Branch Groups and Automata

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This thesis is dedicated to my parents.

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Abstract

The focus of this thesis is finitely generated subgroups of the automorphism group of an infinite spherically homogeneous rooted tree (regular or irregular).

The first chapter introduces the topic and outlines the main results. The second chapter provides definitions of the terminology used, and also some preliminary results.

The third chapter introduces a group that appears to be a promising candidate for a finitely generated group of infinite upper rank with finite upper p-rank for all primes p. It goes on to demonstrate that in fact this group has infinite upper p-rank for all primes p. As a by-product of this construction, we obtain a finitely generated branch group with quotients that are virtually-(free abelian of rank n) for arbitrarily large n.

The fourth chapter gives a complete classification of ternary automata with C_2-action at the root, and a partial classification of ternary automata with C_3-action at the root. The concept of a 'windmill automaton' is introduced in this chapter, and a complete classification of binary windmill automata is given.

The fifth chapter contains a detailed study of the non-abelian ternary automata with C_3-action at the root. It also contains some conjectures about possible isomorphisms between these groups.

Contents

Notation

The following notation will be used throughout:

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ for the natural numbers (that is, the positive integers), the non-negative integers, the integers, and the reals, respectively.

$[i, j]$: for integers i and j this denotes the set of integers n such that $i \leqslant n \leqslant j$.

$\lfloor x \rfloor$, $\lceil x \rceil$: greatest integer $\leqslant x$, least integer $\geqslant x$.

$\underline{d}$ is the constant sequence $(d, d, \ldots)$.

$\mathrm{Sym}(X)$, $\mathrm{Alt}(X)$, $\mathrm{Sym}(n)$, $\mathrm{Alt}(n)$ for the symmetric group on the set X, the alternating group on the set X, the symmetric group of degree n, and the alternating group of degree n, respectively. Permutations are read left-to-right. The action of σ on i is denoted i^σ.

Maps (other than permutations) are written on the left, as in $f(x)$. Thus the composition f followed by g is written $g(f(x))$.

C_n for the cyclic group of order n. We identify C_n with the subgroup of $\mathrm{Sym}(n)$ generated by the cycle $(1, 2, \ldots, n)$.

C_∞, D_∞ for the infinite cyclic group, and the infinite dihedral group, respectively.

The extra-special group of order 27 and exponent 3 is the group defined by the presentation $\big\langle x, y \bigm| x^3 = y^3 = [x, y]^3 = [x, y, x] = [x, y, y] = 1 \big\rangle$.

The extra-special group of order 27 and exponent 9 is the group defined by the presentation $\big\langle x, y \bigm| x^9 = y^3 = 1,\ x^y = x^7 \big\rangle$.

$A \leqslant B$: A is a subgroup of B.

$A \leqslant_f B$: A is a subgroup of finite index in B.

$X^{-1} = \{x^{-1} \mid x \in X\}$; $X^{\pm 1} = X \cup X^{-1}$: defined for a subset X of a group G.

$G' = [G, G]$: the derived subgroup of G.

$G^{\mathrm{ab}} = G/G'$: the abelianization of G.

$G^{(k)} = [G^{(k-1)}, G^{(k-1)}]$: the k-th term in the derived series of $G = G^{(0)}$.

$\gamma_k(G) = [\gamma_{k-1}(G), G]$: the k-th term in the lower central series of $G = \gamma_1(G)$.

$P_k(G) = P_{k-1}(G)^p[P_{k-1}(G), G]$: the k-th term in the lower p-central series (for a fixed prime p) of $G = P_1(G)$. When G is understood, we write P_k instead of $P_k(G)$. For example, G/P_k in place of $G/P_k(G)$. In Chapter 5, we always take $p = 3$.

$x^y = y^{-1}xy$; $[x, y] = x^{-1}y^{-1}xy$; $[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$.

$x^{ny} = (x^n)^y$: defined for an integer n, and group elements x and y.

$S^x = \{s^x \mid s \in S\}$: defined for a subset S and an element x of a group G.

$A^B = \langle a^b \mid a \in A, b \in B \rangle$: defined for subgroups A and B of a group G.

$A \rtimes B$ for the semi-direct product of the normal subgroup A by the subgroup B.

$A \wr B$ for the permutational wreath product of the finite permutation groups A and B. It is equal to $\big(\prod_{i=1}^{n} A_i\big) \rtimes B$ where: each A_i is a copy of A; and B is a subgroup of $\mathrm{Sym}(n)$, acting by permutation of the factors.

$A_m \wr \cdots \wr A_1$ for the permutational wreath product of the finite permutation groups A_i. It is equal to $\big(\prod_{i=1}^{n} B_i\big) \rtimes A_1$ where: each B_i is a copy of $A_m \wr \cdots \wr A_2$; and A_1 is a subgroup of $\mathrm{Sym}(n)$, acting by permutation of the factors.

$G \wr A = \big(\prod_{i=1}^{n} G_i\big) \rtimes A$ where: G is an arbitrary group, with each G_i a copy of G; and A is a subgroup of $\mathrm{Sym}(n)$, acting by permutation of the factors.

$G \wr A_m \wr \cdots \wr A_1 = \big(\prod_{i=1}^{n} H_i\big) \rtimes A_1$ where: each A_i is a finite permutation group; G is an arbitrary group; each H_i is a copy of $G \wr A_m \wr \cdots \wr A_2$; and A_1 is a subgroup of $\mathrm{Sym}(n)$, acting by permutation of the factors.

$C_2 \wr C_\infty$ is the group defined by the presentation $\langle x, y \mid y^2 = 1,\ [y, y^{x^k}] = 1\ \forall k \in \mathbb{N} \rangle$.

T_i; $T_{\underline{d}}$: a subtree of T (see Definition 2.9); the rooted $\underline{d}$-ary tree (see Definition 2.44).

$\dot{\sigma} = \sigma(1, \ldots, 1)$: a rooted automorphism (see Definition 2.21).

$\overline{\sigma} = \sigma(\overline{\sigma}, \ldots, \overline{\sigma})$: the tree automorphism defined in Definition 2.47.

Chapter 1

Introduction

1.1 Subgroups of Finite Index

Let G be a finitely generated group and let n be a natural number. Denote the number of subgroups of index n in G by $a_n(G)$. This is an invariant of the group G; that is, it is well-defined up to isomorphism. It is not immediately clear that G has only finitely many subgroups of index n. A proof of this fact can be found in [H], the first paper to be published on the subject of subgroup growth. The argument is as follows:

Let H be a subgroup of index n in G. Label H with 1 and label the other right cosets of H in G with $2, \ldots, n$ in any order. Observe that G acts on the set of right cosets of H by right multiplication. This gives a homomorphism $\varphi : G \to \mathrm{Sym}(n)$ with transitive image. Observe that $\mathrm{Stab}_{G,\varphi}(1) = H$. Conversely, given a homomorphism $\varphi : G \to \mathrm{Sym}(n)$ with transitive image, the Orbit-Stabilizer Theorem implies that $\mathrm{Stab}_{G,\varphi}(1)$ is a subgroup of index n in G. Define $t_n(G)$ to be the number of homomorphisms $G \to \mathrm{Sym}(n)$ with transitive image. The labelling of the right cosets other than H was arbitrary, so $a_n(G) = t_n(G)/(n-1)!$ by the established correspondence. Since G is finitely generated, there are only finitely many homomorphisms $G \to \mathrm{Sym}(n)$. Therefore, G has only finitely many subgroups of index n.

In the same paper, Marshall Hall Jr also gives a recursive formula for calculating $a_n(G)$, provided one knows the number of homomorphisms $G \to \mathrm{Sym}(n)$ for all n. If F_d is the free group on d generators then the number of homomorphisms $F_d \to \mathrm{Sym}(n)$ is $(n!)^d$. Thus, Hall has found a recursive formula for the number of subgroups of index n in a finitely generated free group.

1.2 Subgroup Growth

In general, it is very hard to gain information about $a_n(G)$ by using Hall's recursive formula. The function $a_n(G)$ can behave very irregularly; for example, there may be infinitely many values of n for which G has no subgroups of index n at all. In order to smooth out the irregularities in $a_n(G)$, one defines the *subgroup growth function* of G to be the summation

$$s_n(G) := \sum_{m \leqslant n} a_m(G).$$

This function can be studied asymptotically as $n \to \infty$. Let $f : \mathbb{N} \to \mathbb{R}$ be a function. A group G is said to have *subgroup growth of type* f if there exist positive constants α and β such that

$$\begin{array}{rcll} s_n(G) & \leqslant & f(n)^{\alpha} & \text{for all } n\text{; and} \\ s_n(G) & \geqslant & f(n)^{\beta} & \text{for infinitely many } n. \end{array}$$

Every finitely generated non-abelian free group has subgroup growth of type n^n. This can be deduced from Hall's recursive formula. Every finitely generated group is a quotient of a finitely generated free group. Therefore, n^n is the fastest possible subgroup growth type in a finitely generated group.

A group G is said to have *polynomial subgroup growth* if G has subgroup growth of type at most n. Between n and n^n there is a whole hierarchy of functions. The current state of knowledge about the subgroup growth of different classes of groups is explained in detail in the book [SG].

Let G be a group and let $d(G)$ denote the minimal size of a generating set for G. The *rank* of G, denoted $\mathrm{rk}(G)$, is defined by

$$\mathrm{rk}(G) := \sup_{H \leqslant G} \{d(H) \mid d(H) < \infty\}.$$

The PSG (Polynomial Subgroup Growth) Theorem of Lubotzky, Mann and Segal (Theorem 5.1 in [SG]) states the following:

> "Let G be a finitely generated residually finite group. Then G has PSG if and only if G is virtually soluble of finite rank."

1.3 Upper p-Rank

Let G be a group. The *upper rank* of G, denoted $\mathrm{ur}(G)$, is defined by

$$\mathrm{ur}(G) := \sup \left\{ \mathrm{rk}\left(\overline{G}\right) \;\middle|\; \overline{G} \text{ is a finite quotient of } G \right\}.$$

Let p be a prime. The *p-rank* of a finite group H, denoted $\mathrm{rk}_p(H)$, is defined to be the rank of a Sylow p-subgroup of H. The *upper p-rank* of G, denoted $\mathrm{ur}_p(G)$, is defined by

$$\mathrm{ur}_p(G) := \sup \left\{ \mathrm{rk}_p \left(\overline{G}\right) \;\middle|\; \overline{G} \text{ is a finite quotient of } G \right\}.$$

Conjecture A in [S3] states the following:

> "Let G be a finitely generated soluble group. If $\mathrm{ur}_p(G)$ is finite for every prime p, then G has finite upper rank."

In [S1] it is shown that if the conjecture is true then there is a gap in the possible subgroup growth types of finitely generated soluble groups between polynomial subgroup growth and type $n^{\log n/(\log\log n)^2}$. This is known as the 'soluble version of the Lubotzky Gap Conjecture', and is the motivation for Chapter 3. The original Lubotzky Gap Conjecture, that there is a gap in the possible subgroup growth types of all finitely generated groups, was refuted by the author's supervisor in [S2]. We try to construct a finitely generated group G that has finite upper p-rank for all primes p such that $\mathrm{ur}_p(G)$ is unbounded as p ranges over all the primes. This implies that G has infinite upper rank. Indeed, if $\mathrm{ur}(G) \leqslant n$ then $\mathrm{ur}_p(G) \leqslant n$ for all primes p. The existence of such a group would be a step towards refuting the soluble version of the Lubotzky Gap Conjecture (indeed, if the group was soluble then it would refute the conjecture). If such a group is found it would certainly be interesting to calculate its subgroup growth.

The finitely generated subgroups of iterated permutational wreath products appeared to the author and his supervisor to be plausible candidates for such a group. The idea is to take a spherically homogeneous rooted tree T (see Definition 2.6) of type $(p_i)_{i\geqslant 0}$ where p_i and p_j are distinct primes whenever $i \neq j$. Consider the subgroup of $\mathrm{Aut}(T)$ defined by $(C_{p_i})_{i\geqslant 0}$ (see Definition 2.39). By Theorem 3.3, it contains a dense two-generator subgroup G (see Definition 2.36 for the definition of the topology). This group is studied in Section 3.1. It is not soluble, but is just non-soluble (see Corollary 3.17). The idea was that G would have upper p_i-rank equal to the number of vertices of level i in the tree. Thus it would have finite upper p-rank for all primes p but the upper p-rank would be unbounded as p ranges over all the primes. Unfortunately, this is not the case. In fact, the group has infinite upper p-rank for all primes p. This is implied by Corollary 3.20. The underlying reason for this phenomenon is that the profinite completion of G is not isomorphic to the closure of G in $\mathrm{Aut}(T)$. This is equivalent to G not having the congruence subgroup property (see Definition 2.33); that is, there are subgroups of finite index in G that do not contain a level stabilizer of G (see Definition 2.31). The two-generator group G is an example of a 'branch group' (see Definition 2.34 and Theorem 3.6).

In Section 3.2, we generalise the construction in Section 3.1. We feel that some progress has been made towards the construction of a finitely generated group G with infinite upper rank and finite upper p-rank for all primes p. In particular, Theorem 3.32 (together with Theorem 2.43) demonstrates that if we can control the quotient of G by $\prod_{\text{level}(\omega)=k} B_k^{'}$ as k ranges over the levels of T then we can control the upper p-rank of G.

1.4 Branch Groups

The definition of a branch group is given in Definition 2.34. The notion of a branch group was coined by Rotislav Grigorchuk (for example, see [NH1]). In the 1980s, Grigorchuk produced a three-generator subgroup of the automorphism group of the rooted binary tree (see Definitions 2.10 and 2.44) with many interesting properties. This group is commonly referred to as the 'First Grigorchuk Group', and is an example of a branch group. The First Grigorchuk Group is an infinite torsion group and has intermediate word growth (see [G] or [Ha] for details). The existence of a finitely generated group with intermediate word growth had been conjectural until then.

Branch groups are of interest in their own right. Indeed, it is known that every finitely generated group can be mapped onto a just infinite group (G is just infinite if all non-trivial quotients of G are finite), and that every just infinite group must belong to one of the following three classes (see [NH2] and the introduction of [NH1]):

(i) branch groups;

(ii) finite extensions of groups of the form $S \times \cdots \times S$ where S is a simple group; and

(iii) finite extensions of groups of the form $L \times \cdots \times L$ where L is a hereditarily just infinite group (L is hereditarily just infinite if every subgroup of finite index in L is just infinite).

Building on his early work on The First Grigorchuk Group, Grigorchuk has produced many papers about finitely generated subgroups of the rooted binary tree. The generators of these groups are recursively defined and he calls them 'automata'. The groups have a 'self-similar' (or fractal) structure in that they contain copies of themselves as restricted level stabilizers (see Definition 2.31). Informally, G contains a normal subgroup K of finite index such that $K \times K \times K \times K$ is a subgroup of K of finite index. The recursive nature of these groups means that they lend themselves to computer calculation. There is a program in GAP called `AutomGrp` which can perform such calculations on a regular tree.

The notion of a directed automorphism (see Definition 2.28) is analogous to that of an automaton. The self-similar nature of Grigorchuk's groups is mimicked in the groups in Chapter 3 (see Theorem 3.8). Since the tree is not regular (see Definition 2.44), the groups we have constructed cannot contain copies of themselves as restricted level stabilizers. They do, however, possess something of a fractal nature. Perhaps they can be said to be morally self-similar.

If G is a branch group then $G/\mathrm{rst}_G(m)'$ is virtually abelian for all $m \in \mathbb{N}_0$ (see Definition 2.31 for the definition of $\mathrm{rst}_G(m)$). This means that it contains an abelian normal subgroup of finite index. Such a normal subgroup is $\mathrm{rst}_G(m)/\mathrm{rst}_G(m)'$. That this has finite index follows from the definition of a branch group. By Theorem 2.43, if N is a non-trivial normal subgroup of a branch group G then G/N is virtually abelian. In the paper [DG], Grigorchuk raises the question of exactly which virtually abelian groups occur as the quotient of some finitely generated branch group. He states that the infinite cyclic group and the infinite dihedral group are the only known infinite examples. We have shown that virtually free-abelian groups of arbitrarily large rank occur (see Corollary 3.21). We have also shown that infinite soluble groups of arbitrarily large derived length occur (take $G/\mathrm{rst}_G(m)'$ with m suitably large - see Corollary 3.16).

1.5 Automata

For many years, Grigorchuk has been attempting to classify three-state binary automata (the classification of two-state binary automata having been completed with the publication of [GZ]). It is worth noting that The First Grigorchuk Group is defined by a five-state binary automaton. In Chapters 4 and 5 we focus on two-state ternary automata (see Definition 2.56). Attempting a complete classification up to group isomorphism is an enormous project. We restrict ourselves to two-state ternary automata with abelian action at the root (see Definition 2.64).

We have achieved a complete classification up to group isomorphism of those groups with C_2-action at the root (see Section 4.4). They are: C_2; $C_2 \times C_2$; C_∞; $C_\infty \times C_\infty$; D_∞; and $C_2 \wr C_\infty$. We have only a partial classification for the groups with C_3-action at the root (see Section 4.5). Specifically, we have: the abelian groups

C_3, $C_3 \times C_3$, and C_∞; and the non-abelian groups $H_1, \ldots, H_{17}$ defined in Definition 4.17. We prove that $H_1, \ldots, H_{17}$ are not abelian (see Lemma 5.3), contain elements of infinite order (Theorem 5.44), and are not free (Theorem 5.45). These non-abelian groups are the focus of our study in Chapter 5. It is not known whether $H_1, \ldots, H_{17}$ are pairwise non-isomorphic. In this direction, we show that H_1, H_3, H_4, H_6, and H_9 are pairwise non-isomorphic (see Corollary 5.9, Theorem 5.18, Corollary 5.38, and Theorem 5.43). Therefore, up to group isomorphism the number of two-state ternary automata with C_3-action at the root is at least eight and at most twenty. Chapter 5 ends with some conjectures about possible isomorphisms between the H_i, and with a summary of the work that has been done towards a classification, up to group isomorphism, of the non-abelian $(2, 3)$-automaton groups with C_3-action at the root.

We introduce the notion of 'level growth type' in Section 2.7. We believe this is a new idea. It proves to be useful in Section 5.4, where we prove that many of the H_i have an infinite cyclic quotient. By Lemma 4.40, any group with an infinite cyclic quotient does not have the congruence subgroup property. In Section 4.6, we introduce the notion of a 'windmill automaton'. We believe that this is also a new idea. During the author's research, it was found that groups defined by windmill automata naturally occur as subgroups of two-state automaton groups (see Remark 4.22). We classify the groups defined by binary windmill automata. We also study a ternary windmill automaton that is of particular interest because it occurs as a normal subgroup of index 3 in H_2.

Chapter 2

Preliminaries

2.1 Spherically Homogeneous Rooted Trees

2.1 Definition.
A *rooted tree* is a tree T with a distinguished vertex ν_ϕ called the *root*. The *level* of a vertex ν is the distance of ν from the root. The level of ν is denoted $\text{level}_T(\nu)$, or simply $\text{level}(\nu)$, when the tree T is understood.

2.2 Definition.
Let T be a rooted tree and $i \in \mathbb{N}_0$. Define $T[i]$, the *truncation* of T to level i, to be the rooted subtree of T with root ν_ϕ whose vertex set consists of all the vertices of level at most i in T.

2.3 Definition.
Let T be a rooted tree. If ν is a vertex of level $i > 0$ then define the *mother* of ν to be the vertex of level $i-1$ on the unique path of length i from the root to ν. The *children* of a vertex ω are those vertices ν such that ω is the mother of ν. We say that ω is a *terminal* vertex if ω has no children. Define $L(T)$ to be the set of integers n such that there exists a non-terminal vertex of level n in T.

2.4 Definition.
Let T be a rooted tree and ν a vertex of T. The vertices of the tree which can be reached from ν by taking a sequence of mothers are called the *ancestors* of ν. The vertices which can be reached from ν by taking a sequence of children are called the *descendants* of ν. We define ν to be a descendant of itself, but not an ancestor of itself.

2.5 Definition.
Let $(n_0, n_1, \ldots, n_{i-1})$ be an i-tuple of integers strictly greater than 1. The *spherically homogeneous rooted tree* of *type* $(n_0, n_1, \ldots, n_{i-1})$ is the finite rooted tree such that:

(i) level(ν) $\leqslant i$ for every vertex ν;

(ii) the vertices of level i are terminal; and

(iii) for every $j \in [0, i-1]$, every vertex of level j has n_j children.

2.6 Definition.
Let $(n_i)_{i \geqslant 0}$ be a sequence of integers strictly greater than 1. The *spherically homogeneous rooted tree* of *type* $(n_i)_{i \geqslant 0}$ is the infinite rooted tree such that, for every $i \in \mathbb{N}$, the truncation of T to level i is the spherically homogeneous rooted tree of type $(n_0, n_1, \ldots, n_{i-1})$.

Note that a spherically homogeneous rooted tree may be either finite or infinite. We use the notation $(n_i)_{i \in L(T)}$ to specify the type of tree, even when it has not been specified if the tree is finite or infinite. This is to be read as a finite tuple if the tree is finite, and as a sequence if the tree is infinite. This allows us to state results that hold true for finite and infinite trees alike without the need for tedious repetition. The integers in $L(T)$ are referred to as the *non-terminal levels* of T.

2.7 Remark.
We think of the tree growing down from the root. Label the edges extending down from a vertex of level i by the integers modulo n_i. When drawn on the page, the edges are labelled left-to-right with the left-most edge labelled 1 and the right-most edge labelled n_i.

2.8 Definition.
Let T be a spherically homogeneous rooted tree. The vertex of level i which can be reached by the sequence $(m_0, m_1, \ldots, m_{i-1})$ of edges is denoted $\nu_{(m_0, m_1, \ldots, m_{i-1})}$. For brevity, the vertices of the first level are denoted $\nu_1, \ldots, \nu_{n_0}$ respectively.

2.9 Definition.
Let T be a spherically homogeneous rooted tree and ν a vertex of T. The subtree formed by the descendants of ν is denoted T_ν. For any two vertices ν and ω of level i, the subtrees T_ν and T_ω are of the same type. The tree of this type is denoted T_i.

We identify T_ν and T_ω with T_i when it is convenient to do so.

2.2 Tree Automorphisms

2.10 Definition.
Let T be a spherically homogeneous rooted tree. An *automorphism* of T is a permutation of the vertices which preserves mothers. The set of automorphisms of T form a group, with composition of permutations as the group multiplication. This group is called the *automorphism group* of T, and is denoted $\mathrm{Aut}(T)$.

2.11 Remark.
Preservation of mothers implies preservation of level. Thus, $\mathrm{Aut}(T[i])$ is naturally isomorphic to the permutational wreath product $\mathrm{Sym}(n_{i-1}) \wr \cdots \wr \mathrm{Sym}(n_1) \wr \mathrm{Sym}(n_0)$, where i is any non-zero level of the tree. Note that $\mathrm{Aut}(T[0])$ is the trivial group. For every non-terminal level i, there is an epimorphism $\mathrm{Aut}(T[i+1]) \to \mathrm{Aut}(T[i])$ given by ignoring the first factor in the wreath product. If T is infinite then $\mathrm{Aut}(T)$ is isomorphic to the projective limit $\varprojlim_{i \geqslant 1} \mathrm{Aut}(T[i])$. Indeed, two tree isomorphisms that agree on all the truncations $T[i]$ of T ($i \in \mathbb{N}$) are the same. The infinite group $\mathrm{Aut}(T)$ is given the profinite topology. This is the topology induced from the discrete topology on the finite groups $\mathrm{Aut}(T[i])$.

2.12 Definition.
Let T be a spherically homogeneous rooted tree, let ν be a vertex of T and let x be a tree automorphism. The image of ν under the action of x (that is, the vertex to which ν is sent by x) is denoted ν^x.

2.13 Definition.
Let T be the spherically homogeneous rooted tree of type $(n_i)_{i \in L(T)}$. For every level i of T, let $\pi_i : \mathrm{Aut}(T) \to \mathrm{Aut}(T[i])$ be the natural surjection. If i is a non-terminal level, we consider $\mathrm{Aut}(T[i])$ to be a subgroup of $\mathrm{Aut}(T[i+1])$ in the natural way. Let x be a tree automorphism and let ν be a non-terminal vertex of T. Define *the action of x at ν* to be the permutation of the children of ν induced by $\pi_i(x^{-1})\pi_{i+1}(x)$ where $i = \mathrm{level}_T(\nu)$. We denote the action of x at ν by $x(\nu)$ and identify it with an element of $\mathrm{Sym}(n_i)$. We also say that *x acts as $x(\nu)$ at ν*.

2.14 Definition.
Let T be a spherically homogeneous rooted tree and let x be an automorphism of T. Define the *portrait* of x to be the labelling of the non-terminal vertices ν of T by the elements $x(\nu)$ that are defined in Definition 2.13.

2.15 Remark.
Note that a tree automorphism can be defined by specifying its portrait.

2.16 Definition.
Let T be a spherically homogeneous rooted tree and let x be an element of $\mathrm{Aut}(T)$. The *support* of x is the set of non-terminal vertices of T at which x does not act as the identity.

2.17 Definition.
Let T be a spherically homogeneous rooted tree and let x be an element of $\mathrm{Aut}(T)$. We say that x has *depth* n if every vertex in the support of x has level at most $n-1$. If there exists $n \in \mathbb{N}$ such that x has depth n then we say that x has *finite depth*; otherwise, x has *infinite depth*.

The identity is the unique tree automorphism of depth 0.

2.18 Remark.
Let T be a spherically homogeneous rooted tree and i a level of T. We identify $\mathrm{Aut}(T[i])$ with the subgroup of $\mathrm{Aut}(T)$ consisting of all tree automorphisms of depth at most i. In this way, we have an ascending chain $\mathrm{Aut}(T[0]) \subseteq \mathrm{Aut}(T[1]) \subseteq \mathrm{Aut}(T[2]) \subseteq \cdots$ of subgroups of $\mathrm{Aut}(T)$. Under this identification, the natural surjection $\pi_i : \mathrm{Aut}(T) \to \mathrm{Aut}(T[i])$ is a projection map; that is, $\pi_i \circ \pi_i = \pi_i$. It is useful to note that $\pi_i \circ \pi_j = \pi_{\min\{i,j\}}$ for all $i, j \in \mathbb{N}_0$.

2.19 Lemma.
Let T be a spherically homogeneous rooted tree, let x and y be tree automorphisms, and let ν be a non-terminal vertex of T. Let xy denote the composition x followed by y. Then $xy\big(\nu\big) = x\big(\nu^{y^{-1}}\big)y\big(\nu\big)$.

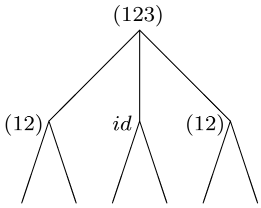
Proof:
Let i denote the level of ν. By definition, $xy(\nu)$ is the permutation of the children of ν induced by $\pi_i((xy)^{-1})\pi_{i+1}(xy)$. We calculate that

$$\begin{aligned}
\pi_i((xy)^{-1})\pi_{i+1}(xy) &= \pi_i(y)^{-1}\pi_i(x)^{-1}\pi_{i+1}(x)\pi_{i+1}(y) \\
&= \pi_i(x)^{-\pi_i(y)}\pi_{i+1}(x)^{\pi_i(y)}\pi_i(y)^{-1}\pi_{i+1}(y) \\
&= \pi_i\big(x^{-\pi_i(y)}\big)\pi_{i+1}\big(x^{\pi_i(y)}\big)\pi_i\big(y^{-1}\big)\pi_{i+1}\big(y\big).
\end{aligned}$$

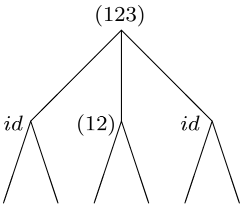
The final equality follows from the fact that $\pi_i \circ \pi_i = \pi_i$ and $\pi_{i+1} \circ \pi_i = \pi_i$, as observed in Remark 2.18. It has thus been shown that $xy(\nu) = x^{\pi_i(y)}(\nu)y(\nu)$, and the result follows from the observation that $x^{\pi_i(y)}(\nu) = x(\nu^{\pi_i(y)^{-1}}) = x(\nu^{y^{-1}})$. $\square$

2.20 Example.
A short example demonstrating the composition of tree automorphisms may be useful. Define elements of $\mathrm{Aut}(T_{(3,2)}) = \mathrm{Sym}(2) \wr \mathrm{Sym}(3)$ by their portraits as follows:

$x :=$

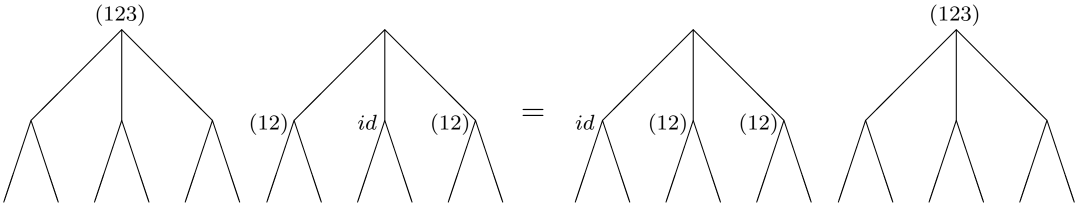


$y :=$



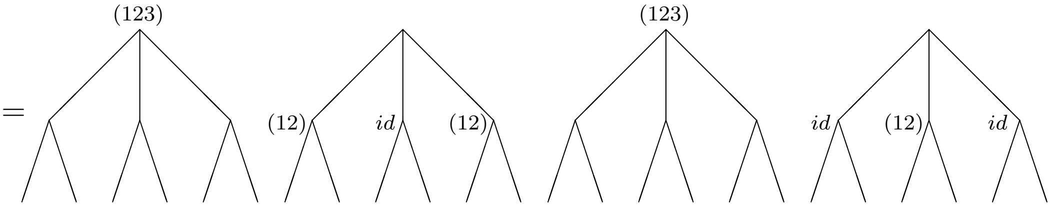
The first thing to understand is how x and y permute the vertices of the second level in $T_{(3,2)}$. In temporary notation, label the vertices of the second level simply 1, 2, 3, 4, 5, 6. Then $x = (136)(245)$ and $y = (146235)$. Observe that

$x =$

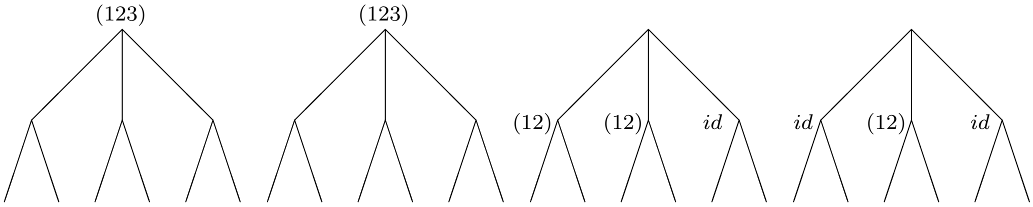


Calculate $xy = (136)(245)(146235) = (153264)$. We now check that the same answer is obtained on the tree.

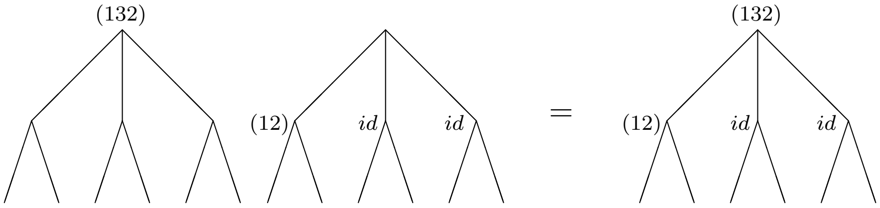
$xy =$



$=$



$=$



This tree automorphism does indeed permute the vertices of the second level by (153264).

2.21 Definition.
Let T be a spherically homogeneous rooted tree. Let n_0 denote the valency of the root of T. Given an element $\beta_0 \in \mathrm{Sym}(n_0)$, the *rooted automorphism* $\dot{\beta}_0 \in \mathrm{Aut}(T)$ is defined to be the tree automorphism which acts as β_0 at the root and as the identity at all other vertices of T.

2.22 Remark.
We use id to denote the identity element of $\mathrm{Sym}(n_i)$ and 1 to denote the identity element of $\mathrm{Aut}(T)$. Thus, $\dot{\mathrm{id}} = 1$. If Γ_i is a subgroup of $\mathrm{Sym}(n_i)$, say $\Gamma_i = \{\gamma_1, \ldots, \gamma_k\}$, then we denote the subgroup $\{\dot{\gamma_1}, \ldots, \dot{\gamma_k}\}$ of $\mathrm{Aut}(T_i)$ by $\dot{\Gamma}_i$.

2.23 Definition.
Let T be a spherically homogeneous rooted tree and let x be an automorphism of T. For each vertex ν, the labelling of the subtree T_ν in the portrait of x defines an element of $\mathrm{Aut}(T_\nu)$. This automorphism of T_ν is denoted x_ν.

2.24 Definition.
Let T be a spherically homogeneous rooted tree, let x be an automorphism of T and let i be a level of T. The *expansion* of x to level i is the labelling of the vertices ν of the finite tree $T[i]$ given by:

(i) if $\mathrm{level}(\nu) < i$ then ν is labelled by $x(\nu)$; and

(ii) if $\mathrm{level}(\nu) = i$ then ν is labelled by x_ν.

2.25 Remark.
Note that a tree automorphism can be defined by its expansion to level i, where i is any chosen level of the tree.

2.26 Definition.
Let T be a spherically homogeneous rooted tree and let x be an automorphism of T. If in the expansion of x to the first level, the terminal vertices of $T[1]$ (namely $\nu_1, \ldots, \nu_{n_0}$) are labelled by $x_1, \ldots, x_{n_0}$, and its root is labelled by σ, then we denote this expansion by $x = \sigma(x_1, \ldots, x_{n_0})$. If $\sigma = \mathrm{id}$ then we simply write $x = (x_1, \ldots, x_{n_0})$.

2.27 Remark.
Note that $\sigma(x_1, \ldots, x_{n_0})$ is equal to the composition of $\dot{\sigma}$ with $(x_1, \ldots, x_{n_0})$. A useful observation is that $\dot{\sigma}(x_1, \ldots, x_{n_0}) = (x_{1^\sigma}, \ldots, x_{n_0^\sigma})\dot{\sigma}$.

2.28 Definition.
Let T be the spherically homogeneous tree of type $(n_i)_{i \in L(T)}$. Fix an element $\beta_i \in \mathrm{Sym}(n_i)$ for each $i \in L(T) \backslash \{0\}$. The *directed automorphism defined by the sequence* (β_i) is the automorphism b of T whose portrait is defined by the following action at the vertices ν of T:

(i) if there exists $j \in L(T) \backslash \{0\}$ such that $\nu = \nu_{(n_0, \ldots, n_{j-2}, 1)}$ then $b(\nu) = \beta_j$; and

(ii) if $\nu \notin \{\nu_{(n_0, \ldots, n_{j-2}, 1)} \mid j \in L(T) \backslash \{0\}\}$ then $b(\nu) = \mathrm{id}$.

In particular, note that $b(\nu_\phi) = \mathrm{id}$, that $b(\nu_1) = \beta_1$, and that $b(\nu_{(n_0,1)}) = \beta_2$.

2.3 Stabilizers and Branch Groups

2.29 Definition.
Let T be a spherically homogeneous rooted tree and let G be a subgroup of $\mathrm{Aut}(T)$. Given a vertex ν, define the *stabilizer* of ν in G to be the subgroup of G consisting of those automorphisms that fix ν. The stabilizer of ν in G is denoted $\mathrm{st}_G(\nu)$, or simply $\mathrm{st}(\nu)$, when the group G is understood. Define the *restricted stabilizer* of ν in G, denoted $\mathrm{rst}_G(\nu)$ or $\mathrm{rst}(\nu)$, to be the subgroup of G consisting of those automorphisms that fix ν and all the vertices that are not descendants of ν.

2.30 Remark.
In order to fix ν, an automorphism must fix all the ancestors of ν. Note that $\mathrm{st}_G(\nu)$ and $\mathrm{rst}_G(\nu)$ are not, in general, normal subgroups of G. We identify $\mathrm{rst}_G(\nu)$ with a subgroup of $\mathrm{Aut}(T_\nu)$ in the natural way.

2.31 Definition.
Let T be a spherically homogeneous rooted tree and let G be a subgroup of $\mathrm{Aut}(T)$. For $i \in \mathbb{N}_0$, define the i-th *level stabilizer* of G by

$$\mathrm{st}_G(i) := \bigcap_{\mathrm{level}(\nu)=i} \mathrm{st}_G(\nu)$$

and the i-th *restricted level stabilizer* of G by

$$\mathrm{rst}_G(i) := \prod_{\mathrm{level}(\nu)=i} \mathrm{rst}_G(\nu).$$

When the group G is understood, we write $\mathrm{st}(i)$ and $\mathrm{rst}(i)$ respectively. In particular, $G/\mathrm{st}(i)$ denotes $G/\mathrm{st}_G(i)$, and $G/\mathrm{rst}(i)$ denotes $G/\mathrm{rst}_G(i)$.

2.32 Remark.
Note that st(i) and rst(i) are normal subgroups of G, and that rst(i) is a subgroup of st(i). We identify st(i) with a subgroup of $\prod_{\text{level}(\nu)=i} \text{Aut}(T_\nu)$ in the natural way.

2.33 Definition.
Let T be a spherically homogeneous rooted tree and let G be a subgroup of Aut(T). We say that G has the *congruence subgroup property* (CSP) if, given any subgroup H of finite index in G, there exists a level i such that H contains $\text{st}_G(i)$.

2.34 Definition.
Let T be a spherically homogeneous rooted tree and let G be a subgroup of Aut(T). We say that G is *spherically transitive* if G acts transitively on every level of T. We say that G is a *branch group* if G is spherically transitive and every restricted level stabilizer of G has finite index in G.

2.35 Remark.
The term 'branch group' was coined by Rotislav Grigorchuk. For example, see [NH1].

Note that a metric may be defined on the vertices of T, the distance between two vertices being the length of the shortest path between them. In this metric space, the vertices of a given level form a sphere with the root at its centre. A spherically transitive subgroup of Aut(T) acts transitively on each such sphere.

As mentioned in Remark 2.11, if T is infinite then Aut(T) is a profinite group and, as such, it has a natural profinite topology induced from the discrete topology on the finite groups Aut($T[i]$).

2.36 Definition.
Let T be an infinite spherically homogeneous rooted tree and let G be a subgroup of Aut(T). The *topology* on G is defined to be the subspace topology induced from the profinite topology on Aut(T). The closure of G in Aut(T) is denoted $\overline{G}$.

2.37 Remark.
The level stabilizers of G form a basis for the neighbourhoods of the identity. For $i \in \mathbb{N}$, there is a natural restriction homomorphism $\pi_i : \text{Aut}(T) \to \text{Aut}(T[i])$ given by considering the action of the tree automorphisms on the finite subtree $T[i]$. Let $x \in \text{Aut}(T)$. Then $x \in \overline{G}$ if and only if $\pi_i(x) \in \pi_i(G)$ for all $i \in \mathbb{N}$. Note that $\text{st}_G(i) = \ker(\pi_i|_G)$. Therefore, $\text{st}_G(i)$ has finite index in G. We identify $G/\text{st}_G(i)$ with a subgroup of Aut($T[i]$) in the natural way.

