

Supplementary Information for Spatio-temporal eco-evolutionary dynamics of prey-predator systems with defended and undefended prey

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S1. SUPPLEMENTARY NOTE 1 : LINEAR STABILITY ANALYSIS OF WITHOUT DIFFUSION EFFECT

In this section, we neglect spatial effects and identify the equilibrium points of the following ordinary differential equations:

$$\begin{aligned}\frac{du}{dt} &= u \left[\frac{\frac{1}{\kappa} - u - v - w}{\rho + \kappa u} - \xi - \frac{\gamma w}{1 + u + v} \right], \\ \frac{dv}{dt} &= v \left[\frac{\frac{1}{\kappa} - u - v - w}{\rho + \kappa \theta v} - \xi - \frac{\gamma w}{\theta(1 + u + v)} \right], \\ \frac{dw}{dt} &= w \left[\left(\frac{\beta_1 u + \beta_2 v}{1 + u + v} \right) \left(\frac{1}{\kappa} - u - v - w \right) - \delta - \mu w \right].\end{aligned}\quad (\text{S1})$$

We also characterize the linear stability of all the equilibrium points. It is straightforward to show that the equilibrium points of Eq. (S1) are:

(i) $Q_0 = (0, 0, 0)$, the extinction equilibrium. Q_0 is locally asymptotically stable if $\xi > \frac{1}{\rho\kappa}$; if $\xi < \frac{1}{\rho\kappa}$, then Q_0 is a saddle point.

(ii) $Q_{b_1} = (u_{++}, 0, 0) = \left(\frac{\frac{1}{\kappa} - \rho\xi}{1 + \kappa\xi}, 0, 0 \right)$. This steady state represents the persistence of undefended prey and the extinction of the defended prey and predators. It exists and is physically realistic where $\rho\kappa\xi < 1$. It is unstable as one of the eigenvalues is $\kappa\xi\theta\left(\frac{\frac{1}{\rho\kappa} - \xi}{1 + \kappa\xi}\right) > 0$.

(iii) $Q_{b_2} = (0, v_{++}, 0) = \left(0, \frac{\frac{1}{\kappa} - \rho\xi}{1 + \kappa\xi\theta}, 0 \right)$. This steady state represents the domination of defended prey, and extinction of the undefended prey and predator. Q_{b_2} exists if $\rho\kappa\xi < 1$ and is always unstable where it exists (one of the eigenvalues is $\kappa\xi\theta\left(\frac{\frac{1}{\rho\kappa} - \xi}{1 + \kappa\xi\theta}\right) > 0$, with $\rho\kappa\xi < 1$.)

(iv) $Q_{p_1} = (u', v', 0)$ where

$$u' = \frac{\theta(\frac{1}{\kappa} - \rho\xi)}{\theta(1 + \kappa\xi) + 1}, \quad v' = \frac{\frac{1}{\kappa} - \rho\xi}{\theta(1 + \kappa\xi) + 1} \quad \text{with} \quad \rho\kappa\xi < 1. \quad (\text{S2})$$

This steady state corresponds to the coexistence of the undefended and defended prey, and extinction of the predators. It is a stable nodal sink if $\delta > \left(\frac{\beta_1 u' + \beta_2 v'}{u' + v' + 1} \right) \left(\frac{\xi(\theta + \rho(1 + \theta))}{(\theta(1 + \kappa\xi) + 1)} \right)$, and an unstable saddle point if $\delta < \left(\frac{\beta_1 u' + \beta_2 v'}{u' + v' + 1} \right) \left(\frac{\xi(\theta + \rho(1 + \theta))}{(\theta(1 + \kappa\xi) + 1)} \right)$.

(v) $Q_{p_2} = (u'', 0, w'')$ where u'' s are the roots of the following cubic polynomial

$$\Gamma_1 u''^3 + \Gamma_2 u''^2 + \Gamma_3 u'' + \Gamma_4 = 0, \quad (\text{S3a})$$

