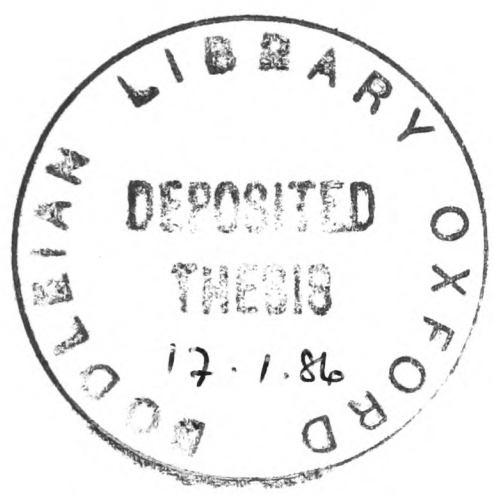
ROBUSTNESS OF MULTIVARIABLE FEEDBACK SYSTEMS:

ANALYSIS AND OPTIMAL DESIGN



Yung Kuan Poo

Oriel College

Submitted in partial fulfilment of the requirements for degree of Doctor of Philosophy at the University of Oxford. Trinity 1985

To my parents

and Irene

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ABSTRACT

The robustness of the stability property of multivariable feedback control systems with respect to model uncertainty is studied and discussed. By introducing a topological notion of arcwise connectivity, existing and new robust stability tests are combined and unified under a common framework. The new switching-type robust stability test is easy to apply, and does not require the nominal and perturbed plants to share the same number of closed right half-plane poles, or zeros, or both. It also highlights the importance of both the sensitivity matrix and the complementary sensitivity matrix in determining the robust stability of a feedback system. More specifically, it is shown that at those frequencies where there is a possibility of an uncertain pole crossing the jw-axis, robust stability is "maximized" by minimizing the maximum singular value of the sensitivity matrix. At frequencies where there is a likelihood of uncertain zeros crossing the imaginary axis, it is then desirable to minimize the maximum singular value of the complementary sensitivity matrix.

A robustness optimization problem is posed as a non-square H^{∞}-optimization problem. All solutions to the optimization problem are derived, and parameterized by the solutions to an "equivalent" two-parameter interpolation problem. Motivated by improvements in disturbance rejection and robust stability, additional optimization objectives are introduced to arrive at the 'best' solution.

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In accordance with the regulations of the University, I hereby declare that this thesis is my own work, except where otherwise indicated, and that it has not been submitted for any degree in this University or elsewhere.

"Computerization is not automation...

Automation is closing the automatic feedback loop."

— Control Engineering, July, 1982.

CONTENTS

		Page
Chapter 1	INTRODUCTION	
1.1	Motivation	
1.1-1	Problem Statement	1
1.1-2	Robustness and Optimality	1
1.1-3	Significance of a Mathematical Theory of Optimum Synthesis	3
1.2	Previous Work and Related Literature	4
1.3	Contribution and Organisation of the Thesis	9
1.4	Notation and Terminology	12
Chapter 2	PRELIMINARIES	
2.1	System Configuration	17
2.2	Design Criteria	20
2.2-1	Stability	22
2.2-2	Sensitivity	28
2.2-3	Performance	35
2.2-4	Integrity	42
2.3	Topology of Linear Time-Invariant Plants	
2.3-1	The Graph Topology	43
2.3-2	Arcwise Connectivity	50
2.4	Continuous and Discontinuous Changes in Nyquist Loci	54
2.5	Concluding Remarks	58

		Page
Chapter 3	REPRESENTATIONS OF UNCERTAINTY AND THEIR IMPLICATIONS IN ROBUSTNESS ANALYSIS	
	3.1 Introduction	60
	3.2 Additive and Multiplicative Perturbations	61
	3.3 Representing Uncertainty by Simultaneous Perturbations	68
	3.4 Multiloop Gain and Phase Margins	71
	3.5 Concluding Remarks	78
	Appendix 3-1 Proof of Theorem 3.3	80
	Appendix 3-2 Proof of Theorem 3.4	81
Chapter 4	ROBUST STABILITY ANALYSIS WITH UNSTRUCTURED UNCERTAINTIES	
	4.1 Introduction	83
	4.2 Nyquist Based Singular Value Robust Stability Tests	86
	4.3 Inverse Nyquist Based Singular Value Robust Stability Tests	93
	4.4 On the Relationship between Lehtomaki's Robust Stability Tests and the Corresponding Inverse Nyquist Based Tests	98
	4.5 Combined Direct/Inverse Nyquist Based Singular Value Robust Stability Test	106
	4.6 Robust Stability Tests Based on Coprime Factorizations	122

Page

4.7 General Switching-Type Robust Stability Tests 125

4.8 Robust Stability Analysis with Actuator and Sensor Failures 131

Appendix 4-1 Proof of Theorem 4.13 136

Chapter 5 ROBUST STABILITY ANALYSIS WITH DISTRIBUTED SYSTEMS

5.1 Introduction 140

5.2 Preliminaries

5.2-1 The $A(\sigma)$ and $B(\sigma)$ Algebras 140

5.2-2 Feedback Stability 143

5.2-3 Characteristic Gain and Inverse Charateristic Gain Loci 145

5.2-4 Poles and Zeros 146

5.3 Direct Nyquist Stability Criteria 146

5.4 Inverse Nyquist Stability Criteria 148

5.5 Robust Stability 155

5.5-1 Nyquist Based Tests 155

5.5-2 Inverse Nyquist Based Tests 155

5.5-3 Nyquist and Inverse Nyquist Combined 157

5.6 Concluding Remarks 159

		Page
Chapter 6	ROBUST STABILITY ANALYSIS WITH STRUCTURED UNCERTAINTIES	
	6.1 Introduction	161
	6.2 Multiple Simultaneous Perturbations	164
	6.2-1 Equivalence in Stability and a Relationship between Nyquist Loci	164
	6.2-2 The Small-μ Theorem with One Nominal Plant Model	167
	6.2-3 The Small-μ Theorem with Several Nominal Plant Models	172
	6.3 Perturbations with Magnitude-Bounded Elements	175
	6.4 Parameter-Dependent Uncertainty	183
	6.5 Concluding Remarks	185
Chapter 7	ROBUST STABILITY WITH NONLINEAR PERTURBATIONS	
	7.1 Introduction	187
	7.2 Preliminaries	188
	7.3 Robust Stability Analysis with a Single Representation of Nonlinear Uncertainty	194
	7.4 Robust Stability Analysis with Stable Perturbations of Coprime Factors	200
	7.5 Robust Stability Analysis with Dual Representations of Nonlinear Uncertainty - Direct and Inverse	

		Page
	Perturbations	202
	7.6 Concluding Remarks	205
Chapter 8	AN H^{∞} MINIMAX APPROACH TO DESIGN	
	8.1 Introduction	207
	8.2 Preliminaries	211
	8.2-1 Basic Definitions and Terminology	212
	8.2-2 The Matrix Nevanlinna-Pick Problem	215
	8.2-3 Optimal Matrix Interpolation	217
	8.3 Optimal Robustness in the Design of Linear Multivariable Systems: A Problem Formulation	219
	8.4 Solution to the Robustness Optimization Problem	223
	8.4-1 Characterization of All Solutions	235
	8.4-2 All-Pass Form Solutions	237
	8.4-3 Conceptual Algorithm	238
	8.4-4 Relaxation of Assumptions on Weighting Matrices and Plant	239
	8.5 Robustness Optimization with Output Regulation	240
	8.6 Best Approximation and Optimal Design	243
	8.7 Concluding Remarks	246
	Appendix 8-1 "Best" Solution to the Robustness Optimization	

Page

Problem 247

Chapter 9 CONCLUSIONS

9.1 Summary 263

9.2 Suggestions for Future Research 264

9.2-1 Non-Square Plants 265

9.2-2 Capturing the Structure of Uncertainty 266

9.2-3 Numerical Implementation of the H^{∞} Optimization 266

APPENDIX A

SINGULAR VALUES OF A MATRIX 267

REFERENCES 269

CHAPTER I

INTRODUCTION

1.1 Motivation

1.1-1 Problem statement

This thesis is motivated by the following engineering problem:

"Given a set of plant models $\{G_p\}$ satisfying certain properties and a set of design specifications, how can we synthesize a linear time invariant feedback controller such that the given specifications are satisfied for all members from the set in an optimal way?"

The motivation is a practical one since uncertainty in some form or other will always be present in a model of an engineering process. For instance, the omission of high frequency dynamics from the model, uncertain parameter values, ageing, and linearization will always be causes of errors. It is therefore absolutely important that any controller designed to give a feedback control system certain desirable properties does so not only on the basis of the nominal plant model but also for any candidate model taken from a set containing some "neighbourhood" of the nominal model. A controller which possesses this property will then be called a robust controller. In practice, $\{G_p\}$ is often (adequately) taken to be linear and time-invariant.

1.1-2 Robustness and optimality

The term robustness as used in this thesis refers to the extent to which a model of an open-loop system may be changed from the nominal design model without causing the overall closed-loop feedback system to lose its desirable properties. Good robustness in this thesis thus

generally includes low sensitivity as well as good robust stability. Nevertheless, stability is the most important property of a feedback control system, and therefore it is most crucial in any feedback design that this property (above all others) be robust.

The word optimality is perhaps more philosophical than mathematical in control system design. In mathematical systems theory, a controller is optimal in connection with a certain cost index if it minimizes the cost with respect to all other admissible controllers. In practice, however, an optimal controller is simply the "most acceptable" controller. The criteria for assessing this "acceptability" are usually mathematically too vague or ill-defined to be readily translated into a cost function for optimization. They will, however, invariably include the following:

* Performance
* Stability
* Sensitivity
* Cost of implementation.

We shall discuss these criteria in detail in later chapters.

The existing gap between theory and practice in control system design is thus arguably attributable to the gap between the mathematical theory of optimality and the physical perception of the "most acceptable" system. If we can develop a mathematical theory of optimal design which appeals to the physical intuition of practising engineers, then we will have succeeded in taking a definite step in closing the "gap". This is the direction in which this thesis is hoped to have contributed.

1.1-3 Significance of a mathematical theory of optimum synthesis

It is sometimes thought necessary to distinguished between the concepts of design and synthesis, [MAC1]. It has been argued that in a synthesis technique one strives to formulate a sharply defined problem with a well founded mathematical solution, whereas in a design technique, one is given a set of manipulative and interpretive tools which will enable one to build up, modify and assess a design put together on the basis of physical reasoning within the guidelines laid down by engineering experience. Thus, the optimal linear quadratic (LQ) control problem, [AND1], and its solution may be viewed as a good example of a synthesis procedure, while the characteristic locus design method, [MAC4], may be viewed as a design technique.

While we will not dispute this distinction, we stress that there is no reason why a good synthesis procedure should not be a good design technique, or why it should not, at least, form part of a good design technique.

A mathematical theory of optimum synthesis which appeals to the physical reasoning and engineering experience of the practitioners could have the following significance:

(i) In practice it is often not easy to make a judgement of what is attainable and what is impossible to achieve even under a set of ideal assumptions. A theory of optimum synthesis could help us to understand what is the "best attainable" performance and hence to eliminate unrealistic ambitions.

(ii) In practice we may be given so much information and criteria that we do not know how to make the best use of it all. An ad hoc approach could lead to disaster. Under such circumstances, a mathematically "optimal" controller which has been derived from a

synthesis procedure which uses a subset of available information may provide a good starting point. An improved design may then be attempted based on this initial design by incorporating more information.

(iii) A synthesis procedure that directly manipulates and optimizes on quantities which carry meaningful engineering interpretations (such as stability margins and sensitivity) could be a very powerful and systematic tool forming part of a design procedure.

1.2 Previous Work and Related Literature

Bode was the first to perform a systematic study on the robustness issue in feedback designs. His classical work, [BOD1], was based on the stability theory of Nyquist, [NYQ1], and was concerned with (i) robust stability through the introduction of gain margin and phase margin as measures of robust stability and (ii) sensitivity through his study on the effects of vanishingly small perturbations on the closed-loop performance of the system and its relation to the magnitudes of the return-ratio ("loop gain") and the return-difference of a feedback design. His celebrated integral formula was the first of its kind to explicitly demonstrate the need for a trade-off between the two (viz., stability margin and sensitivity).

The works of Bode, Evans [EVA1], and Nichols [JAM1] when taken together give engineers a fairly simple yet very powerful approach for designing single-input, single-output control systems. In a way, feedback systems designed by a combination of these classical techniques are inherently robust, because any uncertainty in the plant model may be viewed as a "fuzzy" region around the nominal Nyquist locus.

Sufficiently large gain and phase margins therefore, in a heuristic way, serve to ensure the robust stability of the system when subjected to relatively large perturbations. De-sensitization is achieved by employing sufficiently high gain at low frequencies, and robust stability against nonlinearities is implicitly assured by the circle criterion, [ZAM1].

By the latter half of the fifties, because of escalating performance requirements (which were partly due to the success of the classical techniques in earlier control system designs), engineers had become increasingly interested in designing not only a satisfactory system, but the "best possible" system. Two important publications had appeared in the literature. The first was due to Newton, Gould and Kaiser, [NEW1], who tried to apply the earlier work of Wiener on optimal filtering, [WIE1], to control system design. The second was due to Horowitz, [HOR1], who refined the works of Bode et. al. to what he called a Quantitative Feedback Theory, [HOR2].

Despite the appearance of these two books, the control community was obviously moving in the direction of multivariable control. This movement led to the development of the so called "modern" or state-space approach.

The appeal of the state-space approach may partly be attributed to the inability of the earlier classical control theorists to generalize the works of Nyquist and Bode to the general multivariable case, and partly to the ability of the new approach to generate elegant and optimal solutions to control problems arising in aerospace applications, where realistic problems of time, fuel, or energy minimization could be posed. The state-space approach also leads to a better mathematical formulation of the problems involved, and a better understanding of the control

systems from an internal and structural point of view; see [DAV1], [FRA1], [WOL1], and [WON1], for example. Nevertheless, the state-space approach was found lacking in its ability to include robustness specifications as easily and as naturally as in the classical techniques. This was because the solutions which these modern techniques generate are generally solutions to a sharply defined mathematical problem, which neither explicitly nor implicitly takes into account the factor of uncertain parameters or an uncertain model.

Several suggestions and proposals on how to to circumvent this difficulty were made. The first was to model the uncertainty as noise, [ATH1]. But there was neither a well-founded technique nor a theory to tell a designer how this should be done, [HOR2]. An alternative approach was to use a dual optimal control approach, [FEL1], but sufficient statistics, [AOK1], of the uncertainty would then need to be known. Also, the robustness implications of such an approach were not clear. The third approach was to assume the unknown parameters to lie in a certain set, and a minimax solution instead of the nominal (say LQG) solution would then be calculated, [SWO1]. While this approach did have some appeal from a robustness point of view, it was, unfortunately, in general not easy to find the solutions, except in very simple cases, [SWO1].

The need for an accurate model, the difficulty of translating performance requirements into a cost functional, and the lack of a natural and effective tool for robustness analysis discouraged practising engineers from applying the state-space approach to many industrial control problems in which the model and performance objectives were not well-defined mathematically. Hence, by the late sixties, interest in the frequency domain approach had been revived, and several important works

had appeared, [ROS1]-[ROS4], [MAC3]-[MAC5], [POS1], [YOU1], [BAR1] and [KAL1].

Although these contributions succeeded in extending many of the classical frequency domain concepts to the multivariable case, they failed to address satisfactorily the issue of robustness. For example, in the characteristic locus design technique, a large gain margin and a large phase margin in the characteristic gain loci do not necessarily imply a good robust stability property, because the eigenvalues could be very sensitive if the matrix is ill-conditioned, [WIL1], [POS2].

In his thesis, [SAF1], Safonov introduced a notion of gain margin and phase margin for multivariable systems. Using these definitions, he was able to show, that when a diagonal matrix is used for control weighting, the linear quadratic regulator (LQR) and the Kalman Filter possess a gain margin of $[1/2,\infty)$ and a phase margin of at least 60 degrees in each channel at the plant's input. This work, coupled with earlier research which showed that the LQR exhibits de-sensitization in a mean-square sense, [CRU1], [KRE1], [KWA1], led many to believe that the linear quadratic Gaussian (LQG) approach was the technique for robust control systems design. However, Doyle [DOY1] showed that when a Kalman Filter is combined with the optimal regulator, the system as a whole might exhibit a poor stability margin in spite of the fact that the filter and the regulator possess good gain and phase margins individually. Doyle and Stein, [DOY2], later suggested a sensitivity recovery procedure, due to Kwakernaak [KWA2], and a robust stability ("robustness") recovery procedure, [DOY3], as remedies, but these techniques may be used only if the plant is minimum-phase, and, as we will see in chapter 3, the robustness implication is not clear because of the definitions of the gain and phase margins.

Since the work of Doyle and Stein, [DOY2] and [DOY4], maximum singular value based techniques have emerged as the most popular of tools for robustness analysis of linear multivariable systems, [POS3], [SAF3], [CRU2], [LEH1], [KOU1] and [FRE1]. For example, in [POS3] Postlethwaite et. al. developed a robust stability result for use when both the maximum and minimum principal gains (singular values) and principal phases of the perturbations are known; in [CRU2], Cruz et. al. considered both sensitivity and robust stability in terms of maximum singular values; Lehtomaki et. al., [LEH1], derived a new robust stability test in terms of the inverse of the perturbation and used it to re-derive Safonov's earlier result on the robust stability of LQR; and Safonov et. al., [SAF3], examined the "loop-shaping" of the maximum singular value by means of the LQG approach.

Efforts have also been made to find optimal scaling (weighting) matrices to reduce the conservatism of the spectral norm (maximum singular value) based robust stability tests. In [SAF4], Safonov examined the case of diagonal perturbations, while in [KOU3] Kouvaritakis and Latchman considered the more general case where the magnitude of each element of the perturbation matrix is bounded by a known quantity.

It would be too lenghty an exercise to present in this section a comprehensive review of the literature pertaining to the singular value techniques. However, some of these will be reviewed and discussed in subsequent chapters. For additional information and discussions, the reader is referred to the source-publications given in the reference list.

There are at least two reasons why the singular value techniques have become so popular in the robustness analysis of linear multivariable systems. Firstly, they provide an intuitive and simple multivariable

generalizaion of the notions of "gain", [MAC6], and "distance to the critical point" which were indispensible concepts in the classical techniques for single-input, single-output systems. Secondly, algorithms for the numerical computation of the spectral norm of a matrix are, in general, both efficient and reliable, [LAU1].

Another important direction which has emerged in recent years is the use of the H^{∞} minimax approach (in contrast to the "traditional" H^{2}/LQG approach) for the design of optimal systems. Originating from the seminal work of Zames, [ZAM2], this approach has succeeded in generalizing several important classical concepts in single-input, single-output systems to the multivariable case. We will consider this approach in greater detail in chapter 8.

1.3 Contribution and Organization of the Thesis

Briefly, the basic contribution of this thesis may be viewed as the extension of the existing singular value based robustness analysis techniques to include cases where the number of unstable poles of the open-loop system may be uncertain. The extension leads naturally to a synthesis problem formulation which may appropriately be called the robustness optimization problem in linear multivariable feedback system design. An H^{∞} minimax approach to the solution of the problem is then suggested and examined. The extension and adaptation of the robustness optimization technique to the general design of robust optimal systems is also considered.

The thesis is organized into nine chapters of which this introduction is the first. In chapter 2, we give a brief treatment of

some of the necessary elements of system theory and design. In particular, a discussion on the recently proposed two-degree-of-freedom feedback configuration, [PER1], is included. It is shown that, despite the additional degree of freedom it offers in design, the basic trade-offs in robustness remains the same as those in the classical configurations, [HOR1], [ANT1]. Thus, any robustness results which were developed for the classical single-degree-of-freedom configuration carry readily to the two-degree-of-freedom configuration. We then introduce a topology of (possibly unstable) convolution plants, due to Vidyasagar, Schneider and Francis, [VID1], and the concept of arcwise connectivity, [SIN1], [POS1], [FOO1], in this topology. A discussion of when and how a Nyquist locus may change discontinuously is also included.

Chapter 3 contains a critical examination of the problem of uncertainty representations and its implications in robustness analysis. It is shown that, if the number of outputs of the plant is greater than the number of its inputs, then the input-multiplicative representation is in general inadequate and hence unsuitable for robustness analysis. The implication of this observation is that the guaranteed gain and phase margins of the LQR system do not necessarily guarantee good robust stability.

In chapter 4, we present three types of singular value based robust stability tests, namely, the direct Nyquist based tests, the inverse Nyquist based tests, and the combined direct/inverse Nyquist based tests. The direct Nyquist based tests are not new and stem from the early work of Doyle and Stein, [DOY2], but they are included here for completeness, and also to emphasize the duality they hold with the inverse Nyquist based tests. Both the inverse Nyquist based tests and the combined direct/inverse Nyquist based tests do not require the nominal model and

the perturbed plant models to have the same number of unstable poles. A general switching-type robust stability test is then developed, which includes the combined direct/inverse Nyquist based tests, the graph metric robust stability test, [VID4], and the coprime factors based robust stability test, [VID6], as special cases.

Chapter 5 considers distributed systems: the robust stability results of chapter 4 are extended to include a class of distributed systems, [POS7]. Generalized Nyquist criteria and generalized inverse Nyquist criteria are proved in this chapter. Our formulation of the inverse Nyquist criteria is new in the literature, in the sense that we require neither the plant nor the controller to have full rank. In fact, it will be shown that the direct Nyquist criteria and the inverse Nyquist criteria are equivalent irrespective of whether the plant is invertible.

The results of chapters 4 and 5 are generalized in chapter 6 to allow for simultaneous multiple perturbations. The generalizations represent extensions of Doyle's structured singular value robustness test, [DOY5], [DOY6]. The extensions allow one to admit unstable perturbations, as well as the use of different nominal models to reduce conservatism. Brief discussions of robust stability analysis in the presence of elementwise magnitude-bounded perturbations, and parameter-dependent perturbations, are also included in this chapter.

Since all real systems are necessarily distributed and nonlinear, no practical robustness test may be considered realistic unless it can admit a certain class of nonlinear as well as distributed perturbations. In chapter 7, we examine the robust stability of linear systems subjected to nonlinear perturbations.

In chapter 8, the problem of designing a robust optimal system is considered. Based on the results of earlier chapters, an H^{∞} minimax

formulation of a robustness optimization problem in linear multivariable system synthesis is proposed and a solution provided. An extension and adaptation of the formulation is then proposed to tackle the more general problem of finding the "best" robust optimal system design.

Chapter 9 contains the concluding remarks and suggestions for future research.

Throughout the thesis, the reader will find that, except in chapters 5 and 7, we have confined ourselves to the treatment of lumped linear time-invariant models. Apart from the familiarity they offer, another reason is that these are the most commonly used linear models in practical designs.

1.4 Notation and Terminology

The following notation will be adopted unless otherwise redefined in a specific chapter or section:

Z	The integers
R	The real numbers
C	The complex numbers
Re(s)	The real part of the complex number s
Im(s)	The imaginary part of the complex number s
arg(s)	The argument of the complex number s
$\bar{s}$	The complex conjugate of s
diag[t_1,t_2,..]	Diagonal matrix with elements t_1,t_2,..
Block diag[P_1,P_2,..]	Block diagonal matrix with diagonal blocks P_1,P_2,..
rank(M)	The rank of the matrix M
det(M)	The determinant of the square matrix M

M^H	The complex conjugate transpose of the matrix M; if M is real rational, then $M^H \triangleq M(s)^T$
M^*	The adjoint of the matrix M; if M is real rational, then $M^* \triangleq M(-s)^T$
$tr(M)$	Trace of the square matrix M
$\langle . \mid . \rangle$	The inner product
$\lambda_i(M)$	The ith eigenvalue of the square matrix M
$\lambda_{max}(M)$	The most positive (or least negative) eigenvalue of the hermitian matrix M
$\lambda_{min}(M)$	The most negative (or least positive) eigenvalue of the hermitian matrix M
$\rho(.)$	Spectral radius
$\sigma_i(M)$	The ith singular value of the matrix M; $\sigma_i(M) \triangleq \lambda_i^{\frac{1}{2}}(M^H M)$
$\bar{\sigma}(M)$	The maximum singular value of the matrix M
$\sigma(M)$	The minimum singular value of the matrix M
$\|.\|_p$	pth order norm
$\|M\|$	L_2 induced norm of operator M and the equivalent H^∞ supremum norm of its transfer function matrix
I	The identity matrix
j	$\sqrt{(-1)}$
M^{-1}	Inverse of the matrix M
C^{nxm}	The space of nxm matrices with element in C
R^{nxm}	The space of real nxm matrices
R(s)	The space of real rational functions
$R(s)^{nxm}$	The space of nxm matrices with elements in R(s)
S	The space of real rational proper transfer functions which are analytic in the closed RHP

S^{nxm}	The space of nxm matrices with elements in S
Mat(.)	The space of matrices (of appropriate dimensions) with elements in (.)
H^{∞}	The Hardy space functions bounded and analytic in the open RHP
H_z^{∞}	Hardy space of bounded and analytic functions in the open unit disc
H^2	Hardy space of functions analytic in the open RHP and square integrable on the jω axis
H_z^2	Hardy space of functions analytic in the open unit disc and square integrable on the unit circle
$b \epsilon A$	b is an element of the set A
LHP	Left half complex plane
RHP	Right half complex plane
CRHP	Closed right half complex plane
$\|x\|$	The magnitude of a vector or scalar x. When x is a time function, $\|x(t)\|$ denotes the instantaneous magnitude of x(t) at time t
$A>B$	A-B is positive definite
$A \geq B$	A-B is nonnegative definite
$[\|A\|]$	A real matrix whose elements are equal to the magnitudes of the corresponding elements of A
$[\|A\|] \leq P$	Each element of the real matrix $[\|A\|]$ is smaller than or equal to the corresponding element of the real matrix P

$N(\Omega, f(s), C)$	The number of clockwise encirclements of the point Ω by the locus of $f(s)$ as s traverses the closed contour C in the complex plane in a clockwise sense
$E(s)$	
$\Delta_o(s)$	Direct type of perturbations representing plant
$\Delta_i(s)$	uncertainty
$\tilde{E}(s)$	
$\tilde{\Delta}_o(s)$	Inverse type of perturbations representing plant
$\tilde{\Delta}_i(s)$	uncertainty
$\triangleq$	equal to, by definition
=>	implies
<=	is implied by
<=>	implies, and is implied by
$V(s) \circ W$	Image of W under V(s)
D_D	Direct Nyquist D-contour with indentations to the left of imaginary axis poles, and with a clockwise orientation
D_I	Inverse Nyquist D-contour with indentations to the left of imaginary axis zeros, and with a clockwise orientation
Ω_D	Segment of D_D for which $Re(s) \leq 0$
Ω_I	Segment of D_I for which $Re(s) \leq 0$
sup	Supremum
ess sup	Essential supremum
inf	Infimum
max	Maximum
min	Minimum

r.c.f. Right-coprime factorization

l.c.f. Left-coprime factorization

L_2 Space of square-integrable time (vector) functions

A property of a dynamical system is said to be generic if the systems equipped with this property occupy an open and dense subspace of the space of systems; i.e. if a property is generic, then it is robust against infinitesimally small change in the system.

If **M** is a convolution operator, then **M**(s) denotes its transfer function matrix. Occasionally, the argument s of **M**(s) will be dropped if there is no danger of confusion. Note that $\bar{\sigma}(.)$ and $\underline{\sigma}(.)$ always apply to transfer functions, or frequency response matrices.

A rational function or matrix is said to be stable if it is analytic in $\mathrm{Re}(s)\geq 0$, proper if it is finite at $s=\infty$, and strictly proper if it is zero at $s=\infty$.

A square matrix in Mat(**S**) is unimodular if it has an inverse which is in Mat(**S**). Two matrices in Mat(**S**), **N** and D, with the same number of columns, are right-coprime if every common right-divisor is unimodular; or equivalently, there exists matrices **X** and **Y** ϵ Mat(**S**) such that $XD+YN=I$. Similarly, two matrices in Mat(**S**), $\tilde{\mathbf{N}}$ and $\tilde{D}$, with the same number of rows are left-coprime if and only if there exist $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ ϵ Mat(**S**) such that $\tilde{D}\tilde{Y}+\tilde{N}\tilde{X}=I$. In this thesis, all coprime factorizations are defined in terms of elements from Mat(**S**).

CHAPTER II

PRELIMINARIES

2.1 System Configuration

We will normally be interested in the feedback configuration of figure 2.1, where G, C and F_s are convolution operators representing the plant, the controller and the sensors respectively; $u(t)\epsilon R^m$, $y(t)\epsilon R^l$, $r(t)\epsilon R^p$, $z(t)\epsilon R^l$, $d(t)\epsilon R^l$, and $m(t)\epsilon R^l$ represent the control effort, actual output, reference input, measured output, plant disturbance and measurement noise respectively; and C is given by

$$u(t) = C(r(t),\ z(t))$$
$$= C_r r(t) - C_z z(t) \qquad (2.1)$$

Note that (2.1) is more general than the classical feedback configurations where C is assumed to be in one of the following forms (figures 2.2, 2.3, 2.4 and 2.5):

(a) $u(t) = Kr(t) - Kz(t)$ (2.2)

(b) $u(t) = Kr(t) - KFz(t)$ (2.3)

(c) $u(t) = KPr(t) - Kz(t)$ (2.4)

(d) $u(t) = Pr(t) - Fz(t)$ (2.5)

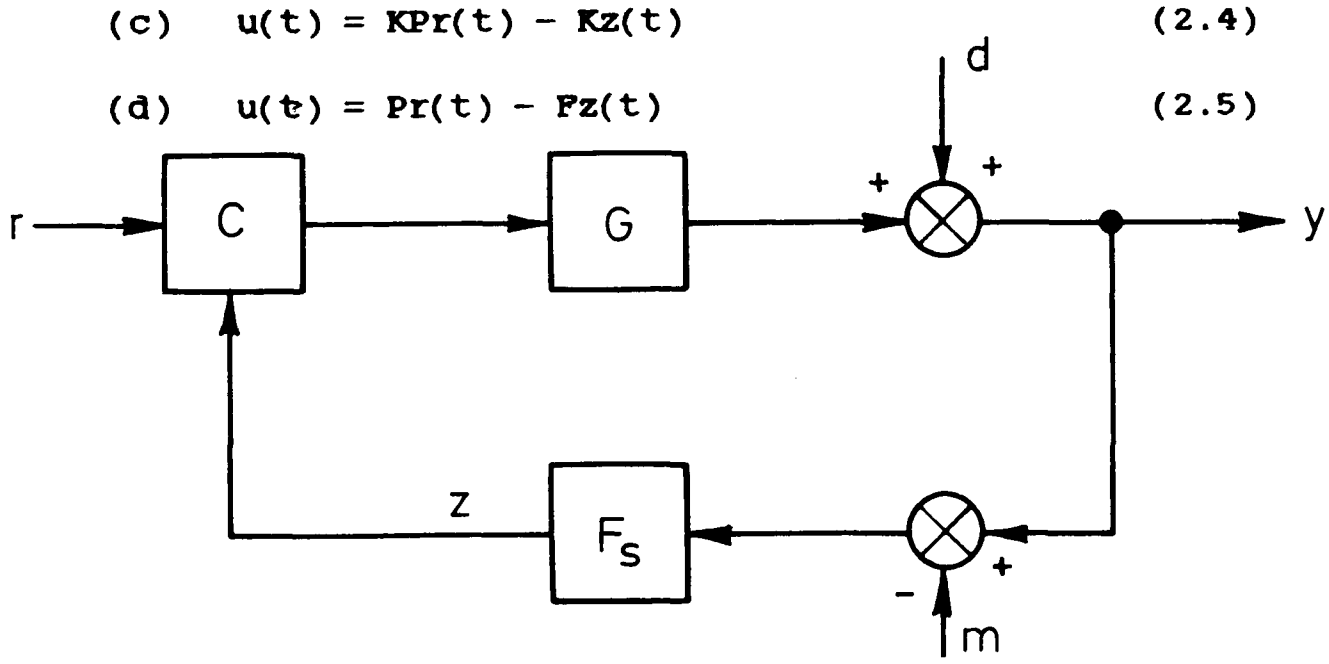


Figure 2.1 Two-degree-of-freedom feedback configuration

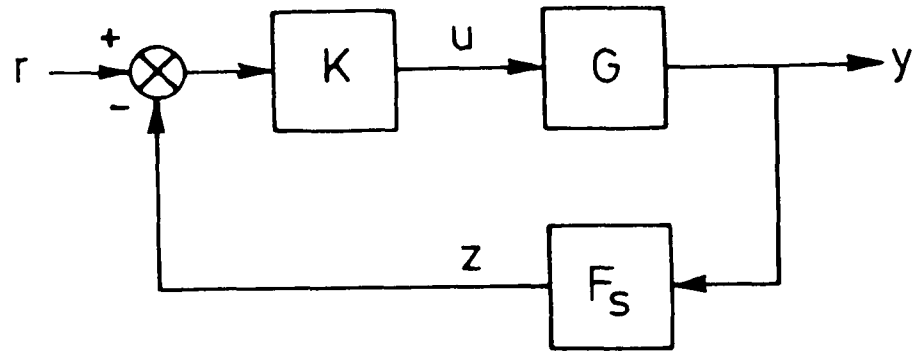


Figure 2.2 Classical feedback configuration (a)

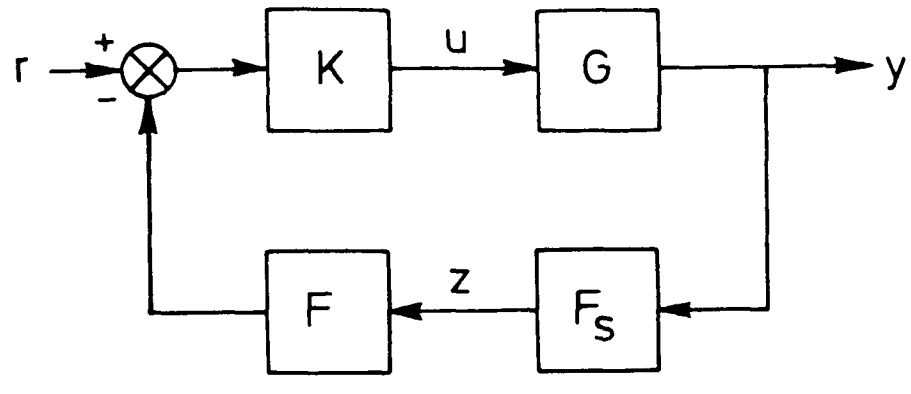


Figure 2.3 Classical feedback configuraton (b)

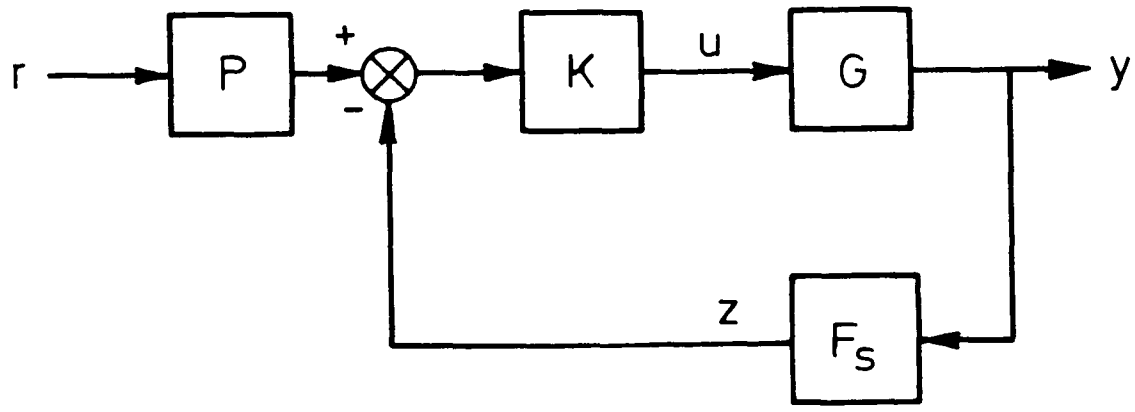


Figure 2.4 Classical feedback configuration (c)

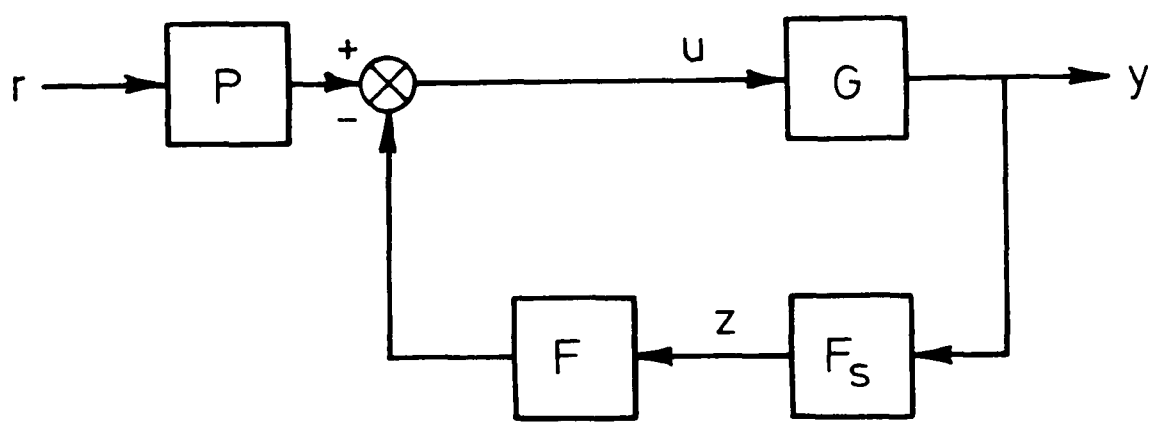


Figure 2.5 classical feedback configuration (d)

It will be interesting to compare (2.1) and (2.5) since the expressions are essentially identical. However, as feedback configurations, figure 2.5 suffers from the constraint that the pre-filter P has to be stable, whereas in figure 2.1 C_r can be slightly more general, as will be shown later.

We will usually not differentiate between a convolution operator A and its Laplace transform (i.e. its transfer function matrix) A(s) unless there is a necessity to do so. We will also use the words system, operator and transfer function (matrix) interchangeably since they usually mean the same thing. For simplicity the arguments of a function are often omitted.

The transfer function matrices G(s) and $F_s(s)$ are assumed to be prescribed in advance, and

$$C(s) = [C_r(s) \quad -C_z(s)] \tag{2.6}$$

is the controller transfer function matrix from the pair of inputs r(s) and z(s) to the control effort u(s). Obviously

$$u = R_r r + R_z F_s (d-m) \tag{2.7}$$

where

$$R_r = (I + C_z F_s G)^{-1} C_r \tag{2.8}$$

and

$$R_z = (I + C_z F_s G)^{-1} C_z \tag{2.9}$$

The closed-loop transfer function matrices from r to y, d to y and m to y are, respectively,

$$H_{yr}(s) = G(s)R_r(s) \tag{2.10}$$

$$S(s) = (I + GC_zF_s)^{-1} \tag{2.11}$$

$$T(s) = (I + GC_zF_s)^{-1}GC_zF_s \tag{2.12}$$

If we assume G, C and F_s to be lumped systems, that is, their Laplace transforms are rational transfer function matrices, then a necessary and sufficient condition for closed-loop stability is that all expressions (2.8) to (2.12) are analytic in the CRHP and are proper.

For practical reasons, we shall assume G to be strictly causal, and C, F to be causal. Thus the well-posedness of the feedback system, [WIL3], and properness of expressions (2.8) to (2.12) are ensured. We shall also assume, as we must, that there are no hidden unstable modes in G, F_s and F_sG.

2.2 Design Criteria

Let R(s) denote the set of all rational functions in s with real coefficients; S the subset of R(s) consisting of proper rational functions whose poles are all in the open left half-plane; and Mat(R(s)) (respectively Mat(S)) the set of matrices with elements in R(s) (respectively S). Then every $F_sG \in$ Mat(R(s)) has coprime factorizations, [DES1], [VID1], [VID2]

$$F_sG = ND^{-1} = \tilde{D}^{-1}\tilde{N} \tag{2.13}$$

where N, $\tilde{N}$, D, $\tilde{D} \in Mat(S)$. It follows that there exists X, $Y \in Mat(S)$, [VID1], [DES2], such that

$$XN + YD = I \tag{2.14}$$

The set of all lumped linear time invariant stabilizing compensators C may then be parameterized by, [YOU2]:

$$C = \{(Y - Q\tilde{N})^{-1}[R \quad X+Q\tilde{D}]\} \tag{2.15}$$

where $R \in Mat(S)$ is arbitrary, and $Q \in Mat(S)$ is arbitrary apart from satisfying the constraint that

$$\det(Y - Q\tilde{N}) \neq 0 \tag{2.16}$$

It is important to note, that under the parameterization of (2.15), C_z is independent of the free parameter R, and expressions (2.9), (2.11) and (2.12) depend only on C_z, not C_r. It follows that the introduction of an additional parameter R (or equivalently C_r) does not give any additional flexibility to trade off the effects of the disturbances d and measurement noise m on the output y. From this point of view, the classical configuration will do just as well as the new two-degree-of-freedom configuration.

However, from a command response point of view, expressions (2.8) and (2.10) are given by:

$$R_r = DR \tag{2.17}$$

$$F_s H_{yr} = NR \qquad (2.18)$$

which are independent of the parameter Q, and hence C_z. We have thus achieved a complete separation in control system design of command response and feedback properties, [PER1].

For design, we may therefore break down our design procedure into two stages:

(i) Synthesize Q, and hence C_z for desirable feedback properties, e.g. disturbance rejection, stability, sensitivity and robust stability.

(ii) Synthesize R, and hence C_r for desirable command response.

In the remainder of this section, we will briefly discuss how these may be achieved.

2.2-1 Stability

Although we have given a parameterization of the set of stabilizing compensators for design purposes, the expression in itself may not have much appeal to a design engineer who is usually trained to think in terms of frequency responses, rather than coprime factorizations. In this subsection we aim to give Nyquist-type stability criteria for the two degree of freedom systems, and hence extend the classical approach to the two degree of freedom design problem.

Theorem 2.1. Let $F_s G$ of figure 2.1 admit the left coprime factorization $F_s G = \tilde{D}^{-1}\tilde{N}$, and C_z the right coprime factorization $C_z = N_{cz} D_{cz}^{-1}$. Then a necessary and sufficient condition for the closed-loop system to be stable is that $[\tilde{D}D_{cz} + \tilde{N}N_{cz}]$ has no zeros in the closed right half-plane,

and C_r is chosen as given by the expression (2.15) with any arbitrary proper and stable R.

Proof The necessity part is obvious because otherwise $[I + F_s GC_z]^{-1}$ would be unstable. To prove the sufficiency part, we have from expression (2.15)

$$(Y - \tilde{Q}\tilde{N})^{-1}(X + \tilde{Q}\tilde{D}) = C_z$$

$$\Leftrightarrow \quad X + \tilde{Q}\tilde{D} = YC_z - \tilde{Q}\tilde{N}C_z$$

$$\Leftrightarrow \quad \tilde{Q}\tilde{D} + \tilde{Q}\tilde{N}C_z = YC_z - X$$

$$\Leftrightarrow \quad Q(\tilde{D} + \tilde{N}N_{cz}D_{cz}^{-1}) = YN_{cz}D_{cz}^{-1} - X$$

$$\Leftrightarrow \quad Q = (YN_{cz} - XD_{cz})(\tilde{D}D_{cz} + \tilde{N}N_{cz})^{-1}$$

Thus Q is stable if $[\tilde{D}D_{cz} + \tilde{N}N_{cz}]^{-1}$ is stable. ※

It should be clear to the reader from coprime factorization theory as applied to single-degree-of-freedom systems, that $[\tilde{D}D_{cz} + \tilde{N}N_{cz}]^{-1}$ is analytic in the CRHP if, and only if, $[I + F_s GC_z]^{-1}$, and hence $[I + GC_z F_s]^{-1}$, is analytic in the CRHP and there are no pole-zero cancellations in the CRHP; that is, if P_{fs}, P_g and P_{cz} are the number of unstable poles of F_s, G and C_z respectively, then $P = P_{fs} + P_g + P_{cz}$, where P denotes the number of unstable poles of the return-ratio matrix $F_s GC_z$. To determine the stability of $[I + F_s GC_z]^{-1}$, one may then apply the standard multivariable Nyquist criterion to $F_s(s)G(s)C_z(s)$, [POS1].

Definition 2.1 The characteristic gains of a rational transfer matrix $M(s) \in R(s)^{nxn}$ are the image under the map s -> (eigenvalues of M(s)) of the clockwise contour D_D consisting of (i) a straight line segment from s=-jr to s=+jr with r arbitrarily large, except for the infinitesimal left indentations around any poles of M(s) with trivial real-part, and

(ii) a semi-circle in the open right half plane joining s=-jr to s=+jr enclosing all the unstable poles of M(s).

Lemma 2.1 (A generalized multivariable Nyquist criterion for C_z). The return-difference matrix $[I + F_s G C_z] \in R(s)^{nxn}$ has an inverse which is analytic in the D_D contour if, and only if, the characteristic gain loci of $F_s G C_z \in R(s)^{nxn}$ encircle the (-1,j0) point, P times anti-clockwise, where P is the number of open-loop poles of $F_s G C_z$ in D_D.

Proof See [POS1], [SMI1]. ∎

The next corollary gives the determinant version of Lemma 2.1

Corollary 2.1 The return-difference matrix $[I + F_s G C_z] \in R(s)^{nxn}$ has an inverse which is analytic in the D_D contour if, and only if, the image of the Nyquist D-contour, D_D, under $\det[I + F_s G C_z] \in R(s)$ encircles the origin P times anti-clockwise, where P is the number of open-loop poles of $F_s G C_z$ in D_D.

Proof Obvious, since the determinant is equal to the product of the eigenvalues. (See also [HSU1]) ∎

We will next turn our attention to an investigation of C_r. Expression (2.18) gives

$$F_s H_{yr} = NR \tag{2.18}$$

$$= F_s G(I + C_z F_s G)^{-1} C_r \tag{2.19}$$

$$= N(\tilde{D}_{CZ} D + \tilde{N}_{CZ} N)^{-1} \tilde{D}_{CZ} C_r \tag{2.20}$$

Compare (2.20) with (2.18), and we see that R is stable if, and only if, $\tilde{D}_{cz}C_r$ is stable. This in turn implies that C_r is part of a stabilizing compensator C if, and only if, it can be written in the form:

$$C_r = \tilde{D}_{cz}^{-1}A \qquad (2.21)$$

where A is any rational matrix analytic in the CRHP such that C_r is proper. We therefore have the following lemma:

Lemma 2.2 Let $(\tilde{N}_{cr}, \tilde{D}_{cr})$, $(\tilde{N}_{cz}, \tilde{D}_{cz})$ be the left coprime factorizations of C_r and C_z respectively. Then there exists $\tilde{A}(s) \in Mat(R(s))$ analytic in the CRHP such that

$$\tilde{D}_{cz}(s) = \tilde{A}(s)\tilde{D}_{cr}(s) \qquad (2.22)$$

Proof We know that $\tilde{D}_{cz}(s)C_r(s)$ is stable from (2.20). This again implies that $\tilde{D}_{cz}\tilde{D}_{cr}^{-1}$ is stable. Thus, we may let

$$\tilde{A}(s) = \tilde{D}_{cz}(s)\tilde{D}_{cr}(s)^{-1} \qquad (2.23)$$

which is analytic in the CRHP. Thus,

$$\tilde{A}\tilde{D}_{cr}\tilde{D}_{cz}^{-1} = I \qquad (2.24)$$

$$<=> \qquad \tilde{A}\tilde{D}_{cr} = \tilde{D}_{cz} \qquad (2.25)$$

which completes the proof. ∎

Lemma 2.2 has important implications in practice because to derive a feasible implementation of the two-parameter compensator, we may define $C = [C_r \; -C_z]$ and let $([\tilde{N}_{c1} \; -\tilde{N}_{c2}], \tilde{D}_c)$ be a left coprime factorization of C. Thus, $C_r = \tilde{D}_c^{-1}\tilde{N}_{c1}$ and $C_z = -\tilde{D}_c^{-1}\tilde{N}_{c2}$. The feedback law (2.7) may then be implemented as shown in figure 2.6, [VID3].

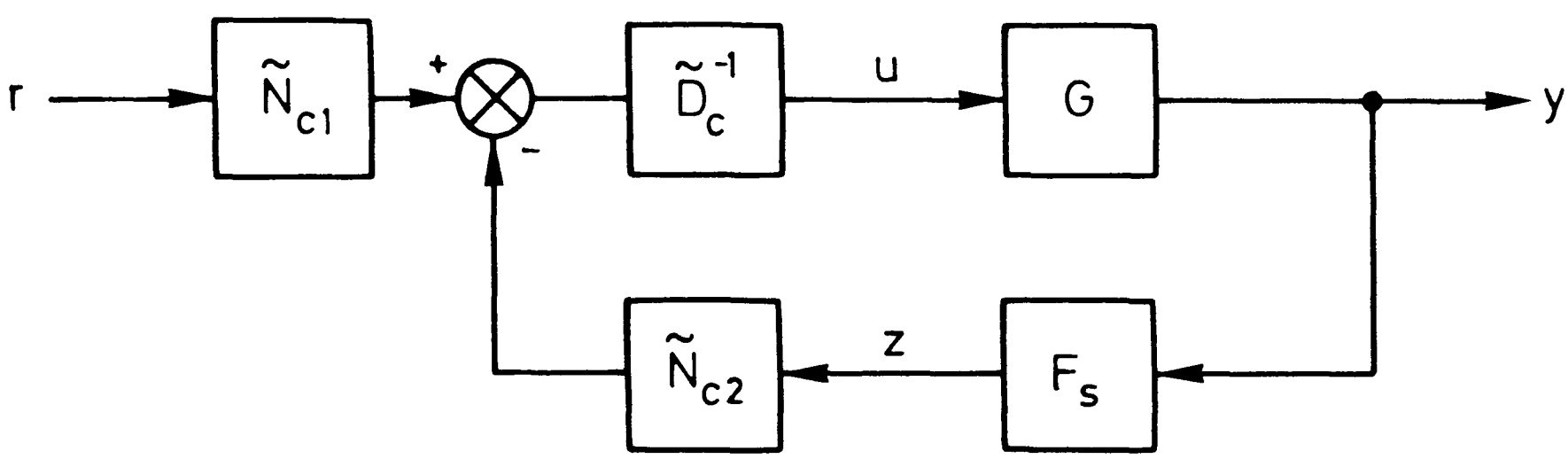


Figure 2.6 Implementation of two-degree-of-freedom feedback configuration

In general, however, the left-coprimeness of $\tilde{D}_c$ and $[\tilde{N}_{c1} \; -\tilde{N}_{c2}]$ does not imply that $\tilde{D}_c$ and $\tilde{N}_{c1}$ are left coprime; i.e. $(\tilde{N}_{c1}, \tilde{D}_c)$ need not be a left coprime factorization of C_r, [VID3]. Similar remarks apply to $\tilde{D}_c$ and $-\tilde{N}_{c2}$. In fact, it may be shown that $\tilde{D}_c$ is a least common left multiple of $\tilde{D}_{cr}$ and $\tilde{D}_{cz}$, and, moreover, if $\tilde{D}_c = \tilde{A}\tilde{D}_{cr} = \tilde{B}\tilde{D}_{cz}$, then $[\tilde{N}_{c1} \; -\tilde{N}_{c2}] = [\tilde{A}\tilde{N}_{cr} \; -\tilde{B}\tilde{N}_{cz}]$, [VID3].

It thus appears that the choice of C_r has a role to play in the robust stability of the system, since $\tilde{D}_c^{-1}$ depends partly on $\tilde{D}_{cr}$ as well. Nevertheless, lemma 2.2 tells us that once C_z is determined, we immediately have

$$\tilde{D}_C = \tilde{A}\tilde{D}_{Cr} = \tilde{D}_{Cz} \qquad (2.26)$$

Note that (2.26) must hold because $\tilde{D}_C$ has to be the least common left multiple of $\tilde{D}_{Cr}$ and $\tilde{D}_{Cz}$, and (2.22) asserts that the right-hand-side of the equality can be satisfied with $\tilde{B} = I$.

Since $\tilde{D}_C = \tilde{D}_{Cz}$ for all stabilizing compensators C, expression (2.26) and figure 2.6 therefor imply that feedback properties of the closed-loop system are solely dependent on the choice of C_z.

Theorem 2.2 Let $F_s G C_z \in \mathbf{R}(s)^{nxn}$ with $F_s G(\infty) = 0$, and let $C_r(s)$, $C_z(s)$ be proper transfer functions in Mat(R(s)). Let P_{fs}, P_g and P_{Cz} denote the number of unstable poles of F_s, G and C_z respectively in D_D. Then the feedback system of figure 2.1 is stable if, and only if:

(i) the Nyquist loci (characteristic gains or determinant) of the return-difference matrix $[I + F_s G C_z]$ do not pass through the origin and encircle it $P_{fs} + P_g + P_{Cz}$ times anti-clockwise;

(ii) C_r and C_z satisfy

$$\nu_\alpha \begin{bmatrix} \tilde{D}_{Cr}(s) \\ \tilde{D}_{Cz}(s) \end{bmatrix} = \nu_\alpha [\tilde{D}_{Cz}(s)] \quad , \ \mathrm{Re}\ \alpha \geq 0 \qquad (2.27)$$

where $\tilde{D}_{Cr}(s)$ and $\tilde{D}_{Cz}(s)$ are the denominator matrices of the left coprime factorizations of $C_r(s)$ and $C_z(s)$ respectively, and $\nu_\alpha[A(s)]$ denotes the total number of zeros of A(s) at α.

Proof (i) <=> $[\tilde{D}_{Cz} D + \tilde{N}_{Cz} N]^{-1}$ stable

(ii) <=> There exists $\tilde{A}$ such that expression (2.22), and hence (2.21) holds, [KAI1].

This completes the proof. ∎

Lemma 2.3 Let $C = [C_r \; -C_z] = \tilde{D}_c^{-1}[\tilde{N}_{c1} \; -\tilde{N}_{c2}]$. Assume that $\tilde{D}_c$, $\tilde{N}_{c2}$ are left coprime, then $\tilde{D}_c$, $[\tilde{N}_{c1} \; -\tilde{N}_{c2}]$ are left coprime for any arbitrary, stable and proper $\tilde{N}_{c1}$.

Proof Note that $\tilde{N}(s)$ and $\tilde{D}(s)$ are left coprime if, and only if, there exists stable and proper rational matrices $\tilde{X}(s)$ and $\tilde{Y}(s)$ such that, [KAI1], [DES2]:

$$\tilde{N}(s)\tilde{X}(s) + \tilde{D}(s)\tilde{Y}(s) = I \qquad (2.28)$$

$\tilde{D}_c$, $-\tilde{N}_{c2}$ are coprime ~~and thus~~ imply that $\tilde{X}_{c2}$ and $\tilde{Y}_{c2}$ exist. To show that $\tilde{D}_c$ and $[\tilde{N}_{c1} \; -\tilde{N}_{c2}]$ are left coprime, let $\tilde{X}_c = [0 \;\; X_{c2}^T]^T$ and $\tilde{Y}_c = \tilde{Y}_{c2}$. This completes the proof. ∎

Note that $[\tilde{N}_{c1} \; -\tilde{N}_{c2}]$ and $\tilde{D}_c$ being left coprime implies that there exists $\tilde{A}$ such that $\tilde{D}_c = \tilde{A}\tilde{D}_{cr}$ since $\tilde{D}_c$ is the least common left multiple of $\tilde{D}_{cr}$ and $\tilde{D}_{cz}$. This again implies that condition (2.27) of theorem 2.2 always holds as long as $\tilde{N}_{c1}$ remains stable, and $\tilde{N}_{cz}$, $\tilde{D}_{cz}$ remain coprime. Thus, if we assume $\tilde{N}_{c1}$ in figure 2.6 to always remain stable (a necessary condition for stability), it suffices to concentrate on C_z to determine robust stability.

2.2-2 Sensitivity

Sensitivity, or differential sensitivity, as it is called in the control community, generally refers to the effects of infinitesimally small perturbations. This is contrasted to the term robustness which usually refers to tolerance of large perturbations. There are numerous methods for sensitivity analysis, both in the time domain and the

frequency domain, [RAD1], [CRU3], but we will only use the comparative sensitivity approach here, [CRU1].

Consider the feedback configuration of figure 2.6. Assume G is now perturbed by a small amount $\tilde{\Delta}_o$ such that :

$$G \dashrightarrow (I + \tilde{\Delta}_o)^{-1}G \quad \triangleq \quad \tilde{G} \tag{2.29}$$

Then H_{yr} is perturbed to $\tilde{H}_{yr}$ where

$$\tilde{H}_{yr} = [I + (I + \tilde{\Delta}_o)^{-1}GC_zF_s]^{-1}(I + \tilde{\Delta}_o)^{-1}G\tilde{D}_{cz}^{-1}\tilde{N}_{cl}$$

$$= [I + S\tilde{\Delta}_o]^{-1}H_{yr} \tag{2.30}$$

where

$$S = [I + GC_zF_s]^{-1}$$

is the <u>sensitivity</u> (function) matrix, and

$$H_{yr} = (I + GC_zF_s)^{-1}G\tilde{D}_{cz}^{-1}$$

is the nominal closed-loop transfer function matrix from the reference signal to the actual output.

Now, assume $\tilde{\Delta}_o$ is vanishingly small, and that the closed-loop system is not de-stabilized, then (2.30) may be approximated by the first order approximation:

$$\tilde{H}_{yr} \approx [I - S\tilde{\Delta}_o]\, H_{yr} \tag{2.31}$$

For any reference input r(t), therefore, a good approximation to the output error of the closed-loop system due to uncertainty $\tilde{\Delta}_o$ is given by:

$$e_c(t) = -S\tilde{\Delta}_o H_{yr} r(t) \qquad (2.32)$$

If the system is being driven by open-loop control (assuming stability), then the corresponding open-loop output error would be given by:

$$e_o(t) = \tilde{\Delta}_o H_{yr} r(t) \qquad (2.33)$$

Thus, the sensitivity property of the closed-loop system may be considered superior to the open-loop system if

$$\int_0^T |e_c(t)|^2 dt \leq \int_0^T |e_o(t)|^2 dt \quad , \quad \text{all } t \geq 0 \qquad (2.34)$$

Note that since control action can only be taken for $t \geq 0$, $e_c(t) = e_o(t)$ for all $t<0$. There is therefore no loss of generality if we assume $e_c(t) = e_o(t) = 0$ for all $t<0$.

In the frequency domain, a sufficient condition for (2.34) to hold is that, [KRI1];

$$\bar{\sigma}[e_c(j\omega)] < \bar{\sigma}[e_o(j\omega)], \quad \text{all } \omega; \qquad (2.35)$$

where $\bar{\sigma}[.]$ denotes the maximum singular value of a matrix at a particular frequency (a vector of m elements is an mx1 matrix). (2.35) is again guaranteed if we have

$$\bar{\sigma}[S\tilde{\Delta}_o] < \bar{\sigma}[\tilde{\Delta}_o], \quad \text{all } \omega; \qquad (2.36)$$

which is implied by

$$\tilde{\Delta}_o(-j\omega)^T S(-j\omega)^T S(j\omega)\tilde{\Delta}_o(j\omega) < \tilde{\Delta}_o(-j\omega)^T \Delta_o(j\omega) \qquad (2.37)$$

Our formulation is slightly different from the conventional approach, [PER2], where Parseval's theorem is used although the result in [PER2] is roughly the same as inequality (2.37). In our approach, the de-sensitization effect is asserted on any finite time interval $t\epsilon[0,T]$, whereas Parseval's theorem holds only when the integral (2.34) is evaluated with $T=\infty$.

