Physica Scripta

OPEN ACCESS

RECEIVED
21 March 2025

REVISED
13 June 2025

ACCEPTED FOR PUBLICATION
24 June 2025

PUBLISHED
1 July 2025

PAPER

Parametric resonance with linear damping: a general formula for the excitation threshold for high orders

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Keywords: Mathieu, parametric, damping

Abstract

We derive a general formula for the excitation threshold of parametric resonances of an oscillator with linear damping from consideration of the asymptotic properties of the Mathieu equation. This provides a good approximation for resonances of order $m \geqslant 2$, and it is especially useful for high-order resonances in systems with light damping for which other approaches are cumbersome. Parametric resonance is ubiquitous in mechanical and electrical systems and its threshold is an important consideration, e.g., for systems that would be damaged by a high amplitude of resonantly excited motion. We present the expressions in a form useful for understanding systems with high quality factors such as trapped atomic ions, micro-mechanical devices and other oscillators, especially those with low dissipation in vacuum. High-order parametric resonances are extremely narrow making direct numerical simulation computationally intensive as well as less insightful.

1. Introduction and outline

Mathieu introduced his famous equation to describe waves or vibrations of elliptical surfaces [1]; the symmetry of an ellipse implies solutions that are periodic in the azimuthal angle ϕ for rotation about a central axis perpendicular to the plane (with period π). This paradigm of periodic second-order differential equations has been studied in great detail and historical developments are outlined in [2]. Many of the known mathematical results are detailed in the NIST Digital Library of Mathematical Functions (DLMF) [3], including an extensive bibliography. The purpose of this work is to show how an analytic formula for the minimum amplitude required to excite a parametric resonance in the presence of linear damping can be derived by considering the exponentially narrow width of the regions, or tongues, of instability (related to Arnold tongues [4]). From the asymptotic properties of series solutions of the Mathieu equation we find a general formula for the threshold excitation that is a good approximation for the second order $m = 2$, and all higher orders. This mathematical result provides clear insight into the general trends for systems with light damping in which high-order parametric resonances, with $m > 20$, are observed, e.g., micro-mechanical oscillators [5] and atomic ions trapped in ultrahigh vacuum [6]. Here, we cast the expressions into a convenient form and investigate error bounds. We also verify this interpretation of the equations by comparison to numerical calculations of eigenvalues to find the boundaries between regions of stability and instability; this is a step beyond previous works. The relevant mathematical expressions were found by Turyn [7] but not expressed in a form that has been widely used in applications. The analysis of parametric excitation of trapped ions by Zhao *et al* [6] found similar expressions for the excitation threshold but with computed coefficients, whereas we give analytic expressions accurate for any order with $m \geqslant 2$. In this work, we combine these separate threads to provide a simple analytic formula that is straightforward to apply. Asymptotic analysis captures fine detail that would otherwise require extensive numerical work, especially for high-order parametric excitation.

