Categorical Symmetries in Quantum Field Theories

Lea E. Bottini
St Anne's College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2025

A Ombri

Abstract

In recent years, the notion of symmetry has greatly expanded, leading to generalised symmetries such as higher-form symmetries, which measure charges of extended operators, and non-invertible symmetries, whose symmetry generators do not have inverses. These are also known as categorical symmetries. This thesis focuses on non-invertible/categorical symmetries, exploring both their abstract mathematical structure and their realisations in physical systems.

The first part develops new methods for constructing such symmetries in spacetime dimensions three and higher. We present a general procedure starting from an invertible symmetry and gauging a specific subgroup of it to produce a non-invertible symmetry. This construction is applicable to many gauge theories in any dimension. Specialising to $(2+1)$d theories, we study the effect of gauging all possible subgroups of an ordinary symmetry, generating a symmetry web of related theories.

The second part examines the physical consequences of non-invertible symmetries, asking whether they can organise and classify phases of matter analogously to what ordinary symmetries do in the celebrated Landau symmetry-breaking theory. We introduce a framework to classify all the possible gapped phases for $(1+1)$d systems with a given non-invertible symmetry. The main tool we use is the so-called the symmetry topological field theory (SymTFT), which also tells us what kinds of order parameters exist in a phase. We further extend this to construct phase transitions with non-invertible symmetries by inputting known phase transitions, such as the critical Ising model. This outlines a broadened version of the classic Landau theory, which we call the categorical Landau paradigm. We also demonstrate its possible realisation in quantum lattice models with a concrete example.

Finally, we apply the techniques developed in the rest of this thesis to tackle a long-standing open problem: finding a conformal field theory (CFT) with the unusual Haagerup symmetry.

Statement of Originality

This thesis is based on results from the following publications, to which the author contributed substantially:

[1] L. Bhardwaj, L. E. Bottini, S. Schafer-Nameki, and A. Tiwari, "**Non-Invertible Higher-Categorical Symmetries**", SciPost Phys. 14 (2023) 007, arXiv:2204.06564.

[2] L. Bhardwaj, L. E. Bottini, S. Schafer-Nameki, and A. Tiwari, "**Non-invertible symmetry webs**", SciPost Phys. 15 no. 4, (2023) 160, arXiv:2212.06842.

[3] L. Bhardwaj, L. E. Bottini, D. Pajer, and S. Schafer-Nameki, "**Categorical Landau Paradigm for Gapped Phases**", Phys. Rev. Lett. 133 no. 16, (2024) 161601, arXiv:2310.03786.

[4] L. Bhardwaj, L. E. Bottini, D. Pajer, and S. Schafer-Nameki, "**Gapped phases with non-invertible symmetries: (1+1)d**", SciPost Phys. 18 no. 1, (2025) 032, arXiv:2310.03784.

[5] L. Bhardwaj, L. E. Bottini, D. Pajer, and S. Schafer-Nameki, "**The Club Sandwich: Gapless Phases and Phase Transitions with Non-Invertible Symmetries**", SciPost Phys. 18 (2025) 156, arXiv:2312.17322.

[6] L. Bhardwaj, L. E. Bottini, S. Schafer-Nameki, and A. Tiwari, "**Illustrating the categorical Landau paradigm in lattice models**", Phys. Rev. B 111 no. 5, (2025) 054432, arXiv:2405.05302.

[7] L. E. Bottini and S. Schafer-Nameki, "**Construction of a Gapless Phase with Haagerup Symmetry**", Phys. Rev. Lett. 134 no. 19, (2025) 191602, arXiv:2410.19040.

During the DPhil, the author also contributed to [8–10], which are not included in this thesis due to space constraints.

Statement of Authorship

The work included in this thesis is the result of a collaboration with other researchers, as outlined in the list of papers above. All collaborators contributed substantially to the development of the ideas and the execution of computations, as well as the actual writing of the manuscripts. While in some cases individual authors may have contributed more to the conceptual development and others more to the concrete computations, the overall workload was split equitably among the co-authors in all the papers [1–7] on which this thesis is based. In particular, I participated in all discussions that shaped the original ideas of the papers, contributed to the formulation of arguments, and carried out a significant share of the concrete computations supporting the results.

Acknowledgements

I would like to start by thanking my supervisor Sakura Schäfer-Nameki for the constant support and encouragement she gave me during my DPhil. In particular, I am very grateful to her for introducing me right from the start to some of the most exciting topics in theoretical physics, for always pushing me to do better, guiding me through all the steps towards becoming an independent researcher, and caring so much about me.

I would also like to thank all the other people I have had the pleasure to collaborate with during my DPhil, in particular Lakshya Bhardwaj, Chiung Hwang, Nick Jones, Daniel Pajer, Sara Pasquetti, Matteo Sacchi and Apoorv Tiwari. I have learned a lot from all of you. I am particularly indebted to Sara Pasquetti for the affection and advice she has always so generously given to me. Thank you also to Paul Fendley and Po-Shen Hsin for agreeing to examine this thesis.

I am also extremely grateful for the wonderful and supporting atmosphere that there is in Oxford between faculty, DPhil students and postdocs, and for all the amazing people with whom I got to share these four years (too many to name!). Thanks for all the happy hours in the common room, pizzas on Thursdays at the Rickety Press, parties in Summertown, cineforums, games of Scopone, and all the memories I will look back on fondly.

In particular, thanks to Pietro Ferrero for being the best office mate in my first year and all the advice since. To Dewi Gould for the hikes in Colorado and the good chats. To Enrico Marchetto for the Italian indie and the Sunday morning radical chic matinée. To Davide De Grandi for the all the climbing and our home in Iffley Road. To Beppe Bogna for being the best friend I could have hoped for, inside and outside the office.

Finally, thank you to all the people that supported me from afar. First of all, my friends in Milano, for patiently trying to understand what I do and for making me feel like I have never left every time we see each other. To my brother Gillo, infallible IT support, climbing coach, and partner in many adventures and mischiefs. To my parents for their sometimes eccentric upbringing, their full love and the values they passed on to me.

And thanks to Nico, for simply being here.

Contents

Chapter 1

Introduction

Symmetries are a foundational concept in modern theoretical physics. They allow us to gain control over complex physical systems by revealing conserved quantities, as formalised by Noether's theorem [11], and they provide powerful tools to access the non-perturbative regime of theories. Not only they are essential in constraining the infrared (IR) dynamics, they are also a fundamental guiding principle to organise phases of matter and predict novel states. Given their crucial role, it is important to ask whether our traditional understanding of symmetries – typically associated with group actions on local operators – captures the full richness of symmetry structures that we can have in quantum field theory (QFT). The recent years have proven that the answer to this question is a clear "no". Indeed, we have witnessed a series of breakthroughs that revolutionised the concept of symmetry and led to what are broadly known as *generalised symmetries*. These have proven an excellent tool to reinterpret already established concepts in a unified perspective and to shed light on new phenomena. For reviews on generalised symmetries, see [12–18].

This radical broadening of the notion of symmetry began from the seminal paper [19], which introduced the so-called *higher-form* symmetries. In contrast to ordinary – or "0-form" – symmetries, which act on point-like local operators, a p-form symmetry acts on p-dimensional extended operators, such as Wilson lines (for $p = 1$) or surface operators (for $p = 2$). These have crucially provided new insights into the dynamics of gauge theories and their phases, notably in relation to dualities and confinement – see e.g. [19–21]. From a condensed matter perspective, they have been used e.g. to enlarge the standard Landau paradigm of spontaneous symmetry breaking (SSB) to include topological order, which can be seen as SSB of a higher-form symmetry [22].

Not long after the introduction of higher-form symmetries, a new generalisation was made to *higher-group* symmetries [23–30], which combine together higher-form symmetries of different degrees into a cohesive mathematical structure known as a higher-group. The simplest example is a 2-group, where a 0-form symmetry is intertwined with a 1-form symmetry.

More recently, a perhaps even more surprising notion was introduced: symmetries that do not correspond to a group action at all, and that have been named *non-invertible* symmetries. Despite having been largely studied in lower space-time dimensions (e.g. as Verlinde lines in 2d conformal field theories (CFTs), or non-abelian anyons in 3d topological quantum phases of matter – see for example [31–56] for a certainly incomplete list of relevant works), their presence in $d \geq 4$ was somehow unexpected. Nevertheless, an array of recent work has demonstrated the ubiquity of non-invertible symmetries in 4d (and higher) QFTs, with numerous examples found using various constructions, as initiated in [1, 57–61]. This indicates that non-invertible symmetries are not a special feature of low-dimensional models, but rather a general and until recently unappreciated feature of QFT.

What unifies all of these notions is a shift in perspective: we should really think of symmetries as generated by topological defects in a theory, which we call *symmetry defects*.[1] This notion is particularly useful for discrete symmetries, which do not have an associated conserved current, but admit a description in terms of the corresponding topological defects. For example, a standard 0-form-symmetry G is implemented by

[1] In this thesis, we will be mostly concerned with relativistic quantum field theories formulated in Euclidian space-time. Therefore the distinction between an operator, placed at a given time, and a defect, stretched along the time direction, is not sharp, and we will use the two terms interchangeably.

codimension-1 defects $D^{(g)}(M_{d-1})$, supported on a M_{d-1} manifold and labelled by a group element $g \in G$. The statement that G is a symmetry – normally described in terms of a unitary operator commuting with the Hamiltonian – becomes that $D^{(g)}(M_{d-1})$ is insensitive to small deformations of M_{d-1}, as long as a charged operator is not crossed in the process; in other words, the dependence of $D^{(g)}(M_{d-1})$ on M_{d-1} is topological. The charge of a local operator under G is measured by inserting the topological defect $D^{(g)}$ along a $(d-1)$-dimensional sphere S^{d-1} surrounding the space-time point where the local operator is placed and subsequently shrinking the topological defect away.[2]

The properties of G are encoded in the corresponding symmetry defects, and notably the group structure corresponds to their *fusion*: if we take two defects parallel and bring them together, they fuse according to the group multiplication law of G

$$D^{(g)}(M_{d-1}) \otimes D^{(h)}(M_{d-1}) = D^{(gh)}(M_{d-1}) \qquad g, h, gh \in G \,. \tag{1.2}$$

A p-form symmetry $\Gamma^{(p)}$ is implemented by an associated symmetry defect which now has codimension $p+1$

$$D^{(\gamma)}(M_{d-p-1}) \qquad \gamma \in \Gamma^{(p)} \,. \tag{1.3}$$

Correspondingly, if a 0-form symmetry measures the charge of a local operator by linking it with a $(d-1)$-dimensional sphere S^{d-1}, the symmetry defect $D^{(\gamma)}(S^{d-p-1})$ for a p-form symmetry supported on a $(d-p-1)$-dimensional sphere S^{d-p-1} naturally links with – and therefore measures the charge of – a p-dimensional extended object.

Also the notion of higher-group symmetry has a neat interpretation in terms of

[2]For a continuous symmetry G, with Lie algebra generator t^a and associated conserved current j^a_μ, and for a local operator transforming in the representation $\mathcal{R}$ under G, this corresponds to the standard Ward Identity

$$\partial^\mu j^a_\mu(x)\mathcal{O}_{\mathcal{R}}(y) = \delta^{(d)}(x-y)t^a_{\mathcal{R}}\mathcal{O}_{\mathcal{R}}(y) \,. \tag{1.1}$$

symmetry defects [25, 27]. For example, a very simple instance of a 2-group arises when a 0-form symmetry G acts by permutation on the set of topological defects associated to a 1-form symmetry $\Gamma^{(1)}$. The symmetry defects $D^{(g)}(M_{d-1})$ for the 0-form symmetry are codimension 1, while the ones $U^{(\gamma)}(M_{d-2})$ for the 1-form symmetry are codimension 2. In terms of these, the 2-group means that when a $U^{(\gamma)}(M_{d-2})$ crosses $D^{(g)}(M_{d-1})$, it emerges transformed as a new element $U^{(\rho_g\cdot\gamma)}(M_{d-2})$, where $\rho_g \cdot \gamma$ denotes the permutation of the 1-form symmetry label γ associated to g.

Now that we have introduced symmetry defects, we can also sharpen what we mean by a non-invertible symmetry: it is a symmetry for which the associated $D^{(a)}(M_n)$ do not obey fusion rules based on a group, but rather a more complicated fusion algebra of the form

$$D^{(a)}(M_n) \otimes D^{(b)}(M_n) = \bigoplus_c N_{ab}^c \, D^{(c)}(M_n)\,. \tag{1.4}$$

Since two topological defects generally fuse into the sum of many others, it can happen that an element $D^{(a)}$ has no inverse $D^{(a^{-1})}$ such that their fusion is the identity operator $D^{(\mathrm{id})}$, hence the name non-invertible symmetries. Notice that in particular these symmetries are not associated to a unitary operator.

The equation (1.4) extends to general space-time dimensions the structure exhibited in 2d by topological defect lines, which form what is known as a *fusion category*. See e.g. [62] for an introduction to the subject. This suggests that ultimately the structure that should describe all the topological defects of various codimension in a d-dimensional theory is a *higher fusion category*. More precisely, a standard d-dimensional theory has topological defects of dimension up to $d-1$, and therefore the associated symmetry category should be a $(d-1)$-category.[3] This has indeed d

[3] Differently from fusion categories, higher fusion categories are a topic still in development in the mathematical literature – with the partial exception of 2-categories, see e.g. [63–66]–which means that in general it is a difficult task to fully assign a $(d-1)$-category to a determined d-dimensional theory. However, a lot on the structure of these categories can be inferred from a bottom up approach, which constructs them using a physical theory and its set of topological defects.

levels, with objects (or 0-morphisms) corresponding to $(d-1)$ topological defects, 1-morphisms corresponding to $(d-2)$ topological defects, and so on, up to $(d-1)$-morphisms corresponding to topological local operators.

While the discovery of mathematical structures within familiar QFTs is of intrinsic interest, the search for categorical symmetries would be of limited physical relevance if it remained a purely formal construction, with no explicit application. After all, as we remarked at the beginning of this introduction, we care so much about symmetries because of the constraints they provide to study the dynamics of a theory, which are exact and non-perturbative, and therefore extremely powerful.

So far, non-invertible symmetries have seen a fruitful application mostly in 2d, in particular in relation to renormalisation group (RG) flows and anomalies, which provide a robust invariant under such RG flows – see for example [4,44,45,53,54,67–69]. It follows from the famous 't Hooft anomaly matching argument [70] that a theory with an 't Hooft anomaly cannot flow to a trivially gapped phase with a single vacuum, as the anomaly has to be matched from the ultraviolet (UV) to the IR: the only options are either a gapless phase (described by some CFT) or a non-trivial gapped phase (described by some topological field theory (TQFT)). For a finite symmetry described by a group G, anomalies in 2d are classified by the cohomology group $H^3(G, U(1))$, with the non-anomalous case corresponding to the trivial element. For symmetries described by a fusion category, an appropriate generalisation is provided in terms of the F-symbols, which relate different but equivalent ways of fusing three lines into a fourth one.[4] It turns out some non-invertible symmetries are intrinsically anomalous. Therefore, if we study an RG flow triggered by a relevant operator which commutes with the non-invertible symmetry, we know that we cannot land on a trivially gapped

[4]F-symbols need to satisfy the famous consistency condition known as pentagon identity. For a given set of fusion rules, the possible solutions to the pentagon identity are finite, something known as Ocneanu's rigidity. The fact that the F-symbols do not have a continuous dependence, but rather come in discrete classes, suggests that they cannot deform continuously under RG flow, i.e. the corresponding anomaly is indeed an RG invariant.

phase. Importantly for RG flows to a massive phase, the possible gapped phases with a non-invertible symmetry can be systematically classified [3, 44, 54]. Some analogous constraints in higher dimensions ruling out the existence of a trivially gapped phase have been explored e.g. in [58, 61, 71–81].

Another fundamental question we can ask is how generalised symmetries organise phases and critical points, analogously to what ordinary symmetries do. The standard and celebrated Landau paradigm [82] tells us that we can label phases of matter according to their symmetries, in particular whether they spontaneously break them or not. Moreover, we can make this quantitative in terms of order parameters, operators charged under symmetry which acquire a non-vanishing vacuum expectation value in some phase and that are vanishing in another. At the critical point we have a second order phase transition.

From a condensed matter viewpoint, the Landau paradigm has already seen an extension to encompass phases without local order parameters. Indeed, the emergence of topological order – where no local order parameter exists, but robust features (e.g. anyonic excitations, ground-state degeneracy on non-trivial manifolds) persist – can be interpreted as spontaneous breaking of a higher-form symmetry. In a similar vein, now that we are equipped with categorical and non-invertible symmetries in general dimensions, it is very natural to ask whether we can upgrade the traditional Landau theory to a *categorical Landau paradigm*. This would capture phases and phase transitions protected by non-invertible symmetries, where we expect to see the absence of conventional order parameters and standard symmetry breaking patterns. The hope is of course that by incorporating more kinds of symmetries, we can account for the full variety of phases and transitions observed in modern quantum systems.

Let us conclude this broad introduction with a paradigmatic example that shows many of the features outlined above, which we wish to explore further in the rest of this thesis. The example is provided by the familiar transverse field Ising model in $(1+1)$d. This is defined on a periodic chain with L sites, with a 2-dimensional qubit at each site. The Hilbert space is therefore 2^L-dimensional and given by the tensor product of the local Hilbert spaces associated to a qubit at each site. The Hamiltonian is given by

$$H = -\sum_{i=1}^{L} \left[g\,\sigma_i^x + \sigma_i^z \sigma_{i+1}^z \right] , \tag{1.5}$$

where σ_i^x and σ_i^z are the usual 2×2 Pauli matrices at site i. We assume without loss of generality that $g \geq 0$. The Hamiltonian (1.5) clearly has a $\mathbb{Z}_2$ symmetry, which is nothing but the usual spin flip symmetry. This is generated by the unitary operator

$$\eta = \prod_{i=1}^{L} \sigma_i^x \, . \tag{1.6}$$

This model provides a realisation of the standard Landau paradigm for $G = \mathbb{Z}_2$, as (1.5) admits two phases, depending on whether the $\mathbb{Z}_2$ symmetry is spontaneously broken or not. In particular, for $g < 1$ the system is in the ordered phase, with two ground states corresponding to having all the spins aligned up or down. Here the $\mathbb{Z}_2$ symmetry is spontaneously broken. Conversely, for $g > 1$ the system is in the disordered phase, with a single ground state and unbroken $\mathbb{Z}_2$ symmetry. At $g = 1$ we have a phase transition between the ordered and disordered phase, which in the continuum limit is described by the 2d Ising CFT with central charge $c = 1/2$. The order parameter is given by the spin (order) operator σ_i^z, which flows to the spin operator σ of the Ising CFT.

Moreover, consider the following transformation of the Hamiltonian (1.5)

$$\sigma_j^z \sigma_{j+1}^z \to \sigma_{j+1}^x \, , \quad \sigma_j^x \to \sigma_j^z \sigma_{j+1}^z \, . \tag{1.7}$$

This is the famous Kramers-Wannier (KW) duality of the Ising model, which exchanges $g \leftrightarrow g^{-1}$, or high and low temperature. Notice that at $g = 1$, however, this becomes an exact symmetry of (1.5). In the continuum limit, this flows to the non-invertible topological line $\mathcal{N}$ of the Ising CFT [31, 38, 83], which has fusion rules

$$\mathcal{N} \otimes \mathcal{N} = 1 \oplus \eta \,. \tag{1.8}$$

Here η is the topological line associated to the $\mathbb{Z}_2$ symmetry of the Ising CFT, satisfying the fusions $\eta \otimes \eta = 1$ and $\eta \otimes \mathcal{N} = \mathcal{N} \otimes \eta = \mathcal{N}$. Altogether,

$$\{1, \eta, \mathcal{N}\} \tag{1.9}$$

generate the so called Ising symmetry, one of the simplest examples of non-invertible symmetry. Under the action of $\mathcal{N}$, the spin operator σ of the Ising CFT is mapped to the disorder operator μ, which has the same conformal dimension but lives in the $\mathbb{Z}_2$ twisted sector. This map between order and disorder operators is an essential feature of non-invertible symmetries, as we will see. The Ising CFT has also a relevant $\mathbb{Z}_2$ symmetric energy operator ϵ, which is odd under the KW duality line $\mathcal{N}$. Adding the relevant deformation ϵ triggers an RG flow to the disordered or ordered phase, depending on the sign of the deformation.

The Ising symmetry is also one of the examples mentioned above of categorical symmetries which are intrinsically anomalous. Indeed, as we will see, it can be realised in a gapped phase with three vacua where the symmetry is fully spontaneously broken, while it is not possible to realise it in a single symmetric vacuum (or also in two vacua). Again, this imposes constraints on RG flows. For example, the tricritical Ising model with $c = 7/10$ admits a deformation by an operator ϵ' commuting with the Ising symmetry. Depending on the sign of the deformation, the theory flows to either the Ising CFT, or to a massive phase with three vacua [84, 85], which is consistent with the analysis of the gapped phases preserving the Ising symmetry.

1.0.1 Outline of the Thesis

This thesis is broadly divided into two main conceptual parts, which we now outline.

Non-invertible symmetries in higher dimensions. The first part of this thesis concerns theories with categorical symmetries in space-time dimensions $d = 3$ and higher. The constructions of non-invertible symmetries in higher-dimensional quantum field theories so far have been of five main types

1. gauging a discrete symmetry H in a theory with symmetry $G \times H$ and a mixed anomaly between G and H [59];
2. considering a theory self-dual under the gauging of a symmetry G and gauging this symmetry only in half of space-time [58, 61];
3. gauging a non-normal finite subgroup of the global symmetry [1, 57];
4. gauging a higher-form symmetry not on the whole space-time but only along a higher codimensional submanifold [60];
5. stacking a lower dimensional symmetric TQFT and gauging the diagonal symmetry of the TQFT and the original theory [86].

In chapter 2 we present the third construction and apply it to gauge theory examples, where the gauging of a 0-form symmetry given by an outer-automorphism of the gauge group gives rise to non-invertible symmetries. This approach is inspired by the one in [43] in 3d (see also [87]), where 0-form global symmetries of TQFTs are gauged. In generalising this to QFTs of arbitrary dimensions, we start with a theory $\mathfrak{T}$ with only invertible symmetries, meaning that the topological defects in $\mathfrak{T}$ form a higher-category, which we will call the *symmetry category* $\mathcal{C}_{\mathfrak{T}}$, that is group-like. We also assume the presence of a 0-form symmetry G which acts as outer-automorphisms, in particular inducing a non-trivial action on the topological defects $D_{(d-p-1)}$ that generate the p-form symmetries. We then gauge G, and study the topological defects that are obtained after gauging. One set of topological operators in the gauged theory $\mathfrak{T}/G$ are the gauge invariant combinations of topological defects $D_{(d-p-1)}$ in the initial category. After gauging the 0-form symmetry, there will be additional topological line operators, that generate the quantum $(d-2)$-form symmetry dual to G. We develop a consistent framework to combine these two sets of defects and determine their fusions. The resulting structure is naturally a higher-category $\mathcal{C}_{\mathfrak{T}/G}$.

Notably, there are also instances where various approaches overlap, and in particular some results obtained with the third approach can also be reproduced with the first method based on mixed anomalies. In chapter 2 we also introduce this latter construction and match the results derived in this way with the non-invertible symmetries obtained via the former categorical approach.

In chapter 3 we focus on $d = 3$ QFTs, whose symmetries are described by a 2-category, and concretely develop the tools to gauge arbitrary 0-form symmetries and determine the 2-category after gauging. An important point is that the initial, "pre-gauged", category is not necessarily invertible. In this sense, this extends the third construction in the list above, which involves gauging non-normal 0-form symmetries starting with only invertible ones, as well as the fifth construction, which considers the gauging of the full 0-form symmetry groups but not of subgroups.

Gauging 0-form symmetries in general 2-categories, in particular in the presence of non-trivial topological lines, is far more subtle for various reasons:

1. More options for implementing the 0-form symmetry: the presence of topological lines results in a much richer way of implementing the 0-form symmetry action. Furthermore, the 1-form symmetry generated by line defects can be gauged on a surface, thus resulting in condensation defects;

2. Symmetry fractionalisation: in the presence of lines, symmetries can fractionalise, which in the categorical setting results in the presence of certain non-trivial

associators, characterised by 4-cocycles. This symmetry fractionalisation results in subtle constraints on the gauging process.

In general, we consider gauging subsequently all the possible subgroups of the 0-form symmetry G, and proceeding in this way we obtain a categorical *symmetry web* of theories related by invertible gauging operations.

The categorical Landau paradigm. In the second part of the thesis, we focus on how categorical symmetries can constraint the IR phases of theories, focusing mostly on $(1+1)$d. We describe a comprehensive framework, the categorical Landau paradigm, to study phases and phase transitions in the presence of non-invertible symmetries.

We start in chapter 4, where we show how to classify all the possible gapped phases of $(1+1)$d theories with a categorical symmetry $\mathcal{S}$. In (1+1)d, many aspects of gapped phases with fusion category symmetries have been discussed over the last few years. The innovations we propose here compared to the existing literature on the topic (see for example [44, 48, 54, 69, 88–91]) are the following:

- We use the SymTFT [92] (and for earlier works, see [50, 93, 94]) to classify $\mathcal{S}$-symmetric gapped phases, arguing that the SymTFT provides a systematic, comprehensive and computationally useful approach for performing the classification. The SymTFT is a $(2+1)$-dimensional TQFT $\mathfrak{Z}(\mathcal{S})$ for a $(1+1)$-dimensional theory $\mathfrak{T}$ with a categorical symmetry $\mathcal{S}$, which is topological and has two boundaries: a topological boundary $\mathfrak{B}^{\text{sym}}_{\mathcal{S}}$, which encodes the symmetries, and a not necessarily topological boundary $\mathfrak{B}^{\text{phys}}_{\mathfrak{T}}$, which depends on the QFT $\mathfrak{T}$. The original theory $\mathfrak{T}$ with the action of the symmetry $\mathcal{S}$ can be recovered as the interval compactification of the SymTFT, also known as the *sandwich construction*. To study gapped phases, we also impose $\mathfrak{B}^{\text{phys}}_{\mathfrak{T}}$ to be a topological boundary condition.

- We identify the (generalised) charges [95] of order parameters for arbitrary $\mathcal{S}$-symmetric gapped phases. In general, we find that such order parameters are mixtures of conventional order parameters (local operators) and string order parameters (twisted sector operators, i.e. operators attached to topological lines), which are forced to coexist in a single irreducible multiplet due to the action of categorical symmetry on these local operators.

- We uncover an interesting physical phenomenon tied closely to spontaneous breaking of non-invertible symmetries, i.e. that spontaneous breaking of non-invertible symmetries can lead to physically distinguishable vacua. This is in stark contrast to spontaneous breaking of invertible symmetries, where all the vacua participating in an irreducible gapped phase are physically indistinguishable.

As we have seen, a fundamental part of the Landau paradigm is to discuss phase transitions. We address this question for categorical symmetries $\mathcal{S}$ in chapter 5, where we construct gapless theories corresponding to phase transitions with non-invertible symmetries, using as input known phase transitions with invertible symmetries. These can be e.g. the critical Ising model (describing the transition between a $\mathbb{Z}_2$ SSB phase and the trivial phase) or the 3-state Potts model (describing the transition between a $\mathbb{Z}_3$ SSB phase and the trivial phase). This requires an extension of the ordinary SymTFT construction by introducing the *club sandwich*, which adds a gapped interface $\mathcal{I}$ between two topological orders. These can be e.g. the SymTFTs $\mathfrak{Z}(\mathcal{S})$ and $\mathfrak{Z}(\mathcal{S}')$ for two symmetries $\mathcal{S}$ and $\mathcal{S}'$, separated by the interface $\mathcal{I}$ and sandwiched on either side between the boundary conditions $\mathfrak{B}^{\text{sym}}$ and $\mathfrak{B}^{\text{phys}}$. Concretely, we can view the club sandwich as providing a map between $\mathcal{S}'$-symmetric theories to $\mathcal{S}$-symmetric theories, with the symmetry $\mathcal{S}'$, as we will see, being an appropriate subcategory of the bigger symmetry $\mathcal{S}$. This is particularly relevant for our application to the study of phase transitions. Indeed, suppose we know a 2d CFT $\mathfrak{C}'$ that sits at the transition between two $\mathcal{S}'$ symmetric gapped phases $\mathfrak{T}'_1$ and $\mathfrak{T}'_2$. Inputting this as right boundary condition $\mathfrak{B}^{\text{phys}}$ of the club sandwich, we can output an $\mathcal{S}$ symmetric CFT $\mathfrak{C}$ at the transition between $\mathcal{S}$-symmetric phases. This allows us to derive phase transitions for larger symmetries by inputting known minimal phase transitions for smaller, "elementary", symmetries.

Chapter 6 studies a specific lattice model that very concretely illustrates the categorical Landau paradigm in action. This lattice model is realised on a tensor product Hilbert space, acted upon by generalised Ising Hamiltonians. These models exhibit four gapped phases, with a commuting projector Hamiltonian within each of them. The ground states cannot be explained as standard SSB phases, but require a non-invertible symmetry, in this case $\mathsf{Rep}(S_3)$, given by the irreducible representations of the group of permutations of three elements S_3. Moreover, by tuning the parameters in the generalised Ising Hamiltonians, we also realise second order phase transitions between such gapped phases. This lattice model provides a UV realisation of the gapped and gapless $\mathsf{Rep}(S_3)$ phases found using the SymTFT framework outlined in the chapters 4 and 5.

Finally, chapter 7 uses the tools developed in the previous chapters to determine a 2d CFT with the exotic Haagerup fusion category symmetry. A general question is whether given any fusion category $\mathcal{S}$, one can always find a 2d CFT with topological lines realising it (a related question is, given any MTC $\mathcal{M}$, to determine the diagonal rational CFT (RCFT) whose Verlinde lines realise $\mathcal{M}$, following the correspondence between RCFTs and MTCs introduced in [31, 32]). This has been a longstanding open problem for the Haagerup symmetry. Numerical evidence for such a CFT was obtained using anyon chain models in [96,97], where a central charge $c \sim 2$ was found and subsequently studies of finite-size effects were performed in [98]. More recently, [99] constructed an integrable spin chain with broken $\mathcal{H}_3$ symmetry, as well as a critical spin chain, where $c \sim 3/2$ (see also [100]). However, the definitive construction

– especially analytic – of a gapless theory with $\mathcal{H}_3$ topological lines remains an open question. In chapter 7 we address this problem and construct a CFT that is $\mathcal{H}_3$-symmetric. In particular, we determine gapped and gapless phases with $\mathcal{H}_3$ symmetry and explore the phase diagram, following the philosophy of the categorical Landau paradigm. We corroborate the continuum analysis by also constructing a lattice model that realises the gapped and gapless phases, following the general approach of [8], which is based on the anyon chain and is tailored to construct the lattice models directly from the SymTFT and club sandwich data. Similar lattice model constructions for fusion category symmetries have appeared in [83, 89, 101–106].

Chapter 2

Higher Categorical Symmetries

In this chapter we will outline a general procedure, applicable in any dimension, which constructs non-invertible symmetries by gauging 0-form sub-symmetries of invertible higher-form and higher-group symmetries. These non-invertible symmetries and their properties, such as the possible gaugings and analogues of ‘t Hooft anomalies, are expected to be encoded in the structure of a higher-category, which can be understood as capturing the local properties of topological defects associated to these symmetries. We can thus call these *higher-categorical symmetries.* The content of this chapter is based on [1].

The structure of the chapter is as follows. In section 2.1 we introduce some notions of higher categorical symmetries. In section 2.2 we describe symmetries localised on topological defects and condensation. In section 2.3 we discuss the category obtained after gauging a 0-form symmetry G in general dimensions. We present some examples of this construction in 3d and 4d in sections 2.4 and 2.5 respectively. Finally, we conclude illustrating an alternative approach based on mixed anomalies from higher groups in section 2.6.

2.1 Symmetries and Higher-Categories

In this section, we review why generalised symmetries are expected to form the mathematical structure of a higher-category.

2.1.1 Symmetries in Terms of Topological Defects

Generalised symmetries of a QFT correspond to the existence of topological defects of various dimensions in the QFT. These topological defects can be genuine or non-genuine. We begin with a discussion of genuine topological defects, that can be defined independently of other higher-dimensional topological defects. A genuine topological defect D_p of dimension-p is a defect operator that can be inserted along any dimension-p sub-manifold Σ_p of the d-dimensional spacetime M_d. The fact that it is topological means the following: consider a correlation function $\langle \cdots D_p(\Sigma_p) \cdots \rangle$ containing D_p, where the dots denote other topological and non-topological defects of various dimensions. Then, we have the equality of correlation functions

$$\langle \cdots D_p(\Sigma_p) \cdots \rangle = \left\langle \cdots D_p \left(\Sigma'_p\right) \cdots \right\rangle , \tag{2.1}$$

where $\left\langle \cdots D_p \left(\Sigma'_p\right) \cdots \right\rangle$ denotes the correlation function obtained by changing the locus of D_p from Σ_p to Σ'_p by a homotopy that does not intersect the loci of other defects participating in the correlation function, and the loci of other defects are not changed.

Non-genuine topological defects can be sub-defects arising at the intersections or junctions of genuine topological defects, see figure 2.1. More generally, non-genuine topological defects arise at the junctions of genuine topological defects and non-genuine topological sub-defects.

So far whatever we have discussed holds true for both discrete and continuous symmetries. A discrete symmetry is one for which the corresponding genuine and

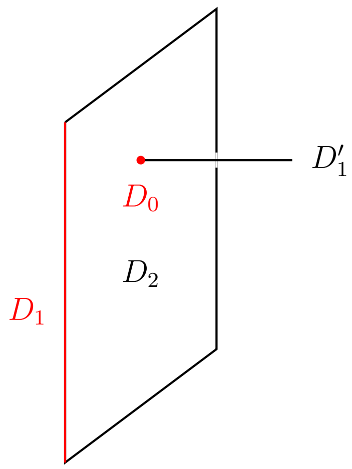


Figure 2.1: Example of non-genuine defects arising at junctions of genuine defects. Here D_1' and D_2 are genuine line and surface defects respectively. D_1 is a non-genuine line defect arising at the end of D_2 and D_0 is a non-genuine local operator that can arise at an end of D_1' along D_2.

non-genuine topological defects are parametrised by discrete parameters. On the other hand, for a continuous symmetry, the corresponding genuine and non-genuine topological defects are parametrised by continuous parameters. For a discrete symmetry, the associated topological defects and their configurations provide full information about the various possible backgrounds for the discrete symmetry that the QFT can be coupled to. However, for a continuous symmetry, the associated topological defects and their configurations only provide information about "flat" backgrounds of the continuous symmetry.

2.1.2 From Topological Defects to Higher-Categories

Symmetry category. From the information about configurations of topological defects in a d-dimensional QFT $\mathfrak{T}$, we can construct a $(d-1)$-category $\mathcal{C}_{\mathfrak{T}}$, which we refer to as the symmetry category of $\mathfrak{T}$. For $d=2$, it is a 1-category, or a standard category. For $d>2$, it is a higher-category.

Recall that a $(d-1)$-category has d levels. At the first level, we have objects of the category, which are also called 0-morphisms. At the second level, we have

1-morphisms between objects. At the third level, we have 2-morphisms between 1-morphisms. Continuing in this fashion, at the i-th level for $2 \leq i \leq d$, we have $(i-1)$-morphisms between $(i-2)$-morphisms.

Objects. The objects of $\mathcal{C}_{\mathfrak{T}}$ correspond to topological codimension-1 defects of $\mathfrak{T}$. We use the same labels D_{d-1} to denote both topological codimension-1 defects and the corresponding objects of $\mathcal{C}_{\mathfrak{T}}$. There is an additive structure on the objects coming from the additive structure on the codimension-1 topological defects. A codimension-1 topological defect $D_{d-1} = \bigoplus_i n_i D_{d-1}^{(i)}$ with $n_i > 0$ is a sum of distinct codimension-1 topological defects $D_{d-1}^{(i)}$. Simple objects are by definition those codimension-1 topological defects that have a single vacuum, or in other words, carry a single topological local operator on their worldvolume.

There is also a product/monoidal structure on the objects coming from fusing codimension-one topological defects, see figure 2.2, where we consider fusing two codimension-1 defects $D_{d-1}^{(1)}$ and $D_{d-1}^{(2)}$. The resulting codimension-1 defect is denoted as $D_{d-1}^{(12)}$, which we represent in equations as

$$D_{d-1}^{(1)} \otimes D_{d-1}^{(2)} = D_{d-1}^{(12)} \tag{2.2}$$

or as

$$D_{d-1}^{(1)}(\Sigma_{d-1}) \otimes D_{d-1}^{(2)}(\Sigma_{d-1}) = D_{d-1}^{(12)}(\Sigma_{d-1}) \tag{2.3}$$

if we want to manifest the codimension-one submanifold Σ_{d-1} of spacetime that the defects wrap.

1-morphisms. The 1-morphisms of $\mathcal{C}_{\mathfrak{T}}$ correspond to topological codimension-2 defects living at the intersection of two topological codimension-1 defects, see figure 2.3.

Two 1-morphisms can be composed to obtain another 1-morphism. Given a 1-

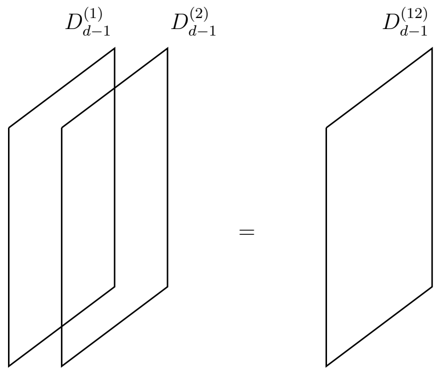


Figure 2.2: Fusion of codimension-1 topological defects that describes a monoidal structure on the objects in the symmetry higher-category.

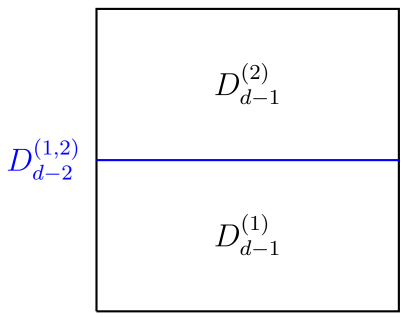


Figure 2.3: A 1-morphism $D^{(1,2)}_{d-2}$ from $D^{(1)}_{d-1}$ to $D^{(2)}_{d-1}$ is a codimension-2 topological defect living between codimension-1 topological defects $D^{(1)}_{d-1}$ and $D^{(2)}_{d-1}$. To specify the direction of the morphism, we need to pick a "time" direction, which is taken to run from bottom to top of the figure.

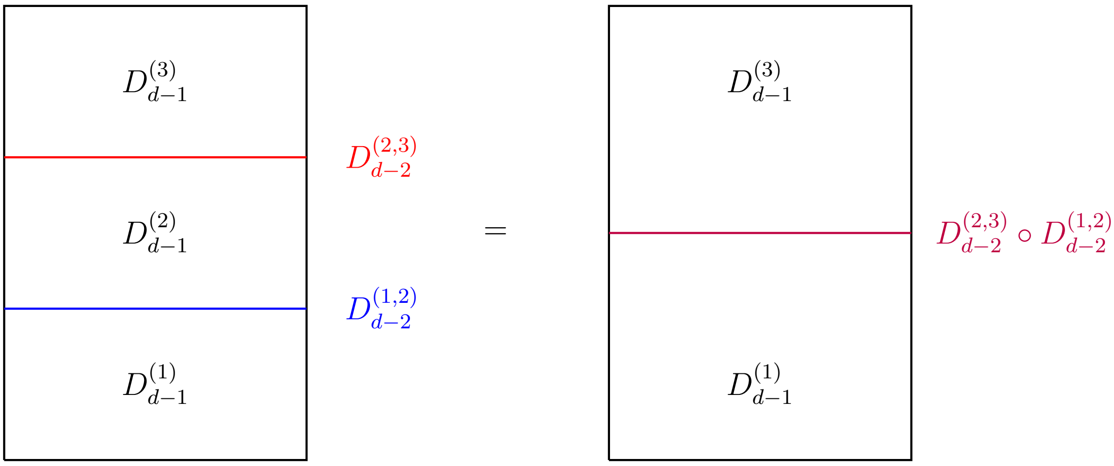


Figure 2.4: Fusing two codimension-2 defects $D^{(1,2)}_{d-2}$ and $D^{(2,3)}_{d-2}$ leads to the defect $D^{(2,3)}_{d-2} \circ D^{(1,2)}_{d-2}$. This is described in the higher-category as a composition of 1-morphisms, and to describe the direction of the morphisms and composition, we need to pick a "time" direction, which is taken to run from bottom to top of the figure.

morphism $D^{(1,2)}_{d-2}$ from $D^{(1)}_{d-1}$ to $D^{(2)}_{d-1}$ and a 1-morphism $D^{(2,3)}_{d-2}$ from $D^{(2)}_{d-1}$ to $D^{(3)}_{d-1}$, we have a 1-morphism

$$D^{(2,3)}_{d-2} \circ D^{(1,2)}_{d-2} \tag{2.4}$$

from $D^{(1)}_{d-1}$ to $D^{(3)}_{d-1}$. See figure 2.4. Changing the time direction in the above fusion leading to composition of morphisms, we obtain a monoidal/fusion structure on 1-morphisms. However, it should be noted that we define this fusion structure only if $\mathcal{C}_{\mathfrak{T}}$ admits 2-morphisms, i.e. if the theory $\mathfrak{T}$ has dimension $d \geq 3$. Given a 1-morphism $D^{(1,2)}_{d-2}$ from $D^{(1)}_{d-1}$ to $D^{(2)}_{d-1}$ and a 1-morphism $D^{(2,3)}_{d-2}$ from $D^{(2)}_{d-1}$ to $D^{(3)}_{d-1}$, we have a 1-morphism

$$D^{(1,2)}_{d-2} \otimes_{D^{(2)}_{d-1}} D^{(2,3)}_{d-2} \tag{2.5}$$

from $D^{(1)}_{d-1}$ to $D^{(3)}_{d-1}$, see figure 2.5. Even though we have

$$D^{(1,2)}_{d-2} \otimes_{D^{(2)}_{d-1}} D^{(2,3)}_{d-2} = D^{(2,3)}_{d-2} \circ D^{(1,2)}_{d-2} \tag{2.6}$$

we use both notions as they have different utilities. For example, we will see later

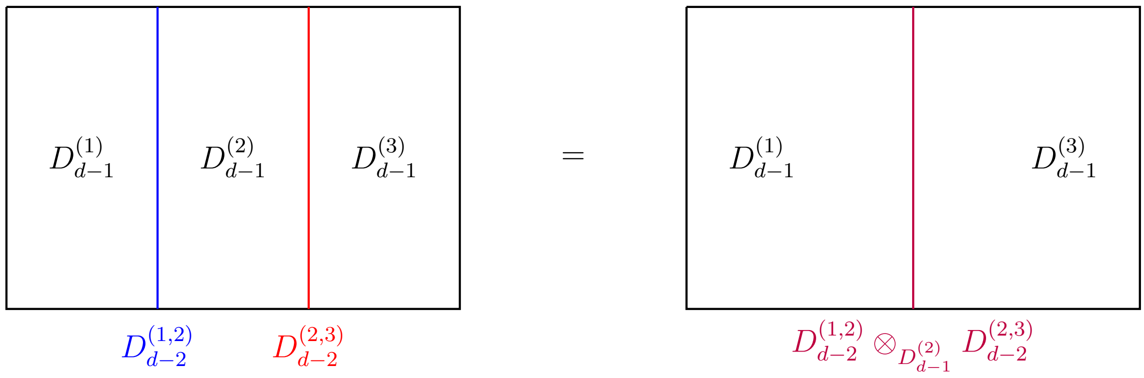


Figure 2.5: Here we have rotated the figure 2.4, while keeping the time direction going from bottom to top. The fusion of $D^{(1,2)}_{d-2}$ and $D^{(2,3)}_{d-2}$ now is represented as a monoidal operation on 1-morphisms. Such a monoidal operation is labeled by objects, as in equation (2.5).

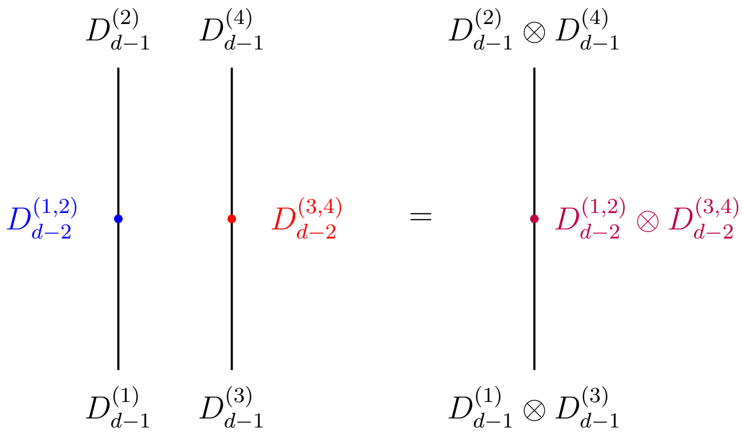


Figure 2.6: The fusion structure $\otimes$ on general codimension-2 topological defects.

that the fusion structure $\otimes_{D_{d-1}}$ on 1-morphisms from D_{d-1} to D_{d-1} descends to a fusion structure on objects of a higher-category of symmetries localised along D_{d-1}.

There is another fusion structure on 1-morphisms, which is defined for any $\mathcal{C}_{\mathfrak{T}}$, irrespective of whether it admits 2-morphisms or not. Given a 1-morphism $D^{(1,2)}_{d-2}$ from $D^{(1)}_{d-1}$ to $D^{(2)}_{d-1}$ and a 1-morphism $D^{(3,4)}_{d-2}$ from $D^{(3)}_{d-1}$ to $D^{(4)}_{d-1}$ constructs a 1-morphism

$$D^{(1,2)}_{d-2} \otimes D^{(3,4)}_{d-2} \tag{2.7}$$

from $D^{(13)}_{d-1}$ to $D^{(24)}_{d-1}$. This fusion operation is described in figure 2.6.

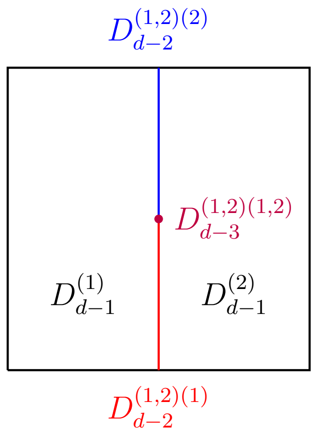


Figure 2.7: A 2-morphism $D_{d-3}^{(1,2)(1,2)}$ between 1-morphisms $D_{d-2}^{(1,2),(1)}$ and $D_{d-2}^{(1,2),(2)}$ (both from $D_{d-1}^{(1)}$ to $D_{d-1}^{(2)}$).

2-morphisms. The 2-morphisms of $\mathcal{C}_{\mathfrak{T}}$ correspond to topological codimension-3 defects living at the intersection of two codimension-2 defects corresponding to 1-morphisms of $\mathcal{C}_{\mathfrak{T}}$, see figure 2.7. 2-morphisms can be similarly composed and admit multiple fusion structures.

Higher-morphisms. Continuing inductively, we define p-morphisms from $p-1$-morphism D_{d-p} to $p-1$-morphism D'_{d-p} of $\mathcal{C}_{\mathfrak{T}}$ as topological codimension-$(p+1)$ defects D_{d-p-1} that live at the intersection of topological codimension-p defects D_{d-p} and D'_{d-p} (with appropriate choices of orientation).

2.2 Localised Symmetries and Condensations

When are two topological defects $D_p^{(1)}$ and $D_p^{(2)}$ related by the so-called condensation? We can define equivalence classes of topological defects[1] that are related to each other by condensations. Pick a representative $D_p^{(1)}$ of such an equivalence class. Then any other defect $D_p^{(2)}$ lying in the equivalence class can be obtained by performing a generalised gauging operation on the worldvolume of $D_p^{(1)}$. The purpose of this

[1]These are also known as 'Schur components' [107].

section is to explain this generalised gauging construction.

To describe the generalised gauging operation, we need to first begin with a discussion of the (higher-)category $\mathcal{C}_{\mathfrak{T},D_p}$ of symmetries localised along the worldvolume of a topological defect D_p (which may be genuine or non-genuine). $\mathcal{C}_{\mathfrak{T},D_p}$ is a $(p-1)$-category describing topological defects that are constrained to live inside D_p, and we refer to it as the *symmetry category of the defect* D_p. In fact, $\mathcal{C}_{\mathfrak{T},D_p}$ can be recognised as a subcategory of the symmetry $(d-1)$-category $\mathcal{C}_{\mathfrak{T}}$ of the theory $\mathfrak{T}$. The defect D_p is itself a $(d-p-1)$-morphism of $\mathcal{C}_{\mathfrak{T}}$. The objects of $\mathcal{C}_{\mathfrak{T},D_p}$ are $(d-p)$-morphisms of $\mathcal{C}_{\mathfrak{T}}$ from D_p to itself. The 1-morphisms of $\mathcal{C}_{\mathfrak{T},D_p}$ are $(d-p+1)$-morphisms of $\mathcal{C}_{\mathfrak{T}}$ going between $(d-p)$-morphisms of $\mathcal{C}_{\mathfrak{T}}$ that are objects of $\mathcal{C}_{\mathfrak{T},D_p}$, and so on. The fusion structure $\otimes$ on $\mathcal{C}_{\mathfrak{T},D_p}$ descends from the fusion structure $\otimes_{D_p}$ on $\mathcal{C}_{\mathfrak{T}}$.

2.2.1 Generalised Gauging

Let us now describe the construction of $D_p^{(2)}$ in terms of $D_p^{(1)}$, when the two defects are related by condensation. We will first discuss the case of $p=2$, where we can be quite concrete.

$D_2^{(2)}$ can be obtained from $D_2^{(1)}$ by performing a generalised gauging [88, 108, 109] of the symmetry $\mathcal{C}_{\mathfrak{T},D_2^{(1)}}$ of $D_2^{(1)}$. The gauging is described by what is known as an algebra inside the 1-category $\mathcal{C}_{\mathfrak{T},D_2^{(1)}}$. The algebra is comprised of the following data:

- First of all, we have an object $A_1^{(1,2)}$ inside $\mathcal{C}_{\mathfrak{T},D_2^{(1)}}$, which implements the gauging procedure to go from $D_2^{(1)}$ to $D_2^{(2)}$.

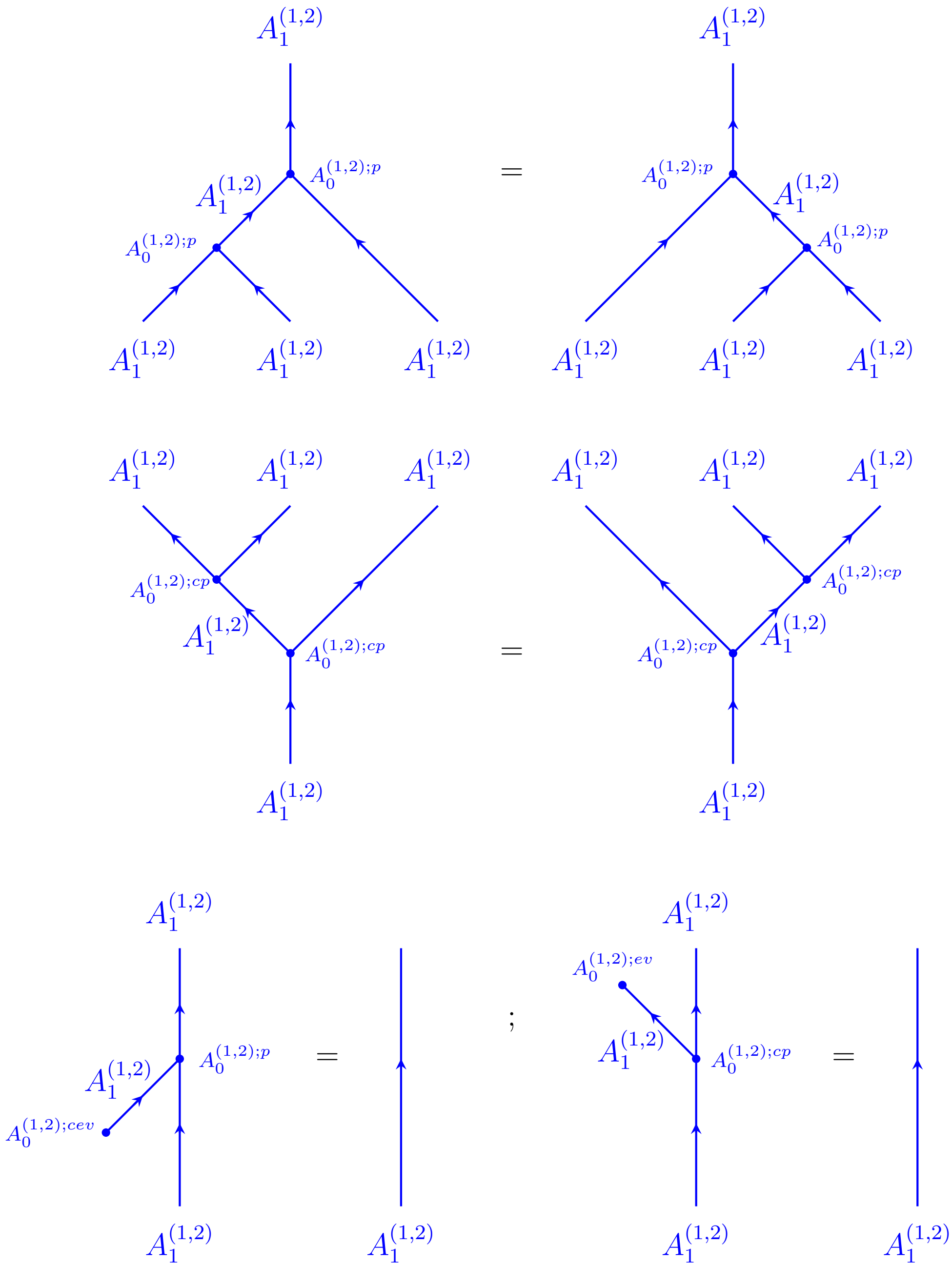


Figure 2.8: Conditions specified by the morphisms comprising the algebra $A^{(1,2)}$.

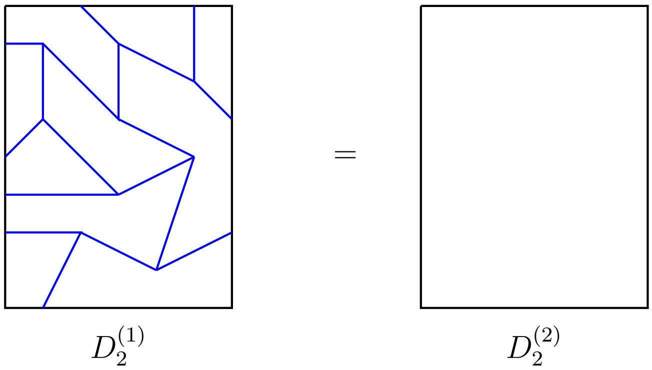


Figure 2.9: The construction of the topological defect $D_2^{(2)}$ by gauging algebra $A^{(1,2)}$ on $D_2^{(1)}$. The blue lines on the left hand side are algebra objects $A_1^{(1,2)}$, while the trivalent junctions are morphisms comprising the algebra.

