Universal Moduli of Parabolic Sheaves on Stable Marked Curves

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A thesis submitted for the degree of
Doctor of Philosophy

Michaelmas Term 2011

To Kris and my parents

Acknowledgements

I am very grateful to my supervisor, Professor Frances Kirwan, for her support and encouragement, her patience and invaluable advice. She suggested studying the universal moduli problem for parabolic sheaves and taught me about GIT and its applications to moduli problems.

I would also like to thank some of my fellow graduate students who I learned a lot from during the last few years: in particular Steve Rayan, Vicky Hoskins, Ben Davison and the other members of the Junior Geometry and Topology seminar.

I am grateful to Dr Tamás Hausel for very helpful comments on previous versions of this thesis.

Finally, I would like to thank the Engineering and Physical Sciences Research Council for their generous funding of my degree: this work was supported by EPSRC grant number GR/T01341/01. I was also supported by a Stipendiary Lectureship from Merton College, University of Oxford.

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The topic of this thesis is the moduli theory of (parabolic) sheaves on stable curves. Using geometric invariant theory (GIT), universal moduli spaces of semistable parabolic sheaves on stable marked curves are constructed: 'universal' indicates that these are moduli spaces of pairs where the underlying marked curve may vary as well as the parabolic sheaf (as in the Pandharipande moduli space for pairs of stable curves and torsion-free sheaves without augmentations).

As an intermediate step in this construction, we construct moduli spaces of semistable parabolic sheaves on flat families of arbitrary projective schemes (of any dimension or singularity type): this is the technical core of this thesis. These moduli spaces are projective, since they are constructed as GIT quotients of projective parameter spaces. The stability condition for parabolic sheaves depends on a choice of polarisation and is derived from the Hilbert-Mumford criterion. It is not quite the same as traditional stability with respect to parabolic Hilbert polynomials, but it is closely related to it, and the resulting moduli spaces are always compactifications of moduli of slope-stable parabolic sheaves. The construction works over algebraically closed fields of arbitrary characteristic.

Contents

Chapter 1

Introduction

The goal of this work is to construct 'universal' moduli of parabolic sheaves on stable marked curves. Here, 'universal' is used in the sense of [Pan96], i.e. for a moduli space of pairs $(X, \mathcal{F})$, where X varies in a particular class of support schemes (e.g. stable curves, marked curves, polarised manifolds) and $\mathcal{F}$ varies in a specified class of bundles / sheaves on X (e.g. semistable vector bundles, sheaves or parabolic sheaves).

1.1 Motivation

Consider the following class of moduli problems: the objects are pairs $(C, \mathcal{E})$, where C is a stable (possibly marked) curve and $\mathcal{E}$ is a torsion-free sheaf on C (possibly equipped with augmentations such as parabolic structures), and two pairs $(C_1, \mathcal{E}_1), (C_2, \mathcal{E}_2)$ are equivalent if there exists an isomorphism $\phi : C_1 \to C_2$ such that $\mathcal{E}_1$ and $\phi^* \mathcal{E}_2$ are S-equivalent. Such a set-up is often called a universal moduli problem of (augmented) sheaves on stable curves, i.e. 'universal' indicates that the underlying (marked) curves are allowed to vary as well as the (augmented) sheaves.

A prototype of such a moduli construction is Pandharipande's universal moduli space $\overline{U_g(e, r)}$ of semistable torsion-free sheaves (of uniform rank r and degree e) on stable curves of genus g [Pan96]: this is a projective variety mapping to $\overline{M}_g$ by the forgetful morphism, and the fibre over $[C] \in \overline{M}_g$ is $M_C(e, r)/\mathrm{Aut}(C)$, where $M_C(e, r)$ is the projective moduli space of semistable torsion-free sheaves of uniform rank r and degree e on C. The notion of stability used by Pandharipande is slope-stability with the degree defined via the Riemann-Roch formula and the rank of a torsion-free sheaf $\mathcal{E}$ given by $\sum_j \omega_j r_j$, where r_j is the rank of $\mathcal{E}$ on the irreducible component C_j of C, and $\omega_j := \deg \omega_C|_{C_j}$. This is Seshadri's generalisation of slope to nodal curves in the case of canonical polarisation: cf. [Ses82] for the first construction of $M_C(e, r)$ in the case of a fixed nodal curve C with arbitrary polarisation. On curves, this is exactly equivalent to Gieseker-p-stability as used by Simpson in his construction of moduli of semistable pure sheaves on arbitrary families of projective schemes [Sim94a], where the reduced Hilbert polynomial p is defined with respect to (a power of) the canonical sheaf ω_C (see lemma 3.2.18). In the rank 1 case, the moduli spaces $M_C(e,1)$ are compactifications of the Jacobian of C; the first comprehensive study of these was carried out by Oda and Seshadri [OS79], defining stability conditions depending on a choice of $\phi \in \partial C_1(\Gamma, \mathbb{Z})$, i.e. a 0-boundary of the (oriented) dual graph Γ of the curve C. In fact, all these compactifications can be obtained as Simpson compactifications for suitable polarisations of C: see [Ale96] for a comparison of the different approaches.

Universal moduli spaces are a natural framework in which to study deformations of M_C, the moduli space of the corresponding (augmented) sheaves on a fixed smooth curve C, as C degenerates to a nodal curve, and in the case of plain torsion-free sheaves without augmentations this approach has yielded interesting results on the geometry and topology of M_C: for example, [NR75] shows finiteness of the automorphism group of the moduli space M_C of stable bundles of coprime rank and degree and calculates $H^3(M_C, \mathbb{Z})$, whereas [Gie84] uses degeneration methods to show the vanishing of the Chern classes $c_i(M_C)$ for $i \geqslant 2g-1$, where M_C is the moduli space of stable rank 2, degree 1 bundles on a curve C of genus g.

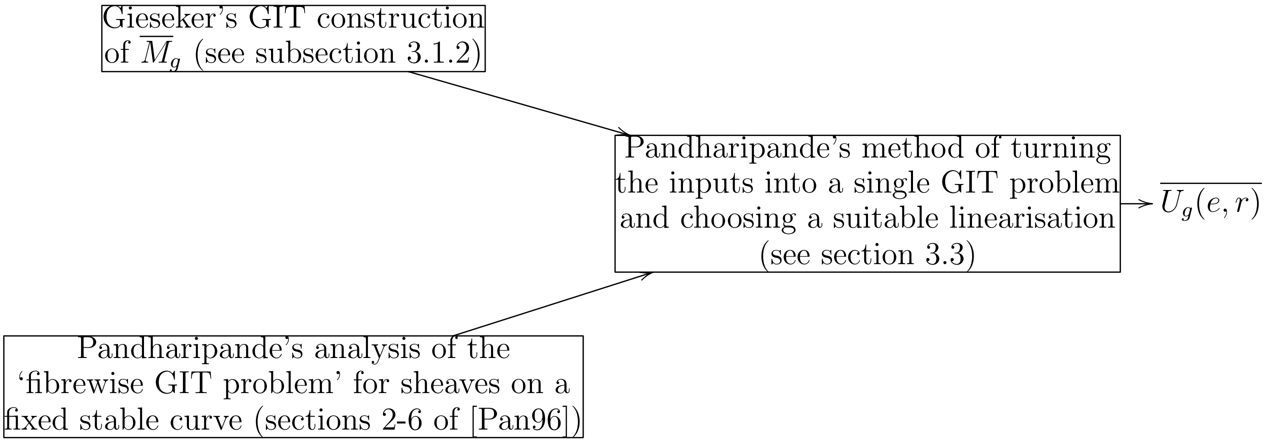
Furthermore, Pandharipande's moduli space $\overline{U_g(e,r)}$ can be thought of as a higher rank version of (the stable locus of) the Hitchin system[1]: if C is a fixed smooth curve, then the Hitchin morphism $N_C(s,d) \to A$ from the moduli space of semistable rank s, degree d Higgs bundles on C to the Hitchin base space (of characteristic polynomials for the Higgs fields) is just a relative Jacobian of the family of spectral curves (which are $s:1$ covers of C constructed from the characteristic polynomials, see [BNR89]), at least over the regular base locus of the Hitchin fibration (i.e. A^{reg}, the set of $a \in A$ whose associated spectral curve C_a is smooth). However, using compactified Jacobians, this may be extended to the elliptic locus A^{ell} (consisting of those $a \in A$ corresponding to integral spectral curves C_a). Compare this to the universal moduli space $\overline{U_g(e,1)}$: as the moduli space $M_C(e,1)$ of torsion-free rank 1 degree e sheaves is a compactification of the Jacobian Jac^e_C, the fibration $\overline{U_g(e,1)} \to \overline{M}_g$ is very similar to the Hitchin fibration, except that the base space parametrises stable curves which are not necessarily obtained as covers of a fixed curve. From this point of view, the universal moduli space of parabolic sheaves constructed here can be regarded as a higher rank version of the parabolic Hitchin system. Of course, there are limitations to this analogy, as there is no reason to expect the total space $\overline{U_g(e,r)}$ to have as rich a geometry as $N_C(s,d) \to A^{\mathrm{ell}}$ which is hyperkähler and an algebraically completely integrable Hamiltonian system. In addition, there is no interpretation of $\overline{U_g(e,r)}$ as a moduli space of (augmented) sheaves on a a single curve C as there is for the total space of the Hitchin system.

[1]I am grateful to Dr Tamás Hausel for pointing out this analogy.

1.2 Construction method

The main result of this thesis is a construction of a universal moduli space $\overline{U_{g,n}(r,d,r^\star_*,\alpha^\star_*)}$ for semistable parabolic sheaves (of uniform rank r, degree d, flag type $r^\star_*$ and parabolic weights $\alpha^\star_*$) on stable marked curves (see chapter 5 for a precise statement). This moduli space is again projective, maps to $\overline{M}_{g,n}$ by forgetting the parabolic sheaves, and its fibre over a fixed stable marked curve $C^\star := (C, x^\star)$ is just $P_{C^\star}(r,d,r^\star_*,\alpha^\star_*)/\mathrm{Aut}(C^\star)$ where $P_{C^\star}(r,d,r^\star_*,\alpha^\star_*)$ is the moduli space of semistable parabolic sheaves (of the same numerical type as before) on $C^\star$, and $\mathrm{Aut}(C^\star)$ consists of the automorphisms of $C^\star$ as a marked curve. In particular, our universal moduli space compactifies the moduli problem of pairs $(C^\star, E^\star_*)$ where $C^\star \in M_{g,n}$ is a smooth marked curve and $E^\star_*$ is a slope-stable parabolic vector bundle of fixed rank, degree and flag type on $C^\star$, i.e. 'universal' refers to the fact that the underlying marked curve varies as well as the parabolic bundle.

The method we employ is based on Pandharipande's construction [Pan96]. Roughly speaking, Pandharipande's approach can be thought of as a method that takes two GIT constructions as input: one for the moduli space of the underlying schemes which support the bundles (these base schemes are stable curves in [Pan96]), and one for the moduli space of sheaves on a fixed support scheme. Pandharipande then provides a framework for bringing these constructions together to build a moduli space for the problem of pairs, essentially by considering sheaves on fibres of U_H over H, where H is the parameter space in the GIT construction for the support schemes (typically some subscheme of a suitable Hilbert scheme) and U_H the universal family over it. The universal moduli space is then constructed as the GIT quotient of some relative Quot scheme of U_H/H by a product of special linear groups, one of which takes care of the redundancy introduced by rigidifying the support schemes, whereas the other reflects the surplus information from rigidifying the bundles or sheaves. A judicious choice of linearisation (weighting the factor of the parameter space which encodes the support schemes heavily in comparison to the bundles part of the parameter space) then allows Pandharipande in sections 7 and 8 of [Pan96] to conclude that the points of the GIT quotient correspond to semistable torsion-free sheaves on stable support schemes (curves), and that orbits correspond precisely to S-equivalence classes of bundles modulo automorphisms of the curves. Thus, we may illustrate the above remark that Pandharipande's work takes two inputs into his framework to produce universal moduli spaces as follows:



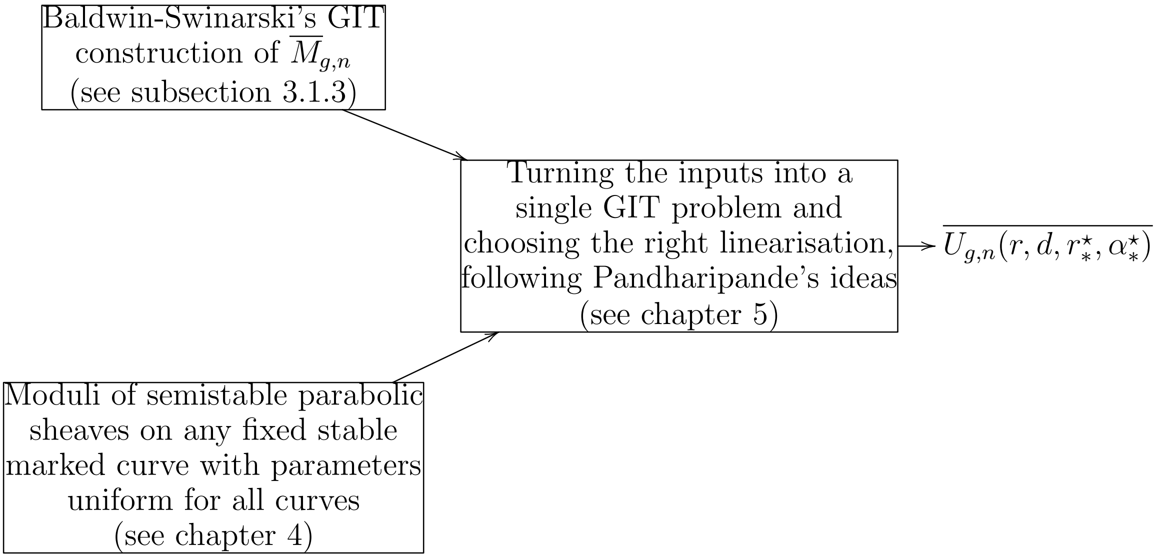
Here, one should note that the two inputs can be black-boxed – only very specific results from the relevant GIT constructions are needed (essentially, a knowledge of the correct parameter spaces, the group action, and the result characterising the GIT-(semi)stable loci in the parameter space, together with the statement that the resulting GIT quotient actually gives a coarse moduli space for the moduli problem under consideration). The difference between the two inputs is that the construction of $\overline{M}_g$ requires no change, whereas in order to incorporate the analysis of the moduli problem of uniform-rank-r pure sheaves on a fixed stable curve, a small but important modification is necessary: stronger boundedness results are required than those obtained by looking at a fixed curve. The parameters involved in this part of the construction (a twisting parameter n exhibiting the sheaves $\mathcal{F}$ as quotients of a fixed sheaf $\mathcal{E} = \Bbbk^{P(n)} \otimes \mathcal{O}(-n)$, allowing the sheaves $\mathcal{F}$ to be parametrised by a suitable Quot scheme Q_n, and a parameter m governing the linearisation of the $\mathrm{SL}_{P(n)}$-action on Q_n) have to be chosen uniformly for all stable curves of genus g. This is the reason why Pandharipande went through the analysis of this part of the problem himself – in the constructions of moduli of sheaves on curves that were available to him (such as [Ses82]), the parameters depended on the fixed curve under consideration: in other words, there was no construction of moduli of semistable torsion-free sheaves on a *family* of curves yet. However, such a construction was provided by Simpson in section 1 of [Sim94a] (actually, this construction is vastly more general, producing moduli spaces of semistable pure sheaves on any family of projective $\Bbbk$-schemes of any dimension or singularity type).

Before trying to generalise [Pan96] to parabolic sheaves on marked curves, I studied Simpson's GIT construction and used it to replace what Pandharipande calls the 'fibrewise GIT problem' in his construction (sections 2-6 of [Pan96]), i.e. his analysis of the moduli problem of uniform-rank-r pure sheaves on a fixed stable curve. There are several reasons for this: Pandharipande's method for this step (invented before the publication of [Sim94a] and thus before Pandharipande was aware of Simpson's ideas in detail) is a little less transparent than Simpson's (in particular, Pandharipande makes no mention of boundedness or m-regularity, even though these concepts essentially govern his construction from the background). More importantly, Simpson's ideas generalise much more easily, both to higher-dimensional support schemes (a possible future direction of generalisation of [Pan96]), but also to bundles and sheaves with augmentations, such as parabolic structures.

How can we utilise Pandharipande's framework for a construction of a universal moduli space $\overline{U_{g,n}(r,d,r_*^\star,\alpha_*^\star)}$ for semistable degree d, rank r, flag type $r_*^\star$, weight $\alpha_*^\star$ parabolic sheaves on (stable) n-pointed curves of genus g? The input of Gieseker's construction of $\overline{M}_g$ in the above diagram has to be replaced by a GIT construction of $\overline{M}_{g,n}$ (such a construction was recently given in [BS08]), and the input of the lower left-hand corner in the above diagram needs to be replaced by a GIT construction of a moduli space $P_{C^\star}(r,d,r_*^\star,\alpha_*^\star)$ of semistable parabolic sheaves on a fixed marked curve $C^\star = (C, x^1, \ldots, x^n)$, or in fact a version of $P_{C^\star}(r,d,r_*^\star,\alpha_*^\star)$ as $C^\star$ varies in a flat family (parametrised by a Hilbert scheme: this is because all parameters in the GIT construction for the lower left-hand corner in the above diagram need to be uniform in varying curves). However, as far as I am aware there is no general construction of a moduli space $P_{C^\star}(r,d,r_*^\star,\alpha_*^\star)$ of semistable parabolic sheaves for $C^\star$ varying in a family (but some special cases of fixed but singular $C^\star$ have been covered before: Narasimhan and Ramadas [NR93] constructed moduli for semistable parabolic sheaves of rank 2 on an irreducible nodal curve, and Sun extended this construction in [Sun00] to higher rank and in [Sun03] to reducible nodal curves, but working with a modified stability condition different from the natural generalisation of parabolic semistability to reducible curves which appears in our universal construction in chapters 4 and 5).

A construction of moduli for semistable parabolic sheaves on a flat family $X \to S$ of projective schemes of arbitrary dimension, based on Simpson's moduli construction [Sim94a], is more conceptual than an ad-hoc treatment of the curves case (and may be of independent interest), so we cover the general situation in chapter 4, even if in chapter 5 we only require the resulting moduli spaces for families of stable (in particular, at worst nodal) marked curves. In fact, this 'fibrewise' construction of moduli of semistable parabolic sheaves on flat families of arbitrary projective schemes (of any dimension and singularity type) is the most technical and involved part of this thesis, taking up all of chapter 4. See section 4.1 for a detailed discussion of previous work on related moduli spaces and an overview of the technical steps needed. Section 4.9 gives the main results of this construction.

With this 'fibrewise' GIT problem solved, the relevant chart for the construction of universal moduli of semistable parabolic sheaves on stable marked curves is then:



This construction of $\overline{U_{g,n}(r,d,r^\star_*,\alpha^\star_*)}$ is completed in chapter 5, where we extend Pandharipande's ideas of weighting the group actions appropriately, so that whenever $C^\star$ is an unstable marked curve, then a pair $(C^\star, \mathcal{E}^\star_*)$ cannot be stable, no matter how well behaved the parabolic sheaf $\mathcal{E}^\star_*$ is.

1.3 Overview of contents and statement of originality

In chapter 2, I give a brief overview of what moduli problems in algebraic geometry are, how they are approached, and how Geometric Invariant Theory (GIT) applied to Hilbert and Quot schemes is a standard technique to solve them. This chapter is strictly introductory in that I give very few proofs in it and review fairly classical material.

I also state the relevant results from GIT (section 2.2) which are needed in later chapters. Section 2.3 introduces Hilbert and Quot schemes as the standard parameter spaces for GIT constructions. I give a summary of their construction, but focus on the properties needed in the later use of Hilb and Quot. In particular, the notions of boundedness and regularity of sheaves, which play a central rôle in the later development, make a first appearance here.

None of the contents of chapter 2 are original – all results can be found in the literature, only the presentation is mine.

Section 3.1 is a rapid introduction to stable curves (with or without markings), their basic properties, and the relevant GIT constructions by Gieseker and Baldwin-Swinarski. As explained above, these constructions are black-boxed when they are later used in the Pandharipande framework, so I restrict myself to listing the most basic steps of these constructions and omit all their technicalities (for example, the Potential Stability Theorem is not even mentioned in subsection 3.1.2).

Section 3.2 is then much longer and more detailed – since I use Simpson's ideas 'uniformly for all curves' in the Pandharipande framework (and as my moduli construction for parabolic sheaves is modelled on Simpson's arguments), I review the construction in some detail, explaining why all choices along the way can be made independently of which stable curve we happen to work on. I also take the opportunity to collect some basic material on pure and (semi)stable sheaves in section 3.2.

Section 3.3 in which I review Pandharipande's construction [Pan96] is briefer again, not least because the set-up was already introduced in subsection 3.2.3, but also because the longest part of Pandharipande's construction (sections 2-6 of [Pan96]) is taken up by the 'fibrewise GIT problem' which is replaced by section 3.2 in this thesis.

Most of chapter 3 is not original: in sections 3.1 and 3.3 I summarise GIT constructions due to Gieseker, Baldwin-Swinarski and Pandharipande because they are relevant to my goal of constructing universal moduli of parabolic sheaves, and again only the presentation is mine. The contents of section 3.2 are due to many contributors – the main GIT construction is Simpson's, of course. However, the comparison between Simpson's and Pandharipande's stability conditions, especially lemmas 3.2.17 and 3.2.18 are my own work. Moreover, the argument in subsection 3.2.4 extending Simpson's GIT analysis from $\widetilde{Q}_n$ (the closure of the locus of torsion-free sheaves) to the whole Quot scheme Q_n is original.

Chapter 4 describes the construction of projective moduli spaces of semistable parabolic sheaves on a flat family $X \to S$ of arbitrary projective schemes (of any dimension or singularity type). This is the technical heart of the thesis, and most of this chapter is original: while the definitions of parabolic sheaves had been extended to this set-up before, I am not aware of any construction of projective moduli for this situation. For a detailed summary of previous work on moduli of parabolic sheaves, a discussion of the approach taken in the construction presented here and a statement of the main results, see section 4.1. The new ingredients needed for this construction were a boundedness result for semistable parabolic sheaves on X/S, a construction of a *projective* parameter space built up from Quot schemes, a suitable projective embedding and linearisation of the group action on the parameter space, and a stability condition for parabolic sheaves that matches the GIT-stability analysis arising from the Hilbert-Mumford criterion.

Chapter 5 finally completes the programme for the construction of universal moduli of parabolic sheaves on stable curves which was described above. Using the same ideas in relative GIT as Pandharipande did, we show that the GIT quotient of the Flag-Quot scheme parametrising parabolic sheaves on the fibres of the universal family in Baldwin-Swinarski's construction of $\overline{M}_{g,n}$ gives the moduli space we were seeking. We should point out that the 'universal family' in question is over a Hilbert scheme (used in the GIT construction of the coarse moduli scheme $\overline{M}_{g,n}$) instead of being the universal family over the DM stack $\overline{\mathcal{M}}_{g,n}$. In fact, if we did work with DM stacks, then existence of the universal moduli space of parabolic sheaves on stable curves would follow directly from applying the results of chapter 4 to the universal (stacky) curve over $S = \overline{\mathcal{M}}_{g,n}$. However, we do not take this approach, since Simpson's construction for moduli of pure sheaves on the fibres of X/S (and by extension also our construction for moduli of parabolic sheaves on the fibres of X/S) do not apply directly to DM stacks as base S. (Possible complications requiring further investigation involve questions of boundedness for sheaves on a family with stacky base.) We do not work with stacks throughout this thesis – all moduli spaces are moduli schemes, and all universal families are over fine moduli schemes (typically over the Hilbert or Quot schemes serving as parameter spaces in GIT constructions).

1.4 Conventions

This introduction concludes with a few conventions concerning notation and terminology that will be used throughout the thesis. Note that individual chapters and sections may have special conventions, these will always be clearly announced. In particular, note subsection 4.1.2 with conventions for chapter 4.

Convention 1.4.1. We will often be working in the category Sch/S of locally noetherian schemes over S (where S is some fixed scheme, sometimes taken to be connected and of finite type over an algebraically closed field $\Bbbk$), and we will use the following shorthand notations for base change: given $X \to S$ and $S' \to S$ in Sch/S, the fibre product will be denoted by $X' := X \times_S S'$ (and similarly for double prime). For any coherent sheaf $\mathcal{F}$ on X, its pullback to X' will be denoted by $\mathcal{F}'$ $(:= (\mathrm{pr}_1)^*\mathcal{F} = (\mathrm{pr}_1)^{-1}\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$, where pr_1 is the projection $X \times_S S' \to X)$. Analogous notations will be used for base changing $X \to S$ and $\mathcal{F}$ on X to $T \to S$ (i.e. $X_T = X \times_S T$ and $\mathcal{F}_T$, respectively). Finally, for $f : X \to S$ proper (in particular for f projective), $H^i(X/S, \mathcal{F})$ will denote the higher direct image sheaf $R^i f_*(\mathcal{F})$ on S (following Simpson's notation in [Sim94a]) – note especially that $H^0(X/S, \mathcal{F})$ is just the direct image of $\mathcal{F}$ under $f : X \to S$.

Convention 1.4.2. Throughout, we follow common notation in the literature in using the sign '$(\leqslant)$' to denote '$\leqslant$' when referring to semistable objects, and '$<$' when referring to stable objects, and similarly for '$(\geqslant)$'. This applies to both GIT-(semi)stable and p- or slope-(semi)stable (parabolic) sheaves. Thus a statement like 'if $\mathcal{F}$ is (semi)stable, then $B(\leqslant)C$' should be read as 'if $\mathcal{F}$ is semistable, then $B \leqslant C$, and if $\mathcal{F}$ is stable, then $B < C$'. In this context, note also that all inequalities between real (or rational) polynomials are with respect to the lexicographic ordering, i.e. $f \leqslant g$ for polynomials $f, g \in \mathbb{Q}[x]$ is a statement on the coefficients of the highest degree term where f and g differ, and thus $f \leqslant g$ is equivalent to saying that $f(m) \leqslant g(m)$ for all sufficiently large $m \in \mathbb{Z}$. This implies that the leading coefficient of f is less than or equal to the leading coefficient of g, but the converse implication is not true: if f and g have the same leading coefficient, then we cannot say whether $f > g$, $f = g$ or $f < g$ without having information on the other coefficients of f and g. For example, $2x + 1$ is strictly less than $2x + 2$ but strictly greater than $2x - 1$. (Two polynomials are equal if and only if all their coefficients are equal.) Note also that in section 4.3 we introduce a lexicographic ordering on polynomials in two variables – see subsection 4.1.2 for more details on this.

Convention 1.4.3. We use the Grothendieck convention of parametrising quotients instead of subspaces by the functors **Grass** and $\mathbb{P}$; therefore we use the spaces $\mathrm{Grass}(V^*)$ (resp. $\mathbb{P}(V^*)$) when talking about subspaces of V.

Convention 1.4.4. Whenever given a morphism $X \to S$ and a point $s \in S$, we will denote by X_s the scheme-theoretic fibre of X over s, i.e. $X_s := X \times_{\mathrm{Spec}\, k(s)} S$ where $\mathrm{Spec}\, k(s) \hookrightarrow S$ is the inclusion of the point s. If $\mathcal{F}$ is any sheaf on X, then $\mathcal{F}_s$ will denote the restriction of $\mathcal{F}$ to the fibre X_s: we write $\mathcal{F}_s := \mathcal{F}|_{X_s}$ as short-hand for the pullback of $\mathcal{F}$ along $X_s \to X$. This should not be confused with a stalk of $\mathcal{F}$, but if $s \in S$ and $\mathcal{F}$ is a sheaf on X, there should be no danger of confusion.

Convention 1.4.5. Throughout, we follow a common abuse of notation in the literature concerning vector bundles: given a fixed finite-dimensional $\Bbbk$-vector space V on a scheme X over a field $\Bbbk$, we let $V \otimes \mathcal{O}_X$ denote the trivial vector bundle on X with fibre V. The main point here is that V is not considered as a sheaf itself, and thus the tensor product in $V \otimes \mathcal{O}_X$ (or more generally in all expressions $V \otimes \mathcal{W}$ for a finite-dimensional $\Bbbk$-vector space V and a coherent $\mathcal{O}_X$-module $\mathcal{W}$) means $\otimes_{\Bbbk}$ and *not* the tensor product of coherent sheaves (i.e. not tensor product over $\mathcal{O}_X$). For example, $V \otimes \mathcal{O}_X(-n)$, a notation that will frequently appear from section 3.2 onwards, means the trivial vector bundle over X with fibre V twisted by $-n$.

In particular, V does *not* mean the constant sheaf $\underline{V}$ on X (i.e. the sheaf that assigns to each Zariski-open $U \subset X$ the space $\underline{V}(U)$ of locally constant maps $U \to V$), which would be more consistent with standard notation as in [Har77], for example. However, all of this is a harmless abuse of notation as it should be understood that all our sheaves are coherent $\mathcal{O}_X$-modules (which we sometimes also refer to as 'coherent sheaves' by abuse of terminology); therefore the constant sheaf will in general not feature in our discussion.

Convention 1.4.6. Unless explicitly stated otherwise, by a *point* of a scheme we will mean a closed point, assumed to be a geometric (i.e. $\Bbbk$-valued) point if we are working over an algebraically closed field $\Bbbk$.

Convention 1.4.7. Throughout this thesis, $\subset$ and $\supset$ will denote weak (i.e. not necessarily proper) inclusions, unless explicitly stated otherwise. In particular, the flags forming quasi-parabolic structures in chapters 4 and 5 may have weak inclusions, i.e. the chains of subspaces or subsheaves need not be strictly decreasing.

Chapter 2

Moduli spaces: general theory and construction methods

In this chapter, we give a brief summary of the basic notions of moduli problems, and we explain one of the fundamental strategies for constructing moduli spaces: the path to coarse moduli spaces via geometric invariant theory (GIT), applied mainly to parameter spaces obtained from Hilbert or Quot schemes. We do not discuss the alternative approach of algebraic stacks as moduli spaces in much detail as all constructions in this thesis are for moduli schemes (whenever moduli stacks exist, we consider the underlying coarse moduli scheme). Basic references are the introductory [New78] (and the more compressed [New09] and [Kir01]) for both sections 2.1 and 2.2, and the foundational [MFK94] (together with [Ses77] which extended the first edition of [Mum65b] to general ground fields) for the technical details skipped over in section 2.2. Other good references (especially on the interplay between GIT and symplectic reduction) are [Tho06] and [Kir84]. For the construction of Hilbert and Quot schemes in section 2.3, the original reference is [Gro62], and modern accounts providing full details are given in [Nit05] and [HL97]; the latter and [Mum66] are also very user-friendly guides to boundedness and Castelnuovo-Mumford regularity of sheaves.

2.1 Moduli problems in algebraic geometry

Moduli problems in algebraic geometry are essentially classification problems: given a collection $\mathcal{A}$ of geometric objects and an equivalence relation $\sim$ on $\mathcal{A}$, we would like to describe the collection $\mathcal{A}/\sim$ of equivalence classes, and not just as a *set*, but as a *space* of some sort. In particular, we would like to study $\mathcal{A}/\sim$ geometrically – this reflects the existence of geometric families of objects: small variations of objects in $\mathcal{A}$ should correspond to nearby points in the space with underlying set $\mathcal{A}/\sim$.

Formally, a moduli problem in algebraic geometry is set up like this: suppose the objects in $\mathcal{A}$ are defined over a base scheme S and $\sim$-equivalent objects lie over the same point in S (e.g. $\mathcal{A}$ could be the collection of all sheaves on the fibres of some morphism $X \to S$ and $\sim$ isomorphism of sheaves on the same fibre), where S is often taken to be of finite type over an algebraically closed field $\Bbbk$. Then we work throughout in the category Sch/S of locally noetherian schemes over S. Given the basic ingredients $\mathcal{A}$ and $\sim$, we define a notion of *families parametrised by* T as well as *equivalence of families parametrised by* T (also denoted by $\sim$, abusing notation mildly) for any $T \to S$ in Sch/S. These concepts should extend $\mathcal{A}$ and $\sim$ in the sense that a family parametrised by a (geometric) point $s : \mathrm{Spec}\ \Bbbk \to S$ should be the same thing as an element of $\mathcal{A}$ lying over $s \in S$, and two families parametrised by $s : \mathrm{Spec}\ \Bbbk \to S$ should be equivalent as families if and only if they are equivalent under $\sim$ as elements of $\mathcal{A}$. A *moduli functor* is then any contravariant functor $\mathbf{F} : \mathsf{Sch}/S \to \mathsf{Sets}$ which is given on objects by

$$\mathbf{F}(T) := \{\text{equivalence classes of families parametrised by } T\},$$

i.e. as well as the information of families and their equivalence, we need a notion of *pullback* of families along any morphism $T' \to T$ in Sch/S, where this pullback should be functorial and respect equivalence of families. Basic examples of moduli functors are the functors **Hilb** and **Quot** defined in section 2.3, and the moduli functor $\mathcal{M}_g$ of smooth curves of genus g over any fixed algebraically closed field $\Bbbk$ (in which case $S = \mathrm{Spec}\ \Bbbk$): this functor is given on objects by

$$\begin{aligned}\mathcal{M}_g(T) \quad = \quad & \{\text{smooth proper morphisms } C \to T \text{ of relative} \\ & \quad \text{dimension 1 with geometric fibres connected of genus } g\}/\!\sim,\end{aligned}$$

where $\sim$ is just isomorphism as schemes over T, and the pullback of the family $C \to T$ along a morphism $T' \to T$ is the usual base change $C' \to T'$, i.e. the fibred product $C' := C \times_T T'$ together with its projection to T'. In a similar spirit, one defines the moduli functors of stable curves, marked curves (see chapter 3 for both of these), vector bundles on a given base, (augmented) sheaves on the fibres of a morphism $X \to S$ (see chapter 4), etc.

In an ideal case, the moduli functor $\mathbf{F}$ is represented by a scheme $M \in \mathsf{Sch}/S$, i.e. $\mathbf{F}$ is naturally isomorphic to $\mathrm{Hom}_S(-, M)$, where Hom_S denotes morphisms in Sch/S. If such M exists, it is automatically unique up to unique isomorphism and we then call M a *fine moduli space* for the moduli functor $\mathbf{F}$ or for the moduli problem $(\mathcal{A}, \sim)$. Note that the points of M lying over $s \in S$ then correspond to $\mathbf{F}(s : \mathrm{Spec}\ \Bbbk \to S)$, i.e. the $\sim$-equivalence classes of $\mathcal{A}$-objects lying over $s \in S$. This explains in which sense a fine moduli space is an answer to our question of how to put a geometric structure on the set of equivalence classes.

Of course, there may be other schemes in Sch/S whose geometric points also correspond to $\mathcal{A}/\!\sim$ (after all, there are lots of non-isomorphic schemes whose geometric points are in bijection to each other, e.g. $\mathbb{A}^1$ and a plane cuspidal cubic, or a fat point and a reduced point). However, in the spirit of the Yoneda lemma, there is more to fine moduli spaces than this bijection: if M represents $\mathbf{F}$, there is an element $U \in \mathbf{F}(M) \cong \mathrm{Hom}_S(M, M)$ corresponding to the identity in $\mathrm{Hom}_S(M, M)$. This family parametrised by M is called the *universal family* since it has the property that any family X parametrised by T is equivalent to the pullback of U via a morphism $f : T \to M$ in Sch/S which is uniquely determined by X: in terms of the geometric points, $t \in T$ is sent to $f(t) := [X_t]$, the point of M corresponding to the equivalence class of X_t, where X_t is the object of $\mathcal{A}$ given by the pullback of X along the inclusion $t : \mathrm{Spec}\, \Bbbk \to T$. (The map f is sometimes referred to as the *classifying map* of the family X.) In particular, morphisms into M correspond exactly to families up to equivalence, and so the schematic structure of M is the best possible for $\mathcal{A}/\sim$.

However, existence of such M (and U) is often too much to ask for: for example, in the common case of $\mathcal{A}$ being a class of abstract schemes over an algebraically closed field $\Bbbk$, families being (flat) morphisms with geometric fibres belonging to $\mathcal{A}$, and $\sim$ given by isomorphism covering the identity on the base, any elements of $\mathcal{A}$ possessing non-trivial automorphisms (of finite order) would automatically make a fine moduli space for $\mathbf{F}$ impossible, and this is essentially the only obstruction to fine moduli spaces (see [Ser06], example 2.6.13(i) and theorem 2.6.15): let $Y \in \mathcal{A}$ be a quasi-projective scheme with non-trivial finite automorphism group $G \subset \mathrm{Aut}(Y)$. We can then construct a family $X \to T$ which is isotrivial (i.e. all fibres are isomorphic to each other, namely to Y) but non-trivial (i.e. not isomorphic to the trivial family $T \times_{\Bbbk} Y \to T$): let T' be any quasi-projective scheme with a free G-action and consider $X' := T' \times_{\Bbbk} Y$ with the diagonal G-action $g \cdot (t, y) := (gt, gy)$ which is still free. Now let $X \to T$ be defined as the quotient of the equivariant map $\mathrm{pr}_1 : X' \to T'$ by G, i.e. $X := X'/G$ and $T := T'/G$ (both well-defined by freeness of the respective G-actions) together with the map $X \to T$ induced by $\mathrm{pr}_1 : T' \times_{\Bbbk} Y \to T'$. Then each fibre of $X \to T$ is isomorphic to Y, but as a family of schemes $X \to T$ is never trivial.

In particular, taking $\mathbf{F}$ to be the moduli functor $\mathcal{M}_g$ of smooth curves of genus $g \geqslant 2$ and fixing $Y = C$ to be any curve with non-trivial automorphisms, the argument above shows that there cannot be a scheme M (with a universal family U, say) representing the moduli functor $\mathcal{M}_g$: otherwise the image of T under the classifying map $f : T \to M$ of the family $X \to T$ would be a single point $[C] \in M$ and so the family $X \to T$ would be isomorphic to the pullback family $f^*U \to T$ which is necessarily trivial as f is constant. Thus, there is no hope for a fine moduli space of all smooth (or stable) curves, at least if $\sim$ is global isomorphism of families.

The situation is much better when considering moduli problems of sub- (or quotient) objects of a fixed object, e.g. subschemes of a fixed scheme (or quotient sheaves of a fixed sheaf): since isomorphisms of such objects must respect the embedding into the ambient object (or the surjection from the fixed ambient object, respectively), the position of all points of such sub-/quotient objects is fixed by the ambient object, and so the objects of such moduli problems do not admit any non-trivial automorphisms (see section 2.3 for what are by far the most important examples of this kind: the Hilbert and Quot functors and their fine moduli spaces).

What are the possible remedies when a moduli functor fails to be representable because of automorphisms? Of course, one could exclude the offending objects from $\mathcal{A}$, i.e. restrict attention to objects without non-trivial automorphisms, but this is usually not very satisfying: those objects with automorphisms are often among the most interesting ones, and moduli spaces of only automorphism-free objects tend to be non-complete. Instead, there are two principal approaches to dealing with the non-existence of fine moduli spaces. One is to weaken what we ask for: we may be able to obtain a scheme whose points correspond to $\mathcal{A}/\sim$ at the price of giving up universal families – this leads to coarse moduli spaces, defined below. The other point of view is that the category Sch/S in which we tried (and failed) to represent $\mathbf{F}$ is simply not large enough, and to study algebraic stacks instead of schemes as a consequence.

There are many advantages to the second approach: stacks often record more information (by their very definition, stacks remember automorphisms of individual objects as well as the equivalence classes, and as a consequence moduli stacks behave very much like fine moduli spaces, i.e. universal families are available) and they may be treated like smooth spaces even when the corresponding schemes have finite quotient singularities. Furthermore, coarse moduli schemes are usually constructed via GIT, thus they still require 'bad' (i.e. unstable) points of the moduli problem to be discarded and $\sim$ to be replaced on strictly semistable objects by a coarser equivalence relation, i.e. more objects of $\mathcal{A}$ than previously intended are identified with each other – this is the notion of S-equivalence (this GIT process will be discussed from page 17 onwards). In contrast, moduli stacks capture all objects of $\mathcal{A}$ (no stability notion is required, even though imposing one may improve the geometric properties of the stack) and exist almost by definition (if we take the route to stacks as 2-functors from Sch/S to the 2-category of groupoids).

On the other hand, stacks require more machinery (in particular for sheaf cohomology and deformation theory), and one technical advantage of coarse moduli schemes is that their GIT construction usually equips them with an ample line bundle which may be difficult to describe on the stack otherwise. In addition, showing that a given moduli stack is Deligne-Mumford (or even just algebraic) requires an atlas which is often constructed just like the parameter space for the associated coarse moduli scheme (see below).

We will not discuss stacks in more detail here as the moduli spaces constructed in this thesis will all be (coarse) moduli schemes. For definitions of stacks and a discussion of their basic properties, good references are (in increasing order of sophistication and scope) [Fan01], [Góm01], [BCE+], and [LMB00]. Mumford's classic paper [Mum65a] is also worth reading: even though he does not mention stacks explicitly, he motivates their use, gives an excellent explanation where they come from, and goes on to calculate the Picard group of the moduli stack $\mathfrak{M}_{1,1}$ of elliptic curves.

From now on, we concentate on coarse moduli spaces.

Definition 2.1.1. A *coarse moduli space* for $\mathbf{F} : \mathsf{Sch}/S \to \mathsf{Sets}$ is a scheme $M \in \mathsf{Sch}/S$ together with a natural transformation $\Psi : \mathbf{F} \to \mathrm{Hom}_S(-, M)$ such that

(a) for every point $s : \mathrm{Spec}\ \Bbbk \to S$, the map $\Psi(s : \mathrm{Spec}\ \Bbbk \to S)$ is a bijection between points of the fibre M_s and $\mathbf{F}(s : \mathrm{Spec}\ \Bbbk \to S)$, the equivalence classes of $\mathcal{A}$-objects lying over $s \in S$; and

(b) Ψ satisfies a universal property with respect to natural transformations from $\mathbf{F}$ to functors of points: for all $N \in \mathsf{Sch}/S$, every natural transformation $\Phi : \mathbf{F} \to \mathrm{Hom}_S(-, N)$ factors uniquely through Ψ.

The universal property (b) ensures that M, if it exists, is unique up to unique isomorphism. Note that every family parametrised by $T \in \mathsf{Sch}/S$ induces a (classifying) map $T \to M$ via Ψ, but not conversely in general since Ψ is not required to be a natural isomorphism.

The basic strategy to construct coarse moduli schemes is a two-step process in the course of which we need to find notions of *(semi)stability* for the equivalence classes $\mathcal{A}/\!\sim$: we will not actually construct coarse moduli for $(\mathcal{A}, \sim)$, but for the closely related moduli problem $(\mathcal{A}^{ss}, \sim_S)$. Here, $\mathcal{A}^{ss} \subset \mathcal{A}$ is the subcollection of semistable objects, and $\sim_S$ is an equivalence relation which agrees with $\sim$ on the stable objects $\mathcal{A}^s \subset \mathcal{A}^{ss}$, but will be coarser than $\sim$ on $\mathcal{A}^{ss} \setminus \mathcal{A}^s$, the strictly semistable objects. Exactly which objects of $\mathcal{A}$ are (semi)stable will be mainly determined by the second step of our process.

The first step is to rigidify the objects of the moduli problem $(\mathcal{A}, \sim)$, i.e. remove any automorphisms by adding extra data, e.g. embeddings of schemes into a fixed ambient space. The aim is to construct a scheme $X \in \mathsf{Sch}/S$ (ideally projective over S), sometimes called a *parameter space* for the moduli problem, together with a family satisfying the local universal property. We say that $V \in \mathbf{F}(X)$ has the *local universal property* for the moduli functor $\mathbf{F}$ if for all $T \in \mathsf{Sch}/S$ every family in $\mathbf{F}(T)$ is (Zariski) locally equivalent under $\sim$ to the pullback of V via some morphism to X, i.e. for any family $W \in \mathbf{F}(T)$ and any point

$t \in T$, there is a neighbourhood U of t in T such that $W|_U$ (the pullback of W via the inclusion $U \hookrightarrow T$) is equivalent to f^*V where $f : U \to X$ is some (not necessarily unique) morphism.

The parameter space X is usually constructed from appropriate Hilbert or Quot schemes over S and will be a (fine!) moduli space for a rigidified version of our moduli problem. However, if we want to ensure that X is not too big, i.e. of finite type over S, we often have to cut down $\mathcal{A}$ by some semistability condition – in general, the collection of equivalence classes may not be bounded. Let us consider a simple example of this phenomenon using some of the concepts introduced in later sections: suppose $\mathcal{A}$ consists of all vector bundles of fixed rank $r \geqslant 2$ and degree d on a smooth curve C of genus $g \geqslant 2$ over $S := \operatorname{Spec} \Bbbk$ and $\sim$ is isomorphism of bundles. Then $\mathcal{A}/\sim$ is unbounded: choose a line bundle $\mathcal{O}_C(1)$ of degree $a > 0$ and write $P(m)$ for the Hilbert polynomial $\chi(V(m)) = ram + d + r(1-g)$ of each $V \in \mathcal{A}$, then each $V \in \mathcal{A}$ is isomorphic to a quotient of $\mathcal{O}_C(-n)^{\oplus P(n)}$ for sufficiently large $n \in \mathbb{N}$, but taking $V = \mathcal{O}_C((d+n)x) \oplus \mathcal{O}_C(-nx) \oplus \mathcal{O}_C^{\oplus r-2}$ for any point $x \in C$ shows that no $n \in \mathbb{N}$ will do for all $V \in \mathcal{A}$. This means that the equivalence classes $\mathcal{A}/\sim$ may be parametrised by

$$\bigcup_{n \in \mathbb{N}} \operatorname{Quot}_{C/\Bbbk}^{\mathcal{O}_C(1)} \left(\mathcal{O}_C(-n)^{\oplus P(n)}, P\right)$$

but no subscheme of finite type over $\Bbbk$ will parametrise all $V \in \mathcal{A}$. On the other hand, if we restrict $\mathcal{A}$ to the slope-semistable vector bundles, a simple Riemann-Roch calculation shows that $n = \frac{r(2g-2)-d}{ra} + 2$ will work. For a precise definition of boundedness, see section 2.3 where we also introduce Hilbert and Quot schemes.

Returning to the general situation, suppose that we have replaced $\mathcal{A}$ by some bounded subcollection $\widetilde{\mathcal{A}}$ so that we have a parameter space X of finite type over S, with local universal family V. Whatever geometric notion of (semi)stability we come up with in the second step of the moduli construction, $\mathcal{A}^{ss}$ will be chosen as a subcollection of $\widetilde{\mathcal{A}}$. In particular, applying the local universal property to the inclusion of a point $s \in S$ shows that for each semistable object $A \in \mathcal{A}^{ss}$ lying over $s \in S$ there should be some $x \in X_s$ such that $V_x \sim A$, i.e. each equivalence class in $\mathcal{A}^{ss}/\sim$ will be represented by (at least) one point of X. The converse will in general not hold: we often aim to get S-projective moduli spaces, and for this purpose it is convenient to start with X projective over S (this is not a necessary, but a sufficient condition as we will see in section 2.2). As a consequence, X may parametrise some of the unstable objects $\widetilde{\mathcal{A}} \setminus \mathcal{A}^{ss}$ as well (since semistability tends to be an open condition), but that is acceptable as long as X parametrises all the semistable objects. In the example above, not all quotients of $\mathcal{O}_C(-n)^{\oplus P(n)}$ parametrised by the Quot scheme $X = \operatorname{Quot}_{C/\Bbbk}^{\mathcal{O}_C(1)}(\mathcal{O}_C(-n)^{\oplus P(n)}, P)$ will be semistable vector bundles – these form an open subscheme of X.

The parameter space X will often come with an action by a linear algebraic group G such that different rigidifications of the same equivalence class in $\widetilde{\mathcal{A}}$ correspond to points in a common G-orbit of X, i.e. the objects V_x and V_y of $\widetilde{\mathcal{A}}$ should be equivalent under $\sim$ if and only if x and y lie in the same G-orbit in X (in particular, G acts along the fibres of $X \to S$, as $\sim$ is assumed to respect fibres over S). In the example of semistable vector bundles on a smooth curve C, any V is rigidified by a surjection $\mathcal{O}_C(-n)^{\oplus P(n)} \twoheadrightarrow V$, and our parameter space $X = \mathrm{Quot}_{C/\Bbbk}^{\mathcal{O}_C(1)}(\mathcal{O}_C(-n)^{\oplus P(n)}, P)$ is acted upon by $G = \mathrm{SL}_{P(n)}$. In another common case, if $\mathcal{A}$ consists of a class of abstract schemes which we rigidify by embedding them into a fixed projective space $\mathbb{P}^N$, then X will be a Hilbert scheme of $\mathbb{P}^N$, with different embeddings identified by the action of $G = \mathrm{SL}_{N+1}$ on X coming from the linear G-action on $\mathbb{P}^N$.

The quotient of the parameter space by the group is then a strong candidate for the moduli space – in fact, we have the following result:

Proposition 2.1.2 ([New78], proposition 2.13)**.** Suppose that $V \in \mathbf{F}(X)$ satisfies the local universal property for the moduli functor $\mathbf{F}$, and assume that a linear algebraic group G acts on X so that any two closed points x and y of X lie in the same G-orbit if and only if $V_x \sim V_y$. Then a coarse moduli space for $\mathbf{F}$ is exactly a categorical quotient of X by G which is also an orbit space.

Here, a *categorical quotient* of a scheme X by a linear algebraic group G acting on X is a G-invariant morphism $\phi : X \to Y$ satisfying a universal property among G-invariant morphisms from X (i.e. all G-invariant morphisms $X \to Z$ factor uniquely through ϕ). If ϕ sends distinct orbits to distinct points of Y, then we call (Y, ϕ) an *orbit space* for the G-action on X. Note that Y is then automatically an S-scheme and ϕ is an S-morphism, since G was assumed to act along the fibres of $X \to S$.

This raises the question of how to construct quotient schemes, or more precisely categorical quotients – to address this problem (especially in the context of moduli constructions) Mumford developed geometric invariant theory (GIT) in [Mum65b], solving this question provided that G is reductive. In this second step of constructing coarse moduli schemes, the process of taking a GIT quotient of X by G, points that behave badly with respect to the G-action (called 'GIT-unstable') need to be removed and the remaining points are either 'GIT-stable' or 'GIT-strictly semistable'. Deciding on GIT-(semi)stability of points depends on a choice of linearisation, i.e. a choice of (S-relatively) very ample line bundle on the parameter space together with an extension of the group action to this line bundle, or equivalently an embedding of X into some projective space $\mathbb{P}^n_S$ such that the G-action extends to a linear action on $\mathbb{P}^n_S$ (see section 2.2). The technical heart of many GIT-style constructions of moduli spaces is to find a linearisation with respect to which objects of $\mathcal{A}$

corresponding to GIT-(semi)stable points of X (i.e. the points surviving in the quotient) can be given a geometrically meaningful and interesting characterisation – this will then serve as the notion of '(semi)stable objects' $\mathcal{A}^{(s)s}$ of the moduli problem $(\mathcal{A}, \sim)$. In particular, the choice of linearisation will affect what part of $\mathcal{A}$ we construct a moduli space for. Effectively, a coarse moduli scheme can be seen as a patch of the moduli stack in the background, where varying the linearisation used in the GIT construction changes which part of the stack is seen by the scheme.

Finally, the GIT quotient of the stable points in X by G will be a coarse moduli space (quasi-projective over S) for the $\sim$-equivalence classes of stable objects $\mathcal{A}^s$, and the GIT quotient of the semistable points in X by G will be a coarse moduli space of the S-equivalence classes (*not* $\sim$-classes!) of semistable objects $\mathcal{A}^{ss}$. Here, two $\sim$-equivalence classes $[V_x], [V_y] \in \mathcal{A}^{ss}/\sim$ are called *S-equivalent* (written $A \sim_S B$ with 'S' for Seshadri – not to be confused with our base scheme S) if the closures of the orbits $G(x)$ and $G(y)$ in the set of semistable points of X intersect, i.e. if and only if the orbits of x and y are identified in the GIT quotient $X//G$ – see section 2.2 for more details on how orbits are collapsed under a GIT quotient. (Note that this description of S-equivalence is fully on the GIT side of the picture, and every moduli construction requires some work to find a geometric interpretation of S-equivalence for strictly semistable objects of $\mathcal{A}$.)

Even more is true: if the parameter space X is projective over S, then the quotient of the semistable locus will be projective over S, giving a natural compactification of the moduli space of stable objects. (If X is arbitrary, then the GIT quotient of the semistable points will still be quasi-projective over S, but may not be projective over S.)

In the next section, we summarise the basic ideas of GIT and justify the claims above about the connection between the GIT quotient of X by G and coarse moduli spaces for the original moduli problem. A description of Hilbert and Quot schemes, the basic parameter spaces involved in such moduli constructions, is delayed to section 2.3.

2.2 Geometric invariant theory

Fix an algebraically closed field $\Bbbk$, and take all schemes to be $\Bbbk$-schemes of finite type for now – we return to the relative situation of the previous section ($X \in \mathsf{Sch}/S$ for some fixed base scheme S) later. Throughout, G is a linear algebraic group over $\Bbbk$, later assumed to be reductive. Recall that G is said to be *reductive* if its radical is isomorphic to a direct product of copies of the multiplicative group $\mathbb{G}_m$. For linear algebraic groups over $\Bbbk = \mathbb{C}$, this is the same as asking that all complex representations of G be completely reducible, i.e. that they split into sums of irreducibles, or equivalently that there be a maximal compact Lie subgroup K of G such that $G = K_{\mathbb{C}}$ is its complexification, i.e. $(\operatorname{Lie} K) \otimes_{\mathbb{R}} \mathbb{C} = \operatorname{Lie} G$.

We give a very brief outline of the basic ideas of GIT, but leave many details to [New78] and [MFK94], see also [Tho06].

Let X be a scheme with a G-action $G \times X \to X$. The basic question motivating geometric invariant theory (GIT) is how to construct a 'good' quotient scheme for this action with properties to be clarified below, but after the discussion of the previous section, a good quotient should at least be a categorical quotient of (some open subscheme of) X by G. The naïve thing to do is to look at the orbit space (i.e. the topological quotient) X/G. However, this often fails to be a separated scheme (or worse, X/G may not even admit any scheme structure making the projection $X \to X/G$ into a morphism). Indeed, if X is a complex projective variety (a case of frequent interest), it is compact in the analytic topology, but many linear algebraic groups G (including all reductive groups over $\Bbbk = \mathbb{C}$) are not, and then any non-trivial G-action on X will not be proper. This leads to non-closed orbits (with lower-dimensional orbits in their closure), so the topological quotient cannot be separated. It turns out that just throwing away orbits of low dimension is not in general good enough either. One of the simplest examples for this problem is the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ given by $\lambda \cdot (x, y) := (\lambda x, \lambda^{-1} y)$. There are three kinds of orbits: firstly, for each $\alpha \in \Bbbk^*$ the conic $\{(x, y) | xy = \alpha\}$, next, the punctured axes $\{(x, 0) | x \in \Bbbk^*\}$ and $\{(0, y) | y \in \Bbbk^*\}$, and finally, the origin $\{(0, 0)\}$. Thus, the orbit space is not separated, even if we leave out the origin, as both punctured axes are limits of the conic orbits as $\alpha \to 0$ (so the topological quotient of $\mathbb{A}^2 \setminus (0, 0)$ will be the punctured affine line with two copies of the origin glued in). In order to get a categorical quotient, we have to combine all three exceptional orbits into one point, giving $\mathbb{A}^1$ as a categorical quotient of $\mathbb{A}^2$ by $\mathbb{G}_m$ with quotient map $(x, y) \mapsto xy$.

Recall that a categorical quotient of X by G is a G-invariant morphism of schemes $\phi : X \to Y$ satisfying a universal property for G-invariant morphisms from X. Thus, categorical quotients are necessarily unique up to unique isomorphism and are functorial, but may not be geometrically meaningful: for the action of $\mathbb{G}_m$ on $\mathbb{A}^n$ by scalar multiplication, the origin lies in the closure of every other orbit, so every $\mathbb{G}_m$-invariant morphism on $\mathbb{A}^n$ must be constant, thus there is no orbit space and the categorical quotient consists of a point. Of course, the right thing to do in this example is to delete the origin (which is GIT-unstable with respect to a well chosen linearisation), giving $\mathbb{P}^{n-1}$ as a categorical quotient and also an orbit space of $\mathbb{G}_m \curvearrowright \mathbb{A}^n \smallsetminus \{(\underline{0})\}$. This is a hint of the general picture: we first need to understand which 'bad' orbits should be discarded before we can hope to form a good quotient.

An important special case of the general GIT quotient construction is when X is projective, so we concentrate on this case first. The general definitions for the case of possibly non-projective X will follow from page 22 onwards. (In particular, note that for projective X we take linearisations to be ample, whereas this requirement is dropped for general X.)

Suppose X is projective. As a first approximation, define a *linearisation* of the G-action on X to be an embedding $X \hookrightarrow \mathbb{P}^n$ into some projective space, together with an extension of the G-action to a linear action on $\mathbb{P}^n$, i.e. an embedding of X into $\mathbb{P}^n$ and a representation $\rho : G \to \mathrm{GL}_{n+1}$ through which the action of G on X factors. Slightly more generally, a linearisation is a choice of ample line bundle $L \to X$ (in the case that L is very ample, this corresponds to the choice of embedding $X \hookrightarrow \mathbb{P}^n$ via $L = \mathcal{O}_{\mathbb{P}^n}(1)|_X$) and a lift of the G-action to a linear action on L. Then the pair (X, L) corresponds to its graded homogeneous coordinate ring

$$A_L(X) := \bigoplus_{k \geq 0} H^0(X, L^{\otimes k}) = \frac{\Bbbk[x_0, \dots, x_n]}{I_X},$$

where I_X is the homogeneous ideal defining X as a subscheme of $\mathbb{P}^n$, and the notation $A_L(X)$ indicates the dependence on the choice of L (or equivalently on the choice of embedding $X \hookrightarrow \mathbb{P}^n$).

The basic idea of GIT is now to consider the ring of G-invariants of X, i.e. the subring $A_L(X)^G$ of elements which are fixed by the G-action on $A_L(X)$ induced from the G-action on L or $\mathbb{A}^{n+1}$. As $A_L(X)$ is a finitely generated graded $\Bbbk$-algebra, $A_L(X)^G$ is also a finitely generated graded $\Bbbk$-algebra if G is reductive (see below for further explanation of this point). Replacing L by $L^{\otimes k}$ for some $k \geq 1$, we may choose generators $\tau_0, \dots, \tau_l$ of degree 1 for $A_L(X)^G$. Then we define the GIT quotient $X/\!/G$ (or, more accurately, $X/\!/_L G$ if we wish to emphasise the linearisation) to be the (polarised) projective subscheme of $\mathbb{P}^l$ corresponding to $A_L(X)^G$, i.e.

$$X/\!/_L G := \mathrm{Proj}\ A_L(X)^G = \mathrm{Proj} \bigoplus_{k \geq 0} H^0(X, L^{\otimes k})^G.$$

The inclusion $A_L(X)^G \subset A_L(X)$ corresponds to a rational map $X \dashrightarrow X/\!/G$ given by $x \mapsto [\tau_0(x) : \cdots : \tau_l(x)]$.

We then get the following commutative diagram:

$$\begin{array}{ccc} X & \dashrightarrow & X/\!/_L G \\ \cup & \nearrow & \\ X^{ss}(L) & & \cup \\ \cup & & \\ X^s(L) & \longrightarrow & X^s(L)/G \end{array} \qquad (2.1)$$

Here, $X^{ss}(L)$ is the set of *GIT-semistable* points (with respect to L), precisely the open (but not necessarily non-empty) subset of X where the rational map $X \dashrightarrow X/\!/_L G$ is defined, i.e. semistable points are those points which the G-invariant sections of L can detect. The elements of $X^s(L)$ are called *GIT-stable* points (with respect to L), and $X^s(L)$ is an open subset of $X^{ss}(L)$ where $X//_L G$ is a geometric quotient, i.e. an orbit space. (Algebraically, a semistable point x is defined to be stable if $A_L(X)^G$ separates orbits near x and the stabiliser of x is finite.) The map $X^{ss}(L) \to X//_L G$ is G-invariant (by construction of $X \dashrightarrow X//_L G$) and surjective, but may in general collapse distinct orbits in X to the same point in $X//_L G$, unless $X^{ss}(L) = X^s(L)$; this gives rise to the notion of S-equivalence alluded to at the end of section 2.1.

Topologically, (semi)stability may be characterised by picking a point $\tilde{x} \in \mathbb{A}^{n+1}$ covering $x \in X \subset \mathbb{P}^n$ (or equivalently a point $\tilde{x} \in L$ in the fibre over x), then

$$\begin{aligned} x \text{ is semistable} \quad &\Longleftrightarrow \quad 0 \notin \overline{G(\tilde{x})}, \\ x \text{ is stable} \quad &\Longleftrightarrow \quad \overline{G(\tilde{x})} = G(\tilde{x}) \text{ in } \mathbb{A}^{n+1} \text{ and } \tilde{x} \text{ has finite stabiliser,} \end{aligned} \tag{2.2}$$

where $G(\tilde{x})$ is the G-orbit of $\tilde{x}$, and finiteness of the stabiliser is equivalent to the orbit of $\tilde{x}$ being of maximal dimension $\dim G$. (For proofs of these claims, see proposition 4.7 and lemma 3.17 in [New78], or proposition 2.1 in [New09].) This description makes it clear that (semi)stability depends on the choice of linearisation. A version of this topological characterisation that is easier to use in practice is given by the Hilbert-Mumford numerical criterion whose key part is the result that (semi)stability for G corresponds to (semi)stability for all one-parameter subgroups $\mathbb{G}_m$ of G simultaneously (see page 25).

Algebraically, the crucial step of the whole construction described above is that the ring of invariants $A_L(X)^G$ is finitely generated (a version of Hilbert's 14th problem). This is where reductivity of G comes into play: Nagata's Theorem (see theorem 3.4 in [New78] for a proof) says that the ring of invariants A^G is finitely generated for any rational[1] action of G on a finitely generated $\Bbbk$-algebra A, provided that G satisfies a technical condition: a linear algebraic group G is said to be *geometrically reductive* over $\Bbbk = \overline{\Bbbk}$ if for every linear action of G on $\Bbbk^n$ and every non-zero G-invariant point $v \in \Bbbk^n$, there is an invariant homogeneous polynomial $f \in \Bbbk[x_1, \ldots, x_n]$ of degree ≥ 1 such that $f(v) \neq 0$. At the time of the first edition of [Mum65b] in 1965, it was only known that geometric reductivity is equivalent to reductivity of G if char $\Bbbk = 0$; Mumford conjectured this to be true over algebraically closed fields $\Bbbk$ of arbitrary characteristic, and the proof of Mumford's Conjecture was completed by Haboush [Hab75] in 1974. Using this result, Seshadri generalised GIT to reductive group actions on schemes over algebraically closed fields of any characteristic [Ses77].