2.38 Remark.
Let H be a subgroup of $\mathrm{Aut}(T)$ containing G. Then G is dense in H if and only if $\pi_i(G) = \pi_i(H)$ for all $i \in \mathbb{N}$.

2.4 Subgroups of $\mathrm{Aut}(T)$

2.39 Definition.
Let T be the spherically homogeneous rooted tree of type $(n_i)_{i\in L(T)}$. For every non-terminal level i of the tree let Γ_i be a subgroup of $\mathrm{Sym}(n_i)$. The *subgroup of* $\mathrm{Aut}(T)$ *defined by* $(\Gamma_i)_{i\in L(T)}$ consists of those automorphisms $x \in \mathrm{Aut}(T)$ such that $x(\nu) \in \Gamma_i$ for every $i \in L(T)$ and every vertex ν of level i.

If T is infinite then the subgroup defined by $(\Gamma_i)_{i\in L(T)}$ is closed in $\mathrm{Aut}(T)$.

2.40 Lemma.
Let T be the spherically homogeneous rooted tree of type $(n_i)_{i\in L(T)}$. For every non-terminal level i let Γ_i be a subgroup of $\mathrm{Sym}(n_i)$. Let H denote the subgroup of $\mathrm{Aut}(T)$ defined by $(\Gamma_i)_{i\in L(T)}$. Fix a non-terminal level k. For $\gamma_k \in \Gamma_k$, let $\overline{\gamma_k}$ denote the image of γ_k under the natural epimorphism $\Gamma_k \to \Gamma_k^{\mathrm{ab}}$. Then the map defined by

$$\begin{array}{rcl} \varphi_k : H & \to & \Gamma_k^{\mathrm{ab}} \\ h & \mapsto & \displaystyle\prod_{\mathrm{level}(\nu)=k} \overline{h(\nu)} \end{array}$$

is a surjective group homomorphism.

Proof:
Let g and h be elements of H. Then

$$\begin{array}{rcl} \varphi_k(gh) & = & \displaystyle\prod_{\mathrm{level}(\nu)=k} \overline{gh(\nu)} = \prod_{\mathrm{level}(\nu)=k} \overline{g(\nu^{h^{-1}})h(\nu)} \\ & = & \displaystyle\prod_{\mathrm{level}(\nu)=k} \overline{g(\nu^{h^{-1}})} \prod_{\mathrm{level}(\nu)=k} \overline{h(\nu)} \\ & = & \displaystyle\prod_{\mathrm{level}(\nu)=k} \overline{g(\nu)} \prod_{\mathrm{level}(\nu)=k} \overline{h(\nu)} \\ & = & \varphi_k(g)\varphi_k(h). \end{array}$$

Therefore, φ_k is a homomorphism. Given an element $\overline{\gamma_k} \in \Gamma_k/\Gamma_k'$, take an automorphism h whose portrait has γ_k at any one chosen vertex of level k and has the identity at all other vertices of the tree. Then $\varphi_k(h) = \overline{\gamma_k}$. Therefore, φ_k is surjective as required. □

The following theorem is weaker than Theorem 2.43 and, as such, could be omitted. However, we include it because the proof is so attractive.

2.41 Theorem (Lemma 13.4.3 in [SG])**.**
Let T be an infinite spherically homogeneous rooted tree. Let G be a spherically transitive subgroup of $\mathrm{Aut}(T)$. If N is a normal subgroup of finite index in G then there exists $m \in \mathbb{N}_0$ such that N contains $\mathrm{rst}_G(m)'$.

Proof: (due to Peter Neumann).
The group G/N is finite, so it contains only finitely many subgroups. Thus, if m is large enough, there exist distinct vertices μ, ν of level m such that $N\mathrm{rst}_G(\mu) = N\mathrm{rst}_G(\nu)$. Therefore,

$$\begin{aligned}\mathrm{rst}_G(\mu)' &= [\mathrm{rst}_G(\mu), \mathrm{rst}_G(\mu)] \\ &\leqslant [N\mathrm{rst}_G(\mu), N\mathrm{rst}_G(\mu)] \\ &= [N\mathrm{rst}_G(\mu), N\mathrm{rst}_G(\nu)] \\ &\leqslant N[\mathrm{rst}_G(\mu), \mathrm{rst}_G(\nu)].\end{aligned}$$

As $\mathrm{rst}_G(\mu)$ and $\mathrm{rst}_G(\nu)$ have disjoint supports, we see that $[\mathrm{rst}_G(\mu), \mathrm{rst}_G(\nu)] = 1$, and so $\mathrm{rst}_G(\mu)' \leqslant N$. As G is spherically transitive, $\mathrm{rst}_G(\mu)'$ is conjugate in G to $\mathrm{rst}_G(\omega)'$ for every vertex ω of level m. Hence

$$\mathrm{rst}_G(m)' = \prod_{\mathrm{level}(\omega)=m} \mathrm{rst}_G(\omega)'$$

is a subgroup of N, as required. □

2.42 Corollary.
Let T be an infinite spherically homogeneous rooted tree and let G be a spherically transitive subgroup of $\mathrm{Aut}(T)$. If H is a subgroup of finite index in G then there exists $m \in \mathbb{N}_0$ such that H contains $\mathrm{rst}_G(m)'$.

Proof:
The normal core of H has finite index in G. Apply the theorem to this normal subgroup. □

2.43 Theorem (Lemma 4 in [S2])**.**
Let T be an infinite spherically homogeneous rooted tree and let G be a spherically transitive subgroup of $\mathrm{Aut}(T)$. If N is a non-trivial normal subgroup of G (not necessarily of finite index) then there exists $m \in \mathbb{N}_0$ such that N contains $\mathrm{rst}_G(m)'$.

Proof: (extracted from Theorem 4 in [NH1]).
Let x be a non-identity element of N. There exists $m \in \mathbb{N}$ such that $x \in \mathrm{st}_G(m-1)$ and $x \notin \mathrm{st}_G(m)$. There exists a vertex μ of level m such that $\mu^x \neq \mu$. Let $a, b \in \mathrm{rst}_G(\mu)$. From the definition of group multiplication in $\mathrm{Aut}(T)$ it follows that

$$[a^{-1}, x] \in \mathrm{rst}_{\mathrm{Aut}_T}(\mu) \times \mathrm{rst}_{\mathrm{Aut}_T}(\mu^x).$$

We calculate that $[a^{-1}, x]_\mu = a_\mu$ (recall from Definitions 2.23 and 2.24 that a_μ is the label of μ in the expansion of a to level(μ)). Therefore, $[a^{-1}, x, b] = [a, b]$. As N is a normal subgroup, $[a^{-1}, x, b]$ is an element of N. Therefore, $N \geqslant \mathrm{rst}_G(\mu)'$. As G is spherically transitive, $\mathrm{rst}_G(\mu)'$ is conjugate to $\mathrm{rst}_G(\omega)'$ for every vertex ω of level m. Hence

$$\mathrm{rst}_G(m)' = \prod_{\mathrm{level}(\omega)=m} \mathrm{rst}_G(\omega)'$$

is a subgroup of N as required. □

2.5 Regular Rooted Trees

2.44 Definition.
Let d be an integer strictly greater than 1. We denote the constant sequence $(d, d, \ldots)$ by $\underline{d}$. The *rooted d-ary tree*, denoted $T_{\underline{d}}$, is defined to be the spherically homogeneous rooted tree of type $\underline{d}$. The *rooted binary tree* is the rooted 2-ary tree, and the *rooted ternary tree* is the rooted 3-ary tree. A *regular rooted tree* is a rooted d-ary tree where d is an unspecified integer strictly greater than 1.

2.45 Definition.
Let d be an integer strictly greater than 1 and let $X = \{1, 2, \ldots, d\}$. Let X^* denote the free multiplicative monoid generated by the elements of X. The elements of X^* are called *words over X*.

2.46 Remark.
The words over X may be naturally identified with the set of vertices of the rooted d-ary tree. Indeed, the words of length n over X are identified with the vertices of level n in the tree. Note that the empty word is identified with the root of the tree.

2.47 Definition.
Let T be the rooted d-ary tree. Given a permutation σ in $\mathrm{Sym}(d)$, define $\overline{\sigma}$ to be the tree automorphism with $\overline{\sigma}(\nu) = \sigma$ for all vertices ν of T; that is, in the portrait of $\overline{\sigma}$, every vertex of T is labelled by σ.

2.48 Remark.

Note that $\overline{\sigma} = \sigma(\overline{\sigma}, \ldots, \overline{\sigma})$, and that conjugating $\mathrm{Aut}(T)$ by $\overline{\sigma}$ corresponds to a renumbering of the alphabet X.

2.49 Definition.

Let T be a regular rooted tree. Let x and y be elements of $\mathrm{Aut}(T)$ and $k \in \mathbb{N}_0$. We say that *x occurs in the expansion of y to level k* if, in the expansion of y to level k, there exists a vertex of level k labelled by x. We say that *x occurs in the expansion of y* if there exists $k \in \mathbb{N}_0$ such that x occurs in the expansion of y to level k.

Note that y occurs in the expansion of y (to level 0).

2.50 Lemma.

Let T be a regular rooted tree and let x be an automorphism of T. Let $k = \mathrm{ord}(x(\nu_\phi))$ and suppose that $k > 1$. If there exists a tree automorphism y and an integer l with $|l| < k$ such that x^{ly} occurs in the expansion of x^k to the first level then x has infinite order.

Proof:

For a contradiction, suppose that x has finite order. Then the order of x must be divisible by k. Suppose that x has order km. In the expansion of x^{km} to the first level there is a vertex labelled by x^{lmy}. As $x^{km} = 1$, we must have that $x^{lmy} = 1$, and hence that $x^{lm} = 1$. However, as $|lm| < km$, this contradicts the claim that $\mathrm{ord}(x) = km$.

□

2.51 Definition.

Let T be a regular rooted tree and S a subset of $\mathrm{Aut}(T)$. We say that S is *closed under expansion* if, given any element s in S, every tree automorphism that occurs in the expansion of s to the first level is an element of S.

2.52 Remark.

If S is closed under expansion and s is an element of S then, in the expansion of s to any level n of the tree, every vertex of level n is labelled by an element of S.

2.53 Lemma.

Let T be a regular rooted tree. Let S be a subset of $\mathrm{Aut}(T)$ closed under expansion. If every element of S acts as the identity at the root then $S = \{1\}$.

Proof:
Let s be an element of S. By hypothesis, s stabilizes pointwise the first level of T. For induction, suppose that s stabilizes pointwise level n of T. Consider the expansion of s to level n. By Remark 2.52, every vertex of level n is labelled by an element of S. By hypothesis, the elements of S act as the identity at the root. Therefore, s stabilizes pointwise level $n+1$ of T. As s stabilizes pointwise every level of the tree, it must be the identity automorphism of T. □

2.54 Theorem.
Let T be the regular d-ary tree. Let x be an automorphism of T that acts as the identity at the root. Let N denote the normal closure of $\langle x\rangle$ in $\mathrm{Aut}(T)$. Suppose that every tree automorphism that occurs in the expansion of x to the first level is an element of N. Then x is the identity automorphism of T.

Proof:
Observe that every element of N acts as the identity at the root. Let $(n_1, \ldots, n_d)$ be the expansion of x to the first level. Let $y = (y_1, \ldots, y_d)\dot{\sigma}$ be any tree automorphism. Then $y^{-1} = \dot{\sigma}^{-1}(y_1^{-1}, \ldots, y_d^{-1})$. Writing $\phi(i) = i^{\sigma^{-1}}$, we see that

$$x^y = \dot{\sigma}^{-1}\big(n_1^{y_1}, \ldots, n_d^{y_d}\big)\dot{\sigma} = \big(n_{\phi(1)}^{y_{\phi(1)}}, \ldots, n_{\phi(d)}^{y_{\phi(d)}}\big).$$

This shows that every tree automorphism that occurs in the expansion of x^y to the first level is an element of N. As N is generated by elements of the form x^y, we have shown that N is closed under expansion. The result follows by Lemma 2.53. □

2.6 Automata

2.55 Definition.
A *finite invertible automaton* A is an ordered quadruple $A = (Q, X, \rho, \tau)$ such that:

(i) Q and X are finite sets of cardinality at least 1 and 2, respectively;

(ii) $\rho : Q \times X \to X$ and $\tau : Q \times X \to Q$ are functions; and

(iii) for each $q \in Q$ the restriction map $\rho_q : X \to X$ given by $\rho_q(x) = \rho(q, x)$ is a permutation of the set X.

In this thesis, we say *automaton* to mean finite invertible automaton.

2.56 Definition.
The elements of Q are called *states*, X is called the *alphabet*, ρ is called the *output map*, and τ is called the *transition map.* If $|Q| = c$ and $|X| = d$, then we call A a (c, d)*-automaton.* The elements of X are denoted $1, 2, \ldots, d$. An automaton with c specified and d unspecified is called a c*-state automaton.* An automaton with d specified and c unspecified is called a d*-ary automaton.* A *binary automaton* is a 2-ary automaton, and a *ternary automaton* is a 3-ary automaton.

2.57 Definition.
Let A be an automaton. For each state q, we define a permutation $\overline{q}$ of the words $w \in X^*$ inductively on the length of w as follows:

(i) if w is the empty word then $w^{\overline{q}} := w$;

(ii) if $w = xu$ for some $x \in X$ and $u \in X^*$ then $w^{\overline{q}} := \rho_q(x)u^{\overline{\tau(q,x)}}$.

2.58 Lemma.
Let $A = (Q, X, \rho, \tau)$ be a (c, d)-automaton and let T be the rooted d-ary tree. Let the elements of Q be denoted $q_1, \ldots, q_c$. Then there exist unique automorphisms $\overline{q_1}, \ldots, \overline{q_c}$ of T satisfying the system

$$\overline{q_i} = \rho_{q_i}\Big(\overline{\tau(q_i, 1)}, \ldots, \overline{\tau(q_i, d)}\Big) \qquad i \in [1, c] \tag{2.1}$$

of equations.

Proof:
Identify the vertices of T with the elements of X^* as in Remark 2.46. For each state q_i, the permutation $\overline{q_i}$ of X^* defined in Definition 2.57 may be seen to preserve both level and ancestry in the tree T. Indeed, its very definition is inductive on the levels T. Therefore, for each state q_i, we have defined an element $\overline{q_i}$ of $\mathrm{Aut}(T)$ and, by definition, $\overline{q_1}, \ldots, \overline{q_c}$ satisfy system (2.1).

In order to prove the uniqueness of $\overline{q_1}, \ldots, \overline{q_c}$, note that system (2.1) forces the action of $\overline{q_i}$ at the root to be ρ_{q_i}. By induction on the level, we see that the action of $\overline{q_i}$ on the truncation of T to any finite level is uniquely determined by system (2.1). However, a tree automorphism is uniquely determined by its action on the finite truncations of T. □

2.59 Remark.
Let A be an automaton. We identify the states q with the associated permutations $\overline{q}$ of X^*, and with the associated automorphisms $\overline{q}$ of T. In an abuse of notation, we

use the same letter q to denote the state, the associated permutation of X^*, and the associated tree automorphism.

2.60 Remark.
Given finite sets Q and X of cardinality $c \geqslant 1$ and $d \geqslant 2$, respectively, we may define an automaton by defining the output map ρ and the transition map τ. In order to define these maps, it is sufficient to specify the restriction maps ρ_q for all $q \in Q$, and to specify $\tau(q, x)$ for all $q \in Q$ and $x \in X$. The tree automorphisms $q = \rho_q\big(\tau(q, 1), \ldots, \tau(q, d)\big)$ of Lemma 2.58 are then specified. Note that the definitions of ρ_q and $\tau(q, x)$ may be inferred from the equation $q = \rho_q\big(\tau(q, 1), \ldots, \tau(q, d)\big)$. Therefore, the system of equations $q = \rho_q\big(\tau(q, 1), \ldots, \tau(q, d)\big)$ (with q ranging over Q) may be taken as a defining set of equations for the automaton.

This observation saves a lot of time when defining automata, so we record it here as a definition.

2.61 Definition.
In order to define the output and transition maps, we shall use the following notation: for each state q, we write

$$q = \rho_q\big(\tau(q, 1), \ldots, \tau(q, d)\big).$$

Note that, by Remark 2.60, the output map ρ and the transition map τ have been uniquely defined by the system of equations $q = \rho_q\big(\tau(q, 1), \ldots, \tau(q, d)\big)$ (with q ranging over Q).

2.62 Definition.
Let A be an automaton. The group of tree automorphisms generated by the states of A is called the *automaton group* defined by A.

We identify the automaton A with the automaton group defined by A. Note that it is generated by $|Q|$ elements.

2.63 Example.
Let A be the $(2, 2)$-automaton with generators $a = \mathrm{id}(b, b)$ and $b = (12)(a, a)$. Let T be the rooted binary tree. In the portrait of a, every vertex of even level in T is labelled by id, and every vertex of odd level is labelled by (12). In the portrait of b, every vertex of even level is labelled by (12), and every vertex of odd level is labelled by id. One checks that $a^2 = b^2 = 1$ and that $ab = ba$. As a group, A is isomorphic to $C_2 \times C_2$.

2.64 Definition.
Let A be a d-ary automaton. Identify $A/\mathrm{st}(1)$ with a subgroup B of $\mathrm{Sym}(d)$ in the natural way. We say that A has *B-action at the root.*

2.65 Remark.
For all $i \in \mathbb{N}_0$ let $\Gamma_i = B$, and let G be the subgroup of $\mathrm{Aut}(T_{\underline{d}})$ defined by $(\Gamma_i)_{i \geqslant 0}$. If A is a d-ary automaton with B-action at the root then A is a subgroup of G.

2.7 Level Growth

2.66 Definition.
Define a partial order $\preceq$ on the set of functions $\mathbb{N}_0 \to \mathbb{R}$ by $f \preceq g$ iff $f(n) = O(g(n))$. Written out in full, this says:

$$f \preceq g \text{ iff there exist } b \in \mathbb{R} \text{ and } N \in \mathbb{N}_0 \text{ such that } |f(n)| \leqslant b|g(n)| \text{ for all } n \geqslant N.$$

If f is equivalent to g under the equivalence relation defined by this partial order then we say that f and g have the same *growth type.* If $f \preceq g$ then we say that f has growth type at most g or, equivalently, that g has growth type at least f. If $f \preceq g$ and $g \not\preceq f$ then we denote this by $f \prec g$ and say that f has growth type strictly less than g or, equivalently, that g has growth type strictly greater than f.

2.67 Definition.
Let $k \in \mathbb{N}$. Define $\ell(\mathrm{id}) := 0$. For every non-identity element $\sigma \in \mathrm{Sym}(k)$, define

$$\ell(\sigma) := \min\{m \mid \sigma \text{ can be expressed as the product of } m \text{ transpositions}\}.$$

2.68 Remark.
If $\sigma \in \mathrm{Sym}(k)$ is equal to the product of disjoint cycles of lengths $l_1, \ldots, l_n$ then $\ell(\sigma) = \sum\limits_{i=1}^{n} (l_i - 1)$.

2.69 Definition.
Let T be a spherically homogeneous rooted tree and let x be an automorphism of T. Define $\ell_x : \mathbb{N}_0 \to \mathbb{N}_0$, the *level growth function* of x, by

$$\ell_x(n) := \sum_{\mathrm{level}_T(\nu) = n} \ell(x(\nu)).$$

Without comment, we also regard ℓ_x as a function $\mathbb{N}_0 \to \mathbb{R}$ in order to speak about its growth type.

2.70 Definition.
Let T be a spherically homogeneous rooted tree and let x be an automorphism of T. Let $f : \mathbb{N}_0 \to \mathbb{R}$ be a function. We say that x has *level growth of type* f if the function ℓ_x has the same growth type as f. We say that x has *exponential level growth* if there exists a real number $c > 1$ such that x has level growth of type at least c^n. We say that x has *polynomial level growth* of *degree* k if x has level growth of type at most n^k.

2.71 Lemma.
Let T be a spherically homogeneous rooted tree. Let x and y be automorphisms of T. Let $f, g : \mathbb{N}_0 \to \mathbb{R}$ be functions. If x has level growth of type at most f, and y has level growth of type at most g, then xy has level growth of type at most $f + g$.

Proof:
It follows from the definition of multiplication in $\mathrm{Aut}(T)$ that $l_{xy}(n) \leqslant l_x(n) + l_y(n)$. $\square$

2.72 Remark.
Note that $f + f$ has the same growth type as f. Also note that $\ell_x = \ell_{x^{-1}}$. Therefore, if x has level growth of type at most f, then x^m has level growth of type at most f for all integers m.

2.73 Lemma.
Let T be a spherically homogeneous rooted tree. Let x be an automorphism of T and let H be a subgroup of $\mathrm{Aut}(T)$. Let $f, g : \mathbb{N}_0 \to \mathbb{R}$ be functions such that g has growth type strictly greater that f. If H is generated by elements of level growth of type at most f, and x has level growth of type at least g, then x is not in H.

Proof:
It follows from Lemma 2.71 and Remark 2.72 that every element of H has level growth of type at most f. $\square$

2.8 Group-Theoretic Definitions and Results

2.74 Definition.
Let F be the free group on $\{x, y\}$ and let w be a word in F. We say that w is a *positive word* if w is an element of the free multiplicative monoid generated by $\{x, y\}$.

That is to say, if w does not contain the letters x^{-1} or y^{-1}.

2.75 Definition.
Let F be the free group on $\{x, y\}$. Let $\pi_x : F \longrightarrow \mathbb{Z}$ be the projection defined by $x \mapsto 1$ and $y \mapsto 0$. Similarly, define the projection $\pi_y : F \longrightarrow \mathbb{Z}$ by $x \mapsto 0$ and $y \mapsto 1$. Let w be a word on $\{x, y\}$. Define the *x-multiplicity* of w to be $\pi_x(w)$ and the *y-multiplicity* of w to be $\pi_y(w)$. Define the *total multiplicity* of w to be $\pi_x(w) + \pi_y(w)$.

Let G be a group generated by two elements a and b. Let $\pi : F \longrightarrow G$ be the projection defined by $x \mapsto a$ and $y \mapsto b$. Let g be an element of G and n an integer. We say that g has *a-multiplicity* n if there exists a word w in F of x-multiplicity n such that $\pi(w) = g$. We say that g has *b-multiplicity* n if there exists a word w in F of y-multiplicity n such that $\pi(w) = g$. We say that g has *total multiplicity* n if there exists a word w in F of total multiplicity n such that $\pi(w) = g$.

2.76 Remark.
Note that a-multiplicity, b-multiplicity, and total multiplicity are not, in general, well-defined elements of $\mathbb{Z}$.

2.77 Lemma.
Let p be a prime and let G be a group generated by two elements x and y. Then $P_2 = \langle x^p, y^p, [x, y]\rangle^G$ and $P_3 = \langle x^{p^2}, y^{p^2}, [x, y]^p, [x, y, x], [x, y, y]\rangle^G$.

Proof:
Observe that $G^pG'/G' = \langle x^pG', y^pG'\rangle$. As $G' = \langle [x, y]\rangle^G$, it follows that

$$G^pG' = \langle x^p, y^p, [x, y]\rangle^G.$$

As P_2 is central modulo $[P_2, G]$, we see that

$$P_2^p[P_2, G]/[P_2, G] = \langle x^{p^2}[P_2, G], y^{p^2}[P_2, G], [x, y]^p[P_2, G]\rangle.$$

We have shown that $P_2 = \langle x^p, y^p, [x, y]\rangle^G$. Thus,

$$[P_2, G] = \langle [x^p, y], [x, y^p], [x, y, x], [x, y, y]\rangle^G.$$

By the commutator identities, $[x, y^p] \equiv [x, y]^p \equiv [x^p, y]$ modulo $\langle [x, y, x], [x, y, y]\rangle^G$. Hence, $[P_2, G] = \langle [x, y]^p, [x, y, x], [x, y, y]\rangle^G$. Therefore,

$$P_2^p[P_2, G] = \langle x^{p^2}, y^{p^2}, [x, y]^p, [x, y, x], [x, y, y]\rangle^G$$

as required. □

Note that G/P_2 is an elementary abelian p-group of rank at most 2.

2.78 Remark.
If F is the free group on $\{x, y\}$ then F/P_3 has order p^5. Indeed, the elements of F/P_3 can be expressed in the normal form $x^iy^j[x, y]^k$ where $i, j \in [1, p^2]$ and $k \in [1, p]$.

Chapter 3

A Branch Group Construction

Let $(n_i)_{i \geqslant 0}$ be a sequence of pairwise coprime integers strictly greater than 1. Note that this hypothesis implies that $n_i \to \infty$ as $i \to \infty$. Let T be the infinite spherically homogeneous rooted tree of type $(n_i)_{i \geqslant 0}$.

3.1 The Cyclic Case

Define the n_i-cycle $\beta_i := (1, 2, \ldots, n_i) \in \mathrm{Sym}(n_i)$ and the cyclic subgroup $\Gamma_i := \langle \beta_i \rangle$. Let H_i be the subgroup of $\mathrm{Aut}(T_i)$ defined by $(\Gamma_j)_{j \geqslant i}$ (see Definition 2.39). Note that H_i is spherically transitive (see Definition 2.34). Let b_i be the directed automorphism (see Definition 2.28) of $\mathrm{Aut}(T_i)$ defined by the sequence $(\beta_j)_{j>i}$. Let $b_i(k) = b_i^{\dot{\beta}_i^k}$, let $G_i = \langle \dot{\beta}_i, b_i \rangle$, and let $B_i = \langle b_i \rangle^{G_i}$. We often drop the subscript 0 in the exposition. That is, $G = G_0$, $B = B_0$ and so on.

3.1 Remark.
Note that $b_i = (\dot{\beta}_{i+1}, 1, \ldots, 1, b_{i+1})$, that $B_i = \mathrm{st}_{G_i}(1)$ and, since $\dot{\beta}_i$ has order n_i, that $B_i = \langle b_i(1), \ldots, b_i(n_i) \rangle$. Calculate that $b_i(1) = (b_{i+1}, \dot{\beta}_{i+1}, 1, \ldots, 1)$, that $b_i(2) = (1, b_{i+1}, \dot{\beta}_{i+1}, 1, \ldots, 1)$, and that $b_i(n_i - 1) = b_i(-1) = (1, \ldots, 1, b_{i+1}, \dot{\beta}_{i+1})$. Observe that $b_i(j) \in G_{i+1} \times \cdots \times G_{i+1}$ (n_i factors). This implies that

$$\mathrm{st}_{G_i}(1) = B_i \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=1} G_{i+1}.$$

Therefore,

$$\mathrm{st}_{G_i}(2) \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=1} \mathrm{st}_{G_{i+1}}(1) = \prod_{\mathrm{level}_{T_i}(\nu)=1} B_{i+1} \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=2} G_{i+2}.$$

Induction on the level shows that, for all $k \in \mathbb{N}$,

$$\mathrm{st}_{G_i}(k) \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=k-1} B_{i+k-1} \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=k} G_{i+k}.$$

Note that $G_i = B_i\dot{\Gamma}_i$ and $B_i \cap \dot{\Gamma}_i = 1$, so $G_i = B_i \rtimes \dot{\Gamma}_i$. Since B_i is a normal subgroup, $[B_i, G_i] = B_i^{'}[B_i, \dot{\Gamma}_i] \leqslant B_i$. Finally, $G_i^{'} = B_i^{'}[B_i, \dot{\Gamma}_i]\dot{\Gamma}_i^{'} = B_i^{'}[B_i, \dot{\Gamma}_i]$ because $\dot{\Gamma}_i$ is abelian.

3.2 Lemma.

Fix an integer $i \in \mathbb{N}_0$. For any $j \in \mathbb{N}$, let $\pi_j : G_i \to \Gamma_{i+j-1} \wr \cdots \wr \Gamma_{i+1} \wr \Gamma_i$ be the natural restriction homomorphism defined by considering the action of G_i on the finite subtree $T_i[j]$. Then π_j is surjective.

Proof:

Observe that $\pi_1(\dot{\beta}_i) = \beta_i$ and $\pi_1(b_i) = 1$. Therefore, $\pi_1(G_i) = \langle \beta_i \rangle = \Gamma_i$.

Observe that $\pi_2(\dot{\beta}_i) = \dot{\beta}_i$. We have $b_i = (\dot{\beta}_{i+1}, 1, \ldots, 1, b_{i+1}) \in \mathrm{st}_{G_i}(1)$ and $b_{i+1} \in \mathrm{st}_{G_{i+1}}(1)$. Thus $\pi_2(b_i) = (\beta_{i+1}, 1, \ldots, 1, 1)$. Therefore, $\pi_2(G_i) = \langle \dot{\beta}_i, (\beta_{i+1}, 1, \ldots, 1) \rangle = \Gamma_{i+1} \wr \Gamma_i$.

Fix $j \geqslant 3$ and suppose, for induction, that π_{j-1} is surjective. By the Chinese Remainder Theorem, there exists an integer r_j such that $r_j \equiv 1$ modulo n_{i+j-1} and $r_j \equiv 0$ modulo $n_{i+j^{'}}$ for all $j^{'} < j-1$. Then $b_i^{r_j} \in \mathrm{st}_{G_i}(j-1)$ and, in the expansion of $b_i^{r_j}$ to level $j-1$, we see that $\nu_{(n_i, n_{i+1}, \ldots, n_{i+j-3}, 1)}$ is labelled by $\dot{\beta}_{i+j-1}$, that $\nu_{(n_i, n_{i+1}, \ldots, n_{i+j-3}, n_{i+j-2})}$ is labelled by $b_{i+j-1}^{r_j}$, and that all other vertices of level $j-1$ in T_i are labelled by 1. As $b_{i+j-1} \in \mathrm{st}_{G_{i+j-1}}(1)$, we see that $\pi_j(b_i^{r_j})$ acts as β_{i+j-1} at exactly one vertex of level $j-1$ in $T_i[j]$ and as the identity at all other vertices in $T_i[j]$.

By induction, $\pi_{j-1}(G_i) = \Gamma_{i+j-2} \wr \cdots \wr \Gamma_{i+1} \wr \Gamma_i$. In particular, $\pi_j(G_i)$ permutes the vertices of level $j-1$ transitively. Given any vertex ν of level $j-1$, we can conjugate $\pi_j(b_i^{r_j})$ by an appropriate element of $\pi_j(G_i)$ to show that $\pi_j(G_i)$ contains the automorphism of $T_i[j]$ acting as β_{i+j-1} at ν and as the identity at all other vertices. We can use these automorphisms to kill off the action of $\pi_j(G_i)$ on the lowest level of $T_i[j]$, showing that $\pi_j(G_i) \geqslant 1 \wr \Gamma_{i+j-2} \wr \cdots \wr \Gamma_{i+1} \wr \Gamma_i$. Therefore, $\pi_j(G_i)$ contains generators of the group $\Gamma_{i+j-1} \wr \Gamma_{i+j-1} \wr \cdots \wr \Gamma_{i+1} \wr \Gamma_i$ as required. $\square$

3.3 Theorem.

Fix an integer $i \in \mathbb{N}_0$. Then G_i is dense in H_i.

Proof:

This is immediate from Lemma 3.2 and Remark 2.37. $\square$

3.4 Lemma.

Fix an integer $i \in \mathbb{N}_0$. Then $B_i^{'} = \left\langle \{[b_i, b_i(k)] : 1 \leqslant k \leqslant n_i\} \right\rangle^{G_i}$. If $n_i \geqslant 3$ then

$[b_i, b_i(k)] \in \mathrm{rst}_{G_i}(1)$ for all k. Specifically:

$$\begin{aligned}
[b_i, b_i(1)] &= \big([\dot{\beta}_{i+1}, b_{i+1}], 1, \ldots, 1\big) \\
[b_i, b_i(-1)] &= \big(1, \ldots, 1, [b_{i+1}, \dot{\beta}_{i+1}]\big) \\
[b_i, b_i(k)] &= 1 \text{ if } k \not\equiv \pm 1 \text{ (modulo } n_i).
\end{aligned}$$

In particular, if $n_i \geqslant 3$ then $B_i^{'} \leqslant \mathrm{rst}_{G_i}(1)$.

Proof:
By Remark 3.1, $B_i = \langle b_i(1), b_i(2), \ldots, b_i(n_i)\rangle$ and so, by the commutator identities,

$$B_i^{'} = \langle \{[b_i(j), b_i(k)] : 1 \leqslant j, k \leqslant n_i\}\rangle^{G_i}.$$

Observe that

$$\Big[b_i(j), b_i(k)\Big] = \Big[b_i^{\dot{\beta}_i^j}, b_i^{\dot{\beta}_i^k}\Big] = \Big[b_i, b_i^{\dot{\beta}_i^{(k-j)}}\Big]^{\dot{\beta}_i^j} = \Big[b_i, b_i(k-j)\Big]^{\dot{\beta}_i^j};$$

therefore,

$$B_i^{'} = \big\langle \{[b_i, b_i(k)] : 1 \leqslant k \leqslant n_i\}\big\rangle^{G_i}.$$

If k is not congruent to 0, 1, or -1 modulo n_i, then b_i and $b_i(k)$ have disjoint supports in their action on the tree T_i, and so $[b_i, b_i(k)] = 1$. The formulae for $[b_i, b_i(1)]$ and $[b_i, b_i(-1)]$ are verified by direct calculation. (Recall from Remark 3.1 that $b_i(1) = (b_{i+1}, \dot{\beta}_{i+1}, 1, \ldots, 1)$ and $b_i(-1) = (1, \ldots, 1, b_{i+1}, \dot{\beta}_{i+1})$.)

We have shown that $B_i^{'}$ is normally generated by elements of the form $[b_i, b_i(k)]$ and that, if $n_i \geqslant 3$, then $[b_i, b_i(k)] \in \mathrm{rst}_{G_i}(1)$. Since $\mathrm{rst}_{G_i}(1)$ is a normal subgroup of G_i, this implies that $B_i^{'} \leqslant \mathrm{rst}_{G_i}(1)$. □

3.5 Lemma.
Fix an integer $i \in \mathbb{N}_0$. If $n_i \geqslant 3$ then

$$B_i^{'} = \prod_{\mathrm{level}_{T_i}(\nu)=1} G_{i+1}^{'}.$$

Proof:
By Remark 3.1,

$$B_i \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=1} G_{i+1};$$

therefore,

$$B_i^{'} \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=1} G_{i+1}^{'}.$$

For the reverse inclusion, calculate directly that

$$\begin{aligned}
[b_i^k, b_i(1)] &= [(\dot{\beta}_{i+1}^k, 1, \ldots, 1, b_{i+1}^k), (b_{i+1}, \dot{\beta}_{i+1}, 1, \ldots, 1)] \\
&= ([\dot{\beta}_{i+1}^k, b_{i+1}], 1, \ldots, 1) \\
&= (b_{i+1}(k)^{-1} b_{i+1}, 1, \ldots, 1); \text{ and} \\
\left[b_i^k, b_i(1)^{-1}\right] &= (b_{i+1}(k) b_{i+1}^{-1}, 1, \ldots, 1).
\end{aligned}$$

The product gives

$$[b_i^k, b_i(1)][b_i^k, b_i(1)^{-1}] = ([b_{i+1}(k), b_{i+1}^{-1}], 1, \ldots, 1).$$

Conjugating by $b_i(1)$ gives

$$\left([b_i^k, b_i(1)][b_i^k, b_i(1)^{-1}]\right)^{b_i(1)} = \left([b_{i+1}(k), b_{i+1}^{-1}]^{b_{i+1}}, 1, \ldots, 1\right) = \left([b_{i+1}, b_{i+1}(k)], 1, \ldots, 1\right).$$

This is an element of $B_i^{'}$. By definition, $G_{i+1} = \langle \dot{\beta}_{i+1}, b_{i+1} \rangle$ and, by Lemma 3.4,

$$B_{i+1}^{'} = \langle \{[b_{i+1}, b_{i+1}(k)] : 1 \leqslant k \leqslant n_{i+1}\} \rangle^{G_{i+1}}.$$

We can conjugate the elements $([b_{i+1}, b_{i+1}(k)], 1, \ldots, 1)$ by any word on $\{b_i, b_i(1)\}$ and still remain in $B_i^{'}$. Now, $b_i = (\dot{\beta}_{i+1}, 1, \ldots, 1, b_{i+1})$ and $b_i(1) = (b_{i+1}, \dot{\beta}_{i+1}, 1, \ldots, 1)$. Therefore,

$$B_i^{'} \geqslant B_{i+1}^{'} \times 1 \times \cdots \times 1.$$

Recall from Remark 3.1 that $G_{i+1}^{'} = B_{i+1}^{'}[B_{i+1}, \dot{\Gamma}_{i+1}]$. The abelian group $G_{i+1}^{'}/B_{i+1}^{'}$ is generated by the set $\{b_{i+1}(j) b_{i+1}(k)^{-1} B_{i+1}^{'} : 1 \leqslant j, k \leqslant n_{i+1}\}$. Calculate that

$$\begin{aligned}
\left[b_i^{j-k}, b_i(1)^{-1}\right]^{b_i^k} &= \left((b_{i+1}(j-k) b_{i+1}^{-1}), 1, \ldots, 1\right)^{b_i^k} \\
&= \left((b_{i+1}(j-k) b_{i+1}^{-1})^{\dot{\beta}_{i+1}^k}, 1, \ldots, 1\right) \\
&= \left(b_{i+1}(j) b_{i+1}(k)^{-1}, 1, \ldots, 1\right)
\end{aligned}$$

This is an element of $B_i^{'}$, which shows that

$$B_i^{'} \geqslant G_{i+1}^{'} \times 1 \times \cdots \times 1.$$

As $B_i^{'}$ is a normal subgroup of G_i, we can conjugate by powers of $\dot{\beta}_i$ to show that

$$B_i^{'} \geqslant G_{i+1}^{'} \times \cdots \times G_{i+1}^{'}.$$

□

3.6 Theorem.
If $n_i \geqslant 3$ for all $i \in \mathbb{N}_0$ then G is a branch group.

Proof:
By Lemma 3.2, G is spherically transitive.