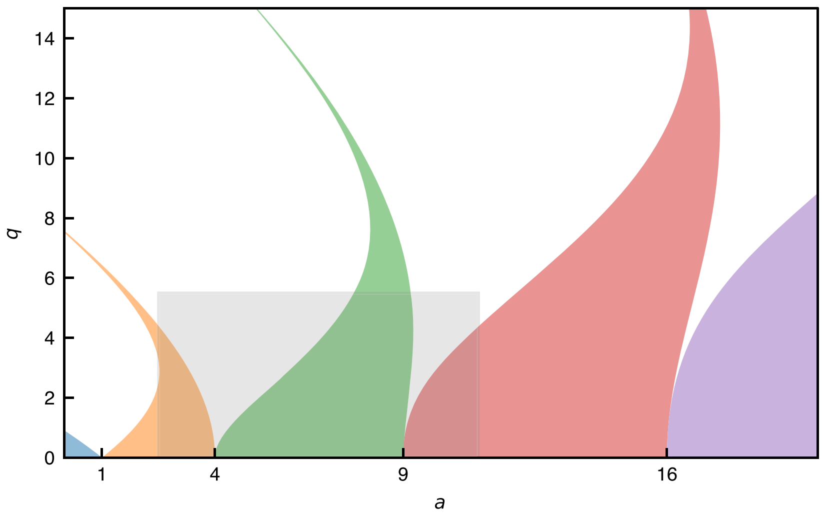

Figure 1. (a) The typical stability diagram for the Mathieu equation (Ince-Strutt plot). The boundaries of the stable (shaded) regions correspond to integer eigenvalues of equation (1). The unstable regions (white) has tongues of instability that reach to $q = 0$ axis at integer values $a = m^2$ where m is order of the resonance; here only the first four resonances are shown. The shaded area is shown in more detail in figure 2.

The Mathieu equation for $y(t)$ has the form

$$\frac{d^2y}{dt^2} + (a - 2q\cos 2t)y = 0, \tag{1}$$

where a and q are two parameters [3] (DLMF equation (28).2.1). Figure 1 shows a plot of the regions of stable solutions of the Mathieu equation, showing only the half-plane $q \geqslant 0$ since stability is independent of the sign of q. For parametric resonance it is convenient to have a as the abscissa and q as the ordinate (an Ince-Strutt plot [2]). The figure illustrates how the analytic formula derived in this paper relates to the stability of a differential equation that has the form of a Mathieu equation plus a linear damping term (which can be reduced to a Mathieu by a change of variables). This result is explained in the rest of this paper as follows. We summarise the usual treatment of the Mathieu equation plus linear damping in section 2, before expounding the derivation of the formula in section 3 which is the core of this paper. The analytic formula supersedes numerical calculations and provides intuitive guidance, however to estimate the error bounds and to plot the stability region in figure 1 we have used the efficient numerical approach described in section 4. In the concluding section 5, we comment on the relationship with other methods for determining the boundaries between stable and unstable regions.

2. The Mathieu equation plus a linear damping term

In this section we set out standard results that are the background of the derivation in the next section. Adding a linear damping term to equation (1) gives the damped Mathieu equation [8],

$$\frac{d^2y}{dt^2} + 2\gamma\frac{dy}{dt} + (a - 2q\cos 2t)y = 0. \tag{2}$$

The substitution $y = \tilde{y}\exp(-\gamma t)$ eliminates the first derivative (for any second-order differential equation with constant coefficients of the 1st and 2nd derivatives) leading to the Mathieu equation

$$\frac{d^2\tilde{y}}{dt^2} + (\tilde{a} - 2q\cos 2t)\tilde{y} = 0, \tag{3}$$

where q is unchanged but $\tilde{a} = a - \gamma^2$. Note, however, that $\tilde{a} \simeq a$ is a good approximation for light damping, e.g., typical values of $\gamma < 10^{-2}$ for which the treatment in this paper is most relevant. For $q = 0$ and $\tilde{a} > 0$ the solutions correspond to damped simple harmonic motion (SHM) at angular frequency $\omega = \sqrt{\tilde{a}}$. Hence the fractional decrement of the amplitude is $2\pi\gamma/\omega$ per oscillation period for this definition of the damping coefficient, i.e., the fractional loss decreases as ω increases [9]. Thus, the solution of the equation with damping is related to a Mathieu equation without damping but *not* the same equation as setting $\gamma = 0$. The effect of this change of variables on the stability regions is shown in the following.