[1] A *rational* G-action on a $\Bbbk$-algebra A is a G-action on A by $\Bbbk$-algebra automorphisms, such that every element of A is contained in a finite-dimensional subspace of A which is G-invariant and upon which G acts by a representation. Any G-action on a projective X linearised by ample L gives rise to a rational G-action on $A_L(X)$.

After this overview, let us briefly state the main results of GIT (for general X): throughout, let $\Bbbk$ be a fixed algebraically closed field of arbitrary characteristic, let all schemes be of finite type over $\Bbbk$, let G be a reductive linear algebraic group over $\Bbbk$ acting on a $\Bbbk$-scheme X which is not necessarily projective. Even though the discussion above (in particular the step of taking Proj of the G-invariant sections of L) really only applies to projective X and ample L, a GIT quotient can be formed for arbitrary X and L by patching together quotients of affine open subschemes. (The following results apply more generally to a reductive group scheme over an arbitrary base S acting on an S-scheme X, but we state them for simplicity in the absolute case $S = \operatorname{Spec} \Bbbk$: this is enough for our purposes because in the relative situation $X \in \mathsf{Sch}/S$ of section 2.1 which we return to shortly, the same $\Bbbk$-group G acts on each fibre of $X \to S$.)

Formally, a *linearisation* of the G-action is a choice of (not necessarily ample) line bundle L on X together with a lift of the G-action on X to a G-action on L which is linear on each fibre. (We often abuse terminology slightly and speak about 'the linearisation (given by) L', suppressing mention of the lift of the G-action to L.) A point $x \in X$ is said to be *(GIT-)semistable with respect to the linearisation* L if for some $r \in \mathbb{N}$ there is a G-invariant section f of $L^{\otimes r}$ such that $f(x) \neq 0$ and $X_f := \{x \in X | f(x) \neq 0\}$ is affine (note that X_f is automatically affine if L is ample and X is projective, which is the most interesting case for us). We call $x \in X$ *(GIT-)stable with respect to the linearisation* L if $\dim G(x) = \dim G$ (equivalently, if the stabiliser of x is finite) and if for some $r \in \mathbb{N}$ there is a G-invariant section f of $L^{\otimes r}$ such that $f(x) \neq 0$, X_f is affine and the action of G on X_f is closed. We usually just call points satisfying these definitions 'GIT-(semi)stable with respect to (G and) L'. Write $X^{(s)s}(L)$ for the GIT-(semi)stable points with respect to L. Recall that $x, y \in X^{ss}(L)$ are called *S-equivalent* if $\overline{G(x)} \cap \overline{G(y)} \cap X^{ss}(L) \neq \emptyset$. (Note that two GIT-stable points x and y are S-equivalent if and only if their G-orbits agree, because GIT-stability of a point x includes the requirement that the G-action on an open affine invariant neighbourhood X_f of x be closed.) Then the formal version of diagram (2.1) is:

Theorem 2.2.1 ([MFK94], theorem 1.10 and remark just before converse 1.12; see [Ses77] for proofs concerning projectivity and S-equivalence; also cf. [New78], theorems 3.21, 3.14, and proposition 3.11)**.** Given a $\Bbbk$-scheme X of finite type, equipped with an action of a reductive linear algebraic group G linearised by a line bundle L, a categorical quotient $\phi : X^{ss}(L) \to Y$ of $X^{ss}(L)$ by G exists. In fact, this is a *good quotient*, i.e. ϕ is a uniform categorical quotient, an affine map, and Y is a quasi-projective $\Bbbk$-scheme. Furthermore, there is an open subscheme Y^s of Y such that $\phi^{-1}(Y^s) = X^s(L)$ and (Y^s, ϕ) is a *geometric quotient*, i.e. a good quotient which is also an orbit space of $X^s(L)$.

For two (strictly) semistable points $x, y \in X^{ss}(L)$, we have $\phi(x) = \phi(y)$ if and only if x and y are S-equivalent, i.e. the closed points of Y correspond to the closed orbits in $X^{ss}(L)$

and for each $x \in X^{ss}(L)$ its image $\phi(x)$ is the point of Y corresponding to the unique closed orbit contained in $\overline{G(x)} \cap X^{ss}(L)$.

If X is projective and L is ample, then $Y \cong \text{Proj } \bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^G$ and thus Y is projective.

We usually write $X//G$ for Y, or $X//_L G$ if we wish to emphasise the linearisation with respect to which the quotient is taken, and we often refer to $X//_L G$ as the GIT quotient of X by G with respect to L, rather than the quotient of $X^{ss}(L)$.

The statement that $\phi : X^{ss}(L) \to Y$ is a *uniform categorical quotient* means that for all flat morphisms $Y' \to Y$ the projection $X^{ss}(L) \times_Y Y' \to Y'$ is a categorical quotient for the induced G-action. If this holds for all morphisms $Y' \to Y$, whether flat or not, then we call Y a *universal categorical quotient*. In characteristic 0, the two notions coincide (see [Ses77], remark 8), but in positive characteristic the GIT quotient will only be uniform categorical.

Note also that this theorem explains why we emphasised the importance of a projective parameter space X in section 2.1: then the GIT quotient serving as the moduli space is automatically projective, provided we work with an ample linearisation L. (However, even if X is not projective, it is sometimes possible to prove projectivity of the moduli space using non-GIT techniques: most importantly in some moduli constructions for vector bundles and sheaves using Langton's method [Lan75], see also 2.B in [HL97].) Given the very mild assumptions on X, quasi-projectivity of the quotient Y in the general case may come as a surprise, but note that $X^{ss}(L)$ is automatically quasi-projective since $L|_{X^{ss}(L)}$ is ample (this follows directly from the definition of semistability: for every $x \in X^{ss}(L)$ there is a section f of $L^{\otimes r}$ for some $r \in \mathbb{N}$ so that $(X^{ss}(L))_f$ is affine).

Observe that the correspondence between closed points of Y and S-equivalence classes of semistable points in X can be refined: there is an intermediate locus $X^{ps}(L)$ between $X^s(L)$ and $X^{ss}(L)$, consisting of the *GIT-polystable* points, i.e. those semistable points whose orbits are closed in $X^{ss}(L)$. Then $Y = X^{ps}(L)/G$ is an orbit space: in every S-equivalence class of semistable orbits, there is a unique polystable orbit. In our opening example at the start of this section ($\mathbb{G}_m$ acting on $\mathbb{A}^2$ with weights 1 and -1), the polystable orbits were the conics $\{(x,y)|xy = \alpha\}$ and the origin. The punctured axes $\{(x,0)|x \in \mathbb{k}^*\}$ and $\{(0,y)|y \in \mathbb{k}^*\}$ are semistable but not polystable, so the point 0 in the GIT quotient $\mathbb{A}^1 = \mathbb{A}^2//\mathbb{G}_m$ can be thought of as either corresponding to the S-equivalence class of all three strictly semistable orbits, or as representing the unique strictly polystable orbit.

We now justify our claims at the end of section 2.1 about how GIT helps in constructing a coarse moduli space for a moduli problem $(\mathcal{A}, \sim)$. Recall that the situation of the last section: S is a $\mathbb{k}$-scheme of finite type, and our parameter space X is a scheme of finite type (possibly even projective) over S, with a reductive group G acting on X and the morphism $X \to S$ being G-invariant. Our first goal is to show that GIT-(semi)stability may be analysed fibrewise. A result in this direction is:

Proposition 2.2.2 ([MFK94], theorem 1.19)**.** Suppose V and Z are $\Bbbk$-schemes which G acts on, and let L be a very ample line bundle on Z linearising the G-action. Suppose V is projective and $f : V \to Z$ is a G-invariant closed embedding. Then $V^{ss} = f^{-1}(Z^{ss})$ and $V^s = f^{-1}(Z^s)$, where GIT-(semi)stability in V and Z are taken with respect to f^*L and L, respectively.

This result will be useful in all our GIT constructions, in particular when we linearise group actions on Quot schemes via the Grothendieck embeddings into Grassmannians (see section 2.3 and chapter 4): then GIT-(semi)stability of a point in the Quot scheme will be equivalent to GIT-(semi)stability of the same point considered in the Grassmannian. The proposition also characterises GIT-(semi)stable points in the relative situation as the GIT-(semi)stable points in the fibres:

Lemma 2.2.3. Suppose $Z \to T$ is projective and invariant for a G-action on Z linearised by a relatively very ample line bundle L on Z. Then for every geometric point $t \in T$, the GIT-(semi)stable points in the fibre Z_t (with respect to the restricted line bundle L_t) are those which are GIT-(semi)stable in Z, i.e. $(Z_t)^{(s)s}(L_t) = (Z^{(s)s}(L))_t$.

Proof. This follows immediately from the previous proposition: take $V = Z_t$. □

We can now justify the claim that the GIT quotient of the parameter space X by G is a coarse moduli space for $(\mathcal{A}^{ss}, \sim_S)$:

Theorem 2.2.4. Given a moduli problem $(\mathcal{A}, \sim)$ with associated moduli functor $\mathbf{F} : \mathsf{Sch}/S \to \mathsf{Sets}$, suppose that we have a parameter space X of finite type over S with local universal family V (for a bounded sub-collection $\widetilde{\mathcal{A}} \subset \mathcal{A}$) satisfying

$$V_x \sim V_y \iff G(x) = G(y)$$

for some reductive group G acting on X (such that the morphism $X \to S$ is G-invariant). Then the GIT quotient of X by G (with respect to any linearisation L) is a coarse moduli scheme for S-equivalence classes of semistable objects of $\widetilde{\mathcal{A}}$ and is projective over S if $X \to S$ is projective and L relatively very ample. Here, an object $A \in \widetilde{\mathcal{A}}$ is called (semi)stable if $A \sim V_x$ for $x \in X^{(s)s}(L)$, and semistable objects $A, B \in \widetilde{\mathcal{A}}$ are called S-equivalent if the corresponding points of X are S-equivalent with respect to the G-action.

This theorem now follows from combining theorem 2.2.1 with an easy generalisation of proposition 2.1.2: the only modification needed in proposition 2.1.2 is that a coarse moduli space for $(\mathcal{A}, \sim_S)$, where $\sim_S$ is S-equivalence rather than the original $\sim$, is exactly a categorical quotient whose points correspond bijectively to S-equivalence classes of G-orbits on X. The GIT quotient of X^s (the locus of stable points in X) is an open subscheme of $X//G$ which by proposition 2.1.2 is a coarse moduli space for $\sim$-classes of stable objects of $\mathcal{A}$: on stable objects, $\sim_S$ is just $\sim$, as the GIT quotient of the stable locus is an orbit space. Finally, note that that the quotient $X//_L G$ is automatically an S-scheme ($X \to S$ is G-invariant, so factors through the quotient $X//_L G$) and if $X \to S$ is projective and L very ample, then lemma 2.2.3 implies that the fibre of $X//_L G$ over $s \in S$ is just the GIT quotient of X_s by G which is projective by theorem 2.2.1.

Of course, the notion of (semi)stability (as well as S-equivalence) depends on the GIT set-up, i.e. the choice of parameter scheme X, the group action, and the linearisation L involved in forming the GIT quotient. In particular, this leads to 'variation of GIT', one of the most interesting aspects of geometric invariant theory: the study of how quotients vary under change of linearisation, and the resulting problem of how moduli spaces for different stability conditions relate to each other. For introductions to this topic, see [Dol03], [Tha96], and [DH98].

Note that an analysis of GIT-(semi)stability for points in the parameter space is difficult to carry out given the definitions above. For proper (in particular, for projective) X and ample L, the situation is much-improved by the Hilbert-Mumford criterion: crucially, it turns out that a point $x \in X$ is GIT-(semi)stable with respect to G and L if and only if it is GIT-(semi)stable with respect to every one-parameter subgroup (1-PS) $\lambda : \mathbb{G}_m \to G$ and L, and an analysis of (semi)stability for these is much simpler. Given a 1-PS $\lambda : \mathbb{G}_m \to G$ and $x \in X$, define $x_0 := \lim_{t \to 0} \lambda(t) \cdot x$ (which exists by properness of X). The group $\lambda(\mathbb{G}_m)$ acts on the fibre L_{x_0} by a character of $\mathbb{G}_m$, say $t \mapsto t^r$ with weight r. Set $\mu^L(x, \lambda) := -r$. Then, based on an idea of Hilbert, Mumford proves:

Theorem 2.2.5 ([MFK94], theorem 2.1)**.** Given a proper $\Bbbk$-scheme X equipped with an action of a reductive linear algebraic group G linearised by an ample line bundle L, a point $x \in X$ is GIT-(semi)stable with respect to G and L if and only if $\mu^L(x, \lambda)(\geqslant)0$ for all non-zero one-parameter subgroups λ of G.

This is just a concise way of saying that GIT-(semi)stability for G is equivalent to GIT-(semi)stability for all 1-PS of G (which is a consequence of reductivity: this ensures that G has enough 1-PS), and that (semi)stability for a 1-PS is easy to describe: $\mu^L(x, \lambda)(\geqslant)0$ is a statement about whether $\lim_{t \to 0} \lambda(t) \cdot \tilde{x}$ exists (if $\tilde{x}$ is any lift of x to L), and if it does, whether it is non-zero; now use the topological description (2.2) of (semi)stability.

Note that μ^L behaves well with respect to conjugation of 1-PS: we have $\mu^L(gx, \lambda) = \mu^L(x, g^{-1}\lambda g)$ for any $g \in G$, so we may always replace λ by a convenient conjugate to simplify calculations.

For a far more detailed description of the technical foundations of GIT, we refer the reader to [New78] and [MFK94]. At the end of this section, we need to quote one result about GIT on Grassmannians which will be required in later chapters, expressing the Hilbert-Mumford criterion for the case of Grassmannians of the form $\mathrm{Grass}(V \otimes W, r)$. Let $\Bbbk$ be an algebraically closed field, let V and W be finite-dimensional $\Bbbk$-vector spaces and consider the Grassmannian of r-dimensional quotients of $V \otimes W$. The group $\mathrm{SL}(V)$ acts naturally on $\mathrm{Grass}(V \otimes W, r)$, and the Plücker embedding gives us a natural linearisation L (corresponding to an $\mathrm{SL}(V)$-equivariant projective embedding) – explicitly, the fibre of the very ample line bundle L over a point of $\mathrm{Grass}(V \otimes W, r)$ corresponding to a quotient $V \otimes W \to B \to 0$ is given by $\Lambda^r B$.

Proposition 2.2.6 ([Sim94a], proposition 1.14)**.** A point of $\mathrm{Grass}(V \otimes W, r)$ corresponding to a quotient $V \otimes W \to B \to 0$ is GIT-(semi)stable for the $\mathrm{SL}(V)$-action with respect to L if and only if, for all non-zero proper subspaces $U < V$, we have

$$\dim B \cdot \dim U (\leqslant) \dim V \cdot \dim \mathrm{Im}(U \otimes W)$$

where $\mathrm{Im}(U \otimes W)$ denotes the image of $U \otimes W \hookrightarrow V \otimes W \to B$.

Proof. The result follows from proposition 4.3 in [MFK94] – note that the signs are reversed as Mumford considers the Grassmannian of subspaces, not quotients. □

2.3 Hilbert and Quot schemes

In [Gro62], Grothendieck gave a construction of Hilbert and Quot schemes, the basic parameter spaces in algebraic geometry. As explained in section 2.1, the standard GIT approach to moduli problems first requires a parameter space (i.e. a space parametrising a family satisfying the local universal property) for a rigidified moduli problem such that the equivalence classes we would like to classify are precisely the orbits for an action of a reductive linear algebraic group on the parameter space. Rigidification means modifying the moduli problem so that objects no longer have non-trivial automorphisms – this is usually achieved by considering sub- or quotient objects of a fixed object, e.g. subschemes of a fixed ambient scheme or quotient sheaves of a fixed sheaf, and these are parametrised by Hilbert and Quot schemes. Thus, the significance of the particular moduli spaces discussed in this section is that in most situations the parameter space in the first step of a GIT construction is given by a suitable Hilbert or Quot scheme.

Intimately related to the search for a parameter space in the first step of a moduli problem $(\mathcal{A}, \sim)$ is the question of boundedness: is the collection $\mathcal{A}$ 'too big' to be parametrised, or can we 'bound' it by a scheme of finite type? This and the twin notion of regularity of sheaves is discussed in subsection 2.3.2; a lot of the notions and results about coherent sheaves introduced there will be used frequently in later chapters.

We mainly follow [Nit05] for the construction of Hilbert and Quot schemes, except that we will not worry about representing the Quot functor when $X \to S$ is only (quasi-) projective in the sense of Grothendieck or Altman-Kleiman instead of the strictly stronger standard notion of (quasi-)projectivity: if S is a noetherian scheme, Grothendieck calls a morphism $X \to S$ *(quasi-)projective* if it factors into a (locally) closed embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$ followed by the projection $\mathbb{P}(\mathcal{E}) \to S$ for some coherent sheaf $\mathcal{E}$ on S; Altman and Kleiman call a morphism $X \to S$ of noetherian schemes *strongly (quasi-)projective* if the same holds for a locally free sheaf $\mathcal{E}$ on X. This is still weaker than the usual notion of (quasi-)projectivity as in [Har77], for example, where a morphism $X \to S$ of noetherian schemes is called (quasi-)projective if it factors into a (locally) closed embedding $X \hookrightarrow \mathbb{P}^n_S$ followed by the projection $\mathbb{P}^n_S \to S$, and this is what we mean by (quasi-)projective morphisms throughout.

We outline the key steps of the construction of Hilbert and Quot schemes for projective $X \to S$, but omit detailed proofs (which are all readily available in [Nit05], [HL97], [Gro62] or [Ser06], for example), except in a few places where we wish to emphasise properties of Quot schemes that are either not explicitly pointed out in [Nit05] or will be of particular importance in later sections. Although it is not much harder, we will not explain what additional arguments are needed to generalise the construction from projective to quasi-projective $X \to S$, since this case will not be needed in what follows.

2.3.1 The Hilbert and Quot functors

Let us first describe the moduli problems which are solved by Hilbert and Quot schemes. Let S be a noetherian scheme; throughout section 2.3, we consider functors on the category Sch/S of locally noetherian schemes over S. For each scheme $X \to S$ of finite type over S and any coherent sheaf $\mathcal{F}$ on X, we define a moduli functor parametrising quotients of $\mathcal{F}$ which are flat and have proper support over S. By thinking of subschemes of X in terms of their ideal sheaves and the resulting quotients of $\mathcal{O}_X$, we get a moduli functor parametrising subschemes of X which are proper and flat over S as a special case. Recall convention 1.4.1 for our notation used for base change.

Definition 2.3.1. Given a locally noetherian scheme $T \in \mathsf{Sch}/S$, a *family of quotients of $\mathcal{F}$ parametrised by T* is a pair $(\mathcal{E}, q)$ where

(a) $\mathcal{E}$ is a coherent sheaf on X_T which is flat over T and whose (schematic) support is proper over T; and

(b) $q : \mathcal{F}_T \twoheadrightarrow \mathcal{E}$ is a surjective morphism of $\mathcal{O}_{X_T}$-modules.

Two families $(\mathcal{E}, q)$ and $(\mathcal{E}', q')$ parametrised by T are said to be *equivalent* if there is an isomorphism $\mathcal{E} \cong \mathcal{E}'$ making the diagram

$$\begin{array}{ccc} \mathcal{F}_T & \xrightarrow{q} & \mathcal{E} \\ \| & & \downarrow \cong \\ \mathcal{F}_T & \xrightarrow{q'} & \mathcal{E}' \end{array}$$

commute – equivalently, $(\mathcal{E}, q) \sim (\mathcal{E}', q')$ if $\ker(q) = \ker(q')$. The equivalence class of $(\mathcal{E}, q)$ will be denoted by $[\mathcal{E}, q]$, or sometimes by $[q : \mathcal{F}_T \twoheadrightarrow \mathcal{E}]$. Then the pullback of $[\mathcal{E}, q]$ under a morphism $T' \to T$ in Sch/S is well-defined (using that properness and flatness are preserved by base-change and that tensor product is right exact), so we get a contravariant functor

$$\begin{aligned} \mathbf{Quot}_{\mathcal{F}/X/S} : \mathsf{Sch}/S &\to \mathsf{Sets} \\ T &\to \{\text{equivalence classes } [\mathcal{E}, q] \text{ of families parametrised by } T\}, \end{aligned}$$

i.e. a moduli functor in the sense of section 2.1, known as the *Quot functor*.

For a non-trivial base scheme S, this functor is sometimes also referred to as a *relative* Quot functor of quotient sheaves on the fibres of X/S – see lemma 2.3.2 and the preceding discussion for a justification of this name. As a special case, note that for $X = S$ the flatness requirement is equivalent to quotients being locally free, and in this case $\mathbf{Quot}_{\mathcal{F}/S/S}$ is known as the (relative) Grassmann functor $\mathbf{Grass}_{\mathcal{F}/S}$. The representability of $\mathbf{Grass}_S$ is a simple consequence of the existence of vector space Grassmannians (with $S = \operatorname{Spec} \Bbbk$): see e.g. example 5.1.5(2) in [Nit05], or example 2.2.3 in [HL97]. In the other extreme case of $S = \operatorname{Spec} \mathbb{Z}$ (or $S = \operatorname{Spec} \Bbbk$, if we are working with respect to a fixed base field $\Bbbk$), the functor is called an *absolute* Quot functor.

Taking $\mathcal{F} = \mathcal{O}_X$, the basic correspondence $Y \subset X \longleftrightarrow \mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}$ between closed subschemes Y and ideal sheaves $\mathcal{I} \subset \mathcal{O}_X$ shows that for any $T \in \mathsf{Sch}/S$ the elements of $\mathbf{Quot}_{\mathcal{O}_X/X/S}(T)$ are the closed subschemes of X_T that are proper and flat over T. In this case the Quot functor is known as the *Hilbert functor* $\mathbf{Hilb}_{X/S} := \mathbf{Quot}_{\mathcal{O}_X/X/S}$ and we refer to the elements of $\mathbf{Hilb}_{X/S}(T)$ as *families of subschemes of X parametrised by T*.

It may sometimes be more natural to think in terms of subsheaves of a fixed sheaf, and the definition of families and their equivalence shows that the data of the equivalence class $[q : \mathcal{F}_T \twoheadrightarrow \mathcal{E}]$ is equivalent to giving the subsheaf $\ker(q) \subset \mathcal{F}_T$. However, we generally view the elements of $\mathbf{Quot}_{\mathcal{F}/X/S}(T)$ as quotients to guarantee functoriality: tensor product is right-exact, so pull-backs of quotient sheaves are quotient sheaves.

Each Quot functor is stratified by *Hilbert polynomials*: if $\mathcal{E}$ is a coherent sheaf of proper support on a scheme Y of finite type over a field $\Bbbk$, and L is a line bundle on Y, then the Euler characteristic of the successive twists $\mathcal{E}(m) := \mathcal{E} \otimes L^{\otimes m}$ defines a function

$$\chi(\mathcal{E}(m)) = \sum_{i \geqslant 0} (-1)^i h^i(\mathcal{E}(m))$$

which is a polynomial in m with rational coefficients, denoted by $P(\mathcal{E}, m)$. For a proof that $P(\mathcal{E})$ is rational polynomial in m (of degree equal to the dimension of Supp $\mathcal{E}$), see lemma 1.2.1 in [HL97] in the case of projective Y and very ample L (which is the situation we will be interested in) or theorem B.7 in [Kle05] for the general case. Of course, there is also a relative version of this concept: let $Y \to R$ be a finite type morphism of noetherian schemes and M a line bundle on Y. Then given a coherent sheaf $\mathcal{E}$ on Y whose schematic support is proper over R, the restriction $\mathcal{E}_r := \mathcal{E}|_{Y_r}$ to the fibre over any $r \in R$ has a Hilbert polynomial $P_r \in \mathbb{Q}[t]$ with respect to the line bundle $M_r := M|_{Y_r}$:

$$P_r(m) := \chi(\mathcal{E}_r(m)) = \sum_{i=0}^{\dim \mathcal{E}_r} (-1)^i \dim_{k(r)} H^i(Y_r, \mathcal{E}_r \otimes M_r^{\otimes m}).$$

If in addition $\mathcal{E}$ is flat over R and M is relatively very ample, then the map $R \to \mathbb{Q}[t]$ given by $r \mapsto P_r$ is locally constant ([Gro63], proposition 7.9.11). Now apply this with $R = T \in \mathsf{Sch}/S$, $Y = X_T$ and $M = L_T$ in the same set-up as before – this shows that for $X \to S$ a finite type morphism of noetherian schemes and L a relatively very ample line bundle on X, the functor $\mathbf{Quot}_{\mathcal{F}/X/S}$ decomposes as

$$\mathbf{Quot}_{\mathcal{F}/X/S} = \coprod_{P \in \mathbb{Q}[t]} \mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}$$

where $\mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}$ is the subfunctor of $\mathbf{Quot}_{\mathcal{F}/X/S}$ which associates to $T \in \mathsf{Sch}/S$ the set of $[\mathcal{E}, q]$ such that for each $t \in T$ the Hilbert polynomial of $\mathcal{E}_t$ with respect to L_t is P.

The Hilbert functor decomposes analogously with respect to any given very ample L: the functor $\mathbf{Hilb}^{L,P}_{X/S} := \mathbf{Quot}^{L,P}_{\mathcal{O}_X/X/S}$ sends T to the set of closed subschemes of X_T which are proper and flat over T and whose fibres over T have Hilbert polynomial P with respect to the restriction of L_T. Similarly, if $X = S$ then the fibres are points, so the Hilbert polynomial is just a number r (independent of L) and the Grassmann functor $\mathbf{Grass}^r_{\mathcal{F}/S} = \mathbf{Quot}^{L,r}_{\mathcal{F}/S/S}$ maps $T \in \mathsf{Sch}/S$ to the set of locally free rank r quotients of $\mathcal{F}_T$.

The aim of this section is to show that if $X \to S$ is (quasi-)projective, L very ample, and $\mathcal{F}$ a coherent quotient of $\mathcal{O}_X(\nu)^{\oplus p}$ for any $\nu \in \mathbb{Z}$ and $p \in \mathbb{N}$, then the Quot functor $\mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}$ is representable, its fine moduli scheme $\mathrm{Quot}^L_{X/S}(\mathcal{F}, P)$ is (quasi-)projective over S, and there is a natural embedding, known as the *Grothendieck embedding*, of Quot and Hilbert schemes into Grassmannians. (If $X \to S$ is projective and S Noetherian, then the condition on $\mathcal{F}$ is vacuous: every coherent sheaf is a quotient of $\mathcal{O}_X(\nu)^{\oplus p}$ for some ν and p – this is a consequence of Serre vanishing and regularity, see subsection 2.3.3.)

At the heart of the construction – at least in the simplified version (due to Mumford, Altman and Kleiman) which we discuss here – are two methods of projective geometry: Castelnuovo-Mumford regularity and flattening stratifications. The concept of regularity (see subsection 2.3.2) is used to construct a natural embedding of **Quot** into an appropriate Grassmann functor, and using flattening stratifications (theorem 2.3.14) one may show that this is a locally closed embedding. If $X \to S$ is projective, the valuative criterion for properness then shows that the embedding is in fact closed, so we obtain a representing scheme Quot which is a closed subscheme of the relevant Grassmannian, hence projective (via Plücker embeddings).

Before starting the construction, we note that Quot functors behave well under base change, so if $X' \to S'$ and $\mathcal{F}'$ on X' are obtained from some $\mathcal{F}$ on $X \to S$ by base change via some morphism $f : S' \to S$, then proving representability of $\mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}$ is sufficient for representability of $\mathbf{Quot}^{L',P}_{\mathcal{F}'/X'/S'}$. Compatibility with base change also shows the precise sense in which $\mathrm{Quot}^{L}_{X/S}(\mathcal{F}, P)$ is *relative* to the base S: the relative Quot scheme $Q := \mathrm{Quot}^{L}_{X/S}(\mathcal{F}, P)$, if it exists, comes equipped with a map $\pi : Q \to S$ down to the base scheme, and the scheme-theoretic fibre $\pi^{-1}(s)$ over any point $s \in S$ is just $\mathrm{Quot}^{L_s}_{X_s/\mathrm{Spec}\, k(s)}(\mathcal{F}_s, P)$, the Quot scheme of quotients of the restriction $\mathcal{F}_s$ to the fibre X_s. Therefore, the relative Quot scheme of $\mathcal{F}$ on X/S is nothing other than the family of Quot schemes of the restriction $\mathcal{F}_s$ to the fibres X_s, and a point $q \in Q$ of the relative Quot scheme can be thought of as a point $\pi(q) = s$ of S together with a quotient of $\mathcal{F}_s$ on the fibre X_s. (As a special case, note that the fibre of a relative Grassmannian $\mathrm{Grass}^r_S(\mathcal{F})$ over a $\Bbbk$-point $s \in S$ is just the classical Grassmannian of rank r vector space quotients of $\mathcal{F}_s$.) Recall convention 1.4.1 for base change notation.

Lemma 2.3.2. Given a morphism $\phi : T \to S$ of noetherian schemes, write $\Phi : \mathsf{Sch}/T \to \mathsf{Sch}/S$ for the covariant functor $\Phi(f : R \to T) = (\phi \circ f : R \to S)$. Let $X \to S$ be of finite type, let $\mathcal{F}$ be a coherent sheaf on X and L a line bundle on X, and pick $P \in \mathbb{Q}[t]$. Then we have a natural isomorphism

$$\mathbf{Quot}^{L,P}_{\mathcal{F}/X/S} \circ \Phi \cong \mathbf{Quot}^{L_T,P}_{\mathcal{F}_T/X_T/T} \tag{2.3}$$

of contravariant functors $\mathsf{Sch}/T \to \mathsf{Sets}$.

Therefore, if $\mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}$ is represented by $\pi : \mathrm{Quot}^{L}_{X/S}(\mathcal{F}, P) \to S$, then $\mathbf{Quot}^{L_T,P}_{\mathcal{F}_T/X_T/T}$ is also representable and

$$\mathrm{Quot}^{L}_{X/S}(\mathcal{F}, P) \times_S T \cong \mathrm{Quot}^{L_T}_{X_T/T}(\mathcal{F}_T, P), \tag{2.4}$$

a natural isomorphism as T-schemes with both schemes representing the same functor. In particular, taking $T = \text{Spec } k(s)$ for any point $s \in S$, this shows that $\pi^{-1}(s)$ is isomorphic to $\text{Quot}^{L_s}_{X_s/\text{Spec } k(s)}(\mathcal{F}_s, P)$ as a $k(s)$-scheme.

Proof. The isomorphism in (2.3) is trivial, noting that flatness and properness are preserved by base change, and (2.4) then follows formally from representability and the universal property of fibred products. $\square$

We conclude this subsection with an observation that, together with the previous lemma, reduces the task of representing $\mathbf{Quot}_{\mathcal{F}/X/S}$ to the absolute case $S = \text{Spec } \mathbb{Z}$ and $X = \mathbb{P}^n$. For formal categorical definitions of open and closed subfunctors, see section VI.1.1 in [EH00]. Note that if a functor $\mathbf{F} : \mathsf{Sch}/S \to \mathsf{Sets}$ is represented by $M \in \mathsf{Sch}/S$, then an open (closed, or locally closed) subfunctor of $\mathbf{F}$ is represented by an open (respectively closed, locally closed) subscheme of M.

Lemma 2.3.3 (cf. [Nit05], lemma 5.17)**.** Let $X \to S$ be a morphism of finite type between noetherian schemes, let $\mathcal{F} \in \mathsf{Coh}(X)$, let L be a line bundle on X, and choose $P \in \mathbb{Q}[t]$.

(a) For any $\nu \in \mathbb{Z}$, tensoring by $L^{\otimes \nu}$ gives a natural isomorphism between $\mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}$ and $\mathbf{Quot}^{L,P[\nu]}_{\mathcal{F}(\nu)/X/S}$, where $P[\nu](m) := P(m+\nu)$.

(b) Let $\iota : X \hookrightarrow Y$ be a closed embedding of noetherian schemes over S (compatible with the maps to S). Then the natural transformation $\mathbf{Quot}^{L,P}_{\mathcal{F}/X/S} \to \mathbf{Quot}^{\iota_* L,P}_{\iota_*\mathcal{F}/Y/S}$ given by considering sheaves $\mathcal{E}$ on X as sheaves $\iota_*\mathcal{E}$ on Y (via extension by zero) is a natural isomorphism.

(c) Let $\phi : \mathcal{F} \twoheadrightarrow \mathcal{G}$ be a surjective homomorphism of coherent sheaves on X. Then the induced natural transformation $\mathbf{Quot}^{L,P}_{\mathcal{G}/X/S} \to \mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}$ is a closed embedding of functors.

(d) In particular, if $X \to S$ is (quasi-)projective and L very ample, i.e. if $X \to S$ factors through a (locally) closed embedding $X \hookrightarrow \mathbb{P}^n_S$ (for some n) such that $L = \mathcal{O}_{\mathbb{P}^n_S}(1)|_X$, and if $\mathbf{Quot}^{\mathcal{O}_{\mathbb{P}^n}(1),P}_{\mathcal{O}^{\oplus p}_{\mathbb{P}^n}/\mathbb{P}^n/\mathbb{Z}}$ is represented by Q (where $\mathbb{P}^n := \mathbb{P}^n_{\mathbb{Z}}$), then for any coherent quotient $\mathcal{F}$ of $\mathcal{O}_X(\nu)^{\oplus p}$, the Quot functor $\mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}$ is represented by a (locally) closed subscheme of $Q \times_{\mathbb{Z}} S$.

Proof. Part (a) is clear: given $T \in \mathsf{Sch}/S$, tensoring $\mathcal{F}_T \twoheadrightarrow \mathcal{E}$ by $L_T^{\otimes \nu}$ gives $\mathcal{F}(\nu)_T \twoheadrightarrow \mathcal{E}(\nu)$ with $\chi(\mathcal{E}(\nu)(m)) = \chi(\mathcal{E}(m+\nu)) = P(m+\nu)$, hence the natural transformation of the Quot functors, which can clearly be inverted by tensoring with $L_T^{\otimes -\nu}$.

Part (b) is also immediate: the extension-by-zero map $\mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}(T) \to \mathbf{Quot}^{\iota_* L,P}_{\iota_*\mathcal{F}/Y/S}(T)$ given by

$$[q : \mathcal{F}_T \twoheadrightarrow \mathcal{E}] \mapsto [\iota_* q : (\iota_*\mathcal{F})_T \twoheadrightarrow \iota_*\mathcal{E}]$$

has restriction to X as inverse, i.e. $\mathbf{Quot}^{\iota_* L,P}_{\iota_*\mathcal{F}/Y/S}(T) \to \mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}(T)$ given by

$$[q : (\iota_*\mathcal{F})_T \twoheadrightarrow \mathcal{E}] \mapsto [\iota^*(q) : (\iota^*\iota_*\mathcal{F})_T = \mathcal{F}_T \twoheadrightarrow \iota^*\mathcal{E}],$$

since any quotient sheaf of $(\iota_*\mathcal{F})_T$ on Y_T is necessarily supported on $X_T \hookrightarrow Y_T$.

Part (c) needs a little more unravelling: we are required to show that for every $T \in \mathsf{Sch}/S$ and every $[\mathcal{E}, q] \in \mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}(T)$ there is a closed subscheme $T' \subset T$ such that for any $U \in \mathsf{Sch}/T$ the pulled back homomorphism $\mathcal{F}_U \twoheadrightarrow \mathcal{E}_U$ factors through the pulled back homomorphism $\phi_U : \mathcal{F}_U \twoheadrightarrow \mathcal{G}_U$ if and only if the map $U \to T$ factors through the closed embedding $T' \hookrightarrow T$. This can be accomplished by taking the vanishing scheme for the map of $\mathcal{O}_X$-modules $\ker\phi \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{E}$ (cf. remark 5.9 in [Nit05]).

Finally, part (d) follows for projective $X \to S$ from (a)-(c): by (b), we may replace X by $\mathbb{P}^n_S$. Lemma 2.3.2 then implies that we may replace S by Spec $\mathbb{Z}$, since $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times S$ and is obtained by base change from $\mathbb{P}^n_{\mathbb{Z}}$. By (a), we may take $\nu = 0$, and by (c), it then suffices to prove representability for $\mathcal{F} = \mathcal{O}^{\oplus p}_{\mathbb{P}^n}$ instead of a general quotient. If $X \to S$ is merely quasi-projective, we need a version of (b) for open subschemes $X \hookrightarrow Y$: then $\mathbf{Quot}^{L|_X,P}_{\mathcal{F}|_X/X/S}$ is an open subfunctor of $\mathbf{Quot}^{L,P}_{\mathcal{F}/Y/S}$ – we omit the proof of this fact as we will only sketch the construction of Quot schemes for projective $X \to S$. □

2.3.2 Boundedness and m-regularity

Good references for the material in this subsection are [Mum66] (lecture 14), [Kle71], [HL97] (section 1.7), and [Nit05] (section 5.2).

Let X be a projective scheme over a field $\Bbbk$ and $\mathcal{O}_X(1)$ a very ample line bundle on X (with respect to which all Hilbert polynomials are considered in the following). In order to construct a moduli space of a collection of sheaves on X, we need to ensure that the collection is not too big to be parametrised, i.e. that there is an algebraic family of sheaves on X (parametrised by a scheme of finite type) which contains our collection. This idea is captured by the notion of boundedness:

Definition 2.3.4 (e.g. [HL97], definition 1.7.5)**.** A collection $\mathfrak{F} = \{[\mathcal{F}_i]\}_{i \in I}$ of isomorphism classes of coherent sheaves on X is called *bounded* if there is a $\Bbbk$-scheme T of finite type and a coherent sheaf $\mathcal{F}$ on $X \times_{\Bbbk} T$ such that $\mathfrak{F} \subset \{[\mathcal{F}_t] \mid t \text{ is a closed point in } T\}$, where as usual $\mathcal{F}_t$ means the restriction of $\mathcal{F}$ to the fibre over t.

In most of the references for this section, the terminology used is '(bounded) *families* of sheaves'. We deliberately try not to use 'family' in this purely set-theoretic sense of 'collection' in order to avoid confusion with the use of 'family' in moduli theory as in section 2.1. In practice, we often talk about a collection of sheaves being bounded, suppressing mention of 'isomorphism classes' – whenever considering boundedness questions, it is implicit that sheaves are only considered up to isomorphism.

Note in particular that in a moduli problem of sheaves on X where a 'family' (in the sense of moduli functors) is defined as a sheaf on $X \times_{\Bbbk} T$, any family of sheaves parametrised by a $\Bbbk$-scheme of finite type is bounded. A significant consequence of boundedness is that it implies m-regularity for a uniform m:

Definition 2.3.5. Let $m \in \mathbb{Z}$. A coherent sheaf $\mathcal{F}$ on X is *m-regular* if

$$H^i(X, \mathcal{F}(m-i)) = 0 \quad \text{for all } i > 0.$$

We say that $\mathcal{F}$ is *regular*, or *Castelnuovo-Mumford regular*, if it is m-regular for some m.

Lemma 2.3.6. Given a collection of sheaves $\mathfrak{F} = \{[\mathcal{F}_i]\}_{i \in I}$ on X, the following are equivalent:

(a) $\mathfrak{F}$ is bounded.

(b) There exists m such that $\mathcal{F}_i$ is m-regular for all $i \in I$, and the set $\{P(\mathcal{F}_i)\}_{i \in I}$ of Hilbert polynomials is finite.

(c) There is a coherent $\mathcal{O}_X$-module $\mathcal{F}$ surjecting onto $\mathcal{F}_i$ for all $i \in I$, and the set $\{P(\mathcal{F}_i)\}_{i \in I}$ of Hilbert polynomials is finite.

Proof. This is the special case $S = \operatorname{Spec} \Bbbk$ of theorem 2.3.9 below, which is proved in [Kle71] (theorem 1.13). □

The point of regularity is revealed by the following lemma:

Lemma 2.3.7. If $\mathcal{F}$ is m-regular, then:

(a) $\mathcal{F}$ is m'-regular for all $m' \geqslant m$, and in particular $H^i(X, \mathcal{F}(m)) = 0$ for all $i > 0$;

(b) $\mathcal{F}(m)$ is generated by its global sections; and

(c) for all $n \geqslant 0$ the natural multiplication maps $H^0(X, \mathcal{F}(m)) \otimes H^0(X, \mathcal{O}_X(n)) \to H^0(X, \mathcal{F}(m+n))$ are surjective.

Proof. [Mum66] (lecture 14) or [Kle71] (proposition 1.3). □

Note in particular that m-regularity gives us a vanishing theorem for the higher cohomology of $\mathcal{F}(m)$, so $h^0(\mathcal{F}(m)) = \chi(\mathcal{F}(m))$, and by (b), we can recover $\mathcal{F}(m)$ from knowing its global sections. This is one of the key ideas in the construction of the Quot scheme: if $\mathcal{F}$ is m-regular, represent $\mathcal{F}(m)$ by the vector space $H^0(X, \mathcal{F}(m))$ which has fixed dimension $P(m)$, where P is the Hilbert polynomial of $\mathcal{F}$. Since we can recover $\mathcal{F}(m)$ from its global sections, and $\mathcal{F}$ from $\mathcal{F}(m)$ by tensoring with $\mathcal{O}_X(-m)$, this allows us to embed the Quot functor into an appropriate Grassmannian, provided that we can find a value of m that works uniformly for all $\mathcal{F}$ which we are interested in. This is where boundedness comes to our help: by Serre vanishing ([Har77], theorem III.5.2), every coherent sheaf on a projective scheme is m-regular for some m, but boundedness will allow us to choose such m uniformly. Hence, boundedness theorems are at the heart of GIT constructions of moduli spaces for a given class of sheaves (see chapter 4 for an example of this in practice) – boundedness is the main technical result in what we identified in section 2.1 as the first step of a moduli construction: the choice of a parameter scheme.

In our moduli constructions (both for parabolic sheaves on a family of projective schemes $X \to S$ and for universal moduli of parabolic sheaves and marked curves) we will need to work in a relative setting, and this requires versions of definition 2.3.4 and lemma 2.3.6 adapted to the case of X defined over a scheme S:

Definition 2.3.8 ([Kle71], definition 1.12)**.** Let S be a noetherian scheme and X an S-scheme of finite type. Let $\mathfrak{F}$ be a *collection of classes of coherent sheaves on the fibres of* X/S, i.e. for each point $s \in S$ and each field extension K of $k(s)$, we are given a collection $\{[\mathcal{F}_{K,j}] | j \in J_K \subset I\}$ of isomorphism classes of coherent sheaves on the scheme X_K, and for K and K' two field extensions of the same $k(s)$ one says that $\mathcal{F}_{K,j_1}$ and $\mathcal{F}_{K',j_2}$ are in the same class if there exist $k(s)$-homomorphisms of K, K' into a common field extension K'' of $k(s)$ such that $\mathcal{F}_{K'',j_1} (:= \mathcal{F}_{K,j_1} \otimes_K K'')$ and $\mathcal{F}_{K'',j_2}$ are isomorphic over $X_{K''}$. Then the collection $\mathfrak{F}$ is said to be *bounded by a coherent sheaf* $\mathcal{F}$ *on* $X_T = X \times_S T$, where T is a scheme of finite type over S, if $\mathfrak{F}$ is contained in the collection of classes of sheaves $\{[\mathcal{F}_t] | t \in T\}$, and $\mathfrak{F}$ is called *bounded* if it is bounded by some $\mathcal{F}$ and T as above.

Theorem 2.3.9 ([Kle71], theorem 1.13)**.** Let S be a noetherian scheme and X a projective S-scheme equipped with an ample line bundle $\mathcal{O}_X(1)$ such that for all $s \in S$ the induced line bundles $\mathcal{O}_{X_s}(1)$ are generated by their global sections (this condition is automatically satisfied if $\mathcal{O}_X(1)$ is relatively very ample). Let $\mathfrak{F}$ be a collection of classes of coherent sheaves on the fibres of X/S. Then the following are equivalent:

(a) $\mathfrak{F}$ is bounded.

(b) There exists m such that every $\mathcal{F}_{K,j} \in \mathfrak{F}$ is m-regular, and the set $\{P(\mathcal{F}_{K,j}) | \mathcal{F}_{K,j} \in \mathfrak{F}\}$ of Hilbert polynomials (where the Hilbert polynomial $P(\mathcal{F}_{K,j})$ is taken with respect to $\mathcal{O}_{X_K}(1)$, the base change of $\mathcal{O}_X(1)$ to X_K) is finite.

(c) There is a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that every $\mathcal{F}_{K,j} \in \mathfrak{F}$ is a quotient of $\mathcal{F}_K$, and the set $\{P(\mathcal{F}_{K,j}) | \mathcal{F}_{K,j} \in \mathfrak{F}\}$ of Hilbert polynomials is finite.

The basic result to guarantee m-regularity in the construction of Quot schemes is the following theorem by Mumford:

Theorem 2.3.10. For all non-negative integers p and n, there is a polynomial $F_{p,n} \in \mathbb{Z}[t_0, \dots, t_n]$ such that for any field $\Bbbk$ (not necessarily algebraically closed, and of any characteristic) and any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_{\Bbbk}$ which is isomorphic to a subsheaf of $\mathcal{O}_{\mathbb{P}^n}^{\oplus p}$ and has Hilbert polynomial

$$\chi(\mathcal{F}(m)) = \sum_{i=0}^{n} a_i \binom{m}{i},$$

$\mathcal{F}$ is m-regular where $m = F_{p,n}(a_0, \dots, a_n)$.

Proof. See for example [Nit05], theorem 5.3, based on [Mum66]. □

Finally, we state general guidelines for dealing with sub- and quotient sheaves of sheaves varying in a bounded collection – this will be useful in our GIT construction for moduli of semistable parabolic sheaves in chapter 4. We are getting slightly ahead of ourselves in as far as this idea already uses the existence of Quot schemes which we are yet to justify in the next section, but this section (whose results on the link between boundedness and regularity are frequently quoted in later chapters) seems the best place to state this principle:

Proposition 2.3.11. Assume that X is a projective S-scheme, S is of finite type over an algebraically closed field $\Bbbk$, and $\mathcal{O}_X(1)$ is a relatively very ample line bundle. Suppose $\mathfrak{F}$ is a bounded collection of sheaves on the fibres of X/S, then:

(a) $\mathfrak{F}$ may be bounded by the universal sheaf on a finite union Q of suitable Quot schemes (in particular, $\mathfrak{F}$ may be bounded by a sheaf which is flat over the parameter scheme Q);

(b) if $\mathfrak{G}$ is a collection of sub- (or quotient) sheaves of elements of $\mathfrak{F}$ such that the elements of $\mathfrak{G}$ have only finitely many Hilbert polynomials, then $\mathfrak{G}$ is also bounded.

Proof. A bounded family $\mathfrak{F}$ is uniformly m-regular for some m, so in particular for every $\mathcal{F} \in \mathfrak{F}$ we have $h^0(\mathcal{F}(m)) = P(\mathcal{F}, m)$ and $\mathcal{F}(m)$ is generated by its global sections, so we can write $\mathcal{F}$ as a quotient

$$V_m \otimes \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{F}$$

for V_m a fixed $\Bbbk$-vector space of dimension $h^0(\mathcal{F}(m)) = P(m)$. Hence, we can parametrise the $\mathcal{F} \in \mathfrak{F}$ by the universal sheaves associated to $Q_P := \mathrm{Quot}_{X/S}^{\mathcal{O}_X(1)}(V_m \otimes \mathcal{O}_X(-m), P)$ for the finitely many polynomials P that occur as Hilbert polynomials $P(\mathcal{F})$ for $\mathcal{F} \in \mathfrak{F}$: the collection $\mathfrak{F}$ is contained in $\{\widetilde{\mathcal{E}}_q | q \in Q\}$, where $\widetilde{\mathcal{E}}$ is the universal sheaf on $X \times_S Q$ with $Q := \coprod_P Q_P$ a scheme of finite type over S. This shows (a).

For (b), use double Quot schemes, i.e. Quot schemes parametrising quotients (or subsheaves) of the universal sheaf $\widetilde{\mathcal{E}}$: if $\mathfrak{G}$ consists of quotients, then it is contained in $\{\widetilde{\mathcal{G}}_q | q \in Q\}$ where $\widetilde{\mathcal{G}}$ is the universal sheaf on $X_Q \times_Q R = X \times_S R$ with

$$R := \coprod_p \mathrm{Quot}_{X_Q/Q}^{\mathcal{O}_{X_Q}(1)}(\widetilde{\mathcal{E}}, p)$$

where p now ranges over the finitely many Hilbert polynomials that occur in $\mathfrak{G}$. □

Note that there are analogous versions of this result saying that standard operations (such as tensor products of sheaves in two bounded families) preserve boundedness. In addition, if $\mathfrak{F}$ and $\mathfrak{G}$ are any two bounded families of sheaves on the fibres of a proper morphism $X \to S$ with noetherian base, then the collection of all kernels, cokernels and images of morphisms $\mathcal{F} \to \mathcal{G}$ for all $\mathcal{F} \in \mathfrak{F}$ and all $\mathcal{G} \in \mathfrak{G}$ is bounded, and the collection of all extension of $\mathcal{F}$ by $\mathcal{G}$ where $\mathcal{F}$ ranges over $\mathfrak{F}$ and $\mathcal{G}$ ranges over $\mathfrak{G}$ is bounded ([Gro62], proposition 1.2).

We conclude this subsection with a specific result on subsheaves of a bounded collection which will again be useful in chapter 4: these subsheaves are themselves bounded under weaker conditions than stated in (b) above. In the statement of this result, we use the terminology of pure and saturated sheaves – see section 4.2 for the definition and basic properties.

Proposition 2.3.12. Let S be a noetherian scheme, $f : X \to S$ a projective morphism with a relatively very ample line bundle $\mathcal{O}_X(1)$, and $\mathfrak{F}$ a bounded collection of purely d-dimensional coherent sheaves on the fibres of X/S. Then consider a collection $\mathfrak{G}$ whose elements are saturated subsheaves $\mathcal{G} \subset \mathcal{F}$ of elements $\mathcal{F} \in \mathfrak{F}$. Write the Hilbert polynomials of $\mathcal{G}$ as

$$P(\mathcal{G}, m) = \frac{a_d(\mathcal{G})}{d!} m^d + \frac{a_{d-1}(\mathcal{G})}{(d-1)!} m^{d-1} + \text{ terms of lower order.}$$

Then $\mathfrak{G}$ is a bounded family if the rational numbers $a_{d-1}(\mathcal{G})$ are bounded below for $\mathcal{G} \in \mathfrak{G}$.

Proof. This is lemma 2.5 in [Gro62], where Grothendieck states the equivalent result for a family of pure quotient sheaves $\mathcal{F} \twoheadrightarrow \mathcal{E}$ with $a_{d-1}(\mathcal{E})$ bounded above. Grothendieck also assumes that the elements $\mathcal{F} \in \mathfrak{F}$ are quotients of a fixed sheaf on X, but by theorem 2.3.9 this can be arranged for every bounded family. □

2.3.3 Constructing fine moduli spaces for Hilb and Quot

The main theorem of section 2.3 is:

Theorem 2.3.13. Let S be a noetherian scheme, $f : X \to S$ a (quasi-)projective morphism with relatively very ample line bundle L (i.e. $X \to S$ factors through a closed embedding $X \hookrightarrow \mathbb{P}^n_S$ for some $n \geqslant 0$ and $L = \mathcal{O}_{\mathbb{P}^n_S}(1)|_X$), let $\mathcal{F}$ be a coherent quotient of $\mathcal{O}_X(\nu)^{\oplus p}$ for some integers $p \geqslant 0$ and ν, and let $P \in \mathbb{Q}[t]$. Then the functor $\mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}$ is represented by a (quasi-)projective S-scheme $\mathrm{Quot}^L_{X/S}(\mathcal{F}, P)$, and it admits projective embeddings which are compositions of *Grothendieck embeddings*

$$\begin{aligned} \psi_m : \mathrm{Quot}^L_{X/S}(\mathcal{F}, P) \quad \hookrightarrow \quad & \mathrm{Grass}_S(\mathcal{O}_S^{\oplus p} \otimes \mathrm{Sym}^m \mathcal{O}_S^{\oplus n+1}, P(m)) \\ & = \mathrm{Grass}_{\mathbb{Z}}(\mathcal{O}_{\mathbb{Z}}^{\oplus p} \otimes \mathrm{Sym}^m \mathcal{O}_{\mathbb{Z}}^{\oplus n+1}, P(m)) \times S \end{aligned} \tag{2.5}$$

(for all sufficiently large m) with the Plücker embeddings of these Grassmannians. Furthermore, the value of $m \gg 0$ to make ψ_m an embedding only depends on n, p and P, but not on X, $\mathcal{F}$ or any other data.

Note that if $f : X \to S$ is projective, then the hypothesis on $\mathcal{F}$ is satisfied by every coherent sheaf on X: by Serre vanishing, $\mathcal{F}$ is m-regular for some m, hence $\mathcal{F}(m)$ is generated by global sections, so $\mathcal{F}$ is a quotient of $H^0(\mathcal{F}(m)) \otimes \mathcal{O}_X(-m)$ which we can identify with $\mathcal{O}_X(-m)^{p(m)}$ as the higher cohomology of $\mathcal{F}(m)$ vanishes, where p is the Hilbert polynomial of $\mathcal{F}$ with respect to $\mathcal{O}_X(1)$.

We now briefly sketch the main ideas of the proof for the case of projective $X \to S$, following [Nit05]. (Another good exposition of this may be found in section 2.2 of [HL97] with the extra hypothesis of S being of finite type over a field $\Bbbk$, which is the case we will be considering in later chapters.) Note first that by lemma 2.3.3, we may take $S = \mathrm{Spec}\,\mathbb{Z}$, $X = \mathbb{P}^n = \mathbb{P}^n_{\mathbb{Z}}$, $L = \mathcal{O}_{\mathbb{P}^n}(1)$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}^{\oplus p}$. By theorem 2.3.10, there exists m_0 (only depending on p, n and P) such that for all $T \in \mathsf{Sch}/S$, all quotients $q : \mathcal{F}_T \twoheadrightarrow \mathcal{E}$ on $\mathbb{P}^n_T$ (with fibrewise Hilbert polynomial P) and all points $t : \mathrm{Spec}\,\Bbbk \hookrightarrow T$, the sheaves $\mathcal{F}_t$, $\mathcal{E}_t$ and $\mathcal{G}_t$ (where $\mathcal{G} := \ker(q)$) are all m_0-regular. In particular, by lemma 2.3.7, for all $m \geqslant m_0$ the higher cohomology of $\mathcal{F}_t(m)$, $\mathcal{E}_t(m)$ and $\mathcal{G}_t(m)$ vanishes and these sheaves are generated by their global sections. The Theorem on Cohomology and Base Change (see theorem 5.10 in [Nit05] and [Har77], section III.12) then implies that $(f_T)_*\mathcal{E}(m)$ is a locally free sheaf of rank $P(m)$ which is a quotient of

$$(f_T)_*\mathcal{F}_T(m) = (f_T)_*\mathcal{O}^{\oplus p}_{\mathbb{P}^n_T}(m) = \mathcal{O}_T^{\oplus p} \otimes \mathrm{Sym}^m \mathcal{O}_T^{\oplus n+1},$$

thus we have a natural transformation

$$\begin{aligned} \psi_m : \mathbf{Quot}^{\mathcal{O}_{\mathbb{P}^n}(1),P}_{\mathcal{O}^{\oplus p}_{\mathbb{P}^n}/\mathbb{P}^n/\mathbb{Z}}(T) \quad & \to \quad \mathbf{Grass}^{P(m)}_{\mathcal{O}^{\oplus p}_{\mathbb{Z}} \otimes \mathrm{Sym}^m \mathcal{O}^{\oplus n+1}_{\mathbb{Z}}/\mathbb{Z}}(T) \\ [q : \mathcal{O}^{\oplus p}_{\mathbb{P}^n_T} \twoheadrightarrow \mathcal{E}] \quad & \mapsto \quad [(f_T)_*(q(m)) : (f_T)_*\mathcal{O}^{\oplus p}_{\mathbb{P}^n_T}(m) \twoheadrightarrow (f_T)_*\mathcal{E}(m)]. \end{aligned} \tag{2.6}$$

This is in fact injective: by lemma 2.3.7(a) and Cohomology and Base Change again, we have a short exact sequence

$$0 \to (f_T)_*\mathcal{G}(m) \to (f_T)_*\mathcal{F}_T(m) \to (f_T)_*\mathcal{E}(m) \to 0 \tag{2.7}$$

and a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (f_T)^*(f_T)_*\mathcal{G}(m) & \longrightarrow & (f_T)^*(f_T)_*\mathcal{F}_T(m) & \longrightarrow & (f_T)^*(f_T)_*\mathcal{E}(m) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G}(m) & \longrightarrow & \mathcal{F}_T(m) & \longrightarrow & \mathcal{E}(m) & \longrightarrow & 0. \end{array} \tag{2.8}$$

By lemma 2.3.7(c) the multiplication maps $(f_T)_*\mathcal{G}(m) \otimes (f_T)_*\mathcal{O}(m'-m) \twoheadrightarrow (f_T)_*\mathcal{G}(m)$ are surjective for all $m' \geqslant m$, so the sequence (2.7) induces an exact sequence

$$(f_T)_*\mathcal{G}(m) \otimes (f_T)_*\mathcal{O}(m'-m) \to (f_T)_*\mathcal{F}_T(m') \to (f_T)_*\mathcal{E}(m') \to 0$$

for all $m' \geqslant m$. Thus, $(f_T)_*\mathcal{G}(m)$ determines $(f_T)_*\mathcal{E}(m')$ for all $m' \geqslant m$ and hence $\mathcal{E}$ itself, and as $(f_T)_*\mathcal{G}(m)$ is the kernel of $(f_T)_*(q(m))$, we see that the quotient sheaf $q : \mathcal{F}_T \twoheadrightarrow \mathcal{E}$ may be recovered from its image $(f_T)_*(q(m))$ in the Grassmannian.

The second step of the argument uses flattening stratifications to show that the natural transformation ψ_m is relatively representable and so the Quot functor may be represented by a locally closed subscheme of the Grassmannian. As flattening stratifications will be useful for our moduli construction in chapter 4, we recall their existence theorem (due to Grothendieck and Mumford):

Theorem 2.3.14. *Let $f : X \to S$ be a projective morphism of noetherian schemes, let $\mathcal{O}_X(1)$ be a relatively very ample line bundle on X and let $\mathcal{F} \in \mathsf{Coh}(X)$. Then the set of Hilbert polynomials $P(\mathcal{F}_s)$ of the restrictions of $\mathcal{F}$ to the fibres X_s is finite, and for each $P \in \mathbb{Q}[t]$ there is a locally closed subscheme $S_P \subset S$ whose points are those $s \in S$ such that $P(\mathcal{F}_s) = P$. In particular, the underlying sets give a partition: $|S| = \coprod_P |S_P|$. Moreover, the scheme structure on each S_P is uniquely determined by a universal property: given any morphism $T \to S$, the pull-back $\mathcal{F}_T$ to $X_T = X \times_S T$ is T-flat if and only if the morphism factors through $\coprod_P S_P \hookrightarrow S$. Finally, the (scheme-theoretic) stratification of S by S_P is upper-semicontinuous: for each fixed polynomial $P' \in \mathbb{Q}[t]$, the union of strata $\coprod_{P \geqslant P'} S_P$ is a closed subscheme of S.*

In the proof of this theorem, the general case $X \hookrightarrow \mathbb{P}^n_S$ may be reduced to the case $n = 0$: in Mumford's words, the stratification $S_P \subset S$ is constructed as the 'greatest common divisor' of the flattening stratifications for the sheaves $f_*\mathcal{F}(N+i)$ for $i \geqslant 0$ on S, for some sufficiently large integer N. Now in the case $n = 0$, the sheaf $\mathcal{F} \in \mathsf{Coh}(S)$ is S-flat if and only if it is locally free, and the required stratification of S is obtained by taking local presentations

$$\mathcal{O}_V^{\oplus r} \xrightarrow{\psi} \mathcal{O}_V^{\oplus e} \longrightarrow \mathcal{F}|_V \longrightarrow 0 \tag{2.9}$$

on open $V \subset S$, and then $\dim_{k(s)} \mathcal{F}_s \otimes k(s) = e - \mathrm{rk}\ (\psi_{ij}(s))$ where (ψ_{ij}) is the $\mathcal{O}_V$-valued matrix given by the morphism ψ. As the rank of a matrix of functions is lower-semicontinuous, $\dim_{k(s)} \mathcal{F}_s \otimes k(s)$ is upper-semicontinuous in $s \in S$, giving rise to our stratification in the case $n = 0$. In fact, as pointed out in [Ser06], the stratification obtained this way arises from the Fitting ideals of $\mathcal{F}$: let $Fitt_k(\mathcal{F})$ be the ideal sheaf locally defined by the $(e-k)$-minors of the matrix (ψ_{ij}), and let $N_k(\mathcal{F})$ denote the closed subscheme of S defined by $Fitt_k(\mathcal{F})$. These definitions are independent of the choice of presentation (2.9) as they satisfy a universal property: given a morphism $T \to S$, we have $\dim_{k(t)} \mathcal{F}_t \otimes k(t) \geqslant k$ for all $t \in T$ if and only if the morphism $T \to S$ factors through $N_k(\mathcal{F}) \hookrightarrow S$. Then $S_e := N_e(\mathcal{F}) \setminus N_{e+1}(\mathcal{F})$ gives us the flattening stratification, and in particular this justifies the last claim in the theorem: $\coprod_{e \geqslant e'} S_e = N_{e'}(\mathcal{F})$ is a closed subscheme of S for each fixed e'. For full details of the proof, see lecture 8 in [Mum66], section 4.2 in [Ser06], and theorem 5.13 in [Nit05].

Returning to the proof of theorem 2.3.13, the Grassmann functor which is the image of the natural transformation (2.6) is representable by the classical Grassmannian scheme $G := \mathrm{Grass}_{\mathbb{Z}}(\mathcal{O}_{\mathbb{Z}}^{\oplus p} \otimes \mathrm{Sym}^m \mathcal{O}_{\mathbb{Z}}^{\oplus n+1}, P(m))$. Let

$$0 \to \mathcal{K} \to \mathcal{O}_G^{\oplus p} \otimes \mathrm{Sym}^m \mathcal{O}_G^{\oplus n+1} \to \mathcal{U} \to 0$$

be the tautological exact sequence on G, i.e. let $\mathcal{U}$ be the universal quotient sheaf parametrised by G and let $\mathcal{K}$ be the associated kernel. We want to identify a locally closed subscheme $Q \subset G$ such that morphisms $\phi : T \to G$ factor through Q if and only if the quotient sequence

$$0 \to \phi^*\mathcal{K} \to \mathcal{O}_T^{\oplus p} \otimes \mathrm{Sym}^m \mathcal{O}_T^{\oplus n+1} \to \phi^*\mathcal{U} \to 0$$

on T is in the image of ψ_m, and by diagram (2.8) this is equivalent to the cokernel of

$$(f_T)^*\phi^*\mathcal{K} \to (f_T)^*\mathcal{O}_T^{\oplus p} \otimes \mathrm{Sym}^m \mathcal{O}_T^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n_T}^{\oplus p} \otimes \mathrm{Sym}^m \mathcal{O}_{\mathbb{P}^n_T}^{\oplus n+1} \tag{2.10}$$

being a T-flat coherent sheaf on $X_T = \mathbb{P}^n_T$ of Hilbert polynomial P. Thus, we may take Q to be G_P, the locally closed subscheme of G given by the flattening stratification for the cokernel of (2.10) with $T = G$.