- Additionally we have the following canonical morphisms

$$\begin{aligned} A_0^{(1,2;p)} &: A_1^{(1,2)} \otimes A_1^{(1,2)} \to A_1^{(1,2)} \\ A_0^{(1,2;cp)} &: A_1^{(1,2)} \to A_1^{(1,2)} \otimes A_1^{(1,2)} \\ A_0^{(1,2;ev)} &: A_1^{(1,2)} \to 1_{D_2^{(1)}} \\ A_0^{(1,2;cev)} &: 1_{D_2^{(1)}} \to A_1^{(1,2)}\,, \end{aligned} \tag{2.8}$$

  satisfying the properties shown in figure 2.8.

The gauging of $\mathcal{C}_{\mathfrak{T},D_2^{(1)}}$ by the algebra

$$A^{(1,2)} = \left\{A_1^{(1,2)}, A_0^{(1,2;p)}, A_0^{(1,2;cp)}, A_0^{(1,2;ev)}, A_0^{(1,2;cev)}\right\} \tag{2.9}$$

is performed by inserting a mesh of topological defects comprised out of the algebra along the full locus of $D_2^{(1)}$, as shown in figure 2.9. We denote the defect with algebra $A^{(1,2)}$ condensed by

$$D_2^{(2)} = \frac{D_2^{(1)}}{A^{(1,2)}}\,. \tag{2.10}$$

An interface $D_1^{(1,2)}$ between the original defect $D_2^{(1)}$ and $D_2^{(2)}$ can be constructed by inserting a mesh of topological defects comprised out of the algebra along only

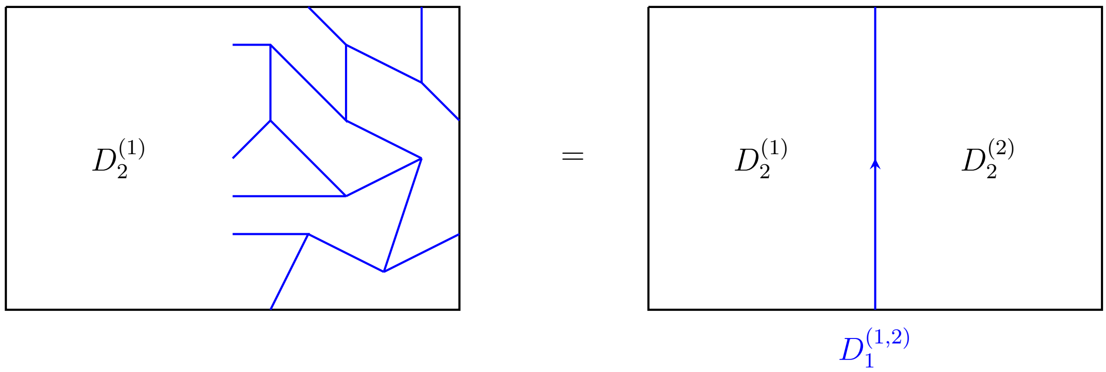


Figure 2.10: The construction of the interface $D_1^{(1,2)}$ using the algebra $A^{(1,2)}$.

half of the locus of $D_1^{(1)}$, as shown in figure 2.10.

Category of lines after condensation. The symmetry category capturing localised symmetries on $D_2^{(2)}$ can be recognised as

$$\mathcal{C}_{\mathfrak{T},D_2^{(2)}} = \mathrm{Bimod}_{A^{(1,2)}}\left(\mathcal{C}_{\mathfrak{T},D_2^{(1)}}\right) , \tag{2.11}$$

which is the category of $A^{(1,2)}$ *bimodules* in $\mathcal{C}_{\mathfrak{T},D_2^{(1)}}$. That is, the topological line operators living on $D_2^{(2)}$ are bimodules of the algebra $A^{(1,2)}$. Such a bimodule $B^{D_2^{(1)}}$ comprises of the following data

$$B^{D_2^{(1)}} = \left\{ B_1^{D_2^{(1)}}, B_0^{D_2^{(1)};lp}, B_0^{D_2^{(1)};rp}, B_0^{D_2^{(1)};lcp}, B_0^{D_2^{(1)};rcp} \right\} , \tag{2.12}$$

where $B_1^{D_2^{(1)}}$ is an object of $\mathcal{C}_{\mathfrak{T},D_2^{(1)}}$, and the other four are morphisms

$$\begin{aligned}
B_0^{D_2^{(1)};lp} &: \; A_1^{(1,2)} \otimes B_1^{D_2^{(1)}} \to B_1^{D_2^{(1)}} \\
B_0^{D_2^{(1)};rp} &: \; B_1^{D_2^{(1)}} \otimes A_1^{(1,2)} \to B_1^{D_2^{(1)}} \\
B_0^{D_2^{(1)};lcp} &: \; B_1^{D_2^{(1)}} \to A_1^{(1,2)} \otimes B_1^{D_2^{(1)}} \\
B_0^{D_2^{(1)};rcp} &: \; B_1^{D_2^{(1)}} \to B_1^{D_2^{(1)}} \otimes A_1^{(1,2)} ,
\end{aligned} \tag{2.13}$$

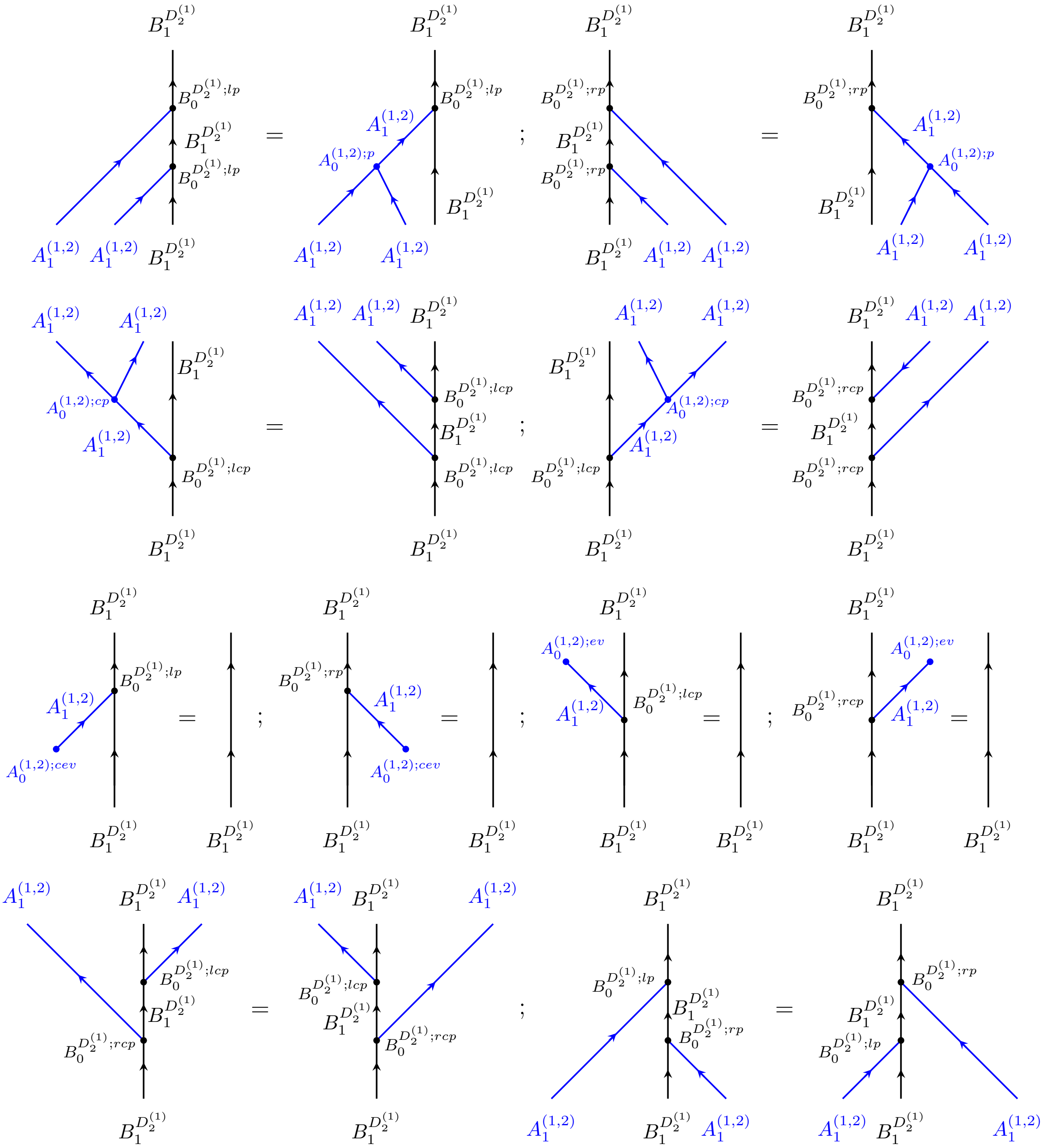


Figure 2.11: Conditions that a bimodule $B^{D_2^{(1)}}$ has to satisfy.

such that these satisfy the properties shown in figure 2.11. See [1, 88] for details regarding morphisms in the category $\mathrm{Bimod}_{A^{(1,2)}}\left(\mathcal{C}_{\mathfrak{T},D_2^{(1)}}\right)$.

The above description for $p = 2$ is expected to generalise to general p. In this chapter we only consider the case of $p = 2$, and hence do not need to develop the theory of generalised gauging for general p, but see [110–114] for prior work in this direction.

2.3 0-Form Gauging of Higher-Categorical Symmetries

In this section, we study a sub-symmetry of $\mathfrak{T}$ given by a $(d-2)$-category $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$ which is a subcategory of the $(d-1)$-category $\mathcal{C}_{\mathfrak{T}}$ capturing the full symmetry of $\mathfrak{T}$. We have a group action on $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$ given by a finite group G, which may be non-abelian. The group action corresponds to a 0-form symmetry of $\mathfrak{T}$ which can be gauged, resulting in the theory $\mathfrak{T}/G$. We describe a construction of the corresponding $(d-2)$-category $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$ of the full symmetry $(d-1)$-category $\mathcal{C}_{\mathfrak{T}/G}$ of the gauged theory $\mathfrak{T}/G$ in terms of the data of $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$ and the action of G on it.

The classes of G that we consider are restricted to be of the form[2]

$$G = \Gamma_1 \rtimes \Gamma_2 \rtimes \cdots \rtimes \Gamma_k \,, \tag{2.14}$$

where Γ_i are abelian groups. This is because we describe the effect of gauging a finite abelian group Γ, and the effect of gauging G can be deduced by sequentially gauging the finite abelian groups Γ_i.

[2]These do not exhaust all non-abelian finite G. An example of a group that cannot be written in this fashion is the group (of order 8) formed by quaternions.

2.3.1 Setup

The category $\mathcal{C}_{\text{id},\mathfrak{T}}$ is the category describing symmetries localised along the codimension-1 identity defect of $\mathfrak{T}$. In other words, $\mathcal{C}_{\text{id},\mathfrak{T}}$ is obtained from the full symmetry category $\mathcal{C}_{\mathfrak{T}}$ by forgetting about non-trivial codimension-1 topological defects.

On the other hand, G is a 0-form symmetry of $\mathfrak{T}$. This means that $\mathcal{C}_{\mathfrak{T}}$ contains objects $D_{d-1}^{(g)}$ parametrised by the elements g of G. We further assume that there are no 1-morphisms between $D_{d-1}^{(g)}$ and $D_{d-1}^{(g')}$ for $g \neq g'$. This condition is equivalent to requiring that there are no topological defects in the twisted sector of the G 0-form symmetry. If this condition is violated, one obtains extra codimension-two (and also higher codimensional) defects in $\mathcal{C}_{\text{id},\mathfrak{T}/G}$ that are not accounted by our procedure discussed below. These extra defects are also known as topological defects lying in non-trivial flux sector, or as topological Gukov-Witten operators. Our procedure can be applied to such cases, but in such cases we only construct a subcategory of the full category $\mathcal{C}_{\text{id},\mathfrak{T}/G}$, which can be understood as the subcategory formed by defects lying in the trivial flux sector.

The tensor product of these objects follows the group operation on G:

$$D_{d-1}^{(g)} \otimes D_{d-1}^{(g')} = D_{d-1}^{(gg')} \,. \tag{2.15}$$

We assume that G does not participate in any higher-group structures and 't Hooft anomalies.

In general G has an action on $\mathcal{C}_{\text{id},\mathfrak{T}}$ as follows. Consider an object D_{d-2} of $\mathcal{C}_{\text{id},\mathfrak{T}}$. An element $g \in G$ sends D_{d-2} to an element $g \cdot D_{d-2}$ of $\mathcal{C}_{\text{id},\mathfrak{T}}$, which can be computed as

$$g \cdot D_{d-2} = D_{d-2}^{(g)} \otimes D_{d-2} \otimes D_{d-2}^{(g^{-1})} \tag{2.16}$$

using the fusion structure on 1-morphisms of the category $\mathcal{C}_{\mathfrak{T}}$, where $D_{d-2}^{(g)}$ is the identity $(d-2)$-dimensional defect on $D_{d-1}^{(g)}$. See figure 2.12.

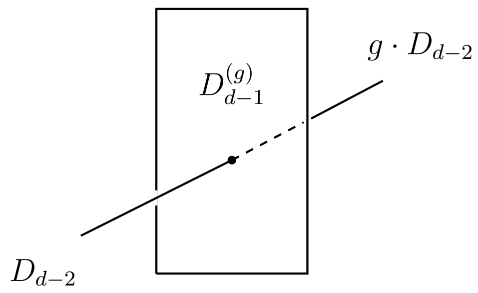


Figure 2.12: Action of the symmetry $g \in G$ on the defects D_{d-2} as in equation (2.16).

2.3.2 Gauging in 3d

We begin the discussion of gauging finite, abelian G from the special case of $d = 3$. This has been considered in the literature earlier [43,87] in the context of 3d TQFTs, and our discussion is mostly a review of these works but now applied to general 3d QFTs that need not be topological. We formulate the discussion such that it is amenable to generalization to higher dimensions. The category $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$ is a standard 1-category describing the genuine topological line defects and the topological local operators living at their junctions. Our task is to determine the 1-category $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$ of the 3d theory $\mathfrak{T}/G$ obtained after gauging the 0-form symmetry G of the 3d theory $\mathfrak{T}$.

Let us begin by discussing the objects of $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$, i.e. the genuine line defects in the theory $\mathfrak{T}/G$. First of all, gauging G produces Wilson line defects for the gauge group G, which are topological as G is finite. These line defects form a sub-category

$$\mathrm{Rep}(G) = \mathrm{Vec}_{\widehat{G}} \tag{2.17}$$

of $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$, where $\mathrm{Vec}_{\widehat{G}}$ is the category of vector spaces graded by elements of the Pontryagin dual $\widehat{G}$ of G. Recall that $\widehat{G}$ is the group formed by irreducible representations of the finite group G (which are all 1-dimensional) under tensor product operation. This subcategory provides objects in $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$ labeled by representations of

G. The irreducible representations of G, i.e. elements of $\widehat{G}$, give rise to simple objects of $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$.

In addition to the G representations, there are objects of $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$ arising from the objects of $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$. However, not every object of $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$ descends to an object of $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$. This is because only those objects of $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$ that are left invariant by the action of G are gauge invariant in the theory $\mathfrak{T}/G$, so only those objects survive as objects of $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$. A simple object $D_1^{(O)}$ of $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$ arising in this way can be described as the object

$$D_1^{(O)} \equiv \bigoplus_{i\in O} D_1^{(i)} \tag{2.18}$$

in the category $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$, where $D_1^{(i)}$ are distinct simple objects of $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$ lying in an orbit O of the G action.

Finally, there are simple objects of $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$ which are mixtures of the two above kinds of simple objects, which can be thought of as objects $D_1^{(O)}$ dressed by Wilson line defects. Concretely, to a simple object $D_1^{(O)}$, we can associate a subgroup G_O of G, which is the stabiliser of any object $D_1^{(i)}$ for $i \in O$. Such an object $D_1^{(O)}$ can be dressed by representations of the stabiliser G_O. Thus the simple objects corresponding to $D_1^{(O)}$ can be represented as $D_1^{(O,R_O)}$, where R_O is an irreducible representation of G_O, or in other words, an element of the Pontryagin dual group $\widehat{G}_O$ of G_O. The bare object $D_1^{(O)}$ is obtained by choosing R_O to be the trivial representation. The simple objects of the subcategory $\mathrm{Rep}(G)$ of $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$ are obtained as special cases of $D_1^{(O,R_O)}$ by taking O to be the orbit formed by the identity object of $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$, for which the stabiliser is the whole group G. We represent these simple objects as $D_1^{(R)}$, where R is an irreducible representation of G. The identity object of the category $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$ is denoted as $D_1^{(\mathrm{id})}$, which is obtained by choosing R to be the trivial representation of G.

Let us now discuss the fusion operation on the objects of $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$. First of all, we have

$$D_1^{(R)} \otimes D_1^{(R')} = D_1^{(RR')}\,, \tag{2.19}$$

where $R, R' \in \widehat{G}$, and $RR' \in \widehat{G}$ is the product of R and R' in $\widehat{G}$. Next, we have

$$D_1^{(O,R_O)} \otimes D_1^{(R')} = D_1^{(O,R_O R'_O)}\,, \tag{2.20}$$

where $R'_O \in \widehat{G}_O$ is the image of $R \in \widehat{G}$ under the surjective homomorphism

$$\pi_O : \ \widehat{G} \to \widehat{G}_O \tag{2.21}$$

dual to the injective homomorphism

$$i_O : \ G_O \to G \tag{2.22}$$

descending from the fact that G_O is a subgroup of G.

The fusions $D_1^{(O,R_O)} \otimes D_1^{(O',R'_{O'})}$ are more complicated. For this purpose, we consider the fate of local operators, i.e. morphisms of $\mathcal{C}_{\text{id},\mathfrak{T}}$ under gauging. Because we have an action of G on $\mathcal{C}_{\text{id},\mathfrak{T}}$, the morphisms (between objects left invariant by G action) can be decomposed into representations of G. After gauging G, a morphism transforming in representation R of G is not gauge invariant, but can be made gauge invariant by attaching a Wilson line in representation R of G. See figure 2.13. This phenomenon provides information about the morphisms of $\mathcal{C}_{\text{id},\mathfrak{T}/G}$, and hence in particular the tensor product structure on objects of $\mathcal{C}_{\text{id},\mathfrak{T}/G}$. In particular, we can relate the morphism space $D_1^{(O,R_O)} \otimes D_1^{(O',R_{O'})} \to D_1^{(O'',R_{O''})}$ after gauging G to the subspace of the morphism space $D_1^{(O)} \otimes D_1^{(O')} \to D_1^{(O'')}$ before gauging G transforming in a specific representation of G determined by the representations $R_O, R_{O'}, R_{O''}$. We will see this in examples.

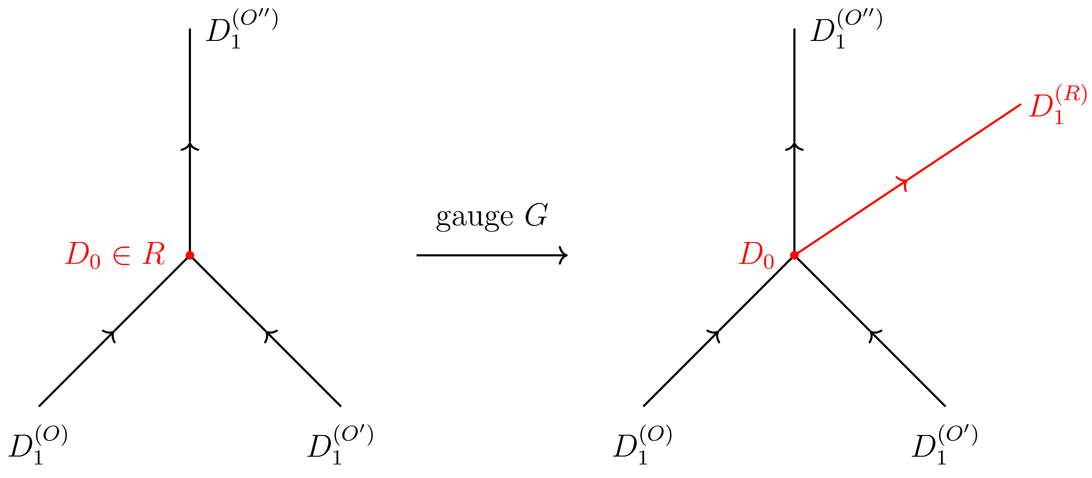


Figure 2.13: A local operator transforming in representation R of G before gauging G, is attached to a Wilson line in representation R of G after gauging G.

2.3.3 Gauging in Higher d and Fusion

We now extend the discussion of the previous subsection to arbitrary $d \geq 4$. The category $\mathcal{C}_{\text{id},\mathfrak{T}}$ before gauging is now a $(d-2)$-category. Our task is to determine the $(d-2)$-category $\mathcal{C}_{\text{id},\mathfrak{T}/G}$ obtained after gauging the 0-form symmetry G of the theory $\mathfrak{T}$.

The objects of $\mathcal{C}_{\text{id},\mathfrak{T}/G}$ are objects of $\mathcal{C}_{\text{id},\mathfrak{T}}$ left invariant by the G action, and objects related to such gauge invariant objects by condensation. Similarly, p-morphisms in $\mathcal{C}_{\text{id},\mathfrak{T}/G}$, $p \neq d-2, d-3$, descend from p-morphisms in $\mathcal{C}_{\text{id},\mathfrak{T}}$ left invariant by the G action, and p-morphisms related to such gauge invariant objects by condensation.

Fusion of two non-condensation defects can create condensation defects. That is, if we consider two p-morphisms of $\mathcal{C}_{\text{id},\mathfrak{T}/G}$ obtained directly as gauge invariant p-morphisms of $\mathcal{C}_{\text{id},\mathfrak{T}}$ without involving any condensation, then the product p-morphism is in general a p-morphism of $\mathcal{C}_{\text{id},\mathfrak{T}/G}$ obtained as a gauge invariant p-morphism of $\mathcal{C}_{\text{id},\mathfrak{T}}$ with an additional condensation on top of it. We will describe how the additional condensation can be determined for surface defects.

We still need to describe $(d-3)$-morphisms and $(d-2)$-morphisms of $\mathcal{C}_{\text{id},\mathfrak{T}/G}$. These correspond respectively to (genuine and non-genuine) topological line defects and topological local operators of the gauged theory $\mathfrak{T}/G$. As in previous subsection, to describe them we need to also incorporate Wilson line defects created by the G gauging. The analysis is a straightforward generalization of the analysis in the previous subsection. First of all, the $(d-3)$-morphisms of $\mathcal{C}_{\text{id},\mathfrak{T}}$ that are left invariant by G action descend to $(d-3)$-morphisms of $\mathcal{C}_{\text{id},\mathfrak{T}/G}$. Thus, a class of simple $(d-3)$-morphisms of $\mathcal{C}_{\text{id},\mathfrak{T}/G}$ are $D_1^{(O)}$ which can be represented in the category $\mathcal{C}_{\text{id},\mathfrak{T}}$ as

$$D_1^{(O)} = \bigoplus_{i \in O} D_1^{(i)} , \tag{2.23}$$

where O is an orbit under G action formed by simple $(d-3)$-morphisms $D_1^{(i)}$ of the category $\mathcal{C}_{\text{id},\mathfrak{T}}$. Other simple $(d-3)$-morphisms of $\mathcal{C}_{\text{id},\mathfrak{T}/G}$ are obtained by dressing $D_1^{(O)}$ by Wilson line defects valued in $\widehat{G}_O$, which is the Pontryagin dual of the stabiliser $G_O \subseteq G$ of the orbit O. These $(d-3)$-morphisms are represented as $D_1^{(O,R_O)}$, with $R_O \in \widehat{G}_O$. The $(d-2)$-morphisms of $\mathcal{C}_{\text{id},\mathfrak{T}/G}$ from $D_1^{(O,R_O)} \otimes D_1^{(O',R_{O'})}$ to $D_1^{(O'',R_{O''})}$ are obtained in terms of $(d-2)$-morphisms of $\mathcal{C}_{\text{id},\mathfrak{T}}$ from $D_1^{(O)} \otimes D_1^{(O')}$ to $D_1^{(O'')}$ as described in the previous subsection.

Fusion of Surface Defects and Condensation. We now provide the key to computing the fusion of topological surface defects in the symmetry category and describe how fusion of surfaces can create condensations.

Consider two surfaces $D_2^{(O)}$ and $D_2^{(O')}$ described by orbits O and O'. In the theory $\mathfrak{T}$ before gauging, they have a fusion rule

$$D_2^{(O)} \otimes D_2^{(O')} = \oplus_i D_2^{(i)} , \tag{2.24}$$

where $D_2^{(i)}$ are simple surfaces. We have line operators describing the 1-morphisms $D_2^{(O)} \otimes D_2^{(O')} \to D_2^{(i)}$ in the category $\mathcal{C}_{\text{id},\mathfrak{T}}$. These line operators organise themselves

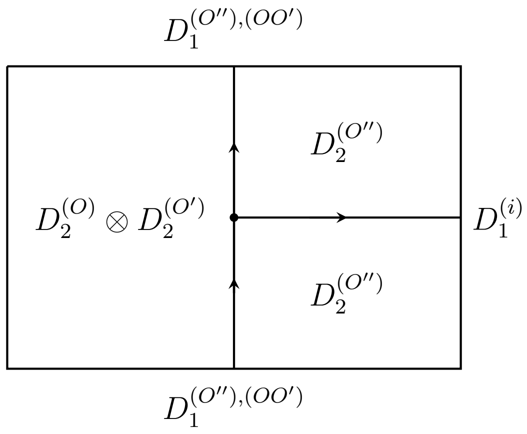


Figure 2.14: The coefficient n_i appearing in (2.26) is the dimension of the vector space of local operators shown in the figure.

into orbits under the G action such that we can write the above equation as

$$D_2^{(O)} \otimes D_2^{(O')} = \oplus_{O''} D_2^{(O'')} \,, \tag{2.25}$$

where O'' are orbits of surfaces. The right hand side $\oplus_{O''} D_2^{(O'')}$ of the above fusion is a representative of the equivalence class under condensation of $D_2^{(O)} \otimes D_2^{(O')}$ in the theory $\mathfrak{T}/G$.

Our task is now to describe the generalised gauging on top of each $D_2^{(O'')}$. This is captured in terms of an algebra $A^{(O''),(OO')}$ in the 1-category of localised symmetries $\mathcal{C}_{\text{id},\mathfrak{T}/G,D_2^{O''}}$, with object $A_1^{(O''),(OO')}$. From the gauging procedure, we obtain a line operator $D_1^{(O''),(OO')}$ describing a 1-morphism $D_2^{(O)} \otimes D_2^{(O')} \to D_2^{(O'')}$ in the category $\mathcal{C}_{\text{id},\mathfrak{T}/G}$. The algebra object can be expressed as

$$A_1^{(O''),(OO')} = \oplus_i n_i D_1^{(i)} \,, \tag{2.26}$$

where $D_1^{(i)}$ are the line operators living on $D_2^{(O'')}$ and n_i is the dimension of the vector space formed by local operators living at the end of $D_1^{(i)}$ along $D_1^{(O''),(OO')}$ as shown in figure 2.14. Thus, the algebra object $A_1^{(O''),(OO')}$ is described by local operators living on $D_1^{(O''),(OO')}$.

In summary the fusion of surfaces is computed as follows:

1. Determine the orbit decomposition of the surface fusion as in (2.25).

2. Compute the algebra object $A_1^{(O''),(OO')}$ which characterises the gauging of the localised symmetry on $D_2^{O''}$. This is computed as in (2.26) in terms of certain 2-morphisms capturing various kinds of local operators living on the line defect describing a 1-morphism $D_2^{(O)} \otimes D_2^{(O')} \to D_2^{(O'')}$.

3. Then the fusion of $D_2^{(O)} \otimes D_2^{(O')}$ will contain a term

$$\frac{D_2^{(O'')}}{A^{(O''),(OO')}} \tag{2.27}$$

which describes the condensation appearing on top of $D_2^{(O'')}$ in terms of the algebra $A^{(O''),(OO')}$, using which one performs the generalised gauging associated to the condensation.

We finally comment that the fusion of condensation defects, which are a central part of the symmetry category, will not be discussed here, as it requires developing further technology. We discuss some examples of this in the next chapter 3. The fusion of condensation defects has appeared e.g. in [61, 86, 115, 116].

2.4 Example: Pure $\mathrm{Pin}^+(4N)$ Gauge Theory in 3d

In this section we provide an example of a 3d theory whose topological line defects and local operators form a non-invertible symmetry described by a standard 1-category. We start with pure $\mathrm{Spin}(4N)$ Yang-Mills theory in 3d, which has a 1-form symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The outer-automorphism of the gauge algebra $\mathfrak{so}(4N)$ provides a $\mathbb{Z}_2$ 0-form symmetry of the theory that exchanges the two $\mathbb{Z}_2$ factors of the 1-form symmetry group. Gauging this 0-form symmetry results in the pure $\mathrm{Pin}^+(4N)$ Yang-Mills theory in 3d, which we show to contain a non-invertible categorical symmetry descending from the 1-form symmetry of $\mathrm{Spin}(4N)$ Yang-Mills theory.

We label the two $\mathbb{Z}_2$ factors in the 1-form symmetry group of $\mathrm{Spin}(4N)$ theory as $\mathbb{Z}_2^{(S)}$ and $\mathbb{Z}_2^{(C)}$ depending on whether the $\mathbb{Z}_2$ leaves the spinor irrep S invariant, or the cospinor irrep C invariant. The diagonal $\mathbb{Z}_2$ factor can be represented as $\mathbb{Z}_2^{(V)}$ as it leaves the vector irrep invariant. Thus, we express the 1-form symmetry group $\Gamma^{(1)}$ as

$$\Gamma^{(1)} = \mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(C)} . \tag{2.28}$$

The outer-automorphism provides a 0-form symmetry group

$$\Gamma^{(0)} = \mathbb{Z}_2^{(0)} , \tag{2.29}$$

which exchanges $\mathbb{Z}_2^{(S)}$ and $\mathbb{Z}_2^{(C)}$, while leaving the $\mathbb{Z}_2^{(V)}$ invariant.

The data of 1-form symmetry can be converted into the data of a 1-category $\mathcal{C}_{\mathrm{Spin}(4N)}$ as follows. The simple objects of $\mathcal{C}_{\mathrm{Spin}(4N)}$ correspond to topological line operators implementing the 1-form symmetry $\Gamma^{(1)}$, and we write the set of simple objects as

$$\mathcal{C}^{\mathrm{ob}}_{\mathrm{Spin}(4N)} = \left\{ D_1^{(\mathrm{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\} , \tag{2.30}$$

where $D_1^{(\mathrm{id})}$ is the identity line, and $D_1^{(i)}$ are the topological line operators corresponding respectively to generators of $\mathbb{Z}_2^{(i)}$ 1-form symmetries. The $\mathbb{Z}_2$ 0-form symmetry acts on the 1-form symmetry generators as

$$D_1^{(S)} \longleftrightarrow D_1^{(C)} . \tag{2.31}$$

and leaves $D_1^{(\mathrm{id})}, D_1^{(V)}$ invariant. The tensor product of these objects follows the group law of $\Gamma^{(1)}$.

Now we gauge $\mathbb{Z}_2$ to obtain a category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ describing topological line defects and local operators of the $\mathrm{Pin}^+(4N)$ theory. A subset of simple objects of $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ arise as objects of $\mathcal{C}_{\mathrm{Spin}(4N)}$ left invariant by the $\mathbb{Z}_2$ outer automorphism action. These are $D_1^{(\mathrm{id})}$, $D_1^{(V)}$ and

$$D_1^{(SC)} := \left(D_1^{(S)} \oplus D_1^{(C)}\right)_{\mathcal{C}_{\mathrm{Spin}}} , \tag{2.32}$$

where the subscript $\mathcal{C}_{\mathrm{Spin}(4N)}$ on the RHS reflects that the object $D_1^{(SC)}$ is decomposed as this direct sum only in the category $\mathcal{C}_{\mathrm{Spin}(4N)}$, but it is a simple object in the category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$.

Other simple objects of $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ are obtained by dressing with Wilson line defects. Note that the stabiliser for $D_1^{(\mathrm{id})}$, $D_1^{(V)}$ is the whole 0-form symmetry group $\mathbb{Z}_2$, while the stabiliser for $D_1^{(SC)}$ is trivial. Thus, we obtain new simple objects of $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ by dressing $D_1^{(\mathrm{id})}$, $D_1^{(V)}$ with the non-trivial irrep of $\mathbb{Z}_2$. We call the resulting simple objects as $D_1^{(-)}$, $D_1^{(V_-)}$ respectively. Thus, the full set of simple objects of $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ is

$$\mathcal{C}^{\mathrm{ob}}_{\mathrm{Pin}^+(4N)} = \left\{D_1^{(\mathrm{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)}\right\} . \tag{2.33}$$

The topological line defects

$$\left\{D_1^{(\mathrm{id})}, D_1^{(-)}\right\} , \tag{2.34}$$

are the Wilson line defects for the gauged $\mathbb{Z}_2$ 0-form symmetry, and hence generate the dual $\mathbb{Z}_2$ 1-form symmetry arising as a result of $\mathbb{Z}_2$ 0-form gauging, with the non-trivial fusion rule (2.19)

$$D_1^{(-)} \otimes D_1^{(-)} = D_1^{(\mathrm{id})} \tag{2.35}$$

following the representation theory of $\mathbb{Z}_2$.

Moreover, from (2.20) we have the following non-trivial fusions

$$\begin{aligned} D_1^{(SC)} \otimes D_1^{(-)} &= D_1^{(SC)} \\ D_1^{(V)} \otimes D_1^{(-)} &= D_1^{(V_-)} \end{aligned} \tag{2.36}$$

in the category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$.

Now we consider the tensor product $D_1^{(SC)} \otimes D_1^{(SC)}$ in the category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$. Notice that in $\mathcal{C}_{\mathrm{Spin}(4N)}$ we have the fusion

$$\left(D_1^{(SC)} \otimes D_1^{(SC)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}} = \left(2D_1^{(\mathrm{id})} \oplus 2D_1^{(V)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}}, \tag{2.37}$$

which implies that, in the category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$, only $D_1^{(\mathrm{id})}$, $D_1^{(-)}$, $D_1^{(V)}$ and $D_1^{(V_-)}$ can appear in the fusion $D_1^{(SC)} \otimes D_1^{(SC)}$. To determine the precise multiplicity of these objects, we need to study the $\mathbb{Z}_2$ representations formed by morphisms from $D_1^{(SC)} \otimes D_1^{(SC)}$ to $D_1^{(\mathrm{id})}$ in $\mathcal{C}_{\mathrm{Spin}(4N)}$ and the $\mathbb{Z}_2$ representations formed by morphisms from $D_1^{(SC)} \otimes D_1^{(SC)}$ to $D_1^{(V)}$ in $\mathcal{C}_{\mathrm{Spin}(4N)}$.

Let us first consider the $\mathcal{C}_{\mathrm{Spin}(4N)}$ morphisms from $D_1^{(SC)} \otimes D_1^{(SC)}$ to $D_1^{(\mathrm{id})}$. There is a 2-dimensional vector space of such morphisms of $\mathcal{C}_{\mathrm{Spin}(4N)}$ spanned by a morphism $D_0^{(S\otimes S,\mathrm{id})}$ from $D_1^{(S)} \otimes D_1^{(S)}$ to $D_1^{(\mathrm{id})}$, and a morphism $D_0^{(C\otimes C,\mathrm{id})}$ from $D_1^{(C)} \otimes D_1^{(C)}$ to $D_1^{(\mathrm{id})}$. It is clear that $\mathbb{Z}_2$ outer automorphism acts as the exchange

$$D_0^{(S\otimes S,\mathrm{id})} \longleftrightarrow D_0^{(C\otimes C,\mathrm{id})}. \tag{2.38}$$

Thus the morphism space decomposes as $1 \oplus 1_-$ under the $\mathbb{Z}_2$ 0-form symmetry. Since there is a single copy of both $\mathbb{Z}_2$ representations, we learn that $D_1^{(SC)} \otimes D_1^{(SC)}$ contains a single copy of $D_1^{(\mathrm{id})}$ and a single copy of $D_1^{(-)}$ in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$. Now consider the $\mathcal{C}_{\mathrm{Spin}(4N)}$ morphisms from $D_1^{(SC)} \otimes D_1^{(SC)}$ to $D_1^{(V)}$. Completely analogously to the above, there is a 2-dimensional vector space of such morphisms which decomposes as $1 \oplus 1_-$ under the $\mathbb{Z}_2$ 0-form symmetry. Since there is a single copy of both $\mathbb{Z}_2$ representations, we learn that $D_1^{(SC)} \otimes D_1^{(SC)}$ contains a single copy of $D_1^{(V)}$ and a single copy of $D_1^{(V_-)}$ in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$. Combining everything, we learn the fusion rule

$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\mathrm{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)} \tag{2.39}$$

of $\mathcal{C}_{\mathrm{Pin}^+(4N)}$. Since the RHS contains objects other than $D_1^{(\mathrm{id})}$, we find that $D_1^{(SC)}$ is a non-invertible topological line defect. Thus, the category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ describes non-invertible symmetries of the $\mathrm{Pin}^+(4N)$ gauge theory.

Finally, we determine the fusions $D_1^{(SC)} \otimes D_1^{(V)}$ and $D_1^{(SC)} \otimes D_1^{(V_-)}$ in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$. Notice that in $\mathcal{C}_{\mathrm{Spin}(4N)}$ we have the fusion

$$\begin{aligned}\left(D_1^{(SC)} \otimes D_1^{(V)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}} = \left((D_1^{(S)} \oplus D_1^{(C)}) \otimes D_1^{(V)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}} &= \left(D_1^{(C)} \oplus D_1^{(S)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}} \\ &= \left(D_1^{(SC)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}}\end{aligned} \tag{2.40}$$

which implies that, in the category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$, only at most a single copy of $D_1^{(SC)}$ can appear in the fusions $D_1^{(SC)} \otimes D_1^{(V)}$ and $D_1^{(SC)} \otimes D_1^{(V_-)}$. Using associativity and (2.39), we learn that

$$\begin{aligned} D_1^{(SC)} \otimes D_1^{(V)} &= D_1^{(SC)} \\ D_1^{(SC)} \otimes D_1^{(V_-)} &= D_1^{(SC)} \end{aligned} \tag{2.41}$$

are the descending fusion rules in the category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$.

From the fusion rules, we observe that the resulting category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ can be recognised as one of the Tambara-Yamagami categories based on the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are four such categories [117] (see also Section 5.5 of [88]), and it is a natural question to ask which one is $\mathcal{C}_{\mathrm{Pin}^+(4N)}$. The difference between the four categories is captured in the data of the associators. We can compute the associators of $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ from the associators of $\mathcal{C}_{\mathrm{Spin}(4N)}$, where the latter associators are trivial as $\mathcal{C}_{\mathrm{Spin}(4N)}$ describes a non-anomalous 1-form symmetry. From this computation, we find that

$$\mathcal{C}_{\mathrm{Pin}^+(4N)} = \mathrm{Rep}(D_8)\,, \tag{2.42}$$

i.e. $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ is the category formed by representations of the dihedral group D_8.

2.5 Example: Pure $\mathrm{Pin}^+(4N)$ Gauge Theory in 4d

In this section we provide an example of 4d theory whose topological surface defects, line defects and local operators form a non-invertible symmetry described by a 2-category. We start with 4d $\mathrm{Spin}(4N)$ pure Yang-Mills theory which has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry on which a $\mathbb{Z}_2$ 0-form symmetry acts non-trivially. Gauging the $\mathbb{Z}_2$ 0-form symmetry leads to the 4d $\mathrm{Pin}^+(4N)$ pure Yang-Mills theory. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry of the $\mathrm{Spin}(4N)$ theory descends to a non-invertible 2-categorical symmetry in the $\mathrm{Pin}^+(4N)$ theory. We discuss the topological defects in the two theories before and after gauging, including their fusion algebra.

The 1-form symmetry of the $\mathrm{Spin}(4N)$ theory is

$$\Gamma^{(1)} = \mathbb{Z}_2^S \times \mathbb{Z}_2^C\,. \tag{2.43}$$

As before, we represent by $\mathbb{Z}_2^V$ the diagonal $\mathbb{Z}_2$ of $\mathbb{Z}_2^S$ and $\mathbb{Z}_2^C$. The theory has a

$$\Gamma^{(0)} = \mathbb{Z}_2 \tag{2.44}$$

outer-automorphism 0-form symmetry which exchanges $\mathbb{Z}_2^S$ and $\mathbb{Z}_2^C$, while leaving $\mathbb{Z}_2^V$ invariant.

The 1-form symmetry $\Gamma^{(1)}$ corresponds to a rather trivial 2-category $\mathcal{C}_{\mathrm{Spin}(4N)}$, whose simple objects are

$$\mathcal{C}^{\text{ob}}_{\text{Spin}(4N)} = \left\{ D_2^{(\text{id})}, D_2^{(S)}, D_2^{(C)}, D_2^{(V)} \right\} , \tag{2.45}$$

where $D_2^{(\text{id})}$ is the identity surface defect, while $D_2^{(i)}$ for $i \in \{S, C, V\}$ is the topological surface defect corresponding to the generator of $\mathbb{Z}_2^i$. The fusion of these surface defects follows the group law of $\Gamma^{(1)}$

$$D_2^{(i)} \otimes D_2^{(j)} = D_2^{(ij)} \tag{2.46}$$

with $ij \in \Gamma^{(1)}$.

There is a single simple 1-endomorphism for each simple object, which we denote as $D_1^{(i)}$ for $i \in \{\text{id}, S, C, V\}$. It can be understood as the identity line defect living on each simple surface defect $D_2^{(i)}$. There are no 1-morphisms between two distinct simple objects. Thus, the full set of simple 1-endomorphisms of simple objects is

$$\mathcal{C}^{\text{1-endo}}_{\text{Spin}(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\} . \tag{2.47}$$

The fusion $\otimes$ for 1-endomorphisms follows the group law of $\Gamma^{(1)}$.

The $\Gamma^{(0)} = \mathbb{Z}_2$ outer-automorphism 0-form symmetry acts on $\mathcal{C}_{\text{Spin}(4N)}$ as

$$D_i^{(S)} \longleftrightarrow D_i^{(C)} \tag{2.48}$$

for each $i \in \{1, 2\}$, while leaving invariant $D_i^{(\text{id})}$ and $D_i^{(V)}$.

We now gauge the outer automorphism $\mathbb{Z}_2$, which results in the $\text{Pin}^+(4N)$ gauge theory. Let us call the resulting symmetry 2-category as $\mathcal{C}_{\text{Pin}^+(4N)}$. The objects of $\mathcal{C}_{\text{Pin}^+(4N)}$ modulo condensations are the objects of $\mathcal{C}_{\text{Spin}(4N)}$ left invariant by the $\mathbb{Z}_2$

action. Thus, the simple objects of $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ modulo condensations are[3]

$$\mathcal{C}^{\mathrm{ob}}_{\mathrm{Pin}^+(4N)} = \left\{ D_2^{(\mathrm{id})}, D_2^{(SC)}, D_2^{(V)} \right\} , \tag{2.49}$$

where

$$D_2^{(SC)} = \left(D_2^{(S)} \oplus D_2^{(C)} \right)_{\mathcal{C}_{\mathrm{Spin}(4N)}} \tag{2.50}$$

as an object of the 2-category $\mathcal{C}_{\mathrm{Spin}(4N)}$.

Now let us discuss 1-morphisms between the objects appearing in the set $\mathcal{C}^{\mathrm{ob}}_{\mathrm{Pin}^+(4N)}$. Since there are no non-identity simple 1-endomorphisms of the identity object in $\mathcal{C}_{\mathrm{Spin}(4N)}$, the simple 1-endomorphisms of the identity object $D_2^{(\mathrm{id})}$ in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ are

$$\left\{ D_1^{(\mathrm{id})}, D_1^{(-)} \right\} , \tag{2.51}$$

which are the Wilson line defects for the gauged $\mathbb{Z}_2$. Similarly, since the simple 1-endomorphism $D_1^{(V)}$ in $\mathcal{C}_{\mathrm{Spin}(4N)}$ is left invariant by the $\mathbb{Z}_2$ action, the simple 1-endomorphisms of the object $D_2^{(V)}$ in the 2-category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ are

$$\left\{ D_1^{(V)}, D_1^{(V-)} \right\} , \tag{2.52}$$

where $D_1^{(V)}$ is the identity 1-endomorphism on $D_2^{(V)}$, and $D_1^{(V-)}$ is obtained by dressing $D_1^{(V)}$ with the non-trivial $\mathbb{Z}_2$ Wilson line. On the other hand, there is only the identity 1-endomorphism $D_1^{(SC)}$ of $D_2^{(SC)}$, which can be expressed as

$$D_1^{(SC)} = \left(D_1^{(S)} \oplus D_1^{(C)} \right)_{\mathcal{C}_{\mathrm{Spin}(4N)}} \tag{2.53}$$

[3]The simple objects of the symmetry category include the condensation defects. We use the notation $\mathcal{C}^{\mathrm{ob}}$ to denote however simple objects modulo condensation, since we will only discuss fusion of these objects, with the condensation defects being discussed elsewhere [86, 115, 116].

as a 1-morphism of $\mathcal{C}_{\text{Spin}(4N)}$. Thus,

$$\mathcal{C}^{\text{1-endo}}_{\text{Pin}^+(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)} \right\} \tag{2.54}$$

are the simple 1-endomorphisms of simple objects $\mathcal{C}^{\text{ob}}_{\text{Pin}^+(4N)}$.

Let us deduce the non-trivial fusion rules of the objects in $\mathcal{C}^{\text{ob}}_{\text{Pin}^+(4N)}$. First of all, we have

$$D_2^{(V)} \otimes D_2^{(V)} = D_2^{(\text{id})} \, . \tag{2.55}$$

This is just the fusion rules of $\mathcal{C}_{\text{Spin}(4N)}$ as there is no $\mathbb{Z}_2$ action on the involved objects. Thus, $D_2^{(V)}$ is an invertible surface defect which can be recognised as generating the $\mathbb{Z}_2$ center 1-form symmetry of the $\text{Pin}^+(4N)$ theory.

To determine the fusion rule $D_2^{(V)} \otimes D_2^{(SC)}$, notice that there are two simple 1-morphisms from the object $D_2^{(V)} \otimes D_2^{(SC)}$ to the object $D_2^{(SC)}$ in the 2-category $\mathcal{C}_{\text{Spin}(4N)}$ and no 1-morphisms from $D_2^{(V)} \otimes D_2^{(SC)}$ to any other object. The two simple 1-morphisms can be recognised as the 1-morphisms

$$\begin{aligned} D_1^{(V,S;C)} &: \; D_2^{(V)} \otimes D_2^{(S)} \to D_2^{(C)} \\ D_1^{(V,C;S)} &: \; D_2^{(V)} \otimes D_2^{(C)} \to D_2^{(S)} \end{aligned} \tag{2.56}$$

in the 2-category $\mathcal{C}_{\text{Spin}(4N)}$. The $\mathbb{Z}_2$ outer-automorphism acts as the exchange

$$D_1^{(V,S;C)} \longleftrightarrow D_1^{(V,C;S)} \tag{2.57}$$

Thus, there is a single simple 1-morphism

$$D_1^{(V,SC;SC)} : \; D_2^{(V)} \otimes D_2^{(SC)} \to D_2^{(SC)} \tag{2.58}$$

in the 2-category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$, which can be expressed as

$$D_1^{(V,SC;SC)} = \left(D_1^{(V,S;C)} \oplus D_1^{(V,C;S)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}} \tag{2.59}$$

in the category $\mathcal{C}_{\mathrm{Spin}(4N)}$, leading to the fusion rule

$$D_2^{(V)} \otimes D_2^{(SC)} = D_2^{(SC)} \tag{2.60}$$

in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$. There is no possibility of condensations arising on the right hand side of the above equation, because there are no non-trivial lines living on $D_2^{(SC)}$ as discussed above.

Now let us discuss $D_2^{(SC)} \otimes D_2^{(SC)}$ in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$. From the corresponding fusion in $\mathcal{C}_{\mathrm{Spin}(4N)}$, we see that only $D_2^{(\mathrm{id})}$ and $D_2^{(V)}$ can appear in this fusion. There are two 1-morphisms $D_2^{(SC)} \otimes D_2^{(SC)} \to D_2^{(\mathrm{id})}$ in $\mathcal{C}_{\mathrm{Spin}(4N)}$, which can be recognised as

$$\begin{aligned} D_1^{(S,S;\mathrm{id})} &: \; D_2^{(S)} \otimes D_2^{(S)} \to D_2^{(\mathrm{id})} \\ D_1^{(C,C;\mathrm{id})} &: \; D_2^{(C)} \otimes D_2^{(C)} \to D_2^{(\mathrm{id})} \,. \end{aligned} \tag{2.61}$$

Similarly, there are two 1-morphisms $D_2^{(SC)} \otimes D_2^{(SC)} \to D_2^{(V)}$ in $\mathcal{C}_{\mathrm{Spin}(4N)}$, which can be recognised as

$$\begin{aligned} D_1^{(S,C;V)} &: \; D_2^{(S)} \otimes D_2^{(C)} \to D_2^{(V)} \\ D_1^{(C,S;V)} &: \; D_2^{(C)} \otimes D_2^{(S)} \to D_2^{(V)} \,. \end{aligned} \tag{2.62}$$

These 1-morphisms are exchanged by the $\mathbb{Z}_2$ action as

$$\begin{aligned} D_1^{(S,S;\mathrm{id})} &\longleftrightarrow D_1^{(C,C;\mathrm{id})} \\ D_1^{(C,S;V)} &\longleftrightarrow D_1^{(S,C;V)} \,. \end{aligned} \tag{2.63}$$

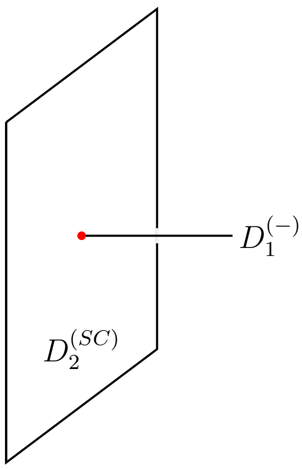


Figure 2.15: The defect configuration describing a 2-morphism from $D_1^{(SC)} \otimes D_1^{(SC)}$ to $D_1^{(-)}$, where $D_1^{(SC)}$ is the identity line living on the surface $D_2^{(SC)}$ shown in the figure.

Thus, we have single simple 1-morphisms

$$\begin{aligned} D_1^{(SC,SC;\mathrm{id})} &: \; D_2^{(SC)} \otimes D_2^{(SC)} \to D_2^{(\mathrm{id})} \\ D_1^{(SC,SC;V)} &: \; D_2^{(SC)} \otimes D_2^{(SC)} \to D_2^{(V)} \end{aligned} \tag{2.64}$$

in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$, and hence following the general analysis of section 2.3.3, we obtain that the fusion rule must take the form

$$D_2^{(SC)} \otimes D_2^{(SC)} = \frac{D_2^{(\mathrm{id})}}{A^{(\mathrm{id})}} \oplus \frac{D_2^{(V)}}{A^{(V)}} \tag{2.65}$$

in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$, where we still need to determine the algebras $A^{(\mathrm{id})}$ and $A^{(V)}$ describing the condensation/gauging on top of the surfaces $D_2^{(\mathrm{id})}$ and $D_2^{(V)}$ respectively. The form of the above fusion rule means that $D_2^{(SC)}$ is a non-invertible surface defect, and hence the 2-category $\mathcal{C}_{\mathrm{Pin}^+(4N)}$ describes a *non-invertible symmetry* of the $\mathrm{Pin}^+(4N)$ theory.

To complete the description of above fusion, we need to determine local operators corresponding to 2-morphisms from $D_1^{(SC)} \otimes D_1^{(SC)}$ to lines $D_1^{(\mathrm{id})}$, $D_1^{(-)}$, $D_1^{(V)}$ and $D_1^{(V-)}$. This is the same as the determination of local operators in the analogous 3d case we considered in the previous section. From the results obtained there, we learn that there is a single dimensional vector space of 2-morphisms from $D_1^{(SC)} \otimes D_1^{(SC)}$ to each of the lines $D_1^{(\mathrm{id})}$, $D_1^{(-)}$, $D_1^{(V)}$ and $D_1^{(V_-)}$. Let us make a side comment here in order to resolve some of the confusing statements found in previous literature: the fact that we have a 2-morphism

$$D_1^{(SC)} \otimes D_1^{(SC)} \to D_1^{(-)} \tag{2.66}$$

means that there is a non-zero local operator lying at the intersection of the genuine line $D_1^{(-)}$ and the genuine surface $D_2^{(SC)}$, since $D_1^{(SC)}$ is the identity line on the surface $D_2^{(SC)}$. See figure 2.15. This looks like a configuration implying that a surface can fuse with itself to give line defects, and could motivate one to introduce fusion rules taking two objects (i.e. surfaces) to a 1-morphism (i.e. a line). However, in the standard definitions of 2-category used in mathematics, the fusion of two objects is always an object. As we have described above, mathematically this figure is interpreted instead as the existence of a particular 2-morphism.

Returning back to our original problem, we have determined the algebra objects to be

$$\begin{aligned} A_1^{(\mathrm{id})} &= D_1^{(\mathrm{id})} \oplus D_1^{(-)} \\ A_1^{(V)} &= D_1^{(V)} \oplus D_1^{(V_-)}\,, \end{aligned} \tag{2.67}$$

which means that we have to gauge the $\mathbb{Z}_2$ 0-form symmetry on $D_2^{(\mathrm{id})}$ generated by $D_1^{(-)}$ and the $\mathbb{Z}_2$ 0-form symmetry localised on $D_2^{(V)}$ generated by $D_1^{(V_-)}$. There is a unique way to perform this gauging as $H^2(B\mathbb{Z}_2, U(1)) = 0$. Consequently the morphisms comprising the algebras $A^{(\mathrm{id})}$ and $A^{(V)}$ are uniquely fixed, and the full fusion rule can be expressed as

$$D_2^{(SC)} \otimes D_2^{(SC)} = \frac{D_2^{(\mathrm{id})}}{\mathbb{Z}_2} \oplus \frac{D_2^{(V)}}{\mathbb{Z}_2}\,. \tag{2.68}$$

The fusions of 1-morphisms are determined in [1].