Fix an integer $k \geqslant 0$. An induction on Lemma 3.5 shows that

$$G \geqslant \prod_{\text{level}(\nu)=k} B_k^{'}.$$

Therefore,

$$\text{rst}_G(k) \cap \text{st}_G(k+1) \geqslant \prod_{\text{level}(\nu)=k} B_k^{'}.$$

By Remark 3.1,

$$\text{st}_G(k+1) \leqslant \prod_{\text{level}(\nu)=k} B_k.$$

Thus, $\text{st}_G(k+1)/(\text{rst}_G(k) \cap \text{st}_G(k+1))$ is a section of the abelian group

$$\prod_{\text{level}(\nu)=k} B_k/B_k^{'}.$$

By Remark 3.1, $B_k = \langle b_k(1), \dots, b_k(n_k)\rangle$ For each j, a suitable power of $b_k(j)$ is contained in $\text{rst}_G(k) \cap \text{st}_G(k+1)$. Therefore,

$$\big(\prod_{\text{level}(\nu)=k} B_k\big)/\big(\text{rst}_G(k) \cap \text{st}_G(k+1)\big)$$

is an abelian group generated by finitely many elements of finite order. It is therefore finite. Hence $\text{rst}_G(k) \cap \text{st}_G(k+1)$ has finite index in $\text{st}_G(k+1)$. By Remark 2.37, $\text{st}_G(k+1)$ has finite index in G. Therefore, $\text{rst}_G(k) \cap \text{st}_G(k+1)$ has finite index in G and, a fortiori, $\text{rst}_G(k)$ has finite index in G. □

3.7 Lemma.
Fix an integer $i \in \mathbb{N}_0$. If $n_i \geqslant 5$ then $G_i^{''} = B_i^{'}$.

Proof:
By 3.1 Remark, $G_i^{'} \leqslant B_i$, and so $G_i^{''} \leqslant B_i^{'}$. Conversely, by Lemma 3.4,

$$B_i^{'} = \langle \{[b_i, b_i(k)] : 1 \leqslant k \leqslant n_i\}\rangle^{G_i}.$$

As G_i'' is a normal subgroup of G_i, it suffices to show that $[b_i, b_i(k)] \in G_i''$ for all k. If k is not congruent to 1 or -1 modulo n_i then $[b_i, b_i(k)] = 1$ by Lemma 3.4. By hypothesis, $n_i \geqslant 5$, so $b_i(3)$ has disjoint support from b_i and from $b_i(1)$. Therefore,

$$\begin{aligned}[][b_i, b_i(1)] &= [b_i(3)^{-1} b_i, b_i(3)^{-1} b_i(1)] \\ &= \big[[\dot\beta_i^3, b_i], b_i(3)^{-1} b_i b_i^{-1} b_i(1)\big] \\ &= \big[[\dot\beta_i^3, b_i], [\dot\beta_i^3, b_i][b_i, \dot\beta_i]\big]\end{aligned}$$

is an element of G_i''. Similarly, $b_i(2)$ has disjoint support from b_i and from $b_i(-1)$ and

$$\begin{aligned}[][b_i, b_i(-1)] &= [b_i(2)^{-1} b_i, b_i(2)^{-1} b_i(-1)] \\ &= \big[[\dot\beta_i^2, b_i], [\dot\beta_i^2, b_i][b_i, \dot\beta_i^{-1}]\big]\end{aligned}$$

is an element of G_i''. □

3.8 Theorem.
Fix integers $i \in \mathbb{N}_0$ and $k \in \mathbb{N}$. If $n_j \geqslant 5$ for all $j \geqslant i$ then

$$G_i^{(k+1)} = \prod_{\mathrm{level}_{T_i}(\nu)=k} G_{i+k}'.$$

Proof:
This follows by induction from Lemmas 3.5 and 3.7. □

3.9 Lemma.
Fix an integer $i \in \mathbb{N}_0$ and let $m \in \mathbb{N}$ be an integer divisible by n_{i+1}. If $n_i \geqslant 3$ then

$$B_i' B_i^m = \prod_{\mathrm{level}_{T_i}(\nu)=1} G_{i+1}' B_{i+1}^m \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=1} B_{i+1}(1).$$

Proof:
By Lemma 3.5,

$$B_i' = \prod_{\mathrm{level}_{T_i(\nu)=1}} G_{i+1}'.$$

Observe that the group $G_{i+1}' B_{i+1}^m / G_{i+1}'$ is abelian and generated by the set of cosets $\{b_{i+1}(1)^m G_{i+1}', \ldots, b_{i+1}(n_{i+1})^m G_{i+1}'\}$. Observe that $b_i(j)^m$ is an element of $\mathrm{rst}_{G_i}(\nu_j)$, and the action on T_{ν_j} is as b_{i+1}^m. This shows that

$$B_i' B_i^m \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=1} G_{i+1}' B_{i+1}^m.$$

However, $b_{i+1} \equiv b_{i+1}(l)$ modulo G'_{i+1} for all l, so we have also shown that

$$B'_i B_i^m \geqslant \prod_{\mathrm{level}_{T_i}(\nu)=1} G'_{i+1} B_{i+1}^m.$$

Recall, from Remark 3.1, that $G'_{i+1} \leqslant B_{i+1}$. Therefore,

$$\prod_{\mathrm{level}_{T_i}(\nu)=1} G'_{i+1} B_{i+1}^m \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=1} B_{i+1}(1).$$

□

3.10 Lemma.
Let $C_\infty := \langle x \rangle$ be the infinite cyclic group. Fix an integer $i \in \mathbb{N}_0$. Recall from Remark 3.1 that $\{b_i(1), \ldots, b_i(n_i)\}$ is a generating set for B_i. Define the map $\vartheta_i : B_i \to C_\infty$ by $w(b_i(1), \ldots, b_i(n_i)) \mapsto w(x, \ldots, x)$ for all words $w(y_1, \ldots, y_{n_i})$ in the free group on the letters $\{y_1, \ldots, y_{n_i}\}$. Then ϑ_i is a surjective group homomorphism.

Proof:
We have to show that ϑ_i is well-defined. It then follows from its definition that ϑ_i is a surjective group homomorphism. Let $w(y_1, \ldots, y_{n_i})$ be any word in the free group on the letters $\{y_1, \ldots, y_{n_i}\}$. In order to show that ϑ_i is well-defined, we have to show that

$$\text{if } w(b_i(1), \ldots, b_i(n_i)) = 1 \in B_i \text{ then } w(x, \ldots, x) = 1 \in C_\infty.$$

To this end, fix $k \in \mathbb{N}$ and consider the action of $b_i(j)$ on the vertices of level k in the tree T_i. The action is as $\dot{\beta}_{i+k}$ at precisely one vertex of level k, and as the identity at all other vertices of level k. There is also an action at vertices of other levels, but this will be immaterial.

Recall the epimorphism $\varphi_k : H_i \to \Gamma_{i+k}/\Gamma'_{i+k}$ of Lemma 2.40. We can write $\Gamma_{i+k}/\Gamma'_{i+k} = \Gamma_{i+k}$ because Γ_{i+k} is abelian. The observation about the action of $b_i(j)$ on the vertices of level k shows that $\varphi_k(b_i(j)) = \beta_{i+k}$. If $w(b_i(1), \ldots, b_i(n_i)) = 1 \in B_i$ then $\varphi_k(w(b_i(1), \ldots, b_i(n_i))) = 1 \in \Gamma_{i+k}$ because φ_k is a homomorphism. Also, $\varphi_k(w(b_i(1), \ldots, b_i(n_i))) = w(\varphi_k(b_i(1)), \ldots, \varphi_k(b_i(n_i))) = w(\beta_{i+k}, \ldots, \beta_{i+k})$. The order of β_{i+k} is n_{i+k} so, if $w(b_i(1), \ldots, b_i(n_i)) = 1 \in B_i$, then the total number of letters (counted with multiplicities) in the word $w(y_1, \ldots, y_{n_i})$ must be divisible by n_{i+k}. Since k was chosen arbitrarily, this must be true for all $k \in \mathbb{N}$. Thus, $w(x, \ldots, x) \in \langle x^{n_{i+k}} \rangle$ for all $k \in \mathbb{N}$. Now, $n_l \geqslant 2$ for all l and $\mathrm{hcf}(n_l, n_{l'}) = 1$ for all $l \neq l'$, so the sequence $(n_{i+k})_{k \geqslant 0}$ is unbounded. Therefore, $w(x, \ldots, x) = 1 \in C_\infty$ as required. □

3.11 Lemma.
Fix an integer $i \in \mathbb{N}_0$. Then $\ker(\vartheta_i) = G_i'$.

Proof:
First, we show that $\ker(\vartheta_i)$ is a normal subgroup of G_i. Let $w(y_1, \dots, y_{n_i})$ be a word in the free group on the letters $\{y_1, \dots, y_{n_i}\}$. Observe that $w(b_i(1), \dots, b_i(n_i))$ is in $\ker(\vartheta_i)$ if and only if $w(x, \dots, x) = 1 \in C_\infty$. Suppose that $w(b_i(1), \dots, b_i(n_i))$ is in $\ker(\vartheta_i)$. Calculate that

$$w\big(b_i(1), b_i(2), \dots, b_i(n_i - 1), b_i(n_i)\big)^{\dot{\beta}_i} = w\big(b_i(2), b_i(3), \dots, b_i(n_i), b_i(1)\big)$$

and observe that this is an element of $\ker(\vartheta_i)$ since $w(x, \dots, x) = 1 \in C_\infty$. Also, $\ker(\vartheta_i)$ is a normal subgroup of B_i, so it is closed under conjugation by b_i. As $G_i = \langle \dot{\beta}_i, b_i \rangle$, we have shown that $\ker(\vartheta_i)$ is a normal subgroup of G_i.

Observe that $G_i' = \langle [b_i, \dot{\beta}_i] \rangle^{G_i}$ and $[b_i, \dot{\beta}_i] = b_i^{-1} b_i(1) \in \ker(\vartheta_i)$. Since $\ker(\vartheta_i)$ is a normal subgroup of G_i, this proves that $G_i' \leqslant \ker(\vartheta_i)$. By Linear Algebra, the abelian group $\ker(\vartheta_i)/G_i'$ is generated by the set $\{b_i(j)^{-1} b_i(j+1) G_i' : 1 \leqslant j \leqslant n_i\}$. Observe that

$$b_i(j)^{-1} b_i(j+1) = [b_i, \dot{\beta}_i]^{\dot{\beta}_i^j}$$

is an element of G_i' for all j. □

3.12 Lemma.
Fix an integer $i \in \mathbb{N}_0$. If $n_i \geqslant 3$ then $\mathrm{rst}_{G_i}(1) = B_i' B_i^{n_{i+1}}$.

Proof:
First, we show that $B_i' B_i^{n_{i+1}} \leqslant \mathrm{rst}_{G_i}(1)$. By Lemma 3.4, $B_i' \leqslant \mathrm{rst}_{G_i}(1)$. Observe that $b_i(j)^{n_{i+1}} \in \mathrm{rst}_{G_i}(1)$ and $B_i' B_i^{n_{i+1}}/B_i' = \langle \{ b_i(j)^{n_{i+1}} B_i' : 1 \leqslant j \leqslant n_i \} \rangle$. Therefore, $B_i' B_i^{n_{i+1}}/B_i' \leqslant \mathrm{rst}_{G_i}(1)/B_i'$, and so $B_i' B_i^{n_{i+1}} \leqslant \mathrm{rst}_{G_i}(1)$ as required.

We now show that $\mathrm{rst}_{G_i}(1) \leqslant B_i' B_i^{n_{i+1}}$. To this end, let $g \in \mathrm{rst}_{G_i}(1)$. Then, certainly $g \in \mathrm{st}_{G_i}(1) = B_i$. There exist integers $e_1, e_2, \dots, e_{n_i}$ such that

$$g \equiv b_i(1)^{e_1} \dots b_i(n_i)^{e_{n_i}} \text{ modulo } B_i^{n_{i+1}} B_i'.$$

We have to show that n_{i+1} divides each of the integers $e_1, e_2, \dots, e_{n_i}$. As already shown, $B_i' B_i^{n_{i+1}} \leqslant \mathrm{rst}_{G_i}(1)$, so certainly

$$g \equiv b_i(1)^{e_1} \dots b_i(n_i)^{e_{n_i}} \text{ modulo } \mathrm{rst}_{G_i}(1).$$

Calculate that

$$b_i(1)^{e_1} \dots b_i(n_i)^{e_{n_i}} = (b_{i+1}^{e_1} \dot{\beta}_{i+1}^{e_{n_i}}, \dot{\beta}_{i+1}^{e_1} b_{i+1}^{e_2}, \dot{\beta}_{i+1}^{e_2} b_{i+1}^{e_3}, \dots, \dot{\beta}_{i+1}^{e_{n_i - 1}} b_1^{e_{n_i}}).$$

This is an element of $\mathrm{rst}_{G_i}(1)$ and, in particular, $(b_{i+1}^{e_1}\dot{\beta}_{i+1}^{e_{n_i}}, 1, \dots, 1) \in \mathrm{rst}_{G_i}(1)$. Repeating the argument shows that there exist integers $f_1, f_2, \dots, f_{n_i}$ such that

$$(b_{i+1}^{e_1}\dot{\beta}_{i+1}^{e_{n_i}}, 1, \dots, 1) \equiv b_i(1)^{f_1} \dots b_i(n_i)^{f_{n_i}} \text{ modulo } B_i^{'}B_i^{n_{i+1}}.$$

As above, $b_i(1)^{f_1} \dots b_i(n_i)^{f_{n_i}} = \left(b_{i+1}^{f_1}\dot{\beta}_{i+1}^{f_{n_i}}, \dot{\beta}_{i+1}^{f_1}b_{i+1}^{f_2}, \dot{\beta}_{i+1}^{f_2}b_{i+1}^{f_3}, \dots, \dot{\beta}_{i+1}^{f_{n_i-1}}b_1^{f_{n_i}}\right)$. Therefore,

$$\left(b_{i+1}^{e_1}\dot{\beta}_{i+1}^{(e_{n_i}-f_{n_i})}b_{i+1}^{-f_1}, b_{i+1}^{-f_2}\dot{\beta}_{i+1}^{-f_1}, b_{i+1}^{-f_3}\dot{\beta}_{i+1}^{-f_2}, \dots, b_1^{-f_{n_i}}\dot{\beta}_{i+1}^{-f_{n_i-1}}\right)$$

is an element of $B_i^{'}B_i^{n_{i+1}}$.

By Lemma 3.9,

$$B_i^{'}B_i^{n_{i+1}} \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=1} \mathrm{st}_{G_{i+1}}(1).$$

Consideration of the second vertex shows that n_{i+1} divides f_1. Consideration of the first vertex shows that $(e_{n_i} - f_{n_i})$ is divisible by n_{i+1}. Thus,

$$b_{i+1}^{e_1}\dot{\beta}_{i+1}^{(e_{n_i}-f_{n_i})}b_{i+1}^{-f_1} = b_{i+1}^{e_1}b_{i+1}^{-f_1} = b_{i+1}^{(e_1-f_1)}.$$

By Lemma 3.9,

$$B_i^{'}B_i^{n_{i+1}} = \prod_{\mathrm{level}_{T_i}(\nu)=1} G_{i+1}^{'}B_{i+1}^{n_{i+1}};$$

therefore, $b_{i+1}^{(e_1-f_1)} \in G_{i+1}^{'}B_{i+1}^{n_{i+1}}$. As $\ker(\vartheta_{i+1}) = G_{i+1}^{'}$, by Lemma 3.11, we see that $\vartheta_{i+1}(G_{i+1}^{'}B_{i+1}^{n_{i+1}}) = \langle x^{n_{i+1}} \rangle$. Therfore, n_{i+1} divides $(e_1 - f_1)$. Since n_{i+1} divides f_1, this implies that n_{i+1} divides e_1.

Similar consideration of the other terms in

$$(b_{i+1}^{e_1}\dot{\beta}_{i+1}^{e_{n_i}}, \dot{\beta}_{i+1}^{e_1}b_{i+1}^{e_2}, \dot{\beta}_{i+1}^{e_2}b_{i+1}^{e_3}, \dots, \dot{\beta}_{i+1}^{e_{n_i-1}}b_1^{e_{n_i}})$$

shows that n_{i+1} divides each of the integers $e_1, e_2, \dots, e_{n_i}$, as required. □

3.13 Lemma.

Fix an integer $k \in \mathbb{N}$ and let $m_k = n_1 n_2 \dots n_k$. If $n_i \geqslant 3$ for all $i \in \mathbb{N}_0$ then

$$\prod_{\mathrm{level}(\nu)=k} G_k^{'}B_k^{m_k} \leqslant \mathrm{rst}_G(k).$$

Proof:

By Lemmas 3.9 and 3.12,

$$\prod_{\mathrm{level}(\nu)=1} G_1^{'}B_1^{m_1} = B_0^{'}B_0^{m_1} = \mathrm{rst}_G(1).$$

For induction, suppose that

$$\prod_{\text{level}(\nu)=k} G_k^{'} B_k^{m_k} \leqslant \text{rst}_G(k).$$

Observe that

$$\prod_{\text{level}(\nu)=k+1} G_{k+1}^{'} B_{k+1}^{m_{k+1}} = \prod_{\text{level}(\nu)=k} \left(\prod_{\text{level}_{T_\nu}(\omega)=1} G_{k+1}^{'} B_{k+1}^{m_{k+1}} \right).$$

By Lemma 3.9, this equals

$$\prod_{\text{level}(\nu)=k} B_k^{'} B_k^{m_{k+1}},$$

which is a subgroup of

$$\prod_{\text{level}(\nu)=k} G_k^{'} B_k^{m_k},$$

and therefore a subgroup of G by the induction hypothesis. Therefore,

$$\prod_{\text{level}(\nu)=k+1} G_{k+1}^{'} B_{k+1}^{m_{k+1}} \leqslant \text{rst}_G(k+1)$$

as required. □

3.14 Lemma.

Fix an integer $k \in \mathbb{N}$. If $n_i \geqslant 3$ for all $i \in \mathbb{N}_0$ then

$$\text{rst}_G(k) \leqslant \prod_{\text{level}(\nu)=k} G_k^{'} B_k^{n_k}.$$

Proof:

By Remark 3.1,

$$\text{st}_G(k-1) \leqslant \prod_{\text{level}(\nu)=k-1} G_{k-1}.$$

As $\text{rst}_G(k) \leqslant \text{rst}_G(k-1) \leqslant \text{st}_G(k-1)$, we see that

$$\text{rst}_G(k) \leqslant \prod_{\text{level}(\nu)=k-1} \text{rst}_{G_{k-1}}(1).$$

Applying Lemmas 3.9 and 3.12, we have

$$\prod_{\text{level}(\nu)=k-1} \text{rst}_{G_{k-1}}(1) = \prod_{\text{level}(\nu)=k-1} B_{k-1}^{'} B_{k-1}^{n_k} = \prod_{\text{level}(\nu)=k} G_k^{'} B_k^{n_k}.$$

□

3.15 Theorem.
Fix an integer $k \in \mathbb{N}$. If $n_0 \geqslant 3$ and $n_i \geqslant 5$ for all $i \in \mathbb{N}$ then

$$\operatorname{rst}_G(k)' = \prod_{\text{level}(\nu)=k} B_k' = \prod_{\text{level}(\nu)=k+1} G_{k+1}'.$$

Proof:
By Lemmas 3.13 and 3.14,

$$\prod_{\text{level}(\nu)=k} G_k' B_k^{n_1 \ldots n_k} \leqslant \operatorname{rst}_G(k) \leqslant \prod_{\text{level}(\nu)=k} G_k' B_k^{n_k};$$

therefore, taking commutators yields

$$\prod_{\text{level}(\nu)=k} \left(G_k' B_k^{n_1 \ldots n_k}\right)' \leqslant \operatorname{rst}_G(k)' \leqslant \prod_{\text{level}(\nu)=k} \left(G_k' B_k^{n_k}\right)'.$$

In order to prove the theorem, we show that $\left(G_k' B_k^{n_1 \ldots n_k}\right)' = B_k' = \left(G_k' B_k^{n_k}\right)'$. To this end, fix an integer $l \in \mathbb{N}$. We will show that $(G_k' B_k^l)' = B_k'$. Now, $(G_k' B_k^l)' = G_k''(B_k^l)'\left[G_k', B_k^l\right]$. Recall from Remark 3.1 that $G_k' \leqslant B_k$. Therefore, $[G_k', B_k^l] \leqslant [B_k, B_k] = B_k'$. Also, $(B_k^l)' \leqslant B_k'$. Recall from Lemma 3.7 that $G_k'' = B_k'$. Therefore, $(G_k' B_k^l)' = B_k'$ as required. Finally,

$$\prod_{\text{level}(\nu)=k} B_k' = \prod_{\text{level}(\nu)=k} \left(\prod_{\text{level}_{T_\nu}(\omega)=1} G_{k+1}' \right) = \prod_{\text{level}(\nu)=k+1} G_{k+1}'$$

by Lemma 3.5. □

3.16 Corollary.
Fix an integer $k \in \mathbb{N}$. If $n_i \geqslant 5$ for all $i \in \mathbb{N}_0$ then $G^{(k+2)} = \operatorname{rst}_G(k)'$.

Proof:
By Theorem 3.8,

$$G^{(k+2)} = \prod_{\text{level}(\nu)=k+1} G_{k+1}'.$$

□

3.17 Corollary.
If $n_i \geqslant 5$ for all $i \in \mathbb{N}_0$ then G is just non-soluble. (Recall that G is just non-soluble if G is not soluble but G/N is soluble for every non-trivial normal subgroup N of G.)

Proof:
By Theorem 3.8,

$$G^{(k+1)} = \prod_{level(\nu)=k} G_k^{'}.$$

Therefore, the derived series of G never terminates, and so G is not soluble. By Theorem 2.43, if N is a non-trivial normal subgroup of G then there exists $k \in \mathbb{N}$ such that N contains $\mathrm{rst}_G(k)^{'}$. This is equal to $G^{(k+2)}$ by Corollary 3.16. Therefore, G/N is soluble. □

3.18 Theorem.
Fix an integer $k \in \mathbb{N}$ and let $m_k = n_1 n_2 \dots n_k$. If $n_0 \geqslant 3$ and $n_i \geqslant 5$ for all $i \in \mathbb{N}$ then

$$\prod_{\mathrm{level}(\nu)=k+1} G_{k+1}^{'} B_{k+1}^{m_{k+1}} / G_{k+1}^{'}$$

is a subgroup of finite index in $\mathrm{rst}_G(k)/\mathrm{rst}_G(k)^{'}$.

Proof:
By Lemma 3.13,

$$\mathrm{rst}_G(k) \geqslant \prod_{\mathrm{level}(\nu)=k} G_k^{'} B_k^{m_k}.$$

As $G_k^{'} B_k^{m_k}$ contains $B_k^{'} B_k^{m_{k+1}}$, we see that

$$\prod_{\mathrm{level}(\nu)=k} G_k^{'} B_k^{m_k} \geqslant \prod_{\mathrm{level}(\nu)=k+1} G_{k+1}^{'} B_{k+1}^{m_{k+1}}$$

by Lemma 3.9. By Theorem 3.15,

$$\mathrm{rst}_G(k)^{'} = \prod_{\mathrm{level}(\nu)=k+1} G_{k+1}^{'}.$$

Therefore,

$$\prod_{\mathrm{level}(\nu)=k+1} G_{k+1}^{'} B_{k+1}^{m_{k+1}} / G_{k+1}^{'} \leqslant \mathrm{rst}_G(k)/\mathrm{rst}_G(k)^{'}.$$

By Lemma 3.14,

$$\mathrm{rst}_G(k) \leqslant \prod_{\mathrm{level}(\nu)=k} G_k^{'} B_k^{n_k}.$$

Since $\dot{\Gamma}_k$ is abelian, $G_k^{'} B_k^{n_k}$ is a subgroup of $\mathrm{st}_{G_k}(1)$. By Remark 3.1,

$$\mathrm{st}_{G_k}(1) \leqslant \prod_{\mathrm{level}_{T_k}(\omega)=1} G_{k+1}.$$

Thus,

$$\mathrm{rst}_G(k) \leqslant \prod_{\mathrm{level}(\nu)=k+1} G_{k+1}$$

and, in particular,

$$\mathrm{rst}_G(k)/\mathrm{rst}_G(k)' \leqslant \prod_{\mathrm{level}(\nu)=k+1} G_{k+1}/G_{k+1}'.$$

Suitable powers of the two generators $\dot{\beta}_k$ and b_k of G_{k+1} are contained in $B_{k+1}^{m_{k+1}}$. Therefore, $G_{k+1}'B_{k+1}^{m_{k+1}}/G_{k+1}'$ has finite index in the abelian group G_{k+1}/G_{k+1}'. Hence

$$\prod_{\mathrm{level}(\nu)=k+1} G_{k+1}'B_{k+1}^{m_{k+1}}/G_{k+1}'$$

has finite index in

$$\prod_{\mathrm{level}(\nu)=k+1} G_{k+1}/G_{k+1}',$$

and therefore certainly has finite index in $\mathrm{rst}_G(k)/\mathrm{rst}_G(k)'$. □

3.19 Lemma.
Fix integers $i \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Then $G_i'B_i^m/G_i'$ is isomorphic to C_∞.

Proof:
This follows from Lemmas 3.10 and 3.11. □

3.20 Corollary.
Fix an integer $k \in \mathbb{N}$ and let $l_k := n_0 n_1 \ldots n_k$. If $n_0 \geqslant 3$ and $n_i \geqslant 5$ for all $i \in \mathbb{N}$ then $G/\mathrm{rst}_G(k)'$ has a subgroup of finite index which is free abelian of rank l_k.

Proof:
Note that l_k is the number of vertices of level $k+1$. By Theorem 3.6, $\mathrm{rst}_G(k)$ has finite index in G. It therefore suffices to show that $\mathrm{rst}_G(k)/\mathrm{rst}_G(k)'$ contains a subgroup of finite index which free abelian of rank l_k. This follows from Theorem 3.18 and Lemma 3.19. □

3.21 Corollary.
Fix an integer $k \in \mathbb{N}$ and let $l_k := n_0 n_1 \ldots n_k$. If $n_0 \geqslant 3$ and $n_i \geqslant 5$ for all $i \in \mathbb{N}$ then $G/\mathrm{rst}_G(k)'$ has a normal subgroup of finite index which is free abelian of rank l_k.

Proof:
Take the normal core of the subgroup in Corollary 3.20. □

3.22 Remark.
Let p be a prime. The free abelian normal subgroup of rank l_k has a quotient that is an elementary abelian p-group of rank l_k. Therefore, the upper p-rank of G is at least l_k. As l_k is unbounded as k ranges over the levels of T, it follows that G has infinite upper p-rank for all primes p, as claimed in Section 1.3.

3.2 The General Case

Fix an integer $l \in \mathbb{N}$. For every $i \in \mathbb{N}_0$ fix a transitive subgroup $\Gamma_i = \langle \beta_{(i,1)}, \ldots, \beta_{(i,l)} \rangle$ of $\mathrm{Sym}(n_i)$. The set of generators $\{\beta_{(i,1)}, \ldots, \beta_{(i,l)}\}$ need not be irredundant and we do allow $\beta_{(i,j)}$ to be the identity. Let H_i be the subgroup of $\mathrm{Aut}(T_i)$ defined by $(\Gamma_j)_{j \geqslant i}$. Note that H_i is spherically-transitive. Let $b_{(i,j)}$ be the directed automorphism defined by the sequence $(\beta_{(k,j)})_{k>i}$. Define G_i to be the subgroup of H_i generated by the set $\{\dot{\beta}_{(i,1)}, \ldots, \dot{\beta}_{(i,l)}\} \cup \{b_{(i,1)}, \ldots, b_{(i,l)}\}$. Let $B_i = \langle b_{(i,1)}, \ldots, b_{(i,l)} \rangle^{G_i}$. We often drop the subscript 0 in the exposition. That is, $G = G_0$, $B = B_0$, $\Gamma = \Gamma_0$ and so on.

3.23 Remark.
Note that $B_i = \mathrm{st}_{G_i}(1)$, the first-level stabilizer of G_i in the tree T_i. In particular, B_i is a normal subgroup of G_i. Observe that $B_i = \langle b_{(i,j)}^{\dot{\gamma}} : j \in [1,l] \text{ and } \dot{\gamma} \in \dot{\Gamma}_i \rangle$. Therefore,

$$B_i' = \left\langle [b_{(i,j)}^{\dot{\gamma}}, b_{(i,j')}^{\dot{\delta}}] : j, j' \in [1,l] \text{ and } \dot{\gamma}, \dot{\delta} \in \dot{\Gamma}_i \right\rangle^{B_i}.$$

Note that

$$\mathrm{st}_{G_i}(1) \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=1} G_{i+1}.$$

Induction on the level shows that, for all $k \in \mathbb{N}$,

$$\mathrm{st}_{G_i}(k) \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=k-1} B_{i+k-1} \leqslant \prod_{\mathrm{level}_{T_i}(\nu)=k} G_{i+k}.$$

Note that $G_i = B_i \dot{\Gamma}_i$ and $B_i \cap \dot{\Gamma}_i = 1$. Therefore, G_i is the semi-direct product of B_i by $\dot{\Gamma}_i$. Since B_i is a normal subgroup, $[B_i, G_i] = B_i'[B_i, \dot{\Gamma}_i]$.

3.24 Lemma.
Fix an integer $i \in \mathbb{N}_0$. For any $k \in \mathbb{N}$, let $\pi_k : G_i \to \Gamma_{i+k-1} \wr \cdots \wr \Gamma_{i+1} \wr \Gamma_i$ be the natural restriction homomorphism defined by considering the action of G_i on the finite subtree $T_i[k]$. Suppose that, for any fixed $j \in [1,l]$ and all distinct i_1, $i_2 > i$, the orders of the elements $\beta_{(i_1,j)}$ and $\beta_{(i_2,j)}$ are coprime. Then π_k is surjective.

Proof:
Observe that $\pi_1(\dot{\beta}_{(i,j)}) = \beta_{(i,j)}$ and $\pi_1(b_{(i,j)}) = 1$. Therefore $\pi_1(G_i) = \Gamma_i$.

Observe that $\pi_2(\dot{\beta}_{(i,j)}) = \dot{\beta}_{(i,j)}$ and that $\pi_2(b_{(i,j)}) = (\beta_{(i+1,j)}, 1, \ldots, 1)$. Therefore, $\pi_2(G_i) = \Gamma_{i+1} \wr \Gamma_i$.

Fix an integer $k \geqslant 3$ and suppose for induction that π_{k-1} is surjective. By the Chinese Remainder Theorem, there exist integers $r_{(k,j)}$ such that $r_{(k,j)} \equiv 1$ modulo $\mathrm{ord}(\beta_{(i+k,j)})$ and $r_{(k,j)} \equiv 0$ modulo $\mathrm{ord}(\beta_{(i+k',j)})$ for all $k' < k-1$. Then $b_{(i,j)}^{r_{(k,j)}} \in \mathrm{st}_{G_i}(k-1)$ and, in the expansion of $b_{(i,j)}^{r_{(k,j)}}$ to level $k-1$, we see that $\nu_{(n_i, n_{i+1}, \ldots, n_{i+k-3}, 1)}$ is labelled by $\dot{\beta}_{(i+k-1,j)}$, that $\nu_{(n_i, n_{i+1}, \ldots, n_{i+k-3}, n_{i+k-2})}$ is labelled by $b_{(i+k-1,j)}^{r_{(k,j)}}$, and that all other vertices of level $k-1$ are labelled by 1. Now, $b_{(i+k-1,j)} \in \mathrm{st}_{G_{i+k-1}}(1)$, so $\pi_k\big(b_{(i,j)}^{r_{(k,j)}}\big)$ acts as $\beta_{(i+k-1,j)}$ at precisely one vertex of level $k-1$ in the tree $T_i[k]$ and as the identity at all other vertices of $T_i[k]$.

As Γ_{i+k-1} is generated by the elements $\beta_{(i+k-1,j)}$, we have shown that

$$\mathrm{rst}_{\pi_k(G_i)}(\nu_{(n_i, n_{i+1}, \ldots, n_{i+k-3}, 1)}) \geqslant \Gamma_{i+k-1}.$$

The assumption that π_{k-1} is surjective means that we can conjugate $\Gamma_{(i+k-1,j)}$ to any other vertex of level $k-1$. Therefore,

$$\pi_k(G_i) \geqslant \prod_{\mathrm{level}_{T_i}(\nu) = k-1} \Gamma_{i+k-1}.$$

We can use this group to kill off the action of $\pi_k(G_i)$ on level $k-1$ of $T_i[k+1]$, showing that $\pi_k(G_i) \geqslant 1 \wr \Gamma_{i+k-1} \wr \cdots \wr \Gamma_{i+1} \wr \Gamma_i$. Therefore $\pi_k(G_i)$ contains the group $\Gamma_{i+k} \wr \Gamma_{i+k-1} \wr \cdots \wr \Gamma_{i+1} \wr \Gamma_i$ as required. □

3.25 Theorem.
Fix an integer $i \in \mathbb{N}_0$. Suppose that, for any fixed $j \in [1, l]$ and all distinct $i_1, i_2 > i$, the orders of the elements $\beta_{(i_1,j)}$ and $\beta_{(i_2,j)}$ are coprime. Then G_i is dense H_i.

Proof:
This is immediate from Remark 2.37 and Lemma 3.24. □

3.26 Lemma.
Fix an integer $i \in \mathbb{N}_0$. Let ν be a vertex of level i and label the children of ν by $\nu_1, \nu_2, \ldots, \nu_{n_i}$. Recall that $\dot{\Gamma}_i$ permutes the vertices $\nu_1, \nu_2, \ldots, \nu_{n_i}$ by rooted automorphisms. If there exists $\dot{\xi} \in \dot{\Gamma}_i$ such that $\nu_{n_i}^{\dot{\xi}} = \nu_{n_i}$ and $\nu_1^{\dot{\xi}} \neq \nu_1$ then

$$B_i' \geqslant \prod_{\mathrm{level}_{T_i}(\omega) = 1} B_{i+1}'.$$

Proof:
It suffices to show that $B'_i \geqslant 1 \times \cdots \times 1 \times B'_{i+1}$ because then we can simply conjugate by appropriate elements of $\dot\Gamma_i$. Recall from Remark 3.23 that

$$B'_{i+1} = \left\langle [b_{(i+1,j)}^{\dot\gamma}, b_{(i+1,j')}^{\dot\delta}] : j, j' \in [1,l] \text{ and } \dot\gamma, \dot\delta \in \dot\Gamma_{i+1} \right\rangle^{B_{i+1}}.$$

Arbitrarily choose $j, j' \in [1,l]$ and $\dot\gamma, \dot\delta \in \dot\Gamma_{i+1}$. It suffices to show that

$$\left(1, \dots, 1, [b_{(i+1,j)}^{\dot\gamma}, b_{(i+1,j')}^{\dot\delta}]\right)$$

is an element of B'_i.

There exists a word w in l formal variables such that $\dot\gamma = w(\dot\beta_{(i+1,1)}, \dots, \dot\beta_{(i+1,l)})$. By transitivity of $\dot\Gamma_i$, there exists $\dot\zeta \in \dot\Gamma_i$ such that $\nu_1^{\dot\zeta} = \nu_{n_i}$. Calculate that

$$b_{(i,j)}^{w(b_{(i,1)}^{\dot\zeta}, \dots, b_{(i,l)}^{\dot\zeta})} = \left(*, 1, \dots, 1, b_{(i+1,j)}^{\dot\gamma}\right)$$

where $*$ denotes a non-identity element of G_{i+1}. Similarly, $\left(*, 1, \dots, 1, b_{(i+1,j')}^{\dot\delta}\right)$ is an element of B_i. Conjugate one of these by $\dot\xi$ in order to move $*$ to a different vertex. Take the commutator to prove that $\left(1, \dots, 1, [b_{(i+1,j)}^{\dot\gamma}, b_{(i+1,j')}^{\dot\delta}]\right)$ is an element of B'_i.
□

3.27 Theorem.
Suppose that the following two conditions are satisfied:

(i) for any fixed $j \in [1,l]$ and all distinct $i_1, i_2 > i$, the orders of the elements $\beta_{(i_1,j)}$ and $\beta_{(i_2,j)}$ are coprime; and

(ii) for all $i \in \mathbb{N}_0$ there exists $\dot\xi_i \in \dot\Gamma_i$ such that $\nu_{n_i}^{\dot\xi_i} = \nu_{n_i}$ and $\nu_1^{\dot\xi_i} \neq \nu_1$.

Then G is a branch group.

Proof:
By Lemma 3.24, G is spherically-transitive. Fix an integer $k \in \mathbb{N}_0$. An induction on Lemma 3.26 shows that

$$G \geqslant \prod_{\text{level}(\omega)=k} B'_k.$$

Therefore,

$$\text{rst}_G(k) \cap \text{st}_G(k+1) \geqslant \prod_{\text{level}(\omega)=k} B'_k.$$

By Remark 3.23,

$$\text{st}_G(k+1) \leqslant \prod_{\text{level}(\omega)=k} B_k.$$

Therefore, $\mathrm{st}_G(k+1)/(\mathrm{rst}_G(k) \cap \mathrm{st}_G(k+1))$ is a section of the abelian group

$$\prod_{\mathrm{level}(\omega)=k} B_k/B_k'.$$

By Remark 3.23, B_k is generated by elements of the form $b_{(k,j)}^{\dot{\gamma}}$ where $j \in [1,l]$ and $\dot{\gamma} \in \dot{\Gamma}_k$. Thus B_k is finitely generated. A suitable power of each of these generators is contained in $\mathrm{rst}_G(k) \cap \mathrm{st}_G(k+1)$. Therefore,

$$\Big(\prod_{\mathrm{level}(\omega)=k} B_k\Big)/\big(\mathrm{rst}_G(k) \cap \mathrm{st}_G(k+1)\big)$$

is a finite abelian group. Hence $\mathrm{rst}_G(k) \cap \mathrm{st}_G(k+1)$ has finite index in $\mathrm{st}_G(k+1)$. By Remark 2.37, $\mathrm{st}_G(k+1)$ has finite index in G. Therefore $\mathrm{rst}_G(k) \cap \mathrm{st}_G(k+1)$ has finite index in G and, a fortiori, $\mathrm{rst}_G(k)$ has finite index in G, as required. □

3.28 Definition.
Define D_i to be the subgroup of B_i generated by all elements of the form $b_{(i,j)}^{-\dot{\gamma}} b_{(i,j)}^{\dot{\delta}}$ where $j \in [1,l]$ and $\dot{\gamma}, \dot{\delta} \in \dot{\Gamma}_i$. Define a subgroup of B_i by $E_i := B_i' D_i$.

3.29 Lemma.
Fix an integer $i \in \mathbb{N}_0$. Then $E_i = [B_i, G_i]$.

Proof:
Let z be an element of $\{b_{(i,1)}, \ldots, b_{(i,l)}\}$ and $\dot{\gamma}, \dot{\delta}$ elements of $\dot{\Gamma}_i$. Observe that $z^{-\dot{\gamma}} z^{\dot{\delta}} = \dot{\gamma}^{-1} z^{-1} \dot{\gamma} \dot{\delta}^{-1} z \dot{\delta} = [\dot{\gamma}, z][z, \dot{\delta}]$ is an element of $[B_i, G_i]$. Therefore, D_i is a subgroup of $[B_i, G_i]$, and so $E_i = B_i' D_i$ is a subgroup of $[B_i, G_i]$.

For the reverse inclusion, it suffices to show that $[B_i, \dot{\Gamma}_i] \leqslant B_i' D_i$ because $[B_i, G_i] = B_i'[B_i, \dot{\Gamma}_i]$ by Remark 3.23. To this end, let x be an element of B_i and $\dot{\gamma}$ an element of $\dot{\Gamma}_i$. Then $[x, \dot{\gamma}] = x^{-1} x^{\dot{\gamma}}$. By Remark 3.23, x can be written as a word in letters of the form $b_{(i,j)}^{\dot{\delta}}$. Modulo B_i', we can rearrange such letters. In particular, each letter $b_{(i,j)}^{-\dot{\delta}}$ can be paired with $b_{(i,j)}^{\dot{\delta}\dot{\gamma}}$. Each of these pairs is an element of D_i. Hence $x^{-1} x^{\dot{\gamma}}$ is congruent, modulo B_i', to an element of D_i. Therefore, $[B_i, G_i]$ is a subgroup of $B_i' D_i = E_i$ as required. □

3.30 Lemma.
Let ν be a vertex of level i and label the children of ν by $\nu_1, \nu_2, \ldots, \nu_{n_i}$. If there exist elements $\dot{\sigma}$ and $\dot{\tau}$ of $\dot{\Gamma}_i$ such that ν_1, $\nu_1^{\dot{\sigma}}$, $\nu_1^{\dot{\tau}}$, ν_{n_i}, $\nu_{n_i}^{\dot{\sigma}}$ and $\nu_{n_i}^{\dot{\tau}}$ are all distinct vertices then $B_i' = E_i'$.

