

# U(1)-invariant special Lagrangian 3-folds. I. Nonsingular solutions

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## 1 Introduction

Special Lagrangian submanifolds (SL  $m$ -folds) are a distinguished class of real  $m$ -dimensional minimal submanifolds in  $\mathbb{C}^m$ , which are calibrated with respect to the  $m$ -form  $\text{Re}(dz_1 \wedge \cdots \wedge dz_m)$ . They can also be defined in (almost) Calabi–Yau manifolds, are important in String Theory, and are expected to play a rôle in the eventual explanation of Mirror Symmetry between Calabi–Yau 3-folds.

This is the first of a suite of three papers [10, 11] studying special Lagrangian 3-folds  $N$  in  $\mathbb{C}^3$  invariant under the U(1)-action

$$e^{i\theta} : (z_1, z_2, z_3) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2, z_3) \quad \text{for } e^{i\theta} \in \text{U}(1). \quad (1)$$

These three papers and [12] are surveyed in [13]. Locally we can write  $N$  as

$$N = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : \begin{aligned} &\text{Im}(z_3) = u(\text{Re}(z_3), \text{Im}(z_1 z_2)), \\ &\text{Re}(z_1 z_2) = v(\text{Re}(z_3), \text{Im}(z_1 z_2)), \quad |z_1|^2 - |z_2|^2 = 2a \end{aligned} \right\}, \quad (2)$$

where  $a \in \mathbb{R}$  and  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$  are differentiable functions. It will be shown that  $N$  is a special Lagrangian 3-fold in  $\mathbb{C}^3$  if and only if  $u, v$  satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2(v^2 + y^2 + a^2)^{1/2} \frac{\partial u}{\partial y}. \quad (3)$$

In fact we have to modify this a bit to allow  $N$  to have singularities, which is one of the main things we are interested in. When  $a \neq 0$  it turns out that  $N$  is always nonsingular, and  $u, v$  are always smooth and satisfy (3) in the usual sense. However, when  $a = 0$ , at points  $(x, 0)$  with  $v(x, 0) = 0$  the factor  $-2(v^2 + y^2 + a^2)^{1/2}$  in (3) becomes zero, and then (3) is no longer elliptic.

Because of this, when  $a = 0$  the appropriate thing to do is to consider *weak solutions* of (3), which may have *singular points*  $(x, 0)$  with  $v(x, 0) = 0$ . At such a point  $u, v$  may not be differentiable, and  $(0, 0, x + iu(x, 0))$  is a singular point of the SL 3-fold  $N$  in  $\mathbb{C}^3$ . Weak solutions of (3) when  $a = 0$  and their singularities will be studied in the sequels [10, 11], and this paper will focus on the nonsingular case when  $a \neq 0$ .

We begin in §2 with an introduction to special Lagrangian geometry, and then §3 summarizes some background material from analysis that we will need later, to do with Hölder spaces of functions and elliptic operators. Section 4 considers special Lagrangian 3-folds invariant under the  $U(1)$ -action (1), shows that they can locally be written in the form (2) where  $u, v$  satisfy (3), and gives an explanation of why (3) is a *nonlinear Cauchy–Riemann equation* in terms of almost Calabi–Yau geometry. Examples of solutions  $u, v$  of (3) are given in §5, and the corresponding SL 3-folds  $N$  in  $\mathbb{C}^3$  described.

Section 6 exploits the fact that (3) is a nonlinear Cauchy–Riemann equation, and so  $u+iv$  is a bit like a holomorphic function of  $x+iy$ . We prove analogues for solutions  $u, v$  of (3) of well-known results in complex analysis, in particular those involving multiplicity of zeroes, and formulae counting zeroes of a holomorphic function in terms of winding numbers.

As an application we show that if  $S, T$  are domains in  $\mathbb{R}^2$  and  $(\hat{u}, \hat{v}) : S \rightarrow T$  are solutions of (3) such that  $\hat{u}, \hat{v}, \frac{\partial \hat{v}}{\partial x}$  and  $\frac{\partial \hat{v}}{\partial y}$  take given values at a point, then there do not exist  $(u, v) : T \rightarrow S^\circ$  satisfying (3) such that  $u, v, \frac{\partial v}{\partial x}$  and  $\frac{\partial v}{\partial y}$  take given values at a point. This will be used in [10] to prove a priori estimates for derivatives of bounded solutions  $u, v$  of (3) on domains in  $\mathbb{R}^2$ , and these in turn will be important in proving the existence of weak solutions of (3) when  $a = 0$ .

In §7 we show that if  $S$  is a domain in  $\mathbb{R}^2$  and  $u, v \in C^1(S)$  satisfy (3), then there exists  $f \in C^2(S)$  with  $\frac{\partial f}{\partial y} = u$  and  $\frac{\partial f}{\partial x} = v$ , unique up to addition of a constant, satisfying

$$\left( \left( \frac{\partial f}{\partial x} \right)^2 + y^2 + a^2 \right)^{-1/2} \frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial y^2} = 0. \quad (4)$$

This is a *second-order quasilinear elliptic equation*. Using results from analysis, we prove existence and uniqueness of solutions of the Dirichlet problem for (4) on strictly convex domains when  $a \neq 0$ . Combining this with the results of §4 gives existence and uniqueness results for nonsingular  $U(1)$ -invariant SL 3-folds in  $\mathbb{C}^3$  satisfying certain boundary conditions.

Section 8 takes a different approach to the same problem. We show that if  $S$  is a domain in  $\mathbb{R}^2$  and  $u, v \in C^2(S)$  satisfy (3), then  $v$  satisfies

$$\frac{\partial}{\partial x} \left[ (v^2 + y^2 + a^2)^{-1/2} \frac{\partial v}{\partial x} \right] + 2 \frac{\partial^2 v}{\partial y^2} = 0. \quad (5)$$

Again, this is a second-order quasilinear elliptic equation, and we can prove existence and uniqueness of solutions of the Dirichlet problem for (5) on domains in  $\mathbb{R}^2$  when  $a \neq 0$ . This gives existence and uniqueness results for nonsingular  $U(1)$ -invariant SL 3-folds in  $\mathbb{C}^3$  satisfying a different kind of boundary condition.

In the sequel [10] we first prove a priori estimates for  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$  and  $\frac{\partial v}{\partial y}$  when  $u, v$  are bounded solutions of (3) on a domain  $S$  in  $\mathbb{R}^2$ , and  $a \neq 0$ . Using these we generalize Theorems 7.6, 7.7, 8.8 and 8.9 below to the case  $a = 0$ , proving existence and uniqueness of *weak* solutions  $f \in C^1(S)$  and  $u, v \in C^0(S)$  to the Dirichlet problems for (4) and (5) on strictly convex domains when  $a = 0$ . This

gives existence and uniqueness results for *singular*  $U(1)$ -invariant SL 3-folds in  $\mathbb{C}^3$  satisfying certain boundary conditions.

The following paper [11] studies these singular solutions  $u, v$  of (3) when  $a = 0$  in more detail. We show that under mild conditions  $u, v$  have only isolated singularities, and these isolated singular points have a *multiplicity*, which is a positive integer, and one of two *types*. We also use our results to construct many *special Lagrangian fibrations* on open subsets of  $\mathbb{C}^3$ . In [12] these are used as local models to study special Lagrangian fibrations of (almost) Calabi–Yau 3-folds, and to draw some conclusions about the *SYZ Conjecture* [15]. All four papers are reviewed briefly in [13].

A fundamental question about compact special Lagrangian 3-folds  $N$  in (almost) Calabi–Yau 3-folds  $M$  is: *how stable are they under large deformations?* Here we mean both deformations of  $N$  in a fixed  $M$ , and what happens to  $N$  as we deform  $M$ . The deformation theory of compact SL 3-folds under *small* deformations is already well understood, and is described in [8, §9] and [9, §5]. But to extend this understanding to large deformations, one needs to take into account singular behaviour.

One possible moral of this paper and its sequels [10, 11] is that *compact SL 3-folds are pretty stable under large deformations*. That is, we have shown existence and uniqueness for (possibly singular)  $U(1)$ -invariant SL 3-folds in  $\mathbb{C}^3$  satisfying certain boundary conditions. This existence and uniqueness is *entirely unaffected* by singularities that develop in the SL 3-folds, which is quite surprising, as one might have expected that when singularities develop the existence and uniqueness properties would break down.

This is encouraging, as both the author’s programme for constructing invariants of almost Calabi–Yau 3-folds in [5] by counting special Lagrangian homology 3-spheres, and proving some version of the SYZ Conjecture [15] in anything other than a fairly weak, limiting form, will require strong stability properties of compact SL 3-folds under large deformations; so these papers may be taken as a small piece of evidence that these two projects may eventually be successful.

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## 2 Special Lagrangian geometry

We now introduce the idea of special Lagrangian submanifolds, in two different geometric contexts. First, in §2.1, we discuss special Lagrangian submanifolds in  $\mathbb{C}^m$ . Then §2.2 considers special Lagrangian submanifolds in *almost Calabi–Yau manifolds*, Kähler manifolds equipped with a holomorphic volume form which generalize the idea of Calabi–Yau manifolds. For an introduction to special Lagrangian geometry, see Harvey and Lawson [4] or the author [8, 9].

## 2.1 Special Lagrangian submanifolds in $\mathbb{C}^m$

We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson [4].

**Definition 2.1.** Let  $(M, g)$  be a Riemannian manifold. An *oriented tangent  $k$ -plane*  $V$  on  $M$  is a vector subspace  $V$  of some tangent space  $T_x M$  to  $M$  with  $\dim V = k$ , equipped with an orientation. If  $V$  is an oriented tangent  $k$ -plane on  $M$  then  $g|_V$  is a Euclidean metric on  $V$ , so combining  $g|_V$  with the orientation on  $V$  gives a natural *volume form*  $\text{vol}_V$  on  $V$ , which is a  $k$ -form on  $V$ .

Now let  $\varphi$  be a closed  $k$ -form on  $M$ . We say that  $\varphi$  is a *calibration* on  $M$  if for every oriented  $k$ -plane  $V$  on  $M$  we have  $\varphi|_V \leq \text{vol}_V$ . Here  $\varphi|_V = \alpha \cdot \text{vol}_V$  for some  $\alpha \in \mathbb{R}$ , and  $\varphi|_V \leq \text{vol}_V$  if  $\alpha \leq 1$ . Let  $N$  be an oriented submanifold of  $M$  with dimension  $k$ . Then each tangent space  $T_x N$  for  $x \in N$  is an oriented tangent  $k$ -plane. We say that  $N$  is a *calibrated submanifold* if  $\varphi|_{T_x N} = \text{vol}_{T_x N}$  for all  $x \in N$ .

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [4, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in  $\mathbb{C}^m$ , taken from [4, §III].

**Definition 2.2.** Let  $\mathbb{C}^m$  have complex coordinates  $(z_1, \dots, z_m)$ , and define a metric  $g$ , a real 2-form  $\omega$  and a complex  $m$ -form  $\Omega$  on  $\mathbb{C}^m$  by

$$g = |dz_1|^2 + \dots + |dz_m|^2, \quad \omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \dots + dz_m \wedge d\bar{z}_m), \quad (6)$$

and  $\Omega = dz_1 \wedge \dots \wedge dz_m$ .

Then  $\text{Re } \Omega$  and  $\text{Im } \Omega$  are real  $m$ -forms on  $\mathbb{C}^m$ . Let  $L$  be an oriented real submanifold of  $\mathbb{C}^m$  of real dimension  $m$ . We say that  $L$  is a *special Lagrangian submanifold* of  $\mathbb{C}^m$ , or *SL  $m$ -fold* for short, if  $L$  is calibrated with respect to  $\text{Re } \Omega$ , in the sense of Definition 2.1.

As in [5, 6] there is a more general definition of special Lagrangian  $m$ -fold involving a *phase*  $e^{i\theta}$ , but we will not use it here. Harvey and Lawson [4, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds.

**Proposition 2.3.** *Let  $L$  be a real  $m$ -dimensional submanifold of  $\mathbb{C}^m$ . Then  $L$  admits an orientation making it into a special Lagrangian submanifold of  $\mathbb{C}^m$  if and only if  $\omega|_L \equiv 0$  and  $\text{Im } \Omega|_L \equiv 0$ .*

An  $m$ -dimensional submanifold  $L$  in  $\mathbb{C}^m$  is called *Lagrangian* if  $\omega|_L \equiv 0$ . Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that  $\text{Im } \Omega|_L \equiv 0$ , which is how they get their name.

Next we give a result characterizing SL 3-planes  $\mathbb{R}^3$  in  $\mathbb{C}^3$ . Define an anti-bilinear cross product  $\times : \mathbb{C}^3 \times \mathbb{C}^3 \rightarrow \mathbb{C}^3$  by

$$(r_1, r_2, r_3) \times (s_1, s_2, s_3) = (\bar{r}_2 \bar{s}_3 - \bar{r}_3 \bar{s}_2, \bar{r}_3 \bar{s}_1 - \bar{r}_1 \bar{s}_3, \bar{r}_1 \bar{s}_2 - \bar{r}_2 \bar{s}_1). \quad (7)$$

It is equivariant under the  $\text{SU}(3)$ -action on  $\mathbb{C}^3$ . Using this notation, we prove

**Proposition 2.4.** *Let  $\mathbf{r}, \mathbf{s} \in \mathbb{C}^3$  be linearly independent over  $\mathbb{R}$ , with  $\omega(\mathbf{r}, \mathbf{s}) = 0$ . Then  $\mathbf{r}, \mathbf{s}$  and  $\mathbf{r} \times \mathbf{s}$  are linearly independent over  $\mathbb{R}$ , and  $\langle \mathbf{r}, \mathbf{s}, \mathbf{r} \times \mathbf{s} \rangle_{\mathbb{R}}$  is the unique special Lagrangian 3-plane in  $\mathbb{C}^3$  containing  $\langle \mathbf{r}, \mathbf{s} \rangle_{\mathbb{R}}$ .*

*Proof.* Explicit calculation using (7) shows that

$$g(\mathbf{r}, \mathbf{r} \times \mathbf{s}) = g(\mathbf{s}, \mathbf{r} \times \mathbf{s}) = 0, \quad (8)$$

$$\omega(\mathbf{r}, \mathbf{r} \times \mathbf{s}) = \omega(\mathbf{s}, \mathbf{r} \times \mathbf{s}) = 0, \quad (9)$$

$$|\mathbf{r} \times \mathbf{s}|^2 = |\mathbf{r}|^2 |\mathbf{s}|^2 - g(\mathbf{r}, \mathbf{s})^2 - \omega(\mathbf{r}, \mathbf{s})^2, \quad (10)$$

$$\text{and } (\text{Im } \Omega)(\mathbf{r}, \mathbf{s}, \mathbf{r} \times \mathbf{s}) = 0, \quad (11)$$

for all  $\mathbf{r}, \mathbf{s} \in \mathbb{C}^3$ . When  $\mathbf{r}, \mathbf{s}$  are linearly independent and  $\omega(\mathbf{r}, \mathbf{s}) = 0$ , equation (8) shows that  $\mathbf{r} \times \mathbf{s}$  is orthogonal to  $\mathbf{r}, \mathbf{s}$ , and (10) that  $|\mathbf{r} \times \mathbf{s}| \neq 0$ . Therefore  $\mathbf{r}, \mathbf{s}$  and  $\mathbf{r} \times \mathbf{s}$  are linearly independent.

Also we have  $\omega(\mathbf{r}, \mathbf{s}) = \omega(\mathbf{r}, \mathbf{r} \times \mathbf{s}) = \omega(\mathbf{s}, \mathbf{r} \times \mathbf{s}) = 0$  by (9), so that  $\langle \mathbf{r}, \mathbf{s}, \mathbf{r} \times \mathbf{s} \rangle_{\mathbb{R}}$  is a *Lagrangian* 3-plane. Then (11) shows that  $\langle \mathbf{r}, \mathbf{s}, \mathbf{r} \times \mathbf{s} \rangle_{\mathbb{R}}$  is a *special* Lagrangian 3-plane, by Proposition 2.3. It is easy to see that this is the only SL 3-plane in  $\mathbb{C}^3$  containing  $\langle \mathbf{r}, \mathbf{s} \rangle_{\mathbb{R}}$ .  $\square$

## 2.2 Almost Calabi–Yau $m$ -folds and SL $m$ -folds

We shall define special Lagrangian submanifolds not just in Calabi–Yau manifolds, as usual, but in the much larger class of *almost Calabi–Yau manifolds*.