This concludes the concludes our sketch of the construction of a locally closed subscheme

$$\mathrm{Quot}^L_{X/S}(\mathcal{F}, P) \subset \mathrm{Grass}_{\mathbb{Z}}(\mathcal{O}_{\mathbb{Z}}^{\oplus p} \otimes_{\mathcal{O}_{\mathbb{Z}}} \mathrm{Sym}^m \mathcal{O}_{\mathbb{Z}}^{\oplus n+1}, P(m)) \times S$$

that represents $\mathbf{Quot}^{L,P}_{\mathcal{F}/X/S}$. Note that via the Plücker embeddings the Quot scheme can be seen as a locally closed subscheme of

$$\mathbb{P}\left(\bigwedge^{P(m)} (\mathcal{O}_{\mathbb{Z}}^{\oplus p} \otimes_{\mathcal{O}_{\mathbb{Z}}} \mathrm{Sym}^m \mathcal{O}_{\mathbb{Z}}^{\oplus n+1}) \right) \times S,$$

so $\mathrm{Quot}^L_{X/S}(\mathcal{F}, P) \to S$ is quasi-projective (in particular separated and of finite type). If X is actually projective over S, a standard application of the valuative criterion for properness now shows the morphism $\mathrm{Quot}^L_{X/S}(\mathcal{F}, P) \to S$ to be proper, hence the embedding into projective space over S given above is closed.

If $f : X \to S$ is projective and $\mathcal{F}$ is any coherent sheaf on X, then the restriction of the Grothendieck embedding (2.6) to the Quot scheme of $\mathcal{F}$ is

$$\psi_m : \mathrm{Quot}^L_{X/S}(\mathcal{F}, \mathcal{O}_X(1)) \to \mathrm{Grass}_S(f_*\mathcal{F}(m), P(m)) \tag{2.11}$$

which on fibres over any point $s \in S$ is given as

$$[q : \mathcal{F}_s \twoheadrightarrow \mathcal{E}] \mapsto [H^0(\mathcal{F}_s(m)) \twoheadrightarrow H^0(\mathcal{E}(m))]. \tag{2.12}$$

Here, the quotient vector spaces $[H^0(\mathcal{F}_s(m)) \twoheadrightarrow H^0(\mathcal{E}(m))]$ lies in the classical Grassmannian $\mathrm{Grass}(H^0(\mathcal{F}_s(m)), P(m))$ which is the fibre of $\mathrm{Grass}_S(f_*\mathcal{F}(m), P(m))$ over $s \in S$.

Note that the Quot and Hilbert schemes constructed in this way really are fine moduli spaces for **Quot** and **Hilb**, since the Grassmann functor is represented by the Grassmannian scheme. In particular, Hilbert and Quot schemes have universal families which we will frequently work with in later sections. For H a Hilbert scheme of curves (of some fixed Hilbert polynomial) in $\mathbb{P}^N_{\Bbbk}$, we often refer to the universal family as the 'universal curve': this is the closed subscheme U_H of $H \times \mathbb{P}^N_{\Bbbk}$ given by $\{([C], x) | x \in C\}$, which is projective over H. Similarly, for a Quot scheme $Q := \mathrm{Quot}^L_{X/S}(\mathcal{F}, P) \to S$ the universal family is a quotient sheaf $\mathcal{F}_Q \hookrightarrow \widetilde{\mathcal{E}}$ on X_Q whose restriction to the fibre $(X_Q)_q = X$ for some point $q \in Q_s$ is the quotient $\mathcal{F}_s \hookrightarrow \mathcal{E}$ parametrised by q. We often refer to $\widetilde{\mathcal{E}}$ as the 'universal sheaf' parametrised by Q.

Chapter 3

Moduli spaces: important examples

3.1 Moduli of stable (marked) curves

In this section, we collect some fundamental facts about nodal (marked) curves, and we sketch very briefly GIT constructions of the moduli space of stable curves (due to Gieseker, see [Gie82]) and the moduli space of stable marked curves (due to Baldwin and Swinarski, [BS08]). We omit proofs completely in these GIT stories and do not enter into detailed descriptions of the strategy for these constructions, either – the stability analysis is quite delicate in each case, and a discussion of how to construct a suitable linearisation that works would lead us too far astray. We can afford to focus on a very brief description of the GIT set-up and a small number of key results from the two constructions, because we will slot them into the framework of Pandharipande's universal moduli constructions later (see section 3.3 and chapter 5) – we essentially black-box the presentations of $\overline{M}_g$ and $\overline{M}_{g,n}$ as GIT quotients for the purpose of universal moduli of semistable sheaves on curves and universal moduli of semistable parabolic sheaves on marked curves, respectively. For more detailed discussions of $\overline{M}_g$ and $\overline{M}_{g,n}$ and how they are constructed, see [HM98] and [Edi00] for the former, and [Mor08] for the latter.

Throughout this section, fix an algebraically closed field $\Bbbk$ of arbitrary characteristic.

3.1.1 Deligne-Mumford stable curves

We begin by defining (Deligne-Mumford) stable curves. They first came up in arithmetic algebraic geometry in the context of reduction and (semi)stable models of curves (see chapter 10 of [Liu02]) but achieved real prominence when Deligne and Mumford succeeded (in [DM69]) in giving a compactification $\overline{M}_g$ of the moduli space of smooth curves of fixed genus, where the boundary points are given a modular interpretation: they correspond precisely to stable curves.

Definition 3.1.1. A *Deligne-Mumford stable curve* of genus g over $\Bbbk$ is a connected reduced projective 1-dimensional $\Bbbk$-scheme C of arithmetic genus g (i.e. $h^1(C, \mathcal{O}_C) = g$) with all singularities (if there are any) being nodes and such that every non-singular rational irreducible component of C meets the other components of C in at least three points and such that every irreducible component of arithmetic genus 1 meets the other components of C in at least one point. *Families of stable curves* of genus g are defined to be proper flat morphisms whose geometric fibres are stable curves of genus g, and two families are equivalent if there is an isomorphism between them covering the identity on the base. This defines the moduli functor $\overline{\mathcal{M}}_g$ of stable curves.

Note that if $g \geqslant 2$, then connectedness ensures that every component of genus 1 must meet the other components in at least one point. The condition on the number of special points on rational and genus 1 components has the purpose of excluding positive-dimensional (i.e. infinite) automorphism groups. It also has the pleasant consequence that the *dualising sheaf* ω_C is ample (in fact, this is equivalent to the given condition on the number of nodes on components of genus 0 or 1), hence we have a canonical (up to taking powers) polarisation for stable curves. If $\omega_C^{\otimes k}$ is very ample and $C \hookrightarrow \mathbb{P}^N$ is a projective embedding such that $\mathcal{O}_{\mathbb{P}^N}(1)|_C \cong \omega_C^{\otimes k}$, it is a common abuse of terminology to refer to C as *pluricanonically (or k-canonically) embedded*, in analogy with the canonical bundle K_C of a smooth curve.

Following [HM98], chapter 3.A, we list a few important properties of ω_C and the genus formula for nodal curves:

Theorem 3.1.2. Let C be a complete nodal curve with $h^1(C, \mathcal{O}_C) = g$ and let $\pi : \widetilde{C} \to C$ be its normalisation. Let $y_1, \ldots, y_\delta$ be the nodes of C, and for each y_j let x_j, z_j be the two points in $\widetilde{C}$ lying over y_j, i.e. $\pi^{-1}(y_j) = \{x_j, z_j\}$. Let $C_1, \ldots, C_\nu$ be the irreducible components of C and write g_i for the geometric genus of C_i (i.e. the genus $h^1(\widetilde{C}_i, \mathcal{O}_{\widetilde{C}_i})$ of its normalisation $\widetilde{C}_i$ obtained by ungluing all the nodes of C_i). Let $\mu = h^0(C, \mathcal{O}_C)$ be the number of connected components of C.

(a) The genus formula:
$$g = \Big(\sum_{i=1}^{\nu} g_i\Big) + \delta - \nu + \mu. \tag{3.1}$$

(b) The dualising sheaf ω_C can be defined explicitly as the subsheaf of $\pi_*(\omega_{\widetilde{C}}(\sum_j x_j + z_j))$ given by pushfowards of rational one-forms η satisfying $\mathrm{Res}_{x_j}(\eta) + \mathrm{Res}_{z_j}(\eta) = 0$ for all j.

(c) Riemann-Roch:
$$h^0(C, E) = (\mathrm{rk}\ \ E)(\mu - g) + \deg\ E + h^0(C, E^\vee \otimes \omega_C) \tag{3.2}$$

for any vector bundle E on C.

(d) $\pi^*\omega_C \cong \omega_{\widetilde{C}}(\sum_j x_j + z_j)$.

(e) ω_C is invertible (hence C is Gorenstein).

(f) deg $\omega_C = 2g-2$ and $h^0(C,\omega_C) = g$.

(g) For an irreducible component C_i of C having δ_i points of intersection with $\overline{C \setminus C_i}$, we have deg $\omega_C|_{C_i} = 2h^1(C_i, \mathcal{O}_{C_i}) - 2 + \delta_i$, and this is positive for all i (equivalently, ω_C is ample) if and only if C satisfies the condition on the number of special points in the definition of stability (i.e. $\delta_i \geqslant 3$ whenever C_i has arithmetic genus 0, and $\delta_i \geqslant 1$ whenever C_i has arithmetic genus 1), so a connected nodal curve of genus $g \geqslant 2$ is stable if and only if ω_C is ample.

(h) Let C be connected, then for any $k \geqslant 3$, $\omega_C^{\otimes k}$ is very ample if and only if C is stable. If C is stable, then $h^0(C, \omega_C^{\otimes k}) = (2k-1)(g-1)$ and for all $k \geqslant 2$, $h^1(C, \omega_C^{\otimes k}) = 0$.

Proof. For the genus formula in (a), consider the exact sequence

$$0 \to \mathcal{O}_C \to \pi_*\mathcal{O}_{\widetilde{C}} \to \bigoplus_{j=1}^{\delta} \mathbb{k}_{y_j} \to 0,$$

where $\mathbb{k}_{y_j}$ denotes the rank 1 skyscraper sheaf at y_j. Take the associated long exact sequence in cohomology, observing that $H^k(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) \cong H^k(C, \pi_*\mathcal{O}_{\widetilde{C}})$ via the Leray spectral sequence and that $\widetilde{C}$ decomposes into connected components as $\widetilde{C} = \coprod_{i=1}^{\nu} \widetilde{C}_i$. This yields

$$\begin{aligned} g := h^1(C,\mathcal{O}_C) &= h^1(\widetilde{C},\mathcal{O}_{\widetilde{C}}) - h^1(C, \bigoplus_{j=1}^{\delta}\mathbb{k}_{y_j}) + h^0(C, \bigoplus_{j=1}^{\delta}\mathbb{k}_{y_j}) - h^0(\widetilde{C},\mathcal{O}_{\widetilde{C}}) + h^0(C,\mathcal{O}_C) \\ &= (\sum_{i=1}^{\nu} h^1(\widetilde{C}_i, \mathcal{O}_{\widetilde{C}_i})) - 0 + (\sum_{j=1}^{\delta} 1) - (\sum_{i=1}^{\nu} h^0(\widetilde{C}_i, \mathcal{O}_{\widetilde{C}_i})) + \mu \\ &= (\sum_{i=1}^{\nu} g_i) + \delta - \nu + \mu. \end{aligned}$$

Note that this is exactly the geometric genus of a smoothing of C: consider the case $\mathbb{k} = \mathbb{C}$ and take a topological point of view; if instead of normalising C we now turn all its nodes (topologically) into 'bottlenecks', then we obtain a smooth (possibly disconnected) surface whose genus can be calculated combinatorially (using the dual graph of C) as suggested in [Vak03], giving the same result as (3.1).

For a proof of (b), see [Har66]. The proof of (c) is as in the smooth case, noting that

$$h^0(C,\mathcal{O}) - h^0(C,\omega_C) = h^0(C,\mathcal{O}) - h^1(C,\mathcal{O}) = \mu - g.$$

Then (3.2) for line bundles follows by an induction argument on the degree (see [Ser88]) and the higher rank version is deduced from this as in the smooth case.

For (d), see [Liu02], lemma 10.3.12.

The remaining properties (e)-(h) are easy consequences of (a)-(d) – we only prove (g) here: write $\omega_i := \deg\ \omega_C|_{C_i}$, and consider a component C_i which has Δ_i nodes joining C_i to itself (counted just once, not with multiplicity 2) and which meets $\overline{C \setminus C_i}$ in δ_i nodes. Then the formula

$$\omega_i = 2h^1(C_i, \mathcal{O}_{C_i}) - 2 + \delta_i, \tag{3.3}$$

can be seen either directly from (b) or, following corollary 10.3.13 in [Liu02], using (d): observing that $\pi^*(\omega_C|_{C_i})$ also has degree ω_i, restrict the isomorphism in (d) to the smooth connected curve $\widetilde{C}_i$ to obtain:

$$\begin{aligned} \omega_i = \deg\ \pi^*(\omega_C|_{C_i}) = \deg\ (\pi^*\omega_C)|_{\widetilde{C}_i} &= \deg\ \omega_{\widetilde{C}_i}\left(\sum_{j:\, x_j \in C_i} x_j + \sum_{j:\, z_j \in C_i} z_j\right) \\ &= (2g_i - 2) + 2\Delta_i + \delta_i \\ &= 2h^1(C_i, \mathcal{O}_{C_i}) - 2 + \delta_i, \end{aligned}$$

where the final step uses the genus formula (a) for the curve C_i: the arithmetic genus of C_i is $g_i + \Delta_i$.

For the equivalence between positivity of degree on each irreducible component and ampleness, see [Liu02], proposition 7.5.5. The remaining statements in (g) are obvious, except the transition from arithmetic to geometric genus of components: note that any irreducible component of arithmetic genus 0 must be non-singular of geometric genus 0, while a component of arithmetic genus 1 can only be singular if it is rational with one node, and for a connected curve C such a component C_i can only fail the condition $\delta_i \geqslant 1$ if $C = C_i$, contradicting the assumption $g \geqslant 2$. □

3.1.2 Gieseker's GIT construction of $\overline{M}_g$

Gieseker constructs the moduli space $\overline{M}_g$, a projective coarse moduli space for the moduli problem of families of stable curves up to isomorphism, for all $g \geqslant 2$ via a GIT construction in [Gie82]. As $\omega_C^{\otimes k}$ is very ample for all $k \geqslant 3$, this pluricanonical embedding exhibits all stable curves of genus g as subschemes of $\mathbb{P}^N_{\Bbbk}$, where $N + 1 = h^0(C, \omega_C^{\otimes k}) = (2k-1)(g-1)$ is given by Riemann-Roch. All curves embedded in this way have Hilbert polynomial $h(t) = k(2g-2)t - g + 1$ and are parametrised by H_g, a locally closed subscheme of the Hilbert scheme $H_{g,d,N}$ of curves of degree $d = 10(2g-2)$ and genus g in $\mathbb{P}^N$. Here, the subset H_g is given by the points of $H_{g,d,N}$ corresponding to k-canonically embedded non-degenerate DM-stable curves of genus g in $\mathbb{P}^N$. (Non-degeneracy means that C is not contained in a proper linear subspace of $\mathbb{P}^N$.) The scheme H_g is a parameter space for a rigidified version of $\overline{\mathcal{M}}_g$: it may be thought of as parametrising the curves C together with an isomorphism $\Bbbk^{N+1} \cong H^0(C, \omega_C^{\otimes k})$. Let $U_H \to H_g$ be the universal family over H_g, then this family has the local universal property for the moduli functor $\overline{\mathcal{M}}_g$ of stable curves. To remove the rigidification data, Gieseker considers the natural action by SL_{N+1} on this Hilbert scheme, linearised by some Grothendieck embedding $\widetilde{\psi}_s$. For our purposes, the main result of [Gie82] is:

Theorem 3.1.3. For $k = 10$, there is some $s_0(g)$ such that for all $s \geqslant s_0(g)$

$$\widetilde{\psi}_s : H_{g,d,N} \hookrightarrow \mathrm{Grass}(H^0(\mathbb{P}^N_\Bbbk, \mathcal{O}_{\mathbb{P}^N}(s)), h(s)),$$

yields a linearisation $\widetilde{\psi}_s^* L$ of the natural SL_{N+1}-action on $H_{g,d,N}$ (where L is the determinant line bundle on the Grassmannian) with respect to which

(a) H_g is contained in the GIT-stable locus; and

(b) H_g is closed in the GIT-semistable locus.

By (b), the GIT quotients $\overline{H}_g//\mathrm{SL}_{N+1}$ and $H_g//\mathrm{SL}_{N+1}$ agree (where $\overline{H}_g$ is the closure of H_g in $H_{g,d,N}$), so $H_g//\mathrm{SL}_{N+1}$ is projective, and by (a) there are no strictly semistable points in this quotient, so the GIT-stable locus maps to a projective quotient. Gieseker studies this GIT quotient $H_g//\mathrm{SL}_{N+1}$ (with respect to $\widetilde{\psi}_s$ for $s \geqslant s_0(g)$) and shows that points in the Hilbert scheme are GIT-stable if and only if they represent stable curves. Since H_g has the local universal property for $\overline{\mathcal{M}}_g$, theorem 2.2.4 then implies that the quotient $H_g//\mathrm{SL}_{N+1}$ gives a coarse moduli space $\overline{M}_g$ for the moduli functor $\overline{\mathcal{M}}_g$ of stable curves.

Even though $\omega_C^{\otimes k}$ is already very ample on stable C for $k \geqslant 3$ (by theorem 3.1.2), Gieseker's construction requires a sufficiently high pluricanonical polarisation (such as by $\omega_C^{\otimes k}$ for $k \geqslant 10$) to avoid cuspidal (hence non-DM-stable) curves becoming GIT-stable.

3.1.3 The GIT construction of $\overline{M}_{g,n}$ by Baldwin-Swinarski

We now turn to curves with n marked points, also known as 'n-pointed curves'.

Definition 3.1.4. A *(Deligne-Mumford) prestable marked curve* of genus g with n marked points is a connected reduced projective 1-dimensional $\Bbbk$-scheme C of arithmetic genus g (i.e. $h^1(C, \mathcal{O}_C) = g$) whose only singularities (if there are any at all) are nodes, together with n distinct non-singular marked points $x^1, \dots, x^n$. A *(Deligne-Mumford) stable marked curve* of genus g with n marked points is a prestable marked curve such that every non-singular rational irreducible component of C has at least three special points and every component of arithmetic genus 1 has at least one special point, where by special points we mean nodes or marked points. A *family of stable curves* of genus g with n marked points is a proper flat morphism $f : C \to B$ together with n sections $\sigma^i : B \to C$, such that the n-pointed geometric fibres $(f^{-1}(b), \sigma^1(b), \dots, \sigma^n(b))$ are stable marked curves of genus g. Two families are equivalent if there is an isomorphism between them covering the identity on the base and compatible with the sections. This defines the moduli functor $\overline{\mathcal{M}}_{g,n}$ of stable marked curves.

As in the previous section, the condition on the number of special points on various components characterises stable marked curves as those prestable ones with finite automorphism groups. In the context of marked curves, the most important sheaf to consider is not the canonical sheaf ω_C but a close cousin keeping track of the marked points: $L := \omega_C(x^1 + \dots + x^n)$. This is the correct replacement for ω_C if we wish to obtain a characterisation of stable marked curves analogous to that of part (g) in theorem 3.1.2, i.e. in terms of ampleness of L: note that the curve underlying a stable marked curve is not necessarily stable itself, so ω_C needs the twist incorporated in L.

Theorem 3.1.5. Let C be a complete nodal curve with $h^1(C, \mathcal{O}_C) = g$ and distinct non-singular marked points $x^1, \dots, x^n$. Then $L := \omega_C(x^1 + \dots + x^n)$ has degree $2g - 2 + n$. If C is connected, then L is ample if and only if $(C, x^1, \dots, x^n)$ is stable.

Proof. Analogous to the proof of (f) and (g) in theorem 3.1.2. □

In [BS08], moduli spaces $\overline{M}_{g,n}(\mathbb{P}^r, d)$ of stable maps (from prestable n-pointed genus g curves into $\mathbb{P}^r$ with degree d) are constructed via GIT, provided $2g - 2 + n + 3d \geqslant 1$. We describe the set-up of this construction only for the special case $r = d = 0$ which gives a projective coarse moduli space $\overline{M}_{g,n}$ of stable n-pointed genus g curves for $2g - 2 + n \geqslant 1$. For $r = d = 0$, the constructions of [BS08] work over any algebraically closed field $\Bbbk$, or indeed over Spec $\mathbb{Z}$, as opposed to the case $r > 0$ where the description given in [BS08] relies on the base field being the complex numbers. (Baldwin shows in [Bal08] that the construction can be modified to work over more general base fields even if $r > 0$.)

The set-up is similar to Gieseker's approach discussed in subsection 3.1.2: given a stable n-pointed curve $(C, x^1, \dots, x^n)$, the line bundle $L := \omega_C(x^1 + \dots + x^n)$ is ample and takes the rôle of the canonical line bundle (where ω_C is again the dualising sheaf, see subsection 3.1.1). There exists a only depending on g and n such that $L^{\otimes a}$ is very ample, so C is a-canonically embedded into $\mathbb{P}^N_{\Bbbk}$ (where $N+1 = h^0(C, L^{\otimes a}) = a(2g-2+n) - g + 1$ is given by Riemann-Roch) as a subscheme with Hilbert polynomial $p(t) = a(2g-2+n)t - g + 1$. (Similarly to the terminology for unpointed curves from subsection 3.1.1, we call a pointed curve $C \subset \mathbb{P}^N$ *a-canonically embedded* if $\mathcal{O}_{\mathbb{P}^N}(1)|_C \cong L^{\otimes a} = \omega_C^{\otimes a}(ax^1 + \dots + ax^n)$.) All such curves are parametrised by the Hilbert scheme $H_{g,e,N}$ of curves of genus g and degree $e = a(2g - 2 + n)$ in $\mathbb{P}^N$. To parametrise the marked points as well, we take the product with n copies of $\mathbb{P}^N$, giving the projective scheme $H_{g,e,N} \times (\mathbb{P}^N)^n$ as a first parameter space.

However, to guarantee that the marked points actually lie on the curves, we need to take a closed subscheme I of $H_{g,e,N} \times (\mathbb{P}^N)^n$ defined by the relevant incidence relations:

$$I := \{(C, x^1, \dots, x^n) \in H_{g,e,N} \times (\mathbb{P}^N)^n \quad | \quad \forall i \;\; x^i \in C\}.$$

The universal curve $\mathrm{pr}_1 : U_H \to H_{g,e,N}$ (where pr_1 is the projection of $U_H \hookrightarrow H_{g,e,N} \times \mathbb{P}^N$ onto $H_{g,e,N}$) can be extended over $H_{g,e,N} \times (\mathbb{P}^N)^n$ by

$$(\mathrm{pr}_1, \mathrm{id}_{(\mathbb{P}^N)^n}) : U_H \times (\mathbb{P}^N)^n \to H_{g,e,N} \times (\mathbb{P}^N)^n,$$

and we denote the restriction of this family to I by $\phi : U_I \to I$. Note that

$$U_I \hookrightarrow I \times \mathbb{P}^N \hookrightarrow H_{g,e,N} \times (\mathbb{P}^N)^n \times \mathbb{P}^N.$$

One may define sections $\sigma^i : I \to U_I$ of ϕ by

$$\sigma^i : (C, x^1, \dots, x^n) \mapsto (C, x^1, \dots, x^n, x^i) \in U_I,$$

making $(\phi : U_I \to I, \sigma^1, \dots, \sigma^n)$ into the universal family of n-pointed genus g curves in $\mathbb{P}^N$ with Hilbert polynomial p.

Finally (corresponding to the passage from $H_{g,d,N}$ to H_g in Gieseker's construction, cf. subsection 3.1.2), define J to be the locally closed subscheme of I consisting of all non-degenerate prestable a-canonically embedded n-pointed curves of genus g. Let $\bar{J}$ be the closure of J in the projective scheme I, and denote the restriction of the universal family $(\phi : U_I \to I, \sigma^1, \dots, \sigma^n)$ to J by $(\phi : U_J \to J, \sigma^1, \dots, \sigma^n)$; we will work with this local universal family for $\overline{\mathcal{M}}_{g,n}$ in chapter 5

The group SL_{N+1} acts diagonally on I, and both J and $\bar{J}$ are invariant subschemes. The moduli space $\overline{M}_{g,n}$ is constructed as the GIT quotient $\overline{J}//\mathrm{SL}_{N+1}$ with respect to a suitable linearisation, and this is approached via the main GIT problem under consideration in [BS08]: the action of SL_{N+1} on I.

Baldwin and Swinarski define (see pp. 23–24 of [BS08]) very ample line bundles $L_{m,m'}$ on I by

$$L_{m,m'} := \left((\psi_m)^* \mathcal{O}_{\mathbb{P}(W_m)}(1) \otimes \bigotimes_{i=1}^{n} p_i^* \mathcal{O}_{\mathbb{P}^N}(m') \right) \Bigg|_I , \tag{3.4}$$

where $W_m := \Lambda^{p(m)} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$, the map $\psi_m : H_{g,e,N} \hookrightarrow \mathbb{P}(W_m)$ is is the composition of the m-th Grothendieck embedding (see subsection 2.3.3) with Plücker embeddings, and p_i is the projection of $H_{g,e,N} \times (\mathbb{P}^N)^n$ onto the i-th $\mathbb{P}^N$-factor. As each of the factors of this bundle is naturally linearised, $L_{m,m'}$ inherits a natural linearisation of the SL_{N+1}-action. For our purposes in chapter 5, the main results of [BS08] are then:

Theorem 3.1.6 ([BS08], theorem 6.3 and propositions 6.8, 6.9)**.** There exist m, m' (depending on g and n only) such that $\bar{J}^{ss}(L_{m,m'}) = \bar{J}^{s}(L_{m,m'}) = J$ and such that $\bar{J}//_{L_{m,m'}}\mathrm{SL}_{N+1} = \overline{M}_{g,n}$ – i.e. we have the analogous results to theorem 3.1.3: with respect to the linearisation given by $L_{m,m'}$,

(a) J is contained in the GIT-stable locus (of I); and

(b) J is closed in the GIT-semistable locus (of I).

The same comments as for theorem 3.1.3 apply: by (b), the GIT quotients $\bar{J}//\mathrm{SL}_{N+1}$ and $J//\mathrm{SL}_{N+1}$ agree, so $J//\mathrm{SL}_{N+1}$ is projective, and by (a) there are no strictly semistable points in this quotient, so the GIT-stable locus maps to a projective quotient which Baldwin and Swinarski show to be a coarse moduli space $\overline{M}_{g,n}$ for the moduli functor $\overline{\mathcal{M}}_{g,n}$ of stable marked curves.

3.2 Moduli of semistable pure sheaves after Simpson

After discussing moduli of (marked) curves (i.e. moduli for the base schemes occuring in the universal moduli problems we are interested in) in the previous section, we now change topic and shift our attention to the second ingredient of universal moduli constructions: the 'fibrewise' moduli problem, i.e. that of sheaves on a fixed base scheme. For now, we consider sheaves without further structure (i.e. no parabolic structures involved yet), and the main construction in this context is due to Simpson [Sim94a]; however, our later arguments for moduli of parabolic sheaves (see chapter 4) are also inspired by the arguments outlined in this section. We first collect some basic notions and properties of pure and (semi)stable sheaves in subsection 3.2.1 before sketching the usual two-step process (described in section 2.1) of constructing their moduli: in subsection 3.2.2 we discuss boundedness and regularity issues leading to the existence of a parameter space. (We will build on these results in chapter 4 when proving that the collection of semistable parabolic sheaves is bounded, too.) Subsection 3.2.3 then describes the GIT set-up for the second step of the construction; here we pay particular attention to how Simpson's set-up relates to Pandharipande's set-up [Pan96] for moduli of semistable torsion-free sheaves on a fixed stable curve (this is what Pandharipande calls his 'fibrewise GIT problem') – one of our main motivations for this section is to replace Pandharipande's argument for the 'fibrewise GIT construction' with Simpson's argument. Finally, subsection 3.2.4 states the results of Simpson's analysis of GIT-(semi)stability, culminating in the construction of a coarse moduli space of semistable sheaves as a GIT quotient.

The aim of Simpson's paper [Sim94a] (and its second part [Sim94b]) is to describe the moduli space M_{B} of representations of the fundamental group of a smooth projective variety X. An important aspect of this is the construction of M_{dR}, the moduli space of vector bundles on X with an integrable connection, and M_{Dol}, the moduli space of Higgs bundles on X. It turns out that M_{B} and M_{Dol} are homeomorphic, whereas M_{B} and M_{dR} are biholomorphic (but not isomorphic in the algebraic category). Simpson exhibits these moduli spaces as special cases of a general construction: he obtains moduli spaces of semistable Λ-modules, a notion modelled on that of a $\mathcal{D}$-module, for Λ a general sheaf of rings of differential operators. In particular, taking $\Lambda = \mathcal{O}_X$, the concept of Λ-modules just reduces to coherent $\mathcal{O}_X$-modules. However, Simpson gives a separate construction of the moduli space of semistable coherent sheaves (on families of projective schemes) in section 1 of [Sim94a], since this construction implies that the moduli space of semistable sheaves is projective, a property that cannot be deduced from the general argument using Λ-modules. It is this specific construction that we are interested in here. In a nutshell, Simpson extended the class of schemes and sheaves for which a moduli space can be formed (previously, moduli constructions were only available for torsion-free sheaves on smooth projective varieties, whereas Simpson deals with pure sheaves on families of arbitrary projective schemes), and he used a different linearisation in his construction, based on Grothendieck's original embedding of the Quot scheme into a single Grassmannian. This linearisation subsequently became standard in moduli constructions for sheaves, replacing the previous method of embedding the Quot scheme (that serves as a parameter space in GIT constructions for such moduli spaces) into Gieseker spaces (see e.g. [Gie77]).

The reason we are interested in this construction is that it may be used to make Pandharipande's construction of universal moduli of vector bundles on curves more transparent (see section 3.3), and it makes a generalisation of Pandharipande's results to universal moduli of parabolic sheaves (and potentially to sheaves on higher-dimensional base schemes) easier. We only need the right set-up to apply Simpson's methods for our purposes: his key result is that for coherent sheaves on a projective $\Bbbk$-scheme X with Hilbert polynomial P parametrised by a suitable Quot scheme (depending on a parameter n) and for a suitable choice of linearisation L_m given by a Grothendieck embedding ψ_m of the Quot scheme into a Grassmannian, p-(semi)stability and GIT-(semi)stability agree, where the parameters n and m only depend on X and P (cf. [Sim94a], theorem 1.19). On the face of it (and as Pandharipande says in [Pan96], albeit when [Sim94a] was only circulated in preprint form), this is not quite good enough for Pandharipande's framework for universal moduli of sheaves on varying stable curves: in that case we need the parameters n and m to be independent of the base curve $X = C$ varying in H_g, an open subset of the Hilbert scheme of curves of given genus and degree in some projective space (cf. subsection 3.1.2). One might call this 'uniform' moduli of sheaves on stable curves, rather than 'universal' moduli: for now, we are still after moduli of sheaves on any *fixed* stable curve as base, but we want all parameters in the construction to be *uniform* across all (10-canonically embedded) stable curves.

This potential complication of achieving uniformity is already dealt with in (the final version of) [Sim94a] as Simpson really considers sheaves on *families* of projective schemes: he works with $X \to S$, a projective morphism with codomain S any connected scheme of finite type over $\Bbbk = \mathbb{C}$, and constructs moduli of semistable sheaves on the fibres of $X \to S$ with fixed Hilbert polynomial. This allows us to obtain n and m that work uniformly for all stable curves (10-canonically embedded in some fixed projective space) by applying Simpson's construction to semistable sheaves on $X = U_H$ over $S = H_g$, where U_H is the universal curve over H_g. We will explain this set-up in more detail in section 3.3, but the idea is that semistable sheaves on U_H/H_g correspond precisely to semistable sheaves on curves C such that $[C] \in H_g$.

There is one aspect of Simpson's construction to be modified before it can be used for the universal moduli space of torsion-free sheaves on stable curves: Simpson works with $\Bbbk = \mathbb{C}$ throughout [Sim94a] and [Sim94b]; however the construction of the moduli space for pure sheaves in section 1 of [Sim94a] is valid for any algebraically closed field $\Bbbk$ of characteristic 0. We comment on the case of positive characteristic in subsection 3.2.2, where modifications are necessary – by results of Langer (see [Lan04a], [Lan04b] and [Lan04c]), the construction can be carried out in positive characteristic, too. Looking at this generalisation is again motivated by our aim to simplify and extend Pandharipande's construction of universal moduli (which works over any algebraically closed field).

Throughout this section, unless stated otherwise, S is a connected separated scheme of finite type over an algebraically closed field $\Bbbk$, and $X \to S$ is a projective morphism with a fixed very ample line bundle $\mathcal{O}_X(1)$.

3.2.1 Pure and (semi)stable sheaves

We briefly recall some basic notions on pure sheaves (see [HL97] for more details):

Definition 3.2.1. Let X be a noetherian scheme. The *dimension* of a coherent sheaf $\mathcal{F}$ on X is the dimension of its support, i.e. $\dim \mathcal{F} := \dim \operatorname{Supp} \mathcal{F}$. A coherent sheaf $\mathcal{F}$ is called *pure* if for all non-zero proper coherent subsheaves $\mathcal{E} \subset \mathcal{F}$ we have $\dim \mathcal{E} = \dim \mathcal{F}$. (Note that pure sheaves of top dimension $\dim \mathcal{F} = \dim X$ are generalisations of torsion-free sheaves: the two notions coincide if X is integral.) The *saturation* of a subsheaf $\mathcal{E} \subset \mathcal{F}$ is the minimal subsheaf $\mathcal{E}^{\text{sat}} \subset \mathcal{F}$ containing $\mathcal{E}$ such that $\mathcal{F}/\mathcal{E}^{\text{sat}}$ is either zero or pure of dimension $\dim \mathcal{F}$. Call a subsheaf $\mathcal{E} \subset \mathcal{F}$ *saturated* if it is its own saturation (compare this to the notion of 'subbundle' vs. 'subsheaf' for a locally free sheaf $\mathcal{F}$).

Definition 3.2.2. Let X be a projective scheme over a field $\Bbbk$, let $\mathcal{O}_X(1)$ be a very ample line bundle on X and $\mathcal{E}$ a coherent sheaf on X of dimension d. Then the *Hilbert polynomial* of $\mathcal{E}$ with respect to $\mathcal{O}_X(1)$ is given by the function $m \mapsto \chi(\mathcal{E}(m))$ – see section 1.2 of

[HL97] for a proof that this function is actually polynomial in m with rational coeffcients. In fact, the degree of $\chi(\mathcal{E}(m))$ is equal to d, and there exist integers $a_0(\mathcal{E}), \ldots, a_d(\mathcal{E})$ such that

$$\chi(\mathcal{E}(m)) = \sum_{k=0}^{d} \frac{a_k(\mathcal{E})}{k!} m^k.$$

We often write $P(\mathcal{E})$ for the Hilbert polynomial $P(\mathcal{E}, m) := \chi(\mathcal{E}(m))$. By Serre vanishing, $\chi(\mathcal{E}(m)) = h^0(\mathcal{E}(m))$ for all sufficiently large m, and in particular $a_d(\mathcal{E})$ is positive if $\mathcal{E} \neq 0$. The coefficient $a_d(\mathcal{E})$ is called the *multiplicity* of $\mathcal{E}$, or sometimes by abuse of terminology $a_d(\mathcal{E})$ is known as the *rank* of $\mathcal{E}$ (with respect to $\mathcal{O}_X(1)$). Thus the rank of a coherent sheaf $\mathcal{E}$ on X is non-zero whenever $\mathcal{E} \neq 0$, not just when $\dim \mathcal{E} = \dim X$. Note that with this definition, the rank of $\mathcal{E}$ (as in [Sim94a], for example) depends on the polarisation $\mathcal{O}_X(1)$ but is always integral, as opposed to the more customary definition (e.g. in [HL97]) of rk $\mathcal{E}$ as $a_d(\mathcal{E})/a_d(\mathcal{O}_X)$. Write $\mu(\mathcal{E})$ for $a_{d-1}(\mathcal{E})/a_d(\mathcal{E})$ and call this the *slope* of $\mathcal{E}$. This number comes up as the highest interesting coefficient (up to the constant factor $1/(d-1)!$) in the *reduced Hilbert polynomial* $p_0(\mathcal{E}) := P(\mathcal{E})/a_d(\mathcal{E})$ (which is defined for $\mathcal{E} \neq 0$).

Recall that the Hilbert polynomial is additive on short exact sequences, and observe that every sheaf $\mathcal{F}$ has a non-negative Hilbert polynomial, i.e. $P(\mathcal{F}) \geqslant 0$ (with respect to the lexicographic ordering on $\mathbb{Q}[x]$) – in particular, we can use this to relate the Hilbert polynomial of a subsheaf and its saturation:

Lemma 3.2.3. Let $\mathcal{F}$ be a coherent sheaf on a projective $\Bbbk$-scheme X, let $d = \dim \mathcal{F}$ and let $\mathcal{E} \subset \mathcal{F}$ be a subsheaf with saturation $\mathcal{E}^{\text{sat}} \subset \mathcal{F}$. Then the Hilbert polynomials of $\mathcal{E}$ and $\mathcal{E}^{\text{sat}}$ are related by

$$P(\mathcal{E}) \leqslant P(\mathcal{E}^{\text{sat}}) \text{ and } a_d(\mathcal{E}) = a_d(\mathcal{E}^{\text{sat}}).$$

Proof. The first statement is true without the saturation hypothesis: whenever $\mathcal{E} \subset \mathcal{E}'$ we have

$$0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'/\mathcal{E} \to 0,$$

thus

$$P(\mathcal{E}') - P(\mathcal{E}) = P(\mathcal{E}'/\mathcal{E}) \geqslant 0.$$

The second statement is immediate from an explicit definition of $\mathcal{E}^{\text{sat}}$ (following [HL97]) as the kernel of the composition of surjections

$$\mathcal{F} \to \mathcal{F}/\mathcal{E} \to (\mathcal{F}/\mathcal{E})/T_{d-1}(\mathcal{F}/\mathcal{E}),$$

where for any sheaf $\mathcal{G}$ we denote by $T_{d-1}(\mathcal{G})$ the maximal subsheaf of $\mathcal{G}$ supported in dimension $d-1$. Then the short exact sequence

$$0 \to \mathcal{E}^{\text{sat}} \to \mathcal{F} \to (\mathcal{F}/\mathcal{E})/T_{d-1}(\mathcal{F}/\mathcal{E}) \to 0$$

yields

$$P(\mathcal{F}) - P(\mathcal{E}^{\mathrm{sat}}) = P((\mathcal{F}/\mathcal{E})/T_{d-1}(\mathcal{F}/\mathcal{E})) = P(\mathcal{F}) - P(\mathcal{E}) - P(T_{d-1}(\mathcal{F}/\mathcal{E})),$$

so $P(\mathcal{E}^{\mathrm{sat}}) - P(\mathcal{E})$ is a polynomial of degree at most $d-1$, hence the degree d coefficients of $P(\mathcal{E}^{\mathrm{sat}})$ and $P(\mathcal{E})$ agree. □

For coherent sheaves, there are two important notions of stability:

Definition 3.2.4. A non-zero coherent sheaf $\mathcal{F}$ is *p-(semi)stable* (or *Gieseker-(semi)stable*) if it is pure and if for any proper non-zero subsheaf $\mathcal{E} \subset \mathcal{F}$ we have $p_0(\mathcal{E})(\leqslant)p_0(\mathcal{F})$. Here, we follow convention 1.4.2 for the symbol $(\leqslant)$ and in using the lexicographic ordering on polynomials (and thus the statement $f \leqslant g$ for two polynomials $f, g \in \mathbb{Q}[t]$ is equivalent to the condition $f(n) \leqslant g(n)\ \forall n \gg 0$).

An equivalent condition for a coherent sheaf $\mathcal{F}$ of dimension d to be p-(semi)stable is that

$$a_d(\mathcal{F})P(\mathcal{E})(\leqslant)a_d(\mathcal{E})P(\mathcal{F}) \tag{3.5}$$

for all proper subsheaves $\mathcal{E} \subset \mathcal{F}$. This is clearly the same definition, except that we avoid explicit mention of purity, since purity is implicit in the modified definition: applying (3.5) to $\mathcal{E} = T_{d-1}(\mathcal{F})$, the maximal subsheaf of $\mathcal{F}$ of dimension $\leqslant d-1$, we see that (3.5) gives us $P(\mathcal{E}) \leqslant 0$ (as $a_d(\mathcal{E}) = 0$ and $a_d(\mathcal{F}) > 0$), so $\mathcal{E} = 0$ and thus $\mathcal{F}$ is pure.

Definition 3.2.5. A coherent sheaf is *μ-(semi)stable* (or *slope-(semi)stable*) if it is pure and for any proper non-zero subsheaf $\mathcal{E} \subset \mathcal{F}$ we have $\mu(\mathcal{E})(\leqslant)\mu(\mathcal{F})$. In general, the hypothesis of purity is sometimes slightly weakened (see p. 26 of [HL97]), but for our purposes, certainly when working over curves, restricting to pure sheaves is good enough.

Since p_0 has leading term $m^d/d!$ for any sheaf of dimension d and $\mu/(d-1)!$ is the coefficient of m^{d-1} in p_0, it is immediate that p-semistability implies μ-semistability, and μ-stability implies p-stability. (This is because of the lexicographic ordering on polynomials – see convention 1.4.2 for an explanation of how an inequality between polynomials such as $p_0(\mathcal{E}) \leqslant p_0(\mathcal{F})$ and an inequality between their degree $d-1$ coefficients $\mu(\mathcal{E}) \leqslant \mu(\mathcal{F})$ differ.)

Note that slope-(semi)stability is somewhat better behaved under standard operations than p-(semi)stability: e.g. $\mu(\mathcal{F}(k)) = \mu(\mathcal{F}) + k$ for all $\mathcal{F}$, so twisting by $\mathcal{O}_X(1)$ does not change μ-(semi)stability of any sheaf. However, we work mainly with p-semistability as it this stability condition for which Simpson constructs complete moduli spaces in arbitrary dimensions.

Definition 3.2.6. As for vector bundles, one has the *Harder-Narasimhan filtration* for any pure sheaf $\mathcal{F}$: the unique filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F} \tag{3.6}$$

such that the quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ of successive terms are p-semistable of pure dimension $d = \dim \mathcal{F}$ and such that $p_0(\mathcal{F}_1/\mathcal{F}_0) > p_0(\mathcal{F}_2/\mathcal{F}_1) > \cdots > p_0(\mathcal{F}_k/\mathcal{F}_{k-1})$. (See [HL97], section 1.3, for proofs of existence and uniqueness.) One caveat is that there is an analogous filtration with μ-semistable terms, and this need not agree with the (p-)Harder-Narasimhan filtration.

For a detailed overview of Harder-Narasimhan (HN) filtrations of pure sheaves, we refer the reader to [HL97]. In chapter 4, we will be using the μ-HN filtration for some of our boundedness arguments. In general, there is a second important class of filtrations for coherent sheaves:

Definition 3.2.7. If $\mathcal{F}$ itself is p-semistable, then one can obtain a *Jordan-Hölder filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}$$

such that all the successive quotients are p-stable of pure dimension $d = \dim \mathcal{F}$ and have reduced Hilbert polynomial $p_0(\mathcal{F})$. This filtration is not unique, but the collection of its Jordan-Hölder factors (with multiplicities) is, i.e. $\mathrm{gr}(\mathcal{F}) = \bigoplus \mathcal{F}_i/\mathcal{F}_{i-1}$ is unique up to isomorphism (see [HL97], section 1.5) – this is the usual Jordan-Hölder theorem applied to the abelian category of p-semistable sheaves of fixed reduced Hilbert polynomial. Again, there is an analogous filtration for μ-semistable sheaves. Two p-semistable sheaves $\mathcal{E}, \mathcal{F}$ with the same reduced Hilbert polynomial are called *S-equivalent* (or *Seshadri-equivalent*) if $\mathrm{gr}(\mathcal{E})$ and $\mathrm{gr}(\mathcal{F})$ are isomorphic.

In this definition, we used an elementary observation: the category of p-semistable sheaves of given dimension and reduced Hilbert polynomial is abelian.

Lemma 3.2.8. Fix any polynomial $p \in \mathbb{Q}[t]$. The category C of p-semistable sheaves with reduced Hilbert polynomial p (together with the zero sheaf) is abelian.

Proof. The category C inherits the properties of an additive category from $\mathsf{Coh}(X)$. What remains to be proved is that C has direct sums and in particular kernels and cokernels of morphisms. These arguments are elementary, using that the (non-reduced) Hilbert polynomial is additive, i.e. for short exact sequences $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, we have $P(\mathcal{F}_2) = P(\mathcal{F}_1) + P(\mathcal{F}_3)$ (and in particular also $a_d(\mathcal{F}_2) = a_d(\mathcal{F}_1) + a_d(\mathcal{F}_3)$). □

A direct consequence is:

Lemma 3.2.9. Stable sheaves are simple, i.e. the only endomorphisms of a p-stable sheaf are scalars.

Proof. Simpson ([Sim94a], pp. 55-56) gives a proof for this with $\mathbb{C}$ as his base field, but exactly the same argument works over any algebraically closed field $\Bbbk$: given a non-zero endomorphism f of a p-stable sheaf $\mathcal{F}$, $\ker f$ and $\operatorname{coker} f$ are zero: by the previous lemma, if $\ker f$ is non-zero, then $\ker f$ has the same reduced Hilbert polynomial as $\mathcal{F}$, contradicting p-stability of $\mathcal{F}$. Similarly, if $\operatorname{coker} f$ is non-zero, then $p_0(\operatorname{coker} f) = p_0(\mathcal{F})$, so

$$p_0(\operatorname{im} f) = \frac{P(\mathcal{F}) - P(\operatorname{coker} f)}{a_d(\mathcal{F}) - a_d(\operatorname{coker} f)} = \frac{P(\mathcal{F})}{a_d(\mathcal{F})} = p_0(\mathcal{F}). \tag{3.7}$$

But since $\operatorname{coker} f \neq 0$ implies $\operatorname{im} f \subsetneq \mathcal{F}$, this again violates p-stability of $\mathcal{F}$. Therefore any non-zero $f : \mathcal{F} \to \mathcal{F}$ is an automorphism, thus $\operatorname{End}(\mathcal{F})$ is a finite-dimensional division algebra D over $\Bbbk$ and so must be $\Bbbk$ itself: an element $x \in D \setminus \Bbbk$ would generate a finite-dimensional, thus algebraic, commutative field extension of $\Bbbk = \overline{\Bbbk}$. □

Hence, p-stable sheaves do not have too many automorphisms, giving a first suggestion of why this is a good class of sheaves to look at when trying to construct coarse moduli spaces.

Finally, we pass to the relative situation: suppose S is a connected separated scheme of finite type over $\operatorname{Spec} \Bbbk$, and consider $X \to S$ projective with a choice of very ample line bundle $\mathcal{O}_X(1)$ made. Note then that for all closed points $s \in S$, the scheme-theoretic fibre X_s is a projective scheme over $\operatorname{Spec} \Bbbk$ with very ample line bundle $\mathcal{O}_{X_s}(1)$, so the above definitions can be applied to it. The definition of (semi)stability is extended to this set-up by demanding flatness and (semi)stability in the above sense for the restriction to each fibre:

Definition 3.2.10 ([Sim94a], p. 58)**.** Given $P \in \mathbb{Q}[t]$ of degree d, a *p-(semi)stable sheaf $\mathcal{F}$ on X/S with Hilbert polynomial P* is a coherent sheaf $\mathcal{F}$ on X, flat over S, such that for each closed point $s \in S$ the restriction of $\mathcal{F}_s$ of $\mathcal{F}$ to X_s is p-(semi)stable and has Hilbert polynomial P with respect to $\mathcal{O}_{X_s}(1)$.

The notion of 'p-(semi)stable sheaves on X_T/T' for any $T \in \mathsf{Sch}/S$ (recall convention 1.4.1 on notation for base change) will play the rôle of families of sheaves on X/S parametrised by T, as we shall see at the start of subsection 3.2.3. For the rest of section 3.2 (and when following [Sim94a]), the reader should take care to distinguish between '(p-semistable) sheaves on X/S' as defined above (these are actually sheaves on all of X, i.e. *families* of sheaves on the family of projective schemes $X \to S$) and '(p-semistable) sheaves on the (geometric) fibres of X/S' (which are sheaves on one particular member of the family $X \to S$) – also cf. definition 2.3.8.

3.2.2 Boundedness of the semistable sheaves

One of the key steps in Simpson's construction is to show that the collection of p-semistable sheaves $\mathcal{F}$ on the fibres of X over S with fixed Hilbert polynomial is bounded, and to deduce upper bounds on the number of global sections of $\mathcal{F}(k)$ in terms of k but independent of $\mathcal{F}$. Here, boundedness is in the relative sense of definition 2.3.8.

Definition 3.2.11. Given a pure sheaf $\mathcal{E}$ on a projective $\mathbb{k}$-scheme Y equipped with a very ample line bundle $\mathcal{O}_Y(1)$ (e.g. $Y = X_s$, a fibre of $X \to S$), define the *maximal slope* $\mu_{\max}(\mathcal{E})$ to be the slope of the maximal slope-destabilising subsheaf of $\mathcal{E}$ (i.e. the first term of its μ-HN filtration) if $\mathcal{E}$ is μ-unstable, and $\mu_{\max}(\mathcal{E}) := \mu(\mathcal{E})$ if $\mathcal{E}$ is μ-semistable. Equivalently, $\mu_{\max}(\mathcal{E})$ is the maximal slope of all non-zero subsheaves of $\mathcal{E}$.

Using boundedness results for torsion-free sheaves due to Maruyama [Mar81] in characteristic 0, Simpson proves

Theorem 3.2.12 ([Sim94a], theorem 1.1)**.** Let X be projective over $\mathbb{k}$, equipped with a very ample line bundle $\mathcal{O}_X(1)$. Fix $b \in \mathbb{R}$. Given $P \in \mathbb{Q}[t]$, the collection of pure sheaves $\mathcal{F}$ on X having Hilbert polynomial P and satisfying $\mu_{\max}(\mathcal{E}) \leqslant b$ is bounded. In particular, taking $b = \mu(\mathcal{F})$, we see that the collection of μ-semistable sheaves on X with Hilbert polynomial P is bounded.

From this, Simpson deduces what turns out to be a cornerstone for his own construction and for how we use his ideas in later sections. Langer [Lan04a], [Lan04b] subsequently extended this result to arbitrary characteristic in order to construct moduli spaces for semistable pure sheaves in this situation.

Theorem 3.2.13 ([Sim94a] corollary 1.6; [Lan04a] theorem 4.2)**.** Let $\mathbb{k}$ be an algebraically closed field of arbitrary characteristic, S a scheme of finite type over $\mathbb{k}$ and $X \to S$ projective with a very ample line bundle $\mathcal{O}_X(1)$. Let $b \in \mathbb{R}$ and $P \in \mathbb{Q}[x]$, then the collection of pure sheaves $\mathcal{E}$ of dimension $\deg P$ on the fibres of X/S with Hilbert polynomial P and $\mu_{\max}(\mathcal{E}) \leqslant b$ is bounded (uniformly in $s \in S$, i.e. in the sense of definition 2.3.8). In particular, the collection of p-semistable sheaves on the fibres of X/S with Hilbert polynomial P is bounded: take $b = \mu(\mathcal{E})$ which is determined by P, and note that p-semistability implies μ-semistability.

This basic boundedness theorem for pure sheaves is needed to set up the GIT construction for moduli of pure sheaves on X/S. A second result required as a key tool in the GIT construction (when identifying the GIT-(semi)stable loci in the Quot scheme with the loci of p-(semi)stable sheaves) is a bound on the number of global sections of a sheaf in terms of its maximal slope (sometimes known as the Le Potier-Simpson estimate) – we also state the more general version due to Langer (who extended the result to positive characteristic and strengthened it in characteristic zero):

Theorem 3.2.14 ([Lan04c] corollary 3.4; [Sim94a] corollary 1.7)**.** (a) For any algebraically closed field $\mathbb{k}$, any projective $\mathbb{k}$-scheme Y equipped with a very ample line bundle $\mathcal{O}_Y(1)$ and any pure sheaf $\mathcal{E}$ of dimension d on Y and multiplicity $a_d(\mathcal{E}) = r$, we have

$$h^0(Y,\mathcal{E}) \leqslant \begin{cases} r \cdot \binom{\mu_{\max}(\mathcal{E}) + r^2 + f(r) + \frac{d-1}{2}}{d} & \text{if } \mu_{\max}(\mathcal{E}) - \frac{d+1}{2} + r^2 \geqslant 0 \\ 0 & \text{if } \mu_{\max}(\mathcal{E}) - \frac{d+1}{2} + r^2 < 0 \end{cases}$$

where $f(n) := -1 + \sum_{i=1}^{n} \frac{1}{i}$ for $n \in \mathbb{N}$, so $f(n) \sim \ln n$.

(b) Given integers r and d, there is an integer B satisfying the following: for any algebraically closed field $\mathbb{k}$ of arbitrary characteristic, any scheme S of finite type over $\mathbb{k}$, any $X \to S$ projective with a very ample line bundle $\mathcal{O}_X(1)$, any closed point $s \in S$ and any purely d-dimensional sheaf $\mathcal{E}$ on X_s with multiplicity $a_d(\mathcal{E}) = r$, we have

$$h^0(X_s,\mathcal{E}(m)) \leqslant \begin{cases} r \cdot (\mu_{\max}(\mathcal{E}) + m + B)^d/d! & \text{if } \mu_{\max}(\mathcal{E}) + m + B \geqslant 0 \\ 0 & \text{if } \mu_{\max}(\mathcal{E}) + m + B \leqslant 0 \end{cases}$$

for any $m \in \mathbb{Z}$.

Proof. Part (a) is corollary 3.4 of [Lan04c]. Part (b) is proved in [Sim94a] for characteristic zero and for the case of μ-semistable $\mathcal{E}$, i.e. with $\mu_{\max}(\mathcal{E}) = \mu(\mathcal{E})$. The more general version stated here follows from (a): for every fibre X_s, any $\mathcal{E}$ on X_s and arbitrary $m \in \mathbb{Z}$, we have $\mu_{\max}(\mathcal{E}(m)) = \mu_{\max}(\mathcal{E}) + m$, so we can take the constant B to be $\lceil r^2 + f(r) + \frac{d-1}{2} \rceil \in \mathbb{N}$ (note that this depends on nothing but r and d, as required): then $\mu_{\max}(\mathcal{E}) + m + B \leqslant 0$ implies $\mu_{\max}(\mathcal{E}) + m + r^2 - \frac{d+1}{2} < 0$, so

$$\begin{aligned} h^0(X_s,\mathcal{E}(m)) &\leqslant \begin{cases} \max\left(r \cdot \binom{\mu_{\max}(\mathcal{E}) + m + B}{d}, 0\right) & \text{if } \mu_{\max}(\mathcal{E}) + m + B \geqslant 0 \\ 0 & \text{if } \mu_{\max}(\mathcal{E}) + m + B \leqslant 0 \end{cases} \\ &\leqslant \begin{cases} r \cdot (\mu_{\max}(\mathcal{E}) + m + B)^d/d! & \text{if } \mu_{\max}(\mathcal{E}) + m + B \geqslant 0 \\ 0 & \text{if } \mu_{\max}(\mathcal{E}) + m + B \leqslant 0, \end{cases} \end{aligned}$$

as required. □

We will use both theorems 3.2.13 and 3.2.14 extensively in our construction of moduli of semistable parabolic sheaves in chapter 4.

3.2.3 Moduli of semistable sheaves on stable curves

As before, let S be a connected separated scheme of finite type over Spec $\Bbbk$, let $X \to S$ be projective, pick a very ample line bundle $\mathcal{O}_X(1)$, fix $P \in \mathbb{Q}[t]$, and let $d := \deg P$. The moduli functor which Simpson investigates is

$$\begin{aligned} \mathbf{M}(\mathcal{O}_X, P) : \mathsf{Sch}/S &\to \mathsf{Sets} \\ S' &\mapsto \{p\text{-semistable sheaves } \mathcal{F} \text{ on } X'/S' \text{ with Hilbert polynomial } P\}/\sim, \end{aligned}$$

where $\sim$ is given by S-equivalence (see definition 3.2.7): $\mathcal{F}_1 \sim \mathcal{F}_2$ if and only if $\mathrm{gr}(\mathcal{F}_1)$ and $\mathrm{gr}(\mathcal{F}_2)$ are isomorphic. The pullback of families (in the sense of chapter 2) is given by the ordinary pullback of sheaves, making $\mathbf{M}(\mathcal{O}_X, P)$ a moduli functor. To construct a coarse moduli space $M(\mathcal{O}_X, P)$, i.e. an S-scheme universally corepresenting this functor, Simpson sets up a suitable Quot scheme as parameter space (depending on a choice of integer n telling us by how much to twist up our sheaves) with an action of $\mathrm{SL}_{P(n)}$. He then linearises the action via a Grothendieck embedding ψ_m (cf. subsection 2.3.3) and shows that for m and n large enough, any point of the Quot scheme is GIT-(semi)stable if and only if the quotient sheaf $\mathcal{F}$ it corresponds to is p-semistable and the quotient map it corresponds to gives an isomorphism between $H^0(X, \mathcal{F}(n))$ and a fixed vector space V. We will ultimately want to take $S = H_g$ and $X = U_H$ the universal curve over H_g (see subsection 3.1.2), but consider general $X \to S$ for now.

For a fixed large $n \in \mathbb{Z}$ (to be chosen appropriately in the construction), set $V := \Bbbk^{P(n)}$ and $\mathcal{W} := \mathcal{O}_X(-n)$ and consider the relative Quot scheme $Q_n := \mathrm{Quot}^{\mathcal{O}_X(1)}_{X/S}(V \otimes \mathcal{W}, P)$; note that convention 1.4.5 applies with respect to the sheaf denoted by $V \otimes \mathcal{W}$. For any $S' \in \mathsf{Sch}/S$, the S'-valued points of Q_n are the pairs $(\mathcal{F}, \alpha)$ with $\mathcal{F}$ a coherent sheaf on X', flat over S' with $\mathcal{F}_s$ having Hilbert polynomial P for every $s \in S'$, and $\alpha : (V \otimes \mathcal{W})' \twoheadrightarrow \mathcal{F}$ a surjection. (See convention 1.4.1 for the notation X', $(V \otimes \mathcal{W})'$, etc.) Note that this is equivalent to demanding that the sections in the image of the map $V \otimes \mathcal{O}_{S'} \to H^0(X'/S', \mathcal{F}(n))$, induced from $\alpha : (V \otimes \mathcal{O}_X(-n))' \to \mathcal{F}$ by first tensoring through by $\mathcal{O}_{X'}(n)$ and then pushing forward to S', generate $\mathcal{F}(n)$. In particular, taking $S' = \mathrm{Spec}\, k(s)$ for any closed point $s \in S$, we can describe the $\Bbbk$-points of Q_n lying over $s \in S$ as pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a coherent sheaf on X_s with Hilbert polynomial P, and (by slight abuse of language) α is a map $V \to H^0(\mathcal{F}(n))$ whose image generates $\mathcal{F}(n)$.

By theorem 3.2.13, the collection of p-semistable sheaves $\mathcal{F}$ on the fibres X_s of X/S with Hilbert polynomial P is bounded, hence such sheaves are n-regular for a uniform n (theorem 2.3.9). Thus, by lemma 2.3.7, $\mathcal{F}(n)$ is generated by its global sections, so is a quotient of $H^0(\mathcal{F}(n)) \times \mathcal{O}_{X_s}(-n)$. Using that n-regularity of $\mathcal{F}$ also implies vanishing of higher cohomology (again by lemma 2.3.7), we may choose an isomorphism of $H^0(\mathcal{F}(n))$ with V, i.e. $\mathcal{F}$ occurs as a point of Q_n lying over $s \in S$.

By subsection 2.3.3 there is M such that for all $m \geqslant M$ we have an embedding ψ_m of Q_n into a Grassmannian, given by a very ample line bundle L_m. The group $\mathrm{SL}(V)$ acts on $Q_n = \mathrm{Quot}^P_{X/S}(V \otimes \mathcal{W}, \mathcal{O}_X(1))$ and the line bundle L_m, giving us a linearisation of the action. For technical reasons (see lemma 1.17 of [Sim94a] and the preceding remark), Simpson restricts attention to $\widetilde{Q}_n$, the closure in Q_n of the set of points such that the quotient sheaf $\mathcal{F}$ is pure. (Compare the passage from the full Quot scheme to its subscheme Q in section 4.6 and the significance of this for lemma 4.8.6.) Then the main theorem in Simpson's GIT construction is:

Theorem 3.2.15 (cf. [Sim94a], theorem 1.19)**.** Given an algebraically closed field $\Bbbk$, a connected separated $\Bbbk$-scheme S of finite type, a projective S-scheme X with very ample $\mathcal{O}_X(1)$, an integer d and a polynomial P of degree d, there exist n_0 and m_0 such that for all $n \geqslant n_0$ and all $m \geqslant m_0$ the following holds: a point $(\mathcal{F}, \alpha)$ in $\widetilde{Q}_n$ is GIT-(semi)stable for the action of $\mathrm{SL}(V)$ with respect to the linearisation determined by the embedding ψ_m if and only if $\mathcal{F}$ is p-(semi)stable of pure dimension d and $\alpha : V \to H^0(\mathcal{F}(n))$ is an isomorphism.

Simpson's construction as it stands requires char $\Bbbk = 0$, but with the modifications outlined in subsection 3.2.2, i.e. with Maruyama's and Simpson's boundedness results replaced by Langer's in case of positive characteristic, Simpson's construction goes through for any $\Bbbk = \bar{\Bbbk}$.

For the purposes of section 3.3 , we would like to apply this result in the following set-up (as in [Pan96]): work over any algebraically closed field $\Bbbk$. Recall from subsection 3.1.2 that any Deligne-Mumford stable curve C of genus $g \geqslant 2$ may be embedded 10-canonically (i.e. using ω_C^{10} where ω_C is the dualising sheaf of C) into $\mathbb{P}^N_{\Bbbk}$ (where $N = 10(2g-2) - g$ by Riemann-Roch). If $\mathcal{F}$ is a sheaf of pure dimension 1 on C and $C_1, \ldots, C_\nu$ are the irreducible components of C, then let r_i be the rank (in the usual sense) of $\mathcal{F}$ at the generic point of C_i (i.e. the rank of the vector bundle $\mathcal{F}|_{(C_i)^{\mathrm{sm}}}$), call $(r_1, \ldots, r_\nu)$ the *multirank* of $\mathcal{F}$, and say that $\mathcal{F}$ is of *uniform rank* r if $r_i = r$ for all i.