Proof:
By definition, E_i is a subgroup of B_i, so $E_i^{'}$ is a subgroup of $B_i^{'}$. The reverse inclusion takes a bit more work.

By Remark 3.23, $B_i^{'}$ is the normal closure in B_i of the subgroup generated by all commutators of the form $[x_1^{\dot\gamma_1}, x_2^{\dot\gamma_2}]$ where $x_1, x_2 \in \{b_{(i,1)}, \dots, b_{(i,l)}\}$ and $\dot\gamma_1, \dot\gamma_2 \in \dot\Gamma_i$. By Lemma 3.29, $E_i^{'}$ is a normal subgroup of G_i. To prove that $B_i^{'}$ is a subgroup of $E_i^{'}$, it therefore suffices to show that any commutator of the form $[x_1^{\dot\gamma_1}, x_2^{\dot\gamma_2}]$ is contained in $E_i^{'}$. In fact, by conjugating by $\dot\gamma_2^{-1}$, it suffices to consider commutators of the form $[x_1^{\dot\gamma}, x_2]$ where $x_1, x_2 \in \{b_{(i,1)}, \dots, b_{(i,l)}\}$ and $\dot\gamma \in \dot\Gamma_i$.

In the expansions of x_1 and x_2 to the first level, the vertices $\nu_2, \dots, \nu_{n_i-1}$ are labelled by 1. There are four cases to consider:

(i) if $x_1^{\dot\gamma}$ and x_2 have disjoint support then $[x_1^{\dot\gamma}, x_2] = 1$;

(ii) if the supports of $x_1^{\dot\gamma}$ and x_2 overlap only in the subtree T_{ν_1}, then there exist elements w_1 and w_2 of G_{i+1} such that $x_1^{\dot\gamma} = (w_1, 1, \dots, 1, w_2, 1, \dots, 1)$, where w_2 is labelling a vertex distinct from ν_{n_i}. Say w_2 is labelling vertex ν_j. By hypothesis, either ν_j is not contained in the set $\{\nu_1^{\dot\sigma}, \nu_{n_i}^{\dot\sigma}\}$, or ν_j is not contained in the set $\{\nu_1^{\dot\tau}, \nu_{n_i}^{\dot\tau}\}$. If ν_j is not in $\{\nu_1^{\dot\sigma}, \nu_{n_i}^{\dot\sigma}\}$ then $[x_1^{\dot\gamma}, x_2] = [x_1^{\dot\gamma}x_1^{-\dot\tau}, x_2x_2^{-\dot\sigma}]$ is an element of $E_i^{'}$. If ν_j is not in $\{\nu_1^{\dot\tau}, \nu_{n_i}^{\dot\tau}\}$ then $[x_1^{\dot\gamma}, x_2] = [x_1^{\dot\gamma}x_1^{-\dot\sigma}, x_2x_2^{-\dot\tau}]$ is an element of $E_i^{'}$. The easiest way to verify this is to consider the supports;

(iii) if the supports of $x_1^{\dot\gamma}$ and x_2 overlap only in the subtree $T_{\nu_{n_i}}$ then we perform a calculation identical to that in case (ii); and

(iv) if the supports of $x_1^{\dot\gamma}$ and x_2 overlap only in the subtrees T_{ν_1} and $T_{\nu_{n_i}}$, then there exist elements w_1 and w_2 of G_{i+1} such that $x_1^{\dot\gamma} = (w_1, 1, \dots, 1, w_2)$. Then $[x_1^{\dot\gamma}, x_2] = [x_1^{\dot\gamma}x_1^{-\dot\sigma}, x_2x_2^{-\dot\tau}]$ is an element of $E_i^{'}$.

The cases are exhausted and the lemma is proved. □

3.31 Lemma.
Fix an integer $i \in \mathbb{N}_0$. Let ν be a vertex of level i and label the children of ν by $\nu_1, \nu_2, \dots, \nu_{n_i}$. If there exists an element $\dot\xi$ of $\dot\Gamma_i$ such that $\nu_{n_i}^{\dot\xi} = \nu_{n_i}$ and $\nu_1^{\dot\xi} \neq \nu_1$ then

$$B_i^{'} \geqslant \prod_{\text{level}_{T_i}(\omega)=1} E_{i+1}.$$

Proof:
By Lemma 3.26,

$$B_i^{'} \geqslant \prod_{\mathrm{level}_{T_i}(\omega)=1} B_{i+1}^{'}.$$

As $E_i = B_i^{'} D_i$, it therefore suffices to show that

$$B_i^{'} \geqslant \prod_{\mathrm{level}_{T_i}(\omega)=1} D_{i+1}.$$

We need only show that $B_i^{'}$ conatins $1 \times \cdots \times 1 \times D_{i+1}$ because then we can simply conjugate by appropriate elements of $\dot{\Gamma}_i$.

We claim that there exists an element $\dot{\zeta}$ of $\dot{\Gamma}_i$ such that $\nu_1^{\dot{\zeta}} = \nu_{n_i}$ and $\nu_{n_i}^{\dot{\zeta}} \neq \nu_1$. By transitivity, there exists $\dot{\varepsilon} \in \dot{\Gamma}_i$ such that $\nu_1^{\dot{\varepsilon}} = \nu_{n_i}$. If $\nu_{n_i}^{\dot{\varepsilon}} \neq \nu_{n_1}$ then the claim is proved. Otherwise, take $\dot{\zeta} = \dot{\varepsilon}\dot{\xi}$.

Recall that D_{i+1} is the subgroup of B_{i+1} generated by all elements of the form $b_{(i+1,j)}^{-\dot{\gamma}} b_{(i+1,j)}^{\dot{\delta}}$ where $j \in [1, l]$ and $\dot{\gamma}, \dot{\delta} \in \dot{\Gamma}_{i+1}$. Let $\dot{\gamma}$ and $\dot{\delta}$ be two arbitrarily chosen elements of $\dot{\Gamma}_{i+1}$. There exist words w_γ and w_δ in l formal variables such that $\dot{\gamma} = w_\gamma(\dot{\beta}_{(i+1,1)}, \ldots, \dot{\beta}_{(i+1,l)})$ and $\dot{\delta} = w_\delta(\dot{\beta}_{(i+1,1)}, \ldots, \dot{\beta}_{(i+1,l)})$. Let $z_\gamma = w_\gamma\big(b_{(i,1)}^{\dot{\zeta}}, \ldots, b_{(i,l)}^{\dot{\zeta}}\big)$ and $z_\delta = w_\delta\big(b_{(i,1)}^{\dot{\zeta}}, \ldots, b_{(i,l)}^{\dot{\zeta}}\big)$. Calculate that

$$z_\gamma = (1, \ldots, 1, w_\gamma(b_{(i+1,1)}, \ldots, b_{(i+1,l)}), 1, \ldots, 1, \dot{\gamma}),$$

and similarly that

$$z_\delta = (1, \ldots, 1, *, 1, \ldots, 1, \dot{\delta})$$

where $*$ denotes an element of B_{i+1}. Therefore, $b_{(i,j)}^{z_\delta} = (\dot{\beta}_{(i+1,j)}, 1, \ldots, 1, b_{(i+1,j)}^{\dot{\delta}})$. In particular, the supports of $z_\delta^{-1} z_\gamma$ and $b_{(i,j)}^{z_\delta}$ only overlap on the subtree $T_{\nu_{n_i}}$. Calculate that

$$[z_\delta^{-1} z_\gamma, b_{(i,j)}^{z_\delta}] = \big(1, \ldots, 1, \big[\dot{\delta}^{-1}\dot{\gamma}, b_{(i+1,j)}^{\dot{\delta}}\big]\big) = \big(1, \ldots, 1, b_{(i+1,j)}^{-\dot{\gamma}} b_{(i+1,j)}^{\dot{\delta}}\big)$$

is an element of $B_i^{'}$. This holds for all $j \in [1, l]$ and all $\dot{\gamma}, \dot{\delta} \in \dot{\Gamma}_{i+1}$. Therefore, $B_i{}'$ contains $1 \times \ldots \times 1 \times D_{i+1}$ as required. □

3.32 Theorem.
Fix an integer $k \in \mathbb{N}$. Suppose that, for all $i \in \mathbb{N}_0$, there exists $\dot{\xi}_i \in \dot{\Gamma}_i$ such that $\nu_{n_i}^{\dot{\xi}_i} = \nu_{n_i}$ and $\nu_1^{\dot{\xi}_i} \neq \nu_1$. Then

$$\mathrm{rst}_G(k) \geqslant \prod_{\mathrm{level}(\omega)=k} E_k.$$

Suppose that, in addition, for all $i \in \mathbb{N}$ there exist elements $\dot{\sigma}_i, \dot{\tau}_i \in \dot{\Gamma}_i$ such that ν_1, $\nu_1^{\dot{\sigma}}$, $\nu_1^{\dot{\tau}}$, ν_{n_i}, $\nu_{n_i}^{\dot{\sigma}}$ and $\nu_{n_i}^{\dot{\tau}}$ are all distinct vertices. Then

$$\mathrm{rst}_G(k)' \geqslant \prod_{\mathrm{level}(\omega)=k} B_k'.$$

Proof:
As E_i contains B_i', it follows by induction from Lemma 3.31 that

$$G \geqslant \prod_{\mathrm{level}(\omega)=k} E_k.$$

If the elements $\dot{\sigma}_i$ and $\dot{\tau}_i$ exist then apply Lemma 3.30. □

3.33 Remark.
The results of this section do not have a natural conclusion: this is as far as the research has come. It is hoped that the results proved so far will help with the construction of a finitely generated group G that has finite upper p-rank for all primes p and such that $\mathrm{ur}_p(G)$ is unbounded as p ranges over all the primes, as mentioned in Section 1.3. In order to calculate $ur_p(G)$ it is necessary to understand the finite quotients of G. In the case of a spherically transitive subgroup G of $\mathrm{Aut}(T)$, this reduces to understanding the finite quotients of $G/\mathrm{rst}_G(k)'$, by Theorem 2.41. As long as the fairly unrestrictive conditions of this section are satisfied by G, we have that $G/\mathrm{rst}_G(k)$ is finite and that $\mathrm{rst}_G(k)'$ contains $\prod_{\mathrm{level}(\omega)=k} B_k'$. Hopefully, for a suitably well chosen example, it will be possible to calculate precisely the groups B_k', as in Section 3.1. It should then be possible, as in that section, to calculate the upper p-rank of G for all primes p.

Chapter 4

Automata

4.1 The Word Problem

4.1 Theorem.
Let A be a finite invertible automaton. Then the automaton group defined by A has recursively soluble word problem.

Proof:
Suppose that A is a (c, d)-automaton with states $q_1, \dots, q_c$. Let F be the free group on the formal letters $x_1, \dots, x_c$. Let W_n denote the set of words of length at most n in F. Let U_n denote the subset $\{w_n(q_1, \dots, q_c) \mid w_n \in W_n\}$ of the group A. Let T be the rooted d-ary tree. We provide an algorithm to check, for any given word w in F, if $w(q_1, \dots, q_c)$ is the identity element of $\mathrm{Aut}(T)$.

Note that U_n is closed under expansion. Fix an element w of W_n and let $s = w(q_1, \dots, q_c)$. Consider the expansion of s to any level k of T. By Remark 2.52, the vertices of level k are labelled by elements of U_n. Let R denote the subset of U_n consisting of those elements that occur in the expansion of s. We see that s is the identity element of $\mathrm{Aut}(T)$ if and only if every element of R acts as the identity at the root.

Define R_0 to be the set containing only s. Inductively define R_{i+1} by adjoining to R_i all elements that occur in the expansion of some element of R_i to the first level. Observe that

$$R = \bigcup_{i \geqslant 0} R_i.$$

As $(R_i)_{i \geqslant 0}$ is an increasing sequence of subsets of the finite set U_n, there exists j such that $R_j = R_{j+1}$. It follows that $R = R_j$.

Let N denote the cardinality of U_n. Then certainly $R_N = R_{N+1}$. Therefore, the set R may be calculated in finite time, as may the action of every element of R at the root. □

4.2 Remark.
Given the length ℓ of the word w, we can give an upper bound on the processing time of the algorithm to determine if $w(q_1, \ldots, q_c)$ is the identity automorphism of T. Indeed, the only calculations involved are the expansion of elements to the first level. This involves composing permutations (at the root) and keeping track of the labels of the vertices of the first level. The labels of the vertices of the first level will always be words of length at most ℓ in $\{q_1, \ldots, q_c\}$. We can calculate the number of words of length at most ℓ in the free group on c formal letters and thus give an upper bound on the processing time required to reach the point when $R_j = R_{j+1}$.

4.2 Classes of Automata

4.3 Definition.
Fix a finite set of states Q, a finite alphabet X and a transition map τ. The *class of automata* defined by the triple (Q, X, τ) is the set of finite invertible automata (Q, X, ρ, τ) where ρ ranges over all possible output maps. That is, ρ ranges over all maps $Q \times X \to X$ such that, for every state q, the restriction map $\rho_q : X \to X$ is a permutation of the set X.

In order to define a specific class, we write $q = \big(\tau(q,1), \ldots, \tau(q,d)\big)$ for each state q. In this way, we have defined the transition map τ, but the output map has not been specified.

4.4 Definition.
Fix a finite set of states Q and a finite alphabet X. Let τ_1 and τ_2 be transition maps. We say that the classes of automata (Q, X, τ_1) and (Q, X, τ_2) are *canonically isomorphic* if there exist bijections $\pi_Q : Q \to Q$ and $\pi_X : X \to X$ such that $\pi_Q\big(\tau_1(\pi_Q(q), \pi_X(x))\big) = \tau_2(q, x)$ for all $q \in Q$ and $x \in X$.

Observe that a canonical isomorphism corresponds to a relabelling of the set of states Q and a renumbering of the alphabet X.

4.5 Remark.
Let C_1 and C_2 be canonically isomorphic classes of automata. The bijections $\pi_Q : Q \to Q$ and $\pi_X : X \to X$ induce a bijection $\Pi : C_1 \to C_2$ such that the automaton

group A_1 is isomorphic (as a group) to $\Pi(A_1)$ for all $A_1 \in C_1$. Therefore, in order to classify all (c, d)-automaton groups, we need only choose one class of automata from each canonical isomorphism class.

4.3 $(2, 3)$-Automata

Fix the set of states $Q = \{a, b\}$, the alphabet $X = \{1, 2, 3\}$, and the rooted ternary tree T. As in Definition 2.61, the two generators of a $(2, 3)$-automaton group are of the form $a = \sigma_a(a_1, a_2, a_3)$ and $b = \sigma_b(b_1, b_2, b_3)$ where $\sigma_a, \sigma_b \in \mathrm{Sym}(X)$ and $a_i, b_i \in Q$.

4.6 Definition.
Write $\big\langle a = (\sigma_a(a_1, a_2, a_3), b = \sigma_b(b_1, b_2, b_3)\big\rangle$ to denote the $(2, 3)$-automaton group generated by $a = \sigma_a(a_1, a_2, a_3)$ and $b = \sigma_b(b_1, b_2, b_3)$.

4.7 Lemma.
Every class of $(2, 3)$-automata is canonically isomorphic to one of thirteen classes.

Proof:
As in the note following Definition 4.3, we write $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ where $a_i, b_i \in Q$. In temporary notation, let $n(a) = \big|\{a_i \mid a_i = b\}\big|$ and $n(b) = \big|\{b_i \mid b_i = a\}\big|$. By relabelling the states and renumbering the alphabet as in Definition 4.4, we may assume, without loss of generality, that $n(a) \geqslant n(b)$ and $a \in \{(a, a, a), (b, a, a), (b, b, a), (b, b, b)\}$. There are seven cases to consider:

(i) if $n(a) + n(b) = 0$ then the only possibility is: $a = (a, a, a)$ and $b = (b, b, b)$;

(ii) if $n(a) + n(b) = 1$ then the only possibility is: $a = (b, a, a)$ and $b = (b, b, b)$;

(iii) if $n(a) + n(b) = 2$ then the possibilities are: $a = (b, b, a)$ and $b = (b, b, b)$; $a = (b, a, a)$ and $b = (a, b, b)$; $a = (b, a, a)$ and $b = (b, a, b)$;

(iv) if $n(a) + n(b) = 3$ then the possibilities are: $a = (b, b, b)$ and $b = (b, b, b)$; $a = (b, b, a)$ and $b = (b, b, a)$; $a = (b, b, a)$ and $b = (a, b, b)$;

(v) if $n(a) + n(b) = 4$ then the possibilities are: $a = (b, b, b)$ and $b = (a, b, b)$; $a = (b, b, a)$ and $b = (a, a, b)$; $a = (b, b, a)$ and $b = (a, b, a)$;

(vi) if $n(a) + n(b) = 5$ then the only possibility is: $a = (b, b, b)$ and $b = (a, a, b)$; and

(vii) if $n(a) + n(b) = 6$ then the only possibility is: $a = (b, b, b)$ and $b = (a, a, a)$.

□

In light of Lemma 4.7, we make the following definition.

4.8 Definition.
Define thirteen classes of $(2,3)$-automata as follows:

Class 1 is defined by $a=(a,a,a)$ and $b=(b,b,b)$;

Class 2 is defined by $a=(b,a,a)$ and $b=(b,b,b)$;

Class 3 is defined by $a=(b,b,a)$ and $b=(b,b,b)$;

Class 4 is defined by $a=(b,a,a)$ and $b=(a,b,b)$;

Class 5 is defined by $a=(b,a,a)$ and $b=(b,a,b)$;

Class 6 is defined by $a=(b,b,b)$ and $b=(b,b,b)$;

Class 7 is defined by $a=(b,b,a)$ and $b=(b,b,a)$;

Class 8 is defined by $a=(b,b,a)$ and $b=(a,b,b)$;

Class 9 is defined by $a=(b,b,b)$ and $b=(a,b,b)$;

Class 10 is defined by $a=(b,b,a)$ and $b=(a,a,b)$;

Class 11 is defined by $a=(b,b,a)$ and $b=(a,b,a)$;

Class 12 is defined by $a=(b,b,b)$ and $b=(a,a,b)$; and

Class 13 is defined by $a=(b,b,b)$ and $b=(a,a,a)$.

4.9 Remark.
It is checked that no two of these thirteen classes are canonically isomorphic.

4.10 Definition.
Let C be a class of $(2,3)$-automata. Suppose that C is defined by $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$. Let $\pi:\{a,b\}\to\{a,b\}$ be the map defined by $a\mapsto b$ and $b\mapsto a$. We say that C is *closed under swapping labels* if $\pi(a_j)=b_j$ and $\pi(b_j)=a_j$ for all j. Let τ be a permutation in $\mathrm{Sym}(3)$. We say that C is *closed under conjugation by* τ if $a_{j^\tau}=a_j$ and $b_{j^\tau}=b_j$ for all j. We say that C is *closed under swapping labels followed by conjugation by* τ if $\pi(a_{j^\tau})=b_j$ and $\pi(b_{j^\tau})=a_j$ for all j. The three terms so defined are collectively referred to as the *symmetries* of class C.

4.11 Remark.
If C is closed under swapping labels then

$$\big\langle a=\sigma_a(a_1,a_2,a_3), b=\sigma_b(b_1,b_2,b_3)\big\rangle=\big\langle a=\sigma_b(a_1,a_2,a_3), b=\sigma_a(b_1,b_2,b_3)\big\rangle.$$

Observe that $a^{\overline{\tau}} = \sigma_a^\tau\big(a^{\overline{\tau}}_{\phi(1)}, a^{\overline{\tau}}_{\phi(2)}, a^{\overline{\tau}}_{\phi(3)}\big)$ and $b^{\overline{\tau}} = \sigma_b^\tau\big(b^{\overline{\tau}}_{\phi(1)}, b^{\overline{\tau}}_{\phi(2)}, b^{\overline{\tau}}_{\phi(3)}\big)$ where $\phi(j) = j^{\tau^{-1}}$. If C is closed under conjugation by τ then conjugating $\mathrm{Aut}(T)$ by $\overline{\tau}$ yields the isomorphism

$$\big\langle a = \sigma_a(a_1, a_2, a_3), b = \sigma_b(b_1, b_2, b_3)\big\rangle \cong \big\langle a = \sigma_a^\tau(a_1, a_2, a_3), b = \sigma_b^\tau(b_1, b_2, b_3)\big\rangle.$$

If C is closed under swapping labels followed by conjugation by τ then

$$\big\langle a = \sigma_a(a_1, a_2, a_3), b = \sigma_b(b_1, b_2, b_3)\big\rangle \cong \big\langle a = \sigma_b^\tau(a_1, a_2, a_3), b = \sigma_a^\tau(b_1, b_2, b_3)\big\rangle.$$

4.12 Remark.
The symmetries of each Class are listed in Appendix A.1.

4.4 $(2, 3)$-Automata with C_2-Action at the Root

Recall the definition of 'action at the root' from Definition 2.64. In this section, we classify the $(2, 3)$-automata with C_2-action at the root, where C_2 may be any one of the three copies of C_2 in $\mathrm{Sym}(3)$. For each automaton so defined, we could relabel the tree so that the copy of C_2 is $\{\mathrm{id}, (12)\}$. However, for consistency we have chosen to classify the automata by Class, and have allowed each copy of C_2 to occur.

4.13 Definition. Let G be the group defined by the presentation

$$\left\langle \alpha, \beta \;\middle|\; (\alpha^{-1}\beta)^2 = 1, \left[\alpha^{-1}\beta, (\alpha^{-1}\beta)^{\alpha^k}\right] = 1 \;\forall k \in \mathbb{N}\right\rangle. \tag{4.1}$$

4.14 Remark.
Recall from the Notation section at the beginning of this thesis that $C_2 \wr C_\infty$ is defined by the presentation

$$\big\langle x, y \mid y^2 = 1, \big[y, y^{x^k}\big] = 1 \;\forall k \in \mathbb{N}\big\rangle.$$

Note that the group G defined by presentation (4.1) is isomorphic to $C_2 \wr C_\infty$ via the map defined by $x \mapsto \alpha$ and $y \mapsto \alpha^{-1}\beta$.

4.15 Lemma.
Define a map $\psi : G \to G$ by $w(\alpha, \beta) \mapsto w(\beta, \alpha)$ for every word $w(x, y)$ in the free group on $\{x, y\}$. Then ψ is an automorphism of G.

Proof:
We show that ψ is a well-defined map. It then follows immediately that ψ is an invertible homomorphism as $\psi \circ \psi$ is the identity map. In order to show that ψ is

well-defined it suffices to show that $(\beta^{-1}\alpha)^2 = 1$, and that $\left[\beta^{-1}\alpha, (\beta^{-1}\alpha)^{\beta^k}\right] = 1$ for every natural number k.

In order to show that $(\beta^{-1}\alpha)^2 = 1$, we note that the relation $(\alpha^{-1}\beta)^2 = 1$ implies that $\beta^{-1}\alpha = \alpha^{-1}\beta$.

To show that $\left[\beta^{-1}\alpha, (\beta^{-1}\alpha)^{\beta^k}\right] = 1$, we prove that $(\alpha^{-1}\beta)^{\alpha^k} = (\alpha^{-1}\beta)^{\beta^k}$ by induction on the natural number k.

First, note that conjugating the relation $\left[\alpha^{-1}\beta, (\alpha^{-1}\beta)^{\alpha}\right] = 1$ by α^{-1} proves that $(\alpha^{-1}\beta)^{\alpha^{-1}}$ commutes with $\alpha^{-1}\beta$. As $(\alpha^{-1}\beta)^{\alpha^{-1}} = \beta\alpha^{-1}$, we see that $\alpha^{-1}\beta$ commutes with $\beta\alpha^{-1}$ and with its inverse $\alpha\beta^{-1}$. That is, $(\alpha^{-1}\beta)^{\alpha\beta^{-1}} = \alpha^{-1}\beta$. Therefore, $(\alpha^{-1}\beta)^{\alpha} = (\alpha^{-1}\beta)^{\beta}$. This provides the base for the induction.

Suppose that $(\alpha^{-1}\beta)^{\alpha^k} = (\alpha^{-1}\beta)^{\beta^k}$. Then $(\alpha^{-1}\beta)^{\alpha^{k+1}} = (\alpha^{-1}\beta)^{\beta^k\alpha}$. To prove that $(\alpha^{-1}\beta)^{\alpha^{k+1}} = (\alpha^{-1}\beta)^{\beta^{k+1}}$, it therefore suffices to show that $(\alpha^{-1}\beta)^{\beta^k\alpha} = (\alpha^{-1}\beta)^{\beta^{k+1}}$. Conjugating the relation $\left[\alpha^{-1}\beta, (\alpha^{-1}\beta)^{\alpha^{k+1}}\right]$ by α^{-1} proves that $(\alpha^{-1}\beta)^{\alpha^{-1}}$ commutes with $(\alpha^{-1}\beta)^{\alpha^k}$. As $(\alpha^{-1}\beta)^{-\alpha^{-1}} = \alpha\beta^{-1}$ and $(\alpha^{-1}\beta)^{\alpha^k} = (\alpha^{-1}\beta)^{\beta^k}$, we have shown that $\alpha\beta^{-1}$ commutes with $(\alpha^{-1}\beta)^{\beta^k}$. That is, $(\alpha^{-1}\beta)^{\beta^k\alpha\beta^{-1}} = (\alpha^{-1}\beta)^{\beta^k}$, and so $(\alpha^{-1}\beta)^{\beta^k\alpha} = (\alpha^{-1}\beta)^{\beta^{k+1}}$ as required. □

4.16 Proposition.
Let a_3 and b_3 be elements of the set $\{a, b\}$ and let H be the $(2, 3)$-automaton defined by $a = (12)(b, a, a_3)$ and $b = (a, b, b_3)$. Then H is isomorphic to the group G given by presentation (4.1) via the map defined by $a \mapsto \alpha$ and $b \mapsto \beta$.

Proof:
It is shown in Section 4 of [GZ] that the $(2, 2)$-automaton defined by $a' = (12)(b', a')$ and $b' = (a', b')$ is isomorphic to G via the map defined by $a' \mapsto \alpha$ and $b' \mapsto \beta$.

Let X denote the alphabet $\{1, 2, 3\}$, let X^* denote the free multiplicative monoid generated by X, and let T be the rooted ternary tree. Observe that the submonoid of X^* consisting of all words on $\{1, 2\}$ is closed under the action of H. Let T' denote the binary subtree of T defined by this submonoid. Let a' and b' denote the automorphisms of T' defined by a and b respectively. Then $a' = (12)(b', a')$ and $b' = (a', b')$. If $w(x, y)$ is a word in the free group on $\{x, y\}$ such that $w(a, b)$ acts as the identity on T, then $w(a', b')$ must certainly act as the identity on T'. In order to prove the lemma, we must prove that if $w(a', b')$ acts as the identity on T' then $w(a, b)$ acts as the identity on T.

Let $w(x, y)$ be a word such that $w(a', b')$ acts as the identity on T'. Consider the expansion of $w(a, b)$ to the first level of T. The label of ν_3 is $w(a_3, b_3)$. We claim that $w(a'_3, b'_3)$ acts as the identity on T'. The claim follows immediately if $a_3 = a$ and

$b_3 = b$. If $a_3 = b$ and $b_3 = a$ then the claim follows from Lemma 4.15. Observe that total multiplicity (α-multiplicity + β-multiplicity) is well-defined in G. This follows from the observation that the total multiplicity of every relation in the presentation of G is zero. In particular, the total multiplicity of any relation in G must be zero. This proves the claim in the case $a_3 = b_3$.

We are now in a position to prove the lemma by induction on the level n. Certainly $w(a, b)$ stabilizes the first level of T. We have shown that if g is a tree automorphism that occurs in the expansion of $w(a, b)$ to the first level then there exists a word $u_g(x, y)$ such that: $g = u_g(a, b)$; and $u_g(a', b')$ acts as the identity on T'.

Suppose that $w(a, b)$ stabilizes level n of T. Furthermore, suppose that, for every tree automorphism g that occurs in the expansion of $w(a, b)$ to the level n there exists a word $u_g(x, y)$ such that: $g = u_g(a, b)$; and $u_g(a', b')$ acts as the identity on T'. Then certainly $w(a, b)$ stabilizes level $n + 1$ of the ternary tree. Consider the expansion of $u_g(a, b)$ to the first level. As $u_g(a', b')$ acts as the identity on T', we see that there exist words $u_1(x, y)$ and $u_2(x, y)$ such that ν_1 and ν_2 are labelled by $u_1(a, b)$ and $u_2(a, b)$ respectively, and $u_1(a', b') = u_2(a', b') = 1$. We see that ν_3 is labelled by $u_g(a_3, b_3)$. That $u_g(a_3', b_3') = 1$ follows from the above claim. Therefore, for every tree automorphism g that occurs in the expansion of $w(a, b)$ to level $n + 1$ there exists a word $u_g(x, y)$ such that: $g = u_g(a, b)$; and $u_g(a', b')$ acts as the identity on T'.

We have proved that $w(a, b)$ stabilizes every level of T. That is, $w(a, b)$ acts as the identity on T as required. □

The Tables

Let T be the rooted ternary tree. Fix the set of states $Q = \{a, b\}$ and the alphabet $X = \{1, 2, 3\}$. As in Definition 2.61, the generators of a $(2, 3)$-automaton take the form $a = \sigma_a(a_1, a_2, a_3)$ and $b = \sigma_b(b_1, b_2, b_3)$ where $\sigma_a, \sigma_b \in \mathrm{Sym}(X)$ and $a_i, b_i \in Q$. We consider the case when σ_a and σ_b generate a group of order 2. If $\sigma_a = \sigma_b$ then the group generated by the automaton is C_2.

We list the groups obtained when exactly one of σ_a or σ_b is the identity. A table is given for each of the thirteen classes in Definition 4.8. The rows are indexed by the generators a and b; as a helpful reminder, the transition map defining the class is specified here. The columns are indexed by the transpositions in $\mathrm{Sym}(X)$. The entry in the row indexed by a and the column index by ξ is the automaton of the appropriate class with $\sigma_a = \xi$ and $\sigma_b = \mathrm{id}$. Similarly, the entry in the row indexed by b and the column index by ξ is the automaton of the appropriate class with $\sigma_a = \mathrm{id}$ and $\sigma_b = \xi$.

The most difficult calculation required for these tables is the proof of Proposition 4.16. The remaining calculations take up a lot of space and quickly become repetitive. They are listed in Appendix A.2.

Class 1	(12)	(13)	(23)
$a = (a, a, a)$	C_2	C_2	C_2
$b = (b, b, b)$	C_2	C_2	C_2

Class 2	(12)	(13)	(23)
$a = (b, a, a)$	C_∞	C_∞	C_2
$b = (b, b, b)$	D_∞	D_∞	$C_2 \times C_2$

Class 3	(12)	(13)	(23)
$a = (b, b, a)$	C_2	C_∞	C_∞
$b = (b, b, b)$	$C_2 \times C_2$	D_∞	D_∞

Class 4	(12)	(13)	(23)
$a = (b, a, a)$	$C_2 \wr C_\infty$	$C_2 \wr C_\infty$	$C_2 \times C_2$
$b = (a, b, b)$	$C_2 \wr C_\infty$	$C_2 \wr C_\infty$	$C_2 \times C_2$

Class 5	(12)	(13)	(23)
$a = (b, a, a)$	$C_2 \wr C_\infty$	$C_\infty \times C_\infty$	D_∞
$b = (b, a, b)$	$C_2 \wr C_\infty$	D_∞	$C_\infty \times C_\infty$

Class 6	(12)	(13)	(23)
$a = (b, b, b)$	C_2	C_2	C_2
$b = (b, b, b)$	$C_2 \times C_2$	$C_2 \times C_2$	$C_2 \times C_2$

Class 7	(12)	(13)	(23)
$a = (b, b, a)$	$C_2 \times C_2$	$C_2 \wr C_\infty$	$C_2 \wr C_\infty$
$b = (b, b, a)$	$C_2 \times C_2$	$C_2 \wr C_\infty$	$C_2 \wr C_\infty$

Class 8	(12)	(13)	(23)
$a = (b, b, a)$	D_∞	$C_2 \wr C_\infty$	$C_\infty \times C_\infty$
$b = (a, b, b)$	C_∞	$C_2 \wr C_\infty$	D_∞

Class 9	(12)	(13)	(23)
$a = (b, b, b)$	D_∞	D_∞	$C_2 \times C_2$
$b = (a, b, b)$	$C_\infty \times C_\infty$	$C_\infty \times C_\infty$	$C_2 \times C_2$

Class 10	(12)	(13)	(23)
$a = (b, b, a)$	$C_2 \times C_2$	$C_2 \wr C_\infty$	$C_2 \wr C_\infty$
$b = (a, a, b)$	$C_2 \times C_2$	$C_2 \wr C_\infty$	$C_2 \wr C_\infty$

Class 11	(12)	(13)	(23)
$a = (b, b, a)$	D_∞	$C_\infty \times C_\infty$	$C_2 \wr C_\infty$
$b = (a, b, a)$	$C_\infty \times C_\infty$	D_∞	$C_2 \wr C_\infty$

Class 12	(12)	(13)	(23)
$a = (b, b, b)$	$C_2 \times C_2$	D_∞	D_∞
$b = (a, a, b)$	$C_2 \times C_2$	$C_\infty \times C_\infty$	$C_\infty \times C_\infty$

Class 13	(12)	(13)	(23)
$a = (b, b, b)$	$C_2 \times C_2$	$C_2 \times C_2$	$C_2 \times C_2$
$b = (a, a, a)$	$C_2 \times C_2$	$C_2 \times C_2$	$C_2 \times C_2$

4.5 $(2, 3)$-Automata with C_3-Action at the Root

Recall the definition of 'action at the root' from Definition 2.64. In this section, we classify the $(2, 3)$-automata with C_3-action at the root. Fix the set of states $Q = \{a, b\}$, the alphabet $X = \{1, 2, 3\}$, and the rooted ternary tree T. As in Definition 2.61, the generators of a $(2, 3)$-automaton take the form $a = \sigma_a(a_1, a_2, a_3)$ and $b = \sigma_b(b_1, b_2, b_3)$ where $\sigma_a, \sigma_b \in \mathrm{Sym}(X)$ and $a_i, b_i \in Q$. Indeed, the automaton may be defined by specifying the generators a and b.

4.17 Definition.

Define $H_1, \ldots, H_{17}$ to be the automaton groups generated by a and b where a and b are respectively as follows:

$H_1 : a = (b, a, a), b = (123)(b, b, b);$

$H_2 : a = (b, b, a), b = (123)(b, b, b);$

$H_3 : a = (b, a, a), b = (123)(a, b, b);$

$H_4 : a = (123)(b, a, a), b = (132)(a, b, b);$

$H_5 : a = (b, a, a), b = (132)(b, a, b);$

$H_6 : a = (b, b, a), b = (123)(b, b, a);$

$H_7 : a = (123)(b, b, a), b = (b, b, a);$

$H_8 : a = (123)(b, b, a), b = (132)(b, b, a);$

$H_9 : a = (b, b, a), b = (132)(a, b, b);$

$H_{10} : a = (123)(b, b, a), b = (a, b, b)$;

$H_{11} : a = (132)(b, b, a), b = (123)(a, b, b)$;

$H_{12} : a = (123)(b, b, b), b = (a, b, b)$;

$H_{13} : a = (123)(b, b, b), b = (132)(a, b, b)$;

$H_{14} : a = (b, b, a), b = (123)(a, a, b)$;

$H_{15} : a = (b, b, a), b = (123)(a, b, a)$;

$H_{16} : a = (123)(b, b, b), b = (a, a, b)$; and

$H_{17} : a = (123)(b, b, b), b = (132)(a, a, b)$.

4.18 Lemma.
Let G_1 and G_2 be the automaton groups generated by c and d, where c and d are defined by the systems of equations $c = (123)(d, c, c), d = (132)(d, d, d)$ and $c = (123)(d, d, c), d = (132)(d, d, d)$ respectively. Then as subgroups of $\mathrm{Aut}(T)$ we have that $G_1 = H_1$ and $G_2 = H_2$.

Proof:
The proof that $G_2 = H_2$ is identical to the proof that $G_1 = H_1$, but we give both calculations for completeness:

(i) Let $c = (123)(d, c, c)$ and $d = (132)(d, d, d)$ be the generators of G_1, and let $a = (b, a, a)$ and $b = (123)(b, b, b)$ be the generators of H_1. Then $dc = (d^2, dc, dc)$ and $d^2 = (123)(d^2, d^2, d^2)$. Thus $a = dc$ and $b = d^2$. Now, $d^3 = (d^3, d^3, d^3) = 1$, so $d^2 = d^{-1}$. Thus $\langle c, d\rangle = \langle dc, d^2\rangle$. That is, $G_1 = H_1$;

(ii) Let $c = (123)(d, d, c)$ and $d = (132)(d, d, d)$ be the generators of G_2, and let $a = (b, b, a)$ and $b = (123)(b, b, b)$ be the generators of H_2. Then $dc = (d^2, d^2, dc)$ and $d^2 = (123)(d^2, d^2, d^2)$. Thus $a = dc$ and $b = d^2$. Now, $d^3 = (d^3, d^3, d^3) = 1$, so $d^2 = d^{-1}$. Thus $\langle c, d\rangle = \langle dc, d^2\rangle$. That is, $G_2 = H_2$.

□

4.19 Lemma.
Let G_4 be the automaton group generated by c and d, where c and d are defined by the system of equations $c = (123)(d, d, c), d = (132)(c, c, d)$. Then G_4 is isomorphic to H_4.

Proof:
Recall the generators of H_4 are $a = (123)(b, a, a)$ and $b = (132)(a, b, b)$. Identify the vertices of even level in $T_{\underline{3}}$ with the vertices of $T_{\underline{9}}$ in the natural way. Thus the vertex $\nu_{(i_1,\ldots,i_{2n})}$ of $T_{\underline{3}}$ is identified with the vertex $\nu_{(j_1,\ldots,j_n)}$ of $T_{\underline{9}}$ where $j_k = 3i_{2k-1} - 3 + i_{2k}$. Under this identification, tree automorphisms of $T_{\underline{3}}$ naturally define tree automorphisms of $T_{\underline{9}}$. We calculate the expansions of a, b, c and d to the first level of $T_{\underline{9}}$:

$$\begin{aligned} a &= (159267348)(a, b, b, b, a, a, b, a, a) \\ b &= (195384276)(b, a, a, a, b, b, a, b, b) \\ c &= (167359248)(c, c, d, c, c, d, d, d, c) \\ d &= (194275386)(d, d, c, d, d, c, c, c, d) \end{aligned}$$

Let π denote the permutation $(194376)(285)$ in $\mathrm{Sym}(9)$. We observe that $a^{\overline{\pi}} = c$ and $b^{\overline{\pi}} = d$. □

4.20 Remark.
Conjugation by $\overline{\pi}$ is simply a renumbering of the tree $T_{\underline{9}}$. Under the natural identification of the vertices of level two in $T_{\underline{3}}$ with the vertices of level one in $T_{\underline{9}}$, we see that the element $(132)\big((13), (13), (13)\big)$ of $\mathrm{Aut}(T_{\underline{3}})/\mathrm{st}(2)$ corresponds to the element π of $\mathrm{Aut}(T_{\underline{9}})/\mathrm{st}(1)$. Therefore the isomorphism of Lemma 4.19 is really just a relabelling of the tree $T_{\underline{3}}$.