**Definition 2.5.** Let  $m \geq 2$ . An *almost Calabi–Yau  $m$ -fold*, or *ACY  $m$ -fold* for short, is a quadruple  $(X, J, \omega, \Omega)$  such that  $(X, J)$  is a  $m$ -dimensional complex manifold,  $\omega$  is the Kähler form of a Kähler metric  $g$  on  $X$ , and  $\Omega$  is a non-vanishing holomorphic  $(m, 0)$ -form on  $X$ .

We call  $(X, J, \omega, \Omega)$  a *Calabi–Yau  $m$ -fold*, or *CY  $m$ -fold* for short, if in addition  $\omega$  and  $\Omega$  satisfy

$$\omega^m / m! = (-1)^{m(m-1)/2} (i/2)^m \Omega \wedge \bar{\Omega}. \quad (12)$$

Then for each  $x \in X$  there exists an isomorphism  $T_x X \cong \mathbb{C}^m$  that identifies  $g_x, \omega_x$  and  $\Omega_x$  with the flat versions  $g, \omega, \Omega$  on  $\mathbb{C}^m$  in (6). Furthermore,  $g$  is Ricci-flat and its holonomy group is a subgroup of  $\text{SU}(m)$ .

This is not the usual definition of a Calabi–Yau manifold, but is essentially equivalent to it. (Usually one also assumes that  $X$  is compact). Next, motivated by Proposition 2.3, we define special Lagrangian submanifolds of almost Calabi–Yau manifolds.

**Definition 2.6.** Let  $(X, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold with metric  $g$ , and  $N$  a real  $m$ -dimensional submanifold of  $X$ . We call  $N$  a *special Lagrangian submanifold*, or *SL  $m$ -fold* for short, if  $\omega|_N \equiv \text{Im } \Omega|_N \equiv 0$ .

The properties of SL  $m$ -folds in almost Calabi–Yau  $m$ -folds are discussed by the author in [8, 9]. The deformation and obstruction theory for *compact*

SL  $m$ -folds in almost Calabi–Yau  $m$ -folds is well understood, and beautifully behaved.

In this paper we will focus exclusively on special Lagrangian 3-folds in  $\mathbb{C}^3$ , and the more general almost Calabi–Yau context will hardly enter our story at all. However, because SL  $m$ -folds in ACY  $m$ -folds are expected to behave locally just like SL  $m$ -folds in  $\mathbb{C}^m$ , our results tell us about SL 3-folds in ACY 3-folds, especially their singular behaviour.

### 3 Background material from analysis

We now briefly summarize some background material we will need for later analytic results. Our principal reference is Gilbarg and Trudinger [1].

#### 3.1 Banach spaces of functions on subsets of $\mathbb{R}^n$

We first define a special class of subsets of  $\mathbb{R}^n$  called *domains*.

**Definition 3.1.** A closed, bounded, contractible subset  $S$  in  $\mathbb{R}^n$  will be called a *domain* if it is a disjoint union  $S = S^\circ \cup \partial S$ , where the *interior*  $S^\circ$  of  $S$  is a connected open set in  $\mathbb{R}^n$  with  $S = \overline{S^\circ}$ , and the *boundary*  $\partial S = S \setminus S^\circ$  is a compact embedded hypersurface in  $\mathbb{R}^n$ .

Here the assumption that  $S$  is contractible is made for simplicity, and will not always be necessary. Note that as they are contractible, domains in  $\mathbb{R}^2$  are automatically diffeomorphic to discs. Next we define some Banach spaces of real functions on  $S$ .

**Definition 3.2.** Let  $S$  be a domain in  $\mathbb{R}^n$ . For each integer  $k \geq 0$ , define  $C^k(S)$  to be the space of continuous functions  $f : S \rightarrow \mathbb{R}$  with  $k$  continuous derivatives, and define the norm  $\|\cdot\|_{C^k}$  on  $C^k(S)$  by  $\|f\|_{C^k} = \sum_{j=0}^k \sup_S |\partial^j f|$ . Then  $C^k(S)$  is a Banach space. Define  $C^\infty(S) = \bigcap_{k=0}^\infty C^k(S)$  to be the set of smooth functions on  $S$ . It is not a Banach space, with its natural topology.

Here  $\partial$  is the vector operator  $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ , where  $(x_1, \dots, x_n)$  are the standard coordinates on  $\mathbb{R}^n$ , so that  $\partial^j f$  maps  $S \rightarrow \bigotimes^k (\mathbb{R}^n)^*$ , and has components  $\frac{\partial^j f}{\partial x_{a_1} \dots \partial x_{a_j}}$  for  $1 \leq a_1, \dots, a_j \leq n$ . The lengths  $|\partial^j f|$  are computed using the standard Euclidean metric on  $\mathbb{R}^n$ .

**Definition 3.3.** For  $k \geq 0$  an integer and  $\alpha \in (0, 1]$ , define the *Hölder space*  $C^{k,\alpha}(S)$  to be the subset of  $f \in C^k(S)$  for which

$$[\partial^k f]_\alpha = \sup_{x \neq y \in S} \frac{|\partial^k f(x) - \partial^k f(y)|}{|x - y|^\alpha}$$

is finite, and define the *Hölder norm* on  $C^{k,\alpha}(S)$  to be  $\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + [\partial^k f]_\alpha$ . Again,  $C^{k,\alpha}(S)$  is a Banach space.

### 3.2 Linear and quasilinear elliptic operators

We begin by defining *second-order linear elliptic operators* on functions.

**Definition 3.4.** Let  $S$  be a domain in  $\mathbb{R}^n$ . A *second-order linear differential operator*  $P$  mapping  $C^{k+2}(S) \rightarrow C^k(S)$  or  $C^{k+2,\alpha}(S) \rightarrow C^{k,\alpha}(S)$  or  $C^\infty(S) \rightarrow C^\infty(S)$  is an operator of the form

$$(Pu)(x) = \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x), \quad (13)$$

where  $a^{ij}$ ,  $b^i$  and  $c$  lie in  $C^k(S)$ , or  $C^{k,\alpha}(S)$ , or  $C^\infty(S)$ , respectively, and  $a^{ij} = a^{ji}$  for all  $i, j = 1, \dots, n$ . We call  $a^{ij}$ ,  $b^i$  and  $c$  the *coefficients* of  $P$ , so that, for instance, we say  $P$  has  $C^{k,\alpha}$  coefficients if  $a^{ij}$ ,  $b^i$  and  $c$  lie in  $C^{k,\alpha}(S)$ . We call  $P$  *elliptic* if the symmetric  $n \times n$  matrix  $(a^{ij})$  is positive definite at every point of  $S$ .

There is a much more general definition of ellipticity for differential operators of other orders, or acting on vectors rather than functions, but we will not need it. One can also define ellipticity for *nonlinear* partial differential operators. We will not do this in general, but only for *quasilinear* differential operators, which are linear in their highest-order derivatives.

**Definition 3.5.** Let  $S$  be a domain in  $\mathbb{R}^n$ . A *second-order quasilinear operator*  $Q : C^2(S) \rightarrow C^0(S)$  is an operator of the form

$$(Qu)(x) = \sum_{i,j=1}^n a^{ij}(x, u, \partial u) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + b(x, u, \partial u), \quad (14)$$

where  $a^{ij}$  and  $b$  are continuous maps  $S \times \mathbb{R} \times (\mathbb{R}^n)^* \rightarrow \mathbb{R}$ , and  $a^{ij} = a^{ji}$  for all  $i, j = 1, \dots, n$ . We call the functions  $a^{ij}$  and  $b$  the *coefficients* of  $Q$ . We call  $Q$  *elliptic* if the symmetric  $n \times n$  matrix  $(a^{ij})$  is positive definite at every point of  $S \times \mathbb{R} \times (\mathbb{R}^n)^*$ .

Elliptic operators have good *regularity properties* in Hölder spaces.

**Theorem 3.6.** Let  $S$  be a domain in  $\mathbb{R}^n$  and  $Q : C^2(S) \rightarrow C^0(S)$  a second-order linear or quasilinear elliptic differential operator. Suppose that  $Qu = f$ , with  $u \in C^2(S)$  and  $f \in C^0(S)$ , and  $u|_{\partial S} = \phi$ , for  $\phi \in C^2(\partial S)$ . Then

- (a) Let  $k \geq 0$  and  $\alpha \in (0, 1)$ , and suppose that  $Q$  has  $C^{k,\alpha}$  coefficients,  $f \in C^{k,\alpha}(S)$ , and  $\phi \in C^{k+2,\alpha}(\partial S)$ . Then  $u \in C^{k+2,\alpha}(S)$ .
- (b) Suppose  $Q$  has smooth coefficients,  $f \in C^\infty(S)$ , and  $\phi \in C^\infty(\partial S)$ . Then  $u \in C^\infty(S)$ .
- (c) Suppose  $f$  and the coefficients of  $Q$  are real analytic in  $S^\circ$ . Then  $u$  is real analytic in  $S^\circ$ .

*Proof.* The linear case of part (a) follows from [1, Th. 6.19, p. 111]. For the quasilinear case, regarding  $u$  as fixed, write

$$Pv = \sum_{i,j=1}^n a^{ij}(x, u, \partial u) \frac{\partial^2 v}{\partial x_i \partial x_j}(x),$$

so that  $P$  is a *linear* elliptic operator. Applying the linear case of (a) to the equation  $Pu = f - b(x, u, \partial u)$ , we can deduce the quasilinear case by induction on  $k$ . Part (b) follows from (a), and part (c) from Morrey [14, §5.7–§5.8].  $\square$

Essentially the theorem says that solutions  $u$  of an elliptic equation  $Pu = f$  on  $S$  are as smooth as possible, given the differentiability of  $f$  and the boundary condition  $\phi$ . For linear elliptic operators  $P$  involving only the derivatives of  $u$  there is a *maximum principle* [1, Th. 3.1, p. 32]:

**Theorem 3.7.** *Let  $S$  be a domain in  $\mathbb{R}^n$  and  $P : C^2(S) \rightarrow C^0(S)$  a second-order linear elliptic differential operator of the form (13), with  $c(x) \equiv 0$ . Suppose  $u \in C^0(S) \cap C^2(S^\circ)$ . If  $Pu \geq 0$  in  $S^\circ$  then the maximum of  $u$  is achieved on  $\partial S$ , and if  $Pu \leq 0$  in  $S^\circ$  then the minimum of  $u$  is achieved on  $\partial S$ .*

### 3.3 Existence results for the Dirichlet problem

We shall now use results from Gilbarg and Trudinger [1] to prove existence results for the Dirichlet problem for two classes of quasilinear elliptic operators, that will be needed in §7 and §8. We begin by defining *strictly convex domains* in  $\mathbb{R}^2$ .

**Definition 3.8.** A domain  $S$  in  $\mathbb{R}^2$  is called *strictly convex* if  $S$  is convex and the curvature of  $\partial S$  is nonzero at every point. So, for example,  $x^2 + y^2 \leq 1$  is strictly convex but  $x^4 + y^4 \leq 1$  is not, as its boundary has zero curvature at  $(\pm 1, 0)$  and  $(0, \pm 1)$ .

Here is our first existence result.

**Theorem 3.9.** *Let  $S$  be a strictly convex domain in  $\mathbb{R}^2$ , and suppose*

$$(Pf)(x) = \sum_{i,j=1}^2 a^{ij}(x, f, \partial f) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \quad (15)$$

*is a second-order quasilinear elliptic operator in  $S$  with  $a^{ij} \in C^\infty(S \times \mathbb{R} \times \mathbb{R}^2)$ . Then whenever  $k \geq 0$ ,  $\alpha \in (0, 1)$  and  $\phi \in C^{k+2, \alpha}(\partial S)$  there exists a solution  $f \in C^{k+2, \alpha}(S)$  of the Dirichlet problem  $Pf = 0$  in  $S$ ,  $f|_{\partial S} = \phi$ . Furthermore  $\|f\|_{C^1} \leq C\|\phi\|_{C^2}$  for some  $C > 0$  depending only on  $S$ .*

*Proof.* It is not difficult to show that as  $S$  is strictly convex there exists  $K > 0$  depending only on  $S$ , such that if  $\phi \in C^2(\partial S)$  then any three distinct points in the graph of  $\phi$  in  $\partial S \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R}$  lie in a unique plane in  $\mathbb{R}^2 \times \mathbb{R}$  with slope



$\mathbf{s} \in (\mathbb{R}^2)^*$  satisfying  $|\mathbf{s}| \leq K\|\phi\|_{C^2}$ . In the notation of [1, p. 310], the boundary data  $\partial S, \phi$  satisfies a *three point condition*.

Now (noting the equivalence of the three point and bounded slope conditions, [1, p. 314]), [1, Th. 12.7, p. 312] is an existence result for the Dirichlet problem for an operator of the form (15) with boundary data satisfying a three point condition. Strengthened as in [1, Remark (4), p. 314], it implies that if  $\phi \in C^{2,\alpha}(\partial S)$  then there exists  $f \in C^{2,\alpha}(S)$  with  $Pf = 0$  in  $S$  and  $f|_{\partial S} = \phi$ , which satisfies  $\|\partial f\|_{C^0} \leq K\|\phi\|_{C^2}$ .

By the maximum principle, Theorem 3.7, the maximum of  $f$  is achieved on  $\partial S$ . Thus  $\|f\|_{C^0} = \|\phi\|_{C^0} \leq \|\phi\|_{C^2}$ . Hence

$$\|f\|_{C^1} = \|f\|_{C^0} + \|\partial f\|_{C^0} \leq (1 + K)\|\phi\|_{C^2} = C\|\phi\|_{C^2},$$

where  $C = 1 + K$  depends only on  $S$ . This establishes the case  $k = 0$  of the theorem. If  $\phi \in C^{k+2,\alpha}(S)$  for  $k > 0$  then  $\phi \in C^{2,\alpha}(S)$ , so by the  $k = 0$  case there exists  $f \in C^{2,\alpha}(S)$  with  $Pf = 0$  and  $f|_{\partial S} = \phi$ . But then Theorem 3.6 shows that  $f \in C^{k+2,\alpha}(S)$ , and the proof is complete.  $\square$

Combining [1, Th. 15.12, p. 382] and Theorem 3.6 gives:

**Theorem 3.10.** *Let  $S$  be a domain in  $\mathbb{R}^n$ , and suppose the quasilinear operator*

$$(Qv)(x) = \sum_{i,j=1}^n a^{ij}(x, v) \frac{\partial^2 v}{\partial x_i \partial x_j}(x) + b(x, v, \partial v) \quad (16)$$

*is elliptic in  $S$  with coefficients  $a^{ij} \in C^\infty(S \times \mathbb{R})$  and  $b \in C^\infty(S \times \mathbb{R} \times \mathbb{R}^n)$  satisfying  $|b(x, v, p)| \leq C|p|^2$  and  $v b(x, v, p) \leq 0$  for all  $(x, v, p) \in S \times \mathbb{R} \times \mathbb{R}^n$  and some  $C > 0$ . Then whenever  $k \geq 0$ ,  $\alpha \in (0, 1)$  and  $\phi \in C^{k+2,\alpha}(\partial S)$  there exists a solution  $v \in C^{k+2,\alpha}(S)$  of the Dirichlet problem  $Qv = 0$  in  $S$ ,  $v|_{\partial S} = \phi$ .*

Note that in both theorems,  $Q$  is not a general second-order quasilinear elliptic operator of the form (14), but has some restrictions on its structure. In particular, (15) has  $n = 2$  and no term  $b(x, f, \partial f)$ , and in (16) the  $a^{ij}$  depend on  $x$  and  $v$  but not on  $\partial v$ , and the sign of  $b$  is restricted.

## 4 A class of $U(1)$ -invariant SL 3-folds in $\mathbb{C}^3$

We will now study special Lagrangian 3-folds  $N$  in  $\mathbb{C}^3$  invariant under the  $U(1)$ -action

$$e^{i\theta} : (z_1, z_2, z_3) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2, z_3) \quad \text{for } e^{i\theta} \in U(1). \quad (17)$$

We shall assume that  $N$  may be written

$$N = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : \begin{aligned} &\text{Im}(z_3) = u(\text{Re}(z_3), \text{Im}(z_1 z_2)), \\ &\text{Re}(z_1 z_2) = v(\text{Re}(z_3), \text{Im}(z_1 z_2)), \quad |z_1|^2 - |z_2|^2 = 2a \end{aligned} \right\}, \quad (18)$$

where  $a \in \mathbb{R}$  and  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$  are continuous functions, which are smooth except perhaps at certain singular points. Here is why we choose to write  $N$  in this form. As the functions  $\operatorname{Re}(z_1 z_2)$ ,  $\operatorname{Im}(z_1 z_2)$ ,  $|z_1|^2 - |z_2|^2$ ,  $\operatorname{Re}(z_3)$  and  $\operatorname{Im}(z_3)$  involved in (18) are  $U(1)$ -invariant,  $N$  is automatically  $U(1)$ -invariant.