2.6 Non-Invertibles from Higher-Groups

In [59], a construction was presented that takes as input a d-dimensional quantum field theory $\mathfrak{T}$ with a certain type of mixed anomaly, i.e. one that is linear in the background field A_{p+1} corresponding to a p-form global symmetry $\Gamma^{(p)}$, and produces as an output a new quantum field theory $\mathfrak{T}'$ which contains $(d-p-1)$-dimensional non-invertible defects. The descendent theory $\mathfrak{T}'$ is obtained by gauging some part of the symmetry structure of $\mathfrak{T}$ that is contained in the complement of $\Gamma^{(p)}$ and also appears manifestly in the anomaly action. Crucially, the anomaly, by definition, poses an obstruction to gauging while preserving all the symmetries, that is alleviated by locally modifying the $\Gamma^{(p)}$ defect. In fact, the local modification is what causes the $\Gamma^{(p)}$ defect to become non-invertible in $\mathfrak{T}'$.

Concretely, let the symmetry structure of $\mathfrak{T}$ be a product of higher-form groups $\mathbb{G}_{\mathfrak{T}} = \prod_{a=0}^{d-2} \Gamma^{(a)}$. Some of the factors $\Gamma^{(a)}$ could be trivial.

It is more convenient to formulate everything in terms of background fields A_{a+1} in terms of which the anomaly action is given by

$$\mathcal{A} = \int_{N_{d+1}} A_{p+1} \cup \xi(A^c_{p+1}), \quad A^c_{p+1} = \{A_{a+1}\}_{a\neq p} \;, \tag{2.69}$$

where

$$\xi \in H^{d-p}\left(\mathbb{G}_{\mathfrak{T}}/\Gamma^{(p)}, \Gamma^{(p)}_{\text{dual}}\right), \quad \xi(A^c_{p+1}) \in H^{d-p}\left(N_{d+1}, \Gamma^{(p)}_{\text{dual}}\right) \quad , \tag{2.70}$$

where $\Gamma^{(p)}_{\text{dual}} := \hom(\Gamma^{(p)}, \mathbb{R}/2\pi\mathbb{Z})$. N_{d+1} is an auxiliary $d+1$-manifold, used to define the anomaly in (2.69), whose boundary is the d-manifold where $\mathfrak{T}$ lives. Since $\mathbb{G}_{\mathfrak{T}}$ is a product of higher groups, the quotient by $\Gamma^{(p)}$ should be understood more technically as taking the quotient on the classifying space $B\mathbb{G}$ which is a Cartesian product of $B^{a+1}\Gamma^{(a)}$. We suppress such technicalities since they make the presentation heavy without adding much content. Let the defect corresponding to $\gamma \in \Gamma^{(p)}$ be denoted as D_γ, and by $D_\gamma(\Sigma_{d-p-1})$ if it is wrapped along a $(d-p-1)$-dimensional sub-manifold Σ_{d-p-1} of M_d. Due to the anomaly, such a defect carries a non-trivial dependence on the background A^c_{p+1} which cannot be localised on Σ_{d-p-1}. Now consider gauging some subgroup of $\mathbb{G}_{\mathfrak{T}}/\Gamma^{(p)}$ on which ξ depends. Doing so, we obtain a gauged theory $\mathfrak{T}'$ in which the defect D_γ becomes ill-defined due to an anomaly. More precisely, it has a dependence on dynamical fields that cannot be localised on Σ_{d-p-1}. This situation can be remedied by a local modification to D_γ, which involves adding a topological field theory $\mathcal{X}_\gamma$ with a $\mathbb{G}_{\mathfrak{T}}/\Gamma^{(p)}$ 't-Hooft anomaly ξ. The defects in the gauged theory correspondingly are modified as

$$D_\gamma \longmapsto \mathcal{N}_\gamma = D_\gamma \mathcal{X}_\gamma. \tag{2.71}$$

Notably any such theory $\mathfrak{T}$ with anomaly (2.69) can, in turn, be obtained from a theory $\mathfrak{T}_0$ with a non-anomalous higher-group symmetry $\mathbb{G}$, which sits in the short exact sequence

$$1 \longrightarrow B^{d-p-1}\Gamma^{(p)}_{\text{dual}} \longrightarrow B\mathbb{G} \longrightarrow B\mathbb{G}_{\mathfrak{T}}/B\Gamma^{(p)} \longrightarrow 1, \tag{2.72}$$

with an extension class ξ. The symmetry structure of $\mathfrak{T}$ is obtained from the symmetry structure of $\mathfrak{T}_0$ by gauging $\Gamma^{(p)}_{\text{dual}}$ in $\mathfrak{T}_0$. In summary, the non-invertibles discussed in [59] can be obtained by starting from a higher group and gauging in two steps.

2.6.1 Non-Invertibles from 2-Groups in Pure 4d $\mathfrak{so}(4N)$ Yang-Mills

Let us consider pure Spin$(4N)$ gauge theory and for concreteness let us work in 4d. The theory has a $\Gamma^{(1)} = \mathbb{Z}_2^{(1),B} \times \mathbb{Z}_2^{(1),C}$ 1-form symmetry and a $\Gamma^{(0)} = \mathbb{Z}_2^{(0)}$ outer-automorphism 0-form symmetry. The two symmetries combine into a 2-group, which in terms of the background gauge fields reads

$$\delta B_2 = A_1 C_2 \,, \tag{2.73}$$

where B_2, C_2 are the backgrounds for the two 1-form symmetry factors and A_1 is the background for the 0-form symmetry.

This 2-group is equivalent to the $\mathbb{Z}_2^{(0)}$ outer-automorphism action exchanging two $\mathbb{Z}_2^{(1)}$ subgroups of the 1-form symmetry, as can be understood pictorially in the following way (see figure 2.16). We denote the subgroups of $\Gamma^{(1)}$ which are exchanged by the $\mathbb{Z}_2^{(0)}$ action by $\mathbb{Z}_2^{(1),S}$ and $\mathbb{Z}_2^{(1),C}$, while $\mathbb{Z}_2^{(1),B}$ denotes the diagonal subgroup.[4] In terms of symmetry defects, the action is the following: if a topological surface defect $D_2^{(C)}$ associated to $\mathbb{Z}_2^{(1),C}$crosses the codimension-1 defect $D_3^{(-)}$ for $\mathbb{Z}_2^{(0)}$, it emerges as the defect $D_2^{(S)}$ associated to $\mathbb{Z}_2^{(1),S}$. Since, following the $\Gamma^{(1)}$ group law, we have that $D_2^{(S)} = D_2^{(B)} \otimes D_2^{(C)}$, we can re-interpret the above action in this way: upon passing $D_2^{(C)}$ through a codimension-1 defect for $\mathbb{Z}_2^{(0)}$, we create a codimension-3 junction from which the defect $D_2^{(B)}$ associated to the $\mathbb{Z}_2^{(1),B}$ subgroup is emitted. The 2-group (2.73) states precisely this: at the junction of $D_3^{(-)}$ (on a 3-cycle Poincaré dual to A_1) and $D_2^{(C)}$ (on a 2-cycle Poincaré dual to C_2), there is a flux for $\mathbb{Z}_2^{(1),B}$ meaning that a non-trivial background B_2 is sourced.

We will show that by gauging various combination of the symmetries appearing in the 2-group (2.73) we can go to different theories that have non-invertible symmetries:

[4] Notice that this was denoted in the previous sections as $\mathbb{Z}_2^{(1),V}$.

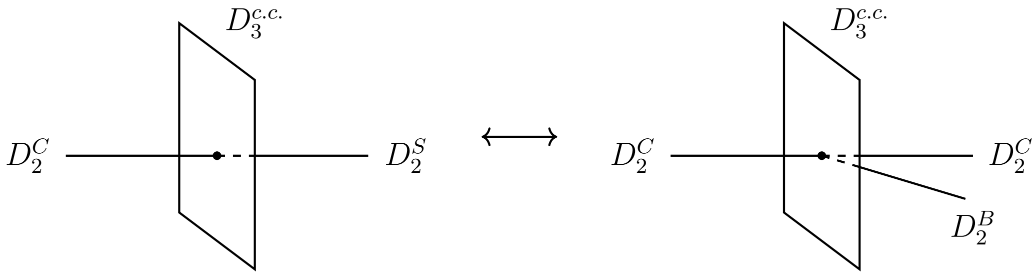


Figure 2.16: Left: exchange action of $\mathbb{Z}_2^{(0)}$ on the 1-form symmetry defects $D_2^{(S)}$ and $D_2^{(C)}$. Right: equivalently, the 1-form symmetry defect $D_2^{(B)}$ associated to the diagonal subgroup is emitted at the junction of $D_3^{(-)}$ and $D_2^{(C)}$. This represents pictorially the 2-group $\delta B_2 = A_1 C_2$.

1. $PO(4N)$ theory:

   gauge B_2, C_2, A_1: we obtain a codimension-2 non-invertible defect;

2. $\mathrm{Pin}^+(4N)$ theory:

   gauge A_1: we obtain a codimension-2 non-invertible defect;

3. $Sc(4N)$ theory:

   gauge C_2: we obtain a codimension-1 non-invertible defect.

A way of deriving this result is to first gauge the $\mathbb{Z}_2^{(1),B}$ subgroup of the 1-form symmetry to go to $SO(4N)$ gauge theory by promoting B_2 to a dynamical field b_2 (see figure 2.17). The $SO(4N)$ theory has an emergent dual 1-form symmetry $\mathbb{Z}_2^{(1),B'}$ (in 4d), whose background we denote by B'_2 and which couples as $\pi \int_{M_4} b_2 B'_2$. Due to the relation (2.73), this coupling is ill-defined, as it has a bulk dependency

$$\mathcal{A} = \pi \int_{M_5} \delta b_2 B'_2 = \pi \int_{M_5} A_1 C_2 B'_2 \,. \tag{2.74}$$

This results in a mixed 't Hooft anomaly for the $SO(4N)$ theory. Using this map from 2-groups to mixed 't Hooft anomalies, the fusion rules can then by derived by following the approach of [59]. We summarise the non-invertible defects that we obtain

- $\mathcal{N}(M_2; B_2')$: non-invertible defect in $PO(4N)$, corresponding to the codimension-2 defect generating $\mathbb{Z}_2^{(1),B'}$;

- $\mathcal{N}(M_2; C_2)$: non-invertible defect in $\mathrm{Pin}^+(4N)$, corresponding to the codimension-2 defect generating $\mathbb{Z}_2^{(1),C}$;

- $\mathcal{N}(M_3; A_1)$: non-invertible defect in $Sc(4N)$, corresponding to the codimension-1 defect generating $\mathbb{Z}_2^{(0)}$.

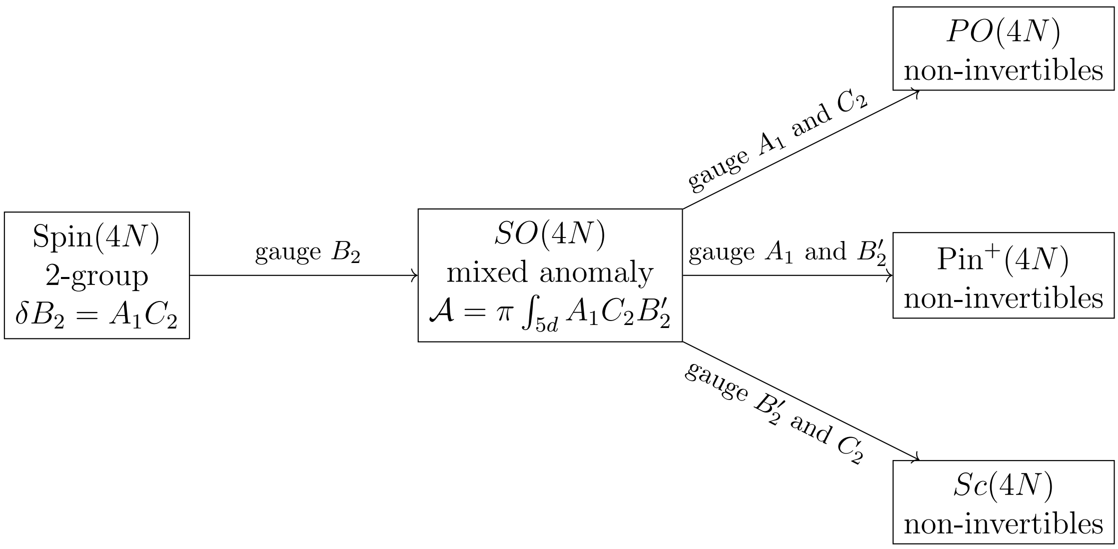


Figure 2.17: Overview of the theories with non-invertible symmetries that we can construct from gauging the 2-group in the Spin($4N$) theory in 4d.

Example: Fusion rules in Pin$^+(4N)$. This can be obtained by gauging B_2' and A_1 in the $SO(4N)$ theory. Gauging $\mathbb{Z}_2^{(1),B'}$ we recover Spin($4N$), and gauging A_1 we obtain Pin$^+(4N)$. Therefore the overall effect of these gaugings is to gauge charge conjugation in Spin($4N$) theory. The fusion algebra that we find is

$$\begin{aligned}
\mathcal{N}(M_2; C_2) \times \mathcal{N}(M_2; C_2) &= \frac{1 + T(M_2)}{|H^0(M_2, \mathbb{Z}_2)|} \sum_{M_1 \in H_1(M_2, \mathbb{Z}_2)} L(M_1) \\
\mathcal{N}(M_2; C_2) \times T(M_2) &= \mathcal{N}(M_2; C_2) \\
\mathcal{N}(M_2; C_2) \times L(M_1) &= \mathcal{N}(M_2; C_2)\,.
\end{aligned} \tag{2.75}$$

Here $T(M_2) = e^{i\pi \oint_{M_2} b_2'}$ is the defect generating the $\mathbb{Z}_2^{(1),B}$ 1-form symmetry and $L(M_1) = e^{i\pi \oint_{M_1} a_1}$ is the defect generating the $\mathbb{Z}_2^{(2)}$ 2-form symmetry dual to $\mathbb{Z}_2^{(0)}$.

This is precisely the theory which we studied in section 2.5 using the higher-categorical approach. In particular

$$
\begin{aligned}
D_2^{(SC)} \quad &\longleftrightarrow \quad \mathcal{N}(M_2; C_2) \\
D_2^{(V)} \quad &\longleftrightarrow \quad T(M_2) \\
D_1^{(-)} \quad &\longleftrightarrow \quad L(M_1)\,,
\end{aligned}
\tag{2.76}
$$

and the identification of the identity surface and lines with $D_2^{(\mathrm{id})}$ and $D_1^{(\mathrm{id})}$, respectively. Note that in (2.75) we use $\times$ and not $\otimes$ as in section 2.5 to distinguish between the somewhat "mixed" fusion algebra, between objects of various dimensions and the 'proper' fusion algebra, in the higher category, that involves only objects and morphisms of the same dimension.

Notice also that the right hand side of the fusion $\mathcal{N}(M_2; C_2) \times \mathcal{N}(M_2; C_2)$ is precisely

$$
\frac{D_2^{(\mathrm{id})}}{\mathbb{Z}_2}\,(M_2) \oplus \frac{D_2^{(V)}}{\mathbb{Z}_2}\,(M_2)
\tag{2.77}
$$

as we found using our approach in section 2.5.

Chapter 3

Non-Invertible Symmetry Webs

This chapter focuses on 3d QFTs, whose symmetries are described by a 2-category, and concretely develops the tools to gauge arbitrary 0-form symmetries and determine the 2-category after gauging. The key here is that the initial, "pre-gauged", category is not necessarily only given in terms of invertible defects. In this sense the procedure we describe here extends the gauging of 0-form symmetries of non-normal subgroups outlined in the previous chapter and the gauging of the full 0-form symmetry group in [86, 116]. The content of this chapter is based on [2].

The structure of the chapter is as follows. In section 3.1 we introduce some key points of our construction by means of the examples $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ vs. $G = \mathbb{Z}_4$. In section 3.2 we discuss the gauging of a general normal subgroup of the 0-form symmetry. We further consider gauging the remaining symmetry in section 3.3. Finally, we comment on the D_8 web for orthogonal gauge theories in section 3.4.

3.1 A paradigmatic example

We start by mentioning some of the salient features by considering briefly the two paradigmatic examples $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G = \mathbb{Z}_4$.

Gauging the full group G results in $2\mathsf{Rep}(G)$ as the symmetry category [86, 116]. At the level of 1-morphisms, i.e. topological lines, $2\mathsf{Rep}(G)$ has a subcategory $\mathsf{Rep}(G)$, describing the 1-form symmetry dual to G. All the objects, i.e. the topological surfaces, are of condensation type. This means they are obtained by putting a mesh of 1-form symmetry lines on a surface, as explained in section 2.2. The simple objects in $2\mathsf{Rep}(G)$ are in one to one correspondence with irreducible 2d TQFTs with symmetry G [86]. Indeed, we could consider stacking a 3d theory with symmetry G with a 2d TQFT with the same symmetry. If we gauge the diagonal G symmetry, we couple the TQFT to the theory, and we can think of the TQFT as giving rise to a 2d topological defect in the gauged theory. 2d TQFTs are essentially trivial and are classified by an integer number n describing the number of vacua. 2d TQFTs with symmetry G are classified by

1. a subgroup $H \subseteq G$ which represents the symmetry which is not spontaneously broken (the TQFT has then $|G/H|$ vacua);
2. a 2-cocyle $\alpha \in H^2(H, U(1))$ describing a possible SPT phase for the unbroken symmetry.

As remarked above, these are in 1-to-1 correspondence with the simple objects of the 2-category $2\mathsf{Rep}(G)$, which we denote accordingly by $D_2^{(G/H,\alpha)}$. The simple objects in the symmetry category $2\mathsf{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ obtained after gauging a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry are then

$$\mathrm{Obj}(\mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}) = \left\{ D_2^{(\mathrm{id})}, D_2^{(-)}, D_2^{(\mathbb{Z}_2^S)}, D_2^{(\mathbb{Z}_2^C)}, D_2^{(\mathbb{Z}_2^V)}, D_2^{(\mathbb{Z}_2 \times \mathbb{Z}_2)} \right\} , \tag{3.1}$$

where $-$ denotes the non trivial element in $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$, and by $\mathbb{Z}_2^i$, $i = S, C, V$, we denote the three $\mathbb{Z}_2$ subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The simple objects in the symmetry category $2\mathsf{Rep}(\mathbb{Z}_4)$ obtained after gauging a $\mathbb{Z}_4$ symmetry are instead

$$\mathrm{Obj}(\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_4}) = \left\{ D_2^{(\mathrm{id})}, D_2^{(\mathbb{Z}_2)}, D_2^{(\mathbb{Z}_4)} \right\} , \tag{3.2}$$

since there is no non-trivial cocycle for $\mathbb{Z}_4$, and only one $\mathbb{Z}_2$ subgroup.

What we wish to understand now is how to obtain the same result by gauging stepwise, i.e. by subsequently gauging $\mathbb{Z}_2$ subgroups. The first puzzle that arises is how the two categories, one coming from $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and the other from $G = \mathbb{Z}_4$, are differentiated after gauging one $\mathbb{Z}_2$ subgroup. The two groups are distinguished in terms of the extension class

$$\epsilon \in H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2\,, \tag{3.3}$$

where the trivial elements describes $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and non-trivial one $G = \mathbb{Z}_4$. Field theoretically, it is clear that a non-trivial (non-split) extension yields a mixed anomaly between the dual 1-form symmetry $\widehat{\mathbb{Z}_2}$ and the remaining $\mathbb{Z}_2$ 0-form symmetry [25]. We will show how this is implemented in the 2-categorical framework, and identify its imprint as a *symmetry fractionalisation* of the 0-form symmetry. This has an extensive history in the math and condensed matter literature (see [43, 118–124] and references therein), which we revisit in the light of fusion 2-categories. This symmetry fractionalisation plays a crucial role in the subsequent gauging of the remaining $\mathbb{Z}_2$, where it results in the absence of certain topological defects for $G = \mathbb{Z}_4$ and lands us correctly on $2\mathsf{Rep}(\mathbb{Z}_4)$ as the final category (which has less simple objects than $2\mathsf{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2)$). This is summarised in figure 3.1.

3.2 Gauging Normal Subgroups

In this section, we start with a 0-form symmetry G and discuss the gauging of a normal subgroup H[1] subgroup of G. We sketch the general procedure and exemplify it with the cases $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $H = \mathbb{Z}_2$ and $G = \mathbb{Z}_4$, $H = \mathbb{Z}_2$.

[1]Non-normal subgroups can be treated using similar techniques. See the previous chapter 2.

$$\boxed{\begin{array}{c}\mathfrak{T}_1\\ \mathsf{2Vec}\left(\mathbb{Z}_2^{(0)}\times\mathbb{Z}_2^{(0)}\right)\end{array}} \xrightarrow{\mathbb{Z}_2} \boxed{\begin{array}{c}\mathfrak{T}_2\\ \mathsf{2Vec}\left(\mathbb{Z}_2^{(0)}\times\mathbb{Z}_2^{(1)}\right)\end{array}} \xrightarrow{\mathbb{Z}_2} \boxed{\begin{array}{c}\mathfrak{T}_3\\ \mathsf{2Rep}\left(\mathbb{Z}_2^{(0)}\times\mathbb{Z}_2^{(0)}\right)\end{array}}$$

$$\boxed{\begin{array}{c}\mathfrak{T}_1\\ \mathsf{2Vec}\left(\mathbb{Z}_4^{(0)}\right)\end{array}} \xrightarrow{\mathbb{Z}_2} \boxed{\begin{array}{c}\mathfrak{T}_2\\ \mathsf{2Vec}^{\omega}\left(\mathbb{Z}_2^{(0)}\times\mathbb{Z}_2^{(1)}\right)\end{array}} \xrightarrow{\mathbb{Z}_2} \boxed{\begin{array}{c}\mathfrak{T}_3\\ \mathsf{2Rep}\left(\mathbb{Z}_4^{(0)}\right)\end{array}}$$

Figure 3.1: Categorical symmetry web for 3d gauge theories starting with the theory $\mathfrak{T}_1$ with 0-form symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ (top) and $\mathbb{Z}_4$ (bottom), and gauging subsequently $\mathbb{Z}_2$. The category in the middle for theory $\mathfrak{T}_2$ has a trivial (top) and non-trivial (bottom) cocycle ω, which is encoded in the group extension class ϵ. This cocycle is key in order to obtain the correct category after gauging the remaining $\mathbb{Z}_2$ to reach theory $\mathfrak{T}_3$.

3.2.1 2-Category Associated to 0-Form Symmetries

Let us begin with the discussion of the initial symmetry 2-category

$$\mathcal{C}_G = \mathsf{2Vec}(G)\,, \tag{3.4}$$

carried by a 3d QFT $\mathfrak{T}_G$ having a non-anomalous 0-form symmetry group G. Throughout this chapter will use the notation

$$D_p^{(a)} \;:\quad p\text{-dimensional topological operators labeled by } a\,. \tag{3.5}$$

The 0-form symmetry is generated by topological codimension 1 operators, and hence $\mathfrak{T}_G$ carries topological surface operators labeled by elements of G, which we denote by $D_2^{(g)}$, $g \in G$. These form the simple objects (upto isomorphism) of $\mathcal{C}_G$ and satisfy group-like fusion rules

$$D_2^{(g)} \otimes D_2^{(h)} = D_2^{(gh)}\,, \qquad g, h \in G\,. \tag{3.6}$$

Each simple object $D_2^{(g)}$ carries a single simple 1-endomorphism (upto isomorphism) in $\mathcal{C}_G$ which is identified with the trivial/identity line $D_1^{(g;\mathrm{id})}$ on the surface

$D_2^{(g)}$. There are no 1-morphisms between the objects $D_2^{(g)}$ and $D_2^{(g')}$ in $\mathcal{C}_G$ for $g \neq g'$. There is a single 2-endomorphism in $\mathcal{C}_G = 2\mathsf{Vec}(G)$ of the 1-morphism $D_1^{(g;\text{id})}$ which corresponds to identity local operators living on the surface $D_2^{(g)}$, and there are no 2-endomorphisms between $D_1^{(g;\text{id})}$ and $D_1^{(g';\text{id})}$ for $g \neq g'$.

Example $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. An example of a 3d QFT with this 0-form symmetry is pure gauge theory with gauge group PSO$(4N)$, where $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is identified with the magnetic 0-form symmetry[2] of this QFT, arising from the fact that PSO$(4N)$ admits the construction

$$\mathrm{PSO}(4N) = \mathrm{Spin}(4N)/\mathbb{Z}_2 \times \mathbb{Z}_2\,, \tag{3.7}$$

in terms of the associated simply connected group Spin$(4N)$.

The corresponding symmetry 2-category is $\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2} = 2\mathsf{Vec}(\mathbb{Z}_2 \times \mathbb{Z}_2)$, with simple objects which we denote by

$$\mathrm{Obj}(\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2}) = \left\{D_2^{(\text{id})}, D_2^{(S)}, D_2^{(C)}, D_2^{(V)}\right\}\,, \tag{3.8}$$

corresponding to the topological surfaces generating the $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Here $D_2^{(\text{id})}$ is the identity object and the order 2 elements are

$$D_2^{(x)} \otimes D_2^{(x)} = D_2^{(\text{id})}\,, \qquad x = S, C, V\,, \tag{3.9}$$

with mutual fusion rule

$$D_2^{(S)} \otimes D_2^{(C)} = D_2^{(V)}\,. \tag{3.10}$$

[2]There is also a $\mathbb{Z}_2$ charge-conjugation type symmetry arising from outer-automorphisms of the Lie algebra $\mathfrak{so}(4N)$, which we do not take into account at the moment. This symmetry will be accounted in section 3.4 where it will enhance the $\mathbb{Z}_2 \times \mathbb{Z}_2$ 0-form symmetry considered here to $D_8 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ 0-form symmetry.

Example $G = \mathbb{Z}_4$. An example of a 3d QFT with this 0-form symmetry is pure gauge theory with gauge group $\mathrm{PSO}(4N+2)$, where $G = \mathbb{Z}_4$ is identified with the magnetic 0-form symmetry of this QFT, arising from the fact that $\mathrm{PSO}(4N+2)$ admits the construction

$$\mathrm{PSO}(4N+2) = \mathrm{Spin}(4N+2)/\mathbb{Z}_4\,, \tag{3.11}$$

in terms of the associated simply connected group $\mathrm{Spin}(4N+2)$.

The associated symmetry category is $\mathcal{C}_{\mathbb{Z}_4} = 2\mathsf{Vec}(\mathbb{Z}_4)$, whose simple objects are

$$\mathrm{Obj}(\mathcal{C}_{\mathbb{Z}_4}) = \left\{ D_2^{(\mathrm{id})}, D_2^{(S)}, D_2^{(V)}, D_2^{(C)} \right\}\,, \tag{3.12}$$

with the non-trivial fusion relations

$$D_2^{(S)} \otimes D_2^{(S)} = D_2^{(V)}\,, \qquad \left(D_2^{(S)}\right)^{\otimes 3} = D_2^{(C)}\,, \qquad \left(D_2^{(S)}\right)^{\otimes 4} = D_2^{(\mathrm{id})}\,. \tag{3.13}$$

3.2.2 Surface Defects After Partial Gauging

Consider gauging a normal subgroup $H \triangleleft G$. The 3d QFT obtained after gauging is labelled as $\mathfrak{T}_{G/H}$. The gauging procedure converts the symmetry 2-category $\mathcal{C}_G$ carried by $\mathfrak{T}_G$ to a symmetry 2-category $\mathcal{C}_{G/H}$ carried by $\mathfrak{T}_{G/H}$, which we want to determine.

The category $\mathcal{C}_{G/H}$ comprises of H-symmetric objects and morphisms in $\mathcal{C}_G$. Let us begin by exploring various ways of making multiples of the identity object $D_2^{(\mathrm{id})}$ H-symmetric. Consider for example making the object $nD_2^{(\mathrm{id})}$ H-symmetric. We need to choose junction lines between $nD_2^{(\mathrm{id})}$ and $D_2^{(h)}$ for all $h \in H$ (see figure 3.2) such that these junction lines can be freely rearranged (without the appearance of any extra phases) along the worldvolume of $nD_2^{(\mathrm{id})}$. By folding the $D_2^{(h)}$ surfaces away, the junction lines can be identified with lines living on the worldvolume of $nD_2^{(\mathrm{id})}$.

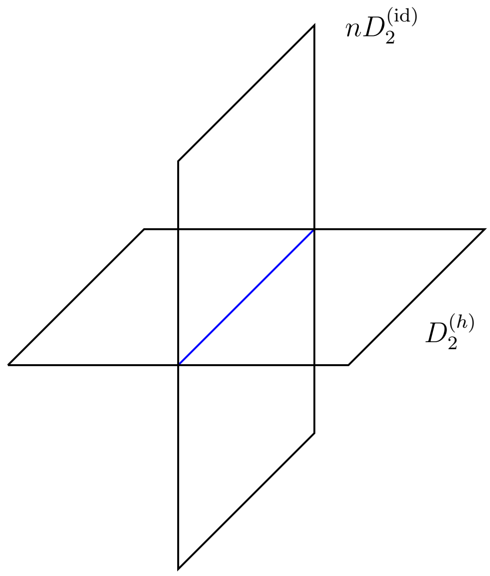


Figure 3.2: In order to make $nD_2^{(\mathrm{id})}$ H-symmetric, we need to choose line operators (shown in blue) living at the junction of $D_2^{(h)}$ for all $h \in H$ and $nD_2^{(\mathrm{id})}$.

Mathematically, these form the multi-fusion 1-category $\mathsf{Mat}_n(\mathsf{Vec})$ formed by $n \times n$ matrices valued in the category Vec of finite-dimensional vector spaces. The (i, j)-th element of such a matrix describes a line from the i-th copy of $D_2^{(\mathrm{id})}$ in $nD_2^{(\mathrm{id})}$ to the j-th copy of $D_2^{(\mathrm{id})}$ in $nD_2^{(\mathrm{id})}$.

Since there are no non-trivial associators for $D_2^{(h)}$ surfaces with $nD_2^{(\mathrm{id})}$, the folding procedure commutes with rearrangements of junctions. Hence, a choice of making $nD_2^{(\mathrm{id})}$ H-symmetric is a choice of lines living on $nD_2^{(\mathrm{id})}$ labelled by elements of H, which can be freely rearranged. Mathematically, this describes a 2-representation of H, whose collection for all possible values of n forms a fusion 2-category $2\mathsf{Rep}(H)$ (see [2] for more details). We thus find a 2-subcategory

$$2\mathsf{Rep}(H) \subseteq \mathcal{C}_{G/H} \,. \tag{3.14}$$

An alternative perspective is obtained by recognizing $nD_2^{(\mathrm{id})}$ as the defect obtained by stacking a 2d TQFT with n trivial vacua on top of $\mathfrak{T}_G$. In fact, all defects arising by stacking 2d TQFTs can be identified with $nD_2^{(\mathrm{id})}$, because all 2d TQFTs are essentially determined by their number of vacua n. Thus, the various ways of making $nD_2^{(\mathrm{id})}$ H-symmetric for various values of n are parametrized by H-symmetric 2d TQFTs, i.e. we can understand the topological surfaces of $\mathfrak{T}_{G/H}$ lying in $2\mathsf{Rep}(H) \subseteq \mathcal{C}_{G/H}$ as being obtained by first stacking an H-symmetric TQFT on top of the spacetime occupied by $\mathfrak{T}_G$, and then gauging the combined/diagonal H symmetry [86].

Now we consider other kinds of defects in $\mathcal{C}_{G/H}$. First, consider making multiples of $D_2^{(h)}$, for $h \in H$, H-symmetric. These lead to objects isomorphic to the objects already contained in the 2-subcategory $2\mathsf{Rep}(H) \subseteq \mathcal{C}_{G/H}$ because we can convert $D_2^{(h)}$ into $D_2^{(\mathrm{id})}$ by multiplication by elements of H. By same argument, we only need to study H-symmetrisation of a single defect $D_2^{(g)}$ for a single element g lying in each coset $k \in K := G/H$. Since we are dealing with a non-anomalous G symmetry, there are no associators for topological defects generating H in the presence of any $D_2^{(g)}$. Consequently, we obtain a copy of $2\mathsf{Rep}(H)$ inside $\mathcal{C}_{G/H}$ for each element $k \in K$.

In total, we learn that the objects of $\mathcal{C}_{G/H}$ are the same as the objects of the 2-category $2\mathsf{Vec}(K) \boxtimes 2\mathsf{Rep}(H)$. However, we will see later that

$$\mathcal{C}_{G/H} \neq 2\mathsf{Vec}(K) \boxtimes 2\mathsf{Rep}(H) \,. \tag{3.15}$$

The equality holds if and only if $G = H \times K$.

We will label the objects of $\mathcal{C}_{G/H}$ as $D_2^{(kR)}$ where $k \in K$ and R a 2-representation of H. In fact, there exists at least one 1-morphism (none of which is an isomorphism) from $D_2^{(kR)}$ to $D_2^{(k)}$ (obtained by choosing the trivial 2-representation) for every choice of 2-representation R. This is often captured by saying that $D_2^{(kR)}$ and $D_2^{(k)}$ lie in the same 'Schur component'. The existence of such a 1-morphism is equivalent to the fact that $D_2^{(kR)}$ can be obtained by condensing/gauging a (possibly non-invertible) symmetry localized along the worldvolume of $D_2^{(k)}$. Thus, Schur components can be thought of as capturing defects modulo condensations. The Schur components of the

2-category $\mathcal{C}_{G/H}$ are parametrized by elements of the group $K = G/H$.

Examples $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $G = \mathbb{Z}_4$. Returning to our main examples, for either $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $G = \mathbb{Z}_4$, consider gauging a $H = \mathbb{Z}_2$ subgroup. For $G = \mathbb{Z}_4$ there is a unique $\mathbb{Z}_2$ subgroup generated by $V \in \mathbb{Z}_4$. On the other hand, for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ there are three possible choices of $\mathbb{Z}_2$ subgroups generated by $x \in \{S, C, V\}$, but they are all equivalent. For maintaining consistency of notation with the $G = \mathbb{Z}_4$ case, we pick the $\mathbb{Z}_2$ subgroup of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by V.

From the above discussion, the objects of $\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$ and $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$ are the same and coincide with the objects of $2\mathsf{Vec}(\mathbb{Z}_2) \boxtimes 2\mathsf{Rep}(\mathbb{Z}_2)$, even though at the level of full 2-categories

$$\begin{aligned} \mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2} &= 2\mathsf{Vec}(\mathbb{Z}_2) \boxtimes 2\mathsf{Rep}(\mathbb{Z}_2) \\ \mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2} &\neq 2\mathsf{Vec}(\mathbb{Z}_2) \boxtimes 2\mathsf{Rep}(\mathbb{Z}_2)\,. \end{aligned} \tag{3.16}$$

Let us describe the objects of $\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$ and $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$ in more detail. Since the discussion is same for both categories, we refer to them together as $\mathcal{C}_{G/\mathbb{Z}_2}$.

First of all, the object $D_2^{(\text{id})}$ of $\mathcal{C}_G$ leads to a single simple object $D_2^{(\text{id})}$ of $\mathcal{C}_{G/\mathbb{Z}_2}$. This is because there is a single 2d $\mathbb{Z}_2$-symmetric SPT phase, namely the trivial one. The object $2D_2^{(\text{id})}$ of $\mathcal{C}_G$ also leads to a single simple object $D_2^{(\mathbb{Z}_2)}$ of $\mathcal{C}_{G/\mathbb{Z}_2}$. The $H = \mathbb{Z}_2$ symmetry on $2D_2^{(\text{id})}$ is generated by a 1-morphism $2D_2^{(\text{id})} \to 2D_2^{(\text{id})}$, which is implemented by a 2×2 matrix of 1-morphisms $D_2^{(\text{id})} \to D_2^{(\text{id})}$, with the (i,j)-th entry of the matrix describing a 1-morphism from the i-th copy of $D_2^{(\text{id})}$ to the j-th copy of $D_2^{(\text{id})}$. For the construction of $D_2^{(\mathbb{Z}_2)}$, the relevant matrix is

$$\begin{pmatrix} 0 & D_1^{(\text{id})} \\ D_1^{(\text{id})} & 0 \end{pmatrix} \;:\quad 2D_2^{(\text{id})} \to 2D_2^{(\text{id})}\,, \tag{3.17}$$

where $D_1^{(\text{id})}$ is the identity line defect in $\mathfrak{T}_G$. The objects $D_2^{(\text{id})}, D_2^{(\mathbb{Z}_2)}$ generate the 2-subcategory $2\mathsf{Rep}(\mathbb{Z}_2) \subseteq \mathcal{C}_{G/\mathbb{Z}_2}$. Similarly, the object $D_2^{(S)}$ of $\mathcal{C}_G$ leads to a single simple object of $\mathcal{C}_{G/\mathbb{Z}_2}$ and the object $2D_2^{(S)}$ of $\mathcal{C}_G$ also leads to a single simple object $D_2^{(S\mathbb{Z}_2)}$ of $\mathcal{C}_{G/\mathbb{Z}_2}$. $D_2^{(S)}, D_2^{(S\mathbb{Z}_2)}$ generate another copy of the 2-subcategory $2\mathsf{Rep}(\mathbb{Z}_2) \subseteq \mathcal{C}_{G/\mathbb{Z}_2}$. In total, we therefore have the simple objects

$$\text{Obj}(\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}) = \text{Obj}(\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}) = \left\{ D_2^{(\text{id})}, D_2^{(\mathbb{Z}_2)}, D_2^{(S)}, D_2^{(S\mathbb{Z}_2)} \right\}. \tag{3.18}$$

3.2.3 Fusion of Surface Defects after Partial Gauging

The fusion of 2-representations converts the set of 2-representations into a twisted Burnside ring as discussed in detail in [86]. This controls the fusion of objects $D_2^{(R)} \in 2\mathsf{Rep}(H) \subseteq \mathcal{C}_{G/H}$. In general, we have

$$D_2^{(k_1 R_1)} \otimes D_2^{(k_2 R_2)} = D_2^{\left(k_1 k_2 \left(R_1^{k_2} R_2\right)\right)}. \tag{3.19}$$

$R_1^{k_2}$ is a 2-representation obtained by applying the action of k_2 on R_1 (see [116] for more details) and $R_1^{k_2} R_2$ is the tensor product 2-representation of $R_1^{k_2}$ and R_2.

Examples $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $G = \mathbb{Z}_4$. Let us determine the fusion rules of simple objects in our examples $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $G = \mathbb{Z}_4$ and $H = \mathbb{Z}_2$. Recall the objects (3.18).

It is easy to see that $D_2^{(\text{id})}$ is the identity object of $\mathcal{C}_{G/\mathbb{Z}_2}$. As $H = \mathbb{Z}_2$ acts trivially on $D_2^{(S)}$, the fusion follows from $\mathcal{C}_G$:

$$D_2^{(S)} \otimes D_2^{(S)} = D_2^{(\text{id})}, \tag{3.20}$$

which implies that $D_2^{(S)}$ generates a $\mathbb{Z}_2$ 0-form symmetry in the theory $\mathfrak{T}_{G/\mathbb{Z}_2}$. On the other hand we have

$$D_2^{(\mathbb{Z}_2)} \otimes D_2^{(\mathbb{Z}_2)} = 2D_2^{(\mathbb{Z}_2)}. \tag{3.21}$$

To understand this fusion rule, we have to understand the combined $\mathbb{Z}_2$ action on the underlying defect $2D_2^{(\text{id})} \otimes 2D_2^{(\text{id})} \cong 4D_2^{(\text{id})} \in \mathcal{C}_G$ which is generated by the tensor product of the matrix of lines

$$\begin{pmatrix} 0 & D_1^{(\text{id})} \\ D_1^{(\text{id})} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & D_1^{(\text{id})} \\ D_1^{(\text{id})} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & D_1^{(\text{id})} \\ 0 & 0 & D_1^{(\text{id})} & 0 \\ 0 & D_1^{(\text{id})} & 0 & 0 \\ D_1^{(\text{id})} & 0 & 0 & 0 \end{pmatrix}, \tag{3.22}$$

In more detail, we can label the underlying $D_2^{(\text{id})} \in \mathcal{C}_G$ objects of $D_2^{(\mathbb{Z}_2)}$ by $D_2^{(\text{id})(i)}$ for $i \in \{0,1\}$. Then we can label the underlying $D_2^{(\text{id})} \in \mathcal{C}_{\mathfrak{T}}$ objects of $D_2^{(\mathbb{Z}_2)} \otimes D_2^{(\mathbb{Z}_2)}$ as $D_2^{(\text{id})(i,j)}$ for $i, j \in \{0,1\}$. The combined $\mathbb{Z}_2$ acts

$$\begin{aligned} i &\to i+1 \ (\text{mod } 2)\,, \\ j &\to j+1 \ (\text{mod } 2)\,. \end{aligned} \tag{3.23}$$

Thus $D_2^{(\text{id})(0,0)}$ and $D_2^{(\text{id})(1,1)}$ are exchanged by $\mathbb{Z}_2$, and $D_2^{(\text{id})(0,1)}$ and $D_2^{(\text{id})(1,0)}$ are exchanged by $\mathbb{Z}_2$, leading again to the fusion rule (3.21).

The remaining fusion rules are[3]

$$\begin{aligned} D_2^{(S\mathbb{Z}_2)} \otimes D_2^{(\mathbb{Z}_2)} &= 2D_2^{(S\mathbb{Z}_2)}\,, \\ D_2^{(S\mathbb{Z}_2)} \otimes D_2^{(S\mathbb{Z}_2)} &= 2D_2^{(\mathbb{Z}_2)}\,, \\ D_2^{(S)} \otimes D_2^{(\mathbb{Z}_2)} &= D_2^{(S\mathbb{Z}_2)}\,, \\ D_2^{(S)} \otimes D_2^{(S\mathbb{Z}_2)} &= D_2^{(\mathbb{Z}_2)}\,. \end{aligned} \tag{3.24}$$

That is, the label S is simply a $K = \mathbb{Z}_2$ grading on the fusion rules, which is because the conjugation action of $K = \mathbb{Z}_2$ on $H = \mathbb{Z}_2$ is trivial for both $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and

[3]Throughout this chapter, when the fusion (or composition) rules are commutative, we only mention fusion rules with a single choice of order.

$G = \mathbb{Z}_4$. Thus, the fusion of objects of $\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2} = \mathsf{2Vec}(\mathbb{Z}_2) \boxtimes \mathsf{2Rep}(\mathbb{Z}_2)$ and $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$ are same. In addition to the fusion of objects, the 1-morphisms as well as their composition and fusion rules are also the same for $\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2} = \mathsf{2Vec}(\mathbb{Z}_2) \boxtimes \mathsf{2Rep}(\mathbb{Z}_2)$ and $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$. Still, we claim that

$$\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2} \neq \mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2} \,. \tag{3.25}$$

What then is the difference between the two categories? The answer lies in the fact that $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$ has some non-trivial associators, which are physically understood as the phenomenon of symmetry fractionalisation. This is the subject of the next subsection.

3.2.4 Symmetry Fractionalisation on Lines

From this subsection onward, we restrict H to be an abelian group.

Data Specifying Embedding of H in G. To understand the distinction between $\mathcal{C}_{G/H}$ and $\mathsf{2Vec}(K) \boxtimes \mathsf{2Rep}(H)$, we need to first explore the following abstract group-theoretic structure. Since H is normal in G, there is a short exact sequence

$$1 \to H \to G \to K \equiv G/H \to 1 \,. \tag{3.26}$$

The extension is characterized by the group cohomology (twisted by the conjugation action of K on H) class

$$\epsilon \in H^2(K, H) \,. \tag{3.27}$$

As a set, we can write G as pairs (h, k) with $h \in H$ and $k \in K$ and product given by

$$(h, k) \times (h', k') = (h(k \circ h')\epsilon(k, k'), kk') \,, \tag{3.28}$$

where $(k \circ h') = kh'k^{-1}$ denotes the action of K on H by conjugation.

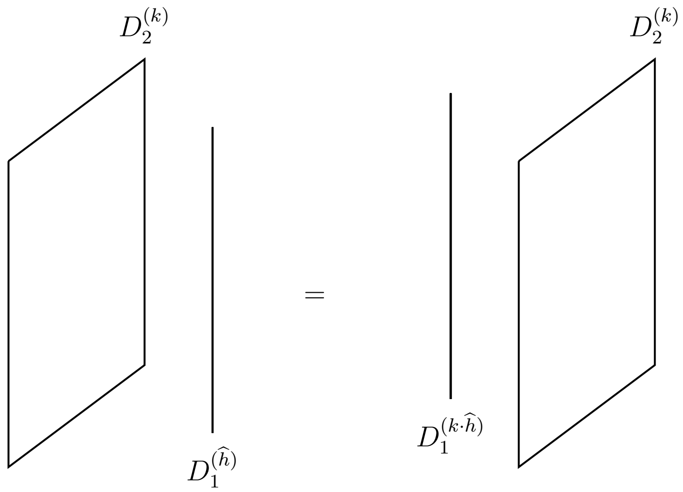


Figure 3.3: A 1-form symmetry generating line $D_1^{(\widehat{h})}$ changes to another 1-form symmetry generating line $D_1^{(k\cdot\widehat{h})}$ upon sliding it across a 0-form symmetry generating surface $D_2^{(k)}$. See equation (3.29).

Split 2-Group. It is well-known that after gauging H 0-form symmetry of $\mathfrak{T}_G$, we obtain in $\mathfrak{T}_{G/H}$ a dual 1-form symmetry with group $\widehat{H}$ which is the Pontryagin dual of H [19] . If there is a non-trivial (conjugation) action of K on H, then we have a non-trivial dual action of K on $\widehat{H}$, implying that the dual 1-form symmetry $\widehat{H}$ and the residual 0-form symmetry K combine to form a non-trivial (split) 2-group symmetry Γ in which the 0-form symmetry has a non-trivial action on the 1-form symmetry. This action is displayed in terms of topological defects generating the 0-form and 1-form symmetries in figure 3.3. In a categorical language, this means that we have the following non-commutativity in the fusion rules for 1-morphisms

$$D_1^{(k;\mathrm{id})} \otimes D_1^{(\widehat{h})} = D_1^{(k\cdot\widehat{h})} \otimes D_1^{(k;\mathrm{id})} \,. \tag{3.29}$$

Here $D_1^{(\widehat{h})}$ for $\widehat{h} \in \widehat{H}$ are simple 1-endomorphisms of $D_2^{(\mathrm{id})}$ in $\mathcal{C}_{G/H}$ corresponding to topological lines generating the $\widehat{H}$ 1-form symmetry, $D_1^{(k;\mathrm{id})}$ are 1-endomorphisms corresponding to identity lines on the surfaces $D_2^{(k)}$, and $k \cdot \widehat{h} \in \widehat{H}$ is the element obtained after the action of k on $\widehat{h}$.

The fusion rule (3.29) already differentiates between $\mathcal{C}_{G/H}$ and $2\mathsf{Vec}(K)\boxtimes 2\mathsf{Rep}(H)$ in special cases. However, when there is a trivial action of K on H as in our two examples $G = \mathbb{Z}_2 \times \mathbb{Z}_2, G = Z_4$ and $H = \mathbb{Z}_2$, this does not provide the required distinction between the two categories. In fact we can modify the problem as follows: what is the distinction between the categories $\mathcal{C}_{G/H}$ and $2\mathsf{Vec}(\Gamma)$, where Γ is the split 2-group generated by K 0-form symmetry acting on $\widehat{H}$ 1-form symmetry? When the action is trivial, we have $2\mathsf{Vec}(\Gamma) = 2\mathsf{Vec}(K) \boxtimes 2\mathsf{Rep}(H)$ and so we reduce to our previous problem.

't Hooft Anomaly and Symmetry Fractionalisation. The categories $\mathcal{C}_{G/H}$ and $2\mathsf{Vec}(\Gamma)$ turn out to be equal only when we can write $G = H \rtimes K$, in which case the extension class ϵ discussed above vanishes. This captures an 't Hooft anomaly for the 2-group Γ [25], which can be expressed as

$$A^*\omega = B_2 \cup a^*\epsilon \,, \tag{3.30}$$

where B_2 is the background field for H 1-form symmetry and a^* is pullback under the background a for K 0-form symmetry. The anomaly $A^*\omega$ descends from

$$\omega \in H^4(\Gamma, U(1)) \,, \tag{3.31}$$

upon pull-back A^* to spacetime, corresponding to a 2-group background A.

The element ω describes non-trivial associators in the category $\mathcal{C}_{G/H}$, distinguishing it from $2\mathsf{Vec}(\Gamma)$. Using a standard notation, we write

$$\mathcal{C}_{G/H} = 2\mathsf{Vec}^{\omega}(\Gamma) \,. \tag{3.32}$$

The anomaly (3.30) describes the fractionalisation of K 0-form symmetry along the

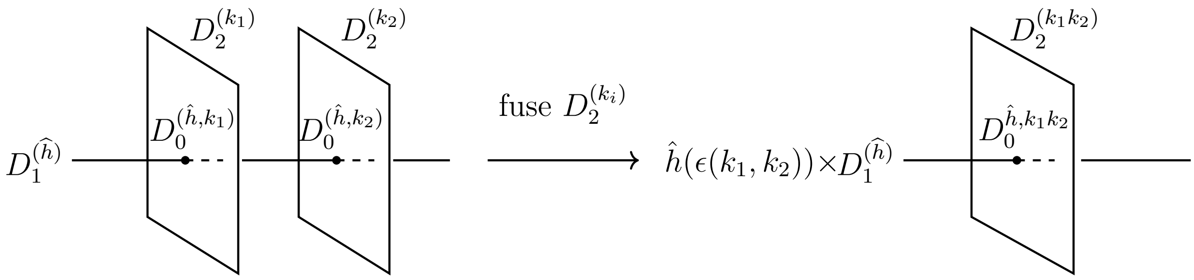


Figure 3.4: Symmetry fractionalisation: the junctions of topological surface operators $D_2^{(k_i)}$ generating the K 0-form symmetry with a fixed line $D_1^{(\widehat{h})}$ generating the 1-form symmetry do not obey K multiplication laws, but instead acquire a projectivity $\widehat{h}\big(\epsilon(k_1,k_2)\big) \in U(1)$.

topological line operators $D_1^{(\widehat{h})}$ generating $\widehat{H}$ 1-form symmetry, see figure 3.4.

Derivation. Let us now provide a derivation of this symmetry fractionalisation phenomenon. First of all, we need to understand the emergence of $\widehat{H}$ 1-form symmetry in the category $\mathcal{C}_{G/H}$. These are simple 1-endomorphisms of the identity object $D_2^{(\mathrm{id})}$ of $\mathcal{C}_{G/H}$. Consequently, they correspond to various ways of making the identity 1-endomorphism $D_1^{(\mathrm{id})}$ of $D_2^{(\mathrm{id})}$ in $\mathcal{C}_G$ symmetric under H. Such different ways correspond to homomorphisms

$$H \longrightarrow \mathrm{End}_{\mathcal{C}_G}(D_1^{(\mathrm{id})}) = \mathbb{C}\,, \tag{3.33}$$

where we used that the vector space $\mathrm{End}_{\mathcal{C}_G}(D_1^{(\mathrm{id})})$ of 2-endomorphisms of the 1-morphism $D_1^{(\mathrm{id})}$ in the 2-category $\mathcal{C}_G = 2\mathsf{Vec}(G)$ is simply $\mathbb{C}$. Such homomorphisms are described by elements of the Pontryagin dual group $\widehat{H}$, resulting in lines $D_1^{(\widehat{h})}$ in the category $\mathcal{C}_{G/H}$.

Now a junction local operator[4] $D_0^{(\widehat{h},k)}$ of $D_2^{(k)}$ with $D_1^{(\widehat{h})}$ in $\mathcal{C}_{G/H}$ can be uplifted

[4]These junction operators are chosen to satisfy $D_0^{(\widehat{h}_1,k)} \otimes_{D_2^{(k)}} D_0^{(\widehat{h}_2,k)} = D_0^{(\widehat{h}_1\widehat{h}_2,k)}$ where $\otimes_{D_2^{(k)}}$ denotes fusion of junction operators along the surface $D_2^{(k)}$. That is these junction operators are chosen to obey $\widehat{H}$ fusion rules for a fixed surface $D_2^{(k)}$, but as we will see they fail to obey K fusion rules for a fixed line $D_1^{(\widehat{h})}$.

to a junction of $D_2^{(1,k)}$ with $D_1^{(\mathrm{id})}$ in $\mathcal{C}_G$, or in particular the identity operator $D_0^{(1,k;\mathrm{id})}$ on $D_2^{(1,k)}$ in $\mathcal{C}_G$. The fusion of $D_0^{(k_1;\widehat{h})}$ with $D_0^{(k_2;\widehat{h})}$ in $\mathcal{C}_{G/H}$ arises from the following fusion in $\mathcal{C}_G$

$$D_2^{(1,k_1)} \otimes D_2^{(1,k_2)} = D_2^{(\epsilon(k_1,k_2),1)} \otimes D_2^{(1,k_1k_2)} \,. \tag{3.34}$$

Note that on the right hand side we have a surface operator $D_2^{(\epsilon(k_1,k_2),1)}$ valued purely in H. In the above definition of $D_1^{(\widehat{h})}$ line in terms of the data of the category $\mathcal{C}_G$, this surface operator acts by a phase $\hat{h}(\epsilon(k_1,k_2))$, leading to the fusion rule

$$D_0^{(\widehat{h},k_1)} \otimes D_0^{(\widehat{h},k_2)} = \widehat{h}(\epsilon(k_1,k_2))\, D_0^{(\widehat{h},k_1k_2)} \,, \tag{3.35}$$

in $\mathcal{C}_{G/H}$.

Examples $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $G = \mathbb{Z}_4$. The difference between trivial and non-trivial extension class is best illustrated in our examples. The relevant extension group is $H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$. The choice $G = \mathbb{Z}_2 \times \mathbb{Z}_2, H = \mathbb{Z}_2$ corresponds to the trivial cohomology class, while the choice $G = \mathbb{Z}_4, H = \mathbb{Z}_2$ corresponds to the non-trivial extension class $\epsilon \in H^2(\mathbb{Z}_2, \mathbb{Z}_2)$.

There is a $\mathbb{Z}_2$ 1-form symmetry generated by an order two line $D_1^{(V)}$ in both $\mathfrak{T}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$ and $\mathfrak{T}_{\mathbb{Z}_4/\mathbb{Z}_2}$. According to our above analysis, the 0-form $\mathbb{Z}_2$ symmetry of $\mathfrak{T}_{G/\mathbb{Z}_2}$ generated by the surface $D_2^{(S)}$ in $\mathcal{C}_{G/\mathbb{Z}_2}$ fractionalizes to a $\mathbb{Z}_4$ 0-form symmetry on the line $D_1^{(V)}$ for $G = \mathbb{Z}_4$, see figure 3.5.