The Tables

A table is given for each of the thirteen classes in Definition 4.8. The rows and columns are indexed by the elements of $\mathrm{Alt}(X)$. The element in the row indexed by ζ and the column indexed by ξ is the automaton group of the appropriate class with $\sigma_a = \zeta$ and $\sigma_b = \xi$. Observe that the groups H_i have been defined by their first appearance in the tables. Further instances of H_i denote isomorphic groups. Some isomorphisms have been proved in Lemmas 4.18 and 4.19. The remaining isomorphisms in the tables are proved in Appendix A.3. Some results about the groups $H_1, \ldots, H_{17}$ are given in Chapter 5.

Class 1	$b = (b, b, b)$		
$a = (a, a, a)$	id	(123)	(132)
id	1	C_3	C_3
(123)	C_3	C_3	C_3
(132)	C_3	C_3	C_3

Class 2	$b = (b, b, b)$		
$a = (b, a, a)$	id	(123)	(132)
id	1	H_1	H_1
(123)	C_∞	C_3	H_1
(132)	C_∞	H_1	C_3

Class 3	$b = (b, b, b)$		
$a = (b, b, a)$	id	(123)	(132)
id	1	H_2	H_2
(123)	C_∞	C_3	H_2
(132)	C_∞	H_2	C_3

Class 4	$b = (a, b, b)$		
$a = (b, a, a)$	id	(123)	(132)
id	1	H_3	H_3
(123)	H_3	C_3	H_4
(132)	H_3	H_4	C_3

Class 5	$b = (b, a, b)$		
$a = (b, a, a)$	id	(123)	(132)
id	1	H_3	H_5
(123)	H_5	C_3	H_4
(132)	H_3	C_∞	C_3

Class 6	$b = (b, b, b)$		
$a = (b, b, b)$	id	(123)	(132)
id	1	$C_3 \times C_3$	$C_3 \times C_3$
(123)	C_3	C_3	$C_3 \times C_3$
(132)	C_3	$C_3 \times C_3$	C_3

Class 7	$b = (b, b, a)$		
$a = (b, b, a)$	id	(123)	(132)
id	1	H_6	H_6
(123)	H_7	C_3	H_8
(132)	H_7	H_8	C_3

Class 8	$b = (a, b, b)$		
$a = (b, b, a)$	id	(123)	(132)
id	1	H_6	H_9
(123)	H_{10}	C_3	H_8
(132)	H_7	H_{11}	C_3

Class 9	$b = (a, b, b)$		
$a = (b, b, b)$	id	(123)	(132)
id	1	C_∞	C_∞
(123)	H_{12}	C_3	H_{13}
(132)	H_{12}	H_{13}	C_3

Class 10	$b = (a, a, b)$		
$a = (b, b, a)$	id	(123)	(132)
id	1	H_{14}	H_{14}
(123)	H_{14}	C_3	H_4
(132)	H_{14}	H_4	C_3

Class 11	$b = (a, b, a)$		
$a = (b, b, a)$	id	(123)	(132)
id	1	H_{15}	H_{14}
(123)	H_{14}	C_3	C_∞
(132)	H_{15}	H_4	C_3

Class 12	$b = (a, a, b)$		
$a = (b, b, b)$	id	(123)	(132)
id	1	C_∞	C_∞
(123)	H_{16}	C_3	H_{17}
(132)	H_{16}	H_{17}	C_3

Class 13	$b = (a, a, a)$		
$a = (b, b, b)$	id	(123)	(132)
id	1	$C_3 \times C_3$	$C_3 \times C_3$
(123)	$C_3 \times C_3$	C_3	C_3
(132)	$C_3 \times C_3$	C_3	C_3

4.6 Windmill Automata

4.21 Definition.

Let c and d be natural numbers with $d \geqslant 2$. Let A be a $(2c+1, d)$-automaton with alphabet X. We say that A is a *windmill automaton* if there exists a labelling $q_1, r_1, \ldots, q_c, r_c, s$ of the states of A such that:

(i) $s = 1$ when interpreted as a tree automorphism;

(ii) $r_i = q_i^{-1}$ when interpreted as tree automorphisms; and

(iii) $\tau(q_i, x), \tau(r_i, x) \in \{q_i, r_i, s\}$ for all $i \in [1, c]$ and all $x \in X$.

4.22 Remark.

If an automaton A has precisely two states, a and b, then the set $\{1, a^{-1}b, b^{-1}a\}$ of tree automorphisms is closed under expansion (see Definition 2.51), as is $\{1, ab^{-1}, ba^{-1}\}$. Therefore, the automaton with $\{1, a^{-1}b, b^{-1}a, ab^{-1}, ba^{-1}\}$ for its set of states (with the output and transition maps induced from A) is a windmill automaton.

The above remark proves that if A is a two-state automaton then the group defined by A contains a subgroup that is defined by a windmill automaton. In the case of the groups defined by the ternary automata H_1 and H_2, it turns out that the subgroup defined by the windmill automaton with states $\{1, a^{-1}b, b^{-1}a, ab^{-1}, ba^{-1}\}$ is a normal subgroup of index 3 (see Proposition 5.39). For an automaton A with more than two states, we sometimes find that A contains a subgroup defined by a windmill automaton. The transition maps have to be chosen carefully in order to arrange this.

4.6.1 Binary Windmill Automata

We classify, up to group isomorphism, the subgroups of $\mathrm{Aut}(T_{\underline{2}})$ that are defined by binary windmill automata.

Let T be the rooted binary tree. Identify C_2 with $\mathrm{Sym}(2)$, and let ε denote the non-identity element of $\mathrm{Sym}(2)$.

4.23 Definition.
Define A to be the six-state binary automaton with states $1, a, b, c, d, e$ defined, using the notation of Definition 2.61, by the following system of equations:
$1 = \mathrm{id}(1,1)$; $a = \varepsilon(1,1)$; $b = \varepsilon(b,b)$; $c = \varepsilon(c,1)$; $d = \varepsilon(e,1)$; and $e = \varepsilon(1,d)$.
Define the tree automorphisms $\gamma := ac$ and $\delta := d^{-1}a$.

We identify the states of A with the tree automorphisms they define. As ed is a conjugate of de, it follows from Theorem 2.54 that $de = (1, ed)$ acts as the identity on T. Therefore, $e = d^{-1}$ and $d = \varepsilon(d^{-1}, 1)$.

4.24 Lemma.
If x is a state of a binary windmill automaton then, when considered as a tree automorphism, x is contained in the set $\{1, a, b, c, c^{-1}, d, d^{-1}\}$ of tree automorphisms.

Proof:
If $x(\nu_\phi) = \mathrm{id}$ then $x = 1$. We check each possible x with $x(\nu_\phi) = \varepsilon$ in turn:
(i) if $x = \varepsilon(1,1)$ then $x = a$;
(ii) if $x = \varepsilon(x,1)$ then $x = c$;
(iii) if $x = \varepsilon(x^{-1},1)$ then $x = d$;
(iv) if $x = \varepsilon(1,x)$ then $cx = (1, cx) = 1$;
(v) if $x = \varepsilon(1,x^{-1})$ then $dx = (1, d^{-1}x^{-1})$, so $dx = 1$ by Theorem 2.54; and
(vi) if $x \in \{\varepsilon(x,x), \varepsilon(x,x^{-1}), \varepsilon(x^{-1},x), \varepsilon(x^{-1},x^{-1})\}$ then $x^2 = 1$ and so $x = b$. □

4.25 Remark.

By Lemma 4.24, in order to classify the groups defined by binary windmill automata, we need only consider groups generated by subsets of $\{a, b, c, d\}$. It is checked that if S is any subset of $\{a, b, c, d\}$ then the subgroup of $\mathrm{Aut}(T)$ generated by S is defined by a windmill automaton.

4.26 Proposition.

(i) The tree automorphisms a and b are involutions; and

(ii) the tree automorphisms c and d have infinite order.

Proof:

(i) Note that both a and b act non-trivially at the root. Thus they are non-identity elements of $\mathrm{Aut}(T)$. Calculate that $a^2 = (1, 1) = 1$ and $b^2 = (b^2, b^2) = 1$.

(ii) Note that $c^2 = (c, c)$. As c acts as ε at the root, it follows from Lemma 2.50 that c has infinite order. Similarly, $d^2 = (d^{-1}, d^{-1})$ and $d(\nu_\phi) = \varepsilon$. Thus d has infinite order by Lemma 2.50. □

4.27 Remark.

As b is an involution and $bc = (bc, b)$, it follows that $(bc)^2 = ((bc)^2, 1) = 1$, and therefore that $c^b = c^{-1}$.

We calculate that $(bd)^2 = ((bd^{-1})^2, 1)$ and $(bd^{-1})^2 = (1, (bd)^2)$. By Lemma 2.53, we have that $(bd)^2 = 1$ and therefore that $d^b = d^{-1}$.

4.28 Remark.

Note that $\gamma = (c, 1) = \gamma^{a^2}$, that $\gamma^a = (1, c)$, that $\gamma^b = (1, c^b) = (1, c^{-1}) = \gamma^{-a}$, and that $\gamma^{ab} = \gamma^{-1}$. Thus, $\langle\gamma\rangle^{\langle a,b\rangle} = \langle\gamma, \gamma^a\rangle$. As c has infinite order, it follows that $\langle\gamma, \gamma^a\rangle$ is naturally isomorphic to $C_\infty \times C_\infty$.

Similarly: $\delta = (d, 1) = \delta^{a^2}$; $\delta^a = (1, d)$; $\delta^b = \delta^{-a}$; $\delta^{ab} = \delta^{-1}$; and $\langle\delta\rangle^{\langle a,b\rangle} = \langle\delta, \delta^a\rangle$ is naturally isomorphic to $C_\infty \times C_\infty$.

4.29 Proposition.

(i) The group generated by a and b is isomorphic to $C_2 \times C_2$;

(ii) the group generated by a and c is equal to $\langle\gamma, \gamma^a\rangle \rtimes \langle a\rangle$;

(iii) the group generated by a and d is equal to $\langle\delta, \delta^a\rangle \rtimes \langle a\rangle$;

(iv) the group generated by b and c is isomorphic to D_∞; and

(v) the group generated by b and d is isomorphic to D_∞.

Proof:

The orders of a, b, c and d are known by Proposition 4.26.

(i) Observe that $ab = (b, b)$ has order 2;
(ii) certainly $\langle a, c\rangle = \langle a, \gamma\rangle$. It follows that the normal closure of $\langle\gamma\rangle$ is $\langle\gamma, \gamma^a\rangle$. Now a acts non-trivially at the root, whereas γ and γ^a stabilize the first level. Therefore the subgroups $\langle\gamma, \gamma^a\rangle$ and $\langle a\rangle$ intersect trivially;
(iii) this is identical to (ii);
(iv) we know that c has infinite order and that b is an involution. We showed that $c^b = c^{-1}$ in Remark 4.27; and
(v) this is identical to (iv). □

There does not seem to be an easy description of the group generated by c and d. We therefore make the following definition:

4.30 Definition.
Define B to be the subgroup of $\mathrm{Aut}(T)$ generated by c and d.

4.31 Remark.
Modulo the second level stabilizer, c has order 4 and c is congruent to d. Identify $\mathrm{Aut}(T)/\mathrm{st}(2)$ with $\mathrm{Aut}(T[2])$. Then $B/\mathrm{st}_B(2) = \{1, \varepsilon(\varepsilon, 1), (\varepsilon, \varepsilon), \varepsilon(1, \varepsilon)\}$. As b is congruent to $\varepsilon(\varepsilon, \varepsilon)$ modulo $\mathrm{st}_{\mathrm{Aut}(T)}(2)$, we see that b is not contained in B.

4.32 Proposition.
(i) The group $\langle a, b, c\rangle$ is equal to $\langle\gamma, \gamma^a\rangle \rtimes \langle a, b\rangle$;
(ii) the group $\langle a, b, d\rangle$ is equal to $\langle\delta, \delta^a\rangle \rtimes \langle a, b\rangle$;
(iii) the group $\langle a, c, d\rangle$ is equal to $\langle\gamma, \gamma^a, \delta, \delta^a\rangle \rtimes \langle a\rangle$; and
(iv) the group $\langle b, c, d\rangle$ is equal to $\langle c, d\rangle \rtimes \langle b\rangle$.

Proof:
(i) Certainly $\langle a, b, c\rangle = \langle a, b, \gamma\rangle$. It follows from Remark 4.28 that the normal closure of $\langle\gamma\rangle$ is equal to $\langle\gamma, \gamma^a\rangle$, and that this is a free abelian group of rank 2. We know that the elements of $\langle a, b\rangle$ have finite order by Proposition 4.29 (i). Therefore, the subgroups $\langle\gamma, \gamma^a\rangle$ and $\langle a, b\rangle$ intersect trivially;
(ii) this is identical to (i);
(iii) certainly $\langle a, c, d\rangle = \langle a, \gamma, \delta\rangle$. The normal closure of $\langle\gamma, \delta\rangle$ is equal to $\langle\gamma, \gamma^a, \delta, \delta^a\rangle$ by Remark 4.28. As a acts non-trivially at the root, $\langle\gamma, \gamma^a, \delta, \delta^a\rangle$ and $\langle a\rangle$ intersect trivially; and
(iv) recall from Remark 4.27 that $c^b = c^{-1}$ and $d^b = d^{-1}$. Thus, $\langle c, d\rangle$ is a normal subgroup of $\langle b, c, d\rangle$. As b has order 2, it follows from Remark 4.31 that $\langle c, d\rangle$ and $\langle b\rangle$ intersect trivially. □

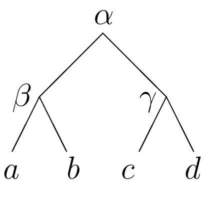
4.33 Remark.
Observe that $\langle \gamma, \gamma^a, \delta, \delta^a \rangle$ stabilizes the first level of T. Therefore, a and b are not contained in $\langle \gamma, \gamma^a, \delta, \delta^a \rangle$. Recall from Remark 4.31 that b is not contained in the group $B = \langle c, d \rangle$. By Remark 4.28, $\langle \gamma, \gamma^a, \delta, \delta^a \rangle$ is equal to $\mathrm{rst}(\nu_1) \times \mathrm{rst}(\nu_2)$, and $\mathrm{rst}(\nu_1) = \mathrm{rst}(\nu_2) = B$ (when identified with subgroups of $\mathrm{Aut}(T_{\nu_1})$ and $\mathrm{Aut}(T_{\nu_2})$ respectively). As $ab = (b, b)$, and b is not contained in B, we see that ab is not contained in $\langle \gamma, \gamma^a, \delta, \delta^a \rangle$.

4.34 Proposition.
The group $\langle a, b, c, d \rangle$ is equal to $\langle \gamma, \gamma^a, \delta, \delta^a \rangle \rtimes \langle a, b \rangle$.

Proof:
Certainly $\langle a, b, c, d \rangle = \langle a, b, \gamma, \delta \rangle$. Recall from Remark 4.28 that $\gamma^b = \gamma^{-a}$ and $\delta^b = \delta^{-a}$. It follows that the normal closure of $\langle \gamma, \delta \rangle$ equals $\langle \gamma, \gamma^a, \delta, \delta^a \rangle$. By Proposition 4.29 (i), we know that $\langle a, b \rangle = \{1, a, b, ab\}$. It follows from Remark 4.33 that the subgroups $\langle a, b \rangle$ and $\langle \gamma, \gamma^a, \delta, \delta^a \rangle$ intersect trivially. $\square$

4.35 Remark.
We have now classified all groups defined by binary windmill automata. They are: 1; C_2; $C_2 \times C_2$; C_∞; D_∞; $C_\infty \wr C_2$; $(C_\infty \times C_\infty) \rtimes (C_2 \times C_2)$; B; $B \rtimes C_2$; $B \wr C_2$; and $(B \times B) \rtimes (C_2 \times C_2)$. The actions in the semi-direct products can be inferred from the statements of the propositions in this subsection (recall from Remark 4.28 that $\langle \gamma, \gamma^a \rangle$ and $\langle \delta, \delta^a \rangle$ are isomorphic to $C_\infty \times C_\infty$, and from Remark 4.33 that $\langle \gamma, \gamma^a, \delta, \delta^a \rangle$ is isomorphic to $B \times B$).

4.36 Definition.
Let x be an element of $\mathrm{Aut}(T_{\underline{2}})$. Suppose that the expansion of x to the second level is given by:



We use the notation $\alpha(\beta, \gamma)(a, b, c, d)$ to denote the expansion of x to the second level and write $x = \alpha(\beta, \gamma)(a, b, c, d)$. If $\alpha = \mathrm{id}$ then, for brevity, we write $x = (\beta, \gamma)(a, b, c, d)$. Similarly, if $\alpha = \beta = \gamma = \mathrm{id}$ then we write $x = (a, b, c, d)$.

Recall the definition of $G \wr C_2 \wr C_2$ for an arbitrary group G from the Notation section at the beginning of this thesis.

4.37 Definition.
Let G be a group. Let $\delta_0\big(\delta_1(g_1,g_2),\delta_2(g_3,g_4)\big)$ be an element of $G\wr C_2\wr C_2$ where $\delta_i \in C_2$ and $g_i \in G$. Mirroring Definition 4.36, we use the notation $\delta_0(\delta_1,\delta_2)(g_1,g_2,g_3,g_4)$ to denote this element. If $\delta_0 = \mathrm{id}$ then we write $(\delta_1,\delta_2)(g_1,g_2,g_3,g_4)$; if $\delta_0=\delta_1=\delta_2=\mathrm{id}$ then we write (g_1,g_2,g_3,g_4).

4.38 Theorem.
The abelianization of B is isomorphic to $C_\infty \times C_\infty$.

Proof:
Recall that $c=\varepsilon(c,1)$, that $d=\varepsilon(d^{-1},1)$, and that B is the group generated by c and d. The expansion of c to the second level is $\varepsilon(\varepsilon,\mathrm{id})(c,1,1,1)$; the expansion of d to the second level is $\varepsilon(\varepsilon,\mathrm{id})(1,d,1,1)$. Let F be the free group on the two generators x and y. Let $\ell\big(w(x,y)\big)$ denote the length of the word $w(x,y)$. Define the homomorphism

$$\begin{aligned}\varphi : F &\rightarrow F\wr C_2\wr C_2\\ x &\mapsto \varepsilon(\varepsilon,\mathrm{id})(x,1,1,1)\\ y &\mapsto \varepsilon(\varepsilon,\mathrm{id})(1,y,1,1)\end{aligned}$$

Let W be the set $\{x^2, xy, yx, y^2, xy^{-1}, x^{-1}y\}$. Any reduced word of length 2ℓ in F can be uniquely expressed as a reduced word of length ℓ on $W^{\pm 1}$. We calculate $\varphi\big(w(x,y)\big)$ for every element $w(x,y)$ of W:

$$\begin{aligned}\varphi(x^2) &= (\varepsilon,\varepsilon)(x,1,x,1) & \varphi(xy) &= (\varepsilon,\varepsilon)(1,y,x,1)\\ \varphi(yx) &= (\varepsilon,\varepsilon)(x,1,1,y) & \varphi(y^2) &= (\varepsilon,\varepsilon)(1,y,1,y)\\ \varphi(xy^{-1}) &= (1,1,y^{-1},x) & \varphi(x^{-1}y) &= (x^{-1},y,1,1)\end{aligned}$$

Let $w(x,y)$ be a word of length ℓ in F. It follows from the above calculations that there exist $\delta_0,\delta_1,\delta_2 \in C_2$ and words $w_1(x,y)$, $w_2(x,y)$, $w_3(x,y)$, $w_4(x,y)$ of length at most $\lceil \ell/2 \rceil$ such that $\varphi\big(w(x,y)\big) = \delta_0(\delta_1,\delta_2)\big(w_1(x,y),w_2(x,y),w_3(x,y),w_4(x,y)\big)$.

Recall from Remark 4.31 that c is congruent to d modulo $\mathrm{st}_B(2)$ and that c has order 4 modulo $\mathrm{st}_B(2)$. It follows that $\mathrm{st}_B(2)$ is equal to the normal closure of $\langle c^4, cd^{-1}\rangle$ in B. Let N be the normal closure of $\langle x^4, xy^{-1}\rangle$ in F. Then F/N is is isomorphic to C_4 with coset representatives 1, x, x^2 and x^3. We use the Schreier method to find a set of free generators for N. For completeness, we provide the details of the calculation in the table below. The column indexed by u contains the elements of the set $\{1,x,x^2,x^3\}$ of coset representatives for N in F, and the column indexed by v contains the generators x and y of F. We let $\overline{uv}$ denote the unique element of $\{1,x,x^2,x^3\}$ such that uv is congruent to $\overline{uv}$ modulo N.

u	v	$\overline{uv}$	$uv(\overline{uv})^{-1}$
1	x	x	1
x	x	x^2	1
x^2	x	x^3	1
x^3	x	1	x^4
1	y	x	yx^{-1}
x	y	x^2	xyx^{-2}
x^2	y	x^3	x^2yx^{-3}
x^3	y	1	x^3y

It follows that $\{x^4, yx^{-1}, xyx^{-2}, x^2yx^{-3}, x^3y\}$ is a free generating set for N.

Let $z_1 = x^4$, let $z_2 = yx^{-1}$, let $z_3 = xyx^{-2}$, let $z_4 = x^2yx^{-3}$, and let $z_5 = x^3y$. Let $w(x, y)$ be an element of F such that $\varphi\big(w(x, y)\big) \in F' \times F' \times F' \times F'$. Rewrite $w(x, y)$ as a word $u(z_1, z_2, z_3, z_4, z_5)$ on $\{z_1, z_2, z_3, z_4, z_5\}$. We calculate the following:

$$\begin{aligned}
\varphi(z_1) &= (x, x, x, x) \\
\varphi(z_2) &= (1, 1, y, x^{-1}) \\
\varphi(z_3) &= (y, x^{-1}, 1, 1) \\
\varphi(z_4) &= (1, 1, x^{-1}, xyx^{-1}) \\
\varphi(z_5) &= (1, xy, x, x)
\end{aligned}$$

Consideration of the first factor of the direct product $F \times F \times F \times F$ shows that the z_1-muliplicity and the z_3-multiplicity of $u(z_1, z_2, z_3, z_4, z_5)$ must be zero; the second factor yields that z_5-multiplicity is zero; the third factor yields that the z_2-multiplicity and the z_4-multiplicity are zero. Therefore, $u(z_1, z_2, z_3, z_4, z_5)$ is an element of N' and, in particular, $w(x, y)$ is an element of F'.

Let $w(x, y)$ be an element of F such that $w(c, d)$ is the identity element of B. Then $w(c, d)$ is in $\mathrm{st}_B(2)$ and so $w(x, y)$ is in N. We have shown that there exist words $w_{(i)}(x, y)$ of length at most $\lceil \ell\big(w(x, y)\big)/2\rceil$ such that $\varphi\big(w(x, y)\big) = \big(w_{(1)}(x, y), w_{(2)}(x, y), w_{(3)}(x, y), w_{(4)}(x, y)\big)$. The expansion of $w(c, d)$ to the second level is $\big(w_{(1)}(c, d), w_{(2)}(c, d), w_{(3)}(c, d), w_{(4)}(c, d)\big)$. As $w(c, d)$ is in $\mathrm{st}_B(4)$, it follows that $w_{(i)}(c, d)$ is in $\mathrm{st}_B(2)$ for each i. Therefore, $w_{(i)}(x, y)$ is in N for each i. Let W_1 be the subset $\{w_{(i)}(x, y) \mid i \in [1, 4]\}$ of N.

We are going to inductively define subsets W_n of N. Let $n \in \mathbb{N}$. Suppose that, for every $m \in [1, n]$, we have defined a subset $W_m = \{w_{(i_1,\ldots,i_m)}(x, y) \mid i_j \in [1, 4]\}$ of N such that the following two conditions are satisified:

(i) $\varphi\big(w_{(i_1,\ldots,i_j)}(x, y)\big) = \big(w_{(i_1,\ldots,i_j,1)}(x, y), w_{(i_1,\ldots,i_j,2)}(x, y), w_{(i_1,\ldots,i_j,3)}(x, y), w_{(i_1,\ldots,i_j,4)}(x, y)\big)$ for all $i_1, \ldots, i_j \in [1, 4]$ and all $j \in [1, n-1]$; and

(ii) $\ell\big(w_{(i_1,\ldots,i_j,i_{j+1})}(x,y)\big) \leqslant \Big\lceil \ell\big(w_{(i_1,\ldots,i_j)}(x,y)\big)/2\Big\rceil$ for all $i_1,\ldots,i_{j+1} \in [1,4]$ and all $j \in [1,n-1]$.

In order to define the subset W_{n+1} of N, we repeat the argument of the previous paragraph for every word $w_{(i_1,\ldots,i_n)}(x,y)$ of W_n in turn. We see that conditions (i) and (ii) are satisfied.

Let n be a natural number strictly greater than $\lceil \log_2\big(\ell(w(x,y))\big)\rceil$. Then every element $w_{(i_1,\ldots,i_n)}(x,y)$ of W_n has length at most 1. Now, $w_{(i_1,\ldots,i_n)}(x,y)$ is in N, and therefore $w_{(i_1,\ldots,i_n)}(c,d)$ is in $\mathrm{st}_B(2)$. As c and d act non-trivially at the root, we would arrive at a contradiction if $w_{(i_1,\ldots,i_n)}(x,y)$ had length 1. Therefore, all of the words in W_n must be of length 0. That is, $W_n = \{1\}$. We have shown that $\varphi^{-1}\big(F^{'} \times F^{'} \times F^{'} \times F^{'}\big)$ is a subgroup of $F^{'}$. Therefore, certainly $\varphi^{-1}(1,1,1,1)$ is a subgroup of $F^{'}$. This provides the base for an induction (working up the levels of the tree two at a time) from which it follows that $w(x,y)$ is in $F^{'}$.

We have shown that if $w(x,y)$ is any word in F such that $w(c,d)$ is the identity in B then $w(x,y)$ is in $F^{'}$. This implies that the natural epimorphism

$$\begin{aligned} \pi : F^{\mathrm{ab}} &\longrightarrow B^{\mathrm{ab}} \\ xF^{'} &\longmapsto cB^{'} \\ yF^{'} &\longmapsto dB^{'} \end{aligned}$$

is an isomorphism. Indeed, if $w(x,y)F^{'}$ is in the kernel of π then there exists $w'(x,y)$ in $F^{'}$ such that $w(c,d)w'(c,d) = 1$. Thus $w(x,y)w'(x,y)$ is in $F^{'}$, and hence so is $w(x,y)$. That is, $w(x,y)F^{'} = F^{'}$. Therefore, B^{ab} is free abelian of rank 2 as required.

□

4.39 Remark.

Contained in the above proof is an algorithm to solve the word problem in B. This algorithm is more efficient than the algorithm given in the proof of Theorem 4.1 which solves the word problem for an arbitrary automaton. The algorithm in the above proof is a generalization of the algorithm given by Lysenok in [L] to solve the word problem for The First Grigorchuk Group. Using his algorithm, Lysenok is able to give a recursive presentation for The First Grigorchuk Group.

4.40 Lemma.

Let G be a subgroup of $\mathrm{Aut}(T_{\underline{d}})$. Suppose that G has a subgroup H of finite index such that H maps onto C_∞. Then G does not have the congruence subgroup property.

Proof:
Let Π denote the set of primes that divide d and let p be a prime not contained in Π. As H maps onto C_∞, there exists a normal subgroup N of H such that H/N is isomorphic to C_p. Suppose, for a contradiction, that N contains $\mathrm{st}_G(i)$ where i is a level of $T_{\underline{d}}$. Then H/N is isomorphic to a section of $\mathrm{Aut}(T[i])$. This contradicts the fact that $\mathrm{Aut}(T[i])$ is a Π-group. $\square$

4.41 Corollary.
If A is an infinite binary windmill automaton then the group defined by A does not have the congruence subgroup property.

Proof:
Recall the classification of binary windmill automata from Remark 4.35. Observe that the infinite groups are either virtually C_∞, virtually $C_\infty \times C_\infty$, virtually B, or virtually $B \times B$. The result follows from Theorem 4.38 and Lemma 4.40. $\square$

4.42 Remark.
It is worth noting that B is not free. In order to show this, it suffices to find a non-trivial relation in its generators $c = \varepsilon(c, 1)$ and $d = \varepsilon(d^{-1}, 1)$. Indeed, this shows that B is not free of rank 2. As B is a two-generator group, if it was free then its rank would be at most 2. It follows from Theorem 4.38 that B is not cyclic. In order to find a relation, we observe that $cdc^{-2} = (d^{-1}c^{-1}, 1)$, that $dcd^{-2} = (cd, 1)$, and therefore that $cdc^{-2}dcd^{-2} = 1$.

4.6.2 A Ternary Windmill Automaton

The problem of classifying, up to group isomorphism, the subgroups of $\mathrm{Aut}(T_{\underline{3}})$ that are defined by ternary windmill automata is far harder than that for binary windmill automata. We do not attempt such a classification here. We give one result, similar to Theorem 4.38, which we use in Section 5.5 to show that H_2 does not have the congruence subgroup property.

4.43 Theorem.
Let c and d be defined by the system of equations $c = (123)(c, 1, 1)$, $d = (123)(1, d, 1)$. Let $G = \langle c, d\rangle$. Then $G^{\mathrm{ab}} = C_\infty \times C_\infty$.

Proof:
The proof is very similar to that given for Theorem 4.38. We provide only the salient points, leaving the details to the reader.

Let F be the free group on the two generators x and y. Identify C_3 with $\mathrm{Alt}(3)$ and let σ denote the permutation (123). Define the homomorphism

$$\begin{aligned} \varphi : F &\to F \wr C_3 \\ x &\mapsto \sigma(x, 1, 1) \\ y &\mapsto \sigma(1, y, 1) \end{aligned}$$

Let W be the set of reduced words of length 3 in F. We check that for all $w \in W$ there exist $\delta \in C_3$ and words w_1, w_2, w_3 of length at most 2 such that $\varphi(w) = \delta(w_1, w_2, w_3)$. For example, $\varphi(xyx^{-1}) = \sigma(xy, 1, x^{-1})$. If follows that if $w(x, y)$ is a word of length $\ell > 2$ then there exist $\delta \in C_3$ and words $w_1(x, y), w_2(x, y), w_3(x, y)$ of length strictly less than ℓ such that $\varphi\big(w(x, y)\big) = \delta\big(w_1(x, y), w_2(x, y), w_3(x, y)\big)$.

Observe that $\mathrm{st}_G(1)$ is equal to the normal closure of $\langle c^3, cd^{-1}\rangle$ in G. Let N be the normal closure of $\langle x^3, xy^{-1}\rangle$ in F. We use the Schreier method to show that N is freely generated by the set $\{x^3, xy^{-1}, x^{-1}y, xyx\}$. We calculate that $\varphi(x^3) = (x, x, x)$, that $\varphi(xy^{-1}) = (y^{-1}, 1, x)$, that $\varphi(x^{-1}y) = (x^{-1}, y, 1)$, and that $\varphi(xyx) = (x, 1, xy)$. From this we deduce that $\varphi^{-1}\big(F', F', F'\big)$ is a subgroup of F'.

We now proceed as in the proof of Theorem 4.38. Indeed, if $w(x, y)$ is a word such that $w(c, d)$ is the identity element of G then by applying φ sufficiently many times we arrive at a set of words $w_i(x, y)$ of length at most 2. As $w(c, d)$ acts as the identity on $T_{\underline{3}}$ and $w_i(c, d)$ occurs in the expansion of $w(c, d)$ we must have that $w_i(c, d)$ acts as the identity on $T_{\underline{3}}$. We check that if $w_i(x, y)$ is a word of length 1 or 2 then $w_i(c, d)$ acts non-trivially on $T_{\underline{3}}$. Therefore, the words $w_i(x, y)$ are all equal to 1. We have shown that $\varphi^{-1}\big(F', F', F'\big)$ is a subgroup of F'. It follows that $w(x, y)$ is in F' as required. □

4.44 Remark.
It is worth noting that G is not free. In order to show this, it suffices to find a non-trivial relation in its generators $c = (123)(c, 1, 1)$ and $d = (123)(1, d, 1)$. This shows that G is not free of rank 2. As G is a two-generator group, if it was free then its rank would be at most 2. It follows from Theorem 4.43 that B is not cyclic. Observe that $dcd^{-2} = (1, c, d^{-1})$, that $c^{-2}dc = (1, c^{-1}, d)$, and therefore that $dcd^{-2}c^{-2}dc$ is a non-trivial relation in G.

Chapter 5

Results on $H_1, \ldots, H_{17}$

We fix some notation for the duration of this chapter:

Let F be the free group of rank 2 with generators x and y;
let $P_k(H)$ denote the k-th term of the lower 3-central series of the group H;
let T denote the rooted ternary tree $T_{\underline{3}}$;
for all $i \in \mathbb{N}_0$ identify $\mathrm{Aut}(T[i])$ with $\mathrm{Aut}(T)/\mathrm{st}(i)$;
let σ denote the permutation (123) in Sym(3);
identify C_3 with the subgroup of Sym(3) generated by σ;
for all $i \in \mathbb{N}_0$ let $\Gamma_i = C_3$;
let G be the subgroup of $\mathrm{Aut}(T)$ defined by $(\Gamma_i)_{i \geqslant 0}$;
for all $i \in \mathbb{N}_0$ identify $G/\mathrm{st}(i)$ with $\Gamma_{i-1} \wr \cdots \wr \Gamma_0$.

5.1 Remark.
Let $i \in [1, 17]$. Recall from Definition 4.17 that H_i has C_3-action at the root. It follows from Remark 2.65 that H_i is a subgroup of G. Note that $G/\mathrm{st}(n)$ is a group of order $3^{(3^n-1)/2}$ for all $n \in \mathbb{N}$. Therefore, $H_i/\mathrm{st}(n)$ is a finite 3-group for all $n \in \mathbb{N}$.

5.2 Lemma.
If x is an element of finite order in G then $\mathrm{ord}(x)$ is a power of 3.

Proof:
Let $n = \mathrm{ord}(x)$. For a contradiction, suppose that n is divisible by a prime $p \neq 3$. Let $m = n/p$. Then x^m is not the identity. Therefore, there exists a level k such that x^m stablizes level k and does not stabilize level $k+1$. Thus, there exists a vertex ν of level k in the tree such that ν is labelled by a 3-cycle in the portrait of x^m, and all the ancestors of ν are labelled by the identity. The p-th power of a 3-cycle is again a 3-cycle. Therefore, $x^n = (x^m)^p$ does not stabilize level $k+1$. This contradicts the assumption that $x^n = 1$. Therefore, n must be a power of 3 as required. □

5.1 Elementary Abelian Quotients

In this section we prove that H_i/P_2 is isomorphic to $C_3 \times C_3$ for all $i \in [1, 17]$.

5.3 Lemma.
Let $i \in [1, 17]$ and let $H = H_i$.

(i) If $i = 4$ then $H/\mathrm{st}(2)$ is isomorphic to the extra-special group of order 27 and exponent 9;

(ii) if $i \neq 4$ then $H/\mathrm{st}(2) = C_3 \wr C_3$.

Proof:
Let a and b be the generators of H defined in Definition 4.17.

(i) Recall that $a = \sigma(b, a, a)$ and $b = \sigma^2(a, b, b)$. Let $u = a$ and $v = a^{-1}b^{-1}$. Then $H_4 = \langle u, v \rangle$. Working modulo the second level stabilizer, $u = \sigma(\sigma^2, \sigma, \sigma)$ and $v = (\sigma^2, \sigma, 1)$. Thus $\mathrm{ord}(u) = 9$ and $\mathrm{ord}(v) = 3$. We calculate that $u^v = u^7 = \sigma(\sigma, 1, 1)$. This proves that $H_4/\mathrm{st}(2)$ is a non-abelian image of the stated group.

(ii) This is a calculation for each group. The calculations can easily be performed using Magma (see Appendix B.2.3), or they may be done by hand. The calculation for H_1 is presented as an example, both in the appendix, and by hand. Recall that $a = (b, a, a)$ and $b = \sigma(b, b, b)$. Working modulo the second level stabilizer, $a = (\sigma, 1, 1)$ and $b = \sigma(\sigma, \sigma, \sigma)$. We observe that $a^b = (1, \sigma, 1)$ and $a^{b^2} = (1, 1, \sigma)$. Thus $ba^{-1}a^{-b}a^{-b^2} = \sigma(1, 1, 1)$. This proves the lemma because $C_3 \wr C_3 = \langle \sigma(1, 1, 1), (\sigma, 1, 1) \rangle$.

□

5.4 Remark.
It follows from Lemma 5.3 that $H_1, \dots, H_{17}$ are not abelian groups.

5.5 Proposition.
Let $i \in [1, 17]$ and let $H = H_i$. Then H/P_2 is isomorphic to $C_3 \times C_3$.

Proof:
By definition, H is generated by two elements. Therefore, $C_3 \times C_3$ maps onto H/P_2.

If $i = 4$ then, by Lemma 5.3(i), $(H/\mathrm{st}(2))/P_2$ has the presentation $\langle u, v | u^3 = v^3 = 1, u^v = u \rangle$ and is therefore isomorphic to $C_3 \times C_3$.

If $i \neq 4$ then, by Lemma 5.3(ii), $(H/\mathrm{st}(2))/P_2$ is equal to $(C_3 \wr C_3)/P_2$. Now $C_3 \wr C_3$ has presentation $\langle x, y | x^3 = y^3 = [x, x^y] = 1\rangle$, so $(C_3 \wr C_3)/P_2$ is $C_3 \times C_3$.

Finally, H/P_2 maps onto $(H/\mathrm{st}(2))/P_2$ because H maps onto $H/\mathrm{st}(2)$. Therefore, H/P_2 maps onto $C_3 \times C_3$. □

5.2 H_4

We prove that if $j \in [1, 17]\backslash\{4\}$ then H_4 and H_j are not isomorphic groups.

5.6 Lemma.

The group given by the presentation $\big\langle u, v \big| u^9 = v^9 = 1, u^v = u^7 \big\rangle$ maps onto H_4/P_3.

Proof:

Recall from Definition 4.17 that $a = \sigma(b, a, a)$ and $b = \sigma^2(a, b, b)$. Observe that $a^{-1} = \sigma^2(a^{-1}, a^{-1}, b^{-1})$ and $b^{-1} = \sigma(b^{-1}, a^{-1}, b^{-1})$. We calculate that $a^{-2}b^2 = \sigma^2(1, a^{-2}b^2, 1)$ and $b^{-1}a = \sigma^2(1, b^{-1}a, 1)$. Thus $a^{-2}b^2 = b^{-1}a$. Let G_4 be the group defined by the presentation $\big\langle c, d \mid c^{-2}d^2 = d^{-1}c \big\rangle$. We know that G_4 maps homomorphically onto H_4. In particular, G_4/P_3 maps onto H_4/P_3.