Also, as in [6, Prop. 4.2], if  $N$  is a connected Lagrangian submanifold of  $\mathbb{C}^m$  invariant under a Lie subgroup  $G$  of the automorphism group  $U(m) \ltimes \mathbb{C}^m$  of  $\mathbb{C}^m$ , then the moment map  $\mu$  of  $G$  is constant on  $N$ . Now the moment map of the  $U(1)$ -action (17) is  $|z_1|^2 - |z_2|^2$ . Thus  $|z_1|^2 - |z_2|^2 = 2a$  for some  $a \in \mathbb{R}$  on any  $U(1)$ -invariant SL 3-fold  $N$  in  $\mathbb{C}^3$ , which is why we have taken  $|z_1|^2 - |z_2|^2 = 2a$  to be one of the equations defining  $N$ .

In the other two equations  $\operatorname{Re}(z_1 z_2) = v(\operatorname{Re}(z_3), \operatorname{Im}(z_1 z_2))$  and  $\operatorname{Im}(z_3) = u(\operatorname{Re}(z_3), \operatorname{Im}(z_1 z_2))$ , what we are doing is regarding the functions  $x = \operatorname{Re}(z_3)$  and  $y = \operatorname{Im}(z_1 z_2)$  as *coordinates* on  $N/U(1)$ , and expressing the other two degrees of freedom  $\operatorname{Re}(z_1 z_2)$  and  $\operatorname{Im}(z_3)$  as functions of  $x$  and  $y$ . Thus we define  $N$  as a kind of graph of the pair of functions  $(u, v)$ .

Note that not every  $U(1)$ -invariant SL 3-fold  $N$  in  $\mathbb{C}^3$  may be written in the form (18). Locally this is generally possible, but globally the functions  $u$  and  $v$  would have to be multi-valued, branched covers of  $\mathbb{R}^2$  for instance. However, we will see that the class of SL 3-folds of this form do have many nice properties, and are interesting both in themselves and for our later applications. So equation (18) should be regarded as more than just an arbitrary choice of coordinate system.

#### 4.1 Finding the equations on $u$ and $v$

We now calculate the conditions on the functions  $u(x, y)$ ,  $v(x, y)$  for the 3-fold  $N$  of (18) to be special Lagrangian.

**Proposition 4.1.** *Let  $S$  be a domain in  $\mathbb{R}^2$  or  $S = \mathbb{R}^2$ , let  $u, v : S \rightarrow \mathbb{R}$  be continuous, and  $a \in \mathbb{R}$ . Define*

$$N = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v(x, y) + iy, \quad z_3 = x + iu(x, y), \quad |z_1|^2 - |z_2|^2 = 2a, \quad (x, y) \in S\}. \quad (19)$$

Then

- (a) *If  $a = 0$ , then  $N$  is a (possibly singular) special Lagrangian 3-fold in  $\mathbb{C}^3$ , with boundary over  $\partial S$ , if  $u, v$  are differentiable and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2(v^2 + y^2)^{1/2} \frac{\partial u}{\partial y}, \quad (20)$$

*except at points  $(x, 0)$  in  $S$  with  $v(x, 0) = 0$ , where  $u, v$  need not be differentiable. The singular points of  $N$  are those of the form  $(0, 0, z_3)$ , where  $z_3 = x + iu(x, 0)$  for  $x \in \mathbb{R}$  with  $v(x, 0) = 0$ .*

- (b) *If  $a \neq 0$ , then  $N$  is a nonsingular SL 3-fold in  $\mathbb{C}^3$ , with boundary over  $\partial S$ , if and only if  $u, v$  are differentiable on all of  $S$  and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2(v^2 + y^2 + a^2)^{1/2} \frac{\partial u}{\partial y}. \quad (21)$$

*Proof.* We shall give the proof for part (a). Part (b) is similar but more complicated, and will be left to the reader. Let  $a = 0$ , let  $N$  be defined by (19), and let  $\mathbf{z} = (z_1, z_2, z_3) \in N$ . For  $\mathbf{z}$  to be a nonsingular point of  $N$ , we need  $u$  and  $v$  to be differentiable at  $(x, y) = (\operatorname{Re}(z_3), \operatorname{Im}(z_1 z_2))$  in  $S$ , and for the derivatives of the three functions

$$\operatorname{Re}(z_1 z_2) - v(\operatorname{Re}(z_3), \operatorname{Im}(z_1 z_2)), \quad \operatorname{Im}(z_3) - u(\operatorname{Re}(z_3), \operatorname{Im}(z_1 z_2)), \quad |z_1|^2 - |z_2|^2$$

on  $\mathbb{C}^3$  to be linearly independent at  $\mathbf{z}$ .

Now if  $z_1 = z_2 = 0$  then  $|z_1|^2 - |z_2|^2$  has zero derivative at  $\mathbf{z}$ . Thus points of the form  $(0, 0, z_3)$  in  $N$  will be singular. Clearly, these occur exactly when  $z_3 = x + iu(x, 0)$  for  $x \in \mathbb{R}$  with  $v(x, 0) = 0$ . Also, as  $|z_1|^2 - |z_2|^2 = 0$ , such points occur in  $N$  only when  $a = 0$ . We shall see that these are the only singular points in  $N$ , provided  $u$  and  $v$  are differentiable.

To prove part (a) we need to show that each  $\mathbf{z} \in N$  not of the form  $(0, 0, z_3)$  is a nonsingular point of  $N$ , and the tangent space  $T_{\mathbf{z}}N$  is a special Lagrangian 3-plane  $\mathbb{R}^3$  in  $\mathbb{C}^3$ . As  $N$  is  $U(1)$ -invariant, it is enough to prove this for one point in each orbit of the  $U(1)$ -action (17). Since  $|z_1| = |z_2|$  on  $N$ , each  $U(1)$ -orbit in  $N$  contains one or two points  $(z_1, z_2, z_3)$  with  $z_1 = z_2$ .

Thus it is enough to show that  $T_{\mathbf{z}}N$  exists and is special Lagrangian for points  $\mathbf{z} = (z_1, z_1, z_3)$  in  $N$  with  $z_1 \neq 0$ . In our next lemma we identify  $T_{\mathbf{z}}N$  at such a point. The proof is elementary, and is left as an exercise.

**Lemma 4.2.** *Let  $\mathbf{z} = (z_1, z_1, z_3) \in N$ , with  $z_1 \neq 0$ . Set  $x = \operatorname{Re}(z_3)$  and  $y = \operatorname{Im}(z_1^2)$ . Then  $N$  is nonsingular at  $\mathbf{z}$ , and  $T_{\mathbf{z}}N = \langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \rangle_{\mathbb{R}}$ , where*

$$\mathbf{p}_1 = (iz_1, -iz_1, 0), \quad (22)$$

$$\mathbf{p}_2 = ((2z_1)^{-1} \frac{\partial v}{\partial x}(x, y), (2z_1)^{-1} \frac{\partial v}{\partial x}(x, y), 1 + i \frac{\partial u}{\partial x}(x, y)) \quad \text{and} \quad (23)$$

$$\mathbf{p}_3 = ((2z_1)^{-1} (\frac{\partial v}{\partial y}(x, y) + i), (2z_1)^{-1} (\frac{\partial v}{\partial y}(x, y) + i), i \frac{\partial u}{\partial y}(x, y)). \quad (24)$$

Now define  $\times : \mathbb{C}^3 \times \mathbb{C}^3 \rightarrow \mathbb{C}^3$  as in (7), and apply Proposition 2.4 with  $\mathbf{r} = \mathbf{p}_1$  and  $\mathbf{s} = \mathbf{p}_2$ . Clearly  $\mathbf{p}_1$  and  $\mathbf{p}_2$  are linearly independent, and  $\omega(\mathbf{p}_1, \mathbf{p}_2) = 0$ . So Proposition 2.4 shows that  $\langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1 \times \mathbf{p}_2 \rangle_{\mathbb{R}}$  is the unique SL 3-plane in  $\mathbb{C}^3$  containing  $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle_{\mathbb{R}}$ .

Therefore  $\langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \rangle_{\mathbb{R}}$  is an SL 3-plane if and only if  $\mathbf{p}_3 \in \langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1 \times \mathbf{p}_2 \rangle_{\mathbb{R}}$ . Combining equations (7), (22) and (23) gives

$$\mathbf{p}_1 \times \mathbf{p}_2 = (\bar{z}_1 (\frac{\partial u}{\partial x} + i), \bar{z}_1 (\frac{\partial u}{\partial x} + i), -i \frac{\partial v}{\partial x}). \quad (25)$$

So suppose  $\mathbf{p}_3 = \alpha \mathbf{p}_1 + \beta \mathbf{p}_2 + \gamma \mathbf{p}_1 \times \mathbf{p}_2$ . As the first two coordinates are equal in  $\mathbf{p}_2, \mathbf{p}_3$  and  $\mathbf{p}_1 \times \mathbf{p}_2$  but not in  $\mathbf{p}_1$ , we see that  $\alpha = 0$ . Taking real parts in the third coordinate gives  $\beta = 0$ . And comparing real multiples of  $i\bar{z}_1$  in the first coordinate shows that  $\gamma = \frac{1}{2}|z_1|^{-2}$ .

Thus  $T_{\mathbf{z}}N$  is special Lagrangian if and only if  $\mathbf{p}_1 \times \mathbf{p}_2 = 2|z_1|^2 \mathbf{p}_3$ . By (24) and (25), this reduces to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2|z_1|^2 \frac{\partial u}{\partial y} \quad \text{at } (x, y). \quad (26)$$

But  $v = \operatorname{Re}(z_1^2)$  and  $y = \operatorname{Im}(z_1^2)$  by (19), so that  $|z_1|^4 = v^2 + y^2$ , and  $|z_1|^2 = (v^2 + y^2)^{1/2}$ . Substituting this into (26) gives equation (20), which proves part (a) of Proposition 4.1. Part (b) is left to the reader.  $\square$

Equations (20) and (21) are *nonlinear versions of the Cauchy–Riemann equations*. For if we replace the factors  $2(v^2 + y^2)^{1/2}$  and  $2(v^2 + y^2 + a^2)^{1/2}$  in (20) and (21) by 1, the equations become

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

which are the conditions for  $u + iv$  to be a holomorphic function of  $x + iy$ . We may therefore expect the solutions of (20) and (21) to have qualitative features in common with solutions of the Cauchy–Riemann equations.

**Proposition 4.3.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and suppose  $u, v \in C^1(S)$  satisfy (21). Then  $u, v$  are real analytic in  $S^\circ$ , and satisfy*

$$\frac{\partial^2 u}{\partial x^2} + 2(v^2 + y^2 + a^2)^{1/2} \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial}{\partial y} \left[ (v^2 + y^2 + a^2)^{1/2} \right] \frac{\partial u}{\partial y} = 0 \quad \text{and} \quad (27)$$

$$(v^2 + y^2 + a^2)^{-1/2} \frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial x} \left[ (v^2 + y^2 + a^2)^{-1/2} \right] \frac{\partial v}{\partial x} = 0 \quad \text{in } S^\circ. \quad (28)$$

*Proof.* One can show that  $u, v$  are real analytic in  $S^\circ$  following Harvey and Lawson [4, Th. III.2.7]. Thus  $v$  is twice continuously differentiable, so that  $\frac{\partial}{\partial x} \left[ \frac{\partial v}{\partial y} \right] = \frac{\partial}{\partial y} \left[ \frac{\partial v}{\partial x} \right]$  in  $S^\circ$ . Using (21) to substitute for  $\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}$  in terms of  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$  gives (27). Equation (28) follows in the same way.  $\square$

Regarding the factors  $(v^2 + y^2 + a^2)^{\pm 1/2}$  as part of the coefficients  $a^{ij}(x), b^i(x)$ , we see that (27) and (28) are second-order linear elliptic equations in  $u$  and  $v$  respectively, of the form (13), with  $c(x) \equiv 0$ . Therefore by the maximum principle, Theorem 3.7, we have:

**Corollary 4.4.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and suppose  $u, v \in C^1(S)$  satisfy (21). Then the maxima and minima of  $u$  and  $v$  are achieved on  $\partial S$ .*

## 4.2 Interpretation using Kähler quotients

We can use an idea due independently to Goldstein [2, §2] and Gross [3, §1] to interpret some features of the above construction. Let  $(X, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold, as in §2.2, and  $G$  a  $k$ -dimensional Lie group acting on  $X$  preserving  $J, \omega, \Omega$ , with Lie algebra  $\mathfrak{g}$ . Suppose the  $G$ -action admits a moment map  $\mu : X \rightarrow \mathfrak{g}^*$ .

Then for each  $c \in Z(\mathfrak{g}^*)$ , the quotient  $M_c = \mu^{-1}(c)/G$  is nonsingular wherever  $G$  acts freely, and has the structure of an almost Calabi–Yau  $(m-k)$ -fold on its nonsingular part. If  $N$  is a connected,  $G$ -invariant SL  $m$ -fold in  $X$ , then  $N \subset \mu^{-1}(c)$  for some  $c \in Z(\mathfrak{g}^*)$ , and  $L = N/G$  is an SL  $(m-k)$ -fold in  $M_c$ .

Conversely, if  $L$  is an SL  $(m-k)$ -fold in  $M_c$  then  $L$  pulls back to an SL  $m$ -fold  $N$  in  $X$ , contained in  $\mu^{-1}(c)$ .

In our case,  $X$  is  $\mathbb{C}^3$  and  $G$  is  $U(1)$ , acting as in (17). Any  $U(1)$ -invariant SL 3-fold  $N$  in  $\mathbb{C}^3$  lies in  $\mu^{-1}(2a)$  for some  $a \in \mathbb{R}$ , where  $\mu(z_1, z_2, z_3) = |z_1|^2 - |z_2|^2$ , and pushes down to an SL 2-fold in  $M_a = \mu^{-1}(2a)/U(1)$ .

Now SL 2-folds in an almost Calabi–Yau 2-fold  $(M, I, \omega, \Omega)$  are the same thing as *pseudoholomorphic curves* in  $M$  with respect to an alternative almost complex structure  $J$  depending on  $I, \omega$  and  $\Omega$ . Thus, finding  $U(1)$ -invariant SL 3-folds  $N$  in  $\mathbb{C}^3$  is equivalent to finding pseudoholomorphic curves  $\Sigma$  in a family of almost complex 2-folds  $M_a$ .

Therefore, it is not surprising that (20) and (21) are nonlinear versions of the Cauchy–Riemann equations. However, this almost complex point of view is not that helpful in understanding the *singular points* of  $N$ , which occur when  $a = v = y = 0$ . For the  $U(1)$ -action on  $\mu^{-1}(0)$  is not free, and thus  $M_0 = \mu^{-1}(0)/U(1)$  is a *singular* almost complex 2-fold.

So the problem is not one of studying singular pseudoholomorphic curves in a nonsingular almost complex 2-fold, which are already very well understood, but of studying pseudoholomorphic curves in a singular almost complex 2-fold, where the almost complex structure itself has unpleasant, non-isolated singularities, which are not at all like the singularities of complex manifolds.

## 5 Examples

By starting with known examples  $N$  of SL 3-folds in  $\mathbb{C}^3$  invariant under the  $U(1)$ -action (17) and solving (18) for  $u$  and  $v$ , we can construct examples of solutions  $u, v$  to equations (20) and (21).

We shall do this with a family of explicit SL 3-folds in  $\mathbb{C}^3$  written down by Harvey and Lawson [4, §III.3.A], and studied in more detail by the author [5, §3]. Let  $a \geq 0$ . Define a subset  $N_a$  in  $\mathbb{C}^3$  by

$$N_a = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - 2a = |z_2|^2 = |z_3|^2, \right. \\ \left. \operatorname{Im}(z_1 z_2 z_3) = 0, \quad \operatorname{Re}(z_1 z_2 z_3) \geq 0 \right\}. \quad (29)$$

By [4, §III.3.A] and [5, §3],  $N_a$  is a nonsingular SL 3-fold diffeomorphic to  $\mathcal{S}^1 \times \mathbb{R}^2$  when  $a > 0$ , and  $N_0$  is an SL  $T^2$ -cone with one singular point at  $(0, 0, 0)$ . We shall show that these SL 3-folds can be written in the form (18).