However, (2.36) only guarantees de-sensitization with respect to the particular perturbation $\tilde{\Delta}_o$. If $\tilde{\Delta}_o$ may be arbitrary, then a sufficient condition for de-sensitization is

$$\sup_\omega \bar{\sigma}[S(j\omega)] < 1 \qquad (2.38)$$

At this juncture, it may be appropriate to point out that the optimal linear quadratic state feedback regulator exhibits de-sensitization in the form of (2.37), but not (2.38). More specifically, it has been shown, [KRE1], [KWA1], that the optimal regulator which is given by the negative feedback controller [KWA1, page 313]:

$$F = R^{-1}B^T P \qquad (2.39)$$

where P is the nonnegative-definite solution of the algebraic Riccati equation

$$0 = C^T Q C - PBR^{-1}B^T P + A^T P + PA \qquad (2.40)$$

exhibits the following property:

$$S(-j\omega)^T W S(j\omega) \leq W$$

where $W = F^T R F$, the plant has state-apace matrices (A,B,C), and Q and R are the weighting matrices in the usual cost function.

In the context of robust-control, it is often necessary to ensure that the output of the plant is not only de-sensitized, but is also bounded within some set, when subjected to relatively large uncertainty. In such cases, we may assume y(t), the nominal output, is one of the members of the set. Let $v_p(t)$ denote the maximum allowable output error due to plant uncertainty. That is, we require that for any admissible reference input r(t),

$$|e_c(t)| < |v_p(t)|, \quad \text{all } t, \text{ all admissible } \tilde{\Delta}_o \qquad (2.41)$$

Because of the difficulty in finding an equivalent frequency domain condition for (2.41), the following weaker condition is often considered instead, [KRI1]:

$$\int_0^t |e_c(t)|^2 dt \leq \int_0^t |v_p(t)|^2 dt \qquad (2.42)$$

It has been pointed out that since (2.42) applies for every $t \geq 0$, and therefore ensures

$$\frac{1}{t}\int_0^t |e_c(t)|^2 dt \leq \frac{1}{t}\int_0^t |v_p(t)|^2 dt, \qquad (2.43)$$

it follows that (2.41) will be satisfied on the average over finite interval [0,t). A sufficient condition for (2.42) is given by

$$\max_{\text{all } \tilde{\Delta}_o} \bar{\sigma}[e_c(j\omega)] < \bar{\sigma}[v_p(j\omega)], \quad \text{all } \omega \qquad (2.44)$$

i.e. $$\max_{\text{all } \tilde{\Delta}_o} [\ \bar{\sigma}[([I+\tilde{\Delta}_o(I+GC_zF_s)^{-1}]^{-1}- I)y(j\omega)]] < \bar{\sigma}[v_p(j\omega)],\quad \text{all } \omega \tag{2.45}$$

<= $$\max_{\text{all } \tilde{\Delta}_o} [\ \bar{\sigma}([I + \tilde{\Delta}_o(I+GC_zF_s)^{-1}]^{-1}- I)] < \frac{\bar{\sigma}\ (v_p(j\omega))}{\bar{\sigma}\ (y(j\omega))} \tag{2.46}$$

Assume that we may find $(I + GC_zF_s)^{-1}$ such that

$$\max_{\text{all } \tilde{\Delta}_o} [\ \bar{\sigma}[\tilde{\Delta}_o(I + GC_zF_s)^{-1}]] \triangleq q(\omega) < 1\ ,\ \text{all } \omega \tag{2.47}$$

then (2.46) is implied by

$$\frac{q(\omega)}{1-q(\omega)} < \frac{\bar{\sigma}\ [v_p(j\omega)]}{\bar{\sigma}\ [y(j\omega)]}\ ,\quad \text{all } \omega \tag{2.48}$$

<= $$q(\omega) < \frac{\bar{\sigma}[v_p(j\omega)]}{\bar{\sigma}[v_p(j\omega)] + \bar{\sigma}[y(j\omega)]} \tag{2.49}$$

To express the allowable output error as a percentage of the nominal output, we let

$$\frac{\bar{\sigma}[v_p(j\omega)]}{\bar{\sigma}\ [y(j\omega)]} < p\ ,\ \text{all } \omega \tag{2.50}$$

then (2.48) leads to

$$q < \frac{p}{1 + p} \quad , \quad \text{all } \omega \tag{2.51}$$

Note that (2.50) is derived with respect to (2.42). If the condition is only required to be satisfied over the infinite frequency interval, i.e. t→∞, then we may apply Parseval's theorem to obtain p as

$$p = \frac{\| v_p(t) \|_2}{\| y(t) \|_2} \tag{2.52}$$

where $\|.\|_2$ is given by

$$\| x(t) \|_2 = \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} x(-j\omega)^T x(j\omega)\, d\omega \right]^{\frac{1}{2}} \tag{2.53}$$

We may also use a row-sum matrix norm, instead of the maximum singular value, in which case a bound may be applied to each of the elements of $y(j\omega)$ and $v_p(j\omega)$. We will, nevertheless, not pursue this any further.

The reader may verify that the sensitivity of the closed-loop system with respect to uncertainty in sensor dynamics is dependent on the complementary sensitivity function matrix, $T = [I + GC_z F_s]^{-1} GC_z F_s$. Thus, to minimize output error due to uncertainty in sensor dynamics, we will like to keep $\bar{\sigma}[T(s)]$ small. But $S + T = I$, hence there is always a need for a trade-off. However, in practice, uncertainties in sensors are usually small at low frequencies where command response of the closed-loop system is important. Thus at these frequencies one can often concentrate on the sensitivity function alone.

2.2-3 Performance

Performance criteria are generally specified as the requirements on a control system to follow commands and reject disturbances.

The command response problem, in a two-degree-of-freedom configuration, is essentially reduced to a filter design and model-matching problem, [WAN1], by expression (2.18). Note that the only restriction on the set of attainable closed-loop transfer function matrices is given by the non-minimum-phase zeros of G(s), since any CRHP zeros of N(s) which are also CRHP zeros of F(s) will be cancelled by $F(s)^{-1}$, assuming the latter exists.

In general, the desired closed-loop response of a system should be chosen to have a sufficiently rapid response, while keeping the overshoot within acceptable bounds to avoid saturation in the plant. In the classical design techniques, the time-domain performance specifications are usually expressed as the response of a system to some standard test input such as the step input, the ramp input or the parabolic input. In "modern" optimal control, the performance criterion may be written in the form of a cost function:

$$\int_0^\infty \| W_1(y - Hr) \|_2^2 + \| W_2 R_r r \|_2^2 \, dt \qquad (2.54)$$

where W_1 and W_2 are some weighting matrices which may be "frequency dependent", [AND2], or, more correctly, convolution operators, and H is the "ideal" transfer matrix which may be unrealizable because of the CRHP zeros of G(s) and the constraint on u(t). The determination of the ideal transfer function matrix, H, is an application-dependent problem, and since it is literally a filter design problem, the standard signal-processing techniques, [LAT1], will be useful. Note that in general it

is the Fourier transform (Laplace transform evaluated on the jω-axis) of the filter that is important in filter design.

Quite often in multivariable control, non-interaction or low-interaction among the outputs is a major objective; that is, the ith output should more or less depend only on the ith input (after the inputs and outputs have been ordered in an appropriate way). Although a direct model-matching approach may be used to construct a pre-filter which gives a desirable transfer function matrix, a Nyquist array type of technique to attain diagonal dominance, [ROS4], may sometimes be useful because of the relatively simple pre-filter it might give. Note we are here talking about using the Nyquist array technique in filter design, not feedback controller design.

Disturbances may enter a closed-loop system with the reference input r(t), or as a plant output disturbance, or measurement noise, or a combination of these. Generally, those disturbances which corrupt the reference input may be filtered out, or minimized, with the pre-filter. We will therefore concentrate on d(t) and m(t) in the remainder of this sub-section.

There are in general three types of signal that can be used to model disturbances, namely, stochastic, [JAZ1], uncertain but bounded, [ZAM2], and deterministic (which may be unstable), [BEN1]. In all these cases, it is often useful and convenient to model d(t) and m(t) as

$$d(t) = W_d n_1 \tag{2.55}$$

$$m(t) = W_m n_2 \tag{2.56}$$

and

$$v = [n_1^T \quad n_2^T]^T \tag{2.57}$$

where W_d and W_m are convolution operators, and n_1, n_2 and v are usually chosen to possess certain "nice" properties. For example, in the

stochastic case, one may model v(t) as a white noise vector with identity power spectral density matrix; in the uncertain but bounded disturbance case, v(t) may be assumed to be any (unknown) vector with norm less than or equal to unity; and, finally, in the deterministic case, v(t) may be modelled as a unit impulse at t=0.

It is well known that the error in the output due to v(t) is given by

$$y_e(t) = Zv \tag{2.58}$$

where

$$Z = [SW_d \quad TW_m] \tag{2.59}$$

and S and T are the sensitivity and complementary sensitivity matrices defined by (2.11) and (2.12) respectively. It is therefore clear, that to minimize the effect of v on the output, Z must be "small" in some appropriate sense. For example, when v is a Gaussian white-noise vector with identity power spectral density matrix, the usual LQG cost functional

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} |y_e(t)|^2 dt \tag{2.60}$$

is minimized by minimizing, [YOU1]:

$$\mathrm{tr} \int_{-\infty}^{+\infty} Z(-j\omega)^T Z(j\omega)\, d\omega \tag{2.61}$$

However, if n_1 and n_2 are unknown, then it is impossible to evaluate (2.60). A meaningful objective would then be to minimize the worst case effect over the whole class of possible $v\in\{v;\ \|v\|\leq 1\}$, that is, to minimize

$$\sup_{\|v\|\leq 1} \int_{-\infty}^{+\infty} |y_e(t)|^2 dt \qquad (2.62)$$

It is well known from linear system theory that (2.62) is minimized if

$$\sup_{w} \bar{\sigma}[Z(j\omega)] \qquad (2.63)$$

is minimized, [ZAM2].

If v(t) is deterministic, then $y_e(t)$ will be deterministic. A straightforward application of Parseval's theorem to $y_e(t)$ then shows that $\int_{-\infty}^{\infty} |y_e(t)|^2 dt$ is minimized if (2.61) is minimized, provided that $y_e(t)$ is stable and square integrable. When d or m may be unstable, the problem of synthesizing a controller such that $y_e(t)$ of the feedback system is stable is known as the output regulation problem, [BEN1]. When regulation is attained despite small perturbations in the plant model, output regulation is said to be robust, [FRA2]. We note that robust regulation is achievable only if the perturbations happen in the plant, and not when uncertainty occurs in the unstable parts of W_d and W_m, or in both the plant and the controller.

Lemma 2.4 Let v(t) be in L_2, then output regulation is attainable if, and only if, there exists a stabilizing controller $C_z(s)$ such that Z(s) of (2.59) is analytic in the CRHP and is proper.

Proof Obviously true. ∎

Although lemma 2.4 is a necessary and sufficient condition for output regulation, it is not sufficient for robust output regulation. Roughly speaking, lemma 2.4 says that in order to achieve output

regulation, the unstable poles of W_d and W_m must be cancelled by the non-minimum-phase zeros of S and T respectively. A CRHP zero of S (or T) may be supplied by an unstable pole (CRHP zero respectively) of either G(s) or $C_z(s)$, assuming $F_s(s)$ stable and minimum-phase for simplicity. Obviously, if it is provided by a CRHP pole (zero respectively) of G(s), then output regulation fails when this pole (zero respectively) is being perturbed to another location.

In what follows, we shall assume $F_s = I$ and W_m stable for simplicity. Let W_d have the left coprime factorization

$$W_d = \tilde{D}_d^{-1} \tilde{N}_d \qquad (2.64)$$

and let α_d be a least degree, proper, stable, scalar rational which contains all the CRHP zeros of $\tilde{D}_d$ (multiplicities counted) as its zeros. Note that since $F_s = I$, (N, D) and $(\tilde{D}, \tilde{N})$ are therefore coprime factorizations of G. The following results characterize the solutions of the robust output regulation problem.

Lemma 2.5, [FRA2]. If C_z is a solution to the robust output regulation problem and $C_z = N_{cz} D_{cz}^{-1}$ is a right-coprime factorization, then $(1/\alpha_d)D_{cz}$ is a proper and stable matrix.

Proof See [FRA2]. ∎

Theorem 2.3 The robust output regulation problem (i) has a solution if, and only if, $\alpha_d I$ and N are left-coprime; (ii) when the existence condition holds, the set of all solutions is given by the set of all stabilizing controllers which make $(1/\alpha_d)SV$ stable, where V is any minimum-phase proper matrix such that $\tilde{D}V$ is stable and minimum-phase.

Proof To prove (i) <=> , see [FRA2];

To prove (ii) <=>, we have

$$(1/\alpha_d)SV = (1/\alpha_d)D_{cz}(\tilde{DD}_{cz} + \tilde{NN}_{cz})^{-1}\tilde{DV} \qquad (2.65)$$

Because $(\tilde{DD}_{cz} + \tilde{NN}_{cz})^{-1}\tilde{DV}$ is stable and minimum-phase, $(1/\alpha_d)D_{cz}$ is stable if, and only if, $(1/\alpha_d)SV$ is stable. Hence, by lemma 2.5, it is a necessary condition. Sufficiency is obvious by letting $D_d^{-1} = (1/\alpha_d)D_s^{-1}$ where D_s^{-1} is stable. ∎

A very interesting and somewhat surprising implication of theorem 2.3 is that the sensitivity function matrix of the closed-loop system must vanish completely at the zeros of α_d, irrespective of whether the zero is in only one or many channels. For example, if GC_z is diagonal, and W_d is diagonal with only one unstable channel, then one would expect that it would be sufficient to have the corresponding element of the diagonal of S vanish for robust output regulation. But theorem 2.3 shows that this is not adequate. The following lemma characterizes the structure of C_z.

Lemma 2.6 If C_z is a solution to the robust output regulation problem, then C_z is of the form

$$C_z(s) = (1/\alpha_d)K(s) \qquad (2.66)$$

where K(s) is free of zeros at the zeros of α_d.

Proof From lemma 2.6, $(1/\alpha_d)\ D_{cz}$ is proper and stable. Thus we may write

$$D_{cz} = \alpha_d V$$

where V(s) is any proper and stable matrix. Now, since N_{cz} and D_{cz} are right-coprime, N_{cz} must be free of zeros at the zeros of α_d. We thus have

$$K(s) = N_{cz}(s)\ V(s)^{-1} \qquad (2.67)$$

which is free of zeros at the zeros of α_d. ∎

Lemma 2.2 is very useful from the a design point of view because it says that to solve a robust output regulation problem, one simply needs to cascade $(1/\alpha_d)I$ in front of the plant and then design a stabilizing controller for $(1/\alpha_d)G$.

Corollary 2.2 A solution to the robust output regulation problem exists if and only if G(s) does not have any zeros at the zeros of $\alpha_d(s)$. If the existence condition holds, then the set of all controllers which solve the output regulation problem for G(s) and $W_d(s)$ is the set of all stabilizing controllers which stabilize $(1/\alpha_d)G(s)$, cascaded with $(1/\alpha_d)I$.

Proof Obvious from lemma 2.2. Note that $N_{cz}V^{-1}$ is a right coprime factorization of K. To prove this, because $\alpha_d V$ and N_{cz} are right coprime, there exists X, Y ϵ Mat(S) such that

$$\alpha_d XV + YN_{cz} = I \qquad (2.68)$$

This show that V and N_{cz} are right coprime because $\alpha_d X$, $YN_{cz} \epsilon$ Mat(S). This completes the proof ∎

The attractive feature of corollary 2.2 is that like the Nyquist criterion, it gives necessary and sufficient conditions for C_z to be a robust output regulator based on G(s) and $W_d(s)$ rather than their coprime factorizations. This may be physically more appealing to practising engineers. Note that $(1/\alpha_d)$ is simply a minimum-phase proper transfer function which contains all the unstable poles of $W_d(s)$ as its poles.

It should be clear to the reader that one may again use the comparative approach of sub-section 2.2-2 to evaluate the relative merit of a feedback design compared to open-loop control (assuming the plant open-loop stable) in the presence of disturbances.

2.2-4 Integrity

Integrity is not usually formulated as a specific design requirement, but a control system designer would normally like to achieve it, if it was possible. When the plant model is known precisely, assessing the integrity of a closed-loop system in the event of a sensor or actuator failure is simple and straightforward, as is indicated by the following theorems:

Theorem 2.4 Assume $F_s = I$, and let s_{ii} denote the ith diagonal element of the sensitivity function matrix $S = [I + GC_z]^{-1}$. If the closed-loop system is stable, then it will remain stable in the event of a sensor failure in loop i if s_{ii} does not possess a CRHP zero.

Proof See [OWE2]. ∎

Theorem 2.5 Assume $F_s = I$, and let $\tilde{s}_{ii}$ denote the ith diagonal element of $\tilde{S} = [I + C_z G]^{-1}$. If the closed-loop system is stable, then it will remain stable in the event of an actuator failure in loop i if $\tilde{s}_{ii}$ does not possess a CRHP zero.

Proof See [OWE2]. ∎

Theorems 2.4 and 2.5 only guarantee integrity in the event of a single sensor or actuator failure in a loop. It cannot cope with the problem of multiple failures. It also assumes the plant model G to be

known precisely. Therefore it is, in general, not suitable for analysing the integrity of a system which is significantly uncertain. We will consider this problem in a later chapter.

It should also be emphasized that the integrity of a closed-loop system may also be improved by a good fault-detection/fault-tolerant design. This is a rich field in its own right and is outside the scope of this thesis. The interested reader is referred to [WIL2], [SYR1], and the references cited therein. These techniques may be used when integrity is of ultimate importance but is not achievable, or is achievable only at the expense of severely deteriorating the performance of the closed-loop system with a linear time-invariant feedback control law.

2.3 Topology of Linear Time-invariant Plants

2.3-1 The Graph topology

In this sub-section, we introduce a topology of linear convolution plants, first defined by Vidyasagar, Schneider and Francis, [VID1]. For simplicity, our treatment of the subject is necessarily selective in the sense that only the essential definitions and facts for the understanding of later chapters are presented, we have restricted ourselves to the case of lumped systems. A more rigorous and complete treatment of the topology can be found in [VID1].

Consider a linear, time-invariant, causal and stable, single-input, single-output system described by its impulse response g(t). From

elementary control theory, we know that for any input u(t), the output y(t) is given by:

$$y(t) = g(t)u(t) \tag{2.69}$$

where multiplication is defined by the operation of convolution. Under this definition of multiplication and taking addition in the usual sense, the set of all causal and stable convolution operators whose impulse responses admit Laplace transforms in **S** form a ring.

The set of all multi-input, multi-output, stable and causal, lumped convolution operators may therefore be treated as matrices with elements in the ring of stable and causal lumped single-input, single-output convolution operators. Thus the set of all n-input, m-output stable and causal convolution operators form a linear space.

We can make each space a normed space by defining an operator-norm for each convolution operator G in the space, given by

$$\|G\| = \sup_{\text{all } x(t)\in L_2} \frac{\|Gx(t)\|}{\|x(t)\|} \tag{2.70}$$

where L_2 denotes the space of appropriately dimensioned square-integrable vector-functions, and

$$\|x(t)\| \triangleq \left(\int_{-\infty}^{\infty} |x(t)|^2 dt \right)^{\frac{1}{2}} \tag{2.71}$$

It is well-known that the above operator-norm coincides with the H^∞ norm of its transfer function matrix G(s) and therefore

$$\|G\| = \sup_{s=j\omega} \{ \bar{\sigma}[G(s)] \} \tag{2.72}$$

For simplicity, we will confine our discussion to the frequency domain. It should be clear to the reader that there is a one-to-one correspondence between the space of causal and stable lumped convolution operators and the space of proper rational transfer function matrices which are analytic in the closed right half-plane. For ease of exposition the superscripts which are used to denote the dimension of a matrix will usually be dropped.

Since $Mat(S)^{mxn}$ is a normed linear space, the concept of a neighbourhood of G, denoted by $N_G(\xi)$, is well-defined; that is, $N_G(\xi)$ is an open-ball around G given by

$$N_G(\xi) \triangleq \{G_p \mid G_p(s) \in Mat(S),\ \sup_{s=j\omega} \bar{\sigma}[G_p(s)-G(s)] < \xi\} \tag{2.73}$$

for some $\xi>0$. (Note that this definition of neighbourhood will be well-defined even if $G_p(s)$ and $G(s)$ are improper as long their difference gives a proper transfer function matrix). Therefore, given $g(s)=1/(s+b)$ and a sequence of transfer functions $g_i(s)=(s+a+\epsilon_i)/(s+a)(s+b)$, we see that $g_i(s)$ (and hence g_i) belongs to $N_{g(s)}(\xi)$ (and hence $N_g(\xi)$) for sufficiently small $|\epsilon_i|$, and $\|g_i - g\| \to 0$ as $|\epsilon_i| \to 0$. Thus, g_i converges to g as $\epsilon_i \to 0$. Hence, we may adopt the view-point that the ratios of polynomials

$$\frac{(s+1)}{(s+2)(s+3)} \quad \text{and} \quad \frac{(s+1)(s+4)}{(s+2)(s+3)(s+4)}$$

represent the same rational transfer function. In other words, we consider them to be the same point in the space of stable transfer functions; and

$$\frac{(s+1)(s+4+\epsilon_1)}{(s+2)(s+3)(s+4+\epsilon_2)}$$

is in the small neighbourhood of $\frac{(s+1)}{(s+2)(s+3)}$ for $|\epsilon_1|$, $|\epsilon_2| << 1$. It is also not difficult to show that if K(s) is a controller which stabilizes G(s), then it will also stabilize all $G_p(s)$ in some neighbourhood $N_G(\xi)$ of G, [FRA4].

The above discussion applies only to stable systems. For unstable systems the operator norm is in general not defined, or is infinite. The neighbourhood of an unstable transfer function cannot therefore be defined in an analogous way, and we are forced to work in an extended topology as discussed in [VID1]. A formal definition of the extended topology is given below, almost exactly as it appears in [VID1].

Definition 2.2 Define a topology τ on Mat(R(s)) as follows. Given $G(s) \in Mat(R(s))$, let $(N,D) \in Mat(S)$ be any right-coprime factorization (r.c.f) on G, and define a neighbourhood of G as the set of all ratios $N_1 D_1^{-1}$, where N_1 and D_1 belong to neighbourhoods of N and D respectively. Define a set in Mat(R(s)) to be open if, and only if, it is an arbitrary union of finite intersections of neighbourhoods.

A direct consequence of the topology defined in definition 2.2 is the concept of a basic neighbourhood of a lumped convolution operator G, defined as follows, [VID3]:

Definition 2.3 Let (N, D) be any r.c.f. of G, and let ξ be any positive number less than $\mu(N, D)$, where $\mu(N, D)$ is a real positive constant such that (N_1, D_1) is also a right-coprime pair and $\det(D_1) \neq 0$ whenever

$$\left\| \begin{bmatrix} N_1 - N \\ D_1 - D \end{bmatrix} \right\| < \mu(N, D).$$

Then the set

$$N_G(N,D;\xi) = \{G_p = N_p D_p^{-1} : \left\| \begin{bmatrix} N_p - N \\ D_p - D \end{bmatrix} \right\| < \xi \} \qquad (2.74)$$

is a basic neighbourhood of G.

Thus, a basic neighbourhood of G merely consists of all plants G_p which have r.c.f.'s (N_p, D_p) which are "close" to an r.c.f. of G. Note that in defining neighbourhoods of N and D, pole-zero cancellations are allowed in the left half complex plane as we have previously seen for stable transfer function matrices. We shall illustrate this point by way of examples:

Example 2.1 Given $G(s) = \frac{1}{s-1}$, we show that $G_1(s) = \frac{(s+1+\epsilon_1)}{(s+1)(s-1+\epsilon_2)}$ belongs to the small neighbourhood G for arbitrarily small ϵ_1 and ϵ_2.

We may therefore choose N = 1/(s+1), D = (s-1)/(s+1) for G(s), and $N_1 = 1/(s+1)$, $D_1 = (s-1+\epsilon_2)/(s+1+\epsilon_1)$ for $G_1(s)$. It is not difficult to show that

$$\left\| \begin{bmatrix} N_1 - N \\ D_1 - D \end{bmatrix} \right\| \to 0 \text{ as } \epsilon_1, \epsilon_2 \to 0.$$

Therefore, $G_1(s)$ is in the small neighbourhood of G(s) for arbitrarily small ϵ_1 and ϵ_2.

Example 2.2 $G(s) = (s-1+\epsilon_1)/(s-1)(s-1+\epsilon_2)$ does not lie in the small neighbourhood of $1/(s-1)$ for arbitrarily small ϵ_1 and ϵ_2. This can be clearly seen as any $N_2(s)$ of a r.c.f. pair (N_2, D_2) of $G_2(s)$ must have a zero at $s = 1-\epsilon_1$, while N does not have this zero. Hence $\|N_2 - N\|$ does not approach zero as $\epsilon_1, \epsilon_2 \to 0$ for any choice of (N_2, D_2). The same argument applies to D_2 and D.

It may be shown, [FRA2], that the extended topology defined above has the following useful properties:

(i) The set of unimodular matrices matrices is open and inversion of a unimodular matrix is a continuous operation.

(ii) Internal stability is a generic (robust) property; that is, internal stability is preserved throughout some open neighbourhood of the plant and the stabilizing controller.

(iii) The closed-loop transfer function matrix from (r,d,n) to (y,z,u) is a continuous function of G(s) and C(s).

(iv) The topology is the weakest one having the properties (ii) and (iii).

(v) The presence of an unstable pole-zero cancellation between G(s) and C(s) is a non-generic property.

Another feature of the topology is that it is a graph topology. By this we mean that two transfer function matrices $G_1(s)$ and $G_2(s)$ are "close" if, and only if, their graphs are "close". To explain this further, suppose $G(s) \in Mat(R(s))$ is a possibly unstable plant and (N, D) is an r.c.f. pair of G. Then it is obvious that G.D is a stable operator. It therefore follows that if u is any bounded signal in the range-space of D, then G.u will (theoretically) produce a bounded output. The graph of a plant G is precisely the set of all bounded input-output

pairs corresponding to G. More specifically, if $G(s) \in Mat(R(s))^{mxn}$, then the graph of G is given by:

$$G_g(G) \triangleq \{(u, y) \mid u(s) \in S^n,\ y(s) \in S^m,\ y=Gu\} \qquad (2.75)$$

Note that the graph of a plant can be very simply parameterized in terms of any r.c.f. (N, D) of G as follows, [VID3]:

$$G_g(G) = \{(Dv, Nv): v(s) \in S^n\} \qquad (2.76)$$

The following definitions should be apparent to the reader:

Definition 2.4, [VID4]. A sequence of transfer function matrices $\{G^{(k)}\}$ in Mat(S) converges to G in Mat(S) (in the sense of the norm) if, and only if, each of the component sequences $\{g_{ij}^{(k)}\}$ converges to g_{ij} (in the sense of the norm).

Definition 2.5, [VID4]. A sequence of transfer function matrices $\{G^{(i)}\}$ in Mat(R(s)) converges to G in Mat(R(s)) in the topology if there exist right-coprime factorizations (N_i, D_i) of the $G^{(i)}$ and (N, D) of G such that $N_i \to N$ and $D_i \to D$ in Mat(S).

We note that according to the above definitions, $1/(1+\epsilon s)$ does not converge to 1 as $\epsilon \to 0$, [FRA2]. However, we have

$$\lim_{\epsilon \to 0} (1+\epsilon s)/(1+s) = 1/(1+s); \qquad (2.77)$$

$$\lim_{\epsilon \to 0} 1/(1+\epsilon s)(s\pm 1) = 1/(s\pm 1); \quad \text{and} \qquad (2.78)$$

$$\lim_{\epsilon \to 0} (1+\epsilon s)/(s+1)(s+2) = 1/(s+1)(s+2) \qquad (2.79)$$

Note that all the above examples involve a pole or a zero movement towards infinity. We will discuss the effects of these movements on the Nyquist loci later. In the mean time, we introduce another topological idea which is of vital importance in later chapters; that is, we shall introduce the notion of an arcwise connected set in the topology.

2.3-2 Arcwise connectivity

We begin with some definitions, [SIN1], [KEL1]:

Definition 2.6 Let X be a topological space. A path in X from x_o to x_1 (origin x_o and end x_1) is a continuous map $\alpha : [0,1] \to X$ such that $\alpha(0)=x_o$ and $\alpha(1)=x_1$.

We note that (by definition) a path is a function, and not a set of points.

Definition 2.7 A space X is arcwise connected if, given any two points x_o and x_1 in X, there exits a path in X with origin x_o and end x_1.

Definition 2.8 Let (Z, U) be a topological space and let $A \subset Z$. Let

$$U_A = \{A \cap V;\ V \in U\}$$

Then U_A is a topology on A, called the relative topology on A.

Definition 2.9 A subset of a topological space is said to be an arcwise connected set if it is arcwise connected in the relative topology.

We can now specialize the above definitions to the extended topology for possibly unstable plants.

Definition 2.10 A path in $Mat(R(s))^{mxn}$ from $G_o = N_o D_o^{-1}$ to $G_1 = N_1 D_1^{-1}$ is a continuous map $\alpha : [0,1] \rightarrow Mat(N,D)$, where $N \epsilon S^{mxn}$, $D \epsilon S^{nxn}$, N and D are right-coprime, and $\alpha(0)=(N_o, D_o)$ and $\alpha(1) = (N_1, D_1)$.

Definition 2.11 A set of transfer function matrices $\{G_p(s)\}$ in $Mat(R(s))$ is said to be arcwise connected in the extended topology if for each $G_p \epsilon \{G_p(s)\}$ there exists an r.c.f. (N_p, D_p) such that for any two members of $\{G_p(s)\}$ there exists a path in $\{G_p(s)\}$ connecting them.

A path has to be continuous. Hence, if we perturb G(s) to $G_p(s)$ along a path, there cannot be any sudden appearance or disappearance of poles or zeros in the Nyquist D-contour, D. However, it should be clear to the reader that pole and zero movements across the imaginary axis are possible when we perturb G(s) to $G_p(s)$ along a path. Pole and zero movement across the point at infinity deserves further consideration, as follows.

In general, N and D being right-coprime implies that either N(s) or D(s) has to be non-strictly proper, i.e. proper but not strictly proper. G(s) is proper therefore implies that D(s) is non-strictly proper. This implies that D(s) cannot have any pole or zero at infinity. Thus, in general we cannot "move" a finite unstable pole of G(s) to infinity by perturbation along a path. Therefore, pole movement between the left and right half-planes via the point at infinity is generally not permitted along a path. However, zero movement across the point at infinity is generally possible.

Example 2.3 $\{(1+\alpha s)/(1+s) : \alpha \in [-1, 1]\}$ is an arcwise connected set.

Example 2.4 $\{1/(1+\alpha s)(1+s) : \alpha \in [-1, 1]\}$ is not an arcwise connected set.

Example 2.5 $\{1/(1-\alpha s)(1+s) : \alpha \in [0,1)\}$ is not an arcwise connected set.

It should be clear to the reader that, from an algebraic geometrical point of view, a set of transfer function matrices is arcwise connected if, and only if, the graphs of its members are connected.

2.3-3 Graph metric

We next introduce a graph metric for possibly unstable plants, due to Vidyasagar, [VID4].

Definition 2.12 An r.c.f. (N, D) of G is normalized if

$$N(s)^*N(s) + D(s)^*D(d) = I, \quad \text{all } s \tag{2.80}$$

Every $G(s) \in Mat(\mathbb{R}(s))$ has a normalized r.c.f., [VID4], which is unique to within right multiplication by an orthogonal matrix.

Definition 2.13, [VID4]. Suppose G_1, $G_2 \in Mat(\mathbb{R}(s))$ have the same dimensions, and let (N_i, D_i), i=1,2, be normalized r.c.f.'s of G_i for i=1,2. Let

$$A_i = \begin{bmatrix} D_i \\ N_i \end{bmatrix}, \quad i = 1, 2 \tag{2.81}$$

and define

$$\delta(G_1, G_2) = \inf_{u \in Mat(S), \|u\| \leq 1} \|A_1 - A_2 U\|, \qquad (2.82)$$

$$d(G_1, G_2) = \max \{ \delta(G_1, G_2), \delta(G_2, G_1) \} \qquad (2.83)$$

Then $d(G_1, G_2)$ is the graph metric on $Mat(\mathbf{R}(s))$.

For a more detailed discussion on the graph metric, the reader is referred to [VID4]. Here we simply mention that it is a metric on $Mat(\mathbf{R}(s))$ assuming values in the interval [0, 1].

At present it is still not clear how one may compute exactly the metric distance between two plants. However, it is possible to give upper and lower bounds, [VID4].

Lemma 2.7, [VID4]. Suppose (N_1, D_1) is a normalized r.c.f. of G_1, (N_2, D_2) is a (not necessarily normalized) r.c.f. of G_2, and let

$$A_1 = [D_1^T \quad N_1^T]^T , \qquad (2.81)$$

$$M_2 = [D_2^T \quad N_2^T]^T , \qquad (2.84)$$

Suppose $\|A_1 - M_2\| \stackrel{\Delta}{=} \gamma < 1$. Then

$$d(G_1, G_2) \leq \frac{2\gamma}{1-\gamma} \qquad (2.85)$$

Proof See [VID4].

A lower bound on the graph metric distance between two plants is given by

$$\delta(G_1, G_2) \geq \inf_{Q \in Mat(S)} \|A_1 - A_2 Q\| \qquad (2.86)$$

We will present a robust stability test which uses this graph metric measure in a later chapter.

2.4 Continuous and Discontinuous Changes in Nyquist Loci

In this section, we briefly examine continuity and discontinuity in the deformation of the Nyquist characteristic gain loci and the Nyquist determinant locus, [POS1], of a transfer function matrix when the latter is subjected to a perturbation.

It is well-known that the eigenvalues of a matrix satisfy a continuity property; that is, if we perturb a frequency response matrix continuously from G_o to G_1, the characteristic gain loci, and hence the deterministic locus, must also change in a continuous manner.

The question is, under what type of perturbations will the change in frequency response be continuous? In particular, we would like to examine the effect of pole and zero movements across the Nyquist D contour on the continuity in the change of the characteristic loci.

Obviously, if there is a zero movement across the $j\omega$-axis, a characteristic locus of the transfer function matrix simply passes through the origin. Thus the changes in both the frequency response and the characteristic gain loci are continuous. Similarly, one may conclude that this continuity property also holds when there is a zero movement across the point at infinity if the transfer function is proper.

The problem is slightly more involved if there is a pole movement across the $j\omega$-axis because a characteristic gain locus then becomes

infinite and the change appears to be discontinuous. However, if one considers mapping the complex plane into the Riemann sphere, or making a change of variable with $s = 1/z$, and then investigating the behaviour of the characteristic gain loci, it is not difficult to convince oneself that the change is indeed continuous in an extended sense. We shall from now on call this type of continuity "continuity in the extended sense." Note that if a pole moves cross the imaginary axis from left to right, it will automatically create an additional anticlockwise encirclement of the origin by the characteristic gain loci because the extended principle of the argument, [POS1], must always hold. More specifically, the characteristic gain loci of a return-ratio matrix $M(s)$ are given by

$$\det[gI - M(s)] = 0 \quad , \quad s \epsilon D_D \qquad (2.87).$$

The total number of clockwise encirclements of the origin by the characteristic gain loci is thus given by the total number of characteristic zeros of $M(s)$ in the D_D-contour minus the total number of characteristic poles of $M(s)$ in the same region. Therefore, a change in the number of the CRHP poles or zeros of $M(s)$ must automatically change the number of encirclements of the origin by the characteristic gain loci. The difference between the effects of a change in the number of zeros and the number of poles is that in the former, the change is accommodated by one of the characteristic gain loci passing through the origin while in the latter case the change is made possible by a characteristic gain locus passing through the point at infinity. Thus, in the case of a zero movement across the $j\omega$-axis or the D_D-contour, the change in the number of encirclements of the origin does not result in a change in the number of encirclements of the critical point $(-1, j0)$. But in the case of a pole movement across the $j\omega$-axis, the creation of an additional anticlockwise encirclement of the origin also automatically

creates an additional anticlockwise encirclement of the critical point. Similarly, whenever there is a pole movement across the $j\omega$-axis from the right to left, an extra clockwise encirclement is created.

A pole movement from the RHP into the LHP, or vice versa, through the point at infinity is impossible or unlikely in practice because it implies a drastic change in an eigenvalue of the system A-matrix which, in turn, is possible only if an infinite change in at least one of the elements (of the A-matrix) occurs. It is also extremely unlikely that one would mistakenly model a far right half-plane pole as a far left half-plane pole. However, it will still be interesting to investigate the effect on the Nyquist loci of modelling a pole which moves continuously from one half-plane into the other via the point at infinity. The significance of this investigation will be discussed in section 4.3 when we consider the inverse Nyquist type robust stability tests. Generally speaking, if a plant is strictly proper, then the change in the characteristic loci will not be continuous when we model a pole movement across the point at infinity. We provide example 2.7 to illustrate this.

Example 2.6 To illustrate how the number of encirclements of the critical point changes as a pole crosses the imaginary axis, consider the Nyquist diagram of

$$G(s) = \frac{1}{s+\epsilon} \qquad (2.87)$$

for different values of ϵ, as shown in figure 2.7. Note how the Nyquist diagram "flips" through infinity as the poles cross the imaginary axis.

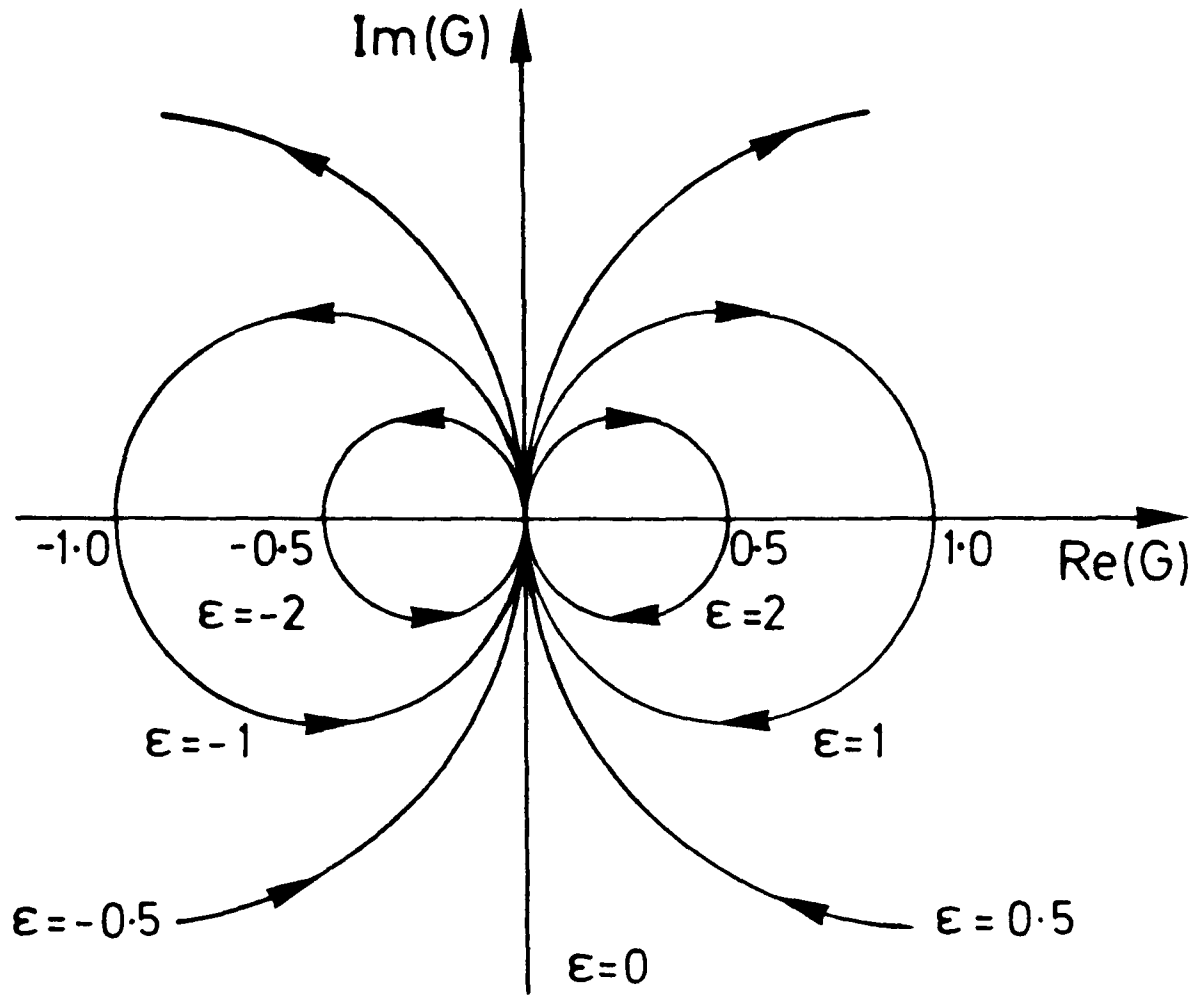


Figure 2.7 Nyquist Diagram of $G(s) = 1/(s+\epsilon)$

Example 2.7 To show that the change in the characteristic gain loci is not always continuous for a proper plant, consider the Nyquist diagram of $1/(1+\alpha s)$, $\alpha \epsilon (-1, 1)$ in figures 2.8 and 2.9. Note that the Nyquist diagram is a circle with diameter on [0, 1] for $\alpha \neq 0$, but when $\alpha = 0$, the Nyquist diagram is simply a point at (1,j0). Note also the abrupt change in the direction (corresponding to increasing frequencies) in the Nyquist diagram.

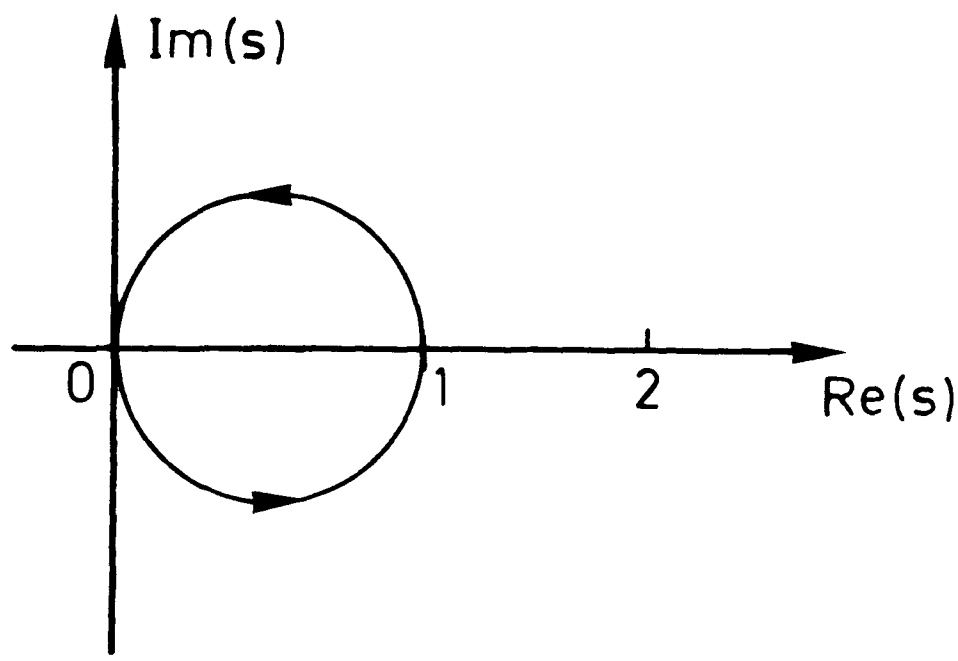


Figure 2.8 Nyquist Diagram of $1/(1+\alpha s)$ for $\alpha < 0$

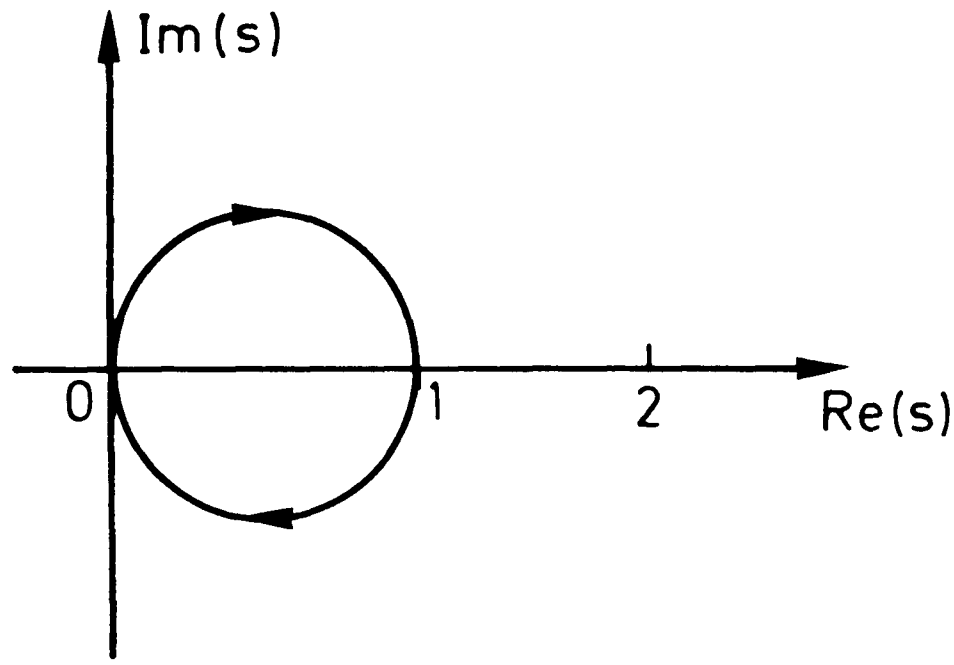


Figure 2.9 Nyquist Diagram of $1/(1+\alpha s)$ for $\alpha>0$

2.5 Concluding Remarks

In this chapter, we have presented a discussion of some of the more basic aspects of system theory and design. In section 2.1 we introduced the two-parameter compensator and showed that a rigorous and complete separation between command response and a desired loop transfer matrix is possible. In section 2.2 we discussed the various design criteria from the point of view of a two-degree-of-freedom feedback configuration and presented a Nyquist type stability criterion. Although the criterion is by necessity slightly more complicated than the classical criterion for single-degree-of-freedom feedback systems, the essential and characteristic features of the criterion are retained; that is, only the open-loop frequency response and the total number of unstable poles of the plant need to be known in order to be able to determine stability. The exact transfer function matrix of the plant or its coprime factorization is not needed. We have also shown that although C_r of the compensator needs to satisfy certain necessary and sufficient conditions

in order that the feedback system be stable, in practice it generally does not have any effect on the robustness of the stability property.

In section 2.3, the graph topology for linear time-invariant systems, [VID1], was introduced and the notion of arcwise connectivity in the graph topology was discussed. We emphasize again that although zero movement between the left and the right half-planes through the point at infinity is possible when we perturb a plant model along a path, pole movement between the two half-planes is strictly restricted to across the $j\omega$-axis.

Finally in section 2.4, we briefly discussed the continuity of the characteristic gain loci when a transfer function matrix was subjected to perturbation. It was shown that when there is a pole moving from the RHP to the LHP, or vice versa, through the point at infinity, the change in the Nyquist loci may be discontinuous. This problem does not arise when pole movement is replaced by zero movement.

CHAPTER III

REPRESENTATIONS OF UNCERTAINTY AND THEIR IMPLICATIONS IN ROBUSTNESS ANALYSIS

3.1 Introduction

Since uncertainty is always present in real systems, a mathematical model can never perfectly describe a physical system. The usefulness of a nominal plant model in control system design is therefore greatly enhanced if it is accompanied by a representation of the uncertainty involved. In robustness analysis, the most widely accepted way of doing this is by defining a set of perturbed plant models in the "neighbourhood" (not necessarily in the sense of the graph topology) of a nominal plant model in the belief that the true plant model lies in this set. This approach has been used extensively by research workers interested in the robustness properties of multivariable control systems, and several representations of uncertainty can be found in [DOY1], [POS3], [SAF3], [CRU2], [LEH1], [OWE1], and [POS4].

The purpose of this chapter is to examine the adequacy of these representations, that is, their ability to represent all possible perturbations of the plant. Our aim is to provide a simple yet critical discussion which will enable the reader to appreciate the problem of selecting an adequate representation of uncertainty in robustness analysis, and the implications of selecting an inadequate representation. The discussion and results presented are thought to be new in the context of robustness analysis.

In section 3.2, we examine the validity of characterizing uncertainty by a single perturbation whose maximum singular value bound is known, [DOY2], [POS3], [SAF3], [CRU2], [LEH1]. It is shown that not

all the perturbation sets are adequate for this purpose. For example, if the plant has more outputs than inputs, which is the usual situation, then uncertainty cannot generally be represented by a multiplicative perturbation at the plant input.

In section 3.3, the results of section 3.2 are extended to include simultaneous input and output perturbations as a means of representing uncertainty. We further examine whether it is always possible to find a stable output perturbation and a corresponding input perturbation which is minimum phase: conditions which would allow us to use the results in [DOY6].

The reliability of multi-loop gain and phase margins, [SAF2] and [LEH1], as measures of robust stability is studied in section 3.4. We argue that because the margins are unsuitable for representing all possible types of uncertainty, they are generally insufficient to guarantee a robust stability property. Some examples are given to illustrate the weakness of the multi-loop gain and phase margins as general measures of robust stability.

3.2 Additive and Multiplicative Perturbations

In this section we examine the validity of characterizing uncertainty in a nominal transfer matrix G(s) by the following forms, [DOY2], [POS3], [SAF3], [CRU2], [LEH1], [OWE1], [POS4]:

$$\{G_p(s)\}_E = \{\, G(s)+E(s) \mid \bar{\sigma}[E(j\omega)\leq e(\omega)\,\} \qquad (3.1)$$

$$\{G_p(s)\}_{\Delta_o} = \{ (I+\Delta_o(s))G(s) \mid \bar{\sigma}[\Delta_o(j\omega)] \leq \delta_o(\omega) \} \quad (3.2)$$

$$\{G_p(s)\}_{\Delta_i} = \{ G(s)(I+\Delta_i(s)) \mid \bar{\sigma}[\Delta_i(j\omega)] \leq \delta_i(\omega) \} \quad (3.3)$$

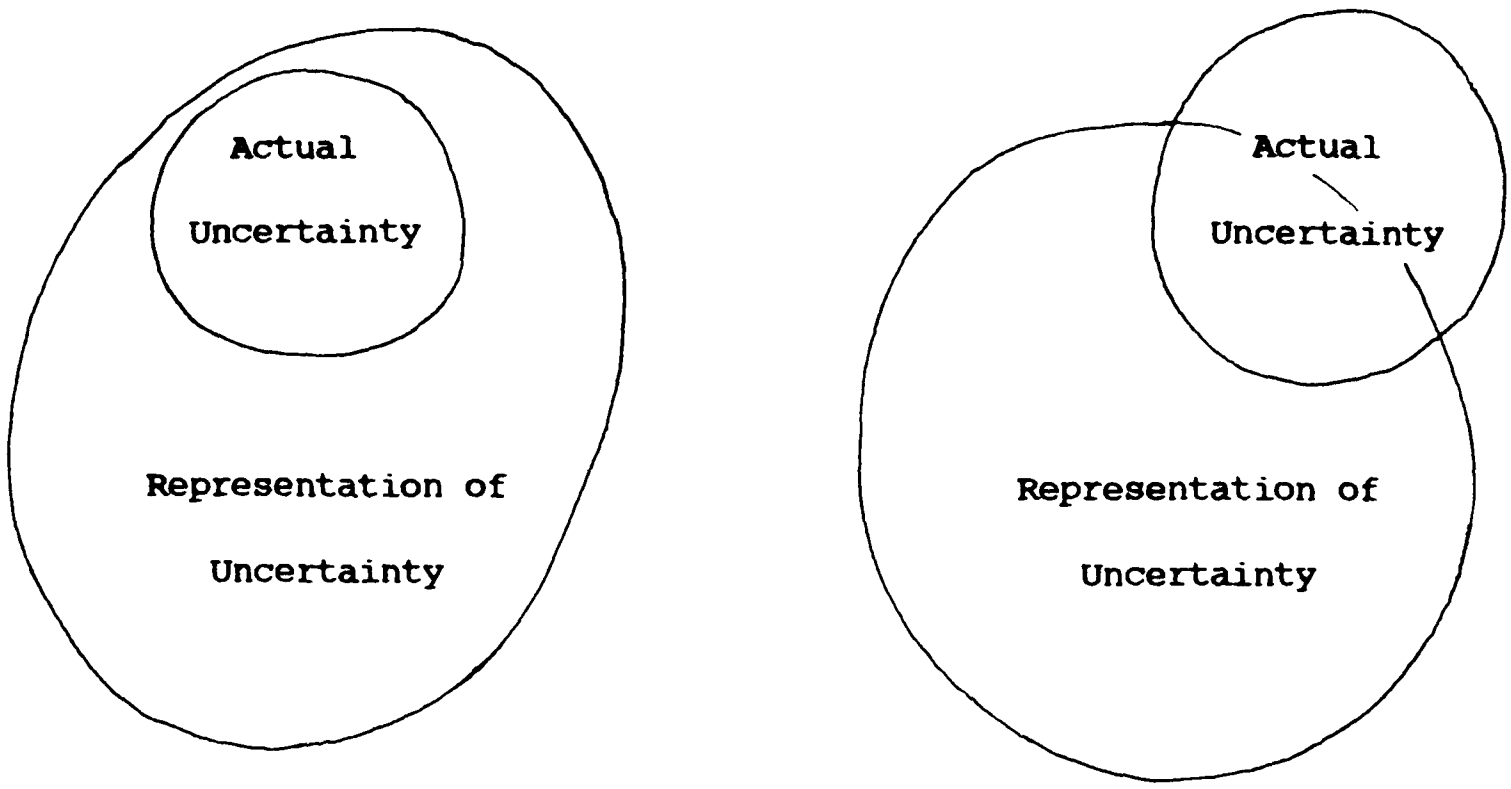
$$\{G_p(s)\}_{\tilde{E}} = \{ (G(s)^+ + \tilde{E}(s))^+ \mid \bar{\sigma}[\tilde{E}(j\omega)] \leq \tilde{e}(\omega) \} \quad (3.4)$$

$$\{G_p(s)\}_{\tilde{\Delta}_o} = \{(I+\tilde{\Delta}_o(s))^{-1}G(s) \mid \bar{\sigma}[\tilde{\Delta}_o(j\omega)] \leq \tilde{\delta}_o(\omega) \} \quad (3.5)$$

$$\{G_p(s)\}_{\tilde{\Delta}_i} = \{(I+\tilde{\Delta}_i(s))^{-1}G(s) \mid \bar{\sigma}[\tilde{\Delta}_i(j\omega) \leq \tilde{\delta}_i(\omega) \} \quad (3.6)$$

where A^+ is the left or right inverse of A, whichever exists, such that if A has more rows than columns, then $A^+A = I$. When both G(s) and $G_p(s)$ are invertible, (3.4)-(3.6) can be interpreted physically as an either additive or fractional changes in the inverse of the plant model. Note that although the representation of uncertainty in [POS3] and [KOU1] are in a slightly different form from those above, the arguments that follow apply to these as well. The reader is also referred to [LEH2] for diagrams of block representations of the various perturbations.

In each of the representations (3.1)-(3.6) a bound on the maximum singular is used to define a set of perturbed plant models. The question that arises is whether the set so defined includes all possible physical perturbations in G(s). If it does not, then the set is clearly inadequate for representing uncertainty (see figure 3.1). If the result of a robustness test is to be valid, the representation of uncertainty used must be an adequate representation.



a. Adequate b. Inadequate

Figure 3.1 Adequate/inadequate representation of uncertainty

Obviously, if the representation and the actual perturbation happen to be at the same point in the loop, then the representation will be adequate, provided that its maximum singular value bound is at least as large as that of the actual perturbation. However, in a real system, uncertainty occurs at several points in the loop and it is often not easy to identify and separate them. For example, when a plant model is obtained by a frequency response test, then all we get are bounds on both the gain and phase of each of the elements of G(s), and it is in general not possible to pin-point whether the uncertainty is due to an input-multiplicative, an output multiplicative, or an additive perturbation. In such a situation, the most practical approach is perhaps to reflect all uncertainty to one point and combine the individual contributions in a single perturbation as in the forms (3.1)-(3.6). It is then important to know whether the chosen representation of uncertainty can adequately represent perturbations from all possible sources in the system.

It is trivially true that given any perturbed plant model G_p and a nominal plant model G, we can always find an uncertainty model E of the same field, such that $G_p = G + E$. Hence the additive perturbation E can be an adequate representation of all different forms of perturbation under consideration. We may therefore adopt the view-point that all perturbations are additive in nature. In other words, no matter what the uncertainties actually are, they will show up as an additive perturbation and robustness against this perturbation implies, and is implied by, robustness against the actual perturbations.

When using the multiplicative forms of perturbations, more care needs to be exercised. To give an example of the difficulties involved, the reader may verify that there is no input multiplicative perturbation which can cause $G(s) = [1/(s+1) \quad 1/(s+2)]^T$ to become $G_p(s) = [1/(s+3) \quad 1/(s+4)]^T$.

The problem of knowing when G(s) can be transformed to $G_p(s)$ will now be considered in more detail. The results are not new and are therefore stated without proof. The same problem has received considerable attention in the area of model matching and system inversion; see [KAI1], [WAN2] and [POR1] for example.

Definition 3.1 Given two mxn matrices A and B, we say A is linearly transformable to B if there exists a linear operator T such that TA=B.

It follows that:

(i) If G is linearly transformable to G_p, i.e. there exists an L_o such that $L_o G \triangleq (I+\Delta_o)G = G_p$, then G and the set $\{\Delta_o\}$ can combine to give an adequate representation of the true plant G_p.

(ii) If G^T is linearly transformable to G_p^T, i.e. there exists an L_i such that $L_i^T G^T \stackrel{\Delta}{=} G_p^T$, then G and the set $\{\Delta_i\}$ can combine to give an adequate representation of the true plant G_p.

Necessary and sufficient conditions for G to be linearly transformable to G_p are given in theorem 3.1 below.

Theorem 3.1 Let G and G_p be two mxn real real rational matrices. Then the following statements are equivalent:

(i) G is linearly transformable to G_p;

(ii) the row-span of G_p is contained in the row-span of G;

(iii) rank G = rank F, where F is the partitioned matrix $[G_p^T \quad G^T]^T$.

Theorem 3.1 implies that $\text{rank}(G_p) \leq \text{rank}(G)$ is a necessary condition for G to be linearly transformable to G_p. Since the nominal model of most physical plants has full column rank, we give the following corollary.