where

$$\begin{aligned}\Gamma_1 &= \kappa\beta_1(\xi - \gamma) + (\kappa\xi + 1)\mu, \\ \Gamma_2 &= (\rho\xi - \rho\gamma + \gamma - 1)\beta_1 + (\kappa\xi + 1)(\beta_1 + 2\mu) - (1 + \kappa\gamma)\delta + (\rho\xi - \frac{1}{\kappa})\mu, \\ \Gamma_3 &= \rho(\xi + \frac{\gamma}{\kappa})\beta_1 - \{(\rho + \kappa)\gamma + 2\}\delta + \{(2\rho + \kappa)\xi - \frac{2}{\kappa} + 1\}\mu, \\ \Gamma_4 &= -[\rho(\gamma\delta - \xi\mu) + \delta + \frac{\mu}{\kappa}], \text{ and } w'' \text{ is given by}\end{aligned}$$

$$w'' = \frac{-\beta_1 u''^2 + \left(\frac{\beta_1}{\kappa} - \delta\right)u'' - \delta}{(\beta_1 + \mu)u'' + \mu}. \quad (\text{S3b})$$

This steady state represents coexistence of undefended prey and the predators and extinction of the defended

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prey. It is always unstable for $\kappa \neq \eta$ as one of the eigenvalues is $\left[\frac{\kappa\xi}{\rho}u'' + \left(\frac{\kappa}{\rho}u'' + \frac{\eta}{\kappa} \right) \frac{\gamma w''}{u''+1} \right] > 0$. If $\kappa = \eta$, (i.e., $\theta = \infty$), then $G_v(Q_{p_2}) = -\xi < 0$ where $G_v = \frac{\partial G}{\partial v}(Q_{p_2})$ is the partial derivative of $G(u, v, w)$ with respect to v . Therefore, for $\kappa = \eta$, Q_{p_2} is locally asymptotically stable if

$$F_u(Q_{p_2}) + H_w(Q_{p_2}) < 0, \quad (\text{S4a})$$

and

$$F_u(Q_{p_2})H_w(Q_{p_2}) > F_w(Q_{p_2})H_u(Q_{p_2}). \quad (\text{S4b})$$

Moreover, Q_{p_2} is a stable node (spiral) if

$$[F_u(Q_{p_2}) - H_w(Q_{p_2})]^2 + 4F_w(Q_{p_2})H_u(Q_{p_2}) > 0 (< 0). \quad (\text{S5})$$

(vi) $Q_{p_3} = (0, v''', w''')$ where

$$w''' = \frac{-\beta_2 v'''^2 + (\frac{\beta_2}{\kappa} - \delta)v''' - \delta}{(\beta_2 + \mu)v''' + \mu}, \quad (\text{S6})$$

and v''' solves the following cubic equation

$$\Gamma'_1 v'''^3 + \Gamma'_2 v'''^2 + \Gamma'_3 v''' + \Gamma'_4 = 0, \quad (\text{S7a})$$

where $\Gamma'_1 = \theta[\kappa\xi\theta(\beta_2 + \mu) + \mu - \kappa\gamma\beta_2]$, $\Gamma'_2 = (2 + \kappa\xi\theta)\theta\mu + (\kappa\gamma + 1)\theta(\frac{\beta_2}{\kappa} - \delta) - \{\frac{1}{\kappa} - (\rho + \kappa\theta)\xi\}\theta(\beta_2 + \mu) - \rho\gamma\beta_2$, $\Gamma'_3 = -[\{\frac{2}{\kappa} - 2\rho\xi - \kappa\xi\theta - 1\}\theta\mu - (\xi\theta + \frac{\gamma}{\kappa})\rho\beta_2 + \{(\rho + \kappa\theta)\gamma + 2\theta\}\delta]$, $\Gamma'_4 = -[(\rho\gamma + \theta)\delta + (\frac{1}{\kappa} - \rho\xi)\theta\mu]$. This steady state represents coexistence of the defended prey and predators and extinction of undefended prey. $Q_{p_3} = (0, v''', w''')$ is locally asymptotically stable if

$$\xi + \frac{\gamma w'''}{v''' + 1} > \frac{1}{\rho} \left(\frac{1}{\kappa} - v''' - w''' \right), \quad (\text{S8a})$$

$$G_v(Q_{p_3}) + H_w(Q_{p_3}) < 0, \quad (\text{S8b})$$

and

$$G_v(Q_{p_3})H_w(Q_{p_3}) > G_w(Q_{p_3})H_v(Q_{p_3}). \quad (\text{S8c})$$

Moreover, Q_{p_3} is a nodal sink if

$$[G_v(Q_{p_3}) - H_w(Q_{p_3})]^2 + 4G_w(Q_{p_3})H_v(Q_{p_3}) \geq 0, \quad (\text{S9})$$

and a spirally stable equilibrium if

$$[G_v(Q_{p_3}) - H_w(Q_{p_3})]^2 + 4G_w(Q_{p_3})H_v(Q_{p_3}) < 0, \quad (\text{S10})$$