Pandharipande works with the following definition of (semi)stability:

Definition 3.2.16 ([Pan96], section 1.1)**.** Let $\Bbbk$ be an algebraically closed field and C a DM-stable curve of genus $g \geqslant 2$. Let ω_i be the degree of the restriction of ω_C to C_i. A pure sheaf $\mathcal{F}$ of dimension 1 and multirank $(r_1, \ldots, r_\nu)$ has *slope (in the sense of Pandharipande)*

$$\mu^P(\mathcal{F}) := \frac{\chi(\mathcal{F})}{\sum_1^\nu r_i \omega_i}, \tag{3.8}$$

and $\mathcal{F}$ is μ^P*-(semi)stable* if it is pure of dimension 1 and for each non-zero proper subsheaf $\mathcal{E} \subset \mathcal{F}$ the inequality

$$\mu^P(\mathcal{E})(\leqslant)\mu^P(\mathcal{F}) \tag{3.9}$$

holds.

Note that this definition is just the standard definition (due to Seshadri) of slope in the case of canonical polarisation (i.e. with respect to the dualising sheaf ω_C). We will see in lemma 3.2.18 that for 10-canonically embedded curves, this is exactly the same as slope (with respect to the very ample line bundle $\mathcal{O}_C(1) := \omega_C^{10}$) in the sense of Simpson as defined in subsection 3.2.1, except that Pandharipande does not adjust for using ω_C^{10} instead of ω_C, hence there will be a constant correction factor of 1/10. Of course, this makes no difference to the notion of μ-(semi)stability.

In order to compare the above definition to the (semi)stability condition which Simpson works with, we first need to decompose pure sheaves of dimension 1 on DM-stable curve into vector bundles and skyscraper components. This lemma is also of interest in its own right as it gives us a more concrete geometric description of what these sheaves look like:

Lemma 3.2.17. Let $\Bbbk$, C and g be as in the definition above. Then for any pure sheaf $\mathcal{F}$ of dimension 1 there is a short exact sequence

$$0 \to \mathcal{F} \to \bigoplus_{i=1}^{\nu} (\iota_i)_* F_i \to \mathcal{G} \to 0 \tag{3.10}$$

where $\iota_i : C_i \hookrightarrow C$ is the inclusion of an irreducible component, F_i is a locally free sheaf on C_i, and $\mathcal{G}$ is a skyscraper sheaf supported in the set of nodes of C.

Proof. Let $\pi : \widetilde{C} \to C$ be the normalisation of C and consider the pullback $\pi^*\mathcal{F}$. Let $\widetilde{\mathcal{F}}$ be the quotient of $\pi^*\mathcal{F}$ by its torsion part:

$$0 \to (\pi^*\mathcal{F})_{\text{tors}} \to \pi^*\mathcal{F} \to \widetilde{\mathcal{F}} \to 0. \tag{3.11}$$

Pushing this sequence down to C, we have

$$0 \to \pi_*((\pi^*\mathcal{F})_{\text{tors}}) \to \pi_*\pi^*\mathcal{F} \to \pi_*\widetilde{\mathcal{F}}. \tag{3.12}$$

Now as the direct image functor and the inverse image functor are adjoints (see [Har77], p. 110), we have a natural isomorphism

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{F}, \pi_*\pi^*\mathcal{F}) = \text{Hom}_{\mathcal{O}_{\widetilde{C}}}(\pi^*\mathcal{F}, \pi^*\mathcal{F}), \tag{3.13}$$

hence the identity map in $\text{Hom}_{\mathcal{O}_{\widetilde{C}}}(\pi^*\mathcal{F}, \pi^*\mathcal{F})$ corresponds to a natural morphism $\eta_{\mathcal{F}} : \mathcal{F} \to \pi_*\pi^*\mathcal{F}$ (the unit of the adjunction, applied to $\mathcal{F}$). Comparing stalks at smooth points (using that π is an isomorphism away from the singular points of C) shows that this natural map has kernel supported in the set of nodes of C, hence $\ker \eta_{\mathcal{F}}$ is a zero-dimensional subsheaf of $\mathcal{F}$, so by purity of $\mathcal{F}$ we have $\ker \eta_{\mathcal{F}} = 0$. Composing $\eta_{\mathcal{F}}$ with $\pi_*\pi^*\mathcal{F} \to \pi_*\widetilde{\mathcal{F}}$ from (3.12), we obtain a map $f : \mathcal{F} \to \pi_*\widetilde{\mathcal{F}}$. But $\pi_*((\pi^*\mathcal{F})_{\text{tors}})$ also has support contained in the set of nodes: since $\mathcal{F}$ is pure, its restriction to $(C_i)^{\mathrm{sm}}$ is pure, hence torsion-free (as these concepts coincide on integral schemes), so $\mathcal{F}_x$ is a torsion-free $\mathcal{O}_x$-module at each smooth point $x \in C$. Now at each smooth point $x \in C$ the local ring $\mathcal{O}_x$ is a regular Noetherian local integral domain of dimension 1, hence a PID (see [AM69], proposition 9.2), so torsion-freeness of $\mathcal{F}_x$ implies freeness by the structure theorem of finitely generated modules over PIDs. Therefore $(\pi^*\mathcal{F})_{\mathrm{tors}}$ is supported in the set of points of $\widetilde{C}$ lying over nodes of C, hence $\pi_*((\pi^*\mathcal{F})_{\mathrm{tors}})$ is supported on the nodes of C. (Note that this also shows why $\mathcal{F}|_{(C_i)^{\mathrm{sm}}}$ is a vector bundle for each i, as claimed before definition 3.2.16: since $\mathcal{F}_x$ is free for each smooth point $x \in C$, every such $x \in C$ has an open neighbourhood U such that $\mathcal{F}|_U$ is locally free, i.e. $\mathcal{F}|_{(C_i)^{\mathrm{sm}}}$ is locally free for each i.)

Now since $\pi_*((\pi^*\mathcal{F})_{\mathrm{tors}})$ is supported on the nodes, (3.12) shows that the map $\pi_*\pi^*\mathcal{F} \to \pi_*\widetilde{\mathcal{F}}$ is injective away from the nodes, and so the same holds for f. Again by purity of $\mathcal{F}$, the zero-dimensional subsheaf $\ker f \subset \mathcal{F}$ must then be zero, so f is injective. But f is also surjective away from the nodes (again because $\mathcal{F}|_{(C_i)^{\mathrm{sm}}}$ is torsion-free), so setting $\mathcal{G} := \operatorname{coker} f$, we obtain an exact sequence

$$0 \to \mathcal{F} \to \pi_*\widetilde{\mathcal{F}} \to \mathcal{G} \to 0. \tag{3.14}$$

Now observe that $\pi_*\widetilde{\mathcal{F}}$ is of the form $\bigoplus_{j=1}^{\nu}(\iota_i)_*F_i$ for some locally free sheaves F_i on C_i, and the assertion follows. □

Lemma 3.2.18. Let $\Bbbk$, C and g be as in the definition above, and assume that C is k-canonically embedded for some integer $k \geqslant 1$, i.e. projectively embedded using the sections of $\omega_C^{\otimes k}$. Then we have

$$\mu(\mathcal{F}) = \frac{1}{k}\mu^P(\mathcal{F}) \tag{3.15}$$

for any pure sheaf $\mathcal{F}$ of dimension 1 (where μ is the slope with respect to the ample line bundle $\mathcal{O}_C(1) = \omega_C^{\otimes k}$), and hence a coherent sheaf on C is μ^P-(semi)stable if and only if it is μ-(semi)stable (in the sense of definition 3.2.5), and this is equivalent to being p-(semi)stable, where the (reduced) Hilbert polynomial is calculated with respect to $\omega_C^{\otimes k}$.

Proof. We need to consider $\chi(\mathcal{F}(n)) = a_1(\mathcal{F})n + a_0(\mathcal{F})$ in order to determine $\mu(\mathcal{F}) = a_0(\mathcal{F})/a_1(\mathcal{F})$. Take a short exact sequence as given by lemma 3.2.17 and tensor it with $\mathcal{O}_C(n)$:

$$0 \to \mathcal{F}(n) \to \bigoplus_{i=1}^{\nu}(\iota_i)_*F_i(n) \to \mathcal{G}(n) \to 0.$$

Here, $F_i(n) := F_i \otimes \mathcal{O}_C(n)|_{C_i}$, and the fact that $(\bigoplus_{i=1}^{\nu}(\iota_i)_*F_i) \otimes \mathcal{O}_C(n) = \bigoplus_{i=1}^{\nu}(\iota_i)_*F_i(n)$ can be verified by comparing stalks at an arbitrary point $x \in C_i$:

$$\begin{aligned} ((\iota_i)_*F_i \otimes \mathcal{O}_C(n))_x &= (F_i)_x \otimes_{\mathcal{O}_{C,x}} \mathcal{O}_C(n)_x \\ &= (F_i)_x \otimes_{\mathcal{O}_{C_i,x}} (\mathcal{O}_C(n)|_{C_i})_x \\ &= ((\iota_i)_*F_i(n))_x. \end{aligned}$$

Now also note that $\mathcal{G}(n) = \mathcal{G}$: a skyscraper sheaf is not changed under tensor product with an invertible sheaf. Then additivity of Euler characteristic gives us

$$\chi(\mathcal{F}(n)) = \chi(\mathcal{G}) + \sum_i \chi(F_i(n)),$$

and by applying Riemann-Roch to each of the locally free $F_i(n)$, we see that

$$\begin{aligned} a_1(\mathcal{F})n + a_0(\mathcal{F}) &= \chi(\mathcal{F} \otimes \mathcal{O}_C(n)) \\ &= \chi(\mathcal{G}) + \sum_i \chi(F_i(n)) \\ &= \chi(\mathcal{G}) + \sum_i \deg(F_i(n)) + r_i(1 - g_i). \end{aligned}$$

Recall that r_i was defined to be the rank of $\mathcal{F}$ at the generic point of C_i, which is the same as $\mathrm{rk}\ F_i = \mathrm{rk}\ F_i(n)$. Calculating $\deg\ F_i(n)$, we have

$$\begin{aligned} \deg\ F_i(n) &= \deg\ F_i + (\mathrm{rk}\ F_i) \cdot \deg\ \mathcal{O}_C(n)|_{C_i} \\ &= d_i + nk\omega_i r_i, \end{aligned}$$

where $d_i := \deg(F_i)$ and the last equality follows from $\mathcal{O}_C(1) = \omega_C^{\otimes k}$, as we have assumed that C is k-canonically embedded. Therefore,

$$a_1(\mathcal{F})n + a_0(\mathcal{F}) = \chi(\mathcal{G}) + \sum_i \left(d_i + nk\omega_i r_i + r_i(1 - g_i)\right), \tag{3.16}$$

and so

$$\begin{aligned} a_1(\mathcal{F}) &= k \sum_i r_i \omega_i, \text{ and} \\ a_0(\mathcal{F}) &= \chi(\mathcal{G}) + \sum_i d_i + r_i(1 - g_i). \end{aligned}$$

Thus,

$$\begin{aligned} \mu(\mathcal{F}) &= \frac{a_0(\mathcal{F})}{a_1(\mathcal{F})} \\ &= \frac{\chi(\mathcal{F}(0))}{k \sum_i r_i \omega_i} \\ &= \frac{1}{k} \frac{\chi(\mathcal{F})}{\sum_i r_i \omega_i} \\ &= \frac{1}{k} \mu^P(\mathcal{F}). \end{aligned}$$

This proves the first assertion. Since μ and μ^P only differ by a constant positive factor, the second assertion follows immediately, and finally μ-(semi)stability is equivalent to p-(semi)stability in this situation since for sheaves $\mathcal{F}$ of pure dimension 1 we have

$$p_0(\mathcal{F}, n) = n + \mu(\mathcal{F}),$$

hence for $0 \neq \mathcal{E} \subsetneq \mathcal{F}$ we have $\mu(\mathcal{E})(\leqslant)\mu(\mathcal{F})$ if and only if $p_0(\mathcal{E})(\leqslant)p_0(\mathcal{F})$. $\square$

Hence, when working on k-canonically embedded stable curves, we shall switch between the two definitions of slope freely, using appropriate notation (μ or μ^P) to indicate which one we refer to, and we shall drop any mention of p, μ or μ^P when talking about (semi)stability as all three notions coincide.

Given a pure sheaf $\mathcal{F}$ of dimension 1 and uniform rank r on a DM-stable curve C of genus g, define the degree of $\mathcal{F}$ by

$$e = \deg(\mathcal{F}) := \chi(\mathcal{F}) - r(1-g), \tag{3.17}$$

extending the usual notion of degree for vector bundles. Now recall from subsection 3.1.2 that we embedded genus $g \geqslant 2$ DM-stable curves 10-canonically into $\mathbb{P}^N_{\Bbbk}$ where $N = 10(2g-2)-g$. Let $d = 10(2g-2)$ and consider the Hilbert scheme $H_{g,d,N}$ of genus g, degree d curves in $\mathbb{P}^N_{\Bbbk}$. Let $H_g \subset H_{g,d,N}$ be the subset of non-degenerate 10-canonical DM-stable curves of genus g (this is a locally closed subscheme of $H_{g,d,N}$). Given a rank $r \geqslant 1$, a degree e and any curve $C \in H_g$, the Hilbert polynomial of degree e, uniform rank r, purely 1-dimensional sheaves $\mathcal{F}$ on C with respect to $\mathcal{O}_{\mathbb{P}^N}(1)|_C \cong \omega_C^{10}$ is

$$f_{e,r}(n) := \chi(\mathcal{F} \otimes \omega_C^{10n}) = e + r(1-g) + r \cdot 10(2g-2)n. \tag{3.18}$$

This can be seen by the same argument as in the proof of lemma 3.2.18: given $\mathcal{F}$, let F_i and $\mathcal{G}$ be as in lemma 3.2.17; then by (3.16) with $k=10$ and $r_i = r$ for all i we have

$$f_{e,r}(t) = \chi(\mathcal{G}) + \sum_i d_i + 10t\omega_i r + r(1-g_i).$$

But by definition of e,

$$e + r(1-g) = \chi(\mathcal{F}) = f_{e,r}(0) = \chi(\mathcal{G}) + \sum_i d_i + r(1-g_i).$$

So in order to prove (3.18), we are left to show that

$$\sum_i \omega_i = 2g-2 \tag{3.19}$$

– but this follows from theorem 3.1.2, parts (g) and (a):

$$\begin{aligned}
\sum_i \omega_i &= \sum_i (2h^1(C_i, \mathcal{O}_{C_i}) - 2 + \delta_i) \\
&= \sum_i (2g_i + 2\Delta_i - 2 + \delta_i) \\
&= 2\Big((\sum_i g_i) + \delta - \nu\Big) \\
&= 2(g-1),
\end{aligned}$$

using the expression (3.3), $2\delta = \sum_i (2\Delta_i + \delta_i)$ by double-counting of all nodes, and the genus formula (3.1) for connected curves C_i and C, respectively.

Now we can state the version of theorem 3.2.15 needed for the 'fibrewise GIT problem' in section 3.3: given $C \in H_g$, let

$$Q_g(C, b, f) := \mathrm{Quot}^{\omega_C^{\otimes 10}}_{C/\mathrm{Spec}\,\Bbbk}(\Bbbk^b \otimes \mathcal{O}_C, f), \tag{3.20}$$

and as usual we have the Grothendieck embedding

$$\psi_m : Q_g(C, b, f) \hookrightarrow \mathrm{Grass}(f(m), (\Bbbk^b \otimes \mathrm{Sym}^m(H^0(C, \omega_C^{10})))^*) \tag{3.21}$$

for all sufficiently large m (cf. subsection 2.3.3). Composing with the Plücker embedding, ψ_m gives us a linearisation of the $\mathrm{SL}_b(\Bbbk)$-action on $Q_g(C, b, f)$.

Theorem 3.2.19 ([Pan96], theorem 2.1.1)**.** Let $g \geqslant 2$ and $r \geqslant 1$ be integers. Then there exist bounds $e(g, r)$ and $m(g, r, e)$ such that for each $e > e(g, r)$ and each $m > m(g, r, e)$ and any curve C with $C \in H_g$, the following holds: a point $(\mathcal{F}, \alpha) \in Q_g(C, b = f_{e,r}(0), f_{e,r})$ is GIT-(semi)stable for the action of $\mathrm{SL}_b(\Bbbk)$ with respect to the linearisation determined by ψ_m if and only if $\mathcal{F}$ is (semi)stable of pure dimension 1 and the map $\alpha : \Bbbk^b \to H^0(C, \mathcal{F})$, which is induced by the quotient map $\Bbbk^b \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{F}$, is an isomorphism.

This is immediate from theorem 3.2.15 for $X = C$, $S = \mathrm{Spec}\,\Bbbk$, $\mathcal{O}_C(1) = \omega_C^{10}$, and $P = f_{e,r}$, *except* that the bounds $e(g, r)$ and $m(g, r, e)$ (corresponding to n_0 and m_0, respectively) are now uniform in $C \in H_g$. Here, the choice of sufficiently high degree e corresponds to the twisting parameter n in [Sim94a] in the following way: from (3.16), we have

$$\begin{aligned} \chi(\mathcal{F}(n)) &= \chi(\mathcal{F} \otimes \omega_C^{\otimes 10n}) \\ &= \chi(\mathcal{G}) + \sum_i d_i + 10n\omega_i r + r(1 - g_i) \\ &= \chi(\mathcal{F}) + 10n \cdot \sum_i r\omega_i \\ &= \chi(\mathcal{F}) + 10nr(2g - 2), \end{aligned}$$

hence twisting by n (i.e. tensoring by ω_C^{10n}) gives a natural correspondence between (semi)stable, uniform rank r, purely 1-dimensional sheaves of degrees e and $e + 10nr(2g - 2)$:

$$e(\mathcal{F}(n)) = \chi(\mathcal{F}(n)) - r(1-g) = \chi(\mathcal{F}) + 10nr(2g-2) - r(1-g) = e(\mathcal{F}) + 10nr(2g-2). \tag{3.22}$$

Also note that $b = f_{e+10nr(2g-2),r}(0) = e + r(1-g) + 10nr(2g-2) = f_{e,r}(n)$, so b is determined by n, once g, r and e are fixed.

Note that there are two slight differences between theorems 3.2.15 (applied to $X = C$, $S = \mathrm{Spec}\,\Bbbk$, $\mathcal{O}_C(1) = \omega_C^{10}$, and $P = f_{e,r}$) and 3.2.19: firstly, Simpson parametrises the sheaves $\mathcal{F}$ themselves, whereas Pandharipande's Quot scheme parametrises $\mathcal{F}(n)$ for sufficiently large n – this manifests itself in the choice of sheaf whose quotients are considered: Pandharipande looks at quotients of $\Bbbk^{P_n(0)} \otimes \mathcal{O}_C$ (where $P_n := f_{e+10nr(2g-2),r}$) with Hilbert polynomial $f_{e+10nr(2g-2),r}$, while Simpson considers quotients of $\Bbbk^{P(n)} \otimes \mathcal{O}_C(-n)$ (where $P := f_{e,r}$) with Hilbert polynomial $f_{e,r}$ (note that, as seen above, $P_n(0) = P(n)$). But as tensoring by $\mathcal{O}_C(-n)$ gives a canonical $\mathrm{SL}_{P(n)}(\Bbbk)$-equivariant isomorphism between those Quot schemes, we may identify the two GIT problems. The second difference is that Simpson only looks at $\widetilde{Q}_n$, the closure of the locus of pure sheaves in the Quot scheme Q_n – this is only for the sake of lemma 1.17 in [Sim94a] which states (in our sitatution on curves) that for any sheaf $\mathcal{F}$ represented by a point of Q_n, there is a sheaf $\mathcal{F}'$ of pure dimension 1 with Hilbert polynomial P and an inclusion $\mathcal{F}/\mathcal{F}_{\mathrm{tors}} \hookrightarrow \mathcal{F}'$. The proof of this lemma given in [Sim94a] might fail for $\mathcal{F}$ lying in irreducible components of Q_n parametrising non-pure sheaves (if any such components exist at all) which is why Simpson imposes the restriction $(\mathcal{F}, \alpha) \in \widetilde{Q}_n$. Simpson goes on to use this lemma when proving that GIT-(semi)stability of $(\mathcal{F}, \alpha)$ implies purity and (semi)stability of $\mathcal{F}$ and invertibility of α. However, for the situation we need to consider ($\dim \mathcal{F} = 1$), we can do without this lemma (as we will show below – see page 68), allowing us to replace $\widetilde{Q}_n$ by Q_n throughout Simpson's proof. Thus, we may prove theorem 3.2.19 by proving theorem 3.2.15, provided that we give bounds n_0, m_0 which are uniform in $C \in H_g$. We claim that a careful analysis of Simpson's construction (together with one application of theorem 3.2.13 to sheaves on fibres of $U_H \to H_g$, where U_H is the universal curve over H_g) in fact yields the uniform bounds required – uniformity of the bounds would also be achieved by applying theorem 3.2.15 directly to $X = U_H$ and $S = H_g$, but since we want to replace $\widetilde{Q}_n$ by Q_n in passing from theorem 3.2.15 to theorem 3.2.19, we state the ingredients of the proof and give full detail where we deviate from Simpson's arguments.

3.2.4 Identification of the GIT-(semi)stable loci

This section sketches the proof of theorem 3.2.19. As explained at the end of subsection 3.2.3, we do so by proving theorem 3.2.15 (with $X = C$, $S = \operatorname{Spec} \Bbbk$, $\mathcal{O}_C(1) = \omega_C^{10}$, and $P = f_{e,r}$) together with the additional assertion that n_0 and m_0 may be chosen uniformly in $C \in H_g$, and with $\widetilde{Q}_n$ replaced by Q_n.

We begin with n-regularity of all semistable sheaves on C, *uniformly in* $C \in H_g$:

Lemma 3.2.20. Let $g \geqslant 2$, $r \geqslant 1$ and $e \geqslant 1$ be integers. Then there is an integer n depending on g, r and e (but independent of C) such that for *any* $C \in H_g$, every semistable sheaf on C with Hilbert polynomial $f_{e,r}$ is n-regular.

Proof. We can identify the collection

$$\{\text{s.s. sheaves on } C \text{ with Hilbert polynomial } f_{e,r} \text{ w.r.t. } \omega_C^{10} = \mathcal{O}_{\mathbb{P}^N}(1)|_C \mid C \in H_g\}$$

with the collection of semistable sheaves on the fibres of U_H over closed points of H_g with Hilbert polynomial $f_{e,r}$ (with respect to the line bundle $\mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^N}(1))$ on U_H, where pr_2 is the projection of $U_H \hookrightarrow H_g \times \mathbb{P}^N$ onto $\mathbb{P}^N$), but the latter collection is bounded by an application of theorem 3.2.13 to $X = U_H$, $S = H_g$ and $\mathcal{O}_X(1) = \mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^N}(1))$. Then uniform n-regularity follows by theorem 2.3.9. □

We now state the key ingredients in Simpson's proof of theorem 3.2.15 (for $X = C$ and $P = f_{e,r}$) as given in [Sim94a], except that we adapt the statements using lemma 3.2.20 to make all choices of constants uniform in $C \in H_g$. First recall that in the paragraph immediately preceding theorem 3.2.15 we chose n such that semistable sheaves on C with given Hilbert polynomial are n-regular, in order to parametrise them by a Quot scheme $Q_n = \mathrm{Quot}_{C/\Bbbk}^{\mathcal{O}_C(1)}(V \otimes \mathcal{O}_C(-n), P)$; here and throughout this subsection, we set $P := f_{e,r}$ and $V = \Bbbk^{P(n)}$. By virtue of lemma 3.2.20 (together with lemma 2.3.6), we can now pick this n simultaneously for all $C \in H_g$ – call this value n_1.

The second step towards choosing n_0 is a technical lemma whose proof we omit (but we prove a parabolic analogue of this in full in chapter 4 in the higher-dimensional situation):

Lemma 3.2.21 (cf. [Sim94a], lemma 1.18)**.** There exists n_2 (depending only on g, e and r) such that for all $C \in H_g$, all $n \geqslant n_2$ and all semistable sheaves $\mathcal{F}$ on C with Hilbert polynomial $P = f_{e,r}$ the following is true: for all non-zero subsheaves $\mathcal{E} \subset \mathcal{F}$ we have

$$\frac{h^0(\mathcal{E}(n))}{a_1(\mathcal{E})} \leqslant \frac{P(n)}{a_1(\mathcal{F})} \tag{3.23}$$

and if equality holds, then $p_0(\mathcal{E}) = p_0(\mathcal{F})$.

We now need to start choosing m_0, i.e. a suitable linearisation L_m ($m \geqslant m_0$), that works for all $C \in H_g$:

Lemma 3.2.22 (cf. [Sim94a], lemma 1.15)**.** Given $n \in \mathbb{N}$, there exists $m_1(n) \in \mathbb{N}$ (depending only on g, e, r and n, but not on C) such that for all $C \in H_g$ and all $m \geqslant m_1$, the following holds: suppose $\mathcal{F}$ is a quotient of $V \otimes \mathcal{O}_C(-n)$ corresponding to a point of $Q_n(C) := \mathrm{Quot}_{C/\Bbbk}^{\omega_C^{10}}(V \otimes \mathcal{O}_C(-n), P)$. For any non-zero subspace $H \subset V$, let $\mathcal{G}_H$ be the subsheaf of $\mathcal{F}$ generated by $H \otimes \mathcal{O}_C(-n)$. Now suppose that for our given $m \in \mathbb{N}$ and for all $0 \neq H \subsetneq V$ we have $P(\mathcal{G}_H, m) > 0$ and

$$\frac{\dim H}{P(\mathcal{G}_H, m)} (\leqslant) \frac{P(n)}{P(m)}.$$

Then the point $[\mathcal{F}]$ of the Quot scheme is GIT-(semi)stable for the $\mathrm{SL}_{P(n)}(\Bbbk)$-action with respect to L_m.

There is a partial converse:

Lemma 3.2.23 (cf. lemma 1.16 in [Sim94a]). Given $n \in \mathbb{N}$, there exists $m_2(n) \in \mathbb{N}$ (depending only on g, e, r and n, but not on C) such that for all $C \in H_g$ and all $m \geqslant m_2(n)$, the following holds: suppose $\mathcal{F}$ is a quotient of $V \otimes \mathcal{O}_C(-n)$ corresponding to a point of $Q_n := \mathrm{Quot}^{\omega_C^{10}}_{C/\Bbbk}(V \otimes \mathcal{O}_C(-n), P)$. For any non-zero subspace $H \subset V$, let $\mathcal{G}_H$ be the subsheaf of $\mathcal{F}$ generated by $H \otimes \mathcal{O}_C(-n)$. Now suppose that $[\mathcal{F}]$ is GIT-semistable for the $\mathrm{SL}_{P(n)}(\Bbbk)$-action with respect to L_m. Then for all $0 \neq H \subset V$ we have $a_1(\mathcal{G}_H) > 0$, $\dim \mathcal{G}_H = \dim \mathcal{F} = 1$, and

$$\frac{\dim H}{a_1(\mathcal{G}_H)} \leqslant \frac{P(n)}{a_1(\mathcal{F})}. \tag{3.24}$$

We will also need the following corollary of lemma 3.2.23:

Corollary 3.2.24. Under the same hypotheses on $\mathcal{F}$ as in lemma 3.2.23 and for $m \geqslant m_2(n)$, let $\mathcal{K}$ be any quotient of $\mathcal{F}$. Then

$$\frac{h^0(\mathcal{K}(n))}{a_1(\mathcal{K})} \geqslant \frac{P(n)}{a_1(\mathcal{F})}. \tag{3.25}$$

Finally, we need the following cohomology bound for sheaves parametrised by our Quot schemes Q_n:

Lemma 3.2.25 (cf. [Pan96], lemma 3.1.1). Fix integers $g \geqslant 2$ and $r \geqslant 1$. For all $C \in H_g$ (i.e. 10-canonically embedded DM stable curves of genus g), all $e \geqslant 1$ and all $n \geqslant 1$, we have $h^1(C, \mathcal{F}(n)) \leqslant (2g-2)^2 rg$ for all $\mathcal{F} \in Q_n$.

Proof. Recall that (by abuse of notation – see convention 1.4.5) the sheaf $V \otimes \mathcal{O}_C(-n)$ denotes the trivial vector bundle of rank $P(n)$ twisted by $-n$, i.e. $\Bbbk^{P(n)} \otimes \mathcal{O}_C(-n)$ is (non-canonically) isomorphic to $\mathcal{O}_C^{\oplus P(n)} \otimes \mathcal{O}_C(-n) = \mathcal{O}_C(-n)^{\oplus P(n)}$, so we have a surjection $\mathcal{O}_C(-n)^{\oplus P(n)} \to \mathcal{F} \to 0$. Thus, $\mathcal{O}_C^{\oplus P(n)} \to \mathcal{F}(n) \to 0$, i.e. $\mathcal{F}(n)$ is generated by its global sections. But $\mathcal{F} \in Q_n$ has Hilbert polynomial $f_{e,r}$ with respect to ω_C^{10}, so if $\mathcal{F}$ has multirank $(r_1, \dots, r_\nu)$, then by comparing (3.16) and (3.18) we have

$$\sum_{i=1}^{\nu} r_i \omega_i = (2g-2)r,$$

so $r_i \leqslant (2g-2)r$ for $1 \leqslant i \leqslant \nu$. Hence, $\mathcal{F} \in Q_n$ has generic rank less than $(2g-2)r$ on each irreducible component of C, and the same holds for our globally generated $\mathcal{F}(n)$: the rank does not change by twisting up, as shown by (3.16), for example. Thus, we have an exact sequence

$$\mathcal{O}_C^{\oplus \nu(2g-2)r} \to \mathcal{F}(n) \to \mathcal{T} \to 0$$

where $\mathcal{T}$ is a sheaf of dimension 0, and ν is the number of irreducible components of C. The associated long exact sequence in cohomology shows that $h^1(C, \mathcal{F}(n)) \leqslant h^1(C, \mathcal{O}_C^{\oplus \nu(2g-2)r})$. Since cohomology commutes with direct sums, we have

$$\begin{aligned} h^1(C, \mathcal{F}(n)) &\leqslant \nu(2g-2)r \cdot h^1(C, \mathcal{O}_C) \\ &\leqslant (2g-2)^2 rg, \end{aligned}$$

because $h^1(C, \mathcal{O}_C) = g$ (by definition of genus for a stable curve), and because ω_C is ample and of degree $2g-2$ for a stable curve, so the number of components is bounded above by $\nu \leqslant 2g-2$. □

With the previous five results in place, we can now put all the pieces together and complete the proof of theorem 3.2.19, i.e. theorem 3.2.15 with m_0 and n_0 uniform in $C \in H_g$ and valid for all of Q_n (rather than just $\widetilde{Q}_n$). We will choose n_0 and m_0 as we go along. Throughout, given n, assume that $m(n)$ is sufficiently large so that the Grothendieck map ψ_m embeds Q_n into a Grassmannian.

<u>'If':</u> Take $[\mathcal{F}, \alpha] \in Q_n$ (i.e. we have a quotient $\alpha : V \otimes \mathcal{O}_C(-n) \to \mathcal{F} \to 0$ such that $\mathcal{F}$ has Hilbert polynomial P with respect to $\mathcal{O}_C(1)$) and assume that $\mathcal{F}$ is (semi)stable and the map $\alpha : V \to H^0(\mathcal{F}(n))$ is an isomorphism; we are required to prove that $[\mathcal{F}, \alpha] \in Q_n$ is GIT-(semi)stable with respect to the linearisation determined by ψ_m. By lemma 3.2.22, it suffices to prove that there is $m(n)$ such that for all $m \geqslant m(n)$ and all non-zero subsheaves $\mathcal{G} \subset \mathcal{F}$ generated by a subspace of $H^0(\mathcal{F}(n))$ (which is identified with V via α), we have

$$\frac{h^0(\mathcal{G}(n))}{P(\mathcal{G}, m)} (\leqslant) \frac{P(n)}{P(m)}. \tag{3.26}$$

Now take $n \geqslant n_0 := \max(n_1, n_2)$ where n_1 is as in lemma 3.2.20, and n_2 is as given by lemma 3.2.21, i.e. such that for all C, all semistable sheaves $\mathcal{F}$ on C with Hilbert polynomial P, and all non-zero subsheaves $\mathcal{G} \subset \mathcal{F}$ we have

$$\frac{h^0(\mathcal{G}(n))}{a_1(\mathcal{G})} \leqslant \frac{P(n)}{a_1(\mathcal{F})} \tag{3.27}$$

and if equality holds, then $p_0(\mathcal{G}) = p_0(\mathcal{F})$. But once $n \geqslant \max(n_1, n_2)$ is fixed, the collection

$$\begin{aligned} \{\mathcal{G} \subset \mathcal{F} \mid \mathcal{G} \text{ generated by a subspace of } H^0(\mathcal{F}(n)), \mathcal{F} \text{ semistable sheaf on } C \\ \text{with Hilbert polynomial } P, C \in H_g\} \end{aligned} \tag{3.28}$$

is bounded – this can either be shown by a double Quot scheme argument (proposition 2.3.11), or we can bound this collection by $Q_n \times \mathrm{Grass}(V^*)$, where $\mathrm{Grass}(V^*)$ is the total Grassmannian of subspaces of V.

In particular, the set of Hilbert polynomials $P(\mathcal{G})$ for $\mathcal{G}$ in (3.28) is finite, so we can increase $m(n)$ further such that for those $\mathcal{G} \subset \mathcal{F}$ where we have strict inequality in (3.27), we get (3.26) with strict inequality for all $m \geqslant m(n)$. This completes the stable case, and the case of non-critical $\mathcal{G} \subset \mathcal{F}$ for strictly semistable $\mathcal{F}$. If equality holds in (3.27), then $\mathcal{G} \subset \mathcal{F}$ is a critical subsheaf, i.e. $p_0(\mathcal{G}) = p_0(\mathcal{F})$. Thus, dividing the left-hand side of (3.27) by $p_0(\mathcal{G}, m) = P(\mathcal{G}, m)/a_1(\mathcal{G})$ and the right-hand side of (3.27) by $p_0(\mathcal{G}, m) = p_0(\mathcal{F}, m) = P(m)/a_1(\mathcal{F})$ gives (3.26) with equality.

'Only if': Increase $m_0(n)$ further to be greater than $m_2(n)$ as given by lemma 3.2.23, and consider $m \geqslant m_0(n)$. Take $[\mathcal{F}, \alpha] \in Q_n$ and assume that $[\mathcal{F}, \alpha] \in Q_n$ is GIT-(semi)stable with respect to the linearisation determined by ψ_m; we are required to prove that $\mathcal{F}$ is (semi)stable and the map $V \to H^0(\mathcal{F}(n))$ induced by α is an isomorphism. At this point, Simpson quotes lemma 1.17 in [Sim94a] which is the reason why he needs to restrict to $\mathcal{F} \in \widetilde{Q}_n$. To get rid of this restriction, we replace Simpson's use of lemma 1.17 by an ad-hoc argument that relies on the fact that our sheaves are based on curves (a careful analysis of the following argument shows that the only form in which the hypothesis $\dim \mathcal{F} = 1$ is used repeatedly is the fact that the torsion subsheaf does not change under twisting, as it is a skyscraper sheaf).

First, notice that by lemma 3.2.23, the map $V \to H^0(\mathcal{F}(n))$ induced by α must be injective for GIT-semistable $[\mathcal{F}, \alpha]$: suppose not, let $H \subset V$ be the non-zero kernel, then $\mathcal{G}_H = 0$, so $a_1(\mathcal{G}_H) = 0$ and we have a contradiction with lemma 3.2.23. Next, we show that $\mathcal{F}$ must be pure and the map $\alpha : V \to H^0(\mathcal{F}(n))$ an isomorphism.

Suppose that $\mathcal{F}$ has a proper non-zero subsheaf $\mathcal{G}$ of dimension 0. Then $\mathcal{G}$ is generated by global sections of $\mathcal{F}(n)$: as $\mathcal{G}(n) \cong \mathcal{G}$ for a skyscraper sheaf and as $\mathcal{F}(n)$ is generated by its global sections for any quotient $\mathcal{F}$ of $V \otimes \mathcal{O}_C(-n)$, we see that $\mathcal{G}$ is generated by $H^0(\mathcal{F}(n))$ as $H^0(\mathcal{F}(n))$ generates $\mathcal{F}(n)_x \supset \mathcal{G}(n)_x = \mathcal{G}_x$ for all $x \in C$. Let G be the subspace of $H^0(\mathcal{F}(n))$ that generates $\mathcal{G}$; as $\mathcal{G} \neq 0$, we also have $G \neq 0$. Now if the image of the map $\alpha : V \to H^0(\mathcal{F}(n))$ meets G, then let H be the preimage of G in V, necessarily $H \neq 0$. In the notation of lemma 3.2.23, we have $\mathcal{G}_H \subset \mathcal{G}$, so the assumption that $\dim \mathcal{G} = 0$ implies $\dim \mathcal{G}_H = 0$ and hence contradicts GIT semistability of $\mathcal{F}$ by lemma 3.2.23. Therefore, for all GIT-(semi)stable $[\mathcal{F}, \alpha] \in Q_n$ and all non-zero subsheaves $\mathcal{G} \subset \mathcal{F}$ of dimension 0, we see that the image of α in $H^0(\mathcal{F}(n))$ does not meet $G \supset H^0(\mathcal{G}(n)) = H^0(\mathcal{G})$, so for all such $\mathcal{F}$ and $\mathcal{G}$ we have

$$h^0(\mathcal{G}) \leqslant h^0(\mathcal{F}(n)) - \dim (\mathrm{Im}\ \alpha) = h^0(\mathcal{F}(n)) - P(n) = h^1(\mathcal{F}(n)), \tag{3.29}$$

using injectivity of α for $\dim (\mathrm{Im}\ \alpha) = P(n)$.

Now let $\mathcal{F}' := \mathcal{F}/\mathcal{F}_{\mathrm{tors}}$, where $\mathcal{F}_{\mathrm{tors}}$ is the maximal subsheaf of $\mathcal{F}$ supported in dimension $< \dim \mathcal{F} = 1$, hence $\mathcal{F}_{\mathrm{tors}}$ is a skyscraper sheaf and $\mathcal{F}/\mathcal{F}_{\mathrm{tors}}$ is a pure sheaf of dimension 1 with Hilbert polynomial $P(\mathcal{F}', n) = \chi(\mathcal{F}(n)) - \chi(\mathcal{F}_{\text{tors}}(n)) = P(n) - h^0(\mathcal{F}_{\text{tors}}) = P(n) - c(\mathcal{F})$ for some non-negative integer $c(\mathcal{F})$, using that skyscraper sheaves are unchanged under twisting. Now observe that

$$c(\mathcal{F}) = h^0(C, \mathcal{F}_{\text{tors}}) \leqslant h^1(\mathcal{F}(n)) \leqslant (2g-2)^2 rg$$

where the first inequality is by equation (3.29), and the second inequality follows from lemma 3.2.25. Thus $0 \leqslant c(\mathcal{F}) \leqslant (2g-2)^2 rg$ and there are only finitely many $c(\mathcal{F})$ occurring for all C, n and $\mathcal{F} \in Q_n$.

By corollary 3.2.24, we have for any quotient $\mathcal{F}' \to \mathcal{G} \to 0$:

$$\frac{h^0(\mathcal{G}(n))}{a_1(\mathcal{G})} \geqslant \frac{P(n)}{a_1(\mathcal{F})}, \tag{3.30}$$

since $\mathcal{G}$ is also a quotient of the GIT-semistable sheaf $\mathcal{F} \in Q_n$. Since $\mathcal{F}'$ is pure, it has a Harder-Narasimham filtration – for each $\mathcal{F}$, let $\mathcal{G}$ be the last step in the Harder-Narasimham filtration of $\mathcal{F}'$, i.e. the quotient of $\mathcal{F}'$ with the smallest non-zero slope $\mu = \mu(\mathcal{G})$. Taking (3.30) together with theorem 3.2.14 applied to the sheaf $\mathcal{G}(n)$, we obtain

$$\frac{P(n)}{a_1(\mathcal{F})} \leqslant \frac{h^0(\mathcal{G}(n))}{a_1(\mathcal{G})} \leqslant \mu(\mathcal{G}) + n + B \tag{3.31}$$

for some integer B depending only on r but not on C, $\mathcal{G}$ or n. Since P is fixed, there is an integer c (independent of C, n and $\mathcal{F}$) such that

$$\frac{P(n)}{a_1(\mathcal{F})} \geqslant n - c \tag{3.32}$$

for all n (e.g. $c := \lceil a_0(\mathcal{F})/a_1(\mathcal{F}) \rceil$ will do). Then $n - c \leqslant \mu(\mathcal{G}) + n + B$, i.e. $\mu(\mathcal{G}) \geqslant -B - c$. Thus we have shown that for all C, n and GIT-semistable $\mathcal{F} \in Q_n$ all quotients of $\mathcal{F}'$ have slope bounded below, hence all non-zero subsheaves of $\mathcal{F}'$ have slope bounded above (independent of n). Recall that the $\mathcal{F}'$ also have only finitely many different Hilbert polynomials (namely $P - c(\mathcal{F})$ where $0 \leqslant c(\mathcal{F}) \leqslant (2g-2)^2 rg$), so by theorem 3.2.13 (applied with $S = H_g$ and $X = U_H$) the sheaves $\mathcal{F}'$ vary in a bounded collection, independent of n. In particular, we may increase n_0 such that the $\mathcal{F}'$ are all n_0-regular, and so for all $n \geqslant n_0$ we have $h^0(\mathcal{F}'(n)) = P(\mathcal{F}', n) = P(n) - c(\mathcal{F})$ and $\mathcal{F}'(n)$ is generated by global sections. Now applying corollary 3.2.24 to the quotient $\mathcal{F}'$ of $\mathcal{F}$, we have

$$\frac{h^0(\mathcal{F}'(n))}{a_1(\mathcal{F})} = \frac{h^0(\mathcal{F}'(n))}{a_1(\mathcal{F}')} \geqslant \frac{P(n)}{a_1(\mathcal{F})}, \tag{3.33}$$

i.e. $P(n) \leqslant h^0(\mathcal{F}'(n)) = P(\mathcal{F}', n) = P(n) - c(\mathcal{F})$ – hence $c(\mathcal{F}) = 0$, so (for GIT-semistable $\mathcal{F} \in Q_n$) $\mathcal{F} = \mathcal{F}'$ is pure and remains in a bounded collection independent of n. We also have $h^0(\mathcal{F}(n)) = P(n)$. Recall that at the start of the 'only if' argument, we had shown that $\alpha : V \to H^0(\mathcal{F}(n))$ is injective, so by counting dimensions it is an isomorphism.

We have shown that $\mathcal{F}$ is pure and α an isomorphism, it remains to show that $\mathcal{F}$ satisfies the (semi)stability condition on subsheaves. Suppose there are any GIT-semistable but not semistable $\mathcal{F}$ – for each such $\mathcal{F}$ choose a quotient corresponding to a destabilising subsheaf, i.e. a quotient $\mathcal{G}$ of $\mathcal{F}$ such that

$$\frac{P(\mathcal{G})}{a_1(\mathcal{G})} < \frac{P}{a_1(\mathcal{F})} \tag{3.34}$$

– since the GIT-semistable $\mathcal{F} \in Q_n$ vary in a bounded collection (uniformly for all C and n) as seen above, and since we can choose the $\mathcal{G}$ to have only a finite number of Hilbert polynomials (as the existence of a Harder-Narasimham filtration for pure $\mathcal{F}$ allows us to pick $\mathcal{G}$ corresponding to maximal destabilising subsheaves, we conclude by proposition 2.3.11 that the $\mathcal{G}$ form a bounded collection, so we may increase n_0 such that for all $n \geqslant n_0$

$$\frac{P(\mathcal{G}, n)}{a_1(\mathcal{G})} < \frac{P(n)}{a_1(\mathcal{F})} \tag{3.35}$$

and also $h^0(\mathcal{G}(n)) = P(\mathcal{G}, n)$. But for GIT-semistable $\mathcal{F}$, this contradicts corollary 3.2.24, hence there are no GIT-semistable $\mathcal{F}$ that are unstable. This completes the 'only if' case for GIT-semistable $\mathcal{F}$.

Suppose that there are GIT-stable $\mathcal{F} \in Q_n$ which are strictly semistable. For each such $\mathcal{F}$, pick $0 \neq \mathcal{E} \subsetneq \mathcal{F}$ with $p_0(\mathcal{E}) = p_0(\mathcal{F})$. Writing $V := \Bbbk^{P(n)}$, $W := H^0(\mathcal{O}_C(m-n))$, $B := H^0(\mathcal{F}(m))$ and $H := H^0(\mathcal{E}(n)) \subset H^0(\mathcal{F}(n)) \cong V$, we apply proposition 2.2.6: as the $\mathcal{F}$ remain in a bounded collection and as there are finitely many choices for $P(\mathcal{E})$ (namely $a_1(\mathcal{E})p_0(\mathcal{F})$ for $1 \leqslant a_1(\mathcal{E}) \leqslant a_1(\mathcal{F})$), by proposition 2.3.11 the $\mathcal{E}$ remain in a bounded collection, so we may increase n_0 and m_0 further such that the $\mathcal{E}$ and $\mathcal{F}$ are both n_0- and m_0-regular. In particular, for all $n \geqslant n_0$ and all $m \geqslant m_0$, $\mathcal{E}(m)$ is globally generated, hence $\operatorname{Im}(H \otimes W) = H^0(\mathcal{E}(m))$ (where $\operatorname{Im}(H \otimes W)$ is the image of $H \otimes W$ in B), and

$$\frac{\dim H}{h^0(\mathcal{E}(m))} = \frac{h^0(\mathcal{E}(n))}{h^0(\mathcal{E}(m))} = \frac{P(\mathcal{E}, n)}{P(\mathcal{E}, m)} = \frac{P(\mathcal{F}, n)}{P(\mathcal{F}, m)} = \frac{P(n)}{h^0(\mathcal{F}(m))} \tag{3.36}$$

(where the third equality uses $p_0(\mathcal{E}) = p_0(\mathcal{F})$). Then

$$\frac{\dim H}{\dim \operatorname{Im}(H \otimes W)} = \frac{\dim V}{\dim B},$$

so by proposition 2.2.6, $\mathcal{F}$ maps under the Grothendieck embedding ψ_m to a point which is not GIT-stable in the Grassmannian, hence by proposition 2.2.2, $\mathcal{F} \in Q_n$ is not GIT-stable in the Quot scheme, contradicting our hypothesis.

This concludes the proof of theorem 3.2.19.

Remark 3.2.26. The above proof of theorem 3.2.19 does not depend on the assumption that the schemes C are 1-dimensional, except for the ad-hoc argument which allows us to consider the GIT problem of $\mathrm{SL}_{P(n)} \curvearrowright Q_n$ instead of $\mathrm{SL}_{P(n)} \curvearrowright \widetilde{Q}_n$ (recall that $\widetilde{Q}_n$ was defined as the closure of the set of points $[\mathcal{F}] \in Q_n$ corresponding to pure sheaves $\mathcal{F}$). Indeed, theorem 3.2.15 was proved by Simpson for families $X \to S$ of projective schemes of arbitrary dimension and thus can be applied to give 'uniform' moduli of sheaves on base schemes Y varying in a bounded family themselves: if the base schemes Y are embedded as subschemes (with some fixed Hilbert polynomial h with respect to $\mathcal{O}_{\mathbb{P}^N}(1)$) of an ambient space $\mathbb{P}^N$, i.e. if the Y range over points in some part H of a Hilbert scheme parametrising subschemes of $\mathbb{P}^N$ with Hilbert polynomial h, take $S = H$ and $X = U_H$ the universal family over H. Then the collection of all semistable sheaves $\mathcal{F}$ on any $Y \in H$ with Hilbert polynomial P (with respect to $\mathcal{O}_{\mathbb{P}^N}(1)|_X$) is the same as the collection of semistable sheaves on the fibres of U_H over closed points of H with Hilbert polynomial P with respect to $\mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^N}(1))$, where pr_2 is the projection of the universal family $U_H \hookrightarrow H \times \mathbb{P}^N$ to $\mathbb{P}^N$. Thus, theorem 3.2.15 can be applied directly to give an analogue of theorem 3.2.19 for higher-dimensional base schemes Y varying in H (and with constants n_0 and m_0 independent of $[Y] \in H$).

Remark 3.2.27. The subscheme of $\widetilde{Q}_n$ corresponding to p-semistable sheaves $\mathcal{E}$ and isomorphisms α (which by theorem 3.2.15 is precisely the GIT-semistable locus of $\widetilde{Q}_n$) satifies the local universal property for the functor $\mathbf{M}(\mathcal{O}_X, P)$, and the group action of $\mathrm{SL}_{P(n)}(\Bbbk)$ on the GIT-semistable locus of $\widetilde{Q}_n$ satisfies the condition of theorem 2.2.4, so Simpson's GIT quotient $\widetilde{Q}_n // \mathrm{SL}_{P(n)}(\Bbbk)$ (with respect to the linearisation determined by the embedding ψ_m) gives the moduli space $M(\mathcal{O}_X, P)$ of semistable sheaves on X/S with Hilbert polynomial P, a coarse moduli space for $\mathbf{M}(\mathcal{O}_X, P)$ which is projective over S. In particular, Simpson identifies the points of the GIT quotient $\widetilde{Q}_n // \mathrm{SL}_{P(n)}(\Bbbk)$ (which a priori correspond to S-equivalence classes in the sense of GIT, see section 2.2) as the S-equivalence classes (in the sense of Jordan-Hölder filtrations, see definition 3.2.7) of semistable sheaves: by theorem 2.2.1, this reduces to showing that the closures of the orbits of two semistable sheaves $\mathcal{F}_1$, $\mathcal{F}_2$ in Q_n intersect if and only if $\mathrm{gr}(\mathcal{F}_1) \cong \mathrm{gr}(\mathcal{F}_2)$, and Simpson proves this using an argument from [Gie77]: write $\mathcal{F}$ as an extension of some of its Jordan-Hölder factors, then there is a family parametrised by $\mathbb{A}^1$ which is isomorphic to $\mathcal{F}$ away from $0 \in \mathbb{A}^1$ and isomorphic to the direct sum of the JH factors at 0. Repeating this process shows that the orbit of $\mathrm{gr}(\mathcal{F})$ lies in the closure of the orbit of $\mathcal{F}$, and is indeed a closed orbit.

3.3 Pandharipande's universal moduli of sheaves on stable curves

In this section, we explain Pandharipande's construction of universal moduli spaces of slope-semistable vector bundles (or torsion-free sheaves of uniform rank, to be more precise) on stable curves. (We continue to use the notation introduced in subsection 3.3.1 from page

58 onwards.) Following [Pan96], we construct such spaces as GIT quotients of a suitable relative Quot scheme by a product of two special linear groups, where one of the factors acts on the bundles part of the problem only, whereas the other factor acts on curves as well as bundles (via the bundle isomorphisms induced by isomorphisms of the curves). There are two (related) aspects of [Pan96] which we modify: we replace Pandharipande's treatment of what he calls the 'fibrewise GIT problem' (i.e. his approach to studying $\mathrm{SL}_{P(n)} \curvearrowright Q_n(C)$ for a fixed curve C, where $Q_n(C)$ is the same Quot scheme of a certain sheaf on C as in the last section) by the Simpson-style construction of section 3.2, since this is more suitable for generalisations. To do this, we change the (global and fibrewise) Quot schemes used by Pandharipande slightly – this is for the sake of compatibility with section 3.2: Pandharipande's relative Quot scheme $Q_g(\mu, n, f)$ parametrises $\mathcal{F}(n)$ where $\mathcal{F}$ ranges over the sheaves which we are interested in, whereas we use a Quot scheme Q_n^{big} (depending on a parameter n) that parametrises the sheaves $\mathcal{F}$ themselves (cf. the observations at the end of subsection 3.2.3 for why these two approaches are equivalent). Throughout this section, we work over Spec $\Bbbk$, where $\Bbbk$ is a fixed algebraically closed field of arbitrary characteristic.

In [Pan96], a compactification for the moduli problem of pairs (C, F) is given, where F is a slope-semistable vector bundle (of degree e and rank r) on a non-singular curve C of genus $g \geqslant 2$. Specifically, Pandharipande constructs a projective variety $\overline{U_g(e,r)}$ functorially parametrising equivalence classes of slope-semistable pure 1-dimensional sheaves on Deligne-Mumford stable curves of genus $g \geqslant 2$ satisfying the following:

- The set $U_g(e, r)$ of equivalence classes of pairs (C, F), where C is a non-singular genus g curve and F is a slope-semistable vector bundle on C, corresponds functorially to a dense open subset of $\overline{U_g(e,r)}$;
- $\overline{U_g(e,r)}$ maps to $\overline{M}_g$ (the moduli space of DM-stable curves) via a morphism η making

$$\begin{array}{ccc} U_g(e,r) & \hookrightarrow & \overline{U_g(e,r)} \\ \downarrow & & \downarrow \eta \\ M_g & \hookrightarrow & \overline{M}_g \end{array}$$

  commute, where $U_g(e, r) \to M_g$ is just the forgetful map (and M_g is the moduli space of smooth curves of genus g);
- for each $[C] \in M_g$, there is a natural isomorphism $\eta^{-1}([C]) \cong \overline{M_C(e,r)}/\mathrm{Aut}(C)$, where $\overline{M_C(e,r)}$ is the coarse moduli space of degree e, rank r slope-semistable vector bundles up to S-equivalence on C. (Here, S-equivalence is defined using Jordan-Hölder filtrations, see definition 3.2.7.)

Recall from definition 3.2.16 that for a pure sheaf $\mathcal{F}$ of dimension 1 on a 10-canonically embedded DM-stable curve C of genus $g \geqslant 2$, Pandharipande defines the slope of $\mathcal{F}$ to be

$$\mu^P(\mathcal{F}) := \frac{\chi(\mathcal{F})}{\sum_1^\nu r_i \omega_i}, \tag{3.37}$$

where ω_i is the degree of the restriction of the dualising sheaf ω_C to C_i (with $C_1, \dots, C_\nu$ being the irreducible components of C) and the tuple $(r_1, \dots, r_\nu)$ is called the multirank of $\mathcal{F}$ (where r_i is the rank of $\mathcal{F}$ at the generic point of C_i). As we have seen in lemma 3.2.18, this is essentially the same as the slope of $\mathcal{F}$ (with respect to $\mathcal{O}_C(1) := \mathcal{O}_{\mathbb{P}^N}(1)|_C \cong \omega_C^{10}$, where $C \hookrightarrow \mathbb{P}^N_{\Bbbk}$ is induced by ω_C^{10}) in the sense of subsection 3.2.1, so as in subsections 3.2.3 and 3.2.4 we shall continue to say that a purely 1-dimensional sheaf $\mathcal{F}$ on C is (semi)stable if satisfies any of the equivalent conditions of μ-(semi)stability, μ^P-(semi)stability, or p-(semi)stability.

3.3.1 The GIT set-up

As in subsections 3.2.3 and 3.2.4, let

$$e = \deg(\mathcal{F}) := \chi(\mathcal{F}) - r(1-g). \tag{3.38}$$

be the degree of purely 1-dimensional sheaves $\mathcal{F}$ of uniform rank r on a DM-stable curve C of genus g and let H_g be the locally closed subscheme of $H_{g,d,N}$ parametrising non-degenerate 10-canonical DM-stable curves of genus g, where $H_{g,d,N}$ is the Hilbert scheme of genus g, degree $d = 10(2g-2)$ curves in $\mathbb{P}^N_{\Bbbk}$, the projective space into which all DM-stable curves of genus g are 10-canonically embedded. Let

$$f_{e,r}(n) := \chi(\mathcal{F} \otimes \omega_C^{10n}) = e + r(1-g) + r \cdot 10(2g-2)n \tag{3.39}$$

be the Hilbert polynomial (with respect to $\mathcal{O}_{\mathbb{P}^N}(1)|_C \cong \omega_C^{10}$) of degree e, uniform rank r, purely 1-dimensional sheaves $\mathcal{F}$ on C for any $C \in H_g$ (see the argument following (3.18) for why this is the correct Hilbert polynomial). We often write P for $f_{e,r}$ (as the degree $e \in \mathbb{Z}$ and the uniform rank $r \geqslant 1$ will be fixed throughout section 3.3).

The compactified moduli space $\overline{U_g(e,r)}$ is obtained via a GIT construction: First, the problem is rigidified by adding isomorphisms $\Bbbk^{N+1} \to H^0(C, \omega_C^{10})$ and $\Bbbk^{P(n)} \to H^0(C, \mathcal{F}(n))$ (for sufficiently large n) to the data of pairs, where $N = 10(2g-2) - g$ by Riemann-Roch, and $P(n) = h^0(\mathcal{F}(n))$, assuming that $\mathcal{F}(n)$ is generated by its global sections and that its higher cohomology vanishes. For sufficiently large n, these assumptions are consequences of semistability: the statement that all semistable sheaves of uniform rank r and degree e are n-regular for some uniform n (lemma 3.2.20) gives us just what we need.

The parameter space for the rigidified pairs is given by the relative Quot scheme

$$Q_n^{\mathrm{big}} := \mathrm{Quot}_{U_H/H_g}^{\mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^N}(1))}(\Bbbk^{P(n)} \otimes \mathcal{O}_{U_H}(-n), P),$$

where $U_H \hookrightarrow H_g \times \mathbb{P}^N$ is the universal curve (i.e. the universal family over the Hilbert scheme $H_{g,d,N}$, restricted to the subset $H_g \subset H_{g,d,N}$), and pr_2 is the projection of U_H onto $\mathbb{P}^N$. Note that since U_H is projective over H_g, theorem 2.3.13 implies that Q_n^{big} is projective over H_g. Let $\pi : Q_n^{\mathrm{big}} \to H_g$ be the map given by the construction of the Quot scheme; recall from lemma 2.3.2 that the fibre of Q_n^{big} over $C \in H_g$ is the Quot scheme

$$Q_n(C) := \mathrm{Quot}_{C/\mathrm{Spec}\,\Bbbk}^{\mathcal{O}_C(1)}(\Bbbk^{P(n)} \otimes \mathcal{O}_C(-n), P),$$

for quotients on the curve C. By the results of section 2.3, we have the closed embedding (Grothendieck embedding)

$$\begin{array}{rcl}\psi_m : Q_n^{\mathrm{big}} & \to & H_g \times_{\mathrm{Spec}\,\Bbbk} \mathrm{Grass}(\Bbbk^{P(n)} \otimes H^0(\mathbb{P}^N_{\Bbbk}, \mathcal{O}_{\mathbb{P}^N}(m-n)), P(m)) \\ {[\mathcal{F}]} & \mapsto & \Big(\pi([\mathcal{F}]), H^0(\mathcal{F}(m))\Big)\end{array}$$

for all sufficiently large m, say $m \geqslant m_0$. Recall from subsection 3.1.2 Gieseker's (semi)stability results for the Grothendieck embedding $\widetilde{\psi}_s$ of the Hilbert scheme $H_{g,d,N}$ for sufficiently large s: if $s \geqslant s_0(g)$, then

$$\widetilde{\psi}_s : H_{g,d,N} \to \mathrm{Grass}(H^0(\mathbb{P}^N_{\Bbbk}, \mathcal{O}_{\mathbb{P}^N}(s)), h(s)),$$

is a linearisation of the natural SL_{N+1}-action on $H_{g,d,N}$ with respect to which

(a) H_g is contained in the GIT-stable locus; and

(b) H_g is closed in the GIT-semistable locus.

Here, $h(s) := ds - g + 1$ is the Hilbert polynomial of subschemes parametrised by $H_{g,d,N}$.

Composing ψ_m with $\widetilde{\psi}_s \times$ id (for $s \geqslant s_0(g)$) and with the relevant Plücker embeddings, we get an embedding

$$j_{s,m} : Q_n^{\mathrm{big}} \to \mathbb{P}\left(\bigwedge^{h(s)} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(s))\right) \times \mathbb{P}\left(\bigwedge^{P(m)} \left(\Bbbk^{P(n)} \otimes H^0(\mathbb{P}^N_{\Bbbk}, \mathcal{O}_{\mathbb{P}^N}(m-n))\right)\right) \tag{3.40}$$

which makes it clear that the natural SL_{N+1}- and $\mathrm{SL}_{P(n)}$-actions commute on Q_n^{big}, hence giving an action of $\mathrm{SL}_{N+1} \times \mathrm{SL}_{P(n)}$ on Q_n^{big}. Define Q_n^r to be the subset of Q_n^{big} corresponding to quotient sheaves $\mathcal{F}$ on curves C such that $\mathcal{F}$ has uniform rank r on C – this is an open and closed subscheme (i.e. a union of connected components) of Q_n^{big} (see lemma 8.1.1 of [Pan96]) and invariant under $\mathrm{SL}_{N+1} \times \mathrm{SL}_{P(n)}$. Then the object of study of this GIT construction is the action of $\mathrm{SL}_{N+1} \times \mathrm{SL}_{P(n)}$ on Q_n^r.

The strategy for solving this GIT problem (due to Pandharipande) involves the following steps: we begin with a result in abstract GIT giving a choice of linearisation of the $\mathrm{SL}_{N+1} \times \mathrm{SL}_{P(n)}$-action for which Q_n^r is entirely contained in the SL_{N+1}-stable locus and closed in the SL_{N+1}-semistable locus – the trick here is to weight the curves part of the problem heavily in comparison to the bundles part. This allows us to import Gieseker's results for the SL_{N+1}-action on H_g from subsection 3.1.2. Next, it is shown that with respect to the same linearisation, the $\mathrm{SL}_{N+1} \times \mathrm{SL}_{P(n)}$-(semi)stable loci equal the $\mathrm{SL}_{P(n)}$-(semi)stable loci, respectively (since $\mathrm{SL}_{P(n)}$ only acts on the second factor in (3.40) corresponding to the bundles part, the weighting is irrelevant to the GIT problem for $\mathrm{SL}_{P(n)}$ alone) – so by choosing n and m large enough to apply the results of section 3.2 to what Pandharipande calls the 'fibrewise GIT problem', i.e. the action of $\mathrm{SL}_{P(n)}$ on the fibre $\pi^{-1}(C) = Q_n(C)$ for any fixed $C \in H_g$, the (semi)stable loci can be identified. We replace Pandharipande's own study of the fibrewise problem, which takes up a large part of his construction (sections 2-6 of [Pan96]), by the procedure of section 3.2, as this makes generalisations to parabolic sheaves (and possibly higher-dimensional base schemes) easier. We also note that the last step of Pandharipande's argument, where he uses results of Seshadri from [Ses82] to characterise which points in the parameter space get identified in the quotient, can be modified using Simpson's ideas so that it also passes from curves to higher-dimensional schemes.

3.3.2 Choice of suitable weights and construction of $\overline{U_g(e,r)}$

We first quote a result from section 7 of [Pan96] on the abstract GIT problem of a special linear group acting on a product of projective spaces, allowing us to analyse the SL_{N+1}-action on Q_n^r, linearised by the embedding (3.40) – note that $\mathrm{SL}_{P(n)}$ only acts on one of the projective spaces in (3.40). We omit the proof of this result, as it does not require any change from that given in [Pan96]; however, we emphasise that this is the crucial tool for universal moduli constructions, allowing us to bring together moduli constructions for the base schemes (e.g. stable curves) with moduli constructions for sheaves (e.g. torsion-free sheaves or parabolic sheaves on a fixed curve). Note that Hu has an alternative approach to universal moduli constructions in [Hu96], but this method only works over $\Bbbk = \mathbb{C}$ as it utilises the connection between GIT and symplectic geometry: Hu studies GIT-(semi)stability via moment maps.