Corollary 3.1 Let G and G_p be two real rational mxn matrices with m≥n and rank(G)=n, then G is linearly transformable to G_p.

In simple terms, corollary 3.1 implies:

(i) If G has full rank and at least as many outputs as inputs, then the output multiplicative perturbation can be an adequate representation of uncertainty.

(ii) If G has full rank and at least as many inputs as outputs, then the input multiplicative perturbation can be an adequate representation of uncertainty.

If we choose to represent uncertainty by one of the inverse forms (i.e. $\bar{E}$, $\bar{\Delta}_i$, or $\bar{\Delta}_o$), then more restrictions have to be observed. In particular:

(i) for the case of $\bar{E}$, both G and G_p have to be invertible or pseudo-invertible (in the sense that they are left or right invertible);

(ii) for $\bar{\Delta}_o$, $(I+\bar{\Delta}_o)$ has to be invertible (i.e. $\det(I+\bar{\Delta}_o)\neq 0$);

(iii) for $\bar{\Delta}_i$, $(I+\bar{\Delta}_i)$ has to be invertible (i.e. $\det(I+\bar{\Delta}_i)\neq 0$).

Theorem 3.2 There exists an mxm matrix L_o which is nonsingular and satisfies $L_oG = G_p$ if, and only if, the two mxn matrices G and G_p have equal rank and G is linearly transformable to G_p.

Corollary 3.2 Let two mxn matrices G and G_p have equal rank, and assume that there ~~either~~ exists an mxm singular matrix L' such that $L'G = G_p$. Then there exists another mxm matrix L which is nonsingular and satisfies $L'G = G_p = LG$.

Corollary 3.2 enables us to make a few remarks concerning the possibility of modelling complete actuator or sensor failures with multiplicative perturbations which are invertible. If G is full rank and has more outputs than inputs, and if there is an actuator failure in one or more of the loops, then G_p will have lower rank than G, and it follows that no invertible L_i or L_o exists such that $GL_i = G_p$ or $L_oG = G_p$. However, if there is a sensor failure in the loops, then we may still be able to find an invertible L_o such that $L_oG = G_p$. Hence robustness tests which utilize the inverse representations would indeed be capable of handling some cases of sensor failure. For example, let

$$G = [1/(s+1) \quad 1/(s+2)]^T$$

and $G_p = [1/(s+1) \quad 0 \quad]^T$

due to a sensor failure in output channel 2. A natural choice of L_o would be

$$L_o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which is not invertible. However, an invertible L_o is given by

$$L_o = \begin{bmatrix} 1 & 0 \\ 1/(s+2) & -1/(s+1) \end{bmatrix}.$$

We end this section by making a few remarks concerning the possibility of representing uncertainty by a stable perturbation. It should be clear to the reader that for a multiplicative perturbation to be stable, the plant would generally need to be both stable and minimum-phase. This is because any uncertain unstable pole of G(s) would have to be cancelled by a CRHP zero of $(I+\Delta_o)$ or $(I+\Delta_i)$ which makes $\tilde{\Delta}_o$ or $\tilde{\Delta}_i$ unstable. The correct unstable pole location would then need to be provided by an unstable pole of $(I+\tilde{\Delta}_o)$ or $(I+\tilde{\Delta}_i)$. Similar remarks apply to uncertain non-minimum-phase zeros. Since the pole and zero locations of a system can never be known exactly in practice, the perturbations are therefore necessarily unstable if the nominal plant is unstable and/or non-minimum-phase. For the additive perturbations, it is easy to verify that E will be stable if both G and G_p are stable, and $\tilde{E}$ will be stable if both G and G_p are minimum-phase. Note that if G is unstable, then E will be unstable even if we know all the unstable poles exactly (i.e. the unstable pole of G_p and G are identical). Similar remarks apply to $\tilde{E}$.

Example 3.1

$$G = 1/(s-1), \quad G_p = 1.1/(s-1)$$

therefore $E = 0.1/(s-1)$ which is unstable.

3.3 Representing Uncertainty by Simultaneous Perturbations

Combinations of the above forms of perturbations, (3.1)-(3.6), have been suggested in [DOY5], in order to represent simultaneous perturbations at the input and output of the plant, and also to cope with changing numbers of both poles and zeros in the CRHP, [DOY6]. A necessary requirement in [DOY5] is that the perturbations (i.e. E, $\tilde{E}$, Δ_o, $\tilde{\Delta}_o$, Δ_i, or $\tilde{\Delta}_i$) have to be stable, [DOY6]. In this section, we examine whether it is possible to represent arbitrary uncertainty by a combination of stable perturbations.

Theorem 3.3 Let G and G_p be two $m \times n$ real rational matrices, then there exists an $m \times m$ matrix L and an $n \times n$ matrix R such that

$$G_p = LGR$$

if, and only if, rank $G \geq$ rank G_p.

Proof See Appendix III-1

Note that the above result is fairly general because it does not require the row-span of G_p to be contained in the row-span of G, nor the column-span of G to be contained in the column-span of G. It thus follows that uncertainty which may not be representable by an input multiplicative perturbation nor an output multiplicative perturbation alone may be represented by a combination of input and output

multiplicative perturbations. However, it does not indicate whether there exists an L and a corresponding R which are either stable, or minimum phase and invertible. To illustrate the difficulty, recall the problems in section 3.2 of representing uncertainty by a single perturbation -- although E can always be an adequate representation, we have:

(i) if G, or G_p, is unstable, then E will generally be an unstable perturbation;

(ii) if G has uncertain CRHP poles (or zeros), then Δ_o (or Δ_i) will be an unstable perturbation in general.

The usefulness of the result of [DOY6] in coping with the problem of changing numbers of poles and zeros in the CRHP thus depends on our ability to decompose an arbitrary uncertainty into several stable perturbations. For this purpose we state the following theorem. Note that for simplicity (and practical common sense), we have assumed G to be of full column rank, but not necessarily square.

Theorem 3.4 Let G and G_p be two mxn real rational matrices with m≥n. Then there exists an mxm real rational matrix L and an nxn real rational matrix R such that

(i) $G_p = LGR$,

(ii) L is stable; and

(iii) R is minimum-phase and invertible.

Proof See Appendix III-2.

Although the theorem states that it is always possible to decompose an arbitrary uncertainty into a stable perturbation ,and a minimum phase and invertible perturbation, such a decomposition is artificial if the

uncertainty happens with a CRHP pole or zero. And it is possible that such an artificial decomposition will give rise to extra conservatism.

Example 3.2 Let $G = 1/(s-1)$ and $G_p = 1/(s-2) \stackrel{\Delta}{=} (I+\Delta_o)G(I+\tilde{\Delta}_i)^{-1}$. A feasible decomposition would be $(I+\Delta_o) = (s-1)/(s+100)$ which is stable, and $(I+\tilde{\Delta}_i)^{-1} = (s+100)/(s-2)$ which is minimum-phase.

It is seen that the decomposition is, in general, non-unique, and it is difficult to perform such a decomposition if the transfer function matrices of the plant and perturbation are not known exactly. Hence it appears that in general it will not be easy to estimate bounds for $\bar{\sigma}[\Delta_o(j\omega)]$ and $\bar{\sigma}[\tilde{\Delta}_i(j\omega)]$.

The decomposition may also lead to a more conservative robustness test when compared with the single perturbation tests of section 3.2. For example, let $G = 1/(s-1)$ and $G_p = 1/(s-1+\epsilon)$ where $|\epsilon| << 1$, then a direct and straightforward way to represent uncertainty would lead to $\Delta_o = L-I \approx 0$, whereas a simultaneous decomposition may give

$$(I + \Delta_o) = \frac{s-1}{s+1000} \qquad \text{and} \qquad (I + \tilde{\Delta}_i) = \frac{s-1+\epsilon}{s+1000}$$

which implies that $\bar{\sigma}[\Delta_o]$ and $\bar{\sigma}[\tilde{\Delta}_i]$ have a magnitude of order 1 at some frequencies.

It therefore appears that although the structured singular value test of [DOY5] is desirable in coping with multiple simultaneous perturbations in the feedback system, using it as a means of handling changing numbers of poles and zeros in the CRHP would not be satisfactory. In fact, it would seem that the requirement that all perturbations be stable could seriously impair the usefulness of the test. A new way of developing the test which does not require the

perturbations to be stable would certainly be desirable. This is undertaken in chapter 6.

3.4 Multiloop Gain and Phase Margins

Multiloop gain and phase margins [SAF2], [LEH1], which characterize the extent to which simultaneous gain or phase perturbations can occur in the control channels of a multivariable system without causing instability, are sometimes used as measures of robust stability, [SOB1]. The margins themselves may be viewed as a special form of multiplicative perturbation given in frequency response terms by

$$\hat{L}_i(j\omega) = I+\hat{\Delta}_i(j\omega) = \text{diag}\{l_i(\omega)e^{j\phi_i(\omega)}\} \qquad (3.7)$$

In a linear quadratic optimal regulator (using diagonal weighting matrix for control) which has a guaranteed gain margin of $(\frac{1}{2}, \infty)$ and a guaranteed phase margin of $(-60^\circ, 60^\circ)$, $L_i(j\omega)$ is either given by $\text{diag}\{l_i(\omega)\}$, where each of the $l_i(\omega)$ can take on any value between $\frac{1}{2}$ and ∞, or by $\text{diag}\{e^{j\phi_i(\omega)}\}$, where the ϕ_i are any values between -60° and $+60^\circ$. These margins are only guaranteed in the input control channels, and therefore $\hat{L}_i(j\omega)$ should be looked upon as a multiplicative perturbation at the plant input. Now the optimal regulator is a full state-feedback system and so the plant will generally have more outputs than inputs. It follows from section 3.2 that if we are to characterize uncertainty in the plant by a multiplicative perturbation then it would be more appropriate to position it at the output of the plant. It would

seem that the apparently large gain and phase margins associated with an optimal regulator may in fact be misleading from the point of view of a robust stability property. We will examine this in detail throughout the remainder of this section.

First note that the multiplicative perturbation $\hat{L}_i$ (or $\hat{\Delta}_i$) used by Lehtomaki et. al. ([LEH1], eqn (2.13)) is in general <u>not</u> the same as the L_i (or Δ_i) used here in section 3.2. Δ_i denotes the "fractional change" in G while $\hat{\Delta}_i$ denotes the fractional change in KG; K(s) being a controller transfer function matrix, or simply a constant matrix in the case of an optimal state-feedback regulator. For single-input, single-output systems Δ_i and $\hat{\Delta}_i$ are equivalent but for multivariable systems they are generally different.

For example, let $G = [1 \quad 1]^T$, $G_p = [1 \quad 2]^T$ and $K = [1 \quad 1]$. The reader may easily verify that there is no L_i which can transform G such that $GL_i = G_p$. However, $KG = 2$ and $KG_p = 3$, giving $\hat{L}_i = 3/2$ in $KG\hat{L}_i = KG_p$.

The example clearly demonstrates that the perturbation $\hat{\Delta}_i$ is dependent upon the choice of the controller K. As a result, if we use the formulae given in [LEH1] to estimate gain and phase margins, i.e.

$$\text{Gain margin} = \frac{1}{1 \pm \alpha} \tag{3.8}$$

and

$$\text{Phase margin} = \pm \cos^{-1}\left[1 - \frac{\alpha^2}{2}\right] \tag{3.9}$$

where

$$\alpha = \min_{\omega} \underline{\sigma}[I + K(j\omega)G(j\omega)] \tag{3.10}$$

then two controllers may give the same gain and phase margins but we cannot conclude that they are equally robust under a given physical

perturbation, say Δ, because one controller may amplify Δ into a much larger $\hat{\Delta}_i$ than the other. For example, assume the actual physical perturbation in G is an output multiplicative perturbation Δ_o then

$$KG(I+\hat{\Delta}_i) = K(I+\Delta_o)G \tag{3.11}$$

and

$$\hat{\Delta}_i = (KG)^{-1}K\Delta_o G \tag{3.12}$$

assuming KG is nonsingular, which shows the dependence of $\bar{\sigma}[\hat{\Delta}_i]$ on K.

For single-input systems the stability condition $\underline{\sigma}[I+K(j\omega)G(j\omega)] > \bar{\sigma}[\hat{\Delta}_i(j\omega)]$, [LEH1], [POS4], means: the distance between the Nyquist locus and the critical point -1 must be greater than the magnitude of $\hat{\Delta}_i(j\omega)$. Geometrically, if for the sake of argument we assume $\bar{\sigma}[\hat{\Delta}_i(j\omega)]$ to be constant and independent of frequency, the condition can then be interpreted as: the Nyquist locus should not enter the circle centred at the critical point -1 with radius $\bar{\sigma}[\hat{\Delta}_i]$. In this case, it is clear that robust stability not only depends on the gain and phase margins of the Nyquist locus, but also the size of the circle representing uncertainty. For single-input, single-output systems the radius of the circle depends only on the size of the plant perturbation, but for multivariable systems, (3.12) shows that the radius of the circle also depends on the choice of controller.

The above discussion is all part of the general problem of representing unstructured uncertainty by a multiplicative perturbation at the input of the plant because if the plant has more outputs than inputs, the input perturbation is generally unsuitable. In the definition of multiloop gain and phase margin, the situation is made worse by restricting the perturbation to be diagonal as well as positioning it at the plant input.

To consider this point further, let $KG = [g_1, g_2, \ldots, g_m]$ where g_i denotes the ith column of KG, then after multiplication by (3.7), we have

$$\hat{KGL}_i = [g_1 l_1 e^{j\phi_1}, g_2 l_2 e^{j\phi_2}, \ldots g_m l_m e^{j\phi_m}] \qquad (3.13)$$

The perturbation causes the elements in the same column of KG to have the same amount of gain and phase change. Though this will be the case when there are changes in actuator dynamics, it is not in general adequate to represent all possible types of linear perturbations in KG. The reader should not confuse this with the fact that given an input $u(j\omega)$ and corresponding outputs $y(j\omega)$ and $y_p(j\omega)$ of KG and KG_p respectively, then $y_p(j\omega)$ can generally be represented as a gain and phase type perturbation of y. That is, there exists a diagonal gain and phase type perturbation $\hat{L}$ such that

$$\hat{L}y = y_p \qquad (3.14).$$

One might think that this being the case, and since a transfer function matrix is simply an input-output description of the system, one ought to be able to find some diagonal gain and phase type perturbation for KG to represent uncertainty. But consider the following:

$$\hat{L}y = y_p \qquad (3.14)$$

$$\Rightarrow \quad \hat{L}KGu = y_p \quad \text{(since } y = KGu \text{ by definition)} \qquad (3.15).$$

$$\Rightarrow \quad \hat{L}KGu = LKGu \text{ (since } y_p = KG_p u = LKGu \text{ by definition)} \qquad (3.16)$$

(Note that $\hat{L}KG$ corresponds to a gain and phase perturbation at the output of K, which is equivalent to the input of G). If L happens to be diagonal, then it is obvious that $\hat{L} = L$. However, if L is not diagonal, then (3.16) will have a solution for $\hat{L}$ if, and only if:

$$(\hat{L}KG - LKG)u = 0 \qquad (3.17)$$

has a solution for $\hat{L}$. It is well-known that (3.17) will have a solution for nontrivial u only if $\det(\hat{L}KG - LKG)=0$. Such a solution will generally depend on the value of u. Hence, if we insist on representing uncertainty by a diagonal gain and phase type perturbation, what we get is a nonlinear perturbation (i.e. one dependent on the value of u). In such cases, a functional analytic approach or the small gain theorem can be used to derive robust stability results, [SAF2], but

(i) in general, such a nonlinear technique would require the perturbation to be stable, and

(ii) some small <u>linear</u> perturbation may be amplified into an arbitrary large, nonlinear, gain and phase type perturbation.

It should be clear to the reader that although the multiloop gain and phase margins are not adequate measures of robust stability, good gain and phase margins at both the input <u>and</u> the output are necessary for robust stability. One needs only to consider uncertainties in the actuators and sensors to realize that small gain and phase margins necessarily indicate a lack of robustness in stability.

The linear quadratic optimal state-feedback regulator is renowned for its guaranteed gain and phase margins of at least $(\frac{1}{2},\infty)$ and $(-60^{\circ}, 60^{\circ})$ respectively. But when the states are not available for feedback and an optimal observer is used to reconstruct them,it has been shown, [DOY3], that robust stability of the combined system can be arbitrarily bad. We now illustrate by examples that even if all the states are

available the regulator may have a very poor robust stability property. The problem is that the multiloop gain and phase margins are not adequate measures of robust stability in general. Note that in all these examples, the weighting matrices R are "diagonal".

Example 3.3, [FOO2]. Consider the optimal regulator

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1.1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \qquad (3.18)$$

$$u = -(-56.611 \quad 61.523)\, x$$

obtained for a weighting R=I on the input, and weighting Q=I on the states. If we break the second output channel and examine the Nyquist diagram (see figure 3.2) we find that the system has very small gain and phase margins at this point.

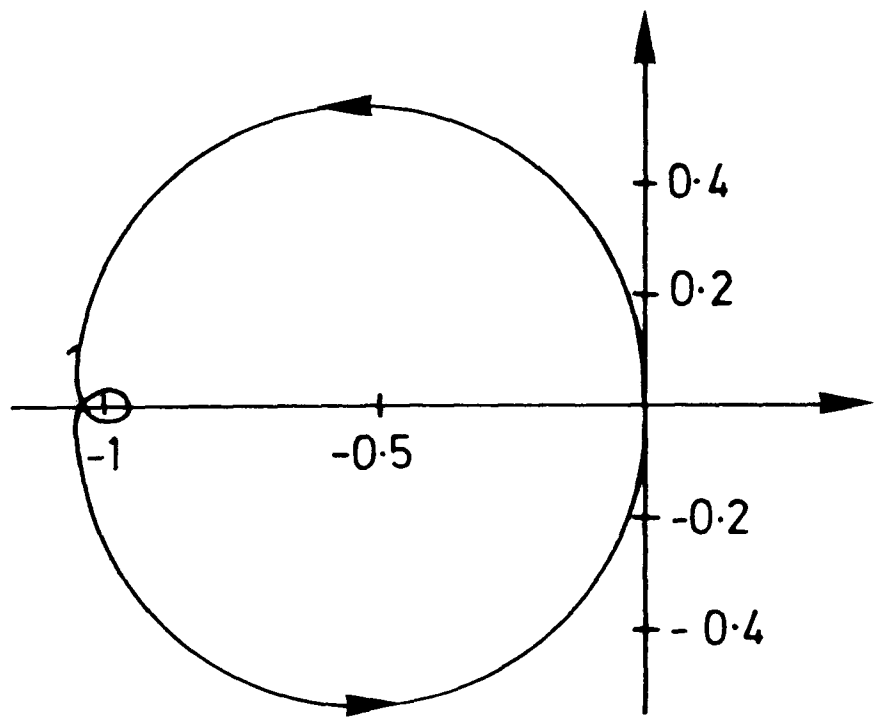


Figure 3.2 Nyquist diagram for output channel 2, example 3.3

Example 3.4, [SOR1]. Consider the optimal regulator

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \quad .$$

Let $R=rI$, and $Q=CC^T$ where $C=[1 \quad 1]^T$, then as $r \to 0$, $B \to [1+\epsilon \quad 1]^T$, and the closed-loop poles of the system are given by

$$p_1 = -\epsilon r^{\frac{1}{2}} + O(1), \qquad p_2 = -(1+2\epsilon)/\epsilon + O(1)$$

where $O(1)$ denotes order of magnitude ϵ^0

which may be made to be unstable by arbitrarily small ϵ.

The above examples may be viewed as extreme cases. For instance in the first example, the lack of robust stability may be attributed to the fact that the plant is very close to being unstabilizable. In the second example, the problem is due to the sensitivity of far LHP zeros which move into the RHP under a slight change in the parameter, [SOR1]. In both cases, the gain margin and phase margin fail to detect such a lack of robust stability. Note that in both examples, the closed-loop systems can indeed be de-stabilized by small asymmetrical perturbations. A closer examination will show that a gain and phase type perturbation at the plant input cannot be used to represent such asymmetrical perturbations. In fact, an input perturbation would require both elements of the transfer function matrix or input matrix B to change by the same factor. This would, of course, rarely be the case in practice. It is therefore no wonder that the input gain and phase margins fail to detect the lack of robust stability in these examples.

It can similarly be argued that multiloop gain and phase margin at the output of the plant are inadequate for representing arbitrary uncertainty. It is also the case that a combination of diagonal gain and phase type perturbations at both the input and output of the plant is not adequate for representing arbitrary uncertainty when $m+n<mn$, where m is the number of plant outputs and n the number of plant inputs. This follows simply on recognizing that mn is the number of transfer function elements which may be subjected to uncertainty, whilst m+n is the number of independent perturbations which can be used to represent uncertainty in the elements. Note also that, in general, relative stability (i.e. the magnitude of the real part of the closed-loop poles) is not a reliable measure of robust stability in multivariable systems. This is because closed-loop poles can be extremely sensitive when subjected to perturbations.

3.5 Concluding Remarks

In this chapter, we have discussed several representations of uncertainty and examined the adequacy of each. We emphasize that a robustness test can only guarantee robustness against uncertainty only if the uncertainty is adequately characterized by its representation.

When uncertainty is known to occur at a particular point in a feedback system then it can be represented by a perturbation at the point in question and robustness assessed against the perturbation. Unfortunately, uncertainty often arises at several points and in many cases the individual uncertainties cannot be identified in the form of individual perturbations. In these circumstances all uncertainties may

be modelled by a single perturbation at a single point in the feedback system. But for the resulting robustness test to be valid the particular perturbation at the point of application must give an adequate representation of uncertainty; i.e. the perturbation must be capable of representing all possible uncertainties. For example, it was shown in section 3.2 that an input multiplicative perturbation cannot be used to represent an arbitrary perturbation in G if the plant has more outputs than inputs.

In section 3.3, we examine the possibility of representing uncertainty by a combination of stable perturbations at the input and output of the plant. It was shown that such a representation is almost always possible, although some doubt was expressed concerning the desirability of such a decomposition for handling pole and zero movements in the CRHP.

It is well known that the robustness of the stability property for a multivariable system cannot be evaluated by observing gain and phase margins one loop at a time. In section 3.4, we have argued that even if the gain and phase margins are evaluated in all loops simultaneously, as in the LQ state-feedback regulator design, they are still unreliable indicators of robust stability. The point was illustrated by two examples both of which had the LQ state-feedback guaranteed gain and phase margins in the input control channels but comparatively poor margins in the output channels.

Appendix III-1

Proof of theorem 3.3

The necessary part is obvious.

To prove the sufficiency part, we let GR = G', then G' is linearly transformable to G_p if, and only if, the row-span of G_p is contained in the row-span of G'. But the row-span of G' is always contained in the row-span of R. Since we can choose R arbitrarily, we can always choose it such that row-span{GR} $\geq$ row-span{G_p}. Note that this is possible if and only if rank G $\geq$ rank G_p. This completes the proof. ∎

Note: It is always possible to choose R such that row-span{GR} $\geq$ row-span{G_p} if, and only if, rank G $\geq$ rank G_p. To demonstrate this, we shall assume rank{G_p} = r_p. Then it is always possible to find an r_p-dimensional subspace of the column-span of G^T. We will use this as the domain, and use the column-span of G_p^T as co-domain, and define an arbitrary isomorphic linear operator R_1^T on these spaces. Note that such an R_1^T always exists because the domain and co-domain are of the same dimension (hence isomorphic) in this case. Define another well-defined operator R_2^T whose domain is (column-span{G^T} $\ominus$ Domain(R_1^T)), and whose range is a subspace of (C^n $\ominus$ Range(R_1^T)). Then R^T is given by $R_1^T \oplus R_2^T$, with R_1^T and R_2^T each defined on their respective domains and co-domains. Since R^T is an operator of finite dimensional spaces, there exists a matrix representation of R.

Appendix III-2

Proof of theorem 3.4

For any real rational transfer function matrix G, we can derive the Smith-McMilliam form

$$G = PMQ.$$

Let Z_G^+ be the set of uncertain CRHP zeros of G and P_{Gp}^+ be the set of uncertain CRHP poles of G_p.

Define $R = Q^{-1}R_mQ$ where R_m is a diagonal matrix containing Z_G^+ and P_{Gp}^+ as its poles such that GR has all the uncertain poles of G_p as its poles and none of the uncertain zeros of G as its zeros. Clearly, R is invertible and minimum-phase since it has no CRHP zeros and rank(GR) equals rank(G). Since GR is of full column-rank, it is linearly transformable to G_p. It remains to be shown that L will be a stable perturbation.

Define $F = \begin{bmatrix} G_p \\ GR \end{bmatrix}$.

Kung and Kaileth, [KAI1], showed that there exists a solution to

$$LA = B$$

with no pole at α if, and only if

$$\nu_\alpha(A) = \nu_\alpha(F) = (\text{number of zeros at } \alpha) - (\text{number of poles at } \alpha)$$

Now since GR has no uncertain zeros of G, any CRHP zero of GR has to be a zero of G_p. Thus, the number of zeros of GR at any location α in the CRHP has to be equal to the number of zeros of F at the same location. (Any zero which is a zero of G_p but not a zero of GR is not a zero of F). Similarly, since $P^+_{GR} = P^+_{Gp}$, the number of poles of F at any location α in the CRHP is the same as the number of CRHP poles of GR at the same location. It follows that

$$\begin{aligned}\sum_{\alpha \in Res \geq 0} \nu_\alpha(GR) &= (\text{no. of zeros of GR at } \alpha) - (\text{no. of poles of GR at } \alpha) \\ &= (\text{no. of zeros of F at } \alpha) - (\text{no. of poles of F at } \alpha) \\ &= \nu_\alpha(F)\end{aligned}$$

Hence L is stable. This completes the proof. ∎

CHAPTER IV

ROBUST STABILITY ANALYSIS WITH UNSTRUCTURED UNCERTAINTIES

4.1 Introduction

In this chapter we present some new singular value based tests for assessing the robustness of the stability property of a feedback design. The tests are more widely applicable than most previously described singular value based tests in the literature, [DOY2], [POS3], [SAF3], [CRU2], [LEH1], and [KOU1].

The main contributions of this chapter are the results given in sections 4.5 and 4.7. The layout of the chapter is as follows. In section 4.2, we use the representations of uncertainty given by (3.1) to (3.3) and derive robust stability tests which are based on a multivariable version of Nyquist's stability criterion. These tests are not new and stem from the early work of Doyle and Stein, [DOY2], but they are included here for completeness and because of the duality they hold with later results in section 4.3. They are applicable providing all members of the set $\{G_p(s)\}$, which of course includes the nominal model $G(s)$, share the same number of unstable poles. There is no restriction on the zeros of any $G_p(s)$ and therefore the tests can accommodate both zero migration between the left and right half-planes, as a result of parameter variations, and the sudden appearance or disappearance of zeros (anywhere in the complex plane), arising from errors in the nominal model.

In section 4.3, we utilize representations of uncertainty (3.4) to (3.6) to derive robust stability tests based on a multivariable version of the inverse Nyquist stability criterion, [WHI1], [POS1]. The tests mirror those in section 4.2, with the role of poles replaced by that of

zeros. The tests are therefore applicable when all members of the set $\{G_p(s)\}$ have the same number of zeros in the closed right-half complex plane, with no restriction on their poles.

The relationship between the robust stability test of Lehtomaki et. al., [LEH1], [LEH3], and the inverse Nyquist based robust stability tests of section 4.3 is considered and established in section 4.4, [POS8]. It will be shown that Lehtomaki's test is simply a special case of the corresponding inverse Nyquist based test. More specifically, we will show that the inverse Nyquist based test will be satisfied if Lehtomaki's test is satisfied, whereas Lehtomaki's test will be satisfied only if the inverse Nyquist based test is satisfied and the nominal and the perturbed plants share the same numbers of poles and zeros in the CRHP.

In section 4.5, the problem of simultaneous pole and zero crossings of the imaginary axis is solved by combining a Nyquist based test from section 4.2 with an inverse based test from section 4.3. The resulting test is applicable to a large class of transfer function matrices $\{G_p(s)\}$ whose frequency responses are known to lie within given bounds and whose CRHP pole and zero structures are known to satisfy certain necessary properties for robust stabilizability. We will say more about these properties in the section, but for now it is sufficient to think of them as ruling out exact or nearly exact pole-zero cancellations in the closed right half complex plane.

Robust stability tests in which uncertainty is represented in the form of perturbations in the coprime factors, [VID4], [VID6], are reviewed in section 4.6 and their relationship with the robust stability test of section 4.5 is established.

In section 4.7, the problem of combining various robust stability tests to form a general switching-type of robust stability test is

examined. The resulting test includes the robust stability tests of sections 4.5 and 4.6 as special cases.

The problem of robustness analysis for loop integrity is briefly addressed in section 4.8, and we give our conclusions for the chapter in section 4.9.

Throughout this chapter we will be interested in the feedback configuration shown in figure 4.1, where G(s) is a nominal model used for design purposes and K(s) is a controller such that the nominal feedback system is stable. The adaptation of the results for application to the more general two-degree-of-freedom configuration is straightforward and should be clear to the reader from the discussions in chapter 2. As before, G(s)K(s) and $G_p(s)K(s)$ are assumed for practical reasons to be strictly proper, (rational) transfer function matrices, and hence $\det[I+G(\infty)K(\infty)]\neq 0$, which is also required for the feedback system to be dynamical. For simplicity, the tests of sections 4.2 to 4.5 are only developed for square and invertible plants. The case of non-square plants and that of non-invertible plants will be postponed until we have proved a generalized inverse Nyquist stability criterion in the next chapter. We stress again that in this chapter we have assumed both the plant and controller to be square and invertible.

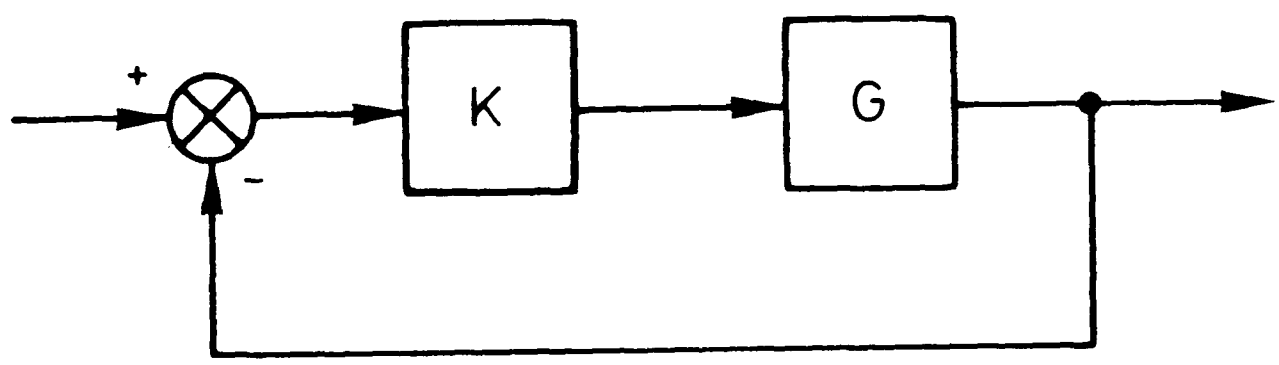


Figure 4.1 Feedback Configuration

4.2 Nyquist Based Singular Value Robust Stability Tests

The robust stability tests of this section are based on a multivariable version of the Nyquist stability criterion, [POS1], [SMI1] and [HSU1]. For the convenience of the reader, the criterion is re-stated as theorem 4.1 below.

Theorem 4.1 The feedback system of figure 4.1 is stable if, and only if, the image of the Nyquist D-contour, D_D, under $\det[I + G(s)K(s)]$ encircles the origin P times anticlockwise, where P is the number of open-loop poles in D_D.

From theorem 4.1, it is easy to see that if $G_p(s)$ is a perturbed plant model with the same number of poles in D_D as $G(s)$, then the corresponding perturbed feedback system will be stable if the image of D_D under $\det[I + G_p(s)K(s)]$ encircles the origin the same number of times as the image of D_D under $\det[I + G(s)K(s)]$. This argument is central to the development of the direct Nyquist criterion based robust stability tests, [DOY2], [LEH1]. We therefore state it as an abstract theorem.

Theorem 4.2 (Fundamental Nyquist based Robust Stability Theorem) Assume the feedback system of figure 4.1 is closed-loop stable, then it will remain stable when the nominal model is replaced by any perturbed model taken from the set $\{G_p\}$ if, for every $G_p \in \{G_p\}$:

(i) $G_p(s)$ and $G(s)$ share the same number of poles in D_D and

(ii) $N(0,\det[I+G_p(s)K(s)],D_D) = N(0,\det[I+G(s)K(s)],D_D)$

where $N(\pi,f(s),C)$ denotes the number of clockwise encirclements of the point π by the locus of $f(s)$ as s traverses the closed contour C in the complex plane in a clockwise direction.

Proof Obvious. ∎

Theorem 4.2 is slightly different from the "fundamental robustness theorem" (theorem 2) of [LEH2] where the argument is based on the continuous deformation of the Nyquist diagram when a parameter α is perturbed continuously from 0 to 1. To facilitate discussions and comparisons in later sections, it is re-stated here as a corollary to theorem 4.2.

Corollary 4.1 Let $G(s,\alpha)$ be a matrix of rational transfer functions with real coefficients which are continuous functions of α for all $\alpha\epsilon[0, 1]$ and for all $s\epsilon D_D$, which satisfies the following two conditions:

$$G(s,0) = G(s) \tag{4.1}$$

and

$$G(s,1) = G_p(s) \tag{4.2}$$

Assume the feedback system of figure 4.1 is closed-loop stable, then it will remain stable when the nominal model is replaced by any perturbed plant model taken from the set $\{G_p\}$ if for every $G_p \epsilon \{G_p\}$:

(i) $G_p(s)$ and $G(s)$ share the same number of poles in D_D, and

(ii) $\det[I + G(s,\alpha)K(s)]\neq 0$ for all $(s,\alpha) \epsilon D_D x [0,1]$.

Proof $G(s,\alpha)$ is continuous in (s,α) implies that $\det[I+G(s,\alpha)K(s)]$ is continuous for all $(s,\alpha) \epsilon D_D x[0,1]$. Hence the locus of $\det[I+G(s,\alpha)K(s)]$ on D_D forms a bounded loop in the complex plane for any $\alpha\epsilon[0,1]$. Thus the only way to change the number of encirclements of the critical point (0,j0) is for the locus to pass through the critical point for some $\alpha\epsilon[0,1]$; that is there exists $\alpha_o \epsilon [0,1]$ such that

$$\det[I + G(s,\alpha_o)K(s)] = 0 \tag{4.3}$$

for some $s \in D_D$. This is prevented by condition (ii), and hence the number of clockwise encirclements of $\det[I + G_p(s)K(s)]$ and $\det[I + G(s)K(s)]$ have to be the same. Application of theorem 4.2 completes the proof.∎

Although theorem 4.2 is fairly general, it is quite useless from a practical point of view unless we can examine the number of encirclements of $\det[I + G_p(s)K(s)]$ for each G_p. Since the set $\{G_p(s)\}$ is almost always uncountable, we therefore need to modify theorem 4.2 in such a way so that it is not necessary to test stability for each individual element of $\{G_p\}$. A simple and useful way of doing this is to use the representations of uncertainty in the form of (3.1) to (3.3); that is, let $\{G_p\}$ be a subset of

$$\{G_p(s)\}_E \triangleq \{ G(s)+E(s) \mid \bar{\sigma}[E(s)] \leq e(s), \text{ all } s \in D_D \} \qquad (4.4)$$

$$\{G_p(s)\}_{\Delta_o} \triangleq \{(I+\Delta_o(s))G(s) \mid \bar{\sigma}[\Delta_o(s)] \leq \delta_o(s), \text{ all } s \in D_D\} \quad (4.5)$$

$$\{G_p(s)\}_{\Delta_i} \triangleq \{G(s)(I+\Delta_i(s)) \mid \bar{\sigma}[\Delta_i(s)] \leq \delta_i(s), \text{ all } s \in D_D\} \quad (4.6)$$

and derive robust stability tests in terms of $\{G_p(s)\}_E$, $\{G_p(s)\}_{\Delta_o}$ or $\{G_p(s)\}_{\Delta_i}$. Note that we have modified the representations slightly in order to avoid the technical difficulty of having poles on the $j\omega$-axis.

Theorem 4.3 The feedback system of figure 4.1 will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the set $\{G_p(s)\}_E$ if:

(i) $G_p(s)$ and $G(s)$ share the same number of poles in D_D, and

(ii) $\bar{\sigma}[K(s)(I + G(s)K(s))^{-1}] < e(s)^{-1}$, all $s \epsilon D_D$.

Proof Given that the nominal feedback system is stable we have by theorem 4.1 that $\det[I + G(s)K(s)] \circ D_D$ encircles the origin P times anticlockwise, where the notation $V(s) \circ W$ denotes the image of W under V(s).

If G(s) is perturbed to $G_p(s) \epsilon \{G_p(s)\}_E$ and both have the same number of poles in D_D, then P is unchanged and by theorem 4.2 the perturbed feedback system will be stable if (and only if) the curves $\det[I + G_p(s)K(s)] \circ D_D$ encircle the origin the same number of times.

Given $G_p = G + E$, where $\bar{\sigma}[E(s)] \leq e(s)$, all $s \epsilon D_D$, it is easy to show that

$$\det[I + G_p(s)K(s)]$$
$$= \det[I+E(s)K(s)(I+G(s)K(s))^{-1}]\det[I+G(s)K(s)] \quad (4.7)$$

So $\det[I+G_p(s)K(s)] \circ D_D$ and $\det[I+G(s)K(s)] \circ D_D$ will encircle the origin the same number of times if

$$\det[I + E(s)K(s)(I+G(s)K(s))^{-1}] \circ D_D$$

does not encircle the origin at all, and this is guaranteed for all allowable E(s) if, and only if

$$\det[I+\gamma E(s)K(s)(I+G(s)K(s))^{-1}] \neq 0 \text{ and is finite, all } (s,\gamma) \epsilon D_D \times [0,1] \quad (4.8)$$

$$\Leftarrow \quad \bar{\sigma}[E(s)K(s)(I+G(s)K(s))^{-1}] < 1, \quad \text{all } s \epsilon D_D \quad (4.9).$$

$$\Leftarrow \quad \bar{\sigma}[K(s)(I+G(s)K(s))^{-1}] < (\bar{\sigma}[E(s)])^{-1}, \text{ all } s \epsilon D_D \quad (4.10).$$

$$\Leftarrow \quad \bar{\sigma}[K(s)(I+G(s)K(s))^{-1}] < \frac{1}{e(s)}, \quad \text{all } s \epsilon D_D \quad (4.11).$$

This completes the proof. ∎

If E is truly unstructured; that is, E(s) is allowed to be any real rational matrix such that $\bar{\sigma}[E(s)] \leq |r(s)| = e(s)$, where r(s) is a scalar real rational function, and G(s)+E(s) has the same number of unstable

poles as G(s), then theorem 4.3 is also necessary in the sense that if the equality is not satisfied at some frequency, then there always exists $G_p(s)$ from the set (4.4) such that the perturbed closed-loop system is unstable. Similar remarks apply to the next two theorems, and the theorems in the next section. However, since real perturbations are rarely truly unstructured, especially at low frequencies, we prefer to look upon them as sufficient conditions. A proof of necessity may be found in [VID3].

Theorem 4.4 The feedback system of figure 4.1 will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the et $\{G_p(s)\}_{\Delta_o}$ if:

(i) $G_p(s)$ and G(s) share the same number of poles in D_D, and

(ii) $\bar{\sigma}[(I+G(s)K(s))^{-1}G(s)K(s)] < \delta_o(s)^{-1}$, all $s\epsilon D_D$.

Proof A proof follows the same pattern as in theorem 4.3, recognizing that

$$\det[I + G_p(s)K(s)]$$
$$= \det[I+G(s)K(s)]\det[I+\Delta_o(s)(I+G(s)K(s))^{-1}G(s)K(s)] \quad (4.12)$$

This completes the proof. ∎

Theorem 4.5 The feedback system of figure 4.1 will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the set $(G_p(s))_{\Delta_i}$ if:

(i) $G_p(s)$ and G(s) share the same number of poles in D_D, and

(ii) $\bar{\sigma}[(I + K(s)G(s))^{-1}K(s)G(s)] < \delta_i(s)$, all $s\epsilon D_D$.

Proof A proof follows the same pattern as in theorem 4.3, recognizing that

$$\det[I+K(s)G_p(s)]$$
$$= \det[I+K(s)G(s)]\det[I+(I+K(s)G(s))^{-1}K(s)G(s)\Delta_i(s)] \quad (4.13)$$

This completes the proof. ∎

In the theorems above it is not necessary to use the large semi-circular part of the D_D in the inequality tests because $G(s)K(s)$ and $G_p(s)K(s)$ are both assumed to be strictly proper causing $\det[I+G(s)K(s)]$ and $\det[I+G_p(s)K(s)]$ to approach 1 as $|s|$ approaches infinity. This, in turn, implies that $\bar{\sigma}[EK(I+GK)^{-1}]$, $\bar{\sigma}[\Delta_o GK(I+GK)^{-1}]$ and $\bar{\sigma}[(I+KG)^{-1}KG\Delta_i]$ approach zero as $|s|$ approaches infinity. It is necessary, however, to use the small indentations which are present if $G(s)$ or $G_p(s)$ have any imaginary axis poles, and consequently we have the problem of estimating each of the bounds $e(s)$, $\delta_o(s)$ and $\delta_i(s)$ around the small indentations.

Another problem which could arise from the direct application of theorem 4.4 or theorem 4.5 is when $G(s)$ has a zero on the $j\omega$-axis and $G_p(s)$ does not have this zero. In such cases, $\Delta_o(s)$ or $\Delta_i(s)$ will have a pole on the $j\omega$-axis which causes $\bar{\sigma}[\Delta_o(s)]$ and $\bar{\sigma}[\Delta_i(s)]$ to be infinite at the particular frequency. This problem is somewhat artificial because if $G(s)$ has a zero on the $j\omega$-axis, then $T(s) \triangleq [I + G(s)K(s)]^{-1}G(s)K(s)$ and $\tilde{T} \triangleq [I + K(s)G(s)]^{-1}K(S)G(s)$ will have a zero at that frequency such that $\bar{\sigma}[\Delta_o(s)T(s)]$ and $\bar{\sigma}[\tilde{T}(s)\Delta_i(s)]$ are bounded on the $j\omega$-axis even though $\bar{\sigma}[\Delta_o(s)]$ and $\bar{\sigma}[\Delta_i(s)]$ are not. A simple way to get around this is to multiply $\Delta_o(s)$ (or $\Delta_i(s)$) by a scaling matrix, say $R(s)$ (or $L(s)$ respectively) which possesses a zero at the appropriate frequency such that both $\bar{\sigma}[\Delta_o(s)R(s)]$ and $\bar{\sigma}[R^{-1}(s)T(s)]$ (or $\bar{\sigma}[L(s)\Delta_i(s)]$ and $\bar{\sigma}[\tilde{T}(s)L_i(s)^{-1}]$) are bounded on the $j\omega$-axis. The inequality tests can then be replaced by the scaled inequality tests:

$$\bar{\sigma}[R(s)^{-1}T(s)] < (\bar{\sigma}[\Delta_o(s)R(s)])^{-1} \qquad (4.14)$$

$$\bar{\sigma}[\tilde{T}(s)L(s)^{-1}] < (\bar{\sigma}[L(s)\Delta_i(s)])^{-1} \qquad (4.15).$$

The above discussion suggests a way of capturing the structure of the perturbations and reducing the conservatism of the singular value robust stability tests. That is, assume we can decompose E(s) (similar remarks apply to $\Delta_o(s)$ and $\Delta_i(s)$) as $E(s) = L(s)\hat{E}(s)R(s)$ where L(s) and R(s) are known and $\hat{E}(s)$ is an unstructured perturbation satisfying $\bar{\sigma}[\hat{E}(s)] < \hat{e}(s)$ for all $s \epsilon D_D$, then the inequality test of theorem 4.3.may be replaced by the following inequality:

$$\bar{\sigma}[R(s)K(s)(I+G(s)K(s))^{-1}L(s)] \leq (\hat{e}(s))^{-1}, \text{ all } s \in D_D. \qquad (4.16)$$

The problem of finding a pair of scaling matrices L(s) and R(s) given a set of perturbations is another important field of research in robustness studies but is unfortunately outside the scope of this chapter. The interested reader is referred to [SAF4], [KOU2] and section 6.3 for further discussions. For simplicity, we shall not use scaling matrices this thesis. Nevertheless, it should be clear to the reader that they should be used when and where appropriate. In general the extensions of the results to include scaling are fairly straightforward.

One of the major disadvantages of theorems 4.3 to theorem 4.5 is that, because $G_p(s)$ and G(s) must have the same number of poles in D_D, they cannot be used when G(s) has any nominal poles whose regions of uncertainty extend across the imaginary axis.

4.3 Inverse Nyquist Based Singular Value Robust Stability Tests

In this section we use three further representations of uncertainty (expressions (3.4) - (3.6)) and present robust stability tests which are based on a multivariable version of the inverse Nyquist stability criterion, [POS1], [LOG1]. The tests mirror those in section 4.2 with the role of poles being replaced by that of zeros.

The representations of uncertainty are in terms of the inverse of a perturbed plant model and are re-stated below.

$$\{G_p(s)\}_{\underset{\sim}{E}} \triangleq \{ (G(s)^{-1} + \tilde{E}(s)) \mid \bar{\sigma}[\tilde{E}(s)] \leq \tilde{e}(s), \text{ all } s \epsilon D_I \} \tag{4.17}$$

$$\{G_p(s)\}_{\underset{\sim}{\Delta_o}} \triangleq \{ (I+\tilde{\Delta}_o(s))^{-1} G(s) \mid \bar{\sigma}[\tilde{\Delta}_o(s)] \leq \tilde{\delta}_o(s), \text{ all } s \epsilon D_I \} \tag{4.18}$$

$$\{G_p(s)\}_{\underset{\sim}{\Delta_i}} \triangleq \{ G(s)(I+\tilde{\Delta}_i(s))^{-1} \mid \bar{\sigma}[\tilde{\Delta}_i(s) \leq \tilde{\delta}_i(s), \text{ all } s \epsilon D_I \} \tag{4.19}$$

where D_I is the standard inverse Nyquist D-contour with indentations to the left of imaginary axis zeros, and with a clockwise orientation.

The appropriate inverse Nyquist stability criterion is given below in theorem 4.6; it may be derived by applying the multivariable inverse Nyquist stability criterion of [POS1] or [LOG1], and recognizing that the determinant of a matrix is simply the product of its eigenvalues.

Theorem 4.6 The feedback system of figure 4.1 is stable if, and only if, the image of the inverse Nyquist D-contour D_I under $\det[I + (G(s)K(s))^{-1}]$ encircles the origin Z times anticlockwise, where Z is the total number of zeros of G(s) and K(s) in D_I.

Theorem 4.6 indicates that if $G_p(s)$ is a perturbed plant model with the same number of zeros in D_I as $G(s)$, then the corresponding perturbed feedback system will be stable if the curves $\det[I + (G(s)K(s))^{-1}]\circ D_I$ and $\det[I+(G_p(s)K(s))^{-1}]\circ D_I$ encircle the origin the same number of times. We therefore have the following abstract inverse Nyquist based robust stability theorem.

Theorem 4.7 (Fundamental Inverse Nyquist based Robust Stability Theorem)

Assume the feedback system of figure 4.1 is closed-loop stable, then it will remain stable when the nominal model is replaced by any perturbed plant model taken from the set $\{G_p(s)\}$ if, for every $G_p(s) \in \{G_p(s)\}$:

(i) $G_p(s)$ and $G(s)$ share the same number of zeros in D_I, and

(ii) $N(0,\det[I+K(s)^{-1}G(s)^{-1}],D_I) = N(0,\det[I+K(s)^{-1}G(s)^{-1}],D_I)$.

Proof Obvious. ∎

The similarity between the arguments leading to theorem 4.7 and that used throughout the previous section should be clear to the reader and for this reason only partial proofs are given for the robust stability theorems stated in this section.

Theorem 4.8 The feedback system of figure 4.1 will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the set $\{G_p(s)\}_{\tilde{E}}$ if :

(i) $G_p(s)$ and $G(s)$ share the same number or zeros in D_I, and

(ii) $\bar{\sigma}[(I + G(s)K(s))^{-1}G(s)] < \tilde{e}(s)$, all $s\epsilon D_I$.

Proof A proof follows the same pattern as in section 4.2, recognizing that both $G(s)$ and $G_p(s)$ are square and invertible, and

$$\det[I + (G_p(s)K(s))^{-1}]$$
$$= \det[I+(GK)^{-1}]\det[I+\tilde{E}(I+GK)^{-1}G] \qquad (4.20)$$

since K(s) is invertible. This completes the proof ∎

Theorem 4.9 The feedback system of figure 4.1 will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the set $\{G_p(s)\}_{\Delta_o}$ if:

(i) $G_p(s)$ and $G(s)$ share the same number of zeros in D_I, and

(ii) $\bar{\sigma}[(I + G(s)K(s))^{-1}] < (\tilde{\delta}_o(s))^{-1}$, all $s \in D_I$.

Proof A proof follows the same pattern as in section 4.2, recognizing that

$$\det[I + (G_p(s)K(s))^{-1}]$$
$$= \det[I+(GK)^{-1}]\det[I+\tilde{\Delta}_o(I+GK)^{-1}] \qquad (4.21)$$

and this completes the proof. ∎

Theorem 4.10 The feedback system of figure 4.1 will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the set $\{G_p(s)\}_{\Delta_i}$ if :

(i) $G_p(s)$ and $G(s)$ share the same number of zeros in D_I, and

(ii) $\bar{\sigma}[(I + K(s)G(s))^{-1}] < (\tilde{\delta}_i(s))^{-1}$, all $s \in D_I$.

Proof A proof follows the same pattern as in section 4.2, recognizing that

$$\det[I + (K(s)G_p(s))^{-1}]$$
$$= \det[I + (KG)^{-1}]\det[I + \tilde{\Delta}_i(I+KG)^{-1}] \qquad (4.22)$$

and this completes the proof. ∎

Note that it is, again, not necessary to use the large semi-circular part of D_I in the above theorems because both $G(s)K(s)$ and $G_p(s)K(s)$ are strictly proper. However, it is interesting to note that a necessary condition for the inequality test of , say, theorem 4.9 is that

$$\lim_{|s|\to\infty} \bar{\sigma}[\tilde{\Delta}_o(s)] < 1.$$

This implies that no pole will be allowed to move across the semi-circular part of the D_I-contour, which is equivalent to saying that the Nyquist loci cannot change in a discontinuous way (see section 2.4) when $G(s)$ is perturbed to $G_p(s)$. To see this , we note that when we perturb $G(s)$ "continuously" to $G_p(s)$ by means of $G(s,\gamma) = (I + \gamma\tilde{\Delta}_o(s))^{-1}G(s)$ where γ is varied from 0 to 1, the poles of $G(s,\gamma)$ move in a root-locus pattern, [POS1]. This is because

$$(I + \gamma\tilde{\Delta}_o(s)) = I + \gamma(L_o(s)^{-1} - I)$$

$$= (1 - \gamma)[I + \frac{\gamma}{1-\gamma} L_o(s)^{-1}] \qquad (4.23)$$

where $L_o(s) \triangleq (I + \tilde{\Delta}_o(s))^{-1}$. Thus, $\bar{\sigma}[\tilde{\Delta}_o(\infty)] < 1$ implies $\underline{\sigma}[I+\gamma\tilde{\Delta}_o(s)]\neq 0$, $\gamma\epsilon[0, 1]$, on the semi-circular part of the D_I contour. This, in turn, implies that $[I + \gamma\tilde{\Delta}_o(s)]$ does not have a zero on the semi-circular part of D_I, $\gamma\epsilon[0, 1]$. Thus, when we vary γ from 0 to 1, no pole crosses the semi-circular part of the D_I contour. It is therefore seen that

$$\lim_{|s|\to\infty} \bar{\sigma}[\tilde{\Delta}_o(s)] < 1$$

is the price we have to pay for not having a condition on the CRHP poles of $G(s)$ and $G_p(s)$ in the inverse Nyquist based singular value tests; that is, the inequality tests implicitly eliminate the possibility of a discontinuous change in the Nyquist locus, and hence the inverse Nyquist locus, when we perturb $G(s)$ to $G_p(s)$ "continuously" by requiring $\lim_{|s|\to\infty} \bar{\sigma}[\tilde{\Delta}_o(s)] < 1$. Clearly, to remove this limitation, some information on the CRHP poles of $G(s)$ and $G_p(s)$ has to be taken into account. This will be the subject of section 4.5.

Another interesting feature of the robust stability tests of this section is that when the multiplicative-type perturbation $[I + \tilde{\Delta}_o(s)]$ is used to characterize uncertainty, the tests are in terms of the maximum singular value of the sensitivity function matrix $S(s) \stackrel{\Delta}{=} [I + G(s)K(s)]^{-1}$. That is, the function normally used to assess the performance of a system with respect to disturbance rejection and sensitivity (chapter 2) is also shown here to be useful for assessing the robust stability of the system. It follows that if the perturbation does not cause a change in the number of zeros in D_I, then good (low) sensitivity and robust stability are compatible objectives in the sense that the smaller we make $\bar{\sigma}[(I+GK)^{-1}]$ over all frequencies, thereby reducing sensitivity, the larger the class of perturbations, $\tilde{\Delta}_o(s)$, the system can tolerate without becoming unstable.

The reader may note the similarity between the above tests and those of Lehtomaki et. al., [LEH1]. But note also that Lehtomaki's tests are based on the direct Nyquist stability criterion and therefore require $G(s)$ and $G_p(s)$ to have the same number of unstable poles. The relationship between Lehtomaki's tests and the tests of this section will be established in the next section.

4.4 On the Relationship Between Lehtomaki's Robust Stability Tests and the corresponding Inverse Nyquist Based Tests

In this section we compare the robust stability test of Lehtomaki et. al., [LEH1], [LEH2], with the inverse Nyquist based robust stability test (theorem 4.10) of the previous section, [POS7]. Although the inequality in each test is the same, the conditions under which each is applicable are different. The inverse Nyquist based test can be used providing the plant and the perturbed plant models share the same number of CRHP zeros, whereas Lehtomaki's test requires the nominal and perturbed plants to share the same number of unstable poles, and the perturbation $L(s) = (I + \tilde{\Delta}(s))^{-1}$ satisfies certain inequalities or an eigenvalue condition. In this section, we will show that these inequalities or eigenvalue condition imposed on $L(s)$ can be satisfied over all frequencies only if $G(s)$ and $G_p(s)$ have the same numbers of poles and zeros in the CRHP. Lehtomaki's test is therefore a special case of the inverse Nyquist based test.

For simplicity we consider the same feedback configuration as that of [LEH1], (see figure 4.2), which is assumed to be stable.

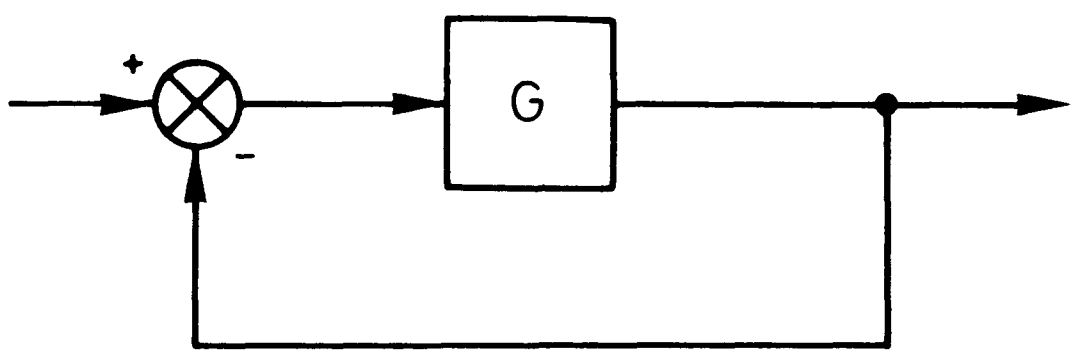


Figure 4.2 Feedback Configuration

A statement of each test is re-stated below for ease of comparison.

Lehtomaki's test, [LEH1]. The feedback system of figure 4.2 will remain stable when G(s) is replaced by G(s)L(s) if:

(a) G(s) and G(s)L(s) have the same number of poles in D_D,

(b) $\underline{\sigma}[I+G(s)] \triangleq \alpha(s) > \bar{\sigma}[L(s)-I]$, for all $s\epsilon\Omega_D$, and

(c) any one of the following is satisfied at each $s\epsilon\Omega_D$:

(i) $\alpha(s) \leq 1$

(ii) $L(s)^H + L(s) \geq 0$

(iii) $4(\alpha(s)^2 - 1)\underline{\sigma}^2[L(s)-I] > \alpha(s)^2\,\bar{\sigma}^2[L(s)^H + L(s) - 2I]$

where Ω_D denotes the indented straightline part of D_D and A^H denotes the complex conjugate transpose of A.

Inverse Nyquist based test. The feedback system of figure 4.2 will remain stable when G(s) is replaced by G(s)L(s) if:

(a) G(s) and G(s)L(s) have the same number of zeros in D_I, and

(b) $\underline{\sigma}[I+G(s)] \triangleq \alpha(s) > \bar{\sigma}[L(s)^{-1} - I]$, for all $s\epsilon\Omega_I$

where Ω_I denote the indented straightline part of D_I.

Note that inequality (b) is the same in each test. We now prove that if inequality (b) and any one of the conditions (i)-(iii) in Lehtomaki's test is satisfied, then G(s) and G(s)L(s) have the same number of zeros in the CRHP.

Because G(s) is assumed to be square and nonsingular, $\underline{\sigma}[G(s_o)]=0$ at $s=s_o$ if, and only if, G(s) has a zero at $s=s_o$. Also if $\underline{\sigma}[G(s_o)L(s_o)]=0$ when $\underline{\sigma}[G(s_o)]\neq 0$, then L(s) must have a zero at $s=s_o$. A similar result can be obtained for poles. Therefore, taking L(s) to represent any unmodelled poles and zeros as well as the uncertainties in the poles and

zeros of G(s), we have that (i) the zeros of L(s) contain the new zeros of G(s)L(s) and the uncertain poles of G(s), and (ii) the poles of L(s) contain the new poles of G(s)L(s) and the uncertain zeros of G(s).