3.2.5 Symmetry Fractionalisation on Condensation Surfaces

We saw in the previous subsection that the residual K 0-form symmetry of $\mathcal{C}_{G/H}$ fractionalizes on lines $D_1^{(\widehat{h})}$ in $\mathcal{C}_{G/H}$ – whose underlying line before H-gauging is the identity line $D_1^{(\mathrm{id})}$ in $\mathcal{C}_G$. Using similar arguments with one extra dimension added (see [2] for more details), we can see that $\mathcal{C}_{G/H}$ fractionalizes on surfaces $D_2^{(R)}$ for 2-

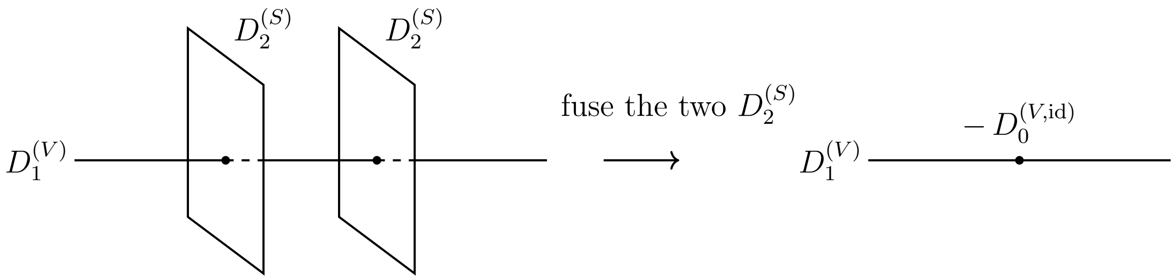


Figure 3.5: Symmetry fractionalisation in $\mathfrak{T}_{\mathbb{Z}_4/\mathbb{Z}_2}$: fusing the junction of line $D_1^{(V)}$ with surface $D_2^{(S)}$ along the line $D_1^{(V)}$ produces the operator $-D_0^{(V,\mathrm{id})}$, where $D_0^{(V,\mathrm{id})}$ is the identity local operator on the line $D_1^{(V)}$. Thus the $\mathbb{Z}_2$ symmetry generated by $D_2^{(S)}$ is fractionalized to $\mathbb{Z}_4$ on $D_1^{(V)}$.

representations R in $\mathcal{C}_{G/H}$ – whose underlying surfaces before H-gauging are multiples of the identity surface $D_2^{(\mathrm{id})}$ in $\mathcal{C}_G$. This is illustrated in figure 3.6.

Example $G = \mathbb{Z}_4$**.** To illustrate this symmetry fractionalisation for condensation defects consider again the gauging of $H = \mathbb{Z}_2 \triangleleft \mathbb{Z}_4$. There is a single condensation surface defect $D_2^{(\mathbb{Z}_2)}$ in $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$ obtained by gauging the $\mathbb{Z}_2$ 1-form symmetry generated by $D_1^{(V)}$ on a two-dimensional surface.

The lines living on $D_2^{(\mathbb{Z}_2)}$ generate a $\mathbb{Z}_2$ 0-form symmetry localised on $D_2^{(\mathbb{Z}_2)}$. Let us call the generator of this localised symmetry[5] as $D_1^{(\mathbb{Z}_2;-)}$. From the point of view of $\mathcal{C}_{\mathbb{Z}_4}$, this line arises as the $\mathbb{Z}_2$-symmetric 1-morphism

$$\begin{pmatrix} 0 & D_1^{(\mathrm{id})} \\ D_1^{(\mathrm{id})} & 0 \end{pmatrix} : \quad 2D_2^{(\mathrm{id})} \to 2D_2^{(\mathrm{id})}\,, \tag{3.36}$$

which is also the 1-morphism (3.17) generating the $\mathbb{Z}_2$ action converting the object $2D_2^{(\mathrm{id})}$ of $\mathcal{C}_{\mathbb{Z}_4}$ into the object $D_2^{(\mathbb{Z}_2)}$ of $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$.

Symmetry fractionalisation on condensation surfaces implies that the square of the junction of $D_2^{(\mathbb{Z}_2)}$ and $D_2^{(S)}$ along $D_2^{(\mathbb{Z}_2)}$ leaves behind the line $D_1^{(\mathbb{Z}_2;-)}$. Thus the $\mathbb{Z}_2$

[5]Note that this line should not identified as the image of the stacking of the bulk line $D_1^{(V)}$ on top of $D_2^{(\mathbb{Z}_2)}$. Actually, $D_1^{(V)}$ becomes the identity line $D_1^{(\mathbb{Z}_2;\mathrm{id})}$ of $D_2^{(\mathbb{Z}_2)}$ under this stacking procedure. Instead, the line $D_1^{(\mathbb{Z}_2;-)}$ is localised/trapped on $D_2^{(\mathbb{Z}_2)}$ and cannot be lifted into the bulk.

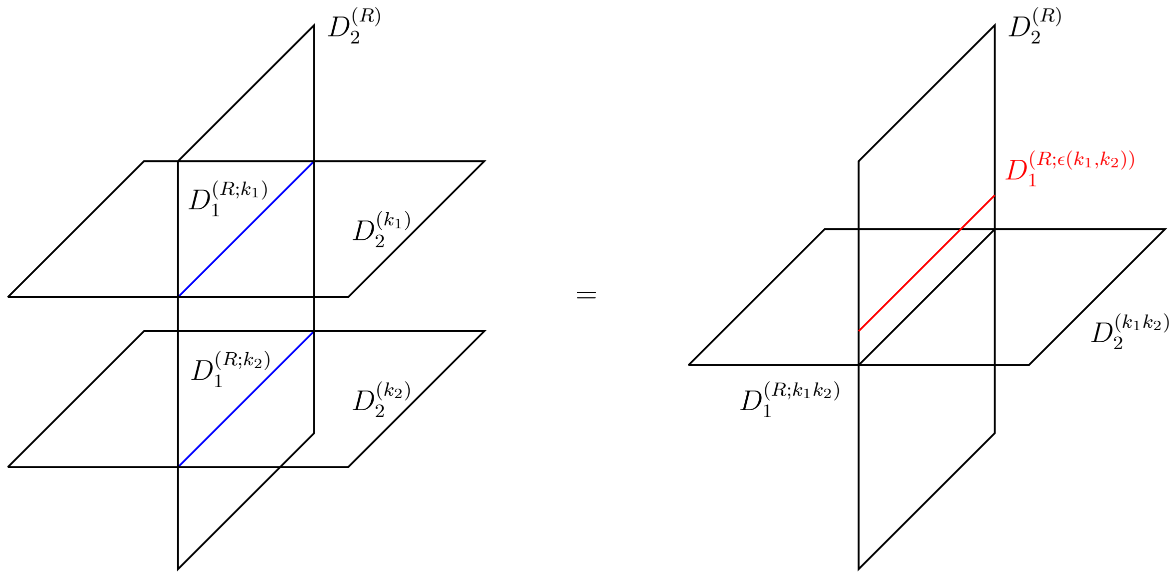


Figure 3.6: The figure depicts symmetry fractionalisation on a surface $D_2^{(R)}$ in $\mathcal{C}_{G/H}$ specified by a 2-representation R of H. Composing two junctions (in blue) of $D_2^{(R)}$ with surfaces $D_2^{(k_i)}$ generating K 0-form symmetry yields an extra line (in red) living on $D_2^{(R)}$. This means that the bulk K 0-form symmetry group fractionalizes to some larger 0-form symmetry group K_R on the surface $D_2^{(R)}$.

0-form symmetry of $\mathfrak{T}_{\mathbb{Z}_4/\mathbb{Z}_2}$ generated by $D_2^{(S)}$ fractionalizes to a $\mathbb{Z}_4$ 0-form symmetry on the worldvolume of $D_2^{(\mathbb{Z}_2)}$ because the line $D_1^{(\mathbb{Z}_2;-)}$ has order two, see figure 3.7. More precisely, there are two simple junction lines $L_1^{\pm}$ that can live at the junction of $D_2^{(\mathbb{Z}_2)}$ and $D_2^{(S)}$. These correspond to the two irreducible representations of the $\mathbb{Z}_2$ 1-form symmetry generated by $D_1^{(V)}$ being gauged to construct $D_2^{(\mathbb{Z}_2)}$. These have the fusion rules

$$\begin{aligned} L_1^{\pm} \otimes_{D_2^{(\mathbb{Z}_2)}} L_1^{\pm} &= D_1^{(\mathbb{Z}_2;-)}\,, \\ L_1^{+} \otimes_{D_2^{(\mathbb{Z}_2)}} L_1^{-} &= D_1^{(\mathbb{Z}_2;\mathrm{id})}\,, \end{aligned} \tag{3.37}$$

which we will use in the analysis of the next section.

3.3 Gauging the Gauged

So far we have generalized the analysis in [86] to perform a partial gauging of a normal subgroup $H \triangleleft G$ which acts as a 0-form symmetry on a 3d theory. This resulted in

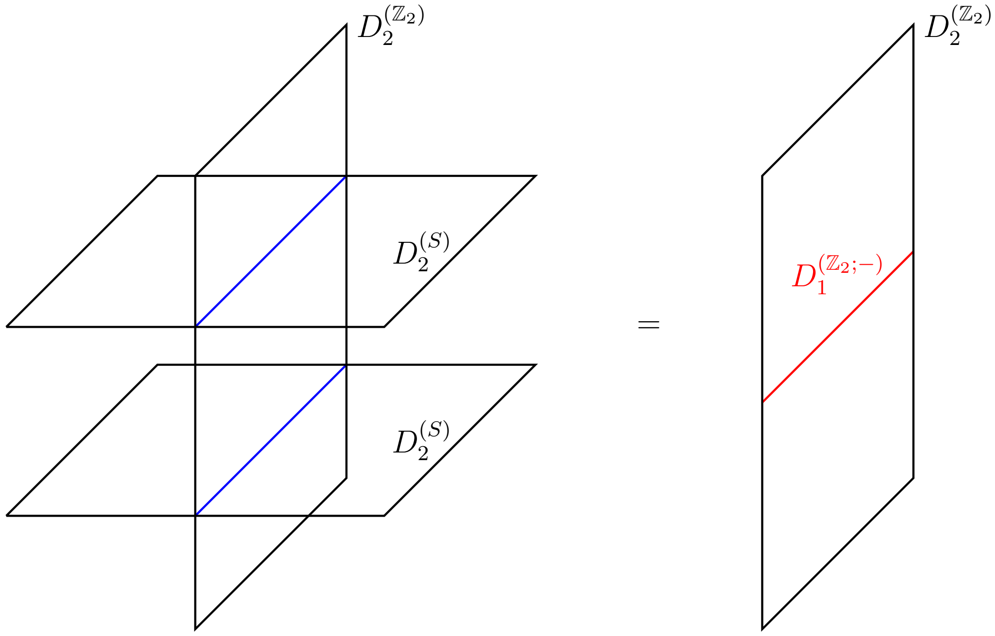


Figure 3.7: The figure depicts symmetry fractionalisation on the condensation surface $D_2^{(\mathbb{Z}_4)}$ in $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$. Composing two junctions (in blue) of $D_2^{(\mathbb{Z}_2)}$ with surfaces $D_2^{(S)}$ generating $\mathbb{Z}_2$ 0-form symmetry yields a line $D_1^{(\mathbb{Z}_2;-)}$ (in red) living on $D_2^{(\mathbb{Z}_2)}$. This means that the bulk $\mathbb{Z}_2$ 0-form symmetry group fractionalizes to $\mathbb{Z}_4$ 0-form symmetry group on the surface $D_2^{(\mathbb{Z}_2)}$. More precisely, the blue lines are either both L_1^+ or both L_1^-, see fusion rules (3.37).

the category

$$\mathcal{C}_{G/H} = 2\mathsf{Vec}^{\omega}(\Gamma)\,, \tag{3.38}$$

where ω is determined in terms of the associated group-cocycle $\epsilon \in H^2(K, H)$ with $K = G/H$, and Γ is a split 2-group containing K 0-form symmetry acting on $\widehat{H}$ 1-form symmetry.

In this section we discuss how to gauge the remaining K 0-form symmetry. This requires developing how the gauging is realised in a setting where the symmetry is not necessarily invertible (for example, in this case there is a non-invertible $2\mathsf{Rep}(H)$ subsymmetry). More generally, one can of course have multiple more steps in this gauging. Once all 0-form symmetries are gauged we expect the symmetry category to be $\mathcal{C}_{G/G} = 2\mathsf{Rep}(G)$. We perform an explicit analysis of G/H gauging for our examples $G = \mathbb{Z}_2 \times \mathbb{Z}_2, H = \mathbb{Z}_2$ and $G = \mathbb{Z}_4, H = \mathbb{Z}_2$ and cross-check our results against this expectation.

3.3.1 Surface Defects After Sequential Gauging

K-invariant Combinations of Surfaces. First of all, since multiplication by $D_2^{(k)}$ relates any $D_2^{(kR)}$ to $D_2^{(R)}$, we only need to focus on making combinations of $D_2^{(R)}$ K-symmetric. Any other K-symmetric combination of objects of $\mathcal{C}_{G/H}$ only produces objects of $\mathcal{C}_G$ isomorphic to the ones descending from K-symmetric combinations of $D_2^{(R)}$. Note that we can only implement the K symmetry if we begin with a multiple of

$$\bigoplus_{R \in K\text{-orbit}} D_2^{(R)}\,, \tag{3.39}$$

because the action of K on H descends to an action of K on 2-representations R of H.

Obstruction: Symmetry Fractionalisation. There are various further obstructions in the K-symmetrisation procedure, which are most cleanly understood in the case when the action of K is trivial on a particular 2-representation R. For the first type of obstruction, consider making a single copy of $D_2^{(R)}$ K-symmetric. If K 0-form symmetry is fractionalized on $D_2^{(R)}$, then it is impossible to K-symmetrize $D_2^{(R)}$. However, this obstruction may be cured by beginning instead with $nD_2^{(R)}$ for $n > 1$.

We can then try to implement the K-symmetry acting as a permutation of the n copies of $D_2^{(R)}$ with different choices for the junction lines $D_1^{(R;k)(\widehat{h}_R)}$. Such a non-trivial combination of junction lines with permutation can defractionalise the 0-form symmetry back to K, making $nD_2^{(R)}$ K-symmetric.

Obstruction: Localised 't Hooft Anomaly and Projective 2-Representations. Another type of obstruction arises even when the K symmetry does not fractionalize on $D_2^{(R)}$. Consider for example the identity defect $D_2^{(\text{id})}$. There is a canonical junction line $D_1^{(\mathrm{id};k)(\mathrm{id})}$ between $D_2^{(k)}$ and $D_2^{(\mathrm{id})}$, which upon folding of the $D_2^{(k)}$ surface reduces to the identity line $D_1^{(\mathrm{id})}$ on $D_2^{(\mathrm{id})}$. Making $nD_2^{(\mathrm{id})}$ K-symmetric by combining $D_1^{(\mathrm{id};k)(\mathrm{id})}$ junctions with permutations of n copies of $D_2^{(\mathrm{id})}$ lead to the K-symmetric defects labeled by n-dimensional 2-representations of K.

Consider performing the same procedure with another junction line

$$D_1^{(\mathrm{id};k)(\rho(k))} := D_1^{(\rho(k))} \otimes D_1^{(\mathrm{id};k)(\mathrm{id})}\,, \tag{3.40}$$

between $D_2^{(k)}$ and $D_2^{(\mathrm{id})}$ obtained by stacking the bulk line $D_1^{(\rho(k))}$ for $\rho(k) \in \widehat{H}$ on top of $D_1^{(\mathrm{id};k)(\mathrm{id})}$. Here ρ is a homomorphism from K to $\widehat{H}$. The junctions $D_1^{(\mathrm{id};k)(\rho(k))}$ satisfy the K fusion rules

$$D_1^{(\mathrm{id};k)(\rho(k))} \otimes_{D_2^{(\mathrm{id})}} D_1^{(\mathrm{id};k')(\rho(k'))} = D_1^{(\mathrm{id};kk')(\rho(kk'))}\,. \tag{3.41}$$

However the 1-category formed by these junctions is not Vec_K but rather $\mathsf{Vec}_K^{\omega_\rho}$ where $\omega_\rho \in H^3(K, U(1))$ provides a possibly non-trivial associator between these junction lines. A representative is

$$\omega_\rho(k_1, k_2, k_3) = \rho(k_3)\,(\epsilon(k_1, k_2)) \in U(1)\,, \tag{3.42}$$

which can be obtained as a consequence of the symmetry fractionalisation of K on $\widehat{H}$ lines.

In other words, the junction lines $D_1^{(\mathrm{id};k)(\rho(k))}$ generate K 0-form symmetry on the two-dimensional worldvolume of $D_2^{(\mathrm{id})}$ which is afflicted with the 't Hooft anomaly ω_ρ. Thus, in order to obtain a K-symmetric surface defect we need to stack $D_2^{(\mathrm{id})}$ with a 2d TQFT carrying K 0-form symmetry with opposite anomaly ω_ρ^{-1}. Such 2d TQFTs are classified by *projective 2-representations*[6] of H in the class ω_ρ. In total, a choice

[6]Projective 2-representations in the class $\omega = 0$ coincide with the usual 2-representations.

of ρ and a projective 2-representation R of K of dimension n in class ω_ρ provides a way of making $nD_2^{(\text{id})}$ in $\mathcal{C}_{G/H}$ K-symmetric.

Example: $G = \mathbb{Z}_2 \times \mathbb{Z}_2$**.** Let us now consider gauging the $K = \mathbb{Z}_2$ 0-form symmetry generated by $D_2^{(S)}$ in our example $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $H = \mathbb{Z}_2$. Since the short exact sequence

$$1 \to H = \mathbb{Z}_2 \to G = \mathbb{Z}_2 \times \mathbb{Z}_2 \to K = \mathbb{Z}_2 \to 1\,, \tag{3.43}$$

splits, none of the obstructions discussed above are relevant, and the analysis is relatively straightforward.

First consider objects of $\mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$ descending from multiples of the identity object $D_2^{(\text{id})}$ of $\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$. Note that there are only two possible homomorphisms

$$\rho: \;\; K = \mathbb{Z}_2 \to \widehat{H} = \mathbb{Z}_2\,, \tag{3.44}$$

namely the trivial and the identity homomorphisms. Since the extension class $\epsilon = 0$, for both homomorphisms we have $\omega_\rho = 0$. Thus, both of them lead to a copy of $2\mathsf{Rep}(\mathbb{Z}_2)$. Let us label the simple objects in $2\mathsf{Rep}(\mathbb{Z}_2) \subseteq \mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$ arising from trivial homomorphism as

$$D_2^{(\text{id})}, \qquad D_2^{(\mathbb{Z}_2^V)}\,, \tag{3.45}$$

where $D_2^{(\text{id})}$ corresponds to the trivial 2-representation and $D_2^{(\mathbb{Z}_2^V)}$ corresponds to the non-trivial irreducible 2-representation of dimension 2. Similarly, let us label the simple objects in $2\mathsf{Rep}(\mathbb{Z}_2) \subseteq \mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$ arising from identity homomorphism as

$$D_2^{(-)}, \qquad D_2^{(\mathbb{Z}_2^{V'})}\,, \tag{3.46}$$

where $D_2^{(-)}$ corresponds to the trivial 2-representation and $D_2^{(\mathbb{Z}_2^{V'})}$ corresponds to the non-trivial irreducible 2-representation. More concretely, we can make $D_2^{(\text{id})}$ symmetric under $\mathbb{Z}_2^S$ by either generating the symmetry using one of the lines $D_1^{(\mathrm{id})}$ or $D_1^{(V)}$. The resulting simple objects of $\mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$ respectively are

$$D_2^{(\mathrm{id})}, \qquad D_2^{(-)}\,. \tag{3.47}$$

Likewise consider making $2D_2^{(\mathrm{id})}$ symmetric under $\mathbb{Z}_2^S$. The 1-endomorphisms of this defect are in $\mathsf{Mat}_{2\times 2}(\mathsf{Rep}(\mathbb{Z}_2))$, where $\mathsf{Rep}(\mathbb{Z}_2)$ is generated by $D_1^{(\mathrm{id})}$ and $D_1^{(V)}$. There are five choices of 1-endomorphism for generating $\mathbb{Z}_2^S$ 0-form symmetry, which give rise to the following objects in $\mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$:

$$\begin{aligned}
\begin{pmatrix} D_1^{(\mathrm{id})} & 0 \\ 0 & D_1^{(\mathrm{id})} \end{pmatrix} &: \qquad D_2^{(\mathrm{id})} \oplus D_2^{(\mathrm{id})}\,, \\
\begin{pmatrix} D_1^{(V)} & 0 \\ 0 & D_1^{(V)} \end{pmatrix} &: \qquad D_2^{(-)} \oplus D_2^{(-)}\,, \\
\begin{pmatrix} D_1^{(\mathrm{id})} & 0 \\ 0 & D_1^{(V)} \end{pmatrix} &: \qquad D_2^{(\mathrm{id})} \oplus D_2^{(-)}\,, \\
\begin{pmatrix} 0 & D_1^{(\mathrm{id})} \\ D_1^{(\mathrm{id})} & 0 \end{pmatrix} &: \qquad D_2^{(\mathbb{Z}_2^V)}\,, \\
\begin{pmatrix} 0 & D_1^{(V)} \\ D_1^{(V)} & 0 \end{pmatrix} &: \qquad D_2^{(\mathbb{Z}_2^{V\prime})} \cong D_2^{(\mathbb{Z}_2^V)}\,.
\end{aligned} \tag{3.48}$$

To see that $D_2^{(\mathbb{Z}_2^{V\prime})} \cong D_2^{(\mathbb{Z}_2^V)}$, note that

$$\begin{pmatrix} D_1^{(\mathrm{id})} & 0 \\ 0 & D_1^{(V)} \end{pmatrix}\,, \tag{3.49}$$

provides a 1-morphism $D_1^{(\mathbb{Z}_2^{V\prime},\mathbb{Z}_2^V)} : D_2^{(\mathbb{Z}_2^{V\prime})} \to D_2^{(\mathbb{Z}_2^V)}$, and also a 1-morphism $D_1^{(\mathbb{Z}_2^V,\mathbb{Z}_2^{V\prime})}$: $D_2^{(\mathbb{Z}_2^V)} \to D_2^{(\mathbb{Z}_2^{V\prime})}$ in $\mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$. The composition of these two 1-morphisms is the identity endomorphism (as the square of the above matrix is the identity matrix) of $D_2^{(\mathbb{Z}_2^V)} \in \mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$. Similar to above, $D_2^{(\mathbb{Z}_2)} \in \mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$ leads to two simple objects

$$D_2^{(\mathbb{Z}_2^S)},\ D_2^{(\mathbb{Z}_2^C)} \in \mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}\,. \tag{3.50}$$

The 0-form symmetry $\mathbb{Z}_2^S$ is implemented by $D_1^{(\mathbb{Z}_2;\mathrm{id})} \in \mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$ for $D_2^{(\mathbb{Z}_2^S)} \in \mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$, and by $D_1^{(\mathbb{Z}_2;-)} \in \mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$ for $D_2^{(\mathbb{Z}_2^C)} \in \mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$. Finally, $2D_2^{(\mathbb{Z}_2)} \in \mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$ leads to a single simple object

$$D_2^{(\mathbb{Z}_2\times\mathbb{Z}_2)} \in \mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}\,, \tag{3.51}$$

for which the $\mathbb{Z}_2^S$ symmetry is implemented by

$$\begin{pmatrix} 0 & D_1^{(\mathbb{Z}_2;\mathrm{id})} \\ D_1^{(\mathbb{Z}_2;\mathrm{id})} & 0 \end{pmatrix}\,. \tag{3.52}$$

In summary the surface defects in the gauged category are built upon the following simple objects

$$\mathrm{Obj}(\mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}) = \left\{ D_2^{(\mathrm{id})}, D_2^{(-)}, D_2^{(\mathbb{Z}_2^S)}, D_2^{(\mathbb{Z}_2^C)}, D_2^{(\mathbb{Z}_2^V)}, D_2^{(\mathbb{Z}_2\times\mathbb{Z}_2)} \right\}\,, \tag{3.53}$$

which exactly reproduce the simple objects of the category $2\mathsf{Rep}(\mathbb{Z}_2\times\mathbb{Z}_2)$ (see section 3.1) obtained by gauging in a single step the full $\mathbb{Z}_2\times\mathbb{Z}_2$ 0-form symmetry.

Example: $G = \mathbb{Z}_4$. Let us now consider the gauging of residual $\mathbb{Z}_2^S$ symmetry in the 2-category $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$ associated to our example $G = \mathbb{Z}_4, H = \mathbb{Z}_2$. We expect the resulting category to be equivalent to $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_4} = 2\mathsf{Rep}(\mathbb{Z}_4)$ obtained by gauging the full $\mathbb{Z}_4$ 0-form symmetry in $\mathcal{C}_{\mathbb{Z}_4}$. Notice that up to the gauging of the first $\mathbb{Z}_2^V$ subgroup, the spectrum of objects and 1-morphisms has been identical irrespective of whether we start from $G = \mathbb{Z}_2\times\mathbb{Z}_2$ or $G = \mathbb{Z}_4$. However, upon the subsequent gauging of the residual $\mathbb{Z}_2^S$ symmetry, we expect to see differences in these spectra, as in one case we should land on the symmetry 2-category $2\mathsf{Rep}(\mathbb{Z}_2\times\mathbb{Z}_2)$ and in the other on $2\mathsf{Rep}(\mathbb{Z}_4)$, and these two 2-categories have different number of simple objects. In particular, $2\mathsf{Rep}(\mathbb{Z}_4)$ has less simple objects than $2\mathsf{Rep}(\mathbb{Z}_2\times\mathbb{Z}_2)$, which means that some of the topological surfaces of the latter will have to be inconsistent in the former. This is due to the obstructions related to symmetry fractionalisations discussed above.

First consider objects of $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_4}$ descending from multiples of the identity object $D_2^{(\text{id})}$ of $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$. We have to consider the trivial and identity homomorphisms

$$\rho: \; K = \mathbb{Z}_2 \to \widehat{H} = \mathbb{Z}_2 \,. \tag{3.54}$$

Since the extension class ϵ is non-trivial, we have for trivial homomorphism $\omega_\rho = 0$, but $\omega_\rho \neq 0 \in H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$ for the identity homomorphism. The trivial homomorphism leads to a copy of $2\mathsf{Rep}(\mathbb{Z}_2) \subseteq \mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$ whose simple objects we label as

$$D_2^{(\text{id})}, \qquad D_2^{(\mathbb{Z}_2^V)}, \tag{3.55}$$

where $D_2^{(\text{id})}$ corresponds to the trivial 2-representation and $D_2^{(\mathbb{Z}_2^V)}$ corresponds to the non-trivial irreducible 2-representation of dimension 2.

Now consider the simple objects descending from the identity homomorphism ρ. There is only a single irreducible projective 2-representations of $\mathbb{Z}_2$ with non-trivial class ω_ρ, since ω_ρ trivialises only on the trivial subgroup of $\mathbb{Z}_2$ and there is only a single choice of trivialisation. Since the choice of subgroup is trivial inside $\mathbb{Z}_2$, the $\mathbb{Z}_2^S$ symmetry is spontaneously broken and the irreducible projective 2-representation has dimension 2. Let us label the corresponding simple object in $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_4}$ as

$$D_2^{(\mathbb{Z}_2^{V'})} . \tag{3.56}$$

Note that we do not obtain an analogue of $D_2^{(-)} \in \mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$ in $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_4}$. In other words, symmetry fractionalisation in $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$ has obstructed the existence of $D_2^{(-)}$ in $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_4}$.

Just as for the previous example, it also turns out here that

$$D_2^{(\mathbb{Z}_2^{V'})} \cong D_2^{(\mathbb{Z}_2^V)} , \tag{3.57}$$

though here the isomorphism is quite complicated. Now let us consider making $D_2^{(\mathbb{Z}_2)}$

symmetric under $\mathbb{Z}_2^S$. This fails already at the level of lines, because as discussed above $\mathbb{Z}_2^S$ fractionalizes to a $\mathbb{Z}_4$ 0-form symmetry on $D_2^{(\mathbb{Z}_2)}$. Thus, for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case, $D_2^{(\mathbb{Z}_2)} \in \mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$ gives rise to two simple objects $D_2^{(\mathbb{Z}_2^S)}$ and $D_2^{(\mathbb{Z}_2^C)}$ in $\mathcal{C}_{\mathbb{Z}_2^2/\mathbb{Z}_2^2}$, but for the $\mathbb{Z}_4$ case we do not get even a single simple object of $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_4}$ from $D_2^{(\mathbb{Z}_2)} \in \mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$.

On the other hand, we can successfully make $2D_2^{(\mathbb{Z}_2)}$ symmetric under $\mathbb{Z}_2^S$ by implementing the $\mathbb{Z}_2^S$ symmetry using the matrix of junction lines

$$\begin{pmatrix} 0 & L_1^+ \\ L_1^- & 0 \end{pmatrix} . \tag{3.58}$$

These junction lines were defined around equation (3.37). This leads to the simple object $D_2^{(\mathbb{Z}_4)}$ of $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_4}$.

In summary we obtain the simple objects

$$\text{Obj}\left(\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_4}\right) = \left\{D_2^{(\text{id})}, D_2^{(\mathbb{Z}_2^V)}, D_2^{(\mathbb{Z}_4)}\right\} , \tag{3.59}$$

of $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_4}$ starting from the 2-category $\mathcal{C}_{\mathbb{Z}_4/\mathbb{Z}_2}$ and gauging its $\mathbb{Z}_2^S$ 0-form symmetry. This is exactly equivalent to the simple objects of $2\mathsf{Rep}(\mathbb{Z}_4)$ (see 3.1).

3.3.2 Fusion of Surface Defects after Sequential Gauging

The fusion of surfaces in $\mathcal{C}_{G/G}$ can be computed in terms of $\mathcal{C}_{G/H}$ information by computing the fusion of underlying surfaces in $\mathcal{C}_{G/H}$ and the fusion of the junction lines implementing K symmetry (which provides the junction lines implementing K symmetry on the fusion of underlying surfaces).

Example: $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. First of all, we clearly have

$$D_2^{(-)} \otimes D_2^{(-)} = D_2^{(\text{id})} , \tag{3.60}$$

because the $\mathbb{Z}_2^S$ symmetry for $D_2^{(-)} \otimes D_2^{(-)}$ is implemented by $D_1^{(V)} \otimes D_1^{(V)} \cong D_1^{(\mathrm{id})}$ in $\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$. Next, we also have

$$D_2^{(-)} \otimes D_2^{(\mathbb{Z}_2^V)} = D_2^{(\mathbb{Z}_2^{V'})} \cong D_2^{(\mathbb{Z}_2^V)}\,, \tag{3.61}$$

because the tensor product of the junctions implementing $\mathbb{Z}_2^S$ symmetry is

$$D_1^{(V)} \otimes \begin{pmatrix} 0 & D_1^{(\mathrm{id})} \\ D_1^{(\mathrm{id})} & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_1^{(V)} \\ D_1^{(V)} & 0 \end{pmatrix}. \tag{3.62}$$

Likewise we find

$$\begin{aligned} D_2^{(-)} \otimes D_2^{(\mathbb{Z}_2^x)} &\cong D_2^{(\mathbb{Z}_2^x)}\,, \qquad x = S, C \\ D_2^{(-)} \otimes D_2^{(\mathbb{Z}_2\times\mathbb{Z}_2)} &\cong D_2^{(\mathbb{Z}_2\times\mathbb{Z}_2)}\,, \end{aligned} \tag{3.63}$$

because $D_1^{(V)}$ line becomes invisible when stacked on top of $D_2^{(\mathbb{Z}_2)}$ in $\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$. The fusion rules

$$D_2^{(\mathbb{Z}_2^V)} \otimes D_2^{(\mathbb{Z}_2^V)} \cong 2D_2^{(\mathbb{Z}_2^V)}\,, \tag{3.64}$$

follow from arguments similar to those leading to the fusion rule (3.21), and

$$D_2^{(\mathbb{Z}_2^x)} \otimes D_2^{(\mathbb{Z}_2^x)} \cong 2D_2^{(\mathbb{Z}_2^x)}\,, \qquad x = S, C\,, \tag{3.65}$$

follows from the fact that $D_1^{(\mathbb{Z}_2;i)} \otimes D_1^{(\mathbb{Z}_2;i)} \in \mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$ is given by $D_1^{(\mathbb{Z}_2;i)}\mathrm{id}_{2\times 2}$ for $i \in \{\mathrm{id}, -\}$ (see also [86], (3.68) and (3.71) for similar arguments). The fusion rule

$$D_2^{(\mathbb{Z}_2^S)} \otimes D_2^{(\mathbb{Z}_2^C)} \cong D_2^{(\mathbb{Z}_2\times\mathbb{Z}_2)}\,, \tag{3.66}$$

follows from the fact that $D_1^{(\mathbb{Z}_2;\mathrm{id})} \otimes D_1^{(\mathbb{Z}_2;-)} \in \mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2}$ is given by 2×2 off-diagonal matrix with both entries $D_1^{(\mathbb{Z}_2;\mathrm{id})}$ (see also [86], (3.69) for similar arguments). Combining the above facts, the reader can easily show the remaining fusion rules

$$D_2^{(\mathbb{Z}_2^V)} \otimes D_2^{(\mathbb{Z}_2^x)} \cong D_2^{(\mathbb{Z}_2 \times \mathbb{Z}_2)} , \qquad x = S, C , \tag{3.67}$$

and

$$D_2^{(\mathbb{Z}_2 \times \mathbb{Z}_2)} \otimes D_2^{(\mathbb{Z}_2 \times \mathbb{Z}_2)} \cong 4 D_2^{(\mathbb{Z}_2 \times \mathbb{Z}_2)} . \tag{3.68}$$

The fusion rules for the $\mathbb{Z}_4$ case can be computed analogously.

3.4 The D_8-Categorical Symmetry Web for 3d Orthogonal Gauge Theories

A rich class of categorical symmetries arises for gauge theories with orthogonal gauge groups, with gauge algebra $\mathfrak{so}(2N)$. Using the techniques described in the previous sections, we can construct the full web of categorical symmetries. We start with the theory which has purely a 0-form symmetry, which in this case is D_8, and is realised in the PSO$(4N)$ and PSO$(4N+2)$ gauge theories. The group D_8 can be presented as follows

$$D_8 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2 = \left\langle a, x \;\middle|\; x^2 = a^4 = 1, xax = a^3 \right\rangle . \tag{3.69}$$

All other gauge theories with gauge groups (including both connected and disconnected ones) having gauge algebras $\mathfrak{so}(4N)$ and $\mathfrak{so}(4N+2)$ can be obtained by gauging a 0-form symmetry subgroup of D_8. We present the categorical symmetry web for $\mathfrak{so}(4N)$ in figure 3.8, which shows well the complexity of these structures. The full detailed description of each of the gauging steps and the resulting symmetry categories can be found in [2].

Let us specify precisely how these symmetries are realised in the 3d gauge theories: perhaps the most familiar theory is the Spin$(4N)$ gauge theory, which has 0-form

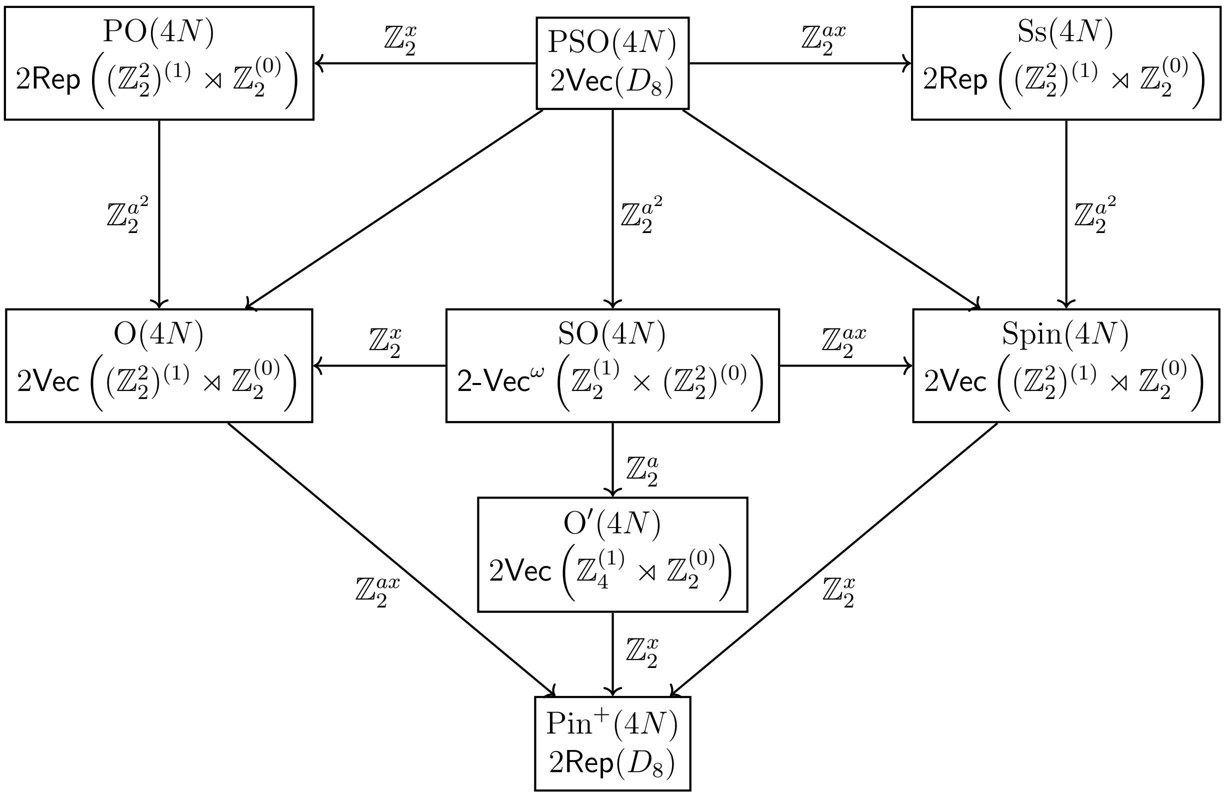


Figure 3.8: Categorical symmetry web for 3d gauge theories with gauge algebra $\mathfrak{so}(4N)$. We label each theory by its global gauge group, and the 2-category which is its symmetry category. The arrows denote gaugings of 0-form symmetries and each arrow is labelled by the subgroup of D_8 that is gauged along that arrow.

symmetry $\mathbb{Z}_2^x$, that acts as outer automorphism, and 1-form symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ that acts (on Wilson lines) as the center[7] of $\mathrm{Spin}(4N)$. Gauging one of the two $\mathbb{Z}_2$ 1-form symmetries leads to two equivalent theories with gauge groups called $\mathrm{Ss}(4N)$ or $\mathrm{Sc}(4N)$, while gauging the diagonal $\mathbb{Z}_2$ subgroup of the 1-form symmetry gives the $\mathrm{SO}(4N)$ theory. Gauging the full 1-form symmetry results in the theory $\mathrm{PSO}(4N)$, which is our starting point for the web. Gauging instead the 0-form symmetry starting with $\mathrm{Spin}(4N)$ results in $\mathrm{Pin}^+(4N)$, which is also the theory obtained by gauging the full D_8 0-form symmetry of the $\mathrm{PSO}(4N)$ theory. The remaining gaugings are shown in figure 3.8.

[7]The two $\mathbb{Z}_2$ in the center are distinguished by the irreducible spinor representation (namely spinor or cospinor) they act on.

Chapter 4

Categorical Landau Paradigm: Gapped Phases

In this chapter we propose a general framework to study the impact of a categorical symmetry $\mathcal{S}$ in the UV on the IR physics, focusing on gapped, i.e. topological, phases.[1] We determine the generalised charges under $\mathcal{S}$ of order parameters for the gapped phases, leading to a categorical Landau paradigm. In particular, here we describe a classification of gapped $\mathcal{S}$-symmetric phases using the SymTFT. Importantly, we restrict to finite $\mathcal{S}$ symmetries.

Many works have studied gapped theories with categorical symmetries in $d = 2$, see e.g. [44, 48, 52, 53, 53, 54, 69, 89, 125], while a comprehensive comparison to the existing literature is provided in [4]. We remark in particular that in $(1 + 1)$d a classification of gapped phases with symmetry $\mathcal{S}$ has been given in terms of module categories over $\mathcal{S}$ [44]. Here we reformulate the problem in an equivalent way in terms of gapped boundaries of the SymTFT, which gives a comprehensive and clean picture of the spontaneous symmetry breaking patterns, order parameters, and relative Euler terms. This approach can moreover be generalised to higher dimensions, see e.g. [79, 80] for $(2 + 1)$d. The content of this chapter is based on [3, 4].

The structure of the chapter is as follows. We begin in section 4.1 with a general

[1] Here we focus on bosonic phases and we also assume that we are working with systems that have emergent Lorentz symmetry in the IR.

review of the SymTFT, which includes a discussion of gapped boundary conditions specialised to $(1+1)$d. Then in section 4.2 we explain how to characterise gapped phases from the SymTFT. In the following section 4.3 we illustrate the above proposal in several examples in (1+1)d, and we conclude in section 4.4 by mentioning a (3+1)d application.

4.1 The Symmetry Topological Field Theory

When studying categorical symmetries, it is particularly useful to invoke the symmetry topological field theory, or SymTFT [50, 92–94]. This separates the physical theory from its symmetries, allowing us to infer theory-independent aspects of the latter. It also provides a unified framework to study symmetries that are related by (generalised) gauging, and it encodes all the generalised charges [95,126] (i.e. local and extended operators that are charged under the categorical symmetry). In particular, two theories related by gauging of a global symmetry share the same SymTFT. Concretely, the SymTFT is a $(d+1)$-dimensional TQFT $\mathfrak{Z}(\mathcal{S})$ for a d-dimensional theory $\mathfrak{T}$ with a categorical symmetry $\mathcal{S}$, which can be constructed by gauging the symmetry $\mathcal{S}$ in $(d+1)$ dimensions. The simplest example of SymTFT arises for abelian group-like (higher form-)symmetries, where it is a BF-theory (or Dijkgraaf-Witten theory), potentially with a twist, for the background gauge field of the symmetry and its dual. In general, however, there is no such a simple action for the SymTFT. Nevertheless, we can still utilise the vast knowledge about the topological defects of the SymTFT to extract the key physical information, as we shall see. Mathematically,the topological defects of the SymTFT form a modular tensor category (MTC) known as the Drinfeld center of $\mathcal{S}$, $\mathcal{Z}(\mathcal{S})$.

The SymTFT has two boundaries: a topological boundary $\mathfrak{B}^{\text{sym}}_{\mathcal{S}}$, which encodes the symmetry $\mathcal{S}$, and a not necessarily topological boundary $\mathfrak{B}^{\text{phys}}_{\mathfrak{T}}$, which is the-

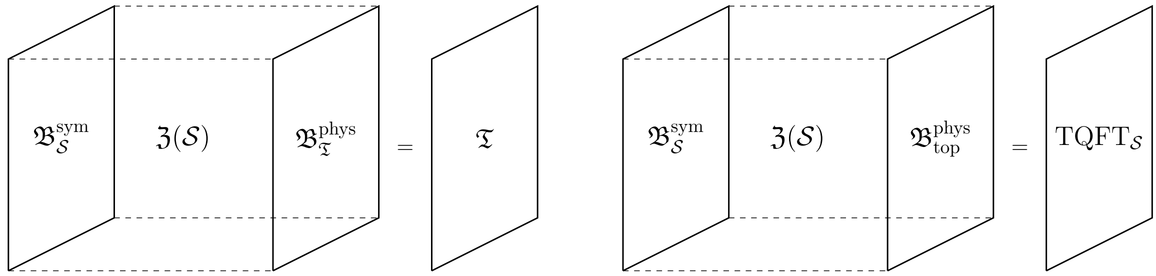


Figure 4.1: The basic SymTFT sandwich. (1) LHS: The d-dimensional theory $\mathfrak{T}$ on the RHS is constructed as the interval compactification of $d+1$-dimensional SymTFT $\mathfrak{Z}(\mathcal{S})$ on the LHS, with two boundary conditions. The gapped, i.e. topological, boundary $\mathfrak{B}^{\text{sym}}_{\mathcal{S}}$ is on the left and the physical, possibly non-topological, boundary $\mathfrak{B}^{\text{phys}}_{\mathfrak{T}}$ is on the right. (2) RHS: Here we focus on sandwich constructions for $\mathcal{S}$-symmetric TQFTs (denoted $\text{TQFT}_{\mathcal{S}}$) in which case the physical boundary $\mathfrak{B}^{\text{phys}}_{\mathfrak{T}}$ is also topological, $\mathfrak{B}^{\text{phys}}_{\text{top}}$ (though not necessarily the same as the symmetry boundary on the left).

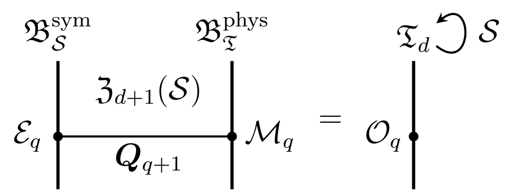
ory specific and encodes its dynamics. In the case of the BF-theory, the topological boundary condition amounts simply to specifying Dirichlet or Neumann for the form-fields. Compactifying the interval direction recovers $\mathfrak{T}$, which is known as the *sandwich construction.* This is shown on the LHS of figure 4.1

We can also consider an "open" version of the sandwich, known as the *quiche.* This consists of the symmetry boundary $\mathfrak{B}^{\text{sym}}_{\mathcal{S}}$ and the topological bulk TQFT $\mathfrak{Z}(\mathcal{S})$, but without the presence of the physical boundary $\mathfrak{B}^{\text{phys}}$ on the RHS.

4.1.1 Order Parameters from the SymTFT

Let us briefly review how order parameters come about from the SymTFT. Any (possibly non-topological) q-dimensional operator $\mathcal{O}_q$ in a d-dimensional $\mathcal{S}$-symmetric QFT $\mathfrak{T}$ is charged under the symmetry only if it somehow interacts with the symmetry boundary. This is possible only if the sandwich construction of $\mathcal{O}_q$ involves a bulk topological $(q+1)$-dimensional operator $\boldsymbol{Q}_{q+1}$ ending on the physical boundary $\mathfrak{B}^{\text{phys}}_{\mathfrak{T}}$

along a (possibly non-topological) q-dimensional operator $\mathcal{M}_q$ as shown below[2]



(4.1)

The end $\mathcal{E}_q$ of $\boldsymbol{Q}_{q+1}$ is a topological q-dimensional operator along the symmetry boundary $\mathfrak{B}^{\text{sym}}_{\mathcal{S}}$, which may be attached to topological operators living on the boundary $\mathfrak{B}^{\text{sym}}_{\mathcal{S}}$ (in which case $\mathcal{O}_q$ is also attached to topological operators in $\mathfrak{T}_d$ generating the symmetry $\mathcal{S}$ and hence lives in twisted sector for the symmetry). The action of the symmetry $\mathcal{S}$ on $\mathcal{O}_q$ is captured in how the bulk operator $\boldsymbol{Q}_{q+1}$ interacts with the symmetry boundary $\mathfrak{B}^{\text{sym}}_{\mathcal{S}}$ (via the end $\mathcal{E}_q$), and hence $\boldsymbol{Q}_{q+1}$ captures the charge of $\mathcal{O}_q$ under $\mathcal{S}$. Note that if there is no end $\mathcal{M}_q$ of a bulk topological operator $\boldsymbol{Q}_{q+1}$ on a physical boundary $\mathfrak{B}^{\text{phys}}_{\mathfrak{T}}$, then there is no q-dimensional operator in the theory $\mathfrak{T}$ carrying the generalised charge $\boldsymbol{Q}_{q+1}$.

4.1.2 Group-Theoretical Non-Invertible Symmetries

As an example of SymTFT construction, we now specialise to (1+1)d and consider a group-theoretical non-invertible symmetry $\mathcal{S}$, i.e. one that can obtained by gauging invertible symmetries. This means that $\mathcal{S}$ shares the same SymTFT with its sibling invertible symmetry $\mathsf{Vec}^{\omega}_{G}$ (where G is the symmetry group and $\omega \in H^3(G, U(1))$ is the ‘t Hooft anomaly)

$$\mathfrak{Z}(\mathcal{S}) \cong \mathfrak{Z}(\mathsf{Vec}^{\omega}_{G}) \,, \tag{4.2}$$

and hence the same set of generalised charges is also shared between them. Nevertheless, what differentiates the SymTFT with two different symmetries is its choice

[2] This is an analogous of figure 4.1, but projected it to one dimension lower in order to be more compact. We will adopt this “reduced” version of picturing the SymTFT also in the following.

of symmetry topological boundary: what we label as Dirichlet boundary condition corresponds to Vec_G^ω symmetry, whereas various Neumann boundary conditions correspond to some parts of the Vec_G^ω symmetry being gauged. Ultimately, having different boundary conditions translates to allowing different topological line defects to condense (terminate) on the boundaries, which in turn affects which associated irreducible multiplets of operators charged under $\mathcal{S}$ are present. Such a multiplet may contain both twisted and untwisted sector operators. From this point on, we restrict our attention to $\omega = 0$. Analogous statements for $\omega \neq 0$ can be found in [4].

Categorically speaking, topological line defects of $\mathfrak{Z}(\mathsf{Vec}_G)$ form a modular fusion category, the Drinfeld center $\mathcal{Z}(\mathcal{S}) \cong \mathcal{Z}(\mathsf{Vec}_G)$. The simple lines (or objects) of $\mathcal{Z}(\mathsf{Vec}_G)$ are

$$\boldsymbol{Q}_{[g],\boldsymbol{R}}\,, \tag{4.3}$$

and can be labelled by a conjugacy class $[g]$ of G and an irreducible representation $\boldsymbol{R}$ of H_g, the centraliser of any element $g \in [g]$. For the trivial conjugacy class $[g] = [\mathrm{id}]$, $\boldsymbol{R}$ are irreducible representations of G. Physically, these lines $\boldsymbol{Q}_{[\mathrm{id}],\boldsymbol{R}}$ are the Wilson lines for the 3d G gauge theory, forming the objects of $\mathsf{Rep}(G)$. On the other hand, a line $\boldsymbol{Q}_{[g],1}$ is a vortex line around which we have a holonomy for the G gauge fields. The remaining lines $\boldsymbol{Q}_{[g],\boldsymbol{R}}$ are mixed lines obtained by dressing vortex lines with Wilson lines.

4.1.3 Gapped Boundaries

To discuss gapped boundaries, we remain in $(1+1)$d, which is the case we are interested in the most and where we can be more concrete. Irreducible topological boundary conditions of $\mathfrak{Z}(\mathcal{S})$ are described by Lagrangian algebras in the category $\mathcal{Z}(\mathcal{S})$. A discussions of Lagrangian algebras can be found in [127–132]. We refer to the deformation class corresponding to a Lagrangian algebra $\mathcal{L}$ as $[\mathfrak{B}](\mathcal{L})$. The deformation class comprises of a (real) one-parameter family of irreducible topological boundary conditions

$$\mathfrak{B}_\lambda(\mathcal{L}), \qquad \lambda \in \mathbb{R}\,. \tag{4.4}$$

These boundaries are related as

$$\mathfrak{B}_\lambda(\mathcal{L}) = \mathfrak{T}_\lambda \boxtimes \mathfrak{B}_0(\mathcal{L}) \tag{4.5}$$

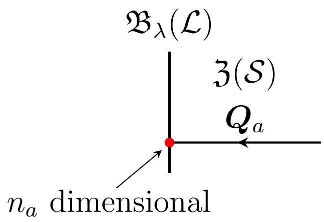
i.e. the boundary $\mathfrak{B}_\lambda(\mathcal{L})$ can be obtained from a reference boundary $\mathfrak{B}_0(\mathcal{L})$ by stacking it with an invertible 2d TQFT $\mathfrak{T}_\lambda$, known as the Euler term [3]. More generally, we have

$$\mathfrak{B}_{\lambda_2}(\mathcal{L}) = \mathfrak{T}_{\lambda_2-\lambda_1} \boxtimes \mathfrak{B}_{\lambda_1}(\mathcal{L})\,. \tag{4.6}$$

The Lagrangian algebra $\mathcal{L}$ can be expressed as

$$\mathcal{L} = \bigoplus_a n_a \boldsymbol{Q}_a \tag{4.7}$$

where the sum is over the simple bulk anyons $\boldsymbol{Q}_a$ in $\mathcal{Z}(\mathcal{S})$. Physically, the presence of a term $n_a\boldsymbol{Q}_a$ in $\mathcal{L}$ means that there is an n_a-dimensional vector space of topological local operators along any corresponding topological boundary $\mathfrak{B}_\lambda(\mathcal{L})$ at which the line $\boldsymbol{Q}_a$ can end



(4.8)

[3]Its partition function on a closed 2d manifold Σ_g (of genus g) is $Z[\mathfrak{T}_\lambda, \Sigma_g] = \exp\left(-\lambda\chi(\Sigma_g)\right)$, where $\chi(\Sigma_g) = 2-2g$ is the Euler characteristic of Σ_g.

Any such algebra has to satisfy the fusion coefficient and dimension constraints,

$$n_a n_b \leq \sum_c N_{ab}^c n_c \quad \text{and} \quad \dim(\mathcal{A}) := \sum_{a\in\mathcal{A}} n_a \dim(\boldsymbol{Q}_a) = \dim^2(\mathcal{S})\,, \tag{4.9}$$

where N_{ab}^c are the fusion coefficients sending $a\otimes b \to c$ in $\mathcal{S}$ and $\dim(\mathcal{S})$ is the quantum dimension of the category, which can be found as

$$\dim(\mathcal{S}) = \sqrt{\sum_a \dim^2(a)}\,, \tag{4.10}$$

where the sum is over all simple objects a of $\mathcal{S}$ and $\dim(a)$ denotes the expectation value of a loop a. Finally, we note that all $\boldsymbol{Q}_a$ participating in $\mathcal{A}$ must be bosons.

There is a canonical Lagrangian algebra $\mathcal{L}_{\mathcal{S}}^{\text{sym}}$ in $\mathcal{Z}(\mathcal{S})$ that describes a symmetry boundary $\mathfrak{B}_{\mathcal{S}}^{\text{sym}}$

$$\mathcal{L}_{\mathcal{S}}^{\text{sym}} = \bigoplus_a n_a^{\text{sym}} \boldsymbol{Q}_a, \qquad n_a^{\text{sym}} \in \mathbb{Z}_{\geq 0} \tag{4.11}$$

where $\boldsymbol{Q}_a$ are simple bulk anyons and n_a is the dimension of morphism space $\text{Hom}_{\mathcal{S}}(F(\boldsymbol{Q}_a), 1)$ in $\mathcal{S}$ between the object $F(\boldsymbol{Q}_a) \in \mathcal{S}$ obtained by applying the forgetful functor $F:\ \mathcal{Z}(\mathcal{S}) \to \mathcal{S}$ and the identity object $1 \in \mathcal{S}$.