Floquet's theorem states that equations with periodic coefficients have solutions of the form $\tilde{y} = e^{\mu t}u(t)$, where $u(t) = u(t + \pi)$ is a periodic function, matching the period of $\cos 2t$ in this case, and μ is the characteristic exponent. The solution of equation (3) is a linear combination of two independent functions:

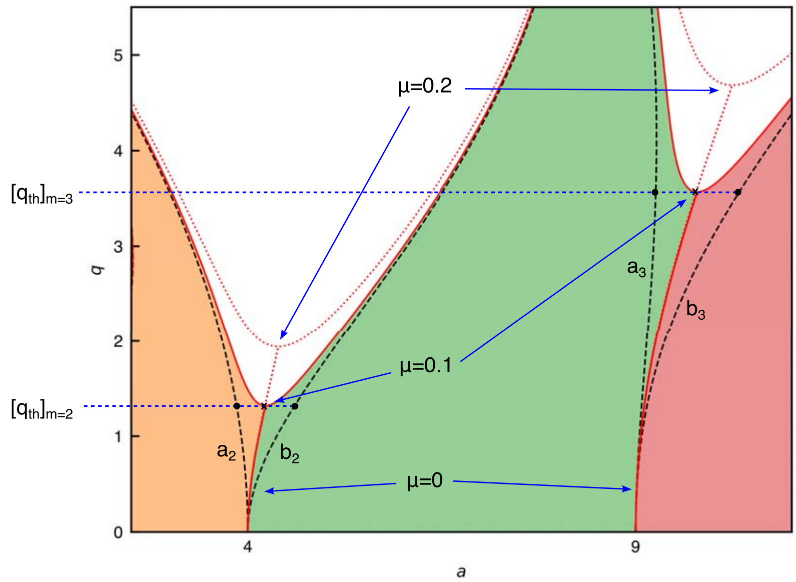

Figure 2. In the presence of damping the tips of the instability tongues do not touch the $q = 0$ axis, i.e., a finite threshold value of the parametric driving is required to excite the resonance. This can be seen from the regions of stability contours for resonances of order $m = 2$ and 3, for three values of the characteristic exponent $\mu = 0, 0.1, 0.2$. The minimum value of q to excite each resonance corresponds to the maximum value of μ along the (horizontal) line of constant q; the points of intersection, marked with X, are approximately midway between the lower and upper boundaries, a_m and b_m respectively, of each resonance tongue. This illustrates the method described in the main text, from McLachlan [11].

$$\tilde{y} = c_1 e^{\mu t} u_1(t) + c_2 e^{-\mu t} u_2(t). \tag{4}$$

Hence the solutions of equation (2) are of the form:

$$y = C_1 e^{(\mu-\gamma)t} u_1(t) + C_2 e^{-(\mu+\gamma)t} u_2(t). \tag{5}$$

There are stable solutions for $\mu \leqslant \gamma$. The methods for determining μ and hence the effect of damping on the first-order resonance ($m = 1$) are well-known [10]. An approach suitable for determining μ for higher order resonances is described in the next section. A pedagogical description of how the solutions of the differential equation depend on μ is given in [8].

3. An analytic expression for the excitation threshold

The characteristic exponent μ in the region of an instability tongue can be determined using an approximation based on the form of the contours of constant μ near the tip of the instability tongue, shown in figure 2, as described in the treatise on the Mathieu equation by McLachlan [11]. A similar plot can be found in [2] with an explanation of the effect of damping on low-order resonances. For the resonance of order m the local maximum value of μ can be expressed in terms of the width of the instability tongue $a_m - b_m$, where $a_m(q)$ and $b_m(q)$ are the values at the upper and lower boundary of the m- th tongue respectively [11] (section 4.91, equation (8));

$$[\ \mu\ ]_{\max} = \frac{a_m - b_m}{4m}. \tag{6}$$

As illustrated on figure 2, the maximum occurs approximately midway between a_m and b_m (also given in [6]). The width of the resonance has the form $a_m - b_m = A_m q^m$. The asymptotic expression for A_m (equivalent to [3] (DLMF equation (28).6.15)), is

