k(M; \pi) \right) / \approx$$

where $(f_1, \dots, f_k; s_1, \dots, s_k) \approx (f_1, \dots, f_{k-1}; s_1, \dots, s_{k-1})$ if $f_k(0) \in M_0$ or $s_k = o|_{\text{im}(f_k)}$. We emphasise that we require the entire section s_k to agree with the zero section rather than just $s_k(m_k) = o(m_k)$. Our choice of definition here is motivated by the construction of the scanning map in the next section. Note that whereas a particle in a configuration vanishes if its label

agrees with the zero section, a component of a tubular configuration vanishes only if the entire section over the image of that component agrees with the zero section. A tubular configuration does not necessarily vanish when the underlying particle in the configuration vanishes. This fact complicates the proof of the next result.

Proposition 2.8. *There is a well defined weak homotopy equivalence $p : E(M, M_0; \pi) \rightarrow C(M, M_0; \pi)$ induced by the projections $p : E_k(M; \pi) \rightarrow C_k(M; \pi)$.*

Proof. Define $p : E(M, M_0; \pi) \rightarrow C(M, M_0; \pi)$ by $p[(f; s)] := [p(f; s)]$ for any representative of the class. This is well defined since if a component of a configuration in $E(M, M_0; \pi)$ vanishes, the underlying particle in $C(M, M_0; \pi)$ does. The number of particles in configurations and tubular configurations induce filtrations

$$\begin{aligned} * &= C^0(M, M_0; \pi) \subset C^1(M, M_0; \pi) \subset \dots \subset C(M, M_0; \pi), & C^n(M, M_0; \pi) &:= \left(\prod_{k=0}^n C_k(M; \pi) \right) / \sim \\ & & \text{and} \\ * &= E^0(M, M_0; \pi) \subset E^1(M, M_0; \pi) \subset \dots \subset E(M, M_0; \pi), & E^n(M, M_0; \pi) &:= \left(\prod_{k=0}^n E_k(M; \pi) \right) / \approx \end{aligned}$$

and p respects the filtrations. Topologically the configuration spaces $C(M, M_0; \pi)$ and $E(M, M_0; \pi)$ are the colimits of the filtration sequences. For each k define the subspaces of configurations in $C_k(M; \pi)$ and $E_k(M; \pi)$ in which at least one particle or component vanishes under the equivalence relation \sim or \approx respectively. More precisely, define

$$\begin{aligned} B_{C_k}(M, M_0; \pi) &:= \{(\mathbf{m}, \mathbf{x}) \in C_k(M; \pi) : m_k \in M_0 \text{ or } x_k = o|_{m_k}\} \\ &\text{and} \\ B_{E_k}(M, M_0; \pi) &:= \{(f, s) \in E_k(M; \pi) : f_k(0) \in M_0 \text{ or } s_k = o|_{\text{im}(f_k)}\}. \end{aligned}$$

For each $k \geq 1$ we have a commutative diagram

$$\begin{array}{ccccc} & & E_k(M; \pi) & \xrightarrow{\quad} & C_k(M; \pi) \\ & \nearrow & \downarrow & & \nearrow \\ B_{E_k}(M, M_0; \pi) & \xrightarrow{\quad} & B_{C_k}(M, M_0; \pi) & & \\ \downarrow & & \downarrow & & \downarrow \\ & \nearrow & E^k(M, M_0; \pi) & \xrightarrow{\quad} & C^k(M, M_0; \pi) \\ E^{k-1}(M, M_0; \pi) & \xrightarrow{\quad} & C^{k-1}(M, M_0; \pi) & & \end{array}$$

in which all the maps from left to right are induced by p , the vertical maps send configurations to their equivalence classes, and the maps from front to back are inclusions. The upper maps from front to back are cofibrations by our well-pointedness condition on the section o , and the left and right faces are homotopy pushout squares. Thus if p induces weak homotopy equivalences on the two upper maps and the front map, it also will induce a weak homotopy equivalence on the lower back map.

By the previous proposition the projections $p : E_k(M; \pi) \rightarrow C_k(M; \pi)$ are weak homotopy equivalences for each k . Note that $E^0(M, M_0; \pi) = * = C^0(M, M_0; \pi)$. Thus, if p induces weak homotopy equivalences $B_{E_k}(M, M_0; \pi) \rightarrow B_{C_k}(M, M_0; \pi)$ for each k we can proceed by induction on k to show that $E^k(M, M_0; \pi) \rightarrow C^k(M, M_0; \pi)$ is a weak homotopy equivalence and hence obtain the result.

To see that $B_{E_k}(M, M_0; \pi) \rightarrow B_{C_k}(M, M_0; \pi)$ is a homotopy equivalence, define subspaces

$$B_{C_k}^o := \{(\mathbf{m}; \mathbf{x}) : x_k = o|_{m_k}\}, B_{C_k}^{M_0} := \{(\mathbf{m}; \mathbf{x}) : m_k \in M_0\} \subset B_{C_k}(M, M_0; \pi)$$

and

$$B_{E_k}^o := \{(f; s) : s_k = o|_{\text{im}(f)}\}, B_{E_k}^{M_0} := \{(f; s) : f_k(0) \in M_0\} \subset B_{E_k}(M, M_0; \pi).$$

There is a commutative diagram

$$\begin{array}{ccccc} & & B_{E_k}^o & \xrightarrow{\quad} & B_{C_k}^o \\ & \nearrow & \downarrow & & \downarrow \\ B_{E_k}^o \cap B_{E_k}^{M_0} & \xrightarrow{\quad} & B_{C_k}^o \cap B_{C_k}^{M_0} & \xrightarrow{\quad} & B_{C_k}^o \\ \downarrow & & \downarrow & & \downarrow \\ B_{E_k}(M, M_0; \pi) & \xrightarrow{\quad} & B_{C_k}(M, M_0; \pi) & \xrightarrow{\quad} & B_{C_k}(M, M_0; \pi) \\ \downarrow & & \downarrow & & \downarrow \\ B_{E_k}^{M_0} & \xrightarrow{\quad} & B_{C_k}^{M_0} & \xrightarrow{\quad} & B_{C_k}^{M_0} \end{array}$$

where the maps from left to right are induced by p and the left and right faces are homotopy pushout squares. Note that $B_{E_k}^o \subset E_k(M; \pi)|_{B_{C_k}^o}$ are subbundles of $E_k(M; \pi) \rightarrow E_k(M)$ with fibres $\Gamma(X|_{\text{im}(f) \setminus \text{im}(f_k)})$ and $\{s \in \Gamma(X|_{\text{im}(f)}) : s_k|_{f_k(0)} = o|_{f_k(0)}\}$ respectively over f . The fibres are homotopy equivalent so $B_{E_k}^o \rightarrow B_{C_k}^o$ is a weak homotopy equivalence. The front maps from left to right are weak homotopy equivalences since $B_{E_k}^{M_0} = E_k(M; \pi)|_{B_{C_k}^{M_0}}$ and $B_{E_k}^o \cap B_{E_k}^{M_0} = B_{E_k}^o|_{B_{C_k}^{M_0}}$ as bundles over $C_k(M; \pi)$. So $B_{E_k}(M, M_0; \pi) \rightarrow B_{C_k}(M, M_0; \pi)$ is a weak homotopy equivalence. \square

Remark 2.9. If Y has the homotopy type of a CW-complex then the spaces $C_k(M; \pi)$ and $C(M, M_0; \pi)$, and their filtration subspaces and quotients have the homotopy types of CW-complexes [McD75] and hence the same holds for tubular configuration spaces. The map in the previous proposition is thus in fact a homotopy equivalence.

3 The scanning map

We define the scanning map for tubular configuration spaces, show that it induces a homotopy equivalence under the usual assumptions and that it is equivariant with respect to the group of those diffeomorphisms of M which can be lifted to an isomorphism of the bundle $\pi : Y \rightarrow M$.