**Theorem 5.1.** *Let  $a \geq 0$ . Then there exist unique  $u_a, v_a : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that*

$$N = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : \operatorname{Im}(z_3) = u_a(\operatorname{Re}(z_3), \operatorname{Im}(z_1 z_2)), \right. \\ \left. \operatorname{Re}(z_1 z_2) = v_a(\operatorname{Re}(z_3), \operatorname{Im}(z_1 z_2)), \quad |z_1|^2 - |z_2|^2 = 2a \right\} \quad (30)$$

*is the special Lagrangian 3-fold  $N_a$  of (29). Furthermore:*

- (a)  $u_a, v_a$  are smooth on  $\mathbb{R}^2$  and satisfy (21), except at  $(0,0)$  when  $a = 0$ , where they are only continuous.
- (b)  $u_a(x, y) < 0$  when  $y > 0$  for all  $x$ , and  $u_a(x, 0) = 0$  for all  $x$ , and  $u_a(x, y) > 0$  when  $y < 0$  for all  $x$ .
- (c)  $v_a(x, y) > 0$  when  $x > 0$  for all  $y$ , and  $v_a(0, y) = 0$  for all  $y$ , and  $v_a(x, y) < 0$  when  $x < 0$  for all  $y$ .
- (d)  $u_a(0, y) = -y(|a| + \sqrt{y^2 + a^2})^{-1/2}$  for all  $y$ .
- (e)  $v_a(x, 0) = x(x^2 + 2|a|)^{1/2}$  for all  $x$ .

*Proof.* For simplicity, we first consider the case  $a = 0$ . Let  $N_0$  be as in (29), let  $(z_1, z_2, z_3) \in N_0$ , and set

$$x = \operatorname{Re}(z_3), \quad y = \operatorname{Im}(z_1 z_2), \quad u = \operatorname{Im} z_3 \quad \text{and} \quad v = \operatorname{Re}(z_1 z_2). \quad (31)$$

Then  $z_3 = x + iu$ , and  $z_1 z_2 = v + iy$ . Thus the first condition  $|z_1|^2 = |z_2|^2 = |z_3|^2$  in (29) becomes

$$|z_1|^2 = |z_2|^2 = x^2 + u^2.$$

Squaring gives  $|z_1 z_2|^2 = (x^2 + u^2)^2$ , so substituting for  $z_1 z_2$  yields

$$v^2 + y^2 = (x^2 + u^2)^2. \quad (32)$$

Similarly, using the expressions for  $z_1 z_2$  and  $z_3$  above, the second and third conditions on  $(z_1, z_2, z_3)$  in (29) become

$$vu + yx = 0 \quad \text{and} \quad vx - yu \geq 0. \quad (33)$$

We will use equations (32) and (33) to prove parts (b) and (c) of the theorem. First suppose  $y = 0$ . Then (33) gives  $vu = 0$ , so  $v = 0$  or  $u = 0$ . If  $v = 0$  then (32) gives  $x^2 + u^2 = 0$ , so  $x = u = 0$ . Thus  $y = 0$  implies  $u = 0$ . Similarly  $u = 0$  implies  $y = 0$ , so  $u = 0$  if and only if  $y = 0$ , as in part (b). In the same way  $v = 0$  if and only if  $x = 0$ , as in part (c).

We claim that the two terms  $vx$  and  $-yu$  in (33) are both nonnegative. If one is zero this is obvious. So suppose both are nonzero, so that  $x, y, u$  and  $v$  are all nonzero. From (33), the signs of three of these terms determine the sign of the fourth. It is easy to verify that for all eight sign possibilities,  $vx$  and  $-yu$  have the same sign. So both are nonnegative by (33). Hence  $yu \leq 0$ , and  $u = 0$  if and only if  $y = 0$ . Clearly, this proves part (b). Part (c) follows in the same way.

Next we shall show that for each pair  $(x, y)$ , there is exactly one pair  $(u, v)$  satisfying (32) and (33). Multiplying (32) by  $u^2$  and replacing  $v^2 u^2$  by  $y^2 x^2$  using (33), we get  $u^6 + 2x^2 u^4 + (x^2 - y^2)u^2 - y^2 x^2 = 0$ . This is a sextic in  $u$ , independent of  $v$ . Putting  $\alpha = u^2$ , it becomes

$$P(\alpha) = \alpha^3 + 2x^2 \alpha^2 + (x^2 - y^2)\alpha - y^2 x^2 = 0.$$

Thus  $u^2$  is a real, nonnegative root of the cubic  $P$ . Divide into cases

- (i)  $x \neq 0, y \neq 0$  and  $P$  has three real roots  $\gamma_1, \gamma_2, \gamma_3$ , not necessarily distinct;
- (ii)  $x \neq 0, y \neq 0$  and  $P$  has one real root  $\gamma$  and a complex conjugate pair of non-real roots  $\delta, \bar{\delta}$ ;
- (iii)  $y = 0$ ; and (iv)  $x = 0$  and  $y \neq 0$ .

We shall show that in cases (i)–(iii), the cubic  $P$  has exactly one real nonnegative root, giving a unique value of  $u^2$ . In case (iv) there are two nonnegative roots, but one can be excluded.

In case (i) we have  $\gamma_1 + \gamma_2 + \gamma_3 = -2x^2 < 0$ , so at least one  $\gamma_j$  is negative. But  $\gamma_1\gamma_2\gamma_3 = y^2x^2 > 0$ , so an even number of  $\gamma_j$  are negative and an odd number positive. The only possibility is that one  $\gamma_j$  is positive and two negative. So  $P$  has exactly one nonnegative root. In case (ii) we have  $\gamma|\delta|^2 = y^2x^2 > 0$ , proving that  $\gamma > 0$ , so  $P$  has exactly one nonnegative root. In case (iii) we have  $P(\alpha) = \alpha(\alpha + x^2)^2$ , with roots 0 and  $-x^2$  (twice), so the only nonnegative root is 0.

In case (iv) we have  $P(\alpha) = \alpha^3 - y^2\alpha$ , with roots  $y, 0$  and  $-y$ . Thus there are two nonnegative roots,  $|y|$  and 0. However, if  $\alpha = 0$  then  $u^2 = 0$ , and  $x^2 = 0$  by assumption, so the right hand side of (32) is zero. But  $y \neq 0$ , so the left hand side is positive, a contradiction. Hence  $\alpha \neq 0$ , and there is one allowable value for  $\alpha$ , which is  $|y|$ .

We have shown that (32) and (33) determine  $u^2$  uniquely, and that there is a solution  $u^2$  for all  $x, y$ . This yields  $u$  up to sign. But part (b) gives the sign of  $u$ , so  $u$  is determined uniquely. If  $u \neq 0$ , equation (33) determines  $v$ . If  $u = 0$  then  $y = 0$  by (b), so (32) gives  $v^2 = x^2$ , and  $v = \pm x$ . The sign of  $v$  is given by (c). Therefore for all pairs  $x, y$ , there are unique solutions  $u, v$  to (32) and (33).

Let us review what we have proved so far. If  $(z_1, z_2, z_3) \in N_0$  and  $x, y, u, v$  are defined by (31), then they satisfy (32) and (33). Also, given any  $x, y$  there exist unique  $u, v$  satisfying (32) and (33). So, putting  $u_0(x, y) = u$  and  $v_0(x, y) = v$  defines the functions  $u_0, v_0$  in the theorem uniquely, and then  $N_0$  is a subset of the 3-fold  $N$  of (30). The converse, that  $N \subseteq N_0$ , follows easily by reversing the argument above, since if  $(z_1, z_2, z_3) \in N$  then (32) and (33) are equivalent to the equations defining  $N_0$ . Hence  $N = N_0$ .

It remains to prove parts (a), (d) and (e). The smoothness in (a) follows directly from (32) and (33), or indirectly from the fact that  $N_0$  is smooth except at  $(0, 0, 0)$ , and  $u_0, v_0$  satisfy (21) where they are smooth by Proposition 4.1. For part (d), set  $x = 0$ . Then  $v = 0$  by (c), so (32) gives  $u^4 = y^2$ . So  $u_0(0, y) = \pm|y|^{1/2}$ , and the sign is determined by (b). Part (e) follows in the same way. This completes the proof for  $a = 0$ .

When  $a \neq 0$ , equation (32) must be replaced by

$$v^2 + y^2 = (x^2 + u^2)(x^2 + u^2 + 2|a|),$$

but the rest of the proof is more-or-less unchanged. □

Here are some remarks on the theorem.

- Let  $a > 0$ . As (21) depends only on  $a^2$ , the functions  $u_a, v_a$  also solve (21) with  $a$  replaced by  $-a$ . The corresponding SL 3-fold is

$$N_{-a} = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 - 2a = |z_3|^2, \right. \\ \left. \operatorname{Im}(z_1 z_2 z_3) = 0, \quad \operatorname{Re}(z_1 z_2 z_3) \geq 0 \right\}.$$

- The SL 3-fold  $N_0$  is a *cone* in  $\mathbb{C}^3$ , so that  $tN_0 = N_0$  for all  $t > 0$ . It follows that the functions  $u_0, v_0$  constructed above satisfy

$$u_0(tx, t^2y) = tu_0(x, y) \quad \text{and} \quad v_0(tx, t^2y) = t^2v_0(x, y) \quad \text{for all } t > 0, \quad (34)$$

a kind of *weighted homogeneity equation*.

- The functions  $u_0, v_0$  in the theorem are not smooth at  $(0, 0)$ . Their behaviour helps us to guess properties of more general singular solutions to (20). For instance,  $u_0(0, y) = y|y|^{-1/2}$  by (d), so  $\frac{\partial u_0}{\partial y}$  is unbounded near  $(0, 0)$ . This will be important when we consider the problem of finding *a priori estimates* for derivatives of solutions  $u, v$  of (20) in [10].

Here are some other explicit examples of solutions to (20) and (21).

**Example 5.2.** Let  $\alpha, \beta, \gamma \in \mathbb{R}$  and define  $u(x, y) = \alpha x + \beta$  and  $v(x, y) = \alpha y + \gamma$ . Then  $u, v$  satisfy (21) for any value of  $a$ .

**Example 5.3.** Let  $S = \mathbb{R}^2$ ,  $u(x, y) = y \tanh x$  and  $v(x, y) = \frac{1}{2}y^2 \operatorname{sech}^2 x - \frac{1}{2} \cosh^2 x$ . Then  $u$  and  $v$  satisfy (20). Equation (19) with  $a = 0$  defines an explicit nonsingular special Lagrangian 3-fold  $N$  in  $\mathbb{C}^3$ . It can be shown that  $N$  is ruled, and arises from Harvey and Lawson's 'austere submanifold' construction [4, §III.3.C] of SL  $m$ -folds in  $\mathbb{C}^m$ , as the normal bundle of a catenoid in  $\mathbb{R}^3$ .

**Example 5.4.** Let  $S = \mathbb{R}^2$ ,  $u(x, y) = |y| - \frac{1}{2} \cosh 2x$  and  $v(x, y) = -y \sinh 2x$ . Then  $u, v$  satisfy (20), except that  $\frac{\partial u}{\partial y}$  is not well-defined on the line  $y = 0$ . So equation (19) defines an explicit special Lagrangian 3-fold  $N$  in  $\mathbb{C}^3$ . It turns out that  $N$  is the union of two nonsingular SL 3-folds intersecting in a real curve, which are constructed in [7, Ex. 7.4] by evolving paraboloids in  $\mathbb{C}^3$ .

## 6 Results using 'winding number' techniques

We will now discuss some results based on the idea of *winding number*.

**Definition 6.1.** Let  $C$  be a compact oriented 1-manifold, and  $\gamma : C \rightarrow \mathbb{R}^2 \setminus \{0\}$  a differentiable map. Then the *winding number of  $\gamma$  about 0 along  $C$*  is  $\frac{1}{2\pi} \int_C \gamma^*(d\theta)$ , where  $d\theta$  is the closed 1-form  $(x dy - y dx)/(x^2 + y^2)$  on  $\mathbb{R}^2 \setminus \{0\}$ .

In fact the winding number is simply the *topological degree* of  $\gamma$ . Thus it is actually well-defined for  $\gamma$  only *continuous*, and is invariant under *continuous deformations* of  $\gamma$ .



The motivation for our results is the following theorem from elementary complex analysis:

**Theorem 6.2.** *Let  $S$  be a domain in  $\mathbb{C}$ , and suppose  $f : S \rightarrow \mathbb{C}$  is a holomorphic function, with  $f \neq 0$  on  $\partial S$ . Then the number of zeroes of  $f$  in  $S^\circ$ , counted with multiplicity, is equal to the winding number of  $f|_{\partial S}$  about 0 along  $\partial S$ .*

As (21) is a nonlinear version of the Cauchy–Riemann equations for holomorphic functions, it is natural to expect that similar results should hold for solutions of (21). We will prove such results.

### 6.1 Winding number results for solutions of (21)

Rather than considering with a single solution  $u, v$  of (21), we shall get more general results by working with two solutions  $u_1, v_1$  and  $u_2, v_2$ , and treating  $(u_1, v_1) - (u_2, v_2)$  like a holomorphic function for which we wish to count the zeroes. Here is the definition of the multiplicity of a zero of  $(u_1, v_1) - (u_2, v_2)$ .

**Definition 6.3.** Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and suppose  $(u_1, v_1)$  and  $(u_2, v_2)$  are solutions of (21) in  $C^1(S)$ . Let  $k \geq 1$  be an integer and  $(b, c) \in S^\circ$ . We say that  $(u_1, v_1) - (u_2, v_2)$  has a zero of multiplicity  $k$  at  $(b, c)$  if  $\partial^j u_1(b, c) = \partial^j u_2(b, c)$  and  $\partial^j v_1(b, c) = \partial^j v_2(b, c)$  for  $j = 0, \dots, k-1$ , but  $\partial^k u_1(b, c) \neq \partial^k u_2(b, c)$  and  $\partial^k v_1(b, c) \neq \partial^k v_2(b, c)$ . Here  $\partial$  is the vector operator  $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ .

The following lemma justifies this definition, by showing that every zero of  $(u_1, v_1) - (u_2, v_2)$  has a unique multiplicity.

**Lemma 6.4.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and let  $(u_1, v_1)$  and  $(u_2, v_2)$  be solutions of (21) in  $C^1(S)$ , with  $(u_1, v_1) \not\equiv (u_2, v_2)$ . Suppose  $(b, c) \in S^\circ$  with  $u_1(b, c) = u_2(b, c)$  and  $v_1(b, c) = v_2(b, c)$ . Then  $(u_1, v_1) - (u_2, v_2)$  has a zero of multiplicity  $k$  at  $(b, c)$  for some unique  $k$ .*

*Proof.* Since  $(u_1, v_1) = (u_2, v_2)$  at one point and  $\partial u_j$  determines  $\partial v_j$  by (21), it is easy to see that  $u_1 \equiv u_2$  if and only if  $v_1 \equiv v_2$ . But  $(u_1, v_1) \not\equiv (u_2, v_2)$  by assumption. Thus  $u_1 \not\equiv u_2$  and  $v_1 \not\equiv v_2$ . By Proposition 4.3,  $u_1, v_1$  and  $u_2, v_2$  are real analytic in  $S^\circ$ , and so they are locally given by their Taylor series at  $(b, c)$ . Thus, if  $\partial^j u_1(b, c) = \partial^j u_2(b, c)$  for all  $j = 0, 1, 2, \dots$  then  $u_1 \equiv u_2$ , a contradiction.

Hence, there exists a unique integer  $k \geq 1$  such that  $\partial^j u_1(b, c) = \partial^j u_2(b, c)$  for  $j = 0, \dots, k-1$ , and  $\partial^k u_1(b, c) \neq \partial^k u_2(b, c)$ . Similarly, there exists a unique  $l \geq 1$  such that  $\partial^j v_1(b, c) = \partial^j v_2(b, c)$  for  $j = 0, \dots, l-1$ , and  $\partial^l v_1(b, c) \neq \partial^l v_2(b, c)$ . But if  $\partial^j u_1(b, c) = \partial^j u_2(b, c)$  and  $\partial^j v_1(b, c) = \partial^j v_2(b, c)$  for  $j = 0, \dots, m-1$ , one can show from (21) that  $\partial^m u_1(b, c) = \partial^m u_2(b, c)$  if and only if  $\partial^m v_1(b, c) = \partial^m v_2(b, c)$ . This implies that  $k = l$ , and the lemma follows.  $\square$

Next we show that near a zero,  $(u_1, v_1) - (u_2, v_2)$  can be modelled by a genuine holomorphic function, to highest order.