Define elements of G_4 by $u := c^{-1}d$ and $v := c$. By definition, the exponent of G_4/P_3 divides 9. In particular, $u^9 \equiv v^9 \equiv 1$ modulo $P_3(G_4)$. We have the relation $c^{-2}d^2 = d^{-1}c$. Thus $c^{-1}d = (cd^{-1})^2$ and, as the exponent of G_4/P_3 divides 9, $c^{-1}d \equiv (cd^{-1})^{-7}$ modulo $P_3(G_4)$. That is, $c^{-1}d \equiv (dc^{-1})^7$. It follows that $d^{-1}c^{-1}dc \equiv (c^{-1}d)^6$ modulo $P_3(G_4)$. That is, $[u, v] \equiv u^6$ modulo $P_3(G_4)$, and so $u^v \equiv u^7$.

We have shown that the group with presentation $\big\langle u, v \big| u^9 = v^9 = 1, u^v = u^7 \big\rangle$ maps onto G_4/P_3, and that G_4/P_3 maps onto H_4/P_3. □

5.7 Theorem.

The group of order 81 given by the presentation $\big\langle u, v \big| u^9 = v^9 = 1, u^v = u^7 \big\rangle$ is isomorphic to H_4/P_3.

Proof:

We show that $(H_4/\mathrm{st}(3))/P_3$ has order 81. This shows that H_4/P_3 has order at least 81. The result then follows by Lemma 5.6.

Let N denote $\mathrm{st}_{H_4}(3)$. Identify H_4/N with a subgroup of $\mathrm{Aut}(T[3])$. Define two elements of H_4/N by $c := ba^{-1}N$ and $d := b^{-1}a^{-1}N$. A calculation shows that bN has order 27 in H_4/N. As $cd^{-1} = b^2N$, it follows that $bN \in \langle c, d\rangle$, and hence that $aN \in \langle c, d\rangle$. Therefore, $H_4/N = \langle c, d\rangle$. A calculation shows that $\mathrm{ord}(c) = 27$, that $\mathrm{ord}(d) = 9$ and that $c^d = c^7$.

Let $e = c^3$ and $f = d^3$. By Lemma 2.77, $P_2\big(H_4/N\big)$ is the normal closure of $\big\langle e, f, [c,d]\big\rangle$ in H_4/N. We have calculated that $c^d = c^7$. Hence $[c,d] = e^2$ and $e^d = e^7$. We compute that $f^c = fe^3$. Therefore, $P_2\big(H_4/N\big) = \big\langle e, f\big\rangle$. We know that $\mathrm{ord}(e) = 9$ and $\mathrm{ord}(f) = 3$. We compute that $[e,f] = 1$ and $f \notin \langle e^3\rangle$. Therefore, $P_2\big(H_4/N\big)$ is isomorphic to $C_3 \times C_9$.

We have already shown that $d^9 = 1$, that $\mathrm{ord}(e) = 9$ and that $[c,d] = e^2$. It follows from Lemma 2.77 that $P_3(H_4/N)$ is equal to the normal closure of $\big\langle e^3, [c,d,c], [c,d,d]\big\rangle$ in H_4/N. Now, $[c,d,c] = [c^6, c] = 1$ and $[c,d,d] = [e^2, d]$. As $e^d = e^7$, it follows that $[e^2, d] = e^{-2}\big(e^d\big)^2 = e^{12} = e^3$. Therefore, $P_3(H_4/N)$ is equal to the normal closure of $\langle e^3\rangle$ in H_4/N. But $\big(e^3\big)^c = e^3$ and $\big(e^3\big)^d = \big(e^d\big)^3 = \big(e^7\big)^3 = e^3$. Thus $P_3\big(H_4/N\big)$ is the subgroup of $P_2\big(H_4/N\big)$ generated by e^3. Thus we have shown that $P_3\big(H_4/N\big)$ is isomorphic to C_3. Therefore, $P_3\big(H_4/N\big)$ has index 9 in $P_2\big(H_4/N\big)$.

It remains to show that $P_2\big(H_4/N\big)$ has index 9 in H_4/N. Recall from the proof of Proposition 5.5 that $\big(H_4/\mathrm{st}(2)\big)/P_2$ has order 9. As $\big(H_4/N\big)$ is a two-generator group, $\big(H_4/N\big)/P_2$ has order at most 9. It follows that $\big(H_4/N\big)/P_2$ has order equal to 9 because it maps onto $\big(H_4/\mathrm{st}(2)\big)/P_2$. Thus $P_2\big(H_4/N\big)$ has index 9 in H_4/N, and so $P_3\big(H_4/N\big)$ has index 81 in H_4/N as required. □

5.8 Lemma.
The group $(C_3 \wr C_3)/P_3$ is isomorphic to the extra-special group of order 27 and exponent 3.

Proof:
Let $H = C_3 \wr C_3$. Define two elements of H by $u := \sigma(1,1,1)$ and $v := (\sigma, 1, 1)$. Then $H = \langle u, v\rangle$ and, by Lemma 2.77, $P_3(H) = \big\langle u^9, v^9, [u,v]^3, [u,v,u], [u,v,v]\big\rangle^H$. As $u^3 = v^3 = 1$, the extra-special group of order 27 and exponent 3 maps onto H/P_3. We calculate that $[u,v,v] = [u,v]^3 = 1$ and $[u,v,u] = (\sigma^2, \sigma^2, \sigma^2)$. This element is central in H. Therefore $P_3(H)$ is the subgroup of order 3 generated by $(\sigma^2, \sigma^2, \sigma^2)$. This implies that H/P_3 has order 27 because H has order 81. □

5.9 Corollary.
Let $j \in [1,17]\backslash\{4\}$. Then H_4 and H_j are not isomorphic groups.

Proof:
It suffices to show that H_4/P_3 and H_j/P_3 are not isomorphic.

By Lemma 5.3(ii) and Lemma 5.8, H_j/P_3 maps onto the extra-special group of order 27 and exponent 3. By Theorem 5.7, any homomorphic image of H_4/P_3 in a group of exponent 3 must be abelian. This follows from the relation $u^v = u^7$. Therefore, H_4/P_3 does not map onto H_j/P_3. □

5.3 H_6, H_7 and H_8

We show that H_6, H_7 and H_8 are (locally finite)-by-(infinite cyclic) groups. This will be sufficient to prove that if $i \in [6,8]$ and $j \in [1,17]\backslash[6,8]$ then H_i and H_j are not isomorphic. It is not known if H_6, H_7 and H_8 are pairwise non-isomorphic.

5.10 Definition.
Let $i \in [6,8]$ and let $H = H_i$ with generators a and b as in Definition 4.17. Define the element $\alpha := ab^{-1}$ and the normal subgroup $A := \langle \alpha \rangle^H$ of H. For every $m \in \mathbb{N}$, define subsets of H by

$$\begin{aligned} X_m &:= \big\{x_1 \ldots x_m \mid x_1, \ldots, x_m \in \{1, a, b\}\big\}; \\ Y_m &:= \big\{y_1 \ldots y_m \mid y_1, \ldots, y_m \in \{1, a^{-1}, b^{-1}\}\big\}; \text{ and} \\ Z_m &:= \big\{xy \mid x \in X_m \text{ and } y \in Y_m\big\}. \end{aligned}$$

5.11 Remark.
Let $x \in X_m$ and $y \in Y_m$. Then there exist permutations π_x, π_y and elements $x_1, x_2, x_3 \in X_m$ and $y_1, y_2, y_3 \in Y_m$ such that $x = \pi_x(x_1, x_2.x_3)$ and $y = \pi_y(y_1, y_2, y_3)$. That is, X_m and Y_m are closed under expansion.

5.12 Lemma.
Let $i \in [6,8]$, let $H = H_i$ with generators a and b as in Definition 4.17, and let $m \in \mathbb{N}$. Then the elements of Z_m have depth m.

Proof:
Observe that ab^{-1} and ba^{-1} have depth 1. This provides the base for an induction on m because $Z_1 = \{1, ab^{-1}, ba^{-1}\}$. Let $z_{m+1} = x_1 \ldots x_{m+1} y_1 \ldots y_{m+1} \in Z_{m+1}$. Then $x_1 \ldots x_m \in X_m$ and $y_2 \ldots y_{m+1} \in Y_m$. Since $x_{m+1} y_1 \in Z_1$, there exists a permutation π_z such that $x_{m+1} y_1 = \dot{\pi}_z$. By Remark 5.11, there exist permutations π_x, π_y and elements $x_1, x_2, x_3 \in X_m$ and $y_1, y_2, y_3 \in Y_m$ such that $x_1 \ldots x_m = \pi_x(x_1, x_2, x_3)$ and $y_2 \ldots y_{m+1} = \pi_y(y_1, y_2, y_3)$. Let $\pi = \pi_z \pi_y$ and $\phi(j) = j^{\pi^{-1}}$. Then $z_{m+1} = \pi_x \pi(x_{\phi(1)} y_1, x_{\phi(2)} y_2, x_{\phi(3)} y_3)$. Now, $x_{\phi(j)} y_j \in Z_m$ has depth m by the induction hypothesis. Therefore, z_{m+1} has depth $m+1$. $\square$

5.13 Theorem.
Let $i \in [6,8]$, let $H = H_i$ with generators a and b as in Definition 4.17, and let A be the normal subgroup of Definition 5.10. Then the group A is locally finite.

Proof:

Observe that $H = \langle \alpha, a \rangle$. Therefore, $A = \left\langle \alpha^{a^n} \middle| n \in \mathbb{Z} \right\rangle$. Let S be a finite subset of A. We have to show that the subgroup of A generated by S is finite. Every element of S can be expressed as a word on $\left\{ \alpha^{a^n} \middle| n \in \mathbb{Z} \right\}$. Let k be a sufficiently large natural number so that every element of $S^{a^{-k}}$ can be expressed as a word in $\left\{ \alpha^{a^{-n}} \middle| n \in \mathbb{N}_0 \right\}$. Since every word has finite length and $S^{a^{-k}}$ is a finite set, there exists a natural number m such that every element of $S^{a^{-k}}$ can be expressed as a word in the finite set $\left\{ \alpha^{a^{-n}} \middle| n \in [0, m] \right\}$.

We have shown that $\left\langle S^{a^{-k}} \right\rangle$ is a subgroup of $\left\langle \alpha^{a^{-n}} \middle| n \in [0, m] \right\rangle$. Since $\left\langle S^{a^{-k}} \right\rangle = \left\langle S \right\rangle^{a^{-k}}$ is isomorphic to $\left\langle S \right\rangle$, the theorem will be proved once we show that $\left\langle \alpha^{a^{-n}} \middle| n \in [0, m] \right\rangle$ is a finite group. Now, $\alpha^{a^{-n}} = a^{n+1} b^{-1} a^{-n}$, so $\alpha^{a^{-n}}$ has depth $n+1$ by Lemma 5.12. Therefore, $\left\langle \alpha^{a^{-n}} \middle| n \in [0, m] \right\rangle$ is isomorphic to a subgroup of the finite group $\mathrm{Aut}\left(T[m+1]\right)$. □

5.14 Corollary.

For $i \in [6, 8]$ the group H_i is (locally finite)-by-(infinite cyclic).

Proof:

By Theorem 5.13, we need only show that $H_i/A \cong C_\infty$. Observe that H_i/A is generated by the coset aA. Certainly every element of a locally finite group has finite order. It therefore suffices to show that a has infinite order. For a contradiction, suppose that $\mathrm{ord}(a) = n$. Let $\beta = b^{-1} a$. Then $\beta = \beta^{a^{-n}} = a^n b^{-1} a^{1-n}$. This has finite depth by Lemma 5.12. However, if $i = 6$ then $\beta = \sigma^2(1, \beta^{-1}, \beta)$, if $i = 7$ then $\beta = \sigma(\beta^{-1}, 1, \beta)$, and if $i = 8$ then $\beta = \sigma^2(1, \beta^{-1}, \beta)$. Therefore, β has infinite depth. □

5.15 Remark.

Let $i \in [6, 8]$ and let $H = H_i$ with generators a and b as in Definition 4.17. It follows from Lemma 4.40 and Corollary 5.14 that H_i does not have the congruence subgroup property. By Proposition 5.5 and Corollary 5.14, the abelianization of H_i maps onto $C_3 \times C_\infty$. As $H = \langle a, ab^{-1} \rangle$ and $(ab^{-1})^3 = 1$, it follows that $H_i^{\mathrm{ab}} = C_3 \times C_\infty$.

5.16 Lemma.

Let $j \in [1, 17] \backslash [6, 11]$ and let $H = H_j$ with generators a and b as in Definition 4.17. Then $[a, b]$ has infinite order. In particular, H_j is not (locally finite)-by-abelian.

Proof:

If H_j was (locally finite)-by-abelian then its derived subgroup would be locally finite and, in particular, all elements of the derived subgroup would have finite order. To show that $[a, b]$ has infinite order is a calculation for each j. As examples, we do the cases $j = 2$ and $j = 17$. The other cases are very similar.

Let $j = 2$. Then $[a, b] = (b^{-2}ab, 1, a^{-1}b)$. We calculate that $a^{-1}b = \sigma(a^{-1}b, 1, 1)$. Let $\beta = a^{-1}b$. Then $\beta^3 = (\beta, \beta, \beta)$. It follows that β has infinite order by Lemma 2.50, and therefore so does $[a, b]$.

Let $j = 17$. Then $[a, b] = (b^{-1}a, [b, a], b^{-1}a^{-1}b^2)$ and $b^{-1}a = \sigma^2(a^{-1}b, 1, a^{-1}b)$. Letting $\beta = b^{-1}a$, we see that $\beta = \sigma^2(\beta^{-1}, 1, \beta^{-1})$ has infinite order by Lemma 2.50. Therefore, $[a, b]$ has infinite order. $\square$

For $j \in [6, 11]$ we find that $[a, b]$ has order 3. GAP was used in order to find the elements in the next lemma (the details of the search may be found in Appendix B.1). Note that the argument to show they are elements of infinite order is a generalization of that used in the proof of Lemma 2.50.

5.17 Lemma.
Let $j \in [9, 11]$ and let $H = H_j$ with generators a and b in Definition 4.17. Then there exists an element β in $H^{'}$ of infinite order. In particular, H_j is not (locally finite)-by-abelian.

Proof:
Let $j = 9$ and let $\beta = b^{-1}a^{-1}b^{-1}a^2b^{-2}ab^2a^{-1}b^{-1}a^{-1}ba^{-1}b^2a$. Observe that β has a-multiplicity zero and b-multiplicity zero. Thus β is an element of $H^{'}$. We calculate that $\beta \in \mathrm{st}(2)\backslash\mathrm{st}(3)$. Thus β^3 stabilizes the third level. A calculation shows that β^{-1} is the label of the vertex $\nu_{(1,1,1)}$ in the expansion of β^3 to the third level. We know by Lemma 5.2 that if $\mathrm{ord}(\beta)$ is finite then it is a power of 3. Suppose that β has order 3^n for some $n \in \mathbb{N}$. As $\beta^{-3^{n-1}}$ occurs in the expansion of β^{3^n}, we must have that $\beta^{3^{n-1}} = 1$. This contradicts the assumption that $\mathrm{ord}(\beta) = 3^n$. Therefore, β has infinite order.

Let $j = 10$ and let $\beta = a^{-2}bab^{-2}ab$. Observe that β has a-multiplicity zero and b-multiplicity zero. Thus β is an element of $H^{'}$. We calculate that $\beta \in \mathrm{st}(1)\backslash\mathrm{st}(2)$. Thus β^3 stabilizes the second level. A calculation shows that $a^{-1}ba^{-1}b^{-1}a^2b^{-1}a^{-1}ba$ is the label of $\nu_{(1,1)}$ in the expansion of β^3 to the second level. Observe that $\beta^{-1} = b^{-1}a^{-1}b^2a^{-1}b^{-1}a^2$ and $\beta^{-[b,a]} = a^{-1}ba^{-1}b^{-1}a^2b^{-1}a^{-1}ba$. Thus a conjugate of β occurs in the expansion of β^3 to the second level. As in the case $j = 9$, we see that if β^{3^n} equals 1 then $\beta^{3^{n-1}}$ equals 1. Therefore, β has infinite order.

Let $j = 11$ and let $\beta = b^{-2}a^{-2}b^2a^{-2}b^4a^{-2}b^{-2}a^3b^{-2}ab^{-2}ab^2a$. Observe that β has a-multiplicity zero and b-multiplicity zero. Thus β is an element of $H^{'}$. We calculate that $\beta \in \mathrm{st}(2)\backslash\mathrm{st}(3)$. Thus β^3 stabilizes the third level. A calculation shows that β^{-1}

is the label of $\nu_{(1,1,1)}$ in the expansion of β^3 to the third level. We now proceed as before to show that β has infinite order. □

5.18 Theorem.
If $i \in [6,8]$ and $j \in [1,17] \backslash [6,8]$ then H_i and H_j are not isomorphic.

Proof:
This follows from Corollary 5.14, Lemma 5.16, and Lemma 5.17. □

5.19 Remark.
The automata defining the groups H_6, H_7 and H_8 are examples of so-called 'reset automata'. That is, they are contained in a class defined by $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ where $a_i = b_i$ for all i. Using the notion of depth, it is proved in [SS] that an infinite group defined by a reset automaton is (locally finite)-by-(infinite cyclic). However, we feel the proof given here is more transparent, and it serves as motivation for Section 5.4.

In the current section, we proved that the normal subgroup A is locally finite by showing the elements of Z_m have finite depth. It then followed that H_6, H_7 and H_8 have infinite cyclic quotients. In Section 5.4, we define sets Z_m in an analogous way for other groups H_i, and prove that the elements of Z_m have polynomial level growth. This can be regarded as a generalization of the idea of finite depth. Indeed, an element of finite depth has polynomial level growth of degree 0. It then follows that the groups H_i under consideration have an infinite cyclic quotient.

5.20 Remark.
Let $i \in [6,8]$ and let $H = H_i$. Let $\alpha = ab^{-1}$ and $A = \langle \alpha \rangle^H$ be the element and normal subgroup of H, respectively, that are defined in Defintion 5.10. We might hope that A is elementary 3-abelian and that H_i is isomorphic to $C_3 \wr C_\infty$. This would be a direct analogue of the case of two-state binary reset automata where it is known that we have $C_2 \wr C_\infty$ (see [GZ] for details). However, this is not the case for our two-state ternary reset automata H_6, H_7 and H_8. For example, in the case of H_6 $\big($recall that $a = (b,b,a)$, that $b = \sigma(b,b,a)$, and hence $\alpha = ab^{-1} = \dot{\sigma}^2\big)$, one checks that $[\alpha, \alpha^a] \neq 1$ so that A is not abelian. Furthermore, A does not have exponent 3; indeed, $\alpha\alpha^a\alpha^{a^4}$ has order 9. We also note: that $\alpha\alpha^a\alpha^{a^5}$ has order 27; that $\alpha\alpha^a\alpha^{a^{10}}$ has order 81; and that $\alpha\alpha^a\alpha^{a^{12}}$ has order 243. It thus seems likely that A is not of finite exponent. All that can be said at this time is that A is a locally finite 3-group.

5.4 Infinite Cyclic Quotients

5.21 Remark.
Let $i \in [1, 17]$ and let $H = H_i$ with generators as defined in Definition 4.17. By Proposition 5.5, H maps onto $C_3 \times C_3$. Therefore, a-multiplicity and b-multiplicity in H (see Definition 2.75) are well-defined modulo 3.

5.22 Lemma.
Let $i \in \{3, 4, 5, 13, 14, 15, 17\}$ and let $H = H_i$ with generators a and b as defined in Definition 4.17. Then $\operatorname{ord}(a) = \operatorname{ord}(b) = \infty$.

Proof:
There exist $\sigma_b \in C_3 \backslash \{\mathrm{id}\}$, $\sigma_a \in C_3 \backslash \{\sigma_b\}$ and $a_i, b_i \in \{a, b\}$ such that $a = \sigma_a(a_1, a_2, a_3)$ and $b = \sigma_b(b_1, b_2, b_3)$. Let $k = \big|\{a_i \mid a_i = b\}\big|$ and $\ell = \big|\{b_i \mid b_i = b\}\big|$. Note that $\ell \in \{1, 2\}$ and $k \not\equiv \ell$ modulo 3.

We show that $b^{3^n} \neq 1$ for all natural numbers n. Then b must have infinite order by Lemma 5.2. We achieve this by proving that, in the expansion of b^{3^n} to the n-th level, there exists a vertex of level n which is labelled by a word w of non-zero b-multiplicity modulo 3. As the sum of the a and b multiplicities of w must be congruent to 0 modulo 3, this implies that the a-multiplicity of w is not congruent to the b-multiplicity of w modulo 3. In particular, $b^{3^n} \notin \operatorname{st}(n+1)$. The proof is by induction on n.

Expand b^3 to the first level. Observe that every vertex of level 1 is labelled by a word of b-multiplicity ℓ modulo 3. This provides the base for the induction.

Expand b^{3^n} to the n-th level. For induction, suppose there exists a vertex ν of level n which is labelled by a word w of b-multiplicity $r \not\equiv 0$ modulo 3. Note that w must therefore have a-multiplicity $-r$ modulo 3.

Expand b^{3^n} to the $(n+1)$-th level. Then ν is labelled by a 3-cycle. Recall from Remark 5.1 that H is a subgroup of G. It follows that b^{3^n} stabilizes the n-th level of T. Therefore, in the expansion of b^{3^n} to the $(n+1)$-th level, the ancestors of ν are labelled by the identity. Let the words labelling the children of ν be denoted w_1, w_2 and w_3 (in any order). As w has b-multiplicity r and a-multiplicity $-r$ (modulo 3), the product $w_1 w_2 w_3$ has b-multiplicity $r\ell - rk$ modulo 3. Since $r \not\equiv 0$ and $k \not\equiv \ell$ modulo 3, the b-multiplicity of the product $w_1 w_2 w_3$ is non-zero modulo 3. Therefore, in the expansion of $b^{3^{n+1}}$ to the $(n+1)$-th level, there exists a vertex of level $n+1$ which is labelled by a word of non-zero b-multiplicity modulo 3.

We have proved that $\operatorname{ord}(b) = \infty$. If $\sigma_a = \mathrm{id}$, observe that $k \neq 0$, so that $b^n = 1$ if and only if $a^n = 1$. Therefore $\operatorname{ord}(a) = \operatorname{ord}(b)$. For H_{13} and H_{17}, we note that

$a^3 = (b^3, b^3, b^3)$, so $\mathrm{ord}(a) = \mathrm{ord}(b)$. For H_4, we repeat the above argument to show that $\mathrm{ord}(a) = \infty$. $\square$

5.23 Definition.
Let $i \in \{3, 4, 5, 13, 14, 15\}$ and let $H = H_i$ with generators a and b as defined in Definition 4.17. For every $m \in \mathbb{N}$, define subsets of H by

$$\begin{aligned} X_m &:= \big\{x_1 \dots x_m | x_1, \dots, x_m \in \{a^{-1}, b^{-1}\}\big\} \\ Y_m &:= \big\{y_1 \dots y_m | y_1, \dots, y_m \in \{a, b\}\big\} \\ Z_m &:= \big\{xy | x \in X_m \text{ and } y \in Y_m\big\} \end{aligned}$$

5.24 Remark.
Let $i \in \{3, 4, 5, 13, 14, 15\}$ and let $H = H_i$ with generators a and b as defined in Definition 4.17. Expand $a^{-1}b$ to the first level. Observe that two of the vertices of the first level are labelled by 1 and that the third vertex is labelled by an element of Z_1. This implies that $a^{-1}b$ has constant level growth. Indeed, it has precisely one 3-cycle on every level of its portrait.

5.25 Lemma.
Let $i \in \{3, 4, 5, 13, 14, 15\}$ and let $H = H_i$ with generators a and b as defined in Definition 4.17. Let $m \in \mathbb{N}$. Then every element of Z_m has polynomial level growth of degree $m - 1$.

Proof:
The proof is by induction on m. By Remark 5.24, $a^{-1}b$ has constant level growth. Its inverse, $b^{-1}a$, has the same growth function. As $Z_1 = \{1, a^{-1}b, b^{-1}a\}$, this provides the base for the induction. Suppose that we have proved the following for every element z_m of Z_m:

(i) z_m has polynomial level growth of degree $m - 1$; and

(ii) in the expansion of z_m to the first level, two of the vertices of the first level are labelled by elements of Z_{m-1} and the third vertex is labelled by an element of Z_m.

Let z be an element of Z_{m+1}. There exist $x_1 \in X_1$, $y_1 \in Y_1$ and $z_m \in Z_m$ such that $z = x_1 z_m y_1$. In the expansions of x_1 and y_1 to the first level, each vertex is labelled by an element of X_1 and Y_1 respectively. Note that $Z_{j+1} = \{x_1 z_j y_1 | x_1 \in X_1, y_1 \in Y_1, z_j \in Z_j\}$. Therefore, in the expansion of z to the first level, two of the vertices of the first level are labelled by elements of Z_m and the third vertex is labelled by an element of Z_{m+1}.

Let $k \in \mathbb{N}$. Note that in the expansion of z to the k-th level, one vertex of level k is labelled by an element of Z_{m+1} and all other vertices of level k are labelled elements of Z_m. It is worth remarking here that Z_j is a subset of Z_{j+1} for all j. This avoids possible confusion arising from the fact that some of the vertices of level k may be labelled by elements of Z_j where $j < m$. As Z_{m+1} is a finite set, there must exist $q, r \in \mathbb{N}$ and $\zeta \in Z_{m+1}$ such that ζ is the only element of Z_{m+1} labelling a vertex of level q in the expansion of z to the q-th level, and ζ is the only element of Z_{m+1} labelling a vertex of level r in the expansion of ζ to the r-th level. We prove that ζ has polynomial level growth of degree m. This proves that z has polynomial level growth of degree m, as in the expansion of z to the q-th level, every vertex is labelled by an element of polynomial level growth of degree m.

By the induction hypothesis, there exist $C > 0$ and $N \in \mathbb{N}$ such that every element of Z_m has level growth function bounded above by Cn^{m-1} for all $n \geqslant N$. Let ℓ be the level growth function of ζ. Using the expansion of ζ to the r-th level, we see that $\ell(n) < \ell(n-r) + 3^r Cn^{m-1}$ for all $n \geqslant r+N$. The term $3^r Cn^{m-1}$ is an upper bound for the contributions of the $3^r - 1$ vertices of the r-th level which are labelled by elements of Z_m. An induction on n proves that there exists $D > 0$ such that $\ell(n) < Dn^m$ for all sufficiently large n. □

5.26 Lemma.
Let $i \in \{3, 5\}$ and $H = H_i$ with generators a and b as defined in Definition 4.17. Then a^N has exponential level growth for all $N \in \mathbb{Z} \backslash \{0\}$.

Proof:
As a^N and a^{-N} have the same level growth function, it suffices to prove the lemma for $N > 0$.

Let ℓ be the level growth function of a^N. As $a^N = (b^N, a^N, a^N)$, we see that $\ell(n+1) \geqslant 2\ell(n)$ for all $n \geqslant 0$. By Lemma 5.22, $b^N \neq 1$. Therefore, there exists $k \in \mathbb{N}_0$ and a vertex ν of level k such that ν is labelled by a non-identity element of C_3 in the portrait of b^N. Thus $\ell(k) > 0$. Therefore, a^N has level growth of type at least 2^n. That is, a^N has exponential level growth. □

5.27 Lemma.
Let F be the free group of rank 2 with generators x and y. Let $i \in \{14, 15\}$ and let $H = H_i$ with generators a and b as defined in Definition 4.17. Let $w(x, y)$ be a positive word in F of length ℓ. Then a^ℓ occurs in the expansion of $w(a, b)$.

Proof:
For each i the proof is by induction on ℓ. If $\ell = 1$ then $w(x,y) \in \{x,y\}$, and so a occurs in the expansion of $w(a,b)$. This provides the base for the induction. Suppose that the lemma holds true for every positive word of length ℓ. Let $w(x,y)$ be a positive word of length $\ell+1$. There exists a positive word $u(x,y)$ of length ℓ and a positive word $v(x,y)$ of length 1 such that $w = uv$. By induction, there exists $k \geqslant 0$ such that a^ℓ occurs in the expansion of $u(a,b)$ level k. The only elements of $\mathrm{Aut}(T)$ that can occur in the expansion of $v(a,b)$ are a and b. Therefore, in the expansion of $w(a,b)$ to level k, at least one of $a^{\ell+1}$ and $a^\ell b$ occurs.

(i) If $i = 14$ then $a = (b,b,a)$ and $b = \sigma(a,a,b)$. We calculate that $a^\ell b = \sigma(a^{\ell+1}, b^\ell a, b^{\ell+1})$;

(ii) if $i = 15$ then $a = (b,b,a)$ and $b = \sigma(a,b,a)$. We calculate that $a^\ell b = \sigma(a^{\ell+1}, b^{\ell+1}, b^\ell a)$.

□

5.28 Lemma.
Let F be the free group of rank 2 with generators x and y. Let $i \in \{4, 13\}$ and let $H = H_i$ with generators a and b as defined in Definition 4.17. Let $w(x,y)$ be a positive word in F of even length 2ℓ. Then $(ab)^\ell$ occurs in the expansion of $w(a,b)$.

Proof:
For each i the proof is by induction on ℓ. Note that if $\ell = 1$ then $w \in \{x^2, xy, yx, y^2\}$. We prove the base case $\ell = 1$ in (i) and (ii) below. Suppose that the lemma holds true for every positive word of length 2ℓ. Let $w(x,y)$ be a positive word of length $2(\ell+1)$. There exists a positive word $u(x,y)$ of length 2ℓ and a positive word $v(x,y)$ of length 2 such that $w = uv$. By induction, there exists $k \geqslant 0$ such that $(ab)^\ell$ occurs in the expansion of $u(a,b)$ level k. The only elements of $\mathrm{Aut}(T)$ that can occur in the expansion of $v(a,b)$ are a^2, ab, ba, and b^2. Therefore, in the expansion of $w(a,b)$ to level k, at least one of $(ab)^\ell a^2$, $(ab)^\ell ab$, $(ab)^\ell ba$, and $(ab)^\ell b^2$ occurs.

(i) If $i = 4$ then $a = \sigma(b,a,a)$ and $b = \sigma^2(a,b,b)$. We see that $a^2 = \sigma^2(ab, ba, a^2)$, that $ba = (b^2, a^2, ba)$, and that $b^2 = \sigma(ba, b^2, ab)$. Thus ab occurs in the expansions of a^2 and b^2. As a^2 occurs in the expansion of ba, and ab occurs in the expansion of a^2, it follows that ab occurs in the expansion of ba. This provides the base for the induction.

Observe that $ab = (a^2, ab, b^2)$ and $(ab)^\ell = (a^{2\ell}, (ab)^\ell, b^{2\ell})$. We calculate that $(ab)^\ell a^2 = \sigma^2\big((ab)^{\ell+1}, b^{2\ell+1}a, a^{2\ell+2}\big)$, that $(ab)^\ell ba = \big(a^{2\ell}ba, (ab)^\ell a^2, b^{2\ell+1}a\big)$, and that $(ab)^\ell b^2 = \sigma\big(b^{2\ell+1}a, a^{2\ell}b^2, (ab)^{\ell+1}\big)$. Thus $(ab)^{\ell+1}$ occurs in the expansions of $(ab)^\ell a^2$ and $(ab)^\ell b^2$. As $(ab)^\ell a^2$ occurs in the expansion of $(ab)^\ell ba$, it follows that $(ab)^{\ell+1}$ occurs in the expansion of $(ab)^\ell ba$;

(ii) if $i = 13$ then $a = \sigma(b, b, b)$ and $b = \sigma^2(a, b, b)$. We see that $a^2 = \sigma^2(b^2, b^2, b^2)$, that $ba = (b^2, ab, b^2)$, and that $b^2 = \sigma(ba, b^2, ab)$. Thus ab occurs in the expansions of ba and b^2. As b^2 occurs in the expansion of a^2, it follows that ab occurs in the expansion of a^2. This provides the base for the induction.

Observe that $ab = (ba, b^2, b^2)$ and $(ab)^\ell = ((ba)^\ell, b^{2\ell}, b^{2\ell})$. We calculate that $(ab)^\ell a^2 = \sigma^2\big(b^{2\ell+2}, b^{2\ell+2}, (ba)^\ell b^2\big)$, that $(ab)^\ell ba = \big((ba)^\ell b^2, b^{2\ell}ab, b^{2\ell+2}\big)$, and that $(ab)^\ell b^2 = \sigma\big(b^{2\ell+1}a, (ba)^\ell b^2, b^{2\ell}ab\big)$. Thus $(ba)^\ell b^2$ occurs in the expansions of $(ab)^\ell a^2$, $(ab)^\ell ba$, and $(ab)^\ell b^2$. It therefore suffices to show that $(ab)^{\ell+1}$ occurs in the expansion of $(ba)^\ell b^2$. We calculate that $(ba)^\ell b^2 = \sigma\big(b^{2\ell+1}a, b^{2\ell+2}, (ab)^{\ell+1}\big)$.

□

5.29 Lemma.
Let F be the free group of rank 2 with generators x and y. Let $i \in \{4, 13, 14, 15\}$ and let $H = H_i$ with generators a and b as defined in Definition 4.17. If $w(x, y)$ is any positive word in F then $w(a, b) \neq 1$.

Proof:
If $i \in \{14, 15\}$ then this follows from Lemmas 5.22 and 5.27. If $i \in \{4, 13\}$ then ab stabilizes the first level and b^2 occurs in the expansion of ab to the first level. As b^2 has infinite order by Lemma 5.22, it follows that ab has infinite order. If w has even length then the result follows by Lemma 5.28. If w has odd length $2\ell + 1$ then it follows from Lemma 5.28 that at least one of $(ab)^\ell a$ and $(ab)^\ell b$ occurs in the expansion of w. Now $(ab)^\ell$ acts as the identity at the root. Therefore, both $(ab)^\ell a$ and $(ab)^\ell b$ do not stabilize the first level. It follows that $w(a, b) \neq 1$. □

5.30 Lemma.
Let $i \in \{4, 13, 14, 15\}$ and let $H = H_i$ with generators a and b as in Definition 4.17. Let N be any natural number. Then b^N does not have polynomial growth.

Proof:
Let W denote the set of positive words of length N in F. Let S denote the subset $\{w(a, b) \mid w(x, y) \in W\}$ of H. Note that S is closed under expansion. Denote the cardinality of S by M. Fix a natural number k such that $3^k > M$.

Choose an element s_1 of S. In the expansion of s_1 to level k, every vertex is labelled by an element of S. As the number of vertices of level k in T is strictly greater than the cardinality of S, there must exist an element s_2 of S which occurs at least twice in the expansion of s_1 to level k. We repeat this argument for s_2, showing that there exists an element s_3 of S which occurs at least twice in the expansion of s_2 to level k. Continuing in this manner, we must eventually find a positive multiple rk of k and an element s_j of S such that s_j occurs at least twice in the expansion of s_j to level rk.

We show that s_j does not have polynomial level growth. Let ℓ denote the level growth function of s_j. As s_j occurs twice in its own expansion to level rk, we must have that $\ell(n) \geqslant 2\ell(n-rk)$ for all $n \geqslant rk$. In order to show that s_j does not have polynomial level growth, it suffices to show that there exists $n \in \mathbb{N}_0$ such that $\ell(n) \geqslant 1$. That is, we have to show that $s_j \neq 1$. This follows from Lemma 5.29.

By Lemma 5.25, $b^{-N}s_j$ has polynomial level growth. If b^N has polynomial level growth then so does $b^N(b^{-N}s_j) = s_j$. But we have shown that s_j does not have polynomial level growth. Therefore, b^N does not have polynomial level growth. $\square$

5.31 Theorem.
Let $i \in \{3,4,5,13,14,15\}$ and let $H = H_i$ with generators a and b as in Definition 4.17. Then H maps onto C_∞.

Proof:
Let $\alpha = a^{-1}b$ and let A be the normal closure of $\langle \alpha \rangle$ in H.

Suppose $i \in \{3,5\}$. We show that a has infinite order modulo A. As $H = \langle \alpha, a \rangle$, it follows that $A = \langle \alpha^{a^k} \mid k \in \mathbb{Z} \rangle$. Let $N \in \mathbb{N}$. For a contradiction, suppose that $a^N \in A$. Then a^N can be expressed as a word on the set $\{\alpha^{a^k} \mid k \in \mathbb{Z}\}$. Conjugating by a sufficiently large power of a, we express a^N as a word on the set $\{\alpha^{a^k} \mid k \in \mathbb{N}_0\}$. By Lemma 5.25, the elements of $\{\alpha^{b^k} \mid k \in \mathbb{N}_0\}$ have polynomial level growth. By Lemma 5.26, a^N has exponential level growth. It follows from Lemma 2.73 that a^N is not an element of the subgroup of H generated by $\{\alpha^{a^k} \mid k \in \mathbb{N}_0\}$. This contradicts the fact that a^N may be expressed as a word on the set $\{\alpha^{a^k} \mid k \in \mathbb{N}_0\}$. Therefore, $a^N \notin A$.

Suppose $i \in \{4,13,14,15\}$. We show that b has infinite order modulo A. This suffices to prove the result. As $H = \langle \alpha, b \rangle$, it follows that $A = \langle \alpha^{b^k} \mid k \in \mathbb{Z} \rangle$. Let $N \in \mathbb{N}$. For a contradiction, suppose that $b^N \in A$. Then b^N can be expressed as a word on the set $\{\alpha^{b^k} \mid k \in \mathbb{Z}\}$. Conjugating by a sufficiently large power of b, we express b^N as a word on the set $\{\alpha^{b^k} \mid k \in \mathbb{N}_0\}$. By Lemma 5.25, the elements of $\{\alpha^{b^k} \mid k \in \mathbb{N}_0\}$ have polynomial level growth. It follows that b^N has polynomial level growth. This contradicts Lemma 5.30. Therefore $b^N \notin A$ as required. $\square$

5.32 Corollary.
If $i \in \{3, 4, 5, 13, 14, 15\}$. then H_i does not have the congruence subgroup property.

Proof:
This follows from Lemma 4.40 and Theorem 5.31. □

5.33 Remark.
Recall from the proof of Lemma 5.6 that the generators $a = \sigma(b, a, a)$ and $b = \sigma^2(a, b, b)$ of H_4 satisfsy the relation $a^{-2}b^2 = b^{-1}a$. This implies that $a^3 \equiv b^3$ modulo $[H_4, H_4]$. It follows from Proposition 5.5 and Theorem 5.31 that the abelianization of H_4 is $C_3 \times C_\infty$.

5.5 H_1, H_2, H_{12} and H_{16}

Let $i \in \{1, 2, 12, 16\}$ and $j \in [1, 17]\backslash\{1, 2, 12, 16, 17\}$. In this section, we prove that H_i is a quotient of $C_3 * C_3$, while the abelianization of H_j is not of exponent 3. In particular, H_i and H_j are not isomorphic. It is not known if H_1, H_2, H_{12}, H_{16} and H_{17} are pairwise non-isomorphic.