$$A_m = \frac{8m}{2^{2m}\, m!(m-1)!}. \tag{7}$$

Instability arises when the exponent $\mu - \gamma \geqslant 0$ in equation (5). For a given damping coefficient γ, the lowest value of q for which $[\ \mu\ ]_{\max} = \gamma$ corresponds to the threshold condition for parametric excitation $q_{\rm th} = [\ \mu\ ]_{\max}$. Hence we find

$$q_{\rm th} = \left(\frac{4m}{A_m}\right)^{1/m} \gamma^{1/m}. \tag{8}$$

Table 1. Example values for the excitation threshold for the orders $m = 2, 4, 8, 16, 32$ and for typical values of damping γ ranging from 10^{-8} to 10^{-2}. The difference between equations (9), (16) and the numerically calculated ones becomes negligible as the order of the resonance increases as long as the damping is small.

resonance	$-\log_{10}(\gamma)$	a_{th}	a_n	$\delta a/a_{th}$ (%)	q_{th}	q_n	$\delta q/q_{th}$ (%)
m = 2	2	4.027	4.024	0.071	0.4021	0.4	0.53
	4	4.0	4.0	0.00075	0.04006	0.04	0.15
	6	4.0	4.0	7.6e-06	0.004006	0.004	0.15
	8	4.0	4.0	1.4e-07	0.0004021	0.0004	0.53
m = 4	2	16.45	16.48	0.16	3.746	3.685	1.7
	4	16.05	16.05	0.0021	1.169	1.165	0.34
	6	16.0	16.0	8.8e-05	0.369	0.3685	0.15
	8	16.0	16.0	8.5e-06	0.1167	0.1165	0.15
m = 8	2	68.03	68.59	0.81	23.43	22.54	4.0
	4	65.27	65.32	0.073	12.81	12.67	1.1
	6	64.4	64.41	0.0071	7.152	7.127	0.34
	8	64.13	64.13	0.00087	4.014	4.008	0.15
m = 16	2	280.5	286.2	2.1	118.9	111.7	6.4
	4	269.8	271.3	0.56	86.44	83.77	3.2
	6	263.7	264.2	0.16	63.87	62.82	1.7
	8	260.4	260.5	0.05	47.53	47.1	0.91
m = 32	2	1145	1186	3.5	541.9	497.6	8.9
	4	1115	1134	1.7	456.1	430.9	5.9
	6	1092	1102	0.88	388.6	373.1	4.1
	8	1075	1080	0.48	332.8	323.1	3.0

Using the asymptotic expression for A_m in equation (7) gives

$$q_{\rm th} = 4\left(m!(m-1)!\frac{\gamma}{2}\right)^{1/m}. \tag{9}$$

This formula has also been derived by Turyn [7] with $\nu \to \gamma$ and $\epsilon \to 2q$. We can recast this into a more understandable form, that is very useful for applications, by using Stirling's formula: $m! \simeq \sqrt{2\pi m}\,(m/e)^m$, hence the product the factorials is $m!(m-1)! \simeq 2\pi(m/e)^{2m}$. This simplification gives

$$q_{\rm th} = \frac{1}{2}B_m\, m^2(\pi\gamma)^{1/m}. \tag{10}$$

The constant of proportionality is

$$B_m = B(1+\epsilon_m) \tag{11}$$

where $B = 8/e^2 \simeq 1.0827$ and the small correction term, $\epsilon_m \ll 1$, is given by

$$\epsilon_m = 1 - \left(\frac{e}{m}\right)^2\left(\frac{m!(m-1)!}{2\pi}\right)^{1/m} \tag{12}$$