**Proposition 6.5.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and let  $(u_1, v_1)$  and  $(u_2, v_2)$  be solutions of (21) in  $C^1(S)$ . Suppose  $(u_1, v_1) - (u_2, v_2)$  has a zero of multiplicity  $k \geq 1$  at  $(b, c)$  in  $S^\circ$ . Then there exists a nonzero complex number  $C$  such that*

$$\begin{aligned} \lambda u_1(x, y) + iv_1(x, y) &= \lambda u_2(x, y) + iv_2(x, y) + C(\lambda(x - b) + i(y - c))^k \\ &\quad + O(|x - b|^{k+1} + |y - c|^{k+1}), \end{aligned} \quad (35)$$

where  $\lambda = \sqrt{2}(v_1(b, c)^2 + c^2 + a^2)^{1/4}$ .

*Proof.* Define polynomials  $p(x, y)$ ,  $q(x, y)$  of order  $k$  by

$$\begin{aligned} p(x, y) &= \sum_{j=0}^k \frac{(x - b)^j (y - c)^{k-j}}{j!(k-j)!} \cdot \frac{\partial^k (u_1 - u_2)}{\partial x^j \partial y^{k-j}}(b, c) \\ \text{and } q(x, y) &= \sum_{j=0}^k \frac{(x - b)^j (y - c)^{k-j}}{j!(k-j)!} \cdot \frac{\partial^k (v_1 - v_2)}{\partial x^j \partial y^{k-j}}(b, c). \end{aligned}$$

Then as  $(u_1, v_1) - (u_2, v_2)$  has a zero of multiplicity  $k$  at  $(b, c)$ , we see that  $p, q$  are nonzero and

$$\begin{aligned} u_1(x, y) &= u_2(x, y) + p(x, y) + O(|x - b|^{k+1} + |y - c|^{k+1}), \\ v_1(x, y) &= v_2(x, y) + q(x, y) + O(|x - b|^{k+1} + |y - c|^{k+1}). \end{aligned} \quad (36)$$

Taking the difference of equation (21) for  $u_1, v_1$  and  $u_2, v_2$ , the highest order terms at  $(b, c)$  imply that

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} \quad \text{and} \quad \frac{\partial q}{\partial x} = -\lambda^2 \frac{\partial p}{\partial y}.$$

But these are the Cauchy–Riemann equations for  $\lambda p + iq$  to be a holomorphic function of  $\lambda x + iy$ . Since  $p, q$  are homogeneous of order  $k$  in  $(x - b), (y - c)$  it follows that  $\lambda p(x, y) + iq(x, y) = C(\lambda(x - b) + i(y - c))^k$  for some  $C \in \mathbb{C}$ , which is nonzero as  $p, q$  are nonzero. Combining this with (36) gives (35).  $\square$

From (35) we see that if  $(x, y)$  is close to  $(b, c)$  in  $S^\circ$  but not equal to it then  $(u_1, v_1) \neq (u_2, v_2)$  at  $(x, y)$ . This proves:

**Corollary 6.6.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and let  $(u_1, v_1)$  and  $(u_2, v_2)$  be solutions of (21) in  $C^1(S)$ , with  $(u_1, v_1) \not\equiv (u_2, v_2)$ . Then the zeroes of  $(u_1, v_1) - (u_2, v_2)$  are isolated in  $S^\circ$ , that is, they have no limit points in  $S^\circ$ .*

*Hence, if  $(u_1, v_1) \neq (u_2, v_2)$  at every point of  $\partial S$ , then  $(u_1, v_1) - (u_2, v_2)$  has finitely many zeroes in  $S$ .*

The last part follows because  $S$  is compact, and the set of zeroes of  $(u_1, v_1) - (u_2, v_2)$  in  $S$  has no limit points. Here is the main result of this section.

**Theorem 6.7.** *Let  $S$  be a domain in  $\mathbb{R}^2$  and  $(u_1, v_1), (u_2, v_2)$  solutions of (21) in  $C^1(S)$  for some  $a \neq 0$ , with  $(u_1, v_1) \neq (u_2, v_2)$  at every point of  $\partial S$ . Then  $(u_1, v_1) - (u_2, v_2)$  has finitely many zeroes in  $S$ . Let there be  $n$  zeroes, with multiplicities  $k_1, \dots, k_n$ . Then the winding number of  $(u_1, v_1) - (u_2, v_2)$  about 0 along  $\partial S$  is  $\sum_{i=1}^n k_i$ .*

*Proof.* Let  $B_\epsilon(x, y)$  denote the open ball of radius  $\epsilon$  about  $(x, y)$  in  $\mathbb{R}^2$ , and  $\bar{B}_\epsilon(x, y)$  its closure. Let  $\gamma_\epsilon(x, y)$  be the circle of radius  $\epsilon$  about  $(x, y)$ , with the natural orientation, and  $\bar{\gamma}_\epsilon(x, y)$  the same circle with the reverse orientation.

By Corollary 6.6 there are finitely many zeroes of  $(u_1, v_1) - (u_2, v_2)$  in  $S$ . Let these be  $(b_1, c_1), \dots, (b_n, c_n)$ , with multiplicities  $k_1, \dots, k_n$  respectively. From (35) we see that if  $\epsilon > 0$  is sufficiently small then the winding number of  $(u_1, v_1) - (u_2, v_2)$  about 0 along  $\gamma_\epsilon(b_i, c_i)$  is  $k_i$ . Choose  $\epsilon_1, \dots, \epsilon_n > 0$  small enough that:

- $\bar{B}_{\epsilon_i}(b_i, c_i)$  lies in  $S^\circ$  for all  $i = 1, \dots, n$ ;
- $\bar{B}_{\epsilon_i}(b_i, c_i) \cap \bar{B}_{\epsilon_k}(b_k, c_k) = \emptyset$  for all  $1 \leq j < k \leq n$ ; and
- the winding number of  $(u_1, v_1) - (u_2, v_2)$  about 0 along  $\gamma_{\epsilon_i}(b_i, c_i)$  is  $k_i$ .

Define  $T = S \setminus \bigcup_{i=1}^n \bar{B}_{\epsilon_i}(b_i, c_i)$ . Then  $(u_1, v_1) - (u_2, v_2)$  has no zeroes in  $T$ . It follows that the winding number of  $(u_1, v_1) - (u_2, v_2)$  about 0 along  $\partial T$  is zero. This can be proved from the definition using Stokes' Theorem, as  $(u_1 - u_2, v_1 - v_2)^*(d\theta)$  is a closed 1-form on  $T$ , so  $\int_{\partial T} (u_1 - u_2, v_1 - v_2)^*(d\theta) = 0$ .

Now  $\partial T$  is the disjoint union of  $\partial S$  and  $\bar{\gamma}_{\epsilon_i}(b_i, c_i)$  for  $i = 1, \dots, n$ . Thus the winding number of  $(u_1, v_1) - (u_2, v_2)$  about 0 along  $\partial T$  is the sum of its winding numbers along  $\partial S$  and  $\bar{\gamma}_{\epsilon_i}(b_i, c_i)$  for  $i = 1, \dots, n$ . But the winding number along  $\bar{\gamma}_{\epsilon_i}(b_i, c_i)$  is  $-k_i$ , as the winding number along  $\gamma_{\epsilon_i}(b_i, c_i)$  is  $k_i$ . Hence the winding number of  $(u_1, v_1) - (u_2, v_2)$  about 0 along  $\partial S$  minus the sum of  $k_1, \dots, k_n$  is zero, as we want.  $\square$

## 6.2 Inverse solutions

Recall from §4 that equation (21) was derived by beginning with a  $U(1)$ -invariant  $SL$  3-fold  $N$ , and defining functions  $x, y, u$  and  $v$  on  $N$  by

$$x = \operatorname{Re}(z_3), \quad u = \operatorname{Im}(z_3), \quad y = \operatorname{Im}(z_1 z_2) \quad \text{and} \quad v = \operatorname{Re}(z_1 z_2)$$

for each  $(z_1, z_2, z_3)$  in  $N$ , which also satisfies  $|z_1|^2 - |z_2|^2 = 2a$ . Locally we can regard  $u, v$  as functions of  $x, y$  (except at branch points), and then the condition that  $N$  be special Lagrangian is equivalent to (21).

Consider the map  $\sigma : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  defined by  $\sigma(z_1, z_2, z_3) = (\bar{z}_1, i\bar{z}_2, i\bar{z}_3)$ . This is an isometry with  $\sigma^*(\operatorname{Re} \Omega) = -\operatorname{Re} \Omega$ , and therefore takes  $SL$  3-folds to  $SL$  3-folds, reversing orientation, as  $SL$  3-folds are calibrated w.r.t.  $\operatorname{Re} \Omega$ . Also,  $\sigma^*(x) = u$ ,  $\sigma^*(u) = x$ ,  $\sigma^*(y) = v$  and  $\sigma^*(v) = y$ , so that  $\sigma$  swaps round  $(x, y)$  and  $(u, v)$ , and  $\sigma$  preserves the equation  $|z_1|^2 - |z_2|^2 = 2a$ .

Therefore, if we regard the  $SL$  3-fold  $N$  as a kind of graph of the function  $(x, y) \mapsto (u, v)$ , the  $SL$  3-fold  $\sigma(N)$  is the 'graph' of the inverse function  $(u, v) \mapsto (x, y)$ . By Proposition 4.1, it follows that  $(u, v)$  satisfies (21) if and only if its inverse satisfies (21), provided a differentiable inverse exists. So we have proved:

**Proposition 6.8.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and let  $u, v \in C^1(S)$  satisfy (21). Define  $T = (u, v)[S]$ , and suppose  $(u, v) : S \rightarrow T$  has a differentiable inverse  $(u', v') : T \rightarrow S$ , for  $u', v' \in C^1(T)$ . Then  $u', v'$  satisfy (21), and the  $U(1)$ -invariant  $SL$  3-folds  $N, N'$  in  $\mathbb{C}^3$  corresponding to  $u, v$  and  $u', v'$  are related by the involution  $(z_1, z_2, z_3) \mapsto (\bar{z}_1, i\bar{z}_2, i\bar{z}_3)$ .*

One can also easily prove the proposition directly, by expressing the derivatives of  $u', v'$  in terms of those of  $u, v$  by matrix inversion, and observing that (21) for  $u, v$  is equivalent to (21) for  $u', v'$ . We can interpret the proposition as an analogue of the fact that the inverses of holomorphic functions are holomorphic.

### 6.3 Nonexistence of $u, v$ with given $u, v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at $(x_0, y_0)$

We shall use the ‘winding number’ results of §6.1 and the ‘inverse solution’ idea of §6.2 to show that when  $S, T$  are domains in  $\mathbb{R}^2$  and  $(\hat{u}, \hat{v}) : S \rightarrow T$  is a solution of (21), then maps  $(u, v) : T \rightarrow S^\circ$  satisfying (21) cannot have certain values of  $u, v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$  at points  $(x_0, y_0)$  in  $T^\circ$ .

**Theorem 6.9.** *Let  $S, T$  be domains in  $\mathbb{R}^2$ . Let  $a \neq 0$ ,  $(\hat{x}_0, \hat{y}_0) \in S^\circ$ ,  $(\hat{u}_0, \hat{v}_0) \in T^\circ$ , and  $(\hat{p}_0, \hat{q}_0) \in \mathbb{R}^2 \setminus \{0\}$ . Suppose  $(\hat{u}, \hat{v}) : S \rightarrow T$  is  $C^1$  and satisfies (21) and*

$$\hat{u}(\hat{x}_0, \hat{y}_0) = \hat{u}_0, \quad \hat{v}(\hat{x}_0, \hat{y}_0) = \hat{v}_0, \quad \frac{\partial \hat{v}}{\partial x}(\hat{x}_0, \hat{y}_0) = \hat{p}_0 \quad \text{and} \quad \frac{\partial \hat{v}}{\partial y}(\hat{x}_0, \hat{y}_0) = \hat{q}_0. \quad (37)$$

Define

$$\begin{aligned} x_0 &= \hat{u}_0, & y_0 &= \hat{v}_0, & u_0 &= \hat{x}_0, & v_0 &= \hat{y}_0, \\ p_0 &= -\frac{\hat{p}_0}{\frac{1}{2}(\hat{v}_0^2 + \hat{y}_0^2 + a^2)^{-1/2}\hat{p}_0^2 + \hat{q}_0^2} \quad \text{and} \quad q_0 = \frac{\hat{q}_0}{\frac{1}{2}(\hat{v}_0^2 + \hat{y}_0^2 + a^2)^{-1/2}\hat{p}_0^2 + \hat{q}_0^2}. \end{aligned} \quad (38)$$

Then there does not exist  $(u, v) : T \rightarrow S^\circ$  which is  $C^1$  and satisfies (21) and

$$u(x_0, y_0) = u_0, \quad v(x_0, y_0) = v_0, \quad \frac{\partial v}{\partial x}(x_0, y_0) = p_0 \quad \text{and} \quad \frac{\partial v}{\partial y}(x_0, y_0) = q_0. \quad (39)$$

*Proof.* Suppose for a contradiction that there exists  $(u, v) : T \rightarrow S^\circ$  which is  $C^1$  and satisfies (21) and (39). Suppose also that  $(\hat{u}, \hat{v}) : S \rightarrow T$  is injective with nowhere vanishing first derivatives. Define  $U = (\hat{u}, \hat{v})(S)$ . Then  $U$  is a domain in  $\mathbb{R}^2$ , and  $(\hat{u}, \hat{v}) : S \rightarrow U$  is an invertible map with differentiable inverse.

Let  $(u', v') : U \rightarrow S$  be the inverse map. Then by Proposition 6.8,  $u', v'$  satisfy (21). As  $(\hat{u}, \hat{v})(\hat{x}_0, \hat{y}_0) = (\hat{u}_0, \hat{v}_0)$  we see that  $(u', v')(x_0, y_0) = (u_0, v_0)$ . Also, since  $\hat{u}, \hat{v}$  satisfy (21) we deduce from (37) that

$$\begin{pmatrix} \frac{\partial \hat{u}}{\partial x} & \frac{\partial \hat{u}}{\partial y} \\ \frac{\partial \hat{v}}{\partial x} & \frac{\partial \hat{v}}{\partial y} \end{pmatrix}(\hat{x}_0, \hat{y}_0) = \begin{pmatrix} \hat{q}_0 & -\frac{1}{2}(\hat{v}_0^2 + \hat{y}_0^2 + a^2)^{-1/2}\hat{p}_0 \\ \hat{p}_0 & \hat{q}_0 \end{pmatrix}.$$

But as  $(u', v')$  is the inverse map of  $(\hat{u}, \hat{v})$  and  $(\hat{u}, \hat{v})(\hat{x}_0, \hat{y}_0) = (x_0, y_0)$  we have

$$\begin{pmatrix} \frac{\partial u'}{\partial x} & \frac{\partial u'}{\partial y} \\ \frac{\partial v'}{\partial x} & \frac{\partial v'}{\partial y} \end{pmatrix} (x_0, y_0) = \begin{pmatrix} \frac{\partial \hat{u}}{\partial x} & \frac{\partial \hat{u}}{\partial y} \\ \frac{\partial \hat{v}}{\partial x} & \frac{\partial \hat{v}}{\partial y} \end{pmatrix}^{-1} (\hat{x}_0, \hat{y}_0).$$

Combining the last two equations and (38) shows that

$$\begin{pmatrix} \frac{\partial u'}{\partial x} & \frac{\partial u'}{\partial y} \\ \frac{\partial v'}{\partial x} & \frac{\partial v'}{\partial y} \end{pmatrix} (x_0, y_0) = \begin{pmatrix} q_0 & -\frac{1}{2}(v_0^2 + y_0^2 + a^2)^{-1/2} p_0 \\ p_0 & q_0 \end{pmatrix}.$$

Comparing this with (39) and remembering that  $u, v$  satisfy (21), we see that at  $(x_0, y_0)$  we have

$$u = u', \quad v = v', \quad \frac{\partial u}{\partial x} = \frac{\partial u'}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u'}{\partial y}, \quad \frac{\partial v}{\partial x} = \frac{\partial v'}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial v'}{\partial y}.$$

Thus,  $(u', v') - (u, v)$  has a zero of multiplicity at least 2 at  $(x_0, y_0)$ , in the sense of Definition 6.3.