4.2 Symmetric Gapped Phases from the SymTFT

As we are interested in *gapped* phases, or TQFTs, with $\mathcal{S}$ symmetry, we now take the physical boundary to also be a topological boundary condition. This SymTFT set-up is shown on the RHS of figure 4.1. We propose that classifying $\mathcal{S}$-symmetric d-dimensional gapped phases, $d = D+1$, by utilising the SymTFT perspective requires the steps that we illustrate below. This is a proposal which we thoroughly check in $d = 2$ and that remains future work for $d > 2$ – see [75, 79] for progress along these lines.

(1) SymTFT and Drinfeld center. Given a symmetry category $\mathcal{S}$, we construct the associated $(d+1)$d SymTFT $\mathfrak{Z}(\mathcal{S})$, which amounts to gauging $\mathcal{S}$ in $(d+1)$d. The SymTFT has topological defects which form the so-called Drinfeld center $\mathscr{Z}(\mathcal{S})$ of $\mathcal{S}$, and play a key physical role as generalised charges.

An example is provided by a $\mathbb{Z}_N$ p-form symmetry in d-dimensions. Its SymTFT is the BF theory

$$S_{\text{SymTFT}} = \frac{2\pi}{N} \int_{M_{d+1}} B_{p+1} \cup \delta C_{d-p-1} \,, \tag{4.12}$$

where B_{p+1} is the background for the p-form symmetry and C_{d-p-1} the one for the dual $(d-p-2)$-form symmetry. Its topological defects, i.e. the elements of the Drinfeld center, include the Wilson surfaces, but the full structure is a braided higher-fusion category (see [95] for more details).

(2) Gapped boundary conditions. We then classify all the irreducible topological boundary conditions of $\mathfrak{Z}(\mathcal{S})$. As we have seen, a topological boundary condition is essentially characterised by a set $\mathcal{L}$ of topological defects that can end on the boundary, i.e. by a "Lagrangian algebra". Topological defects in $\mathcal{L}$ become trivial on the boundary (i.e. they have Dirichlet boundary condition), while the ones not in $\mathcal{L}$ are not trivialised and become symmetry generators (i.e. they have Neumann boundary condition).

(3) Symmetry boundary conditions. To classify $\mathcal{S}$-symmetric gapped phases, we fix the symmetry boundary to be

$$\mathfrak{B}^{\text{sym}}_{\mathcal{S}} = \mathcal{L}_{\mathcal{S}} \,, \tag{4.13}$$

where $\mathcal{L}_{\mathcal{S}}$ is a Lagrangian algebra that realises the symmetry $\mathcal{S}$ on the boundary (generated by the topological defects with Neumann boundary conditions).

(4) Physical boundary conditions. The key difference with the standard SymTFT is the choice of physical boundary $\mathfrak{B}^{\text{phys}}$, which we also take to be topological, and is thus specified by a Lagrangian algebra $\mathcal{L}_{\text{phys}}$

$$\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\text{phys}} \,, \tag{4.14}$$

which determines the topological defects that can end on $\mathfrak{B}^{\text{phys}}$. Upon interval compactification of this SymTFT, we obtain a d-dimensional TQFT. By varying $\mathcal{L}_{\text{phys}}$, while keeping $\mathcal{L}_{\mathcal{S}}$ fixed, we move between different irreducible $\mathcal{S}$-symmetric gapped phases.

(5) Generalised charges as order parameters. For an arbitrary $\mathcal{S}$-symmetric QFT $\mathfrak{T}$, the charges of (extended) operators under $\mathcal{S}$ are captured by topological defects of the SymTFT that can end on its physical boundary $\mathfrak{B}^{\text{phys}}_{\mathfrak{T}}$ [95]. In our context, this implies that $\mathcal{L}_{\text{phys}}$ determines the charges of the operators in the d-dimensional theory under $\mathcal{S}$: $\mathcal{L}_{\text{phys}}$ captures the order parameters for the $\mathcal{S}$-symmetric gapped phase.

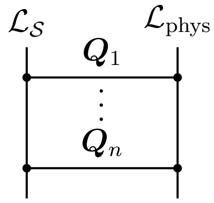
The order parameters for non-invertible symmetries will typically be a mixture of untwisted (conventional order parameters) and twisted-sector (string order parameters) operators, which combine to form irreducible multiplets under $\mathcal{S}$.[4] This provides a generalised, categorical Landau paradigm describing gapped phases for an arbitrary categorical symmetry $\mathcal{S}$.

4.2.1 Classification of (1+1)d Gapped Phases

We now specialise to unitary fusion categories $\mathcal{S}$ in (1+1)d to provide a concrete implementation of the proposal. In this case, we can extend the above program with additional refined properties for general $\mathcal{S}$.

[4] By "twisted-sector" operator, we mean an operator at the end of a topological defect of $\mathcal{S}$.

(6)$^{(1+1)d}$ Vacua. The number of vacua is easily determined by the number of the lines $\boldsymbol{Q}_i$ that can end on both boundaries, i.e. that appear in both $\mathcal{L}_{\mathcal{S}}$ and $\mathcal{L}_{\text{phys}}$. Indeed, these lines can completely end on both boundaries and give rise to a topological local operator after taking the sandwich.



(4.15)

(7)$^{(1+1)d}$ Action of the symmetry $\mathcal{S}$. The action of the symmetry $\mathcal{S}$ on the (1+1)d gapped phase under discussion is specified by line operators $D_1^{(a)}$ of the associated 2d TQFT for each object $a \in \mathcal{S}$, which represent the fusion category $\mathcal{S}$ on the phases. The lines $D_1^{(a)}$ are determined as combinations of line operators of the 2d TQFT that act on the IR local operators realising the order parameters according to their charges under $\mathcal{S}$.

(8)$^{(1+1)d}$ SSB of non-invertibles and Euler terms. A notable phenomenon arises for (1+1)d gapped phases with categorical symmetries: the different vacua may be physically distinguishable as they can carry different Euler terms. Such terms are encoded in the properties of interfaces (which are line defects in 2d) between different vacua. Two vacua carrying different Euler terms are necessarily related by a line $D_1^{(a)}$ implementing a non-invertible symmetry $a \in \mathcal{S}$ on the gapped phase, which is thus spontaneously broken. We conclude that: spontaneous breaking of non-invertible symmetries can lead to physically distinguishable vacua. This is one hallmark of categorical symmetries. This idea has been observed in various dimensions and implemented in examples using different approaches [44, 59, 71, 133].

4.3 Examples in $(1+1)$d

We now apply the framework outlined above to several examples of fusion category symmetry $\mathcal{S}$, including both invertible and non-invertible ones.

4.3.1 $\mathcal{S}$ = Finite Group

For a non-anomalous group symmetry G, with $\mathcal{S} = \mathsf{Vec}_G$, it is well known that (1+1)d gapped phases are a mixture of spontaneously symmetry broken and symmetry protected topological (SPT) phases. These are classified by pairs (H, β), where $H \leq G$ is a subgroup representing the symmetry *unbroken* in one of the vacua v and $\beta \in H^2(H, U(1))$ is the SPT phase for the unbroken H symmetry in v.

This easily follows from the SymTFT $\mathfrak{Z}(\mathsf{Vec}_G)$, which is a 3d Dijkgraaf-Witten (DW) theory with gauge group G. The symmetry boundary is chosen to be a Dirichlet boundary condition (Dir) for the bulk G gauge fields

$$\mathfrak{B}^{\text{sym}}_{\mathsf{Vec}_G} = \mathcal{L}_{\text{Dir}} \,. \tag{4.16}$$

Any other topological boundary condition, labelled $(\text{Neu}(H), \beta)$, is related to this by gauging a subgroup $H \leq G$ with discrete torsion $\beta \in H^2(H, U(1))$. This is realised by imposing Neumann boundary conditions for the H gauge fields, which are labelled by β. By choosing the physical topological boundary to be

$$\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\text{Neu}(H),\beta} \tag{4.17}$$

we obtain the G-symmetric (1+1)d gapped phase associated to the pair (H, β), thus reproducing the expected classification. Let us mention two special cases: $H = 1$, $\beta = 0$ is $\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\text{Dir}}$ and corresponds to the SSB phase for G. The order parameters are untwisted local operators transforming in irreducible representations of G, given by all the Wilson lines of the DW theory. On the other hand $H = G$ and any β, i.e. $\mathfrak{B}^{\rm phys} = \mathcal{L}_{{\rm Neu}(G),\beta}$, characterises the G SPTs. The order parameters are string-like, i.e. topological vortices (or magnetic) lines of the DW theory, dressed by β-dependent Wilson lines.

4.3.2 $\mathcal{S} = \mathsf{Rep}(S_3)$

One of the simplest example of non-invertible symmetry is $\mathsf{Rep}(S_3)$, the fusion category of representations of the permutation group S_3. Its simple objects are irreducible representations: the trivial 1, 1-dimensional sign P and 2-dimensional standard E representations, with non-trivial fusions

$$P \otimes P = 1\,, \quad P \otimes E = E \otimes P = E\,, \quad E \otimes E = 1 \oplus P \oplus E\,. \tag{4.18}$$

$\mathsf{Rep}(S_3)$ is obtained by gauging the non-anomalous $G = S_3$ symmetry. In this sense, $\mathsf{Rep}(S_3)$ is a group-theoretical non-invertible symmetry.

4.3.2.1 $\mathsf{Rep}(S_3)$ SymTFT

We can easily construct the SymTFT, as it is the same of Vec_{S_3}. To do so we first label the elements of S_3 as

$$S_3 = \left\{1, a, a^2, b, ab, a^2 b\right\} \quad \text{with} \quad a^3 = b^2 = 1 \quad \text{and} \quad ab = ba^2\,. \tag{4.19}$$

As described in 4.1.2, for a (non-anomalous) group G in 2d, the topological defects (anyons)[5] of the SymTFT are labelled by two pieces of data: a conjugacy classes $[g]$ of G and irreducible representations (irreps) of the centraliser H_g in G of any element

[5] Here and in the following we use interchangeably the term *anyon* – coming from the condensed matter literature – when referring to a topological line defect in an MTC describing some topological order in $(2+1)$d.

$g \in [g]$. In the case of $G = S_3$, one finds the following conjugacy classes

$$[1] = \{1\} \quad , \quad [a] = \{a, a^2\} \quad , \quad [b] = \{b, ab, a^2b\}\,, \tag{4.20}$$

with corresponding centralisers

$$H_1 = S_3 \quad , \quad H_a = \mathbb{Z}_3 = \{1, a, a^2\} \quad , \quad H_b = \mathbb{Z}_2 = \{1, b\}\,. \tag{4.21}$$

Consequently, the simple topological lines of the SymTFT $\mathfrak{Z}(\mathsf{Vec}_{S_3}) = \mathfrak{Z}(\mathsf{Rep}(S_3))$ are labelled by the conjugacy class and centraliser irreps,

- $[1]$, $\boldsymbol{R} = 1, P, E$. This gives rise to three anyons $1, P, E$;
- $[a]$, $\boldsymbol{R} = 1, \omega, \omega^2$. This gives rise to three anyons $a_1, a_\omega, a_{\omega^2}$;
- $[b]$, $\boldsymbol{R} = +, -$. This gives rise to two anyons $b_\pm$.

Above, P is the sign irrep of S_3, E is the 2d irrep of S_3, $\omega = e^{\pm 2\pi i/3}$ is a $\mathbb{Z}_3$ irrep, and $+, -$ denote the $\mathbb{Z}_2$ irreps.

The topological lines in $\mathfrak{Z}(\mathsf{Vec}_{S_3}) = \mathfrak{Z}(\mathsf{Rep}(S_3))$ that are bosons are

$$1, \qquad P, \qquad E, \qquad a_1, \qquad b_+\,, \tag{4.22}$$

and the possible Lagrangian algebras they form that satisfy the constraints illustrated in 4.1.3 are

$$\begin{aligned} \mathcal{L}_{\text{Dir}} &= 1 \oplus P \oplus 2E \\ \mathcal{L}_{\text{Neu}} &= 1 \oplus a_1 \oplus b_+ \\ \mathcal{L}_{\text{Neu}(\mathbb{Z}_2)} &= 1 \oplus E \oplus b_+ \\ \mathcal{L}_{\text{Neu}(\mathbb{Z}_3)} &= 1 \oplus P \oplus 2a_1\,. \end{aligned} \tag{4.23}$$

As mentioned previously, various kinds of Neumann boundary conditions correspond to taking the Dirichlet boundary condition, $\mathcal{L}_{\text{Dir}}$, and gauging different subgroups of the full group symmetry. From the SymTFT perspective, gauging simply means imposing free/Neumann boundary condition on the subset of gauge fields one wishes to gauge. Here, fully gauging the S_3 symmetry corresponds to $\mathcal{L}_{\text{Neu}}$, whereas partially gauging $\mathbb{Z}_2 \subseteq S_3$, $\mathbb{Z}_3 \subseteq S_3$ results in $\mathcal{L}_{\text{Neu}(\mathbb{Z}_2)}$, $\mathcal{L}_{\text{Neu}(\mathbb{Z}_3)}$ respectively.

4.3.2.2 $\mathsf{Rep}(S_3)$ Gapped Phases

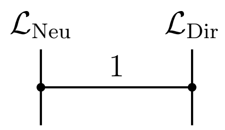
We are now ready to discuss all the gapped phases with $\mathsf{Rep}(S_3)$ symmetry. To implement the $\mathsf{Rep}(S_3)$ symmetry from the SymTFT, we take the symmetry boundary to be

$$\mathfrak{B}^{\text{sym}}_{\mathsf{Rep}(S_3)} = \mathcal{L}_{\text{Neu}} \,. \tag{4.24}$$

Choosing various physical boundaries in (4.23) gives rise to the various $\mathsf{Rep}(S_3)$-symmetric phases. Given four choices of physical boundaries in the present case, one finds four gapped phases, which we study in detail below.

Trivial Phase. By choosing $\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\text{Dir}}$, one finds that only the bulk line 1 can end on both sides of the SymTFT:

$\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\text{Dir}}$:



(4.25)

Hence one finds a phase with a single, trivial (untwisted) local operator which is the *trivial phase* with one vacuum, where $\mathsf{Rep}(S_3)$ symmetry is preserved. We can therefore identify the $\mathsf{Rep}(S_3)$ generators as

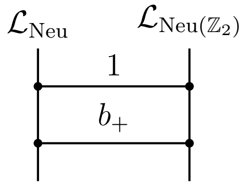
$$P \cong 1 \,, \qquad E \cong 1 \oplus 1 \,, \tag{4.26}$$

where 1 is the identity line operator.

This phase is characterised by the coexistence of two different kinds of order parameters. One type of order parameters are multiplets of operators carrying generalised charge P. Such a multiplet comprises of a local operator in twisted sector for the P symmetry, and hence such an order parameter is a string order parameter. The other type of order parameters are also of string type and carry generalised charge E, which is a multiplet comprising only of twisted sector operators, one in the E-twisted sector and one changing E to P. All the operators discussed above are uncharged under P. See [4] for more details.

$\mathbb{Z}_2$ SSB Phase. By choosing $\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\text{Neu}(\mathbb{Z}_2)}$, we end up with two untwisted sector local operators, which translates into finding two vacua in this phase:

$\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\text{Neu}(\mathbb{Z}_2)}$:



(4.27)

In comparison to the trivial phase, there is now another, non-trivial, untwisted sector local operator associated to b_+ that we label as $\mathcal{O}^b$. Additionally, there are two twisted sector operators $\mathcal{O}^b_\pm$ that belong to the b_+ multiplet and do not participate in the determination of vacua, but become important later as order parameters of the resulting phase.

In general, the n vacua v_i (with $i = 1, 2, \dots, n$) of the TQFT satisfy the relation

$$v_i v_j = \delta_{ij} v_i \,, \tag{4.28}$$

where δ_{ij} is the Kronecker delta. To find such vacua given the untwisted local operators in the theory, one must first obtain their operator algebra, in this case of $\{1, \mathcal{O}^b\}$. The only non-trivial algebra rule we must specify will be of the form

$$\mathcal{O}^b\mathcal{O}^b = \alpha + \beta\mathcal{O}^b, \qquad (\alpha, \beta) \in \mathbb{C}^2 - \{(0,0)\}\,. \tag{4.29}$$

To constrain the algebra we can study the action of P (and E) on $\mathcal{O}^b$,

(4.30)

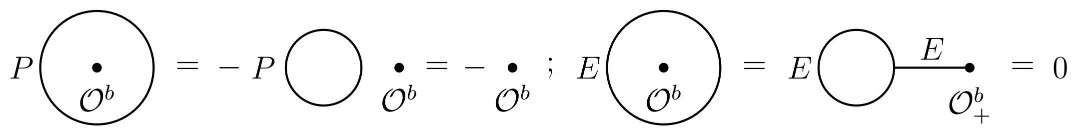
which also confirms that $\mathcal{O}^b$ is charged non-trivially under the $\mathbb{Z}_2$ subsymmetry of $\mathsf{Rep}(S_3)$ generated by P. Consequently, by symmetry, β must vanish and by rescaling $\mathcal{O}^b$ one finds the algebra to be

$$\mathcal{O}^b\mathcal{O}^b = 1\,. \tag{4.31}$$

The two vacua are then idempotent combinations of the identity local operator and $\mathcal{O}^b$,

$$v_0 = \frac{1+\mathcal{O}^b}{2}\,, \qquad v_1 = \frac{1-\mathcal{O}^b}{2}\,. \tag{4.32}$$

Now in order to identify the linking actions of P and E on the vacua, we first have to identify the linking actions on the local operator $\mathcal{O}^b$ which are



(4.33)

These linking actions then translate to the following linking actions on the vacua

$$\begin{aligned} P\bigcirc: &\qquad v_0 \to v_1\,, \qquad v_1 \to v_0\,, \\ E\bigcirc: &\qquad v_0, v_1 \to 1 = v_0 + v_1\,, \end{aligned} \tag{4.34}$$

which shows that there are no relative Euler terms between the two vacua as the linking actions only contain trivial factors for all terms. We can identify the $\mathsf{Rep}(S_3)$ symmetry generators as

$$P \cong 1_{01} \oplus 1_{10}\,, \qquad E \cong 1_{00} \oplus 1_{01} \oplus 1_{10} \oplus 1_{11} \cong 1 \oplus P\,. \tag{4.35}$$

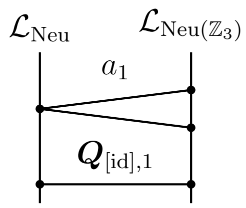
Here and in the following, the notation 1_{ij} denotes the irreducible unit line operator corresponding to an interface from vacuum i to vacuum j.

This phase is the $\mathbb{Z}_2$ *SSB phase* as the $\mathbb{Z}_2$ subgroup symmetry of the overall $\mathsf{Rep}(S_3)$ symmetry is spontaneously broken in both vacua. Note that E acts on a vacuum to generate both vacua and hence is also spontaneously broken. However, the two vacua are physically indistinguishable as far as the action of $\mathsf{Rep}(S_3)$ is concerned.

This phase is characterised by the coexistence of two different kinds of order parameters. One is of mixed-type, has generalised charge b_+, and is a 3-dimensional multiplet of operators: one is untwisted, one is in the twisted sector for E, and one transitions between lines E and P. All of them are charged non-trivially under P. The other is of string-type, has generalised charge E, and is a 2-dimensional multiplet of string operators: one is in the twisted sector for E, while the other transitions between lines E and P. Both are uncharged under P.

$\mathsf{Rep}(S_3)/\mathbb{Z}_2$ **SSB Phase.** By choosing $\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\text{Neu}(\mathbb{Z}_3)}$, one interestingly finds that a_1 on its own gives rise to two untwisted sector local operators, $\mathcal{O}^a_{+,1}$ and $\mathcal{O}^a_{+,2}$, after collapsing the SymTFT sandwich

$\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\text{Neu}(\mathbb{Z}_3)}$:



(4.36)

This happens as the bulk line may end twice on the physical boundary in this case, which can be seen from its Lagrangian algebra (4.23). With the trivial local operator, this phase includes three untwisted local operators and thus three vacua. Additionally, there are two twisted local operators $\mathcal{O}^a_{-,1}$ and $\mathcal{O}^a_{-,2}$ descending from $\mathcal{O}^a_{-}$ of the a_1 multiplet, which become additional order parameters of the phase.

In order to find the operator algebra in this setting, we turn our attention to the $\mathbb{Z}_3 \subseteq S_3$ subgroup symmetry localised on the physical boundary $\mathfrak{B}^{\text{phys}}$. The $\mathbb{Z}_3$ action has the following transformation property

$$\mathcal{O}^a_{+,1} \to \omega\, \mathcal{O}^a_{+,1}\,, \qquad \mathcal{O}^a_{+,2} \to \omega^2\, \mathcal{O}^a_{+,2}\,, \tag{4.37}$$

forcing the algebra to take the form

$$\mathcal{O}^a_{+,1}\mathcal{O}^a_{+,1} = \mathcal{O}^a_{+,2}\,, \qquad \mathcal{O}^a_{+,2}\mathcal{O}^a_{+,2} = \mathcal{O}^a_{+,1}\,, \qquad \mathcal{O}^a_{+,1}\mathcal{O}^a_{+,2} = 1\,, \tag{4.38}$$

after imposing associativity and rescaling $\mathcal{O}^a_{+,1}$ and $\mathcal{O}^a_{+,2}$. This determines the three vacua to be

$$v_i = \frac{1 + \omega^i\, \mathcal{O}^a_{+,1} + \omega^{2i}\, \mathcal{O}^a_{+,2}}{3}, \qquad i \in \{0,1,2\}\,, \qquad \omega = e^{\pm 2\pi i/3}\,. \tag{4.39}$$

The linking action of P on $\mathcal{O}^a_{+,1}$ and $\mathcal{O}^a_{+,2}$ is trivial,

$$P \bigcirc : \qquad \mathcal{O}^a_{+,1} \to \mathcal{O}^a_{+,1}\,, \qquad \mathcal{O}^a_{+,2} \to \mathcal{O}^a_{+,2}\,. \tag{4.40}$$

From this one learns that the symmetry P leaves each vacuum invariant. Since the $\mathbb{Z}_2$ subsymmetry of $\mathsf{Rep}(S_3)$ is unbroken in all three vacua, we refer to this phase as the $\mathsf{Rep}(S_3)/\mathbb{Z}_2$ *SSB phase.*

On the other hand, the linking action of E with the vacua is more interesting as E links with $\mathcal{O}^a_+$ in the following way

$$E \bigcirc \mathcal{O}^a_+ = -\tfrac{1}{2}\, E \bigcirc \; \mathcal{O}^a_+ + \left(\omega + \tfrac{1}{2}\right)\, E \bigcirc\!\overset{P}{\text{——}}\!\mathcal{O}^a_- = -\, \tfrac{2}{2}\, \mathcal{O}^a_+ + 0 = -\, \mathcal{O}^a_+ \tag{4.41}$$

where the second term on the right-hand side vanishes because there are no topological local operators in $\mathsf{Rep}(S_3)$ converting the line P into the identity line.

Thus the only non-trivial linking is

$$E\bigcirc: \qquad \mathcal{O}^a_{+,1} \to -\mathcal{O}^a_{+,1}\,, \qquad \mathcal{O}^a_{+,2} \to -\mathcal{O}^a_{+,2}\,, \tag{4.42}$$

which implies its linking action on the vacua is

$$E\bigcirc: \quad v_0 \to v_1 + v_2\,, \qquad v_1 \to v_2 + v_0\,, \qquad v_2 \to v_0 + v_1\,. \tag{4.43}$$

Notice that there are no relative Euler terms between the three vacua. We can identify the $\mathsf{Rep}(S_3)$ symmetry generators as

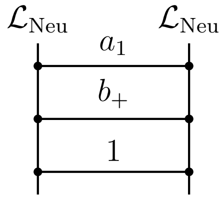
$$\begin{aligned} &P \cong 1_{00} \oplus 1_{11} \oplus 1_{22} \cong 1\,, \\ &E \cong 1_{01} \oplus 1_{02} \oplus 1_{12} \oplus 1_{10} \oplus 1_{20} \oplus 1_{21} \end{aligned} \tag{4.44}$$

Note that E acts on a vacuum to generate the other two vacua, and hence is spontaneously broken. However, all three vacua are physically indistinguishable as far as the action of $\mathsf{Rep}(S_3)$ is concerned.

This phase is again characterised by the coexistence of two types of order parameters. One is a string order parameter discussed above, which carries generalised charge P. The other is of mixed-type, has generalised charge a_1, and is a 2-dimensional multiplet comprising of an untwisted sector local operator and an P-twisted sector operator. The two operators are mixed into each other by the action of E, but are uncharged under P.

$\mathsf{Rep}(S_3)$ SSB Phase: Choosing $\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\text{Neu}}$ results in a phase with three untwisted local operators and thus three vacua:

$$\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\text{Neu}} :$$



(4.45)

We will call these three untwisted local operators 1, $\mathcal{O}^a_+$ and $\mathcal{O}^b$, which descent from 1, a_1, and b_+, respectively. Additionally, there are again some twisted sector operators that we will mention later when we discuss order parameters.

To determine the operator algebra in this case, we first look at the fusion rule of the bulk anyons

$$b_+ \otimes a_1 = b_+ \oplus b_- \tag{4.46}$$

which after noticing that 1 and a_1 are absent on the RHS means that after a possible rescaling of $\mathcal{O}^b$ one must find

$$\mathcal{O}^a_+ \mathcal{O}^b = \mathcal{O}^b \,. \tag{4.47}$$

Similarly, since the bulk fusion

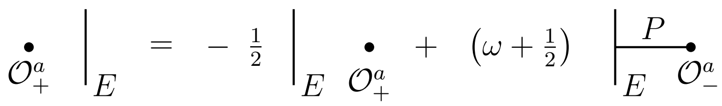
$$a_1 \otimes a_1 = 1 \oplus P \oplus a_1 \tag{4.48}$$

does not contain b_+, we must have

$$\mathcal{O}^a_+ \mathcal{O}^a_+ = \alpha + (1-\alpha)\mathcal{O}^a_+, \qquad \alpha \in \mathbb{C}\,, \tag{4.49}$$

where the relative weight between the two coefficients on the RHS has been set by imposing associativity with (4.47).

To further constrain α, we first establish the action of E on $\mathcal{O}^a_+$,



(4.50)

We can then apply the action of E on (4.49) and by matching the $\mathcal{O}^a_-$ contributions on both sides it can be shown that

$$\mathcal{O}^a_-\mathcal{O}^a_+ = -(1-\alpha)\mathcal{O}^a_- \,. \tag{4.51}$$

Imposing associativity with (4.49) fixes $\alpha = \frac{1}{2}$ (as the other root produces inconsistent results).

For the final product relation, note that the bulk fusion

$$b_+ \otimes b_+ = 1 \oplus E \oplus \bigoplus_{i=0}^{2} a_{\omega^i} \tag{4.52}$$

does not contain b_+, and so imposing associativity and rescaling we obtain

$$\mathcal{O}^b\mathcal{O}^b = \frac{1}{2} + \mathcal{O}^a_+ \,. \tag{4.53}$$

Putting everything together, the operator algebra consists of the following three non-trivial rules

$$\mathcal{O}^a_+\mathcal{O}^b = \mathcal{O}^b \,, \qquad \mathcal{O}^a_+\mathcal{O}^a_+ = \frac{1}{2}(1+\mathcal{O}^a_+) \,, \qquad \mathcal{O}^b\mathcal{O}^b = \frac{1}{2} + \mathcal{O}^a_+ \,, \tag{4.54}$$

from which one can determine the vacua to be

$$v_0 = \frac{2}{3}\left(1-\mathcal{O}^a_+\right) \,, \quad v_1 = \frac{1}{6}\left(1+2\mathcal{O}^a_+ + \sqrt{6}\mathcal{O}^b\right) \,, \quad v_2 = \frac{1}{6}\left(1+2\mathcal{O}^a_+ - \sqrt{6}\mathcal{O}^b\right) \,. \tag{4.55}$$

Similarly to (4.40) and (4.33), we deduce again that the linking action of P is

$$P \bigcirc : \qquad \mathcal{O}^a_+ \to \mathcal{O}^a_+ \,, \qquad \mathcal{O}^b \to -\mathcal{O}^b \tag{4.56}$$

and hence it acts on the vacua as

$$P \bigcirc : \qquad v_0 \to v_0\,, \qquad v_1 \to v_2\,, \qquad v_2 \to v_1\,. \tag{4.57}$$

Thus the present phase decomposes as a sum of a $\mathbb{Z}_2$ SSB phase (formed by vacua v_1 and v_2) and a $\mathbb{Z}_2$ non-SSB phase (formed by vacuum v_0).

The linking action of E on the operators from (4.33) and (4.41) is

$$E \bigcirc : \qquad \mathcal{O}^a_+ \to -\mathcal{O}^a_+\,, \qquad \mathcal{O}^b \to 0 \tag{4.58}$$

while the action on the vacua is

$$\begin{aligned} E \bigcirc : \quad v_0 &\to \frac{2}{3}\left(2 + \mathcal{O}^a_+\right) = v_0 + 2(v_1 + v_2)\,, \\ v_1 &\to \frac{1}{3}\left(1 - \mathcal{O}^a_+\right) = \frac{1}{2} v_0\,, \\ v_2 &\to \frac{1}{3}\left(1 - \mathcal{O}^a_+\right) = \frac{1}{2} v_0\,. \end{aligned} \tag{4.59}$$

Thus, also E is spontaneously broken. We therefore refer to this as the $\mathsf{Rep}(S_3)$ *SSB phase*. We can identify the $\mathsf{Rep}(S_3)$ symmetry generators as

$$\begin{aligned} P &\cong 1_{00} \oplus 1_{12} \oplus 1_{21}\,, \\ E &\cong 1_{00} \oplus 1_{01} \oplus 1_{02} \oplus 1_{10} \oplus 1_{20}\,. \end{aligned} \tag{4.60}$$

Judging from (4.57) and (4.59), one can clearly see there are no relative Euler terms between vacua v_1 and v_2, however, the presence of fractions in (4.59) uncovers relative Euler terms between vacua v_0 and v_1, v_2. The vacuum v_0 is thus physically distinguishable from the vacua v_1 and v_2, as is also apparent from the action of the unique $\mathbb{Z}_2$ subsymmetry P of $\mathsf{Rep}(S_3)$. In this case, the spontaneous breaking of non-invertible symmetry is linked to the appearance of physically indistinguishable vacua!

This phase is again characterised by the coexistence of two types of order parameters. Both are of mixed type and have been discussed above: one of them carries generalised charge a_1, while the other carries generalised charge b_+.

4.3.3 $\mathcal{S} = \mathsf{Ising}$

One of the simplest examples of a non-group-theoretical non-invertible symmetry is the Ising symmetry, which is also the $\mathbb{Z}_2$ Tambara-Yamagami fusion category $\mathsf{TY}(\mathbb{Z}_2)$. The simple objects are 1, η (generating a $\mathbb{Z}_2$ subsymmetry) and the non-invertible $\mathcal{N}$ with non-trivial fusions

$$\eta \otimes \eta = 1\,, \;\; \eta \otimes \mathcal{N} = \mathcal{N} \otimes \eta = \mathcal{N}\,, \quad \mathcal{N} \otimes \mathcal{N} = 1 \oplus \eta\,. \tag{4.61}$$

The SymTFT in this case is the 3d TQFT carrying modular fusion category $\mathcal{Z}(\mathsf{Ising}) = \mathsf{Ising} \boxtimes \overline{\mathsf{Ising}}$ of topological line defects

$$\{1, \eta, \overline{\eta}, \mathcal{N}, \overline{\mathcal{N}}, \mathcal{N}\overline{\mathcal{N}}, \eta\overline{\eta}, \mathcal{N}\overline{\eta}, \eta\overline{\mathcal{N}}\}\,. \tag{4.62}$$

There is only one Lagrangian algebra

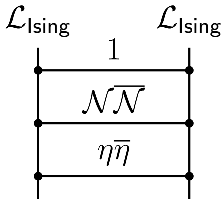
$$\mathcal{L}_{\mathsf{Ising}} = 1 \oplus \eta\overline{\eta} \oplus \mathcal{N}\overline{\mathcal{N}}\,, \tag{4.63}$$

and therefore the SymTFT admits only one topological boundary condition

$$\mathfrak{B}^{\text{sym}}_{\mathsf{Ising}} = \mathcal{L}_{\mathsf{Ising}}\,. \tag{4.64}$$

Hence the only possible configuration is

$$\mathfrak{B}^{\text{phys}} = \mathcal{L}_{\mathsf{Ising}} :$$



(4.65)

This phase has three vacua v_0, v_1, v_2, with the action of Ising given as follows

$$\begin{aligned} \eta \bigcirc : &\quad v_0 \to v_0 \,, \quad v_1 \to v_2 \,, \quad v_2 \to v_1 \,, \\ \mathcal{N} \bigcirc : &\quad v_0 \to \sqrt{2}(v_1 + v_2) \,, \quad v_1, v_2 \to \frac{1}{\sqrt{2}} v_0 . \end{aligned} \tag{4.66}$$

We can therefore identify the Ising generators as

$$\begin{aligned} \eta &\cong 1_{00} \oplus 1_{12} \oplus 1_{21} \,, \\ \mathcal{N} &\cong 1_{01} \oplus 1_{02} \oplus 1_{10} \oplus 1_{20} \,. \end{aligned} \tag{4.67}$$

Similarly to the $\mathsf{Rep}(S_3)$ SSB phase, this Ising-symmetric phase decomposes as a trivial and $\mathbb{Z}_2$ SSB phase under the $\mathbb{Z}_2$, with the two subphases permuted by the non-invertible symmetry $\mathcal{N}$. The three vacua form an irreducible phase under Ising. We call this the **Ising SSB phase**. This is another example where spontaneous breaking of non-invertible symmetry generates physically distinguishable vacua. Despite the similarity with the $\mathsf{Rep}(S_3)$ SSB, however, this phase carries different relative Euler terms, as seen from (4.66).

4.4 A $(3+1)$d Application

In $d > 2$, order parameters are be both local and extended operators. The SymTFT prescription characterises e.g. the IR gapped phases of 4d QFTs with 0-form and

1-form symmetries.[6] A classic instance is $\mathcal{N}=1$ Super-Yang-Mills with $SU(2)$ gauge group. This has a $\mathbb{Z}_4^{(0)}$ spontaneously broken to $\mathbb{Z}_2^{(0)}$, resulting in two vacua. The theory has also a $\mathbb{Z}_2^{(1)}$ 1-form symmetry unbroken in both vacua, as signalled by the Wilson line, its order parameter, having area law. Gauging the 1-form symmetry gives the $SO(3)_+$ theory [7], which also has two vacua, with the difference that $\mathbb{Z}_2^{(1)}$ is spontaneously broken in one vacuum and unbroken in the other. Correspondingly, the order parameters ('t Hooft lines) have perimeter law in one vacuum and area law in the other. The relevant SymTFT is

$$S = \int_{M_5} \frac{2\pi}{4} a_3 \delta a_1 + \frac{2\pi}{2} c_2 \delta b_2 + \frac{2\pi}{4} a_1 b_2 b_2 \,, \tag{4.68}$$

where the last term is due to a mixed anomaly between $\mathbb{Z}_4^{(0)}$ and $\mathbb{Z}_2^{(1)}$. Notice $\mathbb{Z}_2^{(0)} \subset \mathbb{Z}_4^{(0)}$ is anomaly free. The topological defects of the SymTFT are the Wilson "surfaces" $\boldsymbol{Q}_x$, with x a gauge field in S (for details, see [72, 134]).

SymTFT for the IR of the $SU(2)$. On $\mathfrak{B}^{\text{sym}}_{SU(2)}$ we impose Dirichlet boundary conditions for a_1, i.e. $\boldsymbol{Q}_{a_1}$ terminates, while on $\mathfrak{B}^{\text{phys}}$ we choose Neu($\mathbb{Z}_2$), so only $\boldsymbol{Q}^2_{a_1}$, generating $\mathbb{Z}_2^{(0)} \subset \mathbb{Z}_4^{(0)}$, terminates. The only non-trivial line ending on both boundaries is $\boldsymbol{Q}^2_{a_1}$, so after compactification, we obtain one non-trivial untwisted local operator, and hence two vacua $v_\pm$. These are permuted by $\boldsymbol{Q}_{a_3}$, so we have a $\mathbb{Z}_2^{(0)}$ SSB phase. Let us now turn to the 1-form symmetry. We claim $SU(2)$ is realised by choosing for $\boldsymbol{Q}_{b_2}$ Dirichlet boundary conditions on $\mathfrak{B}^{\text{sym}}_{SU(2)}$ and Neumann boundary conditions on $\mathfrak{B}^{\text{phys}}$ (and consequently the opposite for $\boldsymbol{Q}_{c_2}$). Correspondingly, after compactifying we obtain no untwisted line operator form the surfaces ending on $\mathfrak{B}^{\text{phys}}$. This means that in this gapped phase there are no non-trivial line operators in the

[6]We stress that below we do not discuss the most general boundary conditions of the corresponding 5d SymTFTs, but only some examples.

[7]Without loss of generality we can focus on this global variant, with the treatment for $SO(3)_-$, obtained by stacking an additional 1-form symmetry SPT before its gauging, being completely analogous.

IR, and therefore $\mathbb{Z}_2^{(1)}$ is left unbroken.

SymTFT for the IR of the $SO(3)$**.** $SO(3)_+$ can be obtained by turning the 1-form symmetry boundary conditions for $\boldsymbol{Q}_{b_2}$ and $\boldsymbol{Q}_{c_2}$ into Neumann and Dirichlet respectively on $\mathfrak{B}^{\text{sym}}_{SO(3)_+}$. This implies that the 4d gapped phase has one non-trivial untwisted line operator coming from the ends of $\boldsymbol{Q}_{c_2}$. The fact that we have a line in the IR means that $\mathbb{Z}_2^{(1)}$ is spontaneously broken in one vacuum, say v_+. We can transition to the other vacuum v_- by applying $\boldsymbol{Q}_{a_3}$. This turns the line from $\boldsymbol{Q}_{c_2}$ into a twisted-sector line, attached to a magnetic surface $\boldsymbol{Q}_{b_2}$. We then have no non-trivial untwisted line, and we conclude $\mathbb{Z}_2^{(1)}$ is unbroken in v_-.

Chapter 5

Categorical Landau Paradigm: Gapless Phases

The goal of this chapter is to provide a full picture of Landau paradigm in the presence of categorical symmetries by characterising phase transitions. This builds on what we described in the previous chapter 4, where we proposed a framework to study gapped phases with non-invertible symmetries. The study of phase transitions between such gapped phases require us to extend the standard SymTFT construction with the addition of a topological interface, i.e. we promote the usual SymTFT sandwich to a *club sandwich*.[1] Recent works that also use SymTFT techniques to extend the study of gapless phases and phase-transitions – though applied to group-like symmetries – are [125, 135–137]. The content of this chapter is based on [5].

The structure of the chapter is as follows. We begin in section 5.1 by introducing the club quiche, which is an open version of the club sandwich, without the physical boundary on the RHS. In the following section 5.2 we introduce the club sandwich, while in section 5.3 we explain how we can use it to characterise phase transitions. Throughout the chapter, we use $\mathcal{S} = \mathsf{Rep}(S_3)$ as an example.

[1] A club sandwich is comprised of three slides of bread containing two distinct fillings.

5.1 Club Quiche

We define a d-dimensional club quiche $\mathcal{Q}_d$ as a tuple

$$\mathcal{Q}_d = (\mathfrak{B}_d, \mathfrak{Z}_{d+1}, \mathcal{I}_d, \mathfrak{Z}'_{d+1})\,, \tag{5.1}$$

where $\mathfrak{Z}_{d+1}$ and $\mathfrak{Z}'_{d+1}$ are $(d+1)$-dimensional TQFTs, $\mathfrak{B}_d$ is a topological boundary condition of $\mathfrak{Z}_{d+1}$ and $\mathcal{I}_d$ is a topological interface from $\mathfrak{Z}_{d+1}$ to $\mathfrak{Z}'_{d+1}$. We will often use it in the context of SymTFT $\mathfrak{Z}_{d+1}(\mathcal{S})$, where the club quiche takes the form

$$\begin{array}{ccccccc}
\mathfrak{B}^{\text{sym}}_{\mathcal{S}} & & \mathcal{I}_d & & & \mathfrak{B}'_d \circlearrowleft \mathcal{S} & \\
\Bigg| & \mathfrak{Z}_{d+1}(\mathcal{S}) & \Bigg| & \mathfrak{Z}'_{d+1} & = & \Bigg| & \mathfrak{Z}'_{d+1}
\end{array} \tag{5.2}$$

Compactifying the interval occupied by the SymTFT $\mathfrak{Z}_{d+1}(\mathcal{S})$ leads to an $\mathcal{S}$-symmetric quiche

$$\mathcal{Q}^{\mathcal{S}}_d = (\mathfrak{B}'_d, \mathfrak{Z}'_{d+1}) \tag{5.3}$$

with the $\mathcal{S}$-symmetry being realised on a topological boundary $\mathfrak{B}'_d$ of the TQFT $\mathfrak{Z}'_{d+1}$. Conversely, any topological boundary $\mathfrak{B}'_d$ with symmetry $\mathcal{S}$ of $\mathfrak{Z}'_{d+1}$ can be expressed as a club quiche. We have a one-to-one correspondence

$$\begin{gathered}
\{\mathcal{S}\text{-symmetric topological boundaries of } \mathfrak{Z}'_{d+1}\} \\
\updownarrow \\
\{\text{Topological interfaces from SymTFT } \mathfrak{Z}_{d+1}(\mathcal{S}) \text{ to } \mathfrak{Z}'_{d+1}\}
\end{gathered} \tag{5.4}$$

5.1.1 Gapped Boundary Phases with Non-Invertible Symmetries

Physically, such a club quiche can be understood as characterising gapped boundary phases with $\mathcal{S}$-symmetry, where the symmetry is localised completely along the boundary. We now specialise to $d = 2$, where we can characterise an $\mathcal{S}$-symmetric gapped boundary phase as

$$\mathcal{Q}^{\mathcal{S}} = (\mathfrak{B}^{\text{sym}}_{\mathcal{S}}, \mathfrak{Z}(\mathcal{S}), \mathcal{I}, \mathfrak{Z}')\,, \tag{5.5}$$

where $\mathcal{I}$ is a topological interface from the 3d SymTFT $\mathfrak{Z}(\mathcal{S})$ to a 3d TQFT $\mathfrak{Z}'$. By folding, this is the same as a topological boundary condition of the 3d TQFT $\mathfrak{Z}(\mathcal{S}) \boxtimes \bar{\mathfrak{Z}}'$. As we discussed in 4.1.3, such boundary conditions are characterised by Lagrangian algebras in the MTC $\mathcal{Z}(\mathcal{S}) \boxtimes \bar{\mathcal{Z}}'$, where $\mathcal{Z}(\mathcal{S})$ is the Drinfeld center of $\mathcal{S}$ and $\bar{\mathcal{Z}}'$ is the MTC formed by topological lines of $\bar{\mathfrak{Z}}'$. We are thus led to a one-to-one correspondence

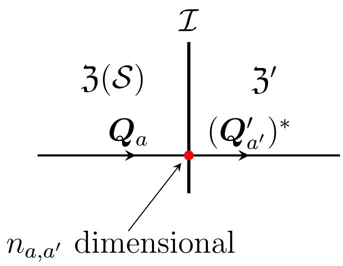
$$\begin{gathered}\{\mathcal{S}\text{-symmetric gapped boundary phases w/ bulk phase } \mathfrak{Z}'\}\\ \updownarrow \\ \big\{\text{Lagrangian algebras in } \mathcal{Z}(\mathcal{S}) \boxtimes \bar{\mathcal{Z}}'\big\}\end{gathered} \tag{5.6}$$

Consider a Lagrangian algebra $\mathcal{L}_{\mathcal{I}}$ of $\mathcal{Z}(\mathcal{S}) \boxtimes \bar{\mathcal{Z}}'$. It can be expressed as

$$\mathcal{L}_{\mathcal{I}} = \bigoplus_{a,a'} n_{a,a'} \left(\boldsymbol{Q}_a, \bar{\boldsymbol{Q}}'_{a'}\right)\,, \tag{5.7}$$

where $\boldsymbol{Q}_a$ are simple anyons in $\mathfrak{Z}(\mathcal{S})$ and $\boldsymbol{Q}'_{a'}$ are simple anyons in $\mathfrak{Z}'$. Let $\mathcal{I}$ be a topological interface. The presence of a term $n_{a,a'}\left(\boldsymbol{Q}_a, \bar{\boldsymbol{Q}}'_{a'}\right)$ in $\mathcal{L}_{\mathcal{I}}$ means that there is an $n_{a,a'}$-dimensional vector space of topological local operators along $\mathcal{I}$ acting as line changing operators from the line $\boldsymbol{Q}_a$ to the line $(\boldsymbol{Q}'_{a'})^*$, which is the dual of $\boldsymbol{Q}'_{a'}$

in the MTC $\mathcal{Z}'$, or in other words the orientation reversed version of it



(5.8)

After contracting $\mathfrak{Z}(\mathcal{S})$, such an operator descends to a topological local operator along the resulting $\mathcal{S}$-symmetric boundary $\mathfrak{B}'$ of $\mathfrak{Z}'$, which is attached to the bulk anyon $(\boldsymbol{Q}'_{a'})^*$, and carries a generalised charge $\boldsymbol{Q}_a$ under the symmetry $\mathcal{S}$ acting on $\mathfrak{B}'$. This may be regarded as an order parameter in the $\boldsymbol{Q}_a$-anyon sector for the resulting $\mathcal{S}$-symmetric (1+1)d gapped boundary phase $\mathcal{Q}_{\mathcal{I}}^{\mathcal{S}}$. Thus, the Lagrangian algebra $\mathcal{L}_{\mathcal{I}}$ captures the order parameters for the associated $\mathcal{S}$-symmetric gapped boundary phase.

Using the Lagrangian algebra $\mathcal{L}_{\mathcal{I}}$, one can also deduce the underlying non-symmetric gapped boundary phase

$$\mathcal{Q}_{\mathcal{I}} = (\mathfrak{B}', \mathfrak{Z}') \ . \tag{5.9}$$

A topological boundary condition $\mathfrak{B}'$ is in general reducible

$$\mathfrak{B}' = \bigoplus_i \mathfrak{B}'_i \,, \tag{5.10}$$

where $\mathfrak{B}'_i$ are irreducible topological boundary conditions of $\mathfrak{Z}'$. Here by irreducible boundary condition we mean a boundary condition that cannot be expressed as a sum of other boundary conditions. We can characterise $\mathfrak{B}'$ in terms of a "Lagrangian algebra" $\mathcal{L}_{\mathfrak{B}'}$

$$\mathcal{L}_{\mathfrak{B}'} := \bigoplus_i \mathcal{L}_{\mathfrak{B}'_i} \,, \tag{5.11}$$

where $\mathcal{L}_{\mathfrak{B}'_i}$ are Lagrangian algebras associated to the topological boundary conditions $\mathfrak{B}'_i$. This is related to $\mathcal{L}_{\mathcal{I}}$ via

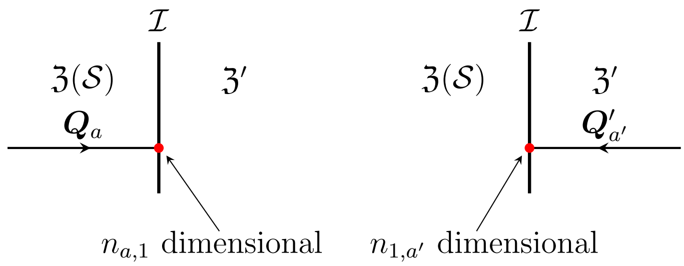
$$\mathcal{L}_{\mathfrak{B}'} = \bigoplus_{a,a'} n^{\text{sym}}_{a^*} n_{a,a'} \boldsymbol{Q}'_{a'} \,, \tag{5.12}$$

where $n_{a,a'}$ are the coefficients appearing in the Lagrangian algebra (5.7) associated to the interface and $n^{\text{sym}}_{a^*}$ is the coefficient for $\boldsymbol{Q}^*_a$ appearing in the Lagrangian algebra (4.11) associated to the symmetry boundary $\mathfrak{B}^{\text{sym}}_{\mathcal{S}}$.

We will further restrict the studies in this chapter to $\mathcal{S}$-symmetric minimal (1+1)d gapped boundary phases. In order to define such phases, note that the Lagrangian algebra $\mathcal{L}_{\mathcal{I}}$ specifies a non-Lagrangian condensable algebra $\mathcal{A}_{\mathcal{Z}(\mathcal{S})}(\mathcal{L}_{\mathcal{I}}) \in \mathcal{Z}(\mathcal{S})$ and a non-Lagrangian condensable algebra $\mathcal{A}_{\mathcal{Z}'}(\mathcal{L}_{\mathcal{I}}) \in \mathcal{Z}'$, which are obtained respectively by restricting $\mathcal{L}_{\mathcal{I}}$ to $\overline{\boldsymbol{Q}'}_{a'} = 1$ or to $\boldsymbol{Q}_a = 1$, i.e.

$$\begin{aligned} \mathcal{A}_{\mathcal{Z}(\mathcal{S})}(\mathcal{L}_{\mathcal{I}}) &= \bigoplus_{a} n_{a,1} \boldsymbol{Q}_a \\ \mathcal{A}_{\mathcal{Z}'}(\mathcal{L}_{\mathcal{I}}) &= \bigoplus_{a'} n_{1,a'} \boldsymbol{Q}'_{a'} \,. \end{aligned} \tag{5.13}$$

These condensable algebras capture the possible ends along a topological interface $\mathcal{I}$ of topological bulk lines from left and right



(5.14)

Note that, if the algebra $\mathcal{A}_{\mathcal{Z}'}(\mathcal{L}_{\mathcal{I}})$ is non-trivial, then we have non-identity topological operators in the $\mathcal{S}$-symmetric gapped boundary phase $\mathcal{Q}^{\mathcal{S}}_{\mathcal{I}}$ that are completely uncharged under $\mathcal{S}$. This is because the $\mathcal{S}$ symmetry is captured by topological lines living on the symmetry boundary $\mathfrak{B}^{\text{sym}}_{\mathcal{S}}$ while a topological operator of the type appearing on the right hand side of (5.14) does not interact with the symmetry boundary. This means that there is physical information in such a phase that is disconnected from the symmetry $\mathcal{S}$, e.g. some of the order parameters for such a phase carry trivial generalised charges under $\mathcal{S}$. We are thus led to define a minimal $\mathcal{S}$-symmetric gapped boundary phase $\mathcal{Q}^{\mathcal{S}}_{\mathcal{I}}$ to be a phase specified by a Lagrangian algebra $\mathcal{L}_{\mathcal{I}} \in \mathcal{Z}(\mathcal{S}) \boxtimes \bar{\mathcal{Z}}'$ whose associated condensable algebra $\mathcal{A}_{\mathcal{Z}'}(\mathcal{L}_{\mathcal{I}}) \in \mathcal{Z}'$ is trivial, i.e. $\mathcal{A}_{\mathcal{Z}'}(\mathcal{L}_{\mathcal{I}}) = 1$.

The minimal $\mathcal{S}$-symmetric (1+1)d gapped boundary phases are classified by condensable (non-maximal/non-Lagrangian) algebras in $\mathcal{Z}(\mathcal{S})$.[2] Pick a condensable algebra $\mathcal{A} \in \mathcal{Z}(\mathcal{S})$, then the associated gapped bulk phase $\mathfrak{Z}'$ is determined by computing local $\mathcal{A}$-modules in $\mathcal{Z}(\mathcal{S})$ [129], which form a "smaller" modular tensor category $\mathcal{Z}(\mathcal{S})/\mathcal{A}$

$$\mathcal{Z}' = \mathcal{Z}(\mathcal{S})/\mathcal{A} \,. \tag{5.15}$$

The topological interface $\mathcal{I}$ is then determined by picking a Lagrangian algebra

$$\mathcal{L}_{\mathcal{I}} \in \mathcal{Z}(\mathcal{S}) \boxtimes \overline{\mathcal{Z}(\mathcal{S})/\mathcal{A}} \tag{5.16}$$

such that the condensable algebra $\mathcal{A}_{\mathcal{Z}(\mathcal{S})}(\mathcal{L}_{\mathcal{I}}) \in \mathcal{Z}(\mathcal{S})$ associated to $\mathcal{L}_{\mathcal{I}}$ is the same as $\mathcal{A}$

$$\mathcal{A}_{\mathcal{Z}(\mathcal{S})}(\mathcal{L}_{\mathcal{I}}) = \mathcal{A} \,. \tag{5.17}$$

There may be multiple possibilities for $\mathcal{L}_{\mathcal{I}}$ satisfying the above condition, but they are all related by the action of some 0-form symmetry of $\mathfrak{Z}'$, which are auto-equivalences that do not change the physical properties of the resulting minimal $\mathcal{S}$-symmetric (1+1)d gapped boundary phase.

[2]These are algebras in the Drinfeld center $\mathcal{Z}(\mathcal{S})$ that do not have to satisfy the dimensionality condition in (4.9).

There are a couple of special choices for the condensable algebra $\mathcal{A}$:

- First of all, we can choose the trivial algebra

$$\mathcal{A} = 1\,, \tag{5.18}$$

The associated gapped bulk phase is simply

$$\mathfrak{Z}' = \mathfrak{Z}(\mathcal{S})\,. \tag{5.19}$$

- We can also choose a Lagrangian algebra

$$\mathcal{A} = \mathcal{L} \in \mathcal{Z}(\mathcal{S})\,. \tag{5.20}$$

In particular, for any $\mathcal{S}$ we have at least the choice $\mathcal{A} = \mathcal{L}_{\mathcal{S}}^{\text{sym}}$ where $\mathcal{L}_{\mathcal{S}}^{\text{sym}}$ is the Lagrangian algebra for the symmetry boundary $\mathfrak{B}_{\mathcal{S}}^{\text{sym}}$. In this case the the associated (2+1)d gapped bulk phase is trivial

$$\mathfrak{Z}' = 1\,. \tag{5.21}$$

5.1.2 $\mathsf{Rep}(S_3)$ Symmetry

We now consider minimal (1+1)d $\mathsf{Rep}(S_3)$-symmetric gapped boundary phases. We recall in this case we have

$$\mathcal{S} = \mathsf{Rep}(S_3) = \{1, P, E\}\,, \tag{5.22}$$

with fusion rules

$$P \otimes E = E \otimes P = E\,, \qquad E^2 = 1 \oplus P \oplus E\,. \tag{5.23}$$

Recall that $\mathsf{Rep}(S_3)$ can be obtained by gauging S_3 symmetry, which implies the two symmetries have the same SymTFT (see section 4.3.2.1)

$$\mathfrak{Z}(\mathsf{Rep}(S_3)) = \mathfrak{Z}(S_3)\,. \tag{5.24}$$

The difference lies in the choice of the symmetry boundary $\mathfrak{B}^{\text{sym}}$, which for $\mathcal{S} = \mathsf{Rep}(S_3)$ we take to be specified by the Lagrangian algebra

$$\mathcal{L}^{\text{sym}}_{\mathsf{Rep}(S_3)} = 1 \oplus a_1 \oplus b_+\,. \tag{5.25}$$

There are three condensable, non-Lagrangian, algebras

$$\mathcal{A}_P = 1 \oplus P, \quad \mathcal{A}_E = 1 \oplus E, \quad \mathcal{A}_{a_1} = 1 \oplus a_1\,, \tag{5.26}$$

which we now study case by case.