(vii) $Q_* = (u_*, v_*, w_*)$. This state represents the coexistence of undefended prey, defended prey, and predator, where

$$w_* = \frac{(\frac{\beta_1}{\kappa} - \delta)u_* + (\frac{\beta_2}{\kappa} - \delta)v_* - (\beta_1 u_*^2 + \beta_2 v_*^2 + (\beta_1 + \beta_2)u_* v_* + \delta)}{(\beta_1 + \mu)u_* + (\beta_2 + \mu)v_* + \mu}, \quad (\text{S11})$$

u_* and v_* simultaneously solve the following equations,

$$\begin{aligned} \Gamma''_1 u_*^3 + \Gamma''_2 u_*^2 + \Gamma''_3 u_* + \Gamma''_4 v_*^3 + \Gamma''_5 v_*^2 + \Gamma''_6 v_* \\ + \Gamma''_7 u_*^2 v_* + \Gamma''_8 u_* v_* + \Gamma''_9 u_* v_*^2 + \Gamma''_{10} = 0, \\ \Gamma''_1 v_*^3 + \Gamma''_2 v_*^2 + \Gamma''_3 v_* + \Gamma''_4 u_*^3 + \Gamma''_5 u_*^2 + \Gamma''_6 u_* \\ + \Gamma''_7 u_* v_*^2 + \Gamma''_8 u_* v_* + \Gamma''_9 u_*^2 v_* + \Gamma''_{10} = 0. \end{aligned} \quad (\text{S12})$$

where $\Gamma''_1 = \kappa(\xi - \gamma)\beta_1 + (\kappa\xi + 1)\mu$, $\Gamma''_2 = [\rho(\xi - \gamma) + \kappa\xi + \gamma]\beta_1 - (\kappa\gamma + 1)\delta + [\rho\xi + 2\kappa\xi - \frac{1}{\kappa} + 2]\mu$, $\Gamma''_3 = \rho(\xi + \frac{\gamma}{\kappa})\beta_1 - \{(\rho + \kappa)\gamma + 2\}\delta + (2\rho\xi + \kappa\xi - \frac{2}{\kappa} + 1)\mu$, $\Gamma''_4 = \mu$, $\Gamma''_5 = \rho(\xi - \gamma)\beta_2 - \delta + (\rho\xi - \frac{1}{\kappa} + 2)\mu$, $\Gamma''_6 = \rho(\xi + \frac{\gamma}{\kappa})\beta_2 - (\rho\gamma + 2)\delta + \{2(\rho\xi - \frac{1}{\kappa}) + 1\}\mu$, $\Gamma''_7 = \kappa(\xi - \gamma)(\beta_1 + \beta_2) + 2(\kappa\xi + 1.5)\mu$, $\Gamma''_8 = \rho(\xi - \gamma)\beta_1 + \{\rho(\xi - \gamma) + \kappa\xi + \gamma\}\beta_2 - (\kappa\gamma + 2)\delta + 2(\rho\xi + \kappa\xi - \frac{1}{\kappa} + 2)\mu$, $\Gamma''_9 = \kappa(\xi - \gamma)\beta_2 + (\kappa\xi + 3)\mu$, $\Gamma''_{10} = -[(\rho\gamma + 1)\delta + (\frac{1}{\kappa} - \rho\xi)\mu]$,