As  $U = (\hat{u}, \hat{v})(S)$  and  $(\hat{u}, \hat{v}) : S \rightarrow T$  we see that  $U \subseteq T$ . Therefore  $(u, v)$  and  $(u', v')$  are both solutions of (21) on the domain  $U$ . Since  $(u', v')$  is an orientation-preserving diffeomorphism  $U \rightarrow S$ , it takes  $\partial U$  to  $\partial S$ , and  $(u', v')|_{\partial U}$  winds once round  $\partial S$  in the positive sense.

Now  $(u, v)$  maps to  $S^\circ$  by assumption, and  $S$  is contractible. Therefore the winding number of  $(u', v') - (u, v)$  about 0 along  $\partial U$  is 1. So by Theorem 6.7 the sum of the zeroes of  $(u', v') - (u, v)$  in  $U^\circ$ , counted with multiplicity, is 1. However, we have already shown that  $(u', v') - (u, v)$  has a zero of multiplicity at least 2 at  $(x_0, y_0)$ , and  $(x_0, y_0) \in U^\circ$  as  $(u_0, v_0) \in S^\circ$ , a contradiction.

This proves the theorem under the additional assumption that  $(\hat{u}, \hat{v}) : S \rightarrow T$  is injective with nowhere vanishing first derivatives. To complete the proof we need to explain how to remove this assumption. We can do this using the Kähler quotient point of view of §4.2. Let  $\Sigma$  be the graph of  $(u, v)$  in  $S \times T$ , swapping round the factors  $S, T$ , and  $\hat{\Sigma}$  the graph of  $(\hat{u}, \hat{v})$  in  $S \times T$ .

We can naturally identify  $S \times T$  with a subset of the Kähler quotient  $M_a$  discussed in §4.2. Thus,  $S \times T$  carries an almost complex structure  $J$ . Since  $\Sigma, \hat{\Sigma}$  are both quotients of  $U(1)$ -invariant  $SL$  3-folds in  $\mathbb{C}^3$ , from §4.2 we see that  $\Sigma, \hat{\Sigma}$  are *pseudo-holomorphic curves* with respect to  $J$ .

Now  $\partial \Sigma \subset S^\circ \times \partial T$  and  $\partial \hat{\Sigma} \subset \partial S \times T$ , and  $\partial \Sigma, \partial \hat{\Sigma}$  wind once round  $\partial T$  and  $\partial S$  respectively. Therefore the algebraic intersection number  $\Sigma \cap \hat{\Sigma}$  is 1. By properties of pseudo-holomorphic curves it follows that  $\Sigma, \hat{\Sigma}$  intersect at only one point, with multiplicity 1. However, the argument above shows that  $\Sigma, \hat{\Sigma}$  intersect with multiplicity at least 2 at  $(u_0, v_0, x_0, y_0)$ , a contradiction, and the theorem is complete.  $\square$

This theorem will be used in [10] to construct a priori estimates for  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$  and  $\frac{\partial v}{\partial y}$  for bounded solutions  $u, v$  of (21).

## 7 Rewriting (21) in terms of a potential $f$

Let  $S$  be a domain in  $\mathbb{R}^2$ , as in Definition 3.1, and fix  $a \neq 0$  in  $\mathbb{R}$ . We shall study differentiable functions  $u, v : S \rightarrow \mathbb{R}$  satisfying equation (21) in  $S$ , and also certain *boundary conditions* on  $\partial S$ . As  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ , we can write  $u, v$  in terms of a *potential*  $f : S \rightarrow \mathbb{R}$  with  $u = \frac{\partial f}{\partial y}$  and  $v = \frac{\partial f}{\partial x}$ .

**Proposition 7.1.** *Let  $S$  be a domain in  $\mathbb{R}^2$  and  $u, v \in C^1(S)$  satisfy (21) for  $a \neq 0$ . Then there exists  $f \in C^2(S)$  with  $\frac{\partial f}{\partial y} = u$ ,  $\frac{\partial f}{\partial x} = v$  and*

$$P(f) = \left( \left( \frac{\partial f}{\partial x} \right)^2 + y^2 + a^2 \right)^{-1/2} \frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial y^2} = 0. \quad (40)$$

*This  $f$  is unique up to addition of a constant,  $f \mapsto f + c$ . Conversely, all solutions of (40) yield solutions of (21).*

*Proof.* Define a 1-form  $\alpha$  on  $S$  by  $\alpha = v(x, y)dx + u(x, y)dy$ . Then  $d\alpha = 0$  as  $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ , so  $\alpha$  is closed. As  $S$  is contractible,  $\alpha$  is exact, and so  $\alpha = df$  for some  $f \in C^2(S)$ , unique up to addition of a constant. Equating coefficients of  $dx$  and  $dy$  in  $\alpha = df$  gives  $\frac{\partial f}{\partial x} = v$ ,  $\frac{\partial f}{\partial y} = u$ . Equation (40) follows by substituting these into the first equation of (21) and multiplying by  $(v^2 + y^2 + a^2)^{-1/2}$ . The converse is easy.  $\square$

Now (40) is a *second-order quasilinear elliptic equation*, so Theorem 3.6 gives:

**Theorem 7.2.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and suppose  $f \in C^2(S)$  satisfies (40) with  $f|_{\partial S} = \phi \in C^2(\partial S)$ . Then  $f$  is real analytic in  $S^\circ$ , and if  $\phi \in C^{k+2, \alpha}(\partial S)$  for  $k \geq 0$  and  $\alpha \in (0, 1)$  then  $f \in C^{k+2, \alpha}(S)$ , and if  $\phi \in C^\infty(\partial S)$  then  $f \in C^\infty(S)$ .*

As (40) is of the form (13) with  $b^i \equiv c \equiv 0$ , by the maximum principle, Theorem 3.7, we deduce:

**Lemma 7.3.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and suppose  $f \in C^2(S)$  is a solution of (40). Then the maximum and minimum of  $f$  are achieved on  $\partial S$ .*

Equation (40) may also be written

$$P(f) = \frac{\partial}{\partial x} \left[ A \left( y, \frac{\partial f}{\partial x} \right) \right] + 2 \frac{\partial^2 f}{\partial y^2} = 0, \quad (41)$$

where  $A(y, v)$  is defined to be

$$A(y, v) = \int_0^v (w^2 + y^2 + a^2)^{-1/2} dw, \text{ so that } \frac{\partial A}{\partial v} = (v^2 + y^2 + a^2)^{-1/2}. \quad (42)$$

Equation (41) is equivalent to (40), but is in *divergence form*.

Calculation shows that we may write  $A$  explicitly as

$$A(y, v) = \log \left[ \frac{(v^2 + y^2 + a^2)^{1/2} + v}{(y^2 + a^2)^{1/2}} \right] = \log \left[ \frac{(y^2 + a^2)^{1/2}}{(v^2 + y^2 + a^2)^{1/2} - v} \right].$$

Note that  $A$  is undefined when  $a = y = 0$ . That is, if  $a = 0$  then  $A$  is undefined along the  $x$ -axis.

## 7.1 Expressing (40) as an Euler–Lagrange equation

We shall show that equation (40) is in fact the *Euler–Lagrange equation* of a certain functional  $I : C^{0,1}(S) \rightarrow \mathbb{R}$ . Fix  $a \neq 0$ , and define a function  $B(y, v)$  by  $B(y, v) = \int_0^v A(w, y)dw$ , so that  $\frac{\partial B}{\partial v}(y, v) = A(y, v)$ . Define a function  $F$  on  $S \times \mathbb{R}^2$  by

$$F(x, y, u, v) = B(y, v) + u^2, \quad (43)$$

and define a functional  $I : C^{0,1}(S) \rightarrow \mathbb{R}$  by

$$I(f) = \int_S F\left(x, y, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) dx dy. \quad (44)$$

The *Euler–Lagrange equation* for  $I$  is

$$\frac{\partial}{\partial x} \left[ \frac{\partial F}{\partial v} \left( x, y, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[ \frac{\partial F}{\partial u} \left( x, y, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \right] = 0.$$

From (43) this becomes

$$\frac{\partial}{\partial x} \left[ \frac{\partial B}{\partial v} \left( y, \frac{\partial f}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[ 2 \frac{\partial f}{\partial y} \right] = 0,$$

and this is equivalent to (41), since  $\frac{\partial B}{\partial v}(y, v) = A(y, v)$ . Thus we have proved:

**Proposition 7.4.** *Equations (40) and (41) are equivalent to the Euler–Lagrange equation of the functional  $I : C^{0,1}(S) \rightarrow \mathbb{R}$  defined in (44).*

We could use this to solve the Dirichlet problem for (40) on  $S$ , by choosing a minimizing sequence  $(f_n)_{n=1}^\infty$  for  $I$  amongst all  $f \in C^{0,1}(S)$  with  $f|_{\partial S} = \phi$  for some  $\phi \in C^{k+2,\alpha}(\partial S)$ , and then showing that  $f_n$  converges to a solution as  $n \rightarrow \infty$ . But we will instead do it by more elementary methods in §7.3.

## 7.2 Super- and subsolutions of (40)

Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and let  $P$  be the operator defined in (40). A function  $f \in C^2(S)$  is called a *supersolution* of (40) if  $P(f) \geq 0$  on  $S$ , and a *subsolution* if  $P(f) \leq 0$ . Sub- and supersolutions  $f, f'$  with  $f \leq f'$  on  $\partial S$  satisfy  $f \leq f'$  on  $S$ .

**Proposition 7.5.** *Let  $S$  be a domain in  $\mathbb{R}^2$  and  $a \neq 0$ . Suppose  $f, f' \in C^2(S)$  satisfy  $P(f) \geq 0$  on  $S$  and  $P(f') \leq 0$  on  $S$ , where  $P$  is defined in (40). If  $f \leq f'$  on  $\partial S$  then  $f \leq f'$  on  $S$ , and if  $f < f'$  on  $\partial S$  then  $f < f'$  on  $S$ .*

*Proof.* Applying the Mean Value Theorem to  $F(z) = (z^2 + y^2 + a^2)^{-1/2}$  on the interval  $[\frac{\partial f}{\partial x}, \frac{\partial f'}{\partial x}]$  we find that

$$\left( \left( \frac{\partial f}{\partial x} \right)^2 + y^2 + a^2 \right)^{-1/2} - \left( \left( \frac{\partial f'}{\partial x} \right)^2 + y^2 + a^2 \right)^{-1/2} = -w(w^2 + y^2 + a^2)^{-3/2} \left( \frac{\partial f}{\partial x} - \frac{\partial f'}{\partial x} \right)$$

for some  $w$  between  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f'}{\partial x}$ . Using (40) to expand  $P(f) - P(f')$  and rearranging then gives

$$\begin{aligned} P(f) - P(f') &= \left( \left( \frac{\partial f}{\partial x} \right)^2 + y^2 + a^2 \right)^{-1/2} \frac{\partial^2}{\partial x^2} (f - f') + 2 \frac{\partial^2}{\partial y^2} (f - f') \\ &\quad - \left( w(w^2 + y^2 + a^2)^{-3/2} \frac{\partial^2 f'}{\partial x^2} \right) \frac{\partial}{\partial x} (f - f'). \end{aligned}$$

We may regard the right hand side of this equation as  $L(f - f')$ , where  $L$  is a *linear elliptic operator* of the form (13) with  $c \equiv 0$ . Thus  $L(f - f') = P(f) - P(f') \geq 0$ , as  $P(f) \geq 0$  and  $P(f') \leq 0$ . So by the maximum principle, Theorem 3.7, the maximum of  $f - f'$  is achieved on  $\partial S$ . Thus, if  $f - f' \leq 0$  on  $\partial S$  then  $f - f' \leq 0$  on  $S$ , and if  $f - f' < 0$  on  $\partial S$  then  $f - f' < 0$  on  $S$ .  $\square$

In particular, if  $P(f) = P(f') = 0$  and  $f|_{\partial S} = f'|_{\partial S}$ , then the proposition implies that  $f \leq f'$  and (exchanging  $f, f'$ ) that  $f' \leq f$ , so that  $f = f'$ . This implies uniqueness of solutions of the Dirichlet problem for (40).

### 7.3 The Dirichlet problem for $f$

Observe that (40) is of the form (15). Therefore Theorem 3.9 applies to give existence for the Dirichlet problem for  $f$ , and an a priori bound for  $\|f\|_{C^1}$ . Combining this with the real analyticity in Theorem 7.2 and the uniqueness following from Proposition 7.5 gives:

**Theorem 7.6.** *Let  $S$  be a strictly convex domain in  $\mathbb{R}^2$ , and let  $a \neq 0$ ,  $k \geq 0$  and  $\alpha \in (0, 1)$ . Then for each  $\phi \in C^{k+2, \alpha}(\partial S)$  there exists a unique solution  $f$  of (40) in  $C^{k+2, \alpha}(S)$  with  $f|_{\partial S} = \phi$ . This  $f$  is real analytic in  $S^\circ$ , and satisfies  $\|f\|_{C^1} \leq C \|\phi\|_{C^2}$ , for some  $C > 0$  depending only on  $S$ .*

Thus, the Dirichlet problem for (40) is uniquely solvable in a strictly convex domain. Combining the theorem with Propositions 4.1 and 7.1, we get an existence and uniqueness result for  $U(1)$ -invariant special Lagrangian 3-folds in  $\mathbb{C}^3$  satisfying certain boundary conditions.

However, solving the Dirichlet problem in a general, nonconvex domain is more difficult, as to get an a priori estimate for  $|\partial f|$  on  $\partial S$  one needs to find super- and subsolutions of (40) satisfying certain equalities and inequalities on  $\partial S$ , and this does not seem easy to do in an elementary way. The point about strictly convex domains is that one can use affine functions as super- and subsolutions to estimate  $|\partial f|$ .

An analogue of Theorem 7.6 for the case  $a = 0$  will be given in [10, Th. 7.1], which shows that (41) has a unique solution  $f \in C^1(S)$  with weak second derivatives, and  $f|_{\partial S} = \phi$ . But  $f$  may have singular points, at which it is only once differentiable.

By looking closely at the proofs of existence and uniqueness of  $f$ , one can show that small changes in  $\phi$  and  $a$  result in small changes in  $f$ , where ‘small’ may be interpreted in the  $C^{k+2, \alpha}$  sense. Hence we may prove:



**Theorem 7.7.** *Let  $S$  be a strictly convex domain in  $\mathbb{R}^2$ ,  $k \geq 0$  and  $\alpha \in (0, 1)$ . Then the map  $C^{k+2,\alpha}(\partial S) \times (\mathbb{R} \setminus \{0\}) \rightarrow C^{k+2,\alpha}(S)$  taking  $(\phi, a) \mapsto f$  is continuous, where  $f$  is the unique solution of (40) with  $f|_{\partial S} = \phi$  constructed in Theorem 7.6.*

Presumably this map  $(\phi, a) \mapsto f$  is also smooth. An extension of Theorem 7.7 to include the case  $a = 0$  is given in [10, Th. 7.2], but with the  $C^1$  rather than the  $C^{k+2,\alpha}$  topology on  $f$ .

## 7.4 Winding number results for potentials

We shall now extend some of the ‘winding number’ results of §6 to the situation of this section. We begin with a definition.

**Definition 7.8.** Let  $S$  be a domain in  $\mathbb{R}^2$ , and  $\phi \in C^2(\partial S)$ . Choose a smooth parametrization

$$\mathbb{R}/2\pi\mathbb{Z} \rightarrow \partial S, \quad \text{written } \theta \mapsto (x(\theta), y(\theta)) \quad \text{for } \theta \in \mathbb{R}/2\pi\mathbb{Z}, \quad (45)$$

and regard  $\phi$  as a function of  $\theta$ . We call  $\phi$  a *Morse function* if  $\frac{d\phi}{d\theta}$  is zero at only finitely many points in  $\partial S$ , and  $\frac{d^2\phi}{d\theta^2}$  is nonzero at each of these points.

It can be shown that this definition is independent of the parametrization (45), and that the Morse functions are an open dense subset of  $C^2(\partial S)$ . Also, each stationary point of  $\phi$  on  $\partial S$  is either a local maximum or a local minimum, as  $\frac{d^2\phi}{d\theta^2} \neq 0$ , and there are the same number of each, so  $\phi$  has exactly  $l$  local maxima and  $l$  local minima for some  $l \geq 1$ .

If  $f \in C^2(S)$  and  $f|_{\partial S}$  is a Morse function, we can relate the winding number of  $\partial f$  round  $\partial S$  to the number of local maxima and minima of  $f$  on  $\partial S$ .