Now if G(s) is continuously perturbed to G(s)L(s) by letting γ go from 0 to 1 in the model $\tilde{G}(s) = G(s)P(s,\gamma)$ where $P(s,\gamma) = (1-\gamma)I + \gamma L(s)$, we find the "movement" of zeros follows a root-locus type pattern because

$$P(s,\gamma) = (1-\gamma)I + \gamma L(s)$$

$$= (1-\gamma)[I + \frac{\gamma}{1-\gamma} L(s)] \qquad (4.24)$$

For example, if we have $G(s) = 1/(s+1)(s+2)$ and $G_p(s) = 1/(s+3)(s+4)$, then the zero-variation of $P(s,\gamma)$ as γ goes from 0 to 1 is as shown in figure 4.3

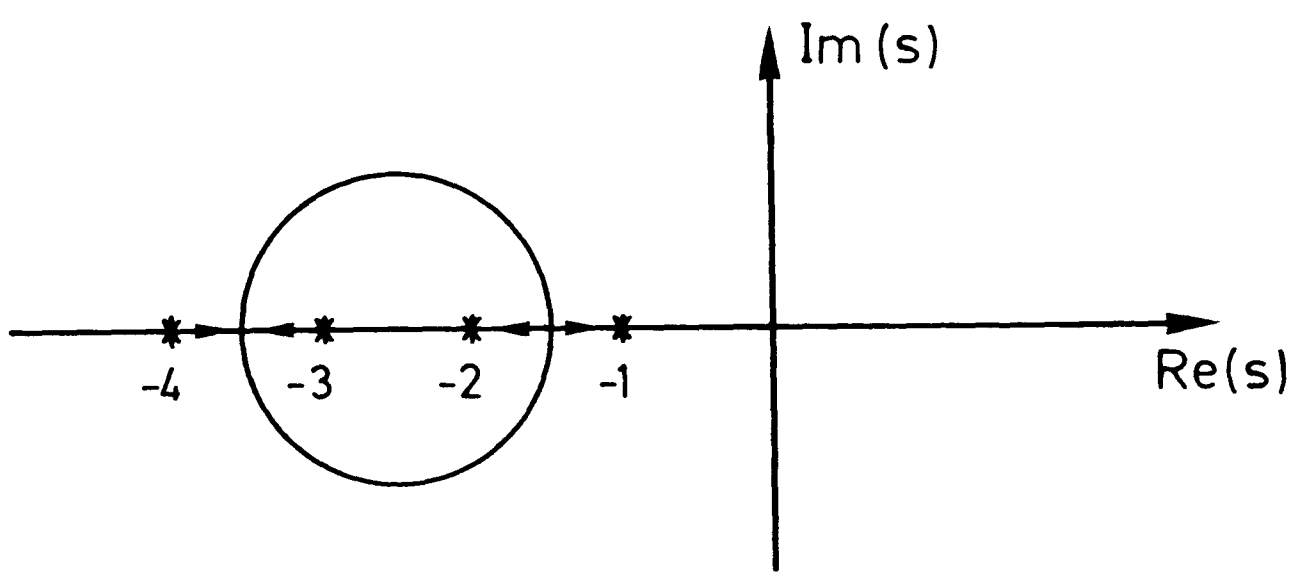


Figure 4.3 Zero variation of $G(s)[I+\gamma(L(s)-I)]$

The conditions (i)-(iii) are obtained in [LEH1] by ensuring that the direct Nyquist contour, i.e. the image of D_D under det[I+G(s)], does not pass through the origin as G(s) is perturbed to $G_p(s)$. That is, each of

the conditions when combined with the condition $\bar{\sigma}[L(s)^{-1} - I] < \alpha(s)$ is sufficient to imply that

$$\bar{\sigma}[P(s,\gamma)^{-1} - I] < \alpha(s) \quad \text{for all } s\epsilon\Omega_D, \text{ and all } \gamma \in [0, 1] \tag{4.25}$$

Obviously, because G(s) is strictly proper,

$$\lim_{|s|\to\infty} \bar{\sigma}[(I + G(s))^{-1}] = 1.$$

Therefore the inequality (b) can be satisfied on the whole indented $j\omega$-axis only if $L(s)^{-1}$ is proper. This is equivalent to requiring that L(s) is <u>not</u> strictly proper. Now, assume for $\gamma = \gamma_o$ there is a zero of $P(s,\gamma_o)$ on the D_D contour. Then $\underline{\sigma}[P(s,\gamma_o)] = 0$ which implies that at that frequency $\bar{\sigma}[P(s,\gamma_o)^{-1} - I]$ is infinite. Hence (4.25) cannot be satisfied for any finite $\alpha(s)$. This leads to a contradiction and therefore we conclude $P(s,\gamma)$ does not have a zero on D_D for any $\gamma\epsilon[0, 1]$.

In two recent papers by Chan and Athans, [CHA1], and Lehtomaki et. al., [LEH3], Lehtomaki's test has been used but conditions (i)-(iii) have been replaced by

(iv) L(s) has no zero or strictly negative real eigenvalues for any $s \in D_D$.

Let us examine this condition in more detail:

L(s) has no eigenvalues on $[-\infty,0]$

<=> L(s) - I has no eigenvalues on $[-\infty,-1]$

<=> $\gamma[L(s) - I]$ has no eigenvalues at -1 for any $\gamma \in [0,1]$

<=> $I + \gamma(L(s) - I)$ has no zero eigenvalue for any $\gamma \in [0,1]$.

Recall $\tilde{G}(s) = G(s)[I + \gamma(L(s)-I)]$ and we see again that condition (iv) also prevents the number of CRHP zeros from changing as G(s) is perturbed to $G_p(s)$ (as was also observed in [LEH3]).

The above arguments therefore establish the fact that condition (i)-(iv) of Lehtomaki's test are sufficient to ensure that the number of CRHP zeros does not change as G(s) is perturbed to G(s)L(s). In the following example, we show that it is <u>not</u> a necessary condition for G(s) and G(s)L(s) to have the same number of zeros in the closed right half-plane.

<u>Example 4.1</u>

$$L(s) = \frac{(s^2+2s+4)(s+8)(s+10^6)(s+10^7)}{(s+\epsilon)(s+4)(s+6)(s^2+1.4s+1)}, \quad 0<\epsilon<<1$$

The zero-locus of $P(s,\gamma)$ when γ varies from 0 to 1 is plotted in figure 4.4. Note that the locus crosses the $j\omega$-axis a number of times and this implies that the Nyquist locus of L(s) crosses the negative real axis a number of times. However, because L(s) is stable and minimum-phase, it follows that G(s)L(s) shares the same numbers of CRHP poles and zeros as G(s).

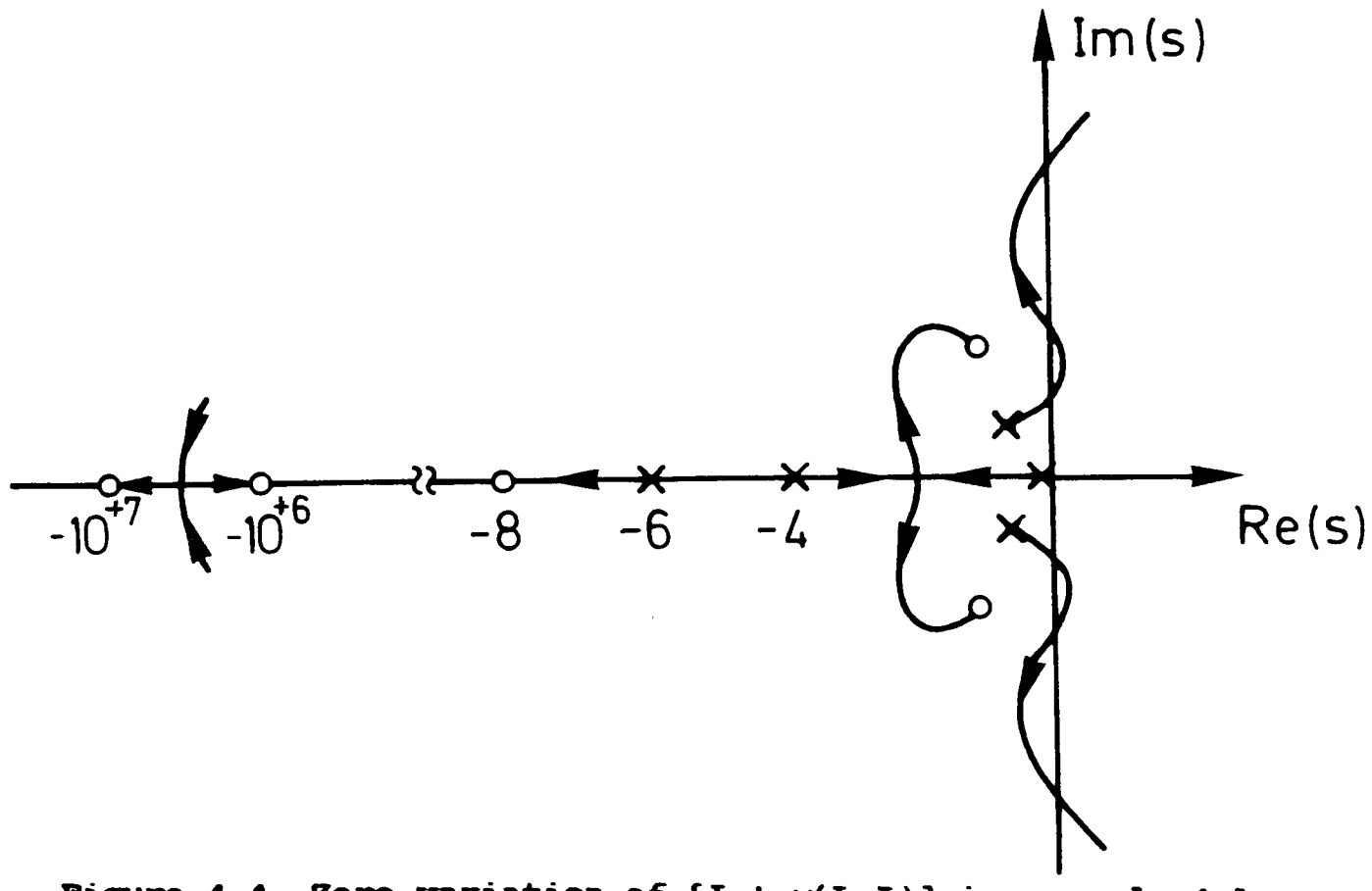


Figure 4.4 Zero variation of $[I + \gamma(L-I)]$ in example 4.1

From a perturbational point of view, conditions (i)-(iii) are derived by ensuring the continuity of

$$G(s,\gamma) = (1-\gamma)G(s) + \gamma G_p(s) \qquad (4.26)$$

and condition (iv) is derived by ensuring the continuity of

$$\tilde{G}(s,\gamma) = [(1-\gamma)G(s)^{-1} + \gamma G_p(s)^{-1}]^{-1} \qquad (4.27)$$

when γ varies from 0 to 1, [LEH2]. As (4.27) is exactly the same as the type of representation of uncertainty used by the inverse Nyquist based robust stability tests of section 4.3, we therefore conclude that Lehtomaki's test with condition (iv) is simply the inverse Nyquist based test with the unnecessary and restrictive requirements that G(s) and G(s)L(s) share the same number of unstable poles, and L(s) satisfies condition (iv). However, conditions (i)-(iii) deserve further attention because they are derived using (4.26) which is the type of representation of uncertainty used in the direct Nyquist based robust stability tests of section 4.2. One may therefore be able to combine the inequality test (b) with robust stability tests derived in section 4.2 to form a switching-type robust stability criterion, as was suggested in [LEH2]. The combined test would not then be a special case of the inverse Nyquist based tests. The reader should not confuse the previous sentence with the results we proved in the first half of this section; that is, when taken alone, Lehtomaki's test is simply a special case of the inverse Nyquist robust stability test because it requires G(s) and $G_p(s)$ to share the same numbers of both poles and zeros, and the inequality is the same for both tests over all frequencies. However, when combined with the

robust stability test of section 4.2, the resulting test would be quite different from the inverse Nyquist based test.The combined test would not require $G(s)$ and $G_p(s)$ to share the same number of CRHP zeros anymore because over those frequencies where $P(s,\gamma)$ is singular, a direct Nyquist-type inequality could be used.

It should be clear to the reader that because of duality, conditions similar to (i)-(iii) in Lehtomaki's test may be imposed on $L(s)^{-1}$ and $\tilde{P}(s,\gamma) \triangleq [(1-\gamma)I + \gamma L(s)^{-1}]$ to derive an inverse Nyquist based switching-type robust stability test. We may therefore close this section by stating the following two theorems.

Theorem 4.11 (A Direct Nyquist based Combined Robust Stability Test) The feedback system of figure 4.2 will remain stable when the nominal plant model is replaced by the perturbed plant model $G(s)L(s)$ if:

(a) $G(s)$ and $G(s)L(s)$ share the same number of poles in D_D,

(b) $\Omega_D = \Omega_{1D}$ union Ω_{2D},

(c) a direct Nyquist type robust stability inequality is satisfied on Ω_{1D}X [0,1], and

(d) an inverse Nyquist type robust stability inequality is satisfied on Ω_{2D}X [0,1] along with any one of the following conditions at each $s\epsilon\Omega_{2D}$:

(i) $\underline{\sigma}[I + G(s)] \leq 1$

(ii) $L(s)^H + L(s) \geq 0$

(iii) $4(\alpha(s)^2-1)\underline{\sigma}^2[L(s)-I] > \alpha(s)^2\,\bar{\sigma}^2[L(s)^H + L(s) - 2I]$

where $\alpha(s) \triangleq \underline{\sigma}[I+G(s)]$

Proof

(c) => $\det[I+G(s,\gamma)]\neq 0$ and is finite on Ω_{1D}X [0, 1].

(d) => $\det[I+G(s,\gamma)]\neq 0$ and is finite on Ω_{2D}X [0, 1].

Thus, det[I+G(s)L(s)] must have the same number of encirclements around the origin as det[I+G(s)]. Apply theorem 4.2 and this completes the proof. ∎

Theorem 4.12 (An Inverse Nyquist based Combined Robust Stability Test)

The feedback system of figure 4.2 will remain stable when the nominal plant is replaced by the perturbed plant model G(s)L(s) if :

(a) G(s) and G(s)L(s) share the same number of zeros in D_I,

(b) $\Omega_I = \Omega_{1I}$ union Ω_{2I},

(c) an inverse Nyquist type robust stability inequality is satisfied on Ω_{1I}X [0, 1], and

(d) a direct Nyquist type robust stability inequality is satisfied on Ω_{2I}X [0, 1] along with any one of the following conditions at each $s\epsilon\Omega_{2I}$:

(i) $\bar{\alpha}(s) \leq 1$

(ii) $(L(S)^{-1})^H + L(s)^{-1} \geq 0$

(iii) $4(\bar{\alpha}(s)^2-1)\underline{\sigma}^2[L(s)^{-1} - I] > \bar{\alpha}(s)^2\,\bar{\sigma}^2[L(s)^{-1} + (L(s)^{-1})^H - 2I]$

where $\bar{\alpha}(s) \triangleq \underline{\sigma}[I + G(s)^{-1}]$.

Proof

(c) => $\det[I+\tilde{G}(s,\gamma)^{-1}]\neq 0$ and is finite on Ω_{1I}X [0,1].

(d) => $\det[I+\tilde{G}(s,\gamma)^{-1}]\neq 0$ and is finite on Ω_{2I}X [0,1].

Thus, $\det[I+L(s)^{-1}G(s)^{-1}]$ must have the same number of encirclements around the origin as $\det[I+G(s)^{-1}]$. Apply theorem 4.7 and this completes the proof. ∎

4.5 Combined Direct/Inverse Nyquist Based Singular Value Robust Stability Test

In this section we present a robust stability test which is a combination of a Nyquist based test from section 4.2, and an inverse Nyquist based test from section 4.3. The virtue of this combination is that $G_p(s)$ need not have the same number of poles or zeros in the closed right half complex plane as the nominal model $G(s)$. The improvement is achieved at the expense of a more refined, yet no less practical, definition of the set $\{G_p(s)\}$.

The test will be derived under the assumption that $\{G_p(s)\}$ is an arcwise connected set in the topology of convolution plants as defined by Vidyasagar, Schneider and Francis, [VID1]. A brief description of the topology and a definition of arcwise connectedness have been given in section 2.3. In what follows, we will briefly describe, from a practical view-point, the class of perturbed plants of interest.

Let $\{G_p(s)\}$ be a set of strictly proper and stabilizable rational transfer function matrices satisfying the following properties:

A. For all $G_p(s) \in \{G_p(s)\}$ the frequency response of each of the elements of $G_p(s)$ is bounded within a certain envelope.

B. If $G_1(s)$ and $G_2(s)$ are two members of $\{G_p(s)\}$, then there exists a map α: $[0,1] \rightarrow \{G_p(j\omega)\}$, $\alpha(0) = G_1(j\omega)$ and $\alpha(1) = G_2(j\omega)$, which is continuous (elementwise) in the extended sense (i.e. on the Riemann sphere).

C. There exists $\beta \geq 0$ such that all $G_p(s) \in \{G_p(s)\}$ have no CRHP poles for $|s| > \beta$.

D. If $G^{(i)}$ is a family in $\{G_p\}$ whose frequency responses converge uniformly to those of $G_o(s)$, then as $i \rightarrow \infty$, $G^{(i)}(s)$ and $G_o(s)$

have the same number of poles and zeros in the Nyquist D-contour, and these poles and zeros of $G^{(i)}(s)$ approach those of $G_o(s)$.

Property A is usually assumed in practice because a system's frequency response matrix can never be known exactly and often the best one can achieve are bounds on each of its elements in gain and phase, [LOH1]. We note that the envelope may enclose the point at infinity and thus does not rule out the possibility of poles crossing the $j\omega$-axis.

Property B assumes the perturbations are linear and are continuous in the frequency response sense; that is, the elements of $\{G_p(s)\}$ generate an arcwise connected frequency response set $\{G_p(j\omega)\}$.

Property C roughly ensures that no far right half-plane poles have been "mistakenly" omitted, or modelled as far left half-plane poles. Note that this property is also necessary if for each $G_p \in \{G_p\}$, there is to exist a Nyquist D contour "large enough" to enclose all the unstable poles of $G_p(s)$.

Property D deserves further consideration, as follows. Consider a sequence of single-input, single-output transfer functions $g^{(i)}(s)$ and a nominal model $g_o(s)$, all of which are stable. Then we may define

$$g^{(i)}(j\omega) \to g_o(j\omega) \tag{4.28}$$

if, and only if,

$$\sup_{\omega} |g^{(i)}(j\omega) - g_o(j\omega)| \to 0 \quad \text{uniformly} \tag{4.29}$$

as $i \to \infty$.

Under such a notion of convergence, feedback stability is a generic (robust) property in the sense that any controller which stabilizes $g_o(s)$ will also stabilize $g^{(i)}(s)$ for very large i. Unfortunately, this is not generally true if the plant is unstable; see [FRA4] for a detailed discussion. However, if we assume that $g^{(i)}(s)$ and $g_o(s)$ have the same number of unstable poles, then the above notion of convergence ensures that feedback stability is again a generic property.

Similarly, if we assume that $g^{(i)}(s)$ and $g_o(s)$ have the same number of non-minimum-phase zeros, then we can define

$$g^{(i)}(j\omega) \to g_o(j\omega) \tag{4.30}$$

if, and only if,

$$\sup_{\omega} |g^{(i)}(j\omega)^{-1} - g_o(j\omega)^{-1}| \to 0 \quad \text{uniformly} \tag{4.31}$$

as $i \to \infty$.

Under such a definition, feedback stabilization is again generic in the sense that a controller which stabilizes $g_o(s)$ will also stabilizes $g^{(i)}(s)$ for very large i. Note that in this case we do <u>not</u> require the number of unstable poles of $g_o(s)$ and $g^{(i)}(s)$ to be the same.

We are now in a position to define what we mean by convergence in frequency response. Note that to avoid the technical difficulties which arise when there are pole and zero movements across the Nyquist D-contour, we have considered mapping onto the Riemann sphere.

Definition 4.1 The frequency responses of a sequence $g^{(i)}(s)$ converge to $g_o(j\omega)$ if, and only if, as $i \to \infty$,

$$\max\left(\sup_{\omega} \partial[\phi(g^{(i)}(j\omega)), \phi(g_o(j\omega))],\ \sup_{\omega} \partial[\phi(g^{(i)}(j\omega)^{-1}), \phi(g_o(j\omega)^{-1})]\right) \tag{4.32}$$

converges to 0 uniformly, where ϕ denotes stereographic projection onto the Riemann sphere, and $\partial[.,.]$ denotes the distance between two points on the sphere.

For multivariable systems, we define $G^{(i)}(j\omega) \to G_o(j\omega)$ if, and only if, each element of $G^{(i)}(j\omega)$ converges to the corresponding element of $G_o(j\omega)$.

At the expense of mathematical rigour and to facilitate later discussions we now give a more intuitive and physically appealing view of property D. We begin with some terminology.

Terminology 4.1 Two frequency responses $g_1(j\omega)$ and $g_2(j\omega)$ are said to be almost identical if, and only if, there exists a real nonnegative constant ϵ, which is almost indistinguishable from zero such that

$$\max\left(\sup_{\omega} |g_1(j\omega) - g_2(j\omega)|,\ \sup_{\omega} |g_1(j\omega)^{-1} - g_2(j\omega)^{-1}|\right) < \epsilon \tag{4.33}.$$

If there are any poles and zeros on the imaginary axis, then (4.33) is evaluated around infinitesimally small indentations into the left half-plane.

Roughly speaking, terminology 4.1 says that two frequency responses are almost identical if, and only if, their gains and phases are almost identical at each frequency.

Terminology 4.2 Two transfer functions $g_1(s)$ and $g_2(s)$ are said to have almost identical pole structures in a region if, and only if, in that region:

(i) $g_1(s)$ and $g_2(s)$ have the same number of poles (multiplicities included), and

(ii) [Let $S = \{s_1, s_2, \ldots, s_n\}$ be the set of complex frequencies where $g_1(s)$ has poles (multiplicities counted as separate poles), and $Z = \{z_1, z_2, \ldots, z_n\}$ be the set of complex frequencies at which $g_2(s)$ has poles.] There exists a one-to-one corresponding match between S and Z, such that if (s_i, z_i) is a pair under the match, there exists a real nonnegative constant δ, which is almost indistinguishable from zero, such that

$$|s_i - z_i| < \delta, \quad \text{all } i. \tag{4.34}$$

Similar terminology can be used to describe the zeros of two transfer functions, and the extension to multivariable systems is achieved by considering the corresponding "diagonal" Smith-McMillian forms.

Equipped with the above definitions, we re-state property D as :

D'. If any two members of $\{G_p\}$ have almost identical frequency responses (elementwise), then they must have almost identical pole and zero structures in the D-contour.

In properties D' and D, we consider the Nyquist D-contour rather than the CRHP because $G_1(s)$ may happen to have a pole or zero on the $j\omega$-axis while $G_2(s)$ has the corresponding pole or zero very nearby but in the open LHP plane. In such a situation $G_1(s)$ and $G_2(s)$ would not even have the same number of poles or zeros in the CRHP, but a suitably indented D-contour will enable us to avoid such technical difficulties.

It should be clear that property D' implies property D. It may also be shown that property D' is a necessary assumption for the set $\{G_p\}$ to be robustly stabilizable, as is given by the following theorem.

Theorem 4.13 Let $G_1(s)$ and $G_2(s)$ be two strictly proper mxm nonsingular and stabilizable rational transfer function matrices with almost identical frequency responses, then the following statements are equivalent:

(i) $G_1(s)$ and $G_2(s)$ are simultaneously stabilizable, [VID5], [SAE1].

(ii) $G_1(s)$ and $G_2(s)$ have the same number of poles and the same number of zeros in the Nyquist D contour.

(iii) $G_1(s)$ and $G_2(s)$ have almost identical pole and zero structures in the Nyquist D contour.

Proof (Given in appendix IV-1). ∎

Theorem 4.13 tells us that two transfer function matrices which have almost identical frequency responses ($\epsilon \to 0$) yet different pole and zero structures ($\delta \not\to 0$) cannot be simultaneously stabilizable. Therefore, if $\{G_p\}$ contains at least two members which violate property D', then the set $\{G_p\}$ cannot possibly be robustly stabilizable.

The following proposition states that the set $\{G_p\}$ which satisfies properties A to D is an arcwise connected set in the graph topology of finite dimensional convolution plants.

Proposition 4.1 Let $\{G_p(s)\}$ be ~~the~~ set of ~~all~~ stabilizable, strictly proper, transfer function matrices in Mat(R(s)) satisfying properties A to D. Then $\{G_p(s)\}$ is an arcwise connected set of stabilizable transfer function matrices in the graph topology of transfer functions.

Proof For any two members of $\{G_p(s)\}$, say $G_o(s)$ and $G_1(s)$, property B asserts that we can find a subset of $\{G_p(s)\}$ whose frequency responses are connected in the extended sense. Now since all the $G_p(s)$'s are stabilizable, it follows that all $G_p(s)$'s admit coprime factorizations. Properties C and D ensure that as we continuously perturb $G_o(s)$ to $G_1(s)$, continuity in frequency responses implies continuity in coprime factors. Hence $G_o(s)$ and $G_1(s)$ are arcwise connected. But $G_o(s)$ and $G_1(s)$ are arbitrary, and so the whole set $\{G_p(s)\}$ is arcwise connected. ∎

(Note that property A is not used in the proof because it is simply a consequence of property B. It is stated as a property to provide the reader with a better pictorial view of the set $\{G_p(s)\}$).

We note that properties A to D are fairly general and do not rule out the possibility of a zero movement crossing the point at infinity. Neither does property D'. For example, consider $\{g_o(s)\} = \{1/(1+s)^2\}$, $\{g_1(s)\} = \{(1+\epsilon s)/(1+s)^2;\ \epsilon>0\}$ and $\{g_2(s)\} = \{(1-\epsilon s)/(1+s)^2;\ \epsilon>0\}$. Then it is clear that (i) $\{g_o(s)\}$, $\{g_1(s)\}$ and $\{g_2(s)\}$ are simultaneously stabilizable, and (ii) the union of $\{g_o(s)\}$, $\{g_1(s)\}$ and $\{g_2(s)\}$, denoted by $\{g_p(s)\}$, is an arcwise connected set. However, it may not be clear that the example does not violate property D' (and hence

theorem 4.13). Any two members of $\{g_p(s)\}$ which have almost identical frequency responses must come from the same subset $\{g_1(s)\}$ or $\{g_2(s)\}$ because the frequency responses of $\{g_o(s)\}$, $\{g_1(j\omega)\}$ and $\{g_2(j\omega)\}$ are not considered "almost identical" in the terms of terminology 4.1 even though ϵ may be very small. This awkwardness is the price we pay for the 'advantage' of not having to work with the Riemann sphere.

We can now state our main result.

Theorem 4.14 Let $\{G_p(s)\}$ be an arcwise connected set of strictly proper transfer function matrices in the graph topology of finite dimensional convolution plants. Assume that for every member $G_p(s)$ in $\{G_p(s)\}$ there exists either $\Delta_o(j\omega)$, $\tilde{\Delta}_o(j\omega)$, or both, such that

$$G_p(j\omega) = (I + \Delta_o(j\omega))G(j\omega) = (I + \tilde{\Delta}_o(j\omega))^{-1}G(j\omega), \quad \text{all } \omega$$

where $G(s)$ is a member of $\{G_p(s)\}$, known as the nominal model, such that the closed-loop system of figure 4.1 is stable.

Define:

$$\delta_o(\omega) = \sup_{\text{all } G_p \in \{G_p\}} \{\bar{\sigma}[\Delta_o(j\omega)]\}$$

$$\tilde{\delta}_o(\omega) = \sup_{\text{all } G_p \in \{G_p\}} \{\bar{\sigma}[\tilde{\Delta}_o(j\omega)]\}$$

Then under these conditions, the perturbed closed-loop system will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the set $\{G_p\}$ if, at each $\omega \in [0,\infty]$, either

(i) $\bar{\sigma}[T(j\omega)] < (\delta_o(\omega))^{-1}$,

or

(ii) $\bar{\sigma}[S(j\omega)] < (\tilde{\delta}_o(\omega))^{-1}$.

where $S=(I+GK)^{-1}$ and $T=(I+GK)^{-1}GK$ are the sensitivity function and the complementary sensitivity function matrices respectively.

Proof For any $G_p(s) \in \{G_p(s)\}$ we can find a path connecting $G(s)$ and $G_p(s)$ such that at any point on the path, say $G_p'(s)$, $G_p'(s)$ is a member of $\{G_p(s)\}$ and $G_p'(s) = (I + \Delta_o'(s))G(s) = (I + \tilde{\Delta}_o'(s))^{-1}G(s)$ where $\bar{\sigma}[\Delta_o'(j\omega)] \leq \delta_o(\omega)$ and $\bar{\sigma}[\tilde{\Delta}_o'(j\omega)] \leq \tilde{\delta}_o(\omega)$.

Conditions (i) and (ii) ensure that as we move along the path perturbing $G(s)$ to $G_p(s)$ continuously, the Nyquist-type locus of $\det[I+G_p'(j\omega)K(j\omega)]$ and the inverse Nyquist-type locus of $\det[I+(G_p'(j\omega)K(j\omega))^{-1}]$ never pass through the origin. Equivalently, they ensure that the characteristic gain loci and the inverse characteristic gain loci, [POS1], do not pass through the (-1,j0) point. The derivation of these conditions (as seen in sections 4.2 and 4.3) is based on satisfying the condition $\det[I + G_p'(s)K(s)]\neq 0$ for all s on the Nyquist and inverse Nyquist D-contours and for all $G_p'(s)$ on the path between and including $G(s)$ and $G_p(s)$. Note that there is no need to consider the large semi-circular part of the D-contour because $G_p'(s)K(s)$ is assumed to be strictly proper.

Now if a pole moves across the imaginary axis during the perturbation process, it will automatically create an additional anticlockwise (or clockwise) encirclement of the critical point by the Nyquist loci if the movement is from left to right (or from right to left, respectively). This is exactly what is required for stability to be maintained.

It is clear that zero movement across the D-contour produces the same effect on the inverse Nyquist loci as pole movement does on the Nyquist loci. Note that if there is a zero movement across the semi-circular part of the D_I contour, then $\bar{\sigma}[\tilde{\Delta}_o(j\infty)] > 1$ so we will be forced

to use the direct Nyquist-type inequality (i) at very high frequency, and if the inequality is satisfied, the Nyquist loci, and hence the inverse loci, will not pass through the critical point.

To summarize the above, whenever a pole (zero) moves across the imaginary axis from the LHP to the RHP it creates an additional anticlockwise encirclement of the critical point by the Nyquist (inverse Nyquist) loci; and whenever the movement is from the RHP into the LHP an extra clockwise encirclement is created.

It follows that the perturbed feedback system will be stable if, and only if, the Nyquist (and hence also the inverse Nyquist) loci do not pass through the critical point, and this is ensured if at each frequency either test (i) or test (ii) is satisfied. ※

We have argued that whenever a pole moves across the imaginary axis, the Nyquist determinant locus moves through infinity thereby maintaining the correct number of encirclements for stability. Similarly, if a zero moves across the imaginary axis, the inverse Nyquist locus moves through infinity again creating or destroying the extra encirclement required for stability. However, it is not difficult to see that over frequencies for which $\underline{\sigma}[I + (G(j\omega)K(j\omega))^{-1}] > \bar{\sigma}[\Delta_o(j\omega)]$ the Nyquist locus is prevented from passing through infinity as well as the critical point. Consequently the condition $\underline{\sigma}[I + (G(j\omega)K(j\omega))^{-1}] > \bar{\sigma}[\Delta_o(j\omega)]$ cannot be satisfied at frequencies for which poles cross the imaginary axis, and in fact $\bar{\sigma}[\Delta_o(j\omega)]$ would be infinite at these frequencies. Similarly at frequencies for which zeros cross the imaginary axis $\bar{\sigma}[\tilde{\Delta}_o(j\omega)]$ is infinite, and the condition $\underline{\sigma}[I + G(j\omega)K(j\omega)] > \bar{\sigma}[\tilde{\Delta}_o(j\omega)]$ cannot be satisfied. We therefore state the following corollary to theorem 4.14.

Corollary 4.2 Let $\{\omega_p\}$ define the set of frequencies over which it is possible for poles to cross the imaginary axis. Similarly, let $\{\omega_z\}$ define the frequencies over which it is possible for zeros to cross the imaginary axis. We will assume that the intersection of $\{\omega_p\}$ and $\{\omega_z\}$ is the empty set, and define a set $\{\omega_e\}$ such that the intersection of $\{\omega_e\}$, $\{\omega_p\}$ and $\{\omega_z\}$ is again the empty set, and the union of $\{\omega_e\}$, $\{\omega_p\}$ and $\{\omega_z\}$ is $[0,\infty]$. Then the perturbed feedback system of figure 4.1 is stable with G(s) replaced by any member from $\{G_p(s)\}$ if :

(i) $\bar{\sigma}[S(j\omega)] < (\tilde{\delta}_o(\omega))^{-1}$, all $\omega \in \{\omega_p\}$,

(ii) $\bar{\sigma}[T(j\omega)] < (\delta_o(\omega))^{-1}$, all $\omega \in \{\omega_z\}$,

and

(iii) at each $\omega \in \{\omega_e\}$,

either $\bar{\sigma}[S(j\omega)] < (\tilde{\delta}_o(\omega))^{-1}$

or $\bar{\sigma}[T(j\omega)] < (\delta_o(\omega))^{-1}$.

Proof Direct consequence of theorem 4.14 and above discussion. ∎

The usefulness of this corollary is that it highlights very clearly over which frequencies good sensitivity and robust stability are compatible design objectives, and over which frequencies a compromise has to be made. This, however, does not imply that there always exists a controller such that a particular switching criterion is satisfied. Thus trade-offs among different frequencies may still be necessary.

Note that in both theorem 4.14 and its corollary we need not concern ourselves with small indentations around imaginary axis poles and zeros, for in the vicinity of an imaginary axis pole we use the inverse Nyquist based test, and in the vicinity of an imaginary axis zero we use the

direct Nyquist based test. However, as mentioned in section 4.2, if G(s) has an imaginary axis zero and $G_p(s)$ does not have this zero on the jω-axis, then $\bar{\sigma}[\Delta_o(j\omega)]$ will be infinite at the particular frequency. It therefore follows that some form of scaling will be necessary in these cases.

It should be clear to the reader that we may use more than two representations of uncertainty in order to define an even more refined set $\{G_p(s)\}$. For example, if each of the perturbations $\Delta_o(s)$, $\tilde{\Delta}_o(s)$, $\Delta_i(s)$, $\tilde{\Delta}_i(s)$, E(s) and $\tilde{E}(s)$ models an adequate representation of uncertainty, [chapter 3], then theorem 4.14 may be extended to include six corresponding inequality tests only one of which needs be satisfied for each $\omega \in [0,\infty]$ for stability to be maintained. It follows that theorem 4.14 and its extensions may be used to reduce conservatism, even in cases where there is no likelihood of pole or zero movement across the imaginary axis.

Before closing this section, we provide the following example to illustrate the application of theorem 4.14 to a multivariable feedback system.

Example 4.2 Let

$$G(s)K(s) = \frac{1}{4a}\begin{bmatrix} b+c & b-c \\ b-c & b+c \end{bmatrix} \qquad (4.35)$$

where

$$a = (1+0.01s)(1+0.1s)(1+0.5s)(1+s)(1+0.001s+0.01s^2)$$

$$b = (1+0.5s)(1+s)^2(1+0.001s+0.01s^2)$$

$$c = (1+0.01s)(1+0.1s)^2(1+0.667s)(1+10s)$$

Note that this model is both stable and minimum-phase.

We will assume that bounds for $\Delta_o(j\omega)$ and $\tilde{\Delta}_o(j\omega)$ have been identified and are as shown in figures 4.3 to 4.6. Note that $\delta_o(\omega)$ approaches infinity in the vicinity of $\omega = 10$ rs^{-1} indicating the possibility of pole migration across the imaginary axis near to this frequency. Also, $\tilde{\delta}_o(\omega)$ approaches infinity as ω approaches zero indicating the possibility of zero migration at low frequencies. We will assume that $\tilde{\bar{\delta}}_o(\omega)$ approaches infinity as ω approaches infinity, which would be the case if in the nominal model we had omitted more poles than zeros.

Figure 4.3a shows that for $0.01 \leq \omega \leq 1.0$,

$$\underline{\sigma}[I + (G(j\omega)K(j\omega))^{-1}] > \delta_o(\omega)$$

and therefore the robust stability test is satisfied in this region. Note that $\underline{\sigma}[I + G(j\omega)K(j\omega)]$ is actually smaller than $\tilde{\bar{\delta}}_o(\omega)$ in part of this frequency range (see figure 4.3b) but because the direct test is satisfied we can ignore this fact.

From figures 4.3b, 4.4 and 4.5a we conclude that

$$\underline{\sigma}[I + G(j\omega)K(j\omega)] > \tilde{\delta}_o(\omega), \quad \text{all } \omega \in [1, 12]$$

and from figures 4.6 and 4.7 it may be verified that for $12 \leq \omega \leq 1000$,

$$\underline{\sigma}[I + (G(j\omega)K(j\omega))^{-1}] > \delta_o(\omega)$$

and therefore the robust stability test of theorem 4.1 is satisfied for all $\omega \in [0.01, 1000]$.

Note that $\tilde{\delta}_o(\omega)$ approaches infinity as ω approaches infinity and therefore we are forced into using the direct test at high frequencies.

The following is an example of a perturbed return-ratio which is included in the set of plants for which stability is maintained; the reader may verify that $\bar{\sigma}[\Delta_o(j\omega)] \leq \delta_o(\omega)$ and $\bar{\sigma}[\tilde{\Delta}_o(j\omega)] \leq \tilde{\delta}_o(\omega)$ for all ω.

$$G_p(s)K(s) = \frac{1}{4a_p}\begin{bmatrix} b_p + c_p & b_p - c_p \\ b_p - c_p & b_p + c_p \end{bmatrix}$$

where

$$a_p = (1+0.001s)(1+0.01s)(1+0.1s)(1+0.5s)(1-0.001s+0.01s^2)$$

$$b_p = (1+0.5s)(1+s)^2(1 - 0.001s + 0.01s^2)$$

$$c_p = (1+0.001s)(1+0.01s)(1+0.1s)^2(1 + 0.667s)(10s - 1)$$

Note that there is a pair of unstable complex poles at approximately $s = 0.05 \pm 10j$, a non-minimum-phase zero at $s = 0.1$, and when compared with the nominal model we see that there is also an extra pole at $s = -1000$.

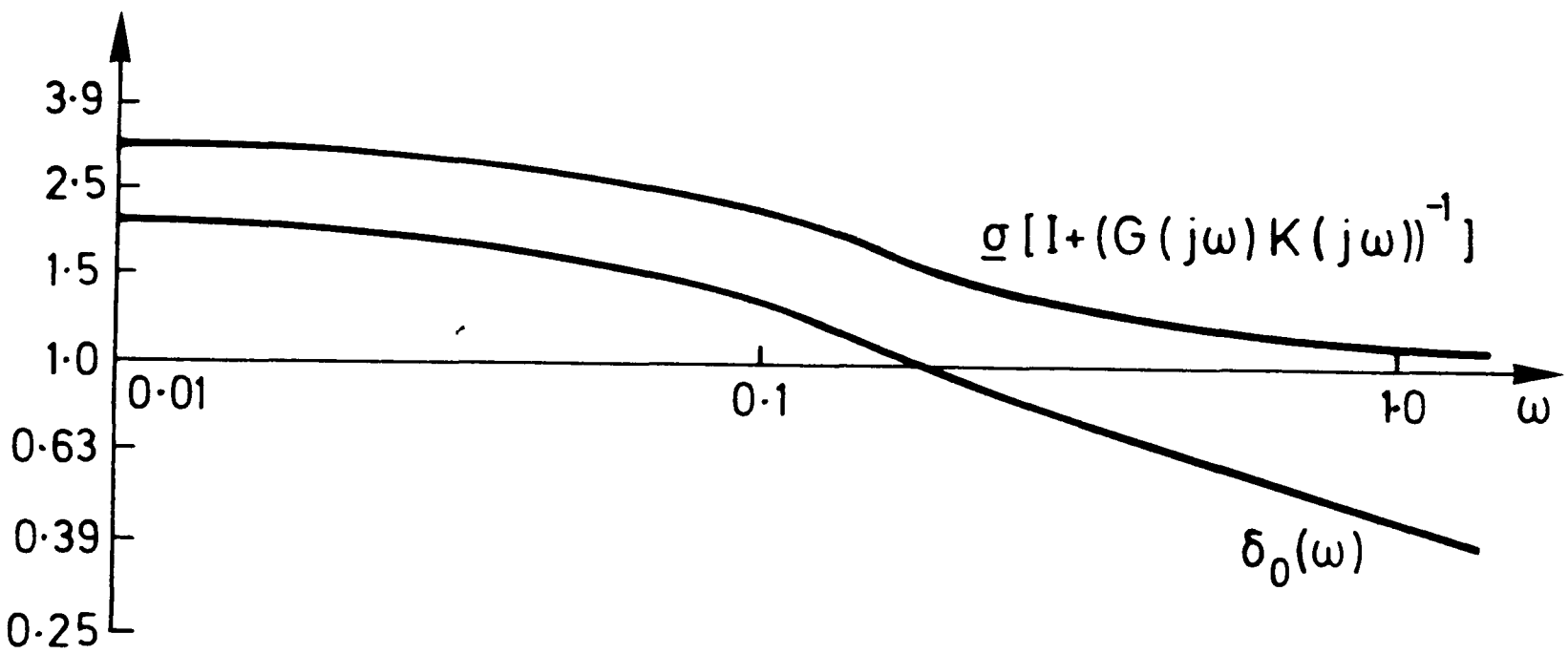


Figure 4.3a Direct Nyquist robust stability test, example 4.2, $\omega \in [0.01, 1.0]$

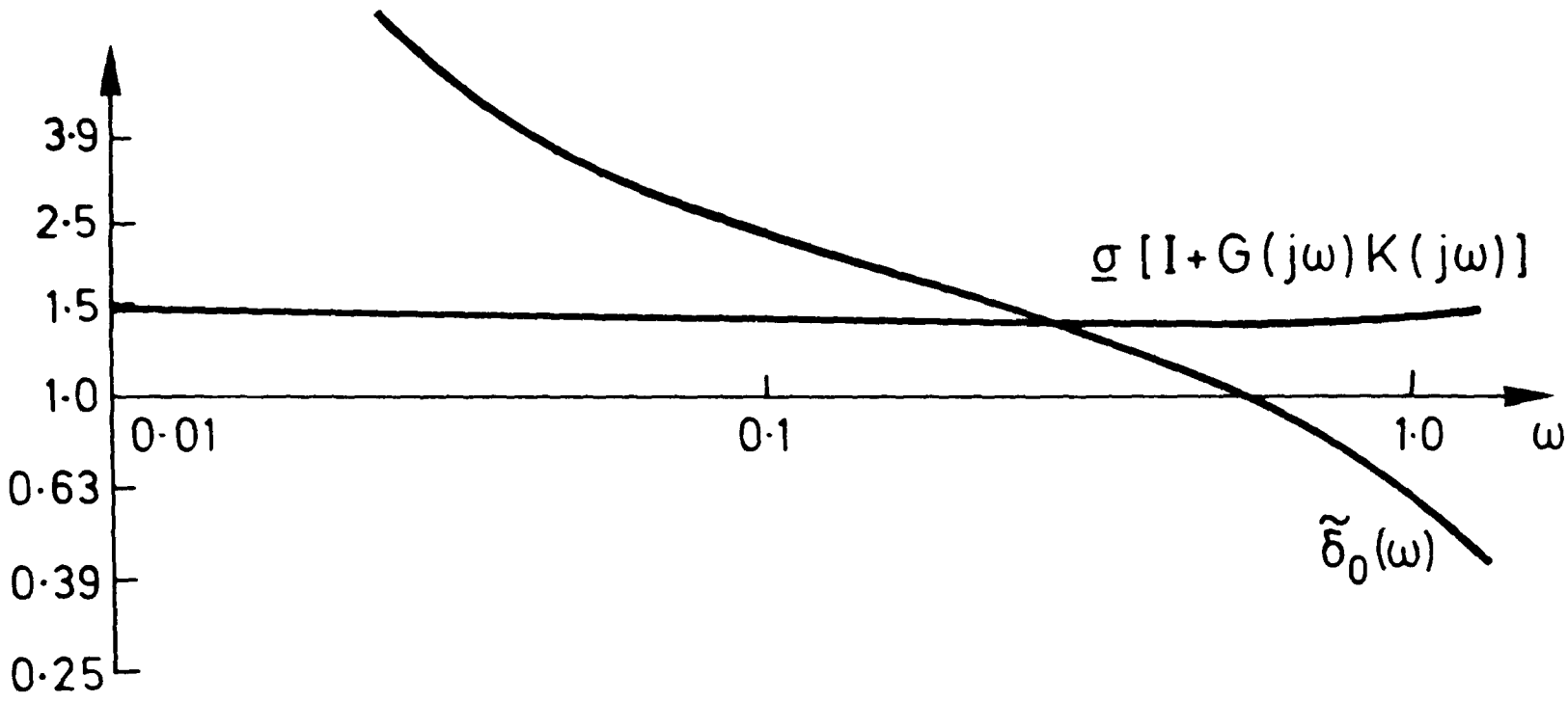


Figure 4.3b Inverse Nyquist robust stability test,

example 4.2, $\omega \in [0.01, 1.0]$

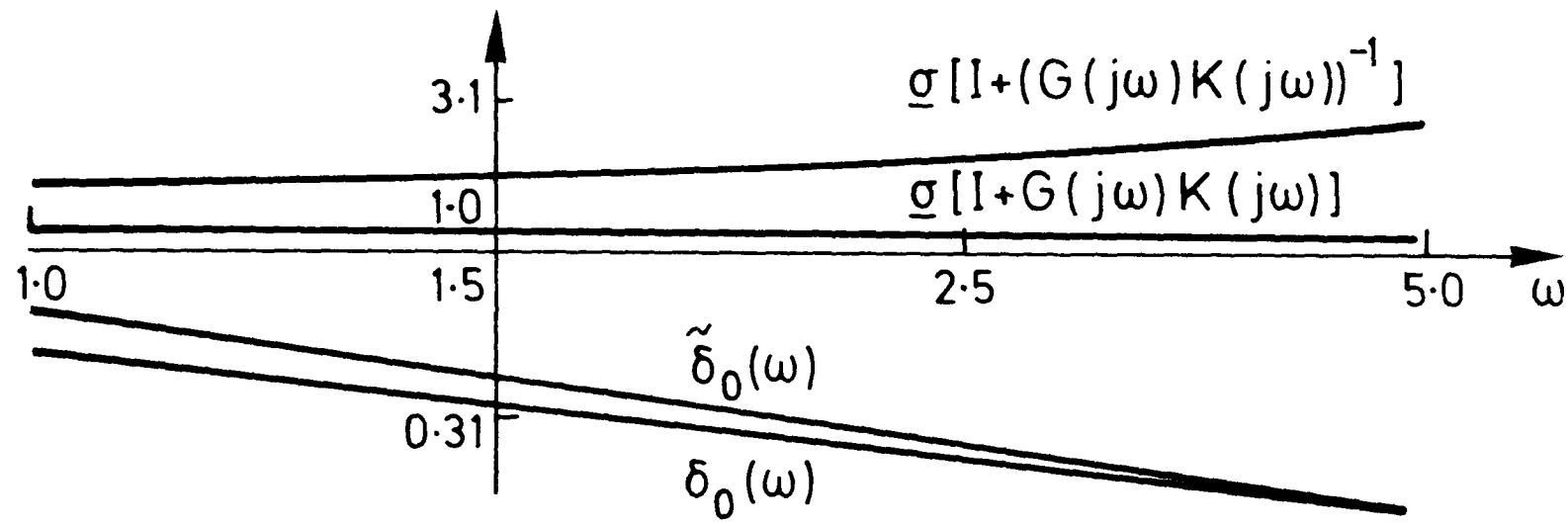


Figure 4.4 Direct and inverse tests, example 4.2, $\omega \in [1, 5]$

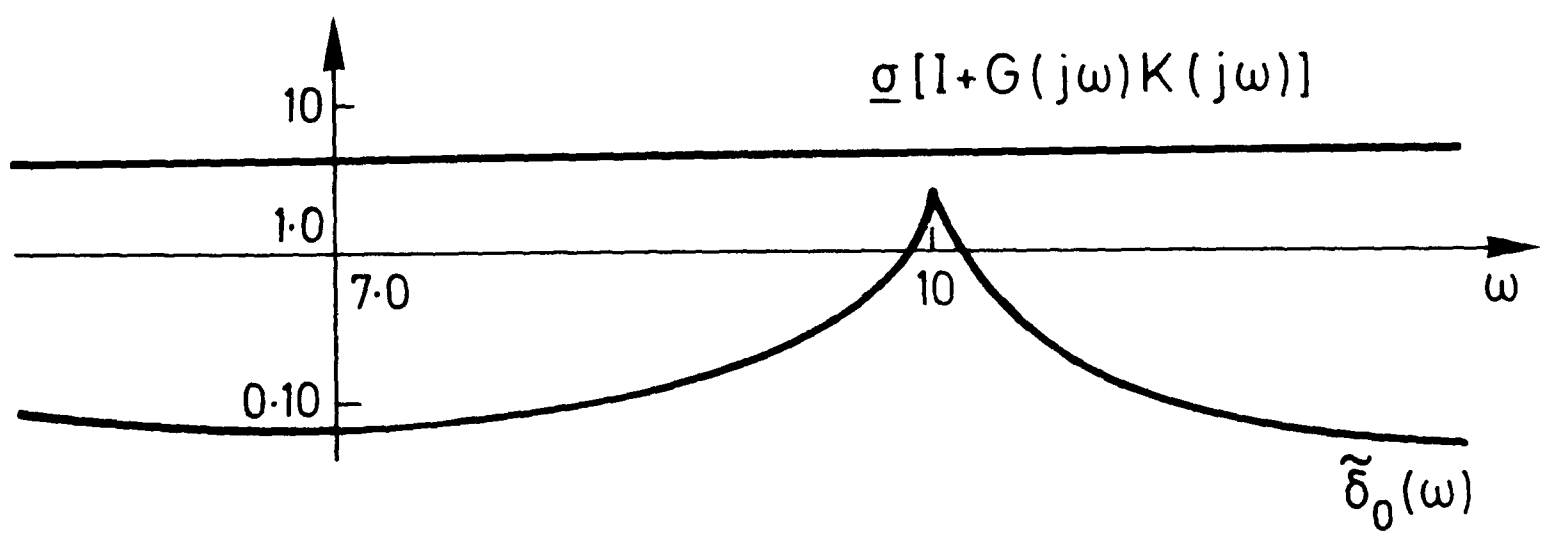


Figure 4.5a Inverse Nyquist based test, example 4.2, $\omega \in [5, 12]$

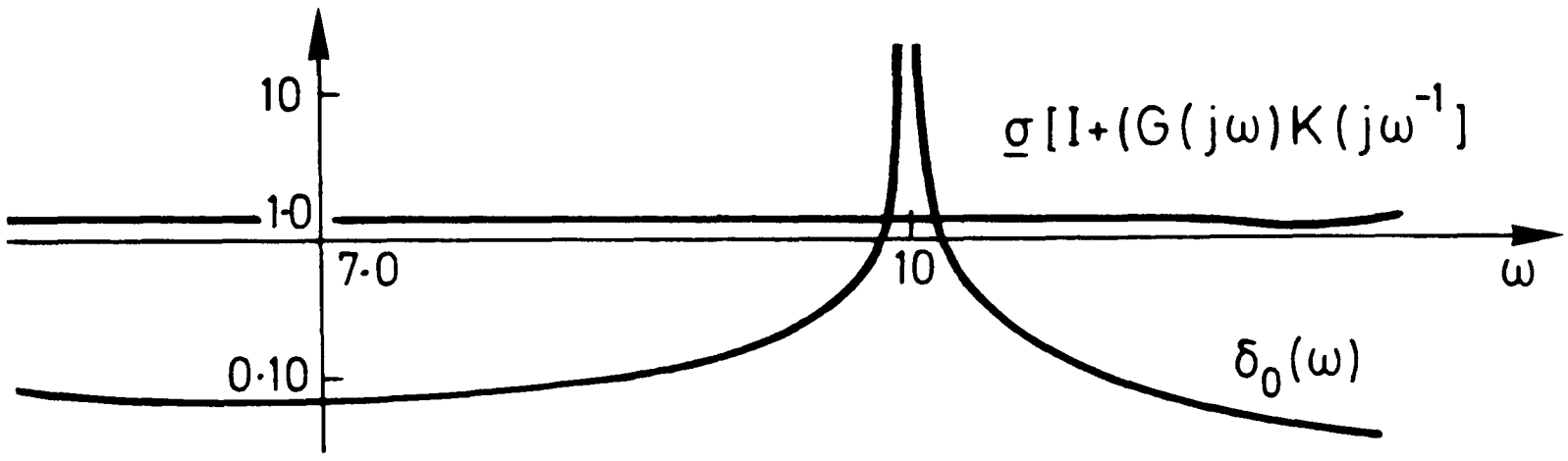


Figure 4.5b Direct Nyquist based test, example 4.2, $\omega \in [5, 12]$

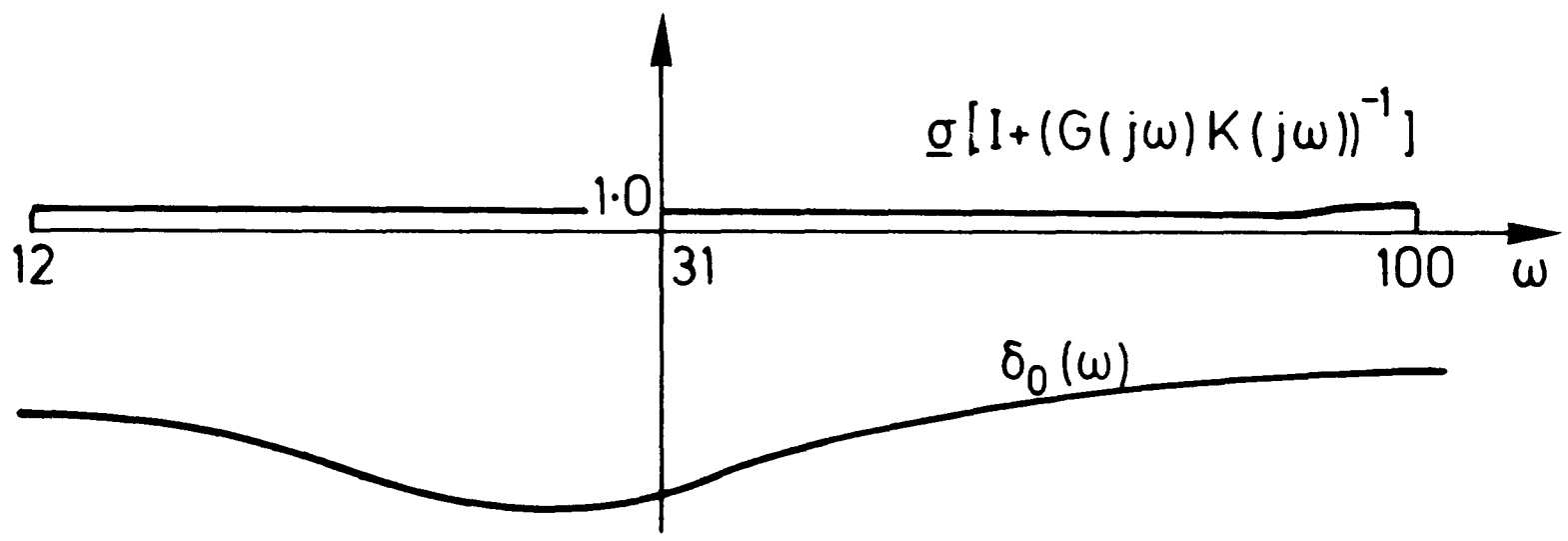


Figure 4.6 Direct Nyquist based test, example 4.2, $\omega \in [12,100]$

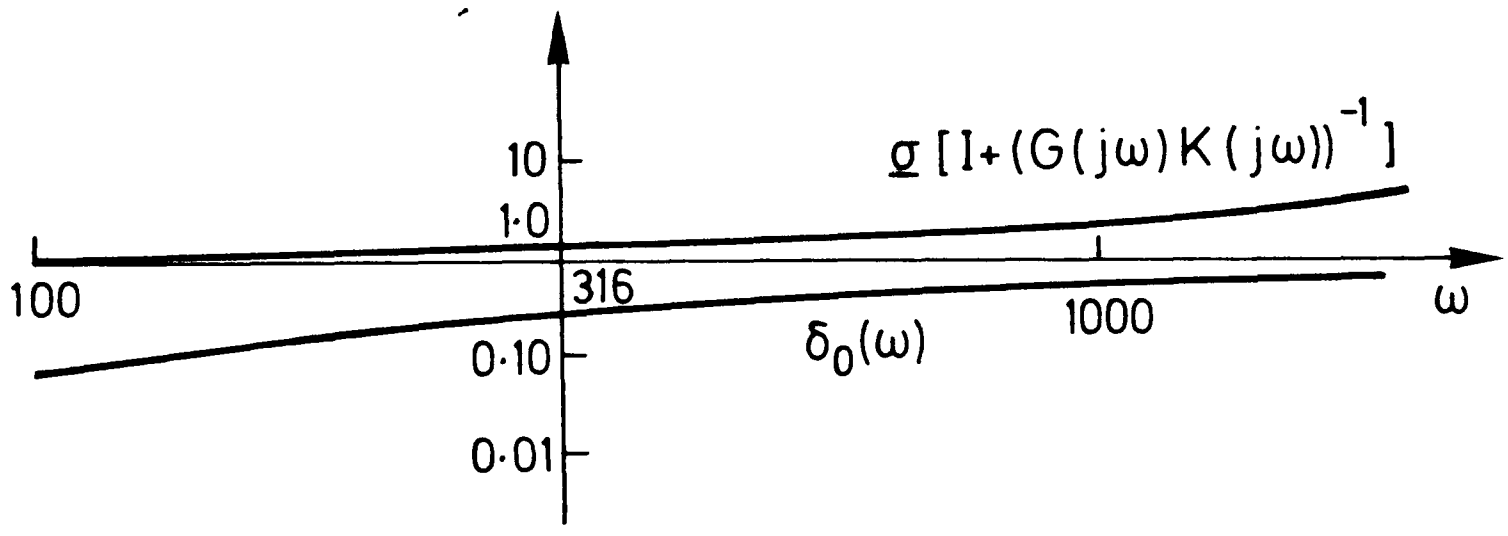


Figure 4.7 Direct Nyquist based test, example 4.2, $\omega \in [100,1000$ plus]

4.6 Robust Stability Tests based on Coprime Factorizations

In this section, we introduce two more models of plant uncertainties and present robust stability tests which utilize these representations. Let $G(s) = N(s)D(s)^{-1}$ where $N(s)$ and $D(s)$ are right-coprime, stable and real-rational matrices. Then the perturbed plant sets are modelled as :

$$\{G_p(s)\}_{sf} \triangleq \{G_p(s)=N_p(s)D_p(s)^{-1}: \bar{\sigma}\begin{bmatrix} N_p(s) - N(s) \\ D_p(s) - D(s) \end{bmatrix} \leq r(s), \text{ all } s=j\omega\} \tag{4.36}$$

and

$$\{G_p(s)\}_{gm} \triangleq \{G_p(s) : d(G, G_p) < r\} \tag{4.37}$$

where r is a positive real number, and $d(.,.)$ denote the graph metric distance described in section 2.3. We will call uncertainty of the form (4.36) stable-factor uncertainty, and uncertainty of the type (4.37) graph metric uncertainty, [VID6].