$$\simeq \frac{1}{6m^2}, \tag{13}$$

where the approximation in equation (13) is obtained from equation (12) by considering Stirling's formula to two orders [3] (equation (5).11.3), which gives a correction factor of $1 + 1/(12m)$ times the single-order formula for $m!$, used above for equation (10). Example numerical values from equation (12) are: $\{\epsilon_2, \epsilon_4, \epsilon_6\} = \{0.042\,207, 0.010\,450, 0.004\,636\}$, for $m = 2, 4, 6$. The difference between equations (12) and (13) is negligible: $\epsilon_m - 1/(6m^2) = \{0.000\,540, 0.000\,033, 0.000\,006\}$, for $m = 2, 4, 6$. Values for the parameters of interest for different excitation orders are shown in table 1. In summary, the correction is small: $\epsilon_2 \simeq 4\%$ dropping to $\epsilon_4 \simeq 1\%$ and even smaller for higher orders. Such small corrections may be unimportant in practical applications so that it is sufficiently accurate to assume $B_m \simeq B \simeq 1$ is constant, for such that equation (10) reduces to a simple form

$$2q_{\rm th} \simeq m^2(\pi\gamma)^{1/m}. \tag{14}$$

The dominant influence on $2q_{\rm th}$ is its dependence on m^2 and $(\pi\gamma)^{1/m} \to 1$ (slowly) as $m \to \infty$. This expression is expected to be valid below the line $2q = a$ that marks the boundary between predominately stable and unstable regions (shown in figure 3) and $a_m \simeq m^2$ for light damping. The eigenvalues of the Mathieu equation ([3] (section 28.6 *Expansions for Small q*)) are

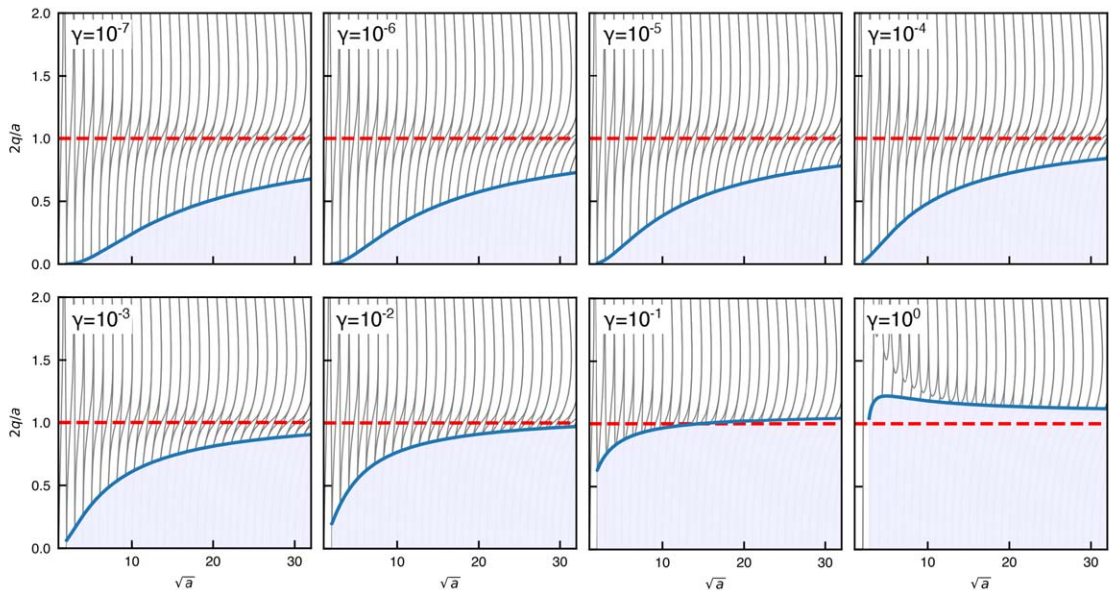

Figure 3. Stability diagrams for various values of γ showing $2q/a$ over $\sqrt{a} \equiv m$. The stable and unstable regions have approximately the same width along the line $2q/a = 1$ (red dashed line) which can be related to the results in [13]. The critical boundary (blue line) for the excitation threshold q_{th}, as given by equation (9), renders the shaded blue region stable and disallow the tongues of instability to reach the $q = 0$ axis. Increasing values of γ enlarge the stable region. For values of $\gamma > 10^{-1}$, equation (9) does not approximate the threshold as well, underestimating for low orders and overestimating it for higher ones.