3.1 Definition and homotopy equivalence

Let $M_0 \subset M$ and (Y, π, o) be as defined in the previous section, and let $n = \dim(M)$. Denote by $\dot{T}M$ the fibrewise one point compactification of the tangent bundle of M and let $\dot{T}_z M$ denote the fibre of $\dot{T}M$ over a point $z \in M$. Let $\dot{\tau}_\pi := \dot{T}M \wedge_M Y$ be the fibrewise smash product of $\dot{T}M$ with the bundle Y , where the base-point in each fibre of Y is determined by the zero-section o . Then, if X denotes a typical fibre of π , $\dot{\tau}_\pi$ is a $\Sigma^n X$ -bundle over M with a canonical section $* : M \rightarrow \dot{\tau}_\pi$. Let $\Gamma_c(M, M_0; \pi)$ denote the space of sections of $\dot{\tau}_\pi$ which are defined on $M \setminus M_0$ and which agree with $*$ outside a compact set of M . In this section we define a scanning map $\gamma : E(M, M_0; \pi) \rightarrow \Gamma_c(M, M_0; \pi)$.

We begin by constructing a sequence of maps $\gamma_k : E_k(M; \pi) \rightarrow \Gamma_c(M, M_0; \pi)$. Intuitively, our scanning map will send a tubular configuration $(f; s) \in E_k(M; \pi)$ to a section defined by s on

the Y component and by a modification of f^{-1} over M on the $\dot{T}M$ component, sending points outside of $\text{im}(f)$ to the compactification points in the appropriate fibres.

Lemma 3.1. *[Lee03] Let V be a finite dimensional vector space. Given a vector $v \in V$, there is a canonical linear isomorphism $V \rightarrow T_v V$. Moreover, for any finite dimensional vector space U and any linear map $L : V \rightarrow U$ the following diagram commutes.*

$$\begin{array}{ccc} V & \xrightarrow{\cong} & T_v V \\ L \downarrow & & \downarrow d_v L \\ W & \xrightarrow{\cong} & T_{Lv} W \end{array}$$

□

For any $(f, s) \in E_k(M; \pi)$ and $z \in \text{im}(f)$, $f^{-1}(z) \in T_{m_i} M$ for some i . For each z and each f let $\phi_{f,z} : T_{m_i} M \rightarrow T_{f^{-1}(z)}(T_{m_i} M)$ be the canonical isomorphism. $\phi_{f,z}$ varies continuously in f and z . Let $(f, s) \in E_k(M; \pi)$. Having removed any components (f_i, s_i) with $m_i \in M_0$ if necessary, we define a section of $\dot{\tau}_\pi$ by

$$z \mapsto \begin{cases} (d_{f^{-1}(z)} f \circ \phi_{f,z} \circ f^{-1}(z), s(z)) & \text{if } z \in \text{im}(f) \cap (M \setminus M_0) \\ *_z & \text{otherwise.} \end{cases}$$

Although the above construction gives a well-defined map, it is not continuous. The problem is that the contribution of f_i in the section of $\dot{\tau}_\pi$ suddenly vanishes as m_i reaches M_0 . We introduce the following modification of the above map for components (f_i, s_i) with m_i in a collar neighbourhood of ∂M_0 in M which makes sure that for a fixed $z \in \text{im}(f_i)$ its image under the section goes to the point at infinity in $\dot{T}_z M$ as m_i gets close to the boundary of M_0 in M . To this end we fix a collar and choose a metric on M such that the chosen collar has width 1. Let $\delta(m_i)$ be the distance of m_i to M_0 . We now multiply the above formula by $\exp(\frac{1}{\delta(m_i)})$ when $z \in \text{im}(f_i)$ and m_i is in the collar. Note that we only use the metric on the collar, and indeed only the induced distance function to M_0 . The resulting map γ is now continuous.

Remark 3.2. The composition

$$C(M, M_0; \pi) \xrightarrow{\sigma} E(M, M_0; \pi) \xrightarrow{\gamma} \Gamma_c(M, M_0; \pi)$$

is homotopic to the scanning map of McDuff [McD75] when $\pi : Y \rightarrow M$ is a trivial X -bundle. If we compose with a reflection $(t_1, \dots, t_n; x) \mapsto (-t_1, \dots, -t_{n-1}, -t_n; x)$ in each fibre of $\dot{\tau}_\pi$ it is homotopic to the scanning map of Bökigheimer and Madsen [BM88] when π is trivial and the extension due to Hesselholt [Hes92] for arbitrary π .

Theorem 3.3. *The scanning map $\gamma : E(M, M_0; \pi) \rightarrow \Gamma_c(M, M_0; \pi)$ is a weak homotopy equivalence if the pair (M, M_0) or the typical fibre X of π is 0-connected. Moreover, if X has the homotopy type of a CW-complex then it is a homotopy equivalence.*

Proof. By Remark 3.2 we have a homotopy commutative diagram

$$\begin{array}{ccc} E(M, M_0; \pi) & \xrightarrow{\gamma} & \Gamma_c(M, M_0; \pi) \\ \simeq \uparrow & \nearrow \simeq & \\ C(M, M_0; \pi) & & \end{array}$$

factoring the ordinary scanning map, which is a weak homotopy equivalence; see [Hes92] for the case with twisted coefficients. By proposition 2.8, the left hand map is a weak homotopy equivalence. Hence, γ is one too. If Y has the homotopy type of CW-complex so do all the three spaces in the diagram and hence the weak homotopy equivalences are homotopy equivalences. \square

3.2 Equivariance

Let $\text{Diff}(M)$ be the group of diffeomorphisms of M . We define actions on configuration spaces as follows. Let $\phi \in \text{Diff}(M)$, then $\phi \cdot (m_1, \dots, m_k) := (\phi(m_1), \dots, \phi(m_k))$ for $(m_1, \dots, m_k) \in \tilde{C}_k(M)$ or $C_k(M)$ and $\phi \cdot f := \phi \circ f \circ d\phi^{-1}$ for $f \in \tilde{E}_k(M)$ or $E_k(M)$. With these actions the projections p and \tilde{p} are $\text{Diff}(M)$ -equivariant non-equivariant homotopy equivalences.

This action of the diffeomorphism group extends to configuration spaces with labels when $Y = M \times X$ is a trivial bundle. However, in general diffeomorphisms of M do not lift to automorphisms of the labelling bundle Y . Instead we need to consider the automorphism group of Y and the induced diffeomorphisms on the base space M .

Let $\pi_1 : Y_1 \rightarrow M$ and $\pi_2 : Y_2 \rightarrow M$ be fibre bundles over M . We consider bundle isomorphisms $\alpha : Y_1 \rightarrow Y_2$ that induce diffeomorphisms α_M on the base and denote isomorphic bundles by $Y_1 \cong Y_2$. If the isomorphism induces the identity on the base we call it an equivalence and denote equivalent bundles by $Y_1 \equiv Y_2$.

Lemma 3.4. *Let $\text{Aut}(\pi)$ denote the group of automorphisms of $\pi : Y \rightarrow M$ that restrict to diffeomorphisms of M . Then the homomorphism $\text{Aut}(\pi) \rightarrow \text{Diff}(M)$, $\alpha \mapsto \alpha_M$ has image*

$$\text{Diff}(M; \pi) := \{\beta \in \text{Diff}(M) : (\beta_M)^*Y \equiv Y\}.$$

Proof. Let $\beta \in \text{Diff}(M; \pi)$ and $\tilde{\beta} : \beta^*Y \rightarrow Y$ be the map completing the pullback square. Let $f : Y \rightarrow \beta^*Y$ be a bundle equivalence, then $\tilde{\beta} \circ f$ is an automorphism of Y whose image under ρ is β . Conversely let $\alpha \in \text{Aut}(\pi)$, and let $\tilde{\alpha}_M : (\alpha_M)^*Y \rightarrow Y$ be the map completing the pullback square, then $\alpha^{-1} \circ \tilde{\alpha}_M : (\alpha_M)^*Y \rightarrow Y$ is an equivalence. \square

Remark 3.5. In general, $\text{Diff}(M; \pi) \subsetneq \text{Diff}(M)$. For example, consider the Hopf bundle over S^2 and the antipodal map on the base. The Chern class of the Hopf bundle is $+1$ but the Chern class of the pullback is -1 so they cannot be equivalent. However, when Y is trivial or some other natural bundle such as a tangent bundle, $\text{Diff}(M; \pi) = \text{Diff}(M)$ and there is a lift $\text{Diff}(M) \rightarrow \text{Aut}(\pi)$. For any bundle, $\text{Diff}(M; \pi)$ contains $\text{Diff}_0(M)$, the connected component of the identity, since homotopic maps induce equivalent pullbacks.