Theorem 4.15 The feedback system of figure 4.1 will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the set $\{G_p(s)\}_{sf}$ if :

$$\bar{\sigma}[\ \tilde{N}_K(j\omega) \quad \tilde{D}_K(j\omega)\ ] < \frac{1}{r(j\omega)}$$

where $(\tilde{N}_K(s),\ \tilde{D}_K(s))$ are left-coprime factorizations of $K(s)$ such that $\tilde{D}_K(s)D(s) + \tilde{N}_K(s)N(s) = I$.

Proof The feedback system of figure 4.1 is stable under G_p if, and only if, $\det[\tilde{D}_K(s)D_p(s) + \tilde{N}_K(s)N_p(s)]$ has no zero in the CRHP. But

$$\det[\tilde{D}_K(s)D_p(s) + \tilde{N}_K(s)N_p(s)]$$

$$= \det[\tilde{D}_K(s)(D_p(s)-D(s)) + \tilde{N}_K(s)(N_p(s)-N(s)) + \tilde{N}_K(s)N(s) + \tilde{D}_K(s)D(s)]$$

$$= \det[I + [\tilde{N}_K(s) \quad \tilde{D}_K(s)]\begin{bmatrix} N_p(s) - N(s) \\ D_p(s) - D(s) \end{bmatrix}] \qquad (4.38)$$

because $\tilde{D}_K(s)D(s) + \tilde{N}_K(s)N(s) = I$. Now, by the principle of the argument, $\det[\tilde{D}_K(s)D_p(s) + \tilde{N}_K(s)N_p(s)]$ does not have a zero in the CRHP if, and only if, its Nyquist-type locus does not encircle the origin. This is implied by:

$$\bar{\sigma}[\tilde{N}_K(s) \quad \tilde{D}_K(s)] < \frac{1}{r(s)}, \quad \text{all } s=j\omega.$$

Note that it is not necessary to use the large semi-circular part of the D-contour because $G_p(s)K(s)$ is strictly proper. ∎

A major disadvantage of the robust stability test of theorem 4.15 is that the test is dependent not only on G(s), but also on a particular right-coprime factorization of G(s). Thus, given a pair of plant models G(s) and $G_p(s)$, we may find two pairs of coprime factorizations (N(s), D(s)) and ($N_p(s)$, $D_p(s)$) which fail the test. But this does not necessarily imply that we cannot find other pairs of coprime factorizations for G(s) and $G_p(s)$ such that the test is satisfied. However, the test does offer the advantage that it can handle the case when both a pole and a zero are likely to cross the $j\omega$-axis in the same frequency range.

We next state the graph metric robust stability test of Vidyasagar, [VID4].

Theorem 4.16 The feedback system of figure 4.1 will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the set $\{G_p(s)\}_{gm}$ if :

$$\sup_{\omega} \bar{\sigma}[M(s)] < \frac{1}{d(G_p(s),\ G(s))}$$

where

$$M(s) = \begin{bmatrix} -G(s)K(s)(I+G(s)K(s))^{-1} & -G(s)(I+G(s)K(s))^{-1} \\ K(s)(I + G(s)K(s))^{-1} & (I + K(s)G(s))^{-1} \end{bmatrix}$$

Proof See [VID4]. ∎

One major advantage of the test of theorem 4.16 is, of course, that it does not depend on a particular coprime factorization. The major disadvantage is that, at the present state of knowledge, the graph metric distance between two plants cannot be computed exactly.

A common property shared by the robust stability tests of this section, and the combined direct/inverse Nyquist test of section 4.5 is that the set of perturbed plant models $\{G_p(s)\}$ used by each is arcwise connected. It is therefore possible to combine all these tests to form a general switching-type robust stability criterion. This is done in the next section.

4.7 General Switching-Type Robust Stability Tests

A switching-type robust stability test was first developed by Postlethwaite et. al., [POS3], in the principal gains and principal phases robust stability test where they combined the small gain test with a small phase test. Later, Lehtomaki, [LEH2], proposed combining a robust stability test which uses the structure of the uncertainty, [LEH3], with one which did not, in order to reduce conservatism. These switching-type criteria are conceptually quite different from the combined direct/inverse Nyquist robust stability test in that central to the developments of the former tests is an argument based on the continuity on the part of $G(s,\gamma)$ or $\tilde{G}(s,\gamma)$. Such continuity is usually artificial. For example, let $\{G_p(s)\} = \{1/(s-a);\ 1 \leq a \leq 2\}$. Then, if the direct type perturbation is used, we have

$$G(s,\gamma) = \frac{s-2+\gamma}{(s-1)(s-2)} \quad ; \quad 0 \leq \gamma \leq 1$$

which is <u>not</u> a member of $\{G_p(s)\}$ for any γ on $(0,1)$. Note that since $G(s,\gamma)$ has different number of unstable poles from $G(s)$, it follows that if the singular value inequality test is satisfied, then $K(s)$ which stabilizes $G(s)$ cannot possibly stabilize $G(s,\gamma)$ for $\gamma \in (0,1)$. It thus appears that the <u>convexity with respect to the nominal</u> property suggested in [BAR2, page 27] is invalid in general, except in cases where the perturbations are stable.

On the other hand, the combined direct/inverse Nyquist robust stability test of section 4.5 assumes continuity on the part of $\{G_p(s)\}$. From a practical point of view, verifying that the set $\{G_p(s)\}$ satisfies a certain set of properties would appear to be a more feasible task than verifying that the perturbations (say $L(s)$ where $G_p(s) = G(s)L(s)$)

satisfy certain conditions. From a theoretical point of view, the approach of section 4.5 also seems to offer more potential for developing less conservative robust stability tests with a wider scope of application. For this purpose, we state the following conceptual switching-type robust stability theorem.

Theorem 4.17 (Fundamental Robust Stability Test based on Arcwise Connectivity). Let $\{G_p(s)\}$ be an arcwise connected set of strictly proper rational transfer function matrices in the graph topology. Then the feedback system of figure 4.1 will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the set $\{G_p(s)\}$ if, at each frequency $\omega \in [0,\infty]$, there exists some test ascertaining that the Nyquist (and hence the inverse Nyquist) loci of all $G_p(s) \in \{G_p(s)\}$ do not pass through the critical point.

Proof Because $\{G_p(s)\}$ is arcwise connected, it follows that for any $G_p(s) \in \{G_p(s)\}$, we can find a path in $\{G_p(s)\}$ connecting $G_p(s)$ and $G(s)$. Since any change in the Nyquist (and hence the inverse Nyquist) loci has to be continuous in the extended sense when we perturb $G(s)$ to $G_p(s)$ along a path, it follows that the number of encirclements corresponding to $G_p(s)$ has to be correct if the loci do not pass through the critical point. Thus the feedback system is stable under $G_p(s)$. But $G_p(s)$ is arbitrary, so theorem 4.17 holds. This completes the proof. ∎

Theorem 4.17 is very general, and provides a powerful framework within which the robust stability tests of section 4.5 and section 4.6 can be suitably combined. The possibility of combining theorem 4.14 and theorem 4.15 to form part of a general switching-type robust stability

test is apparent. To accommodate theorem 4.16 in the switching criterion, it would be useful to make a slight modification, as follows.

Let $G_i(s) = N_i(s)D_i(s)^{-1}$, $i = 1, 2$, where (N_i, D_i) is a normalized r.c.f. of $G_i(s)$. Let

$$A_i = \begin{bmatrix} D_i \\ N_i \end{bmatrix}, \quad i = 1, 2 \tag{4.39}$$

Define

$$\delta_\omega(G_1, G_2) = \inf_{U \in Mat(S), \|U\| \leq 1} \bar{\sigma}[A_1(j\omega) - A_2(j\omega)U(j\omega)] \tag{4.40}$$

and

$$d_\omega(G_1, G_2) = \max \{ \delta_\omega(G_1, G_2), \delta_\omega(G_2, G_1)\} \tag{4.41}$$

(Note that d_ω is in general a frequency-dependent quantity). It is then not difficult to show, using a slightly modified version of the proof given in [VID4] for theorem 4.16, that $\det[I + G_p(j\omega)K(j\omega)]$ does not pass through the origin at $\omega=\omega_o$ if

$$d_\omega(G_1, G_2)\, \bar{\sigma}[M(s)] < 1 \tag{4.42}$$

at $s = j\omega_o$, where $M(s)$ has been given in theorem 4.16.

Other inequality tests which can be readily accommodated into the general switching criterion include those using principal gains and phases, [POS3], those using projection model error structures, [LEH3], matrix measure tests, [BAR2], and tests using matrix bilinear transformations, [BAR2]. For completeness, and the convenience of the reader, we re-state some of these tests below.

Definition 4.2, [POS3]. Any square complex matrix P can be represented by the polar decompositions :

$$P = U H_R \tag{4.43}$$

$$P = H_L U \tag{4.44}$$

where U is unitary and H_R, H_L are positive semi-definite hermitian matrices, then

(i) the principal gains (equivalently singular values) of P are the eigenvalues of the hermitian part of its polar decomposition; and

(ii) the principal phase are the arguments of the eigenvalues of the unitary part of its polar decomposition.

Lemma 4.1 (small phase conditions), [KOU1]. Let $A(j\omega)$ and $B(j\omega)$ have principal phases

$$\theta_1(\omega) \geq \theta_2(\omega) \geq \cdots \geq \theta_m(\omega) \tag{4.45}$$

and

$$\phi_1(\omega) \geq \phi_2(\omega) \geq \cdots \geq \phi_m(\omega) \tag{4.46}$$

respectively. Define the condition numbers

$$c_1(\omega) = \frac{\bar{\sigma}[A(j\omega)]}{\underline{\sigma}[A(j\omega)]} \tag{4.47}$$

and

$$c_2(\omega) = \frac{\bar{\sigma}[B(j\omega)]}{\underline{\sigma}[B(j\omega)]} \tag{4.48}$$

and also define

$$\psi_1(\omega) \triangleq \tan^{-1}\left[\frac{(c_1(\omega)-1)(c_2(\omega)-1)}{2-(c_1(\omega)-1)(c_2(\omega)-1)}\right] \tag{4.49}$$

then $\det[I + A(j\omega)B(j\omega)] \neq 0$ at $\omega=\omega_o$ if the following conditions hold :

(i) $\{\theta_i(\omega)\}$ and $\{\theta_i(\omega)+\epsilon_j(\omega)\}$ have spread less than π for i,j=1,.., m.

(ii) $(c_1(\omega)-1)(c_2(\omega)-1) < 2$

(iii) $(\theta_1(\omega) + \phi_1(\omega)) - (\theta_m(\omega) - \phi_m(\omega)) < \pi$, and

(iv) $\theta_1(\omega) + \phi_1(\omega) + \psi_1(\omega) < \pi$, and

$\theta_m(\omega) + \phi_m(\omega) - \psi_1(\omega) > -\pi$

at $\omega = \omega_o$.

Proof See [KOU1]. ∎

Lemma 4.2 (projection error conditions), [LEH3]. Let $A(j\omega)$ have singular values

$$\bar{\sigma}[A(j\omega)] = \sigma_1(\omega) > \sigma_2(\omega) > \cdots > \sigma_n = \underline{\sigma}[A(j\omega)]$$

then $\det[I + A(s)B(s)] \neq 0$ at $s = j\omega_o$ if the following conditions hold

(i) $A(j\omega)$ has singular value decomposition $A(j\omega) = U(j\omega)\Sigma(j\omega)V(j\omega)^H = \sum_i \sigma_i(\omega)u_i(j\omega)v_i(j\omega)^H$.

(ii) $\bar{\sigma}[B(j\omega)] < [\sigma_1^{-1}(\omega)\sigma_2^{-1}(\omega) + b(\omega)[\sigma_1^{-1}(\omega) - \sigma_2^{-1}(\omega)]]^{\frac{1}{2}}$

(iii) $|< u_1(j\omega)v_1(j\omega)^H, B(j\omega) >| \leq b(\omega) < \sigma_1(\omega)$

at $\omega = \omega_o$, where $\langle X,Y \rangle = \mathrm{tr}[X^H Y]$ denotes the usual matrix inner product.

Proof See [LEH3]. ∎

Lemma 4.3 (spectral measure conditions). Define the maximum and minimum measures of a matrix X as

$$\bar{\mu}_r(X) = \lambda_{max}[(X + X^H)/2] \tag{4.50}$$

and

$$\underline{\mu}_r(X) = \lambda_{min}[(X + X^H)/2] \tag{4.51}.$$

Then $\det[I+A(j\omega)B(j\omega)] \neq 0$ at $\omega = \omega_o$ if, either

(i) $\bar{\mu}_r(A(j\omega)^{-1}) < 0$ and $\bar{\mu}_r(A(j\omega)^{-1}) + \bar{\mu}_r(B(j\omega)) < 0$ at $\omega = \omega_o$

or

(ii) $\underline{\mu}_r(A(j\omega)^{-1}) > 0$ and $\underline{\mu}_r(A(j\omega)^{-1}) + \underline{\mu}_r(B(j\omega)) > 0$ at $\omega = \omega_o$.

Proof

$$[I + A(j\omega)B(j\omega)] = A(j\omega)[A(j\omega)^{-1} + B(j\omega)].$$

Now, by the first half of condition (i) (or condition (ii)), we have $\det[A(j\omega_o)] \neq 0$. Applying the following inequalities for matrix measures, [DES1], [BAR2], shows that $\det[A(j\omega_o)^{-1} + B(j\omega_o)] \neq 0$:

$$\bar{\mu}_r(X + Y) \leq \bar{\mu}_r(X) + \bar{\mu}_r(Y) \qquad (4.52)$$

$$\underline{\mu}_r(X + Y) \geq \underline{\mu}_r(X) + \underline{\mu}_r(Y) \qquad (4.53).$$

This completes the proof. ∎

Lemma 4.4 (imaginary singular value conditions). Define the maximum and minimum imaginary singular values of a matrix X as, [AMI1]:

$$\bar{\mu}_i(X) = \lambda_{max}[(X - X^H)/2j]$$

and

$$\underline{\mu}_i(X) = \lambda_{min}[(X - X^H)/2j]$$

respectively. Then $\det[I + A(j\omega_o)B(j\omega_o)] \neq 0$ if, either

(i) $\bar{\mu}_i(A(j\omega_o)^{-1}) < 0$ and $\bar{\mu}_i(A(j\omega_o)^{-1}) + \bar{\mu}_i(B(j\omega_o)) < 0$;

or

(ii) $\underline{\mu}_i(A(j\omega_o)^{-1}) > 0$ and $\underline{\mu}_i(A(j\omega_o)^{-1}) + \underline{\mu}_i(B(j\omega_o)) > 0$.

Proof

Similar to lemma 4.3. (Note that $\mu_r(-jX) = \mu_i(X)$). ∎

Lemma 4.5 (positivity conditions), [LEH2]. The matrix $[I + A(s)B(s)]$ is nonsingular at $s=j\omega_o$ if

(i) $A(j\omega_o) + A(j\omega_o)^H \geq 0$; and

(ii) $B(j\omega_o) + B(j\omega_o)^H > 0$.

Proof See [LEH2]. ∎

Lemma 4.6 (bilinear transformation condition) The matrix [I + A(s)B(s)] is nonsingular at $s=j\omega_o$ if

$$\bar{\sigma}[(I+A(j\omega_o))^{-1}(I-A(j\omega_o))] < \{\bar{\sigma}[(I-B(j\omega_o))(I+B(j\omega_o))^{-1}]\}^{-1}$$

Proof See [BAR2], [LEH2]. ∎

Lemma 4.7 (minimum singular value condition) The matrix [I + A(s)B(s)] is nonsingular at $s = j\omega_o$ if

$$\underline{\sigma}[A(j\omega_o)]\underline{\sigma}[B(j\omega_o)] > 1.$$

Proof Obvious. ∎

Note that the above lemmas can be applied to $\det[I + G_p(s)K(s)]$ itself, or some transformed version of it (as in the case of the maximum singular value tests). Which is the best to use will depend on the type of information available at the particular frequency.

4.8 Robust Stability Analysis with Actuator and Sensor Failures

A well-known singular value robust stability test for determining stability in the presence of one or more actuator failures has been given by Doyle and Stein, [DOY2], as

$$\underline{\sigma}[I + (K(s)G(s))^{-1}] > 1 , \quad \text{all } s = j\omega. \qquad (4.54)$$

Similarly, one may show that the feedback system will remain stable in the presence of one or more sensor failures if the following inequality is satisfied for all ω :

$$\underline{\sigma}[I + (G(s)K(s))^{-1}] > 1, \quad \text{all } s = j\omega. \tag{4.55}$$

Unfortunately, it may be shown that both (4.54) and (4.55) cannot be satisfied if the plant or the controller is open-loop unstable. This is because in order to satisfy either of the inequalities, the characteristic gains λ_{GK} of K(s)G(s) (and hence G(s)K(s)) must satisfy

$$\left|\frac{\lambda_{GK}}{1+\lambda_{GK}}\right| < 1 \tag{4.56}$$

$$<=> \qquad |\lambda_{GK}| < |1 + \lambda_{GK}| \tag{4.57}$$

(4.57) can be satisfied only if all the characteristic gain loci of K(s)G(s) are confined in the plane Re(s) > -1/2, which is impossible if G(s)K(s) has unstable poles. Another point we emphasize is that (4.54) and (4.55) guarantee robust stability only when the perturbations are due to actuator or sensor failures alone, or when the combined effects of plant uncertainty plus the actuator or sensor failure produce a perturbation representation with singular value less than or equal to 1 for all frequencies.

To cope with the possibility of actuator or sensor failure plus significant uncertainties in the plant model, we may proceed as follows. Let L(i) denote a diagonal matrix with 1 on its diagonal except the ith diagonal element which is zero. Then $L(i)[I+\Delta_o(s)]G(s)$ denotes the set of uncertain plant models in the presence of sensor failure in loop i. We may therefore state the following lemma.

Lemma 4.8 (Robust Stability with Sensor Failure in Loop i) The feedback system of figure 4.1 with $G(s)$ replaced by any $G_p(s)$ (including $G(s)$ itself) from $\{G_p(s)\}_{\Delta_o}$ will remain stable in the presence of a sensor failure in loop i if :

(i) $G_p(s)$ and $G(s)$ share the same number of unstable poles;

(ii) $\det[I + L(i)G(s)K(s)]$ encircles the origin the same number of times as $\det[I + G(s)K(s)]$ as s traverses the D_D contour; and

(iii) $\bar{\sigma}[(I+G(s)K(s)L(i)^{-1})G(s)K(s)L(i)] < (\delta_o(\omega))^{-1}$, all $s \in \Omega_D$.

Proof (ii) is necessary and sufficient for the nominal feedback system to remain stable in the presence of the sensor failure. (iii) implies the number of encirclements of the origin by $\det[I + L(i)G_p(s)K(s)]$ is the same as the number of encirclements by $\det[I + L(i)G(s)K(s)]$ because

$$\begin{aligned}
&\det[I + L(i)G_p(s)K(s)] \\
&= \det[I + (I + \Delta_o(s))G(s)K(s)L(i)] \\
&= \det[I+\Delta_o G(s)K(s)L(i)(I+G(s)K(s)L(i))^{-1}]\det[I+G(s)K(s)L(i)].
\end{aligned}$$

This completes the proof. ∎

In lemma 4.8, it is not necessary to worry about whether the sensor failure has created an unobservable or undetectable mode because the nominal feedback system is stable, and this implies that we already have the correct number of unstable poles.

It should be clear to the reader that we may apply a similar approach to investigate the robust stability of the uncertain feedback system when subjected to multiple sensor or actuator failures. However, as the number of allowable component failures increases, the test becomes more complex and tedious if all possible combinations are to be tested.

It should also be clear to reader that if $\{G_p(s)\}$ is an arcwise connected set, then a switching-type robust stability test may be used.

4.9 Concluding Remarks

In this chapter we have presented three kinds of singular value based robust stability tests:

(i) the direct Nyquist based tests,

(ii) the inverse Nyquist based tests, and

(iii) the combined direct and inverse Nyquist based tests.

The direct tests are applicable when the plant and perturbed plant models (representing uncertainty) have the same number of poles in the direct D-contour. They can therefore cope with zero movement across the imaginary axis as a result of parameter variations, and the sudden appearance or disappearance of zeros (anywhere in the complex plane) arising from large errors in the nominal model.

The inverse Nyquist based tests are applicable when the plant and perturbed plant models have the same number of zeros in the inverse Nyquist D-contour, and can therefore cope with the sudden appearance or disappearance of poles as well as their movement across the imaginary axis.

In section 4.5, the problem of simultaneous pole and zero crossing of the imaginary axis (and zero crossing of the point of infinity) is solved by combining a Nyquist based test from section 4.2 with an inverse Nyquist based test from section 4.3. The resulting singular value test, which we believe to be one of the main contributions of this chapter, is applicable to a large class of transfer function matrices $\{G_p(s)\}$ whose

frequency responses are known to lie within given bounds in gain and phase. For a set of perturbed plant models which satisfies this property, we show that, if certain verifiable conditions are satisfied then the set is an arcwise connected set in the graph topology of transfer function matrices. The tests are easy to apply and convey useful information for design; that is, the tests indicate over which frequencies good sensitivity and robust stability are compatible objectives, and over which frequencies a compromise has to be made. It follows that if there are no uncertain zeros which can cross the imaginary axis at a particular frequency, then good sensitivity and robust stability are compatible objectives at that frequency.

The importance of the two functions :

$$\bar{\sigma}[S(j\omega)] \quad \text{and} \quad \bar{\sigma}[T(j\omega)]$$

in characterizing the sensitivity and robust stability properties of a feedback design suggests that a fruitful line of research might be to consider designs based on the minimization of

$$\max \left(\bar{\sigma}[W_1(j\omega)S(j\omega)],\ \bar{\sigma}[W_2(j\omega)T(j\omega)]\right) \qquad (4.58)$$

where W_1 and W_2 are weighting functions. Note that the minimization of the supremum over all frequencies of the first term is the minimax sensitivity minimization problem, a problem which has received considerable attention of late, e.g. [ZAM2].

A general switching-type robust stability test is given in section 4.7. The test may be viewed as the generalization and unification of the robust stability test of section 4.5 with those based on coprime factorizations, [VID4], [VID6]. It has been recently argued that the topology induced by the gap-metric must be identical to the graph

topology, [ELS1]. Thus it seems that the gap-metric robust stability test may be considered a special case of theorem 4.17 as well. More specifically, we may consider theorem 4.17 as an "idea" or concept, and the combined direct/inverse Nyquist robust stability test, the graph metric test, the gap-metric test, and the coprime-factor based robust stability test as different tools for putting the concept to work. (From an algebraic geometrical view-point, all these tests can be considered as based on homotopical arguments on the Riemann sphere, which is different from the individual direct and inverse Nyquist based tests which are based on homotopy in the complex plane).

Finally, we remark that the constraint on the non-minimum-phase zeros of $\{G_p(s)\}$ in properties D and D', as well as theorem 4.13 (section 4.5), is in general necessary only in the case of square and nonsingular plants. For non-square plants, which generically do not have any zeros, [KOU5], two plants can be very close topologically and yet have quite different zero structures in the CRHP. (See appendix IV-1 for an example).

Appendix IV-1

Proof of theorem 4.13

(i) <=> (ii) If $G_1(s)$ and $G_2(s)$ have almost identical frequency responses, then they are simultaneously stabilizable if, and only if, they have the same numbers of poles and zeros in the Nyquist D-contour. This is because for any arbitrary $K(s)$ which stabilizes both $G_1(s)$ and $G_2(s)$, $G_1(s)K(s)$ and $G_2(s)K(s)$ have almost identical frequency responses, and therefore, almost identical Nyquist and inverse Nyquist loci. For

stability, the Nyquist loci, and the inverse Nyquist loci, must have the correct numbers of encirclements.

(ii) <=> (iii) Let $g_1(s)$ and $g_2(s)$ be two simultaneously stabilizable single-input, single-output transfer functions, and let $k(s)$ be a controller which stabilizes both. Then the map from $g_i(j\omega)$, i=1, 2 to $(1+g_i(j\omega)k(j\omega))^{-1}$ is an invertible map. This means that given any $g_i(j\omega)$ and $k(j\omega)$, $(1 + g_i(j\omega)k(j\omega))^{-1}$ is uniquely determined. Conversely, given $k(j\omega)$ and any $(1 + g_i(j\omega)k(j\omega))^{-1}$, there exists a unique $g_i(j\omega)$ which gives the frequency response of $(1 + g_i k)^{-1}$. (Note that this does not necessarily imply there exists a unique $g_i(s)$). it follows that $g_1(s)$ and $g_2(s)$ having almost identical frequency responses implies $(1+g_1(s)k(s))^{-1}$ and $(1+g_2(s)k(s))^{-1}$, too, have almost identical frequency responses. Now since $k(s)$ stabilizes both $g_1(s)$ and $g_2(s)$, $(1 + g_i k)^{-1}$, $i = 1, 2$, are analytic in the CRHP. Thus by the maximum modulus principle,

$$J \triangleq \sup_{s=j\omega} [(1 + g_2(j\omega)k(j\omega))^{-1} - (1 + g_1(j\omega)k(j\omega))^{-1}] \tag{4.59}$$

$$= \sup_{Re(s)\geq 0} [(1 + g_2(j\omega)k(j\omega))^{-1} - (1 + g_1(j\omega)k(j\omega))^{-1}] \tag{4.60}.$$

Since $g_1(s)$ and $g_2(s)$ have almost identical frequency responses, it follows from (4.59) that $J\approx 0$. Thus from (4.60)

$$[1 + g_2(j\omega)k(j\omega)]^{-1} \approx [1 + g_1(j\omega)k(j\omega)]^{-1}, \quad \text{all } Re(s)\geq 0. \tag{4.61}$$

It thus follows from Hurwitz's theorem, [WAL1], that $[1 + g_2 k]^{-1}$ and $[1 + g_1 k]^{-1}$ must have the same number of zeros (multiplicities included) in the CRHP. Dividing the CRHP into small regions and applying Hurwitz's

theorem again shows that they must have an almost identical zero structure in the CRHP. Note that the zeros of $(1 + g_i k)^{-1}$ are the poles of $g_i(s)$ and $k(s)$. (4.61), therefore, implies that the pole structure of $g_1(s)$ must be almost identical to that of $g_2(s)$ in the CRHP.

Similarly, since

$$(1 + g_i k)^{-1} g_i k = 1 - (1 + g_i k)^{-1}, \quad i = 1,2 \qquad (4.62)$$

we have

$$(1 + g_1 k)^{-1} g_1 k \approx (1 + g_2 k)^{-1} g_2 k, \quad \mathrm{Re}(s) \geq 0 \qquad (4.63)$$

Because the zeros of $(1 + g_i k)^{-1} g_i k$ are the zeros of $g_i(s)$ plus the zeros of $k(s)$, it follows from Hurwitz's theorem again that $g_1(s)$ and $g_2(s)$ must have almost identical zero structure in the CRHP.

In the multivariable case, for any two transfer matrices $G_1(s)$ and $G_2(s)$, given that they have the same number of poles and zeros in the CRHP and almost identical frequency responses, implies that each of their corresponding elements must be almost identical to each other topologically. (For transfer function matrices which have 'pole-zero cancellations' in some of the elements, we may consider a suitable 'extension' to treat those with a perfect cancellation as being 'close' to those with an almost exact cancellation at the same location). In fact, $G_1(s)$ and $G_2(s)$ being stabilizable rules out any possibility of unstable pole-zero cancellations in $G_i(s)$, i=1,2. Because the Smith-McMillian structure of a given transfer matrix is unique, $G_1(s)$ and $G_2(s)$ having almost identical pole and zero structures elementwise implies they have almost identical pole and zero structures in the CRHP. Note that $G_1(s)$ and $G_2(s)$ having almost identical pole and zero structures in the

CRHP implies they have the same numbers of poles and zeros in the CRHP, by definition. This completes the proof. ∎

We note that if G_1 and G_2 are <u>not</u> square and nonsingular, then theorem 4.13 does not necessarily hold, this is because two rational matrices may be very close topologically and yet have very different zero structure in the CRHP. For example, let

$$G_1(s) = [(s-2)/(s+3) \quad (s-2)/(s+4)]$$

and

$$G_2(s) = [(s-2+\epsilon)/(s+3) \quad (s-2)/(s+4)].$$

Then it is easy to show that G_1 has a zero at $s=2$ while G_2 does not have this zero. Nonetheless both G_1 and G_2 are simultaneously stabilizable for small enough $\epsilon > 0$. It thus follows that the constraint on the non-minimum-phase zeros in both properties D and D' may be relaxed in general. This is desirable because unstable poles are, in practice, generally easier to identify than non-minmum-phase zeros.

CHAPTER V

ROBUST STABILITY ANALYSIS WITH DISTRIBUTED SYSTEMS

5.1 Introduction

The purpose of this chapter is to extend the results of the previous chapter to include two classes of distributed systems, [POS7]. We will only discuss one in detail because the results obtained are applicable to the other with only very slight modifications.

The chapter is organized into six sections, of which this introduction is the first. Section 5.2 contains necessary preliminaries, including definitions of poles and zeros, and the $\mathbf{A}(\sigma)$ and $\mathbf{B}(\sigma)$ algebras introduced by Callier and Desoer, [CAL1]. A new definition of the inverse Nyquist characteristic gain loci is also given to enable us to extend the inverse Nyquist stability criterion to include singular plants. In section 5.3, we present Nyquist-type stability criteria based on the work of Desoer et. al., [CAL1], [DES4], [CHE1]; inverse Nyquist-type criteria are given in section 5.4. The main aim of the chapter is reached in section 5.5 where the robust stability tests of the previous chapter are extended to the $\mathbf{B}(0)$-class of distributed systems, and to non-square plants. We give our conclusions in section 5.6 where we also identify another class of distributed systems for which the robust stability tests are applicable.

5.2 Preliminaries

5.2-1 The $\mathbf{A}(\sigma)$ and $\mathbf{B}(\sigma)$ Algebras

We will mainly adopt the notation used in [DES4], [CHE3]. Let

$$f(t) = \begin{cases} 0 & , \ t<0 \\ f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t - t_i), & t \geq 0 \end{cases} \qquad (5.1)$$

where

$$\int_0^{\infty} |f_a(t)| e^{-\sigma t} dt < \infty,$$

$$t_0 = 0, \ t_i > 0 \text{ for all } i > 0,$$

$$\sum_{i=0}^{\infty} |f_i| e^{-\sigma t_i} < \infty.$$

$\delta(.,t_i)$ is the Dirac delta distribution applied at t_i, and $\sigma \in \mathbf{R}$, the field of real numbers.

$\mathbf{A}(\sigma)$ is the set of all generalized functions (distributions) defined by (5.1).

$f \in \mathbf{A}_{-}(\sigma)$ if, and only if, for some $\sigma_1 < \sigma$, $f \in \mathbf{A}(\sigma_1)$.

$f(s)$ denotes the Laplace transform of f, and $\mathbf{A}(\sigma)$ denotes the set of all transfer functions $f(s)$ such that $f \in \mathbf{A}(\sigma)$. Similarly, $\hat{\mathbf{A}}_{-}(\sigma) \triangleq \{f(s): f \in \mathbf{A}_{-}(\sigma)\}$.

Note that $\mathbf{A}(0)$ can be interpreted as a set of L_1-stable impulse responses, and $\hat{\mathbf{A}}(0)$ a set of L_1- stable transfer functions.

Let $\hat{\mathbf{A}}_{-}^{\infty}(\sigma)$ denote the subset of $\hat{\mathbf{A}}_{-}(\sigma)$ consisting of those $f(s)$ that are bounded away from zero at infinity in $C_{\sigma+}$, the CRHP $\mathrm{Re}(s) \geq \sigma$. $\mathbf{A}(\sigma)^{mxn}$ denotes the set of mxn matrices of distributions in $\mathbf{A}(\sigma)$. $\mathbf{A}_{-}(\sigma)^{mxn}$ is similarly defined.

If $G \in \mathbf{A}(0)^{nxm}$, then $u \in L_p^m$ implies $Gu \in L_p^n$, where L_p denotes the class of time functions satisfying

$$\left(\int_0^{\infty} |f(t)|^p dt \right)^{1/p} < \infty \qquad (5.2)$$

and L_∞ is given by

$$\operatorname*{ess\,sup}_{t \geq 0} |f(t)| < \infty. \qquad (5.3).$$

Thus, if the closed-loop impulse response of a feedback system has all its elements in $\hat{A}(0)$, then the system is closed-loop stable for all $u \in L_p$, $1 \leq p \leq \infty$. For more information on $\hat{A}(\sigma)$, the reader is referred to [CAL1].

$\hat{B}(\sigma) \triangleq [\hat{A}_-(\sigma)][\hat{A}^\infty_-(\sigma)]^{-1}$ is the set of quotients $f(s) = n(s)/d(s)$ where $n(s) \in \hat{A}_-(\sigma)$ and $d(s) \in \hat{A}^\infty_-(\sigma)$. $\hat{B}(\sigma)^{m\times n}$ denotes the set of $m\times n$ transfer function matrices with elements in $\hat{B}(\sigma)$. Any transfer function matrix M in $\hat{B}(\sigma)^{m\times n}$ admits a right-coprime factorization $M = ND^{-1}$ and a left coprime factorization $M = \tilde{D}^{-1}\tilde{N}$ where

$$N(s),\ \tilde{N}(s) \in \hat{A}_-(\sigma)^{m\times n},\quad D(s),\ \tilde{D}(s) \in \hat{A}_-(\sigma)^{n\times n}.$$

$\det(D(s))$ and $\det(\tilde{D}(s)) \in \hat{A}^\infty_-(\sigma)$; and (N,D), $(\tilde{N}, \tilde{D})$ are right and left-coprimes respectively, [VID3].

A detailed discussion of the properties of functions in $\hat{B}(\sigma)$ can be found in [CAL1]. For our purpose here, we simply list some of these properties (without proof) to provide a more intuitive view of the class of functions involved.

(i) Let f be a Laplace transformable distribution with support on R_+. Then $f(s) \in \hat{B}(\sigma_o)$ if, and only if, there exists some $\sigma_1 < \sigma_o$ such that

$$f(s) = r(s) + g(s), \quad \text{for all } s \in C_{\sigma_1+} \qquad (5.4)$$

where

a) $g(s) \in \mathbf{A}_{-}(\sigma_o)$,

b) $r(s)$ is a strictly proper rational function which is zero if, and only if, $f(s) \in \hat{\mathbf{A}}_{-}(\sigma_o)$.

(ii) For all $f_1(s)$, $f_2(s) \in \hat{\mathbf{B}}(\sigma_o)$, $\alpha \in C$, the following algebra holds:

$$\alpha f_1(s) \in \hat{\mathbf{B}}(\sigma_o),$$

$$f_1(s) + f_2(s) \in \hat{\mathbf{B}}(\sigma_o),$$

$$f_1(s)f_2(s) = f_2(s)f_1(s) \in \hat{\mathbf{B}}(\sigma_o) \qquad (5.5)$$

(iii) $f(s) \in \hat{\mathbf{B}}(\sigma_o)$ has an inverse in $\hat{\mathbf{B}}(\sigma_o)$ if, and only if, there exists a $\sigma_1 < \sigma_o$ such that $f(s)$ is bounded away from zero at infinity in C_{σ_1+}.

(iv) If $f(s) \in \hat{\mathbf{B}}(\sigma_o)$, then it has only a finite number of poles in C_{σ_o+}, and a finite number of zeros in any compact set of C_{σ_o+}.

(v) $f(s) \in \hat{\mathbf{B}}(\sigma_o)$ is meromorphic in the half-plane $Re(s) \geq \sigma_o$.

5.2-2 Feedback stability

We will consider the linear time-invariant feedback system shown in figure 5.1, consisting of a plant G and controller K, where

$$G(s) \in \hat{\mathbf{B}}(\sigma)^{m \times n} \text{ for some } \sigma \leq 0,\ m \geq n \qquad (5.6)$$

and

$$K(s) \in \hat{\mathbf{B}}(\sigma)^{n \times m}.$$

$G(s)$, $K(s)$ and $G(s)K(s)$ have r.c.f's $N_G D_G^{-1}$, $N_K D_K^{-1}$ and $N_{GK} D_{GK}^{-1}$ respectively.

Let

$$H_{eu} = \begin{bmatrix} (I + GK)^{-1} & -G(I+KG)^{-1} \\ K(I+GK)^{-1} & (I + KG)^{-1} \end{bmatrix} \qquad (5.7)$$

be the input-output map of the system $u \rightarrow e$, where $u = (u_1, u_2)^T$, $e = (e_1, e_2)^T$. $H_{eu}(s)$ is the transfer function matrix of H_{eu}.

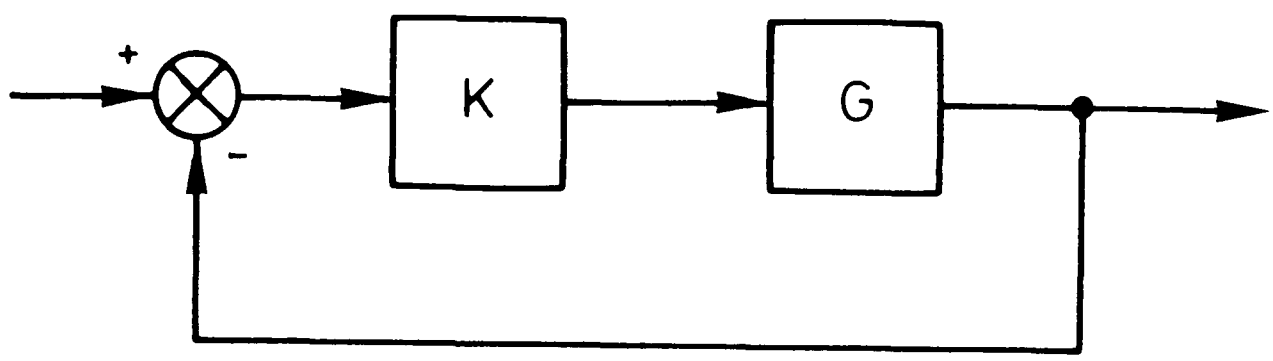


Figure 5.1 Feedback configuration

Definition 5.1 [CHE3]. The feedback system of figure 5.1 is $\mathbf{A}_-(\sigma)$-stable if, and only if, $H_{eu} \in \mathbf{A}_-(\sigma)^{(n+2m)\times(n+2m)}$.

Normally σ would be taken to be zero, but the extra flexibility in the definition allows a measure of relative stability to be included (if desired) by choosing σ negative.

Proposition 5.1 Let G be an m-input, n-output Laplace transformable convolution operator. Then $G(s) \in \hat{\mathbf{B}}(0)^{n\times m}$ if there exists $K(s) \in \hat{\mathbf{B}}(0)^{m\times n}$ such that $H_{eu} \in \mathbf{A}(0)^{(n+2m)\times(n+2m)}$.

Proof Let K which stabilizes G admit the left-coprime factorization $(\tilde{N}_K, \tilde{D}_K)$. H_{eu} is $\mathbf{A}_-(0)$-stable, implies that $G(I+KG)^{-1}$ and $(I+KG)^{-1}$ are in $\mathrm{Mat}(\mathbf{A}_-(0))$. Now let $N = G(I+KG)^{-1}\tilde{D}_K^{-1}$ and $D = (I+KG)^{-1}\tilde{D}_K^{-1}$ which gives

$$G = ND^{-1} \tag{5.8}$$

Note that $N \in \mathrm{Mat}(\mathbf{A}_-(0))$ and $D \in \mathrm{Mat}(\mathbf{A}_-^{\infty}(0))$. To show that N and D are right-coprime, we need to show that there exists $X, Y \in \mathrm{Mat}(\mathbf{A}_-(0))$ such that

$$XN + YD = I \qquad (5.9)$$

This is given by $X = \tilde{N}_K$ and $Y = \tilde{D}_K$, and hence the proof is completed.∎

Note that the converse of proposition 5.1 will also be true if G(s) is strictly proper, [VID1]. The class of transfer function matrices in $Mat(\hat{B}(0))$ is therefore a fairly general one in practice.

For consistency with the case of lumped systems in the previous chapter, we will assume all plant models are such that their transfer function matrices vanish identically in $C_{\sigma+}$ as $|s| \to \infty$.

5.2-3 Characteristic gain and inverse characteristic gain loci

The characteristic gain loci of a transfer function matrix $M(s) \in \hat{B}(\sigma)^{mxm}$ are the image under the map s → (eigenvalues of M(s)) of the clockwise contour $D(\sigma,r)$ consisting, with r arbitrary large, of (i) a straight line segment from $(\sigma - jr)$ to $(\sigma + jr)$ except for the usual infinitesimal left indentations around any poles of M(s) with real part equal to σ, and (ii) a semi-circle in $C_{\sigma+}$ joining $(\sigma + jr)$ to $(\sigma - jr)$.

The inverse characteristic gain loci of M(s) are defined as the reciprocal of the nontrivial characteristic gain loci, except that along the straight line segment of $D(\sigma, r)$ there is no need to indent around poles since the inverse loci will then be finite, but we do construct infinitesimal left indentations around any zeros of the characteristic gain loci.

It has already been established in [DES4] that the characteristic gain loci form closed curves, and so the inverse characteristic gain loci must do likewise. Note that M(s) may be singular (i.e. det[M(s)]≡0) in the definitions above.

5.2-4 Poles and zeros

The poles of G, K and GK are defined as the isolated zeros of $X_G \triangleq \det(D_G)$, $X_G \triangleq \det(D_K)$, and $X_{GK} \triangleq \det(D_{GK})$ respectively. The number of poles (including multiplicities) of G in $C_{\sigma+}$ is denoted by P_G; P_K and P_{GK} are similarly defined. We say there is a $C_{\sigma+}$ pole-zero cancellation in GK if $P_G + P_K \neq P_{GK}$.

As for lumped systems, [POS1], [SMI1], we will define the characteristic poles (zeros) of GK as the poles (zeros, respectively) of the eigenvalue function $\lambda(s)$ defined by the characteristic equation $\det[\lambda I - G(s)K(s)] = 0$ (or equivalently $\det[\lambda D_{GK}(s) - N_{GK}(s)] = 0$). The number of characteristic zeros (counting multiplicities) of GK in $C_{\sigma+}$ is denoted by Z.

A pole of GK which is not a characteristic pole is called a fixed pole. A detailed discussion of fixed poles for rational transfer function matrices has been given in [SMI1]; the extensions to the case of $B(\sigma)$ is straightforward. Note that a fixed pole of GK is also a pole of the feedback system of figure 5.1, and would be a fixed pole of $(I+GK)^{-1}GK$.

5.3 Direct Nyquist Stability Criteria

In this section, we present four stability theorems derived from the work of Desoer et. al., [CAL2], [DES4]. The aim is to arrive at a stability test in terms of the characteristic gain loci as an aid to the development of the a robust stability test in section 5.5.

Theorem 5.1 The feedback system of figure 5.1 is $\mathbf{A}_{-}(\sigma)$-stable if, and only if:

(i) $\det[I+G(s)K(s)]$ has no $C_{\sigma+}$ zeros, and

(ii) the $C_{\sigma+}$ poles of $\det[I+G(s)K(s)]$ are exactly the $C_{\sigma+}$ zeros of $X_G X_K$.

Proof See the proof to theorem 1 in [CHE3]. ∎

Theorem 5.2 The feedback system of figure 5.1 is $\mathbf{A}_{-}(\sigma)$-stable if, and only if:

(i) $\inf_{\mathrm{Re}(s)\geq\sigma} \det[I+G(s)K(s)] > 0$, and

(ii) there are no $C_{\sigma+}$ pole-zero cancellations in GK.

Proof (i) <=> (i), theorem 5.1, straightforwardly.

(ii) <=> (ii), theorem 5.1, by definition.

(i) and (ii) <=> Theorem 5.1 <=> stability.

This completes the proof. ∎

Theorem 5.3 The feedback system of figure 5.1 is $\mathbf{A}_{-}(\sigma)$-stable if, and only if:

(i) the characteristic gain loci of G(s)K(s) do not go through -1,

(ii) the characteristic gain loci of G(s)K(s) encircle -1, P_{GK} times anticlockwise, and

(iii) there are no $C_{\sigma+}$ pole-zero cancellations in GK.

Proof (i) and (ii) <=> (i), theorem 5.2, as shown in [DES4]. ∎

Theorem 5.4 The feedback system of figure 5.1 is $\mathbf{A}_{-}(\sigma)$-stable if, and only if, the characteristic gain loci of GK :

(i) do not go through -1, and

(ii) encircle -1, $P_G + P_K$ times anticlockwise.

Proof (ii) <=> (ii) and (iii), theorem 5.3, straightforwardly, and hence this completes the proof. ∎

Theorems 5.3 and 5.4 provide tests which can be easily used to determine feedback stability and form the basis of the robust stability test to be presented later.

5.4 Inverse Nyquist Stability Criteria

Consider the identity :

$$(I+GK)^{-1}GK + (I+GK)^{-1} = I \ . \qquad (5.10)$$

We will denote the first term on the left (the sensitivity matrix) by S and the second term (the complementary sensitivity matrix) by T. From theorem 5.2, condition (i), a necessary condition for feedback stability is that S is analytic in $C_{\sigma+}$, which is equivalently to the requirement that T is analytic in $C_{\sigma+}$. But T is analytic in $C_{\sigma+}$ if, and only if, it has no characteristic poles and no fixed poles in $C_{\sigma+}$, and hence we have the following theorem.

Theorem 5.5 The feedback system of figure 5.1 will be $A_(\sigma)$-stable if, and only if :

(i) there are no characteristic poles of T in $C_{\sigma+}$,

(ii) there are no fixed poles of T in $C_{\sigma+}$, and

(iii) there are no $C_{\sigma+}$ pole-zero cancellations in GK.

Proof (i) and (ii) <=> (i), theorem 5.2, as argued above.

(iii) same as (iii), theorem 5.2.

(i), (ii) and (iii) <=> theorem 5.2 <=> stability.

This completes the proof. ∎

In the following lemma, we show that there is a one-to-one correspondence between the inverse characteristic gain loci of T and those of GK. The result is analogous to the relationship between the characteristic gain loci of the return difference, I + GK, and those of GK.

Lemma 5.1 The number of anticlockwise encirclements of -1 by the inverse characteristic gain loci of GK is equal to the number of anticlockwise encirclements of the origin by the inverse characteristic gain loci of T.

Proof Let $\lambda_i[.]$ denote the ith eigenvalue of [.]. Then

$$\lambda_i[T] = 1 - \lambda_i[S] = 1 - (1 + \lambda_i[G(s)K(s)])^{-1}, \quad \lambda_i[GK] \neq -1,$$
$$= \lambda_i[G(s)K(s)](1 + \lambda_i[G(s)K(s))^{-1}.$$

Hence

$$\frac{1}{\lambda_i[T(s)]} = 1 + \frac{1}{\lambda_i[G(s)K(s)]}, \quad \lambda_i[G(s)K(s)] \neq 0,$$

and hence the lemma follows naturally. ∎

The following lemma shows that the characteristic zeros of T are identical to those of GK.

Lemma 5.2 The characteristic zeros of T and GK are the same.

Proof Consider the characteristic equation for T, i.e.

$$\det[\lambda(s)I - (I + G(s)K(s))^{-1}G(s)K(s)] = 0. \qquad (5.11)$$

Then, assuming as we must that $\det[(I+G(s)K(s))^{-1}]\neq 0$, solutions of (5.11) with $\lambda\to 0$ are equivalent to solutions of

$$\det[\lambda(s)(I+G(s)K(s)) - G(s)K(s)] = 0, \quad \lambda\to 0 \qquad (5.12)$$

i.e. $$\det[\lambda(s) - (1-\lambda(s))G(s)K(s)] = 0, \quad \lambda\to 0.$$

Now, to investigate the characteristic structure of (5.12), we re-write it as

$$\det[\lambda_1(s) - (1 - \lambda_2(s))G(s)K(s)] , \quad \lambda_1 = \lambda_2 \to 0.$$

i.e. $$\det[\lambda_1(s)I - G(s)K(s)] = 0 , \ \lambda_1(s)\to 0 \qquad (5.13)$$

Hence lemma 5.2. ∎

We can now state and prove an inverse Nyquist stability criterion.

Theorem 5.6 The feedback system of figure 5.1 will be $\mathbf{A}_{-}(\sigma)$- stable if, and only if :

(i) the inverse characteristic gain loci of GK do not go through -1,

(ii) the inverse characteristic gain loci of GK encircle -1, Z times anticlockwise,

(iii) there are no fixed poles of GK in $C_{\sigma+}$, and

(iv) there are no $C_{\sigma+}$ pole-zero cancellations in GK.

Proof Consider the eigenvalue function of T defined by the equation $\det[\lambda I - T(s)] = 0$. Because these functions are meromorphic in $C_{\sigma+}$ we can apply the principle of the argument as in [POS1] or [VAL1] to show that the characteristic gain loci of T will encircle the origin Z times clockwise if, and only if, there are no characteristic poles of T in $C_{\sigma+}$, (condition (i) of theorem 5.5). Recall that Z is the number of the characteristic zeros in $C_{\sigma+}$.

But the number of clockwise encirclements of the origin by the characteristic gain loci of T is equal to the number of anticlockwise

encirclements of the origin by the inverse characteristic gain loci of T. Therefore, theorem 5.6 follows from theorem 5.5, using lemmas 5.1 and 5.2, and the fact that the fixed poles of T are the same as the fixed poles of GK. This completes the proof. ※

We next provide another fairly well-known alternative statement of the inverse Nyquist stability criterion.

Theorem 5.7 The feedback system of figure 5.1 will be $\mathbf{A}_{-}(\sigma)$-stable if, and only if :

(i) the net sum of anticlockwise encirclements of the critical point (-1,j0) by the set of inverse characteristic gain loci of G(s)K(s), minus the net sum of anticlockwise encirclements of the origin by the set of inverse characteristic gain loci of G(s)K(s), is equal to the number of CRHP characteristic poles of G(s)K(s),

(ii) there are no fixed poles of GK in $C_{\sigma+}$, and

(iii) there are no $C_{\sigma+}$ pole-zero cancellation in GK.

Proof Conditions (ii) and (iii) are the conditions (iii) and (iv) of theorem 5.6. So all we have to prove is (i) <=> (i) and (ii) of theorem 5.6.

Let $N(\Omega)$ denote the number of anticlockwise encirclements of the point Ω by the inverse characteristic gain loci of GK. Then applying the principle of the argument to G(s)K(s), we have

$$N(0, j0) = Z - P \qquad (5.14)$$

where P denote the number of characteristic poles of GK in $C_{\sigma+}$. Applying the principle of the argument to T(s), as in theorem 5.6, gives

$$N(-1, j0) = Z - \pi_c \qquad (5.15)$$

where π_c denotes the number of closed-loop characteristic poles of the feedback system. Subtract (5.14) from (5.15), we have

$$N(-1, j0) - N(0, j0) = p - \pi_c \qquad (5.16).$$

Thus condition (i) follows naturally, and this completes the proof. %

It is usually not very convenient to count the number of characteristic poles. Hence it would be more desirable to have an inverse criterion based on $P_G + P_K$, rather than P. Hence we give the following theorem.

Theorem 5.8 The feedback system of figure 5.1 will be $A_(\sigma)$-stable if, and only if, the net sum of anticlockwise encirclements of the critical point (-1, j0) by the set of inverse characteristic gain loci of G(s)K(s), minus the net sum of anticlockwise encirclements of the origin by the set of inverse characteristic gain loci of G(s)K(s), is equal to $P_G + P_K$.

Proof Let

$$P_G + P_K - P = e \qquad (5.17)$$

That is, e accounts for all the "missing poles" due to pole-zero cancellations or fixed poles. Then (6.16) may be re-written as

$$N(-1, j0) - N(0,j0) = P_G + P_K - e - \pi_c \qquad (5.18).$$

Necessity is then obvious because $\pi_c + e$ would otherwise be nonzero which implies $\pi_c \neq 0$ and/or $e \neq 0$.

To prove sufficiency, we note that if

$$N(-1, j0) - N(0, j0) = P_G + P_K$$

then $$e + \pi_c = 0 \qquad (5.19).$$

But both e and π_c are nonnegative integers, we therefore get

$$e = 0 \quad \text{and} \quad \pi_c = 0. \qquad (5.20)$$

This completes the proof. ∎

Theorem 5.8 suggests how we might "count the number of zeros" in order to obtain a simpler statement of theorem 5.6. This is done in definition 5.2

Definition 5.2 The number of $C_{\sigma+}$ root-locus zeros of GK, denoted by Z_{GK}, is defined as

$$Z_{GK} = Z + e \tag{5.21}$$

where $e = P_G + P_K - P$.

Intuitively speaking, definition 5.2 says that if α is a $C_{\sigma+}$ pole of G or K, but not a characteristic pole of GK, then we must consider it as being "cancelled" by some $C_{\sigma+}$ zero of GK. Note that the set of zeros defined by Z_{GK} are the "root-locus" zeros, [SMI1], of GK because they determine the infinite-gain positions of the $C_{\sigma+}$ finite closed-loop poles of GK.

Equipped with this new definition of zeros, we may re-state theorem 5.6 in a simpler way, which also shows the duality it has with the direct Nyquist stability criterion.

Theorem 5.9 The feedback system of figure 5.1 will be $\mathbb{A}_{-}(\sigma)$-stable if, and only if, the net sum of anticlockwise encirclements of critical point (-1, j0) by the set of inverse characteristic gain loci of G(s)K(s) is equal to Z_{GK}, the number of $C_{\sigma+}$ root-locus zeros of GK.

Proof From the principle of the argument, we have

$$N(-1, j0) = Z - \pi_c$$

$$= (Z + e) - (\pi_c + e) \qquad (5.22)$$

But $(\pi_c + e)$ is simply the total number of the closed-loop poles of the feedback system in $C_{\sigma+}$ because any cancelled or fixed poles of GK will show up as closed-loop poles. We therefore have

$$N(-1, j0) = Z_{GK} - P_{CL} \qquad (5.23)$$

where P_{CL} denotes the total number of closed-loop poles. Theorem 5.9 thus follows naturally from (5.23). This completes the proof. ■

To the best of our knowledge, the definition and formulation of the inverse Nyquist stability criteria in this section is new in the literature. To-date, all other existing versions of the multivariable inverse Nyquist criteria define the inverse Nyquist loci as the Nyquist loci of the inverse of the return-ratio transfer function matrix, [POS1], [LOG1], [STE1]. Thus if the plant is rank deficient, then the inverse Nyquist criterion would not be applicable. This is not the case in our formulation. In fact, since the inverse Nyquist loci are simply the reciprocal of the nontrivial Nyquist loci, the existence of a Nyquist stability criterion invariably implies, and is implied by, the existence of a corresponding inverse Nyquist stability criterion. The two stability criteria are therefore equivalent mathematically, although as the number of the $C_{\sigma+}$ characteristic zeros of GK becomes large, the usefulness of the inverse criteria diminishes accordingly. Nevertheless, we will make use of the inverse criteria in the next section where we are interested in maintaining feedback stability under plant perturbations.

5.5 Robust Stability

The purpose of this section is to extend the robust stability results of chapter 4 to include the class of $\hat{B}(0)^{nxm}$ transfer function matrices. For simplicity we take $\sigma=0$, and to avoid unnecessary repetitions, we re-state only the combined direct/inverse Nyquist based singular value test as a robust stability theorem for $\hat{B}(0)^{nxm}$. Note that unlike in the previous chapter, we do not assume the plant to be square or nonsingular in this section.

5.5-1 Nyquist based tests

Since the direct Nyquist stability criterion is essentially the same when extended to the class of $\hat{B}(0)^{mxm}$ transfer function matrices, the Nyquist based robust stability tests are also essentially identical. The non-squareness of the plant will have little effect except that if $G(s)K(s)$ is singular then we must avoid using $\underline{\sigma}[I + (G(s)K(s))^{-1}]$, replacing it by

$$\{\bar{\sigma}[(I+G(s)K(s))^{-1}G(s)K(s)]\}^{-1}$$

where appropriate.

5.5-2 Inverse Nyquist based tests

The effects of nonsquare plants on the inverse Nyquist based robust stability tests are slightly more involved than they are on the direct tests. First, in the representations of uncertainty, any inverses must be replaced by pseudo inverses, for example $G(s)$ becomes $G(s)^{+}$, where $G(s)^{+}G(s) = I$. Then Z_{GK}, the number of $C_{\sigma+}$ root-locus zeros of GK, rather than Z_G, the number of transmission zeros of G, has to be assumed constant. (Note that Z_G may be a non-generic number, i.e. two non-square

transfer function matrices can be arbitrarily close topologically and yet have different numbers of transmission zeros).

Consider then the inverse Nyquist loci of $G_p(s)K(s)$, or $K(s)G_p(s)$. They will not pass through the critical point (-1, j0) if, and only if, the Nyquist loci $G_p(s)K(s)$ (equivalently $K(s)G_p(s)$) do not pass through the critical point. This is again equivalent to the Nyquist locus of $\det[I + G_p(s)K(s)]$ not passing through the origin; that is

$$\det[I + G_p(s)K(s)] \neq 0, \quad \text{all } s \in D_I \tag{5.24}$$

and conditions for (5.24) are given by lemmas 5.3 and 5.4.

Lemma 5.3 Assume

$$G_p(s) = (I + \tilde{\Delta}_o(s))^{-1}G(s) = L_o(s)G(s)$$

where $(I + \tilde{\Delta}_o(s))^{-1} = L_o(s)$; that is $L_o(s)$ is assumed to be invertible. Then

$$\det[I + G_p(s)K(s)] \neq 0, \quad \text{all } s \in D_I$$

if $\underline{\sigma}[I + G(s)K(s)] > \bar{\sigma}[L_o(s)^{-1} - I] \triangleq \bar{\sigma}[\tilde{\Delta}_o(s)]$, all $s \in D_I$.

Proof Following [LEH1], we write

$$I + G_p(s)K(s) = I + L_o(s)G(s)K(s)$$

$$= L_o(I+GK)[I+(I+GK)^{-1}(L_o^{-1} - I)]$$

Now $I + G(s)K(s)$ is invertible in the closed right half plane because the nominal feedback system is stable, and $L_o(s)$ is invertible on D_I by assumption. Therefore $I + G_p(s)K(s)$ is invertible on all $s \in D_I$ if

$$\bar{\sigma}[I + G(s)K(s)] > \bar{\sigma}[\tilde{\Delta}_o(s)], \quad \text{all } s \in D_I \tag{5.15}$$

which is ensured by

$$\underline{\sigma}[I + G(s)K(s)] > \bar{\sigma}[\tilde{\Delta}_o(s)], \quad \text{all } s \in D_I.$$

This completes the proof. ∎

Lemma 5.4 Assume $G(s)$ has an inverse $G(s)^+$, and

$$G_p(s)^+ = G)s)^+ + \tilde{E}(s).$$

Then,

$$\det[I + G_p(s)K(s)] \neq 0, \text{ all } s \in D_I$$

if

$$\bar{\sigma}[(I + G(s)K(s))^{-1}G(s)] < \frac{1}{\bar{\sigma}[\tilde{E}(s)]}, \quad \text{all } s \in D_I.$$

Proof

$$\tilde{E}(s) = G(s)^+\tilde{\Delta}_o(s),$$

or equivalently

$$G(s)\tilde{E}(s) = \tilde{\Delta}_o(s) = L(s)^{-1} - I \qquad (5.16).$$

Substitution of (5.16) into 5.15) completes the proof. ∎

It is therefore seen that the inverse Nyquist based tests of section 4.3 are directly extendable to non-square plants. The extension to $\hat{B}(0)^{nxm}$ systems is obvious.