5.34 Lemma.
Let $i \in \{1, 2, 12, 16\}$ and let $H = H_i$ with generators a and b as in Definition 4.17. Then $\operatorname{ord}(a) = \operatorname{ord}(b) = 3$. Thus H_i is a quotient of $C_3 * C_3$.

Proof:
This is a calculation for each i. As an example, let $i = 12$. We calculate that $a^3 = (b^3, b^3, b^3)$ and $b^3 = (a^3, b^3, b^3)$. Therefore, $a^3 = b^3 = 1$. □

5.35 Lemma.
Let $i \in \{1, 2, 12, 16\}$. Then the abelianisation of H_i is isomorphic to $C_3 \times C_3$.

Proof:
This follows from Proposition 5.5 and Lemma 5.34. □

5.36 Theorem.
Let $j \in [3, 8] \cup [13, 15]$. Then H_j has infinite abelianization.

Proof:
This follows from Corollary 5.14 and Theorem 5.31. □

5.37 Lemma.
Let $j \in [9, 11]$. Then H_j^{ab} maps onto $C_3 \times C_9$.

Proof:
We calculate the action of the generators of H_j on the vertices of level 4. This gives generators for a subgroup of Sym(81) which is isomorphic to $H_j/\mathrm{st}(4)$. The computer code to calculate the abelianization of this permutation group is contained in Appendix B.2. The abelianization of $H_j/\mathrm{st}(4)$ is isomorphic to $C_3 \times C_9$. □

5.38 Corollary.
If $i \in \{1, 2, 12, 16\}$ and $j \in [1, 17]\backslash\{1, 2, 12, 16, 17\}$ then H_i and H_j are not isomorphic.

Proof:
This follows from Lemma 5.35, Theorem 5.36 and Lemma 5.37. □

5.39 Proposition.
Let $i \in \{1, 2\}$ and let $H = H_i$ with generators a and b as in Definition 4.17. Then there exists a normal subgroup G_i of index 3 in H_i such that G_i is defined by a windmill automaton.

Proof:
Let $c = ba^{-1}$ and let $G = \langle c\rangle^H$. We show that G is defined by a windmill automaton and that G has index 3 in H.

If $i = 1$ then $a = (b, a, a)$ and $b = \sigma(b, b, b)$. We calculate that $c = \sigma(1, c, c)$, that $c^b = \sigma(c^b, 1, c^b)$, that $c^{b^2} = \sigma(c^{b^2}, c^{b^2}, 1)$, and that $cc^bc^{b^2} = \mathrm{id}(cc^bc^{b^2}, cc^bc^{b^2}, 1) = 1$. As $H = \langle b, c\rangle$ and b has order 3 it follows that $G = \langle c, c^b\rangle$. Let $d = c^b$. Then $d = \sigma(d, 1, d)$. Therefore, $\{1, c, d\}$ is the set of states of a windmill automaton. It remains to show that G has index 3 in H. The index is at most 3 because $\mathrm{ord}(b) = 3$. Observe that c and d have level growth function 2.2^n, and that b has level growth function 2.3^n. It follows from Lemma 2.73 that b is not in G.

If $i = 2$ then $a = (b, b, a)$ and $b = \sigma(b, b, b)$. We calculate that $c = \sigma(1, 1, c)$, that $c^b = \sigma(c^b, 1, 1)$, that $c^{b^2} = \sigma(1, c^{b^2}, 1)$, and that $cc^bc^{b^2} = \mathrm{id}(1, cc^bc^{b^2}, 1) = 1$. As before, it follows that $G = \langle c, c^b\rangle$. Let $d = c^b$. Then $d = \sigma(d, 1, 1)$. Therefore, $\{1, c, d\}$ is the set of states of a windmill automaton. It remains to show that G has index 3 in H. The index is at most 3 because $\mathrm{ord}(b) = 3$. Observe that c and d have constant level growth, and that b has exponential level growth. It follows from Lemma 2.73 that b is not in G. □

We refer the reader to the comments after Remark 4.22 where we stated, but did not prove, Proposition 5.39. Note that $c^{-1} = ab^{-1}$, $c^b = a^{-1}b$, and that $c^{-b} = b^{-1}a$. Therefore, $\langle c\rangle^H = \langle 1, a^{-1}b, b^{-1}a, ab^{-1}, ba^{-1}\rangle$.

5.40 Remark.

Consider the subgroup $G_2 = \langle c \rangle^{H_2}$ of H_2 defined in Proposition 5.39. Observe that it is generated by the elements $c^b = \sigma(c^b, 1, 1)$ and $c^{b^2} = \sigma(1, c^{b^2}, 1)$. Therefore, G_2 is equal to the group G of Theorem 4.43. If follows from Lemma 4.40 that H_2 does not have the congruence subgroup property.

5.6 H_3, H_9, H_{10}, H_{11} and H_{14}

Let $i \in [9, 11]$ and $j \in \{3, 14\}$. In this section, we prove that H_i is a quotient of $C_3 * C_\infty$ and that H_j is not a quotient of $C_3 * C_\infty$. In particular, H_i and H_j are not isomorphic. It is not known if H_9, H_{10}, and H_{11} are pairwise non-isomorphic, nor is it not known if H_3 and H_{14} are isomorphic.

5.41 Lemma.

If $i \in [9, 11]$ then H_i is a quotient of $C_3 * C_\infty$.

Proof:

Let $H = H_i$ with generators a and b as in Definition 4.17. Let $\beta = ab^{-1}$. As $H = \langle a, \beta \rangle$, it suffices to show that $\beta^3 = 1$. We calculate the expansion of β to the first level for each i:

if $i = 9$ then $\beta = \sigma(\beta, \beta^{-1}, 1)$;

if $i = 10$ then $\beta = \sigma(\beta^{-1}, 1, \beta)$; and

if $i = 11$ then $\beta = \sigma(1, \beta, \beta^{-1})$.

In each case, $\beta^3 = \mathrm{id}(1, 1, 1) = 1$ as required. □

5.42 Lemma.

Let $j \in \{3, 14\}$. Then H_j is not a quotient of $C_3 * C_\infty$.

Proof:

Suppose that H_j is a quotient of $C_3 * C_\infty$. Then $H_j/\mathrm{st}(5)$ is a quotient of $C_3 * C_\infty$, and so $\big(H_j/\mathrm{st}(5)\big)/P_4$ is a quotient of $(C_3 * C_\infty)/P_4$. We show that this is not the case. As $H_j/\mathrm{st}(5)$ is a finite group and $C_3 * C_\infty$ is a finitely presented group, there are terminating algorithms to calculate the lower p-central factors of these groups. The calculations are done using the Magma computer package. The details may be found in Appendix B.2. We summarise the proof below.

Let $H = H_j$ with generators a and b as in Definition 4.17. We calculate the actions of a and b on the vertices of level 5 in T. To save labour, this is done using the GAP computer package. We then have two generators for a subgroup of Sym(243) that is isomorphic to $H_j/\mathrm{st}(5)$. We calculate presentations for the groups $\big(H_j/\mathrm{st}(5)\big)/P_4$ and $(C_3 * C_\infty)/P_4$. From these presentations, we calculate that $(C_3 * C_\infty)/P_4$ has order 2187, that $\big(H_j/\mathrm{st}(5)\big)/P_4$ has order 729, and that $(C_3 * C_\infty)/P_4$ has thirteen normal subgroups of order 3. For each such normal subgroup N, we check that the quotient of $(C_3 * C_\infty)/P_4$ by N is not isomorphic to $\big(H_j/\mathrm{st}(5)\big)/P_4$. □

5.43 Theorem.
Let $i \in [9, 11]$ and $j \in \{3, 14\}$. Then H_i and H_j are not isomorphic.

Proof:
This follows from Lemmas 5.41 and 5.42. □

5.7 Other Results and Conjectures

Recall from Remark 5.4 that the groups $H_1, \ldots, H_{17}$ are not abelian. In this section, we prove that $H_1, \ldots, H_{17}$ are not free, and that they contain elements of infinite order. We go on to define the notion of 'two-state duality' for two-state automata, and conjecture that some of the groups $H_1, \ldots, H_{17}$ are isomorphic to their two-state duals.

5.44 Theorem.
For all $i \in [1, 17]$ there exists an element of infinite order in H_i.

Proof:
For $i \in [6, 8]$ this follows from Corollary 5.14; for $i \in [1, 17]\backslash[6, 11]$ it follows from Lemma 5.16; for $i \in [9, 11]$ it follows from Lemma 5.17. □

5.45 Theorem.
The groups $H_1, \ldots, H_{17}$ are not free.

Proof:
Let $H = H_i$ with generators a and b as in Definition 4.17. We prove the result by finding a non-trivial relation in a and b. For a contradiction, suppose that H is free. As H is a two-generator group, the existence of a non-trivial relation in a and b implies that H must have rank stricly less than 2. That is, H must be cyclic. Recall from Remark 5.4 that H is not abelian. This is a contradiction. Therefore, if there exists a non-trivial relation in a and b then H is not free.

If $i \in \{1, 2, 12, 16\}$ then we have the relation $a^3 = 1$ (see Lemma 5.34). If $i = 4$ then we have the relation $a^{-2}b^2 = b^{-1}a$ (see the proof of Lemma 5.6). If $i \in [6, 11]$ then we have the relation $(ab^{-1})^3 = 1$. This is readily checked for each group. For example, if $i = 11$ then $a = (132)(b, b, a)$ and $b = (123)(a, b, b)$, and so $ab^{-1} = (123)(1, ab^{-1}, ba^{-1})$ and $(ab^{-1})^3 = \mathrm{id}(1, 1, 1) = 1$.

For the other groups, we employ the following method in order to find a non-trivial relation in a and b. First, we calculate a generating set for $\mathrm{st}_H(1)$ by using the Schreier method (details of a calculation using the Schreier method are given in the proof of Theorem 4.38, and we omit them here). As $\mathrm{st}_H(1)$ is a subgroup of index 3 in the two-generator group H, the Schreier method yields a set of four generators for $\mathrm{st}_H(1)$. Using these generators, we search for words w_1, w_2, and w_3 on $\{a, b\}$ that act trivially on the subtrees T_{ν_1}, T_{ν_2}, and T_{ν_3} respectively. The triple commutator $[w_1, w_2, w_3]$ must therefore act trivially on the entire tree T. It then remains to check that $[w_1, w_2, w_3]$ is a non-trivial relation. We present all the details for H_3, and summarise for the other groups.

If $i = 3$ then $a = (b, a, a)$ and $b = (123)(a, b, b)$. Observe that $\mathrm{st}_H(1)$ is equal to $\langle a, b^3\rangle^H$, and that a set of coset representatives for $\mathrm{st}_H(1)$ in H is $\{1, b, b^2\}$. The Schreier method yields that $\mathrm{st}_H(1) = \langle a, a^b, a^{b^{-1}}, b^3\rangle$. We give the expansions of the generators of $\mathrm{st}_H(1)$ to the first level:

$$\begin{aligned}
a &= (b, a, a)\\
a^b &= (a, b, b^{-1}ab)\\
a^{b^{-1}} &= (bab^{-1}, bab^{-1}, aba^{-1})\\
b^3 &= (b^2a, bab, ab^2)
\end{aligned}$$

It follows that $a^{-2}b^3a^{-b}$ acts trivially on T_{ν_1}, that $a^{-b}a^{b^{-1}}a^ba^{-1}$ acts trivially on T_{ν_2}, and that $a^{-1}b^3a^{-1}a^{-2b^{-1}}a$ acts trivially on T_{ν_3}. For completeness, we calculate their expansions to the first level:

$$\begin{aligned}
a^{-2}b^3a^{-b} &= (1, a^{-2}ba, a^{-1}ba^{-1}b)\\
a^{-b}a^{b^{-1}}a^ba^{-1} &= (a^{-1}bab^{-1}ab^{-1}, 1, b^{-1}a^{-1}baba^{-1}b^{-1}aba^{-1})\\
a^{-1}b^3a^{-1}a^{-2b^{-1}}a &= (ba^{-1}, a^{-1}baba^{-1}ba^{-2}b^{-1}a, 1)
\end{aligned}$$

It follows that $\left[a^{-2}b^3a^{-b}, a^{-b}a^{b^{-1}}a^ba^{-1}, a^{-1}b^3a^{-1}a^{-2b^{-1}}a\right] = 1$. We check that this is a non-trivial relation (the use of a computer is recommended, though not strictly essential).

For the remaining groups, we list words w_1, w_2, and w_3 on $\{a, b\}$ that act trivially on the subtrees T_{ν_1}, T_{ν_2}, and T_{ν_3} respectively. It follows that $[w_1, w_2, w_3]$ is a relation in H_i. It remains to check that $[w_1, w_2, w_3]$ is a non-trivial relation. We leave the details to the reader.

i	w_1	w_2	w_3
5	$[a,b]^b$	$[a,b]$	$[a,b]^{b^2}$
13	$[a,b]^a$	$[a,b]^{a^2}$	$[a,b]$
14	$a^{-b}a^{b^{-1}}a^b a^{-1}$	$a^{-1}b^3a^{-1}a^{-2b^{-1}}a$	$a^{-2}b^3a^{-b}$
15	$[a,b]^{b^2}$	$[a,b]$	$[a,b]^b$
17	$[a^3,b]^a$	$[a^3,b]^{a^2}$	$[a^3,b]$

□

5.46 Definition.
Let A be a two-state d-ary automaton with generators $a = \sigma_a(a_1, \ldots, a_d)$ and $b = \sigma_b(b_1, \ldots, b_d)$ where $\sigma_a, \sigma_b \in \mathrm{Sym}(d)$ and $a_i, b_i \in \{a, b\}$. Define the *two-state dual* of A to be the two-state d-ary automaton with generators $a = \sigma_a(b_1, \ldots, b_d)$ and $b = \sigma_b(a_1, \ldots, a_d)$.

Two-state duality may be a good place to look for isomorphisms between two-state automata. For example, recall that $H_4 = \langle a = (123)(b,a,a), b = (132)(a,b,b)\rangle$. The two-state dual of H_4 is $\langle a = (123)(a,b,b), b = (132)(b,a,a)\rangle$. Conjugating the generators by $(\overline{132})$ (that is, relabelling the tree), we obtain $\langle a = (123)(b,b,a), b = (132)(a,a,b)\rangle$. It is shown in Lemma 4.19 that this group is isomorphic to H_4. Therefore, H_4 is isomorphic to its two-state dual.

5.47 Remark.
Let F be the free group on two generators x and y. Observe that the subgroup of $F \wr C_3$ generated by $a := (123)(y,x,x)$ and $b := (132)(x,y,y)$ maps epimorphically onto H_4. Similarly, the subgroup of $F \wr C_3$ generated by $a := (123)(x,y,y)$ and $b := (132)(y,x,x)$ maps epimorphically onto the two-state dual of H_4. The two subgroups of $F \wr C_3$ so defined are naturally isomorphic. It is this argument that leads us to believe that other two-state automaton groups may be isomorphic to their two-state duals.

As a word of caution, we give a simple example demonstrating that two-state automaton groups are not always isomorphic to their two-state duals. Recall from the tables in Section 4.5 that the automaton group $\langle a = (a,a,a), b = (123)(b,b,b)\rangle$ is isomorphic to C_3, while its two-state dual $\langle a = (b,b,b), b = (123)(a,a,a)\rangle$ is isomorphic to $C_3 \times C_3$.

Are H_6 and H_7 Isomorphic?

The two-state dual of H_6 has generators $a = (a, a, b)$ and $b = (123)(a, a, b)$. Swapping the labels a and b, we obtain $a = (123)(b, b, a)$ and $b = (b, b, a)$. These are the generators of H_7 given in Definition 4.17. Thus, the two-state dual of H_6 is equal to H_7.

5.48 Conjecture.
Let $H_6 = \langle a = (b, b, a), b = (123)(b, b, a)\rangle$ and $H_7 = \langle c = (c, c, d), d = (123)(c, c, d)\rangle$. Let $\varphi : H_6 \to H_7$ be the map defined by $\varphi\big(w(a, b)\big) = w(c, d)$ for all $w(x, y) \in F$. We conjecture that φ is an isomorphism.

One possible attack on this conjecture is to prove that the given map induces an isomorphism $\varphi_n : H_6/\mathrm{st}(n) \to H_7/\mathrm{st}(n)$ for all $n \in \mathbb{N}$. This suffices to prove the conjecture. Indeed, if φ is not an isomorphism then there exists a word $w(x, y)$ such that at least one of the following holds true:

(i) $w(a, b) = 1$ and $w(c, d) \neq 1$;

(ii) $w(a, b) \neq 1$ and $w(c, d) = 1$.

If $w(c, d) \neq 1$ then there exists $n \in \mathbb{N}$ such that $w(c, d) \notin \mathrm{st}(n)$. If in addition $w(a, b) = 1$ then the map $\varphi_n : H_6/\mathrm{st}(n) \to H_7/\mathrm{st}(n)$ is not well-defined. Similarly, if $w(a, b) \neq 1$ and $w(c, d) = 1$ then there exists $n \in \mathbb{N}$ such that the map $\varphi_n : H_6/\mathrm{st}(n) \to H_7/\mathrm{st}(n)$ has a non-trivial kernel.

5.49 Definition.
Let x be an automorphism of T and let n be a level of T. Define the *slice of x at level n*, denoted x_n, to be the tree automorphism with the following portrait:

(i) if ν is a vertex of level n then $x_n(\nu) = x(\nu)$;

(ii) if ν is a vertex not of level n then $x_n(\nu) = \mathrm{id}$.

Observe that $a_0 = c_0 = 1$, and that $a_n = b_n$ and $c_n = d_n$ for all $n \in \mathbb{N}$. Let $B_n = \langle b_0, \ldots, b_n\rangle$ and $D_n = \langle d_0 \ldots d_n\rangle$. To prove that φ_{n+1} is an isomorphism it would suffice to show that the map $\vartheta_n : B_n \to D_n$ induced by $b_i \mapsto d_i$ is an isomorphism. Indeed, one observes that $a \equiv b_1 b_2 \ldots b_n$ modulo $\mathrm{st}(n+1)$, that $b \equiv b_0 b_1 \ldots b_n$ modulo $\mathrm{st}(n+1)$, that $c \equiv d_1 d_2 \ldots d_n$ modulo $\mathrm{st}(n+1)$, and that $d \equiv d_0 d_1 \ldots d_n$ modulo $\mathrm{st}(n+1)$. It is quickly proved that ϑ_0 and ϑ_1 are isomorphisms.

To prove that ϑ_n is an isomorphism, we assume that ϑ_{n-1} is an isomorphism and use the fact that $B_n = \langle b_n \rangle^{B_{n-1}} \rtimes B_{n-1}$ and $D_n = \langle d_n \rangle^{D_{n-1}} \rtimes D_{n-1}$. We observe that the centralizer of b_n in B_n is B_{n-2}, and that the centralizer of d_n in D_n is D_{n-2}. Therefore, $\langle b_n \rangle^{B_{n-1}} = \langle b_n \rangle^{B_{n-2}}$ and $\langle d_n \rangle^{D_{n-1}} = \langle d_n \rangle^{D_{n-2}}$.

Observe that $\langle b_n \rangle^{B_{n-2}}$ and $\langle d_n \rangle^{D_{n-2}}$ are, by construction, elementary abelian 3-groups. Thus, there exists a subset S_{n-2} of B_{n-2} such that $\{b_n^s \mid s \in S_{n-2}\}$ freely generates $\langle b_n \rangle^{B_{n-2}}$ as an elementary abelian 3-group. If we can prove that there exists such a subset S_{n-2} with the property that $\left\{d_n^{\vartheta_{n-2}(s)} \mid s \in S_{n-2}\right\}$ is a free generating set for $\langle d_n \rangle^{D_{n-2}}$ then ϑ_n will be proved to be an isomorphism. This follows from simply writing down presentations for the semi-direct products $\langle b_n \rangle^{B_{n-2}} \rtimes B_{n-1}$ and $\langle d_n \rangle^{D_{n-2}} \rtimes D_{n-1}$.

5.50 Remark.
A similar argument on slices might prove that H_6 and H_8 are isomorphic. The isomorphism would send the element $a^{-1}b = \dot{\sigma}$ of H_6 to the element $a^{-1}b = \dot{\sigma}$ of H_8, and the element b of H_6 to the element a of H_8.

Other Two-State Duals

Observe that the two-state dual of H_3 is isomorphic to H_{14} (conjugate the generators of H_{14} by $(\overline{123})$ to obtain the generators of the two-state dual of H_3), and that the two-state dual of H_5 is isomorphic to H_{15} (invert the generators of H_{15} and then conjugate by $(\overline{123})$ to obtain the generators of the two-state dual of H_5),

We conjecture that the map from H_3 to H_{14} sending a to a and b to b is an isomorphism, and that the map from H_5 to H_{15} sending a to a^{-1} and b to b^{-1} is an isomorphism.

Recall from Definition 4.17 that $H_2 = \langle a = (b, b, a), b = (123)(b, b, b) \rangle$. The two-state dual of H_2 is $\langle c = (c, c, d), d = (123)(c, c, c) \rangle$, and it is isomorphic to H_{12}. We remark that the map induced $a \mapsto c$ and $b \mapsto d$ is not an isomorphism. Indeed, we have found a word $w(x, y)$ such that $w(a, b) = 1$ and $w(c, d) \neq 1$. We explain briefly how the word was found.

Recall from Remark 5.47 that there is a two-generator subgroup of $F \wr C_3$ that maps onto both H_4 and its two-state dual, the epimorphisms mapping generators to generators. Similarly, there is a two-generator subgroup J of $F \wr C_3$ that maps onto both H_2 and its two-state dual, the epimorphisms mapping generators to generators. Any relation in J is necessarily a relation in both H_2 and its two-state dual. In order to find the word $w(x, y)$, we searched for a relation in H_2 that is not a relation in J.

We found such a relation by using a triple commutator as in the proof of Theorem 5.45. Define elements of F as follows:

$$\begin{aligned} t &:= x^{y^{-1}xy}x^{-1} \\ u &:= x^{yx^{-1}y^{-1}}x^{-y^{-1}xy} \\ v &:= xx^{-yx^{-1}y^{-1}} \\ w &:= [t, u, v] \end{aligned}$$

Then $w(a, b) = 1$ and $w(c, d) \neq 1$. Indeed, $w(c, d)$ does not stabilize the fifth level of T. This leads us to conjecture that H_2 is not isomorphic to its two-state dual.

What is the Locally Finite 3-Group?

It is shown in Corollary 5.14 that H_6, H_7 and H_8 are (locally finite)-by-(infinite cyclic). As mentioned in Remark 5.20, the locally finite bit is a non-abelian 3-group that we suspect has infinite exponent. In the binary case, it is known that we obtain $C_2 \wr C_\infty$ so that the locally finite bit is elementary 2-abelian. An interesting research problem is to attempt to find a nice description of our locally finite 3-groups, and thus a nice description of H_6, H_7 and H_8. It may turn out that it is the same locally finite 3-group in each case, and this may in turn show that H_6 and H_7 (and perhaps H_8) are isomorphic.

5.8 Summary of the Classification So Far

In this brief section, we summarise the work of this chapter towards providing a complete list, up to group isomorphism, of the non-abelian $(2, 3)$-automaton groups with C_3-action at the root.

There are at most seventeen non-isomorphic groups, labelled $H_1, \ldots, H_{17}$. It is known that the following subsets of $\{H_1, \ldots, H_{17}\}$ are disjoint, in the sense that groups in different sets are not isomorphic:

$\{H_4\}$;
$\{H_6, H_7, H_8\}$;
$\{H_1, H_2, H_{12}, H_{16}\}$;
$\{H_3, H_{14}\}$; and
$\{H_9, H_{10}, H_{11}\}$.

Little is known about H_5, H_{13}, H_{15} and H_{17}. However, it is known that they are distinct from H_4, H_6, H_7, H_8 and, except in the case of H_{17}, that they are distinct from H_1, H_2, H_{12}, H_{16}.

When calculating the two-state dual of H_i, it is often necessary to relabel the states $\{a, b\}$ and renumber the alphabet $\{1, 2, 3\}$ in order to find the H_j to which H_i is dual. We perform this tedious task without comment, and provide a list of the two-state duals:

H_1 and H_{16} are dual;
H_2 and H_{12} are dual;
H_3 and H_{14} are dual;
H_4 is self-dual;
H_5 and H_{15} are dual;
H_6 and H_7 are dual;
H_8 is self-dual;
H_9 and H_{10} are dual; and
H_{11} is self-dual.

At first glance, it appears that H_{13} and H_{17} are not dual to any H_i. However, it follows from Lemma 4.18 that:

H_{13} is dual to a group isomorphic to H_2; and
H_{17} is dual to a group isomorphic to H_1.

It is felt that duality is a promising place to look for possible isomorphisms, particularly in the case of H_6 and H_7, as mentioned in Section 5.7. A look at our partial classification will confirm that the only known case of H_i and its dual not being isomorphic is: H_{13} is not isomorphic to H_2. However, this is an artificial case as H_{13} is not the 'natural' dual of H_2, rather the dual of a group that happens to be isomorphic to H_2.

Appendix A

Calculation of the Tables

A.1 Class Symmetries

We list the symmetries of each Class in Definition 4.8:

Class 1 is closed under swapping labels. Class 1 is closed under conjugation by any given permutation in Sym(3);

Class 2 is closed under conjugation by (23);

Class 3 is closed under conjugation by (12);

Class 4 is closed under swapping labels and under conjugation by (23);

Class 5 is closed under swapping labels followed by conjugation by (12);

Class 6 is closed under conjugation by any given permutation in Sym(3);

Class 7 is closed under conjugation by (12);

Class 8 has no symmetries;

Class 9 is closed under conjugation by (23);

Class 10 is closed under swapping labels and under conjugation by (12);

Class 11 is closed under swapping labels followed by conjugation by (23);

Class 12 is closed under conjugation by (12);

Class 13 is closed under swapping labels. Class 13 is closed under conjugation by any given permutation in Sym(3).

A.2 Calculations for C_2-Action

We provide the calculation of the tables in Section 4.4.

Class 1

Class 1 is defined by $a = (a, a, a)$ and $b = (b, b, b)$. Therefore, $a = \overline{\sigma_a}$ and $b = \overline{\sigma_b}$. It follows that $\langle a, b \rangle$ is isomorphic to $\langle \sigma_a, \sigma_b \rangle$, a subgroup of Sym(3).

Class 2

Class 2 is defined by $a = (b, a, a)$ and $b = (b, b, b)$. The columns indexed by (12) and (13) are identical because Class 2 is closed under conjugation by (23).

(i) Let $a = (12)(b, a, a)$ and $b = (b, b, b)$. Then $b = 1$ and $a = (12)(1, a, a)$. Hence $a^2 = (a, a, a^2)$ and we see that a has infinite order by Lemma 2.50. Thus $\langle a, b \rangle \cong C_\infty$.

(ii) Let $a = (23)(b, a, a)$ and $b = (b, b, b)$. Then $b = 1$ and $a = (23)(1, a, a)$. Hence $a^2 = (1, a^2, a^2) = 1$. Thus $\langle a, b \rangle \cong C_2$.

(iii) Let $a = (b, a, a)$ and $b = (12)(b, b, b)$. We see that $b^2 = (b^2, b^2, b^2) = 1$, that $a^2 = (b^2, a^2, a^2) = 1$, and that $ab = (12)(ab, b^2, ab) = (12)(ab, 1, ab)$. Hence $(ab)^2 = (ab, ab, (ab)^2)$, and ab has infinite order by Lemma 2.50. Therefore, $\langle a, b \rangle \cong D_\infty$.

(iv) Let $a = (b, a, a)$ and $b = (23)(b, b, b)$. We see that $b^2 = (b^2, b^2, b^2) = 1$, that $a^2 = (b^2, a^2, a^2) = 1$, that $ab = (23)(b^2, ab, ab) = (23)(1, ab, ab)$, and that $(ab)^2 = (1, (ab)^2, (ab)^2) = 1$. Consideration modulo st(2) shows that 1, a, b, and ab are distinct tree automorphisms. Thus $\langle a, b \rangle \cong C_2 \times C_2$.

Class 3

Class 3 is defined by $a = (b, b, a)$ and $b = (b, b, b)$. The columns indexed by (13) and (23) are identical because Class 3 is closed under conjugation by (12).

(i) Let $a = (12)(b, b, a)$ and $b = (b, b, b)$. Then $b = 1$ and $a^2 = (1, 1, a^2) = 1$. Thus $\langle a, b \rangle \cong C_2$.

(ii) Let $a = (13)(b, b, a)$ and $b = (b, b, b)$. Then $b = 1$ and $a^2 = (a, 1, a)$. By Lemma 2.50, a has infinite order. Thus $\langle a, b \rangle \cong C_\infty$.

(iii) Let $a = (b, b, a)$ and $b = (12)(b, b, b)$. We see that $b^2 = (b^2, b^2, b^2) = 1$, that $a^2 = (b^2, b^2, a^2) = 1$, that $ab = (12)(1, 1, ab)$, and that $(ab)^2 = (1, 1, (ab)^2) = 1$. Consideration modulo st(2) shows that 1, a, b, and ab are distinct tree automorphisms. Thus $\langle a, b \rangle \cong C_2 \times C_2$.

(iv) Let $a = (b, b, a)$ and $b = (13)(b, b, b)$. Then $b^2 = (b^2, b^2, b^2) = 1$ and $a^2 = (b^2, b^2, a^2) = 1$. Now, $ab = (13)(ab, b^2, b^2) = (13)(ab, 1, 1)$ and $(ab)^2 = (ab, 1, ab)$, so ab has infinite order by Lemma 2.50. Thus $\langle a, b\rangle \cong D_\infty$.

Class 4

Class 4 is defined by $a = (b, a, a)$ and $b = (a, b, b)$. The two rows of the table are identical because Class 4 is closed under swapping labels. The columns indexed by (12) and (13) are identical because Class 4 is closed under conjugation by (23).

(i) If $a = (12)(b, a, a)$ and $b = (a, b, b)$ then $\langle a, b\rangle$ is isomorphic to $C_2 \wr C_\infty$ by Lemma 4.16 and Remark 4.14.

(ii) Let $a = (23)(b, a, a)$ and $b = (a, b, b)$. Then $a^2 = (b^2, a^2, a^2)$ and $b^2 = (a^2, b^2, b^2)$ so $a^2 = b^2 = 1$. Also, $ab = (23)(ba, ab, ab)$ and $(ab)^2 = ((ba)^2, (ab)^2, (ab)^2)$. As ba is a conjugate of ab, we see that $(ab)^2 = 1$ by Theorem 2.54. Consideration modulo st(2) shows that 1, a, b, and ab are distinct tree automorphisms. Thus $\langle a, b\rangle \cong C_2 \times C_2$.

Class 5

Class 5 is defined by $a = (b, a, a)$ and $b = (b, a, b)$. Class 5 is closed under swapping labels followed by conjugation by (12). Therefore, we have the isomorphisms:

$$\begin{aligned}
\langle a = (12)(b, a, a), b = (b, a, b)\rangle &\cong \langle a = (b, a, a), b = (12)(b, a, b)\rangle;\\
\langle a = (13)(b, a, a), b = (b, a, b)\rangle &\cong \langle a = (b, a, a), b = (23)(b, a, b)\rangle;\ \text{and}\\
\langle a = (23)(b, a, a), b = (b, a, b)\rangle &\cong \langle a = (b, a, a), b = (13)(b, a, b)\rangle.
\end{aligned}$$

(i) Let $a = (12)(b, a, a)$ and $b = (b, a, b)$. Then $a^{-\overline{(12)}} = (12)\big(b^{-\overline{(12)}}, a^{-\overline{(12)}}, a^{-\overline{(12)}}\big)$ and $b^{-\overline{(12)}} = \big(a^{-\overline{(12)}}, b^{-\overline{(12)}}, b^{-\overline{(12)}}\big)$. Let $c = a^{-\overline{(12)}}$ and $d = b^{-\overline{(12)}}$. Then $c = (12)(d, c, c)$ and $d = (c, d, d)$. We have shown $\langle a, b\rangle \cong \langle c, d\rangle$, and we know that $\langle c, d\rangle \cong C_2 \wr C_\infty$ by Lemma 4.16 and Remark 4.14.

(ii) Let $a = (13)(b, a, a)$ and $b = (b, a, b)$. Then $[a, b] = (1, 1, [a, b]) = 1$, so $\langle a, b\rangle$ is abelian. We aim to prove it is free abelian of rank 2. Suppose there exists a relation $a^m = b^n$ with $m \neq 0$. Choose the relation with m positive of minimal size. Because of the action at the root, m must be even. Let $l = m/2$ and observe that $a^m = ((ab)^l, a^m, (ba)^l) = (a^l b^l, a^m, a^l b^l)$ and that $b^n = (b^n, a^n, b^n)$. Comparing the actions on T_{ν_1}, we see that $a^l = b^{n-l}$. This contradicts the minimality of m. In particular, $a^m \neq 1$ for all $m > 0$, and so $\operatorname{ord}(a) = \infty$. As $b^n = (b^n, a^n, b^n)$, we see that $\operatorname{ord}(b) = \operatorname{ord}(a)$.

(iii) Let $a = (23)(b, a, a)$ and $b = (b, a, b)$. Then $a^2 = (b^2, a^2, a^2)$ and $b^2 = (b^2, a^2, b^2)$ so $a^2 = b^2 = 1$. Also, $ab = (23)(b^2, a^2, ab) = (23)(1, 1, ab)$ and $(ab)^2 = (1, ab, ab)$, so ab has infinite order by Lemma 2.50. Thus $\langle a, b\rangle \cong D_\infty$.

Class 6

Class 6 is defined by $a = (b, b, b)$ and $b = (b, b, b)$. All three columns of the table are identical because Class 6 is closed under conjugation by (123).

(i) If $a = (12)(b, b, b)$ and $b = (b, b, b)$ then $b = 1$ and $a = \dot{(12)}$ is an involution.

(ii) If $a = (b, b, b)$ and $b = (12)(b, b, b)$ then $a^2 = b^2 = (ab)^2 = 1$. We see that 1, a, b and ab are all distinct by comparing them modulo st(2).

Class 7

Class 7 is defined by $a = (b, b, a)$ and $b = (b, b, a)$. The columns indexed by (13) and (23) are identical because Class 7 is closed under conjugation by (12).

(i) Let $a = (12)(b, b, a)$ and $b = (b, b, a)$. Then $a^2 = (b^2, b^2, a^2) = b^2 = 1$ and $ab = \dot{(12)}$. We see that 1, a, b and ab are all distinct by comparing them modulo st(2).

(ii) If $a = (b, b, a)$ and $b = (12)(b, b, a)$ then $a^2 = b^2 = (ab)^2 = 1$. We see that 1, a, b and ab are all distinct by comparing them modulo st(2).

(iii) Let $a = (13)(b, b, a)$ and $b = (b, b, a)$. Then $a^{-(\overline{123})} = (12)\big(b^{-(\overline{123})}, a^{-(\overline{123})}, b^{-(\overline{123})}\big)$ and $b^{-(\overline{123})} = \big(a^{-(\overline{123})}, b^{-(\overline{123})}, b^{-(\overline{123})}\big)$. By Lemma 4.16 and Remark 4.14, we have that $\langle a, b\rangle$ is isomorphic to $C_2 \wr C_\infty$.

(iv) Let $a = (b, b, a)$ and $b = (13)(b, b, a)$. Swap the labels so that $a = (13)(a, a, b)$ and $b = (a, a, b)$. Then $a^{-(\overline{23})} = (12)\big(b^{-(\overline{132})}, a^{-(\overline{132})}, a^{-(\overline{132})}\big)$ and $b^{-(\overline{132})} = \big(a^{-(\overline{132})}, b^{-(\overline{132})}, a^{-(\overline{132})}\big)$. By Lemma 4.16 and Remark 4.14, we have that $\langle a, b\rangle$ is isomorphic to $C_2 \wr C_\infty$.

Class 8

Class 8 is defined by $a = (b, b, a)$ and $b = (a, b, b)$.

(i) Let $a = (12)(b, b, a)$ and $b = (a, b, b)$. Then $a^2 = b^2 = 1$ and $ab = (12)(ba, 1, ab)$. As ab and ba are conjugates, and $(ab)^2 = (ba, ba, (ab)^2)$, it follows by Lemma 2.50 that $\operatorname{ord}(ab) = \infty$. Thus $\langle a, b\rangle \cong D_\infty$.

(ii) Let $a = (13)(b, b, a)$ and $b = (a, b, b)$. Then $a^{-(\overline{13})} = (13)\big(b^{-(\overline{13})}, b^{-(\overline{13})}, a^{-(\overline{13})}\big)$ and $b^{-(\overline{13})} = \big(b^{-(\overline{13})}, b^{-(\overline{13})}, a^{-(\overline{13})}\big)$. Therefore, $\langle a, b\rangle$ is isomorphic to the group defined by the generators $a = (13)(b, b, a)$ and $b = (b, b, a)$. This group has been dealt with in Class 7.

(iii) Let $a = (23)(b, b, a)$ and $b = (a, b, b)$. Then $[a, b] = ([b, a], 1, [a, b])$ is equal to 1 by Theorem 2.54. Suppose there exists a relation $a^m = b^n$ with $m \neq 0$. Choose the relation with m positive of minimal size. Because of the action at the root, m must be even. Let $l = m/2$ and observe that $a^m = (b^m, (ab)^l, (ba)^l) = (b^m, a^l b^l, a^l b^l)$ and that $b^n = (a^n, b^n, b^n)$. Comparing the actions on T_{ν_2}, we see that $a^l = b^{n-l}$. This contradicts the minimality of m. In particular, $a^m \neq 1$ for all $m > 0$ so $\operatorname{ord}(a) = \infty$. As $b^n = (a^n, b^n, b^n)$, we see that $\operatorname{ord}(b) = \operatorname{ord}(a)$. Thus $\langle a, b\rangle \cong C_\infty \times C_\infty$.

(iv) Let $a = (b, b, a)$ and $b = (12)(a, b, b)$. Then $[a, b] = ([b, a], 1, [a, b]) = 1$ and $ab^2 = (b^2 a, bab, ab^2) = (ab^2, ab^2, ab^2) = 1$. Also, $a^2 = (b^2, b^2, a^2) = (a^{-1}, a^{-1}, a^2)$ so a has infinite order by Lemma 2.50. Therefore, $\langle a, b\rangle = \langle b\rangle \cong C_\infty$.

(v) Let $a = (b, b, a)$ and $b = (13)(a, b, b)$. Then $a^{-1} = (b^{-1}, b^{-1}, a^{-1})$ and $b^{-1} = (13) = (b^{-1}, b^{-1}, a^{-1})$. This group has been dealt with in Class 7.

(vi) Let $a = (b, b, a)$ and $b = (23)(a, b, b)$. Then $a^2 = b^2 = 1$, and $ab = (23)(ba, ab, 1)$ has infinite order by Lemma 2.50. Therefore, $\langle a, b\rangle \cong D_\infty$.

Class 9

Class 9 is defined by $a = (b, b, b)$ and $b = (a, b, b)$. The columns indexed by (12) and (13) are identical because conjugation by (23) preserves this class.

(i) Let $a = (12)(b, b, b)$ and $b = (a, b, b)$. Then $a^2 = b^2 = 1$ and $ab = (12)(ba, 1, 1)$ has infinite order by Lemma 2.50.