Let $\text{Aut}(\pi|_{M \setminus \text{int}(M_0)})$ denote the group of bundle automorphisms of π restricted to M without the interior of M_0 . Let $\text{Aut}^o(\pi|_{M \setminus \text{int}(M_0)})$ denote its subgroup of bundle automorphisms that preserve the zero section. We will assume that the diffeomorphisms restrict on an ϵ -collar of ∂M_0 in M to a product of a diffeomorphism on the boundary and the identity on $[0, \epsilon]$. Let $\alpha \in \text{Aut}^o(\pi|_{M \setminus \text{int}(M_0)})$, then we define actions as follows:

- $\alpha \cdot (\mathbf{m}; \mathbf{x}) := (\alpha_M(\mathbf{m}); \alpha \circ \mathbf{x} \circ \alpha_M^{-1})$ for $(\mathbf{m}; \mathbf{x}) \in C_k(M; \pi)$ and $\mathbf{m} \in M \setminus M_0$
- $\alpha \cdot (f; s) := (\alpha_M \circ f \circ d\alpha_M^{-1}; \alpha \circ s \circ \alpha_M^{-1})$ for $(f; s) \in E_k(M; \pi)$ and $\mathbf{m} \in M \setminus M_0$
- $\alpha \cdot \sigma := (d\alpha_M \wedge_W \alpha) \circ \sigma \circ \alpha_M^{-1}$ for $\sigma \in \Gamma_c(M, M_0; \pi)$.

The action on an equivalence class in $C(M, M_0; \pi)$ or $E(M, M_0; \pi)$ is given by the equivalence class of the image of any representative under the action. This is well-defined as α preserves the zero section.

Proposition 3.6. *The projection $p : E(M, M_0; \pi) \rightarrow C(M, M_0; \pi)$ is $\text{Aut}^o(\pi|_{M \setminus \text{int}(M_0)})$ -equivariant.*

Proof. It follows immediately from the definitions that the projections $E_k(M; \pi) \rightarrow C_k(M; \pi)$ restricted to such configurations with $\mathbf{m} \in M \setminus M_0$ are equivariant. We obtain the result by noting that the actions are well behaved with respect to equivalence classes. \square

Remark 3.7. Although the projection is equivariant, the homotopy inverse $\sigma : C(M, M_0; \pi) \rightarrow E(M, M_0; \pi)$, which defined by the exponential map and open ε -disks around each particle, depends on choosing a metric and is not preserved under diffeomorphisms.

Theorem 3.8. *The scanning map $\gamma : E(M, M_0; \pi) \rightarrow \Gamma_c(M, M_0; \pi)$ is $\text{Aut}^o(\pi|_{M \setminus \text{int}(M_0)})$ -equivariant.*

Proof. Consider the scanning map $\gamma : E_k(M; \pi) \rightarrow \Gamma_c(M, M_0; \pi)$ for some $k \geq 0$. Choose a point $z \in M \setminus M_0$, an automorphism $\alpha \in \text{Aut}^o(\pi|_{M \setminus \text{int}(M_0)})$ and a tubular configuration $(f; s)$ with underlying configuration $(\mathbf{m}; \mathbf{x})$. We want to show

$$(\alpha \cdot \gamma_k(f; s))(z) = \gamma_k(\alpha \cdot (f; s))(z).$$

Suppose $\alpha_M^{-1}(z) \in \text{im}(f_i)$ for some i . By the definition of the action on $\Gamma_c(M, M_0; \pi)$ we have

$$(\alpha \cdot \gamma_k(f; s))(z) = \left(d_{\alpha_M^{-1}(z)} \alpha_M \circ d_{f_i^{-1} \circ \alpha_M^{-1}(z)} f_i \circ \phi_{f, \alpha_M^{-1}(z)} \circ f_i^{-1} \circ \alpha_M^{-1}(z), \alpha \circ s \circ \alpha_M^{-1}(z) \right).$$

We calculate $\gamma_k(\alpha \cdot (f; s))(z)$ componentwise:

- (i) $(\alpha \cdot f_i)^{-1} = d_{m_i} \alpha_M \circ f_i^{-1} \circ \alpha_M^{-1}$
- (ii) by Lemma 3.1 there is a commutative diagram

$$\begin{array}{ccc} T_{m_i} M & \xrightarrow{\phi_{f, \alpha_M^{-1}(z)}} & T_{f_i^{-1} \circ \alpha_M^{-1}(z)} T_{m_i} M \\ d_{m_i} \alpha_M \downarrow & & \downarrow d_{f_i^{-1} \circ \alpha_M^{-1}(z)} d_{m_i} \alpha_M \\ T_{\alpha_M(m_i)} M & \xrightarrow{\phi_{\alpha \cdot f, z}} & T_{(\alpha \cdot f_i)^{-1}(z)} T_{\alpha_M(m_i)} M \end{array}$$

so we have $\phi_{\alpha \cdot f, z} = d_{f_i^{-1} \circ \alpha_M^{-1}(z)} d_{m_i} \alpha_M \circ \phi_{f, \alpha_M^{-1}(z)} \circ (d_{m_i} \alpha_M)^{-1}$

$$(iii) \quad d_{(\alpha \cdot f_i)^{-1}(z)} (\alpha \cdot f_i) = d_{\alpha_M^{-1}(z)} \alpha_M \circ d_{f_i^{-1} \circ \alpha_M^{-1}(z)} f_i \circ \left(d_{f_i^{-1} \circ \alpha_M^{-1}(z)} d_{m_i} \alpha_M \right)^{-1}.$$

Putting these together we have

$$\gamma_k(\alpha \cdot (f; s))(z) = \left(d_{\alpha_M^{-1}(z)} \alpha_M \circ d_{f_i^{-1} \circ \alpha_M^{-1}(z)} f_i \circ \phi_{f, \alpha_M^{-1}(z)} \circ f_i^{-1} \circ \alpha_M^{-1}(z), \alpha \circ s \circ \alpha_M^{-1}(z) \right).$$

In our computations above we have implicitly assumed that none of the factors of $(f; s)$ have midpoint in the collar of ∂M_0 in M . We leave it to the reader to check that the multiplicative factor $\exp(\frac{1}{\delta(m_i)})$ does not interfere with the equivariance argument as our diffeomorphisms factor as products on the collar by assumption and the distance of a point in the collar to M_0 is preserved. \square

Combining Theorems 3.8 and 3.3 gives the following corollary.

Corollary 3.9. *Let (M, M_0) or a typical fibre X of π be 0-connected. If a topological group G acts via a homomorphism $G \rightarrow \text{Aut}^o(\pi|_{M \setminus \text{int}(M_0)})$, then there are weak homotopy equivalences of the reduced Borel constructions*

$$EG \ltimes_G C(M, M_0; \pi) \simeq EG \ltimes_G E(M, M_0; \pi) \simeq EG \ltimes_G \Gamma_c(M, M_0; \pi)$$

which are homotopy equivalences if Y has the homotopy type of a CW-complex.

Remark 3.10. Let $Y = M \times X$ be a trivial bundle and let G be a compact Lie group acting on M through isometries and on X fixing the base-point. Then the ordinary scanning map

$$\gamma \circ \sigma : C(M, M_0; X) \longrightarrow \Gamma(M, M_0; X)$$

is G -equivariant. Compare [BM88].

4 Stable splittings

In this section we construct Snaith type splittings via diffeomorphism equivariant maps. As a result we generalise and strengthen the results of Bødigheimer and Madsen [BM88]. As an application we also derive an equivariant stable split injection for configuration spaces and diffeomorphisms of manifolds with marked points.

Constructions and assumptions: Let $M_0 \subset M$ and $\pi : Y \rightarrow M$ be as before. Recall M is connected without boundary and M_0 is closed as a subset of M . For each $k \geq 1$ let $C^k = C^k(M, M_0; \pi)$ be the filtrations of $C = C(M, M_0; \pi)$. For each $k \geq 1$ let $D^k = D^k(M, M_0; \pi) := C^k / C^{k-1}$ be the filtration quotient and let $V := \bigvee_{i \geq 1} D^i$ be the wedge sum with k th filtration $V_k := \bigvee_{i=1}^k D^i$.

Throughout subsections 4.1, 4.2, 4.3 and 4.4 we will assume that (M, M_0) or the typical fibre X of π is 0-connected. Then also C, C^k, D^k, V, V_k are all path connected as well. We also note that if X has the homotopy type of a CW-complex then so do all these spaces.