**Proposition 7.9.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , and  $f \in C^2(S)$  with  $f|_{\partial S} = \phi$ . Suppose that  $\partial f \neq 0$  at each point of  $\partial S$  and the winding number of  $\partial f$  about 0 along  $\partial S$  is  $k$ , and that  $\phi \in C^2(\partial S)$  is a Morse function with  $l$  local maxima and  $l$  local minima for some  $l \geq 1$ . Then  $1 - l \leq k \leq 1 + l$ .*

*Proof.* Choose a smooth, positively oriented parametrization for  $\partial S$  as in (45). Let the  $l$  local maxima of  $\phi$  be at  $\theta = \alpha_j$  and the  $l$  local minima at  $\theta = \beta_j$ , where  $\alpha_1, \dots, \alpha_l$  and  $\beta_1, \dots, \beta_l$  lie in  $\mathbb{R}/2\pi\mathbb{Z}$ , and are arranged in the cyclic order  $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \beta_l, \alpha_{l+1} = \alpha_1$ . Define  $(\gamma, \delta) : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}^2 \setminus \{0\}$  by  $(\gamma, \delta) = \frac{d}{d\theta}(x(\theta), y(\theta))$ , so that  $(\gamma, \delta)(\theta)$  is tangent to  $\partial S$  at  $(x(\theta), y(\theta))$ .

Then  $\frac{d\phi}{d\theta} = \gamma \frac{\partial f}{\partial x} + \delta \frac{\partial f}{\partial y}$  on  $\partial S$ . Therefore we have

$$\gamma \frac{\partial f}{\partial x} + \delta \frac{\partial f}{\partial y} \begin{cases} = 0, & \theta = \alpha_j \text{ or } \theta = \beta_j, \quad j = 1, \dots, l, \\ < 0, & \alpha_j < \theta < \beta_j, \quad j = 1, \dots, l, \\ > 0, & \beta_j < \theta < \alpha_{j+1}, \quad j = 1, \dots, l, \end{cases} \quad (46)$$

using the cyclic order on  $\mathbb{R}/2\pi\mathbb{Z}$ . Also, as  $\partial f$  is nonzero at each point of  $\partial S$ , we see that if  $\theta = \alpha_j$  or  $\beta_j$  then  $\delta \frac{\partial f}{\partial x} - \gamma \frac{\partial f}{\partial y} \neq 0$ . Define

$$\eta_j = \begin{cases} 1, & \delta(\alpha_j) \frac{\partial f}{\partial x}(x(\alpha_j), y(\alpha_j)) - \gamma(\alpha_j) \frac{\partial f}{\partial y}(x(\alpha_j), y(\alpha_j)) < 0, \\ 0, & \delta(\alpha_j) \frac{\partial f}{\partial x}(x(\alpha_j), y(\alpha_j)) - \gamma(\alpha_j) \frac{\partial f}{\partial y}(x(\alpha_j), y(\alpha_j)) > 0, \end{cases} \quad (47)$$

$$\text{and } \zeta_j = \begin{cases} -1, & \delta(\beta_j) \frac{\partial f}{\partial x}(x(\beta_j), y(\beta_j)) - \gamma(\beta_j) \frac{\partial f}{\partial y}(x(\beta_j), y(\beta_j)) < 0, \\ 0, & \delta(\beta_j) \frac{\partial f}{\partial x}(x(\beta_j), y(\beta_j)) - \gamma(\beta_j) \frac{\partial f}{\partial y}(x(\beta_j), y(\beta_j)) > 0. \end{cases} \quad (48)$$

Now we can use equations (46)–(48) to compare the winding numbers of  $(\delta, -\gamma)$  and  $\partial f$  about 0 along  $\partial S$ , as they tell us when the direction of  $\partial f$  crosses that of  $\pm(\delta, -\gamma)$ . But the winding number of  $(\delta, -\gamma)$  about 0 along  $\partial S$  is 1, as it is an outward normal vector to  $\partial S$ . Using this it is easy to show that the winding number of  $\partial f$  about 0 along  $\partial S$  is  $k = 1 + \sum_{j=1}^l \eta_j + \sum_{j=1}^l \zeta_j$ . As  $\eta_j$  is 0 or 1 and  $\zeta_j$  is 0 or  $-1$ , we see that  $1 - l \leq k \leq 1 + l$ .  $\square$

Here is the main result of this subsection:

**Theorem 7.10.** *Let  $S$  be a domain in  $\mathbb{R}^2$  and  $f_1, f_2 \in C^2(S)$  satisfy (40) for  $a \neq 0$  with  $f_j|_{\partial S} = \phi_j$ . Set  $u_j = \frac{\partial f_j}{\partial y}$  and  $v_j = \frac{\partial f_j}{\partial x}$ , so that  $u_j, v_j \in C^1(S)$  satisfy (21). Suppose  $\phi_1 - \phi_2$  is a Morse function on  $\partial S$ , with  $l$  local maxima and  $l$  local minima. Then  $(u_1, v_1) - (u_2, v_2)$  has  $n$  zeroes in  $S^\circ$  with multiplicities  $k_1, \dots, k_n$  and  $m$  zeroes on  $\partial S$ , where  $\sum_{i=1}^n k_i + m \leq l - 1$ .*

*Proof.* First suppose, for simplicity, that  $(u_1, v_1) \neq (u_2, v_2)$  at every point of  $\partial S$ . Then  $m = 0$ , and the theorem in this case follows from Theorem 6.7 and Propositions 7.1 and 7.9, noting that

$$\partial(f_1 - f_2) = \left( \frac{\partial}{\partial x}(f_1 - f_2), \frac{\partial}{\partial y}(f_1 - f_2) \right) = (v_1 - v_2, u_1 - u_2),$$

so that the winding number of  $\partial(f_1 - f_2)$  about 0 along  $\partial S$  is  $-\sum_{i=1}^n k_i$ , by Theorem 6.7. It remains to prove the result in the case when  $(u_1, v_1) = (u_2, v_2)$  at  $m \geq 1$  points  $(x_0, y_0)$  in  $\partial S$ .

Then  $\partial(f_1 - f_2) = 0$  at  $(x_0, y_0)$ , so  $(x_0, y_0)$  must be one of the  $l$  local maxima or  $l$  local minima of  $\phi_1 - \phi_2$ . Thus  $m$  is finite. Furthermore, as  $\phi_1 - \phi_2$  is a Morse function  $\frac{d^2}{d\theta^2}(\phi_1 - \phi_2) \neq 0$  at  $(x_0, y_0)$ , which implies that  $\partial(u_1, v_1) \neq \partial(u_2, v_2)$  at  $(x_0, y_0)$ , and therefore  $(x_0, y_0)$  is an *isolated* zero of  $(u_1, v_1) - (u_2, v_2)$ . By Corollary 6.6 and compactness of  $S$  we deduce that  $(u_1, v_1) - (u_2, v_2)$  has finitely many zeroes in  $S^\circ$ , so we can suppose there are  $n$  zeroes, with multiplicities  $k_1, \dots, k_n$ .

For  $\epsilon \geq 0$ , define  $S_\epsilon$  to be the subset of  $(x, y) \in S$  with distance at least  $\epsilon$  from  $\partial S$ , so that  $S_0 = S$ . Choose  $\epsilon > 0$  sufficiently small that  $S_\epsilon$  is a domain, and  $S_\epsilon^\circ$  contains all the  $n$  zeroes of  $(u_1, v_1) - (u_2, v_2)$  in  $S^\circ$ , and  $f|_{\partial S_\epsilon}$  is also a Morse function with  $l$  local maxima and  $l$  local minima. It is easy to see that this is possible. Then  $(u_1, v_1) \neq (u_2, v_2)$  at every point of  $\partial S_\epsilon$ , as the zeroes of  $(u_1, v_1) - (u_2, v_2)$  in  $S^\circ$  lie in  $S_\epsilon^\circ$ .

Let  $k$  be the winding number of  $\partial(f_1 - f_2)$  about 0 along  $\partial S_\epsilon$ . Then Proposition 7.9 shows that  $1 - l \leq k \leq 1 + l$ . However, we can improve the result in this

case. Recall that  $\partial(f_1 - f_2) = 0$  at  $m$  out of the  $2l$  local maxima and minima of  $\phi_1 - \phi_2$  on  $\partial S$ . Using (40) we can show that if  $\theta = \alpha_j$  is one of these  $m$  points then  $\eta_j = 1$  in (47) at the corresponding local maximum in  $\partial S_\epsilon$ , and if  $\theta = \beta_j$  is one of the  $m$  points then  $\zeta_j = 0$  in (48) at the corresponding local minimum in  $\partial S_\epsilon$ . Thus, the proof of Proposition 7.9 shows that  $1 - l + m \leq k \leq 1 + l$ . But applying Theorem 6.7 gives  $k = -\sum_{i=1}^n k_i$ , and the theorem follows.  $\square$

The theorem can be used in conjunction with Theorem 7.6, the solution of the Dirichlet problem for  $f$  on a strictly convex domain. In this case, we would know  $\phi_1, \phi_2$  explicitly, but would otherwise know little about the  $f_j, u_j$  or  $v_j$ . The theorem tells us something about  $(u_1, v_1)$  and  $(u_2, v_2)$ , using only the boundary data  $\phi_1, \phi_2$ .

Using Theorems 7.6 and 7.7 we can drop the condition that  $\phi_1 - \phi_2$  is Morse, requiring instead that it has only finitely many local maxima and minima.

**Theorem 7.11.** *Let  $S$  be a strictly convex domain in  $\mathbb{R}^2$ , let  $a \neq 0$ ,  $\alpha \in (0, 1)$ , and  $f_1, f_2 \in C^{2,\alpha}(S)$  satisfy (40) with  $f_j|_{\partial S} = \phi_j$ . Set  $u_j = \frac{\partial f_j}{\partial y}$  and  $v_j = \frac{\partial f_j}{\partial x}$ , so that  $u_j, v_j \in C^{1,\alpha}(S)$  satisfy (21). Suppose  $\phi_1 - \phi_2$  has exactly  $l$  local maxima and  $l$  local minima on  $\partial S$ . Then  $(u_1, v_1) - (u_2, v_2)$  has  $n$  zeroes in  $S^\circ$  with multiplicities  $k_1, \dots, k_n$ , where  $\sum_{i=1}^n k_i \leq l - 1$ .*

*Proof.* It is not difficult to construct a smooth family  $\phi_1^t \in C^{2,\alpha}(\partial S)$  for  $t \in (0, 1]$ , such that  $\phi_1^t \rightarrow \phi_1$  as  $t \rightarrow 0_+$ , and  $\phi_1^t$  is a Morse function with  $l$  local maxima and  $l$  local minima, at the same points as  $\phi_1$ . Let  $f_1^t$  be the solution of (40) given by Theorem 7.6 with  $f_1^t|_{\partial S} = \phi_1^t$ , and set  $u_1^t = \frac{\partial f_1^t}{\partial y}$  and  $v_1^t = \frac{\partial f_1^t}{\partial x}$ .

Then the sum of the zeroes of  $(u_1^t, v_1^t) - (u_2, v_2)$  in  $S^\circ$  with multiplicity is no more than  $l - 1$  for all  $t \in (0, 1]$ , by Theorem 7.10. Also  $(u_1^t, v_1^t) \rightarrow (u_1, v_1)$  in  $C^{1,\alpha}(S)$  as  $t \rightarrow 0_+$  by Theorem 7.7. Combining these using a limiting argument we find that the sum of the zeroes of  $(u_1, v_1) - (u_2, v_2)$  in  $S^\circ$  with multiplicity is no more than  $l - 1$ .  $\square$

Note that the theorem does not limit the number of zeroes of  $(u_1, v_1) - (u_2, v_2)$  on  $\partial S$ , which can appear at any stationary point of  $\phi_1 - \phi_2$ .

## 8 Another approach to solving (21)

In Proposition 4.3 we showed that if  $S$  is a domain in  $\mathbb{R}^2$  and  $u, v \in C^1(S)$  satisfy (21) then  $v$  satisfies (28) in  $S^\circ$ . Conversely, if  $v \in C^2(S)$  satisfies (28) then using (21) to find  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ , it is easy to show that as  $S$  is contractible there exists  $u \in C^2(S)$ , unique up to addition of a constant, such that  $u, v$  satisfy (21). In this way we prove:

**Proposition 8.1.** *Let  $S$  be a domain in  $\mathbb{R}^2$  and  $u, v \in C^2(S)$  satisfy (21) for  $a \neq 0$ . Then*

$$Q(v) = \frac{\partial}{\partial x} \left[ (v^2 + y^2 + a^2)^{-1/2} \frac{\partial v}{\partial x} \right] + 2 \frac{\partial^2 v}{\partial y^2} = 0. \quad (49)$$

Conversely, if  $v \in C^2(S)$  satisfies (49) then there exists  $u \in C^2(S)$ , unique up to addition of a constant  $u \mapsto u + c$ , such that  $u, v$  satisfy (21).

Equation (49) is a *second-order quasilinear elliptic equation* upon  $v$ . It is also in *divergence form*. By elliptic regularity, Theorem 3.6, we get:

**Proposition 8.2.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and suppose  $v \in C^2(S)$  is a solution of (49) with  $v|_{\partial S} = \phi$  for some  $\phi \in C^2(\partial S)$ . Then  $v$  is real analytic in  $S^\circ$ , and if  $\phi \in C^{k+2,\alpha}(\partial S)$  for  $k \geq 0$  and  $\alpha \in (0, 1)$  then  $v \in C^{k+2,\alpha}(S)$ , and if  $\phi \in C^\infty(\partial S)$  then  $v \in C^\infty(S)$ .*

Taking the derivative in (49) gives the equivalent

$$(v^2 + y^2 + a^2)^{-1/2} \frac{\partial^2 v}{\partial x^2} - \frac{v}{(v^2 + y^2 + a^2)^{3/2}} \left( \frac{\partial v}{\partial x} \right)^2 + 2 \frac{\partial^2 v}{\partial y^2} = 0. \quad (50)$$

This is of the form (13) with  $c = 0$ . Therefore by the maximum principle, Theorem 3.7, we have:

**Lemma 8.3.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and suppose  $v \in C^2(S)$  is a solution of (49). Then the maximum and minimum of  $v$  are achieved on  $\partial S$ .*

## 8.1 Super- and subsolutions of (49)

We now carry out the programme of §7.2 for equation (49).

**Proposition 8.4.** *Let  $T$  be a closed, bounded subset of  $\mathbb{R}^2$  whose boundary  $\partial T = T \setminus T^\circ$  is a piecewise-smooth closed curve, and let  $a \neq 0$ . Suppose  $v, v' \in C^2(T)$  satisfy  $Q(v) \geq 0$ ,  $Q(v') \leq 0$  and  $v \geq v'$  on  $T$ , where  $Q$  is defined in (49), and  $v = v'$  on  $\partial T$ . Then  $v = v'$  on  $T$ .*

*Proof.* Choose  $C > 0$  such that  $y^2 \leq C$  on  $T$ . Then we have

$$\begin{aligned} 0 &\geq - \int_T (C - y^2) [Q(v) - Q(v')] dx dy \\ &= - \int_T (C - y^2) \left[ \frac{\partial}{\partial x} \left( (v^2 + y^2 + a^2)^{-1/2} \frac{\partial v}{\partial x} - ((v')^2 + y^2 + a^2)^{-1/2} \frac{\partial v'}{\partial x} \right) \right. \\ &\quad \left. + 2 \frac{\partial^2}{\partial y^2} (v - v') \right] dx dy \\ &= \int_{\partial T} (C - y^2) (v^2 + y^2 + a^2)^{-1/2} \left( - \frac{\partial}{\partial x} (v - v') \right) dy \\ &\quad + 2 \int_{\partial T} (C - y^2) \frac{\partial}{\partial y} (v - v') dx + 4 \int_T (v - v') dx dy, \end{aligned} \quad (51)$$

using integration by parts, and the fact that  $v = v'$  on  $\partial T$ .

We claim that all three integrals on the final line of (51) are nonnegative. For the first integral, as  $v - v' = 0$  on  $\partial T$  and  $v - v' \geq 0$  on  $T$ , we see that if  $(x, y) \in \partial T$  and  $w$  is a vector in  $\mathbb{R}^2$  pointing outwards from  $T$  at  $(x, y)$  then

$\partial_w(v - v')|_{(x,y)} \leq 0$ , and if  $w$  points inwards from  $T$  then  $\partial_w(v - v')|_{(x,y)} \geq 0$ . But  $w = \frac{\partial}{\partial x}$  points outwards from  $T$  at  $(x, y)$  if and only if  $dy|_{\partial T}$  is a positive 1-form on  $\partial T$  at  $(x, y)$ , with the natural orientation on  $\partial T$ .