5.5-3 Nyquist and inverse Nyquist combined.

Because the class of rational functions is a subclass of the $\hat{B}(0)$ functions, and because addition and multiplication of $\hat{B}(0)$ functions in the CRHP are similar to addition and multiplication of rational functions, we can certainly expand the topology of section 2.3 to include $\hat{B}(0)$ functions. (For more information, the reader is referred to [VID1]). Hence, all the robust stability tests based on arcwise connectivity are straightforwardly extendable to include the class of $\hat{B}(0)$ functions. For illustration, we state theorem 5.7 which is a $\hat{B}(0)$ version of the combined direct/inverse Nyquist based singular value robust stability test with output multiplicative perturbations.

Theorem 5.7 Let $\{G_p(s)\}$ be an arcwise connected set of transfer function matrices in the graph topology of (possibly unstable) plants in $\mathbf{B}(0)^{nxm}$, and $G_p(\infty) = 0$ for all $G_p \in \{G_p\}$.

Assume that for every member $G_p(s) \in \{G_p(s)\}$, there exists $\Delta_o(j\omega)$ and $\tilde{\Delta}_o(j\omega)$ such that

$$G_p(j\omega) = [I + \Delta_o(j\omega)]G(j\omega) = [I + \tilde{\Delta}_o(j\omega)]^{-1}G(j\omega)$$

for all ω, where $G(s)$ is a member of $\{G_p(s)\}$, known as the nominal plant model, such that the feedback system of figure 5.1 is $\mathbf{A}_-(0)$-stable.

Define

$$\delta_o(\omega) = \sup_{G_p \in \{G_p\}} \{\bar{\sigma}[\Delta_o(j\omega)]\} ,$$

$$\tilde{\delta}_o(\omega) = \sup_{G_p \in \{G_p\}} \{\bar{\sigma}[\tilde{\Delta}_o(j\omega)]\} .$$

Then under these conditions the perturbed feedback system is $\mathbf{A}_-(0)$-stable for any member $G_p(s)$ from $\{G_p(s)\}$ if at each $\omega \in [0,\infty]$, either

(i) $\bar{\sigma}[T(j\omega)] < \dfrac{1}{\delta_o(\omega)}$, or

(ii) $\bar{\sigma}[S(j\omega)] < \dfrac{1}{\tilde{\delta}_o(\omega)}$.

Proof Similar to the case for lumped systems. ※

Note that it is not necessary to use the semi-circular part because we have assumed

$$\lim_{|s|\to\infty \text{ in } C_{0+}} \{G_p(s)\} = \{0\} \qquad (5.17).$$

There is also no need to test for unstable fixed poles or pole-zero cancellations because if there is a $G_p(s)$ in $\{G_p(s)\}$ for which these occur then there will always be another model arbitrarily close to $G_p(s)$ for which they do not. And as the frequency responses of these neighbouring models will be almost identical the switching criterion will (by necessity) fail.

5.6 Concluding Remarks

In this chapter, we have extended the robust stability results of chapter 4 to include the class of $\mathbf{B}(0)$ transfer function matrices. To achieve this aim, we have used the stability results of Desoer et. al. [DES4], [CHE3], and a new generalized inverse Nyquist stability criterion which is applicable to rank deficient plant models.

The $\mathbf{A}_{-}(0)$ algebra is a convolution algebra consisting of the Laplace transforms of impulse responses that have an L_1-stable part, together with an L_1 sequence of delayed impulses. Because of the well-known inequality

$$\|w \cdot u\|_p \leq \|w\|_1 \|u\|_p , \quad p \in [1,\infty] \qquad (5.18).$$

This class of systems is nice in the sense that a system which is $\mathbf{A}_{-}(0)$ stable is also L_p-stable for all $p \in [1,\infty]$. In general, however, the Nyquist stability criterion, and hence the inverse Nyquist stability

criterion, is applicable to a wider class of transfer function matrices. For example, if we restricted our signals to be in L_2, then it is possible to prove the Nyquist criterion (and hence the inverse Nyquist criterion) for transfer function matrices with elements in the quotient field of $\hat{M}_-(L_2)[\hat{M}_-^{\infty}(L_2)]^{-1}$ where $M_-(L_2)$ denotes the subspace of all causal convolution operators: $L_2 \rightarrow L_2$, whose impulse responses admit Laplace transforms which are analytic in the CRHP, and $\hat{M}_-^{\infty}(L_2)$ is the subset of $\hat{M}_-(L_2)$ transfer functions which are bounded away from zero in the CRHP as $|s| \rightarrow \infty$. The norm of this class of convolution operators may be defined as

$$||G|| \triangleq \sup_{u \in L_2} \frac{||Gu||_2}{||u||_2} = \sup_{s=j\omega} \bar{\sigma}[G(s)]$$

which is identical to the case of lumped systems. As in the case of the $B(0)$ algebra, one may define an extended topology on this class of systems. The robust stability results presented in this and the previous chapters will then be applicable to systems with transfer function matrices in this new class.

CHAPTER VI

ROBUST STABILITY ANALYSIS WITH STRUCTURED UNCERTAINTIES

6.1 Introduction

In the last two chapters, we have presented tests for determining the robust stability of a feedback system when subjected to a perturbation characterized by a bound on the maximum singular value of its frequency response. While this method of representing uncertainty turns out to be convenient in many situations, in other circumstances more may be known about the nature and structure of the perturbations. For example, the extent of uncertainties in actuators, sensors and some of the plant parameters can often be estimated especially at low frequencies. Uncertainties may also be known to occur at different points in the loop. In such cases, a straightforward application of the robust stability tests discussed earlier may be quite conservative. In this chapter, we therefore consider systems with structured uncertainties.

It is well-known that robust stability tests applied at one point in the feedback loop at a time may be unsuitable if the perturbations are known to occur simultaneously at different locations in the loop, [DOY5], [DOY6], [FRE1]. To cope with this problem, Doyle et. al. have developed the small-μ test (or the structured singular value test), [DOY5], [DOY6], which provides necessary and sufficient conditions for the robust stability of an equivalent system with a block diagonal perturbation. Unfortunately in their test, the perturbations are required to be stable, [DOY6]. It has been shown in chapter 3 that such a requirement does not rule out the application of the test to situations in which an individual perturbation may be unstable because it is almost

always possible to decompose the actual perturbation into two fictitious yet stable perturbations. But, even so, as is also shown in chapter 3, such a decomposition (especially an "optimal" one) may be difficult to find and is likely to introduce more conservatism into the analysis.

Section 6.2 of this chapter sets out to apply the concept of arcwise connectivity to develop robust stability tests which will overcome the above-mentioned problem. The results may be viewed as the combination and extension of the tests in [DOY5] and the combined direct/inverse Nyquist based test reported in chapter 4. The main results of the section are theorems 6.1 to 6.4. Theorem 6.1 is a direct extension of the small-μ theorem given in [DOY6] but without the requirement that the perturbations representing uncertainty are stable. As discussed in chapter 3, we believe such a relaxation to be highly desirable because it eliminates the conservatism induced by having to decompose an unstable perturbation into stable perturbations, and allows for a more natural application of the test. Theorem 6.2 is the small-μ version of the combined direct/inverse Nyquist based singular-value test. It thus accommodates a more refined definition of the perturbation set, and hence may be less conservative than the test of theorem 6.1. The representation of uncertainty is further refined in theorems 6.3 and 6.4 by means of singular-value-bounded perturbations centred around several different nominal models. The corresponding robust stability tests can be looked upon as the least conservative amongst those characterized by singular-value-bounds.

Section 6.3 examines the problem of element-wise magnitude-bounded perturbations. Similar problems have been studied by Safonov, [SAF4], Owens and Chotai, [OWE3], and Yeh et. al., [YEH1]. However, our results are mainly inspired by the work of Kouvaritakis and Latchman, [KOU2],

[KOU3], [KOU4], where necessary and sufficient conditions for robust stability are obtained using non-similarity scaling matrices. Our results represent an alternative approach for deriving these necessary and sufficient conditions and suggest an alternative way of computing the associated optimal non-similarity scaling matrices. We also show that the optimal non-similarity scaling matrices are in fact two "worst case" optimal similarity scaling matrices.

In section 6.4, we briefly examine the problem of parameter-dependent uncertainty. We acknowledge that our efforts in this direction are still at a preliminary stage and so should be looked upon as an outline for possible future research, rather than a description of matured research.

In sections 6.3 and 6.4 we restrict our attention to the case of additive perturbations. The extension to multiplicative perturbations is straightforward and should be clear to the reader. We stress that all the tests reported in this chapter can be brought together to serve as part of our general switching robust stability criterion (section 4.7). For ease of exposition, all transfer functions are assumed to be rational.

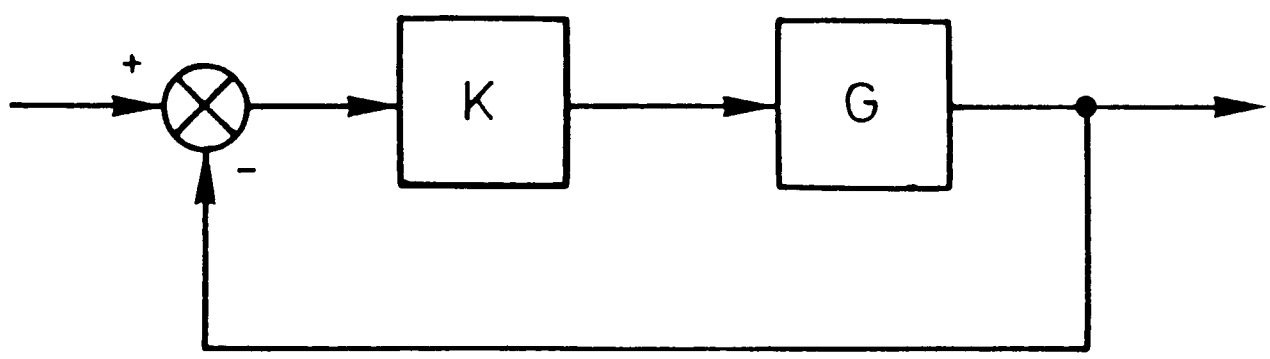


Figure 6.1 Feedback configuration

6.2 Multiple Simultaneous Perturbations

6.2-1 Equivalence in stability and a relationship between Nyquist loci

The usefulness of the small-μ test rests upon the fact that perturbations occurring at different points in the loop can be collected and represented as a single block-diagonal perturbation occurring at one location in a large feedback loop. That is, if G_p in figure 6.1 is given by

$$G_p = (I + \Delta_o)G(I + \Delta_i) \qquad (6.1)$$

where G is the nominal plant model then we may represent the feedback system by the "equivalent" system shown in figure 6.2, where the block-structured matrix

$$M = \begin{bmatrix} (I+KG)^{-1}KG & (I+KG)^{-1}K \\ -(I+GK)^{-1}G & (I+GK)^{-1}GK \end{bmatrix} \qquad (6.2)$$

represent all interactions of the feedback loop with all its uncertainties, and

$$\Delta = \begin{bmatrix} \Delta_i & 0 \\ 0 & \Delta_o \end{bmatrix} \qquad (6.3).$$

Note that the feedback system of figure 6.2 is equivalent to that of figure 6.1 in the sense of bounded-input bounded-output stability if, and only if, Δ is stable. To illustrate this point let

$$G(s) = \frac{(s-1)}{(s+10)(s+20)} \qquad (6.4)$$

$$K(s) = \frac{1}{100} \tag{6.5}$$

and

$$G_p(s) = \frac{s-1+\epsilon}{(s+10)(s+20)} = \frac{s-1}{(s+10)(s+20)} \cdot \left(1+\frac{\epsilon}{s-1}\right) \tag{6.6}$$

that is, we assume $\Delta_i(s) = \epsilon/(s-1)$ and $\Delta_o(s) = 0$. Then it is intuitively clear that, since the nominal feedback system is stable, the feedback system will remain stable when G is replaced by G_p if ϵ is sufficiently small. However, for this example $M = (I + KG)^{-1}KG$ has a non-minimum-phase zero at s=1, and Δ has a pole at the same location. It follows that the feedback system of figure 6.2 cannot be stable.

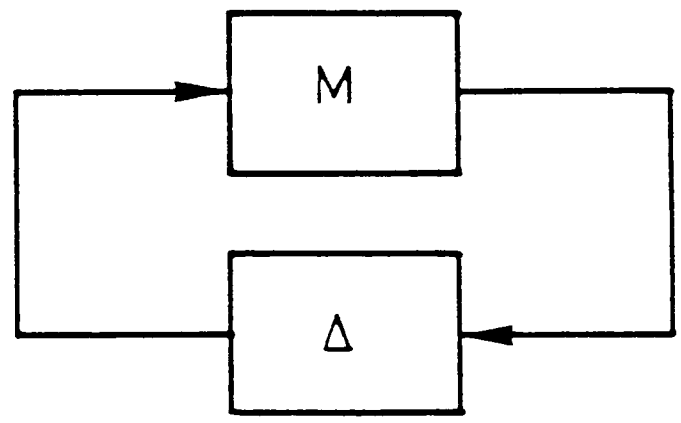


Figure 6.2 "Equivalent" perturbed feedback system

The above discussion has shown that if we are to remove the requirement that Δ be stable, then figures 6.1 and 6.2 cannot be regarded as equivalent for stability analysis. For this reason we now establish a new relationship between them.

Proposition 6.1 Assume that the nominal feedback system of figure 6.1 is stable. Then the feedback system of figure 6.2 will have a closed-loop pole on the $j\omega$-axis if the perturbed feedback system of figure 6.1 has a closed-loop pole on the $j\omega$-axis.

Proof Obvious, since the block-structured matrix M is simply an alternative representation of the transfer function relationships between the various point in the loop, [DES5]. ∎

In general, the feedback system of figure 6.2 will have a closed-loop pole on the jω-axis if, and only if, either of the following conditions is satisfied:

(i) The Nyquist diagram (determinant or eigenloci) of MΔ passes through the critical point.

(ii) There is a jω-axis pole-zero cancellation in MΔ.

A jω-axis pole-zero cancellation would occur in MΔ if GK has an uncertain pole or zero on the jω-axis, as suggested by the example above, equations (6.4)-(6.6). Such a cancellation, however, is inevitable and does not imply that the perturbed feedback system of figure 6.1 has a closed-loop pole on the jω-axis. Another possibility is that the perturbed system of figure 6.1 has a pole-zero cancellation on the jω-axis. But in either case, it is impossible to detect the presence of such closed-loop poles by examining the Nyquist loci. We can therefore state the following corollary to proposition 6.1.

Corollary 6.1 If the Nyquist locus of the feedback system of figure 6.2 does not pass through the critical point, then neither does the Nyquist locus of the perturbed feedback system of figure 6.1.

In conclusion then, the perturbed feedback system will remain stable (regardless of the stability of Δ) if the Nyquist locus of the system in figure 6.2 does not pass through the critical point.

6.2-2 The small-μ theorem with one nominal plant model

A formal definition of the structured singular value, μ^{-1}, and a list of its properties are given in [DOY5] and [DOY6]. For our purpose here, it suffices to know that, [DOY5]:

$$\mu(M) = \begin{cases} 0 \quad \text{if no } \Delta \epsilon \chi_{\infty} \text{ solves } \det(I+M\Delta)=0 \\ \left(\min_{\Delta \epsilon \chi_{\infty}} \{ \bar{\sigma}[\Delta] \mid \det(I+M\Delta)=0 \} \right)^{-1} \end{cases} \qquad (6.6)$$

where χ_{∞} is the set of block-diagonal matrices with no restriction on the matrix norm.

From the definition of $\mu(.)$, it is clear that given a block-diagonal perturbation matrix Δ with $\bar{\sigma}[\Delta(j\omega)] < \delta(\omega)$, the Nyquist locus of the feedback system in figure 6.2 will not pass through the critical point if

$$\frac{1}{\mu(M)} > \delta(\omega) \; ; \quad \text{all } \omega \qquad (6.7)$$

Note that in their report, [DOY6], Doyle et. al. recommended that scaling be done because unscaled singular value bounds provided, in general, poor estimates for μ. (Related references on the computations of μ are [DOY5], [SAF4], [SAF6] and [KOU4]). However, for simplicity and ease of exposition, we shall not use scaling here.

In chapters 2 and 4, it has been shown that when the plant of figure 6.1 is being perturbed from G to G_p along a path, the number of encirclements of the Nyquist locus can change only in one of two ways : either

(i) by passing through infinity; or

(ii) by passing through the critical point.

Situation (i) happens when, and only when, there is a pole moving across the $j\omega$-axis, and the change is always such that stability is maintained. Hence, for robust stability, one needs only prevent the Nyquist locus from passing through the critical point.

Theorem 6.1 Let $\{G_p\}$ be an arcwise connected set of strictly proper transfer function matrices. Also, for every member $G_p(s)$ from $\{G_p(s)\}$, let there exist $\Delta_o(j\omega)$ and $\Delta_i(j\omega)$ such that

$$G_p(j\omega) = (I + \Delta_o(j\omega))\, G(j\omega)\, (I + \Delta_i(j\omega)) ,$$

where $\bar{\sigma}[\Delta_o(j\omega)] < \delta_o(\omega)$ and $\bar{\sigma}[\Delta_i(j\omega)] < \delta_i(j\omega)$; and where $G(s)$ is also a member of $\{G_p\}$, known as the nominal model, such that the closed-loop system of figure 6.1 is stable. Then the perturbed closed-loop system of figure 6.1 with $G(s)$ replaced by any $G_p \in \{G_p\}$ is stable if :

$$\text{(i)} \quad \mu(M\Delta) < 1 , \quad \text{all } \omega , \text{ all } \Delta \tag{6.8}$$

or

$$\text{(ii)} \quad \mu(M) < \frac{1}{\bar{\sigma}[\Delta]} , \text{ all } \omega , \text{ all } \Delta \tag{6.9}$$

or

$$\text{(iii)} \quad \mu(M) < \frac{1}{\max(\delta_o, \delta_i)} \tag{6.10}$$

where

$$M = \begin{bmatrix} (I+KG)^{-1}KG & (I+KG)^{-1}K \\ -(I+GK)^{-1}G & (I+GK)^{-1}GK \end{bmatrix} \tag{6.11}$$

and

$$\Delta = \begin{bmatrix} \Delta_i & 0 \\ 0 & \Delta_o \end{bmatrix} \tag{6.12}$$

Proof Any of the inequalities above implies that $\det[I+M\Delta] \neq 0$. This, in turn, implies that the Nyquist locus of the feedback system of figure 6.1

will not pass through the critical point for all $G_p \in \{G_p\}$. Now, since $\{G_p\}$ is an arcwise connected set, applying theorem 4.17 completes the proof. ∎

The reader should be aware that in theorem 6.1, we do not make any assumption as to the number of poles or zeros of the plant and perturbed plant models. Neither do we assume the perturbations to be stable. However, if G has an uncertain pole which may move across the jω-axis, then either $\bar{\sigma}[\Delta_o(j\omega)]$ or $\bar{\sigma}[\Delta_i(j\omega)]$ will be unbounded at that crossing frequency. Thus, the robust stability test of theorem 6.1 cannot possibly be satisfied for all frequencies. In such situations, it is necessary to introduce an alternative representation of uncertainty, as was done in chapter 4. If we assume the pole crossing is due to an output perturbation, then the following theorem is applicable.

Theorem 6.2 Let $\{G_p(s)\}$ be an arcwise connected set of strictly proper transfer function matrices. Also for every member $G_p(s)$ from $\{G_p(s)\}$, let there exist $\Delta_o(j\omega)$, $\bar{\Delta}_o(j\omega)$ and $\Delta_i(j\omega)$, $\bar{\sigma}[\Delta_o(j\omega)] < \delta_o(\omega)$, $\bar{\sigma}[\bar{\Delta}_o(j\omega)] < \bar{\delta}_o(\omega)$ and $\bar{\sigma}[\Delta_i(j\omega)] < \delta_i(\omega)$ such that $G_p(j\omega) = [I+\Delta_o(j\omega)]G(j\omega)[I+\Delta_i(j\omega)] = [I+\bar{\Delta}_o(j\omega)]^{-1}G(j\omega)[I+\Delta_i(j\omega)]$, where the nominal model G(s) is also a member from $\{G_p(s)\}$ such that the closed-loop system of figure 6.1 is stable. Then the perturbed closed-loop system of figure 6.1 with G(s) replaced by any $G_p(s) \in \{G_p(s)\}$ is stable if, at each $\omega \in [0,\infty]$, either of the following inequalities is satisfied :

$$\text{(i)} \quad \mu[M_1\Delta_1] < 1 \quad , \text{ all } \Delta_1 \tag{6.13}$$

or

$$\text{(ii)} \quad \mu[M_2\Delta_2] < 1 \quad , \text{ all } \Delta_2 \tag{6.14}$$

where

$$M_1 = \begin{bmatrix} (I+KG)^{-1}KG & (I+KG)^{-1}K \\ -(I+GK)^{-1}G & (I+GK)^{-1}GK \end{bmatrix} \qquad (6.15)$$

$$\Delta_1 = \begin{bmatrix} \Delta_i & 0 \\ 0 & \Delta_o \end{bmatrix} \qquad (6.16)$$

$$M_2 = \begin{bmatrix} (I+KG)^{-1}KG & (I+KG)^{-1}K \\ (I+GK)^{-1}G & (I+GK)^{-1} \end{bmatrix} \qquad (6.17)$$

and

$$\Delta_2 = \begin{bmatrix} \Delta_o & 0 \\ 0 & \tilde{\Delta}_o \end{bmatrix} \qquad (6.18)$$

Proof Similar to theorem 6.1. ∎

We note that in theorem 6.2, Δ_o and $\tilde{\Delta}_o$ are simply different representations of the same perturbation. Of course, we may represent the input perturbation by two different representations too, and in such a situation we would have four inequalities, only one of which needs be satisfied at each frequency for stability. Hence, robust stability tests similar to theorem 6.2 may be used to reduce conservatism even when there is no possibility of poles or zeros crossing the $j\omega$-axis. Indeed, if we have three perturbations in the loop (e.g. input-multiplicative, output-multiplicative, and additive), we would have a maximum of eight inequalities to work with.

The robust stability tests of this section will handle perturbations in plant and controller with equal ease, since the multiplication of two arcwise connected sets, assuming no unstable pole-zero cancellations, is an arcwise connected set. In other words, if both $\{G_p(s)\}$ and $\{K_p(s)\}$ are arcwise connected sets, then the collection of all possible

multiplications of members from $\{G_p(s)\}$ and $\{K_p(s)\}$: $\{G_p(s)K_p(s)\} \triangleq \{G_p(s)K_p(s) : G_p \in \{G_p\}, K_p \in \{K_p\}\}$ forms an arcwise connected set if there are no unstable pole-zero cancellations in all products $G_p(s)K_p(s)$.

Nevertheless, it is not necessary to assume that there are no CRHP pole-zero cancellations in all G_pK_p's in order to apply the robust stability test. To illustrate that this is the case, we note that if $\{G_cK_c\}$ denotes the set of all possible members of $\{G_pK_p\}$ with CRHP pole-zero cancellations, then $\{G_pK_p\} \ominus \{G_cK_c\}$ is an arcwise connected set, provided $\{G_p\}$ and $\{K_p\}$ are arcwise connected sets. Thus, if we assume there is an unstable pole-zero cancellation in G_cK_c, where G_c is a member of $\{G_p\}$ and K_c a member of $\{K_p\}$, then, in the small neighbourhood of G_c, we may finds another member of $\{G_p\}$, say G_d, such that the frequency response of G_d is almost identical to that of G_c and such that there is no CRHP pole-zero cancellation in G_dK_c. It follows that the Nyquist loci of G_dK_c and G_cK_c must be almost identical. Since the closed-loop system is unstable under G_cK_c, and hence G_dK_c because we are working in a topology in which feedback stabilization is a generic property, the Nyquist loci of G_cK_c and G_dK_c must have the wrong number of encirclements. It follows that the robust stability test cannot be satisfied for the arcwise connected set $\{G_pK_p\} \ominus \{G_cK_c\}$ where G_dK_c is a member. Thus, any instability caused by an unstable pole-zero cancellation in G_pK_p must be detectable by the robust stability test, subject to the condition that both $\{G_p\}$ and $\{K_p\}$ are arcwise connected sets.

6.2-3 The small-μ theorem with several nominal plant models

In theorem 6.2, a more refined definition of the uncertainty was achieved by using the intersection of two or more singular-value-bounded perturbations. However, some unnecessary conservatism may still be present due to the fact that only one nominal model was used throughout the whole frequency range $[0,\infty]$. More specifically, let $\{G_p\}$ be an arbitrary arcwise connected set of transfer functions, then in defining Δ_o, E etc., we are intuitively "drawing" balls centred on G each of which encloses the whole set $\{G_p\}$. For a given perturbation, one may therefore regard the best choice of the nominal model as the one which gives the smallest value of $\bar{\sigma}[.]$ for all frequencies. Unfortunately, due to the structure of the perturbations, a nominal model which minimizes $\bar{\sigma}[\Delta_o(s)]$ will not normally minimize $\bar{\sigma}[\tilde{\Delta}_o(s)]$. Hence, in choosing the best nominal model, one is forced to make a trade-off between different representations of uncertainty.

Such a difficulty may be removed, or minimized, if we are allowed to use different nominal models for the different representations of uncertainty. For example, let G_1 be the best choice of nominal model when $\{\Delta_o\}$ is used to represent uncertainty and G_2 be the least conservative selection when $\{\tilde{\Delta}_o\}$ is used. Then we may conclude that $\{G_p\}$ is a subset of the intersection of $\{G_1, \Delta_o\}$ and $\{G_2, \tilde{\Delta}_o\}$, assuming $\{G_1, \Delta_o\}$ and $\{G_2, \tilde{\Delta}_o\}$ are both adequate representations of the set $\{G_p\}$. Because G_1 and G_2 are the best possible choice for the particular representation, it follows that the intersection of $\{G_1,\Delta_o\}$ and $\{G_2,\tilde{\Delta}_o\}$ is a subset of the intersection of $\{G, \Delta_o\}$ and $\{G, \tilde{\Delta}_o\}$, because in the worst case we may choose $G_1 = G_2 = G$. Thus a robust stability test which allows for the use of different nominal models at different frequencies will be less conservative than the robust stability test of theorem 6.2.

In the remainder of this section, we will use U to represent the set of singular-value-bounded perturbations. For example, if we use the simultaneous input-multiplicative, output-multiplicative and additive perturbations as a representation of uncertainty, then we have $U_1 = \{\Delta_i, \Delta_o, E\}$, $U_2 = \{\Delta_i, \tilde{\Delta}_o, E\}$, $U_3 = \{\tilde{\Delta}_i, \tilde{\Delta}_o, E\}$ etc, with the corresponding block-structured matrices M_1, M_2, M_3 etc. We will say $\{M_j, U_j\}$ is an adequate representation of $\{G_p\}$ if, and only if, the set $\{G_j, U_j\}$ is an adequate representation of $\{G_p\}$. Note that G_j does not necessarily equal G_K for $j \neq k$.

Theorem 6.3 Let $\{G_p\}$ be an arcwise connected set of strictly proper transfer function matrices. Also let there exist combinations $\{G_j, U_j\}$, $j = 1,\ldots,n$, which are adequate representations of the set $\{G_p\}$ and assume that the closed-loop system is stable for all of the nominal models, G_j, which are also members of $\{G_p\}$. Then the perturbed closed-loop system is stable with G replaced by any $G_p \in \{G_p\}$, if at each $\omega \in [0,\infty]$, at least one of the following inequalities is satisfied :

$$\mu[M_j \Delta_j] < 1 \ , \quad j = 1,\ldots,n \ , \ \text{all } \Delta_j \qquad (6.19)$$

where Δ_j is the block-diagonal representation of U_j, and M_j the corresponding block-structured matrix.

Proof Similar to theorem 6.1 ∎

We may even relax the conditions further by not requiring the G_j to be members of $\{G_p\}$, or topologically close to each other. The frequency response criterion is sufficient to establish robust stability as is illustrated by the following theorem :

Theorem 6.4 Let $\{G_p(s)\}$ be an arcwise connected set of transfer function matrices. Also let there exist $\{G_m(j\omega), U_m(j\omega)\}$, $m = 1,\ldots,n$, which are adequate representations of the frequency responses of $\{G_p(s)\}$ in the sense that the frequency response of $\{G_m(j\omega), U_m(j\omega)\}$ bounds the frequency response of each member of $\{G_p(s)\}$. Let $\det[I+G_m(j\omega)K(j\omega)]\neq 0$, all m, all ω, and assume there exists a $G(s) \in \{G_p(s)\}$, known as the nominal model, such that the nominal closed-loop system is stable. Under these conditions, the perturbed closed-loop system is stable with $G(s)$ replaced by any $G_p(s) \in \{G_p(s)\}$, if for each $\omega \in [0,\infty]$, at least one of the following inequalities is satisfied :

$$\mu(M_m \Delta_m) < 1, \quad m = 1,\ldots,n \text{ , all } \Delta_m \qquad (6.20)$$

where Δ_m is the block-diagonal representation of U_m, and M_m the corresponding block-structured matrix.

Proof Similar to theorem 6.1. ∎

Note that in theorem 6.4, we do not require the $G_m(s)$ to be members of $\{G_p(s)\}$. This enables us to use the best frequency-response approximation without any restriction on the pole-zero structure of $G_m(s)$. For example, let $\{G_p(s)\} = \{1/(s-a) : 1\leq a\leq 2\}$, then the frequency response approximation which gives the smallest additive error is given by $G_m(s) = \frac{1}{2}\{\frac{1}{s-1} + \frac{1}{s-2}\}$ which does not have the same numbers of CRHP poles and CRHP zeros as $\{G_p(s)\}$. Hence theorem 6.4 allows one the freedom to use a best nominal model in the frequency response sense. This is desirable because the magnitudes of the singular-value-bounded perturbations are dependent on the frequency response of the nominal plant, rather than the model itself.

However, from a design point of view, theorems 6.2 and 6.3 are potentially more useful because they suggest a way of posing the robust

design problem as an optimization problem : that is, they suggest minimizing a cost function of the form

$$C = \max_i \left(\mu[W_i(\omega)M_i(K(s))] \right) \qquad (6.21)$$

where the W_i's denote frequency-dependent weighting functions, and the $M_i(K(s))$'s are appropriate block-structured matrices with, perhaps, a matrix corresponding to a fictitious perturbation for performance requirements, [SAF7]. Note that equation (6.21) is closely related to the design scheme proposed in [SAF7].

6.3 Perturbations with Magnitude-Bounded Elements

In this section we examine the problem where the perturbation class is characterized by an upper bound on the absolute value of each of the elements of the perturbation matrix. (Different elements may have different upper bounds). The results of this section are mainly inspired by the original work of Kouvaritakis and Latchman, [KOU2], [KOU3] and [KOU4]. However, our approach is slightly different from theirs where non-similarity scaling matrices are obtained by explicitly differentiating the maximum singular values of certain matrices. Instead we show that the pair of non-similarity scaling matrices are simply two similarity scaling matrices. Our result thus provides an alternative derivatiion of their result and establishes a relationship between optimal non-similarity scaling and similarity scaling matrices.

Let $[|\Delta|]$ denote the matrix whose elements are equal to the magnitude of the corresponding elements of Δ. We will write $[|\Delta|] \leq P$, P

a real matrix, if and only if the magnitude of each element of $[|\Delta|]$ is less than or equal to the corresponding elements of P. Thus, we may characterize our class of perturbed plant models by

$$\{G_p\}_+ \triangleq \{G_p | G_p(s)=G(s)+E(s),\ [|E(j\omega)|]<P\} \qquad (6.22)$$

Let $M(j\omega) = K(j\omega)(I+G(j\omega)K(j\omega))^{-1}$, we then obtain the following robust stability theorem, [KOU2].

Theorem 6.5 Let L and R be two real positive diagonal matrices, then there exists no $E(j\omega_o)$ such that $[|E(j\omega_o)|]\leq P$ and $\det[I+E(j\omega_o)M(j\omega_o)]=0$ if

$$\bar{\sigma}[LP(\omega_o)R]\bar{\sigma}[R^{-1}M(j\omega_o)L^{-1}] < 1 \qquad (6.23)$$

Proof As a consequence of a property of nonnegative matrices, we have, [HUN2] :

$$\bar{\sigma}[E(j\omega)] \leq \bar{\sigma}[|E(j\omega)|] \leq \bar{\sigma}[P(\omega)] \qquad (6.24)$$

Now if L and R are two positive diagonal real matrices, then

$$\bar{\sigma}[LE(j\omega)R] \leq \bar{\sigma}[|LE(j\omega)R|] \leq \bar{\sigma}[LP(\omega)R] \qquad (6.25)$$

It thus follows that $\det[I+E(s)K(s)(I+G(s)K(s))^{-1}]\neq 0$ at $s=j\omega_o$ if there exists L and R such that

$$\bar{\sigma}[LP(\omega_o)R]\bar{\sigma}[R^{-1}M(j\omega_o)L^{-1}] < 1 \qquad (6.26)$$

where

$$M(j\omega) = K(j\omega)(I+G(j\omega)K(j\omega))^{-1} \qquad (6.27). \quad \text{※}$$

The optimal choices of L and R are thus those which minimize the left-hand-side of equation (6.26), [KOU2].

Lemma 6.1 [KOU3]. The spectral radius of any matrix $A \in C^{nxn}$ is equal to the maximum singular value of A, if and only if, the left and right principal singular vectors (i.e. singular vectors associated with the largest singular value) of A are aligned (i.e. one is equal to the other multiplied by an arbitrary constant $e^{j\theta}$).

Proof See [Kou3], lemma 4.1. ▨

In the rest of this section, μ denotes the structured singular value, [DOY5], with respect to diagonal perturbations. We can then state the following lemma which may already be apparent to the reader from [DOY5] or [KOU4].

Lemma 6.2 For any complex matrix $A \in C^{nxn}$, $\bar{\sigma}[A] = \mu[A]$ (for diagonal perturbations) if, and only if, A is characterized by the equalization of the moduli of the corresponding elements of left and right singular vectors associated with the maximum singular value of A.

Proof With no loss of generality, we assume that $\bar{\sigma}[A] = 1$. The extension of the proof to cases where $\bar{\sigma}[A] \neq 1$ is straightforward. Now if A is characterized by the equalization of the moduli of the corresponding elements of the left and right singular vectors associated with the maximum singular value of A, then there exists $S = \mathrm{diag}[e^{j\theta_1}, \ldots, e^{j\theta_n}]$ such that the left and the right principal singular vectors of AS are aligned. Thus $\rho[AS] = 1$ and we have $\mu[A] \geq 1$. But $\mu[A] \leq \bar{\sigma}[A]$.

Thus $\bar{\sigma}[A] = 1 \Rightarrow \mu[A] = 1$. Conversely, if A is characterized by the non-equalization of the moduli of the corresponding elements of the left and right singular vectors associated with the maximum singular value of A, then there exists <u>no</u> $S = \text{diag}[e^{j\theta_i}]$ such that the left and right principal vectors are aligned. Hence $\rho[A\Delta] < 1$ for all diagonal Δ where $\bar{\sigma}[\Delta] \leq 1$. Thus $\mu[A] < 1$. But $\bar{\sigma}[A]=1$ by assumption and we have $\bar{\sigma}[A] \neq \mu[A]$. ∎

The generalization of lemma 6.2 to the block-diagonal cases is straightforward but outside the scope of this section. The reader is referred to [DOY5] and [KOU4] for more detailed discussions.

A consequence of lemmas 6.1 and 6.2 is the following two lemmas which establish the existence of a de-stabilizing perturbation.

<u>Lemma 6.3</u> If $\inf_{L} \bar{\sigma}[LP(\omega_o)M(j\omega_o)L^{-1}] = \mu[M] \geq 1$, then there exists $E(j\omega_o)$ such that $[|E(j\omega_o)|] \leq P(\omega_o)$ and $\det[I+E(j\omega_o)K(j\omega_o)(I+G(j\omega_o)K(j\omega_o))^{-1}] = 0$.

<u>Proof</u> From lemma 6.2, there exists

$$S = \text{diag}[e^{j\theta_1}, \cdots, e^{j\theta_n}]$$

such that the left and right principal singular vectors of $SLP(\omega_o)M(j\omega_o)L^{-1}$ are aligned. It follows that the spectral radius of $SP(\omega_o)M(j\omega_o)$ is greater than or equal to 1. Thus there exists $\gamma \in (0,1]$ and $\theta \in (-\pi, \pi]$ such that at least one of the eigenvalues of $\gamma e^{j\theta} SP(\omega_o)M(j\omega_o)$ is equal to -1. Let $E(j\omega_o) = \gamma e^{j\theta} SP(\omega_o)M(j\omega_o)$; this completes the proof. ∎

<u>Lemma 6.4</u> If $\inf_{R} \bar{\sigma}[R^{-1}M(j\omega_o)P(\omega_o)R] \geq 1$, then there exists $E(j\omega_o)$ such that $[|E(j\omega_o)|] \leq P(\omega_o)$ and $\det[I+E(j\omega_o)K(j\omega_o)(I+G(j\omega_o)K(j\omega_o))^{-1}] = 0$.

Proof Similar to lemma 6.3. ∎

Lemmas 6.3 and 6.4 give necessary conditions for the existence of scaling matrices L and R such that (6.25) may be satisfied. However, they are in general not sufficient. The gap between the magnitudes of $\bar{\sigma}[LP(\omega_o)R]\bar{\sigma}[R^{-1}M(j\omega_o)L^{-1}]$ and $\max(\bar{\sigma}[LP(\omega_o)M(j\omega_o)L^{-1}],\ \bar{\sigma}[R^{-1}M(j\omega_o)P(\omega_o)R])$ may therefore be conveniently viewed as a measure of conservatism in failing to obtain scaling to give necessary and sufficient conditions for robust stability. In an attempt to narrow this gap, we may introduce two diagonal matrices

$$S_1 = \mathrm{diag}[e^{j\theta_1}, \cdots, e^{j\theta_n}] \tag{6.28}$$

and

$$S_2 = \mathrm{diag}[e^{j\phi_1}, \cdots, e^{j\phi_m}] \tag{6.29}$$

and consider the following two optimization problems:

$$\inf_{L} \sup_{S_1,S_2} \bar{\sigma}[LS_1P(\omega_o)S_2M(j\omega_o)L^{-1}] \tag{6.30}$$

and

$$\inf_{R} \sup_{S_1,S_2} \bar{\sigma}[R^{-1}M(j\omega_o)S_1P(\omega_o)S_2R] \tag{6.31}$$

respectively. Obviously, if any of the above two inequalities is greater than or equal to 1, then robust stability cannot be guaranteed.

We next turn our attention to the solution of the optimization (6.30) and (6.31).

Proposition 6.2 Let R be the solution to the optimization problem

$$\inf_{R} \bar{\sigma}[R^{-1}M(j\omega_o)S_1P(\omega_o)S_2R] \tag{6.32}$$

and let S_2' be such that

$$\bar{\sigma}[R^{-1}M(j\omega_o)S_1P(\omega_o)S_2R] = \rho[M(j\omega_o)S_1P(\omega_o)S_2'] \triangleq r_1 \tag{6.33}$$

where $\rho(.)$ denotes the spectral radius of $(.)$. Then

$$\inf_{L} \bar{\sigma}[LS_1P(\omega_o)S_2'M(j\omega_o)L^{-1}] \geq r_1 \tag{6.34}$$

Proof Obvious since $\bar{\sigma}[.] \geq \rho[.]$ in general. ∎

Proposition 6.2 suggests a non-descending direction for searching for S_2 which solves (6.30) and (6.31). It is clear that we may search for S_1 by considering the optimization problem

$$\inf_{L} \bar{\sigma}[LP(\omega_o)S_2M(j\omega_o)L^{-1}] \tag{6.35}$$

and then find S_1 such that

$$\bar{\sigma}[LS_1P(\omega_o)S_2M(j\omega_o)L^{-1}] = \rho[LS_1P(\omega_o)S_2M(j\omega_o)L^{-1}].$$

Thus by alternately solving (6.32) and (6.35), we are moving in a non-descending direction. The next few propositions state that we are not only moving in a non-descending direction, but indeed in an ascending direction towards the optimal L and R.

Proposition 6.3 In proposition 6.2, (6.34) holds with equality only if

$$\bar{\sigma}[LP(\omega_o)R]\bar{\sigma}[R^{-1}M(j\omega_o)L^{-1}] = r_1 \qquad (6.36)$$

Proof Assume (6.34) holds with equality, then we have

$$r_1 \triangleq \bar{\sigma}[LS_1PS_2'ML^{-1}] = \bar{\sigma}[R^{-1}MS_1PS_2'R] \qquad \text{(by assumption)}$$

$$= \rho[LS_1PS_2'ML^{-1}] \qquad \text{(from (6.33))}$$

$$= \bar{\sigma}[(LS_1PS_2'R)(R^{-1}ML^{-1})]$$

$$= \bar{\sigma}[(R^{-1}ML^{-1})(LS_1PS_2'R)] \qquad (6.37)$$

Let $A = R^{-1}ML^{-1}$, $B = LS_1PS_2'R$. Then (6.37) is equivalent to

$$\bar{\sigma}(AB) = \bar{\sigma}(BA) = \rho(AB) \qquad (6.38)$$

But it is well-known that

$$\bar{\sigma}(A)\bar{\sigma}(B) = \rho(AB) \qquad (6.39)$$

$$\langle\!\rightarrow \quad \bar{\sigma}(A)\bar{\sigma}(B) = \bar{\sigma}(AB) = \bar{\sigma}(BA) \qquad (6.40).$$

Substituting (6.39) into (6.40) gives

$$\bar{\sigma}(A)\bar{\sigma}(B) = \rho(AB)$$

$$\langle\!\rightarrow \quad \rho(AB) = \bar{\sigma}(AB) = \bar{\sigma}(BA)\ . \qquad (6.41)$$

Combining (6.41) and (6.38) then implies that

$$\bar{\sigma}[LS_1PS_2'R]\ \bar{\sigma}[R^{-1}ML^{-1}] = r_1 \qquad (6.42)$$

But S_1 and S_2' are diagonal, and therefore they commute with L and R respectively. They are also unitary and therefore removing them from (6.42) completes the proof. ▨

Thus, it is seen that our algorithm does indeed generate a sequence of suboptimal L's and R's which converge to the solution of (6.42). The next proposition states that the sequence actually leads to the optimal solution.

Theorem 6.6 If there exists L, R, S_1, and S_2 such that $\bar{\sigma}[LPR]\bar{\sigma}[R^{-1}ML^{-1}] = \bar{\sigma}[LS_1PS_2ML^{-1}] = \bar{\sigma}[R^{-1}MS_1PS_2R] = \mu[PS_2M] = \mu[MS_1P] \triangleq r$. Then there exists no $E(j\omega_o)$ such that $[|E(j\omega_o)|] \leq P(\omega_o)$ and $\det[I+E(j\omega_o)M(j\omega_o)]=0$ if, and only if :

$$r < 1 \qquad (6.43)$$

Proof Necessity is obvious because

$$\bar{\sigma}[LS_1PS_2ML^{-1}] < 1$$

and

$$\bar{\sigma}[R^{-1}MS_1PS_2R] < 1$$

are necessary conditions for robust stability. Sufficiency follows from the fact that

$$\bar{\sigma}[LPR]\,\bar{\sigma}[R^{-1}ML^{-1}] < 1$$

is a sufficient condition for robust stability. ▨

Finally, in passing, we note that the usefulness of lemma 6.2 needs not necessarily be restricted to the use of similarity scalings. For example, suppose that we find $\inf_L \bar{\sigma}[LML^{-1}] \neq \mu[M]$ (by lemma 6.2) and suppose that we can find a real diagonal (or, more generally an arbitrary complex) matrix D such that $\inf_L \bar{\sigma}[LMDL^{-1}] = \mu[MD]$ (again, by lemma 6.2), then we have $\mu[M] \leq \mu[MD]\bar{\sigma}[D^{-1}]$ and $\mu[M] \geq \mu[MD]\underline{\sigma}[D^{-1}]$. The problem of

how one can best find such a D may be a subject of interest for future research.

6.4 Parameter-dependent Uncertainty

In the previous section, uncertainty is characterized by a perturbation matrix whose elements are each bounded by some real quantity. A consequence of this type of characterization is that the phase information is either lost, or not used. For example, let

$$G_p(s) = \frac{1}{s+a} \; ; \qquad a \in (0,1]. \tag{6.44}$$

Then it is clear that $G_p(j0)$ is real for all a. Therefore modelling $E(j0)$ as an arbitrary magnitude-bounded perturbation would inevitably introduce some conservatism. One approach which overcomes this problem is to consider parameter-dependent uncertainty.

Let

$$\{G_p(s)\}_{\lambda,E} = \{G_p(s,\lambda)+E(s) \mid \lambda \in \mathbf{A},\ \bar{\sigma}[E(s)] \leq e(s),\ \text{all } s \in D_D\} \tag{6.45}$$

where λ is a vector of parameters, and $\mathbf{A}$ a set of all real vectors whose components each lie in some given compact interval. It should then be clear to the reader that the following robust stability result holds :

Theorem 6.7 The feedback system of figure 6.1 will remain stable when the nominal plant model is replaced by any perturbed plant model taken from the set $\{G_p(s)\}_{\lambda,E}$ if :

(i) $G_p(s)$ and $G(s)$ share the same number of poles in D_D, and

$$\text{(ii)} \quad \sup_{\lambda \in \mathbf{A}} \bar{\sigma}[K(s)(I + G(s,\lambda)K(s))^{-1}] < \frac{1}{e(s)} \ , \ \text{all } s \in D_D.$$

Proof Obvious and omitted. ∎

Condition (ii) if theorem 6.7 is equivalent to :

$$\sup_{\lambda \in \mathbf{A}} \left(\sup_{\omega} \bar{\sigma}[e(s)K(s)(I + G(s,\lambda)K(s))^{-1}] \right) < 1. \qquad (6.45)$$

It is therefore seen that the problem of robust stability analysis with respect to both structured and unstructured perturbations may be solved by finding the optimum (maximum) value of the left hand side of (6.45).

Unfortunately, it seems that the optimization problem with respect to (6.45) is extremely complex and indeed non-convex, [POL2]. Thus, except in the single-input, single-output case where a majorization procedure, [POL2], may be used to derive a sufficient condition, the above problem remains unsolved at present.

One possible approach, albeit heuristic, is to derive a fairly good, but not necessarily sufficient, estimate of robust stability by choosing a finite number of frequency points and solving the above optimization at each of the chosen frequencies. If the inequality is satisfied with a relatively good margin at each frequency, then we may be fairly confident that the feedback system is robustly stable. Note that since uncertainties tend to become unstructured at high frequencies, we need only choose frequencies in the low frequency range. Thus, provided that the frequency range involved is not too wide, a sufficiently large number of sampled frequencies can be chosen to give a good picture of the robustness of the stability property in the given frequency range.

We remark that, in general, applying a gradient ascent type method and beginning with the nominal $\bar{\sigma}[e(s)K(s)(I+G(s)K(s))^{-1}]$ at $s=j\omega_o$ will

not necessarily lead to a global maxima. It therefore seems necessary to apply the ascent method to each of the singular values. But when two or more singular values are equal, a mechanism to distinguish one from another has to be devised. In general, equalization of the singular values of a rational function matrix along a line is a non-generic property; that is, singular values are generically equal only at isolated points, [MAC8]. It therefore appears that we may index the singular values by their analytic branches, [MAC8]. More specifically, let $\bar{\sigma}_i(.)$ denote the ith analytic branch of the singular values, then we may try to find

$$r(\omega_o) = \max_i \left[\sup_{\lambda \in \mathbf{A}} \bar{\sigma}_i [e(j\omega_o)K(j\omega_o)(I+G(j\omega_o,\lambda)K(j\omega_o))^{-1}] \right] \tag{6.46}$$

The robust stability test is then considered satisfied at $s=j\omega_o$ if $r(\omega_o)<1$. Of course, more research has to be done before we can find out whether this approach is feasible, and whether the $r(\omega_o)$ so obtained is indeed the global maximum (at $s=j\omega_o$) for

$$\sup_{\lambda \in \mathbf{A}} \bar{\sigma}[e(j\omega_o)K(j\omega_o)(I+G(j\omega_o,\lambda)K(j\omega_o))^{-1}] \tag{6.47}.$$

6.5 Concluding Remarks

In this chapter, we have given several extensions of the small-μ theorem, none of which requires the perturbations representing uncertainty to be stable. We believe the inclusion in the small-μ test of possibly unstable perturbations greatly enhances its usefulness by allowing a more natural application of the test. For example, one may identify $(\Delta_o, \tilde{\Delta}_o)$ as uncertainty in the sensors, $(\Delta_i, \tilde{\Delta}_i)$ as uncertainty

in the actuators, and $(E, \tilde{E})$ as uncertainty in the plant's parameters, regardless of whether the perturbations are stable or not.

By allowing switching between a direct and an inverse Nyquist based test a more natural framework for coping with changing numbers of right half-plane poles and zeros is obtained than that proposed in [DOY6]. The ability to switch between different nominal models as well as between different kinds of tests is aimed at reducing conservatism.

The chapter also considers the problem of robust stability analysis in the presence of element-wise magnitude-bounded perturbations. An alternative derivation of the results on optimal non-similarity scaling matrices is also suggested. Our approach establishes the relationship between the optimal similarity and non-similarity scaling matrices.

The problem of parameter-dependent perturbations is considered in section 6.4. We suggest a heuristic approach to the approximate solution of the problem. However, more research has to be done before we can comment on the feasibility of the approach.

CHAPTER VII

ROBUST STABILITY WITH NONLINEAR PERTURBATIONS

7.1 Introduction

It can be argued that all real systems are essentially distributed and nonlinear in nature. A linear time-invariant lumped system model is therefore at best an approximation of a physical system. For this reason, it is important that all realistic analysis and synthesis technique are extendable to at least some classes of distributed and nonlinear systems. In chapter 5, we have shown that the robust stability results developed in this thesis are indeed directly extendable to a certain class of distributed systems. The purpose of this chapter is to consider the effects of nonlinear perturbations.

We will be interested in the robust stability of linear time-invariant systems when subjected to nonlinear and possibly time-varying perturbations. The singular value methods are well-suited to this since there is a wealth of stability analysis techniques for nonlinear systems in the setting of normed spaces, [DES1], [WIL3], [ZAM1], [SAN1], [SAF5] and [MEE1]. For example, Lehtomaki, [LEH2], considers robust stability analysis of linear systems subjected to diagonal, memoryless and stable nonlinear perturbations, and Daniel and Kouvaritakis, [DAN2], examine the robust stability of open-loop stable linear feedback systems when subjected to sector bounded, static and stable nonlinear perturbations. In all these papers, the basic tool is the circle criterion of Zames, [ZAM1], and its multivariable generalizations, [MEE1], [SAF5]. Because the circle criterion only admits a nonlinearity which is stable, it is apparent why all these techniques require the perturbations to be stable.

In practice, many plants are open-loop unstable, and therefore it is reasonable to expect that any imprecision in the model will result in a growing error signal between the true plant output and the model output. In such situations the perturbation representing uncertainty will be unstable. It is therefore desirable that some sort of robust stability results be derived for this type of uncertainty.

The chapter is organized as follows. In section 7.2, the underlying mathematics is outlined, and some basic definitions and theorems presented. In section 7.3, robustness results are derived for testing stability when only a single representation of uncertainty is used. In section 7.4, a robust stability test is derived in terms of stable perturbations of the factors in a right-coprime factorization of the plant. In section 7.5, the combined direct/inverse Nyquist based singular value robust stability test of chapter 4 is extended to include nonlinear perturbations, and we conclude this chapter in section 7.6.

7.2 Preliminaries

We will assume that all signals are zero for $t<0$ and adopt the general formulation of Zames, [ZAM1].

L_2^m: An m-tuple function $x(t)$ defined in the interval $t\epsilon[0,\infty]$ belongs to L_2^m if :

$$\int_0^\infty x(t)^T x(t)\, dt < \infty$$

The norm of $x(t)$ in L_2^m is denoted by

$$\|x\|_2 = \left(\int_0^\infty x(t)^T x(t)\, dt \right)^{\frac{1}{2}} \qquad (7.1)$$

where x^T denotes the transpose of x.

L_{2e}^m: An extension of L_2^m square integrable functions. A function x(t) belongs to L_{2e}^m if every finite time truncation of x(t) belongs to L_2^m. An extended norm denoted by $||x||_{2e}$ is assigned to each x(t) in L_{2e}^m as follows :

$$||x||_{2e} = ||x||_2 \qquad \text{if } x \in L_2^m$$

$$||x||_{2e} = \infty \qquad \text{if } x \notin L_2^m \qquad (7.2)$$

Because there is little chance of confusion, the superscipt m in L_{2e}^m and the subsript 2e in $||\cdot||_{2e}$ will be dropped from now on.

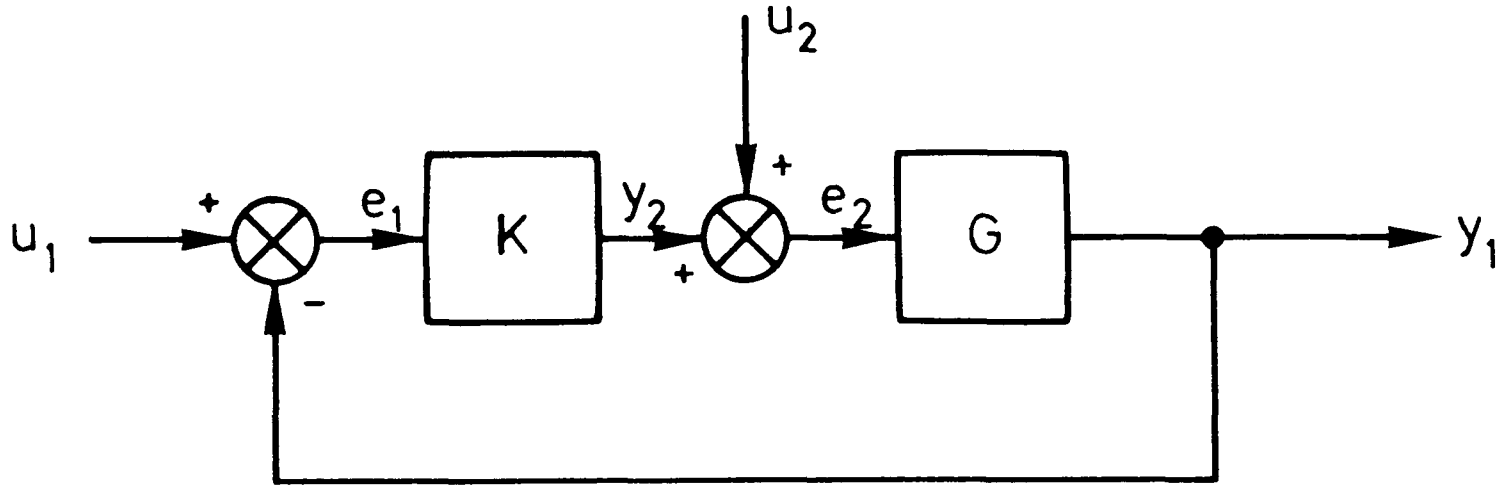


Figure 7.1 Feedback configuration

Definition 7.1 The feedback system of figure 7.1 is said to be stable if

(i) it is well-posed, [WIL3], and

(ii) all inputs $u_i \in L_2$ lead to errors and outputs e_i, y_i, i=1,2, in L_2.

A mathematical model is well-posed if it can adequately represent a physical system; all models and operators will therefore be assumed to be well-posed throughout this chapter.

It is well-known that even in the purely linear time-invariant case, the Nyquist and the inverse Nyquist criteria, and hence the singular value robust stability tests, are applicable to only a certain class of systems, that is systems whose transfer functions are meromorphic in the CRHP. Let B_{2e} denote the subclass of L_{2e} functions whose Laplace transforms are meromorphic in the CRHP; B_2 is similarly defined. We will assume in this chapter that all operators, linear or nonlinear, are defined on $B_{2e} \rightarrow B_{2e}$.

For simplicity, we will take the origin to be the equilibrium point, and assume that $\Sigma \cdot 0 = 0$ for all operators Σ. Adopting the view-point of [ZAM2], the words "operator", and "system" will be used interchangeably. Thus the possibility of any internally unstable states that are inaccessible from the input but which may affect the output, or that cannot be detected from the output, is not admitted.

Definition 7.2 An operator Σ is said to be rational, if it has a representation $\Sigma = ND^{-1}$, where N and D are stable, causal operators and D^{-1} is causal.

The class of rational operators is a fairly general one in practical situations. For example, if we assume K to be an arbitrary controller such that the feedback system of figure 7.1 is stable, then it follows that both $(I+KG)^{-1}$ and $G(I+KG)^{-1}$ are stable. Thus, if we let

$$N = G(I+KG)^{-1} \tag{7.3}$$

$$D = (I+KG)^{-1} \tag{7.4}$$

then we may write

$$G = ND^{-1} \tag{7.5}$$

We therefore see that (roughly speaking) any stabilizable operator may be written as a rational operator, [HAM1].

Theorem 7.1 [VID2]. For the operator $\Sigma : B_{2e} \to B_{2e}$, define

$$D(\Sigma) = \{x \in B_2 : \Sigma x \in B_2\}$$

$$R(\Sigma) = \{x \in B_2 : x = \Sigma y \text{ for some } y \in B_2\}.$$

Suppose there exists a one-to-one mapping $D : B_{2e} \to B_{2e}$ such that $R(D) = D(\Sigma)$, and define $N : B_{2e} \to B_{2e}$ by

$$Nx = \Sigma \cdot Dx$$

Then the system $e = u - Fy$, $y = \Sigma e$, where the operator $F : B_{2e} \to B_{2e}$ corresponds to a constant matrix, is B_2- stable if, and only if, the restricted map $(D + FN) : B_2 \to B_2$ has an inverse $(D+FN)^{-1} : B_2 \to B_2$.

Proof See [VID2]. ∎

Corollary 7.1 The feedback system described in theorem 7.1 (with rational $\Sigma = ND^{-1}$) is stable if, and only if, the restricted map $(I+F\Sigma) : D(\Sigma) \to B_2$ has an inverse $(I+F\Sigma)^{-1} : B_2 \to D(\Sigma)$.

Proof Because the restricted map : $B_2 \to D(\Sigma)$ is one-to-one and onto, it follows that for $x \in D(\Sigma)$, we have

$$\Sigma x = ND^{-1}x$$

and therefore $(I+F\Sigma) = (I+FND^{-1}) = (D+FN)D^{-1}$

To prove =>, assume $(D+FN) : B_2 \to B_2$ is invertible in B_2, then $(D+FN)D^{-1} : D(\Sigma) \to B_2$ has an inverse from $B_2 \to D(\Sigma)$ because $D(D+FN)^{-1} : B_2 \to R(D)$ and

$$(D+FN)D^{-1}D(D+FN)^{-1} = I : B_2 \to B_2.$$

Thus $(I+F\Sigma)^{-1} = D(D+FN)^{-1} : B_2 \to D(\Sigma)$.