Throughout this section we let G be a topological group which acts via a homomorphism $G \rightarrow \text{Aut}^o(\pi|_{M \setminus \text{int}(M_0)})$. We will suppress this homomorphism in our notation. Note that the G action on C induces an action on C^k, D^k and hence on V_k .

4.1 Equivariant splittings in homology

Before we construct equivariant stable splittings for mapping spaces we prove the following weaker statement in homology in virtue of the simplicity of its proof.

Theorem 4.1. *There exist a $\pi_0 G$ -equivariant isomorphism*

$$\tilde{H}_*(C(M, M_0; \pi)) \xrightarrow{\cong} \bigoplus_{k \geq 1} \tilde{H}_*(D^k).$$

Proof. Given a pointed space A we denote the infinite symmetric product of A by $\text{SP}(A)$. Elements of $\text{SP}(A)$ are finite formal sums $\sum k_i a_i$ where $a_i \in A$ and $k_i \in \mathbb{N}$ for each i with the one relation that makes the base point the zero of the monoid, see [DT58]. Let ξ_α be

a configuration in C and write it as a formal sum $\xi_\alpha = \sum_{i \in \alpha} m_i x_i$ where α is a finite set, $m_i \in M$ and $x_i \in \pi^{-1}(m_i)$. Given a subset $\beta \subseteq \alpha$ of size k there is a subconfiguration $\xi_\beta := \sum_{i \in \beta} m_i x_i \in C^k$. Let $\tilde{\xi}_\beta$ denote its image under the composition $C^k \rightarrow D^k \hookrightarrow V_k$. Define a power set map

$$\begin{aligned} \sigma : C &\longrightarrow \mathrm{SP}(V) \\ \xi_\alpha &\longmapsto \sum_{\beta \subseteq \alpha} \tilde{\xi}_\beta \end{aligned}$$

and extend it to a map $\sigma : \mathrm{SP}(C) \rightarrow \mathrm{SP}(V)$ by $\sum k_\alpha \xi_\alpha \mapsto \sum k_\alpha \sigma(\xi_\alpha)$. This is a pointed G -map that respects the filtrations and restricts to a pointed G -map $\mathrm{SP}(C^k) \rightarrow \mathrm{SP}(V_k)$.

The infinite symmetric product functor sends cofibrations to quasifibrations so for each $k \geq 1$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{SP}(C^{k-1}) & \xrightarrow{\sigma} & \mathrm{SP}(V_{k-1}) \\ \downarrow & & \downarrow \\ \mathrm{SP}(C^k) & \xrightarrow{\sigma} & \mathrm{SP}(V_k) \\ \downarrow & & \downarrow \\ \mathrm{SP}(C^k/C^{k-1}) & \xrightarrow{\sigma=\mathrm{id}} & \mathrm{SP}(V_k/V_{k-1}) \end{array}$$

in which the vertical sequences are quasifibrations and the lower horizontal map is the identity $\mathrm{SP}(D^k) \rightarrow \mathrm{SP}(D^k)$. Starting with $C^1 = V_1$ we proceed by induction on k to see that σ induces isomorphisms

$$\pi_* \mathrm{SP}(C(M, M_0; \pi)) \xrightarrow{\cong} \pi_* \mathrm{SP}\left(\bigvee_{k \geq 1} D^k\right).$$

By the Dold-Thom theorem [DT58] this is precisely the homology isomorphism we require. To finish the proof note that the connected component of the identity $G_0 \subset G$ acts trivially on homology so the isomorphism is equivariant under the actions of $\pi_0 G = G/G_0$. \square

Combining this with the results in section 3 we have proved the following corollary.

Corollary 4.2. *There exists a $\pi_0 G$ -equivariant isomorphism*

$$\tilde{H}_*(\Gamma_c(M, M_0; \pi)) \xrightarrow{\cong} \bigoplus_{k \geq 1} \tilde{H}_*(D^k). \quad \square$$

The above proof serves as a model for the proofs of the stronger theorems in the following sections. The main difficulty in each case is to construct a suitable G -equivariant power set map.

4.2 Stable splittings of reduced Borel constructions

Bödigheimer and Madsen's power set map for Borel constructions in [BM88] relies crucially on the fact that the groups they consider are compact of Lie type. In particular they use that one can embed the configuration space $C(M)$ G -equivariantly into a finite dimensional G -representation. Instead we use here suitably chosen models for EG which allow us to define the

power map on Borel constructions for any G acting on M smoothly and on π . This provides a geometric model of the power set map. We need some extra assumptions for the next theorem. We will assume that M is the interior of a compact manifold \bar{M} and π is the restriction of a bundle over \bar{M} . Furthermore, we will assume that G is a subgroup of bundle maps which extend to the bundle on \bar{M} .

Theorem 4.3. *There is a weak homotopy equivalence of suspension spectra*

$$\Sigma^\infty(EG \ltimes_G C(M, M_0; \pi)) \longrightarrow \Sigma^\infty \bigvee_{k \geq 1} EG \ltimes_G D^k.$$

Proof. As a consequence of Whitney's embedding theorem, the embedding space $\text{Emb}(\bar{M}; \mathbb{R}^\infty)$ is contractible. It admits a free $\text{Diff}(\bar{M})$ -action. Choose a model for EG and replace it with $EG \times \text{Emb}(\bar{M}; \mathbb{R}^\infty)$ equipped with the diagonal action where $g \cdot h := h \circ (g^{-1})$ for $g \in G$ and an embedding $h : \bar{M} \rightarrow \mathbb{R}^\infty$. This is again a model for EG and we denote elements of this EG by pairs (u, h) .

Given an embedding $h : \bar{M} \rightarrow \mathbb{R}^\infty$ we define an associated embedding $\hat{h} : \coprod C_k(M) \rightarrow \mathbb{R}^\infty$ as follows. For each $k \geq 1$ and any N , $C_k(\mathbb{R}^N)$ is a finite dimensional, smooth manifold. Thus there is an embedding $C_k(\mathbb{R}^N) \hookrightarrow \mathbb{R}^{L_N}$ for some L_N . Considering $C_k(\mathbb{R}^N)$ as a submanifold of $C_k(\mathbb{R}^{N+1})$ we can extend this embedding to an embedding of $C_k(\mathbb{R}^{N+1}) \hookrightarrow \mathbb{R}^{L_{N+1}}$ for some $L_{N+1} \geq L_N$. Proceeding like this, in the limit we will have constructed an embedding

$$\mu_k : C_k(\mathbb{R}^\infty) \hookrightarrow \mathbb{R}^\infty.$$

Choosing an injection $\coprod_{k \geq 1} \mathbb{R}^\infty \hookrightarrow \mathbb{R}^\infty$ we can construct from μ_k an injection

$$\mu : \coprod_{k \geq 1} C_k(\mathbb{R}^\infty) \hookrightarrow \mathbb{R}^\infty.$$

Given now any embedding $h : \bar{M} \hookrightarrow \mathbb{R}^\infty$, the embedding \hat{h} is defined as the composition

$$\hat{h} : \coprod_k C_k(M) \xrightarrow{C_k h} \coprod_k C_k(\mathbb{R}^\infty) \xrightarrow{\mu} \mathbb{R}^\infty.$$

The action of G on M cannot be extended to an action on \mathbb{R}^∞ , and in particular \hat{h} cannot be made G -equivariant. Nevertheless, the construction is G -equivariant in the sense that

$$g \cdot \hat{h} = \hat{h} \circ g^{-1} = \widehat{h \circ g^{-1}} = \widehat{g \cdot h}.$$