$$a_m = m^2 + \gamma^2 + \frac{1}{2(m^2 - 1)}q^2 \ldots \tag{15}$$

This series expansion for small values of q is from *Algebraic methods to compute Mathieu functions*, Frenkel and Portugal [12] based on recursive differential equations; they find closed form expressions to provide efficient methods for numerical evaluation, which aligns with our aim in this note. Substituting the expression for $2q_{\rm th}$ from equation (14) gives

$$a_m = m^2\left[1 + \frac{\gamma^2}{m^2} + \frac{m^2}{m^2 - 1} \cdot \frac{B_m^2}{8} \cdot (\pi\gamma)^{2/m}\right]. \tag{16}$$

Taking $\gamma = 10^{-3}$ and $m = 16$ as an example, we can neglect $\gamma^2/m^2 = 3 \times 10^{-9}$ and the predominant part of the third term inside the brackets is $(\pi\gamma)^{1/8}/8 \simeq 0.06$. Of significance is the difference $a_m(q) - m^2$ relative to the spacing of the orders; we find $(a_m - m^2)/(a_{m+1} - a_m) \simeq 0.5$, corresponding to half an order which is significant.

The key result of this paper is the expression in equation (10) for the excitation threshold of the m- th resonance. On physical grounds, these expressions are expected to be good approximations in the region of the $(a, q) -$ plane where the tongues of instability are narrow, i.e., well below the critical line $2q = a$. It is implicitly assumed that the damping coefficient γ is 'sufficiently small' that the instability tongues cut deeply into the region $2q < a$. We emphasise that equation (10) is entirely equivalent to equation (9) if the correction ϵ_m in equation (12) is applied. If only the lowest order term of the Stirling approximation is used then ϵ_m gives the error. The radii of convergence of the various series expansions in powers q^m that lead to equation (7) are discussed in [3] (section 28.6), including numerically tabulated values; it is conjectured ([3] (equation (28).6.20)) that these radii are proportional to m^2 which is intuitively reasonable since for $2q_m \leqslant m^2 \simeq a_m$ the tongues of instability are narrower than the stable regions between them. See also comments on convergence relating to equation (15) in Frenkel and Portugal [12]. Thus, although the argument used to find equation (6) is stated in McLachlan [11] to be valid for $q \ll a$ (where $q > 0$ is real) it is useful up to $2q \lesssim a$ with a level of uncertainty sufficient for calculations of the stability of physical systems. The approximation assumes narrow tongues of instability. The condition is that the tongues have widths $(A_m q^m)$ small compared to their spacing (found from the gap between resonances $a_{m+1} - a_m \simeq 2m + 1$) is $A_m q^m < 2m + 1$. For $m \gg 1$, this implies $q < (\pi m/2)^{1/m}(2\,m/e)^2$ which is satisfied if $2q < a \simeq m^2$.

A discussion of the tongues of instability for the more general case of Hill equations, of which the Mathieu equation is an example, is given by Levy and Keller [14]. That the threshold for parametric excitation of an m- th-order resonance is proportional to $\gamma^{1/m}$ is stated in the textbook of Landau and Lifshitz [15]. Our calculation of compact expressions for the constant of proportionally, by combining known mathematical results, is useful for the many applications of the Mathieu equation with a damping term, where the tips of the instability tongues do not touch the $q = 0$ axes, so that a finite threshold value of the parametric driving is required to excite resonance.

4. Numerical calculations and estimation of error bounds

For completeness we include details of the numerical method used to calculate the numerical values for the excitation threshold. The traditional solution to the Mathieu-Hill equation uses the Hill determinant [16]. First $y(t)$ is expanded as a Fourier series, which yields a recurrence relation for the Fourier coefficient, in therm of a and q, that can be cast in a determinant form. The stability regions are then calculated by solving the determinant or using continuous fractions [17]. An alternative method, that captures accurately the boundaries between stable and unstable regions, is transforming equation (2) into algebraic eigenvalue equations using Fourier differentiation matrices [18] that can be directly diagonalised.