To prove <=, assume $(I+F\Sigma)^{-1}: B_2 \to D(\Sigma)$ exists, then $D^{-1}(I+F\Sigma)^{-1}: B_2 \to B_2$ is an invertible map because

$$(I+F\Sigma)DD^{-1}(I+F\Sigma)^{-1} = I : B_2 \to B_2.$$

It is then not difficult to show that

$$D^{-1}(I+F\Sigma)^{-1} = (D+FN)^{-1}.$$

This completes the proof. ∎

Although the expression $(I+F\Sigma)^{-1}$ determines the stability of the feedback system explicitly, in general it is difficult to evaluate. Exceptional cases are when Σ is linear and time-invariant, or when Σ is nonlinear but scalar (or diagonal) and memoryless in which case inversion is merely a reflection of the graph of $I+F\Sigma$ about the 45° line. In the general case, it is sometimes possible to establish the existence of $(I+F\Sigma)^{-1}: L_2 \to D(\Sigma)$ without explicitly evaluating it. A simple example is when F and Σ satisfy the incremental small gain condition, [DES1].

We will now define a conic sector and describe briefly how it can be used to represent plant uncertainty.

<u>Definition 7.3</u> Let $\phi : R^n \to R^n$ with $\phi(0) = 0$. We say that $\phi \in$ sector$[k_1,k_2]$ if, and only if, $k_1e^2 \leq e\phi(e) \leq k_2e^2$, or equivalently

$$\langle \phi(e) - k_1e \mid \phi(e) - k_2e \rangle \leq 0, \quad \text{all } e \in B_{2e}$$

or

$$\|\phi(e) - c(e)\|^2 \leq r^2\|e\|^2, \quad \text{all } e \in B_{2e}$$

with

$$c = (k_1 + k_2)/2$$

$$r = (k_1 - k_2)/2$$

and $\langle .\mid . \rangle$ denotes the usual inner product.

Thus, when we model an operator $G_p = (I+\Delta)G$, where G is a nominal plant model and Δ a stable perturbation, we are implicitly assuming that G_p lies within a certain ~~conic~~ sector with "centre" G. (See [SAP5] for a detailed discussion of the subject). How about $G_p = (I+\tilde{\Delta})^{-1}G$?

Let $(I+\Delta) \triangleq (I+\tilde{\Delta})^{-1}$. Then $(I+\tilde{\Delta})$ bounded within the conic sector $[\tilde{c}_1, \tilde{c}_2]$ means :

$$< u - \tilde{c}_1 y \mid u - \tilde{c}_2 y > \leq 0, \; u = (I+\tilde{\Delta})y, \text{ all } y \in B_{2e}$$

$$\Leftrightarrow \quad < \tilde{c}_1 y - u \mid \tilde{c}_2 y - u > \leq 0$$

$$\Leftrightarrow \quad \tilde{c}_1\tilde{c}_2 < y - \frac{u}{\tilde{c}_1} \mid y - \frac{u}{\tilde{c}_2} > \leq 0 \qquad (7.6)$$

Two cases are possible :

Case 1 $\tilde{c}_1 > \tilde{c}_2 > 0$.

Then

$$(2.7) \Leftrightarrow < y - \frac{u}{\tilde{c}_1} \mid y - \frac{u}{\tilde{c}_2} > \leq 0.$$

i.e. $(I+\Delta)$ is bounded inside the conic sector $[1/\tilde{c}_1, 1/\tilde{c}_2]$.

Case 2 $\tilde{c}_1 > 0 > \tilde{c}_2$.

Then

$$(2.7) \Leftrightarrow < y - \frac{u}{\tilde{c}_1} \mid y - \frac{u}{\tilde{c}_2} > \geq 0.$$

i.e. $(I+\Delta)$ is bounded outside the conic sector $[1/\tilde{c}_1, 1/\tilde{c}_2]$.

We see that if case 2 happens, then conceptually the direct and inverse perturbations combine to give what one may describe as a "hollow" cone. In this case it certainly represents a finer definition of the nonlinearity. Even if case 1 occurs, it may still be possible to obtain a more refined definition of the nonlinearity if it is multivariable and dynamical as in the case of linear time-invariant systems. We note that the representation of the nonlinearity can often be further refined by

scaling , [SAF5]. However, for ease of exposition, we will not consider scaling here. We will also assume all operators to be square and invertible on $B_{2e} \rightarrow B_{2e}$.

7.3 Robust Stability Analysis with a Single Representation of Nonlinear Uncertainty

In practice, one can often represent a plant in the form of

$$G_p = G_n G_l \qquad (7.7)$$

where G_n is a nonlinear but stable operator, and the operator G_l is linear, time-invariant but possibly unstable. In these cases, an alternative to theorem 2.1 and its corollary on the stability of a nonlinear feedback system can be established using a loop-transformation and the small gain theorem.

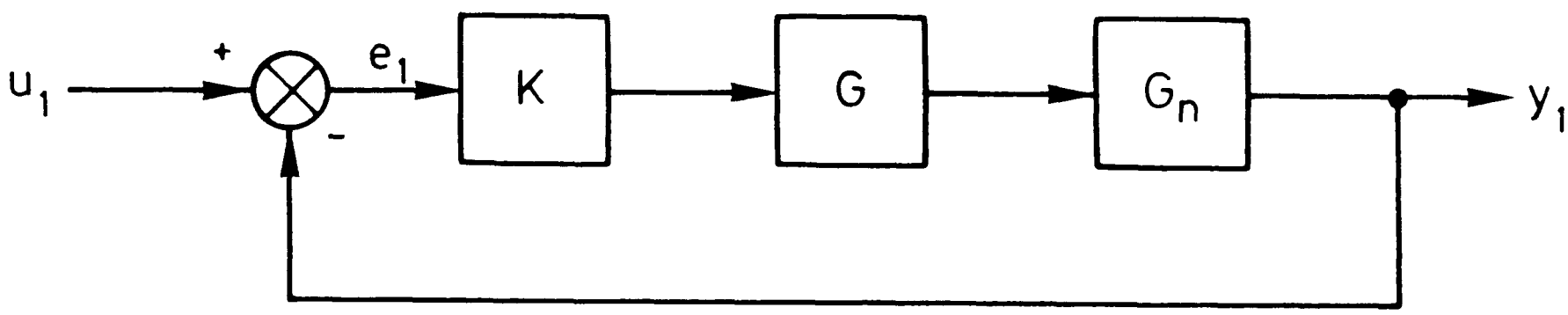


Figure 7.2 Perturbed feedback configuration

Theorem 7.2 The feedback system shown in figure 7.2 where G is a convolution operator and $G_n(.) \triangleq (I+\Delta)$ a stable nonlinear operator is stable if

a) it is stable with $G_n = I$, and

b) for all vectors x

$$||\Delta x|| < \alpha ||x||$$

where $\frac{1}{\alpha} \triangleq \sup_{\omega} \bar{\sigma}[(I+GK)^{-1}GK(j\omega)]$.

Proof Using a similar transformation technique to that in [DOY5] and section 6.2, we obtain the "equivalent" feedback system of figures 7.3 and 7.4 where $M = (I+GK)^{-1}GK$.

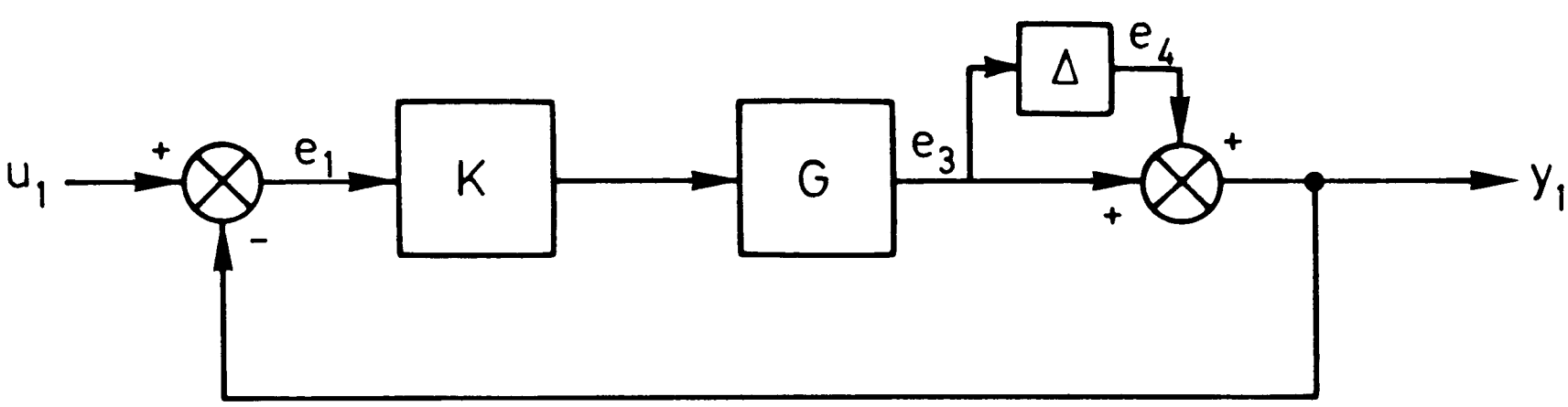


Figure 7.3 Representation of perturbed system

Condition (a) implies that M is stable. Therefore if condition (b) is also satisfied, the small gain theorem will be satisfied for figure 7.4. It thus follows that $e_3, e_4 \in B_2$. But $y_1 = e_3 + e_4$. Therefore $y_1 \in B_2$, and hence $e_1 \in B_2$ if $u_1 \in B_2$. Now if K is stable, then $y_2, e_2 \in B_2$ follows straightforwardly if $u_2 \in B_2$. If K is unstable then two cases can happen:

Case 1 $Ke_1 = y_2 \in B_2$. Then it follows that $e_2 = y_2 + u_2$ belongs to B_2 if $u_2 \in B_2$. Thus we are done.

Case 2 $Ke_1 = y_2$ is not a member of B_2. Then $e_2 = y_2 + u_2$ is not a member of B_2 if $u_2 \in B_2$. Now since $e_1(s)$ is analytic in the CRHP, the only

unstable poles of $y_2(s)$, and hence $e_2(s)$ are the unstable poles of K(s). But the nominal feedback system with Δ=0 is stable. It therefore follows that there are no unstable pole-zero cancellations in G(s)K(s). Thus any unstable mode of y_2 due to the unstable poles of K(s) must also show up at the output of G; that is, $e_3(s)$ has to contain these unstable poles as well. But e_3 is stable by conditions (a) and (b). Thus this leads to a contradiction and we conclude that case 2 cannot occur.

This completes the proof. ∎

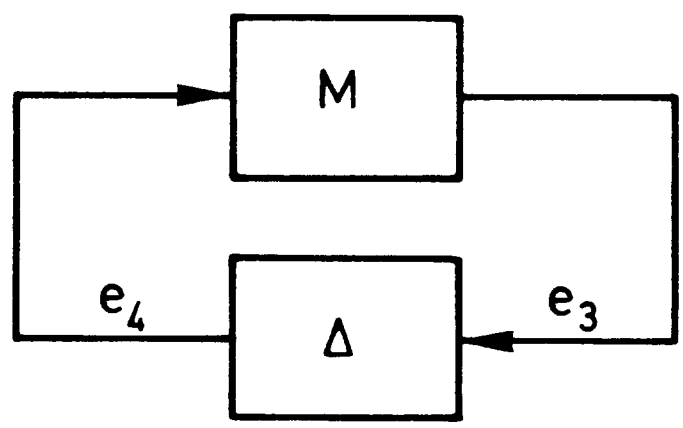


Figure 7.4 Transformed representaion

We note that theorem 7.2 is simply a more general statement of a robust stability 'result reported in [SAN2] where K is assumed to be I, the identity operator.

The next theorem may be viewed as a generalization of the robust stability result reported in [OWE4] where K was assumed to be square, invertible and minimum-phase. [OWE4] actually considered an additive type perturbation, but a similar approach could easily be applied to the case of a multiplicative perturbation.

Theorem 7.3 The closed-loop system of figure 7.2 is stable with $G_n = (I+\tilde{\Delta})^{-1}$ if :

a) it is stable with $G_n = I$, and

b) for all vectors x

$$||\tilde{\Delta}x|| < \tilde{\alpha}||x||$$

where $\frac{1}{\tilde{\alpha}} \triangleq \sup_{\omega} \bar{\sigma}[(I + G(j\omega)K(j\omega))^{-1}] \geq 1$

Proof

Re-draw figure 7.2 as figure 7.5, and then figure 7.4 with $M = (I+GK)^{-1}$ and Δ replaced by $\tilde{\Delta}$. Using similar arguments as those employed in theorem 7.2 completes the proof. ∎

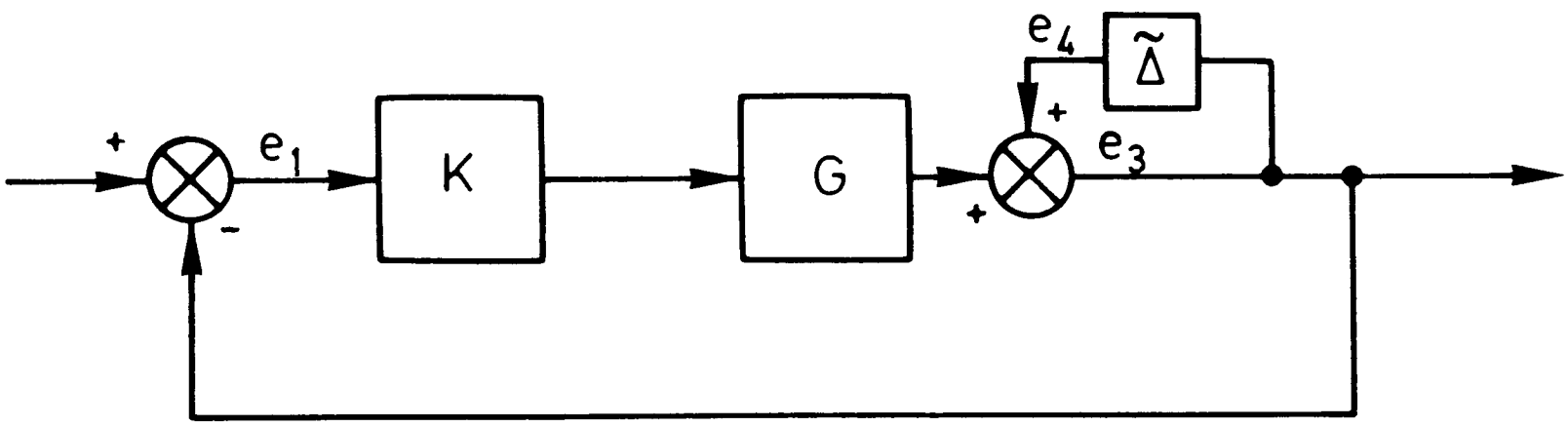


Figure 7.5 Perturbed feedback system

For more generality, one may represent a nonlinear plant in the form of

$$G_p \triangleq G_n \times \{G_l\} \qquad (7.8)$$

where the notation is used to imply that for every $u \in B_{2e}$, there exists $G_l \in \{G_l\}$ such that

$$G_p u = G_n \cdot G_l u \qquad (7.9)$$

Clearly, the closed-loop system under G_p will be stable if it is stable under all product sets $G_n G_1$, for all $G_1 \in \{G_1\}$.

Now, if we write $G_n = (I+\Delta)$ and $G_1 = (I+\Delta_1)G$, then

$$G_p = (I+\Delta)(I+\Delta_1)G \triangleq (I+\Delta_o)G \triangleq (I+\tilde{\Delta}_o)^{-1}G$$

where Δ is nonlinear and stable; Δ_1 is linear, time-invariant but possibly unstable; and Δ_o and $\tilde{\Delta}_o$ are nonlinear and possibly unstable. For the purpose of robustness analysis, it would be most desirable if robust stability tests could be derived in terms of Δ_o and $\tilde{\Delta}_o$. Unfortunately, there seems no simple means of doing this at present.

<u>Theorem 7.4</u> Let $\{G_n\} = \{I + \Delta\}$ where

$$\sup_x \|\Delta x\| / \|x\| \leq \delta_n$$

Assume

$$\{G_p\} \triangleq \{G_n\} \times \{G_1\}$$

where $\{G_1(s)\}$ is a set of strictly proper rational transfer function matrices satisfying

$$G_1(s) = (I + \tilde{\Delta}_1(s))^{-1}G(s), \quad G(s) \in \{G_1(s)\}$$

and

$$\bar{\sigma}[\tilde{\Delta}_1(s)] \leq \tilde{\delta}(s), \qquad s=j\omega.$$

Then the closed-loop system of figure 7.1 remains stable when G is replaced by $G_p \in \{G_p\}$ if

(a) the nominal feedback system (i.e. $G_1 = G$, $G_n = I$) is stable,

(b) $G_1(s)$ and $G(s)$ have the same number of CRHP zeros, and

(c) $\delta_n \bar{\sigma}[T(s)] + \tilde{\delta}_1 \bar{\sigma}[S(s)] < 1$, all $s = j\omega$

where

$$S \triangleq (I + GK)^{-1}$$

$$T \triangleq (I + GK)^{-1}GK.$$

Proof (a), (b) and (c) imply, by theorem 4.9, that the perturbed system is stable under all $G_1 \in \{G_1\}$ if $G_n = I$; that is $(I+G_1K)^{-1}G_1K$ is stable for all $G_1 \in \{G_1\}$. Now, from theorem 7.2, when $G \neq I$, then the nonlinearly perturbed system is stable if

$$\sup_{\text{all } G_1 \in \{G_1\}} \left[\sup_{\omega} \bar{\sigma}[(I+G_1K)^{-1}G_1K(j\omega)] \right] < \frac{1}{\delta_n}$$

$$\Leftrightarrow \quad \sup_{\text{all } \tilde{\Delta}_1} \left[\sup_{\omega} \bar{\sigma}[(I + S\tilde{\Delta}_1)^{-1}T(j\omega)] \right] < \frac{1}{\delta_n}$$

$$\Leftarrow \quad \sup_{\text{all } \tilde{\Delta}_1} [\bar{\sigma}[(I + S\tilde{\Delta}_1)^{-1}] < 1/(\delta_n \bar{\sigma}[T] \quad , \text{ all } \omega,$$

$$\Leftrightarrow \quad \inf_{\text{all } \tilde{\Delta}_1} [\underline{\sigma}[I + S\tilde{\Delta}_1]] > \delta_n \bar{\sigma}[T] \quad , \text{ all } \omega,$$

$$\Leftarrow \quad 1 - [\bar{\sigma}[S] \cdot \sup_{\tilde{\Delta}_1(s)} \bar{\sigma}[\tilde{\Delta}_1(s)]] > \delta_n \bar{\sigma}[T] \quad , \text{ all } \omega,$$

$$\Leftarrow \quad [\delta_n \bar{\sigma}[T] + \tilde{\delta}_1 \bar{\sigma}[S]] < 1 \quad , \text{ all } \omega.$$

This completes the proof. ∎

Similarly, if we apply the stability result of theorem 7.3, then we have the following robust stability theorem.

Theorem 7.5 Let $\{G_n\} = \{(I+\tilde{\Delta})^{-1}\}$ where

$$\sup_{x} \frac{||\tilde{\Delta}(x)||}{||x||} \leq \tilde{\delta}_n.$$

Assume

$$\{G_p\} \triangleq \{G_n\} \times \{G_1\}$$

where $\{G_l(s)\}$ is a set of strictly proper rational transfer function matrices satisfying

$$G_l(s) = (I + \Delta_l(s))G(s)$$

and

$$\bar{\sigma}[\Delta_l(s)] \leq \delta_l(s), \quad s=j\omega.$$

Then the closed-loop system of figure 7.1 will remain stable when G is replaced by $G_p \in \{G_p\}$ if

(a) the nominal feedback system is stable,

(b) $G_l(s)$ and $G(s)$ share the same number of unstable poles, and

(c) $\tilde{\delta}_n \cdot \bar{\sigma}[S(s)] + \delta_l \cdot \bar{\sigma}[T(s)] < 1$, all $s=j\omega$.

Proof similar to that for theorem 7.5 but making use of theorem 4.4 and theorem 7.3. ∎

Note that if $\delta_n = \tilde{\delta}_n = 0$, then theorems 7.4 and 7.5 reduce to the inverse Nyquist and direct Nyquist tests of chapter 4 respectively.

7.4 Robust Stability Analysis with Stable Perturbations of Coprime Factors

Let

$$G = N_G D_G^{-1} \text{ and}$$

$$N_G D_G^{-1} \rightarrow (N_G + N_E)(D_G + D_E)^{-1} \quad . \qquad (7,10)$$

In order to use theorem 7.1 and corollary 7.1, we let, as in [VID2]:

$$\Sigma = \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix}, \quad F = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

then

$$D+FN \rightarrow \begin{bmatrix} D_G & N_K \\ -N_G & D_K \end{bmatrix} + \begin{bmatrix} D_E & 0 \\ -N_E & 0 \end{bmatrix}$$

$$= \begin{bmatrix} D_G & N_K \\ -N_G & D_K \end{bmatrix} \left[I + \begin{bmatrix} D_G^{-1}(I+KG)^{-1} & -D_G(I+KG)^{-1}K \\ D_K^{-1}(I+GK)^{-1}G & D_K^{-1}(I+KG)^{-1} \end{bmatrix} \begin{bmatrix} D_E & 0 \\ -N_E & 0 \end{bmatrix} \right]$$

$$= \begin{bmatrix} D_G & N_K \\ -N_G & D_K \end{bmatrix} \begin{bmatrix} I+D_G^{-1}(I+KG)^{-1}D_E+D_G^{-1}(I+KG)^{-1}KN_E & 0 \\ D_K^{-1}(I+GK)^{-1}GD_E-D_K^{-1}(I+GK)^{-1}N_E & I \end{bmatrix}$$

Because $\begin{bmatrix} D_G & N_K \\ -N_G & D_K \end{bmatrix}$ is invertible by assumption (that the nominal feedback system is stable), (7.11) is invertible $B_2 \rightarrow B_2$ if, and only if, the second term is invertible $B_2 \rightarrow B_2$. This again is invertible if, and only if,

$$I + D_G^{-1}(I+KG)^{-1}D_E + D_G^{-1}(I+KG)^{-1}KN_E$$

is invertible $B_2 \rightarrow B_2$. This will be ensured if

$$\| \begin{bmatrix} D_G^{-1}(I+KG)^{-1} & D_G^{-1}(I+KG)^{-1}K \end{bmatrix} \begin{bmatrix} D_E \\ N_E \end{bmatrix} \| < 1 \qquad (7.12)$$

We then have the following robust stability theorem.

Theorem 7.6 Let

$$\{G_p\} \triangleq \{(N_G+N_E)(D_G+D_E)^{-1} : \| \begin{bmatrix} N_E \\ D_E \end{bmatrix} \| \leq r \}$$

Assume that the nominal feedback system of figure 7.1 is stable. Then it will remain stable when G is replaced by any $G_p \in \{G_p\}$ if

$$\| [\tilde{D}_K \quad \tilde{N}_K] \|_r < 1$$

where $(\tilde{N}_K, \tilde{D}_K)$ is a left coprime factorization of K satisfying $\tilde{D}_K D_G + \tilde{N}_K N_G = I$.

Proof Substitute $\tilde{D}_K D_G + \tilde{N}_K N_G = I$ into (7.12) and the theorem follows naturally. ∎

Note that theorem 7.6 is very similar to the linear time-invariant case (theorem 4.15), except that because N_E and D_E are now nonlinear, the operator-norm rather than the frequency-dependent maximum singular value has to be used. It also follows that theorem 7.6 suffers from the same draw-back as theorem 4.15; that is, the robust stability test is dependent on the choice of coprime factorization of G.

7.5 Robust Stability Analysis with Dual Representations of Nonlinear Uncertainty — Direct and Inverse Perturbations

Recall that an assumption for the applicability of the combined direct/inverse Nyquist robust stability test is that the perturbed plant model set is arcwise connected in the graph topology. In the linear convolution case, when the perturbations are occurring in the form of stable perturbations in coprime factors, the collection of all products $(N + N_E)(D + D_E)^{-1}$ where $\| [N_E^T \quad D_E^T]^T \| < r$ is an arcwise connected set. Although one may define convergence of nonlinear rational operators in

the same way as in the case of convolution operators, the problem is actually very much more complicated. For example, in the convolution case, it is possible simply to examine the frequency response and pole-zero structure of $\{G_p\}$ to determine whether the perturbations are continuous in the coprime factors, [chapter 4]. There appears to be no simple means of doing this for nonlinear systems.

Nevertheless, it is still possible to develop a switching-type robust stability test for nonlinear perturbations.

Theorem 7.7 Let $\{G_p\} \triangleq \{G_n\} \; X \; \{G_1\}$ where for all $x \in B_{2e}$,

$$1-\delta_n \leq \frac{||G_n(x)||}{||x||} \leq \frac{1}{1-\tilde{\delta}_n} \quad ;$$

where $\frac{1}{2}(1-\delta_n + \frac{1}{1-\tilde{\delta}_n}) = 1$, $0 \leq \tilde{\delta}_n < 1$, and $\{G_1(s)\}$ is a set of strictly proper rational transfer function matrices satisfying

$$G_1(s) = (I+\Delta_1(s))G(s) = (I+\tilde{\Delta}_1(s))^{-1}G(s)$$

$$\bar{\sigma}[\Delta_1(s)] \leq \delta_1(s) \quad , \quad \text{and}$$

$$\bar{\sigma}[\tilde{\Delta}_1(s)] \leq \tilde{\delta}_1(s) \quad , \quad \text{all } s=j\omega.$$

If $\{G_1\}$ is arcwise connected in the graph topology of convolution plants, then the closed-loop system of figure 7.1 will remain stable when G is replaced by any $G_p \in \{G_p\}$ if for each $\omega \in [0,\infty]$, either

$$1 > \delta_n \cdot \bar{\sigma}[(I+GK)^{-1}GK] + \tilde{\delta}_1 \cdot \bar{\sigma}[(I+GK)^{-1}]$$

or

$$1 > \tilde{\delta}_n \cdot \bar{\sigma}[(I+GK)^{-1}] + \delta_1 \cdot \bar{\sigma}[(I+GK)^{-1}GK]$$

Proof

First, it is easy to show that

$$\frac{1}{1-\tilde{\delta}_n} + (1 - \delta_n) = 2 \iff \delta_n = \frac{\tilde{\delta}_n}{1-\tilde{\delta}_n} \tag{7.13}$$

It then follows that

$$\begin{aligned} \bar{\sigma}[S] < \frac{1}{\tilde{\delta}_n} &\iff \bar{\sigma}[I - T] < \frac{1}{\tilde{\delta}_n} \\ &\iff 1 + \bar{\sigma}[T] < \frac{1}{\tilde{\delta}_n} \\ &\iff \bar{\sigma}[T] < \frac{1-\tilde{\delta}_n}{\tilde{\delta}_n} \\ &\iff \bar{\sigma}[T] < \frac{1}{\delta_n} \end{aligned} \tag{7.14}$$

Thus, the small gain condition $\bar{\sigma}[S] < 1/\tilde{\delta}_n$ is satisfied for all frequencies if at each frequency, either of the inequalities below is satisfied :

$$\sup_{G_1 \in \{G_1\}} [\sup_{\omega} \bar{\sigma}[S]] < \frac{1}{\tilde{\delta}_n} \tag{7.15}$$

or

$$\sup_{G_1 \in \{G_1\}} [\sup_{\omega} \bar{\sigma}[T]] < \frac{1}{\delta_n} \tag{7.16}$$

(7.15) is implied by the second inequality of theorem 7.7, and (7.16) is implied by the first inequality of theorem 7.7. This completes the proof. ▨

The reader may verify that theorem 7.7 also holds if G_n satisfies the following conic sector condition instead:

$$\frac{1}{1+\tilde{\delta}_n} \leq \frac{||G_n(x)||}{||x||} \leq 1+\delta_n$$

where $\frac{1}{2}(1+\delta_n + \frac{1}{1+\tilde{\delta}}) = 1$ and $\delta_n, \tilde{\delta}_n > 0$.

7.6 Concluding Remarks

In many situations, it has been proposed for design purposes that a nonlinear plant may be replaced by a set of linear plants, [HOR3]. Of course, in order to be able to do this, the set of linear plants must satisfy certain nice properties; that is, elements of the set must have the same number of unstable poles, or non-minimum-phase zeros, or be arcwise connected; otherwise even the linear convolution problem becomes intractable. (Note that the simultaneous stabilization problem, [VID5], for convolution plants has not been solved computationally). It should also be pointed out that if one attempts to reduce the effects of nonlinearities by high gain feedback, especially at low frequencies, then one must reasonably be sure that there is no possibility of zero movement across the $j\omega$-axis in this frequency range.

Given $G_p = (N_G + N_E)(D_G + D_E)^{-1}$, the Frechet derivative of G_p evaluated at $u(t) = \tilde{u}(t)$ provides a linear description of G_p when $u(t) = \tilde{u}(t)$; that is, we have

$$\partial G_p \tilde{u} = (N_G + \partial N_E)\cdot(D_G + \partial D_E)^{-1}\,\tilde{u} \qquad (7.17)$$

where $\partial(.)$ denotes the Frechet derivative with respect to the input u, and the derivatives are evaluated at $u = \tilde{u}$. Thus, if the derivatives are continuous with respect to u(t), we will have an arcwise connected set of linearized models. Note that each u(t) may be visualized as a point in the space B_{2e}, thus the concept of continuity in the derivatives is well-defined.

It is also interesting to note that both the sensitivity and complementary sensitivity terms, S and T respectively, appear in the robust stability tests. This suggests that a realistic robust optimal

design scheme should allow a designer to minimize a cost function which is a combination of S and T. This is the subject of the next chapter.

CHAPTER VIII

AN H^{∞} MINIMAX APPROACH TO DESIGN

8.1 Introduction

The problem of robust linear multivariable system design has attracted considerable interest from research workers in recent years. Roughly speaking, there are four popular approaches to the problem : the frequency-domain extended single-input, single output approach, [HOR4], [DAN1], [ARK1]; the LQR/LQG approach, [DOY2], [SAF3], [DAN4], [HAR2]; the semi-infinite (nonlinear programming) optimization based approach, [POL1], [MAY2]; and the H^{∞} minimax approach, [ZAM6], [FRA7], [DOY7], and [FOO3]. Despite some claims and counter-claims, it appears to us that no one technique or approach can be regarded as universally superior. Nonetheless, if the maximum singular value is chosen as the measure of robustness which we wish to optimize, then the H^{∞} approach as introduced by Zames, [ZAM2], would seem to provide the most natural framework within which an appropriate robustness optimization problem can be posed.

Since the publication of [ZAM2], numerous uses of H^{∞}- optimization methods have appeared in the control literature, [FRA6], [CHA2], [SAF9], [SAF10], [CHE6], [KWA3] and [FOO3]. One reason for this activity is that the approach appears to be more effective than the linear quadratic approach in characterizing robustness with respect to plant variations. Another reason is that while a quadratic performance index (in terms of its expected value) may be a meaningful cost function when the inputs and disturbances are known precisely (in a statistical sense), in many situations it is only possible to assign signals to particular classes. In which case, it may be more meaningful to minimize the maximum error that can occur over the particular class of signals. Also, from the

point of view of achieving a robust design it would appear that "loop-shaping" the maximum singular value is easier to carry out using H^∞-optimization than the linear quadratic approach.

Earlier research on the H^∞ approach has largely concentrated on the problem of minimizing the H^∞-norm (supremum of maximum singular value over frequency) of a weighted sensitivity matrix or a weighted complementary sensitivity matrix, [ZAM6], [FRA6], [CHA2], [SAF9]. Let $S(s)$ denote the sensitivity function matrix $[I + G(s)K(s)]^{-1}$ for a unity feedback system with nominal plant model $G(s)$ and controller $K(s)$; the complementary sensitivity is then $T(s) = I - S(s)$. The minimax problem considered in [ZAM6], [FRA6], [CHA2] and [SAF9] is the minimization , over the set of all stabilizing controllers, of either

$$||S(j\omega)V(j\omega)|| \triangleq \sup_{\omega} \bar{\sigma}[S(j\omega)V(j\omega)] \qquad (8.1)$$

or

$$||T(j\omega)W(j\omega)|| \triangleq \sup_{\omega} \bar{\sigma}[T(j\omega)W(j\omega)] \qquad (8.2)$$

where V and W are stable, frequency dependent weighting function matrices.

For practical control, however, solutions to the minimization of (8.1) or (8.2) suffer serious drawbacks. For example, cost function (8.1) does not include bandwidth and power constraints and therefore its minimization leads to designs with infinite bandwidth and controllers which are improper. If cost function (8.2) is used then a trivial controller may result, that is, one with no feedback at all. In both cases, arbitrary approximations have to be introduced to obtain practical designs.

Another serious disadvantage of these two optimization procedures as design tools is that neither provides the designer with a simple way of making a compromise between a small S(s) (to reduce the effects of disturbances) and a small T(s) (to reduce the effects of measurement noise). It is also clear from a consideration of robust stability that the minimization of (8.1) and (8.2) alone would be unsatisfactory if there were any uncertain plant poles and zeros close to the imaginary axis, [chapter 4]. Therefore, as argued in chapter 4, it would seem more sensible to consider a combination of cost functions (8.1) and (8.2). The purpose of this chapter is to consider such a problem. A similar problem has been considered by Kwakernaak, [KWA3], but his solution involves the numerical solution of a set of nonlinear equations, and using a Newton-Raphson method he reported encountering some difficulties with convergence.

It should be said that even if we were exclusively interested in minimizing the sensitivity function or the complementary sensitivity function, then it would still be logical to consider minimizing a combination of the two for the following reasons :

Consider the perturbed sensitivity matrix

$$S_p(s) \triangleq [I + (I + \Delta(s))G(s)K(s)]^{-1}$$
$$= [I + \Delta(s)T(s)]^{-1}S(s) \qquad (8.3)$$

where $\Delta(s)$ is an unstructured plant perturbation characterized by a bound on the maximum singular value of its frequency response. From this we can see that the sensitivity of $\bar{\sigma}[S(j\omega)]$ is dependent on the value of $\bar{\sigma}[T(j\omega)]$. Similarly , we may consider perturbing the complementary sensitivity function to

$$T_p(s) \triangleq [I + (I+\tilde{\Delta}(s))^{-1}G(s)K(s)]^{-1}(I+\tilde{\Delta}(s))^{-1}G(s)K(s)$$
$$= [I + \tilde{\Delta}(s)S(s)]^{-1}T(s) \qquad (8.4)$$

where $\tilde{\Delta}(s)$ is an unstructured "inverse" plant perturbation, again characterized by a bound on $\bar{\sigma}[\tilde{\Delta}(j\omega)]$. This shows that the sensitivity of $\bar{\sigma}[T(j\omega)]$ is dependent on $\bar{\sigma}[S(j\omega)]$. Now it is well known that a stabilizing controller that minimizes (8.1) (or (8.2)) is non-unique, and therefore we can afford the luxury of searching for a controller that not only minimizes the cost but gives a sensitivity function (or complementary sensitivity function) that is itself insensitive to perturbations. Optimizing a combination of cost functions (8.1) and (8.2) could help to achieve this aim.

The organization of the chapter is as follows. In section 8.2, we briefly review some of the basic concepts and definitions in the theory of Hardy Spaces which will be needed later in the chapter. In section 8.3, we propose a robustness optimization problem in the form of

$$\min_{K(s)} \{ \sup_{\omega} \bar{\sigma}\begin{bmatrix} W_1(j\omega)S(j\omega) \\ W_2(j\omega)T(j\omega) \end{bmatrix} \} \qquad (8.5)$$

A characterization of all solutions to the minimization of (8.5) over the set of all stabilizing controllers is given in section 8.4. The problem of robustness optimization with output regulation is briefly considered in section 8.5. Because exact and robust pole-zero cancellation in the CRHP is impossible in practice, we will consider only the case where the unstable poles of the disturbances are on the $j\omega$-axis; this includes most types of disturbance considered in classical designs (e.g. step, ramp, sinusoidal). The adaptation of the H^∞ approach to the design of more

general robust optimal feedback systems is proposed in section 8.6. Conclusions are given in section 8.7.

One appendix is included in which we propose a "best" solution, among the set of all solutions, to the robustness optimization problem.

For simplicity, we will assume the plant is square and nonsingular throughout the chapter. The extension of the design technique to non-square and/or rank-deficient plants is mathematically straightforward and is therefore omitted.

We are grateful to Prof B.A. Francis for bringing to our attention some related work of his own, [FRA8], and that of Doyle, [DOY7]. Francis' problem is mathematically the same as ours in section 8.3 but the motivation and solution are different. Doyle's problem is more general but again the motivation and solution are different. Our work was inspired by a pre-publication version of [CHA2] and was completed independently of [FRA8] and [DOY7].

8.2 Preliminaries

In this section we present some concepts and results from the theory of Hardy Spaces. For ease of exposition and economy of space, the results will be stated without proof. The interested reader is referred to the standard references [DUR1], [HOR1] and [KOO1] for details and clarifications. A summary of the concepts which are relevant to control applications can also be found in [FRA5].

8.2-1 Basic definitions and terminology

For each $p \in [1, \infty)$, the Hardy space H^p consists of all functions $h(s)$ which are analytic in the open right half-plane and with the property that

$$||h(s)||_p \triangleq \sup_{\alpha>0} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |h(\alpha+j\omega)|^p d\omega \right]^{1/p} < \infty \qquad (8.6)$$

The set H^∞ consists of all functions $h(s)$ analytic in the open RHP such that

$$||h(s)||_\infty \triangleq ||h(s)|| = \sup_{\alpha>0} |h(\alpha+j\omega)| < \infty \qquad (8.7)$$

We shall only be interested in H^2 and H^∞ in this chapter.

A rational function belongs to H^2 if, and only if, it has no poles in the CRHP and it is zero at $|s| = \infty$. A rational function belongs to H^∞ if, and only if, it is analytic and of bounded modulus in CRHP.

For each h in H^2 and H^∞ the limit

$$\lim_{\alpha \downarrow 0} f(\alpha+j\omega) \quad , \quad \alpha \geq 0 \qquad (8.8)$$

exists for almost all ω. Thus, (8.6) and (8.7) may be replaced by their boundary functions, that is

$$||h(s)||_2 \triangleq \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |h(j\omega)|^2 d\omega \right]^{\frac{1}{2}} \qquad (8.9)$$

and

$$||h(s)|| \triangleq \operatorname*{ess\,sup}_{\omega} |h(j\omega)| \qquad (8.10)$$

H^2 is a Hilbert space under the usual inner product

$$\langle f,g\rangle \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} f(-j\omega)g(j\omega)\,d\omega \qquad (8.11)$$

According to the theorems of Paley-Wiener and Parseval, there is an isometric isomorphism between a (time) function in $L_2[0,\infty)$ and its Laplace transform in H^2. Every function in H^2 thus represents a causal L_2-stable time function on the time interval $[0,\infty)$.

Each function in H^∞ represent a multiplier (or multiplication operator) on H^2. That is, for all $h \in H^\infty$, $hg \in H^2$ if $g \in H^2$. The norm of the multiplier is given by the H^∞- norm of the function, that is

$$\begin{aligned}\|h(s)\| &= \sup_{\omega} |h(j\omega)| \\ &\triangleq \sup\{\|h(s)g(s)\| : g \in H^2,\ \|g\|_2 \le 1\}\end{aligned} \qquad (8.12)$$

For the matrix (or vector) function H(s), the norms are defined as follows :

$$\|H(s)\|_2 \triangleq \frac{1}{2\pi}\,\mathrm{tr}\left[\int_{-\infty}^{\infty} H(-j\omega)^T H(j\omega)\,d\omega\right]^{\frac{1}{2}} \qquad (8.13)$$

and

$$\begin{aligned}\|H(s)\|_\infty \triangleq \|H(s)\| &= \sup_{\omega} \bar{\sigma}[H(j\omega)] \\ &= \sup\{\|H(s)g(s)\|_2 : g\in H^2,\ \|g\|_2 \le 1\}\end{aligned} \qquad (8.14)$$

where g(s) is a vector function.

A real-rational matrix M(s) is :

Proper if it does not have a pole at infinity, and

strictly proper if in addition $M(\infty) = 0$.

Stable if its elements are analytic in the closed right half-plane.

Minimum phase if does not lose rank in the closed right half-plane (If $M(s)$ is square and non-singular, this is equivalent to saying that $M(s)^{-1}$ is stable.)

All-pass if $M(s)^{*}M(s) = I$ for all $s=j\omega$, where $A^{*}(s)$ denotes $A(-s)^{T}$.

All-pass form if $M(s)^{*}M(s) = \rho I$, $\rho \in \mathbf{R}$, for all $s=j\omega$.

Inner if it is all-pass and stable.

Outer if it is stable, proper and has an inverse which is analytic in the open right half-plane.

Any stable, proper, real-rational matrix $M(s)$ have inner-outer factorization of the form $M(s) = M_i(s)M_o(s) = \tilde{M}_o(s)\tilde{M}_i(s)$ where $M_i(s)$, $\tilde{M}_i(s)$ are inner, $M_o(s)$, $\tilde{M}_o(s)$ are outer, and $(M_i(s), M_o(s))$, $(\tilde{M}_i(s),\tilde{M}_o(s))$ are unique up to multiplication by a constant unitary matrix. If $M(s)$ has full rank at all points on the extended $j\omega$-axis, then its outer factor has an inverse which is stable.

If $U(s)$ is square inner, then its transpose $U(s)^{T}$ and its determinant $\det[U(s)]$ are also inner. It may be verified that an inner matrix is a norm-preserving multiplier in $\mathrm{Mat}(H^{\infty})$.

It is often convenient to transform the CRHP into the closed unit disc by the norm-preserving isomorphic transformation

$$s = \frac{1 - z}{1 + z}$$

The inverse transform is then given by

$$z = \frac{1 - s}{1 + s}.$$

We shall denote the set of bounded analytic function in the unit disc by H_z^∞. The norm of a function in H_z^∞ is given by

$$\|M(z)\| \triangleq \sup_{|z|\leq 1} \bar{\sigma}[M(z)]$$

$$= \sup_{\theta\in[0,2\pi)} \bar{\sigma}[M(e^{j\theta})] \qquad (8.15).$$

8.2-2 The matrix Nevanlinna-Pick problem

The main result of the matrix Nevanlinna-Pick problem, [DEL1], may be summarized by the following theorem and algorithm :

Theorem 8.1 Let $z_1, \ldots, z_n$ be distinct points in the open unit disc (i.e. $|z_i|<1$, all i), and $F_1, \ldots, F_n$ be constant complex matrices with $\bar{\sigma}[F_i] < 1$, all i. Then there exists a $\Phi(z) \in Mat(H_z^\infty)$ such that $\|\Phi\|\leq 1$ and $\Phi(z_i) = F_i$, all i, if and only if the partitioned matrix

$$P = \begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & & \vdots \\ P_{n1} & \cdots & P_{nn} \end{bmatrix} ,$$

where

$$P_{ij} = \frac{1}{1-\bar{z}_i z_j} (I + F_i F_j^H)$$

and $\bar{z}_i$ denotes the complex conjugate of z_i, is nonnegative definite.

Proof See [DEL1]. ∎

The proof of theorem 8.1 leads to an algorithm for constructing $\Phi(z)$ as follows, [DEL1]

(1) Let $y(\xi,z)$, where ξ is a complex constant, be defined as :

$$y(\xi,z) \triangleq \begin{cases} (\xi-z)/(1-\bar{\xi}z)\cdot \dfrac{|\xi|}{\xi} & \text{if } \xi \neq 0 \\ -z & \text{if } \xi = 0 \end{cases} \qquad (8.16)$$

and

$$L(E) \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$= \begin{bmatrix} (I-EE^H)^{\frac{1}{2}} & -(I-EE^H)^{-\frac{1}{2}}E \\ -(I-E^HE)^{-\frac{1}{2}}E^H & (I-E^HE)^{-\frac{1}{2}} \end{bmatrix} \qquad (8.17)$$

where E is a constant matrix with $\bar{\sigma}(E) < 1$ and $M^{\frac{1}{2}}$ is the hermitian square root of M. (i.e. if $M = RR^H$, we denote $R = M^{\frac{1}{2}}$).

Define the linear fractional transformation

$$T_E(X) = (AX + B)(CX + D)^{-1} \qquad (8.18)$$

where X is a constant matrix. Note that for each X with $\bar{\sigma}(X)\leq 1$, $T_E(X)$ is well-defined and, [VID3],

$$\bar{\sigma}[T_E(X)] \leq 1 \quad (\text{resp. } < 1) \quad <=> \quad \bar{\sigma}[X] \leq 1 \ (\text{resp. } < 1)$$

$$T_{E,y(\xi)}(X) \triangleq y(\xi,z)^{-1}(AX+B)(CX+D)^{-1} \qquad (8.19)$$

which gives

$$T^{-1}_{E,y(\xi)}(Y) = [y(\xi,z)YC - A]^{-1}[B - y(\xi,z)YD] \qquad (8.20)$$

(2) Find $\psi_1(z)$ in H^∞ such that $\psi_1(z_1) = F_1$ and $||\psi_1(z_1)|| < 1$. A simple choice is given by $\psi_1(z) = F_1$.

(3) Iteratively find $\psi_i(z)$ as follows :

$$\psi_2(z) \triangleq T^{-1}_{F_2,y(z_2)}[\psi_1(z)].$$

$$\psi_3(z) \triangleq T^{-1}_{F_3,y(z_3)}[\psi_2(z)].$$

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$$\psi_n(z) \triangleq T^{-1}_{F_n,y(z_n)}[\psi_{n-1}(z)] \qquad (8.21).$$

(4) $\Phi(z)$ is then given by

$$\Phi(z) = \psi_n(z) \qquad (8.22)$$

This completes the algorithm.

Note that if $\psi_1(z)$ is rational, then $\Phi(z)$ is rational.

8.2-3 Optimal matrix interpolation

Given $s_1, \ldots, s_m$ which are distinct points in the open LHP, and constant complex matrices $F_1, \ldots, F_m$, what is the smallest value of μ such that there exists a $\Phi(z) \in \mathrm{Mat}(H^\infty)$ satisfying $||\Phi|| \leq \mu$ and $\Phi(z_i) =$

F_i, all i ? This is the optimal matrix interpolation problem and the following theorem gives a computable answer.

Theorem 8.2 [CHA2]. Suppose $s_1, \ldots, s_m$ are distinct points in the open left half-plane, and $F_1, \ldots, F_m$ are constant rxs complex matrices. Define

$$A \triangleq \text{Block Diag } (F_1, \ldots, F_m) \in C^{msxmr} \tag{8.23}$$

$$B \triangleq (B_{ij}) \ , \ i,j = 1, \ldots, m \tag{8.24}$$

where

$$B_{ij} \triangleq \frac{1}{s_i + \bar{s}_j} \cdot I_r \tag{8.25}$$

I_r is the rxr identity matrix. Then

$$\mu_o \triangleq \inf_{\Phi(s) \in Mat(H_z^\infty)} \{\mu = ||\Phi|| : \Phi(s_i) = F_i, \quad \text{all i}\}$$

$$= \sup \{\lambda^{-\frac{1}{2}} : \det(ABA^H - \lambda B) = 0\} \tag{8.26}$$

Proof See [CHA2]. ∎

Thus, the problem of computing μ_o is equivalent to a generalized eigenvalue problem.

Once μ_o is computed, the problem of constructing an optimum $\Phi(s)$ with the infimum norm can then be solved by using the Nevanlinna-Pick

algorithm to construct $\tilde{\Phi}(s)$ which satisfies $\tilde{\Phi}(s_i) = \frac{1}{\mu_o} F_i$, all i, and $||\tilde{\Phi}(s_i)|| \leq 1$. $\Phi(s)$ is then given by $\mu_o \tilde{\Phi}(s)$.

8.3 Optimal Robustness in the Design of Linear Multivariable Systems : A Problem Formulation

It has been shown in chapter 2 that the sensitivity of the response of a closed-loop system with respect to plant variations is dependent on the size of the maximum singular value of the sensitivity matrix S(s) and the sensitivity with respect to sensor uncertainty is dependent on the size of the maximum singular value of the complementary sensitivity matrix T(s). It is also shown in chapter 4 that robust stability may be characterized by both $\bar{\sigma}[S(j\omega)]$ and $\bar{\sigma}[T(j\omega)]$. In particular, it is shown that at frequencies for which there are uncertain zeros close to the imaginary axis, robust stability is "maximized" by minimizing $\bar{\sigma}[T(j\omega)]$. Similarly, if there are uncertain poles close to the imaginary axis then over these frequencies it is important to minimize $\bar{\sigma}[S(j\omega)]$ for robust stability. Note also, that because S(s) determines the effect of disturbances on the plant output, we would normally want $\bar{\sigma}[S(j\omega)]$ to be small for good load disturbance rejection, and that because T(s) determines the transmittance of measurement noise to the output we would normally want $\bar{\sigma}[T(j\omega)]$ to be small over the noise frequencies.

The above arguments lead us to an examination of the following optimization problem : find a stabilizing controller K(s) which minimizes the cost function

$$\max \{ \|W_1(j\omega)S(j\omega)\|, \|W_2(j\omega)T(j\omega)\| \}. \qquad (8.27)$$

Loosely speaking, W_1 should be "large" at frequencies where performance (accuracy, disturbance rejection, etc.) is important and where there are no uncertain zeros close to the imaginary axis. W_2 should be "large" at frequencies where measurement noise is significant and where there are no uncertain poles close to the imaginary axis. As in [KWA3], a consideration of (8.27) indicates that choosing $W_2(s)$ to be improper tends to force the solution (if one exists) to give a proper controller, and allows control over the bandwidth of the closed-loop system. We note that the ability to control and restrict the bandwidth of a closed-loop system is very important from a practical design point of view because measurement noise will invariably creep in at high frequencies, and sensor dynamics will become more uncertain. It has also been shown that for a system of a given pole-zero geometry, the tendency to saturate increases with the bandwidth, [ASH1], [TOW1]. The cost of feedback, in Horowitz's opinion, is thus "precisely in the bandwidths of the compensation", [HOR2], and one should therefore not design a feedback system with wider bandwidth than necessary to achieve good sensitivity and disturbance rejection in the operating frequency range, unless considerations on robust stability dictate otherwise. The interested reader is referred to [HOR1] for a comprehensive discussion on the trade-offs between the benefits and cost of feedback in terms of gain and bandwidth.

Unfortunately, at the present state of knowledge, it is not known how to solve the optimization of (8.27) and so, as an alternative, we propose solving the related problem : find a stabilizing controller $K(s)$ which minimizes

$$||Z(j\omega)|| \qquad (8.28)$$

where

$$Z(j\omega) = \begin{bmatrix} W_1(j\omega)S(j\omega) \\ \cdots\cdots\cdots \\ W_2(j\omega)T(j\omega) \end{bmatrix} \qquad (8.29)$$

The relationship between the two functionals (8.27) and (8.28) is given by

$$||Z|| \geq \max \{ ||W_1 S||, ||W_2 T|| \} . \qquad (8.30)$$

We see, therefore, that minimizing functional (8.28) has the effect of minimizing (8.27), especially if W_1 and W_2 are a pair of almost ideal switching-type weighting matrices, that is for all $\omega \in [0, \infty]$ W_1 and W_2 satisfy the following property approximately:

$$||W_1|| \cdot ||W_2|| = 0$$

and

$$||W_1|| + ||W_2|| \neq 0 \qquad (8.31).$$

Minimizing (8.28) also has an effect on the other singular values of S(s) and T(s) as stated in the following proposition:

Proposition 8.1 Let $\sigma_i(.)$ denote the ith singular-value of (.) where

$$\bar{\sigma}(.) = \sigma_1(.) \geq \sigma_2 \ldots \geq \sigma_m = \underline{\sigma}(.) \quad .$$

Then

$$||Z||^2 \geq \max_i [\sigma_i^2(W_1 S(j\omega)) + \sigma_{m-i+1}^2(W_2 T(j\omega))] \qquad (8.32).$$

Proof

$$Z^* Z = S^* W_1^* W_1 S + T^* W_2^* W_2 T \qquad (8.33).$$

Note that Z^*Z, $S^*W_1^*W_1S$ and $T^*W_2^*W_2T$ are all hermitian and nonnegative definite.

Now assume $\bar{\sigma}[Z(j\omega)] = \lambda_o(j\omega)$, then we have

$$\lambda_o^2 I \geq S^*W_1^*W_1S + T^*W_2^*W_2T \ , \quad \text{all } \omega, \tag{8.34}$$

where $A \geq B$ denotes $A - B$ is nonnegative definite.

But

$$(8.34) \iff \lambda_o^2 I - T^*W_2^*W_2T \geq S^*W_1^*W_1S$$

$$\Rightarrow \sigma_i[\lambda_o^2 I - T^*W_2^*W_2T] \geq \sigma_i[S^*W_1^*W_1S]$$

But

$$\sigma_i^2[\lambda_o^2 I - T^*W_2^*W_2T] = \lambda_o^2 - \sigma_{m-i+1}^2[T^*W_2^*W_2T]$$

because $T^*W_2^*W_2T$ is hermitian. We therefore have

$$\lambda_o^2 \geq \sigma_i^2[W_1S(j\omega)] + \sigma_{m-i+1}^2[W_2T] \ , \text{ all } i, \text{ all } \omega.$$

This completes the proof. ∎

Finally, we note that if $W_1(s)$ and $W_2(s)$ are chosen to be scalars given by

$$W_1(j\omega) = \frac{1}{\epsilon(j\omega)} I \tag{8.35}$$

and

$$W_2(j\omega) = \delta_o(j\omega) I \tag{8.36}.$$

Then

$$\|Z\| < 1$$

gives

$$\bar{\sigma}[S(j\omega)] < |\epsilon(j\omega)| \tag{8.37}$$

and

$$\bar{\sigma}[T(j\omega)] < \frac{1}{\delta_o(\omega)} \tag{8.38}$$

which is closely related to the problem considered in [OYO2].

8.4 Solution to the Robustness Optimization Problem

We begin by making the following assumptions :

(i) $W_1(s)$, $W_2(s)$, $G(s)$, and $K(s)$ are real-rational square mxm matrices;

(ii) $G(s)$ is strictly proper;

(iii) $G(s)$ has no poles on the $j\omega$-axis, and rank $(G(j\omega)) = m$ for all ω;

(iv) $W_1(s)$ and $W_2(s)$ are nonsingular, minimum-phase and analytic in the CRHP; and

(v) $W_1(s)$ is proper and $W_2(s)$ is such that $W_2(s)G(s)$ is proper.

Let G have left and right-coprime factorizations given by

$$G(s) = N(s)D(s)^{-1} = \tilde{D}(s)^{-1}\tilde{N}(s) \tag{8.39}$$

and $X, Y, \tilde{X}, \tilde{Y} \in Mat(S)$ such that

$$XN + YD = I \tag{8.40}$$

and

$$\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I \tag{8.41}.$$

The set of all proper, real-rational, stabilizing controllers can then be parameterized as, [YOU1] :

$$K(s) = (\tilde{X} + DQ)(\tilde{Y} - NQ)^{-1} \qquad (8.42)$$

where Q is any mxm, stable, proper and real-rational matrix.

Substitution of (8.42) and (8.29), and use of (8.39) - (8.41) yields, [FRA6] :

$$Z = \begin{bmatrix} W_1\tilde{Y} & \tilde{D} \\ W_2(I-\tilde{Y}\tilde{D}) & \end{bmatrix} + \begin{bmatrix} -W_1N \\ W_2N \end{bmatrix} Q\tilde{D} \qquad (8.43)$$

$$\triangleq H + \Pi Q\tilde{D} \qquad (8.44)$$

where

$$H \triangleq \begin{bmatrix} W_1\tilde{Y} & \tilde{D} \\ W_2(I-\tilde{Y}\tilde{D}) & \end{bmatrix} \qquad (8.45)$$

and

$$\Pi \triangleq \begin{bmatrix} -W_1N \\ W_2N \end{bmatrix} \quad . \qquad (8.55)$$

A difficulty with minimizing ||Z|| arises from the non-squareness of Z, being 2mxm. A non-square H^{∞}- optimization problem has been considered by Chang and Pearson, [CHA2], who minimize (introducing here some terminology of our own) ||E(s)|| over all stable and proper L(s), where

$$E(s) = J(s) + \Phi L(s) \qquad (8.47);$$

$E(s)$ and $J(s)$ are stable and proper but not necessarily square, and $\phi(s)$ is a rational, scalar, inner function; i.e. $\phi(s)$ is a stable rational function such that $|\phi(j\omega)| = 1$. In their problem $L(s)$ is allowed to be any proper stable transfer function matrix of appropriate dimensions, whereas if we equate our problem to theirs, $Q(s)$ is restricted to have the form determined by

$$\phi(s)L(s) = \Pi(s)Q(s)\tilde{D}(s) \qquad (8.48)$$

where Π is $2m \times m$, Q and $\tilde{D}$ are each $m \times m$, and L would be $2m \times m$. It is well-known that not every $2m \times m$ matrix can be decomposed into the form of (8.48). The approach of Chang and Pearson is therefore not directly applicable for finding Q in our problem. Instead we propose to reduce our problem to a simpler one.

Let U be an all-pass matrix which makes

$$U\Pi = \begin{bmatrix} \hat{\Pi}_1 \\ \cdots \\ 0 \end{bmatrix} \qquad (8.49)$$

where $\hat{\Pi}_1$ is an $m \times m$ matrix. The following proposition establishes the existence of U.

Proposition 8.2 U of (8.49) exists.

Proof For any $2m \times m$ real-rational matrix $\Pi(s)$, of normal rank m, there exists a factorization of the form

$$\Pi(s) = V(s)M(s) \qquad (8.50)$$

where $V(s)$ is a $2m \times m$ real-rational paraconjugate unitary matrix (i.e. $(V(-s)^T V(s) = I$), and $M(s)$ is $m \times m$, real-rational and analytic together

with its inverse in the open right half-plane. This result is readily derived from a more general result of Youla, [YOU4, page 186].

Let F(s) be any arbitrary mx2m real-rational matrix of normal rank m such that

$$F(s)\Pi(s) = 0 \qquad (8.51).$$

F(s) certainly exists, see for example [KAI1, page 455] for a detailed discussion.

Also by Youla's result F(s) can be factorized as

$$F(s) = A(s)B(s) \qquad (8.52)$$

where B(s) is an mx2m real-rational paraconjugate unitary matrix (i.e. $B(s)B(-s)^T = I$), and A(s) is mxm, real-rational and analytic together with its inverse in the open right half-plane.