We can now define the power-set map in this setting. Choose a configuration $\xi_\alpha = \sum_{i \in \alpha} m_i x_i \in C$ and define subconfigurations $\xi_\beta \in C^k$ and $\bar{\xi}_\beta \in V$ as in the previous section. Let $m_\beta := \sum_{i \in \beta} m_i \in C_k(M)$ be the associated unlabelled subconfiguration and define

$$\begin{aligned} \sigma : EG \ltimes_G C(M, M_0; \pi) &\longrightarrow C(\mathbb{R}^\infty; EG \ltimes_G V) \\ (u, h, \xi_\alpha) &\longmapsto \sum_{\beta \subseteq \alpha} \hat{h}(m_\beta)(u, h, \bar{\xi}_\beta) \end{aligned}$$

where the target is the configuration space of unordered particles in \mathbb{R}^∞ with labels in the trivial $EG \ltimes_G V$ -bundle. The formula for σ gives a well defined map $EG \times C \rightarrow C(\mathbb{R}^\infty; EG \times V)$. To see that it is well defined on the half smash product note that the base-point in C is the empty configuration ξ_\emptyset which has no subconfigurations so σ maps (u, h, ξ_\emptyset) to the empty

configuration in the target for any $(u, h) \in EG$. To check that σ is G -equivariant recall that $g \cdot \xi_\alpha = \sum g(m_i)(g \circ x_i \circ g^{-1})$. The G -action on the target is trivial on \mathbb{R}^∞ and given by the diagonal action on the labels. Hence,

$$\begin{aligned}\sigma(g \cdot (u, h, \xi_\alpha)) &= \sum_{\beta \subseteq \alpha} \widehat{h \circ g^{-1}} \left(\sum_{i \in \beta} g(m_i) \right) (g \cdot u, h \circ g^{-1}, \overline{g \cdot \xi_\beta}) \\ g \cdot \sigma(u, h, \xi_\alpha) &= \sum_{\beta \subseteq \alpha} \widehat{h}(m_\beta) (g \cdot u, h \circ g^{-1}, g \cdot \bar{\xi}_\beta)\end{aligned}$$

By definition $g \cdot \bar{\xi}_\beta = \overline{g \cdot \xi_\beta}$ and

$$\widehat{h \circ g^{-1}} \left(\sum_{i \in \beta} g(m_i) \right) = \mu \left(\sum_{i \in \beta} h \circ g^{-1} \circ g(m_i) \right) = \mu \left(\sum_{i \in \beta} h(m_i) \right) = \widehat{h}(m_\beta).$$

So σ is well defined.

Composing the power set map with the scanning map yields the map

$$EG \ltimes_G C \xrightarrow{\sigma} C(\mathbb{R}^\infty; EG \ltimes_G V) \longrightarrow \Omega^\infty \Sigma^\infty(EG \ltimes_G V)$$

which respects the filtrations of C and V . The cofibration sequences of the filtrations of C and V induce fibration sequences of suspension spectra which fit into commutative diagrams

$$\begin{array}{ccc}\Sigma^\infty(EG \ltimes_G C^{k-1}) & \xrightarrow{p} & \Sigma^\infty(EG \ltimes V_{k-1}) \\ \downarrow & & \downarrow \\ \Sigma^\infty(EG \ltimes_G C^k) & \xrightarrow{p} & \Sigma^\infty(EG \ltimes V_k) \\ \downarrow & & \downarrow \\ \Sigma^\infty(EG \ltimes_G C^k / C^{k-1}) & \xrightarrow{p \simeq \text{id}} & \Sigma^\infty(EG \ltimes V_k / V_{k-1})\end{array}$$

for each $k \geq 1$ and where p is the adjoint of the composition above. The lower horizontal map is homotopic to the identity since it is the adjoint of the inclusion $EG \ltimes_G D^k \hookrightarrow \Omega^\infty \Sigma^\infty(EG \ltimes_G D^k)$. Starting with $C^1 = V_1$ we proceed by induction on k to obtain a weak homotopy equivalence of suspension spectra and the result follows from the observation that $EG \ltimes_G \bigvee_{k \geq 1} D^k = \bigvee_{k \geq 1} EG \ltimes_G D^k$. \square

4.3 Equivariant stable splittings

The purpose of this section is to construct a G -equivariant power-set map which induces stable G -equivariant splittings for configuration spaces. It will be completely natural and avoid having to choose an embedding μ as in the proof of Theorem 4.3. It also gives a stronger result.

While the stable splitting in the previous section made use of the configuration space $C(\mathbb{R}^\infty; A)$ as a model for the free infinite loop space $Q(A) := \varinjlim \Omega^n \Sigma^n A$, here we will make use of the Γ^+ -construction of Barratt and Eccles. The key idea is to replace the Σ_k -orbits of $\tilde{C}_k(M; \pi)$ by the homotopy Σ_k -orbits. Recall the following constructions from [BE74a].

For a discrete group H , let $W.H$ denote its homogeneous bar construction and let $EH := |W.H|$ be its geometric realisation. Group homomorphisms $H \rightarrow K$ induce continuous maps $EH \rightarrow$

EK , and in particular the inclusion $\Sigma_{k-1} \hookrightarrow \Sigma_k$ induces an inclusion map $E\Sigma_{k-1} \hookrightarrow E\Sigma_k$. Let α be any strictly monotone map from $\{1, \dots, m\}$ to $\{1, \dots, k\}$ and $\rho \in \Sigma_k$. We denote by $\rho_*(\alpha)$ the strictly monotone map from $\{1, \dots, m\}$ onto the image of $\rho\alpha$. Define the restriction map $\alpha^* : \Sigma_k \rightarrow \Sigma_m$ so that $\alpha^*(\rho)$ is the unique map such that $\rho_*(\alpha)\alpha^*(\rho) = \rho\alpha$. α^* is not a homomorphism but satisfies $\alpha^*(\rho\tau) = \alpha^*(\rho)\tau$ for all $\tau \in \Sigma_m$. Given a well-pointed space A , the Γ^+ construction on A is defined as

$$\Gamma^+(A) := \left(\prod_{k \geq 0} E\Sigma_k \times_{\Sigma_k} A^k \right) / \sim$$

where for any strictly monotone map α we have $(w; a_1, \dots, a_k) \sim (\alpha^*(w); a_{\alpha(1)}, \dots, a_{\alpha(m)})$ if $a_i = *$ for all i not in the image of α . Writing $R = \text{incl}^*$ for the restriction map of the standard inclusion incl of $\{1, \dots, k-1\}$ to $\{1, \dots, k\}$, this equivalence relation is generated by the relations $(w; a_1, \dots, a_k) \sim (R(w); a_1, \dots, a_{k-1})$ if $a_k = *$. Γ^+ is an endofunctor on the category of well-pointed topological spaces. For any space A there is an inclusion map $\iota_A : A \hookrightarrow \Gamma^+(A)$ identifying A with $\Sigma_1 \times_{\Sigma_1} A \subset \Gamma^+(A)$ and a structure map $h_A : \Gamma^+\Gamma^+(A) \rightarrow \Gamma^+(A)$ making the triple (Γ^+, ι, h) into a monad. The spaces $\Gamma^+(A)$ are the free Γ^+ -algebras.

We define the Borel configuration space of k unordered particles in M with twisted labels in π as

$$\mathcal{C}_k(M; \pi) := E\Sigma_k \times_{\Sigma_k} \tilde{\mathcal{C}}_k(M; \pi).$$

The Borel configuration space with particles vanishing in M_0 is then

$$\mathcal{C}(M, M_0; \pi) := \left(\prod_{k \geq 0} \mathcal{C}_k(M; \pi) \right) / \sim$$

where $(w; m_1, \dots, m_k; x_1, \dots, x_k) \sim (R(w); m_1, \dots, m_{k-1}; x_1, \dots, x_{k-1})$ if $m_k \in M_0$ or $x_k = o|_{m_k}$. The Borel configuration spaces admit filtrations $\mathcal{C}^k := \prod_{i=0}^k \mathcal{C}_i / \sim$ analogous to the ordinary configuration spaces. We make use of the filtration quotients $\mathcal{D}^k := \mathcal{C}^k / \mathcal{C}^{k-1}$. Consider the sequence of spaces $\tilde{D}^k = \tilde{\mathcal{C}}^k(M, M_0; \pi) / \equiv$ where $(m_1, \dots, m_k; x_1, \dots, x_k) \equiv *$ if $m_i \in M_0$ or $x_i = o|_{m_i}$ for some i . These spaces are the filtration quotients of the ordered configuration space with vanishing particles. The filtration quotients of the Borel configuration spaces can then be identified as $\mathcal{D}^k = E\Sigma_k \times_{\Sigma_k} \tilde{D}^k$.

Proposition 4.4. *The projection $P : \mathcal{C}(M, M_0; \pi) \rightarrow C(M, M_0; \pi)$ and the restriction to the filtration quotients $P : \mathcal{D}^k \rightarrow D^k$ are weak homotopy equivalences.*

Proof. The projection $E\Sigma_k \times \tilde{\mathcal{C}}_k(M; \pi) \rightarrow \tilde{\mathcal{C}}_k(M; \pi)$ is a Σ_k -equivariant homotopy equivalence. As Σ_k acts freely on the source and target, it induces a homotopy equivalence on orbit spaces $\mathcal{C}_k(M; \pi) \xrightarrow{\sim} C_k(M; \pi)$. Together they give a well defined map $P : \mathcal{C}(M, M_0; \pi) \rightarrow C(M, M_0; \pi)$ of filtered spaces.