5.1.2.1 Condensable algebra $\mathcal{A}_E$

In this case, we have that $\mathfrak{Z}'$ is the $\mathbb{Z}_2$ DW theory (i.e. the toric code)

$$\mathcal{Z}' = \mathcal{Z}/\mathcal{A}_E = \mathcal{Z}(\mathbb{Z}_2) = \{1, e, m, f\}\,. \tag{5.27}$$

A Lagrangian algebra in $\mathcal{Z}(S_3) \boxtimes \bar{\mathcal{Z}}'$ which completes $\mathcal{A}_E$ is

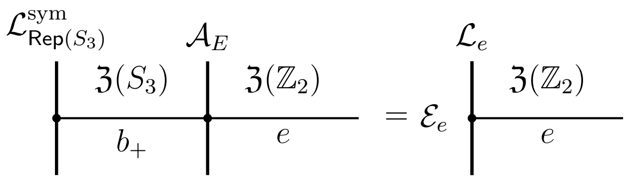
$$\mathcal{L}_{\mathcal{I}} = 1 \oplus E \oplus P\bar{m} \oplus E\bar{m} \oplus b_-\bar{f} \oplus b_+\bar{e}\,. \tag{5.28}$$

The Lagrangian algebra associated to the underlying non-symmetric gapped boundary phase is

$$\mathcal{L}_{\mathfrak{B}'} = 1 \oplus e = \mathcal{L}_e\,, \tag{5.29}$$

which means that $\mathfrak{B}'$ is an irreducible topological boundary associated to electric Lagrangian algebra $\mathcal{L}_e$. The topological lines on $\mathfrak{B}'$ are $\{1, \eta\}$, where η generates the $\mathbb{Z}_2$ symmetry localised on the boundary.

The club quiche is:



(5.30)

The $\mathsf{Rep}(S_3)$ generators linking action on the boundary operators descends from the corresponding action on the ends of the $\mathcal{Z}(\mathsf{Vec}_{S_3})$ anyons and is given by

$$P: \quad \begin{array}{c} \mathcal{E}_e \to -\mathcal{E}_e, \\ \\ 1 \to 1, \end{array} \qquad\qquad E: \quad \begin{array}{c} \mathcal{E}_e \to 0, \\ \\ 1 \to 2\,. \end{array} \tag{5.31}$$

where 1 denotes the identity local operator along $\mathfrak{B}'$. Comparing with the action of lines living on $\mathfrak{B}'$, we find that the $\mathsf{Rep}(S_3)$ symmetry is realised on $\mathfrak{B}'$ by lines

$$\phi(P) = \eta \quad , \quad \phi(E) = 1 \oplus \eta\,. \tag{5.32}$$

Mathematically, we have provided a pivotal tensor functor

$$\phi: \ \mathsf{Rep}(S_3) \to \mathsf{Vec}_{\mathbb{Z}_2} \tag{5.33}$$

where $\mathsf{Vec}_{\mathbb{Z}_2}$ is the fusion category formed by topological lines living on the boundary $\mathfrak{B}'$.

5.1.2.2 Condensable algebra $\mathcal{A}_P$

In this case, we have that $\mathfrak{Z}'$ is the $\mathbb{Z}_3$ DW theory with anyon content

$$\begin{aligned} \mathcal{Z}' &= \mathcal{Z}/\mathcal{A}_P = \mathcal{Z}(\mathsf{Vec}_{\mathbb{Z}_3}) \\ &= \{1, e, e^2, m, m^2, em, e^2m, em^2, e^2m^2\}\,. \end{aligned} \tag{5.34}$$

A Lagrangian algebra in $\mathcal{Z}(S_3) \boxtimes \bar{\mathcal{Z}}'$ which completes $\mathcal{A}_P$ is

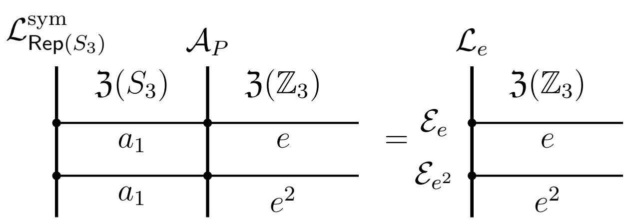
$$\begin{aligned} \mathcal{L}_{\mathcal{I}} =& 1 \oplus P \oplus a_1\bar{e} \oplus a_1\bar{e}^2 \oplus a_{\omega^2}\bar{e}\bar{m} \oplus \\ & \oplus a_{\omega^2}\bar{e}^2\bar{m}^2 \oplus E\bar{m} \oplus E\bar{m}^2 \oplus a_{\omega}\bar{e}^2\bar{m} \oplus a_{\omega}\bar{e}\bar{m}^2\,. \end{aligned} \tag{5.35}$$

From this we see that the Lagrangian algebra associated to the underlying non-symmetric gapped boundary phase is

$$\mathcal{L}_{\mathfrak{B}'} = 1 \oplus e \oplus e^2 = \mathcal{L}_e\,, \tag{5.36}$$

which means that $\mathfrak{B}'$ is an irreducible topological boundary associated to electric Lagrangian algebra $\mathcal{L}_e$. The topological lines on $\mathfrak{B}'$ are $\{1, \eta, \eta^2\}$, where η is the generators of the $\mathbb{Z}_3$ symmetry localised along $\mathfrak{B}'$.

The operators coming from the club quiche are:



(5.37)

The $\mathsf{Rep}(S_3)$ generators linking action on the boundary operators is given by

$$\begin{array}{cccc} & 1 \to 1 & & 1 \to 2 \\ P: & \mathcal{E}_e \to \mathcal{E}_e & E: & \mathcal{E}_e \to -\mathcal{E}_e \\ & \mathcal{E}_{e^2} \to \mathcal{E}_{e^2} & & \mathcal{E}_{e^2} \to -\mathcal{E}_{e^2} \,. \end{array} \tag{5.38}$$

which implies the lines implementing $\mathsf{Rep}(S_3)$ symmetry on $\mathfrak{B}'$ are

$$\phi(P) = 1 \quad , \quad \phi(E) = \eta \oplus \eta^2 \,. \tag{5.39}$$

Mathematically, we have provided a pivotal tensor functor

$$\phi: \ \mathsf{Rep}(S_3) \to \mathsf{Vec}_{\mathbb{Z}_3} \,, \tag{5.40}$$

where $\mathsf{Vec}_{\mathbb{Z}_3}$ is the fusion category formed by topological lines living on the boundary $\mathfrak{B}'$.

5.1.2.3 Condensable algebra $\mathcal{A}_{a_1}$

In this case, $\mathfrak{Z}'$ is again the toric code

$$\mathcal{Z}' = \mathcal{Z}/\mathcal{A}_{a_1} = \mathcal{Z}(\mathsf{Vec}_{\mathbb{Z}_2}) = \{1, e, m, f\} \,. \tag{5.41}$$

A Lagrangian algebra in $\mathcal{Z}(S_3) \boxtimes \bar{\mathcal{Z}}'$ which completes $\mathcal{A}_E$ is

$$\mathcal{L}_I = 1 \oplus a_1 \oplus 1_-\bar{e} \oplus a_1\bar{e} \oplus b_+\bar{m} \oplus b_-\bar{f} \,. \tag{5.42}$$

From this we see that the decomposable algebra associated to the underlying non-symmetric gapped boundary is

$$\mathcal{L}_{\mathfrak{B}'} = (1 \oplus e) \oplus (1 \oplus m) = \mathcal{L}_e \oplus \mathcal{L}_m \,, \tag{5.43}$$

i.e. $\mathfrak{B}'$ is a reducible boundary of the form

$$\mathfrak{B}' = \mathfrak{B}_e \oplus \mathfrak{B}_m \,, \tag{5.44}$$

where $\mathfrak{B}_e$ is an irreducible topological boundary with associated Lagrangian algebra $\mathcal{L}_e = 1 \oplus e$ and $\mathfrak{B}_m$ is an irreducible topological boundary with associated Lagrangian algebra $\mathcal{L}_m = 1 \oplus m$.

The topological line operators on $\mathfrak{B}'$ are

$$1_{ii}, \qquad \eta_{ii}, \qquad \mathcal{N}_{ij} \,, \tag{5.45}$$

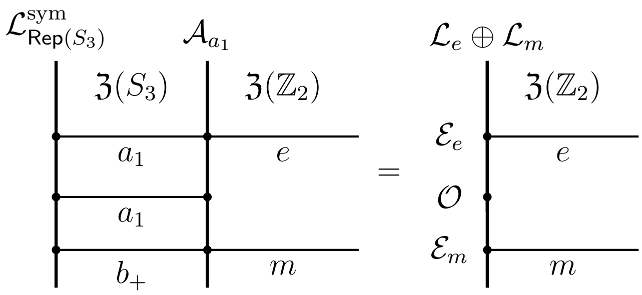
where $i, j \in \{e, m\}$ and $i \neq j$. The line 1_{ii} is the identity line on boundary $\mathfrak{B}_i$, the line η_{ii} is the generator of the $\mathbb{Z}_2$ symmetry localised along the boundary $\mathfrak{B}_i$, and the line $\mathcal{N}_{ij}$ changes the boundary $\mathfrak{B}_i$ to the boundary $\mathfrak{B}_j$, with fusion rules

$$\begin{aligned} \eta_{ii} \otimes \mathcal{N}_{ij} = \mathcal{N}_{ij} \otimes \eta_{jj} &= \mathcal{N}_{ij} \\ \mathcal{N}_{ij} \otimes \mathcal{N}_{ji} &= 1_{ii} \oplus \eta_{ii} \,. \end{aligned} \tag{5.46}$$

Note the non-invertibility of the boundary changing line operators $\mathcal{N}_{ij}$. Note also that the general linking actions of boundary changing lines are

$$\begin{aligned} \mathcal{N}_{em} :\; v_e \to \sqrt{2} e^{-\lambda} v_m, \qquad \mathcal{E}_e \to 0 \\ \mathcal{N}_{me} :\; v_m \to \sqrt{2} e^{\lambda} v_e, \qquad \mathcal{E}_m \to 0 \end{aligned} \tag{5.47}$$

for some $\lambda \in \mathbb{R}$ which captures the relative Euler term between the two boundaries $\mathfrak{B}_e$ and $\mathfrak{B}_m$. In the above equations, v_i is the identity operator in the boundary $\mathfrak{B}_i$ and $\mathcal{E}_i$ are the operators coming from the club quiche



$$(5.48)$$

The products of these operators are fixed to be

$$\begin{aligned} \mathcal{O}^2 &= 2 - \mathcal{O}\,, \quad & \mathcal{O}\mathcal{E}_e &= \mathcal{E}_e\,, \quad & \mathcal{E}_e^2 &= (2+\mathcal{O})/3\,, \\ \mathcal{E}_e\mathcal{E}_m &= 0\,, \quad & \mathcal{O}\mathcal{E}_m &= -2\mathcal{E}_m\,, \quad & \mathcal{E}_m^2 &= (1-\mathcal{O})/3\,. \end{aligned} \tag{5.49}$$

The identity operators along the two boundaries $\mathfrak{B}_e$ and $\mathfrak{B}_m$ can be identified respectively as

$$v_e = \frac{1}{3}(2+\mathcal{O})\,, \qquad v_m = \frac{1}{3}(1-\mathcal{O}) \tag{5.50}$$

The linking actions of $\mathsf{Rep}(S_3)$ symmetry generators are

$$\begin{aligned} &P:\ 1 \to 1, \qquad \mathcal{O} \to \mathcal{O}, \qquad \mathcal{E}_e \to \mathcal{E}_e, \qquad \mathcal{E}_m \to -\mathcal{E}_m, \\ &E:\ 1 \to 2, \qquad \mathcal{O} \to -\mathcal{O}, \qquad \mathcal{E}_e \to -\mathcal{E}_e, \qquad \mathcal{E}_m \to 0\,. \end{aligned} \tag{5.51}$$

where $1 = v_e + v_m$ denotes the identity operator along the reducible boundary $\mathfrak{B}'$. This implies the following linking actions

$$\begin{aligned} &P:\ v_e \to v_e, \qquad & v_m \to v_m \\ &E:\ v_e \to 2v_m + v_e, \qquad & v_m \to v_e\,. \end{aligned} \tag{5.52}$$

Therefore the lines realising the $\mathsf{Rep}(S_3)$ symmetry along $\mathfrak{B}'$ are

$$\phi(P) = 1_{ee} \oplus \eta_{mm}, \quad \phi(E) = \mathcal{N}_{em} \oplus \mathcal{N}_{me} \oplus \eta_{ee} \tag{5.53}$$

with the relative Euler term between $\mathfrak{B}_e$ and $\mathfrak{B}_m$ fixed such that the linking actions of boundary changing lines are

$$\mathcal{N}_{em} : \; v_e \to 2v_m, \qquad \mathcal{N}_{me} : \; v_m \to v_e \,. \tag{5.54}$$

Mathematically, we have provided information about a pivotal tensor functor

$$\phi : \; \mathsf{Rep}(S_3) \to \mathsf{Ising}_{2\times 2}^{\sqrt{2}} \,, \tag{5.55}$$

where $\mathsf{Ising}_{2\times 2}^{e^{-\lambda}}$ is what we call the pivotal multi-fusion category formed by topological lines (5.45) with the quantum dimensions of $\mathcal{N}_{ij}$ lines given by their linking actions on v_i described in (5.54).

5.2 Club Sandwich

Just like closing a quiche leads to a sandwich, closing a club quiche leads to a club sandwich. More precisely, a d-dimensional club sandwich $\mathfrak{S}_d$ is obtained by supplying a d-dimensional club quiche (5.1) on the right with a (possibly non-topological) boundary condition $\mathfrak{B}^{\text{phys}}_{\mathfrak{T}}$ of the $(d+1)$-dimensional TQFT $\mathfrak{Z}'_{d+1}$. We apply this to $\mathfrak{Z} = \mathfrak{Z}(\mathcal{S})$, i.e. in the context of the SymTFT, and we also specialise $\mathfrak{Z}'_{d+1}$ to be the SymTFT associated to a symmetry $\mathcal{S}'$, $\mathfrak{Z}'_{d+1}(\mathcal{S}')$. The club sandwich then becomes

$$\begin{array}{c|c|c|c c c|}
\mathfrak{B}^{\text{sym}}_{\mathcal{S}} & & \mathcal{I}_d & & \mathfrak{B}^{\text{phys}}_{\mathfrak{T}} & & \mathfrak{T}_d \circlearrowleft \mathcal{S} \\
& \mathfrak{Z}_{d+1}(\mathcal{S}) & & \mathfrak{Z}'_{d+1}(\mathcal{S}') & & = &
\end{array} \tag{5.56}$$

The full interval compactification shown on the right hand side involves collapsing both intervals occupied by $\mathfrak{Z}_{d+1}(\mathcal{S})$ and $\mathfrak{Z}'_{d+1}(\mathcal{S}')$, respectively. The result is an $\mathcal{S}$-symmetric QFT $\mathfrak{T}_d$. Thus, a club sandwich $\mathfrak{S}_d$ can be viewed as a machine mapping (possibly non-topological) $\mathcal{S}'$-symmetric d-dimensional QFT $\mathfrak{T}_d^{\mathcal{S}'}$ to $\mathcal{S}$-symmetric d-dimensional QFTs $\mathfrak{T}_d^{\mathcal{S}}$. We can think about this map as a generalised KT (short for Kennedy-Tasaki [138, 139]) transformation

$$\mathcal{K}_{\mathcal{I}_d}^{\mathcal{S},\mathcal{S}'} : \{\mathcal{S}'\text{-symmetric QFTs}\} \to \{\mathcal{S}\text{-symmetric QFTs}\} . \tag{5.57}$$

We now specialise to 2d. We define a minimal KT transformation as one arising from a club sandwich whose interface $\mathcal{I}$ is determined by a minimal Lagrangian algebra $\mathcal{L}_{\mathcal{I}}$. At the level of generalised charges, a minimal 2d KT transformation works as follows. A property of a minimal Lagrangian algebra $\mathcal{L}_{\mathcal{I}} \in \mathcal{Z}(\mathcal{S}) \boxtimes \bar{\mathcal{Z}}(\mathcal{S}')$ is that given a simple anyon $\bar{\boldsymbol{Q}}'_{a'} \in \bar{\mathcal{Z}}(\mathcal{S}')$, there always exists at least one term in $\mathcal{L}_{\mathcal{I}}$ of the form

$$n_{aa'} \boldsymbol{Q}_a \bar{\boldsymbol{Q}}'_{a'} \in \mathcal{L}_{\mathcal{I}} \tag{5.58}$$

for some simple anyon $\boldsymbol{Q}_a \in \mathcal{Z}(\mathcal{S})$ and $n_{aa'} > 0$. This means that a simple anyon $\boldsymbol{Q}'_{a'}$ of $\mathcal{Z}(\mathcal{S}')$ can always be converted into some anyon of $\mathcal{Z}(\mathcal{S})$ as it passes through the interface $\mathcal{I}$ from the right to the left. That is, a minimal interface $\mathcal{I}$ converts each irreducible generalised charge of $\mathcal{S}'$ into a (possibly reducible) generalised charge of $\mathcal{S}$. This map between generalised charges is mathematically encoded in a functor

$$F_{\mathcal{I}} : \mathcal{Z}(\mathcal{S}') \to \mathcal{Z}(\mathcal{S}) , \tag{5.59}$$

which takes the form

$$F_{\mathcal{I}}(\boldsymbol{Q}'_{a'}) = \bigoplus_a n_{aa'} \boldsymbol{Q}^*_a , \tag{5.60}$$

where the coefficients $n_{aa'}$ appear in the Lagrangian algebra $\mathcal{L}_{\mathcal{I}}$ as in (5.7).

5.3 Phase Transitions: New from Old

One of the key applications of the club sandwich construction is the study of phase transitions between gapped phases with categorical symmetries. This comprises a central aspect of the categorical Landau paradigm that we developed in the previous chapter 4. For group-like Vec_G symmetries in 2d this was discussed in [140] using a similar SymTFT description.

In this setup, the interface $\mathcal{I}$ is specified by a condensable algebra which is the intersection of two Lagrangian algebras. In particular, a phase transition between two gapped phases, which are defined by the Lagrangian algebras $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, is characterised by $\mathcal{A}_{1,2} = \mathcal{L}_1 \cap \mathcal{L}_2$, which simply means the set of topological lines that are both in $\mathcal{L}_1$ and $\mathcal{L}_2$, provided this maximal common subalgebra is condensable. Higher order phase transitions can occur when three or more such gapped phases meet. Here we extend this to include non-invertible symmetries such as $\mathsf{Rep}(S_3)$. Notice however that our setup does not require the categorical symmetry to be group-theoretical. Examples of intrinsically non-invertible phase transitions are presented in [5].

5.3.1 General Setup

Suppose that we know an irreducible $\mathcal{S}'$-symmetric 2d CFT $\mathfrak{T}_C^{\mathcal{S}'}$ admitting a relevant operator $\mathcal{O}'$ that is uncharged under $\mathcal{S}'$ (i.e. an $\mathcal{S}'$-symmetric local operator), such that deforming the CFT with $+\mathcal{O}'$ leads to an $\mathcal{S}'$-symmetric 2d TQFT $\mathfrak{T}_1^{\mathcal{S}'}$, and deforming the CFT with $-\mathcal{O}'$ leads to another $\mathcal{S}'$-symmetric TQFT $\mathfrak{T}_2^{\mathcal{S}'}$

$$\mathfrak{T}_2^{\mathcal{S}'} \xleftarrow{-\mathcal{O}'} \mathfrak{T}_C^{\mathcal{S}'} \xrightarrow{+\mathcal{O}'} \mathfrak{T}_1^{\mathcal{S}'} \tag{5.61}$$

In such a situation, $\mathfrak{T}_C^{\mathcal{S}'}$ is referred to as a phase transition between the $\mathcal{S}'$-symmetric gapped phases $\mathfrak{T}_1^{\mathcal{S}'}$ and $\mathfrak{T}_2^{\mathcal{S}'}$.

By applying a minimal KT transformation $\mathcal{K}_{\mathcal{I}}^{\mathcal{S},\mathcal{S}'}$, we obtain an $\mathcal{S}$-symmetric 2d CFT $\mathfrak{T}_C^{\mathcal{S}}$ acting as a phase transition between two $\mathcal{S}$-symmetric gapped phases $\mathfrak{T}_1^{\mathcal{S}}$ and $\mathfrak{T}_2^{\mathcal{S}}$:

$$\mathfrak{T}_2^{\mathcal{S}} \xleftarrow{\;-\mathcal{O}\;} \mathfrak{T}_C^{\mathcal{S}} \xrightarrow{\;+\mathcal{O}\;} \mathfrak{T}_1^{\mathcal{S}} \tag{5.62}$$

This is quite useful as a minimal KT transformation maps $\mathcal{S}'$-symmetric theories to $\mathcal{S}$-symmetric theories, where $\mathcal{S}$ is morally larger than $\mathcal{S}'$

$$\text{“}\,\mathcal{S} > \mathcal{S}'\,\text{”}\,. \tag{5.63}$$

One can then begin with a small enough $\mathcal{S}'$ like $\mathbb{Z}_2$, for which a transition is well-known, and iteratively apply KT transformations to generate new phase transitions for larger symmetries, which may not be invertible in general.

5.3.2 Input Phase Transitions

Let us now discuss a couple of known phase transitions that we will use to construct new phase transitions by applying KT transformations to them.

The Critical Ising Model. The first one is a $\mathbb{Z}_2$-symmetric transition provided by the 2d Ising CFT. The $\mathbb{Z}_2$ symmetry is the spin flip symmetry, which we label as η. We will focus on three special operators in the Ising CFT, namely the order/spin operator σ, the disorder operator μ, and the energy operator ϵ. The order operator σ is an untwisted sector local operator (i.e. a local operator unattached to any line defect) that is charged non-trivially under the $\mathbb{Z}_2$ symmetry η

$$\eta:\ \sigma \to -\sigma\,. \tag{5.64}$$

On the other hand, the disorder operator μ is an η-twisted sector operator, i.e. it is attached to the line η generating the $\mathbb{Z}_2$ symmetry. Additionally it carries trivial $\mathbb{Z}_2$ charge

$$\eta: \ \mu \to \mu \,. \tag{5.65}$$

Finally, the energy operator ϵ is also an untwisted operator, which is also uncharged under the $\mathbb{Z}_2$ symmetry

$$\eta: \ \epsilon \to \epsilon \,. \tag{5.66}$$

Deforming the Ising CFT by the energy operator ϵ leads to $\mathbb{Z}_2$-symmetric gapped phases in the IR, which are different depending on the sign of the deformation. The two gapped phases are

$$\begin{aligned} \mathfrak{T}_1^{\mathbb{Z}_2} &= \mathbb{Z}_2 \text{ SSB phase for } \mathbb{Z}_2 \text{ symmetry} \\ \mathfrak{T}_2^{\mathbb{Z}_2} &= \text{Trivial gapped phase for } \mathbb{Z}_2 \text{ symmetry} \end{aligned} \tag{5.67}$$

The spin operator σ of the Ising CFT acquires a non-zero vacuum expectation value (vev) in the two vacua of the $\mathbb{Z}_2$ SSB phase. It does not acquire a non-zero vacuum expectation value (vev) in the trivial phase, where on the other hand the disorder operator μ acquires a non-zero vev. An order parameter in a non-trivial twisted sector is also referred to as a string order parameter.

3-State Potts Model. We also consider a $\mathbb{Z}_3$-symmetric phase transition, which is realised by the 2d CFT known as the three-state Potts model with $c = 4/5$. Our focus will only be on the $\mathbb{Z}_3$ symmetry of the CFT, but more generally it is well known that the CFT admits a fusion category symmetry with 16 simple objects. See [38, 53] for more details about this. We will call the $\mathbb{Z}_3$ symmetry generating line η and focus on five special operators in the CFT (that are all primary fields)

$$\sigma\,, \quad \sigma^*\,, \quad \mu\,, \quad \mu^*\,, \quad \epsilon\,, \tag{5.68}$$

where σ is an untwisted local operator with charge 1 under $\mathbb{Z}_3$, σ^* is an untwisted local operator with charge 2 under $\mathbb{Z}_3$, μ is an η-twisted sector operator uncharged under $\mathbb{Z}_3$, μ^* is an η^2-twisted sector operator uncharged under $\mathbb{Z}_3$, and ϵ is an untwisted relevant operator uncharged under $\mathbb{Z}_3$. Collectively, the action of η on these operators is

$$\eta:\ \sigma \to \omega\sigma, \quad \sigma^* \to \omega^2\sigma^*, \quad (\mu, \mu^*, \epsilon) \to (\mu, \mu^*, \epsilon)\,. \tag{5.69}$$

Deforming the CFT by the relevant operator ϵ leads to $\mathbb{Z}_3$-symmetric gapped phases in the IR, which are different depending on the sign $\pm\epsilon$ of the deformation. The two gapped phases are[3]

$$\begin{aligned}
\mathfrak{T}_1^{\mathbb{Z}_3} &= \mathbb{Z}_3 \text{ SSB phase for } \mathbb{Z}_3 \text{ symmetry} \\
\mathfrak{T}_2^{\mathbb{Z}_3} &= \text{Trivial gapped phase for } \mathbb{Z}_3 \text{ symmetry}
\end{aligned} \tag{5.70}$$

The operators σ and σ^* of the CFT acquire non-zero vevs in the three vacua of the $\mathbb{Z}_3$ SSB phase, which has no string order parameters. Instead for the trivial phase the disorder operators μ, μ^* acquire non-zero vevs.

5.3.3 Phase Transitions for $\mathsf{Rep}(S_3)$

We now determine the phase transitions between $\mathsf{Rep}(S_3)$ symmetric gapped phases discussed in 4.3.2. We do this by using the KT transformations specified by the condensable algebras (5.26), i.e. by inputting appropriate physical boundary conditions

[3]The CFT actually admits an S_3 symmetry under which ϵ is uncharged. We could thus also regard the 3-state Potts transition as also an S_3-symmetric transition between $\mathbb{Z}_3$ SSB and trivial phases for S_3 symmetry. The additional $\mathbb{Z}_2$ symmetry does not get spontaneously broken on either side of the transition.

to close the club quiches analysed in 5.1.2.

5.3.3.1 $\mathsf{Rep}(S_3)/\mathbb{Z}_2$ SSB and Trivial Phases

Consider the $\mathsf{Rep}(S_3)$ club quiche (5.37) associated to the algebra $\mathcal{A}_P$ studied in 5.1.2.2. The associated KT transformation converts a $\mathbb{Z}_3$ symmetry into am$\mathsf{Rep}(S_3)$ symmetry

$$\mathcal{S}' = \mathbb{Z}_3 \quad \longrightarrow \quad \mathcal{S} = \mathsf{Rep}(S_3)\,. \tag{5.71}$$

From the expression for the Lagrangian algebra $\mathcal{L}_{\mathcal{I}}$ in (5.35) we observe that the map $\mathcal{Z}(\mathbb{Z}_3) \to \mathcal{Z}(\mathsf{Rep}(S_3))$ of generalised charges is

$$\begin{aligned} &1 \to 1 \oplus P, && m \to E, && m^2 \to E \\ &e \to a_1, && em \to a_\omega, && em^2 \to a_{\omega^2} \\ &e^2 \to a_1, && e^2m \to a_{\omega^2}, && e^2m^2 \to a_\omega \end{aligned} \tag{5.72}$$

The idea is to now input the 3-state Potts CFT to obtain a $\mathsf{Rep}(S_3)$-symmetric phase transition described below. Using the map (5.72), we can quickly deduce that the $\mathsf{Rep}(S_3)$-symmetric gapped phases obtained after applying KT transformation on $\mathbb{Z}_3$ SSB and trivial phases for $\mathbb{Z}_3$ symmetry are described respectively by the physical Lagrangian algebras

$$\begin{aligned} \mathcal{L}^{\text{phys}}_{\mathfrak{T}_1^{\mathcal{S}}} &= 1 \oplus P \oplus 2a_1 \\ \mathcal{L}^{\text{phys}}_{\mathfrak{T}_2^{\mathcal{S}}} &= 1 \oplus P \oplus 2E\,, \end{aligned} \tag{5.73}$$

which correspond to the following $\mathsf{Rep}(S_3)$-symmetric gapped phases respectively

$$\begin{aligned} \mathfrak{T}_1^{\mathcal{S}} &= \mathsf{Rep}(S_3)/\mathbb{Z}_2 \text{ SSB phase for } \mathsf{Rep}(S_3) \text{ symmetry} \\ \mathfrak{T}_2^{\mathcal{S}} &= \text{Trivial phase for } \mathsf{Rep}(S_3) \text{ symmetry} \end{aligned} \tag{5.74}$$

The $\mathsf{Rep}(S_3)$-symmetric phase transition is simply the 3-state Potts CFT regarded as a $\mathsf{Rep}(S_3)$ symmetric theory

$$\mathfrak{T}_C^{\mathcal{S}} = \text{3-Potts} \circlearrowleft \mathsf{Rep}(S_3) \tag{5.75}$$

with the generators of $\mathsf{Rep}(S_3)$ being realised as

$$P = 1, \qquad E = \eta \oplus \eta^2 \,, \tag{5.76}$$

as follows from (5.39). The relevant operator responsible for the $\mathsf{Rep}(S_3)$-symmetric transition is ϵ.

5.3.3.2 $\mathbb{Z}_2$ SSB and Trivial Phases

Consider the $\mathsf{Rep}(S_3)$ club quiche (5.30) associated to the algebra $\mathcal{A}_E$ studied in 5.1.2.1. The associated KT transformation converts a $\mathbb{Z}_2$ symmetry into a $\mathsf{Rep}(S_3)$ symmetry

$$\mathcal{S}' = \mathbb{Z}_2 \quad \longrightarrow \quad \mathcal{S} = \mathsf{Rep}(S_3) \,. \tag{5.77}$$

From the expression for the Lagrangian algebra $\mathcal{L}_{\mathcal{I}}$ in (5.28) we observe that the map $\mathcal{Z}(\mathbb{Z}_2) \to \mathcal{Z}(\mathsf{Rep}(S_3))$ of generalised charges is

$$1 \to 1 \oplus E, \quad m \to P \oplus E, \quad e \to b_+, \quad em \to b_- \,. \tag{5.78}$$

We now input the Ising CFT to obtain the $\mathsf{Rep}(S_3)$-symmetric phase transition described below. Using the map (5.78), we can quickly deduce that the $\mathsf{Rep}(S_3)$-symmetric gapped phases obtained after applying KT transformation on $\mathbb{Z}_2$ SSB and trivial phases for $\mathbb{Z}_2$ symmetry are described respectively by the physical Lagrangian algebras

$$\begin{aligned}\mathcal{L}^{\text{phys}}_{\mathfrak{T}_1^{\mathcal{S}}} &= 1 \oplus E \oplus b_+ \\ \mathcal{L}^{\text{phys}}_{\mathfrak{T}_2^{\mathcal{S}}} &= 1 \oplus P \oplus 2E\,,\end{aligned} \tag{5.79}$$

which correspond to the following $\mathsf{Rep}(S_3)$-symmetric gapped phases respectively

$$\begin{aligned}\mathfrak{T}_1^{\mathcal{S}} &= \mathbb{Z}_2 \text{ SSB phase for } \mathsf{Rep}(S_3) \text{ symmetry} \\ \mathfrak{T}_2^{\mathcal{S}} &= \text{Trivial phase for } \mathsf{Rep}(S_3) \text{ symmetry}\end{aligned} \tag{5.80}$$

The $\mathsf{Rep}(S_3)$-symmetric phase transition is simply the Ising CFT regarded as a $\mathsf{Rep}(S_3)$ symmetric theory

$$\mathfrak{T}_C^{\mathcal{S}} = \text{Ising} \circlearrowleft \mathsf{Rep}(S_3) \tag{5.81}$$

with the generators of $\mathsf{Rep}(S_3)$ being realised as

$$P = \eta, \qquad E = 1 \oplus \eta\,, \tag{5.82}$$

as follows from (5.32). The relevant operator responsible for the $\mathsf{Rep}(S_3)$-symmetric transition is ϵ.

5.3.3.3 $\mathsf{Rep}(S_3)/\mathbb{Z}_2$ SSB and $\mathsf{Rep}(S_3)$ SSB Phases

Consider the $\mathsf{Rep}(S_3)$ club quiche (5.48) associated to the algebra $\mathcal{A}_{a_1}$ studied in section 5.1.2.3. The associated KT transformation converts a $\mathbb{Z}_2$ symmetry into a $\mathsf{Rep}(S_3)$ symmetry

$$\mathcal{S}' = \mathbb{Z}_2 \quad \longrightarrow \quad \mathcal{S} = \mathsf{Rep}(S_3)\,. \tag{5.83}$$

From the expression for the Lagrangian algebra $\mathcal{L}_{\mathcal{I}}$ in (5.42) we observe that the map $\mathcal{Z}(\mathbb{Z}_2) \to \mathcal{Z}(\mathsf{Rep}(S_3))$ of generalised charges is

$$1 \to 1 \oplus a_1, \quad e \to P \oplus a_1, \quad m \to b_+, \quad em \to b_-\,. \tag{5.84}$$

We again input the Ising CFT to obtain a $\mathsf{Rep}(S_3)$-symmetric phase transition described below. Using the map (5.84), we can quickly deduce that the $\mathsf{Rep}(S_3)$-symmetric gapped phases obtained after applying KT transformation on $\mathbb{Z}_2$ SSB and trivial phases for $\mathbb{Z}_2$ symmetry are described respectively by the physical Lagrangian algebras

$$\begin{aligned}\mathcal{L}^{\text{phys}}_{\mathfrak{T}_1^{\mathcal{S}}} &= 1 \oplus P \oplus 2a_1 \\ \mathcal{L}^{\text{phys}}_{\mathfrak{T}_2^{\mathcal{S}}} &= 1 \oplus a_1 \oplus b_+\end{aligned} \tag{5.85}$$

which correspond to the following $\mathsf{Rep}(S_3)$-symmetric gapped phases respectively

$$\begin{aligned}\mathfrak{T}_1^{\mathcal{S}} &= \mathsf{Rep}(S_3)/\mathbb{Z}_2 \text{ SSB phase for } \mathsf{Rep}(S_3) \text{ symmetry} \\ \mathfrak{T}_2^{\mathcal{S}} &= \mathsf{Rep}(S_3) \text{ SSB phase for } \mathsf{Rep}(S_3) \text{ symmetry}\end{aligned} \tag{5.86}$$

The $\mathsf{Rep}(S_3)$-symmetric phase transition $\mathfrak{T}_C^{\mathcal{S}}$ obtained after applying the KT transformation to the Ising phase transition can be expressed as follows. It comprises of two universes

$$(\mathfrak{T}_C^{\mathcal{S}})_e = \text{Ising}_e\,, \quad (\mathfrak{T}_C^{\mathcal{S}})_m = (\text{Ising}/\mathbb{Z}_2)_{\sqrt{2}} = (\text{Ising}_m)_{\sqrt{2}}\,, \tag{5.87}$$

where we have used the well-known isomorphism of Ising/$\mathbb{Z}_2$ with Ising to express it as another copy Ising_m of the Ising CFT. Note that there is relative Euler term between the two Ising universes with the linking action of universe changing lines

$$1_{em}:\ \text{id}_e \to \sqrt{2}\text{id}_m \qquad 1_{me}:\ \text{id}_m \to \text{id}_e/\sqrt{2} \tag{5.88}$$

on the identity local operators id_e and id_m of Ising_e and Ising_m respectively. These linking actions reproduce

$$\mathcal{N}_{em}:\ \text{id}_e \to 2\text{id}_m \qquad \mathcal{N}_{me}:\ \text{id}_m \to \text{id}_e\,, \tag{5.89}$$

which follows from (5.54), as these lines can be expressed as

$$\begin{aligned}
\mathcal{N}_{em} &= \mathcal{N}_{ee} \otimes 1_{em} = 1_{em} \otimes \mathcal{N}_{mm} \\
\mathcal{N}_{me} &= \mathcal{N}_{mm} \otimes 1_{me} = 1_{me} \otimes \mathcal{N}_{ee} \,,
\end{aligned} \tag{5.90}$$

where $\mathcal{N}_{ee}$ and $\mathcal{N}_{mm}$ are Kramers-Wannier duality defects of Ising_e and Ising_m respectively, whose quantum dimensions are $\sqrt{2}$, i.e.

$$\mathcal{N}_{ee}: \ \text{id}_e \to \sqrt{2}\text{id}_e \,, \qquad \mathcal{N}_{mm}: \ \text{id}_m \to \sqrt{2}\text{id}_m \,. \tag{5.91}$$

The schematic form of $\mathfrak{T}_C^{\mathcal{S}}$ is thus

$$\mathfrak{T}_C^{\mathcal{S}} \quad = \quad \begin{tikzcd} \text{Ising}_e \arrow[loop left, "E"] \arrow[r, bend right, leftrightarrow, "E"'] & \oplus(\text{Ising}_m)_{\sqrt{2}} \arrow[loop right, "P"] \end{tikzcd} \tag{5.92}$$

The $\mathsf{Rep}(S_3)$ symmetry is realised as

$$P = 1_{ee} \oplus \eta_{mm} \,, \quad E = \mathcal{N}_{em} \oplus \mathcal{N}_{me} \oplus \eta_{ee} \,, \tag{5.93}$$

where η_{ee} and η_{mm} are the $\mathbb{Z}_2$ symmetries of Ising_e and Ising_m respectively, following (5.53).

The relevant operator responsible for the transition is $\epsilon_e - \epsilon_m$. Adding this operator with positive sign sends Ising_e to $\mathbb{Z}_2$ SSB phase and Ising_m to $\widehat{\mathbb{Z}}_2$ trivial phase, and hence we land on the $\mathsf{Rep}(S_3)/\mathbb{Z}_2$ SSB phase discussed above. On the other hand, adding this operator with negative sign sends Ising_e to $\mathbb{Z}_2$ trivial phase and Ising_m to $\widehat{\mathbb{Z}}_2$ SSB phase, and hence we land on the $\mathsf{Rep}(S_3)$ SSB phase discussed above.

Chapter 6

Lattice Models

In this chapter we present a lattice model realised on a tensor product Hilbert space, acted upon by generalised Ising Hamiltonians. These models exhibit four gapped phases, with a commuting projector Hamiltonian within each of them. The ground states cannot be explained as standard SSB phases, but require a non-invertible symmetry, in this case the $\mathsf{Rep}(S_3)$ which we discussed in chapter 4. Moreover, by tuning the parameters in the generalised Ising Hamiltonians, we also realise the second order quantum phase transitions discussed in chapter 5. This lattice model provides therefore a concrete microscopic realisation of the gapped and gapless $\mathsf{Rep}(S_3)$ phases found above using the SymTFT approach. We remark that lattice models whose phases can be characterised in terms of a $\mathsf{Rep}(S_3)$ symmetry have been discussed also in [141, 142], with the key difference that such models are realised on constrained Hilbert spaces, and not on a tensor product Hilbert space like the one presented here. A more systematic discussion of gapped and gapless phases in (1+1)d lattice models is given in [8], based on the anyon chain construction [83, 89, 101, 104–106]. The content of this chapter is based on [6].

The structure of the chapter is as follows. We begin in section 6.1 by introducing the lattice models and their ground states. In the following section 6.2, we analyse the ground states in terms of a $\mathsf{Rep}(S_3)$ symmetry of the lattice model. Finally, we study phase transitions in section 6.3.

6.1 The Model

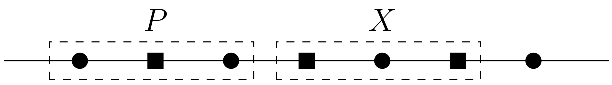
We label the sites by integers and half-integers. On each integer site $j \in \mathbb{Z}$ we have a qutrit realizing $\mathbb{C}^3$, and on half-integer sites $j+\frac{1}{2}$, a qubit realizing $\mathbb{C}^2$. A basis state of the full system is labelled as

$$|\vec{p},\vec{q}\rangle = |\cdots, p_j, q_{j+\frac{1}{2}}, p_{j+1}, q_{j+\frac{3}{2}}, \cdots\rangle \tag{6.1}$$

with $p_j \in \{0,1,2\}$ and $q_{j+\frac{1}{2}} \in \{0,1\}$. Consider a space of generalised Ising Hamiltonians, on a length L lattice, comprising of terms with 3-site interactions of the following form:

$$H = -\sum_{j=1}^{L}\sum_{I=0}^{5}\left[\lambda_I P^{(I)}_{j-\frac{1}{2}} + t_I X^{(I)}_j\right] , \tag{6.2}$$

where the operators act on $(\mathbb{C}^3 \otimes \mathbb{C}^2)^L$ as



(6.3)

Restricted to the qubits (squares), the P and X operators implement a disordering and ordering (in the x-basis) respectively, while restricted to the qutrits (dots), these implement an ordering and disordering respectively:

$$\begin{aligned} P^{(2p+q)}_{j+\frac{1}{2}} &= \frac{1}{6}\left[1+(-1)^q\sigma^z_{j+\frac{1}{2}}\right]\left[\sum_{n=0}^{2}\omega^{-pn}Z^n_j Z^{(2q-1)n}_{j+1}\right] , \\ X^{(2p+q)}_j &= (X_j)^p\left(\sigma^x_{j-\frac{1}{2}}\Gamma_j\sigma^x_{j+\frac{1}{2}}\right)^q , \end{aligned} \tag{6.4}$$

for $p \in \{0,1,2\}$, $q \in \{0,1\}$ and $\omega = \exp(2\pi i/3)$. The local operators $\sigma^\mu_{j+1/2}$ are the usual Pauli operators, whereas the operators acting on the qutrit degrees of freedom are $Z = \mathrm{diag}\,(1, \omega, \omega^2)$ and

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6.5}$$

These models exhibit four different gapped phases, which each have a commuting projector Hamiltonian realising ground states that are (unitarily equivalent to) tensor product states. These can be used to extract the universal properties of the phases, which we now describe.

6.1.1 Phase I with One Ground State

A representative commuting projector Hamiltonian within this gapped phase is provided by setting[1] $\lambda_I = 3t_I = 1$ for $I = 0, 2, 4$ and $\lambda_I = t_I = 0$ otherwise in (6.2). Doing so, the Hamiltonian simplifies to

$$H_1 = -\sum_j \left[\frac{1 + \sigma^z_{j+\frac{1}{2}}}{2} + \frac{1 + X_j + X_j^2}{3} \right]. \tag{6.6}$$

The first term projects onto $\sigma^z = +1$ state for each qubit and the second term projects onto the $X = +1$ state for each qutrit. Thus, the ground state is a product state

$$|\mathrm{GS}_1\rangle = \bigotimes_j \left| X_j = 1\,, \sigma^z_{j+\frac{1}{2}} = 1 \right\rangle = \frac{1}{3^{L/2}} \sum_{\vec{p}} \left|\vec{p}, \vec{0}\right\rangle. \tag{6.7}$$

[1]We have chosen to normalise the terms such that each term is separately a projection operator, i.e. it has eigenvalues 0 and 1.

6.1.2 Phase II with Two Ground States

A commuting projector Hamiltonian is given by setting $\lambda_I = 6t_I = 1$ for all I in (6.2). We note that $\sum_I P^{(I)} = 1$, therefore the Hamiltonian simplifies to

$$H_2 = -\sum_j \frac{1}{6}\left(1 + X_j + X_j^2\right)\left(1 + \sigma^x_{j-\frac{1}{2}}\Gamma_j\sigma^x_{j+\frac{1}{2}}\right) . \tag{6.8}$$

The terms within the two parenthesis commute with one another therefore we may first project onto the qutrit states with $X_j = 1$, energetically satisfying the operator in the first parenthesis. Since $\Gamma_j = 1$ on the qutrit state with $X_j = 1$, we effectively need to satisfy $\sigma^x_{j-1/2}\sigma^x_{j+1/2} = 1$ for each j. There are two ground states

$$|\mathrm{GS}_2, \pm\rangle = \frac{1}{6^{L/2}}\sum_{\vec{p},\vec{q}}(\pm 1)^{\sum_j q_{j+\frac{1}{2}}}\left|\vec{p},\vec{q}\right\rangle . \tag{6.9}$$

6.1.3 Phase III with Three Ground States

For this phase we set $\lambda_I = t_I = \delta_{I,0}$, resulting in

$$H_3 = -\sum_j \frac{1}{6}\left[1 + \sigma^z_{j+\frac{1}{2}}\right]\left[\sum_{n=0}^{2} Z_j^n Z_{j+1}^{-n}\right] , \tag{6.10}$$

which simultaneously projects onto the $\sigma^z_{j+1/2} = 1$ qubit states and $Z_j Z_{j+1}^{-1} = 1$ quitrit states for all j. We thus find three ground states labelled by $n \in \{0, 1, 2\}$

$$|\mathrm{GS}_3, n\rangle = \bigotimes_j \left| Z_j = e^{\frac{2\pi i n}{3}}, \sigma^z_{j+\frac{1}{2}} = 1\right\rangle = \left|\vec{n}\,,\vec{0}\right\rangle . \tag{6.11}$$

6.1.4 Phase IV with Three Ground States

Finally, consider $\lambda_I = 2t_I = \delta_{I,0} + \delta_{I,1}$, resulting in

$$H_4 = -\frac{1}{2}\sum_j \left(1 + \sigma^x_{j-\frac{1}{2}}\Gamma_j \sigma^x_{j+\frac{1}{2}}\right) - \frac{1}{6}\sum_j \sum_{\alpha=\pm1} \left[1 + \alpha\sigma^z_{j+\frac{1}{2}}\right] \left[\sum_{n=0}^{2} Z^n_j Z^{-\alpha n}_{j+1}\right] . \tag{6.12}$$

This has three ground states

$$\begin{aligned}
|\mathrm{GS}_4, 0\rangle &= \frac{1}{2^{L/2}} \sum_{\vec{q}} |\vec{0}\,, \vec{q}\rangle \,, \\
|\mathrm{GS}_4, 1\rangle &= \frac{1}{2^{L/2}} \sum_{\vec{q}} (-1)^{\sum_j q_{j+\frac{1}{2}}} |\vec{0}\,, \vec{q}\rangle \,, \\
|\mathrm{GS}_4, 2\rangle &= \frac{1}{2^{L/2}} {\sum_{\vec{p},\vec{q}}}' |\vec{p}\,, \vec{q}\rangle \,,
\end{aligned} \tag{6.13}$$

where $\sum'$ sums over $p_j \neq 0$ and $p_j + p_{j-1} \bmod 2 = q_{j-\frac{1}{2}}$.

This can be seen as follows. The first and second terms in (6.12) commute, so they can be diagonalised simultaneously. We first consider the $+1$ eigenspace of the second term in (6.12). Notice that this enforces a relation between the variables p_{j-1}, $q_{j+\frac{1}{2}}$ and p_{j+1}, namely

$$\begin{cases} p_{j+1} - p_j = 0 \quad \text{if} \quad q_{j+\frac{1}{2}} = 0 \\ p_{j+1} + p_j = 0 \;(\mathrm{mod}\ 3) \quad \text{if} \quad q_{j+\frac{1}{2}} = 1 \,. \end{cases} \tag{6.14}$$

We first consider states $|\vec{p}, \vec{q}\rangle$ such that $p_j \neq 0$ and $p_j + p_{j-1} = q_{j-\frac{1}{2}}$ (mod 2). It is easy to check that all of these states form a single orbit under the action of the first term in (6.12). Therefore we can identify a first ground state of the Hamiltonian as

$${\sum_{\vec{p},\vec{q}}}' |\vec{p}\,, \vec{q}\rangle \quad , \quad p_j \neq 0 \,, p_j + p_{j-1} = q_{j-\frac{1}{2}} \bmod 2 \tag{6.15}$$

with the appropriate normalisation, where $\sum'_{\vec{p},\vec{q}}$ denotes a sum restricted precisely to states satisfying the above condition. We can then consider states that have $p_j = 0$ for every site j. Among these, all the $|\vec{0}\,, \vec{q}\rangle$ states such that $\sum_j q_{j+\frac{1}{2}} = 0 \pmod 2$ are permuted among each other by the first term in the Hamiltonian (6.12). Similarly, the states such that $\sum_j q_{j+\frac{1}{2}} = 1 \pmod 2$ form an orbit under the action of the first term. We can then define two eigenstates of (6.12) as

$$\begin{aligned} |+\rangle &\sim \sum_{\vec{q}} |\vec{0}, \vec{q}\rangle \quad , \quad \sum_j q_{j+\frac{1}{2}} = 0 \bmod 2 \\ |-\rangle &\sim \sum_{\vec{q}} |\vec{0}, \vec{q}\rangle \quad , \quad \sum_j q_{j+\frac{1}{2}} = 1 \bmod 2\,. \end{aligned} \tag{6.16}$$

with the appropriate normalisation. Notice that $|+\rangle$ and $|-\rangle$ are respectively even and odd under the $\mathbb{Z}_2$ symmetry generated by

$$P = \prod_j \sigma^z_{j+\frac{1}{2}}\,. \tag{6.17}$$

We therefore observe that the $\mathbb{Z}_2$ symmetry must be spontaneously broken in the gapped phase realised by the ground states of this Hamiltonian. Then the two ground states exchanged by the $\mathbb{Z}_2$ symmetry are given by the two combinations

$$\frac{|+\rangle \pm |-\rangle}{2}\,. \tag{6.18}$$

Therefore we find the other two ground states of H_4 as

$$\begin{aligned} &\sum_{\vec{q}} |\vec{0}\,, \vec{q}\rangle\,, \\ &\sum_{\vec{q}} (-1)^{\sum_j q_{j+\frac{1}{2}}} |\vec{0}\,, \vec{q}\rangle\,, \end{aligned} \tag{6.19}$$

with the appropriate normalisation. This gives (6.13).

6.2 Analysis of Gapped Phases

Typically, degenerate ground states in a gapped phase can be explained in terms of spontaneous breaking of a symmetry. There is an obvious $\mathbb{Z}_2$ symmetry of (6.2) that measures the total spin parity of all the qubits, generated by the unitary

$$P = \prod_j \sigma^z_{j+\frac{1}{2}} \, . \tag{6.20}$$

The gapped phases I and II can be explained in terms of spontaneous breaking of this $\mathbb{Z}_2$ symmetry: $|\mathrm{GS}_1\rangle$ preserves it, whereas the states $|\mathrm{GS}_2, \pm\rangle$ are degenerate and $\mathbb{Z}_2$ exchanges them.[2]

What is the explanation for the three-fold degenerate ground states in gapped phases III and IV? Although there is a $\mathbb{Z}_3$ symmetry of the commuting projector Hamiltonian H_3 generated by the unitary operator $\prod_j Z_j$, other Hamiltonians near H_3 in the parameter space of models (6.2) explicitly break this $\mathbb{Z}_3$ symmetry, without lifting the three-fold degeneracy of the ground state subspace, and thus is not the explanation. We will show that this space of Hamiltonians (6.2) exhibits a $\mathsf{Rep}(S_3)$ non-invertible symmetry, whose spontaneous breaking explains the gapped phases III and IV. Recall that the symmetry generators are E and P, as in (4.18).

In order for a system to realise $\mathsf{Rep}(S_3)$ symmetry, both these operators P and E have to commute with the Hamiltonian. Within our model, the P operator is realised as in (6.20), while the E symmetry generator is

$$\begin{aligned} E &= \frac{1}{2}\Big(1 + \prod_j \sigma^z_{j+\frac{1}{2}}\Big)\left(T_1 + T_2\right) \, , \\ T_s &= \frac{1}{2}\prod_{j=1}^{L}\sum_{n=1,2}\left[\Big(1 + (-1)^{n+1}\prod_{i=0}^{j-1}\sigma^z_{i+\frac{1}{2}}\Big)X_j^{ns}\right] . \end{aligned} \tag{6.21}$$

[2]Strictly speaking, these two ground states are degenerate only in the infinite size/thermodynamic limit as at finite volume, one may add $\mathbb{Z}_2$-symmetric terms to the Hamiltonian to create an energy gap between symmetric and anti-symmetric combinations of the ground states.

The operator P clearly commutes with the Hamiltonian (6.2). Let us then show that the symmetry operator E defined in (6.21) also commutes with H. For convenience, we define the operators

$$Q_j^{\pm} = \frac{1}{2}\left(1 \pm \prod_{i=0}^{j-1} \sigma^z_{i+\frac{1}{2}}\right) , \tag{6.22}$$

so that we can rewrite T_s in (6.21) as

$$T_s = \prod_{j=1}^{L} \left[Q_j^+ X_j^s + Q_j^- X_j^{2s}\right] . \tag{6.23}$$

Let us start by considering the term

$$X_j^{(2I+s)} = (X_j)^I (\sigma^x_{j-\frac{1}{2}} \Gamma_j \sigma^x_{j+\frac{1}{2}})^s \tag{6.24}$$

in the Hamiltonian. The X_j operators clearly commute with each other, so we only need to care about the possible non-commutativity of σ^z with σ^x, as well as Γ and X^s, when the two operators are at the same lattice site. In particular, the terms in T_s that overlap on the lattice with

$$\sigma^x_{j-\frac{1}{2}} \Gamma_j \sigma^x_{j+\frac{1}{2}} \tag{6.25}$$

are

$$\left(Q_j^+ X_j^s + Q_j^- X_j^{2s}\right)\left(Q_{j+1}^+ X_{j+1}^s + Q_{j+1}^- X_{j+1}^{2s}\right) \dots . \tag{6.26}$$

First of all, it is easy to show that the $\left(Q_k^+ X_k^s + Q_k^- X_k^{2s}\right)$ operators commute with (6.25) when $k \geq j+1$. This is because in this instance there is no overlap between Γ_j, which acts as $1 \leftrightarrow 2$ on the p_j variable, and $X_k^{s,2s}$, which act on p_k. Moreover, in this case every $Q_k^{\pm}$ contains both $\sigma^z_{j+\frac{1}{2}}$ and $\sigma^z_{j-\frac{1}{2}}$, which implies its commutation with a simultaneous flip of the $q_{j+\frac{1}{2}}$ and $q_{j-\frac{1}{2}}$ variables due to the action of (6.25).