Let V, W, Z be finite-dimensional $\Bbbk$-vector spaces. Given two homomorphisms

$$\begin{aligned} \zeta : \mathrm{SL}(V) &\to \mathrm{SL}(Z) \\ \text{and} \quad \omega : \mathrm{SL}(V) &\to \mathrm{SL}(W), \end{aligned}$$

we have $\mathrm{SL}(V)$-actions on $\mathbb{P}(W)$ and $\mathbb{P}(Z)$ together with natural linearisations. For the induced $\mathrm{SL}(V)$-action on $\mathbb{P}(Z)\times\mathbb{P}(W)$, the linearisations $\mathcal{O}_{\mathbb{P}(Z)}(a)\otimes\mathcal{O}_{\mathbb{P}(W)}(b)$ are indexed by $(a,b)\in\mathbb{Z}^2_{>0}$ (these are all ample linearisations since $\mathrm{Pic}(\mathbb{P}(Z)\times\mathbb{P}(W))=\mathbb{Z}\oplus\mathbb{Z}$). Write $\mathrm{pr}_Z:\mathbb{P}(Z)\times\mathbb{P}(W)\to\mathbb{P}(Z)$ for the projection onto the first factor, and use $[a,b]$ as a subscript to indicate the linearisation of $\mathbb{P}(Z)\times\mathbb{P}(W)$ with respect to which GIT-(semi)stability is described (for the $\mathrm{SL}(V)$-actions on $\mathbb{P}(Z)$ and $\mathbb{P}(W)$ individually, we use the linearisations given by ζ and ω).

Proposition 3.3.1 ([Pan96], propositions 7.1.1 and 7.1.2)**.** There are integers $k_s(\zeta,\omega)$ and $k_{ss}(\zeta,\omega)$ such that for all $k\geqslant k_s$

$$\mathrm{pr}_Z^{-1}(\mathbb{P}(Z)^s)\subset(\mathbb{P}(Z)\times\mathbb{P}(W))^s_{[k,1]} \tag{3.41}$$

and for all $k\geqslant k_{ss}$

$$(\mathbb{P}(Z)\times\mathbb{P}(W))^{ss}_{[k,1]}\subset\mathrm{pr}_Z^{-1}(\mathbb{P}(Z)^{ss}). \tag{3.42}$$

These facts are proved by Pandharipande via explicit calculations involving the Hilbert-Mumford criterion and notions of stability depending on choices of parameters – the proof is elementary, but takes up a few pages (see section 7 of [Pan96]). Note the simple but elegant idea here: faced with a group action on a product of projective spaces, weighting the linearisation heavily on the first factor means that pairs $(z,w)\in\mathbb{P}(Z)\times\mathbb{P}(W)$ with z stable are necessarily stable as pairs, and pairs (z,w) are automatically unstable whenever z is unstable. In the present context, this is why Pandharipande's universal moduli space only includes pairs (C,E) with the underlying curves already DM-stable – if C is not DM-stable (hence not GIT-semistable in Gieseker's construction, see part (a) of theorem 3.1.3), then the pair cannot be stable, no matter how well behaved E is.

Now, for the rest of this section, use the superscripts s and ss to denote GIT-(semi)stability with respect to $\mathrm{SL}_{N+1}\times\mathrm{SL}_{P(n)}$, primed superscripts s' and ss' to indicate GIT-(semi)stability with respect to SL_{N+1}, and double primes s'' and ss'' to denote GIT-(semi)stability with respect to $\mathrm{SL}_{P(n)}$. Set $s:=s_0(g)$ to be the constant from theorem 3.1.3 needed to guarantee that Gieseker's GIT analysis of the linearisation given by the Grothendieck embedding $\widetilde{\psi}_s$ holds. Pick m and n large enough so that theorem 3.2.19 about the fibrewise GIT problem holds, and let

$$Z := \bigwedge^{h(s)} H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(s))$$

$$\text{and}\quad W := \bigwedge^{P(m)}\Big(\Bbbk^{P(n)}\otimes H^0(\mathbb{P}^N_{\Bbbk},\mathcal{O}_{\mathbb{P}^N}(m-n))\Big),$$

so $\mathrm{SL}(V) := \mathrm{SL}_{N+1}$ acts on $\mathbb{P}(Z) \times \mathbb{P}(W)$ in the way described above. Let $k := \max(k_s, k_{ss})$ given by proposition 3.3.1 for this set-up. Then we have

$$\mathrm{pr}_Z^{-1}(\mathbb{P}(Z)^{s'}) \subset (\mathbb{P}(Z) \times \mathbb{P}(W))^{s'}_{[k,1]} \tag{3.43}$$
$$\text{and} \quad (\mathbb{P}(Z) \times \mathbb{P}(W))^{ss'}_{[k,1]} \subset \mathrm{pr}_Z^{-1}(\mathbb{P}(Z)^{ss'}). \tag{3.44}$$

From (a) in subsection 3.3.1, we have $H_g \subset \mathbb{P}(Z)^{s'}$, so from (3.43), we have

$$Q_n^r \subset H_g \times \mathbb{P}(W) \subset (\mathbb{P}(Z) \times \mathbb{P}(W))^{s'}_{[k,1]}. \tag{3.45}$$

Condition (b) of subsection 3.3.1 says that H_g is closed in $\mathbb{P}(Z)^{ss'}$, so by continuity of pr_Z, $H_g \times \mathbb{P}(W)$ is closed in $\mathrm{pr}_Z^{-1}(\mathbb{P}(Z)^{ss'})$. But as Q_n^r is projective over H_g, it is closed in $H_g \times \mathbb{P}(W)$, thus closed in $\mathrm{pr}_Z^{-1}(\mathbb{P}(Z)^{ss'})$, so in particular

$$Q_n^r \text{ is closed in } (\mathbb{P}(Z) \times \mathbb{P}(W))^{ss'}_{[k,1]}, \tag{3.46}$$

which is a subset of $\mathrm{pr}_Z^{-1}(\mathbb{P}(Z)^{ss'})$ by (3.44).

It follows immediately from the general definitions that if a point x is GIT-(semi)stable with respect to an action by some group G, then it is GIT-(semi)stable with respect to any subgroup $H \leqslant G$ (and the same linearisation as for G), so

$$(\mathbb{P}(Z) \times \mathbb{P}(W))^{ss}_{[k,1]} \subset (\mathbb{P}(Z) \times \mathbb{P}(W))^{ss'}_{[k,1]},$$

which together with (3.46) shows that

$$(Q_n^r)^{ss}_{[k,1]} \text{ is closed in } (\mathbb{P}(Z) \times \mathbb{P}(W))^{ss}_{[k,1]}. \tag{3.47}$$

Hence, if we define

$$\overline{U_g(e,r)} := (Q_n^r)^{ss}_{[k,1]}//(\mathrm{SL}_{N+1} \times \mathrm{SL}_{P(n)}), \tag{3.48}$$

then $\overline{U_g(e,r)}$ is a projective $\Bbbk$-scheme by (3.47) and theorem 2.2.1.

This completes the first step of the plan mentioned at the end of subsection 3.3.1. Pandharipande's next step is the observation that the GIT-(semi)stable loci in Q_n^r with respect to $\mathrm{SL}_{N+1} \times \mathrm{SL}_{P(n)}$ coincide with those for $\mathrm{SL}_{P(n)}$: by the same general remark about GIT-(semi)stability with respect to subgroups as above, we certainly have

$$(Q_n^r)^{s}_{[k,1]} \quad \subset \quad (Q_n^r)^{s''}_{[k,1]} \tag{3.49}$$
$$\text{and} \quad (Q_n^r)^{ss}_{[k,1]} \quad \subset \quad (Q_n^r)^{ss''}_{[k,1]}. \tag{3.50}$$

In fact, these are equalities, as Pandharipande proves in proposition 8.2.1 of [Pan96] by an explicit application of the Hilbert-Mumford criterion using only (3.45) – so in particular for our universal moduli construction in chapter 5, these equalities also hold provided we have an analogue of (3.45). Together with theorem 3.2.19, this concludes the study of the GIT-(semi)stable loci (for $n \geqslant n_0$, $m \geqslant m_0$), showing that the points of $(Q_n^r)_{[k,1]}^{(s)s}$ are precisely the $[\mathcal{F}, \alpha]$ with $\mathcal{F}$ a purely 1-dimensional slope-(semi)stable quotient $\Bbbk^{P(n)} \otimes \mathcal{O}_C(-n) \to \mathcal{F} \to 0$ of uniform rank r and degree e on a DM-stable 10-canonical curve $C \subset \mathbb{P}^N$ of genus g, and $\alpha : \Bbbk^{P(n)} \otimes H^0(C, \mathcal{O}_C(-n)) \to H^0(C, \mathcal{F})$ an isomorphism. (Note that we are implicitly using lemma 2.2.3 here: the GIT-(semi)stable points in Q_n^r lying over $C \in H_g$ are the exactly the GIT-(semi)stable points in $Q_n^r(C)$, the fibre of Q_n^r over C, and those are the ones characterised by theorem 3.2.19).

It remains to identify the points of $\overline{U_g(e,r)}$ in terms of the $\mathrm{SL}_{N+1} \times \mathrm{SL}_{P(n)}$-orbits on $(Q_n^r)^{ss}$. A simple argument of Pandharipande (pp. 456 – 457 of [Pan96]) shows that if $\bar{\xi} \in (Q_n^r)_{[k,1]}^{ss}$ lies in the closure of the $\mathrm{SL}_{N+1} \times \mathrm{SL}_{P(n)}$-orbit of $\xi \in (Q_n^r)_{[k,1]}^{ss}$, then the SL_{N+1}-orbit of $\bar{\xi}$ intersects the closure of the $\mathrm{SL}_{P(n)}$-orbit of ξ. But by Simpson's results for the fibrewise GIT problem (see remark 3.2.27), the closures of the $\mathrm{SL}_{P(n)}$-orbits of two sheaves $\mathcal{F}_1$, $\mathcal{F}_2$ over the same curve C intersect if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are S-equivalent (Pandharipande quotes results of Seshadri in [Ses82] to get the same result, but these only appear to be valid for curves, whereas Simpson's arguments hold in arbitrary dimension), so taking account of the SL_{N+1}-action as well, we arrive at the following result:

Theorem 3.3.2 ([Pan96], theorem 8.2.1)**.** The closed points of the projective $\Bbbk$-scheme $\overline{U_g(e,r)}$ correspond to aut-equivalence classes of slope-semistable purely 1-dimensional sheaves of uniform rank r and degree e on DM-stable curves of genus g, where two S-equivalence classes of sheaves on the same curve C are said to be *aut-equivalent* if they differ by an automorphism of C, and the slope of a sheaf $\mathcal{F}$ on a (possibly reducible stable) curve C is given by (3.37), i.e. using the invertible canonical sheaf to define slope and Hilbert polynomial (cf. lemma 3.2.18).

In particular, the fibre of $\overline{U_g(e,r)}$ over any fixed curve $[C] \in \overline{M}_g$ is just $\overline{M_C(e,r)}/\mathrm{Aut}(C)$ where $\overline{M_C(e,r)}$ is the moduli space of pure 1-dimensional (i.e. 'torsion-free') sheaves of uniform rank r and degree e on C.

In section 9 of [Pan96], Pandharipande completes the proof that $\overline{U_g(e,r)}$ is a coarse moduli space for the moduli problem of pairs under consideration (theorem 9.1.1 of [Pan96] shows the required universal property, using that the Quot scheme Q_n^r clearly satisfies the local universal property for families of pairs $(C, \mathcal{F})$). In the last sections of the paper, some basic geometric properties (irreducibility and normality) of $\overline{U_g(e,r)}$ are derived, and it is shown that in the rank 1 case, Pandharipande's coarse moduli space coincides with the compactification of the universal Picard variety which had been obtained earlier in [Cap94].

Chapter 4

Moduli of parabolic sheaves on a family of projective schemes

4.1 Background and overview

Parabolic bundles on smooth curves with marked points were introduced by Mehta and Seshadri to extend the Narasimhan-Seshadri correspondence to open Riemann surfaces: this classical correspondence states that for a smooth projective complex algebraic curve (equivalently, a compact Riemann surface) C of genus $g \geqslant 2$, isomorphism classes of unitary representations of the fundamental group of C correspond to S-equivalence classes of semistable (holomorphic) degree 0 vector bundles on C, and the former are irreducible whenever the latter are stable. Thinking about how to extend this theorem to the case of non-compact Riemann surfaces $C \setminus \{x^1, \dots, x^n\}$ led to the definition of parabolic bundles [MS80]: a parabolic bundle $E^\star_*$ on a smooth projective marked curve $(C, x^1, \dots, x^n)$ is a vector bundle E on C together with a (partial) flag[1]

$$E(x^i) = F^i_1(E) \supset F^i_2(E) \supset \cdots \supset F^i_{l_i+1}(E) = 0 \tag{4.1}$$

of subspaces in the fibre at each point $x^i \in C$, and a collection of weights $0 \leqslant \alpha^i_1 < \cdots < \alpha^i_{l_i} < 1$ determining a notion of parabolic degree

$$\text{par–deg } E := \deg E + \sum_{i=1}^{n} \sum_{j=1}^{l_i} \alpha^i_j (\dim F^i_j(E) - \dim F^i_{j+1}(E)) \tag{4.2}$$

and thus a slope-stability condition for these objects. Mehta and Seshadri constructed moduli spaces for parabolic bundles of fixed flag type (i.e. dimension vector for the flag), fixed parabolic degree and parabolic weights, and showed that isomorphism classes of unitary representations of $\pi_1(C \setminus \{x^1, \dots, x^n\})$ with fixed holonomy around the punctures x^i given

[1] Recall convention 1.4.7: inclusions denoted by $\subset$ and $\supset$ will be not necessarily strict, unless explictly stated otherwise. In particular, we may have $F^i_j(E) = F^i_{j+1}(E)$ for some j in (4.1).

by the α_j^i (with repetitions/multiplicities) correspond to S-equivalence classes of semistable parabolic degree 0 bundles of weight α_j^i (and flag type given by the multiplicities of the α_j^i) on $(C, x^1, \ldots, x^n)$, and irreducibility of the representation corresponds to stability of the parabolic bundle again.

The concept of parabolic bundles was subsequently extended to higher-dimensional smooth projective varieties [Bho92], [MY92], [Yok93] marked by divisors, and to arbitrary projective schemes [Ina00]. As in the moduli theory of sheaves without augmentations [Sim94a], one needs to consider pure sheaves instead of vector bundles to obtain complete moduli spaces when working on projective schemes of any dimension and singularity type. The notion of parabolic structure generalises naturally to this context: given a projective scheme X marked by an effective Cartier divisor D, a 'parabolic sheaf' means a pure coherent sheaf $\mathcal{E}$ (whose support may have strictly smaller dimension than X) such that $\dim(D \cap \operatorname{Supp} \mathcal{E}) < \dim \operatorname{Supp} \mathcal{E}$, together with a filtration by subsheaves whose last term is $\mathcal{E}(-D)$ and with a collection of parabolic weights (see definition 4.2.1). Among the moduli constructions available to date, the most general (as far as the support schemes are concerned) is Inaba's moduli space [Ina00] of stable parabolic sheaves on a flat family $X \to S$ of projective schemes (with a base S only assumed to be noetherian, not necessarily over an algebraically closed field) marked by a single relative effective Cartier divisor.

In this chapter, we content ourselves with a flat family $X \to S$ of arbitrary projective schemes *with base S of finite type over an algebraically closed field* $\Bbbk$ (of any characteristic), but we allow markings by any finite number of divisors. The aim of this chapter is to construct S-projective coarse moduli spaces for semistable parabolic sheaves (of fixed numerical type, i.e. parabolic weights, Hilbert polynomial, and flag type) on such families. In [Ina00], Inaba deals with stable parabolic sheaves only, giving a coarse moduli space which is at best quasi-projective over S (even this is not clear as the moduli space is constructed as an inductive limit of quasi-projective schemes). We extend and simplify Inaba's method by following some of the ideas in Simpson's construction [Sim94a] of moduli of semistable pure sheaves more closely and in particular by strengthening the boundedness results which Inaba works with (and for this purpose we impose the extra assumption that S be finite type over $\Bbbk$).

More specifically, Inaba works with an auxiliary notion of e-stability: let X be a projective scheme polarised by a very ample line bundle $\mathcal{O}_X(1)$. Then [Ina00] calls a d-dimensional parabolic sheaf e-stable if it is stable (i.e. p_0-stable, where p_0 is the reduced parabolic Hilbert polynomial with respect to $\mathcal{O}_X(1)$, see definitions 4.3.1 and 4.3.4) and if in addition its restriction to the intersection of any $d-1$ general divisors in the linear system $|\mathcal{O}_X(1)|$ satisfies an modified slope-stability concept depending on $e \in \mathbb{Z}$. In [Ina00] only the collection of e-stable parabolic sheaves of given numerical type is shown to be bounded. We strengthen this by proving that the collection of all semistable parabolic sheaves on X/S of fixed numerical type is bounded (see section 4.5) and hence can be parametrised by a Flag-Quot scheme. The strategy for our boundedness result is to reduce the problem for parabolic sheaves to that for the underlying coherent sheaves: we use intersection theory to show that for all semistable parabolic sheaves of fixed numerical type the underlying coherent sheaf has uniformly bounded μ-Harder-Narasimhan type (i.e. the slope of all subsheaves is bounded above uniformly for this collection), which by results of Simpson [Sim94a] and Langer [Lan04a] (for characteristic zero and $p > 0$, respectively) implies that the collection of underlying coherent sheaves is bounded (see proposition 4.3.7 and theorem 4.5.2 for the details of this argument).

The lack of a stronger boundedness statement results in an indirect construction of the moduli space in [Ina00]: Inaba obtains a moduli space M^e of e-stable parabolic sheaves for any $e \in \mathbb{Z}_{>0}$ as a GIT quotient and then shows that $M := \varinjlim M^e$ is a coarse moduli space for stable parabolic sheaves. Our construction is a substantial simplification of this in that the inductive limit argument (and the auxiliary concept of e-stability) is no longer needed: instead, we find a suitable Flag-Quot scheme in section 4.6 as our parameter space for all semistable parabolic sheaves of fixed numerical type on X/S. This parameter space (more precisely, a subscheme of it) solves a rigidified version of the moduli problem: it classifies parabolic sheaves arising as quotients of a fixed sheaf (or, equivalently, parabolic sheaves $\mathcal{E}_*^\star$ with a rigidification of $H^0(\mathcal{E}(m))$ for sufficiently large m). To remove the rigidification data, we consider a natural action by a special linear group on this Flag-Quot scheme (which identifies points corresponding to isomorphic parabolic sheaves). We linearise this action in section 4.7 by embedding the Flag-Quot scheme into a product of Grassmannians (using Grothendieck's original projective embedding of the Quot scheme, as employed in Simpson's construction [Sim94a] of moduli of pure sheaves) and by choosing the weights of the group action on these Grassmannians according to the parabolic weights. The resulting GIT quotient is our moduli space, which is then automatically projective over S since it is constructed as the quotient of a parameter space which is projective over S. For this reason, our Flag-Quot scheme differs from the parameter space constructed in [Ina00]: we avoid various open conditions imposed by Inaba (such as purity of the sheaf and vanishing of higher cohomology) as we wish to work with a parameter space which is projective over S.

For this reason, our Flag-Quot scheme also parametrises some objects which are not pure parabolic sheaves. Thus, our analysis of the GIT-(semi)stable locus in section 4.8 must demonstrate that a point of the Flag-Quot scheme is GIT-semistable only if the underlying sheaf is actually pure. More significantly, avoiding open conditions in the construction of the Flag-Quot scheme means that we need to choose our projective embedding of the Flag-Quot scheme very carefully. As a result, the Hilbert-Mumford criterion for the Grassmannians in which our Flag-Quot scheme is embedded gives rise to a stability condition for parabolic sheaves which differs from Inaba's p_0-stability: we need to work with a modified parabolic Hilbert polynomial par–P_2 (definition 4.3.11) and refer to the resulting stability condition as p_2-stability. Even on smooth curves, this is not in general the same stability condition as ordinary slope-stability, but it is less likely to produce strictly semistable objects, and leads to compact moduli spaces even in higher dimensions. Whenever there are no strictly slope-semistable objects, our new stability condition agrees with traditional slope-stability, but we demonstrate in remark 4.3.13 the difference between slope-stability and p_2-stability.

After having established the most suitable stability condition, we use the Hilbert-Mumford criterion to identify the GIT-(semi)stable locus on the Flag-Quot scheme with the locus of p_2-(semi)stable parabolic sheaves (the arguments involved are similar in spirit to those in [Sim94a]). Inaba [Ina00] shows that stable parabolic sheaves are GIT-stable; we also show the converse and extend the result to the semistable locus.

It is not clear in general how the strictly semistable orbits in the GIT quotient are identified: I have not been able to prove that S-equivalence can be defined via graded objects associated to Jordan-Hölder filtrations. Even for traditional p_0-stability of parabolic sheaves, this is a little subtle but can be done: see proposition 4.4.3 where we show that p_0-semistable pure parabolic sheaves of fixed parabolic Hilbert polynomials form an abelian category. (Curiously, we can only prove this result under the assumption that the lowest weight α_1^i is positive for at least one divisor D^i, or alternatively assuming that the intersection of all the effective divisors D^i is empty.) However, our GIT quotient identifies orbits of p_2-semistable parabolic sheaves with each other if their closures meet in the semistable set. It would be desirable to have a more intrinsic description of this GIT-S-equivalence purely in terms of filtrations of parabolic sheaves.

Throughout this chapter, we allow parabolic structures at any number of divisors. We feel it is worth treating the general case: it has been argued in some of the previous moduli constructions for parabolic sheaves that the categories of parabolic sheaves are equivalent whether dealing with separate divisors $D^1, \dots, D^n$ or with a single divisor $D = \sum_{i=1}^n D^i$. However, this does not appear to be accurate: parabolic structures at the individual divisors indeed give rise to a parabolic structure at D, but not conversely in general. This may be illustrated most clearly in the case of vector bundles on a smooth curve marked by points $x^1, \dots, x^n$: a parabolic structure at $D := x^1 + \cdots + x^n$ may involve 'mixing' of fibres over distinct marked points (see remark 4.2.2 for more details).

4.1.1 Statement of main result

The set-up in this chapter is as follows: let $X \to S$ be a flat family of projective schemes (parametrised by a connected scheme S of finite type over an algebraically closed field $\Bbbk$ of any characteristic) marked by finitely many relative effective Cartier divisors D^i and polarised by a relatively very ample line bundle $\mathcal{O}_X(1)$. The main result (theorem 4.9.2) is: given a numerical polynomial $H \in \mathbb{Q}[x]$ of any degree $d \leqslant \dim X/S$ (where $\dim X/S$ is the constant dimension of the fibres of $X \to S$) and numerical polynomials $H^i_j \in \mathbb{Q}[x]$ of degree $d-1$, there exists a coarse moduli space $P^{H,H^\star_*,\alpha^\star_*,ss}_{D^\star/X/S}$ of $\sim$-equivalence classes of p_2-semistable pure d-dimensional parabolic sheaves on the fibres of $X \to S$ (with parabolic structures at the D^i, fixed Hilbert polynomial H and flag type $H^\star_*$ with respect to $\mathcal{O}_X(1)$, and rational parabolic weights $\alpha^\star_*$). Here, by the 'flag type' of a parabolic structure on a sheaf $\mathcal{E}$ at a divisor D^i we mean the Hilbert polynomials H^i_j of the quotients $\mathcal{E}/F^i_j(\mathcal{E})$ where $F^i_j(\mathcal{E})$ are the coherent subsheaves of $\mathcal{E}$ forming the quasi-parabolic structure on $\mathcal{E}$ at D^i. Furthermore, $P^{H,H^\star_*,\alpha^\star_*,ss}_{D^\star/X/S}$ is projective over S, and the equivalence relation $\sim$ is S-equivalence induced by GIT. However, on p_2-stable parabolic sheaves the equivalence relation $\sim$ is always isomorphism, and so $P^{H,H^\star_*,\alpha^\star_*,ss}_{D^\star/X/S}$ contains an open subscheme $P^{H,H^\star_*,\alpha^\star_*,s}_{D^\star/X/S}$ which is a coarse moduli space for isomorphism classes of stable parabolic sheaves.

4.1.2 Conventions and notation

Throughout this chapter, $\Bbbk$ is an algebraically closed field of any characteristic and all schemes are assumed to be of finite type over $\Bbbk$. By a point of a scheme we mean a closed (and therefore geometric, i.e. closed $\Bbbk$-valued) point unless stated otherwise, and for any scheme T the phrase '$t \in T$' includes the tacit statement that t be a closed point of T. Similarly, by a fibre of a morphism $f : X \to S$ we will mean a geometric fibre (i.e. the scheme-theoretic fibre over a closed $\Bbbk$-point of S). We will frequently identify an effective Cartier divisor and the associated locally principal closed subscheme. A relative effective Cartier divisor on X/S is an effective Cartier divisor on X which is flat over S.

For X a projective scheme over $\Bbbk$ and $\mathcal{E}$ a coherent sheaf on X, write $h^k(\mathcal{E}) := \dim_\Bbbk H^k(X, \mathcal{E})$ and denote the Euler characteristic of $\mathcal{E}$ by $\chi(\mathcal{E}) := \sum_k (-1)^k h^k(\mathcal{E})$. If $\mathcal{O}_X(1)$ is a very ample line bundle on X, abbreviate $\mathcal{E}(m) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes m}$ for any integer m.

Given a polynomial $f \in \mathbb{Q}[x]$ and an integer m, we write $f[m]$ for the polynomial f shifted by m, i.e. $f[m](x) := f(x+m)$. Recall also convention 1.4.2 on the lexicographic ordering for polynomials which is used throughout this chapter. Note that for p_2-stability, we employ a lexicographic ordering on polynomials with two coefficients: in $\mathbb{Q}[x,y]$ the variable y is considered greater than x, so for example xy is strictly greater than x^2, and $y^2 + xy$ is strictly greater than $y^2 + x^5$. See the discussion just before proposition 4.3.12 for details of this.

Frequently in this chapter, S will denote our base scheme (which is assumed to be connected and of finite type over a fixed algebraically closed field $\Bbbk$ of arbitrary characteristic). Given a morphism $X \to S$ and an element T of the category Sch/S of locally noetherian schemes over S, we denote the fibred product $X \times_S T$ by X_T, and we write $\mathcal{E}_T$ for the pull-back of a sheaf $\mathcal{E} \in \mathsf{Coh}(X)$ to X_T. For any $t \in T$, we write X_t for the geometric fibre of X_T over t, and for any T-flat coherent sheaf $\mathcal{E}$ on X_T, we write $\mathcal{E}_t$ for the restriction of $\mathcal{E}$ to the scheme X_t, i.e. $\mathcal{E}_t := \mathcal{E} \otimes k(t)$. If X is equipped with a relative effective Cartier divisor D, we denote the induced relative effective Cartier divisor on X_T by D_T, and write D_t for the (absolute) effective Cartier divisor on the scheme X_t. If we are also given a relatively very ample line bundle $\mathcal{O}_X(1)$ on X/S, we write $\mathcal{O}_{X_T}(1)$ for the pull-back of $\mathcal{O}_X(1)$ to X_T, and $\mathcal{O}_{X_t}(1)$ for its restriction to X_t, and any (parabolic) Hilbert polynomials of $\mathcal{E}_t$ are calculated with respect to $\mathcal{O}_{X_t}(1)$. In particular, if $\mathcal{F}$ is a sheaf on a fibre X_t, then $\mathcal{F}(m)$ is shorthand for $\mathcal{F} \otimes_{\mathcal{O}_{X_t}} \mathcal{O}_{X_t}(1)^{\otimes m}$.

4.2 Preliminaries on parabolic sheaves

Recall the definitions and basic results on pure sheaves (subsection 3.2.1): we work with these throughout this chapter. In this section, we begin describing parabolic sheaves, and all definitions following in this section are closely based on [MY92], [Yok93] and [Ina00], with modifications concerning reference to more than one divisor. Throughout this section, let X be a fixed projective scheme over an algebraically closed field $\Bbbk$ and $D^1, \dots, D^n$ effective Cartier divisors (not necessarily reduced or distinct) on X.

Definition 4.2.1 (cf. [Ina00])**.** Let $\mathcal{E}$ be a pure coherent $\mathcal{O}_X$-module of dimension d such that

$$\dim(D^i \cap \operatorname{Supp} \mathcal{E}) < \dim \operatorname{Supp} \mathcal{E} \tag{4.3}$$

for $1 \leqslant i \leqslant n$ (note that this condition is vacuous if we consider pure sheaves of top dimension, i.e. for $d = \dim X$). This is equivalent to requiring that no irreducible component of $\operatorname{Supp} \mathcal{E}$ be contained in the support of D^i (and so for all components Y of $\operatorname{Supp} \mathcal{E}$, the divisor D^i does restrict to an effective Cartier divisor on Y unless $D^i \cap Y = \emptyset$).

A *quasi-parabolic structure* on $\mathcal{E}$ at D^i is a filtration[2]

$$\mathcal{E} = F_1^i(\mathcal{E}) \supset F_2^i(\mathcal{E}) \supset \cdots \supset F_{l_i+1}^i(\mathcal{E}) = \mathcal{E}(-D^i) \tag{4.4}$$

by coherent sheaves, where $\mathcal{E}(-D^i)$ is the image of the natural morphism $\mathcal{E} \otimes \mathcal{O}_X(-D^i) \to \mathcal{E}$ which is injective by purity of $\mathcal{E}$ and the support assumption (4.3). A *parabolic structure* on $\mathcal{E}$ at D^i is a quasi-parabolic structure together with a sequence of real numbers $0 \leqslant \alpha_1^i < \alpha_2^i < \cdots < \alpha_{l_i}^i < 1$ called *(parabolic) weights*. Given these data, call l_i the *length* of the (quasi-)parabolic structure at D^i. If the divisors D^i are understood, we often omit

[2]Recall convention 1.4.7: inclusions denoted by $\subset$ and $\supset$ will be not necessarily strict, unless explictly stated otherwise. In particular, we may have $F_j^i(\mathcal{E}) = F_{j+1}^i(\mathcal{E})$ for some j in (4.4). Also note that if $D^{i_1} = D^{i_2}$, we allow consideration of *different* filtrations $F_*^{i_1}(\mathcal{E})$ and $F_*^{i_2}(\mathcal{E})$ at this divisor.

them from the notation by calling a triple $(\mathcal{E}, F_*^\star(\mathcal{E}), \alpha_*^\star)$ satisfying the conditions above a *parabolic sheaf* on X. We sometimes also abbreviate such a triple by $\mathcal{E}_*^\star$ (this is not ambiguous as $\mathcal{E}$ can be given n filtrations by real numbers from which its flag structure at each D^i may be recovered – see remark 4.2.3). Thus, a parabolic sheaf $\mathcal{E}_*^\star = (\mathcal{E}, F_*^\star(\mathcal{E}), \alpha_*^\star)$ consists of an underlying coherent sheaf $\mathcal{E}$ assumed to be pure and satisfying the support condition (4.3) at each divisor D^i, together with a filtration $F_*^i(\mathcal{E})$ as in (4.4) at each D^i, and a weight sequence α_*^i at each D^i.

For the purposes of defining parabolic degree and parabolic Hilbert polynomials, it is convenient to have a notation for the successive quotients in the filtration $F_*^i(\mathcal{E})$ for each i: we sometimes write $G_j^i(\mathcal{E})$ for $F_j^i(\mathcal{E})/F_{j+1}^i(\mathcal{E})$. We will see in definition 4.3.1 that α_j^i should be regarded as weighting the quotient $G_j^i(\mathcal{E})$, explaining why we write the weight α_j^i underneath the inclusion $F_j^i(\mathcal{E}) \supset F_{j+1}^i(\mathcal{E})$ in expressions such as (4.5) and (4.6).

Note that since $\mathcal{E}(-D^i)$ is equal to $\mathcal{E}$ away from the divisor D^i, we are really specifying a flag of subsheaves of $\mathcal{E}|_{D^i}$: just take the quotients of $F_*^i(\mathcal{E})$ by $\mathcal{E}(-D^i)$. Thus, in the case of X a smooth curve, D^i given by a single marked point x_i, and $\mathcal{E}$ pure of dimension 1 (i.e. locally free), the filtration $F_*^i(\mathcal{E})$ in $\mathcal{E}$ corresponds to a (partial) flag structure in the fibre of the associated vector bundle at x_i and we recover the classical notion (4.1) of parabolic structure.

Remark 4.2.2. Some previous papers on parabolic sheaves have restricted attention to parabolic structures at a single divisor D, arguing that parabolic structures at effective divisors $D^1, \dots, D^n$ can be combined to a single parabolic structure at $D := \sum_{i=1}^n D^i$: given a filtration and weights

$$\begin{array}{ccccccccccccc} \mathcal{E} & = & F_1^i(\mathcal{E}) & \supset & F_2^i(\mathcal{E}) & \supset & \dots & \supset & \dots & \supset & F_{l_i+1}^i(\mathcal{E}) & = & \mathcal{E}(-D^i) \\ & 0 & \leqslant & \alpha_1^i & < & \alpha_2^i & < & \dots & < & \alpha_{l_i}^i & < & 1 & \end{array} \tag{4.5}$$

at D^i (for $1 \leqslant i \leqslant n$), order the α_j^i in a single increasing sequence α_k (discarding any repetitions) and then use the the filtrations at the divisors D^i to define a single filtration

$$\begin{array}{ccccccccccccc} \mathcal{E} & = & F_1(\mathcal{E}) & \supset & F_2(\mathcal{E}) & \supset & \dots & \supset & \dots & \supset & F_{l+1}(\mathcal{E}) & = & \mathcal{E}(-D) \\ & 0 & \leqslant & \alpha_1 & < & \alpha_2 & < & \dots & < & \alpha_l & < & 1 & \end{array} \tag{4.6}$$

at $D := \sum_{i=1}^n D^i$ by setting $F_k(\mathcal{E}) := \cap_{i=1}^m F_{j_i}^i(\mathcal{E})$ where j_i is minimal such that $\alpha_k \leqslant \alpha_{j_i}^i$ (and $j_i = l_i + 1$ if $\alpha_k > \alpha_{l_i}^i$).

This indeed produces a parabolic structure at D, and by restricting it to each divisor D^i we may recover the individual parabolic structure at D^i. However, not every parabolic structure at $D = \sum_{i=1}^n D^i$ arises from parabolic structures at the individual D^i, and this is why we insist on working with $D^1, \dots, D^n$. This problem is most clearly seen in the case of locally free sheaves on a smooth curve X, equipped with parabolic structures at $D^i = x_i$

distinct points for $i = 1, 2$: for example, consider length 2 parabolic structures at x_1, x_2, with the same weights $\alpha_j := \alpha_j^i$ at both points, i.e.

$$\begin{array}{ccccccccc} \mathcal{E} & = & F_1^i(\mathcal{E}) & \supset & F_2^i(\mathcal{E}) & \supset & F_3^i(\mathcal{E}) & = & \mathcal{E}(-x_i) \\ & 0 & \leqslant & \alpha_1 & < & \alpha_2 & < & 1 & \end{array}$$

for $i = 1, 2$. Now let E be the vector bundle whose sheaf of sections is $\mathcal{E}$. As explained before this remark, we may think of these parabolic structures as (partial) flags in the fibre $E(x_i)$ as in (4.1):

$$\begin{array}{ccccccccc} E(x_i) & = & F_1^i(E) & \supset & F_2^i(E) & \supset & F_3^i(E) & = & 0 \\ & 0 & \leqslant & \alpha_1 & < & \alpha_2 & < & 1. & \end{array} \tag{4.7}$$

Then the method described above produces a single parabolic structure at the divisor $D = x_1 + x_2$ which in turn is equivalent to a flag inside the direct sum of the fibres at x_1 and x_2:

$$\begin{array}{ccccccccc} E(x_1) \oplus E(x_2) & = & F_1(E) & \supset & F_2(E) & \supset & F_3(E) & = & 0 \\ & 0 & \leqslant & \alpha_1 & < & \alpha_2 & < & 1, & \end{array} \tag{4.8}$$

where $F_2(E) = F_2^1(E) \oplus F_2^2(E)$. However, not every parabolic structure as in (4.8) arises from separate flags (4.7) at x_1, x_2: in the flag (4.8) there might be 'mixing' between the fibres at x_1 and x_2 if $F_2(E)$ does not decompose as a direct sum of subspaces of $E(x_1)$ and $E(x_2)$. In fact, such structures where flags mix information about the fibres at several marked points on a smooth curve (sometimes also known as *generalised parabolic bundles*, first studied by Bhosle, see in particular [Bho99]), give normalisations and sometimes even desingularisations of moduli spaces of (parabolic) sheaves on nodal curves C via such generalised parabolic bundles on the normalised curve $\widetilde{C}$. On the other hand, if we wish to keep the option of describing separate parabolic structures at marked points x_i on curves (as we do, particularly with a view to constructing universal moduli spaces for 'classical' parabolic sheaves on curves), we need to study separate parabolic structures at divisors $D^1, \dots, D^n$.

The method producing a single parabolic structure at D from separate parabolic structures at D^i as explained above may seem a little cumbersome, and indeed there is a cleaner way to describe this procedure, using an interpretation of parabolic sheaves as (a particular class of) filtered sheaves – this idea originated in [Sim90] and was used in [MY92] and [Ina00] to simplify the definitions of sub- and quotient objects, morphisms and parabolic Hilbert polynomial (which we will also do, cf. definitions 4.2.4 and 4.3.1), as well as proving a useful tool in the homological algebra of parabolic sheaves [Yok95]:

Remark 4.2.3. Given an effective Cartier divisor D^i on X, let $(\mathcal{E}, F_*^i(\mathcal{E}), \alpha_*^i)$ be a sheaf with parabolic structure at D^i as in definition 4.2.1. For all $\alpha \in \mathbb{R}$, let

$$\mathcal{E}_\alpha^i := F_j^i(\mathcal{E})(-\lfloor \alpha \rfloor D^i),$$

where j is chosen such that $\alpha^i_{j-1} < \alpha - \lfloor \alpha \rfloor \leqslant \alpha^i_j$, and $\alpha^i_{l_i+1} := 1$ and $\alpha^i_0 := \alpha^i_{l_i} - 1$. This gives an $\mathbb{R}$-indexed filtration

$$(\iota_i)_*(\iota_i)^*\mathcal{E} = \bigcup_{k\in\mathbb{Z}} \mathcal{E}(-kD^i) = \bigcup_{\alpha\in\mathbb{R}} \mathcal{E}^i_\alpha \supset \cdots \supset \mathcal{E}^i_\alpha \supset \mathcal{E}^i_\beta \supset \dots,$$

where ι_i is the inclusion $X \setminus D^i \hookrightarrow X$. This filtration is

(a) decreasing: $\alpha \leqslant \beta \Longrightarrow \mathcal{E}^i_\alpha \supset \mathcal{E}^i_\beta$,

(b) uniformly lower semi-continuous: for sufficiently small $\epsilon > 0$ (independent of α, once $\alpha^i_1, \dots, \alpha^i_{l_i}$ are fixed), we have $\mathcal{E}^i_{\alpha-\epsilon} = \mathcal{E}^i_\alpha$ for all $\alpha \in \mathbb{R}$ (and in particular the filtration has only finitely many terms when restricted to $\alpha \in [0,1]$),

(c) 1-periodic with twist $-\otimes_{\mathcal{O}_X} \mathcal{O}_X(-D^i)$: for all $\alpha \in \mathbb{R}$ we have $\mathcal{E}^i_{\alpha+1} = \mathcal{E}^i_\alpha(-D^i)$, and

(d) centred at $\mathcal{E}$, i.e. $\mathcal{E}^i_0 = \mathcal{E}$.

Conversely, any filtration $\bigcup_{\alpha\in\mathbb{R}} \mathcal{E}^i_\alpha$ of $\bigcup_{k\in\mathbb{Z}} \mathcal{E}(-kD^i)$ which satisfies (a)-(d) is induced by a unique parabolic structure on $\mathcal{E}$ at D^i (thus, we may abbreviate the parabolic structure $(\mathcal{E}, F^i_*(\mathcal{E}), \alpha^i_*)$ by $\mathcal{E}^i_*$ as all the parabolic data are encoded in the filtration $\mathcal{E}^i_\alpha$, and the whole parabolic sheaf with its structures at all D^i is abbreviated to $\mathcal{E}^\star_*$): note that for $\alpha \in [0,1]$, we simply recover our original filtration with jump points determined by the weights:

$$\mathcal{E}^i_\alpha = \begin{cases} F^i_1(\mathcal{E}) = \mathcal{E} & \text{if } 0 \leqslant \alpha \leqslant \alpha^i_1 \\ F^i_j(\mathcal{E}) & \text{if } \alpha^i_{j-1} < \alpha \leqslant \alpha^i_j, \text{ for all } 2 \leqslant j \leqslant l_i, \\ F^i_{l_i+1}(\mathcal{E}) = \mathcal{E}(-D^i) & \text{if } \alpha^i_{l_i} < \alpha \leqslant \alpha^i_{l_i+1} = 1. \end{cases} \tag{4.9}$$

Setting $\mathcal{E}_\alpha := \bigcap_{i=1}^n \mathcal{E}^i_\alpha$ defines a single parabolic structure on $\mathcal{E}$ at $D = \sum_{i=1}^n D^i$ from which the individual parabolic structures may be recovered, and this matches the procedure described in remark 4.2.2, but for the reasons explained above we will generally keep working with separate parabolic structures $\mathcal{E}^i_\alpha$ at each D^i and write $\mathcal{E}^\star_*$ as shorthand for the parabolic sheaf with its structures $\mathcal{E}^i_*$ at $D^1, \dots, D^n$.

The notion of filtered sheaf used here to rephrase the definition of a parabolic structure is weakened in [Yok95] by only requiring a morphism $\mathcal{E}^i_\beta \to \mathcal{E}^i_\alpha$, not necessarily injective, whenever $\alpha \leqslant \beta$. The merit of this generalisation is that it produces an abelian category, whereas parabolic sheaves in the usual sense as above do not form an abelian category, by the same argument showing that filtered groups do not form an abelian category (see II.5.17 in [GM03]): consider the identity map on a sheaf equipped with a relatively finer filtration on the domain, this is a morphism with zero kernel and cokernel (in the sheaf-theoretic sense) which is not an isomorphism of parabolic sheaves. However, parabolic kernels and cokernels can be defined for morphisms between p_0-semistable parabolic sheaves having the same parabolic Hilbert polynomial: see proposition 4.4.3.

Fix effective Cartier divisors $D^1, \dots, D^n$ on a projective $\Bbbk$-scheme X; in the following definitions all parabolic structures will be with respect to these divisors and we will often suppress mention of the D^i.

We will freely switch between the two descriptions of a parabolic sheaf given in definition 4.2.1 and in remark 4.2.3. In particular, it is more convenient to write down parabolic morphisms, sub- and quotient objects using the interpretation as a filtered sheaf, especially when the domain and codomain have parabolic structures of different weights or even lengths:

Definition 4.2.4 ([Yok93], [Ina00])**.** Given parabolic sheaves $\mathcal{E}_*^\star$ and $\mathcal{F}_*^\star$, a morphism of $\mathcal{O}_X$-modules $f : \mathcal{E} \to \mathcal{F}$ is said to be a *parabolic homomorphism* if $f(\mathcal{E}_\alpha^i) \subset \mathcal{F}_\alpha^i$ for all $\alpha \in [0,1]$ (hence for all $\alpha \geqslant 0$) and all $1 \leqslant i \leqslant n$. (This looks slightly stronger – because more continuous – than the discrete condition of preserving the filtrations, but note that if the parabolic structures of $\mathcal{E}_*^i$ and $\mathcal{F}_*^i$ at D^i have the same length and weights, then this just says that $f(\mathcal{E}_j^i) \subset \mathcal{F}_j^i$ for all $1 \le j \le l_i + 1$.)

A *parabolic subsheaf* of $\mathcal{F}_*^\star$ is a parabolic homomorphism $\mathcal{E}_*^\star \to \mathcal{F}_*^\star$ which is injective as a morphism $\mathcal{E} \to \mathcal{F}$ of $\mathcal{O}_X$-modules (i.e. a parabolic sheaf $\mathcal{E}_*^\star$ with $\mathcal{E}_\alpha^i \subset \mathcal{F}_\alpha^i$ for all real $\alpha \geqslant 0$ and all $1 \leqslant i \leqslant n$). Note that given a parabolic sheaf $\mathcal{F}_*^\star$ and any proper saturated coherent subsheaf $\mathcal{E}$ of $\mathcal{F}$, then $\mathcal{E}$ can be considered as a parabolic subsheaf by giving it the *induced parabolic structure*: this is the maximal parabolic structure on $\mathcal{E}$ making it a parabolic subsheaf of $\mathcal{F}_*^\star$. Concretely, put $\mathcal{E}_\alpha^i = \mathcal{F}_\alpha^i \cap \mathcal{E}$ for all real $\alpha \geqslant 0$ and all $1 \leqslant i \leqslant n$, i.e. consider $\mathcal{E}$ with the same weights as $\mathcal{F}_*^\star$ and with the induced filtrations $F_j^i(\mathcal{E}) := F_j^i(\mathcal{F}) \cap \mathcal{E}$ for each i. (The saturation hypothesis is required here to ensure that $\mathcal{F}(-D^i) \cap \mathcal{E} = \mathcal{E}(-D^i)$.)

Similarly, a *quotient parabolic sheaf* of $\mathcal{F}_*^\star$ is a parabolic homomorphism $\mathcal{F}_*^\star \to \mathcal{G}_*^\star$ which is surjective as a morphism $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules. There is again a notion of induced structure: given a parabolic sheaf $\mathcal{F}_*^\star$ and any surjective morphism of $\mathcal{O}_X$-modules $f : \mathcal{F} \twoheadrightarrow \mathcal{G}$ with $\mathcal{G}$ pure of dimension $d = \dim \mathcal{F}$, the *induced parabolic structure* on $\mathcal{G}$ is the minimal parabolic structure making $\mathcal{G}$ a parabolic quotient of $\mathcal{F}_*^\star$, i.e. $\mathcal{G}_\alpha^i := f(\mathcal{F}_\alpha^i)$ for all real $\alpha \geqslant 0$ and all $1 \leqslant i \leqslant n$. (Note that in this situation $\mathcal{G}$ automatically satisfies the support condition (4.3) in definition 4.2.1: as $\mathcal{F}$ surjects onto $\mathcal{G}$, we have $\operatorname{Supp} \mathcal{G} \subset \operatorname{Supp} \mathcal{F}$, and therefore $\dim(D^i \cap \operatorname{Supp} \mathcal{G}) \leqslant \dim(D^i \cap \operatorname{Supp} \mathcal{F}) < \dim \mathcal{F} = \dim \mathcal{G}$.)

Remark 4.2.5. The definition of parabolic sub- and quotient sheaves given above is that of [Yok93], [Yok95] and [Ina00] which is weaker than that of [MS80] and [MY92]. However, this will not affect the moduli spaces we construct: when testing (semi)stability, it suffices to consider saturated subsheaves with the induced parabolic structure (as we will explain in remark 4.3.5), and these always satisfy the original, stronger definition of parabolic subsheaves.

Definition 4.2.6 ([MY92])**.** Suppose $\mathcal{E}^\star_*, \mathcal{F}^\star_*, \mathcal{G}^\star_*$ are parabolic sheaves of the same dimension d (or possibly zero sheaves). Given parabolic homomorphisms $\mathcal{E}^\star_* \to \mathcal{F}^\star_*$ and $\mathcal{F}^\star_* \to \mathcal{G}^\star_*$, we say that

$$0 \to \mathcal{E}^\star_* \to \mathcal{F}^\star_* \to \mathcal{G}^\star_* \to 0$$

is *exact* if

$$0 \to \mathcal{E}^i_\alpha \to \mathcal{F}^i_\alpha \to \mathcal{G}^i_\alpha \to 0$$

is exact for all real $\alpha \geqslant 0$ and all $1 \leqslant i \leqslant n$. This is equivalent to the underlying sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

being exact and in addition $\mathcal{E}^\star_* \subset \mathcal{F}^\star_*$ and $\mathcal{F}^\star_* \twoheadrightarrow \mathcal{G}^\star_*$ having the induced parabolic structures. Note that it makes sense to talk about the induced structure on $\mathcal{E}$ as pure d-dimensionality (or vanishing) of $\mathcal{G} \cong \mathcal{F}/\mathcal{E}$ means that $\mathcal{E} \subset \mathcal{F}$ is saturated.

4.3 Stability notions

We continue with a projective scheme X over an algebraically closed field $\Bbbk$, effective Cartier divisors $D^1, \ldots, D^n$ on X, and we also fix a very ample line bundle $\mathcal{O}_X(1)$ on X. All parabolic sheaves $\mathcal{E}^\star_*$ on X have parabolic structures at the D^i and all Hilbert polynomials will be with respect to $\mathcal{O}_X(1)$.

In order to talk about (semi)stable parabolic sheaves, we need suitable notions of Hilbert polynomial and slope, taking account of the parabolic structure by involving the quotients $G^i_j(\mathcal{E})$ of successive terms in the filtrations. Note that the following definitions bring the weights $\alpha^\star_*$ into the stability condition; thus, once the Hilbert polynomial of $\mathcal{E}$ and the flag type – i.e. the length l_i of the quasi-parabolic structure at D^i and the Hilbert polynomials of all quotients $\mathcal{E}/F^i_j(\mathcal{E})$ – are fixed, morally we may think of the sheaves equipped with filtrations as the underlying objects of the moduli problem, whereas the choice of weights $\alpha^\star_*$ influences the stability condition and reflects a choice of linearisation in the 'variation of GIT' point of view – see [BH95], [Tha96]. (However, one should bear in mind that the weights *do* play a second rôle once we simultaneously consider parabolic sheaves of varying flag types – the weights influence the notion of parabolic morphism between quasi-parabolic sheaves of different flag type, see definition 4.2.4. This is in contrast to the category of quasi-parabolic sheaves of fixed flag type which may be considered independent of the weights α^i_j – here, only the stability condition will depend on the weights.)

Definition 4.3.1 (cf. [Ina00])**.** Let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Then the *parabolic Hilbert polynomial* of $\mathcal{E}^\star_*$ with respect to $\mathcal{O}_X(1)$ is defined to be

$$\text{par–}P_0(\mathcal{E}^\star_*) := \frac{1}{n}\sum_{i=1}^{n}\left(P(\mathcal{E}(-D^i)) + \sum_{j=1}^{l_i}\alpha^i_j P(G^i_j(\mathcal{E}))\right),$$

where $P(\mathcal{F}) = P(\mathcal{F}, m)$ denotes the ordinary Hilbert polynomial $\chi(\mathcal{F}(m))$ with respect to $\mathcal{O}_X(1)$. As in the case of sheaves without additional structure, the correct invariant to consider in the definition of (semi)stability is not this Hilbert polynomial but its reduced form, i.e. the normalised version of par–$P_0(\mathcal{E}_*^\star)$ having leading term only depending on $d = \dim \mathcal{E}$: define the *reduced parabolic Hilbert polynomial* of a non-zero parabolic sheaf $\mathcal{E}_*^\star$ to be

$$\text{par–}p_0(\mathcal{E}_*^\star) := \frac{\text{par–}P_0(\mathcal{E}_*^\star)}{a_d(\mathcal{E})},$$

where $a_d(\mathcal{E})$ is the multiplicity of $\mathcal{E}$ (see definition 3.2.2).

The leading coefficient of par–$P_0(\mathcal{E}_*^\star)$ is still $a_d(\mathcal{E})/d!$, as the following lemma shows, so par–p_0 is indeed normalised:

Lemma 4.3.2. Let $\mathcal{E}_*^\star$ be a parabolic sheaf of dimension d. Then par–$P_0(\mathcal{E}_*^\star)$ and $P(\mathcal{E})$ have the same leading terms, i.e. both are of degree d and have the form

$$\frac{a_d(\mathcal{E})}{d!} m^d + \text{ terms of lower order.}$$

Proof. For each $1 \leqslant i \leqslant n$, we have a short exact sequence

$$0 \to \mathcal{E}(-D^i) \to \mathcal{E} \to \mathcal{E}|_{D^i} \to 0,$$

so by additivity of Hilbert polynomials

$$P(\mathcal{E}(-D^i)) = P(\mathcal{E}) - P(\mathcal{E}|_{D^i})$$

and by the support assumption (4.3) $\mathcal{E}|_{D^i}$ is a sheaf of dimension strictly less than d, so the leading (degree d) terms of $P(\mathcal{E}(-D^i))$ and $P(\mathcal{E})$ agree.

Now for each $1 \leqslant j \leqslant l_i$ the sheaves $F^i_j(\mathcal{E})$ and $F^i_{j+1}(\mathcal{E})$ coincide away from D^i, so the quotient $G^i_j(\mathcal{E})$ is supported on $D^i \cap \operatorname{Supp} \mathcal{E}$. Thus, by (4.3) again, $G^i_j(\mathcal{E})$ has dimension strictly less than d. Therefore, the $P(G^i_j(\mathcal{E}))$ do not contribute to the degree d term of

$$\text{par–}P_0(\mathcal{E}_*^\star) = \frac{1}{n}\sum_{i=1}^{n}\left(P(\mathcal{E}(-D^i)) + \sum_{j=1}^{l_i}\alpha^i_j P(G^i_j(\mathcal{E}))\right)$$

and so the leading term of par–$P_0(\mathcal{E}_*^\star)$ equals

$$\frac{1}{n}\sum_{i=1}^{n}\big(\text{leading term of } P(\mathcal{E}(-D^i), m)\big) = \frac{a_d(\mathcal{E})}{d!} m^d. \qquad \square$$

The definition of the parabolic Hilbert polynomial is easily rewritten in terms of the filtrations $\mathcal{E}^i_\alpha$ defined for each i in remark 4.2.3:

$$\text{par–}P_0(\mathcal{E}_*^\star) = \frac{1}{n}\sum_{i=1}^{n}\int_0^1 P(\mathcal{E}^i_\alpha)\,\mathrm{d}\alpha \tag{4.10}$$

since by (4.9) we have

$$\begin{aligned}\int_0^1 P(\mathcal{E}^i_\alpha)\,\mathrm{d}\alpha &= \alpha^i_1 P(F^i_1(\mathcal{E})) + \sum_{j=2}^{l_i+1}(\alpha^i_j - \alpha^i_{j-1})P(F^i_j(\mathcal{E}))\\ &= \alpha^i_{l_i+1}P(F^i_{l_i+1}(\mathcal{E})) + \sum_{j=1}^{l_i}\alpha^i_j\Big(P(F^i_j(\mathcal{E})) - P(F^i_{j+1}(\mathcal{E}))\Big)\\ &= P(\mathcal{E}(-D^i)) + \sum_{j=1}^{l_i}\alpha^i_j P(G^i_j(\mathcal{E})),\end{aligned}$$

where as before $\alpha^i_{l_i+1} := 1$.

Lemma 4.3.3. The parabolic Hilbert polynomial par–P_0 is additive on short exact sequences of parabolic sheaves.

Proof. Exactness of

$$0 \to \mathcal{E}^\star_* \to \mathcal{F}^\star_* \to \mathcal{G}^\star_* \to 0$$

is defined to mean that

$$0 \to \mathcal{E}^i_\alpha \to \mathcal{F}^i_\alpha \to \mathcal{G}^i_\alpha \to 0$$

is exact for all real $\alpha \geqslant 0$ and all i. Using additivity of the ordinary Hilbert polynomial for these sequences, the result now follows from (4.10). □

Definition 4.3.4. A parabolic sheaf $\mathcal{F}^\star_*$ is *p_0-(semi)stable* if for every proper non-zero parabolic subsheaf $\mathcal{E}^\star_*$ of $\mathcal{F}^\star_*$ we have

$$\text{par–}p_0(\mathcal{E}^\star_*)(\leqslant)\text{par–}p_0(\mathcal{F}^\star_*), \tag{4.11}$$

where $(\leqslant)$ means $\leqslant$ for semistability and $<$ for stability, and the inequality of polynomials is with respect to the lexicographic ordering on coefficients, equivalent to requiring that par–$p_0(\mathcal{E}^\star_*, m)(\leqslant)$par–$p_0(\mathcal{F}^\star_*, m)$ for all sufficiently large $m \in \mathbb{Z}$. Recall from definition 4.3.1 that par–p_0 is the *reduced* parabolic Hilbert polynomial. If $\mathcal{E}^\star_*$ is semistable but not stable, we call it *strictly p_0-semistable.*

Remark 4.3.5. It suffices to check (4.11) for saturated subsheaves $\mathcal{E} \subset \mathcal{F}$ carrying the induced parabolic structure (cf. definition 4.2.4): let $\mathcal{E}^\star_* \subset \mathcal{F}^\star_*$ be any non-zero proper parabolic subsheaf, write $\mathcal{E}^{\text{sat}} \subset \mathcal{F}$ for the saturation of $\mathcal{E} \subset \mathcal{F}$, and let $(\mathcal{E}^{\text{sat}})^\star_* \subset \mathcal{F}^\star_*$ be the induced parabolic structure on $\mathcal{E}^{\text{sat}}$. Then by maximality of the induced parabolic structure, we have

$$\mathcal{E}^i_\alpha \subset (\mathcal{E}^{\text{sat}})^i_\alpha = \mathcal{E}^{\text{sat}} \cap \mathcal{F}^i_\alpha$$

for all $\alpha \geqslant 0$ and all $1 \leqslant i \leqslant n$. Thus, $P(\mathcal{E}^i_\alpha) \leqslant P((\mathcal{E}^{\mathrm{sat}})^i_\alpha)$ and by lemma 3.2.3 we have $a_d(\mathcal{E}) = a_d(\mathcal{E}^{\mathrm{sat}})$, so using (4.10) we arrive at

$$\begin{aligned}
\text{par–}p_0(\mathcal{E}^\star_*) &= \frac{\text{par–}P_0(\mathcal{E}^\star_*)}{a_d(\mathcal{E})} \\
&= \frac{1}{n}\sum_{i=1}^n \int_0^1 \frac{P(\mathcal{E}^i_\alpha)}{a_d(\mathcal{E})}\,\mathrm{d}\alpha \\
&\leqslant \frac{1}{n}\sum_{i=1}^n \int_0^1 \frac{P((\mathcal{E}^{\mathrm{sat}})^i_\alpha)}{a_d(\mathcal{E}^{\mathrm{sat}})}\,\mathrm{d}\alpha \\
&= \text{par–}p_0((\mathcal{E}^{\mathrm{sat}})^\star_*).
\end{aligned}$$

In particular, if $\text{par–}p_0((\mathcal{E}^{\mathrm{sat}})^\star_*)(\leqslant)\text{par–}p_0(\mathcal{F}^\star_*)$, then $\text{par–}p_0(\mathcal{E}^\star_*)(\leqslant)\text{par–}p_0(\mathcal{F}^\star_*)$.

As for sheaves without augmentations, we could also define stability for parabolic sheaves with respect to parabolic slope instead of using the reduced parabolic Hilbert polynomial as above. While these notions coincide in dimension 1, only p_0-semistability leads to complete moduli spaces in higher dimensions, at least on a smooth projective variety (see [Yok93]). However, we still discuss the parabolic slope here as it plays a key rôle in proving boundedness of the parabolic semistable sheaves of fixed numerical type (see section 4.5), even if parabolic semistability is defined using the modifications of the parabolic Hilbert polynomial which we introduce below (see definitions 4.3.9 and 4.3.11).

Definition 4.3.6. Let $\mathcal{E}^\star_*$ be a parabolic sheaf of dimension d on X. The *parabolic degree* of $\mathcal{E}^\star_*$ is given by the coefficient of $m^{d-1}/(d-1)!$ in the parabolic Hilbert polynomial of $\mathcal{E}^\star_*$, i.e.

$$\text{par–deg}(\mathcal{E}^\star_*) := \frac{1}{n}\sum_{i=1}^n \left(a_{d-1}(\mathcal{E}(-D^i)) + \sum_{j=1}^{l_i} \alpha^i_j a_{d-1}(G^i_j(\mathcal{E})) \right),$$

where a_{d-1} is as in definition 3.2.2, i.e. the coefficient of $m^{d-1}/(d-1)!$ in the (ordinary) Hilbert polynomial. If $\mathcal{E}^\star_* \neq 0$, its *parabolic slope* is

$$\text{par–}\mu(\mathcal{E}^\star_*) := \text{par–deg}(\mathcal{E}^\star_*)/a_d(\mathcal{E}),$$

and we call a parabolic sheaf $\mathcal{F}^\star_*$ *slope-(semi)stable* or *μ-(semi)stable* if for every proper non-zero parabolic subsheaf $\mathcal{E}^\star_*$ of $\mathcal{F}^\star_*$ the relation

$$\text{par–}\mu(\mathcal{E}^\star_*)(\leqslant)\text{par–}\mu(\mathcal{F}^\star_*)$$

holds.

Note that the above definition of parabolic degree differs slightly from the definition in [MY92] where $a_{d-1}(\mathcal{E})$ is used in place of $a_{d-1}(\mathcal{E}(-D^i))$. As the following proposition shows, the difference between these two definitions of parabolic degree can be bounded, and the choice we have made is more natural in that parabolic p_0-semistability will immediately imply parabolic slope-semistability. (In the case of parabolic vector bundles on marked curves, the relation between our definition of parabolic slope and the traditional notion of parabolic slope derived from (4.2) is analogous to the relation between $\mu(\mathcal{E}) = a_{d-1}(\mathcal{E})/a_d(\mathcal{E})$ and $\deg E/\mathrm{rk}\ E$.)

The following proposition will be the key to our boundedness argument in section 4.5:

Proposition 4.3.7. The parabolic slope has the following properties:

(a) For any parabolic sheaf $\mathcal{E}^\star_*$, we have implications:

$$\mathcal{E}^\star_* \text{ is } p_0\text{-semistable} \implies \mathcal{E}^\star_* \text{ is } \mu\text{-semistable},$$
$$\mathcal{E}^\star_* \text{ is } \mu\text{-stable} \implies \mathcal{E}^\star_* \text{ is } p_0\text{-stable}.$$

(b) The parabolic slope can be rewritten as:

$$\text{par–}\mu(\mathcal{E}^\star_*) = \frac{1}{n}\sum_{i=1}^{n}\int_0^1 \mu(\mathcal{E}^i_\alpha)\ \mathrm{d}\alpha.$$

(c) Given a positive integer $a \in \mathbb{N}$, there is $b \in \mathbb{Q}$ (only depending on a, d, the divisors $D^1, \ldots, D^n$, and the polarisation $\mathcal{O}_X(1)$) such that for all d-dimensional parabolic sheaves $\mathcal{E}^\star_*$ with multiplicity $a_d(\mathcal{E}) \leqslant a$, we have

$$\mu(\mathcal{E}) - b \leqslant \text{par–}\mu(\mathcal{E}^\star_*) \leqslant \mu(\mathcal{E}).$$

Proof. By lemma 4.3.2, both $\mathcal{E}^\star_*$ and all its non-zero parabolic subsheaves $\mathcal{F}^\star_*$ have reduced parabolic Hilbert polynomial with leading term $m^d/d!$, and the coefficient of $m^{d-1}/(d-1)!$ in par–$p_0(\mathcal{E})$ is par–$\mu(\mathcal{E}^\star_*)$, so part (a) follows since p_0-(semi)stability is defined with respect to the lexicographic ordering on polynomials.