$\Gamma'''_1 = \theta[\kappa(\theta\xi - \gamma)\beta_2 + (\kappa\xi\theta + 1)\mu]$, $\Gamma'''_2 = [\rho(\xi\theta - \gamma) + \theta(\gamma + \kappa\xi\theta)]\beta_2 - \theta(\kappa\gamma + 1)\delta + \theta(\rho\xi + 2\kappa\xi\theta - \frac{1}{\kappa} + 2)\mu$, $\Gamma'''_3 = \rho\beta_2(\xi\theta + \frac{\gamma}{\kappa}) - \{(\rho + \kappa\theta)\gamma + 2\theta\}\delta + \theta(2\rho\xi + \kappa\xi\theta - \frac{2}{\kappa} + 1)\mu$, $\Gamma'''_4 = \theta\mu$, $\Gamma'''_5 = \rho(\xi\theta - \gamma)\beta_1 - \theta\delta + \theta(\rho\xi - \frac{1}{\kappa} + 2)\mu$, $\Gamma'''_6 = \rho(\xi\theta + \frac{\gamma}{\kappa})\beta_1 - (\rho\gamma + 2\theta)\delta + 2\theta(\rho\xi - \frac{1}{\kappa} + 0.5)\mu$, $\Gamma'''_7 = \theta[\kappa(\xi\theta - \gamma)(\beta_1 + \beta_2) + 2(\kappa\xi\theta + 1.5)\mu]$, $\Gamma'''_8 = [\rho(\xi\theta - \gamma) + (\kappa\xi\theta + \gamma)\theta]\beta_1 + \rho(\xi\theta - \gamma)\beta_2 - (\kappa\gamma + 2)\theta\delta + 2\theta(\rho\xi + \kappa\xi\theta - \frac{1}{\kappa} + 2)\mu$, $\Gamma'''_9 = \kappa\theta(\xi\theta - \gamma)\beta_1 + \theta(\kappa\xi\theta + 3)\mu$, $\Gamma'''_{10} = -[(\rho\gamma + \theta)\delta + \theta(\frac{1}{\kappa} - \rho\xi)\mu]$, and

(S13)

It is locally asymptotically stable if

$$\Xi = [F_u + G_v + H_w]_{(u_*, v_*, w_*)} < 0, \quad (\text{S14a})$$

$$\begin{aligned} \Upsilon = [F_u(G_v H_w - G_w H_v) - F_v(G_u H_w - G_w H_u) \\ + F_w(G_u H_v - G_v H_u)]_{(u_*, v_*, w_*)} < 0, \end{aligned} \quad (\text{S14b})$$

and

$$\Omega = [G_v H_w - G_w H_v + F_u H_w - F_w H_u + F_u G_v - F_v G_u]_{(u_*, v_*, w_*)} > \frac{\Upsilon}{\Xi}. \quad (\text{S14c})$$

In the stability conditions (S4a) to (S14c), $F_u = \frac{\partial F}{\partial u}(Q_*)$, F_v, F_w, G_u, G_v, G_w and H_u, H_v, H_w are the partial derivatives of $F(u, v, w)$, $G(u, v, w)$ and $H(u, v, w)$ with respect to u, v and w , respectively.

S2. SUPPLEMENTARY NOTE 2 : ONLY PREDATOR'S MOVEMENT

In order to examine the role of prey mobility in shaping spatial structures, we now consider a limiting case

Table S1. Summary of the equilibrium points, their existence, and linear stability conditions for Eq. (S1)

Equilibrium point	Existence condition	Linear stability
(i) $Q_0 = (0, 0, 0)$	Always exists	$\rho\kappa\xi > 1$
(ii) $Q_{b_1} = \left(\frac{\frac{1}{\kappa} - \rho\xi}{1 + \kappa\xi}, 0, 0\right)$	$\rho\kappa\xi < 1$	Always unstable
(iii) $Q_{b_2} = \left(0, \frac{\frac{1}{\kappa} - \rho\xi}{1 + \kappa\xi\theta}, 0\right)$	$\rho\kappa\xi < 1$	Always unstable
(iv) $Q_{p_1} = (u', v', 0)$ (see (S2))	$\rho\kappa\xi < 1$	$\delta > \left(\frac{\beta_1 u' + \beta_2 v'}{u' + v' + 1}\right) \left(\frac{1}{\kappa} - u' - v'\right)$
(v) $Q_{p_2} = (u'', 0, w'')$ (see (S3a) and (S3b))	$u'', w'' > 0$ & $u'' + w'' < \frac{1}{\kappa}$	See (S4a), (S4b), (S5)
(vi) $Q_{p_3} = (0, v''', w''')$ (see (S7a) and (S6))	$v''', w''' > 0$ & $v''' + w''' < \frac{1}{\kappa}$	See (S8a), (S8b), (S8c), (S9)
(vii) $Q_* = (u_*, v_*, w_*)$ (see (S12) and (S13))	$u_*, v_*, w_* > 0$ & $u_* + v_* + w_* < \frac{1}{\kappa}$	See (S14a), (S14b), (S14c)