Therefore, we only need to worry about the first term in (6.26), which involves $X_j^{s,2s}$ and $Q_j^{\pm}$. Notice that since

$$Q_j^{\pm} = 1 \pm \sigma^z_{\frac{1}{2}} \dots \sigma^z_{j-\frac{1}{2}}\,, \tag{6.27}$$

acting first on a state $|\vec{p},\vec{q}\rangle$ with (6.25) effectively changes $Q_j^+ \to Q_j^-$. The fact that the full operator T_s commutes with this term follows then since, as one can check,

$$\Gamma_j X_j |p_j\rangle = X_j^2 \Gamma_j |p_j\rangle \quad , \quad \Gamma_j X_j^2 |p_j\rangle = X_j \Gamma_j |p_j\rangle\,. \tag{6.28}$$

Now let us consider the second term in the Hamiltonian (6.2), namely

$$P^{(2I+s)}_{j+\frac{1}{2}} = \frac{1}{6}\left[1 + (-1)^s \sigma^z_{j+\frac{1}{2}}\right]\left[\sum_{n=0}^{2} \omega^{-In} Z_j^n Z_{j+1}^{(2s-1)n}\right]\,. \tag{6.29}$$

Let us focus on $P^{(0)}_{j+\frac{1}{2}}$ and $P^{(1)}_{j+\frac{1}{2}}$, with the other terms being analogous. We first observe that considering the $+1$ eigenspace of these operators enforces a relation between the variables p_{j-1}, $q_{j+\frac{1}{2}}$ and p_{j+1}, namely

$$\begin{cases} p_{j+1} - p_j = 0 \quad \text{if} \quad q_{j+\frac{1}{2}} = 0 \\ p_{j+1} + p_j = 0 \pmod 3 \quad \text{if} \quad q_{j+\frac{1}{2}} = 1\,. \end{cases} \tag{6.30}$$

To show that E commutes with $P_{j+\frac{1}{2}}$, we essentially need to show that its action respects these relations. Let us consider e.g. T_1 for concreteness. The relevant terms are

$$(Q_j^+ X_j + Q_j^- X_j^2)(Q_{j+1}^+ X_{j+1} + Q_{j+1}^- X_{j+1}^2)\,. \tag{6.31}$$

We first consider the case $q_{j+\frac{1}{2}} = 0$. The first term in (6.31) acts on a state $|\vec{p},\vec{q}\rangle$ as $p_j \to p_j + k$, with $k = 1$ if $Q_j^+ = +1$ and $k = 2$ if $Q_j^- = 1$. Moreover, notice that since $q_{j+1} = 0$, the total parity evaluated by $Q_{j+1}^{\pm}$ does not change and therefore also the second term acts by exactly the same shift $p_{j+1} \to p_{j+1} + k$. Therefore the relation $p_{j+1} - p_j = 0$ is preserved. Now let us consider the case $q_{j+\frac{1}{2}} = 1$. Again the term in the first parenthesis in (6.31) acts as $p_j \to p_j + k$ depending if either Q_j^+ or Q_j^- is non-zero. However, since $q_{j+\frac{1}{2}} = 1$, now $Q_{j+1}^{\pm}$ detects precisely the opposite parity. Therefore the second term now acts as $p_{j+1} \to p_{j+1} + 2k \pmod 3$. This means that also in this case the relation $p_{j+1} + p_j = 0 \pmod 3$ is preserved. Therefore T_1 commutes with $P^{(0)}_{j+\frac{1}{2}}$, $P^{(1)}_{j+\frac{1}{2}}$. The other cases are completely analogous.

6.2.1 $\mathsf{Rep}(S_3)$ Action on Gapped Phases

We now show that the ground states of the four gapped phases form irreducible representations[3] of the $\mathsf{Rep}(S_3)$ symmetry, implying that the phases, including III and IV, can all be explained by spontaneous breaking patterns of $\mathsf{Rep}(S_3)$ symmetry. All the four possible symmetry breaking patterns for $\mathsf{Rep}(S_3)$ discussed from the point of view of SymTFT in chapter 4 are realised in our model.

The ground state of phase I is invariant under the action of $\mathsf{Rep}(S_3)$ (up to scalars), and is thus the trivial phase for the $\mathsf{Rep}(S_3)$ symmetry. In contrast, the two ground states of phase II are exchanged by P and the action of E is

$$E|\mathrm{GS}_2, \pm\rangle = |\mathrm{GS}_2, +\rangle + |\mathrm{GS}_2, -\rangle \,. \tag{6.32}$$

Next, the ground states of phase III are invariant under P, but transform into each other by the E action as

$$E|\mathrm{GS}_3, n\rangle = \sum_{m=1,2} \big|\mathrm{GS}_3, n+m \pmod 3\big\rangle \,. \tag{6.33}$$

Finally, the action of P exchanges ground states $|\mathrm{GS}_4, 0\rangle$ and $|\mathrm{GS}_4, 1\rangle$ of phase IV,

[3] More precisely, the action of $\mathsf{Rep}(S_3)$ is irreducible on the ground states exhibiting cluster decomposition in the infinite volume limit (such ground states are also known as vacua). That is, one can generate all vacua starting from any one vacuum and acting on it by $\mathsf{Rep}(S_3)$ generators. All the ground states of gapped phases displayed in this chapter chapter are actually vacuum states.

while leaving $|\mathrm{GS}_4, 2\rangle$ invariant, and the action of E is as follows

$$\begin{aligned} E|\mathrm{GS}_4, 0\rangle &= E|\mathrm{GS}_4, 1\rangle = |\mathrm{GS}_4, 2\rangle \\ E|\mathrm{GS}_4, 2\rangle &= |\mathrm{GS}_4, 0\rangle + |\mathrm{GS}_4, 1\rangle + |\mathrm{GS}_4, 2\rangle\,. \end{aligned} \tag{6.34}$$

Thus, the ground state degeneracy of both phases III and IV can be explained in terms of spontaneous breaking of the non-invertible symmetry E. The two phases are additionally distinguished by the fact that phase IV has ground states that also spontaneously break $\mathbb{Z}_2$ subsymmetry P, but all ground states of phase III preserve P. Phases III and IV were referred to as $\mathsf{Rep}(S_3)/\mathbb{Z}_2$ SSB and $\mathsf{Rep}(S_3)$ SSB phases respectively in the previous chapter 4, section 4.3.2.

6.2.2 Order Parameters

The ground states of the four gapped phases can be distinguished by expectation values of the following two local order parameters

$$O_{q,j+\frac{1}{2}} = \sigma^x_{j+\frac{1}{2}}\,, \quad O_{p,j} = Z_j\,. \tag{6.35}$$

We can easily compute their expectation values in the various ground states, which are

$$\begin{aligned} \langle \mathrm{GS}_1|O_q|\mathrm{GS}_1\rangle &= 0 & \langle \mathrm{GS}_1|O_p|\mathrm{GS}_1\rangle &= 0 \\ \langle \mathrm{GS}_2, \pm|O_q|\mathrm{GS}_2, \pm\rangle &= \pm 1 & \langle \mathrm{GS}_2, \pm|O_p|\mathrm{GS}_2, \pm\rangle &= 0 \\ \langle \mathrm{GS}_3, n|O_q|\mathrm{GS}_3, n\rangle &= 0 & \langle \mathrm{GS}_3, n|O_p|\mathrm{GS}_3, n\rangle &= e^{\frac{2\pi i n}{3}} \\ \langle \mathrm{GS}_4, 0|O_q|\mathrm{GS}_4, 0\rangle &= 1 & \langle \mathrm{GS}_4, 0|O_p|\mathrm{GS}_4, 0\rangle &= 1 \\ \langle \mathrm{GS}_4, 1|O_q|\mathrm{GS}_4, 1\rangle &= -1 & \langle \mathrm{GS}_4, 1|O_p|\mathrm{GS}_4, 1\rangle &= 1 \\ \langle \mathrm{GS}_4, 2|O_q|\mathrm{GS}_4, 2\rangle &= 0 & \langle \mathrm{GS}_4, 2|O_p|\mathrm{GS}_4, 2\rangle &= -1/2\,. \end{aligned} \tag{6.36}$$

The condensation of O_q, which is charged under the $\mathbb{Z}_2$ subsymmetry $UO_qU^{-1} = -O_q$, characterises the spontaneous breaking of P. On the other hand, in phases III and IV, O_p is charged under E and its condensation characterises spontaneous breaking of E.

6.3 Phase Transitions

To study phase transitions, we consider the simplest model of a one-parameter interpolation between two commuting projector Hamiltonians. For the transition between Phases I and J this is

$$H_{I,J}(\lambda) = \lambda H_I + (1-\lambda) H_J \,. \tag{6.37}$$

One already encounters some interesting transitions in this simple space of models. For $H_{1,2}$, the low energy physics is within the $X_j = 1$ subspace. Noting that Γ_j acts identically within this space, one obtains

$$H_{1,2}(\lambda) \approx -\frac{1}{2} \sum_j \left[\lambda \sigma^z_{j+\frac{1}{2}} + (1-\lambda) \sigma^x_{j-\frac{1}{2}} \sigma^x_{j+\frac{1}{2}} \right] \,, \tag{6.38}$$

where $\approx$ denotes that the Hamiltonian on the right hand side only describes the low energy physics of $H_{1,2}(\lambda)$. The critical Ising model describing the transition between gapped phases I and II is at $\lambda = 1/2$, while the $\lambda > 1/2$ and $\lambda < 1/2$ regions corresponds to gapped phases I and II where the P symmetry is preserved and spontaneously broken, respectively. The operator O_q becomes the spin operator of the Ising model, which is the well-known order parameter for this transition.

Similarly, for $H_{1,3}$, the low-energy physics lies in the $\sigma^z = 1$ subspace, in which we find the 3-state Potts model spin chain Hamiltonian

$$H_{1,3}(\lambda) \approx -\frac{1}{3} \sum_j \sum_{n=0}^{2} \left[\lambda X_j^n + (1-\lambda) Z_j^n Z_{j+1}^{-n} \right] \,. \tag{6.39}$$

This model has an emergent $\mathbb{Z}_3$ symmetry generated by $\eta = \prod_j X_j$, which is explained by the fact that at low energies, i.e. in the $\sigma^z = 1$ subspace, the E symmetry operator in (6.21) decomposes into $\eta + \eta^2$. The corresponding $\mathbb{Z}_3$ breaking transition occurs at $\lambda = 1/2$. The operator O_p becomes the spin operator of the Potts model, which is the standard order parameter for this transition.

Lastly, we discuss the transition between the two phases with three ground states modelled by $H_{3,4}$. Note that the Hamiltonian $H_{3,4}$ block decomposes into two state spaces V_1 and V_2. V_1 is spanned by a basis $|\vec{0}, \vec{q}\rangle$, while V_2 is spanned by states $|\vec{p}, \vec{q}\rangle$ such that $p_j \neq 0$ and $q_{j+1/2} = p_{j+1} + p_j \mod 2$. In the V_1 subspace, $Z_j = \Gamma_j = 1$ for all j, therefore

$$H_{3,4}(\lambda)\Big|_{V_1} = -\frac{1}{2}\sum_j \left[\lambda\sigma^z_{j+\frac{1}{2}} + (1-\lambda)\sigma^x_{j-\frac{1}{2}}\sigma^x_{j+\frac{1}{2}}\right] . \tag{6.40}$$

For V_2, we define effective qubits $\widetilde{\sigma}^\mu_j$ such that the states $p_j = 1, 2$ are $\widetilde{\sigma}^z_j$ eigenstates with eigenvalues $+1$ and -1 respectively. In terms of these

$$H_{3,4}(\lambda)\Big|_{V_2} = -\frac{1}{2}\sum_j \left[\lambda\widetilde{\sigma}^z_j\widetilde{\sigma}^z_{j+1} + (1-\lambda)\widetilde{\sigma}^x_j\right] . \tag{6.41}$$

Note that the P symmetry acts trivially within V_2 and as the $\mathbb{Z}_2$ symmetry measuring spin parity within V_1. The action of the E symmetry is more interesting as it maps between the dynamically disconnected state spaces V_1 and V_2 according to (for details see the subsection 6.3.2)

$$E|_{V_1} = \mathcal{N}_{12} , \quad E|_{V_2} = \mathcal{N}_{21} + P_2 , \tag{6.42}$$

where $\mathcal{N}_{12}$ maps V_1 to V_2 and acts on operators as

$$\mathcal{N}_{12} : \left(\sigma^z_{j+\frac{1}{2}} , \sigma^x_{j-\frac{1}{2}}\sigma^x_{j+\frac{1}{2}}\right) \longmapsto \left(\widetilde{\sigma}^z_j\widetilde{\sigma}^z_{j+1} , \widetilde{\sigma}^x_j\right) , \tag{6.43}$$

which is precisely the familiar Kramers-Wannier duality map. $\mathcal{N}_{21}$ implements the inverse map sending a state $|\vec{p},\vec{q}\rangle \in V_2$ to $|\vec{0},\vec{q}\rangle \in V_1$ while P_2 is a $\mathbb{Z}_2$ symmetry operation that acts within V_2 as $p_j \to -p_j$ mod 3, which may be understood as the symmetry dual (under gauging) to P. These satisfy the following operator relations

$$\mathcal{N}_{21}\mathcal{N}_{12} = 1 + P\,, \qquad \mathcal{N}_{12}\mathcal{N}_{21} = 1 + P_2\,. \tag{6.44}$$

Enforced by the E symmetry action, the Hamiltonians (6.40) and (6.41) are precisely related by a Kramers-Wannier duality or equivalently a $\mathbb{Z}_2$ gauging. The transition at $\lambda = 1/2$ is in the Ising$\oplus$Ising universality class. Note that the degeneracy between the two gapless Ising states can only be explained by breaking of the non-invertible symmetry E, and hence is beyond the standard Landau paradigm.

O_q becomes the spin operator in the first copy of Ising (i.e. in V_1), while it vanishes in V_2. Meanwhile, O_p becomes the identity in V_1 and $\exp\{2\pi i\widetilde{\sigma}^z/3\}$ in V_2. $\lambda < 1/2$ is the $\mathsf{Rep}(S_3)/\mathbb{Z}_2$ SSB phase with three ground states on which P acts trivially, while $\lambda > 1/2$ is the $\mathsf{Rep}(S_3)$ SSB phase with three ground states, two of which are the P breaking ferromagnetic states in V_1 and the third is the P invariant ground state in V_2.

We summarise what we found in figure 6.1.

6.3.1 Symmetry Protected Criticality

The Ising$\oplus$Ising transition described above lies in a gapless phase exhibiting symmetry protected criticality [143–149]. Any $\mathsf{Rep}(S_3)$ symmetric deformation of a gapless system lying in this phase can only trigger renormalisation group flows that lead to infrared phases of the form $\mathrm{T} \oplus \mathrm{T}/\mathbb{Z}_2$, where T is a $\mathbb{Z}_2$ symmetric theory and $\mathrm{T}/\mathbb{Z}_2$ is the theory obtained after gauging this $\mathbb{Z}_2$ symmetry. The only possible gapped deformations are obtained by choosing T to be a $\mathbb{Z}_2$ symmetric phase. Choosing T to

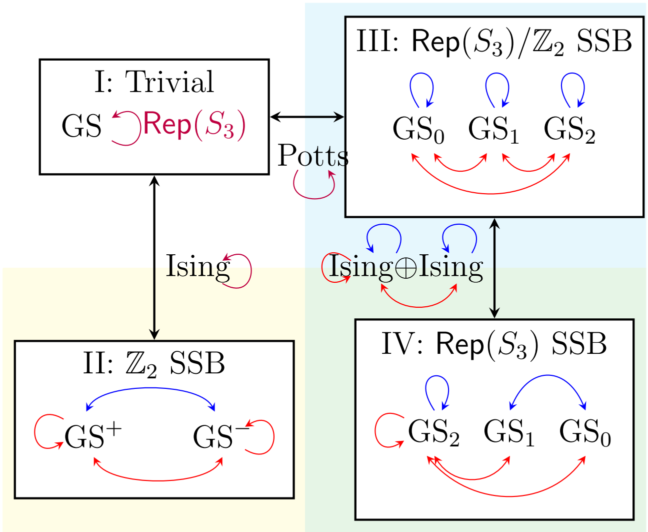


Figure 6.1: The Hamiltonian (6.2) has four gapped phases. These are explained by the $\mathsf{Rep}(S_3)$ non-invertible symmetry breaking, whose action on the gapped ground states (GS) and gapless states is shown in blue (for symmetry operator P) and red (for symmetry operator E), with purple showing the full $\mathsf{Rep}(S_3)$ action. The phase transitions are indicated by black arrows. Non-zero vevs of order parameters are shaded yellow (for O_q) and blue (for O_p), with their intersection region shaded green.

be paramangetic or ferromagnetic ($\mathbb{Z}_2$ SSB) phase leads respectively to $\mathsf{Rep}(S_3)/\mathbb{Z}_2$ and $\mathsf{Rep}(S_3)$ SSB phases. Consequently, either the system remains gapless with two gapless states, or it becomes gapped with three gapped states. Thus, it is not possible to break criticality without inducing additional order, making this into an intrinsically gapless SSB (igSSB) phase introduced in [149].[4]

6.3.2 E Action on the igSSB

Let us finally provide details on how the symmetry $E \in \mathsf{Rep}(S_3)$ acts on the model $H_{3,4}(\lambda)$ describing the transition between the $\mathsf{Rep}(S_3)$ SSB and $\mathsf{Rep}(S_3)/\mathbb{Z}_2$ SSB gapped phases. As described before, the low lying spectrum within this phase contains two dynamically disconnected state spaces V_1 and V_2. The state space V_1 spanned by

[4]These are analogues of intrinsically gapless SPT (igSPT) phases where there is a single gapless state and it is not possible to gap it without inducing order, i.e. every gapped deformation has multiple degenerate gapped ground states.

states of the form $|\vec{0},\vec{q}\rangle$ can further be decomposed as

$$V_1 = V_1^+ \oplus V_1^- \,, \tag{6.45}$$

where V_1^s is the space with a s eigenvalue under P. More specifically, the states with $\sum_j q_{j+\frac{1}{2}} = 0 \bmod 2$ lie in V_1^+, while the states $\sum_j q_{j+\frac{1}{2}} = 1 \bmod 2$ lie in V_1^-. It can be seen from Eq. (6.21) that V_1^- is in the kernel of the E symmetry operator, while a state in V_1^+ transforms as

$$E|\vec{0},\vec{q}\rangle = |\vec{p}_1(\vec{q}),\vec{q}\rangle + |\vec{p}_2(\vec{q}),\vec{q}\rangle \quad \in \; V_2 \,, \tag{6.46}$$

where $\vec{p}_1(\vec{q})$ and $\vec{p}_2(\vec{q})$ are two configurations of the qutrit degrees of freedom such that $p_j \neq 0$ and $q_{j+1/2} = p_{j+1} + p_j \bmod 2$. Since E maps states in V_1 to states completely lying within V_2, we denote

$$E\Big|_{V_1} = \mathcal{N}_{12} \,, \tag{6.47}$$

The map $\mathcal{N}_{12}$ sends a state with $q_{j+\frac{1}{2}} = 0$, i.e. $\sigma^z_{j+\frac{1}{2}} = 1$, to the sum of states with $p_j = p_{j+1}$, i.e. $\widetilde{\sigma}^z_j\widetilde{\sigma}^z_{j+1} = 1$. Similarly, $\mathcal{N}_{12}$ sends a state with $q_{j+\frac{1}{2}} = 1$, i.e. $\sigma^z_{j+\frac{1}{2}} = -1$, to the sum of states with $p_j = -p_{j+1}$, i.e. $\widetilde{\sigma}^z_j\widetilde{\sigma}^z_{j+1} = -1$. One can verify that $\mathcal{N}_{12}$ maps $\sigma^x_{j-\frac{1}{2}}\sigma^x_{j+\frac{1}{2}}$ to $\widetilde{\sigma}^x_j$. To summarise $\mathcal{N}_{12}$ implements the familiar Kramers-Wannier map on operators

$$\mathcal{N}_{12} : \left(\sigma^z_{j+\frac{1}{2}} \,, \sigma^x_{j-\frac{1}{2}}\sigma^x_{j+\frac{1}{2}}\right) \longmapsto \left(\widetilde{\sigma}^z_j\widetilde{\sigma}^z_{j+1} \,, \widetilde{\sigma}^x_j\right) \,. \tag{6.48}$$

Note that $\vec{p}_1(\vec{q})$ and $\vec{p}_2(\vec{q})$ in eq. (6.46) are related by inverting all p_j to $-p_j$. We may therefore define a $\mathbb{Z}_2$ operation $P_2 = \prod_j \Gamma_j$ acting within V_2 that implements

$$P_2|\vec{p},\vec{q}\rangle = |-\vec{p},\vec{q}\rangle \,. \tag{6.49}$$

It then follows from Eq. (6.21) that a state in V_2 transforms under E as

$$E|\vec{p},\vec{q}\rangle = |\vec{0},\vec{q}\rangle + P_2|\vec{p},\vec{q}\rangle =: \mathcal{N}_{21}|\vec{p},\vec{q}\rangle + P_2|\vec{p},\vec{q}\rangle\,. \tag{6.50}$$

To summarise, the E symmetry is realised as

$$E\Big|_{V_1\oplus V_2} = \mathcal{N}_{12} + \mathcal{N}_{21} + P_2\,, \tag{6.51}$$

where $\mathcal{N}_{12}$ and $\mathcal{N}_{21}$ satisfy the relations (6.44).

Chapter 7

Construction of Gapless Phase with Haagerup Symmetry

It has been a longstanding open question to find a 2d CFT with topological defect lines realising the exotic Haagerup symmetry. Despite numerical evidence for it [96, 97], the analytic construction of this putative CFT remains elusive. In this chapter we do not aim to solve this problem here either, at least not under the assumption that the 2d CFT has a single universe/vacuum. However, we provide the construction of *a* CFT that is $\mathcal{H}_3$-symmetric. In particular, we follow the philosophy of the categorical Landau paradigm and explore the phase diagram, determining gapped and gapless phases with $\mathcal{H}_3$ symmetry. The content of this chapter is based on [7].

The structure of the chapter is as follows. We begin in section 7.1 by introducing the Haagerup $\mathcal{H}_3$ category and its Drinfeld centre $\mathcal{Z}(\mathcal{H}_3)$. In section 7.2, we discuss gapped phases with $\mathcal{H}_3$ symmetry. In section 7.3, we discuss $\mathcal{H}_3$ gapless phases and determine a phase transition. Finally, we conclude in section 7.4 with a lattice model analysis which corroborates the continuum results.

7.1 Haagerup Symmetry

We begin with a description of the Haagerup fusion category and its Drinfeld center, which is key to our analysis of phases. The Haagerup fusion category $\mathcal{H}_3$, which has six simple objects

$$\{1, \alpha, \alpha^2, \rho, \alpha\rho, \alpha^2\rho\}\,. \tag{7.1}$$

The non-trivial fusion rules are

$$\alpha^3 = 1\,, \quad \alpha\rho = \rho\alpha^2\,, \quad \rho^2 = 1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho\,. \tag{7.2}$$

In particular, $\{1, \alpha, \alpha^2\}$ form an invertible $\mathbb{Z}_3$ subcategory, while ρ is the generator of a non-invertible symmetry. Notice also that due to the non-commutativity of the fusion rules, this category does not admit braiding.

The quantum dimensions of the simple objects are as follows

$$d_1 = d_\alpha = d_{\alpha^2} = 1 \quad , \quad d_\rho = d_{\alpha\rho} = d_{\alpha^2\rho} = d\,, \tag{7.3}$$

with

$$d = \frac{1}{2}\left(3 + \sqrt{13}\right)\,. \tag{7.4}$$

The F-symbols for this fusion category are discussed in [150, 151]. We remark that $\mathcal{H}_3$ is not one of the Haagerup fusion categories immediately descending from the Haagerup subfactor [152,153], which are denoted $\mathcal{H}_1$ and $\mathcal{H}_2$, but is Morita equivalent to those, and was discovered in [154]. In particular, the fusion category $\mathcal{H}_1$ has four simple objects, all non-invertible, and can be obtained from $\mathcal{H}_3$ by gauging the algebra object $\mathcal{A}_1 = 1 \oplus \rho \oplus \alpha\rho$, while $\mathcal{H}_2$ has the same simple objects and fusion ring as $\mathcal{H}_3$ and can be obtained from it by gauging the invertible $\mathbb{Z}_3$ sub-symmetry via the algebra object $\mathcal{A}_2 = 1 \oplus \alpha \oplus \alpha^2$.

7.1.1 The Drinfeld center $\mathcal{Z}(\mathcal{H}_3)$

We now consider the $\mathcal{H}_3$ Drinfeld center $\mathcal{Z}(\mathcal{H}_3)$ (see [155, 156]). This has twelve simple objects

$$\{1, \pi_1, \pi_2, \sigma_1, \sigma_2, \sigma_3, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}\,, \tag{7.5}$$

which have respectively quantum dimensions of

$$\{1, 3d+1, 3d+2, 3d+2, 3d+2, 3d+2, 3d, 3d, 3d, 3d, 3d, 3d\} \tag{7.6}$$

and spins

$$\left\{1, 1, 1, 1, e^{\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}, e^{\frac{4i\pi}{13}}, e^{-\frac{4i\pi}{13}}, e^{\frac{10i\pi}{13}}, e^{-\frac{10i\pi}{13}}, e^{\frac{12i\pi}{13}}, e^{-\frac{12i\pi}{13}}\right\}\,. \tag{7.7}$$

Notice there are three non-trivial bosons σ_1 , π_1 and π_2.

There are three irreducible topological boundary conditions of $\mathcal{Z}(\mathcal{H}_3)$, or equivalently three Lagrangian algebras in the Drinfeld center, which we compute to be

$$\begin{aligned}
\mathcal{L}_1 &= 1 \oplus \pi_1 \oplus 2\sigma_1 \\
\mathcal{L}_2 &= 1 \oplus \pi_1 \oplus \pi_2 \oplus \sigma_1 \\
\mathcal{L}_3 &= 1 \oplus \pi_1 \oplus 2\pi_2\,.
\end{aligned} \tag{7.8}$$

Among these, the Lagrangian algebra $\mathcal{L}_3$ is the one realising the Haagerup $\mathcal{H}_3$ fusion category symmetry on the boundary

$$\mathcal{L}_3 = \mathcal{L}_{\text{sym}}\,. \tag{7.9}$$

7.1.2 Haagerup Fusion Ring Representations

Let us recall the possible representations of the Haagerup fusion ring, following [68]. Since the two generators ρ and α do not commute, there is no basis of local operators in which their action can be simultaneously diagonalised. If we choose a basis that diagonalises the $\mathbb{Z}_3$ action, the irreducible representations of $\mathcal{H}_3$ are:

- a 1-dimensional representation 1^+, where α acts trivially and ρ acts as $d = \frac{3+\sqrt{13}}{2}$;
- a 1-dimensional representation 1^-, where α acts trivially and ρ acts as $-d^{-1} = \frac{3-\sqrt{13}}{2}$;
- a 2-dimensional representation $\mathbf{2}$, where α acts as $\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$ and ρ acts as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We remark that in the above by "action" we mean the linking action of ρ and α on local operators, obtained by encircling a local operator with a line and shrinking it to zero size to obtain a new local operator.

7.2 Gapped Phases

We will now study the gapped phases with Haagerup symmetry using the SymTFT approach discussed in chapter 4.

To study $\mathcal{H}_3$ symmetric phases, we select the symmetry boundary $\mathcal{L}_{\text{sym}} = \mathcal{L}_3$. According to the above prescription, we then obtain three possible gapped phases corresponding to the three choices $\mathfrak{B}^{\text{phys}} = \mathcal{L}_i$, $i = 1, 2, 3$. We remark that the results derived in this section using the SymTFT approach have been already derived using different methods in [54, 68]. For us the SymTFT perspective will be key to then subsequently constructing the gapless phase.

We present now a schematic discussion of each of the three gapped phases. More details on the derivation can be found in the supplementary material of [7].

7.2.1 $\mathbb{Z}_3$ Unbroken Phase: Φ_2^{--}

We start by analysing the simplest gapped phase with $\mathcal{H}_3$ symmetry, which is obtained by selecting $\mathcal{L}_1$ as the physical boundary

$$\mathcal{L}_{\text{phys}} = \mathcal{L}_1 \,. \tag{7.10}$$

In this case, only the 1 and π_1 lines can terminate on both boundaries of the SymTFT, and the resulting TQFT has two vacua v_0 and v_1. The action of the $\mathbb{Z}_3$ symmetry generators is trivial on both of them

$$\alpha : \quad v_0 \to v_0 \quad , \quad v_1 \to v_1 \tag{7.11}$$

and therefore we can represent the line operator as

$$\alpha = \alpha^2 \cong 1 = 1_{00} \oplus 1_{11} \,. \tag{7.12}$$

The action of the non-invertible generator ρ is

$$\rho : v_0 \to d^{-1}\, v_1 \quad , \quad v_1 \to d\, v_0 + 3v_1 \,. \tag{7.13}$$

Notice in particular the presence of non-trivial Euler terms encoded in the factors of d, which typically characterise a non-invertible symmetry action in the case where the vacua are physically distinguishable [4]. We can represent ρ as the line operator

$$\rho \cong 1_{01} \oplus 1_{10} \oplus 3\, 1_{11}\,. \tag{7.14}$$

This gapped phase can be characterised in terms of the vev of a local operator $\mathcal{O}_1$ transforming in the 1^- representation of $\mathcal{H}_3$. This local operator $\mathcal{O}_1$ belongs to a multiplet of the $\mathcal{H}_3$ action labelled by the anyon π_1 which contains also $\alpha^i\rho$-twisted sector operators, $i = 0, 1, 2$. This phase is moreover characterised by the condensation of a purely string order parameter which sits in a multiplet labeled by the bulk anyon σ_1, comprising of operators in the $\mathbb{Z}_3$ subsymmetry twisted sector and $\alpha^i\rho$-twisted sector, $i = 0, 1, 2$.

7.2.2 $\mathbb{Z}_3$ SSB + $\mathbb{Z}_3$ Trivial Phase: $\Phi_4^{\pm,\mp}$

We now move to consider the choice $\mathcal{L}_2$ for the physical boundary

$$\mathcal{L}_{\text{phys}} = \mathcal{L}_2. \tag{7.15}$$

In this case the lines 1, π_1 and π_2 can terminate on both boundaries, the latter with multiplicity 2, and threfore the resulting TQFT has four vacua v_0, v_1, v_2 and v_3. The $\mathcal{H}_3$ symmetry action on them is as follows. The invertible $\mathbb{Z}_3$ symmetry generator α permutes the first three vacua and leaves v_3 fixed, and therefore we can identify

$$\alpha \cong 1_{01} \oplus 1_{12} \oplus 1_{20} \oplus 1_{33}\,. \tag{7.16}$$

Notice that from the point of view of the invertible $\mathbb{Z}_3$ symmetry, this phase decomposes as the sum of a $\mathbb{Z}_3$ SSB phase and a $\mathbb{Z}_3$ trivial phase. The non-invertible generator ρ acts in a more complicated manner

$$\begin{aligned}\rho: \qquad & v_0 \to v_0 + a^{-1} v_3 \\ & v_1 \to v_2 + a^{-1} v_3 \\ & v_2 \to v_1 + a^{-1} v_3 \\ & v_3 \to a(v_0 + v_1 + v_2) + 2v_3 \,.\end{aligned} \tag{7.17}$$

with $a = \frac{1+\sqrt{13}}{2}$. In summary, we can identify ρ as

$$\rho \cong 1_{00} \oplus 1_{03} \oplus 1_{12} \oplus 1_{13} \oplus 1_{21} \oplus 1_{23} \oplus 1_{30} \oplus 1_{31} \oplus 1_{32} \oplus 2\, 1_{33} \,. \tag{7.18}$$

Notice again the presence of non-trivial relative Euler terms between the first three vacua and v_3 encoded in the a factors.

This phase can be characterised in terms of the condensation of a local operator $\mathcal{O}_1$ transforming in the 1^- representations of Haagerup and local operators $\mathcal{O}_2^1$ and $\mathcal{O}_2^1$ transforming as a doublet of the **2** representation of $\mathcal{H}_3$. $\mathcal{O}_1$ sits again in a $\mathcal{H}_3$ multiplet labeled by the anyon π_1, while $\mathcal{O}_2^i$ sit in a $\mathcal{H}_3$ multiplet labeled by the anyon π_2. Both of these also involve $\alpha^i\rho$-twisted operators. Finally, this phase also has a string order parameter labeled by the anyon σ_1.

7.2.3 $\mathcal{H}_3$ SSB Phase: Φ_6^{++}

A gapped phases where the full $\mathcal{H}_3$ symmetry is spontaneously broken is obtained by selecting $\mathcal{L}_3$ as physical boundary

$$\mathcal{L}_{\text{phys}} = \mathcal{L}_3 \,. \tag{7.19}$$

The resulting TQFT has six vacua v_i, $i = 1, \ldots, 6$, as follows from the fact that the π_1 and π_2 lines can terminate on both boundaries, the latter with multiplicity two on each boundary. Let us consider the $\mathcal{H}_3$ symmetry action of the vacua. We start with the $\mathbb{Z}_3$ invertible symmetry, which permutes the first set of vacua v_0, v_1 and v_2 among themselves, as well as the second set of vacua v_3, v_4 and v_5. We can therefore identify

$$\alpha \cong 1_{01} \oplus 1_{12} \oplus 1_{20} \oplus 1_{34} \oplus 1_{45} \oplus 1_{53} \,. \tag{7.20}$$

The action of the non-invertible generator ρ on the vacua is given by

$$\begin{aligned} \rho: \qquad & v_0 \to v_0 + v_1 + v_2 + d\, v_3 \quad , \quad v_1 \to v_0 + v_1 + v_2 + d\, v_5 \\ & v_2 \to v_0 + v_1 + v_2 + d\, v_4 \quad , \quad v_3 \to d^{-1} v_0 \\ & v_4 \to d^{-1} v_2 \quad , \quad v_5 \to d^{-1} v_1 \,, \end{aligned} \tag{7.21}$$

where once again we see the presence of non-trivial Euler counterterms. We can identify ρ as the sum of the following lines in the TQFT

$$\begin{aligned} \rho \cong\ & 1_{00} \oplus 1_{01} \oplus 1_{02} \oplus 1_{03} \oplus 1_{10} \oplus 1_{11} \oplus 1_{12} \oplus 1_{15} \oplus \\ & \oplus 1_{20} \oplus 1_{21} \oplus 1_{22} \oplus 1_{24} \oplus 1_{30} \oplus 1_{42} \oplus 1_{51} \,. \end{aligned} \tag{7.22}$$

This phase can be characterised in terms of the condensation of the local operators $\mathcal{O}_1$, transforming in the 1^- representation, $\mathcal{O}_2^{1,1}$ and $\mathcal{O}_2^{2,1}$, transforming in the **2** representation, and another doublet $\mathcal{O}_2^{1,2}$ and $\mathcal{O}_2^{2,2}$ in the **2**. There are no multiplets involving purely string order parameters condensing in this phase.

7.3 Gapless Phases and Phase Diagram

We now consider the possible gapless phases with Haagerup symmetry. As explained in chapter 5, these can be characterised in a SymTFT approach via condensable algebras $\mathcal{A}$ in the Drinfeld center $\mathcal{Z}(\mathcal{S})$. More precisely, every condensable algebra in $\mathcal{Z}(\mathcal{S})$ defines a $\mathcal{S}$-symmetric phase, with Lagrangian algebras corresponding to

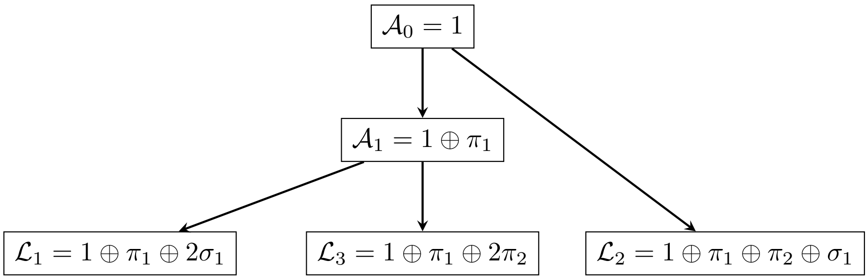


Figure 7.1: Hasse diagram of gapped and gapless phases for the Haagerup symmetry. Each entry corresponds to a condensable algebra of the Drinfeld center $\mathcal{Z}(\mathcal{H}_3)$. The bottom row represents the gapped phases, given by Lagrangian algebras. $\mathcal{L}_3$ is the symmetry Lagrangian algebra. Using the Lagrangians $\mathcal{L}_i$ as physical boundary conditions in the SymTFT results in gapped phases with $2, 4, 6$ vacua, respectively, for $i = 1, 2, 3$. The gapless phase we will construct is determined by $\mathcal{A}_1$, which characterises a phase transition between the gapped phases $\mathcal{L}_1$ and $\mathcal{L}_3$: $\mathcal{A}_1 = \mathcal{L}_1 \cap \mathcal{L}_3$. The algebra $\mathcal{A}_0$ represents the universal gapless phases of the Haagerup symmetry.

gapped phases and non-Lagrangian condensable algebras to gapless phases. Moreover, condensable algebras can be arranged into a Hasse diagram where the partial order is given by inclusion, i.e. $\mathcal{A}_1 < \mathcal{A}_2$ if $\mathcal{A}_1$ is a subalgebra of $\mathcal{A}_2$ [149]. This is shown in figure 7.1 for $\mathcal{H}_3$.

We therefore compute all the possible non-Lagrangian condensable algebras in $\mathcal{Z}(\mathcal{H}_3)$. We find that there is only one non-trivial algebra that is non-Lagrangian and satisfies the necessary conditions for condensability, namely we have only the two possibilities

$$\begin{aligned} \mathcal{A}_0 &= 1 \\ \mathcal{A}_1 &= 1 \oplus \pi_1 \,. \end{aligned} \tag{7.23}$$

We start by making some observations. First of all, notice that $\mathcal{A}_1 = \mathcal{L}_3 \cap \mathcal{L}_1$, so that we expect the gapless phase corresponding to $\mathcal{A}_1$ to be a standard critical point describing the phase transition between the 6 vacua gapped phase and the 2 vacua gapped phase, specified by the Lagrangian algebras $\mathcal{L}_3$ and $\mathcal{L}_1$ respectively. We will construct explicitly this CFT in the following. Notice also that $\mathcal{A}_1$ is not a subalgebra of $\mathcal{L}_2$, which contains only the trivial $\mathcal{A}_0$. Moreover, notice that the other intersections between the Lagrangian algebras, namely $\mathcal{L}_1 \cap \mathcal{L}_2 = 1 \oplus \pi_1 \oplus \sigma_1$ and $\mathcal{L}_3 \cap \mathcal{L}_2 = 1 \oplus \pi_1 \oplus \pi_2$, are not condensable. Thus from the SymTFT point of view we can only infer the existence of a second order Haagerup symmetric transition between the 2 and 6 vacua gapped phases, but not between the 6 and the 4 vacua gapped phases (and similarly between 4 and 2).

7.3.1 Phase Transitions from the SymTFT

We can now determine $\mathcal{S}$-symmetric phase transitions using the club quiche and club sandwich introduced in chapter 5.

7.3.2 The $\mathcal{A}_1$ Club Quiche

We start by considering the condensation of the non-Lagrangian algebra $\mathcal{A}_1 = 1 \oplus \pi_1$ in $\mathcal{Z}(\mathcal{H}_3)$. The total quantum dimension of $\mathcal{Z}(\mathcal{H}_3)$ is $3(3d+2)$ and the quantum dimension of $\mathcal{A}_1$ is $(3d+2)$, so the quantum dimension of the reduced topological order is

$$D(\mathcal{Z}(\mathcal{H}_3)/\mathcal{A}_1) = D(\mathcal{Z}(\mathcal{H}_3))/D(\mathcal{A}_1) = 3\,. \tag{7.24}$$

This, together with the fact that the anyons in $\mathcal{H}_3$ braiding trivially with π_1 are

$$\{1, \pi_1, \pi_2, \sigma_1, \sigma_2, \sigma_3\}\,, \tag{7.25}$$

which have topological spins respectively

$$\{1, 1, 1, 1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}\,, \tag{7.26}$$

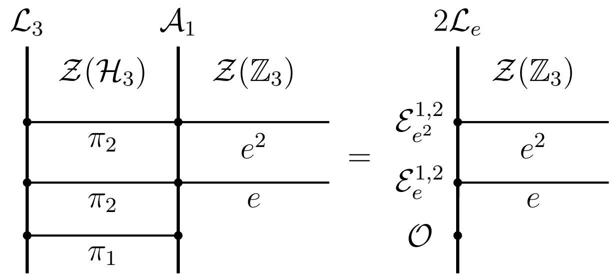
leads us naturally to conjecture the reduced topological order $\mathcal{Z}'$ to be the $\mathbb{Z}_3$ Dijkgraaf-Witten theory, with anyon content

$$\mathcal{Z}' = \mathcal{Z}(\mathbb{Z}_3) = \{1, e, e^2, m, m^2, em, e^2m, em^2, e^2m^2\}\,. \tag{7.27}$$

In fact, this is the only such dimension 3 topological order with suitable spins for the anyons [157]. Moreover, this can be checked by constructing equivalently a Lagrangian algebra in the folded theory $\mathcal{Z}(\mathcal{H}_3) \boxtimes \overline{\mathcal{Z}(\mathbb{Z}_3)}$ which completes $\mathcal{A}_1$

$$\mathcal{L} = 1 \oplus \pi_1 \oplus \pi_2(\overline{e} \oplus \overline{e}^2) \oplus \sigma_1(\overline{m} \oplus \overline{m}^2) \oplus \sigma_2(\overline{em}^2 \oplus \overline{e}^2\overline{m}) \oplus \sigma_3(\overline{em} \oplus \overline{e}^2\overline{m}^2)\,. \tag{7.28}$$

This is unique up to the automorphism of $\mathcal{Z}(\mathbb{Z}_3)$ that exchanges e and m. The club quiche picture we obtain is then the following:



(7.29)

By collapsing the symmetry boundary of $\mathcal{Z}(\mathcal{H}_3)$ with the interface specified by $\mathcal{A}_1$, we obtain a gapped boundary of $\mathcal{Z}(\mathbb{Z}_3)$. The Lagrangian algebra defining this gapped boundary can be determined by looking at the lines ending on $\mathcal{L}_3$ and following what they become in $\mathcal{Z}(\mathbb{Z}_3)$. From the picture above we see that the boundary $\mathfrak{B}'$ of the reduced topological order is

$$\mathfrak{B}' = \mathfrak{B}^e_0 \oplus \mathfrak{B}^e_1\,, \tag{7.30}$$

i.e. the sum of two electric boundary conditions, as specified by the Lagrangian algebra

$$\mathcal{L}_{\mathfrak{B}'} = 2(1 \oplus e \oplus e^2)\,. \tag{7.31}$$

In particular, the boundary is reducible. The topological lines living on $\mathfrak{B}'$ form a multi-fusion category which we call $\mathrm{Mat}_2(\mathbb{Z}_3)$. Its simple objects are

$$1_{ij}\,, \eta_{ij}\,, \eta_{ij}^2 \quad i,j \in \{0,1\}\,. \tag{7.32}$$

In particular, $\{1_{ii}, \eta_{ii}, \eta_{ii}^2\}$ are the lines generating the $\mathbb{Z}_3$ symmetry on $\mathfrak{B}_i^e$, while the other lines for $i \neq j$ are boundary changing operators.

Now let us discuss the boundary operators. Notice we get two possible operators $\mathcal{E}_e^{1,2}$ and $\mathcal{E}_{e^2}^{1,2}$ because π_2 ends with multiplicity 2 on the boundary specified by $\mathcal{L}_3$. The product of boundary operators is determined using constraints such as the symmetry action, the product of bulk lines, and associativity. The symmetry action is given by

$$\begin{aligned} \alpha: &\qquad \mathcal{O} \to \mathcal{O}\,, \quad \mathcal{E}_{e^i}^1 \to \omega \mathcal{E}_{e^i}^1\,, \quad \mathcal{E}_{e^i}^2 \to \omega^2 \mathcal{E}_{e^i}^2 \\ \rho: &\qquad \mathcal{O} \to -d^{-1}\mathcal{O}\,, \quad \mathcal{E}_{e^i}^1 \leftrightarrow \mathcal{E}_{e^i}^2\,. \end{aligned} \tag{7.33}$$

Using this, we can determine the products of local operators to be

$$\begin{aligned} &\mathcal{O}^2 = 1 + 3\mathcal{O} && \mathcal{E}_e^1 \mathcal{E}_e^1 = \mathcal{E}_{e^2}^2 \\ &\mathcal{O}\mathcal{E}_e^1 = d\,\mathcal{E}_e^1 && \mathcal{E}_e^2 \mathcal{E}_e^2 = \mathcal{E}_{e^2}^1 \\ &\mathcal{O}\mathcal{E}_e^2 = -d^{-1}\mathcal{E}_e^2 && \mathcal{E}_e^1 \mathcal{E}_e^2 = 0\,. \end{aligned} \tag{7.34}$$

Notice in particular this implies we can focus on only $\mathcal{E}_e^{1,2}$, as the operators $\mathcal{E}_{e^2}^{1,2}$ are determined in terms of them. We construct local operators

$$v_0 = \frac{d^{-1} + \mathcal{O}}{\sqrt{13}} \quad , \quad v_1 = \frac{d - \mathcal{O}}{\sqrt{13}}\,. \tag{7.35}$$

We can interpret v_0 as the identity local operator along the irreducible boundary $\mathfrak{B}^e_0$, and similarly v_1 as the identity along $\mathfrak{B}^e_1$. This follows from the fact that v_0 and v_1 are orthogonal idempotents, satisfying $v_i v_j = \delta_{ij} v_j$. Moreover, $\mathcal{E}^1_e$ is the end of the e anyon along $\mathfrak{B}^e_0$, while $\mathcal{E}^2_e$ is the end of the e anyon along $\mathfrak{B}^e_1$, as we have

$$\begin{aligned} &\mathcal{E}^1_e v_0 = \mathcal{E}^1_e\,, \quad \mathcal{E}^2_e v_0 = 0\,, \quad \mathcal{E}^1_e v_1 = 0 \\ &\mathcal{E}^2_e v_1 = \mathcal{E}^2_e\,, \quad \mathcal{E}^1_e \mathcal{E}^2_e = 0\,. \end{aligned} \tag{7.36}$$

This implies that indeed v_0 (v_1) acts as the identity for $\mathcal{E}^1_e$ ($\mathcal{E}^2_e$), and that the overlap between the two boundaries is trivial.

We are now ready to discuss how the Haagerup symmetry acts on the boundary $\mathfrak{B}'$, using (7.33). We see that α acts within each of the two irreducible boundaries as

$$\alpha \simeq \eta_{00} \oplus \eta^2_{11}\,. \tag{7.37}$$

The non-invertible generator ρ instead maps between the two boundaries, since it sends

$$v_0 \to d^{-1} v_1\,, \quad v_1 \to d\, v_0 + 3 v_1\,, \tag{7.38}$$

while also switching $\mathcal{E}^1_{e^i}$ with $\mathcal{E}^2_{e^i}$, from which we deduce

$$\rho \simeq 1_{01} \oplus 1_{10} \oplus 1_{11} \oplus \eta_{11} \oplus \eta^2_{11}\,. \tag{7.39}$$

Notice there is a relative Euler counterterm $e^{-\lambda} = d^{-1}$ between $\mathfrak{B}^e_0$ and $\mathfrak{B}^e_1$. One can easily check that all the relations defining $\mathcal{H}_3$ are satisfied, for example the non-commutativity

$$\alpha\rho = \eta_{01} \oplus \eta^2_{10} \oplus 1_{11} \oplus \eta_{11} \oplus \eta^2_{11} = \rho\alpha^2. \tag{7.40}$$

This provides mathematically a functor from $\mathcal{H}_3$ to $\mathrm{Mat}_2(\mathbb{Z}_3)$, the category formed by the lines on the reducible gapped boundary $\mathfrak{B}'$.

7.3.3 Gapless Phase with Haagerup Symmetry

We are now ready to construct a gapless phase with Haagerup symmetry using the club sandwich. We remark that

$$\mathcal{L}_1 \cap \mathcal{L}_3 = \mathcal{A}_1 \,, \tag{7.41}$$

so that we expect this gapless phase to be a standard second order phase transitions at the critical point between the 2 vacua and the 6 vacua gapped phases.

Since the reduced topological order is $\mathcal{Z}' = \mathcal{Z}(\mathbb{Z}_3)$, the input phase transition has to be $\mathbb{Z}_3$ symmetric. For a $\mathbb{Z}_3$ symmetry, there are two possible symmetric gapped phases classified by the Lagrangian algebras

$$\begin{aligned} \mathcal{L}_e &= 1 \oplus e \oplus e^2 \\ \mathcal{L}_m &= 1 \oplus m \oplus m^2 \,. \end{aligned} \tag{7.42}$$

in $\mathcal{Z}(\mathbb{Z}_3)$. Here $\mathcal{L}_e$ is also the reference symmetry boundary. The gapped phase corresponding to $\mathcal{L}_e$ is the $\mathbb{Z}_3$ SSB phase with three vacua $v_0\,, v_1\,, v_2$, upon which the $\mathbb{Z}_3$ generator η acts as a permutation

$$\eta = 1_{01} \oplus 1_{12} \oplus 1_{20} : \quad \begin{tikzcd} v_0 \arrow[r, bend left] & v_1 \arrow[r, bend left] & v_2 \arrow[ll, bend right, "\eta"'] \end{tikzcd} \tag{7.43}$$

The phase corresponding to $\mathcal{L}_m$ is the trivial phase with a single vacuum, upon which the symmetry acts trivially $\eta = 1$. The simplest CFT at the phase transition is the 3-state Potts (3SP) minimal model with central charge $c = 4/5$ discussed in chapter 5. The two gapped phases can be obtained by deforming the 3SP by the relevant $\mathbb{Z}_3$ symmetric operator ϵ of conformal dimension $(2/5, 2/5)$. The CFT flows to one of the two $\mathbb{Z}_3$ phases depending on the sign of the deformation, as discussed e.g. in [158]. In particular, the deformation of the 3SP model

$$H_{\mathbf{3SP}} + \beta \int \epsilon(x) \tag{7.44}$$

flows to the disordered ($\mathbb{Z}_3$-trivial) phase for $\beta < 0$ and ordered ($\mathbb{Z}_3$-SSB) phase for $\beta > 0$.

We can now feed the 3SP model in as the physical boundary of the club quiche (7.29). This outputs the following $\mathcal{H}_3$-symmetric theory

$$\begin{tikzcd} \mathbf{3SP}_0 \arrow[loop left, "\mathbb{Z}_3"] \arrow[rr, bend right, "\rho"'] & \oplus & \mathbf{3SP}_1 \arrow[loop right, "{\mathbb{Z}_3, \rho}"] \arrow[ll, bend right, "\rho"'] \end{tikzcd} \tag{7.45}$$

We have two copies of the 3SP, constituting two universes which are dynamically decoupled but connected by the action of the $\mathcal{H}_3$ symmetry. We denote by η_{ii}, $\eta_{ii}^3 = 1_{ii}$, the generator of the $\mathbb{Z}_3$ symmetry within each 3SP_i, and by 1_{ij}, with $i \neq j$, a universe changing operator. We know from the club quiche analysis that the $\mathcal{H}_3$ acts on the full CFT $3\text{SP}_0 \oplus 3\text{SP}_1$ as

$$\begin{aligned} \alpha &= \eta_{00} \oplus \eta_{11}^2 \\ \rho &= 1_{01} \oplus 1_{10} \oplus 1_{11} \oplus \eta_{11} \oplus \eta_{11}^2 \,. \end{aligned} \tag{7.46}$$

We now want to show that this CFT admits relevant deformations to the 2 and 6 vacua $\mathcal{H}_3$ symmetric gapped phases, i.e. that it is indeed the phase transition between them. We start by considering the deformation $\epsilon_0 + \epsilon_1$. Adding this relevant deformation with the $-$ sign, both copies of 3SP flow to the trivially symmetric gapped phase. Let us denote the 2 vacua of each trivial phase by v_0 and v_1 respectively. In each of these, the $\mathbb{Z}_3$ symmetry is completely trivial, which implies

$$\eta_{ii} = 1_{ii} \,. \tag{7.47}$$

Therefore, using (7.46), the $\mathcal{H}_3$ symmetry generators descend to

$$\begin{aligned} \alpha &= 1_{00} \oplus 1_{11} \\ \rho &= 1_{01} \oplus 1_{10} \oplus 3\, 1_{11} \,. \end{aligned} \tag{7.48}$$

This precisely reproduces the symmetry action we discussed for the $\mathcal{H}_3$ symmetric gapped phases with 2 vacua. Now consider the opposite + sign deformation. Both 3SP now flow to the $\mathbb{Z}_3$ SSB phase. Therefore we obtain 3 vacua for each 3SP, and 6 in total. Let us denote by v_0, v_1, v_2 the three vacua of the gapped phase to which $3\mathrm{SP}_1$ flows, and by v_3, v_4, v_5 the 3 vacua for $3\mathrm{SP}_0$. In this case, the $\mathbb{Z}_3$ symmetry is spontaneously broken and therefore it is represented on the vacua as

$$\begin{aligned} \eta_{11} &= 1_{01} \oplus 1_{12} \oplus 1_{20} \\ \eta_{00} &= 1_{34} \oplus 1_{45} \oplus 1_{53} \,. \end{aligned} \tag{7.49}$$

Therefore, using (7.46), the symmetry generators can be represented on the six vacua as

$$\begin{aligned} \alpha &= 1_{02} \oplus 1_{10} \oplus 1_{21} \oplus 1_{34} \oplus 1_{45} \oplus 1_{53} \\ \rho &= 1_{03} \oplus 1_{14} \oplus 1_{25} \oplus 1_{30} \oplus 1_{41} \oplus 1_{52} \oplus 1_{00} \oplus 1_{11} \\ &\quad + 1_{22} \oplus 1_{01} \oplus 1_{12} \oplus 1_{20} \oplus 1_{02} + 1_{21} \oplus 1_{10} \,. \end{aligned} \tag{7.50}$$

This reproduces exactly the symmetry action we discussed for the full $\mathcal{H}_3$ symmetric gapped phase with 6 vacua.[1]

[1]Provided we switch the labelling $1 \leftrightarrow 2$ everywhere, but this is allowed since the naming of the vacua is arbitrary.

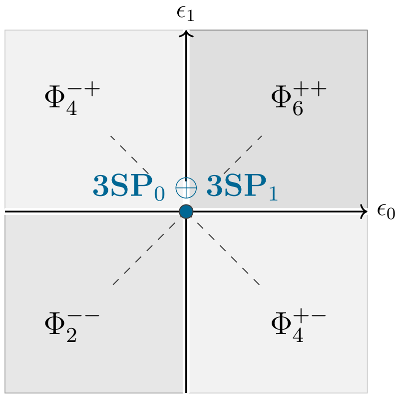


Figure 7.2: Proposed phase diagram of the Haagerup-symmetric theories. We have a critical model at the origin (•), given by two copies of the three-state Potts model $3SP_0 \oplus 3SP_1$. This has relevant deformations to the gapped phases $\Phi_n^{\pm,\pm}$, where $(\pm, \pm)$ indicates the sign of the deformation (ϵ_0, ϵ_1) and n is the number of vacua in the gapped phase. The relevant deformation for each $3SP_i$ model gives 3 vacua for $\epsilon_i > 0$ ($\mathbb{Z}_3$ SSB) and 1 vacuum for $\epsilon_i < 0$ ($\mathbb{Z}_3$ trivial phase). There is a second order Haagerup-symmetric phase transition between the Φ_6^{++} and Φ_2^{--} phases. From the critical point, there are furthermore deformations to 4 vacua gapped phases.