Assuming a period of 2π for the solutions, we define equidistant Fourier nodes $k, j \in \{1, \ldots, N\}$ with spacing $h = 2\pi/N$. The Fourier differentiation matrices, for even N, are the $N \times N$ Toeplitz matrices:

$$D_{kj}^{(1)} = \begin{cases} 0 & k = j \\ \frac{1}{2}(-1)^{k-j}\cot\frac{(k-j)h}{2} & k \neq j \end{cases}$$
$$D_{kj}^{(2)} = \begin{cases} -\frac{\pi^2}{3h^2} - \frac{1}{6} & k = j \\ -\frac{1}{2}(-1)^{k-j}\csc^2\frac{(k-j)h}{2} & k \neq j \end{cases} \tag{17}$$

where $D_{kj}^{(1)}$ and $D_{kj}^{(2)}$ correspond to the first and second derivatives respectively. Using $C = \mathrm{diag}(\cos 2t_k)$ we can approximate equation (2) as

$$(2qC - D^{(2)} - 2\gamma D^{(1)})\mathbf{y} = a\mathbf{y}, \tag{18}$$

where $\mathbf{y}$ is the vector of approximate eigenfunction values $y(t_k)$. The eigenvalue problem can now be solved and it gives N values of a for each q. These values are used in table 1 for comparison with the analytic expressions.

5. Conclusion

The work that we have described in this paper was stimulated by research into trapped ions in a Paul trap, which is a type of electrodynamic trap with an oscillating quadrupole electric field. In that context, the Mathieu equation is applied 'twice'; firstly to explain the operation of the trap itself in terms of a harmonic pseudo-potential and secondly to explain parametric resonances of charged particles confined in this pseudo-potential [19, 20]. Recently, a Paul trap with an electric quadrupole field oscillating at two frequencies has been used to simultaneously confine atomic ions and a charged nanoparticle [21]. Parametric excitation of ions in a standard Paul trap has been investigated previously: Razvi *et al* studied the first few resonances [22], and Zhao *et al* report experimental results supported by theoretical work [6]. These two papers provided a stimulus for this paper.

The Mathieu equation is a paradigm for parametric excitation in a wide variety of systems. As well as oscillators with light damping, there are many other examples in which the Mathieu equation arises in physics and engineering science [5]. The mathematical result that we have presented has implications for understanding the stability of physical systems and engineered structures, however high order parametric resonances in real systems are generally very weak and may be affected by other processes such as nonlinear mixing. Nevertheless, having a prediction for the excitation threshold for linear damping with which to compare can provide important insight. A comprehensive review of many applications of the Mathieu and related equations in Engineering can be found in [2]. The Mathieu equation, and the more general Hill's equation (of which other examples are the equations of Whittaker-Hill and Ince) are also described in [3] (section 28). The Schrödinger equation of a particle in a periodic potential falls into this class of equations.

By extricating the methodology from the specific case we provide simple mathematical formulae that are applicable to the general phenomenon of parametric resonance. Trapped ions provide a good example of a small system with low intrinsic damping in vacuum. The parametric instability of ships at sea being tossed about by waves has been investigated using the damped Mathieu equation [23]. This paper represents a drop in the ocean of literature since Mathieu's work on systems with elliptical symmetry published in 1868 [1], but it does provide a straightforward analytic formula with a clear physical interpretation which is useful in applications that might otherwise require numerical calculations with a very fine grid of points to map out the instability regions that are extremely narrow for high-order resonances.

Acknowledgments

We acknowledge useful discussions with Dr E Bentine.

Data availability statement

No new data were created or analysed in this study.

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