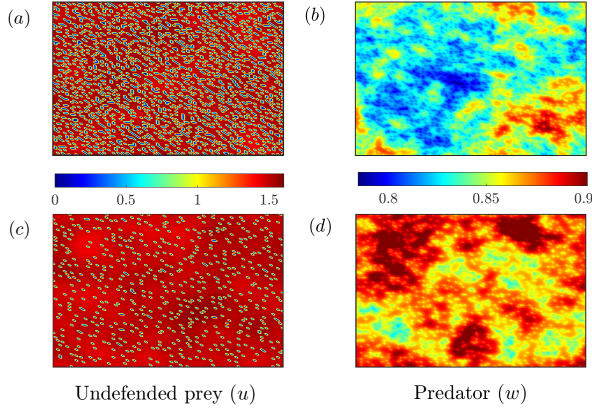


Figure S1. **Spatio-temporal plot by varying μ .** We observe the shift of spatio-temporal pattern setting $D_u = D_v = 0$, and $D_w = 1$, taking two distinct values of the predators' decaying rate (a, b) $\mu = 0.8$ and (c, d) $\mu = 0.92$ where the first and second columns represent the behavior of undefended prey and predator, respectively.

where both undefended and defended prey are completely immobile, *i.e.*, $D_u = D_v = 0$, while predators retain diffusive movement as per stated previously. This scenario corresponds to situations where prey species are spatially fixed due to sessile life forms (e.g., terrestrial plants rooted in soil, corals fixed to reef substrates, or benthic invertebrates such as mussels and barnacles). At the same time, predators are capable of moving freely across the habitat [1]. In such real-world cases, predators must actively search for prey patches, and prey-predator encounters are determined almost entirely by predator foraging ranges and prey spatial arrangement rather than prey movement. By eliminating prey movement in our model, any emergent spatial patterns arise solely from predator redistribution in response to local prey availability and the eco-evolutionary feedback between defended and undefended prey [2].

Figure S1 illustrates the resulting spatiotemporal distributions of undefended prey (u) and predator (w) densities for two representative values of the evolutionary

parameter μ specifically, $\mu = 0.8$ for Fig. S1(a,b) and $\mu = 0.92$ for Fig. S1(c,d) while keeping all other system parameters identical to those used in Fig. 4. In the absence of prey mobility, undefended prey remain permanently fixed at their initial spatial positions, and spatial heterogeneity is generated exclusively by predator movement and localized consumption dynamics.

For $\mu = 0.8$ in Fig. S1(a,b), the prey field [panel (a)] shows a relatively uniform background dominated by high density of undefended prey areas (red-yellow regions), punctuated by small localized voids. The corresponding predator distribution [panel (b)] forms fine grained, mosaic-like patterns, with high density predator clusters (red patches) embedded in a matrix of moderate and low-density regions (yellow-blue), reflecting localized aggregation near prey-rich sites and depletion in prey-poor zone.

As μ increases to 0.92 [Fig. S1(c,d)], the undefended prey distribution [Fig. S1(c)] remains largely homogeneous in density, as expected from the absence of prey movement, but predator spatial organization [Fig. S1(d)] undergoes a qualitative change: the fine-grained structure gives way to larger, more sharply segregated high-density predator zones (deep red) surrounded by extensive low-density predator areas (blue-cyan). This indicates that evolutionary shifts in the undefended prey composition can alter predator foraging success and territorial occupancy patterns, even when prey are immobile.

Such dynamics have real ecological analogues. For example, in coral reef ecosystems [3], immobile coral polyps (some chemically defended, others undefended) experience spatially heterogeneous grazing by mobile predators such as parrotfish, whose feeding territories reflect both prey quality and defense levels. Similarly, in intertidal mussel beds, predatory whelks exhibit patchy distribution patterns that depend on the proportion of defended mussels (e.g., thicker shells or chemical deterrents) within a fixed spatial matrix. In terrestrial contexts, stationary plants with varying levels of chemical defense against herbivores often produce predator/herbivore distributions characterized by patchy clustering in defense-poor zones and avoidance of defense-rich areas.

This special case thus provides a baseline for under-

standing the role of prey dispersal in the full model: when prey are stationary, predator pattern formation is entirely dictated by eco-evolutionary feedback and the initial prey landscape, whereas allowing prey mobility (as in the general case) introduces additional dynamic mixing, spatial coupling, and potentially faster or more transient pattern reorganization.

SUPPLEMENTARY REFERENCES

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