7.3.4 Phase Diagram

Overall, we find the phase diagram in figure 7.2. Starting with the theory (7.45), all combinations of relevant deformations are allowed and give rise to various gapped phases with $n = 2, 4, 6$ vacua, labelled by $\Phi_n^{\epsilon_0,\epsilon_1}$. These arise from the relevant deformation of each of the 3SP models to a single or three vacua gapped phase. The transition between Φ_2^{--} and Φ_6^{++} is second order, likewise between Φ_4^{+-} and Φ_4^{-+}. All the transitions are $\mathbb{Z}_3$ symmetric. Moreover, the Φ_2^{--} to Φ_6^{++} transition arises from the club sandwich and is therefore fully $\mathcal{H}_3$ symmetric. One can indeed check using (7.46) that $\alpha(\epsilon_0 + \epsilon_1) = \epsilon_0 + \epsilon_1$ and $\rho(\epsilon_0 + \epsilon_1) = d(\epsilon_0 + \epsilon_1)$. This comes from the fact that ϵ_0 and ϵ_1 are $\mathbb{Z}_3$ symmetric, so that (7.39) becomes $\rho = 1_{01} \oplus 1_{10} \oplus 3\,1_{11}$, and the use of the Euler terms shown in (7.38). This approach does not say anything directly about the 4 vacua gapped phases. However, it is suggestive that the CFT (7.45) admits a deformation to 4 gapped vacua by taking $\epsilon_0 = -\epsilon_1$, which decomposes as a $\mathbb{Z}_3$ SSB phase and a trivial phase.

We also note that the CFT (7.45) can be thought of as the stacking of the 3SP CFT with the Haagerup symmetric TQFT with 2 vacua. Of course any CFT times an $\mathcal{H}_3$-symmetric TQFT would trivially give a gapless phase with $\mathcal{H}_3$ symmetry. However, the key distinction is that the theory (7.45) correctly models the phase transitions between the gapped phases with Haagerup symmetry.

7.4 Haagerup Symmetric Lattice Models

To complement the continuum analysis, we now construct UV lattice models with Haagerup symmetry that flow to the gapped and gapless phases we discussed above from the SymTFT perspective. We focus on a class of models known as anyon chains [83,89,101–106], which can be defined using an input fusion category that determines naturally the symmetry $\mathcal{S}$ of the model.

7.4.1 Anyon Chains for Gapped and Gapless Phases

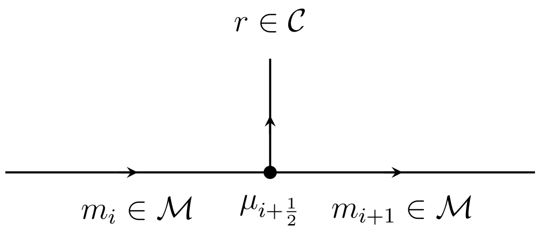
Let us recap the main steps of the construction, following [8]. The input data entering the definition of the anyon chain model is:

- An input fusion category $\mathcal{C}$, which should in general be distinguished from the symmetry fusion category $\mathcal{S}$.
- A $\mathcal{C}$-module category $\mathcal{M}$. The symmetry $\mathcal{S}$ is determined in terms of $\mathcal{C}$ and $\mathcal{M}$ as

$$\mathcal{S} = \mathcal{C}^*_{\mathcal{M}} = \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \,, \tag{7.51}$$

  which is the category formed by $\mathcal{C}$-module functors from $\mathcal{M}$ to $\mathcal{M}$.
- An object r in $\mathcal{C}$, which in general is taken to be a non-simple object.

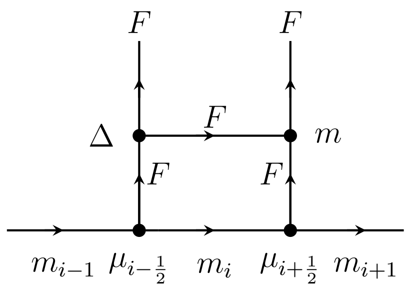
The basic constituent for the lattice model is then a block of the following form



(7.52)

where m_i and m_{i+1} are simple objects in the module category $\mathcal{M}$, and $\mu_{i+\frac{1}{2}} \in \text{Hom}(m_i, r \otimes m_{i+1})$ is a basis vector in the morphism space formed by r ending between m_i and m_{i+1}. The full Hilbert space of the model is constructed by concatenating such basic blocks. We typically use periodic boundary conditions and identify $m_1 = m_{L+1}$, where L is the length of the chain. The lattice model naturally possesses a $\mathcal{S}$ symmetry, as topological lines from $\mathcal{S}$ can be fused from below using the fact that $\mathcal{M}$ is also a right module category over $\mathcal{S}$.

We can realise all the possible gapped phases for $\mathcal{S}$ as the ground states of specific Hamiltonians acting on the anyon chain. In particular, let us recall there is a one-to-one correspondence between Lagrangian algebras of the Drinfeld center $\mathcal{Z}(\mathcal{S})$ and Frobenius algebra objects in $\mathcal{S}$ (they both determine a generalised gauging of all or part of the fusion category symmetry $\mathcal{S}$). To realise the gapped phase specified by a certain Lagrangian algebra $\mathcal{L}$, we then pick the corresponding Frobenius algebra object F and consider the operator H_i^F



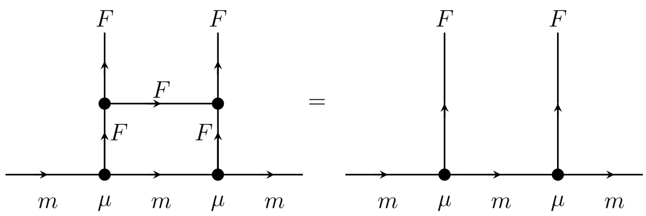
(7.53)

acting at site i. In the above, m and Δ are the multiplication and co-multiplication of the Frobenius algebra F. The full Hamiltonian is

$$H^F = -\sum_j H_j^F \,. \tag{7.54}$$

Notice this intuitively commutes with the symmetry $\mathcal{S}$, as the symmetry lines are fused from below while the Hamiltonian operator acts from above on the anyon chain.

Ground states of this Hamiltonian can simply be identified with left modules $\mathcal{K}$ for the Frobenius algebra F. Such a module is a (not necessarily simple) object $m \in \mathcal{M}$ along with a morphism $\mu \in \mathrm{Hom}(m, F \otimes m)$ satisfying a series of properties (see e.g. equation (3.7) in [8]) which imply



(7.55)

It follows that a state constructed out of an F-module is a $+1$ eigenstate of all projectors H_j^F and hence a ground state. Notice that in the case $\mathcal{M}$ is the regular module, $\mathcal{K}$ becomes a standard left module over the algebra F [89]

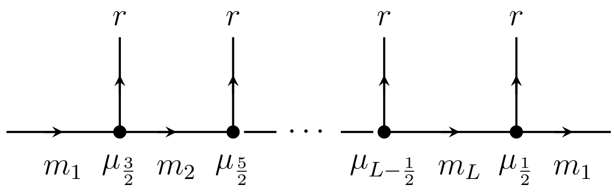
$$\mathcal{K} \in \mathsf{Mod}_{\mathcal{C}}(F) \,. \tag{7.56}$$

We now apply this logic to construct the Haagerup symmetric gapped phases from the anyon chain: the choice of input data is

$$\mathcal{C} = \mathcal{M} = \mathcal{S} = \mathcal{H}_3 \,, \tag{7.57}$$

i.e. the module category $\mathcal{M}$ is the regular module for $\mathcal{C}$ and this gives rise to the symmetry $\mathcal{S} = \mathcal{C}^*_{\mathcal{M}} = \mathcal{H}_3$. The Hilbert space of the model on a lattice of length L

with periodic boundary conditions is spanned by states corresponding to the fusion trees



(7.58)

where

$$r = \bigoplus_{l \in \mathcal{H}_3} l \,. \tag{7.59}$$

Note this element is usually called ρ, which is not a suitable notation in the present context.

We can construct three commuting projector Hamiltonians whose ground states realise each of the three Haagerup symmetric gapped phases. These are labelled by Frobenius algebras in the input category $\mathcal{C} = \mathcal{H}_3$, which are

$$F_1 = 1\,, \qquad F_2 = 1 \oplus \alpha \oplus \alpha^2\,, \qquad F_3 = 1 \oplus \rho \oplus \alpha\rho\,. \tag{7.60}$$

In particular, we find the following:

- $F_1 \Leftrightarrow \mathcal{H}_3$ SSB phase (6 vacua)
- $F_2 \Leftrightarrow \mathcal{H}_3/\mathbb{Z}_3$ SSB phase (2 vacua)
- $F_3 \Leftrightarrow \mathbb{Z}_3$ SSB $\oplus$ trivial phase (4 vacua) .

7.4.2 Phase Transitions for $\mathcal{H}_3$

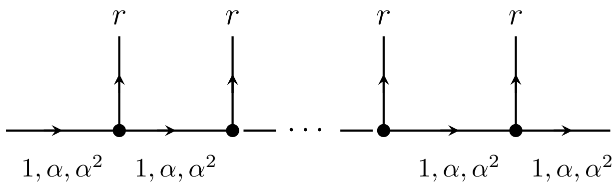
To realise the phase transition between the 6 vacua case and the 2 vacua case, we use the input choices

$$\mathcal{C} = \mathcal{M} = \mathcal{S} = \mathcal{H}_3\,, \tag{7.61}$$

which remain the same as for the gapped phases, but we restrict r to be

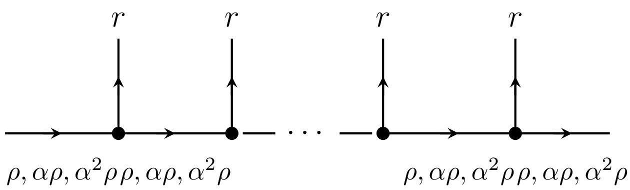
$$r = 1 \oplus \alpha \oplus \alpha^2 \,. \tag{7.62}$$

Restricting to this r leads to a direct sum decomposition of the original Hilbert space into state spaces V_0 and V_1 spanned by



(7.63)

and



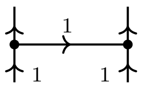
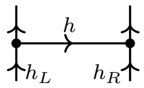
(7.64)

We notice these correspond to two anyon chains defined using the input data $\mathcal{C}' = \mathcal{M}' = \mathbb{Z}_3$, which have $\mathcal{S}' = \mathbb{Z}_3$ symmetry. Correspondingly, the original module category $\mathcal{M} = \mathcal{H}_3$ decomposes as a $\mathbb{Z}_3$ module category as

$$\mathcal{M} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \,. \tag{7.65}$$

Both V_0 and V_1 are therefore tensor product spaces of local qutrits $|q_i\rangle_{0,1}$ assigned to integer sites, where $q_i = 0, 1, 2$ depending on whether $m_i = 1, \alpha, \alpha^2$ for V_0 and $m_i = \rho, \alpha\rho, \alpha^2\rho$ for V_1 respectively. We denote a basis state as $|\vec{q}\rangle$.

We now consider the Hamiltonian given by

$$\mathcal{H} = -\sum_j \Bigg[$$

$$+ \frac{\lambda}{3} \sum_{\substack{h,h_L,h_R \in \\ \{1,\alpha,\alpha^2\}}}$$

$$\Bigg]_j \,, \tag{7.66}$$

where the two terms inside the braket correspond to (7.53) with $F = 1$ and $F = 1 \oplus \alpha \oplus \alpha^2$ respectively. Written in a more familiar language, the above Hamiltonian reads

$$\mathcal{H} = -\sum_j \left(\frac{1 + Z_{j-1}Z_j^\dagger + Z_{j-1}^\dagger Z_j}{3} \right) \left(\frac{1 + Z_j Z_{j+1}^\dagger + Z_j^\dagger Z_{j+1}}{3} \right) + \lambda \left(\frac{1 + X_j + X_j^2}{3} \right) , \tag{7.67}$$

where Z_j and X_j are the standard $\mathbb{Z}_3$ clock and shift operators

$$\begin{aligned} Z_j |q_j\rangle &= \omega^j |q_j\rangle \,, \quad \omega = e^{2\pi i/3} \\ X_j |q_j\rangle &= |q_j + 1 \bmod 3\rangle \,. \end{aligned} \tag{7.68}$$

Since all the building blocks of this Hamiltonian decompose into mutually commuting projectors, we can instead study the simpler Hamiltonian

$$\mathcal{H}_{\mathrm{3SP}} = -\frac{1}{3} \sum_j \left(1 + Z_j Z_{j+1}^\dagger + Z_j^\dagger Z_{j+1} \right) + \lambda \left(1 + X_j + X_j^2 \right) \tag{7.69}$$

This is the usual quantum three-state Potts (3SP) model Hamiltonian (up to a shift). $\mathcal{H}_{\mathrm{3SP}}$ realizes a $\mathbb{Z}_3$ symmetric trivial phase, a $\mathbb{Z}_3$ SSB phase and the 3SP CFT at $\lambda = 1$ [159], giving a transition between the two phases. On $V_0 \oplus V_1$, it acts block-diagonally, so that we obtain two decoupled sectors, with $\lambda = 1$ realising the CFT

$$\mathrm{3SP}_0 \oplus \mathrm{3SP}_1 \,. \tag{7.70}$$

Let us now compute the $\mathcal{H}_3$ symmetry action, recalling that this is obtained by fusing a symmetry line from below on the chain. The invertible generator α clearly acts within each space as a $\mathbb{Z}_3$ shift symmetry, i.e. $\alpha = \eta_{00} \oplus \eta_{11}^2$, where $\eta = \prod_j X_j$, and the subscript indicates whether it acts on V_0 or V_1. The non-invertible generator ρ has a more interesting action, as it maps between V_0 and V_1, as well as acting within V_1, as $\rho = 1_{01} \oplus 1_{10} \oplus 1_{11} \oplus \eta_{11} \oplus \eta_{11}^2$. Here 1_{ij}, $i, j = 0, 1$, sends a state $|\vec{q}\rangle_i$ in V_i to the corresponding state $|\vec{q}\rangle_j$ in V_j.

There are two possible Frobenius algebras we can consider given our choice of r, namely $F_1' = 1$ and $F_2' = 1 \oplus \alpha \oplus \alpha^2$. The Hamiltonian corresponding to F_1' acting on $V_0 \oplus V_1$ has 6 ground states

$$
\begin{aligned}
|GS, \alpha^i\rangle &= |\alpha^i, \cdots, \alpha^i\rangle \,, \qquad i = 0, 1, 2 \\
|GS, \alpha^i\rho\rangle &= |\alpha^i\rho, \cdots, \alpha^i\rho\rangle \,, \qquad i = 0, 1, 2 \,.
\end{aligned}
\tag{7.71}
$$

α clearly permutes the first set of vacua among themselves

$$
\begin{aligned}
\alpha : \quad & |GS, \alpha^i\rangle \to |GS, \alpha^{i+1}\rangle \\
& |GS, \alpha^i\rho\rangle \to |GS, \alpha^{i-1}\rho\rangle \,,
\end{aligned}
\tag{7.72}
$$

while ρ acts between the two sets of ground states

$$
\begin{aligned}
\rho : \quad & |GS, \alpha^i\rangle \to |GS, \alpha^i\rho\rangle \\
& |GS, \alpha^i\rho\rangle \to |GS, \alpha^i\rangle + \sum_{j=0}^{2} |GS, \alpha^j\rho\rangle \,.
\end{aligned}
\tag{7.73}
$$

This fully reproduces the continuum results (7.20) and (7.22), therefore giving the $\mathcal{H}_3$ SSB phase. The Hamiltonian corresponding to F_2' acting on $V_0 \oplus V_1$ has 2 ground states

$$
\begin{aligned}
|GS, 0\rangle &= \frac{1}{3^{L/2}} \sum_{\vec{g}} |\vec{g}\rangle \,, \qquad g_i = \{1, \alpha, \alpha^2\} \\
|GS, 1\rangle &= \frac{1}{3^{L/2}} \sum_{\vec{g}} |\vec{g}\rangle \,, \qquad g_i = \{\rho, \alpha\rho, \alpha^2\rho\}
\end{aligned}
\tag{7.74}
$$

These are clearly left invariant by α, and are permuted by ρ as

$$\rho : \quad |GS,0\rangle \to |GS,1\rangle \ , \ |GS,1\rangle \to |GS,0\rangle + 3|GS,1\rangle \ , \tag{7.75}$$

reproducing (7.12), (7.14) and thus giving the $\mathbb{Z}_3$ unbroken phase. The model (7.66) then realises the Haagerup 6 vacua phase for $\lambda < 1$ and the Haagerup 2 vacua phase for $\lambda > 1$:

$$\underset{(\mathrm{SSB}_{\mathbb{Z}_3} \oplus \mathrm{SSB}_{\mathbb{Z}_3})}{\overset{\text{Six vacua}}{\longrightarrow}} \underset{\lambda = 1}{\overset{3\mathrm{SP}_0 \oplus 3\mathrm{SP}_1}{\bullet}} \underset{(\mathrm{Triv} \oplus \mathrm{Triv})}{\overset{\text{Two vacua}}{\longrightarrow}} \lambda \tag{7.76}$$

This confirms the continuum SymTFT analysis that the phase transition between the two gapped phases with Haagerup symmetry Φ_6^{++} and Φ_2^{--} is indeed the CFT given by (7.45).

Chapter 8

Conclusions

In this thesis, we examined non-invertible and higher-categorical symmetries in quantum field theories, combining formal constructions with more explicit examples.

In higher dimensions ($d \geq 3$), we presented a general approach to constructing non-invertible symmetries through gauging discrete 0-form subgroups which act non-trivially on some higher-form symmetry. Focusing on $d = 3$, we studied webs of theories connected by sequential gauging, highlighting the rich categorical structures that can emerge.

In lower dimensions, we focused on the implications of categorical symmetries for infrared physics, particularly in $(1 + 1)$d. We introduced the SymTFT as a tool to systematically study gapped phases and phase transitions, leading to a categorical generalisation of the Landau paradigm. We illustrated these ideas through an explicit lattice model and constructed a conformal field theory with Haagerup symmetry, addressing a longstanding open question.

There are several natural directions for future work. An obvious one is to extend the categorical Landau framework beyond two dimensions, and to understand to what extent similar tools can be applied in $(2+1)$d and $(3+1)$d. Some recent works in this direction, based on the framework developed here in $(1 + 1)$d, are [75, 76, 79, 80]. We also remark that an analogous story can be carried out in parallel in lattice models, see e.g. [160, 161].

In general, it would be very interesting to further explore what a SymTFT analysis of phases can teach us about the IR dynamics of quantum field theories. In gauge theories in $(3+1)$d, e.g., the spontaneous symmetry breaking of 1-form symmetry is related to deconfinement, and the absence of a trivially gapped phase would signal an obstruction to having a confining phase.

Another interesting direction is to extend the categorical Landau paradigm to incorporate space-time symmetries, which have been so far fairly overlooked compared to their internal counterpart, despite being fundamental symmetries of many physical systems.

Regarding the construction of the Haagerup symmetric CFT (7.45), we expect this to have potentially interesting consequences in determining the single universe Haagerup CFT. E.g., one could try to apply constraints from the holographic modular bootstrap [125, 162] to access this single-universe model. A conjecture motivated by the present work is that the single-universe CFT could be the $c = 1$ compact boson tensored with the 3SP model. This would be compatible with the numerical results on $c \sim 2$ in [96, 97], but it remains to be seen how the full $\mathcal{H}_3$ symmetry acts.

On the mathematical side, further development of higher fusion categories could help clarify and strengthen some of the constructions used here.

In general, categorical symmetries have recently emerged as a crucial tool, which is furthermore at the intersections of many different fields, such as high-energy physics, the mathematics of category theory, condensed matter physics and quantum information. The advance in this area is an extremely promising avenue to deepen our knowledge of physics via the powerful constraints of symmetries.

Bibliography

[1] L. Bhardwaj, L. E. Bottini, S. Schafer-Nameki, and A. Tiwari, "Non-Invertible Higher-Categorical Symmetries," *SciPost Phys.* **14** (2023) 007, arXiv:2204.06564 [hep-th].

[2] L. Bhardwaj, L. E. Bottini, S. Schafer-Nameki, and A. Tiwari, "Non-invertible symmetry webs," *SciPost Phys.* **15** no. 4, (2023) 160, arXiv:2212.06842 [hep-th].

[3] L. Bhardwaj, L. E. Bottini, D. Pajer, and S. Schafer-Nameki, "Categorical Landau Paradigm for Gapped Phases," *Phys. Rev. Lett.* **133** no. 16, (2024) 161601, arXiv:2310.03786 [cond-mat.str-el].

[4] L. Bhardwaj, L. E. Bottini, D. Pajer, and S. Schäfer-Nameki, "Gapped phases with non-invertible symmetries: (1+1)d," *SciPost Phys.* **18** no. 1, (2025) 032, arXiv:2310.03784 [hep-th].

[5] L. Bhardwaj, L. E. Bottini, D. Pajer, and S. Schafer-Nameki, "The Club Sandwich: Gapless Phases and Phase Transitions with Non-Invertible Symmetries," *SciPost Phys.* **18** (2025) 156, arXiv:2312.17322 [hep-th].

[6] L. Bhardwaj, L. E. Bottini, S. Schafer-Nameki, and A. Tiwari, "Illustrating the categorical Landau paradigm in lattice models," *Phys. Rev. B* **111** no. 5, (2025) 054432, arXiv:2405.05302 [cond-mat.str-el].

[7] L. E. Bottini and S. Schafer-Nameki, "Construction of a Gapless Phase with Haagerup Symmetry," *Phys. Rev. Lett.* **134** no. 19, (2025) 191602, arXiv:2410.19040 [hep-th].

[8] L. Bhardwaj, L. E. Bottini, S. Schafer-Nameki, and A. Tiwari, "Lattice Models for Phases and Transitions with Non-Invertible Symmetries," arXiv:2405.05964 [cond-mat.str-el].

[9] L. E. Bottini, C. Hwang, S. Pasquetti, and M. Sacchi, "4d S-duality wall and SL(2, $\mathbb{Z}$) relations," *JHEP* **03** (2022) 035, `arXiv:2110.08001 [hep-th]`.

[10] L. E. Bottini, C. Hwang, S. Pasquetti, and M. Sacchi, "Dualities from dualities: the sequential deconfinement technique," *JHEP* **05** (2022) 069, `arXiv:2201.11090 [hep-th]`.

[11] E. Noether, "Invariante variationsprobleme," *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1918** (1918) 235–257.

[12] L. Bhardwaj, L. E. Bottini, L. Fraser-Taliente, L. Gladden, D. S. W. Gould, A. Platschorre, and H. Tillim, "Lectures on generalized symmetries," *Phys. Rept.* **1051** (2024) 1–87, `arXiv:2307.07547 [hep-th]`.

[13] T. D. Brennan and S. Hong, "Introduction to Generalized Global Symmetries in QFT and Particle Physics," `arXiv:2306.00912 [hep-ph]`.

[14] R. Luo, Q.-R. Wang, and Y.-N. Wang, "Lecture notes on generalized symmetries and applications," *Phys. Rept.* **1065** (2024) 1–43, `arXiv:2307.09215 [hep-th]`.

[15] P. R. S. Gomes, "An introduction to higher-form symmetries," *SciPost Phys. Lect. Notes* **74** (2023) 1, `arXiv:2303.01817 [hep-th]`.

[16] S.-H. Shao, "What's Done Cannot Be Undone: TASI Lectures on Non-Invertible Symmetries," `arXiv:2308.00747 [hep-th]`.

[17] S. Schafer-Nameki, "ICTP lectures on (non-)invertible generalized symmetries," *Phys. Rept.* **1063** (2024) 1–55, `arXiv:2305.18296 [hep-th]`.

[18] D. Costa *et al.*, "Simons Lectures on Categorical Symmetries," 11, 2024. `arXiv:2411.09082 [math-ph]`.

[19] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, "Generalized Global Symmetries," *JHEP* **02** (2015) 172, `arXiv:1412.5148 [hep-th]`.

[20] O. Aharony, N. Seiberg, and Y. Tachikawa, "Reading between the lines of four-dimensional gauge theories," *JHEP* **08** (2013) 115, `arXiv:1305.0318 [hep-th]`.

[21] D. Gaiotto, A. Kapustin, Z. Komargodski, and N. Seiberg, "Theta, Time Reversal, and Temperature," *JHEP* **05** (2017) 091, `arXiv:1703.00501 [hep-th]`.

[22] J. McGreevy, "Generalized Symmetries in Condensed Matter," *Ann. Rev. Condensed Matter Phys.* **14** (2023) 57–82, `arXiv:2204.03045 [cond-mat.str-el]`.

[23] A. Kapustin and R. Thorngren, "Higher Symmetry and Gapped Phases of Gauge Theories," *Prog. Math.* **324** (2017) 177–202, `arXiv:1309.4721 [hep-th]`.

[24] E. Sharpe, "Notes on generalized global symmetries in QFT," *Fortsch. Phys.* **63** (2015) 659–682, `arXiv:1508.04770 [hep-th]`.

[25] Y. Tachikawa, "On gauging finite subgroups," *SciPost Phys.* **8** no. 1, (2020) 015, `arXiv:1712.09542 [hep-th]`.

[26] C. Córdova, T. T. Dumitrescu, and K. Intriligator, "Exploring 2-Group Global Symmetries," *JHEP* **02** (2019) 184, `arXiv:1802.04790 [hep-th]`.

[27] F. Benini, C. Córdova, and P.-S. Hsin, "On 2-Group Global Symmetries and their Anomalies," *JHEP* **03** (2019) 118, `arXiv:1803.09336 [hep-th]`.

[28] C. Cordova, T. T. Dumitrescu, and K. Intriligator, "2-Group Global Symmetries and Anomalies in Six-Dimensional Quantum Field Theories," *JHEP* **04** (2021) 252, `arXiv:2009.00138 [hep-th]`.

[29] Y. Lee, K. Ohmori, and Y. Tachikawa, "Matching higher symmetries across Intriligator-Seiberg duality," *JHEP* **10** (2021) 114, `arXiv:2108.05369 [hep-th]`.

[30] P.-S. Hsin and H. T. Lam, "Discrete theta angles, symmetries and anomalies," *SciPost Phys.* **10** no. 2, (2021) 032, `arXiv:2007.05915 [hep-th]`.

[31] E. P. Verlinde, "Fusion Rules and Modular Transformations in 2D Conformal Field Theory," *Nucl. Phys. B* **300** (1988) 360–376.

[32] G. W. Moore and N. Seiberg, "Classical and Quantum Conformal Field Theory," *Commun. Math. Phys.* **123** (1989) 177.

[33] G. W. Moore and N. Read, "Nonabelions in the fractional quantum Hall effect," *Nucl. Phys. B* **360** (1991) 362–396.

[34] X. G. Wen, "Non-abelian statistics in the fractional quantum hall states," *Phys. Rev. Lett.* **66** (Feb, 1991) 802–805.

[35] V. B. Petkova and J. B. Zuber, "Generalized twisted partition functions," *Phys. Lett. B* **504** (2001) 157–164, `arXiv:hep-th/0011021`.

[36] J. Fuchs, I. Runkel, and C. Schweigert, "TFT construction of RCFT correlators 1. Partition functions," *Nucl. Phys. B* **646** (2002) 353–497, `arXiv:hep-th/0204148`.

[37] C. Bachas and M. Gaberdiel, "Loop operators and the Kondo problem," *JHEP* **11** (2004) 065, `arXiv:hep-th/0411067`.

[38] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, "Kramers-Wannier duality from conformal defects," *Phys. Rev. Lett.* **93** (2004) 070601, `arXiv:cond-mat/0404051`.

[39] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, "Duality and defects in rational conformal field theory," *Nucl. Phys. B* **763** (2007) 354–430, `arXiv:hep-th/0607247`.

[40] J. Fuchs, M. R. Gaberdiel, I. Runkel, and C. Schweigert, "Topological defects for the free boson CFT," *J. Phys. A* **40** (2007) 11403, `arXiv:0705.3129 [hep-th]`.

[41] C. Bachas and S. Monnier, "Defect loops in gauged Wess-Zumino-Witten models," *JHEP* **02** (2010) 003, `arXiv:0911.1562 [hep-th]`.

[42] A. Kapustin and N. Saulina, "Surface operators in 3d Topological Field Theory and 2d Rational Conformal Field Theory," `arXiv:1012.0911 [hep-th]`.

[43] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, "Symmetry Fractionalization, Defects, and Gauging of Topological Phases," *Phys. Rev. B* **100** no. 11, (2019) 115147, `arXiv:1410.4540 [cond-mat.str-el]`.

[44] R. Thorngren and Y. Wang, "Fusion category symmetry. Part I. Anomaly in-flow and gapped phases," *JHEP* **04** (2024) 132, `arXiv:1912.02817 [hep-th]`.

[45] R. Thorngren and Y. Wang, "Fusion Category Symmetry II: Categoriosities at $c = 1$ and Beyond," arXiv:2106.12577 [hep-th].

[46] Y.-H. Lin and S.-H. Shao, "Duality Defect of the Monster CFT," *J. Phys. A* **54** no. 6, (2021) 065201, arXiv:1911.00042 [hep-th].

[47] D. Gaiotto, J. H. Lee, and J. Wu, "Integrable Kondo problems," *JHEP* **04** (2021) 268, arXiv:2003.06694 [hep-th].

[48] Z. Komargodski, K. Ohmori, K. Roumpedakis, and S. Seifnashri, "Symmetries and strings of adjoint QCD_2," *JHEP* **03** (2021) 103, arXiv:2008.07567 [hep-th].

[49] D. Gaiotto, J. H. Lee, B. Vicedo, and J. Wu, "Kondo line defects and affine Gaudin models," *JHEP* **01** (2022) 175, arXiv:2010.07325 [hep-th].

[50] D. Gaiotto and J. Kulp, "Orbifold groupoids," *JHEP* **02** (2021) 132, arXiv:2008.05960 [hep-th].

[51] I. M. Burbano, J. Kulp, and J. Neuser, "Duality defects in E_8," *JHEP* **10** (2022) 186, arXiv:2112.14323 [hep-th].

[52] Y.-H. Lin, M. Okada, S. Seifnashri, and Y. Tachikawa, "Asymptotic density of states in 2d CFTs with non-invertible symmetries," *JHEP* **03** (2023) 094, arXiv:2208.05495 [hep-th].

[53] C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, "Topological Defect Lines and Renormalization Group Flows in Two Dimensions," *JHEP* **01** (2019) 026, arXiv:1802.04445 [hep-th].

[54] T.-C. Huang, Y.-H. Lin, and S. Seifnashri, "Construction of two-dimensional topological field theories with non-invertible symmetries," *JHEP* **12** (2021) 028, arXiv:2110.02958 [hep-th].

[55] J. Kaidi, Z. Komargodski, K. Ohmori, S. Seifnashri, and S.-H. Shao, "Higher central charges and topological boundaries in 2+1-dimensional TQFTs," *SciPost Phys.* **13** no. 3, (2022) 067, arXiv:2107.13091 [hep-th].

[56] M. Nguyen, Y. Tanizaki, and M. Ünsal, "Semi-Abelian gauge theories, non-invertible symmetries, and string tensions beyond N-ality," *JHEP* **03** (2021) 238, arXiv:2101.02227 [hep-th].

[57] B. Heidenreich, J. McNamara, M. Montero, M. Reece, T. Rudelius, and I. Valenzuela, “Non-invertible global symmetries and completeness of the spectrum,” *JHEP* **09** (2021) 203, `arXiv:2104.07036 [hep-th]`.

[58] Y. Choi, C. Cordova, P.-S. Hsin, H. T. Lam, and S.-H. Shao, “Noninvertible duality defects in 3+1 dimensions,” *Phys. Rev. D* **105** no. 12, (2022) 125016, `arXiv:2111.01139 [hep-th]`.

[59] J. Kaidi, K. Ohmori, and Y. Zheng, “Kramers-Wannier-like Duality Defects in (3+1)D Gauge Theories,” *Phys. Rev. Lett.* **128** no. 11, (2022) 111601, `arXiv:2111.01141 [hep-th]`.

[60] K. Roumpedakis, S. Seifnashri, and S.-H. Shao, “Higher Gauging and Non-invertible Condensation Defects,” *Commun. Math. Phys.* **401** no. 3, (2023) 3043–3107, `arXiv:2204.02407 [hep-th]`.

[61] Y. Choi, C. Cordova, P.-S. Hsin, H. T. Lam, and S.-H. Shao, “Non-invertible Condensation, Duality, and Triality Defects in 3+1 Dimensions,” *Commun. Math. Phys.* **402** no. 1, (2023) 489–542, `arXiv:2204.09025 [hep-th]`.

[62] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, vol. 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015. `https://doi.org/10.1090/surv/205`.

[63] C. L. Douglas and D. J. Reutter, “Fusion 2-categories and a state-sum invariant for 4-manifolds,” *arXiv preprint arXiv:1812.11933* (2018) .

[64] T. D. Décoppet, “On the drinfeld centers of fusion 2-categories,” *arXiv preprint arXiv:2211.04917* (2022) .

[65] T. D. Décoppet, P. Huston, T. Johnson-Freyd, D. Nikshych, D. Penneys, J. Plavnik, D. Reutter, and M. Yu, “The Classification of Fusion 2-Categories,” `arXiv:2411.05907 [math.CT]`.

[66] T. D. Décoppet and H. Xu, “Local modules in braided monoidal 2-categories,” *J. Math. Phys.* **65** no. 6, (2024) 061702, `arXiv:2307.02843 [math.CT]`.

[67] C. Zhang and C. Córdova, “Anomalies of (1+1)-dimensional categorical symmetries,” *Phys. Rev. B* **110** no. 3, (2024) 035155, `arXiv:2304.01262 [cond-mat.str-el]`.

[68] T.-C. Huang and Y.-H. Lin, "Topological field theory with Haagerup symmetry," *J. Math. Phys.* **63** no. 4, (2022) 042306, arXiv:2102.05664 [hep-th].

[69] K. Inamura, "Topological field theories and symmetry protected topological phases with fusion category symmetries," *JHEP* **05** (2021) 204, arXiv:2103.15588 [cond-mat.str-el].

[70] G. 't Hooft, "Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking," *NATO Sci. Ser. B* **59** (1980) 135–157.

[71] A. Apte, C. Cordova, and H. T. Lam, "Obstructions to gapped phases from noninvertible symmetries," *Phys. Rev. B* **108** no. 4, (2023) 045134, arXiv:2212.14605 [hep-th].

[72] J. Kaidi, E. Nardoni, G. Zafrir, and Y. Zheng, "Symmetry TFTs and anomalies of non-invertible symmetries," *JHEP* **10** (2023) 053, arXiv:2301.07112 [hep-th].

[73] J. A. Damia, R. Argurio, F. Benini, S. Benvenuti, C. Copetti, and L. Tizzano, "Non-invertible symmetries along 4d RG flows," arXiv:2305.17084 [hep-th].

[74] C. Cordova, P.-S. Hsin, and C. Zhang, "Anomalies of non-invertible symmetries in (3+1)d," *SciPost Phys.* **17** no. 5, (2024) 131, arXiv:2308.11706 [hep-th].

[75] L. Bhardwaj, D. Pajer, S. Schafer-Nameki, A. Tiwari, A. Warman, and J. Wu, "Gapped Phases in (2+1)d with Non-Invertible Symmetries: Part I," arXiv:2408.05266 [hep-th].

[76] A. Antinucci, C. Copetti, and S. Schafer-Nameki, "SymTFT for (3+1)d Gapless SPTs and Obstructions to Confinement," *SciPost Phys.* **18** (2025) 114, arXiv:2408.05585 [hep-th].

[77] A. Antinucci, C. Copetti, G. Galati, and G. Rizi, "Topological constraints on defect dynamics," *Phys. Rev. D* **111** no. 6, (2025) 065025, arXiv:2412.18652 [hep-th].

[78] C. Copetti, "Defect Charges, Gapped Boundary Conditions, and the Symmetry TFT," arXiv:2408.01490 [hep-th].

[79] L. Bhardwaj, S. Schafer-Nameki, A. Tiwari, and A. Warman, "Gapped Phases in (2+1)d with Non-Invertible Symmetries: Part II," `arXiv:2502.20440 [hep-th]`.

[80] L. Bhardwaj, Y. Gai, S.-J. Huang, K. Inamura, S. Schafer-Nameki, A. Tiwari, and A. Warman, "Gapless Phases in (2+1)d with Non-Invertible Symmetries," `arXiv:2503.12699 [cond-mat.str-el]`.

[81] P.-S. Hsin, R. Kobayashi, and C. Zhang, "Anomalies of Coset Non-Invertible Symmetries," `arXiv:2503.00105 [cond-mat.str-el]`.

[82] L. D. Landau, "On the theory of phase transitions," *Zh. Eksp. Teor. Fiz.* **7** (1937) 19–32.

[83] D. Aasen, R. S. K. Mong, and P. Fendley, "Topological Defects on the Lattice I: The Ising model," *J. Phys. A* **49** no. 35, (2016) 354001, `arXiv:1601.07185 [cond-mat.stat-mech]`.

[84] A. B. Zamolodchikov, "From tricritical Ising to critical Ising by thermodynamic Bethe ansatz," *Nucl. Phys. B* **358** (1991) 524–546.

[85] D. A. Huse, "Exact exponents for infinitely many new multicritical points," *Phys. Rev. B* **30** (1984) 3908–3915.

[86] L. Bhardwaj, S. Schafer-Nameki, and J. Wu, "Universal Non-Invertible Symmetries," *Fortsch. Phys.* **70** no. 11, (2022) 2200143, `arXiv:2208.05973 [hep-th]`.

[87] J. C. Teo, T. L. Hughes, and E. Fradkin, "Theory of twist liquids: gauging an anyonic symmetry," *Annals of Physics* **360** (2015) 349–445.

[88] L. Bhardwaj and Y. Tachikawa, "On finite symmetries and their gauging in two dimensions," *JHEP* **03** (2018) 189, `arXiv:1704.02330 [hep-th]`.

[89] K. Inamura, "On lattice models of gapped phases with fusion category symmetries," *JHEP* **03** (2022) 036, `arXiv:2110.12882 [cond-mat.str-el]`.

[90] K. Inamura, "Fermionization of fusion category symmetries in 1+1 dimensions," *JHEP* **10** (2023) 101, `arXiv:2206.13159 [cond-mat.str-el]`.

[91] L. Kong, T. Lan, X.-G. Wen, Z.-H. Zhang, and H. Zheng, "Classification of topological phases with finite internal symmetries in all dimensions," *JHEP* **09** (2020) 093, arXiv:2003.08898 [math-ph].

[92] D. S. Freed, G. W. Moore, and C. Teleman, "Topological symmetry in quantum field theory," arXiv:2209.07471 [hep-th].

[93] W. Ji and X.-G. Wen, "Categorical symmetry and noninvertible anomaly in symmetry-breaking and topological phase transitions," *Phys. Rev. Res.* **2** no. 3, (2020) 033417, arXiv:1912.13492 [cond-mat.str-el].

[94] F. Apruzzi, F. Bonetti, I. n. García Etxebarria, S. S. Hosseini, and S. Schafer-Nameki, "Symmetry TFTs from String Theory," *Commun. Math. Phys.* **402** no. 1, (2023) 895–949, arXiv:2112.02092 [hep-th].

[95] L. Bhardwaj and S. Schafer-Nameki, "Generalized Charges, Part II: Non-Invertible Symmetries and the Symmetry TFT," arXiv:2305.17159 [hep-th].

[96] T.-C. Huang, Y.-H. Lin, K. Ohmori, Y. Tachikawa, and M. Tezuka, "Numerical Evidence for a Haagerup Conformal Field Theory," *Phys. Rev. Lett.* **128** no. 23, (2022) 231603, arXiv:2110.03008 [cond-mat.stat-mech].

[97] R. Vanhove, L. Lootens, M. Van Damme, R. Wolf, T. J. Osborne, J. Haegeman, and F. Verstraete, "Critical Lattice Model for a Haagerup Conformal Field Theory," *Phys. Rev. Lett.* **128** no. 23, (2022) 231602, arXiv:2110.03532 [cond-mat.stat-mech].

[98] Y. Liu, Y. Zou, and S. Ryu, "Operator fusion from wave-function overlap: Universal finite-size corrections and application to the Haagerup model," *Phys. Rev. B* **107** no. 15, (2023) 155124, arXiv:2203.14992 [cond-mat.str-el].

[99] L. Corcoran and M. de Leeuw, "Integrable and critical Haagerup spin chains," arXiv:2410.16356 [cond-mat.stat-mech].

[100] L.-Y. Hung, K. Ji, C. Shen, Y. Wan, and Y. Zhao, "A 2D-CFT Factory: Critical Lattice Models from Competing Anyon Condensation Processes in SymTO/SymTFT," arXiv:2506.05324 [cond-mat.str-el].

[101] A. Feiguin, S. Trebst, A. W. W. Ludwig, M. Troyer, A. Kitaev, Z. Wang, and M. H. Freedman, "Interacting anyons in topological quantum liquids: The golden chain," *Phys. Rev. Lett.* **98** (2007) 160409, `arXiv:cond-mat/0612341`.

[102] S. Trebst, M. Troyer, Z. Wang, and A. W. W. Ludwig, "A short introduction to Fibonacci anyon models," *arXiv e-prints* (Feb., 2009) arXiv:0902.3275, `arXiv:0902.3275 [cond-mat.stat-mech]`.

[103] M. Buican and A. Gromov, "Anyonic Chains, Topological Defects, and Conformal Field Theory," *Commun. Math. Phys.* **356** no. 3, (2017) 1017–1056, `arXiv:1701.02800 [hep-th]`.

[104] D. Aasen, P. Fendley, and R. S. K. Mong, "Topological Defects on the Lattice: Dualities and Degeneracies," `arXiv:2008.08598 [cond-mat.stat-mech]`.

[105] L. Lootens, C. Delcamp, G. Ortiz, and F. Verstraete, "Dualities in One-Dimensional Quantum Lattice Models: Symmetric Hamiltonians and Matrix Product Operator Intertwiners," *PRX Quantum* **4** no. 2, (2023) 020357, `arXiv:2112.09091 [quant-ph]`.

[106] L. Lootens, C. Delcamp, and F. Verstraete, "Dualities in One-Dimensional Quantum Lattice Models: Topological Sectors," *PRX Quantum* **5** no. 1, (2024) 010338, `arXiv:2211.03777 [quant-ph]`.

[107] D. Reutter, "Talk at Perimeter Institute, 2022: Higher S-matrices and higher modular categories, https://pirsa.org/22060017,".

[108] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, "Defect lines, dualities, and generalised orbifolds," in *16th International Congress on Mathematical Physics*. 9, 2009. `arXiv:0909.5013 [math-ph]`.

[109] N. Carqueville and I. Runkel, "Orbifold completion of defect bicategories," *Quantum Topol.* **7** (2016) 203, `arXiv:1210.6363 [math.QA]`.

[110] N. Carqueville, I. Runkel, and G. Schaumann, "Orbifolds of n-dimensional defect TQFTs," *Geom. Topol.* **23** (2019) 781–864, `arXiv:1705.06085 [math.QA]`.

[111] N. Carqueville, I. Runkel, and G. Schaumann, "Orbifolds of Reshetikhin-Turaev TQFTs," *Theor. Appl. Categor.* **35** (2020) 513–561, `arXiv:1809.01483 [math.QA]`.

[112] V. Mulevičius and I. Runkel, "Constructing modular categories from orbifold data," *Quantum Topol.* **13** no. 3, (2023) 459–523, `arXiv:2002.00663 [math.QA]`.

[113] N. Carqueville, V. Mulevicius, I. Runkel, G. Schaumann, and D. Scherl, "Orbifold graph TQFTs," `arXiv:2101.02482 [math.QA]`.

[114] N. Carqueville, V. Mulevicius, I. Runkel, G. Schaumann, and D. Scherl, "Reshetikhin–Turaev TQFTs Close Under Generalised Orbifolds," *Commun. Math. Phys.* **405** no. 10, (2024) 242, `arXiv:2109.04754 [math.QA]`.

[115] L. Lin, D. G. Robbins, and E. Sharpe, "Decomposition, Condensation Defects, and Fusion," *Fortsch. Phys.* **70** no. 11, (2022) 2200130, `arXiv:2208.05982 [hep-th]`.

[116] T. Bartsch, M. Bullimore, A. E. V. Ferrari, and J. Pearson, "Non-invertible symmetries and higher representation theory I," *SciPost Phys.* **17** no. 1, (2024) 015, `arXiv:2208.05993 [hep-th]`.

[117] D. Tambara and S. Yamagami, "Tensor categories with fusion rules of self-duality for finite abelian groups," *J. Algebra* **209** (1998) 692–707.

[118] P. Etingof, D. Nikshych, and V. Ostrik, "Fusion categories and homotopy theory," *Quantum topology* **1** no. 3, (2010) 209–273.

[119] A. M. Essin and M. Hermele, "Classifying fractionalization: Symmetry classification of gapped $\mathbb{Z}_2$ spin liquids in two dimensions," *Phys. Rev. B* **87** no. 10, (2013) 104406, `arXiv:1212.0593 [cond-mat.str-el]`.

[120] N. Tarantino, N. H. Lindner, and L. Fidkowski, "Symmetry fractionalization and twist defects," *New Journal of Physics* **18** no. 3, (Mar., 2016) 035006, `arXiv:1506.06754 [cond-mat.str-el]`.

[121] X. Chen, "Symmetry Fractionalization in Two Dimensional Topological Phases," *arXiv e-prints* (June, 2016) arXiv:1606.07569, `arXiv:1606.07569 [cond-mat.str-el]`.

[122] Y. Qi and M. Cheng, "Classification of symmetry fractionalization in gapped $\mathbb{Z}_2$ spin liquids," *arXiv e-prints* (June, 2016) arXiv:1606.04544, `arXiv:1606.04544 [cond-mat.str-el]`.

[123] M. Barkeshli and M. Cheng, "Relative Anomalies in (2+1)D Symmetry Enriched Topological States," *SciPost Phys.* **8** (2020) 028, arXiv:1906.10691 [cond-mat.str-el].

[124] H. Moradi, O. M. Aksoy, J. H. Bardarson, and A. Tiwari, "Symmetry fractionalization, mixed-anomalies and dualities in quantum spin models with generalized symmetries," arXiv:2307.01266 [cond-mat.str-el].

[125] A. Chatterjee and X.-G. Wen, "Holographic theory for continuous phase transitions: Emergence and symmetry protection of gaplessness," *Phys. Rev. B* **108** no. 7, (2023) 075105, arXiv:2205.06244 [cond-mat.str-el].

[126] L. Bhardwaj and S. Schafer-Nameki, "Generalized charges, part I: Invertible symmetries and higher representations," *SciPost Phys.* **16** no. 4, (2024) 093, arXiv:2304.02660 [hep-th].

[127] L. Kong and I. Runkel, "Cardy algebras and sewing constraints. I.," *Commun. Math. Phys.* **292** (2009) 871–912, arXiv:0807.3356 [math.QA].

[128] A. Davydov, M. Müger, D. Nikshych, and V. Ostrik, "The witt group of non-degenerate braided fusion categories," *Journal für die reine und angewandte Mathematik (Crelles Journal)* **2013** no. 677, (2013) 135–177.

[129] L. Kong, "Anyon condensation and tensor categories," *Nucl. Phys. B* **886** (2014) 436–482, arXiv:1307.8244 [cond-mat.str-el].

[130] A. Davydov and D. Nikshych, "Braided picard groups and graded extensions of braided tensor categories," *Selecta Mathematica* **27** no. 4, (2021) 65.

[131] T. Johnson-Freyd and D. Reutter, "Minimal nondegenerate extensions," arXiv:2105.15167 [math.QA].

[132] J. Zhao, J.-Q. Lou, Z.-H. Zhang, L.-Y. Hung, L. Kong, and Y. Tian, "String Condensations in 3+1D and Lagrangian Algebras," arXiv:2208.07865 [cond-mat.str-el].

[133] J. A. Damia, R. Argurio, and E. Garcia-Valdecasas, "Non-invertible defects in 5d, boundaries and holography," *SciPost Phys.* **14** no. 4, (2023) 067, arXiv:2207.02831 [hep-th].

[134] F. Apruzzi, I. Bah, F. Bonetti, and S. Schafer-Nameki, "Noninvertible Symmetries from Holography and Branes," *Phys. Rev. Lett.* **130** no. 12, (2023) 121601, arXiv:2208.07373 [hep-th].

[135] S.-J. Huang and M. Cheng, "Topological holography, quantum criticality, and boundary states," arXiv:2310.16878 [cond-mat.str-el].

[136] C. Fechisin, N. Tantivasadakarn, and V. V. Albert, "Noninvertible Symmetry-Protected Topological Order in a Group-Based Cluster State," *Phys. Rev. X* **15** no. 1, (2025) 011058, arXiv:2312.09272 [cond-mat.str-el].

[137] R. Wen and A. C. Potter, "Classification of 1+1D gapless symmetry protected phases via topological holography," *Phys. Rev. B* **111** no. 11, (2025) 115161, arXiv:2311.00050 [cond-mat.str-el].

[138] T. Kennedy and H. Tasaki, "Hidden symmetry breaking and the Haldane phase inS=1 quantum spin chains," *Commun. Math. Phys.* **147** no. 3, (1992) 431–484.

[139] T. Kennedy and H. Tasaki, "Hidden Z2×Z2 symmetry breaking in Haldane-gap antiferromagnets," *Phys. Rev. B* **45** no. 1, (1992) 304.

[140] A. Chatterjee, W. Ji, and X.-G. Wen, "Emergent generalized symmetry and maximal symmetry-topological-order," arXiv:2212.14432 [cond-mat.str-el].

[141] L. Eck and P. Fendley, "From the XXZ chain to the integrable Rydberg-blockade ladder via non-invertible duality defects," *SciPost Phys.* **16** no. 5, (2024) 127, arXiv:2302.14081 [cond-mat.stat-mech].

[142] L. Eck and P. Fendley, "Critical lines and ordered phases in a rydberg-blockade ladder," *Physical Review B* **108** no. 12, (2023) 125135.

[143] T. Scaffidi, D. E. Parker, and R. Vasseur, "Gapless symmetry-protected topological order," *Physical Review X* **7** no. 4, (2017) 041048.

[144] R. Verresen, R. Thorngren, N. G. Jones, and F. Pollmann, "Gapless Topological Phases and Symmetry-Enriched Quantum Criticality," *Phys. Rev. X* **11** no. 4, (2021) 041059, arXiv:1905.06969 [cond-mat.str-el].

[145] R. Thorngren, A. Vishwanath, and R. Verresen, "Intrinsically gapless topological phases," *Phys. Rev. B* **104** no. 7, (2021) 075132, `arXiv:2008.06638 [cond-mat.str-el]`.

[146] R. Wen and A. C. Potter, "Bulk-boundary correspondence for intrinsically gapless symmetry-protected topological phases from group cohomology," *Phys. Rev. B* **107** no. 24, (2023) 245127, `arXiv:2208.09001 [cond-mat.str-el]`.

[147] L. Li, M. Oshikawa, and Y. Zheng, "Intrinsically/Purely Gapless-SPT from Non-Invertible Duality Transformations," `arXiv:2307.04788 [cond-mat.str-el]`.

[148] L. Li, M. Oshikawa, and Y. Zheng, "Decorated Defect Construction of Gapless-SPT States," `arXiv:2204.03131 [cond-mat.str-el]`.

[149] L. Bhardwaj, D. Pajer, S. Schafer-Nameki, and A. Warman, "Hasse Diagrams for Gapless SPT and SSB Phases with Non-Invertible Symmetries," `arXiv:2403.00905 [cond-mat.str-el]`.

[150] T.-C. Huang and Y.-H. Lin, "The F-Symbols for Transparent Haagerup-Izumi Categories with $G = \mathbb{Z}_{2n+1}$," `arXiv:2007.00670 [math.CT]`.

[151] T. J. Osborne, D. E. Stiegemann, and R. Wolf, "The F-symbols for the H_3 fusion category," *arXiv preprint arXiv:1906.01322* (2019) .

[152] U. Haagerup, "Principal graphs of subfactors in the index range $4 < [M : N] < 3 + \sqrt{2}$," *Subfactors (Proceedings of Taniguchi Symposium, Kyuzeso, 1993), pp. 1–38* **1** (1994) .

[153] M. Asaeda and U. Haagerup, "Exotic subfactors of finite depth with Jones indices $(5 + \sqrt{13})/2$ and $(5 + \sqrt{17})/2$," *Communications in mathematical physics* **202** no. 1, (1999) 1–63.

[154] P. Grossman and N. Snyder, "Quantum subgroups of the Haagerup fusion categories," *Communications in Mathematical Physics* **311** (2012) 617–643.

[155] M. Izumi, "The structure of sectors associated with Longo-Rehren inclusions. II: Examples," *Rev. Math. Phys.* **13** (2001) 603–674.

[156] S.-M. Hong, E. Rowell, and Z. Wang, "On exotic modular tensor categories," *Communications in Contemporary Mathematics* **10** no. supp01, (2008) 1049–1074.

[157] S.-H. Ng, E. C. Rowell, and X.-G. Wen, "Classification of modular data up to rank 11," arXiv:2308.09670 [math.QA].

[158] E. O'Brien, E. Vernier, and P. Fendley, ""Not-A", representation symmetry-protected topological, and Potts phases in an S3-invariant chain," *Physical Review B* **101** no. 23, (2020) 235108.

[159] R. S. K. Mong, D. J. Clarke, J. Alicea, N. H. Lindner, and P. Fendley, "Parafermionic conformal field theory on the lattice," *J. Phys. A* **47** no. 45, (2014) 452001, arXiv:1406.0846 [cond-mat.stat-mech].

[160] A. Warman, F. Yang, A. Tiwari, H. Pichler, and S. Schafer-Nameki, "Categorical Symmetries in Spin Models with Atom Arrays," arXiv:2412.15024 [cond-mat.str-el].

[161] K. Inamura, S.-J. Huang, A. Tiwari, and S. Schafer-Nameki, "(2+1)d Lattice Models and Tensor Networks for Gapped Phases with Categorical Symmetry," arXiv:2506.09177 [cond-mat.str-el].

[162] W. Ji and X.-G. Wen, "Non-invertible anomalies and mapping-class-group transformation of anomalous partition functions," *Phys. Rev. Research.* **1** (2019) 033054, arXiv:1905.13279 [cond-mat.str-el].