Let $\tilde{B}_{C_k}(M, M_0; \pi) \subset \tilde{\mathcal{C}}_k(M; \pi)$ be the subspace of ordered configurations such that for some i either $m_i \in M_0$ or $x_i = o|_{m_i}$. Recall from the proof of Proposition 2.8 that the filtration $C^k(M, M_0; \pi)$ is obtained as the pushout of the diagram $C^{k-1}(M, M_0; \pi) \leftarrow B_{C_k}(M, M_0; \pi) \rightarrow C_k(M; \pi)$ where $B_{C_k}(M, M_0; \pi) = \tilde{B}_{C_k}(M, M_0; \pi) / \Sigma_k$. Similarly the filtrations of the Borel configuration spaces are obtained from the previous filtration as the pushout along a subspace $B_{\mathcal{C}_k}(M, M_0; \pi) := E\Sigma_k \times_{\Sigma_k} \tilde{B}_{C_k}(M, M_0; \pi)$. The projection induces a homotopy equivalence $B_{\mathcal{C}_k}(M, M_0; \pi) \xrightarrow{\sim} B_{C_k}(M, M_0; \pi)$ and a map of homotopy pushout squares. The result then follows by induction as in the proof of Proposition 2.8. The result for the filtration quotients is proved similarly. \square

Remark 4.5. The maps in the previous proposition are G -equivariant.

Proposition 4.6. *There is a G -equivariant power-set map $\sigma : \mathcal{C}(M, M_0; \pi) \rightarrow \Gamma^+(V)$ of filtered spaces.*

Proof. Fix $k \geq 0$. Let α be an ordered set of cardinality k , for each $0 \leq i \leq k$ let $\mathcal{P}_i(\alpha)$ be the set of (unordered) subsets of α of size i . As α is ordered $\mathcal{P}_i(\alpha)$ has an induced lexicographical ordering and we may list its elements as: $\beta_1, \dots, \beta_{\binom{k}{i}}$.

Let $\xi_\alpha = \sum_{j \in \alpha} m_j x_j$ be an ordered configuration in $\tilde{C}_k(M; \pi)$. For each $\beta \in \mathcal{P}_i(\alpha)$ let $\xi_\beta = \sum_{j \in \beta} m_j x_j \in C_i(M; \pi)$ be an unordered subconfiguration of ξ_α and let $\bar{\xi}_\beta$ be its image under the composition $C_k \rightarrow C^k \rightarrow D^k \rightarrow V_k$. For each i define a map

$$\begin{aligned} \tilde{C}_k(M; \pi) &\longrightarrow V_k^{\binom{k}{i}} \\ \xi_\alpha &\longmapsto \left(\bar{\xi}_{\beta_1}, \dots, \bar{\xi}_{\beta_{\binom{k}{i}}} \right). \end{aligned}$$

Σ_k naturally acts on $\mathcal{P}_i(\alpha)$ since permuting the elements in α maps a subset of size i to another subset of size i . This action defines a homomorphism $\Sigma_k \rightarrow \Sigma_{\binom{k}{i}}$. With this action, the above map is Σ_k -equivariant. For each k the direct sum of these homomorphisms defines a homomorphism $\phi_k : \Sigma_k \rightarrow \Sigma_{\binom{k}{0}} \oplus \Sigma_{\binom{k}{1}} \oplus \dots \oplus \Sigma_{\binom{k}{k}} \subset \Sigma_{2^k}$. Together with the maps $\tilde{C}_k(M; \pi) \rightarrow V_k^{\binom{k}{i}}$ this defines

$$\sigma_k : \mathcal{C}_k(M; \pi) = E\Sigma_k \times_{\Sigma_k} \tilde{C}_k(M; \pi) \longrightarrow E\Sigma_{2^k} \times_{\Sigma_{2^k}} V_k^{2^k}.$$

We will now show that the σ_k 's fit together to define a map $\sigma : \mathcal{C}(M; \pi) \rightarrow \Gamma^+(V)$. To see that σ respects the equivalence relations on each side, choose two equivalent Borel configurations. We may assume they are of the form $(w; m_1, \dots, m_k; x_1, \dots, x_k)$ and its reduction $(R(w); m_1, \dots, m_{k-1}; x_1, \dots, x_{k-1})$. Since $x_k = o|_{m_k}$ or $m_k \in M_0$, any subconfiguration containing $(m_k; x_k)$ is mapped to the base-point in the appropriate copy of $V_k \subset V_k^{2^k}$. The defining relation of Γ^+ identifies a point in V^n with one in V^{n-1} if a coordinate lies in the base-point of V . Precisely 2^{k-1} subconfigurations contain the particle $(m_k; x_k)$. The standard inclusion $\text{incl} : \{1, \dots, k-1\} \rightarrow \{1, \dots, k\}$ induces the standard inclusion $\Sigma_{\binom{k-1}{i}} \rightarrow \Sigma_{\binom{k}{i}}$ as we order lexicographically. Let $\mathcal{P}(\text{incl}) : \Sigma_{2^{k-1}} \rightarrow \Sigma_{2^k}$ denote the induced map defined by the equation $\phi_k \text{incl} = \mathcal{P}(\text{incl}) \phi_{k-1}$. So noting that the square

$$\begin{array}{ccc} \Sigma_k & \xrightarrow{\phi_k} & \Sigma_{2^k} \\ R=\text{incl}^* \downarrow & & \downarrow \mathcal{P}(\text{incl})^* \\ \Sigma_{k-1} & \xrightarrow{\phi_{k-1}} & \Sigma_{2^{k-1}} \end{array}$$

commutes we have

$$\sigma_k(w; m_1, \dots, m_k; x_1, \dots, x_k) \sim \sigma_{k-1}(R(w); m_1, \dots, m_{k-1}; x_1, \dots, x_{k-1}).$$

To see that σ is a map of filtered spaces, note that by definition σ maps \mathcal{C}^k to $\Gamma^+(V_k) \subset \Gamma^+(V)$ where $V_k := \bigvee_{i=1}^k D^k$. Finally, the construction is natural with respect to any inclusions of the underlying fibration π into another fibration, and in particular it is G -equivariant. \square

Theorem 4.7. *There is a stable homotopy equivalence*

$$\mathcal{C}(M, M_0; \pi) \simeq \bigvee_{k \geq 1} D^k(M, M_0; \pi)$$

induced by a G -equivariant map $\bar{\sigma} : \Gamma^+(\mathcal{C}) \rightarrow \Gamma^+(V)$.

Proof. Let $\bar{\sigma}$ denote the composition $\Gamma^+(\mathcal{C}(M, M_0; \pi)) \xrightarrow{\Gamma^+\sigma} \Gamma^+\Gamma^+(V) \xrightarrow{h_V} \Gamma^+(V)$ and note that it restricts to a map of the filtrations and thus induces a map $\mathcal{D}(\bar{\sigma})$ on filtration quotients. $\bar{\sigma}$ is G -equivariant as both the Γ^+ -construction on a map and the construction h_V are natural. Hence, so is $\mathcal{D}(\bar{\sigma})$. We note now that the image of the filtration quotient \mathcal{D}^k under $\mathcal{D}(\bar{\sigma})$ is contained in $E\Sigma_1 \times_{\Sigma_1} D^k = D^k$ and hence restricted to \mathcal{D}^k it factors as $\mathcal{D}^k \rightarrow D^k \xrightarrow{\iota_{D^k}} \Gamma^+(D^k)$ where the first map is the weak homotopy equivalence in Proposition 4.4. Thus $\mathcal{D}(\bar{\sigma})$ factors as the composition

$$\Gamma^+(\mathcal{D}^k) \xrightarrow{\Gamma^+P} \Gamma^+(D^k) \xrightarrow{\Gamma^+\iota_{D^k}} \Gamma^+\Gamma^+(D^k) \xrightarrow{h_{D^k}} \Gamma^+(D^k).$$

The first map is a homotopy equivalence since Γ^+ is a homotopy functor. Also, the composition $h_{D^k} \circ \Gamma^+\iota_{D^k}$ is the identity by virtue of Γ^+ being a monad. It follows that $\mathcal{D}(\bar{\sigma}) : \Gamma^+(\mathcal{D}^k) \rightarrow \Gamma^+(D^k)$ is a homotopy equivalence.