Part (b) is an immediate consequence of (4.10), noting that $a_d(\mathcal{E}) = a_d(\mathcal{E}^i_\alpha)$ for all $\alpha \in [0,1]$ and all i: since $\mathcal{E} \supset \mathcal{E}^i_\alpha \supset \mathcal{E}(-D^i)$ for all $\alpha \in [0,1]$ and all i, each $\mathcal{E}^i_\alpha$ agrees with $\mathcal{E}$ away from D^i and their quotient is supported on $D^i \cap \mathrm{Supp}\ \mathcal{E}$ which has dimension less than or equal to $d-1$ by the support assumption (4.3), thus the Hilbert polynomials of $\mathcal{E}$ and $\mathcal{E}^i_\alpha$ only differ in degree strictly less than d, and so $a_d(\mathcal{E}) = a_d(\mathcal{E}^i_\alpha)$.

Finally, for (c), note that by (b) we have

$$\mu(\mathcal{E}) - \text{par–}\mu(\mathcal{E}^\star_*) = \frac{1}{a_d(\mathcal{E})}\frac{1}{n}\sum_{i=1}^{n}\int_0^1 \Big(a_{d-1}(\mathcal{E}) - a_{d-1}(\mathcal{E}^i_\alpha)\Big)\ \mathrm{d}\alpha. \tag{4.12}$$

Now $\mathcal{E}(-D^i) \subset \mathcal{E}^i_\alpha \subset \mathcal{E}$ for $\alpha \in [0,1]$, so we have $P(\mathcal{E}(-D^i)) \leqslant P(\mathcal{E}^i_\alpha) \leqslant P(\mathcal{E})$, but as the Hilbert polynomials $P(\mathcal{E}(-D^i))$, $P(\mathcal{E}^i_\alpha)$, and $P(\mathcal{E})$ only differ in degree $\leqslant d-1$, this implies

$$a_{d-1}(\mathcal{E}(-D^i)) \leqslant a_{d-1}(\mathcal{E}^i_\alpha) \leqslant a_{d-1}(\mathcal{E})$$

for all $\alpha \in [0,1]$ and all i. Now (4.12) yields

$$0 \leqslant \mu(\mathcal{E}) - \text{par–}\mu(\mathcal{E}^\star_*) \leqslant \frac{1}{a_d(\mathcal{E})}\frac{1}{n}\sum_{i=1}^{n}\Big(a_{d-1}(\mathcal{E}) - a_{d-1}(\mathcal{E}(-D^i))\Big), \tag{4.13}$$

in particular showing one of the inequalities in (c).

For the other inequality in (c), we need to compare the Hilbert polynomials of $\mathcal{E}$ and $\mathcal{E}(-D^i)$ using a little intersection theory: first, recall from [Ful98], 18.3.6, the following formula for Hilbert polynomials: for any coherent sheaf $\mathcal{F}$ on a complete n-dimensional scheme Y and any line bundles $L_1, \dots, L_r$ on Y, we have

$$\chi(Y, \mathcal{F} \otimes L_1^{\otimes m_1} \otimes \cdots \otimes L_r^{\otimes m_r}) = \sum_{k=0}^{n} \frac{1}{k!}\int_Y (m_1 x_1 + \cdots + m_r x_r)^k \cap \tau_{Y,k}(\mathcal{F}),$$

where $x_i := c_1(L_i)$ is the first Chern class (defined as a graded degree -1 morphism $c_1(L_i) \cap - : A_*Y \to A_*Y$), and $\tau_Y : K_0 Y \to A_*Y_{\mathbb{Q}}$ is the generalised Riemann-Roch homomorphism from the Grothendieck group of coherent sheaves to the rational group of cycles modulo rational equivalence, and $\tau_{Y,k}(\mathcal{F})$ is the component of $\tau_Y(\mathcal{F})$ in $A_k Y_{\mathbb{Q}}$, the group of k-dimensional cycles. (Recall also that $\int_Y - : A_0 Y_{\mathbb{Q}} \to \mathbb{Q}$ is the degree map.)

Apply this formula to $\mathcal{F} = \mathcal{E}$ on $Y = \operatorname{Supp} \mathcal{E}$ with $L_1 = \mathcal{O}_X(1)|_Y$, $L_2 = \mathcal{O}_X(-D^i)|_Y$, $m_1 = m$, and m_2 zero or one, respectively – as both $\mathcal{E}$ and $\mathcal{E}(-D^i)$ are supported on Y, their Euler characteristics may be calculated on Y and we obtain:

$$\begin{aligned} P(\mathcal{E}, m) &= \frac{m^d}{d!}\int_Y x^d \cap \tau_{Y,d}(\mathcal{E}) + \frac{m^{d-1}}{(d-1)!}\int_Y x^{d-1} \cap \tau_{Y,d-1}(\mathcal{E}) + \text{ lower order terms,} \\ P(\mathcal{E}(-D^i), m) &= \frac{m^d}{d!}\int_Y x^d \cap \tau_{Y,d}(\mathcal{E}) \\ &\quad + \frac{m^{d-1}}{(d-1)!}\Big(\int_Y x^{d-1} \cap \tau_{Y,d-1}(\mathcal{E}) + \int_Y x^{d-1} y \cap \tau_{Y,d}(\mathcal{E})\Big) + \text{ l. o. t.,} \end{aligned}$$

where x, y are the first Chern classes of $\mathcal{O}_X(1)|_Y$ and $\mathcal{O}_X(-D^i)|_Y$, respectively. Thus, the difference of the terms in degree $d-1$ is

$$a_{d-1}(\mathcal{E}) - a_{d-1}(\mathcal{E}(-D^i)) = -\int_Y x^{d-1} y \cap \tau_{Y,d}(\mathcal{E}).$$

Now denote the (reduced schemes underlying the) irreducible components of Y by Y_r for $1 \leqslant r \leqslant R$, say – note that because $\mathcal{E}$ is pure, its support Y must be pure of dimension d, so that all Y_r are d-dimensional, and thus $A_d Y$ is the free abelian group on the cycles

$[Y_1], \ldots, [Y_r]$. Hence there are $n_r(\mathcal{E}) \in \mathbb{Q}$ such that $\tau_{Y,d}(\mathcal{E}) = \sum_1^R n_r(\mathcal{E})[Y_r]$. Recall that τ_Y has the *rational* Chow group $A_*Y_{\mathbb{Q}} := A_*Y \otimes_{\mathbb{Z}} \mathbb{Q}$ as codomain, so a priori the n_r are rational numbers (in fact, $n_r(\mathcal{E}) = \mathrm{length}_{\mathcal{O}_{X,Y_r}}(\mathcal{E}_{Y_r})$, but we will not need this observation in the argument that follows).

Then we have

$$\begin{aligned} a_{d-1}(\mathcal{E}) - a_{d-1}(\mathcal{E}(-D^i)) &= -\sum_{r=1}^R n_r(\mathcal{E}) \int_{Y_r} x^{d-1} y \cap [Y_r], \\ &= \sum_{r=1}^R n_r(\mathcal{E}) \int_{Y_r} x^{d-1} z \cap [Y_r] \end{aligned}$$

where we abuse notation by continuing to write x, y for the first Chern classes of $\mathcal{O}_X(1)$ and $\mathcal{O}_X(-D^i)$, even when restricted to Y_r, and we write $z := -y = c_1(\mathcal{O}_X(D^i))$.

In this expression, let $b_r := \int_{Y_r} x^{d-1} z \cap [Y_r]$, and observe that $z \cap [Y_r] = [D^i|_{Y_r}]$ by the construction of Chern classes, so the term b_r may be thought of as the degree of the divisor $D^i|_{Y_r}$ with respect to $\mathcal{O}_X(1)|_{Y_r}$. Note that unless $D^i \cap Y_r = \emptyset$ (in which case $b_r = 0$), the restriction $D^i|_{Y_r}$ is genuinely a divisor on Y_r by our support assumption (4.3) and it is exactly the intersection product $D^i \cdot Y_r$.

Now even though the support scheme Y of $\mathcal{E}$ may move around X, we can bound the degree of its irreducible components Y_r, and thus also the degree of the intersection $D^i \cdot Y_r$, using the multiplicity $a_d(\mathcal{E})$: let P be the Hilbert polynomial of $\mathcal{E}$. For sufficiently large M, the sheaf $\mathcal{E}(M)$ is globally generated and has vanishing higher cohomology, so $h^0(\mathcal{E}(M)) = P(M) > 0$. Any non-zero global section $s \in H^0(X, \mathcal{E}(M))$ gives a morphism $s : \mathcal{O}_X \to \mathcal{E}(M)$. Let $V := \mathrm{Supp}\ s$; as $\mathcal{E}$ is pure, so is $\mathcal{E}(M)$ and V must be purely d-dimensional, a union of irreducible components of $Y = \mathrm{Supp}\ \mathcal{E}$. Since $\mathcal{E}(M)$ is generated by its global sections, for each component Y_r of $Y = \mathrm{Supp}\ \mathcal{E}$ there is a section $s \in H^0(X, \mathcal{E}(M))$ such that $Y_r \subset \mathrm{Supp}\ s =: V$. Then s induces an injection $\mathcal{O}_V \hookrightarrow \mathcal{E}(M)$, and thus $P(\mathcal{O}_V) \leqslant P(\mathcal{E}(M)) = P[M]$, where we write $P[M]$ for the polynomial P shifted by M, i.e. $P[M](m) = P(m + M)$. Note that the leading terms of P and $P[M]$ agree and the ordering on polynomials is lexicographic, so $a_d(\mathcal{O}_V)$, the degree of V with respect to the polarisation $\mathcal{O}_X(1)$, is bounded by $a_d(\mathcal{E}(M)) = a_d(\mathcal{E})$, thus $a_d(\mathcal{O}_V) \leqslant a$. But $a_d(\mathcal{O}_V)$ is the sum of the positive integers $a_d(\mathcal{O}_{Y_r})$ for all r such that $Y_r \subset V$. Thus, the degree $a_d(\mathcal{O}_{Y_r})$ of each irreducible component Y_r of $Y = \mathrm{Supp}\ \mathcal{E}$ is bounded above by a, and since each $a_d(\mathcal{O}_{Y_r})$ is positive, this also gives us a bound on the number of irreducible components of $\mathrm{Supp}\ \mathcal{E}$: we have $R \leqslant a$.

Returning to

$$b_r = \int_{Y_r} x^{d-1} z \cap [Y_r] = \deg_{\mathcal{O}_X(1)}(D^i \cdot Y_r),$$

we may apply Bezout's theorem ([Ful98], 8.4), after using the very ample line bundle $\mathcal{O}_X(1)$ to embed X in a projective space $\mathbb{P}^N_{\Bbbk}$: this gives a bound

$$b_r = \deg_{\mathcal{O}_X(1)}(D^i) \cdot \deg_{\mathcal{O}_X(1)}(Y_r) = \deg_{\mathcal{O}_X(1)}(D^i) \cdot a_d(\mathcal{O}_{Y_r}) \leqslant \deg_{\mathcal{O}_X(1)}(D^i) \cdot a. \tag{4.14}$$

Here, we have used that

$$\deg_{\mathcal{O}_X(1)}(Y_r) = \int_X x^d \cap [Y_r] = a_d(\mathcal{O}_{Y_r}),$$

since $\tau_{X,d}(\mathcal{O}_{Y_r}) = [Y_r]$ by theorem 18.3(5) in [Ful98].

In particular, this shows that

$$a_{d-1}(\mathcal{E}) - a_{d-1}(\mathcal{E}(-D^i)) = \sum_{r=1}^{R} b_r n_r(\mathcal{E}) \tag{4.15}$$

is a linear form in the parameters n_r, where the number of terms R is bounded above by a and each coefficient b_r of the linear form is bounded uniformly by (4.14). To bound the value of this linear form above, it is sufficient to show now that the n_r vary in a bounded subset of $\mathbb{Q}^R$. Note that each n_r must be non-negative as the leading coefficient of $P(\mathcal{E}|_{Y_r})$ is

$$\frac{1}{d!}\int_{Y_r} x^d \cap \tau_{Y_r,d}(\mathcal{E}) = \frac{1}{d!} n_r(\mathcal{E}) \int_{Y_r} x^d \cap [Y_r].$$

Now consider

$$a \geqslant a_d(\mathcal{E}) = \int_Y x^d \cap \tau_{Y,d}(\mathcal{E}) = \sum_{r=1}^{R} n_r(\mathcal{E}) \int_{Y_r} x^d \cap [Y_r]$$

and recall that $\int_{Y_r} x^d \cap [Y_r] = a_d(\mathcal{O}_{Y_r})$ are positive integers, as $\mathcal{O}_X(1)$ is very ample. Thus, each n_r is bounded above by a, so for $\mathcal{E}$ with $a_d(\mathcal{E}) \leqslant a$ the possible values of the parameters $(n_1(\mathcal{E}), \ldots, n_R(\mathcal{E}))$ form a bounded subset of $\mathbb{Q}^R$: for all $1 \leqslant r \leqslant R$ we have $0 \leqslant n_r(\mathcal{E}) \leqslant a$. Then for all such $\mathcal{E}$, the linear form (4.15) is bounded above, i.e. there is $b_i \in \mathbb{Q}$ such that

$$a_{d-1}(\mathcal{E}) - a_{d-1}(\mathcal{E}(-D^i)) \leqslant b_i, \tag{4.16}$$

where the bound b_i is independent of $\mathcal{E}$, provided that $a_d(\mathcal{E}) \leqslant a$: explicitly, we may take

$$\begin{aligned} a_{d-1}(\mathcal{E}) - a_{d-1}(\mathcal{E}(-D^i)) &= \sum_{r=1}^{R} b_r n_r(\mathcal{E}) \\ &\leqslant \sum_{r=1}^{R} \big(\deg_{\mathcal{O}_X(1)}(D^i) \cdot a\big) a \\ &= R\big(\deg_{\mathcal{O}_X(1)}(D^i) \cdot a\big) a \\ &\leqslant \deg_{\mathcal{O}_X(1)}(D^i) \cdot a^3 =: b_i. \end{aligned}$$

Finally, our assertion (c) in the proposition follows from (4.13) and (4.16) by taking

$$b := \frac{1}{n}\sum_{i=1}^{n} b_i,$$

giving

$$\begin{aligned}\mu(\mathcal{E}) - \text{par–}\mu(\mathcal{E}_*^\star) &\leqslant \frac{1}{n}\sum_{i=1}^{n}\frac{a_{d-1}(\mathcal{E}) - a_{d-1}(\mathcal{E}(-D^i))}{a_d(\mathcal{E})} \\ &\leqslant \frac{1}{n}\sum_{i=1}^{n}\Big(a_{d-1}(\mathcal{E}) - a_{d-1}(\mathcal{E}(-D^i))\Big) \leqslant b. \end{aligned}$$ □

Note that the argument in (c) relating parabolic and ordinary slope simplifies significantly in the case of smooth X and top-dimensional pure (i.e. torsion-free) $\mathcal{E}$, as in [MY92] and [Yok93]: then $\deg \mathcal{E}(-D^i) = \text{rk}\, \mathcal{E} \cdot \deg \mathcal{O}_X(-D^i) + \deg \mathcal{E}$, where all degrees are taken with respect to $\mathcal{O}_X(1)$. However, in our set-up the support scheme of $\mathcal{E}$ is allowed to be a subscheme of positive codimension moving around X and may indeed be reducible with some, but not all of its components meeting D^i, making the study of $\mathcal{E}|_{D^i}$ more subtle.

Finally, we now introduce two modifications of the parabolic Hilbert polynomial par–P_0 which have no counterpart in [MY92], [Yok93] and [Ina00]: these definitions are really made with the hindsight of the GIT analysis presented in section 4.8 when they will be motivated. First, we need to understand how the parabolic Hilbert polynomial of $\mathcal{E}_*^\star$ is related to basic numerical invariants of $\mathcal{E}$ and $\mathcal{E}_\alpha^i$: suppose $\mathcal{E}$ has Hilbert polynomial $H \in \mathbb{Q}[x]$ of degree d, and $\mathcal{E}/F_j^i(\mathcal{E})$ has Hilbert polynomial $H_j^i \in \mathbb{Q}[x]$ of degree $d-1$ (for $1 \leqslant i \leqslant n$ and $2 \leqslant j \leqslant l_i + 1$). We often refer to $(H, H_*^\star, \alpha_*^\star)$ as the *numerical type* of $\mathcal{E}_*^\star$ and we think of the $H_*^\star$ as the 'flag type' of the (quasi-)parabolic sheaf. Note that the numerical type of $\mathcal{E}_*^\star$ determines its parabolic Hilbert polynomial (but not conversely, in general). We record the precise relation here since it will be important in the analysis of GIT-(semi)stability (section 4.8):

Lemma 4.3.8. Given a parabolic sheaf $\mathcal{E}_*^\star$ of numerical type $(H, H_*^\star, \alpha_*^\star)$, we have

$$\text{par–}P_0(\mathcal{E}_*^\star) = H - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=2}^{l_i+1}\epsilon_j^i H_j^i$$

where $\epsilon_j^i := \alpha_j^i - \alpha_{j-1}^i$ for $2 \leqslant j \leqslant l_i + 1$ and where, as usual, $\alpha_{l_i+1}^i := 1$.

Proof. Recall that $F^i_{l_i+1}(\mathcal{E}) = \mathcal{E}(-D^i)$ and $F^i_1(\mathcal{E}) = \mathcal{E}$. Thus,

$$\begin{aligned}
\text{par–}P_0(\mathcal{E}^\star_*) &= \frac{1}{n}\sum_{i=1}^{n}\left(\alpha^i_{l_i+1}P(F^i_{l_i+1}(\mathcal{E})) + \sum_{j=1}^{l_i}\alpha^i_j\left[P(F^i_j(\mathcal{E})) - P(F^i_{j+1}(\mathcal{E}))\right]\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\alpha^i_1 P(F^i_1(\mathcal{E})) + \sum_{j=2}^{l_i+1}(\alpha^i_j - \alpha^i_{j-1})P(F^i_j(\mathcal{E}))\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\alpha^i_{l_i+1}P(\mathcal{E}) + \sum_{j=2}^{l_i+1}(\alpha^i_j - \alpha^i_{j-1})\left[P(F^i_j(\mathcal{E})) - P(\mathcal{E})\right]\right)\\
&= P(\mathcal{E}) - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=2}^{l_i+1}\epsilon^i_j\left[P(\mathcal{E}) - P(F^i_j(\mathcal{E}))\right].
\end{aligned}$$

□

We are now ready to define the first modification of the parabolic Hilbert polynomial leading towards a stability condition corresponding to that provided by our GIT analysis in sections 4.7 and 4.8:

Definition 4.3.9. Given a parabolic sheaf $\mathcal{E}^\star_*$ of numerical type $(H, H^\star_*, \alpha^\star_*)$, we define

$$\text{par–}P_1(\mathcal{W}^\star_* \subset \mathcal{E}^\star_*, x) := P(\mathcal{W}, x) - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=2}^{l_i+1}\epsilon^i_j\frac{a_{d-1}(\mathcal{W}/F^i_j(\mathcal{W}))}{a_{d-1}(H^i_j)}H^i_j(x) \tag{4.17}$$

for any parabolic subsheaf $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$, where $a_{d-1}(H^i_j)$ denotes the leading coefficient of $H^i_j/(d-1)!$ and ϵ^i_j is defined as in lemma 4.3.8. We write

$$\text{par–}p_1(\mathcal{W}^\star_* \subset \mathcal{E}^\star_*) := \text{par–}P_1(\mathcal{W}^\star_* \subset \mathcal{E}^\star_*)/a_d(\mathcal{W})$$

and we say that $\mathcal{E}^\star_*$ is p_1-(semi)stable if for all non-zero parabolic subsheaves $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$ we have

$$\text{par–}p_1(\mathcal{W}^\star_* \subset \mathcal{E}^\star_*)(\leqslant)\text{par–}p_0(\mathcal{E}^\star_*), \tag{4.18}$$

where as usual this inequality of polynomials is with respect to the lexicographic ordering in $\mathbb{Q}[x]$, equivalent to asking that $\text{par–}p_1(\mathcal{W}^\star_* \subset \mathcal{E}^\star_*, m)(\leqslant)\text{par–}p_0(\mathcal{E}^\star_*, m)$ for all sufficiently large $m \in \mathbb{Z}$.

Observe that the definition of $\text{par–}P_1(\mathcal{W}^\star_* \subset \mathcal{E}^\star_*)$ depends on both $\mathcal{W}^\star_*$ and $\mathcal{E}^\star_*$, explaining the notation, and note that for any parabolic sheaf $\mathcal{E}^\star_*$ we have

$$\text{par–}P_1(\mathcal{E}^\star_* \subset \mathcal{E}^\star_*) = \text{par–}P_0(\mathcal{E}^\star_*),$$

using lemma 4.3.8. We examine the relation between parabolic p_0-(semi)stability and p_1-(semi)stability:

Proposition 4.3.10. Let $\mathcal{E}^\star_*$ be a parabolic sheaf on X.

(a) If $\mathcal{E}_*^\star$ is parabolic p_0-(semi)stable and if for each i, j the coherent sheaf $\mathcal{E}/F_j^i(\mathcal{E})$ (supported on the divisor D^i) is p-(semi)stable, then $\mathcal{E}_*^\star$ is parabolic p_1-(semi)stable.

(b) The two leading coefficients (in degree d and degree $d-1$) of par–$P_0(\mathcal{W}_*^\star)$ and par–$P_1(\mathcal{W}_*^\star \subset \mathcal{E}_*^\star)$ agree for every subsheaf $\mathcal{W}_*^\star \subset \mathcal{E}_*^\star$. In particular, we have implications

$$\begin{aligned} \mathcal{E}_*^\star \text{ is } p_1\text{-semistable } &\Longrightarrow \mathcal{E}_*^\star \text{ is } \mu\text{-semistable}, \\ \mathcal{E}_*^\star \text{ is } \mu\text{-stable } &\Longrightarrow \mathcal{E}_*^\star \text{ is } p_1\text{-stable}. \end{aligned}$$

(c) If X is a projective curve (not necessarily smooth or irreducible), then parabolic p_0-(semi)stability and p_1-(semi)stability agree.

Proof. For (a), it is enough to check that for a parabolic subsheaf $\mathcal{W}_*^\star \subset \mathcal{E}_*^\star$ with the induced parabolic structure $F_j^i(\mathcal{W}) = F_j^i(\mathcal{E}) \cap \mathcal{W}$ the criterion (4.18) holds: this is since the induced structure is maximal, so it minimises the quotients $\mathcal{W}/F_j^i(\mathcal{W})$, therefore maximising par–$P_1(\mathcal{W}_*^\star \subset \mathcal{E}_*^\star, x, y)$ among all parabolic structures of $\mathcal{W}$. In this case, $\mathcal{W}/F_j^i(\mathcal{W})$ is a subsheaf of $\mathcal{E}/F_j^i(\mathcal{E})$, so p-(semi)stability of the coherent sheaf $\mathcal{E}/F_j^i(\mathcal{E})$ gives

$$\frac{P(\mathcal{W}/F_j^i(\mathcal{W}))}{a_{d-1}(\mathcal{W}/F_j^i(\mathcal{W}))} (\leqslant) \frac{H_j^i}{a_{d-1}(H_j^i)},$$

hence

$$\begin{aligned} \text{par–}P_1(\mathcal{W}_*^\star \subset \mathcal{E}_*^\star) &= P(\mathcal{W}) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=2}^{l_i+1} \epsilon_j^i \frac{a_{d-1}(\mathcal{W}/F_j^i(\mathcal{W}))}{a_{d-1}(H_j^i)} H_j^i \\ &(\leqslant) P(\mathcal{W}) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=2}^{l_i+1} \epsilon_j^i P(\mathcal{W}/F_j^i(\mathcal{W})) = \text{par–}P_0(\mathcal{W}_*^\star) \end{aligned}$$

where the last equality is just lemma 4.3.8 applied to $\mathcal{W}_*^\star$. Together with p_0-(semi)stability of $\mathcal{E}_*^\star$ this implies

$$\text{par–}p_1(\mathcal{W}_*^\star \subset \mathcal{E}_*^\star) = \frac{\text{par–}P_1(\mathcal{W}_*^\star \subset \mathcal{E}_*^\star)}{a_d(\mathcal{W})} (\leqslant) \frac{\text{par–}P_0(\mathcal{W}_*^\star)}{a_d(\mathcal{W})} (\leqslant) \text{par–}p_0(\mathcal{E}_*^\star).$$

For part (b) note that H_j^i has degree $d-1$, so by (4.17) the leading (degree d) coefficients of par–$P_1(\mathcal{W}_*^\star \subset \mathcal{E}_*^\star)$ and $P(\mathcal{W})$ agree, and are thus equal to the leading coefficient of par–$P_0(\mathcal{W}_*^\star)$ by lemma 4.3.2. In degree $d-1$, the definition (4.17) gives

$$\begin{aligned} a_{d-1}(\text{par–}P_1(\mathcal{W}_*^\star \subset \mathcal{E}_*^\star)) &= a_{d-1}(\mathcal{W}) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=2}^{l_i+1} \epsilon_j^i \frac{a_{d-1}(\mathcal{W}/F_j^i(\mathcal{W}))}{a_{d-1}(H_j^i)} a_{d-1}(H_j^i) \\ &= a_{d-1}(\mathcal{W}) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=2}^{l_i+1} \epsilon_j^i a_{d-1}(\mathcal{W}/F_j^i(\mathcal{W})) = \text{par–deg}(\mathcal{W}_*^\star), \end{aligned}$$

where the last equality follows from lemma 4.3.8. The implications in (b) then follow in the same way as proposition 4.3.7(a): the parabolic slope is the highest non-trivial coefficient of par–p_1. This also immediately gives part (c): if $\dim X = 1$ then both p_0- and p_1-(semi)stability just reduce to parabolic slope-(semi)stability. □

In particular, this proposition shows that in the widely studied case of parabolic sheaves on curves, this stability notion agrees with the traditional one. Moreover, part (b) in particular shows that any parabolic slope-stable $\mathcal{E}_*^\star$ is then also p_1-stable, demonstrating that the stability condition proposed here is not vacuous.

Note that the parabolic structure of $\mathcal{W}_*^\star \subset \mathcal{E}_*^\star$ only enters into p_1-(semi)stability through the degree $d-1$ coefficient $a_{d-1}(\mathcal{W}/F_j^i(\mathcal{W}))$ of $P(\mathcal{W}/F_j^i(\mathcal{W}))$. In higher dimension we lose too much information this way, so we refine p_1-(semi)stability further: everywhere we see the leading coefficient of a Hilbert polynomial in (4.17), we replace it by the full Hilbert polynomial, but in a *different* variable y. The idea is that y takes values very large compared to x, so terms of top degree in y dominate and hence (4.17) may be thought of as a truncation (or as the dominant terms in an asymptotic expansion) of the definition we are about to give. This stability condition arises out of the GIT analysis in section 4.8.

Definition 4.3.11. Given a parabolic sheaf $\mathcal{E}_*^\star$ of numerical type $(H, H_*^\star, \alpha_*^\star)$, define a rational function in two variables x, y by

$$\text{par–}P_2(\mathcal{W}_*^\star \subset \mathcal{E}_*^\star, x, y) := P(\mathcal{W}, x) - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=2}^{l_i+1} \epsilon_j^i \frac{P(\mathcal{W}/F_j^i(\mathcal{W}), y)}{H_j^i(y)} H_j^i(x) \qquad (4.19)$$

for any parabolic subsheaf $\mathcal{W}_*^\star \subset \mathcal{E}_*^\star$, where ϵ_j^i is defined as in lemma 4.3.8.

We write

$$\text{par–}p_2(\mathcal{W}_*^\star \subset \mathcal{E}_*^\star, x, y) := \frac{\text{par–}P_2(\mathcal{W}_*^\star \subset \mathcal{E}_*^\star, x, y)}{\chi(\mathcal{W}(y))}$$

and we say that $\mathcal{E}_*^\star$ is p_2-*(semi)stable* if for all non-zero parabolic subsheaves $\mathcal{W}_*^\star \subset \mathcal{E}_*^\star$ there is $M \in \mathbb{Z}$ such that for all $m \geqslant M$ there is $K(m) \in \mathbb{Z}$ such that for all $k \geqslant K$ we have

$$\text{par–}p_2(\mathcal{W}_*^\star \subset \mathcal{E}_*^\star, m, k)(\leqslant)\frac{\text{par–}P_0(\mathcal{E}_*^\star, m)}{H(k)}, \qquad (4.20)$$

observing that for any parabolic sheaf $\mathcal{E}_*^\star$ we have $\text{par–}P_2(\mathcal{E}_*^\star \subset \mathcal{E}_*^\star, x, y) = \text{par–}P_0(\mathcal{E}_*^\star, x)$, by lemma 4.3.8.

As usual, this corresponds to asking for an inequality of polynomials with respect to a lexicographic ordering: if we multiply out all denominators in (4.19), the condition (4.20) is equivalent to

$$H(y)\left(\chi(\mathcal{W}(x))\prod_{i,j}H^i_j(y)-\frac{1}{n}\sum_{i,j}\epsilon^i_j\chi(\mathcal{W}/F^i_j(\mathcal{W})(y))H^i_j(x)\prod_{(a,b)\neq(i,j)}H^a_b(y)\right) \\ (\leqslant)\chi(\mathcal{W}(y))\text{par–}P_0(\mathcal{E}^\star_*,x)\prod_{i,j}H^i_j(y), \tag{4.21}$$

where the inequality is with respect to the lexicographic ordering in $\mathbb{Q}[x,y]$ that regards y as greater than x, i.e. $y^{i_1}x^{j_1}\geqslant y^{i_2}x^{j_2}$ if $i_1>i_2$, or if both $i_1=i_2$ and $j_1\geqslant j_2$. This ordering corresponds to the fact that (4.20) is required to hold for sufficiently large values m of x and for very large values k (depending on m) of y, i.e. in the asymptotic expansion the terms of top degree in y dominate.

The following result shows that p_2-(semi)stability indeed refines p_1-(semi)stability:

Proposition 4.3.12. For any parabolic sheaf $\mathcal{E}^\star_*$ on X we have implications

$$\begin{aligned} &\mathcal{E}^\star_* \text{ is } p_2\text{-semistable} \implies \mathcal{E}^\star_* \text{ is } p_1\text{-semistable} \implies \mathcal{E}^\star_* \text{ is } \mu\text{-semistable},\\ &\mathcal{E}^\star_* \text{ is } \mu\text{-stable} \implies \mathcal{E}^\star_* \text{ is } p_1\text{-stable} \implies \mathcal{E}^\star_* \text{ is } p_2\text{-stable}.\end{aligned}$$

Proof. Consider the terms in (4.21) which are dominant under the lexicographic ordering, i.e. the terms of top degree in y: these are

$$a_d(\mathcal{E})\left(\chi(\mathcal{W}(x))\prod_{i,j}a_{d-1}(H^i_j)-\frac{1}{n}\sum_{i,j}\epsilon^i_j a_{d-1}(\mathcal{W}/F^i_j(\mathcal{W}))H^i_j(x)\prod_{(a,b)\neq(i,j)}a_{d-1}(H^a_b)\right) \tag{4.22}$$

on the left-hand side of the inequality, and

$$a_d(\mathcal{W})\text{par–}P_0(\mathcal{E}^\star_*,x)\prod_{i,j}a_{d-1}(H^i_j) \tag{4.23}$$

on the right-hand side of (4.21). Thus, if weak inequality holds in (4.21), then we have in particular

$$a_d(\mathcal{E})\left(\chi(\mathcal{W}(x))\prod_{i,j}a_{d-1}(H^i_j)-\frac{1}{n}\sum_{i,j}\epsilon^i_j a_{d-1}(\mathcal{W}/F^i_j(\mathcal{W}))H^i_j(x)\prod_{(a,b)\neq(i,j)}a_{d-1}(H^a_b)\right) \\ \leqslant a_d(\mathcal{W})\text{par–}P_0(\mathcal{E}^\star_*,x)\prod_{i,j}a_{d-1}(H^i_j)$$

which is just (4.18) with weak inequality. Conversely, strict inequality in (4.18) says that (4.22) is strictly less than (4.23), implying that (4.21) holds with strict inequality. This proves the implications between p_1- and p_2-(semi)stability, and the implications involving slope-(semi)stability are part (b) of proposition 4.3.10. □

In particular, this shows that any parabolic slope-stable $\mathcal{E}_*^\star$ is also p_2-stable, guaranteeing non-emptyness of the moduli spaces we construct in this chapter.

Remark 4.3.13. Note that p_2-(semi)stability differs from the traditional notion of parabolic (semi)stability on curves: if $d = 1$, then the Hilbert polynomials of all quotients $\mathcal{E}/F_j^i(\mathcal{E})$ and $\mathcal{W}/F_j^i(\mathcal{W})$ are constants, in particular $H_j^i(y) = H_j^i(x)$ and so these terms all cancel from (4.21) and the p_2-(semi)stability condition reads

$$\begin{aligned}&(a_1(\mathcal{E})y + a_0(\mathcal{E}))\left(a_1(\mathcal{W})x + a_0(\mathcal{W}) - \frac{1}{n}\sum_{i,j}\epsilon_j^i a_0(\mathcal{W}/F_j^i(\mathcal{W}))\right)\\ &(\leqslant)\,(a_1(\mathcal{W})y + a_0(\mathcal{W}))\left(a_1(\mathcal{E})x + a_0(\mathcal{E}) - \frac{1}{n}\sum_{i,j}\epsilon_j^i a_0(\mathcal{E}/F_j^i(\mathcal{E}))\right),\end{aligned} \tag{4.24}$$

where the term in parentheses on the right-hand side is par$-P_0(\mathcal{E}_*^\star, x)$.

The top term is $a_1(\mathcal{E})a_1(\mathcal{W})xy$ on both sides of (4.24). Recall that the next term is y, followed by x, followed by the constants. Thus, p_2-semistability (4.24) is equivalent to

(a) either

$$a_1(\mathcal{E})\left(a_0(\mathcal{W}) - \frac{1}{n}\sum_{i,j}\epsilon_j^i a_0(\mathcal{W}/F_j^i(\mathcal{W}))\right) < a_1(\mathcal{W})\left(a_0(\mathcal{E}) - \frac{1}{n}\sum_{i,j}\epsilon_j^i a_0(\mathcal{E}/F_j^i(\mathcal{E}))\right),$$

i.e. parabolic slope-stability with respect to $\mathcal{W}_*^\star$,

(b) or par$-\mu(\mathcal{W}_*^\star) =$ par$-\mu(\mathcal{E}_*^\star)$ *and*

$$a_0(\mathcal{E})a_1(\mathcal{W}) < a_0(\mathcal{W})a_1(\mathcal{E}),$$

i.e. $\mu(\mathcal{E}) < \mu(\mathcal{W})$ (sic!),

(c) or par$-\mu(\mathcal{W}) =$ par$-\mu(\mathcal{E})$ and $\mu(\mathcal{W}) = \mu(\mathcal{E})$ and

$$a_0(\mathcal{E})\left(a_0(\mathcal{W}) - \frac{1}{n}\sum_{i,j}\epsilon_j^i a_0(\mathcal{W}/F_j^i(\mathcal{W}))\right) \leqslant a_0(\mathcal{W})\left(a_0(\mathcal{E}) - \frac{1}{n}\sum_{i,j}\epsilon_j^i a_0(\mathcal{E}/F_j^i(\mathcal{E}))\right),$$

i.e.

$$a_0(\mathcal{W})\sum_{i,j}\epsilon_j^i a_0(\mathcal{E}/F_j^i(\mathcal{E})) \leqslant a_0(\mathcal{E})\sum_{i,j}\epsilon_j^i a_0(\mathcal{W}/F_j^i(\mathcal{W})).$$

However, note that (c) does not actually impose an extra condition, given equality in (a) and (b): if par$-\mu(\mathcal{W}_*^\star) =$ par$-\mu(\mathcal{E}_*^\star)$ and $\mu(\mathcal{W}) = \mu(\mathcal{E})$, then we have

$$a_1(\mathcal{E})\left(a_0(\mathcal{W}) - \frac{1}{n}\sum_{i,j}\epsilon_j^i a_0(\mathcal{W}/F_j^i(\mathcal{W}))\right) = a_1(\mathcal{W})\left(a_0(\mathcal{E}) - \frac{1}{n}\sum_{i,j}\epsilon_j^i a_0(\mathcal{E}/F_j^i(\mathcal{E}))\right),$$

as well as $a_0(\mathcal{E})a_1(\mathcal{W}) = a_0(\mathcal{W})a_1(\mathcal{E})$, implying

$$a_1(\mathcal{E}) \sum_{i,j} \epsilon_j^i a_0(\mathcal{W}/F_j^i(\mathcal{W})) = a_1(\mathcal{W}) \sum_{i,j} \epsilon_j^i a_0(\mathcal{E}/F_j^i(\mathcal{E})). \tag{4.25}$$

Now if $\mu(\mathcal{E}) = \mu(\mathcal{W})$ is non-zero, we have

$$a_1(\mathcal{E}) = \frac{a_0(\mathcal{E})}{\mu(\mathcal{E})}$$

and

$$a_1(\mathcal{W}) = \frac{a_0(\mathcal{W})}{\mu(\mathcal{W})} = \frac{a_0(\mathcal{W})}{\mu(\mathcal{E})},$$

so (4.25) is equivalent to

$$a_0(\mathcal{E}) \sum_{i,j} \epsilon_j^i a_0(\mathcal{W}/F_j^i(\mathcal{W})) = a_0(\mathcal{W}) \sum_{i,j} \epsilon_j^i a_0(\mathcal{E}/F_j^i(\mathcal{E})),$$

i.e. condition (c) with equality. On the other hand, if $\mu(\mathcal{E}) = \mu(\mathcal{W}) = 0$, then $a_0(\mathcal{W}) = 0 = a_0(\mathcal{E})$, so we trivially have equality in (c) too.

In summary, p_2-semistability (4.24) on curves is equivalent to

(i) par–$\mu(\mathcal{W}_*^\star) <$ par–$\mu(\mathcal{E}_*^\star)$; or

(ii) par–$\mu(\mathcal{W}_*^\star) =$ par–$\mu(\mathcal{E}_*^\star)$ and $\mu(\mathcal{E}) \leqslant \mu(\mathcal{W})$ (sic!),

and p_2-stability is the same but with strict inequality $\mu(\mathcal{E}) < \mu(\mathcal{W})$ in (ii). Hence, p_2-(semi)stability on curves (even smooth curves) is a refinement of slope-(semi)stability: slope-stable $\mathcal{E}_*^\star$ are p_2-stable, conversely p_2-semistable $\mathcal{E}_*^\star$ are slope-semistable. The stability conditions agree whenever there are no strictly slope-semistable parabolic sheaves, but in general they will differ (and par–p_2 will produce fewer strictly semistable objects than parabolic slope).

We give an example of how (ii) may occur with strict inequality, showing the difference between the stability conditions: let $X = \mathbb{P}^1$ and consider a divisor given by a single point $D = x$. Let $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}_1 \oplus \mathcal{O}_2$ (where $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$ and the index serves to distinguish the summands) with parabolic structure

$$\begin{aligned} F_1(\mathcal{E}) &= \mathcal{E}, \\ F_2(\mathcal{E}) &= \Big(\mathcal{O}(1) \oplus \mathcal{O}_1\Big)(-x) \oplus \mathcal{O}_2, \\ F_3(\mathcal{E}) &= \mathcal{E}(-x), \end{aligned}$$

and weights $0 \leqslant \alpha_1 < \alpha_2 < 1$ such that $\alpha_2 = \frac{1}{2} + \alpha_1$. Let $\mathcal{W} = \mathcal{O}(1) \oplus \mathcal{O}_1$ with the induced parabolic structure $F_1(\mathcal{W}) = \mathcal{W}$ and $F_2(\mathcal{W}) = F_3(\mathcal{W}) = \mathcal{W}(-x)$. We claim that par–$\mu(\mathcal{W}) =$ par–$\mu(\mathcal{E})$ but $\mu(\mathcal{W}) > \mu(\mathcal{E})$:

$$\mu(\mathcal{E}) = \frac{\chi(\mathcal{E})}{\operatorname{rk} \mathcal{E}} = \frac{4}{3},$$

and

$$\mu(\mathcal{W}) = \frac{\chi(\mathcal{W})}{\text{rk}\,\mathcal{W}} = \frac{3}{2},$$

whereas

$$\begin{aligned}\text{par--}\mu(\mathcal{E}) &= \frac{1}{\text{rk}\,(\mathcal{E})}\Big(\chi(\mathcal{E}) - (\alpha_2 - \alpha_1)\chi(\mathcal{E}/F_2(\mathcal{E})) - (1-\alpha_2)\chi(\mathcal{E}/F_3(\mathcal{E}))\Big)\\ &= \frac{1}{3}\Big(4 - \frac{1}{2}\cdot 2 - (1-\alpha_2)\cdot 3\Big)\\ &= \alpha_2,\end{aligned}$$

and

$$\begin{aligned}\text{par--}\mu(\mathcal{W}) &= \frac{1}{\text{rk}\,(\mathcal{W})}\Big(\chi(\mathcal{W}) - (\alpha_2 - \alpha_1)\chi(\mathcal{W}/F_2(\mathcal{W})) - (1-\alpha_2)\chi(\mathcal{W}/F_3(\mathcal{W}))\Big)\\ &= \frac{1}{2}\Big(3 - \frac{1}{2}\cdot 2 - (1-\alpha_2)\cdot 2\Big)\\ &= \alpha_2.\end{aligned}$$

4.4 Families in the relative setting

We now leave the absolute situtation (of a projective scheme X over $\Bbbk$) and consider from here on the relative situation of a flat family of projective schemes: for the remainder of the chapter, let S be a (connected) scheme of finite type over a fixed algebraically closed field $\Bbbk$ and consider a flat family of projective schemes varying over the base S, i.e. let $X \to S$ be a flat projective morphism. Pick a relatively very ample line bundle $\mathcal{O}_X(1)$ and relative effective Cartier divisors D^i (for $1 \leqslant i \leqslant n$) on X. Let Sch/S be the category of locally noetherian schemes over S, and recall the following conventions: for each $T \in \mathsf{Sch}/S$ we let $X_T := X \times_S T$. Given a closed $\Bbbk$-point $t \in T$, write X_t for the fibre of X_T over t, and for any T-flat $\mathcal{E} \in \mathsf{Coh}(X_T)$, write $\mathcal{E}_t$ for the restriction of $\mathcal{E}$ to the scheme X_t, i.e. $\mathcal{E}_t := \mathcal{E} \otimes k(t)$. The relative effective Cartier divisor on X_T induced by D^i is denoted by $(D^i)_T$, and the (absolute) effective Cartier divisor on the scheme X_t given by restriction of $(D^i)_T$ is denoted by $(D^i)_t$. Similarly, we write $\mathcal{O}_{X_T}(1)$ for the pull-back of $\mathcal{O}_X(1)$ to X_T, and $\mathcal{O}_{X_t}(1)$ for its restriction to X_t.

The results of the previous sections will be applied to parabolic sheaves on the geometric fibres of $X_T \to T$ or $X \to S$.

The objects of our moduli problem in the relative set-up are families of (semistable) parabolic sheaves on X/S parametrised by a scheme $T \in \mathsf{Sch}/S$:

Definition 4.4.1 ([Ina00] definitions 1.9 and 1.10)**.** For any $T \in \mathsf{Sch}/S$, a *flat family of parabolic sheaves* on X_T/T (in the following also referred to as a 'flat family' on X_T/T) is a triple $\mathcal{E}^\star_* := (\mathcal{E}, F^\star_*(\mathcal{E}), \alpha^\star_*)$ of

(a) a T-flat coherent sheaf $\mathcal{E}$ on X_T,

(b) for each $1 \leqslant i \leqslant n$ a filtration

$$\mathcal{E} = F_1^i(\mathcal{E}) \supset \cdots \supset F_{l_i+1}^i(\mathcal{E}) = \mathcal{E}(-(D^i)_T)$$

by coherent sheaves such that for each i, j the quotient $\mathcal{E}/F_j^i(\mathcal{E})$ is T-flat, and

(c) real numbers α_j^i with $0 \leqslant \alpha_1^i < \alpha_2^i < \cdots < \alpha_{l_i}^i < 1$,

such that for all $t \in T$ the restriction of $\mathcal{E}_*^\star$ to X_t is a parabolic sheaf as in definition 4.2.1. In particular, $\mathcal{E}_t$ is pure and satisfies the support conditions (4.3) with respect to the divisors $(D^i)_t$ on X_t. (Moreover, for all $t \in T$ the morphisms $\mathcal{E}_t \otimes \mathcal{O}_{X_t}(-(D^i)_t) \to \mathcal{E}_t$ are injective by the support conditions, and using [Gro66], 11.3.7, this implies that $\mathcal{E} \otimes \mathcal{O}_X(-(D^i)_T) \to \mathcal{E}$ is injective for each i.)

Now fix numerical polynomials $H, H_j^i \in \mathbb{Q}[x]$ (for $1 \leqslant i \leqslant n$ and $2 \leqslant j \leqslant l_i + 1$), where $d := \deg H$ and $\deg H_j^i = d - 1$ for all i, j. Then the basic moduli functor of interest in this chapter is

$$\mathbf{Par}_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star} : \mathsf{Sch}/S \to \mathsf{Sets}, \tag{4.26}$$

defined by

$$\mathbf{Par}_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star}(T) := \{\text{flat families } \mathcal{E}_*^\star \text{ on } X_T/T \text{ s.t. for all } t \in T \text{ the sheaf } \mathcal{E}_t \text{ has Hilbert polynomial } H \text{ and } (\mathcal{E}/F_j^i(\mathcal{E}))_t \text{ has Hilbert polynomial } H_j^i \text{ for all } i, j\}/\sim,$$

where $\sim$, *isomorphism of families* over T, is defined by writing $\mathcal{E}_*^\star \sim \mathcal{F}_*^\star$ if there is a line bundle L on T such that

$$\mathcal{E}_*^\star \otimes_{\mathcal{O}_{X_T}} (\mathrm{pr}_2)^* L \cong \mathcal{F}_*^\star$$

(i.e. $\mathcal{E} \otimes_{\mathcal{O}_{X_T}} (\mathrm{pr}_2)^* L \cong \mathcal{F}$ via an isomorphism respecting the filtrations), where $\mathrm{pr}_2 : X_T = X \times_S T \to T$ is second projection.

This is a contravariant functor $\mathsf{Sch}/S \to \mathsf{Sets}$: for any morphism $g : T' \to T$ in Sch/S,

$$\mathbf{Par}_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star}(g) : \mathbf{Par}_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star}(T) \to \mathbf{Par}_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star}(T')$$

is just given by pull-back g^*.

As in the absolute case, we often refer to $(H, H_*^\star, \alpha_*^\star)$ as the *numerical type* of a flat family $\mathcal{E}_*^\star$.

The moduli functor for which we would like to construct a coarse moduli space is given by first passing from $\mathbf{Par}_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star}$ to the subfunctor of *semistable* parabolic sheaves on X/S, and then from $\sim$ to the coarser notion of S-equivalence. However, at this stage we have no proof showing that Jordan-Hölder filtrations are well-defined for p_2-semistable parabolic families of fixed numerical type, so we postpone the definition of our moduli functor to section 4.9. We do, however, have a proof demonstrating that p_0-semistable objects form an abelian category, hence admit Jordan-Hölder filtrations. For this argument, we will need to assume that

$$\text{either } \alpha_1^i > 0 \text{ for some } i, \text{ or } \bigcap_i D^i = \emptyset. \tag{4.27}$$

Definition 4.4.2. A flat family $\mathcal{E}_*^\star$ on X_T/T is called *(semi)stable* if for each $t \in T$ the parabolic sheaf given by restricting $\mathcal{E}_*^\star$ to X_t is p_2-(semi)stable as in definition 4.3.11, where the modified reduced parabolic Hilbert polynomial par–p_2 is taken with respect to $\mathcal{O}_{X_t}(1)$. Similarly, we may define a flat family to be p_0-*(semi)stable* if its restriction to each fibre is p_0-(semi)stable.

Given a p_0-semistable family $\mathcal{E}_*^\star$ on X_T/T, there is a filtration

$$\mathcal{E} = \mathrm{JH}^1(\mathcal{E}) \supset \mathrm{JH}^2(\mathcal{E}) \supset \cdots \supset \mathrm{JH}^{K+1}(\mathcal{E}) = 0$$

by T-flat saturated subsheaves whose restriction at each $t \in T$ yields a Jordan-Hölder filtration of $(\mathcal{E}_t)_*^\star$, i.e. for each $1 \leqslant k \leqslant K$ the sheaf $(\mathrm{JH}^k(\mathcal{E})_t/\mathrm{JH}^{k+1}(\mathcal{E})_t)_*^\star$ with its induced parabolic structure is parabolic p_0-stable and has the same reduced parabolic Hilbert polynomial as $(\mathcal{E}_t)_*^\star$. This filtration is not unique, but the associated graded

$$\mathrm{gr}(\mathcal{E}_*^\star) := \bigoplus_{k=1}^{K} (\mathrm{JH}^k(\mathcal{E})/\mathrm{JH}^{k+1}(\mathcal{E}))_*^\star$$

is unique up to isomorphism of families, i.e. up to $\sim$-equivalence, by the Jordan-Hölder theorem.

Two p_0-semistable families $\mathcal{E}_*^\star, \mathcal{F}_*^\star$ on X_T/T are then called *S-equivalent* if the graded objects of their JH filtrations agree in families, i.e. if $\mathrm{gr}(\mathcal{E}_*^\star) \sim \mathrm{gr}(\mathcal{F}_*^\star)$. If this holds, we write $\mathcal{E}_*^\star \sim_S \mathcal{F}_*^\star$. The 'S' in S-equivalence indicates that this notion is due to Seshadri (in the case of vector bundles on smooth curves) and should not be confused with $\mathcal{E}_*^\star \sim \mathcal{F}_*^\star$ for isomorphism of families over the base scheme S. However, observe that for stable families S-equivalence reduces to isomorphism of families.

Note that we cannot apply the Jordan-Hölder theorem directly to the category of all parabolic sheaves (which is not abelian as we have seen in remark 4.2.3) to show that $\mathrm{gr}(\mathcal{E}_*^\star)$ is well-defined up to isomorphism in families; however, under the assumption (4.27) the subcategory of p_0-semistable parabolic sheaves of fixed reduced parabolic Hilbert polynomial is abelian and so the Jordan-Hölder theorem does apply to it:

Proposition 4.4.3. Given $P \in \mathbb{R}[x]$ and weight sequences $\alpha_*^\star$ satisfying (4.27), the p_0-semistable parabolic sheaves on X_T/T of fixed reduced parabolic Hilbert polynomial P and fixed parabolic weights $\alpha_*^\star$ (together with the zero sheaf) form a full abelian subcategory of the category of all parabolic sheaves on X_T/T.

Proof. In this proof, '(semi)stable' will mean p_0-(semi)stable throughout.

Parabolic sheaves (of weight $\alpha_*^\star$) form an additive category; we need to show that the full subcategory under consideration admits kernels, cokernels, and canonical decompositions of morphisms, i.e. the isomorphism theorem identifying coimage and image of any morphism holds. It suffices to prove these statements in the absolute case: let $\mathcal{E}_*^\star, \mathcal{F}_*^\star$ be p-semistable parabolic sheaves on X_t with $\text{par--}p_0(\mathcal{E}_*^\star) = \text{par--}p_0(\mathcal{F}_*^\star) = P$, and write $d := \deg P$ for the common dimension of $\mathcal{E}$ and $\mathcal{F}$.

Consider a parabolic homomorphism $f : \mathcal{E}_*^\star \to \mathcal{F}_*^\star$, i.e. a morphism $f : \mathcal{E} \to \mathcal{F}$ such that $f(\mathcal{E}_\alpha^i) \subset \mathcal{F}_\alpha^i$ for all $1 \leqslant i \leqslant n$ and all $\alpha \in [0,1]$. The kernel of f as a morphism of $\mathcal{O}$-modules is a coherent subsheaf $\ker f \subset \mathcal{E}$ which inherits a unique parabolic structure:

$$(\ker f)_\alpha^i := (\ker f) \cap \mathcal{E}_\alpha^i.$$

Note that this is the induced parabolic structure which is well-defined as $\ker f$ is a saturated subsheaf of $\mathcal{E}$: the quotient $\mathcal{E}/\ker f$ is isomorphic to $\text{im } f$, a subsheaf of $\mathcal{F}$ which is therefore either zero or pure of dimension $d = \dim \mathcal{E}$.

By semistability of $\mathcal{E}_*^\star$, we have $\text{par--}p_0((\ker f)_*^\star) \leqslant \text{par--}p_0(\mathcal{E}_*^\star)$, and for the reverse inequality, let $(\text{im } f)_*^\star$ be the image of f as a morphism of $\mathcal{O}$-modules, equipped with the induced structure as a parabolic quotient of $\mathcal{E}_*^\star$. Then

$$0 \to (\ker f)_*^\star \to \mathcal{E}_*^\star \to (\text{im } f)_*^\star \to 0 \tag{4.28}$$

is a short exact sequence of parabolic sheaves, so by lemma 4.3.3 we have $\text{par--}P_0(\mathcal{E}_*^\star) = \text{par--}P_0((\ker f)_*^\star) + \text{par--}P_0((\text{im } f)_*^\star)$, which by lemma 4.3.2 implies $a_d(\mathcal{E}) = a_d(\ker f) + a_d(\text{im } f)$. These two equations together yield

$$\begin{aligned}\text{par--}p_0((\ker f)_*^\star) = \frac{\text{par--}P_0((\ker f)_*^\star)}{a_d(\ker f)} &= \frac{\text{par--}P_0(\mathcal{E}_*^\star) - \text{par--}P_0((\text{im } f)_*^\star)}{a_d(\mathcal{E}) - a_d(\text{im } f)} \\ &\geqslant \frac{\text{par--}P_0(\mathcal{E}_*^\star)}{a_d(\mathcal{E})} = \text{par--}p_0(\mathcal{E}_*^\star),\end{aligned}$$

where the inequality is a consequence of semistability of $\mathcal{F}_*^\star$:

$$\frac{\text{par--}P_0((\text{im } f)_*^\star)}{a_d(\text{im } f)} = \text{par--}p_0((\text{im } f)_*^\star) \leqslant \text{par--}p_0(\mathcal{F}_*^\star) = \text{par--}p_0(\mathcal{E}_*^\star) = \frac{\text{par--}P_0(\mathcal{E}_*^\star)}{a_d(\mathcal{E})}.$$

This shows that $\text{par--}p_0((\ker f)_*^\star) = \text{par--}p_0(\mathcal{E}_*^\star) = P$, and as any parabolic subsheaf of $(\ker f)_*^\star$ is a parabolic subsheaf of $\mathcal{E}_*^\star$, we see that $(\ker f)_*^\star$ is automatically semistable.

Defining the image of f as a parabolic homomorphism requires more caution: as the example at the end of remark 4.2.3 illustrates, a parabolic homomorphism f between arbitrary parabolic sheaves need not be strict, i.e. the sheaf-theoretic image of f may be equipped with two parabolic structures (the induced filtration as a subsheaf of $\mathcal{F}$ or as a quotient of $\mathcal{E}$) which need not coincide, preventing a definition of parabolic image that works in general. However, these parabolic structures do agree if $\mathcal{E}_*^\star$ and $\mathcal{F}_*^\star$ are both semistable with par–$p_0 = P$ and if (4.27) holds: as above, let $(\mathrm{im}\ f)_*^\star$ be the sheaf-theoretic image im f, equipped with the induced structure making it a quotient parabolic sheaf of $\mathcal{E}_*^\star$, i.e. $(\mathrm{im}\ f)_\alpha^i := f(\mathcal{E}_\alpha^i)$. Let $(\text{sat–im}\ f)_*^\star$ be the saturation of im f as a subsheaf of $\mathcal{F}$, equipped with the induced structure making it a parabolic subsheaf of $\mathcal{F}_*^\star$, i.e. $(\text{sat–im}\ f)_\alpha^i := (\mathrm{im}\ f)^{\mathrm{sat}} \cap \mathcal{F}_\alpha^i$. Then the argument in remark 4.3.5 shows that we have $(\mathrm{im}\ f)_\alpha^i \subset (\text{sat–im}\ f)_\alpha^i$ for $1 \leqslant i \leqslant n$ and all $\alpha \in [0,1]$, and as $a_d(\mathrm{im}\ f) = a_d((\mathrm{im}\ f)^{\mathrm{sat}})$ we thus have

$$\begin{aligned}\text{par–}p_0((\mathrm{im}\ f)_*^\star) &= \frac{1}{n \cdot a_d(\mathrm{im}\ f)} \sum_{i=1}^n \int_0^1 P((\mathrm{im}\ f)_\alpha^i)\ \mathrm{d}\alpha \\ &\leqslant \frac{1}{n \cdot a_d((\mathrm{im}\ f)^{\mathrm{sat}})} \sum_{i=1}^n \int_0^1 P((\text{sat–im}\ f)_\alpha^i)\ \mathrm{d}\alpha \\ &= \text{par–}p_0((\text{sat–im}\ f)_*^\star).\end{aligned} \tag{4.29}$$

We claim that equality holds in par–$p_0((\mathrm{im}\ f)_*^\star) \leqslant$ par–$p_0((\text{sat–im}\ f)_*^\star)$ if and only if im f is a saturated subsheaf of $\mathcal{F}$ and the two induced parabolic structures agree: suppose equality holds in (4.29), then $P((\mathrm{im}\ f)_\alpha^i) = P((\text{sat–im}\ f)_\alpha^i)$ for all i and for all $\alpha > \alpha_1^i$ (since the differences between successive weights are positive), and so $(\mathrm{im}\ f)_\alpha^i = (\text{sat–im}\ f)_\alpha^i$ for all such α. In other words, the filtrations of im f and $(\mathrm{im}\ f)^{\mathrm{sat}}$ at each D^i will agree from the second term onwards, but we have not yet shown that the first terms, i.e. the sheaves im f and $(\mathrm{im}\ f)^{\mathrm{sat}}$ themselves, are equal. However, if $\alpha_1^i > 0$ for some i, then we must also have im $f = (\mathrm{im}\ f)^{\mathrm{sat}}$ as their Hilbert polynomials now contribute with positive weight to (4.29). On the other hand, we already know that

$$(\mathrm{im}\ f)(-D^i) = (\mathrm{im}\ f)_1^i = (\text{sat–im}\ f)_1^i = (\mathrm{im}\ f)^{\mathrm{sat}}(-D^i) \tag{4.30}$$

for all i, so im f and $(\mathrm{im}\ f)^{\mathrm{sat}}$ agree away from each divisor D^i. If $\alpha_1^i = 0$ for all i, then by (4.27) we have $\cap_i D^i = \emptyset$ and our claim is proved.

Now, by additivity of parabolic Hilbert polynomials in (4.28), we have again

$$\begin{aligned}\text{par–}p_0((\mathrm{im}\ f)_*^\star) &= \frac{\text{par–}P_0((\mathrm{im}\ f)_*^\star)}{a_d(\mathrm{im}\ f)} \\ &= \frac{\text{par–}P_0(\mathcal{E}_*^\star) - \text{par–}P_0((\ker f)_*^\star)}{a_d(\mathcal{E}) - a_d(\ker f)} = \frac{\text{par–}P_0(\mathcal{E}_*^\star)}{a_d(\mathcal{E})} = \text{par–}p_0(\mathcal{E}_*^\star),\end{aligned}$$

using that

$$\frac{\text{par–}P_0((\ker f)_*^\star)}{a_d(\ker f)} = \frac{\text{par–}P_0(\mathcal{E}_*^\star)}{a_d(\mathcal{E})},$$

as shown above. On the other hand, par–$p_0((\text{sat–im } f)^\star_*) \leqslant$ par–$p_0(\mathcal{F}^\star_*)$ by semistability of $\mathcal{F}^\star_*$, so

$$\text{par–}p_0(\mathcal{E}^\star_*) = \text{par–}p_0((\text{im } f)^\star_*) \leqslant \text{par–}p_0((\text{sat–im } f)^\star_*) \leqslant \text{par–}p_0(\mathcal{F}^\star_*) = \text{par–}p_0(\mathcal{E}^\star_*)$$

and we must have equality throughout, thus $(\text{im } f)^\star_* = (\text{sat–im } f)^\star_*$ by the claim, and par–$p_0((\text{im } f)^\star_*) = P$. Viewing parabolic subsheaves of $(\text{im } f)^\star_*$ as parabolic subsheaves of $\mathcal{F}^\star_*$ shows that $(\text{im } f)^\star_*$ is then also semistable. Similar arguments show that the cokernel of f may be equipped with a unique parabolic structure, and that this parabolic sheaf is also semistable of reduced parabolic Hilbert polynomial P. Finally, the short exact sequence (4.28) gives the isomorphism theorem in our full subcategory, which is therefore abelian. □

Remark 4.4.4. We had to impose assumption (4.27) only to prove the claim that equality in (4.29) implies that im f is saturated – in the GIT construction of the moduli space, this assumption will play no further rôle. We have no proof for the proposition above without assuming (4.27), and indeed this reflects a similar phenomenon observed by Yokogawa and Bhosle using different methods: in [Yok93], the moduli space of (p_0-)semistable parabolic sheaves (on a family $X \to S$ of *smooth* projective schemes) is constructed as a GIT quotient of a parameter space which is not projective over S, but a posteriori the moduli space may be shown to be S-projective using Langton's method [Lan75], provided that the lowest parabolic weight is positive. In [Bho99], Bhosle studies (in our language) purely 1-dimensional parabolic sheaves on disconnected smooth curves C with divisors $D^i = x_i + z_i$, where x_i and z_i are points on distinct components of C, and parabolic structures of length 2 at each D^i with parabolic weights $\alpha^i_2 = 1$ for all i and $\alpha^i_1 = \alpha$ for all i, where $\alpha \in [0, 1)$. Bhosle observes that if $\alpha = 0$, then p_0-semistable (or equivalently μ-semistable) parabolic sheaves of fixed parabolic slope do not form an abelian category: cokernels of morphisms between them may have torsion supported on $\bigcup_i D^i$, i.e. im f and $(\text{im } f)^{\text{sat}}$ may differ on $\bigcup_i D^i$. Thus, our alternative assumption that $\bigcap_i D^i = \emptyset$ would then not be enough to give proposition 4.4.3, but note that the situation of [Bho99] is not quite covered by our assumptions as Bhosle allows the highest parabolic weight α^i_2 to be 1: in our proof above, having $\alpha^i_{l_i} = 1$ would mean that we could no longer conclude (4.30) from equality in (4.29).

Note that the subcategory considered in the proposition is noetherian and artinian, and its simple objects are precisely the stable parabolic sheaves of par–$p_0 = P$, so any p_0-semistable parabolic sheaf admits a JH filtration.