Hence  $-\frac{\partial}{\partial x}(v - v')dy|_{\partial T}$  is a nonnegative 1-form on  $\partial T$ , and the first integral on the final line of (51) is nonnegative. Similarly  $\frac{\partial}{\partial y}(v - v')dx|_{\partial T}$  is a nonnegative 1-form, so the second integral is nonnegative, and the third integral is nonnegative as  $v - v' \geq 0$ . But the sum of the three is nonpositive by (51). Thus all three integrals are zero, and  $\int_T(v - v')dx dy = 0$ . As  $v - v' \geq 0$  and  $v, v'$  are continuous, this implies that  $v = v'$  on  $T$ .  $\square$

Using this we can prove an analogue of Proposition 7.5 for (49). The restriction to real analytic  $v, v'$  is not really necessary, but simplifies the proof.

**Proposition 8.5.** *Let  $S$  be a domain in  $\mathbb{R}^2$  and  $a \neq 0$ . Suppose  $v, v' \in C^2(S)$  are real analytic in  $S^\circ$  and satisfy  $v \leq v'$  on  $\partial S$ ,  $Q(v) \geq 0$  on  $S$  and  $Q(v') \leq 0$  on  $S$ , where  $Q$  is defined in (49). Then  $v \leq v'$  on  $S$ .*

*Proof.* Define  $T^\circ$  to be the subset of  $S^\circ$  on which  $v > v'$ , and  $T$  to be the closure of  $T^\circ$ . Suppose for a contradiction that  $T$  is nonempty. Then  $v > v'$  on  $T^\circ$  and  $v = v'$  on  $\partial T$ . As  $v, v'$  are real analytic in  $S^\circ$  by assumption, it follows that  $T$  has piecewise-smooth boundary. Applying Proposition 8.4 then shows that  $v = v'$  on  $T$ , a contradiction. Hence  $T$  is empty, and  $v \leq v'$  on  $S$ .  $\square$

If  $v, v' \in C^2(S)$  satisfy (49) then  $Q(v) = Q(v') = 0$  and  $v, v'$  are real analytic in  $S^\circ$  by Proposition 8.2. So we have:

**Corollary 8.6.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and suppose  $v, v' \in C^2(S)$  satisfy (49) on  $S$ . If  $v \leq v'$  on  $\partial S$  then  $v \leq v'$  on  $S$ .*

In particular, if  $v|_{\partial S} = v'|_{\partial S}$  this gives  $v \leq v'$  and  $v' \leq v$  on  $S$ , so that  $v = v'$ . This implies uniqueness of solutions of the Dirichlet problem for (49). Here is an analogue of Corollary 8.6 but with strict inequalities, proved using a different method.

**Proposition 8.7.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and suppose  $v, v' \in C^2(S)$  satisfy (49) on  $S$ . If  $v < v'$  on  $\partial S$  then  $v < v'$  on  $S$ .*

*Proof.* Suppose for a contradiction that there exists  $(b, c) \in S^\circ$  with  $v(b, c) = v'(b, c)$ . By Proposition 8.1 there exist  $u, u' \in C^2(S)$ , unique up to addition of constants, such that  $u, v$  and  $u', v'$  satisfy (21). Choose the constants such that  $u(b, c) = u'(b, c)$ .

Now apply Theorem 6.7 to  $(u, v)$  and  $(u', v')$ . As  $v < v'$  on  $\partial S$ , the winding number of  $(u, v) - (u', v')$  about 0 along  $\partial S$  is zero, since  $(u, v) - (u', v')$  is confined to a half-plane and cannot go round  $(0, 0)$ . But  $(u, v) - (u', v')$  has at least one zero in  $S^\circ$ , at  $(b, c)$ . This is a contradiction. Therefore  $v \neq v'$  in  $S^\circ$ , and by continuity and connectedness we have  $v < v'$  on  $S$ .  $\square$

## 8.2 The Dirichlet problem for $v$

We now show that the Dirichlet problem for  $v$  is uniquely solvable in arbitrary domains  $S$  in  $\mathbb{R}^2$ .

**Theorem 8.8.** *Let  $S$  be a domain in  $\mathbb{R}^2$ . Then whenever  $a \neq 0$ ,  $k \geq 0$ ,  $\alpha \in (0, 1)$  and  $\phi \in C^{k+2, \alpha}(\partial S)$  there exists a unique solution  $v \in C^{k+2, \alpha}(S)$  of (49) with  $v|_{\partial S} = \phi$ . Fix a basepoint  $(x_0, y_0) \in S$ . Then there exists a unique  $u \in C^{k+2, \alpha}(S)$  with  $u(x_0, y_0) = 0$  such that  $u, v$  satisfy (21). Furthermore,  $u, v$  are real analytic in  $S^\circ$ .*

*Proof.* Observe that the operator  $Q$  of (49) is of the form (16), with coefficients  $a^{ij}$  depending on  $y$  and  $v$  but not on  $\partial v$ , and

$$b((x, y), v, \partial v) = -\frac{v}{(v^2 + y^2 + a^2)^{3/2}} \left( \frac{\partial v}{\partial x} \right)^2.$$

As  $|v(v^2 + y^2 + a^2)^{-3/2}| \leq a^{-2}$  the condition  $|b(x, u, p)| \leq C|p|^2$  in Theorem 3.10 holds with  $C = a^{-2}$ , and the condition  $v b((x, y), v, p) \leq 0$  for all  $((x, y), v, p) \in S \times \mathbb{R} \times \mathbb{R}^2$  also clearly holds.

Thus Theorem 3.10 applies, and there exists  $v$  in  $C^{k+2, \alpha}(S)$  satisfying (49) with  $v|_{\partial S} = \phi$ . Corollary 8.6 shows that  $v$  is unique. Using the condition  $u(x_0, y_0) = 0$  to fix the additive constant, Proposition 8.1 shows that there exists a unique  $u \in C^2(S)$  with  $u(x_0, y_0) = 0$  such that  $u, v$  satisfy (21). But (21) shows that  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in C^{k+1, \alpha}(S)$  as  $v \in C^{k+2, \alpha}(S)$ , so  $u \in C^{k+2, \alpha}(S)$ . Finally, Proposition 4.3 shows that  $u, v$  are real analytic in  $S^\circ$ .  $\square$

Combining the theorem with Proposition 4.1, we again get an existence and uniqueness result for  $U(1)$ -invariant SL 3-folds in  $\mathbb{C}^3$  satisfying certain boundary conditions, but different boundary conditions to those in §7.3. An analogue of Theorem 8.8 for the case  $a = 0$  will be given in [10, Th. 6.1], which shows that (49) has a unique *weak* solution  $v \in C^0(S)$  with  $v|_{\partial S} = \phi$ .

In Theorem 7.6 we restricted  $S$  to be a strictly convex domain, but Theorem 8.8 works for general domains. The basic reason for this is that in the Dirichlet problem for  $v$  we automatically get an a priori estimate for  $\|v\|_{C^0}$ , which implies positive upper and lower a priori bounds for  $(v^2 + y^2 + a^2)^{-1/2}$ .

Hence, in the Dirichlet problem for  $v$  we know in advance that (49) is *uniformly elliptic*. However, in the Dirichlet problem for  $f$  we need an a priori bound for  $\|\frac{\partial f}{\partial x}\|_{C^0}$  to make (40) uniformly elliptic, and we assume  $S$  is strictly convex to prove such a bound.

By analogy with Theorem 7.7, we can also prove:

**Theorem 8.9.** *Let  $S$  be a domain in  $\mathbb{R}^2$ ,  $k \geq 0$ ,  $\alpha \in (0, 1)$  and  $(x_0, y_0) \in S$ . Then the map  $C^{k+2, \alpha}(\partial D) \times (\mathbb{R} \setminus \{0\}) \rightarrow C^{k+2, \alpha}(D)^2$  taking  $(\phi, a) \mapsto (u, v)$  is continuous, where  $(u, v)$  is the unique solution of (21) with  $v|_{\partial D} = \phi$  and  $u(x_0, y_0) = 0$  constructed in Theorem 8.8.*

Presumably the map  $(\phi, a) \mapsto (u, v)$  is also smooth. An extension of Theorem 8.9 to include the case  $a = 0$  is given in [10, Th. 6.2], but with the  $C^0$  rather than the  $C^{k+2, \alpha}$  topology on  $u, v$ .

### 8.3 Winding number results for $v$

As in §7.4, we will now extend some of the ‘winding number’ results of §6 to the situation of this section. Here is the analogue of Morse function for  $v$ .

**Definition 8.10.** Let  $S$  be a domain in  $\mathbb{R}^2$ , and let  $v \in C^1(\partial S)$ . Choose a smooth, positively oriented parametrization  $\theta$  for  $\partial S$  as in (45), and regard  $v$  as a function of  $\theta$ . We call  $v$  *transverse* if  $v = 0$  at only finitely many points in  $\partial S$ , and  $\frac{dv}{d\theta} \neq 0$  at each of these points.

This definition is independent of parametrization  $\theta$ , and transverse functions are an open dense subset of  $C^1(\partial S)$ . Also, each zero of  $v$  is either *increasing*, with  $\frac{dv}{d\theta} > 0$ , or *decreasing*, with  $\frac{dv}{d\theta} < 0$ , and there are the same number of each, so  $f$  has exactly  $l$  increasing and  $l$  decreasing zeroes for some  $l \geq 0$ .

Here is the analogue of Theorem 7.10, with a similar proof.

**Theorem 8.11.** Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and let  $u_1, v_1 \in C^1(S)$  and  $u_2, v_2 \in C^1(S)$  be solutions of (21). Suppose that  $(v_1 - v_2)|_{\partial S}$  is transverse with  $2l$  zeroes. Then  $(u_1, v_1) - (u_2, v_2)$  has  $n$  zeroes in  $S^\circ$  with multiplicities  $k_1, \dots, k_n$  and  $m$  zeroes on  $\partial S$ , where  $n, m \geq 0$  and  $k_i \geq 1$ , and  $\sum_{i=1}^n k_i + m \leq l$ .

*Proof.* Suppose  $(u_1, v_1) = (u_2, v_2)$  at  $(x_0, y_0)$  in  $\partial S$ . Then  $v_1 - v_2 = 0$  at  $(x_0, y_0)$ , so  $(x_0, y_0)$  must be one of the  $2l$  zeroes of  $v_1 - v_2$  on  $\partial S$ . Thus  $m$  is finite. Let  $m_1 \geq 0$  of the  $m$  zeroes of  $(u_1, v_1) - (u_2, v_2)$  on  $\partial S$  be *increasing* zeroes of  $v_1 - v_2$  on  $\partial S$ , and  $m_2 \geq 0$  be *decreasing* zeroes, where  $m_1 + m_2 = m$ . As in the proof of Theorem 7.10 we find that  $(u_1, v_1) - (u_2, v_2)$  has finitely many zeroes in  $S^\circ$ , so let there be  $n$  zeroes, with multiplicities  $k_1, \dots, k_n$ .

Let  $\epsilon > 0$  be small. Then  $(u_1 + \epsilon, v_1)$  also satisfies (21), so we can consider the zeroes of  $(u_1 + \epsilon, v_1) - (u_2, v_2)$  in  $S$ . One can use the ideas of §6 to show that close to the  $n$  zeroes of  $(u_1, v_1) - (u_2, v_2)$  in  $S^\circ$  there will be  $n' \geq n$  zeroes of  $(u_1 + \epsilon, v_1) - (u_2, v_2)$  with multiplicities  $k'_1, \dots, k'_{n'}$ , where  $\sum_{i=1}^{n'} k'_i = \sum_{i=1}^n k_i$ .

If  $(x_0, y_0)$  is one of the  $m_1$  increasing zeroes of  $v_1 - v_2$  on  $\partial S$ , as  $\epsilon > 0$  is small one can show that  $(u_1 + \epsilon, v_1) - (u_2, v_2)$  has a zero in  $S^\circ$  near  $(x_0, y_0)$ . If  $(x_0, y_0)$  is one of the  $m_2$  decreasing zeroes of  $v_1 - v_2$  on  $\partial S$ , there is no zero of  $(u_1 + \epsilon, v_1) - (u_2, v_2)$  in  $S^\circ$  near  $(x_0, y_0)$ . So, by Theorem 6.7 the winding number of  $(u_1 + \epsilon, v_1) - (u_2, v_2)$  about 0 along  $\partial S$  is  $k' = \sum_{i=1}^{n'} k'_i + m_1$ .

Now  $(u_1 + \epsilon, v_1) - (u_2, v_2)$  crosses the  $x$ -axis exactly  $2l$  times on  $\partial S$ , at the zeroes of  $v_1 - v_2$ . However, at the  $m_2$  decreasing zeroes  $(u_1 + \epsilon, v_1) - (u_2, v_2)$  crosses the  $x$ -axis at  $(\epsilon, 0)$  in the negative sense winding round 0. So it is not difficult to see that the winding number  $k'$  satisfies  $k' \leq l - m_2$ . The theorem then follows from the equations  $k' = \sum_{i=1}^{n'} k'_i + m_1$ ,  $\sum_{i=1}^{n'} k'_i = \sum_{i=1}^n k_i$  and  $m_1 + m_2 = m$ .  $\square$

The theorem can be used in conjunction with Theorem 8.8, the solution of the Dirichlet problem for  $v$ . In this case, we would know  $v_1, v_2$  on  $\partial S$  explicitly, but would otherwise know little about the  $u_j$  or  $v_j$ . The theorem tells us something about  $(u_1, v_1)$  and  $(u_2, v_2)$ , using only the known boundary values of  $v_1, v_2$ .

## 8.4 A maximum principle for $\frac{\partial v}{\partial x}$

Finally we show that if  $v$  satisfies (49) on  $S$  then  $|\frac{\partial v}{\partial x}|$  is maximum on  $\partial S$ .

**Proposition 8.12.** *Let  $S$  be a domain in  $\mathbb{R}^2$ , let  $a \neq 0$ , and suppose  $v \in C^2(S)$  satisfies (49). Then the maximum of  $|\frac{\partial v}{\partial x}|$  is achieved on  $\partial S$ .*

*Proof.* As  $v$  satisfies (49) it is real analytic in  $S^\circ$  by Proposition 8.2, and satisfies (50). Taking the derivative  $\frac{\partial}{\partial x}$  of (50) in  $S^\circ$  and rearranging gives

$$L\left(\frac{\partial v}{\partial x}\right) = (v^2 + y^2 + a^2)^{-3/2} \left(\frac{\partial v}{\partial x}\right)^3, \quad \text{where}$$

$$L(g) = (v^2 + y^2 + a^2)^{-1/2} \frac{\partial^2 g}{\partial x^2} + 2 \frac{\partial^2 g}{\partial y^2} - 3(v^2 + y^2 + a^2)^{-3/2} v \frac{\partial v}{\partial x} \cdot \frac{\partial g}{\partial x}.$$

Then  $L$  is a linear elliptic operator of the form (13), with  $c(x) \equiv 0$ .

Suppose that  $\frac{\partial v}{\partial x}$  has a positive maximum achieved at  $(x_0, y_0) \in S^\circ$ , with  $\frac{\partial v}{\partial x}(x_0, y_0) = M > 0$  say, and  $\frac{\partial v}{\partial x} < M$  on  $\partial S$ . Let  $\epsilon \in (0, M)$  be generic and small enough that  $\frac{\partial v}{\partial x} < M - \epsilon$  on  $\partial S$ , and define  $T = \{(x, y) \in S : \frac{\partial v}{\partial x} \geq M - \epsilon\}$ . Then as  $\epsilon$  is generic  $T$  lies in  $S^\circ$  and is compact with smooth boundary, and  $\frac{\partial v}{\partial x} = M - \epsilon$  on  $\partial T$ . Also  $L(\frac{\partial v}{\partial x}) > 0$  on  $T$ , as  $\frac{\partial v}{\partial x} \geq M - \epsilon > 0$  on  $T$ .

Applying Theorem 3.7 shows that the maximum of  $\frac{\partial v}{\partial x}$  on  $T$  is achieved on  $\partial T$ . This contradicts  $(x_0, y_0) \in T^\circ$ ,  $\frac{\partial v}{\partial x}(x_0, y_0) = M$  and  $\frac{\partial v}{\partial x} = M - \epsilon$  on  $\partial T$ . Thus, if  $\frac{\partial v}{\partial x}$  has a positive maximum it is achieved on  $\partial S$ . Similarly, if  $\frac{\partial v}{\partial x}$  has a negative minimum it is achieved on  $\partial S$ . Thus the maximum of  $|\frac{\partial v}{\partial x}|$  is achieved on  $\partial S$ .  $\square$

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