The Γ^+ construction converts cofibrations to fibrations so for each k we can form a commutative diagram in which the vertical sequences are fibrations.

$$\begin{array}{ccc} \Gamma^+(\mathcal{C}^{k-1}) & \xrightarrow{\bar{\sigma}} & \Gamma^+(V_{k-1}) \\ \downarrow & & \downarrow \\ \Gamma^+(\mathcal{C}^k) & \xrightarrow{\bar{\sigma}} & \Gamma^+(V_k) \\ \downarrow & & \downarrow \\ \Gamma^+(\mathcal{D}^k) & \xrightarrow{\mathcal{D}(\bar{\sigma})} & \Gamma^+(D^k) \end{array}$$

By the above discussion the bottom arrow is always a homotopy equivalence. Since $\mathcal{C}^1 = \mathcal{D}^1$ and $C^1 = D^1 = V_1$ we proceed by induction on k to obtain a homotopy equivalence $\Gamma^+(\mathcal{C}) \simeq \Gamma^+(V)$.

By the results of [BE74b], a connected (or group-like) Γ^+ -algebra is an infinite loop space and a morphism of such Γ^+ -algebras is a map of infinite loop spaces. Moreover, the free Γ^+ -algebras correspond to free infinite loop spaces if the underlying space is path connected and has the homotopy type of a CW-complex. By our assumptions on (M, M_0) and X all the spaces considered are path connected and have the homotopy types of CW-complexes. Furthermore, by definition $\bar{\sigma}$ and its restrictions are morphisms of Γ^+ -algebras. Thus the statement of the theorem follows. \square

4.4 Induced splittings for mapping spaces

Combining the stable splitting of Theorem 4.7 with the fact that the scanning map is equivariant, Theorem 3.8, we immediately deduce the following generalisation of Theorem 1.1.

Theorem 4.8. *There is a zigzag of G -equivariant maps*

$$\Gamma_c(M, M_0; \pi) \longleftarrow E(M, M_0; \pi) \longrightarrow C(M, M_0; \pi) \longleftarrow \mathcal{C}(M, M_0; \pi) \longrightarrow \bigvee_{k \geq 1} D^k(M, M_0; \pi)$$

each of which is a stable homotopy equivalence. The zigzag induces a weak homotopy equivalence

$$\Omega^\infty \Sigma^\infty(EG \ltimes_G \Gamma_c(M, M_0; \pi)) \simeq \prod_{k \geq 1} \Omega^\infty \Sigma^\infty(EG \ltimes_G D^k(M, M_0; \pi)).$$

□

We now shift emphasis from configuration spaces to mapping spaces and adopt the notation \bar{D}^k for the k th filtration quotient of

$$C(M \setminus M_0; \pi).$$

When M is parallelisable and π is a trivial bundle, the section space in the above theorem is simply a space of compactly supported maps. But even when M is not parallelisable, we can use configuration spaces with twisted coefficients in the normal bundle of M to ‘untwist’ the section spaces so that they become again mapping spaces.

Corollary 4.9. *For N large enough, there is a homotopy equivalence*

$$\Omega^\infty \Sigma^\infty(EG \ltimes_G \text{Map}_c(M, M_0; \Sigma^N X, *)) \simeq \prod_{k \geq 1} \Omega^\infty \Sigma^\infty(EG \ltimes_G \bar{D}^k).$$

Proof. Let ν be a bundle such that $TM \oplus \nu$ is a trivial bundle. (To construct such a ν note that M can be embedded in \mathbb{R}^N for some N large enough. ν may be taken to be the normal bundle of the embedding.) Let $\pi : Y = \dot{\nu} \wedge_M (M \times X) \rightarrow M$ be the fibre bundle obtained by taking the fibrewise smash product of X with the fibrewise one-point compactification of ν , equipped with the canonical section at infinity. Note that

$$\tau_\pi|_M = \dot{T}M \wedge_M \dot{\nu} \wedge_M (M \times X) \cong \Sigma^N X \times M$$

The result is now a special case of the previous theorem. □

Example 4.10. If M is stably parallelisable, then for some $N \in \mathbb{N}$ there exists a trivial vector bundle ν such that $TM \oplus \nu$ is the trivial N -plane bundle on M . As ν is trivial, it admits an action of the full diffeomorphism group of M . Thus for $G \subset \text{Diff}(M)$ there is a homotopy equivalence

$$\Omega^\infty \Sigma^\infty(EG \ltimes_G \text{Map}(M; \Sigma^N X)) \simeq \prod_{k \geq 1} \Omega^\infty \Sigma^\infty(EG \ltimes_G \bar{D}^k).$$

When M is parallelisable we may take N equal to the dimension of M . For a compact Lie group G and assuming further that M is G -parallelisable, this recovers the main theorem of [BM88].

Example 4.11. When M is the open disc D in \mathbb{R}^n and M_0 empty, we see that the Snaith stable splitting of $\Omega^n \Sigma^n(X)$ is equivariant under the action of $\text{Diff}(D)$.

Example 4.12. When M is the n -sphere $S^n \subset \mathbb{R}^{n+1}$ and M_0 empty, then the normal bundle ν is the one dimensional trivial bundle. Thus for $\pi : Y = \dot{\nu} \wedge_M (M \times X) \simeq M \times \Sigma X \rightarrow M$

$$C(S^n, \emptyset; \pi) \simeq \text{Map}(S^n; \Sigma^{n+1} X).$$

This provides a model for the free higher loop space and a stable splitting, both of which are compatible with the $\text{Diff}(S^n)$ -action.

4.5 Stable injectivity for configurations and diffeomorphisms

We will need to consider manifolds with non-empty boundary. More precisely, let M a connected manifold with $\partial M \neq \emptyset$ and let $M_0 = \emptyset$. We define the configuration space of M to be that of its interior $\text{int } M$ that is

$$C_k(M) := C_k(\text{int } M) \text{ and } E_k(M) := E_k(\text{int } M).$$

Assume further that $\pi : Y \rightarrow M$ has typical fibre X which is path connected, and put $\pi_+ = \pi \amalg \text{id}_M : Y \amalg M \rightarrow M$ be the X_+ -bundle associated to π with canonical section the identity on M . Note that in this situation neither (M, M_0) nor X_+ are connected. We have

$$C(M, M_0; \pi_+) = C(M; \pi_+) = \coprod_{k \geq 0} C_k(M; \pi),$$

and though the scanning map is well-defined, it is no longer a (weak) homotopy equivalence. Similarly, the power set maps are still well-defined. But as the filtration quotient D^k is just C_k the splitting theorems are trivially true. Nevertheless, we will now apply the power set maps to analyse the relation between the spaces $C_k(M; \pi)$ as k grows.

We first define inclusion maps that allow us to think of the configurations of k points as a subspace of the configuration space of $k+1$ points. Let $M \subset M^+ = M \cup \partial M \times [0, 1)$ and extend π to M^+ by pulling back along the natural projection $M^+ \rightarrow M$. Fix points $z_0 \in M^+ \setminus M$ and $x_0 \in \pi^{-1}(z_0)$. We define a stabilisation map

$$b : C_k(M; \pi) \longrightarrow C_{k+1}(M^+; \pi) \longrightarrow C_{k+1}(M; \pi)$$

where the first map adds the particle (z_0, x_0) to any configuration and the second map is an injective homotopy equivalence induced by a map that isotopes M^+ into M mapping $\partial M \times [0, 1)$ into a collar of ∂M in M while at the same time leaving the complement of the collar fixed.

Let $\text{Diff}(M)$ be the group of diffeomorphisms of M . Without changing its homotopy type we assume here that on an ϵ -collar of ∂M the diffeomorphisms restrict to a product of a diffeomorphisms of ∂M and the identity on $(-\epsilon, 0]$. Let G be any subgroup. The following result follows immediately from these definitions.

Proposition 4.13. *The inclusion $b : C_k(M; \pi) \rightarrow C_{k+1}(M; \pi)$ is G -equivariant.* \square

The fact that the map b is stably split injective is well-known. The point of the next result is that this can be done G -equivariantly and that therefore we have stable splittings of Borel constructions.