(ii) Let $a = (23)(b, b, b)$ and $b = (a, b, b)$. Then $a^2 = b^2 = 1$ and $ab = (23)(ba, 1, 1)$ is an involution by Theorem 2.54. Consideration of the group modulo $\mathrm{st}(2)$ proves that this it is isomorphic to $C_2 \times C_2$.

(iii) If $a = (b, b, b)$ and $b = (12)(a, b, b)$ then $[a, b] = ([b, a], 1, 1) = 1$ by Lemma 2.50. We will show that $\langle a, b\rangle$ is free abelian of rank two. We see that $a^n = 1$ if and only if $b^n = 1$ because $a^n = (b^n, b^n, b^n)$. Thus $\mathrm{ord}(a) = \mathrm{ord}(b)$. Observe that $a^{-1}b = (12)(b^{-1}a, 1, 1)$ has infinite order by Lemma 2.50. As $(a^{-1}b)^n = a^{-n}b^n$, it follows that $\mathrm{ord}(a) = \mathrm{ord}(b) = \infty$. Suppose there exists a relation $a^m = b^n$. Then $m = n$ because a and b have infinite order. Choose the relation with n positive of minimal size. Because of the action of b at the root, n must be even. Let $l = n/2$ and observe that $a^n = (b^n, b^n, b^n)$ and that $b^n = ((ba)^l, (ab)^l, b^n)$. Comparing the actions on T_{ν_1}, we see that $a^l = b^l$. This contradicts the minimality of n. Therefore, $\langle a, b\rangle \cong C_\infty$.

(iv) Let $a = (b, b, b)$ and $b = (23)(a, b, b)$. Then $a^2 = b^2 = 1$, and $ab = (23)(ba, 1, 1)$ is an involution by Theorem 2.54. Consideration modulo the second level stabilizer shows that this group is isomorphic to $C_2 \times C_2$.

Class 10

Class 10 is defined by $a = (b, b, a)$ and $b = (a, a, b)$. The two rows of the table are identical because Class 10 is closed under swapping labels. The columns indexed by (13) and (23) are identical because conjugation by (12) preserves this class.

(i) Let $a = (12)(b, b, a)$ and $b = (a, a, b)$. Then $a^2 = b^2 = 1$ and $ab = (12)(ba, ba, ab)$ is an involution by Theorem 2.54. Consideration modulo the second level stabilizer shows that this group is isomorphic to $C_2 \times C_2$.

(ii) Let $a = (13)(b, b, a)$ and $b = (a, a, b)$. Then $a^{(23)} = (12)\big(b^{(23)}, a^{(23)}, b^{(23)}\big)$ and $b^{(23)} = \big(a^{(23)}, b^{(23)}, a^{(23)}\big)$. Thus $\langle a, b\rangle$ is isomorphic to $C_2 \wr C_\infty$ by Lemma 4.16 and Remark 4.14.

Class 11

Class 11 is defined by $a = (b, b, a)$ and $b = (a, b, a)$. Class 11 is closed under swapping labels followed by conjugation by (23). Therefore, we have the isomorphisms:

$$\begin{aligned}
\langle a = (12)(b, b, a), b = (a, b, a)\rangle &\cong \langle a = (b, b, a), b = (13)(a, b, a)\rangle; \\
\langle a = (13)(b, b, a), b = (a, b, a)\rangle &\cong \langle a = (b, b, a), b = (12)(a, b, a)\rangle; \text{ and} \\
\langle a = (23)(b, b, a), b = (a, b, a)\rangle &\cong \langle a = (b, b, a), b = (23)(a, b, a)\rangle.
\end{aligned}$$

(i) Let $a = (12)(b, b, a)$ and $b = (a, b, a)$. Then $a^2 = b^2 = 1$, and $ab = (12)(ba, 1, 1)$ has infinite order by Lemma 2.50. Therefore, $\langle a, b\rangle \cong D_\infty$.

(ii) Let $a = (13)(b, b, a)$ and $b = (a, b, a)$. Then $[a, b] = ([b, a], 1, 1) = 1$ by Theorem 2.54. We see that $a^{-1}b = (13)(1, 1, b^{-1}a)$ has infinite order by Lemma 2.50. As $b^n = (a^n, b^n, a^n)$ for all integers n, we see that $\text{ord}(a) = \text{ord}(b)$. If $a^n = b^n = 1$ then $(a^{-1}b)^n = a^{-n}b^n = 1$. Therefore, a and b have infinite order. Suppose there exists a relation $a^m = b^n$. Then $m = n$ because a and b have infinite order. Because of the action at the root, n must be even. Let $l = n/2$ and observe that $a^n = ((ab)^l, b^n, (ba)^l) = (a^lb^l, b^n, a^lb^l)$ and that $b^n = (a^n, b^n, a^n)$. Comparing the actions on T_{ν_1}, we see that $a^l = b^l$. This implies that $(a^{-1}b)^l = 1$. As $a^{-1}b$ has infinite order, we have shown that $l = 0$. Thus $n = 0$. Therefore, $\langle a, b\rangle \cong C_\infty \times C_\infty$.

(iii) Let $a = (23)(b, b, a)$ and $b = (a, b, a)$. Then $a^{-(\overline{23})} = (23)\big(b^{-(\overline{23})}, b^{-(\overline{23})}, a^{-(\overline{23})}\big)$ and $b^{-(\overline{23})} = \big(a^{-(\overline{23})}, a^{-(\overline{23})}, b^{-(\overline{23})}\big)$. Let $c = a^{-(\overline{23})}$ and $d = b^{-(\overline{23})}$. Then $\langle a, b\rangle$ is isomorphic to $\langle c, d\rangle$, a group dealt with in Class 10.

Class 12

Class 12 is defined by $a = (b, b, b)$ and $b = (a, a, b)$. The columns indexed by (13) and (23) are identical because Class 12 is closed under conjugation by (12).

(i) Let $a = (12)(b, b, b)$ and $b = (a, a, b)$. Then $a^2 = b^2 = 1$, and $ab = (12)(ba, ba, 1)$ is an involution by Theorem 2.54. Consideration modulo the second level stabilizer shows that $\langle a, b\rangle \cong C_2 \times C_2$.

(ii) Let $a = (b, b, b)$ and $b = (12)(a, a, b)$. Then $a^2 = b^2 = 1$, and $ab = (12)(ba, ba, 1)$ is an involution by Theorem 2.54. Consideration modulo the second level stabilizer shows that $\langle a, b\rangle \cong C_2 \times C_2$.

(iii) Let $a = (13)(b, b, b)$ and $b = (a, a, b)$. Then $a^2 = b^2 = 1$, and $ab = (13)(ba, ba, 1)$ has infinite order by Lemma 2.50. Thus $\langle a, b\rangle \cong D_\infty$.

(iv) Let $a = (b, b, b)$ and $b = (13)(a, a, b)$. Then $[a, b] = ([b, a], [b, a], 1)$ is the identity automorphism by Theorem 2.54. By Lemma 2.50, $a^{-1}b = (13)(b^{-1}a, b^{-1}a, 1)$ has infinite order. As $a^n = (b^n, b^n, b^n)$ for all integers n, we see that $\mathrm{ord}(a) = \mathrm{ord}(b)$. If $a^n = b^n = 1$ then $(a^{-1}b)^n = a^{-n}b^n = 1$. Therefore, a and b have infinite order. Suppose that there exists a relation $a^m = b^n$. Then $m = n$ because a and b have infinite order. Because of the action at the root, n must be even. Let $l = n/2$ and observe that $a^n = (b^n, b^n, b^n)$ and that $b^n = ((ba)^l, a^n, (ab)^l) = (a^lb^l, a^n, a^lb^l)$. Comparing the actions on T_{ν_1}, we see that $a^l = b^l$. This implies that $(a^{-1}b)^l = 1$. As $a^{-1}b$ has infinite order, we have shown that $l = 0$. Thus $n = 0$. Therefore, $\langle a, b\rangle \cong C_\infty \times C_\infty$.

Class 13

Class 13 is defined by $a = (b, b, b)$ and $b = (a, a, a)$. The two rows of the table are identical because Class 13 is closed under swapping labels. All three columns of the table are identical because Class 13 is closed under conjugation by (123).

Let $a = (12)(b, b, b)$ and $b = (a, a, a)$. Then $a^2 = b^2 = 1$. We see that $ab = (12)(ba, ba, ba)$ is an involution by Theorem 2.54. Consideration modulo the second level stabilizer shows that $\langle a, b\rangle \cong C_2 \times C_2$.

A.3 Calculations for C_3-Action

We provide the calculation of the tables in Section 4.5. If $\sigma_a = \sigma_b = 1$ then $\langle a, b\rangle = 1$. If $\sigma_a = \sigma_b \neq 1$ then $a = b = \overline{\sigma_a}$ and so $\langle a, b\rangle \cong C_3$. The amount of work to be done when $\sigma_a \neq \sigma_b$ is cut down by the symmetries listed in Section A.1. We refer the reader to Remark 4.11 for the isomorphisms implied by the symmetries of each Class. The calculations, other than checking the symmetries, are listed below.

Class 1

Class 1 is defined by $a = (a, a, a)$ and $b = (b, b, b)$. Therefore, $a = \overline{\sigma_a}$ and $b = \overline{\sigma_b}$. It follows that $\langle a, b\rangle$ is isomorphic to $\langle \sigma_a, \sigma_b\rangle$, a subgroup of $\mathrm{Sym}(3)$.

Class 2

Class 2 is defined by $a = (b, a, a)$ and $b = (b, b, b)$.

If $a = (123)(b, a, a)$ and $b = (b, b, b)$ then $b = 1$ and $a^3 = (a^2, a^2, a^2)$. By Lemma 2.50, a has infinite order.

The other isomorphisms follow from Lemma 4.18 and the symmetries of this Class.

Class 3

Class 3 is defined by $a = (b, b, a)$ and $b = (b, b, b)$.

If $a = (123)(b, b, a)$ and $b = (b, b, b)$ then $b = 1$ and $a^3 = (a, a, a)$. By Lemma 2.50, a has infinite order.

The other isomorphisms follow from Lemma 4.18 and the symmetries of this Class.

Class 4

There is nothing to prove other than the existence of the symmetries of this class.

Class 5

Class 5 is defined by $a = (b, a, a)$ and $b = (b, a, b)$.

(i) Let $a = (b, a, a)$ and $b = (123)(b, a, b)$. Then $a^{-(\overline{23})} = \big(b^{-(\overline{23})}, a^{-(\overline{23})}, a^{-(\overline{23})}\big)$ and $b^{-(\overline{23})} = (123)\big(a^{-(\overline{23})}, b^{-(\overline{23})}, b^{-(\overline{23})}\big)$. Let $c = a^{-(\overline{23})}$ and $d = b^{-(\overline{23})}$. Then $c = (d, c, c)$ and $d = (123)(c, d, d)$. Therefore, $\langle a, b\rangle \cong \langle c, d\rangle = H_3$.

(ii) Let $a = (123)(b, a, a)$ and $b = (132)(b, a, b)$. Then
$a^{-(\overline{13})} = (123)\big(b^{-(\overline{13})}, a^{-(\overline{13})}, a^{-(\overline{13})}\big)$ and $b^{-(\overline{13})} = (132)\big(a^{-(\overline{13})}, b^{-(\overline{13})}, b^{-(\overline{13})}\big)$.
Let $c = a^{-(\overline{13})}$ and $d = b^{-(\overline{13})}$. Then $c = (123)(d, c, c)$ and $d = (132)(c, d, d)$. Therefore, $\langle a, b\rangle \cong \langle c, d\rangle = H_4$.

(iii) Let $a = (132)(b, a, a)$ and $b = (123)(b, a, b)$. We calculate that $ab = (ab, ba, ab)$ and note that $ab = 1$ by Theorem 2.54. Thus $a = (132)(a^{-1}, a, a)$ and $a^3 = (a, a, a)$. By Lemma 2.50, a has infinite order, and so $\langle a, b\rangle \cong C_\infty$.

Class 6

Class 6 is defined by $a = (b, b, b)$ and $b = (b, b, b)$. Let σ denote the permutation (123).

(i) Let $a = (b, b, b)$ and $b = (123)(b, b, b)$. Then $b = \overline{\sigma}$ and $a = (\overline{\sigma}, \overline{\sigma}, \overline{\sigma})$. We see that $a^3 = b^3 = 1$, and that a and b commute. Thus $\langle a, b\rangle$ is a quotient of $C_3 \times C_3$. We check that $\langle a, b\rangle/\mathrm{st}(2)$ has order 9.

(ii) Let $a = (123)(b, b, b)$ and $b = (132)(b, b, b)$. Then $b = \overline{\sigma}^2$ and $a = \overline{\sigma}(\overline{\sigma}^2, \overline{\sigma}^2, \overline{\sigma}^2)$. It follows as in (i) that $\langle a, b\rangle \cong C_3 \times C_3$.

(iii) Let $a = (123)(b, b, b)$ and $b = (b, b, b)$. Then $b = 1$ and $a = \dot{\sigma}$.

Class 7

There is nothing to prove other than the existence of the symmetry of this class.

Class 8

Class 8 is defined by $a = (b, b, a)$ and $b = (a, b, b)$.

(i) Let $a = (b, b, a)$ and $b = (123)(a, b, b)$. Then $a^{-(\overline{12})} = \big(b^{-(\overline{12})}, b^{-(\overline{12})}, a^{-(\overline{12})}\big)$ and $b^{-(\overline{12})} = (123)\big(b^{-(\overline{12})}, b^{-(\overline{12})}, a^{-(\overline{12})}\big)$. Let $c = a^{-(\overline{12})}$ and $d = b^{-(\overline{12})}$. Then $c = (d, d, c)$ and $d = (123)(d, d, c)$. Therefore, $\langle a, b\rangle \cong \langle c, d\rangle = H_6$.

(ii) Let $a = (123)(b, b, a)$ and $b = (132)(a, b, b)$. Then
$a^{-(\overline{23})} = (123)\big(b^{-(\overline{23})}, b^{-(\overline{23})}, a^{-(\overline{23})}\big)$ and $b^{-(\overline{23})} = (132)\big(b^{-(\overline{23})}, b^{-(\overline{23})}, a^{-(\overline{23})}\big)$.
Let $c = a^{-(\overline{23})}$ and $d = b^{-(\overline{23})}$. Then $c = (123)(d, d, c)$ and $d = (132)(d, d, c)$. Therefore, $\langle a, b\rangle \cong \langle c, d\rangle = H_8$.

(iii) Let $a = (132)(b, b, a)$ and $b = (a, b, b)$. Then
$a^{-(\overline{132})} = (123)\big(b^{-(\overline{132})}, b^{-(\overline{132})}, a^{-(\overline{132})}\big)$ and $b^{-(\overline{132})} = \big(b^{-(\overline{132})}, b^{-(\overline{132})}, a^{-(\overline{132})}\big)$.
Let $c = a^{-(\overline{132})}$ and $d = b^{-(\overline{132})}$. Then $c = (123)(d, d, c)$ and $d = (d, d, c)$. Therefore, $\langle a, b\rangle \cong \langle c, d\rangle = H_7$.

Class 9

Class 9 is defined by $a = (b, b, b)$ and $b = (a, b, b)$.

Let $a = (b, b, b)$ and $b = (123)(a, b, b)$. We calculate that $[a, b] = ([b, a], 1, 1)$. By Theorem 2.54, $[a, b] = 1$. Therefore, $ab^3 = (b^3a, b^2ab, bab^2) = (ab^3, ab^3, ab^3) = 1$. Finally, $a^3 = (b^3, b^3, b^3) = (a^{-1}, a^{-1}, a^{-1})$, and so a has infinite order by Lemma 2.50. Thus, $\langle a, b\rangle \cong C_\infty$.

Class 10

We refer to Lemma 4.19 for the isomorphism with H_4. Nothing but the existence of the symmetries of this Class remains to be proved.

Class 11

Class 11 is defined by $a = (b, b, a)$ and $b = (a, b, a)$.

(i) Let $a = (b, b, a)$ and $b = (132)(a, b, a)$. Then $a^{-1} = (b^{-1}, b^{-1}, a^{-1})$ and $b^{-1} = (123)(a^{-1}, a^{-1}, b^{-1})$. Let $c = a^{-1}$ and $d = b^{-1}$. Then $c = (d, d, c)$ and $d = (123)(c, c, d)$. Thus, $\langle a, b\rangle \cong \langle c, d\rangle = H_{14}$.

(ii) Let $a = (123)(b, b, a)$ and $b = (132)(a, b, a)$. We see that $ab = (ba, ab, ba)$ and, by Theorem 2.54, that $ab = 1$. It follows that $a^3 = (bab, ab^2, b^2a) = (a^{-1}, a^{-1}, a^{-1})$. Thus a has infinite order by Lemma 2.50. Hence $\langle a, b\rangle \cong C_\infty$.

(iii) Let $a = (132)(b, b, a)$ and $b = (123)(a, b, a)$. Then
$a^{-(\overline{132})} = (123)\big(b^{-(\overline{132})}, b^{-(\overline{132})}, a^{-(\overline{132})}\big)$ and
$b^{-(\overline{132})} = (132)\big(a^{-(\overline{132})}, a^{-(\overline{132})}, b^{-(\overline{132})}\big)$.
Let $c = a^{-(\overline{132})}$ and $d = b^{-(\overline{132})}$. Then $c = (123)(d, d, c)$ and $d = (132)(c, c, d)$.
Hence $\langle a, b\rangle \cong \langle c, d\rangle = H_4$.

Class 12

Class 12 is defined by $a = (b, b, b)$ and $b = (a, a, b)$.

Let $a = (b, b, b)$ and $b = (123)(a, a, b)$. We see that $[a, b] = ([b, a], [b, a], 1)$ and, by Theorem 2.54, that $[a, b] = 1$. Therefore, $a^2b^3 = (b^2aba, b^3a^2, b^2a^2b) = (a^2b^3, a^2b^3, a^2b^3)$ and $a^2b^3 = 1$ by Lemma 2.50. It follows that $a = (ab)^3$ and $b = (ab)^{-2}$. Hence $\langle a, b\rangle$ cyclic, generated by the element ab. Observe that $a^3 = (b^3, b^3, b^3) = (a^{-2}, a^{-2}, a^{-2})$ and hence that a has infinite order by Lemma 2.50. We have shown that $\langle a, b\rangle$ is infinite and cyclic.

Class 13

Class 13 is defined by $a = (b, b, b)$ and $b = (a, a, a)$.

(i) Let $a = (b, b, b)$ and $b = (123)(a, a, a)$. Observe that $[a, b] = ([b, a], [b, a], [b, a])$ is equal to 1 by Theorem 2.54. As $a^3 = (b^3, b^3, b^3)$ and $b^3 = (a^3, a^3, a^3)$ we see that $a^3 = b^3 = 1$. Thus $C_3 \times C_3$ maps onto $\langle a, b\rangle$. We check that $\langle a, b\rangle/\mathrm{st}(2)$ has order 9.

(ii) Let $a = (123)(b, b, b)$ and $b = (132)(a, a, a)$. Then $ab = (ba, ba, ba)$ is equal to 1 by Theorem 2.54. Thus $a^3 = (a^{-3}, a^{-3}, a^{-3}) = 1$ and $\langle a, b\rangle \cong C_3$.

Appendix B

Computer Programs

B.1 GAP

The AutomGrp package must be loaded into GAP in order for the commands in this section to work. We use the package AutomGrp 1.0. Other versions of AutomGrp have slightly different commands, which may be found in the online help file. In the exposition below, our input commands are prefixed by `gap>`. The machine's output has been copied and pasted under the input.

We begin by defining an automaton group. Recall from Definition 4.17 that the generators of H_9 are defined by the equations $a = (b, b, a), b = (132)(a, b, b)$. We define this group in GAP with the command

```
gap> H9:=AutomatonGroup("a=(b,b,a),b=(b,a,b)(1,3,2)");
```

Note that the permutation (132) has been placed on the right, and that (a, b, b) has been replaced with (b, a, b). By Remark 2.27, $(132)(a, b, b) = (b, a, b)(132)$. Therefore, the same tree automorphisms a and b have been defined.

We can ask GAP to find relations in the generators a and b with the command `FindGroupRelations`. This runs a program which does not, in general, terminate. The program can be forced to quit by pressing Ctrl-C.

```
gap> FindGroupRelations(H9);
b*a^-1*b*a^-1*b*a^-1
b*a*b^-1*a^-1*b*a*b^-1*a^-1*b*a*b^-1*a^-1
```

Thus GAP has found the relations $(ba^{-1})^3$ and $(bab^{-1}a^{-1})^3$ in H_9.

GAP labels the vertices of level n in the tree $T_{\underline{d}}$ left-to-right by $1, 2, \ldots, d^n$. Thus the vertex $\nu_{(i_1,\ldots,i_n)}$ is identified with $d^{n-1}(i_1-1)+d^{n-2}(i_2-1)+\ldots+d(i_{n-1}-1)+i_{n-1}$. GAP can calculate the action of a given element of a d-ary automaton on the vertices of any given level of $T_{\underline{d}}$. As an example, we calculate the action of $a^2ba^{-1} \in H_9$ on the second level of $T_{\underline{3}}$.

```
gap> PermOnLevel(a^2*b*a^-1,2);
(1,7,4,3,9,6,2,8,5)
```

GAP can calculate the expansion of a given element of a d-ary automaton to any given level of $T_{\underline{d}}$. As an example, we calculate the expansion of $a^2ba^{-1} \in H_9$ to the second level of $T_{\underline{3}}$.

```
gap> Decompose(a^2*b*a^-1,2);
(b^2*a*b^-1, a*b, b*a*b*a^-1, b^2, a*b*a*b^-1, b*a*b*a^-1, b^2, b^2,
a^2)(1,7,4,3,9,6,2,8,5)
```

The `Order` command runs a program, which may not terminate, to find the order of a given element of an automaton group. For example, we calculate the orders of the elements $[a, b]$ and $[a, b^2]$ of H_9.

```
gap> Order(a^-1*b^-1*a*b);
3
gap> Order(a^-1*b^-2*a*b^2);
infinity
```

We use the command `SetInfoLevel` in order to obtain the details of the argument that has been used to prove that $[a, b^2]$ has infinite order.

```
gap> SetInfoLevel(InfoAutomGrp,3);
gap> Order(a^-1*b^-2*a*b^2);
#I  (a^-1*b^-2*a*b^2)^9 has b^-1*a^-1*b^-1*a^2*b^-2*a*b^2*a^-1*b^-1*a^
-1*b*a^-1*b^2*a as a section at vertex [ 1, 1, 1, 1, 1 ]
#I  (b^-1*a^-1*b^-1*a^2*b^-2*a*b^2*a^-1*b^-1*a^-1*b*a^-1*b^2*a)^
3 has conjugate of a^-1*b^-2*a*b^-1*a*b*a*b^-2*a^-1*b^2*a^
-2*b*a*b as a section at vertex [ 1, 1, 1 ]
infinity
```

The output informs us that $[a, b^2]^9$ stabilizes the vertex $\nu_{(1,1,1,1,1)}$, and that this vertex is labelled by $b^{-1}a^{-1}b^{-1}a^2b^{-2}ab^2a^{-1}b^{-1}a^{-1}ba^{-1}b^2a$ in the expansion of $[a, b^2]^9$ to the fifth level. Let $\beta = b^{-1}a^{-1}b^{-1}a^2b^{-2}ab^2a^{-1}b^{-1}a^{-1}ba^{-1}b^2a$. The output informs us that β^3 stabilizes the vertex $\nu_{(1,1,1)}$, and that this vertex is labelled by a conjugate of $\beta^{-1} = a^{-1}b^{-2}ab^{-1}abab^{-2}a^{-1}b^2a^{-2}bab$ in the expansion of β^3 to the third level. The argument that β has infinite order is similar to the argument given in the proof of Lemma 2.50. It follows immediately that $[a, b^2]$ has infinite order.

Note that β is the element of H_9 that was used in Lemma 5.17 to prove that $H_9^{'}$ is not locally finite. As remarked before the statement of that lemma, we list here the details of our search for elements of infinite order in $H_{10}^{'}$ and $H_{11}^{'}$.

```
gap> H10:=AutomatonGroup("a=(b,a,b)(1,2,3),b=(a,b,b)");
< a, b >
gap> Order(a^-1*b^-1*a*b);
3
gap> Order(a^-1*b^-2*a*b^2);
#I  (a^-1*b^-2*a*b^2)^3 has a^-2*b*a*b^-2*a*b as a section at vertex
[ 1, 1 ]
#I  (a^-2*b*a*b^-2*a*b)^3 has conjugate of b*a^-1*b^-1*a^2*b^-1*a^
-1*b as a section at vertex [ 1, 1 ]
infinity
gap> H11:=AutomatonGroup("a=(a,b,b)(1,3,2),b=(b,b,a)(1,2,3)");
< a, b >
gap> Order(a^-1*b^-1*a*b);
3
gap> Order(a^-1*b^-2*a*b^2);
3
gap> Order(a^-2*b^-2*a^2*b^2);
#I  (a^-2*b^-2*a^2*b^2)^9 has b^-2*a^-2*b^2*a^-2*b^4*a^-2*b^-2*a^3*b^
-2*a*b^-2*a*b^2*a as a section at vertex [ 1, 1, 1, 1, 1 ]
#I  (b^-2*a^-2*b^2*a^-2*b^4*a^-2*b^-2*a^3*b^-2*a*b^-2*a*b^2*a)^
3 has conjugate of a^-1*b^-2*a^-1*b^2*a^-1*b^2*a^-3*b^2*a^2*b^-4*a^
2*b^-2*a^2*b^2 as a section at vertex [ 1, 1, 1 ]
infinity
```

B.2 Magma

We present the details of the calculations in Lemmas 5.37, 5.42 and 5.3. We freely use GAP to aid us, the commands having been explained in the previous section. In the exposition below, the input commands in Magma are prefixed by `>`, the input commands in GAP are prefixed `gap>`, and the machine's output has been copied and pasted below the input.

B.2.1 Calculations for Lemma 5.37

We calculate the action of the generators of H_9 on the fourth level of $T_{\underline{3}}$ using GAP.

```
gap> H9:=AutomatonGroup("a=(b,b,a),b=(b,a,b)(1,3,2)");
< a, b >
gap> PermOnLevel(a,4);
(1,27,14,4,19,18,9,23,11)(2,25,15,5,20,16,7,24,12)(3,26,13,6,21,17,
8,22,10)(28,54,41,31,46,45,36,50,38)(29,52,42,32,47,43,34,51,39)
(30,53,40,33,48,44,35,49,37)(55,63,59,56,61,60,57,62,58)(64,72,68,
65,70,69,66,71,67)(73,75,74)(76,78,77)
gap> PermOnLevel(b,4);
(1,81,41,11,55,54,27,68,31)(2,79,42,12,56,52,25,69,32)(3,80,40,10,
57,53,26,67,33)(4,73,45,14,58,46,21,71,35)(5,74,43,15,59,47,19,72,
36)(6,75,44,13,60,48,20,70,34)(7,78,39,17,62,49,24,66,29)(8,76,37,
18,63,50,22,64,30)(9,77,38,16,61,51,23,65,28)
```

Using Magma, we define the subgroup of Sym(81) generated by these two permutations, thus obtaining a group isomorphic to $H_9/\mathrm{st}(4)$.

```
> H9:=PermutationGroup<81|
> (1,27,14,4,19,18,9,23,11)(2,25,15,5,20,16,7,24,12)(3,26,13,6,21,17,
> 8,22,10)(28,54,41,31,46,45,36,50,38)(29,52,42,32,47,43,34,51,39)
> (30,53,40,33,48,44,35,49,37)(55,63,59,56,61,60,57,62,58)(64,72,68,
> 65,70,69,66,71,67)(73,75,74)(76,78,77),
> (1,81,41,11,55,54,27,68,31)(2,79,42,12,56,52,25,69,32)(3,80,40,10,
> 57,53,26,67,33)(4,73,45,14,58,46,21,71,35)(5,74,43,15,59,47,19,72,
> 36)(6,75,44,13,60,48,20,70,34)(7,78,39,17,62,49,24,66,29)(8,76,37,
> 18,63,50,22,64,30)(9,77,38,16,61,51,23,65,28)>;
```

We use the command `PCGroup` in order to find a presentation for $H_9/\mathrm{st}(4)$. This creates a so-called "power-conjugate" presentation, which allows for very efficient calculation in p-groups (recall from Remark 5.1 that $H_i/\mathrm{st}(n)$ is a 3-group for all $n \in \mathbb{N}$ and all $i \in [1, 17]$).

```
> H9pc:=PCGroup(H9);
```

The presentation of $H_9/\mathrm{st}(4)$ is stored in the computer's memory. Should we wish to see the presentation, we input

```
> H9pc;
```

The output is extremely long and there is no need to reproduce the presentation of $H_9/\mathrm{st}(4)$ here. We use the command `AbelianQuotient` to calculate the abelianization of $H_9/\mathrm{st}(4)$.

```
> AbelianQuotient(H9pc);
Abelian Group isomorphic to Z/3 + Z/9
Defined on 2 generators
Relations:
    3*$.1 = 0
    9*$.2 = 0
```

For completeness, we list the details for H_{10} and H_{11}.

```
gap> H10:=AutomatonGroup("a=(b,a,b)(1,2,3),b=(a,b,b)");
< a, b >
gap> PermOnLevel(a,4);PermOnLevel(b,4);
(1,31,67,13,40,80,26,53,62,2,32,68,14,41,81,27,54,63,3,33,69,15,
42,79,25,52,61)(4,35,71,17,44,74,21,46,56,5,36,72,18,45,75,19,47,
57,6,34,70,16,43,73,20,48,55)(7,28,65,12,37,76,22,49,58,8,29,66,
10,38,77,23,50,59,9,30,64,11,39,78,24,51,60)
(1,11,23,5,14,27,9,18,21)(2,12,24,6,15,25,7,16,19)(3,10,22,4,13,
26,8,17,20)(28,31,35,29,32,36,30,33,34)(37,38,39)(46,47,48)(55,58,
62,56,59,63,57,60,61)(64,65,66)(73,74,75)
> H10:=PermutationGroup<81|
> (1,31,67,13,40,80,26,53,62,2,32,68,14,41,81,27,54,63,3,33,69,15,
> 42,79,25,52,61)(4,35,71,17,44,74,21,46,56,5,36,72,18,45,75,19,47,
> 57,6,34,70,16,43,73,20,48,55)(7,28,65,12,37,76,22,49,58,8,29,66,
> 10,38,77,23,50,59,9,30,64,11,39,78,24,51,60),
> (1,11,23,5,14,27,9,18,21)(2,12,24,6,15,25,7,16,19)(3,10,22,4,13,
> 26,8,17,20)(28,31,35,29,32,36,30,33,34)(37,38,39)(46,47,48)(55,58,
> 62,56,59,63,57,60,61)(64,65,66)(73,74,75)>;
> H10pc:=PCGroup(H10);
> AbelianQuotient(H10pc);
Abelian Group isomorphic to Z/3 + Z/9
Defined on 2 generators
Relations:
```

```
    3*$.1 = 0
    9*$.2 = 0
gap> H11:=AutomatonGroup("a=(a,b,b)(1,3,2),b=(b,b,a)(1,2,3)");
< a, b >
gap> PermOnLevel(a,4);PermOnLevel(b,4);
(1,81,31,17,55,41,27,65,51)(2,79,32,18,56,42,25,66,49)(3,80,33,16,
57,40,26,64,50)(4,74,34,12,58,44,19,68,54)(5,75,35,10,59,45,20,69,
52)(6,73,36,11,60,43,21,67,53)(7,77,30,13,62,37,23,72,47)(8,78,28,
14,63,38,24,70,48)(9,76,29,15,61,39,22,71,46)
(1,41,81,11,51,55,27,31,71)(2,42,79,12,49,56,25,32,72)(3,40,80,10,
50,57,26,33,70)(4,44,73,14,54,58,20,34,66)(5,45,74,15,52,59,21,35,
64)(6,43,75,13,53,60,19,36,65)(7,39,76,17,46,63,22,29,69)(8,37,77,
18,47,61,23,30,67)(9,38,78,16,48,62,24,28,68)
> H11:=PermutationGroup<81|
> (1,81,31,17,55,41,27,65,51)(2,79,32,18,56,42,25,66,49)(3,80,33,16,
> 57,40,26,64,50)(4,74,34,12,58,44,19,68,54)(5,75,35,10,59,45,20,69,
> 52)(6,73,36,11,60,43,21,67,53)(7,77,30,13,62,37,23,72,47)(8,78,28,
> 14,63,38,24,70,48)(9,76,29,15,61,39,22,71,46),
> (1,41,81,11,51,55,27,31,71)(2,42,79,12,49,56,25,32,72)(3,40,80,10,
> 50,57,26,33,70)(4,44,73,14,54,58,20,34,66)(5,45,74,15,52,59,21,35,
> 64)(6,43,75,13,53,60,19,36,65)(7,39,76,17,46,63,22,29,69)(8,37,77,
> 18,47,61,23,30,67)(9,38,78,16,48,62,24,28,68)>;
> H11pc:=PCGroup(H11);
> AbelianQuotient(H11pc);
Abelian Group isomorphic to Z/3 + Z/9
Defined on 2 generators
Relations:
    3*$.1 = 0
    9*$.2 = 0
```

B.2.2 Calculations for Lemma 5.42

Define G to be $C_3 * C_\infty$. A presentation for G is $\langle x, y \mid x^3 = 1\rangle$. We input this presentation into Magma using the command

```
> G:=Group<x,y|x^3=1>;
```

For a given prime p, the quotient G/P_n may be calculated with the command `pQuotient(G,p,n-1)`. We take $p = 3$ and $n = 4$, obtaining a presentation for G/P_4.

```
> pQuotient(G,3,3);
GrpPC of order 2187 = 3^7
PC-Relations:
    $.2^3 = $.4,
    $.4^3 = $.7,
    $.2^$.1 = $.2 * $.3,
    $.3^$.1 = $.3 * $.5,
    $.3^$.2 = $.3 * $.6
```

The output is a so-called 'power-conjugate' presentation of G. The details of what this means are unimportant here. We offer a very brief explanation: there are seven generators (because the group has order 3^7); the generators are labelled $\$.1, \ldots, \$.7$; the generator $\$.i$ has order 3 unless it is explicitly stated that $(\$.i)^3 = \$.j$; the generators $\$.i$ and $\$.j$ commute unless $(\$.i)^{\$.j}$ is explicitly expressed as a word on the generators.

We define `Gp` to be G/P_4, and calculate its order (note that we already know its order from the power-conjugate presentation given above).

```
> Gp:=pQuotient(G,3,3);
> Order(Gp);
2187
```

Let X denote the set of normal subgroups of G/P_4. We calculate the set X and its cardinality.

```
> X:=NormalSubgroups(Gp);
> #X;
68
```

The elements of X are labelled X[j] for $j \in [1, 68]$, and may be accessed with the command `X[j]‘subgroup`. For example, X[20] is a normal subgroup of G/P_4 isomorphic to $C_3 \times C_3$:

```
> X[20]‘subgroup;
GrpPC of order 9 = 3^2
PC-Relations:
    $.1^3 = Id($),
    $.2^3 = Id($)
```

We find the elements of X that have order 3.

```
> for j in [1..68] do
for> if(Order(X[j]‘subgroup) eq 3) then j;
for|if> end if;end for;
2
3
4
5
6
7
8
9
10
11
12
13
14
```

That is, X[j] has order 3 if and only if $j \in [2, 14]$.

The `IdentifyGroup` command will establish an isomorphism, should one exist, between a given finite group and a group in Magma's database of small groups. In particular, Magma has a database of all groups of order 729. There are 504 groups of order 729. These are labelled $< 729,\ 1 >, \ldots, < 729,\ 504 >$ respectively. They may be accessed with the command `SmallGroup(729,j)`. For example,

```
> SmallGroup(729,11);
GrpPC of order 729 = 3^6
PC-Relations:
    $.1^3 = $.4,
    $.2^3 = $.5^2,
    $.2^$.1 = $.2 * $.3,
    $.3^$.1 = $.3 * $.5,
    $.3^$.2 = $.3 * $.6
```

We calculate the isomorphism type of every quotient of G/P_4 of order 729 with the following command.

```
> for j in [2..14] do IdentifyGroup(Gp/X[j]`subgroup); end for;
<729, 13>
<729, 4>
<729, 7>
<729, 7>
<729, 7>
<729, 9>
<729, 19>
<729, 16>
<729, 13>
<729, 13>
<729, 13>
<729, 13>
<729, 13>
```

As in Subsection B.2.1, we use GAP to find the action of the generators of H_3 and H_{14} on the fifth level of $T_{\underline{3}}$. These permutations are used in order to define subgroups H3 and H14 of Sym(243) in Magma. We use the `PCGroup` command in Magma to obtain groups H3pc and H14pc, isomorphic to $H_3/\mathrm{st}(5)$ and $H_{14}/\mathrm{st}(5)$ respectively, and given by power-conjugate presentations. In order to avoid tedious repetition, the details of these calculations have been omitted. Note that, on today's machines, the program to create the groups H3pc and H14pc takes several minutes to run.

```
> pQuotient(H3pc,3,3);
GrpPC of order 729 = 3^6
PC-Relations:
    $.1^3 = $.4,
    $.2^3 = $.4 * $.5 * $.6^2,
    $.2^$.1 = $.2 * $.3,
    $.3^$.1 = $.3 * $.5,
    $.3^$.2 = $.3 * $.6
> IdentifyGroup(pQuotient(H3pc,3,3));
<729, 11>
> pQuotient(H14pc,3,3);
GrpPC of order 729 = 3^6
PC-Relations:
    $.1^3 = $.4,
```

```
    $.2^3 = $.4 * $.5 * $.6^2,
    $.2^$.1 = $.2 * $.3,
    $.3^$.1 = $.3 * $.5,
    $.3^$.2 = $.3 * $.6
> IdentifyGroup(pQuotient(H14pc,3,3));
<729, 11>
```

Observe that $< 729,\ 11 >$ does not appear on the above list of quotients of G/P_4. We have therefore shown that $\big(H_3/\mathrm{st}(5)\big)/P_4$ and $\big(H_{14}/\mathrm{st}(5)\big)/P_4$ are not quotients of $(C_3 * C_\infty)/P_4$, as required.

B.2.3 Calculations for Lemma 5.3

Recall that the generators of H_1 are defined by the equations $a = (b, a, a)$ and $b = (123)(b, b, b)$. We input H_1 into GAP and calculate the actions of its generators on the second level.

```
gap> H1:=AutomatonGroup("a=(b,a,a),b=(b,b,b)(1,2,3)");
< a, b >
gap> PermOnLevel(a,2);PermOnLevel(b,2);
(1,2,3)
(1,5,9)(2,6,7)(3,4,8)
```

Thus, the actions of a and b on the nine vertices of the second level are as the permutations (123) and (159)(267)(348), respectively. Using Magma, we define `H` to be the subgroup of Sym(9) generated by these two permutations, and calculate its order.

```
> H:=PermutationGroup<9|(1,2,3),(1,5,9)(2,6,7)(3,4,8)>;
> Order(H);
81
```

By construction, `H` is naturally isomorphic to the subgroup $H_1/\mathrm{st}(2)$ of $C_3 \wr C_3$. As $C_3 \wr C_3$ and `H` both have order 81, it follows that $H_1/\mathrm{st}(2) = C_3 \wr C_3$, as required.

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