However, we are really interested in p_2-(semi)stability as this turns out to be the condition matching GIT-(semi)stability in our construction in section 4.8. We will define the relevant moduli functor in section 4.9.

4.5 Boundedness of the semistable parabolic sheaves

Every moduli construction via GIT begins with a boundedness result: we need to know that the objects of our moduli problem may be parametrised by a scheme of finite type before we can attempt to construct the moduli space as a GIT quotient (see subsection 2.3.2 for definitions and basic results on boundedness and Castelnuovo-Mumford regularity).

As explained in section 4.1, Inaba only shows that the parabolic e-stable sheaves of fixed numerical type on X/S form a bounded collection (see section 2 of [Ina00]). Recall from section 4.1 that Inaba's set-up is more general than ours: the base S is only assumed to be noetherian, whereas we take S to be of finite type over an algebraically closed field $\Bbbk$. Under this assumption, we obtain a boundedness result for all semistable parabolic sheaves of fixed numerical type by reducing the problem to boundedness for the underlying pure sheaves: using proposition 4.3.7 we show that parabolic semistability of $\mathcal{E}_*^\star$ implies a bound on the (non-parabolic) slope of all non-zero subsheaves of $\mathcal{E}$, so the boundedness results of Simpson and Langer for pure sheaves of bounded μ-HN type (section 3.2.2) apply.

We combine proposition 4.3.7 with theorem 3.2.13 to obtain the boundedness result for semistable parabolic sheaves of given numerical type on X/S. Continue with the same hypotheses and notation as in section 4.4:

Definition 4.5.1. Fix rational polynomials H and H_j^i (for $1 \leqslant i \leqslant n$ and $2 \leqslant j \leqslant l_i + 1$), where $d := \deg H$ and $\deg H_j^i = d - 1$, and weights $\alpha_*^\star$. Then write $F(H, H_*^\star, \alpha_*^\star)$ for the collection of p_2-*semistable* parabolic sheaves $\mathcal{E}_*^\star$ *on the fibres* X_s of X/S with parabolic structures at the divisors $(D^i)_s$, such that $\mathcal{E}_*^\star$ has numerical type $(H, H_*^\star, \alpha_*^\star)$.

Theorem 4.5.2. Given numerical type $(H, H_*^\star, \alpha_*^\star)$, consider the collection of pure coherent sheaves $\mathcal{E}$ (on the fibres of X/S) admitting a parabolic structure such that $\mathcal{E}_*^\star \in F(H, H_*^\star, \alpha_*^\star)$. This collection is bounded.

Proof. Let $\mathcal{E}_*^\star \in F(H, H_*^\star, \alpha_*^\star)$ be a p_2-semistable parabolic sheaf on X_s, say. We aim to show that there is a constant $b \in \mathbb{R}$, independent of the choice of $\mathcal{E}_*^\star$ in $F(H, H_*^\star, \alpha_*^\star)$, such that $\mu(\mathcal{F}) \leqslant b$ for all non-zero subsheaves $\mathcal{F} \subset \mathcal{E}$. We may assume that $\mathcal{F}$ is saturated as $\mu(\mathcal{F}) \leqslant \mu(\mathcal{F}^{\mathrm{sat}})$ by lemma 3.2.3. Thus, we have the induced structure on $\mathcal{F}$ making it a parabolic subsheaf $\mathcal{F}_*^\star$ of $\mathcal{E}_*^\star$. Now $a_d(\mathcal{F}) \leqslant a_d(\mathcal{E})$, so by proposition 4.3.7 we have $b' \in \mathbb{Q}$ (only depending on d and $a := a_d(\mathcal{E})$, which are determined by H, and on the divisors $(D^i)_s$) such that

$$\begin{array}{rcccl} \mu(\mathcal{F}) - b' & \leqslant & \text{par-}\mu(\mathcal{F}_*^\star) & \leqslant & \mu(\mathcal{F}) \\ \text{and} \quad \mu(\mathcal{E}) - b' & \leqslant & \text{par-}\mu(\mathcal{E}_*^\star) & \leqslant & \mu(\mathcal{E}). \end{array}$$

The proof of proposition 4.3.7 shows us that b' only depends on the divisors $(D^i)_s$ through $\deg_{\mathcal{O}_{X_s}(1)}((D^i)_s)$, and these degrees are bounded above as s ranges over S: by flatness of D^i, the Hilbert polynomials of $(D^i)_s$ are locally constant in $s \in S$. Thus, b' may be chosen independent of $s \in S$.

Proposition 4.3.12 tells us that $\mathcal{E}^\star_*$ is μ-semistable, so par–$\mu(\mathcal{F}^\star_*) \leqslant$ par–$\mu(\mathcal{E}^\star_*)$. Thus,

$$\mu(\mathcal{F}) \leqslant \text{par–}\mu(\mathcal{F}^\star_*) + b' \leqslant \text{par–}\mu(\mathcal{E}^\star_*) + b' \leqslant \mu(\mathcal{E}) + b'$$

and the result now follows from theorem 3.2.13 with $b := \mu(\mathcal{E}) + b'$, noting that both $\mu(\mathcal{E})$ and b' are determined by H. □

4.6 Construction of the parameter space

It follows immediately from theorem 4.5.2 that for all $\mathcal{E}^\star_* \in F(H, H^\star_*, \alpha^\star_*)$ the underlying sheaf $\mathcal{E}$ (on X_s, say) is m-regular for a uniform integer m:

Corollary 4.6.1. Given a numerical type $(H, H^\star_*, \alpha^\star_*)$, there is $m \in \mathbb{N}$ such that for all $\mathcal{E}^\star_* \in F(H, H^\star_*, \alpha^\star_*)$

(a) $\mathcal{E}(m)$ is generated by its global sections;

(b) $H^k(\mathcal{E}(m)) = 0$ for all $k > 0$; and

(c) the multiplication maps $H^0(\mathcal{E}(m)) \otimes H^0(\mathcal{O}_{X_s}(l)) \to H^0(\mathcal{E}(m+l))$ are surjective for all $l \geqslant 0$.

Proof. Combine theorems 4.5.2 and 2.3.9 with lemma 2.3.7. □

Now (b) implies that $h^0(\mathcal{E}(m)) = \chi(\mathcal{E}(m)) = H(m)$, and thus by (a) each such $\mathcal{E}$ may be written as a quotient of $V_m \otimes \mathcal{O}_{X_s}(-m)$, where V_m is a fixed $\Bbbk$-vector space of rank $H(m)$ (from here on, $V_m \otimes -$ denotes $V_m \otimes_\Bbbk -$, i.e. $V_m \otimes \mathcal{O}_X$ is the trivial vector bundle on X with fibre V_m). In other words, for all $\mathcal{E}^\star_* \in F(H, H^\star_*, \alpha^\star_*)$ the underlying sheaves $\mathcal{E}$ are parametrised by the relative Quot scheme

$$\mathrm{Quot}^{\mathcal{O}_X(1)}_{X/S}(V_m \otimes \mathcal{O}_X(-m), H)$$

whose points correspond to triples $[s, \mathcal{F}, \beta]$ of a point $s \in S$ and an isomorphism class of a quotient $\beta : V_m \otimes \mathcal{O}_{X_s}(-m) \twoheadrightarrow \mathcal{F}$ on X_s having Hilbert polynomial H with respect to $\mathcal{O}_{X_s}(1)$. (Two quotients β and β' are considered isomorphic whenever $\ker \beta = \ker \beta'$.) Equivalently and sometimes more conveniently (this is Simpson's interpretation [Sim94a] of the Quot scheme), we may think of the points of this Quot scheme as corresponding to triples $[s, \mathcal{F}, \gamma]$ of a point $s \in S$, a coherent sheaf $\mathcal{F}$ on X_s having Hilbert polynomial H with respect to $\mathcal{O}_{X_s}(1)$, and a morphism $\gamma : V_m \to H^0(\mathcal{F}(m))$ whose image generates $\mathcal{F}(m)$, and two triples $[s, \mathcal{F}, \gamma]$ and $[s, \mathcal{F}', \gamma']$ are equivalent if there is an isomorphism $\mathcal{F} \cong \mathcal{F}'$ compatible with γ and γ'.

For reasons that will become clear in section 4.8 (see lemma 4.8.6 in particular), we will not work with the entire Quot scheme but with a union of some of its irreducible components: set

$$Q = Q_m := \mathrm{Quot}^{\mathcal{O}_X(1)}_{X/S}(V_m \otimes \mathcal{O}_X(-m), H, d) \subset \mathrm{Quot}^{\mathcal{O}_X(1)}_{X/S}(V_m \otimes \mathcal{O}_X(-m), H) \qquad (4.31)$$

to be the closure of the open subscheme corresponding to purely d-dimensional quotients. As Simpson states in [Sim94a], there may be a problem in the GIT analysis if the Quot scheme has irreducible components all of whose points correspond to sheaves which are not pure (it is not clear how to show such points are GIT-unstable – see lemma 4.8.6 for how this matters), so we discard these components (if they exist) now.

Note that this parameter space depends on a choice of m (and we write Q_m for Q whenever we wish to emphasise this point): as $\mathcal{E}$ is also m'-regular for all $m' \geqslant m$, we may replace m by any $m' \geqslant m$ to get a different Quot scheme $Q_{m'}$ which still parametrises all $\mathcal{E}$ with $\mathcal{E}^\star_* \in F(H, H^\star_*, \alpha^\star_*)$.

Recall the following basic properties of Quot schemes:

Lemma 4.6.2. Given $T \in \mathsf{Sch}/S$, a numerical polynomial $P \in \mathbb{Q}[x]$ and a sheaf $\mathcal{H} \in \mathsf{Coh}(X)$, there is an isomorphism

$$\mathrm{Quot}^{\mathcal{O}_{X_T}(1)}_{X_T/T}(\mathcal{H}_T, P) \cong T \times_S \mathrm{Quot}^{\mathcal{O}_X(1)}_{X/S}(\mathcal{H}, P)$$

induced by a natural isomorphism of the respective Quot functors. Given $P \in \mathbb{Q}[x]$ and a surjection of coherent sheaves $\mathcal{E} \twoheadrightarrow \mathcal{F}$ on X, the Quot scheme $\mathrm{Quot}^{\mathcal{O}_X(1)}_{X/S}(\mathcal{F}, P)$ is naturally a closed subscheme of $\mathrm{Quot}^{\mathcal{O}_X(1)}_{X/S}(\mathcal{E}, P)$, induced by a closed embedding of the Quot functors.

Proof. These statements follow easily from the definition of the Quot functor, see lemma 2.3.2 and lemma 2.3.3. □

Starting from the Quot scheme Q, we now build a scheme parametrising all $\mathcal{E}^\star_* \in F(H, H^\star_*, \alpha^\star_*)$, i.e. we incorporate the flag structures (our construction of the parameter space is similar to Inaba's argument, except that we avoid enforcing the various open conditions of [Ina00]: we seek a parameter space which is projective over S to ensure projectivity of the GIT quotient). Suppose $\mathcal{E}^\star_*$ is a parabolic sheaf on X_s and consider the quasi-parabolic structure at $(D^i)_s$:

$$\mathcal{E} = F^i_1(\mathcal{E}) \supset F^i_2(\mathcal{E}) \supset \cdots \supset F^i_{l_i+1}(\mathcal{E}) = \mathcal{E}(-(D^i)_s)$$

This is equivalent to a series of surjections

$$\mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{E}(-(D^i)_s) = \mathcal{E}/F^i_{l_i+1}(\mathcal{E}) \twoheadrightarrow \mathcal{E}/F^i_{l_i}(\mathcal{E}) \twoheadrightarrow \mathcal{E}/F^i_{l_i-1}(\mathcal{E}) \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{E}/F^i_2(\mathcal{E}) \qquad (4.32)$$

where the quotient $\mathcal{E}/F^i_j(\mathcal{E})$ has Hilbert polynomial H^i_j for $2 \leqslant j \leqslant l_i + 1$. Thus, we may parametrise $\mathcal{E}$ (as a quotient of $V_m \otimes \mathcal{O}_{X_s}(-m)$) and its quasi-parabolic structure at $(D^i)_s$ by a Flag scheme built out of a series of Quot schemes (cf. [HL97], section 2.A); for $2 \leqslant j \leqslant l_i + 1$ we define schemes S^i_j and X^i_j (projective over S^i_j) and a coherent S^i_j-flat sheaf $\mathcal{H}^i_j \in \mathsf{Coh}(X^i_j)$ inductively:

Base step: For $j = l_i+1$, add the surjection $\mathcal{E} \twoheadrightarrow \mathcal{E}/F^i_{l_i+1}(\mathcal{E})$ to the data parametrised by the scheme Q which already encodes the quotient $V_m \otimes \mathcal{O}_{X_s}(-m) \twoheadrightarrow \mathcal{E}$. Note that this step is not necessary from the point of view of parametrising the quasi-parabolic structure (4.32) as $F^i_{l_i+1}(\mathcal{E})$ is fixed to be $\mathcal{E}(-(D^i)_s)$. However, recall from definition 4.3.11 that the Hilbert polynomial $H^i_{l_i+1}$ contributes to par–$P_2(\mathcal{W}^\star_* \subset \mathcal{E}^\star_*)$ for any parabolic subsheaf $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$, and since $P(\mathcal{E})$ does not determine $P(\mathcal{E}(-(D^i)_s))$, we need to encode this quotient $\mathcal{E}/F^i_{l_i+1}(\mathcal{E})$ in our Flag scheme in order to measure its contribution to (semi)stability of $\mathcal{E}^\star_*$.

First, restrict Q to a closed subscheme whose points over $s \in S$ correspond to sheaves $\mathcal{E}$ on X_s satisfying $P(\mathcal{E}(-(D^i)_s)) \geqslant H - H^i_{l_i+1}$: let $\widetilde{\mathcal{E}}$ be the universal quotient sheaf on $X_Q := X \times_S Q$, and write $(D^i)_Q$ for the Q-relative effective Cartier divisor on X_Q induced by $D^i \subset X$. Let $\widetilde{\mathcal{E}}(-(D^i)_Q)$ be the image of the natural morphism $\widetilde{\mathcal{E}} \otimes \mathcal{O}_{X_Q}(-(D^i)_Q) \to \widetilde{\mathcal{E}}$. As $X_Q \to Q$ is projective and as the pull-back $\mathcal{O}_{X_Q}(1)$ of $\mathcal{O}_X(1)$ is very ample, there exists a flattening stratification for $\widetilde{\mathcal{E}}(-(D^i)_Q)$ over Q: by theorem 2.3.14, there are finitely many pairwise disjoint locally closed subschemes $Q_g \subset Q$ whose union is Q, such that $q \in Q_g$ if and only if the restriction of $\widetilde{\mathcal{E}}(-(D^i)_Q)$ to $(X_Q)_q$ has Hilbert polynomial g, i.e. if and only if q corresponds to a triple $[s, \mathcal{E}, \gamma]$ such that the sheaf $\mathcal{E}(-(D^i)_s)$ on X_s has Hilbert polynomial g with respect to $\mathcal{O}_{X_s}(1)$. The Q_g are uniquely determined (as schemes) by a universal property they satisfy: the pull-back of $\widetilde{\mathcal{E}}(-(D^i)_Q)$ via any S-morphism $T \to Q$ is flat if and only if the morphism factors through $\coprod_g Q_g \hookrightarrow Q$. Most importantly for us, for each polynomial h, the union of strata $\coprod_{g \geqslant h} Q_g$, is a closed subscheme of Q.

Let $h = H - H^i_{l_i+1}$, and set

$$\widetilde{Q} := (\coprod_{g \geqslant h} Q_g) \times_Q \mathrm{Quot}^{\mathcal{O}_{X_Q}(1)}_{X_Q/Q}(\widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}(-(D^i)_Q), H^i_{l_i+1}), \tag{4.33}$$

where Q is the Quot scheme (4.31) encoding $\mathcal{E}$ as a quotient sheaf, and $\widetilde{\mathcal{E}}$, X_Q and $(D^i)_Q$ are as in the previous paragraph. Equivalently, $\widetilde{Q}$ is the Quot scheme of $\widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}(-(D^i)_Q)$ on the restriction of $X_Q \to Q$ to the locus over the closed subscheme $\coprod_{g \geqslant h} Q_g \subset Q$. Therefore, $\widetilde{Q}$ is projective over S (this map is the composition of the projective maps $\widetilde{Q} \to \coprod_{g \geqslant h} Q_g$ and $\coprod_{g \geqslant h} Q_g \hookrightarrow Q \to S$). The points of $\widetilde{Q}$ parametrise all sequences of surjections

$$V_m \otimes \mathcal{O}_{X_s}(-m) \twoheadrightarrow \mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{E}(-(D^i)_s) \twoheadrightarrow \mathcal{E}/F^i_{l_i+1}(\mathcal{E}) \tag{4.34}$$

with $P(\mathcal{E}) = H$, $P(\mathcal{E}(-(D^i)_s)) \geqslant H - H^i_{l_i+1}$ and $P(\mathcal{E}/F^i_{l_i+1}(\mathcal{E})) = H^i_{l_i+1}$. But since $\mathcal{E}(-(D^i)_s) \subset F^i_{l_i+1}(\mathcal{E})$ we must have

$$P(\mathcal{E}(-(D^i)_s)) \leqslant P(F^i_{l_i+1}(\mathcal{E})) = H - H^i_{l_i+1},$$

so equality must hold and we actually have $\mathcal{E}(-(D^i)_s) = F^i_{l_i+1}(\mathcal{E})$. Note that $\mathcal{E}$ then also satisfies the support conditions (4.3): we have

$$\dim\Big((D^i)_s \,\cap\, \operatorname{Supp}\mathcal{E}\Big) = \dim\operatorname{Supp}\mathcal{E}/\mathcal{E}(-(D^i)_s) = \deg H^i_{l_i+1} < \deg H = \dim\operatorname{Supp}\mathcal{E}. \tag{4.35}$$

Now set

$$\begin{aligned} S^i_{l_i+1} &:= \widetilde{Q}, \\ X^i_{l_i+1} &:= (X_Q)_{S^i_{l_i+1}} = X_Q \times_Q S^i_{l_i+1} = X \times_S S^i_{l_i+1}, \\ \mathcal{H}^i_{l_i+1} &:= \text{the universal quotient sheaf on } X^i_{l_i+1}. \end{aligned}$$

Note that

$$V_m \otimes \mathcal{O}_{X_Q}(-m) \twoheadrightarrow \widetilde{\mathcal{E}} \twoheadrightarrow \widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}(-(D^i)_Q)$$

on X_Q, so by lemma 4.6.2 (and since $\coprod_{g\geqslant h} Q_g \subset Q$ is closed) we have a closed embedding

$$S^i_{l_i+1} = \widetilde{Q} \hookrightarrow \operatorname{Quot}^{\mathcal{O}_{X_Q}(1)}_{X_Q/Q}(V_m \otimes \mathcal{O}_{X_Q}(-m), H^i_{l_i+1}) \cong Q \times_S \operatorname{Quot}^{\mathcal{O}_X(1)}_{X/S}(V_m \otimes \mathcal{O}_X(-m), H^i_{l_i+1}).$$

Observe also that $\mathcal{H}^i_{l_i+1}$ is in particular a quotient of $V_m \otimes \mathcal{O}_{X^i_{l_i+1}}(-m)$ on $X^i_{l_i+1}$, and that its restriction to the fibre of $X^i_{l_i+1}$ over a point $s \in S^i_{l_i+1}$ corresponding to a sequence (4.34) is just $\mathcal{E}/F^i_{l_i+1}(\mathcal{E})$.

Inductive step: Given $2 < j \leqslant l_i + 1$, suppose $X^i_j \to S^i_j$ and $\mathcal{H}^i_j \in \mathsf{Coh}(X^i_j)$ have already been constructed, with S^i_j parametrising sequences of surjections

$$V_m \otimes \mathcal{O}_{X_s}(-m) \twoheadrightarrow \mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{E}(-(D^i)_s) = \mathcal{E}/F^i_{l_i+1}(\mathcal{E}) \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{E}/F^i_j(\mathcal{E}), \tag{4.36}$$

with $X^i_j = X \times_S S^i_j$, and with $\mathcal{H}^i_j$ a quotient of $V_m \otimes \mathcal{O}_{X^i_j}(-m)$. Furthermore, suppose that $\mathcal{H}^i_j$ is a universal sheaf whose restriction to the fibre of X^i_j over a point $s \in S^i_j$ corresponding to a sequence (4.36) is just $\mathcal{E}/F^i_j(\mathcal{E})$.

Then set

$$\begin{aligned} S^i_{j-1} &:= \operatorname{Quot}^{\mathcal{O}_{X^i_j}(1)}_{X^i_j/S^i_j}(\mathcal{H}^i_j, H^i_{j-1}), \\ X^i_{j-1} &:= X^i_j \times_{S^i_j} S^i_{j-1} = X \times_S S^i_{j-1}, \\ \mathcal{H}^i_{j-1} &:= \text{the universal quotient sheaf on } X^i_{j-1}. \end{aligned}$$

Now S^i_{j-1} parametrises sequences

$$V_m \otimes \mathcal{O}_{X_s}(-m) \twoheadrightarrow \mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{E}(-(D^i)_s) = \mathcal{E}/F^i_{l_i+1}(\mathcal{E}) \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{E}/F^i_{j-1}(\mathcal{E}),$$

and by lemma 4.6.2 we have a closed embedding

$$S^i_{j-1} \hookrightarrow \mathrm{Quot}^{\mathcal{O}_{X^i_j}(1)}_{X^i_j/S^i_j}(V_m \otimes \mathcal{O}_{X^i_j}(-m), H^i_{j-1}) \cong S^i_j \times_S \mathrm{Quot}^{\mathcal{O}_X(1)}_{X/S}(V_m \otimes \mathcal{O}_X(-m), H^i_{j-1}).$$

This completes the inductive step of the construction.

In particular, S^i_2 is a parameter space for sheaves $\mathcal{E}$ (as quotients of $V_m \otimes \mathcal{O}_{X_s}(-m)$) together with a quasi-parabolic structure (4.32) at $(D^i)_s$, and we have a closed embedding

$$S^i_2 \hookrightarrow Q \times_S Q^i_{l_i+1} \times_S \cdots \times_S Q^i_2, \tag{4.37}$$

where $Q^i_j := \mathrm{Quot}_{X/S}(V_m \otimes \mathcal{O}_X(-m), H^i_j)$.

Finally, we need to combine the S^i_2 (for $1 \leqslant i \leqslant n$) into a single space parametrising $\mathcal{E}$ and its quasi-parabolic structures at $(D^1)_s, \ldots, (D^n)_s$ simultaneously. Taking the naïve product of the S^i_2 cannot work as a point $(s^1, \ldots, s^n) \in S^1_2 \times \cdots \times S^n_2$ corresponds to n quasi-parabolic sheaves whose underlying coherent sheaves need not coincide. However, note that each S^i_2 comes with a projective morphism to Q (the composition of (4.37) with projection to Q) which in terms of the modular interpretation just corresponds to forgetting the quasi-parabolic structure at $(D^i)_s$ and only remembering the quotient $[V_m \otimes \mathcal{O}_{X_s}(-m) \twoheadrightarrow \mathcal{E}]$. Thus, we may take the fibred product of the S^i_2 over Q as our parameter space: set

$$F_m = \text{Flag–Quot}^{\mathcal{O}_X(1)}_{D^\star/X/S}(V_m \otimes \mathcal{O}_X(-m), H, H^\star_*) := S^1_2 \times_Q S^2_2 \times_Q \cdots \times_Q S^n_2, \tag{4.38}$$

where the subscript in F_m again denotes the dependence on the regularity parameter m. This scheme represents a rigidification of the moduli functor (4.26) we are interested in: more precisely, for any $T \in \mathsf{Sch}/S$, the T-valued points of F_m are in bijection with equivalence classes of pairs $[\mathcal{E}^\star_*, \gamma]$, where

- $\mathcal{E}^\star_*$ is a T-flat sheaf $\mathcal{E}$ on X_T (corresponding to a T-valued point in Q_m, the closure of the locus of points in the Quot scheme corresponding to pure sheaves) together with a filtration
$$\mathcal{E} = F^i_1(\mathcal{E}) \supset F^i_2(\mathcal{E}) \supset \cdots \supset F^i_{l_i+1}(\mathcal{E}) = \mathcal{E}(-(D^i)_T) \tag{4.39}$$
whose restriction to the fibre X_t over any $t \in T$ has Hilbert polynomial H and flag type $H^\star_*$ with respect to $\mathcal{O}_{X_t}(1)$, i.e. $P(\mathcal{E}_t) = H$ and $P((\mathcal{E}/F^i_j(\mathcal{E}))_t) = H^i_j$. This includes flat families of quasi-parabolic sheaves on X_T/T in the sense of definitions 4.4.1 and 4.2.1, but also other objects which do not qualify as quasi-parabolic sheaves as $\mathcal{E}_t$ may not be pure. We will show in section 4.8 that points of F_m not corresponding to pure quasi-parabolic sheaves are GIT-unstable with respect to a suitable linearisation.

- γ is a morphism $V_m \otimes \mathcal{O}_T \to (f_T)_*\mathcal{E}(m)$ whose image globally generates $\mathcal{E}(m)$, where $f_T : X_T \to T$ is the base change of $f : X \to S$. Again, section 4.8 will show that $(\mathcal{E}^\star_*, \gamma)$ is GIT-unstable whenever γ fails to be an isomorphism.

- $[\mathcal{E}^\star_*, \gamma]$ and $[\mathcal{F}^\star_*, \delta]$ are equivalent if there is an isomorphism $\mathcal{E}^\star_* \cong \mathcal{F}^\star_*$ compatible with γ and δ.

In particular, points of F_m lying over $s \in S$ correspond to $[\mathcal{E}^\star_*, \gamma]$ with $\mathcal{E}^\star_*$ a (not necessarily pure) quasi-parabolic sheaf (corresponding to a point in Q_m, the union of the pure components of the Quot scheme) of type $(H, H^\star_*)$ on X_s such that $\mathcal{E}^\star_*$ satisfies the support condition at each divisor by (4.35), and $\gamma : V_m \to H^0(X_s, \mathcal{E}(m))$ a morphism whose image generates $\mathcal{E}(m)$.

4.7 Group action, linearisation and GIT on Grassmannians

We are now ready to set up the GIT construction that will result in the coarse moduli space of semistable parabolic sheaves of type $(H, H^\star_*, \alpha^\star_*)$ on X/S: essentially, the problem is to remove the rigidification data of the morphism γ from F_m, and to do so we consider the natural $\mathrm{SL}(V_m)$-action on F_m factoring through the $\mathrm{SL}(V_m)$-action on V_m. (Note that the irreducible components of the Quot scheme which we selected in (4.31) are preserved by this group action: the components are determined by the sheaf $\mathcal{E}$, whereas $\mathrm{SL}(V_m)$ acts on the quotient map $V_m \otimes \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{E}$ only.) The linearisation for this action will be based on the projective embedding exhibited in Grothendieck's original construction [Gro62] of the Quot scheme and utilised to great effect in Simpson's construction of moduli of pure sheaves [Sim94a]: given projective $f : X \to S$ with very ample $\mathcal{O}_X(1)$, any $\mathcal{H} \in \mathsf{Coh}(X)$ and $P \in \mathbb{Q}[x]$, there are natural morphisms ψ_k from the Quot scheme to a relative Grassmannian (i.e. a Quot scheme of locally free quotients), and for sufficiently large $k \in \mathbb{N}$ these are closed embeddings. Explicitly, for any $T \in \mathsf{Sch}/S$, the morphism ψ_k is given on T-valued points by

$$\begin{array}{rccc} \psi_k : & \mathrm{Quot}^{\mathcal{O}_X(1)}_{X/S}(\mathcal{H}, P)(T) & \to & \mathrm{Grass}_S(f_*\mathcal{H}(k), P(k))(T) \\ & [\mathcal{H}_T \twoheadrightarrow \mathcal{F}] & \mapsto & [(f_T)_*\mathcal{H}_T(k) \twoheadrightarrow (f_T)_*\mathcal{F}(k)] \end{array}$$

and this is a closed embedding for $k \gg 0$: the sheaves $\mathcal{H}_t$ (for all $t \in T$) and all quotients $\mathcal{F}_t$ and kernels $\ker(\mathcal{H}_t \twoheadrightarrow \mathcal{F}_t)$ form a bounded family, so are k-regular for $k \gg 0$. Then $(f_T)_*\mathcal{F}(k)$ is locally free of rank $P(k)$, and the analogue of corollary 4.6.1(c) implies that ψ_k is an immersion: recall this construction from subsection 2.3.3.

Taking $T \to S$ to be the inclusion of a point $s \in S$, this says that ψ_k maps a quotient sheaf $[\mathcal{H}_s \twoheadrightarrow \mathcal{F}]$ to the vector space quotient $[H^0(\mathcal{H}_s(k)) \twoheadrightarrow H^0(\mathcal{F}(k))]$ lying in the Grassmannian $\mathrm{Grass}(H^0(\mathcal{H}_s(k)), P(k))$ which is the fibre of $\mathrm{Grass}_S(f_*\mathcal{H}(k), P(k))$ over $s \in S$.

If k is sufficiently large so that $f_*\mathcal{H}(k)$ is locally free, then we have the Plücker embedding

$$\mathrm{Grass}_S(f_*\mathcal{H}(k), P(k)) \hookrightarrow \mathbb{P}_S(\Lambda^{P(k)} f_*\mathcal{H}(k))$$

as a closed subscheme, and the corresponding ample line bundle on the relative Grassmannian is the determinant line bundle. Pulling back the determinant line bundle by ψ_k, we obtain a very ample line bundle L_k on $\mathrm{Quot}_{X/S}^{\mathcal{O}_X(1)}(\mathcal{H}, P)$ whose fibre at a point corresponding to a quotient $[\mathcal{H}_s \twoheadrightarrow \mathcal{F}]$ is $\Lambda^{P(k)} H^0(X_s, \mathcal{F}(k))$. In our case $\mathcal{H} = V_m \otimes \mathcal{O}_X(-m)$, so $\mathrm{SL}(V_m)$ acts on L_k as it is the pull-back of

$$\mathcal{O}_{\mathbb{P}_S(\Lambda^{P(k)} V_m \otimes f_*\mathcal{O}(k-m))}(1)$$

and the action is compatible with the $\mathrm{SL}(V_m)$-equivariant embedding ψ_k. This is the $\mathrm{SL}(V_m)$-linearisation for the Quot scheme which Simpson works with in [Sim94a] and which forms the basis for our linearisation of the $\mathrm{SL}(V_m)$-action on the Flag-Quot scheme F_m.

By (4.37) and (4.38), we have a closed embedding of F_m into a fibred product of Quot schemes on X/S (all of which depend on m):

$$F_m \hookrightarrow Q \times_S (Q^1_{l_1+1} \times_S \cdots \times_S Q^1_2) \times_S \cdots \times_S (Q^n_{l_n+1} \times_S \cdots \times_S Q^n_2). \tag{4.40}$$

If we choose k sufficiently large (depending on m) to make ψ_k a closed embedding for Q and all Q^i_j, we arrive at a closed embedding of the Flag scheme into a product of relative Grassmannians which we also denote by ψ_k:

$$\psi_k : F_m \hookrightarrow \mathrm{Gr}_S(m, k),$$

where

$$\begin{aligned} \mathrm{Gr}_S(m,k) &:= \mathrm{Gr} \times_S (\mathrm{Gr}^1_{l_1+1} \times_S \cdots \times_S \mathrm{Gr}^1_2) \times_S \cdots \times_S (\mathrm{Gr}^n_{l_n+1} \times_S \cdots \times_S \mathrm{Gr}^n_2), \qquad (4.41)\\ \mathrm{Gr} &:= \mathrm{Grass}_S(V_m \otimes f_*\mathcal{O}_X(k-m), H(k)),\\ \mathrm{Gr}^i_j &:= \mathrm{Grass}_S(V_m \otimes f_*\mathcal{O}_X(k-m), H^i_j(k)), \end{aligned}$$

and f is our projective morphism $X \to S$. Explicitly, a point of F_m (in the fibre over $s \in S$) corresponding to

$$\left[\Big(V_m \otimes \mathcal{O}_{X_s}(-m) \twoheadrightarrow \mathcal{E}\Big), \Big(V_m \otimes \mathcal{O}_{X_s}(-m) \twoheadrightarrow \mathcal{E}/F^i_j(\mathcal{E})\Big)_{i,j}\right] \tag{4.42}$$

is mapped by ψ_k to

$$\begin{aligned} \Big[\Big(V_m \otimes H^0(\mathcal{O}_{X_s}(k-m)) &\twoheadrightarrow H^0(\mathcal{E}(k))\Big),\\ &\Big(V_m \otimes H^0(\mathcal{O}_{X_s}(k-m)) \twoheadrightarrow H^0(\mathcal{E}/F^i_j(\mathcal{E})(k))\Big)_{i,j}\Big], \end{aligned} \tag{4.43}$$

and ψ_k is clearly $\mathrm{SL}(V_m)$-equivariant.

Remark 4.7.1. This differs from the embedding and linearisation chosen in [Ina00] where ψ_k (for $k \gg 0$) is only applied to the Q-factor in (4.40), while ψ_m works for the open subschemes of Q^i_j into which Inaba's flag scheme maps under (4.40): this is because of the open conditions (see [Ina00], p. 125) enforced in the construction of the parameter space (such as vanishing of the higher cohomology of $\mathcal{E}/F^i_j(\mathcal{E})(m)$). As a result, Inaba's flag scheme can be embedded in

$$\mathrm{Gr} \times_S (\mathrm{Gr}^1_{l_1+1}(m) \times_S \cdots \times_S \mathrm{Gr}^1_2(m)) \times_S \cdots \times_S (\mathrm{Gr}^n_{l_n+1}(m) \times_S \cdots \times_S \mathrm{Gr}^n_2(m)),$$

where

$$\begin{aligned} \mathrm{Gr} &:= \mathrm{Grass}_S(V_m \otimes f_*\mathcal{O}_X(k-m), H(k)), \\ \mathrm{Gr}^i_j(m) &:= \mathrm{Grass}_S(V_m \otimes \mathcal{O}_S, H^i_j(m)), \end{aligned}$$

using our notational conventions which differ slightly from [Ina00]. In particular, note that a point (4.42) of F_m would be mapped to

$$\left[\Big(V_m \otimes H^0(\mathcal{O}_{X_s}(k-m)) \twoheadrightarrow H^0(\mathcal{E}(k))\Big), \Big(V_m \otimes H^0(\mathcal{O}_{X_s}) \twoheadrightarrow H^0(\mathcal{E}/F^i_j(\mathcal{E})(m))\Big)_{i,j}\right].$$

Note that in this expression the parabolic structures are twisted by m, rather than by k as in (4.43). This explains the difference between Inaba's criterion for GIT-(semi)stability (proposition 3.3 in [Ina00]) and our criterion below and the simpler linearisation weights used by Inaba. Ultimately, this difference between the embeddings of the parameter space leads to the different parabolic stability condition we arrive at. While one might regard Inaba's stability (which is parabolic p_0-stability) as simpler than our notion of p_2-stability, we need to work with this more complicated notion – this is the price we pay for compactifying the moduli space: for this purpose, it is essential to keep working with a *projective* parameter space, embedded as a *closed* subscheme of Grassmannians.

Returning to our situation, we now describe the linearisation of the group action. As ψ_k embeds F_m in a product $\mathrm{Gr}_S(m,k)$ of Grassmannians, there is a range of $\mathrm{SL}(V_m)$-linearisations corresponding to different relative weightings of the factors in (4.41). Write $\mathcal{O}_{\mathrm{Gr}}(1)$ and $\mathcal{O}_{\mathrm{Gr}^i_j}(1)$ for the determinant line bundles on the factors of $\mathrm{Gr}_S(m,k)$. Given positive rational numbers β and β^i_j (which we occasionally refer to as the *linearisation weights*), let

$$L_{\beta,\beta^i_j} := \mathcal{O}_{\mathrm{Gr}}(\beta) \otimes \bigotimes_{i=1}^{n} \bigotimes_{j=2}^{l_i+1} \mathcal{O}_{\mathrm{Gr}^i_j}(\beta^i_j), \tag{4.44}$$

a $\mathbb{Q}$-invertible sheaf on $\mathrm{Gr}_S(m,k)$. Then for $a \gg 0$, we have a very ample line bundle $L^{\otimes a}_{\beta,\beta^i_j}$ linearising the $\mathrm{SL}(V_m)$-action on $\mathrm{Gr}_S(m,k)$, and $\psi_k^* L^{\otimes a}_{\beta,\beta^i_j}$ is our $\mathrm{SL}(V_m)$-linearisation on F_m

(for suitable weights). By a mild abuse of language, we will refer to L_{β,β^i_j} and $\psi_k^* L_{\beta,\beta^i_j}$ as the respective linearisations since the choice of a does not affect GIT-(semi)stability.

Before we state the numerical criterion for the $\mathrm{SL}(V_m)$-action on the product of Grassmannians, we recall that GIT-(semi)stability on F_m and $\mathrm{Gr}_S(m,k)$ may be analysed fibrewise over S: this is lemma 2.2.3. Thus, as $\mathrm{SL}(V_m)$ preserves the fibres of F_m and $\mathrm{Gr}_S(m,k)$ over S, we may assume that $S = \mathrm{Spec}\,\Bbbk$ when determining the GIT-(semi)stable loci, and we will do so throughout the following section. Then $\mathrm{Gr}_S(m,k)$ reduces to

$$\begin{aligned}\mathrm{Gr}(m,k) :=& \mathrm{Grass}(V_m \otimes H^0(\mathcal{O}_X(k-m)), H(k)) \\ &\times_{\Bbbk} \prod_{i=1}^{n} \prod_{j=2}^{l_i+1} \mathrm{Grass}(V_m \otimes H^0(\mathcal{O}_X(k-m)), H^i_j(k)),\end{aligned} \tag{4.45}$$

a product of vector space Grassmannians over $\Bbbk$.

Proposition 4.7.2. A point of $\mathrm{Gr}(m,k)$ corresponding to

$$\left[\Big(V_m \otimes H^0(\mathcal{O}_X(k-m)) \twoheadrightarrow E\Big), \Big(V_m \otimes H^0(\mathcal{O}_X(k-m)) \twoheadrightarrow E^i_j\Big)^{1\leqslant i\leqslant n}_{2\leqslant j\leqslant l_i+1}\right]$$

is GIT-(semi)stable with respect to the $\mathrm{SL}(V_m)$-linearisation L_{β,β^i_j} given by (4.44) if and only if for all non-zero proper subspaces $U < V_m$ we have

$$\Big(\beta H(k) + \sum_{i=1}^{n}\sum_{j=2}^{l_i+1} \beta^i_j H^i_j(k)\Big) \cdot \dim U (\leqslant) H(m) \cdot \Big(\beta \dim W + \sum_{i=1}^{n}\sum_{j=2}^{l_i+1} \beta^i_j \dim W^i_j\Big), \tag{4.46}$$

where W and W^i_j denote the image of $U \otimes H^0(\mathcal{O}_X(k-m))$ in the quotient E and E^i_j, respectively.

Proof. This is the Hilbert-Mumford criterion for products of Grassmannians: proposition 3.2 in [Ina00] proves this for integral linearisation weights. Our $\mathrm{SL}(V_m)$-linearisation is given by $L^{\otimes a}$ for $a \gg 0$ which has linearisation weights $a\beta$ and $a\beta^i_j$, so dividing all terms by a gives the statement above. The result also follows from proposition 2.2.6 (the version for a single Grassmannian, taking into account that V_m was defined as a $\Bbbk$-vector space of dimension $H(m)$), combined with the fact that the numerical stability function $\mu^\bullet(x,\lambda) : \mathrm{Pic}^{\mathrm{SL}(V_m)}(\mathrm{Gr}(m,k)) \to \mathbb{Z}$ is a group homomorphism when considered as a function in $\mathrm{SL}(V_m)$-linearised line bundles (this is property ii) after definition 2.2 in [MFK94]), i.e. for any point $x \in \mathrm{Gr}(m,k)$ and any one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{SL}(V_m)$ we have

$$\mu^{L_{\beta,\beta^i_j}}(x,\lambda) = \beta\mu^{\mathcal{O}_{\mathrm{Gr}}(1)}(x,\lambda) + \sum_{i,j} \beta^i_j \mu^{\mathcal{O}_{\mathrm{Gr}^i_j}(1)}(x,\lambda). \qquad \square$$

Recall from proposition 2.2.2 that a point in F_m is $\mathrm{SL}(V_m)$-(semi)stable with respect to $\psi_k^* L_{\beta,\beta^i_j}$ if and only if its image under ψ_k is $\mathrm{SL}(V_m)$-semistable in $\mathrm{Gr}(m,k)$ with respect to L_{β,β^i_j}. Thus, proposition 4.7.2 applies to the analysis of GIT-(semi)stability in F_m.

4.8 The GIT-(semi)stable loci

Using the numerical stability criterion in proposition 4.7.2, we now need to decide what linearisation weights β, β^i_j we should choose in (4.44). Let us try to motivate the choice of weights we make: suppose the point $[E, (E^i_j)]$ of $\mathrm{Gr}(m,k)$ considered in proposition 4.7.2 lies in the image of $\psi_k : F_m \hookrightarrow \mathrm{Gr}(m,k)$, i.e. there is a point of F_m corresponding to a (not necessarily pure) parabolic sheaf $\mathcal{E}^\star_*$ together with a morphism $\gamma : V_m \to H^0(\mathcal{E}(m))$, such that $E = H^0(\mathcal{E}(k))$ and $E^i_j = H^0(\mathcal{E}/F^i_j(\mathcal{E})(k))$. In this situation, note that any subspace $U < V_m$ considered in the GIT-stability criterion (4.46) induces a parabolic subsheaf

$$\mathcal{W}^\star_* := \mathrm{im}\,(U \otimes \mathcal{O}_X(-m) \to \mathcal{E}^\star_*)$$

of $\mathcal{E}^\star_*$ (we will describe this in more detail in lemma 4.8.2). If k is sufficiently large (depending on m), then $\dim W = h^0(\mathcal{W}(k)) = \chi(\mathcal{W}(k))$ and $\dim W^i_j = h^0(\mathcal{W}/F^i_j(\mathcal{W})(k)) = \chi(\mathcal{W}/F^i_j(\mathcal{W})(k))$, where W, W^i_j are as in (4.46). If furthermore γ is an isomorphism, then $\dim U = h^0(\mathcal{W}(m))$. Thus, the numerical criterion (4.46) for GIT-(semi)stability with respect to $\psi_k^* L_{\beta,\beta^i_j}$ becomes

$$\Big(\beta H(k) + \sum_{i,j} \beta^i_j H^i_j(k)\Big) h^0(\mathcal{W}(m)) (\leqslant) H(m) \cdot \Big(\beta\chi(\mathcal{W}(k)) + \sum_{i,j} \beta^i_j \chi(\mathcal{W}/F^i_j(\mathcal{W})(k))\Big) \quad (4.47)$$

for sufficiently large k and all $[\mathcal{E}^\star_*, \gamma] \in F_m$ with γ an isomorphism. Compare this with the parabolic Hilbert polynomial

$$P := \mathrm{par}\text{–}P_0(\mathcal{E}^\star_*) = \mathrm{par}\text{–}P_1(\mathcal{E}^\star_* \subset \mathcal{E}^\star_*) = \mathrm{par}\text{–}P_2(\mathcal{E}^\star_* \subset \mathcal{E}^\star_*)$$

calculated with respect to the parabolic weights $\alpha^\star_*$, which by by lemma 4.3.8 equals

$$P = H - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=2}^{l_i+1} \epsilon^i_j H^i_j \quad (4.48)$$

where $\epsilon^i_j := \alpha^i_j - \alpha^i_{j-1}$ and $H = P(\mathcal{E})$, $H^i_j = P(\mathcal{E}/F^i_j(\mathcal{E}))$. Remembering that we must choose β and β^i_j to be *positive* rational numbers (possibly depending on m and k) to obtain a very ample linearisation (which prevents us from choosing $\beta^i_j = -\epsilon^i_j/n$, for example), a comparison of (4.47) and (4.48) now suggests that we take

$$\begin{aligned} \beta &= \frac{P(m)}{H(k)}, \\ \beta^i_j &= \frac{\epsilon^i_j}{n} \frac{H^i_j(m)}{H^i_j(k)}, \end{aligned} \quad (4.49)$$

which gives us a linearisation (4.44), provided all ϵ^i_j are rational, and because $\alpha^i_{l_i+1} := 1$ this is equivalent to all parabolic weights $\alpha^\star_*$ being rational. (There is no loss of generality in this assumption: a variation of parabolic stability argument as in [MS80] shows that the walls and chambers in the space of $\mathrm{SL}(V_m)$-linearisations are defined over $\mathbb{Q}$, and by a corresponding variation-of-GIT argument [Tha96] the weights $\alpha_*^\star$ may be moved to rational values without altering the (semi)stable objects.)

With these linearisation weights, (4.47) now reads

$$\begin{aligned}&\Big(P(m)+\frac{1}{n}\sum_{i,j}\epsilon_j^i H_j^i(m)\Big)h^0(\mathcal{W}(m))\\ &\qquad(\leqslant) H(m)\cdot\Big(\frac{P(m)}{H(k)}\chi(\mathcal{W}(k))+\sum_{i,j}\frac{\epsilon_j^i}{n}\frac{H_j^i(m)}{H_j^i(k)}\chi(\mathcal{W}/F_j^i(\mathcal{W})(k))\Big)\end{aligned} \tag{4.50}$$

for sufficiently large k and for all $[\mathcal{E}_*^\star,\gamma]\in F_m$ with γ an isomorphism. But the bracket-term on the left-hand side is $H(m)$ by (4.48), giving

$$h^0(\mathcal{W}(m))-\frac{1}{n}\sum_{i,j}\epsilon_j^i\frac{H_j^i(m)}{H_j^i(k)}\chi(\mathcal{W}/F_j^i(\mathcal{W})(k))(\leqslant)\frac{P(m)}{H(k)}\chi(\mathcal{W}(k)) \tag{4.51}$$

as GIT-(semi)stability criterion, and this in fact motivates our definition 4.3.11 of p_2-(semi)stability.

After this heuristic argument, we now formally identify the GIT-(semi)stable loci on F_m for the $\mathrm{SL}(V_m)$-action with the linearisation above. Our goal is the following result:

Theorem 4.8.1. Fix a numerical type $(H,H_*^\star,\alpha_*^\star)$ with all parabolic weights $\alpha_*^\star$ rational numbers. Then there exists $M\in\mathbb{N}$ such that for all $m\geqslant M$ there is $K(m)$ so that for all $k\geqslant K$, a point $[\mathcal{E}_*^\star,\gamma]$ of the Flag-Quot scheme F_m is $\mathrm{SL}(V_m)$-(semi)stable with respect to the linearisation $\psi_k^* L_{\beta,\beta_j^i}$ and linearisation weights (4.49) if and only if

(a) $\mathcal{E}_*^\star$ is a p_2-(semi)stable parabolic sheaf (in particular $\mathcal{E}$ is pure), and

(b) $\gamma: V_m\to H^0(\mathcal{E}(m))$ is an isomorphism.

The key ingredients for the proof are proposition 4.7.2 and theorem 3.2.14: the strategy is similar to Simpson's identification of the the GIT-(semi)stable loci on the Quot scheme in his construction [Sim94a] of moduli for pure coherent sheaves on a projective scheme, and we adapt his arguments to the parabolic situation.

Note that we will continue to simplify notation by assuming that $S=\operatorname{Spec}\Bbbk$ when analysing GIT-(semi)stability: as explained before proposition 4.7.2, the GIT-(semi)stable loci of F_m are compatible with restricting to geometric fibres over S. The constants M and $K(m)$ chosen in the proof and the auxiliary lemmas are easily seen to be independent of which fibre over S we deal with: the required results, such as the closed embedding

$\psi_k : F_m \hookrightarrow \mathrm{Gr}_S(m,k)$ and k-regularity of certain bounded collections of sub- and quotient sheaves, work uniformly on S.

Throughout this section, write P for the parabolic Hilbert polynomial determined by $(H, H^\star_*, \alpha^\star_*)$ as in (4.48), and given $m \in \mathbb{N}$ we always take $k(m) \in \mathbb{N}$ to be sufficiently large so that $\psi_k : F_m \hookrightarrow \mathrm{Gr}(m,k)$ is a closed embedding. We begin by interpreting proposition 4.7.2 if our weights are chosen as in (4.49).

Lemma 4.8.2. Given $m \in \mathbb{N}$, there exists $K_1(m) \in \mathbb{N}$ such that for all $k \geqslant K_1$ we have the following criterion for GIT-(semi)stability: consider a point of F_m corresponding to

$$\left[\Big(V_m \otimes \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{E}\Big), \Big(V_m \otimes \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{E}/F^i_j(\mathcal{E})\Big)^{1\leqslant i \leqslant n}_{2\leqslant j \leqslant l_i+1}\right],$$

and for any subspace $U \leqslant V_m$ write $\mathcal{W}^\star_*$ for the (not necessarily pure) parabolic subsheaf of $\mathcal{E}^\star_*$ generated by U, i.e. let $\mathcal{W}$ be the image of $U \otimes \mathcal{O}_X(-m)$ in $\mathcal{E}$, and let $\mathcal{W}/F^i_j(\mathcal{W})$ be the image of $U \otimes \mathcal{O}_X(-m)$ in $\mathcal{E}/F^i_j(\mathcal{E})$. Then the point of F_m corresponding to $\mathcal{E}^\star_*$ is $\mathrm{SL}(V_m)$-(semi)stable with respect to the linearisation $\psi_k^* L_{\beta,\beta^i_j}$ with weights β, β^i_j as in (4.49) if and only if for all non-zero proper subspaces $U < V_m$ we have

$$\dim U (\leqslant) P(m) \frac{\chi(\mathcal{W}(k))}{H(k)} + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=2}^{l_i+1} \epsilon^i_j H^i_j(m) \frac{\chi(\mathcal{W}/F^i_j(\mathcal{W})(k))}{H^i_j(k)}. \tag{4.52}$$

Proof. This follows from proposition 4.7.2 in the same way as lemma 1.15 and 1.16 in [Sim94a] follow from the Hilbert-Mumford criterion. We have given most of the argument in our heuristic discussion of how the linearisation weights are chosen at the start of the section, but we give the full argument here for the sake of completeness.

For all points $\mathcal{E}^\star_*$ of F_m and all $U \leqslant V_m$, the sheaves $\mathcal{W}$ and $\mathcal{W}/F^i_j(\mathcal{W})$ belong to a bounded family (for fixed m): they may be parametrised by $F_m \times \mathrm{Grass}(V_m^*)$, a $\Bbbk$-scheme of finite type, where $\mathrm{Grass}(V_m^*)$ is the total Grassmannian of all subspaces of V_m. Similarly, the kernels

$$\begin{aligned} \mathcal{K} &\hookrightarrow U \otimes \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{W} \\ \mathcal{K}^i_j &\hookrightarrow U \otimes \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{W}/F^i_j(\mathcal{W}) \end{aligned} \tag{4.53}$$

belong to a bounded family, so we may choose $K_1(m)$ such that all $\mathcal{W}, \mathcal{W}/F^i_j(\mathcal{W}), \mathcal{K}$ and $\mathcal{K}^i_j$ are K_1-regular. In particular, for all $k \geqslant K_1$

$$\begin{aligned} h^0(\mathcal{W}(k)) &= \chi(\mathcal{W}(k)), \\ h^0(\mathcal{W}/F^i_j(\mathcal{W})(k)) &= \chi(\mathcal{W}/F^i_j(\mathcal{W})(k)) \end{aligned} \tag{4.54}$$

and

$$h^1(\mathcal{K}(k)) = h^1(\mathcal{K}^i_j(k)) = 0.$$

Now write

$$W := \operatorname{im}\Big(U \otimes H^0(\mathcal{O}_X(k-m)) \to H^0(\mathcal{W}(k)) \subset H^0(\mathcal{E}(k))\Big),$$
$$W_j^i := \operatorname{im}\Big(U \otimes H^0(\mathcal{O}_X(k-m)) \to H^0(\mathcal{W}/F_j^i(\mathcal{W})(k)) \subset H^0(\mathcal{E}/F_j^i(\mathcal{E})(k))\Big).$$

Twisting the short exact sequences (4.53) by $\mathcal{O}_X(k)$ and taking cohomology yields

$$U \otimes H^0(\mathcal{O}_X(k-m)) \to H^0(\mathcal{W}(k)) \to H^1(\mathcal{K}(k)) = 0$$
$$\text{and}\quad U \otimes H^0(\mathcal{O}_X(k-m)) \to H^0(\mathcal{W}/F_j^i(\mathcal{W})(k)) \to H^1(\mathcal{K}_j^i(k)) = 0,$$

so $\dim W = h^0(\mathcal{W}(k))$ and $\dim W_j^i = h^0(\mathcal{W}/F_j^i(\mathcal{W})(k))$. Using this together with (4.54), the GIT-stability criterion of proposition 4.7.2 with weights (4.49) becomes

$$\Big(P(m) + \frac{1}{n}\sum_{i,j}\epsilon_j^i H_j^i(m)\Big)\cdot \dim U$$
$$(\leqslant) H(m)\cdot\Big(\frac{P(m)}{H(k)}\chi(\mathcal{W}(k)) + \sum_{i,j}\frac{\epsilon_j^i}{n}\frac{H_j^i(m)}{H_j^i(k)}\chi(\mathcal{W}/F_j^i(\mathcal{W})(k))\Big)$$

and the bracketed term on the left-hand side of this inequality is $H(m)$ by (4.48), giving our result. □

Remark 4.8.3. Note that $\mathcal{E}$ is not assumed to be pure in lemma 4.8.2, i.e. $\mathcal{E}_*^\star$ may not be a parabolic sheaf in the strict sense. However, if $\mathcal{E}$ is pure, then clearly all non-zero subsheaves $\mathcal{W} \subset \mathcal{E}$ are actually pure themselves, and therefore all $\mathcal{W}_*^\star$ in the statement of the lemma are honest parabolic subsheaves of $\mathcal{E}_*^\star$.

Furthermore, all $\mathcal{W}_*^\star \subset \mathcal{E}_*^\star$ generated by subspaces $U < V_m$ have the *induced* parabolic structure: $\mathcal{W}$ and $\mathcal{W}/F_j^i(\mathcal{W})$ are defined by the following diagram with commutative squares:

$$\begin{array}{ccccc} V_m \otimes \mathcal{O}_X(-m) & \twoheadrightarrow & \mathcal{E} & \twoheadrightarrow & \mathcal{E}/F_j^i(\mathcal{E}) \\ \uparrow & & \uparrow & & \uparrow \\ U \otimes \mathcal{O}_X(-m) & \twoheadrightarrow & \mathcal{W} & \twoheadrightarrow & \mathcal{W}/F_j^i(\mathcal{W}). \end{array}$$

Thus,

$$F_j^i(\mathcal{W}) = \ker(\mathcal{W} \to \mathcal{E}/F_j^i(\mathcal{E})) = \mathcal{W} \cap F_j^i(\mathcal{E}) \tag{4.55}$$

and $\mathcal{W}/F_j^i(\mathcal{W}) \hookrightarrow \mathcal{E}/F_j^i(\mathcal{E})$, so

$$\chi(\mathcal{W}/F_j^i(\mathcal{W})(k)) \leqslant \chi(\mathcal{E}/F_j^i(\mathcal{E})(k)) = H_j^i(k) \tag{4.56}$$

for all sufficiently large k.

The result of the previous lemma has an important consequence for any $[\mathcal{E}_*^\star, \gamma]$ corresponding to a GIT-semistable point of F_m:

Lemma 4.8.4. Given $m \in \mathbb{N}$, there exists $K_3(m) \in \mathbb{N}$ such that for all $k \geqslant K_3$ we have the following consequence of GIT-semistability: if a (not necessarily pure) parabolic sheaf $\mathcal{E}_*^\star$, given as a quotient of $V_m \otimes \mathcal{O}_X(-m)$, corresponds to a point of F_m which is $\mathrm{SL}(V_m)$-semistable with respect to the linearisation $\psi_k^* L_{\beta,\beta_j^i}$ with weights β, β_j^i as in (4.49), then for all non-zero subspaces $U \leqslant V_m$ we have

$$\dim U \leqslant P(m)\frac{a_d(\mathcal{W})}{a_d(\mathcal{E})} + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=2}^{l_i+1} \epsilon_j^i H_j^i(m)\frac{a_{d-1}(\mathcal{W}/F_j^i(\mathcal{W}))}{a_{d-1}(H_j^i)}, \tag{4.57}$$

where $\mathcal{W}$ is the subsheaf of $\mathcal{E}$ generated by $U \otimes \mathcal{O}_X(-m)$.

Under the same assumption of GIT-semistability for $[\mathcal{E}_*^\star, \gamma] \in F_m$ and $k \geqslant K_3$, consider any parabolic quotient sheaf $\mathcal{E}_*^\star \twoheadrightarrow \mathcal{G}_*^\star$ with induced parabolic structure. Then we have

$$\dim J \geqslant P(m)\frac{a_d(\mathcal{G})}{a_d(\mathcal{E})} + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=2}^{l_i+1} \epsilon_j^i H_j^i(m)\frac{a_{d-1}(\mathcal{G}/F_j^i(\mathcal{G}))}{a_{d-1}(H_j^i)}, \tag{4.58}$$

where J is the image of the map $V_m \to H^0(\mathcal{G}(m))$.

In particular, taking $\mathcal{G} = \mathcal{E}$ shows that $\gamma : V_m \to H^0(\mathcal{E}(m))$ is injective if $[\mathcal{E}_*^\star, \gamma]$ is GIT-semistable with respect to $\psi_k^* L_{\beta,\beta_j^i}$ for $k \geqslant K_3$.

Proof. This corresponds to lemma 1.16 in [Sim94a] and follows from the Hilbert-Mumford criterion in the same way: suppose (4.57) fails for a proper subspace $U < V_m$, i.e.

$$\dim U > P(m)\frac{a_d(\mathcal{W})}{a_d(\mathcal{E})} + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=2}^{l_i+1} \epsilon_j^i H_j^i(m)\frac{a_{d-1}(\mathcal{W}/F_j^i(\mathcal{W}))}{a_{d-1}(H_j^i)}.$$

Suppose $\dim \mathcal{W} = d$. Then $a_d(\mathcal{E})/d!$ and $a_d(\mathcal{W})/d!$ are the leading coefficients of H and $P(\mathcal{W})$, respectively, while $a_{d-1}(\mathcal{W}/F_j^i(\mathcal{W}))/(d-1)!$ and $a_{d-1}(H_j^i)/(d-1)!$ are the leading coefficients of $P(\mathcal{W}/F_j^i(\mathcal{W}))$ and H_j^i, respectively. Thus, there is $K(m) \in \mathbb{N}$ such that for all $k \geqslant K$

$$\dim U > P(m)\frac{\chi(\mathcal{W}(k))}{H(k)} + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=2}^{l_i+1} \epsilon_j^i H_j^i(m)\frac{\chi(\mathcal{W}/F_j^i(\mathcal{W})(k))}{H_j^i(k)}. \tag{4.59}$$

If $\dim \mathcal{W} < d$, then $\deg P(\mathcal{W}) < \deg H = d$ and we obtain the same conclusion (4.59).

As U and $\mathcal{W}_*^\star$ range over bounded collections (parametrised by $\mathrm{Grass}(V_m^*)$ and $F_m \times \mathrm{Grass}(V_m^*)$, respectively), there is an integer $K(m) \in \mathbb{N}$ such that for all $U < V_m$ violating (4.57), the statement (4.59) holds. Take $K_3(m)$ to be the maximum of $K(m)$ and $K_1(m)$ as given by lemma 4.8.2: then (4.59) shows that the point $[\mathcal{E}_*^\star, \gamma] \in F_m$ is GIT-unstable for all $k \geqslant K_3$, by lemma 4.8.2. This concludes the proof of the first statement.

For the second statement of the lemma, consider a the quotient $\mathcal{E}^\star_* \twoheadrightarrow \mathcal{G}^\star_*$ with induced parabolic structure. Let $U \leqslant V_m$ be the kernel of the composition $V_m \to H^0(\mathcal{E}(m)) \to H^0(\mathcal{G}(m))$, and let J be the image. Then

$$H(m) = \dim V_m = \dim U + \dim J,$$

and if $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$ is the subsheaf generated by $U \otimes \mathcal{O}_X(-m)$, then the composition

$$\mathcal{W} \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{G}$$

is zero, so $P(\mathcal{W}) \leqslant H - P(\mathcal{G})$ and in particular

$$a_d(\mathcal{W}) \leqslant a_d(\mathcal{E}) - a_d(\mathcal{G}). \tag{4.60}$$

Furthermore, by remark 4.8.3, $\mathcal{W}^\star_*$ has the induced parabolic structure, so $\mathcal{W}/F^i_j(\mathcal{W})$ injects into $\mathcal{E}/F^i_j(\mathcal{E})$. As $\mathcal{G}^\star_*$ has the induced quotient structure, we have well-defined maps

$$\mathcal{W}/F^i_j(\mathcal{W}) \hookrightarrow \mathcal{E}/F^i_j(\mathcal{E}) \twoheadrightarrow \mathcal{G}/F^i_j(\mathcal{G})$$

whose composition is also zero, so

$$a_{d-1}(\mathcal{W}/F^i_j(\mathcal{W})) \leqslant a_{d-1}(\mathcal{E}/F^i_j(\mathcal{E})) - a_{d-1}(\mathcal{G}/F^i_j(\mathcal{G})) = a_{d-1}(H^i_j) - a_{d-1}(\mathcal{G}/F^i_j(\mathcal{G})). \tag{4.61}$$

Therefore we have

$$\begin{aligned}
\dim J &= H(m) - \dim U \\
&\geqslant H(m) - P(m)\frac{a_d(\mathcal{W})}{a_d(\mathcal{E})} - \frac{1}{n}\sum_{i,j}\epsilon^i_j H^i_j(m)\frac{a_{d-1}(\mathcal{W}/F^i_j(\mathcal{W}))}{a_{d-1}(H^i_j)} \qquad (4.62)\\
&\geqslant H(m) - P(m)\frac{a_d(\mathcal{E}) - a_d(\mathcal{G})}{a_d(\mathcal{E})} - \frac{1}{n}\sum_{i,j}\epsilon^i_j H^i_j(m)\frac{a_{d-1}(H^i_j) - a_{d-1}(\mathcal{G}/F^i_j(\mathcal{G}))}{a_{d-1}(H^i_j)} \qquad (4.63)\\
&= H(m) - P(m) - \frac{1}{n}\sum_{i,j}\epsilon^i_j H^i_j(m) + P(m)\frac{a_d(\mathcal{G})}{a_d(\mathcal{E})} + \frac{1}{n}\sum_{i,j}\epsilon^i_j H^i_j(m)\frac{a_{d-1}(\mathcal{G}/F^i_j(\mathcal{G}))}{a_{d-1}(H^i_j)}\\
&= P(m)\frac{a_d(\mathcal{G})}{a_d(\mathcal{E})} + \frac{1}{n}\sum_{i,j}\epsilon^i_j H^i_j(m)\frac{a_{d-1}(\mathcal{G}/F^i_j(\mathcal{G}))}{a_{d-1}(H^i_j)},
\end{aligned}$$

giving (4.58). Here, (4.62) follows by (4.57), and (4.63) uses (4.60) and (4.61).