Theorem 4.14. *The inclusion maps $b : C_k(M; \pi) \rightarrow C_{k+1}(M; \pi)$ are G -equivariantly, stably split injective. In particular, the induced maps*

$$EG \times_G C_k(M; \pi) \longrightarrow EG \times_G C_{k+1}(M; \pi)$$

are stably split injective.

Proof. Let $C^k = C_k(M; \pi)$. Then via the map b we consider C^{k-1} to be a subspace of C^k and $C = \lim_k C_k(M; \pi)$ is a filtered space with filtration quotient $D^k = C^k / C^{k-1}$. Let V_k and V be the wedge products of these. Since the inclusion map b is G -equivariant, G acts naturally on D^k and hence V_k and V .

As in section 4.3 we can also define Borel configuration spaces \mathcal{C}^k , \mathcal{C} , etc. and a power set map

$$\sigma_k : \mathcal{C}^k = E\Sigma_k \times_{\Sigma_k} \tilde{C}_k(M; \pi) \longrightarrow E\Sigma_{2^k} \times_{\Sigma_{2^k}} V_k^{2^k}$$

in this setting. As the action of Σ_k on \tilde{C}_k is free, \mathcal{C}^k is (G -equivariantly) homotopy equivalent to C^k and hence \mathcal{C} to C . The arguments used in the proofs of Proposition 4.6 and Theorem 4.7 go through verbatim to show that the σ_k fit together to give a well-defined map of free Γ^+ -spaces

$$\bar{\sigma} : \Gamma^+(\mathcal{C}) \longrightarrow \Gamma^+(V)$$

which is G -equivariant, filtration preserving and a homotopy equivalence. As both \mathcal{C} and V are connected, $\bar{\sigma}$ is homotopy equivalent to a map of free infinite loop spaces. This proves the fact that \mathcal{C} and V are stably homotopy equivalent.

Collapsing D^k to a point defines a G -equivariant splitting of the inclusion $V_k \rightarrow V_{k+1}$. Via $\bar{\sigma}$ we conclude that stably $\mathcal{C}^k \rightarrow \mathcal{C}^{k+1}$ and hence $b : C^k \rightarrow C^{k+1}$ are also split injective. \square

Let $M \setminus \mathbf{k}$ denote the manifold M with a set of k points deleted from the interior of M (away from the collar of the boundary). Using the same maps on the underlying manifolds as in the definition of b we can define a homomorphism of diffeomorphism groups

$$\bar{b} : \text{Diff}(M \setminus \mathbf{k}) \longrightarrow \text{Diff}(M^+ \setminus \mathbf{k} + \mathbf{1}) \longrightarrow \text{Diff}(M \setminus \mathbf{k} + \mathbf{1}).$$

Theorem 4.15. *The map $\bar{b} : B \text{Diff}(M \setminus \mathbf{k}) \longrightarrow B \text{Diff}(M \setminus \mathbf{k} + \mathbf{1})$ is stably split injective. In particular the induced map in homology is a split injection in all degrees.* \square

Proof. Fix k distinct points $\mathbf{k} = \{z_1, \dots, z_k\}$ in the interior of M . Simultaneous evaluation of diffeomorphisms on these points defines a fibre bundle

$$\text{Diff}(M \setminus \mathbf{k}) \longrightarrow \text{Diff}(M) \longrightarrow C_k(M)$$

and hence a homotopy fibre sequence of classifying spaces

$$C_k(M) \longrightarrow B \text{Diff}(M \setminus \mathbf{k}) \longrightarrow B \text{Diff}(M).$$

Here we may take as a model for the total space of the latter sequence the Borel construction on $C_k(M)$ by the following general construction: given a topological group H with a closed subgroup K the short exact sequence $K \rightarrow H \rightarrow H/K$ gives rise to a fibre bundle $H/K \rightarrow EH \times_H H/K \simeq EH/K \rightarrow EH/H = BH$ where EH is a contractible space with a proper, free H -action; the total space of the latter is a model for BK .

With this identification, the result is now a special case of Theorem 4.14. \square

Remark 4.16. When M is an orientable surface $F_{g,n}^k$ of genus $g \geq 1$ with $n \geq 1$ boundary components and k punctures this specialises to a result in [BT01]. The proof in [BT01] uses the geometric construction of the power set map. Unfortunately, as defined there it is not equivariant. The construction needs to be replaced by the one in section 4.2 here.

In a forthcoming paper [Til] we will expand on the work here and show amongst other things that the maps \bar{b} are furthermore isomorphisms in homology in degrees $\leq k/2$ thus establishing a homology stability criteria for arbitrary manifolds and punctures.

References

- [AAB80] Ahmed M. Abd-Allah and Ronald Brown, *A compact-open topology on partial maps with open domain*, J. London Math. Soc. (2) **21** (1980), no. 3, 480–486.
- [BB78] Peter I. Booth and Ronald Brown, *Spaces of partial maps, fibred mapping spaces and the compact-open topology*, General Topology and Appl. **8** (1978), no. 2, 181–195.
- [BE74a] M. G. Barratt and Peter J. Eccles, Γ^+ -structures. I. A free group functor for stable homotopy theory, Topology **13** (1974), 25–45.
- [BE74b] ———, Γ^+ -structures. II. A recognition principle for infinite loop spaces, Topology **13** (1974), 113–126.
- [BM88] C.-F. Bödigheimer and I. Madsen, *Homotopy quotients of mapping spaces and their stable splitting*, Quart. J. Math. Oxford **39** (1988), no. 156, 401–409.
- [Böd87] C.-F. Bödigheimer, *Stable splittings of mapping spaces*, Algebraic topology (Seattle, Wash., 1985), Lecture Notes in Math., vol. 1286, Springer, Berlin, 1987, pp. 174–187.
- [BT01] Carl-Friedrich Bödigheimer and Ulrike Tillmann, *Stripping and splitting decorated mapping class groups*, Cohomological methods in homotopy theory (Bellaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 47–57.
- [DT58] Albrecht Dold and René Thom, *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. **67** (1958), 239–281.
- [Fra] John Francis, *Factorization homology of topological manifolds*, Preprint; arXiv:1206.5522.
- [Gal11] Søren Galatius, *Stable homology of automorphism groups of free groups*, Ann. of Math. (2) **173** (2011), no. 2, 705–768.
- [God08] Veronique Godin, *Higher string topology operations*, arXiv:0711.4859v2 (2008).
- [GW99] Thomas G. Goodwillie and Michael Weiss, *Embeddings from the point of view of immersion theory. II*, Geom. Topol. **3** (1999), 103–118 (electronic).
- [Hes92] Lars Hesselholt, *A homotopy theoretical derivation of $Q \operatorname{Map}(K, -)_+$* , Math. Scand. **70** (1992), no. 2, 193–203.
- [Lee03] John M. Lee, *Introduction to smooth manifolds*, Graduate Texts in Mathematics, vol. 218, Springer-Verlag, New York, 2003.
- [Lur] Jacob Lurie, *Higher algebra*, Preprint; available at <http://www.math.harvard.edu/~lurie/>.
- [May72] J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin, 1972, Lectures Notes in Mathematics, Vol. 271.
- [McD75] Dusa McDuff, *Configuration spaces of positive and negative particles*, Topology **14** (1975), 91–107.
- [MW07] Ib Madsen and Michael Weiss, *The stable moduli space of Riemann surfaces: Mumford’s conjecture*, Ann. of Math. (2) **165** (2007), no. 3, 843–941.
- [Pal] Martin Palmer, *Homology stability for configurations of submanifolds*, in DPhil-Thesis, Oxford 2013.

- [Sal01] Paolo Salvatore, *Configuration spaces with summable labels*, Cohomological methods in homotopy theory (Bellaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 375–395.
- [Seg73] Graeme Segal, *Configuration-spaces and iterated loop-spaces*, Invent. Math. **21** (1973), 213–221.
- [Sna74] V. P. Snaith, *A stable decomposition of $\Omega^n S^n X$* , J. London Math. Soc. (2) **7** (1974), 577–583.
- [Til] Ulrike Tillmann, *Homology stability for symmetric diffeomorphisms and mapping classes*, in preparation.
- [Til12] ———, *Spaces of graphs and surfaces: on the work of Søren Galatius*, Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 1, 73–90.

Mathematical Institute
Oxford University
24-29 St Giles Street
Oxford OX1 3LB
UK

manthorpe@maths.ox.ac.uk and *tillmann@maths.ox.ac.uk*