Finally, taking $\mathcal{G}^\star_* = \mathcal{E}^\star_*$ yields

$$\dim \operatorname{im} \gamma = \dim J \geqslant P(m)\frac{a_d(\mathcal{E})}{a_d(\mathcal{E})} + \frac{1}{n}\sum_{i,j}\epsilon^i_j H^i_j(m)\frac{a_{d-1}(\mathcal{E}/F^i_j(\mathcal{E}))}{a_{d-1}(H^i_j)} = H(m) = \dim V_m,$$

showing injectivity of γ. □

Lemma 4.8.5. Fix a numerical type $(H, H^\star_*, \alpha^\star_*)$. There is $M_2 \in \mathbb{N}$ such that for all $m \geqslant M_2$ there is $K_2(m) \in \mathbb{N}$ so that for all $k \geqslant K_2$ the following consequence of parabolic p_2-semistability holds: for all $[\mathcal{E}^\star_*, \gamma] \in F_m$ such that $\mathcal{E}^\star_*$ is p_2-semistable, and for all non-zero proper parabolic subsheaves $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$ generated by subspaces $U < V_m$, we have

$$h^0(\mathcal{W}(m)) - \frac{1}{n}\sum_{i,j} \epsilon^i_j H^i_j(m) \frac{\chi(\mathcal{W}/F^i_j(\mathcal{W})(k))}{H^i_j(k)} \leqslant P(m)\frac{\chi(\mathcal{W}(k))}{H(k)}. \tag{4.64}$$

If equality holds for any $m \geqslant M_2$ and any $k \geqslant K_2(m)$, then equality holds for all m and k, so in particular $\mathcal{E}^\star_*$ is strictly p_2-semistable.

Proof. This is a parabolic analogue of lemma 1.18 in [Sim94a]: the Le Potier-Simpson estimate (theorem 3.2.14) is a crucial ingredient in the parabolic case too.

We split the proof into several steps that trace the logical (even though not necessarily the heuristically most natural) path towards choosing the constants M_2 and K_2.

Step 1: Pick an integer A such that for all $m \geqslant A$

$$(m-A)^d/d! < \frac{1}{a_d(\mathcal{E})}P(m). \tag{4.65}$$

Note that $a_d(\mathcal{E})$ is determined by H and independent of $\mathcal{E}$, and recall that P as given by (4.48) is of degree d with leading term $a_d(\mathcal{E})m^d/d!$, so we can choose such A.

Step 2: There is an integer B such that for all $r \leqslant a_d(\mathcal{E})$, all $s \in S$, and all μ-semistable coherent sheaves $\mathcal{D}$ on X_s of pure dimension d and multiplicity $a_d(\mathcal{D}) = r$, the following holds:

$$h^0(X_s, \mathcal{D}(m)) \leqslant \begin{cases} 0 & \text{if } \mu(\mathcal{D}) + m + B \leqslant 0 \\ r(\mu(\mathcal{D}) + m + B)^d/d! & \text{if } \mu(\mathcal{D}) + m + B \geqslant 0. \end{cases} \tag{4.66}$$

For all non-negative integers $r \leqslant a_d(\mathcal{E})$, use theorem 3.2.14(b) to choose $B(r)$ satisfying this inequality for μ-semistable sheaves of pure dimension d and multiplicity exactly r. Then observe that taking B to be the maximum of these finitely many $B(r)$ does what we want: for any semistable $\mathcal{D}$ of multiplicity $r \leqslant a_d(\mathcal{E})$, we have $B(r) \leqslant B$ and so:

$$\begin{aligned} h^0(X_s, \mathcal{D}(m)) &\leqslant \begin{cases} 0 & \text{if } \mu(\mathcal{D}) + m + B(r) \leqslant 0 \\ r(\mu(\mathcal{D}) + m + B(r))^d/d! & \text{if } \mu(\mathcal{D}) + m + B(r) \geqslant 0 \end{cases} \\ &\leqslant \begin{cases} 0 & \text{if } \mu(\mathcal{D}) + m + B \leqslant 0 \\ \max(0, r(\mu(\mathcal{D}) + m + B)^d/d!) & \text{if } \mu(\mathcal{D}) + m + B \geqslant 0 \end{cases} \\ &\leqslant \begin{cases} 0 & \text{if } \mu(\mathcal{D}) + m + B \leqslant 0 \\ r(\mu(\mathcal{D}) + m + B)^d/d!) & \text{if } \mu(\mathcal{D}) + m + B \geqslant 0. \end{cases} \end{aligned}$$

Note that B only depends on d.

Step 3: By the proof of theorem 4.5.2, there is a constant b (determined by H only) such that any p_2-semistable $\mathcal{E}^\star_*$ satisfies $\mu_{\max}(\mathcal{E}) \leqslant b$, where $\mu_{\max}(\mathcal{E})$ is the slope of the maximal slope-destabilising subsheaf of $\mathcal{E}$ (i.e. the first term of its μ-HN filtration).

Pick $C = C(A, B, b, a_d(\mathcal{E})) \geqslant A$ such that for all non-negative integers $r \leqslant a_d(\mathcal{E})$, all $\nu \leqslant \min(b, b - C)$ and all $m \geqslant C$ we have

$$\left.\begin{array}{ll} 0 & \text{if } b + m + B \leqslant 0 \\ (r-1)(b+m+B)^d/d! & \text{if } \nu \leqslant -m - B \leqslant b \\ (r-1)(b+m+B)^d/d! + (\nu+m+B)^d/d! & \text{if } \nu + m + B \geqslant 0 \end{array}\right\} \leqslant r(m-A)^d/d! \tag{4.67}$$

This is again a simple numerical relation: the first condition is automatically satisfied by all $m \geqslant C$ provided we choose $C \geqslant A$. The second condition is

$$(r-1)(m+b+B)^d/d! \leqslant r(m-A)^d/d!$$

which, by comparing the leading terms in m, will be true for each of the finitely many integers r and all sufficiently large m, say for $m \geqslant C(r)$. Then take the maximum of the $C(r)$ for $0 \leqslant r \leqslant a_d(\mathcal{E})$. Finally, the third and last condition is satisfied if for all $0 \leqslant r \leqslant a_d(\mathcal{E})$

$$(r-1)(b+m+B)^d/d! + (b+m+B-C)^d/d! \leqslant r(m-A)^d/d!,$$

using that $\nu \leqslant b - C$. In this case, the degree d terms in m agree, so we compare the terms of degree $d-1$: on the left-hand side, this is

$$((r-1)(b+B) + (b+B-C))\, m^{d-1}/(d-1)! = (r(b+B) - C)\, m^{d-1}/(d-1)!$$

which is less than the corresponding term on the right-hand side

$$-rAm^{d-1}/(d-1)!$$

provided that $r(b+B) - C \leqslant -rA$, i.e. for $C \geqslant r(b+A+B)$. So we only need to pick $C \geqslant a_d(\mathcal{E})(b+B+A)$ since $r \leqslant a_d(\mathcal{E})$.

Step 4: Pick any $s \in S$, any p_2-semistable parabolic sheaf $\mathcal{E}^\star_*$ on X_s with type $(H, H^\star_*, \alpha^\star_*)$ and any non-zero parabolic subsheaf $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$. By the proof of theorem 4.5.2, p_2-semistability of $\mathcal{E}^\star_*$ implies that there is a constant b such that $\mu_{\max}(\mathcal{E}) \leqslant b$, where b is determined by H only.

Let $\mathcal{W}_i$ be the terms in the μ-HN filtration of $\mathcal{W}$, set $\mathcal{D}_i := \mathcal{W}_i/\mathcal{W}_{i-1}$ to be the μ-semistable quotient sheaves, put $r_i := a_d(\mathcal{D}_i)$ and $\mu_i := \mu(\mathcal{D}_i)$. We have

$$\mu_i \leqslant \mu_1 = \mu(\mathcal{W}_1) \leqslant \mu_{\max}(\mathcal{E}) \leqslant b$$

for all i, where the first inequality is due to the defining properties of the HN filtration, and the second inequality follows by the definition of $\mu_{\max}(\mathcal{E})$, applied to the subsheaf $\mathcal{W}_1 \subset \mathcal{E}$.

We also have $\sum_i r_i = a_d(\mathcal{W})$, so in particular $r_i \leqslant a_d(\mathcal{W}) \leqslant a_d(\mathcal{E})$ for all i. Then by (4.66), we have for all i:

$$h^0(X_s, \mathcal{D}_i(m)) \leqslant \begin{cases} 0 & \text{if } \mu_i + m + B \leqslant 0 \\ r_i(\mu_i + m + B)^d/d! & \text{if } \mu_i + m + B \geqslant 0. \end{cases} \tag{4.68}$$

Let $\nu := \min_i(\mu_i) \leqslant b$. Since

$$h^0(\mathcal{W}(m)) = \sum_i (h^0(\mathcal{W}_i(m)) - h^0(\mathcal{W}_{i-1}(m))) \leqslant \sum_i h^0(\mathcal{D}_i(m)),$$

we obtain

$$h^0(\mathcal{W}(m)) \leqslant \sum_i \left\{ \begin{array}{ll} 0 & \text{if } \mu_i + m + B \leqslant 0 \\ r_i(\mu_i + m + B)^d/d! & \text{if } \mu_i + m + B \geqslant 0 \end{array} \right\} \tag{4.69}$$

$$\leqslant \begin{cases} 0 & \text{if } b + m + B \leqslant 0 \\ (a_d(\mathcal{W}) - 1)(b + m + B)^d/d! & \text{if } \nu + m + B \leqslant 0 \leqslant b + m + B \\ (a_d(\mathcal{W}) - 1)(b + m + B)^d/d! + (\nu + m + B)^d/d! & \text{if } \nu + m + B \geqslant 0, \end{cases} \tag{4.70}$$

where the first line in (4.70) holds because $b + m + B \leqslant 0$ implies that $\mu_i + m + B \leqslant 0$ for all i (as $\mu_i \leqslant b$); the second line of (4.70) is true because $\mu_i + m + B \leqslant b + m + B$ for all i, but $\nu + m + B \leqslant 0$ implies that at least one of the terms in (4.69) vanishes (and for that term we have of course $r_i \geqslant 1$); the third line of (4.70) is finally obtained from the estimate in the second line by adding the missing term $(\nu + m + B)^d/d!$ again.

Let $M_2 = C$, where C is given by step 3, and increase M_2 to ensure that $H^i_j(m) > 0$ for all $m \geqslant M_2$. Recall from the proof of lemma 4.8.2 that for any fixed $m \geqslant M$, the collection of parabolic subsheaves $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$ generated by subspaces $U < V_m$ is bounded by $F_m \times \mathrm{Grass}(V^*_m)$, so the collection of Hilbert polynomials $P(\mathcal{W})$ and $P(\mathcal{W}/F^i_j(\mathcal{W}))$ for the subsheaves under consideration is finite. Thus, we may pick $K_2(m) \in \mathbb{N}$ so all these $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$ are K_2-regular, so in particular $\chi(\mathcal{W}/F^i_j(\mathcal{W})(k)) > 0$ for $k \geqslant K_2$. Increase K_2 further to ensure that $H^i_j(k) > 0$ for all $k \geqslant K_2$.

We are now done for all parabolic subsheaves $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$ with $\nu(\mathcal{W}) \leqslant b - C$: for such $\mathcal{W}$, taking (4.70), (4.67), and (4.65) together gives us

$$h^0(\mathcal{W}(m)) < \frac{a_d(\mathcal{W})}{a_d(\mathcal{E})} P(m)$$

for all $m \geqslant M_2$. Then increase $K_2(m)$ further so that (for the finitely many Hilbert polynomials $P(\mathcal{W})$ occurring in our bounded collection of $\mathcal{W}^\star_*$) we have

$$h^0(\mathcal{W}(m)) < \frac{\chi(\mathcal{W}(k))}{H(k)} P(m) \tag{4.71}$$

for all $m \geqslant M_2$ and all $k \geqslant K_2(m)$. After subtracting non-negative terms from the left-hand side of (4.71) we get the conclusion (4.64) with strict inequality. However, this only concludes the argument for parabolic subsheaves $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$ with $\nu(\mathcal{W}) \leqslant b - C$.

Step 5: We are left to deal with those subsheaves $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$ with $\nu(\mathcal{W}) > b - C$. The collection of sheaves

$$\begin{aligned}\{\mathcal{W}^\star_* \mid \mathcal{W}^\star_* \subset \mathcal{E}^\star_* &\text{ for some } p_2\text{-semistable } \mathcal{E}^\star_* \text{ on some fibre } X_s, \text{ with } \mathcal{E}^\star_* \text{ of type} \\ &(H, H^\star_*, \alpha^\star_*), \text{ with } \nu(\mathcal{W}) \geqslant b - C \text{ and with } \mathcal{W} \subset \mathcal{E} \text{ a saturated subsheaf }\}\end{aligned} \tag{4.72}$$

is bounded by proposition 2.3.12, since the sheaves $\mathcal{E}$ admitting a p_2-semistable parabolic structure $\mathcal{E}^\star_*$ of fixed type $(H, H^\star_*, \alpha^\star_*)$ form a bounded collection by theorem 4.5.2. Strictly speaking, this only shows that the underlying sheaves $\mathcal{W}$ vary in a bounded collection, but we may assume that $\mathcal{W}^\star_*$ carries the induced parabolic substructure $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$ as this minimises the quotients $\mathcal{W}/F^i_j(\mathcal{W})$, so it increases the left-hand side of (4.64).

Note that (4.72) is bounded *independent of* m, as opposed to the collection considered in the previous step. Then we may increase M_2 further so that all elements $\mathcal{W}$ of (4.72) are M_2-regular, in particular we get $h^0(\mathcal{W}(m)) = \chi(\mathcal{W}(m))$ for $m \geqslant M_2$.

By p_2-semistability of the sheaves $\mathcal{E}^\star_*$ in (4.72), we have, for each polynomial par–$P_2(\mathcal{W}^\star_*, x, y)$ of a parabolic sheaf $\mathcal{W}^\star_*$ in the set (4.72), an integer $M(\mathcal{W}^\star_*) \geqslant M_2$ such that for all $m \geqslant M(\mathcal{W}^\star_*)$

$$h^0(\mathcal{W}(m)) - \frac{1}{n}\sum_{i,j} \epsilon^i_j H^i_j(m) \frac{\chi(\mathcal{W}/F^i_j(\mathcal{W})(k))}{H^i_j(k)} \leqslant P(m)\frac{\chi(\mathcal{W}(k))}{\chi(\mathcal{E}(k))} \tag{4.73}$$

for all sufficiently large k, depending on m. But as (4.72) is bounded, the set of Hilbert polynomials of $\mathcal{W}$ and $\mathcal{W}/F^i_j(\mathcal{W})$ in (4.72) is finite by theorem 2.3.9, and $M(\mathcal{W}^\star_*)$ only depends on $\mathcal{W}^\star_*$ through these Hilbert polynomials, so we may increase M_2 and $K_2(m)$ further to get (4.73) for all $m \geqslant M_2$, all $k \geqslant K_2$ and all $\mathcal{W}^\star_*$ simultaneously. This gives the desired conclusion (4.64) for all $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$ as in (4.72).

Again because there are only finitely many Hilbert polynomials of $\mathcal{W}$ and $\mathcal{W}/F^i_j(\mathcal{W})$ to consider, we may increase M_2 and K_2 so that if equality holds in (4.73) for any $m \geqslant M_2$ and any $k \geqslant K_2$, then the equality holds for all m and all k, i.e. the subsheaf $\mathcal{W}^\star_*$ makes $\mathcal{E}^\star_*$ strictly p_2-semistable.

It remains to deal with the $\mathcal{W}^\star_* \subset \mathcal{E}^\star_*$ with $\nu(\mathcal{W}) > b - C$ which are not elements of (4.72), i.e. the non-saturated $\mathcal{W} \subset \mathcal{E}$. We have $h^0(\mathcal{W}(m)) \leqslant h^0(\mathcal{W}^{\text{sat}}(m))$ for all m, and as before we may assume that $\mathcal{W}$ carries the induced parabolic structure which minimises the terms $a_{d-1}(\mathcal{W}/F^i_j(\mathcal{W}))$, so we get (4.64) for $\mathcal{W}^\star_*$ by the above argument for $(\mathcal{W}^{\text{sat}})^\star_*$. Finally, if equality holds in (4.64) for $\mathcal{W}^\star_*$, then $h^0(\mathcal{W}(m)) = h^0(\mathcal{W}^{\text{sat}}(m))$. But by our choice of M_2 at the beginning of this step, $\mathcal{W}^{\text{sat}}$ is M_2-regular. In particular, $\mathcal{W}^{\text{sat}}(m)$ is globally generated for $m \geqslant M_2$, so if $h^0(\mathcal{W}(m)) = h^0(\mathcal{W}^{\text{sat}}(m))$, then $\mathcal{W} = \mathcal{W}^{\text{sat}}$, and so by the argument above equality holds in (4.64) for $\mathcal{W}^\star_*$ and all m, k. □

We will need the following lemma in order to prove that for each GIT-semistable $[\mathcal{E}^\star_*, \gamma] \in F_m$ the underlying sheaf $\mathcal{E}$ is pure. This result is the only reason why we removed certain irreducible components from the full Quot scheme in (4.31) when we constructed our parameter space F_m.

Lemma 4.8.6 (cf. lemma 1.17 in [Sim94a])**.** Given a (not necessarily pure) parabolic sheaf $\mathcal{E}_*^\star$ corresponding to a point of F_m, there is a parabolic sheaf $\mathcal{F}_*^\star$ (in particular, $\mathcal{F}$ is pure) with the same parabolic type $(H, H_*^\star, \alpha_*^\star)$ and thus the same parabolic Hilbert polynomial P as $\mathcal{E}_*^\star$, such that $(\mathcal{E}/T_{d-1}(\mathcal{E}))_*^\star$ injects into $\mathcal{F}_*^\star$. Here, $T_{d-1}(\mathcal{E})$ is the torsion subsheaf of $\mathcal{E}$ (i.e. the maximal subsheaf supported in dimension $< d$), and the quotient $(\mathcal{E}/T_{d-1}(\mathcal{E}))_*^\star$ is pure of dimension d, hence carries the induced parabolic structure.

Proof. This is the parabolic analogue of lemma 1.17 in [Sim94a] which itself is based on ideas of [Gie77]. Recall that Q was defined in (4.31) as the Quot scheme with certain irreducible components removed: those components whose points correspond to sheaves which are not pure. The result Simpson proves is this: let $\mathcal{E}$ be a sheaf corresponding to a point of Q. Then there is a pure sheaf $\mathcal{F}$ with Hilbert polynomial H and an injection $\mathcal{E}/T_{d-1}(\mathcal{E}) \hookrightarrow \mathcal{F}$.

Simpson's argument is very geometric, using the fact that $[\mathcal{E}] \in Q$ cannot not be 'too far' from a pure sheaf, so we can deform it into one: take a curve C in Q with a distinguished point $0 \in C$ corresponding to $\mathcal{E}$, and the generic point of C mapping into the open subscheme of Q parametrising pure sheaves. Then we may take the universal sheaf on $X \times C$ which restricts to $\mathcal{E}$ on $X \times \{0\}$. Restrict the universal sheaf to the complement of the support of $T_{d-1}(\mathcal{E})$, push it forward to all of $X \times C$, and restrict to $X \times \{0\}$ – this gives the sheaf $\mathcal{F}$ whose existence is asserted. See [Sim94a] for the full details.

We can now easily adapt this to our parabolic situation: if $\mathcal{E}_*^\star$ corresponds to a point of F_m, then $\mathcal{E}$ corresponds to a point of Q. The quotient sheaf $\mathcal{E}/T_{d-1}(\mathcal{E})$ is pure of dimension d, so we may give it the induced parabolic structure $(\mathcal{E}/T_{d-1}(\mathcal{E}))_*^\star$ as a quotient of $\mathcal{E}_*^\star$, i.e.

$$F_j^i(\mathcal{E}/T_{d-1}(\mathcal{E})) := F_j^i(\mathcal{E})/(T_{d-1}(\mathcal{E}) \cap F_j^i(\mathcal{E})) = F_j^i(\mathcal{E})/T_{d-1}(F_j^i(\mathcal{E})).$$

Thus, we can apply Simpson's result to $F_j^i(\mathcal{E}/T_{d-1}(\mathcal{E}))$ too, embedding it in a pure sheaf $F_j^i(\mathcal{F})$ of Hilbert polynomial equal to $P(F_j^i(\mathcal{E})) = H - H_j^i$. But we can do this for all i, j simultaneously, following Simpson's argument above – just take a curve C in F_m with distinguished point at $\mathcal{E}_*^\star$ and generic point in the locus of F_m parametrising pure parabolic sheaves. As the torsion subsheaves of $F_j^i(\mathcal{E})$ are all contained in $T_{d-1}(\mathcal{E})$, we can again restrict the universal parabolic sheaf on $X \times C$ away from the support of $T_{d-1}(\mathcal{E})$, giving us pure $\mathcal{F}$ as above equipped with a parabolic structure $\mathcal{F}_*^\star$ such that $(\mathcal{E}/T_{d-1}(\mathcal{E}))_*^\star \hookrightarrow \mathcal{F}_*^\star$. Finally, we check that the parabolic types of $\mathcal{E}_*^\star$ and $\mathcal{F}_*^\star$ agree: we already know that $P(\mathcal{F}) = H$, and for the quotients we have

$$P(\mathcal{F}/F_j^i(\mathcal{F})) = H - P(F_j^i(\mathcal{F})) = H - (H - H_j^i) = H_j^i,$$

as required. □

Proof of theorem 4.8.1. We begin by proving the 'if' part of the theorem: suppose that $\mathcal{E}_*^\star$ is a p_2-(semi)stable parabolic sheaf (in particular, $\mathcal{E}$ is pure) and $\gamma : V_m \to H^0(\mathcal{E}(m))$ an isomorphism; we aim to show that the corresponding point of F_m is GIT-(semi)stable. By lemma 4.8.2, it suffices to show that there is $K(m) \geqslant K_1(m)$ such that for all $k \geqslant K$ and all parabolic subsheaves $\mathcal{W}_*^\star \subset \mathcal{E}_*^\star$ generated by non-zero proper subspaces $U \cong H^0(\mathcal{W}(m))$ of $V_m \cong H^0(\mathcal{E}(m))$, we have

$$h^0(\mathcal{W}(m)) - \frac{1}{n}\sum_{i,j} \epsilon_j^i H_j^i(m) \frac{\chi(\mathcal{W}/F_j^i(\mathcal{W})(k))}{H_j^i(k)} (\leqslant) P(m)\frac{\chi(\mathcal{W}(k))}{H(k)}. \tag{4.74}$$

But this is exactly (4.64), so take $M = M_2$ and $K(m) = \max(K_1(m), K_2(m))$, where M_2 and $K_2(m)$ are given by lemma 4.8.5. Thus, $[\mathcal{E}_*^\star, \gamma]$ is GIT-semistable if $\mathcal{E}_*^\star$ is p_2-semistable and if γ is an isomorphism. Now if $[\mathcal{E}_*^\star, \gamma]$ is a strictly GIT-semistable point of F_m with respect to $\psi_k^* L_{\beta,\beta_j^i}$, then equality must hold in (4.74) for some globally generated $\mathcal{W}_*^\star \subset \mathcal{E}_*^\star$ and the given values of m and k. But by lemma 4.8.5, this implies that $\mathcal{E}_*^\star$ is strictly p_2-semistable in this case. Taking the contrapositive, we see that $[\mathcal{E}_*^\star, \gamma]$ is GIT-stable if $\mathcal{E}_*^\star$ is p_2-stable and if γ is an isomorphism. This concludes the 'if' part of the proof.

Now consider the 'only if' statement of the theorem: increase $K(m)$ so that $K(m) \geqslant K_3(m)$ as given by lemma 4.8.4. Let $[\mathcal{E}_*^\star, \gamma] \in F_m$ be a GIT-semistable point with respect to $\psi_k^* L_{\beta,\beta_j^i}$. Then lemma 4.8.4 tells us that γ must be injective.

We first show that $\mathcal{E}$ is pure: let $T_{d-1}(\mathcal{E})$ be the torsion subsheaf of $\mathcal{E}$ and let $\mathcal{F}_*^\star$ be as in lemma 4.8.6. We aim to show that the collection of $\mathcal{F}_*^\star$ is bounded for all $m \geqslant M$ simultaneously. Since the type of $\mathcal{F}_*^\star$ is fixed to be $(H, H_*^\star, \alpha_*^\star)$, by theorem 3.2.13 it suffices to show that the slope of any quotient of $\mathcal{F}$ is bounded below by a constant not depending on m. To do so, we first increase M to guarantee $H_j^i(m) \geqslant 0$ for all $m \geqslant M$.

As $\mathcal{F}$ is a pure sheaf, it admits a μ-HN filtration. Let $\mathcal{G}$ be the last step in the μ-HN filtration of $\mathcal{F}$, i.e. the quotient $\mathcal{F} \twoheadrightarrow \mathcal{G}$ of minimal slope, so $\nu(\mathcal{F}) = \mu(\mathcal{G})$ in the terminology of the proof of lemma 4.8.5. Give $\mathcal{G}$ the induced parabolic structure to make it a quotient $\mathcal{F}_*^\star \twoheadrightarrow \mathcal{G}_*^\star$. Then by lemma 4.8.4 we have

$$\begin{aligned}
\frac{h^0(\mathcal{G}(m))}{a_d(\mathcal{G})} &\geqslant \frac{\dim J}{a_d(\mathcal{G})} \\
&\geqslant P(m)\frac{1}{a_d(\mathcal{E})} + \frac{1}{n}\sum_{i,j} \epsilon_j^i H_j^i(m) \frac{a_{d-1}(\mathcal{G}/F_j^i(\mathcal{G}))}{a_d(\mathcal{G})} \frac{1}{a_{d-1}(H_j^i)} \\
&\geqslant P(m)\frac{1}{a_d(\mathcal{E})}
\end{aligned} \tag{4.75}$$

where J is the image of the map $V_m \to H^0(\mathcal{G}(m))$, and the last step uses the fact that $H_j^i(m) \geqslant 0$ for $m \geqslant M$. But $P(m)/a_d(\mathcal{E})$ is a fixed polynomial (for all $\mathcal{E}_*^\star$) with leading term $m^d/d!$, so there is a constant C and we may increase M such that for all $m \geqslant M$

$$\frac{P(m)}{a_d(\mathcal{E})} \geqslant \frac{(m-C)^d}{d!}.$$

Together with (4.75) this gives

$$\frac{h^0(\mathcal{G}(m))}{a_d(\mathcal{G})} \geqslant \frac{(m-C)^d}{d!}$$

for all $m \geqslant M$. On the other hand, by the Le Potier-Simpson estimate (theorem 3.2.14), we have a constant B, only depending on $a_d(\mathcal{G})$ and d, such that

$$\frac{1}{d!}(\mu(\mathcal{G}) + m + B)^d \geqslant \frac{h^0(\mathcal{G}(m))}{a_d(\mathcal{G})}.$$

As $a_d(\mathcal{G}) \leqslant a_d(\mathcal{F}) = a_d(\mathcal{E})$, only finitely many values of $a_d(\mathcal{G})$ occur and we may choose a single constant B that works for all $\mathcal{G}$ simultaneously. Then the last two inequalities together imply that $\mu(\mathcal{G}) \geqslant -B-C$. This constant is independent of m, concluding the proof of the claim that the parabolic sheaves $\mathcal{F}^\star_*$ vary in a bounded family (uniformly for all m).

As a consequence, we may increase M further so that all the $\mathcal{F}^\star_*$ are M-regular. Thus, for $m \geqslant M$ we have

$$\begin{aligned}
&h^0(\mathcal{F}(m)) = H(m),\\
&h^0(\mathcal{F}/F^i_j(\mathcal{F})(m)) = H^i_j(m)\\
&\text{and } \mathcal{F}(m), \mathcal{F}/F^i_j(\mathcal{F})(m) \text{ are globally generated}
\end{aligned}$$

for all such $\mathcal{F}^\star_*$. Now consider $\mathcal{E}/T_{d-1}(\mathcal{E}) \hookrightarrow \mathcal{F}$, giving

$$H(m) = h^0(\mathcal{F}(m)) \geqslant h^0(\mathcal{E}/T_{d-1}(\mathcal{E})(m)).$$

Applying the conclusion (4.58) of lemma 4.8.4 to the quotient $\mathcal{G} := \mathcal{E}/T_{d-1}(\mathcal{E})$ equipped with the induced parabolic structure, we have

$$\begin{aligned}
h^0(\mathcal{G}(m)) &\geqslant P(m)\frac{a_d(\mathcal{G})}{a_d(\mathcal{E})} + \frac{1}{n}\sum_{i,j} \epsilon^i_j H^i_j(m) \frac{a_{d-1}(\mathcal{G}/F^i_j(\mathcal{G}))}{a_{d-1}(H^i_j)}\\
&\geqslant P(m) + \frac{1}{n}\sum_{i,j} \epsilon^i_j H^i_j(m) = H(m) = h^0(\mathcal{F}(m)).
\end{aligned}$$

Given that $\mathcal{G} = \mathcal{E}/T_{d-1}(\mathcal{E}) \subset \mathcal{F}$, this means that $\mathcal{F}(m)$ is generated by $H^0(\mathcal{E}/T_{d-1}(\mathcal{E})(m))$, so $\mathcal{E}/T_{d-1}(\mathcal{E}) = \mathcal{F}$. But as $\mathcal{E}$ and $\mathcal{F}$ have the same Hilbert polynomial by construction in lemma 4.8.6, so $T_{d-1}(\mathcal{E}) = 0$ and $\mathcal{E}$ is pure.

Note that this also shows that the sheaves $\mathcal{E} = \mathcal{F}$ vary in a bounded family uniformly for all m, and

$$h^0(\mathcal{E}(m)) = H(m),$$
$$h^0(\mathcal{E}/F^i_j(\mathcal{E})(m)) = H^i_j(m)$$

for all $m \geqslant M$ and all GIT-semistable $[\mathcal{E}^\star_*, \gamma]$. By lemma 4.8.4, we already knew that the map $\gamma : V_m \to H^0(\mathcal{E}(m))$ is injective for GIT-semistable $[\mathcal{E}^\star_*, \gamma]$, and by comparing the dimensions it must now be an isomorphism.

We now show that $\mathcal{E}^\star_*$ is p_2-(semi)stable if $[\mathcal{E}^\star_*, \gamma] \in F_m$ is GIT-(semi)stable. If $\mathcal{E}^\star_*$ is p_2-unstable, then there is $\mathcal{F}^\star_* \subset \mathcal{E}^\star_*$ such that

$$H(y)\text{par–}P_2(\mathcal{F}^\star_* \subset \mathcal{E}^\star_*, x, y) > \chi(\mathcal{F}(y))P(x) \tag{4.76}$$

for x sufficiently large and y sufficiently large (depending on x). Since $V_m \cong H^0(\mathcal{E}(m))$, we may write $\mathcal{F}^\star_* \subset \mathcal{E}^\star_*$ as the image of $U \otimes \mathcal{O}_X(-m) \to \mathcal{E}^\star_*$, where $U := H^0(\mathcal{F}(m))$.

By fixing one destabilising $\mathcal{F}^\star_* \subset \mathcal{E}^\star_*$ for each p_2-unstable $\mathcal{E}^\star_*$ in our bounded family of GIT-semistable parabolic sheaves, we may arrange for the $\mathcal{F}^\star_*$ to vary in a bounded family independent of m. Thus, we may increase M and $K(m)$ further so that for all $m \geqslant M$ we have $\chi(\mathcal{F}(m)) = h^0(\mathcal{F}(m)) = \dim U$ for all such $\mathcal{F}$, and such that (4.76) holds for $x = m$ and all $y = k \geqslant K$, i.e. we have

$$\dim U - \frac{1}{n}\sum_{i,j} \epsilon^i_j \frac{\chi(\mathcal{F}/F^i_j(\mathcal{F})(k))}{H^i_j(k)} H^i_j(m) > P(m)\frac{\chi(\mathcal{F}(k))}{H(k)}, \tag{4.77}$$

so lemma 4.8.2 shows that $[\mathcal{E}^\star_*, \gamma] \in F_m$ could not have been GIT-semistable in the first place.

Now take an element $\mathcal{E}^\star_*$ of the bounded collection of GIT-semistable and strictly p_2-semistable parabolic sheaves. Then there is a proper non-zero parabolic subsheaf $\mathcal{F}^\star_* \subset \mathcal{E}^\star_*$ such that (4.76) holds with equality, so by the same argument we may increase M and K to obtain (4.77) with equality. Then lemma 4.8.2 shows that $[\mathcal{E}^\star_*, \gamma] \in F_m$ must be strictly GIT-semistable.

□

4.9 Existence of the moduli space and its basic properties

We are still due to give a definition of the stable and semistable moduli functors. Recall the definition (4.26) of $\mathbf{Par}^{H,H^\star_*,\alpha^\star_*}_{D^\star/X/S}$.

Definition 4.9.1. Given numerical polynomials $H, H^i_j \in \mathbb{Q}[x]$ (for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant l_i$), where $d := \deg H$ and $\deg H^i_j < d$ for all i, j, let

$$\mathbf{Par}^{H,H^\star_*,\alpha^\star_*,s}_{D^\star/X/S} : \mathsf{Sch}/S \to \mathsf{Sets}$$

be the subfunctor of $\mathbf{Par}^{H,H^\star_*,\alpha^\star_*}_{D^\star/X/S}$ of all stable flat families $\mathcal{E}^\star_*$ up to $\sim$-equivalence, and let

$$\mathbf{Par}^{H,H^\star_*,\alpha^\star_*,ss}_{D^\star/X/S} : \mathsf{Sch}/S \to \mathsf{Sets}$$

be defined by

$$\mathbf{Par}^{H,H^\star_*,\alpha^\star_*,ss}_{D^\star/X/S}(T) := \{\mathcal{E}^\star_* \in \mathbf{Par}^{H,H^\star_*,\alpha^\star_*}_{D^\star/X/S}(T) \mid \mathcal{E}^\star_* \text{ is semistable}\}/\sim_S$$

where S-equivalence $\mathcal{E}^\star_* \sim_S \mathcal{F}^\star_*$ of two families (parametrised by a scheme T) is defined by $\overline{G(x)} \cap \overline{G(y)} \cap (F_m(T))^{ss}(\psi^*_k L_{\beta,\beta^i_j}) \neq \emptyset$, where $G := \mathrm{SL}(V_m)(T)$ acts on the T-valued points of F_m as in sections 4.7 and 4.8, and $x, y \in F_m(T)$ correspond to $\mathcal{E}^\star_*$ and $\mathcal{F}^\star_*$, respectively.

Clearly, this definition of S-equivalence (which is just restating the definitions of section 2.2) is highly unsatisfactory and a more intrinsic definition is needed.

The main result of this chapter is:

Theorem 4.9.2. Fix an algebraically closed field $\mathbb{k}$ of arbitrary characteristic, a scheme S of finite type over $\mathbb{k}$, a flat projective morphism $f : X \to S$ of schemes, relative effective Cartier divisors $D^1, \ldots, D^n$ on X, a relatively very ample line bundle $\mathcal{O}_X(1)$, and a numerical type $(H, H^\star_*, \alpha^\star_*)$ with rational parabolic weights $\alpha^\star_*$ and polynomials $H, H^i_j \in \mathbb{Q}[x]$ such that $d := \deg H$ and $\deg H^i_j = d - 1$. Then there exists a coarse moduli space $P^{H,H^\star_*,\alpha^\star_*,ss}_{D^\star/X/S}$ of S-equivalence classes of p_2-semistable pure d-dimensional parabolic sheaves on the fibres of $X \to S$ (with parabolic structures at the divisors D^i, Hilbert polynomial H and flag type $H^\star_*$ with respect to $\mathcal{O}_X(1)$, and parabolic p_2-(semi)stability measured with respect to the weights $\alpha^\star_*$). Furthermore, $P^{H,H^\star_*,\alpha^\star_*,ss}_{D^\star/X/S}$ is projective over S. More precisely:

(a) The closed points of $P^{H,H^\star_*,\alpha^\star_*,ss}_{D^\star/X/S}$ lying over $s \in S$ correspond exactly to S-equivalence classes of p_2-semistable parabolic sheaves of type $(H, H^\star_*, \alpha^\star_*)$ on X_s (where S-equivalence is defined in the sense of GIT).

(b) There is an open subscheme $P^{H,H^\star_*,\alpha^\star_*,s}_{D^\star/X/S}$ whose closed points over $s \in S$ correspond exactly to isomophism classes of p_2-stable parabolic sheaves of type $(H, H^\star_*, \alpha^\star_*)$ on X_s.

(c) $P^{H,H^\star_*,\alpha^\star_*,ss}_{D^\star/X/S}$ is a coarse moduli space for the functor $\mathbf{Par}^{H,H^\star_*,\alpha^\star_*,ss}_{D^\star/X/S}$, and $P^{H,H^\star_*,\alpha^\star_*,s}_{D^\star/X/S}$ is a coarse moduli space for $\mathbf{Par}^{H,H^\star_*,\alpha^\star_*,s}_{D^\star/X/S}$.

(d) The fibre of $P^{H,H^\star_*,\alpha^\star_*,ss}_{D^\star/X/S}$ over $s \in S$ is the coarse moduli space $P^{H,H^\star_*,\alpha^\star_*,ss}_{(D^s)_\star/X_s/\mathbb{k}}$ of S-equivalence classes of p_2-semistable pure d-dimensional parabolic sheaves on the fibre X_s, with parabolic structures at the divisors $(D^1)_s, \ldots, (D^n)_s$.

Proof. First, use corollary 4.6.1 to choose M such that all p_2-semistable sheaves $\mathcal{E}_*^\star$ of type $(H, H_*^\star, \alpha_*^\star)$ on the fibres of $X \to S$ are among the parabolic sheaves parametrised by points of F_m for $m \geqslant M$. Then define the semistable (respectively stable) moduli space as the GIT quotient (respectively orbit space)

$$P_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star,ss} := (F_m)^{\mathrm{ss}}(\psi_k^* L_{\beta,\beta_j^i})//\mathrm{SL}(V_m)$$
$$P_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star,s} := (F_m)^{\mathrm{s}}(\psi_k^* L_{\beta,\beta_j^i})/\mathrm{SL}(V_m)$$

with weights β, β_j^i as in (4.49) and with m and $k(m)$ sufficiently large, as determined by theorem 4.8.1. Since the morphism $F_m \to S$ is projective and $\mathrm{SL}(V_m)$-invariant, and since the line bundle $\psi_k^* L_{\beta,\beta_j^i}$ (or more precisely some power of this $\mathbb{Q}$-invertible sheaf) is very ample, projectivity of the semistable quotient follows from GIT: see theorem 2.2.1.

Now statements (a) and (b) follow from theorems 2.2.1 and 4.8.1. For (c), note that since the Flag-Quot scheme F_m is built from Quot schemes which themselves are fine moduli spaces, there is a universal family $\mathcal{U}_*^\star$ on F_m (universal for the rigidified moduli problem described at the end of section 4.6). Now given any flat family $\mathcal{E}_*^\star$ of (semi)stable parabolic sheaves on X_T/T for some $T \in \mathsf{Sch}/S$, we first note that by corollary 4.6.1 the restriction of $\mathcal{E}_*^\star$ to any fibre X_t is m-regular, so the family $\mathcal{E}_*^\star$, together with a framing of $(f_T)_*\mathcal{E}(m)$, is locally given by a pull-back of the universal family $\mathcal{U}_*^\star$ via some morphism into F_m, and by theorem 4.8.1 in fact via some morphism into the GIT-(semi)stable locus $(F_m)^{(\mathrm{s})\mathrm{s}}$. Moreover, two points $x, y \in F_m$ lie in an $\mathrm{SL}(V_m)$-orbit if and only if the parabolic sheaves $(\mathcal{U}_x)_*^\star$ and $(\mathcal{U}_y)_*^\star$ are isomorphic, since any two framings of $(\mathcal{U}_x)_*^\star$ given by morphisms $V_m \to H^0(\mathcal{U}_x(m))$ are related by an element of $\mathrm{GL}(V_m)$, but $\mathbb{G}_m \subset \mathrm{GL}(V_m)$ acts trivially on the Flag scheme F_m, so the parabolic sheaves $(\mathcal{U}_x)_*^\star$ and $(\mathcal{U}_y)_*^\star$ are isomorphic if and only if $x, y \in F_m$ lie in a common $\mathrm{PGL}(V_m)$-orbit, and the $\mathrm{PGL}(V_m)$-orbits on $F_m \hookrightarrow \mathrm{Gr}_S(m, k)$ agree with the $\mathrm{SL}(V_m)$-orbits. (The reason we consider the $\mathrm{SL}(V_m)$-action instead of the $\mathrm{PGL}(V_m)$-action is that the latter is in general not linearisable. On the other hand, the $\mathrm{GL}(V_m)$-action is clearly linearisable, but all its stabilisers are infinite, so there would be no stable points in the quotient.)

The GIT quotients $(F_m)^{\mathrm{ss}}//\mathrm{SL}(V_m)$ and $(F_m)^{\mathrm{s}}/\mathrm{SL}(V_m)$ are categorical quotients of the corresponding parameter spaces, so by theorem 2.2.4 we then have a natural transformation from $\mathbf{Par}_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star,ss}$ to the functor of points of $(F_m)^{\mathrm{ss}}//\mathrm{SL}(V_m)$, and a natural transformation from $\mathbf{Par}_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star,s}$ to the the functor of points of $(F_m)^{\mathrm{s}}/\mathrm{SL}(V_m)$, and both transformations satisfy the universal property required for coarse moduli spaces (see definition 2.1.1). Finally, statements (a) and (b) give us the bijections making the categorical quotients $P_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star,ss}$ and $P_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star,s}$ into coarse moduli spaces:

$$\mathbf{Par}_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star,ss}(\mathrm{Spec}\,\Bbbk) = \mathrm{Hom}(\mathrm{Spec}\,\Bbbk, P_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star,ss})$$
$$\mathbf{Par}_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star,s}(\mathrm{Spec}\,\Bbbk) = \mathrm{Hom}(\mathrm{Spec}\,\Bbbk, P_{D^\star/X/S}^{H,H_*^\star,\alpha_*^\star,s})$$

for any given closed point $s : \operatorname{Spec} \Bbbk \hookrightarrow S$, so the categorical quotients $P^{H,H^{\star}_{*},\alpha^{\star}_{*},ss}_{D^{\star}/X/S}$ and $P^{H,H^{\star}_{*},\alpha^{\star}_{*},s}_{D^{\star}/X/S}$ are coarse moduli spaces.

Part (d) is immediate as the fibre of F_m over $s \in S$ is the Flag-Quot scheme for parabolic sheaves on X_s, and the $\mathrm{SL}(V_m)$-action is fibrewise, so we may repeat the argument of (c) for fixed $s \in S$. □

Chapter 5

Universal moduli of semistable parabolic sheaves on stable marked curves

We now finish the construction of a a projective coarse moduli space $\overline{U_{g,n}(r,d,r^\star_*,\alpha^\star_*)}$ for the problem of pairs $((C,x^1,\ldots,x^n),\mathcal{E}^\star_*)$, where $(C,x^1,\ldots,x^n)$ is a stable n-pointed curve of genus g (with $2g+n\geq 3$), and $\mathcal{E}^\star_*$ is a p_2-semistable parabolic torsion-free (i.e. purely 1-dimensional) sheaf over $(C,x^1,\ldots,x^n)$, where $\mathcal{E}$ has uniform rank r on each irreducible component of C, degree d, rational parabolic weights

$$\alpha^\star_* = (\alpha^i_j : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant l_i),$$

flag type $r^i_j := \text{rk } \mathcal{E}/F^i_j(\mathcal{E})$, and parabolic structures at the divisors $D^i := x^i$ (no mixing between fibres allowed).

For definitions and notation on stable curves, see section 3.1 (particularly subsection 3.1.3 for marked curves), and for definitions, terminology and conventions about parabolic sheaves, see chapter 4. Fix an algebraically closed field $\Bbbk$ of arbitrary characteristic and take all curves to be defined over $\Bbbk$.

Given numerical type $(r,d,r^\star_*,\alpha^\star_*)$ as above, let us first convert this into the language of the last chapter: by the same argument as after (3.18), combined with theorem 3.1.5, we find that a torsion-free sheaf $\mathcal{E}$ of uniform rank r and degree d on $(C,x^1,\ldots,x^n)$ has Hilbert polynomial

$$H(t) := \chi(\mathcal{E}\otimes L^{\otimes at}) = a(2g-2+n)rt + d + r(1-g)$$

with respect to the very ample line bundle $L^{\otimes a}$ defined in subsection 3.1.3. Similarly, we find

$$H^i_j(t) := \chi(\mathcal{E}/F^i_j(\mathcal{E})\otimes L^{\otimes at}) = r^i_j.$$

In analogy to Pandharipande's universal moduli space for torsion-free sheaves on (unmarked) curves as constructed in section 3.3, our moduli space should map to $\overline{M}_{g,n}$ via the forgetful map η, and the fibre $\eta^{-1}([(C,x^1,\dots,x^n)])$ over any smooth n-pointed curve $[(C,x^1,\dots,x^n)]\in M_{g,n}$ should be naturally isomorphic to

$$P^{H,H^\star_*,\alpha^\star_*,ss}_{x^\star/C/\mathrm{Spec}\,\Bbbk}/\mathrm{Aut}((C,x^1,\dots,x^n)),$$

where $P^{H,H^\star_*,\alpha^\star_*,ss}_{x^\star/C/\mathrm{Spec}\,\Bbbk}$ is the coarse moduli space of uniform rank r, degree d, weight $\alpha^\star_*$, p_2-semistable parabolic vector bundles on C up to (a suitable notion of) S-equivalence. These latter spaces have been constructed in the last chapter.

The method of constructing our universal moduli spaces here is to follow the framework of Pandharipande's construction [Pan96], as outlined in the introduction, replacing the input from Gieseker's construction of $\overline{M}_g$ by Baldwin-Swinarski's construction of $\overline{M}_{g,n}$, and substituting our GIT construction of moduli of semistable parabolic vector bundles on a flat family of nodal curves for the 'fibrewise GIT problem' in [Pan96]. Pandharipande's ideas on relative GIT (proposition 3.3.1) generalise easily to the following situation: consider a reductive algebraic $\Bbbk$-group G acting on proper $\Bbbk$-schemes X and Y of finite type, with M and L linearisations of the respective G-action. Use superscripts '(s)s' to denote GIT-(semi)stability with respect to the stated linearisation. Write $\mathrm{pr}_X : X\times Y\to X$ and $\mathrm{pr}_Y : X\times Y\to Y$ for the projections.

Proposition 5.1.1 (cf. proposition 3.3.1 and also proposition 2.18 in [MFK94])**.** There are integers b_s and b_{ss} such that for all $b\geqslant b_s$

$$\mathrm{pr}_X^{-1}(X^s(M))\subset (X\times Y)^s(\mathrm{pr}_X^*M^{\otimes b}\otimes \mathrm{pr}_Y^*L) \tag{5.1}$$

and for all $b\geqslant b_{ss}$

$$(X\times Y)^{ss}(\mathrm{pr}_X^*M^{\otimes b}\otimes \mathrm{pr}_Y^*L)\subset \mathrm{pr}_X^{-1}(X^{ss}(M)). \tag{5.2}$$

This is the same simple elegant idea used by Pandharipande in [Pan96] and mentioned in [Mum65b] already: weighting the first factor of a product heavily means that a pair (x,y) is automatically stable whenever x is, and (x,y) is unstable whenever x is. We closely follow Pandharipande's strategy in combining the two GIT actions.

Now return to our situation: recall the local universal family $(\phi : U_J\to J,\sigma^1,\dots,\sigma^n)$ for $\overline{\mathcal{M}}_{g,n}$ from section 3.1.3. The parameter space J is defined as a locally closed subscheme of the incidence scheme

$$I\hookrightarrow H_{g,e,N}\times(\mathbb{P}^N)^n$$

which is acted upon diagonally by SL_{N+1} and linearised by (3.4) – we call this linearisation now M, for ease of notation. Call $\bar{J}$ the closure of J in I.

The local universal family $\phi : U_J\to J$ comes with sections σ^i, so define for each $1\leqslant i\leqslant n$ an effective relative Cartier divisor D^i on U_J locally by the equation $x=\sigma^i(\phi(x))$: then the marked fibre of U_J over a point of J which corresponds to a marked curve $[C, x^1, \ldots, x^n]$ is just this marked curve itself.

Apply the construction of the last chapter to $X = U_J$, $S = J$ and $D^i = (x = \sigma^i(\phi(x)))$: this gives us a Flag-Quot scheme F_m which for sufficiently large m parametrises all the (rigidified) parabolic sheaves on the fibres of U_J/J with numerical type $(H, H^\star_*)$. The Flag scheme is acted upon by $\mathrm{SL}(V_m)$, and by the construction of F_m (section 4.6), there is a $\mathrm{SL}(V_m)$-invariant morphism $F_m \to J$. On the other hand, SL_{N+1} acts on the full Flag scheme: the parabolic sheaves are acted on through curve automorphisms.

Before proceeding, we remove the connected components of F_m corresponding to parabolic sheaves of non-uniform rank (i.e. we discard any points of F_m corresponding to pairs of a reducible (marked) curve C and a rigidified parabolic sheaf $\mathcal{E}^\star_*$ on C having different ranks on the irreducible components of C). This is compatible with both group actions: any parabolic sheaf in the same $\mathrm{SL}_{N+1} \times \mathrm{SL}(V_m)$-orbit as $[C, \mathcal{E}^\star_*]$ will also be of non-uniform rank.

The embedding ψ_k of the Flag-Quot scheme constructed in section 4.7 maps F_m into $J \times \mathrm{Gr}(m, k)$, where $\mathrm{Gr}(m, k)$ was the product of absolute Grassmannians given by (4.45). On $\mathrm{Gr}(m, k)$ we had our linearisation L_{β, β^i_j} as in (4.44).

Applying proposition 5.1.1 to $X = \bar{J}$, M as in (3.4), $Y = \mathrm{Gr}(m, k)$ (for sufficiently large k as required by theorem 4.9.2) and $L := \psi_k^* L_{\beta, \beta^i_j}$, we obtain

$$\mathrm{pr}_{\bar{J}}^{-1}(\bar{J}^s(M)) \subset (\bar{J} \times \mathrm{Gr}(m, k))^s(\mathrm{pr}_{\bar{J}}^* M^{\otimes b} \otimes \mathrm{pr}_{\mathrm{Gr}(m,k)}^* L) \tag{5.3}$$

and

$$(\bar{J} \times \mathrm{Gr}(m, k))^{ss}(\mathrm{pr}_{\bar{J}}^* M^{\otimes b} \otimes \mathrm{pr}_{\mathrm{Gr}(m,k)}^* L) \subset \mathrm{pr}_{\bar{J}}^{-1}(\bar{J}^{ss}(M)) \tag{5.4}$$

for b sufficiently large, where all (semi)stability is with respect to the SL_{N+1}-action only (these are the symmetries of the curves which also act on the bundles). Now (5.3) together with part (a) of theorem 3.1.6 shows that

$$J \times \mathrm{Gr}(m, k) \subset (\bar{J} \times \mathrm{Gr}(m, k))^s(\mathrm{pr}_{\bar{J}}^* M^{\otimes b} \otimes \mathrm{pr}_{\mathrm{Gr}(m,k)}^* L), \tag{5.5}$$

so

$$F_m \subset (\bar{J} \times \mathrm{Gr}(m, k))^s(\mathrm{pr}_{\bar{J}}^* M^{\otimes b} \otimes \mathrm{pr}_{\mathrm{Gr}(m,k)}^* L) \tag{5.6}$$

and all points in our Flag-Quot scheme are SL_{N+1}-stable.

On the other hand, by part (b) of theorem 3.1.6 and (5.4), we see that $J \times \mathrm{Gr}(m, k)$ is closed in $(\bar{J} \times \mathrm{Gr}(m, k))^{ss}(\mathrm{pr}_{\bar{J}}^* M^{\otimes b} \otimes \mathrm{pr}_{\mathrm{Gr}(m,k)}^* L)$, so F_m is closed in

$$(\bar{J} \times \mathrm{Gr}(m, k))^{ss}(\mathrm{pr}_{\bar{J}}^* M^{\otimes b} \otimes \mathrm{pr}_{\mathrm{Gr}(m,k)}^* L),$$

as it is projective over J. If F_m is closed in the semistable locus for the SL_{N+1}-action, then its semistable part with respect to the $\mathrm{SL}_{N+1} \times \mathrm{SL}(V_m)$-action is in particular closed in the semistable locus of $\bar{J} \times \mathrm{Gr}(m,k)$ for the product action, and we may define

$$\overline{U_{g,n}(r,d,r_*^\star,\alpha_*^\star)} := F_m//\mathrm{SL}_{N+1} \times \mathrm{SL}(V_m) \tag{5.7}$$

with respect to the linearisation $\mathrm{pr}_{\bar{J}}^* M^{\otimes b} \otimes \mathrm{pr}_{\mathrm{Gr}(m,k)}^* L$ for SL_{N+1} and the linearisation L for $\mathrm{SL}(V_m)$. The quotient is necessarily a projective variety, and it now remains to show that the (semi)stable points for the product action agree with the $\mathrm{SL}(V_m)$-(semi)stable points, so that the points of the quotient correspond to p_2-(semi)stable parabolic sheaves (by theorem 4.9.2) on stable curves. Firstly, it is clear that the GIT-(semi)stable points for the $\mathrm{SL}_{N+1} \times \mathrm{SL}(V_m)$-action lie in the $\mathrm{SL}(V_m)$-(semi)stable locus (restriction to a subgroup always preserves (semi)stability as we are just testing the Hilbert-Mumford criterion for fewer one-parameter subgroups). However, using the same strategy as Pandharipande, equality of these loci can be shown:

Proposition 5.1.2 (cf. proposition 8.2.1 in [Pan96])**.** A point of F_m is $\mathrm{SL}(V_m)$-(semi)stable with respect to the linearisation L if and only if it is GIT-(semi)stable for the $\mathrm{SL}_{N+1} \times \mathrm{SL}(V_m)$-action with respect to the linearisation $(\mathrm{pr}_{\bar{J}}^* M^{\otimes b} \otimes \mathrm{pr}_{\mathrm{Gr}(m,k)}^* L)|_{F_m}$.

The proof of this result is an explicit calculation using the Hilbert-Mumford criterion as well as (5.5) and (5.6). The key point is that *all* points of F_m are SL_{N+1}-stable, so the numerical criterion shows that only $\mathrm{SL}(V_m)$-(semi)stability matters.

Combining this result with theorem 4.8.1, we see that a point of F_m corresponding to a rigidified stable marked curve $[C, x^1, \ldots, x^n] \in J$ and a rigidified parabolic sheaf $[\mathcal{E}_*^\star, \gamma]$ on $[C, x^1, \ldots, x^n]$ is $\mathrm{SL}_{N+1} \times \mathrm{SL}(V_m)$-(semi)stable with respect to the linearisation $(\mathrm{pr}_{\bar{J}}^* M^{\otimes b} \otimes \mathrm{pr}_{\mathrm{Gr}(m,k)}^* L)|_{F_m}$ if and only if $\mathcal{E}_*^\star$ is p_2-(semi)stable and $\gamma : V_m \to H^0(\mathcal{E}(m))$ is an isomorphism. It remains to explain how orbits in F_m are identified with each other in the GIT quotient (and this argument is also analogous to section 8 in [Pan96]): let $y := [D^\star, \mathcal{F}_*^\star, \delta]$ be a point of F_m lying in the closure of the $\mathrm{SL}_{N+1} \times \mathrm{SL}(V_m)$-orbit of $x := [C^\star, \mathcal{E}_*^\star, \gamma]$. Pick a one-parameter subgroup $\lambda = (\lambda_1, \lambda_2) : \mathbb{G}_m \to \mathrm{SL}_{N+1} \times \mathrm{SL}(V_m)$ such that $\lim_{t\to 0} \lambda(t) \cdot x = y$. In particular, $\lim_{t\to 0} \lambda_1(t) \cdot [C^\star] = [D^\star]$ in J. By theorem 3.1.6(a), J is contained in the stable locus of the SL_{N+1}-action, so $[C^\star]$ and $[D^\star]$ lie in a common SL_{N+1}-orbit and the one-parameter subgroup λ_1 may be extended over 0 to yield $\lambda_1(0) \in \mathrm{SL}_{N+1}$ such that $\lambda_1(0) \cdot [C^\star] = [D^\star]$.

On the other hand, the fibre of the Flag-Quot scheme F_m over the pointed curve $C^\star$ is projective, so the morphism $\mu : \mathbb{G}_m \to F_m(C^\star)$ given by $\mu(t) = \lambda_2(t) \cdot x$ also extends over 0. Write $[\mathcal{G}_*^\star, \epsilon]$ for the rigidified parabolic sheaf on $C^\star$ given by $\mu(0)$, noting that $\mathcal{G}_*^\star$ may not equal $\mathcal{E}_*^\star$ since $[\mathcal{G}_*^\star, \epsilon]$ only lies in the closure of the $\mathrm{SL}(V_m)$-orbit of $[\mathcal{E}_*^\star, \gamma]$, rather than in the orbit itself.

Write $\lambda_1 \cdot \mu : \mathbb{G}_m \to F_m$ for the map given by $t \mapsto \lambda_1(t) \cdot \mu(t)$ (which also extends over 0 since both λ_1 and μ do). Then we have

$$\begin{aligned}\lambda_1(0) \cdot [C^\star, \mathcal{G}_*^\star, \epsilon] &= \lambda_1(0) \cdot \mu(0)\\ &= (\lambda_1 \cdot \mu)(0)\\ &= \lim_{t\to 0}(\lambda_1(t) \cdot \lambda_2(t) \cdot [C^\star, \mathcal{E}_*^\star, \gamma])\\ &= \lim_{t\to 0}(\lambda(t) \cdot x) = y.\end{aligned}$$

Thus, $[C^\star, \mathcal{G}_*^\star, \epsilon]$ lies in the SL_{N+1}-orbit of $y = [D^\star, \mathcal{F}_*^\star, \delta]$, so $C^\star$ and $D^\star$ are isomorphic marked curves (and the corresponding element of SL_{N+1} is an automorphism of the marked curve). By construction $[C^\star, \mathcal{G}_*^\star, \epsilon]$ also lies in the closure of the $\mathrm{SL}(V_m)$-orbit of $x = [C^\star, \mathcal{E}_*^\star, \gamma]$, so x and y are identified in the $\mathrm{SL}_{N+1} \times \mathrm{SL}(V_m)$-quotient if and only if they correspond to isomorphic marked curves $C^\star \cong D^\star$ and the parabolic sheaf $\mathcal{F}_*^\star$ differs from a parabolic sheaf $\mathcal{G}_*^\star$ which is S-equivalent to $\mathcal{E}_*^\star$ by an automorphism of $C^\star$.

Thus, we arrive at our main theorem:

Theorem 5.1.3. Let g and n be non-negative integers (such that $2g + n \geq 3$), let $r \in \mathbb{N}$, $d \in \mathbb{Z}$, $l_i \in \mathbb{N}$ (for $1 \leqslant i \leqslant n$), $r_j^i \in \mathbb{Z}_{\geqslant 0}$ (for $1 \leqslant i \leqslant n$ and $2 \leqslant j \leqslant l_i$) and let

$$0 \leqslant \alpha_1^i < \alpha_2^i < \cdots < \alpha_{l_i}^i < 1$$

be rational numbers for $1 \leqslant i \leqslant n$.

Then the projective GIT quotient $\overline{U_{g,n}(r, d, r_*^\star, \alpha_*^\star)}$ constructed in (5.7) with respect to the linearisation $(\mathrm{pr}_{\bar{J}}^* M^{\otimes b} \otimes \mathrm{pr}_{\mathrm{Gr}(m,k)}^* L)|_{F_m}$ is a coarse moduli space of aut-equivalence classes of pairs $[C^\star, \mathcal{E}_*^\star]$ where $C^\star = (C, x^1, \ldots, x^n)$ is a stable n-marked curve of genus g, and $\mathcal{E}_*^\star$ is a p_2-semistable torsion-free parabolic sheaf (with parabolic structures $F_*^i(\mathcal{E})$ at the marked points $D^i = x^i$) of uniform rank r on each component of C, degree d, flag type $\mathrm{rk}\ \mathcal{E}/F_j^i(\mathcal{E}) = r_j^i$ and parabolic weights α_j^i at x^i. Here, $[C^\star, \mathcal{E}_*^\star]$ and $[D^\star, \mathcal{F}_*^\star]$ are *aut-equivalent* if there is an isomorphism $\phi : C^\star \cong D^\star$ of marked curves and $\mathcal{E}_*^\star, \phi^*\mathcal{F}_*^\star$ are S-equivalent[1] parabolic sheaves on $C^\star$. Furthermore:

(a) There is a projective forgetful map $\eta : \overline{U_{g,n}(r, d, r_*^\star, \alpha_*^\star)} \to \overline{M}_{g,n}$ whose fibre over a marked curve $C^\star$ is the coarse moduli space of p_2-semistable parabolic sheaves (of given numerical type) on $C^\star$ up to S-equivalence, modulo the group of automorphisms of $C^\star$.

[1] in the sense of definition 4.9.1 and theorem 4.9.2.

(b) There is an open subscheme of $\overline{U_{g,n}(r,d,r_*^\star,\alpha_*^\star)}$ whose closed points correspond to aut-isomorphism classes of pairs $[C^\star, \mathcal{E}_*^\star]$ where $C^\star$ is a stable marked curve and $\mathcal{E}_*^\star$ is a p_2-stable parabolic sheaf of given numerical type on $C^\star$, and two pairs $[C^\star, \mathcal{E}_*^\star]$ and $[D^\star, \mathcal{F}_*^\star]$ are *aut-isomorphic* if there is an isomorphism $\phi : C^\star \cong D^\star$ of marked curves and $\mathcal{E}_*^\star$, $\phi^*\mathcal{F}_*^\star$ are isomorphic parabolic sheaves.

Proof. Most of these results follow from the arguments preceding the theorem in this chapter, and from theorem 4.9.2. It only remains to justify that $\overline{U_{g,n}(r,d,r_*^\star,\alpha_*^\star)}$ is a coarse moduli space and that its forgetful map η is projective. The former follows from the local universal property of the Flag scheme F_m (cf. the parallel argument in the proof of theorem 4.9.2) together with the observation that two points of F_m are in a common $\mathrm{SL}_{N+1} \times \mathrm{SL}(V_m)$-orbit if and only if the underlying pairs $[C^\star, \mathcal{E}_*^\star]$ and $[D^\star, \mathcal{F}_*^\star]$ are aut-isomorphic.

The morphism $\eta : \overline{U_{g,n}(r,d,r_*^\star,\alpha_*^\star)} \to \overline{M}_{g,n}$ arises from the $SL(V_m)$-invariant morphism $F_m \to J$, and since the latter is projective, the GIT quotient of F_m is also projective over the GIT quotient of J (see theorem 2.2.1). □

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