Clearly :

$$B(s)V(s) = 0 \qquad (8.53)$$

$$V(-s)^T V(s) = I \qquad (8.54)$$

$$B(s)B(-s)^T = I \qquad (8.55).$$

Thus, to satisfy (8.49), we may choose

$$U(s) = \begin{bmatrix} V(-s)^T \\ \cdots\cdots \\ B(s) \end{bmatrix} \qquad (8.56)$$

which gives $U(s)U(-s)^T = I$ and because U(s) is square and nonsingular, $U(-s)^T U(s) = I$. It follows that $U(j\omega)U(-j\omega)^T = U(-j\omega)^T U(j\omega) = I$ and therefore U(s) is the desired all-pass matrix. ∎

Because U is all-pass we have $\bar{\sigma}[Z]$ equal to $\bar{\sigma}[UZ]$. But U is not necessarily stable (although a stable U always exists) and therefore UZ is not necessarily an element of $Mat(H^\infty)$. However, Z is stable so that if $p_1, \ldots, p_n$ are the unstable poles of U then UZ can be expressed as

$$UZ = \Theta(\hat{H} + \hat{\Pi}Q\tilde{D}) \tag{8.57}$$

where

$$\Theta(s) = \frac{(s+p_1)(s+p_2)\ldots(s+p_n)}{(s-p_1)(s-p_2)\ldots(s-p_n)} \tag{8.58}$$

is all-pass, and $\hat{H}$, $\hat{\Pi}$ are each stable and given by $\Theta\hat{H} = UH$ and $\Theta\hat{\Pi} = U\Pi$. It follows that

$$||Z|| = ||\hat{H} + \hat{\Pi}Q\tilde{D}|| \quad . \tag{8.59}$$

Let $\hat{H}$ and $\hat{\Pi}$ be decomposed as

$$\hat{H} = \begin{bmatrix} H_1 \\ \cdots \\ H_2 \end{bmatrix} \quad , \quad \hat{\Pi} = \begin{bmatrix} \hat{\Pi}_1 \\ \cdots \\ 0 \end{bmatrix} \tag{8.60}$$

where H_1, H_2 and Π_1 are mxm, then using (8.49),

$$\hat{Z} \triangleq \hat{H} + \hat{\Pi}Q\tilde{D} = \begin{bmatrix} H_1 + \hat{\Pi}_1 Q\tilde{D} \\ \cdots\cdots\cdots \\ H_2 \end{bmatrix} \triangleq \begin{bmatrix} F(Q) \\ \cdots\cdots \\ H_2 \end{bmatrix} \quad . \tag{8.61}$$

$\hat{Z}$ is particularly nice because its lower half does not depend on Q.

Consider the expansion for $\hat{Z}^*\hat{Z}$, that is

$$\hat{Z}^*\hat{Z} = F(Q)^*F(Q) + H_2{}^*H_2. \tag{8.62}$$

A solution to the minimization of $||Z||$, equivalently $||\hat{Z}||$, is known to have an all-pass form (see corollary 8.2 later), and therefore we may assume that

$$\hat{Z}_o^* \hat{Z}_o = \lambda^2 I \qquad (8.63)$$

where $\hat{Z}_o$ is the optimum $\hat{Z}$, and $||\hat{Z}_o|| = \lambda$. Substituting this optimal form for $\hat{Z}$ in (8.62) and rearranging, gives

$$F(Q)^* F(Q) = \lambda^2 I - H_2^* H_2 \triangleq \Sigma^* \Sigma \quad . \qquad (8.64)$$

By introducing Σ we see that the minimization of $||Z||$ is equivalent to a model-matching type problem of finding a stable Q such that

$$H_1 + \hat{\Pi}_1 Q \tilde{D} = U_1 \Sigma \qquad (8.65)$$

where U_1 is an arbitrary all-pass mxm matrix such that $U_1\Sigma$ is stable.

Now factorize Σ as

$$\Sigma = \Sigma_i \Sigma_o \qquad (8.66)$$

where Σ_i is inner and Σ_o is outer. Then

$$H_1 + \hat{\Pi}_1 Q \tilde{D} = U_i \Sigma_o \qquad (8.67)$$

where

$$U_i \triangleq U_1 \Sigma_i \qquad (8.68)$$

is inner, and

$$U_i = H_1 \Sigma_o^{-1} + \hat{\Pi}_1 Q \tilde{D} \Sigma_o^{-1} \quad . \qquad (8.69)$$

The factorization has therefore transformed the model-matching problem to one of finding a stable Q such that U_i is inner. Note that if $\lambda = ||H_2||$, then two situations are possible :

Case 1 Σ is rank deficient and therefore does not have an inverse. In this case we increase λ by an arbitrarily small amount so that the situation no longer applies.

Case 2 Σ^{-1} exists but has poles on the jω-axis; Σ_o^{-1} will clearly have the same jω-axis poles and is therefore unstable. Herein lies the difficulty because (8.69) no longer represents an H^∞- approximation problem, [FRA8]. In this case, we let

$$\Sigma_o^{-1} = \rho\tilde{\Sigma}_o^{-1} \tag{8.70}$$

where ρ^{-1} is a least degree scalar, proper, stable, real-rational function which is free of zeros in the open RHP, and which contains all the jω-axis poles of Σ_o^{-1} (multiplicities included) as its zeros. $\tilde{\Sigma}_o^{-1}$ is thus a stable, real-rational matrix.

Equation (8.69) can now be re-written as

$$U_i = \rho(H_1\tilde{\Sigma}_o^{-1} + \tilde{\Pi}_1 Q \tilde{D} \tilde{\Sigma}_o^{-1}) \tag{8.71}$$

where $\rho = 1$ if Σ_o^{-1} is stable. To enable us to use the result of section 8.2, we perform the following transformations, [FRA6], [CHA2] :

Let

$$\hat{\Pi}_1 = \hat{\Pi}_i \hat{\Pi}_o \tag{8.72}$$

$$\tilde{D\Sigma}_o^{-1} = (\tilde{D\Sigma}_o^{-1})_o(\tilde{D\Sigma}_o^{-1})_i \tag{8.73}$$

and

$$\tilde{Q} = \hat{\Pi}_o Q (\tilde{D} \tilde{\Sigma}_o^{-1})_o \tag{8.74}$$

where $\hat{\Pi}_i$, $(\tilde{D\Sigma}_o^{-1})_i$ are inner, and $\hat{\Pi}_o$, $(\tilde{D\Sigma}_o^{-1})_o$ are outer. (8.71) then becomes

$$U_i = \rho[H_1 \tilde{\Sigma}_o^{-1} + \hat{\Pi}_i \tilde{Q} (\tilde{D} \tilde{\Sigma}_o^{-1})_i] \qquad (8.75).$$

Define

$$\Phi = \Phi_1 \Phi_2 \qquad (8.76)$$

where $\Phi_1 =$ det $\hat{\Pi}_i$ and $\Phi_2 =$ det$(\tilde{D}\tilde{\Sigma}_o^{-1})_i$. Then Φ is an inner function and $\Phi_1 \hat{\Pi}_i^{-1}$ and $\Phi_2 (\tilde{D}\tilde{\Sigma}_o^{-1})_i^{-1}$ are inner matrices, [CHA2]. Define

$$\Phi = \Phi \hat{\Pi}_i^{-1} U_i (\tilde{D} \tilde{\Sigma}_o^{-1})_i^{-1} \qquad (8.77)$$

$$H_a = \Phi \hat{\Pi}_i^{-1} H_1 \tilde{\Sigma}_o^{-1} (\tilde{D} \tilde{\Sigma}_o^{-1})_i^{-1} \qquad (8.78)$$

and (8.75) becomes

$$\Phi = \rho(H_a + \Phi \tilde{Q}) \qquad (8.79)$$

where $H_a \in$ Mat($\mathbb{RH}^\infty$), ($\mathbb{RH}^\infty$ denotes the set of real-rational H^∞ functions) and Φ is inner and real-rational.

For simplicity, we shall assume that Φ has only simple open RHP zeros. (The case of repeated zeros has been treated in [OYO1] and [WAN3], and the reader is referred there for further details). It is then easy to show that $\tilde{Q} \in \mathbb{RH}^\infty$ if, and only if, Φ is real-rational and satisfies

$$\Phi(s_i) = \rho H_a(s_i) \qquad (8.80)$$

where s_i, $i = 1, 2, \ldots,$ are the open RHP zeros of Φ. If the optimum λ is known, the problem of finding Q is thus transformed to the H^∞-interpolation problem of finding a real-rational Φ which satisfies (8.80)

and $\|\Phi\| \leq 1$, (which will be proved later), which may be solved by the Nevanlinna-Pick algorithm.

The following four lemmas and theorem tell us how to choose λ iteratively to arrive at the optimum solution.

Lemma 8.1 Let λ be the minimum value of $\|Z\|$ over all stable Q, then

$$\lambda \leq \|H_2\| \tag{8.81}$$

Proof Follows simply from (8.62), because

$$\lambda^2 = \bar{\sigma}[F(Q)^* F(Q) + H_2{}^* H_2] \geq \bar{\sigma}[H_2{}^* H_2] \tag{8.82}$$

This completes the proof. ∎

Lemma 8.2

$$\lambda \leq \left\| \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \right\| \tag{8.83}$$

Proof Follows simply by letting $Q = 0$. ∎

Lemma 8.1 emphasizes that if H_2 is improper then the minimization of $\|Z\|$ does not have a finite solution and an optimal Q does not exist. However, the assumptions on W_1 and W_2 ensure that $\|Z\|$ is finite. This follows from

$$\|Z\|^2 \leq \|W_1 S\|^2 + \|W_2 T\|^2 \tag{8.84}.$$

The terms on the right-hand side are finite because of the assumptions on W_1 and W_2, and therefore $||Z||$ is also finite. The optimal Q is that which gives the smallest finite value and therefore clearly exists.

The next lemma tells us when the choice of λ is too large.

Lemma 8.3 Let $\lambda > ||H_2||_\infty$. Then if μ, as defined in (8.85) below, is smaller than 1, there exists $\lambda' < \lambda$ such that (8.69) has a solution.

$$\mu \stackrel{\Delta}{=} \min_{Q \text{ stable}} || H_1 \Sigma_o^{-1} + \hat{\Pi}_1 Q \tilde{D} \Sigma_o^{-1} ||_\infty \qquad (8.85).$$

Proof Note that (8.85) is equivalent to (8.86)

$$\mu \stackrel{\Delta}{=} \min_{\Phi \text{ stable}} \{ ||\Phi||_\infty : \Phi(s_i) = \rho H_a(s_i) \} \qquad (8.86)$$

Because $\mu < 1$, it follows that there exists some Q such that (8.69) is satisfied. The Q that solves (8.69) does not satisfy (8.85) (note $\mu \neq 1$) and therefore there exists a real number $e > 0$, such that

$$\lambda^2 I - eI - H_2^* H_2 \stackrel{\Delta}{=} \Sigma'^* \Sigma' \qquad (8.87)$$

and if Σ' has an inner-outer factorization, $\Sigma' = \Sigma'_i \Sigma'_o$, with Σ'^{-1}_o stable, then

$$\min_{Q \text{ stable}} ||H_1 \Sigma'^{-1}_o + \Pi_1 Q \tilde{D} \Sigma'^{-1}_o||_\infty \stackrel{\Delta}{=} \mu' \leq 1 \qquad (8.88)$$

This implies that (8.69) has a solution with $\Sigma = \Sigma'$, or equivalently $\lambda' = \sqrt{(\lambda^2 - e)}$. This completes the proof. ∎

Lemma 8.4 If μ in (8.85) is greater than 1, then there is no solution to (8.69).

Proof Because $\mu > 1$, no Q exists which solves $||U_i|| = 1$. ※

As the minimization of $||Z||$ is known to have a finite solution, $\mu>1$ implies that the chosen λ is too small.

A consequence of lemma 8.3 and 8.4 is the following theorem which characterizes the optimum λ.

Theorem 8.3 $\lambda > ||H_2||$ is the minimum H^∞- norm of Z if, and only if, $\mu=1$.

Proof $\lambda > ||H_2||$ and $\mu < 1$ implies λ too large, by lemma 8.3.

$\lambda > ||H_2||$ and $\mu > 1$ implies λ too small, by lemma 8.4.

It remains to be proved that if λ is larger than the optimum value, λ_o, then μ cannot possibly be equal to 1. Now assume that $\mu = 1$ when $\lambda > \lambda_o$. Then since $\lambda = \lambda_o$ is also a solution. It follows that there exists $Q = Q_o$ such that

$$\lambda_o^2 I - H_2^* H_2 = F(Q_o)^* F(Q_o).$$

$$\Leftrightarrow \quad \lambda^2 I - (\lambda^2 I - \lambda_o^2 I) - H_2^* H_2 = F(Q_o)^* F(Q_o)$$

$$\Leftrightarrow \quad \lambda^2 I - H_2^* H_2 = F(Q_o)^* F(Q_o) + (\lambda^2 I - \lambda_o^2 I) = F(Q)^* F(Q) = \Sigma^* \Sigma$$

It then follows that $||F(Q_o)\Sigma_o^{-1}|| < ||F(Q)\Sigma_o^{-1}||$. Thus $||F(Q_o)\Sigma_o^{-1}||<1$. But by assumption

$$\mu = \inf_Q ||F(Q)\Sigma_o^{-1}|| = 1$$

This leads to a contradiction and completes the proof. ∎

If $\lambda = \| H_2 \|$, then the following theorem is applicable.

Theorem 8.4 If Σ_o^{-1} exists, then $\lambda = \| H_2 \|$ is the minimum H^∞-norm of Z if, and only if, $\mu \leq 1$.

Proof If we cannot find Q such that $\mu \leq 1$, then it follows that there is no Q which solves (8.69). Conversely, if we can find Q such that $\mu \leq 1$, then from (8.86) it is clear that we can find $\tilde{Q}$ which gives U_i inner. Thus the existence of a solution is guaranteed. Because $\lambda = \|H_2\|$ is the smallest possible (by lemma 8.1), it is clearly the optimum solution. This completes the proof. ∎

Next, we turn to the stability of Q.

Proposition 8.2 If Φ satisfies interpolation constraints (8.80) and ϕ has only simple zeros, then both $\tilde{Q}$ and Q are stable.

Proof We have from (8.79)

$$\Phi = \rho(H_a + \phi\tilde{Q})$$

$$\Leftrightarrow \qquad \tilde{Q} = \phi^{-1}(\rho^{-1}\Phi - H_a) \qquad (8.89)$$

$\Phi(s_i) = \rho H_a(s_i)$ thus ensures that $\tilde{Q}(s_i)$ is bounded, and hence $\tilde{Q}$ is stable. Now we have

$$\tilde{Q} = \hat{\Pi}_o Q \, (\tilde{D} \, \tilde{\Sigma}_o^{-1})_o$$

$$\Leftrightarrow \qquad Q = \hat{\Pi}_o^{-1}\tilde{Q}\,(\tilde{D}\,\tilde{\Sigma}_o^{-1})_o^{-1} \tag{8.90}$$

$\hat{\Pi}_o^{-1}$ is stable by assumption. Because $\tilde{D}$ is free of poles and zeros on the $j\omega$-axis any unstable poles of $(\tilde{D}\,\tilde{\Sigma}_o^{-1})_o^{-1}$ must be due to $\tilde{\Sigma}_o$. But substituting (8.90) into (8.89) gives

$$Q = \Phi^{-1}\hat{\Pi}_o^{-1}(\rho^{-1}\Phi - H_a)(\tilde{D}\,\tilde{\Sigma}_o^{-1})_o^{-1} \tag{8.91}$$

At the unstable poles of $(\tilde{D}\tilde{\Sigma}_o^{-1})_o^{-1}$, say $s=j\omega_i$, which are also the $j\omega$-axis zeros of ρ^{-1}, we have

$$Q(jw_i) = -\Phi^{-1}\,\hat{\Pi}_o^{-1}\,H_a(j\omega_i)(\tilde{D}\,\tilde{\Sigma}_o^{-1})_o^{-1}$$

But

$$H_a = \Phi\,\hat{\Pi}_i^{-1}H_1\Sigma_o^{-1}(\tilde{D}\,\tilde{\Sigma}_o^{-1})_i^{-1}$$

from (8.78). Therefore we have

$$Q(j\omega_i) = \hat{\Pi}^{-1}(j\omega_i)H_1(j\omega_1)\tilde{D}(j\omega_i)^{-1}$$

which is bounded on the $j\omega$-axis. Thus Q is stable. ∎

8.4-1 Characterization of all solutions

We are now in a position to characterize the set of all solutions.

Theorem 8.5 Given the optimum λ, any stable Q which gives $\|\Phi\|_\infty \le 1$ is a solution. Moreover, Q can be characterized by the following two-parameter interpolation problem:

$$U_i = \begin{bmatrix} F(Q\,)\Sigma_o^{-1} \\ P\Sigma_o^{-1} \end{bmatrix}$$

where $P \in H^\infty$ is any matrix which makes U_i an inner matrix, and Σ_o is the outer factor of Σ, where $\Sigma^*\Sigma \triangleq \lambda_o^2 I - H_2^* H_2$.

Proof Given λ, any solution to $\hat{Z}$ can be written as

$$\hat{Z}^*\hat{Z} = \lambda^2 I - P^*P \tag{8.92}$$

where

$$\sup_\omega \bar{\sigma}[P] \leq \lambda \tag{8.93}$$

and

$$\inf_\omega \bar{\sigma}[P] = 0 \ . \tag{8.94}$$

The corresponding (interpolation) problem is then equivalent to that of finding a Q which solves

$$F(Q)^*F(Q) = \lambda^2 I - H_2^* H_2 - P^*P \tag{8.95}$$

or $$F(Q)^*F(Q) + P^*P = \lambda^2 I - H_2^* H_2 \triangleq \Sigma^*\Sigma \tag{8.96}.$$

And this is equivalent to finding a Q such that

$$\begin{bmatrix} F(Q) \\ \cdots \\ P \end{bmatrix} = U_1 \Sigma \tag{8.97}$$

where U_1 is a 2mxm all-pass matrix satisfying $U_1(-j\omega)^T U_1(j\omega) = I$, and $U_1\Sigma$ is stable.

Proceeding as in (8.66), we factor Σ as

$$\Sigma = \Sigma_i \Sigma_o \tag{8.98}$$

where Σ_i is inner and Σ_o is outer. Then (8.97) is equivalent to finding a Q such that

$$\begin{bmatrix} F(Q)\Sigma_o^{-1} \\ \cdots\cdots\cdots \\ P\Sigma_o^{-1} \end{bmatrix} = U_i \qquad (8.99)$$

where $U_i = U_1 \Sigma_i$.

For simplicity we will assume that Σ_o^{-1} is stable; the alternative situation can be handled as before. Because Σ_o is stable and minimum-phase, P is stable if, and only if, $P\Sigma_o^{-1}$ is stable. Thus all solutions may be characterized by a two-parameter (namely $\bar{P} = P\Sigma_o^{-1}$ and Q) interpolation problem; that is, find $\bar{P}$ (equivalently P) and $Q \in H^\infty$ such that

$$\begin{bmatrix} (H_1 + \hat{\Pi}_1 Q \tilde{D})\Sigma_o^{-1} \\ \cdots\cdots\cdots\cdots \\ P\Sigma_o^{-1} \end{bmatrix} = U_i \qquad (8.100)$$

where $U_i(-j\omega)^T U_i(j\omega) = I$.

It should be clear to the reader that any Q which gives

$$\{(H_1 + \hat{\Pi}_i Q \tilde{D})\Sigma_o^{-1}\}^* \{(H_1 + \hat{\Pi}_1 Q \tilde{D})\Sigma_o^{-1}\} \leq I$$

on the $j\omega$-axis is a solution because any "missing part" can be made up $(P\Sigma_o^{-1})^*(P\Sigma_o^{-1})$. This completes the proof. ※

8.4-2 All-pass form solutions

A corollary of theorem 8.5 is that an all-pass form solution always exists, if a solution exists (as was asserted earlier).

Corollary 8.2 An all-pass form solution to the minimization of $\|Z\|$ exists if, and only if, a finite solution exists.

Proof It is obvious that if an all-pass form solution exists, then a finite solution exists because the all-pass form solution is a finite solution. To prove the converse, consider the square H^∞- optimization problem

$$\min_{Q \text{ stable}} \| F(Q)\Sigma_o^{-1} \| \stackrel{\Delta}{=} \mu \qquad (8.101)$$

It is well-known that if $\mu \leq 1$, then an all-pass solution for $F(Q)\Sigma_o^{-1}$ always exists, [SAF10], [GLO1]. In which case, we can choose P=0, the zero matrix, to give a solution to the non-square interpolation problem (8.100), and hence an all-pass form solution for Z. ∎

8.4-3 Conceptual algorithm

To summarize the design procedure which the analysis of the previous subsection suggests we outline a conceptual algorithm.

1. Find D, N, $\tilde{D}$, $\tilde{N}$, $\tilde{X}$, $\tilde{Y}$, and hence H and Π.
2. Find U, and hence Π_1, $\hat{\Pi}$ and $\hat{H}$.
3. Decompose $\hat{H}$ into $[H_1^T \quad H_2^T]^T$. and hence H_2.
4. Choose $\lambda = \|H_2\| + C$ where C is an arbitrary nonnegative real number.
5. Find Σ and factorize : $\Sigma = \Sigma_i \Sigma_o$.
6. Compute μ.
7. If $\mu = 1$, then compute Q by constructing Φ.
8. If $\mu < 1$, then λ is too large and so is decreased.
9. If $\mu > 1$, then λ is too small and so is increased.
10. If $\lambda \neq \| H_2 \|$, go to 6 and repeat.
11. If $\lambda = \| H_2 \|$ continue
12. Find Σ

13. If Σ is rank deficient, increase λ by an arbitrary small amount and compute μ. If $\mu < 1$, then compute Q which gives an arbitrarily close suboptimal solution. Otherwise go to 9 and repeat.
14. If Σ^{-1} exists, factorize Σ into $\Sigma_i\Sigma_o$, and find ρ, $\tilde{\Sigma}_i$ and $\tilde{\Sigma}_o$. Compute μ. If $\mu \leq 1$, compute Q which gives the optimum solution, otherwise go to 9 and repeat.

We note that in general, a binary search procedure for finding λ can be incorporated into the above algorithm. An alternative search algorithm is given in the appendix.

8.4-4 Relaxation of assumptions on weighting matrices and plant

We make a few passing remarks on the relaxation of some of the assumptions on W_1, W_2 and G. First, we note that in general if the interpolation problem (8.80) is solvable with $\|\Phi\| \leq 1$, then the existence of a Q is guaranteed irrespective of whether $H_1\Sigma_o^{-1}$ is proper or not. It thus follows that it may be possible to relax the assumption that W_2G is proper. Also, some of the techniques outlined in this section will be useful in handling cases where W_1 or W_2 have poles on the $j\omega$-axis, a situation which we will explore in the next section. Lastly, we mention that a method of handling $j\omega$-axis poles and zeros has been developed by Francis and Zames, [FRA7]. Although we remark that a plant having poles and/or zeros on the $j\omega$-axis may be considered a non-generic property and thus sometimes it may appear easier to simply "move" these offending poles and zeros by a small distance into the open left of right half-plane, we also give caution because such an approach could lead to numerical difficulties. In fact, the failure of the Nevanlinna-Pick algorithm to handle the cases where there are poles and/or zeros on the

jω-axis suggests that the algorithm could be numerically ill-conditioned if there are poles and zeros close to the jω-axis. A time domain approach such as Doyle's, [DOY7], or Glover's, [GLO1], may therefore be more attractive from a numerical point of view.

8.5 Robustness Optimization with Output Regulation

In this section, we briefly discuss the problem of robustness optimization with output regulation. A similar problem has been considered by Wang and Peason, [WAN3]. Our approach is slightly different in that we simply extend our result of the previous section to allow for unstable weighting matrices.

Strictly speaking, a more appropriate way to approach the problem of output regulation is, as shown in chapter 2, to consider Z of the form:

$$Z = [\ SW_1 \quad TW_2\ ] \qquad (8.102).$$

However, to be consistent with the previous section, we will still consider Z of the form given in (8.30). It should be clear to the reader that (8.102) can be solved by a similar method. We also note that, in general, Z of the form (8.102) is analytic in the CRHP if, and only if, Z of (8.30) is analytic in the CRHP.

As mentioned in the introductory section, robust exact unstable pole-zero cancellation is impossible in practice. Therefore the only unstable poles which we will consider permissible in the disturbance are the jω-axis poles. For simplicity, we assume the measurement noise to be stable, and therefore only $W_1(s)$ is allowed to have poles on the jω-axis. It also follows that G(s) and $W_1(s)$ are an admissible pair, (that is,

both stabilization and output regulation, or robust output regulation, can be achieved), because of the assumption that G(s) does not have any zero on the $j\omega$-axis. Nonetheless, we caution that if G(s) has an uncertain zero which may move onto the $j\omega$-axis, then forcing the controller to place a pole at that particular frequency (in order to regulate an unstable mode of D(s) at that frequency) could lead to an unstable pole-zero cancellation in the perturbed system. For this reason, we will assume that $W_1(s)$ and $G_p(s)$, all $G_p(s) \in \{G_p(s)\}$, are admissible pairs as well.

We point out that both output regulation and robust output regulation (and tracking, which is mathematically the same as the output regulation problem) can be handled in the same framework; the only difference is that, while output regulation calls for the use of unstable weitghting matrices, robust output regulation demands weighting matrices of the form $\alpha(s)^{-1}W(s)$, where $\alpha(s)$ is a least degree, stable and proper real-rational scalar function containing all the unstable modes of the disturbances as its zeros, and W(s) is free of zeros in the CRHP.

We begin by assuming that $W_1(s)$, $W_2(s)$ and G(s) satisfy assumptions (i)-(v) of the previous section with one exception — that is, $W_1(s)$ may contain $j\omega$-axis poles but the McMillian degree of each unstable pole of $[W_1(s)^T \quad W_2(s)^T]^T$ must be the same as the McMillian degree of each unstable pole of $W_d(s)$. Then write

$$W_1(s) = \frac{1}{\alpha(s)} \tilde{W}_1(s) \tag{8.103}$$

and

$$W_2(s) = \frac{1}{\alpha(s)} \tilde{W}_2(s) \tag{8.104}$$

where $\tilde{W}_1(s)$ and $\tilde{W}_2(s)$ are stable rational matrices. We then have

$$Z = \frac{1}{\alpha}\tilde{Z} \tag{8.105}$$

where

$$\tilde{Z} = \begin{bmatrix} \tilde{W}_1 S \\ \tilde{W}_2 T \end{bmatrix} \tag{8.106}.$$

Applying a similar transformation technique to $\tilde{Z}$ as was done to Z in the previous section, we obtain (c.f. (8.61)):

$$\hat{Z} = \frac{1}{\alpha}\left[\hat{H} + \hat{\Pi}Q\tilde{D}\right] = \frac{1}{\alpha}\begin{bmatrix} H_1 + \hat{\Pi}_1 Q\tilde{D} \\ H_2 \end{bmatrix} = \frac{1}{\alpha}\begin{bmatrix} F(Q) \\ H_2 \end{bmatrix} \tag{8.107}.$$

Now, proceeding as before, we consider the expansion of $\hat{Z}^*\hat{Z}$, that is

$$\hat{Z}^*\hat{Z} = \frac{1}{\alpha^*\alpha}\left[F(Q)^*F(Q) + H_2^*H_2\right] \tag{8.108}.$$

Let $\hat{Z}^*\hat{Z} = \lambda^2 I$ which gives

$$\frac{1}{\alpha^*\alpha}F(Q)^*F(Q) = \lambda^2 I - \frac{1}{\alpha^*\alpha}H_2^*H_2 = \Sigma^*\Sigma \tag{8.109}$$

Note that, from an examination of (8.109), if $\alpha^{-1}H_2$ is unstable then the optimization does not have a solution. However, our assumption ensures that a stabilizing and regulating controller exists and therefore $\alpha^{-1}H_2$ cannot be unstable. It follows that Σ is stable in (8.109). We thus finally have (as in (8.69) and (8.71))

$$U_i = \frac{1}{\alpha}\left(H_1\Sigma_o^{-1} + \hat{\Pi}_1 Q\,\tilde{D}\,\Sigma_o^{-1}\right) \tag{8.110}$$

$$= \frac{\rho}{\alpha} (H_1 \tilde{\Sigma}_o^{-1} + \hat{\Pi}_1 Q \tilde{D} \tilde{\Sigma}_o^{-1}) \qquad (8.111).$$

Expression (8.111) is mathematically identical to (8.71) and thus can be solved by using the same method as in the previous section.

8.6 Best Approximation and Optimal Design

In the previous two sections, we have developed a synthesis procedure which enables the designer to synthesize a feedback system. We note that when both $W_1(s)$ and $W_2(s)$ are chosen to be scalar functions, the procedure permits the designer to loop-shape and trade-off between a large $\bar{\sigma}[S]$ and a large $\bar{\sigma}[T]$. This is very similar to the classical approach in single-input, single-output system design when one strives to strike the "best" balance between good sensitivity at low frequencies and good stability margins at high frequencies. However, the additional flexibility offered by the non-scalar W_1 and W_2 permits one to capture the structure of uncertainties such as plant disturbance (previous section) or plant uncertainty. For example, assume we can characterize "directional" plant uncertainty in the following form, [SAF8], [DOY6] :

$$\Delta_o(s) = L(s)\Delta(s)R(s) \qquad (8.112)$$

where $L(s)$, $R(s)$ are real-rational transfer function matrices, and $\Delta(s)$ is an "unstructured" perturbation satisfying $\bar{\sigma}[\Delta(s)] \leq \delta(s)$, $\delta(s)$ a real-rational function. Then optimizing $\|Z\|$ with Z of the form

$$Z(s) = W_L(s)T(s)W_R(s) \qquad (8.113)$$

where

$$W_L(s) = \delta(s)R(s) \qquad (8.114)$$

and

$$W_R(s) = L(s) \qquad (8.115)$$

gives the "best" robust stability property against perturbations from the class defined by (8.112).

Unfortunately, the more general H^∞ optimization problem, such as one with Z of the form

$$Z = \begin{bmatrix} W_1 S & W_3 \\ W_2 T & W_4 \end{bmatrix}$$

has yet to be solved. So an alternative approach may need to be devised.

In practice, it is reasonable to expect that many designs obtained by the procedure in the previous two sections will not be viewed as the "best" by the designer. For example, at very low frequencies (which usually includes the frequency range of the command or reference signal), uncertainty in the plant model may be insignificant, and the presence of broad-band plant disturbance and measurement noise may be fairly accurately modelled as Gaussian white or coloured noises. In such cases it may be suitable to optimize the performance in the mean-square sense. Also, at frequencies where there is a possiblility of both pole and zero "movements" across the $j\omega$-axis, it may be more sensible to optimize

$$\inf_{Q} \sup_{\omega\epsilon[\omega_1,\omega_2]} \bar{\sigma}[A + QB] \qquad (8.116)$$

where

$$A = [Y \quad X]r \qquad (8.117)$$

and

$$B = [-\tilde{N} \quad \tilde{D}]r \tag{8.118}$$

which is derived from theorem 4.15.

The H^{∞} minimax approach, which is essentially a uniform approximation procedure, provides an excellent tool for trading-off the various types of optimization criteria which are important in different frequency ranges. For example, by using an LQG approach, we can find what the ideal sensitivity function matrix, and hence the complementary sensitivity fuction matrix, should be in the frequency range $\omega \in [0,\omega_1]$. At higher frequencies where noise statistics become uncertain and plant uncertainty becomes large, we will want to minimize $\bar{\sigma}[S]$ and $\bar{\sigma}[T]$. Thus, in its most general form, we may pose an optimal design problem in the form of

$$\inf_{\text{stabilizing } K(s)} ||Z|| \tag{8.119}$$

where

$$Z = \begin{bmatrix} W_1(S - S_1) \\ W_2(T - T_1) \\ W_3(S - S_2) \\ W_4(T - T_2) \\ \vdots \end{bmatrix} \tag{8.120}$$

S_1, S_2, ... (resp. T_1, T_2, ...) respresent the "ideal" sensitivity functions (resp. complementary sensitivity functions) in mutually exclusively frequency ranges. Note that if $\bar{\sigma}[S]$ and $\bar{\sigma}[T]$ are important, then the ideal sensitivity and complementary sensitivity matrices are the zero matrices. Thus, by appropriately choose the weighting matrices

which emphasize the relative importance of achieving the "ideal function" in the respective frequency range, one will be able to obtain an "optimal compromise" among the different goals in the various frequency ranges. We note that the idea of trying to design a feedback system which has a sensitivity function (or closed-loop response) which matches an ideal function in a best approximation sense is not new (see [BOD1], [EDM1] and [HUN1], for example), but from a design point of view, the H^{∞}- norm seems to be most suitable because it is precisely the L_2- induced operator-norm.

Mathematically, optimization of (8.119) and (8.120) is the same as the optimization problems of the previous two sections. For this reason, we will not discuss its solution.

8.7 Concluding Remarks

In this chapter, an attempt has been made to pose realistic fedback design problems, based on the minimization of a weighted combination of the sensitivity and complementary sensitivity matrices. By including both functions it is possible to properly characterize robust stability, and to compromise in a simple way between the effect of a disturbance at the plant output and the effect of measurement noise.

The robustness optimization problem is formulated as a non-square H^{∞} minimax optimization problem. We characterize all solutions to the optimization problem as a two-parameter interpolation problem. From this characterization it is easy to show that an all-pass form solution always exists as long as a solution exists..

In appendix VIII-1, we study the possibility of using non all-pass form solutions. A nested optimization criterion is proposed to enable us to search for the "best" solution from the set of all (non-unique) solutions. An algorihtm for finding a solution to the nested optimization problem is then proposed based on the newly developed strenghtened Nevanlinna-Pick algorithm, [YOU5].

APPENDIX VIII-1

"Best" Solution to the Robustness Optimization Problem.

In this appendix, we consider how non all-pass form solutions might be exploited. Motivated by the possiblity of improvements in disturbance rejection and robust stability, we introduce more optimization objectives to arrive at a "best" solution.

VIII-1 "Best" solution -- a formulation

In order to identify the "best" solution from all possible solutions, one needs to introduce more optimization objectives. For example, one could consider that because

$$\int_{-j\infty}^{j\infty} \mathrm{tr}\ Z^{*}Z\ ds \qquad \text{(VIII-1)}$$

may be looked upon as "average energy" (the LQG criterion), a good additional optimization would be to find a Q such that (VIII-1) is minimized subject to the constraint that

$$\sup_{\omega} \bar{\sigma}[Z] = \lambda_1 \qquad \text{(VIII-2)}$$

where λ_1 is the optimal minimax solution for $\|Z\|$. Unfortunately, a solution to the above problem does not always exist, except in special cases, because the H^∞- minimax optimization usually has a non-strictly-proper solution for the largest singular value. An exception to this is when there exists a Q such that

$$F(Q)\Sigma_o^{-1} = 0 \qquad \text{(VIII-3)}$$

and H_2 is strictly proper. Then the Q which solves (VIII-3) gives both the H^∞ and H^2 optimal solutions.

An alternative criterion arises quite naturally if we consider the vector gain of Z as follows, [MAC6]. Any unit input vector $u(j\omega)$ may be expressed as a linear combination of the orthonormal system of eigenvectors v_i of Z^*Z. That is

$$u = \sum_{i=1}^{m} c_i v_i \quad ; \quad \sum_{1}^{m} |c_i|^2 = 1 \qquad \text{(VIII-4)}$$

where $\{c_i : i = 1, \ldots, m\}$ is an appropriate set of co-ordinate determining constants, and therefore we have

$$\frac{u^* Z^* Z\, u}{u^* u} = \frac{\sum_{i=1}^{m} \sigma_i^2 |c_i|^2}{\sum_{k=1}^{m} |c_k|^2} \qquad \text{(VIII-5)}$$

where $\bar{\sigma} = \sigma_1 \geq \sigma_2 \cdots \geq \sigma_m = \underline{\sigma}$ denote the singular values of Z.

Thus, the worst case (maximum) vector gain of Z^*Z occurs when $|c_1|=1$ and $|c_i|=0$, $i \neq 1$. Note that one of the motivations for using the H^∞

approach rather than the LQG approach is to minimize the worst case effect, [ZAM], [ZAM6], [FRA6], [CHA2], [FRA8].

Now, assume that u cannot happen in the direction of v_1; that is we constrain $c_1 = 0$. Then the worst case is determined by σ_2. It thus makes sense to minimize the second largest singular value as well. The argument will be clearer if for the moment we consider Z in the form

$$Z = [SW_1 \quad TW_2] \tag{VIII-6}$$

because then Zu, where $u(s) = [d(s)^T \quad m(s)^T]^T$, d denotes disturbances at the plant output and m denotes measurement noise in the feedback path, determines the combined effect of d and n on the output.

In fact, the most general approach is to solve the following nested optimization problem :

$$\begin{aligned} &\min_{Q} \sup_{\omega} \sigma_m[Z] \\ &\text{subject to } \sup_{\omega} \sigma_{m-1}[Z] \text{ is minimized} \\ &\qquad \text{subject to } \sup_{\omega} \sigma_{m-2}[Z] \text{ is minimized} \\ &\qquad\qquad \vdots \\ &\qquad\qquad \text{subject to } \sup_{\omega} \sigma_1[Z] \text{ is minimized.} \end{aligned} \tag{VIII-7}$$

Solving problem (VIII-7) also has a good effect on the robustness of the stability property. This follows from Lehtomaki et. al., [LEH3], who demonstrate the importance of minimizing the second largest singular

value when the perturbation representing uncertainty is constrained not to occur in the direction of the singular vector associated with the largest singular value. If the perturbation is constrained not to lie in the first and second principal directions, the third singular value will become significant. By induction, we therefore expect (VIII-7) to give the most robustly stable system when subject to perturbations characterized by singular values and singular vectors.

If Z is square, or can be reduced to the form studied by Chang and Pearson, [CHA2], eqn (8.47), then the following strenghtened Nevanlinna-Pick algorithm, due to Young, [YOU5], will be applicable directly.

VIII-2 The Strenghtened Nevanlinna-Pick Problem

First we introduce a few definitions and some notation. For simplicity, we consider H_Z^∞ instead of H^∞. Write

$$s_j^\infty(G) \triangleq \sup_{z \in D} \sigma_j(G(z)) \qquad \text{(VIII-8)}$$

where D denotes the unit disc.

Define the sequence

$$s^\infty(G) = (s_1^\infty(G),\ s_2^\infty(G),\ \cdots) \qquad \text{(VIII-9)}$$

Note that the infinite sequence has at most m non-zero terms. Of course, $s_1^\infty = \|G\|$. The strenghtened Nevanlinna-Pick problem is to find a solution of the Nevanlinna-Pick problem which minimizes not only $\|G\|$ but also $s^\infty(G)$, with respect to the lexicographic ordering. When the

matrices involoved are square mxm, we have the following theorem which is a special case of theorem 2 in [YOU5]. Note that the algorithm gives successively the optimum singular values (largest first) and the associated singular vectors. In what follows, ∂D denotes the unit circle.

Theorem 8.6, [YOU5]. Let $F \in H_z^\infty$ and let $B \in H_z^\infty$ be an inner matrix such that $B^*F \in H_z^\infty + C^{mxm}$. The minimum of $s^\infty(G)$ over all G in the coset $F + BH_z^\infty$, with respect to lexicographic order, is attained for a unique element G in the coset. This G can be obtained by the following algorithm.

(i) Let K be closed subspace of L_m^2 such that $K \supseteq FH_z^2 + BH_z^2$.

(ii) Define an integer $r \geq 1$, subspaces V_j of L_m^2 and K_j, E_j, W_j of L_m^2, operators $T_j : V_j \to W_j$, $1 \leq j \leq r$, positive number t_j and vectors $v_j \in V_j$, $w_j \in W_j$, $1 \leq j \leq r$, inductively as follows.

(iii) Let $V_1 \stackrel{\Delta}{=} H_z^2$, $K_1 \stackrel{\Delta}{=} K$, $E_1 \stackrel{\Delta}{=} BH_z^2$ and $W_1 = K_1 \ominus E_1$. Let

$$T_1 = P_1 M_F : V_1 \to W_1$$

where M_F: $V_1 \to L_m^2$ is the operation of multiplication by F and P_1: $L_m^2 \to W_1$ is the orthognal projection operator.

(iv) If $T_1 = 0$, then $r = 0$ and the desired function G is identically zero. Otherwise, let $t_1 = \|T_1\| > 0$, and let $v_1 \in V_1$, $w_1 \in W_1$ be unit vectors such that $T_1 v_1 = t_1 w_1$.

(v) Now, suppose that V_i, K_i, E_i, W_i, v_i, w_i and t_i have been defined for $1 \leq i \leq j-1$. Let V_j be the orthogonal projection of V_{j-1} onto the pointwise orthogonal complement of v_{j-1} in L_m^2. Let K_j be the closure of the orthogonal projection of K_{j-1} onto the pointwise

orthogonal complement of w_{j-1} in L_m^2. Let E_j be the pointwise orthogonal complement of w_{j-1} in E_{j-1}, and let $W_j = K_j \ominus E_j$. Let

$$T_j = P_j M_F \quad : \quad V_j \to W_j$$

where $M_F : V_j \to L_m^2$ is a multiplication operator and $P_j : L_m^2 \to W_j$ is the orthogonal projection.

(vi) If $T_j = 0$ then let $r=j-1$ (in which case the construction stops). Otherwise let $t_j = \|T_j\|$ and let $v_j \in V_j$, $w_j \in W_j$ be unit vectors such that $T_j v_j = t_j w_j$.

(vii) The construction terminates, with $r \leq m$. The minimizing function G is given by

$$G(z) = \sum_{j=1}^{r} \frac{t_j}{(\bar{\sigma}[v_j(z)])^2} \; w_j(z) \; v_j(z)^*$$

for almost all $z \in \partial D$. Moreover, $s_j(G(z))$ is constant almost everywhere and equal to t_j on ∂D, so that

$$s^{\infty}(G) = (t_1, t_2, \cdots, t_r, 0, 0 \cdots)$$

Proof See [YOU5]. ※

VIII-3 Solution to the nested optimization problem

Following Young's notation we write problem (VIII-7) as

$$\min_{Q} s^{\infty}[Z] \qquad \text{(VIII-7)}$$

We may then state the following lemma which provides a necessary conditon for the solution to the nested optimization problem (VIII-7).

Lemma 8.5 Let Z_o be a solution to (VIII-7), and let Σ be given by

$$Z_o^* Z_o - H_2^* H_2 = \Sigma^* \Sigma \qquad \text{(VIII-10)}$$

Let Σ have inner-outer factorization $\Sigma = \Sigma_i \Sigma_o$ and let Φ be the solution to

$$\Phi = F(Q)\Sigma_o^{-1} : Q \text{ solves } \min_Q s^\infty[F(Q)\Sigma_o^{-1}] \qquad \text{(VIII-11)}$$

Then Φ must satisfy

$$\sup_\omega \sigma_i[\Phi] = 1 \quad , \text{ all } i. \qquad \text{(VIII-12)}$$

Proof Obviously $||\Phi|| > 1$ cannot happen because Z_o is a solution by assumption.

Now assume $||\Phi|| = 1$ but that Φ does not satisfy (VIII-12). Then we may let P be such that

$$(P\Sigma_o^{-1})^*(P\Sigma_o^{-1}) + \Phi^*\Phi = I \qquad \text{(VIII-13)}$$

on the $j\omega$-axis in which case $Z_1 \in H^\infty$ given by

$$Z_1^* Z_1 = Z_o^* Z_o - P^* P \qquad \text{(VIII-14)}$$

is a 'better' solution than Z_o to (VIII-7). This contradicts the assumption that Z_o actually solves (VIII-7). Hence Φ must satisfy (VIII-12) This completes the proof. ∎

We next consider the first stage of the nested optimization problem; that is

$$\min_{Q} \sup_{\omega} \sigma_2[Z]$$
$$\text{subject to } \sup_{\omega} \bar{\sigma}[Z] \text{ is minimized} \qquad \text{(VIII-15)}.$$

Let $\min_{Q} \sup_{\omega} \bar{\sigma}[Z] = \lambda_1$ and let the solution to (VIII-15) be Z, where

$$Z^*Z = \lambda_1^2 I - P^*P \ . \qquad \text{(VIII-16)}$$

Then as in the main chapter, the problem becomes equivalent to that of finding a Q which satisfies

$$\lambda_1^2 I - H_2^* H_2 - P^*P = F(Q)^*F(Q) \qquad \text{(VIII-17)}$$

where $\inf_{\omega} \underline{\sigma}[P] = 0$ and $\bar{\sigma}[P] \leq \lambda_1$. Note that the condition $\inf_{\omega} \underline{\sigma}[P] = 0$ can often be replaced by the condition $\underline{\sigma}[P] = 0$ because the Q's which solve $\mu = 1$ usually gives solutions of the form $\bar{\sigma}[F(Q)\Sigma_o^{-1}] = 1$, all ω. This implies that $\underline{\sigma}[P\Sigma_o^{-1}] = 0$, all ω. But Σ_o^{-1} has full rank for all ω. It therefore follows that $\underline{\sigma}[P] = 0$, all ω.

Let the singular (eigen) values of P^*P be $\gamma_1^2 \geq \gamma_2^2 \geq \cdots \geq \gamma_m^2$ where by necessity $\inf_{\omega} \gamma_m = 0$. Then the singular values of $\lambda_1^2 I - P^*P$ are $\lambda_1^2 - \gamma_m^2$, $\lambda_1^2 - \gamma_{m-1}^2$, ... , $\lambda_1^2 - \gamma_1^2$ in descending order. Thus minimizing $\sigma_2[Z^*Z] = \sigma_2[\lambda_1^2 I - P^*P]$ subject to $\bar{\sigma}[Z] = \lambda_1$ is equivalent to maximizing the second smallest singular value of P^*P subject to $\inf_{\omega} \gamma_m = 0$.

The following lemma establishes the form that P can take in (VIII-16) to give a solution to problem (VIII-15).

Proposition 8.3 If $\inf_{\omega} \sigma_{m-1}[P] = \gamma_{m-1}$ is a solution to the optimization problem (VIII-15), then $P \in H^{\infty}$ of the form

$$P(j\omega)^*P(j\omega) = J(j\omega)\Gamma J(j\omega)^* \qquad \text{(VIII-18)}$$

where

$$\Gamma = \begin{bmatrix} \gamma_{m-1}^2 & \cdots & 0 \\ & \ddots & \\ & \gamma_{m-1}^2 & \\ 0 & & \gamma_{m-1}^2 \end{bmatrix} \qquad \text{(VIII-19)}$$

and

$$J(j\omega)^* J(j\omega) = I \qquad \text{(VIII-20)}$$

provides a solution to (VIII-15) via (VIII-16).

Proof Assume $P = P_1$ is an H^∞ solution to (VIII-15) with singular value decomposition $J\Gamma J^*$. We first consider the simpler case in which there is a rational transfer function matrix J_R such that $J_R(j\omega) = J(j\omega)$. Then we can write

$$P_1^* P_1 = J_R \Gamma J_R^* + R^* R \quad . \qquad \text{(VIII-21)}$$

Note that R^*R always exists because on the $j\omega$-axis $P_1^* P_1 - J_R \Gamma J_R$ is nonnegative-definite.

Substituting for the optimal P in (VIII-17) gives

$$\lambda_1^2 I - H_2^* H_2 - J_R \Gamma J_R^* = F(Q)^* F(Q) + R^* R \qquad \text{(VIII-22)}$$

or

$$\lambda_1^2 I - H_2^* H_2 - J_R \Gamma J_R^* = F(Q)^* F(Q) + R^* R \qquad \text{(VIII-23).}$$

As in the main chapter therefore, the problem can be transformed into the interpolation problem of finding a Q such that

$$\begin{bmatrix} F(Q)\Sigma_o^{-1} \\ \cdots\cdots \\ R\Sigma_o^{-1} \end{bmatrix} = U_i \quad . \qquad \text{(VIII-24)}$$

Now since a solution exists by assumption, it follows that

$$\inf_{Q} \|F(Q)\Sigma_o^{-1}\| = 1 \qquad \text{(VIII-25)}$$

=> we can find Q such that $F(Q)\Sigma_o^{-1}$ is all-pass

=> we can choose R = 0.

In the general case where J is not rational we note that because P_1 is stable, $P_1^* P_1$ is analytic on the $j\omega$-axis. The singular values of P_1 on the $j\omega$-axis are therefore branches of analytic functions; they are simply the characteristic gain loci of the rational matrix $P_1^* P_1$. Likewise, the singular vectors of P_1 are branches of analytic functions defined on the same Riemann surface as the corresponding singular values. A consequence of this analyticity is that each singular value and each element of the singular vectors may be approximated as closely as desired by a rational function. The error of approximation will converge to zero as the degree n of each rational function is allowed to increase to infinity. The lemma therefore holds in the limit as $n \to \infty$. ∎

Proposition 8.3 leads us to suggest the following procedure for solving problem (VIII-15).

Suppose that for $\lambda = \lambda_1$ we have $\inf_{Q} \|F(Q)\Sigma_o^{-1}\| = 1$ (using, for example, the strengthened Nevanlinna-Pick algorithm of Young, [YOU5]). From [YOU5] we can also find the singular vector of $F(Q)\Sigma_o^{-1}$ associated with the maximum singular value and hence u_1, the singular vector of Z corresponding to $\lambda = \lambda_1$ as follows.

Let v be the eigenvector of $(F(Q)\Sigma_o^{-1})^*(F(Q)\Sigma_o^{-1})$ associated with the irreducible eigenvalue ($\mu = 1$). Then we have

$$v^*(F(Q)\Sigma_o^{-1})^*(F(Q)\Sigma_o^{-1})v = 1 \qquad \text{(VIII-26)}$$

on the $j\omega$-axis. From theorem 8.5 we then have

$$v^*(P\Sigma_o^{-1})^*(P\Sigma_o^{-1})v = 0 \ . \qquad \text{(VIII-27)}$$

Write $\Sigma_o^{-1}v = x$ we then obtain

$$x^* P^* P x = 0 \qquad \text{(VIII-28)}$$

where P is in the form given by proposition 8.3. This implies that the eigenvector associated with the smallest (trivial) eigenvalue of P^*P must align with x. But this eigenvector is also the eigenvector associated with the maximum eigenvalue of $Z^*Z = \lambda_1^2 I - P^*P$. It therefore follows that u_1 must align with x.

If $P_1^* P_1$ has the form given by proposition 8.3 then we can always write

$$Z_o^* Z_o = \lambda_1^2 I - P^* P$$

$$= \lambda_1^2 I - J \begin{bmatrix} \gamma_{m-1}^2 & & & 0 \\ & \ddots & & \\ & & \gamma_{m-1}^2 & \\ 0 & & & \gamma_{m-1}^2 \end{bmatrix} J^*$$

$$= J \begin{bmatrix} \lambda_2^2 & & & 0 \\ & \ddots & & \\ & & \lambda_1^2 \lambda_2^2 & \\ 0 & & & \lambda_2^2 \end{bmatrix} J^* \quad , \ \lambda_2^2 = \lambda_1^2 - \gamma_{m-1}^2$$

$$= J \begin{bmatrix} \lambda_2^2 & & & 0 \\ & \ddots & & \\ & & \lambda_1^2 - \lambda_2^2 + \lambda_2^2 & \\ & & & \lambda_2^2 \\ 0 & & & \ddots \\ & & & & \lambda_2^2 \end{bmatrix} J^*$$

$$= \lambda_2^2 I + J \begin{bmatrix} 0 & & & 0 \\ & \ddots & & \\ & & \lambda_1^2 - \lambda_2^2 & \\ & & & 0 \\ 0 & & & \ddots \\ & & & 0 \end{bmatrix} J^*$$

$$= \lambda_2^2 I + (\lambda_1^2 - \lambda_2^2) u_1 u_1^* \qquad \text{(VIII-29)}$$

λ_2 is the second largest singular value of Z. We therefore decrease λ_2 until the minimum is reached. The following lemma characterizes the minimum λ_2.

Lemma 8.6 Assume that $Z_o^* Z_o$ is of the form given by proposition 8.3. Then λ_2 is minimum if, and only if,

$$\inf_Q \sup_\omega \sigma_2[F(Q)\Sigma_o^{-1}] = 1 \qquad \text{(VIII-30)}$$

where Σ_o is the outer factor of Σ which is given by

$$Z_o^* Z_o - H_2^* H_2 = \lambda_2^2 I + (\lambda_1^2 - \lambda_2^2) u_1 u_1^* - H_2^* H_2$$

$$\triangleq \Sigma^* \Sigma \qquad \text{(VIII-31)}$$

Proof Necessity is obvious from lemma 8.5. To prove sufficiency, assume that λ_2 is not a minimum. Then there exists $\epsilon > 0$ such that

$$\lambda_2^2 I - \epsilon^2 I + (\lambda_1^2 - \lambda_2^2 + \epsilon^2) u_1 u_1^* \qquad \text{(VIII-32)}$$

is a solution for $Z_o^* Z_o$. Thus we have

$$\lambda_2^2 I + (\lambda_1^2 - \lambda_2^2) u_1 u_1^* - H_2^* H_2 = F(Q)^* F(Q) + \epsilon^2 I - \epsilon^2 u_1 u_1^* \qquad \text{(VIII-33)}$$

Because $\epsilon^2 I - \epsilon^2 u_1 u_1^*$ has rank m-1, it follows that $F(Q)\Sigma_o^{-1}$ cannot have two singular values equal to 1. It follows that if (VIII-30) is satisfied then λ_2 is the minimum. ∎

Having found the optimal value of λ_2, the corresponding singular vector of Z, u_2, can be determined. We may then repeat the above process to find the minimum value of the third singular value. By induction, the procedure can be repeated to solve the complete nested optimization problem (VIII-7).

We next outline a conceptual algorithm for solving the nested optimization problem. For ease of understanding, we have attempted to present the algorithm in a straightforward and naive manner. As a result, we would not expect the algorithm to be very efficient from a computational point of view.

Conceptual algorithm

1. Determine $\lambda_1 = \min_Q \sup_\omega \sigma_1[Z]$ and the corresponding singular vector u_1 making use of Young's algorithm.

2. Let $Z_o^* Z_o = \lambda_1^2 I - P_1^* P_1$

$$= \lambda_1^2 I - J_1 \begin{bmatrix} \gamma_{m-1}^2 & & & 0 \\ & \ddots & & \\ & & \gamma_{m-1}^2 & \\ 0 & & & \gamma_{m-1}^2 \end{bmatrix} J_1^*$$

$$= (\lambda_1^2 - \lambda_2^2) u_1 u_1^* + \lambda_2^2 I ,$$

where $\lambda_2^2 = \lambda_1^2 - \gamma_{m-1}^2$.

3. Let $\lambda_2 = \lambda_1$ and decrease until $\min_Q \sup_\omega \sigma_2[F(Q)\Sigma_o^{-1}]=1$ then λ_2 is equal to $\min_Q \sup_\omega \sigma_2[Z]$ subject to $\min_Q \sup_\omega \sigma_1[Z] = \lambda_1$.

4. Calculate the corresponding singular vector, u_2, of Z.

5. Let

$$Z_o^* Z_o = (\lambda_1^2 - \lambda_2^2)u_1u_1^* + \lambda_2^2 I - P_2^* P_2$$

$$= (\lambda_1^2 - \lambda_2^2)u_1u_1^* + \lambda_2^2 I - J_2 \begin{bmatrix} \gamma_{m-2}^2 & & & 0 \\ & \ddots & & \\ 0 & & \gamma_{m-2}^2 & \\ & & & \ddots \\ & & & \gamma_{m-2}^2 \end{bmatrix} J_2^*$$

$$= (\lambda_1^2 - \lambda_2^2)u_1u_1^* + (\lambda_2^2 - \lambda_3^2)u_1u_1^* + (\lambda_2^2 - \lambda_3^2)u_2u_2^* + \lambda_3^2 I$$

where $\lambda_3^2 = \lambda_2^2 - \gamma_{m-2}^2$.

6. Let $\lambda_3 = \lambda_2$ and decrease until $\inf_Q \sup_\omega \sigma_3[F(Q)\Sigma_o^{-1}] = 1$, then λ_3 is equal to $\min_Q \sup_\omega \sigma_3[Z]$ subject to $\min_Q \sup_\omega \sigma_2[Z] = \lambda_2$ and $\min_Q \sup_\omega \bar{\sigma}[Z] = \lambda_1$.

7. Repeat the above process until the full nested optimization problem is solved.

VIII-4 Uniqueness of the 'best' solution

In this section we make some remarks on the uniqueness of the solution to the nested optimization problem. Because the strengthened Nevanlinna-Pick problem has a unique solution (which, in our case, has to be all-pass), it implies that the "correction term", $P\Sigma_o^{-1}$, has to be zero. We therefore conclude that the solution to the nested optimization problem is unique in most cases, and that all singular values of the 'best' Z^*Z are constant (i.e. independent of frequency). Exceptional cases may occur when there exists some ω_o such that $\Sigma(j\omega_o)$ is singular. For example, assume the system is single-input, single-output and there exists a Q which solves

$$F(Q) = 0.$$

Then $Z^*Z = H_2^*H_2$ is obviously a solution. However, from corollary 8.2, $Z^*Z = \lambda^2$ where $\lambda = \| H_2 \|$ is also a solution.

VIII-5 Rationality of the 'best' solution

Because the strengthened Nevanlinna-Pick solution to $F(Q)\Sigma_o^{-1}$ has constant singular values, it follows that the associated singular vectors are, in general, rational functions. To see that this is the case, note that the singular vectors, v_i, of $F(Q)\Sigma_o^{-1}$ are given by

$$[\sigma_i^2 I - (F(Q)\Sigma_o^{-1})^*(F(Q)\Sigma_o^{-1})]v_i = 0.$$

Since σ_i is real and constant, it follows that

$$[\sigma_i^2 I - (F(Q)\Sigma_o^{-1})^*(F(Q)\Sigma_o^{-1})]$$

is real-rational if $F(Q)\Sigma_o^{-1}$ is real-rational. Thus v_i is real-rational.

v_i is real-rational implies u_i is real-rational, we therefore conclude that the best solution is real-rational.

VIII-6 Search algorithm for the optimal λ's

In this subsection we propose a search algorithm which may be combined with the binary search algorithm to improve the speed of convergence. For simplicity, we will consider only the largest singular value, i.e. λ_1. It should clear that the same approach can be repeatedly applied to the other singular values.

Assume $\mu = \inf_{Q} \|F(Q)\Sigma_o^{-1}\| < 1$, then we know that λ is not optimal. The problem is, how much should we decrease λ? Let $F(Q)\Sigma_o^{-1}$ be in all-pass form. Then we have

$$(F(Q)\Sigma_o^{-1})^*(F(Q)\Sigma_o^{-1}) = \mu^2 I$$

$$\Leftrightarrow \quad (P\Sigma_o^{-1})^*(P\Sigma_o^{-1}) = (1-\mu^2)I$$

$$\Leftrightarrow \quad P^*P = (1-\mu^2)\Sigma_o^*\Sigma_o$$

$$\Leftrightarrow \quad \underline{\sigma}[P] = (1-\mu^2)\underline{\sigma}[\Sigma_o]$$

$$\Leftrightarrow \quad \underline{\sigma}[P] = (1-\mu^2)\underline{\sigma}[\Sigma] \qquad \text{(VIII-34)}$$

(VIII-34) gives a suggestion for the minimum amount by which λ should be decreased, that is

$$\Delta\lambda = \inf_{\omega} (1-\mu^2)\underline{\sigma}[\Sigma].$$

CHAPTER IX

CONCLUSIONS

9.1 Summary

This thesis has presented new results on the robustness analysis and design of multivariable feedback systems. A main theme of the thesis is how, by introducing a topological concept of arcwise connectivity, many of the existing, as well as new, robustness results can be suitably combined and unified. The concept is intuitive if one considers stereographic projection onto the Riemann sphere and then applies the theory of homotopy. The new combined robustness test does not assume the nominal and perturbed plant models to share the same number of CRHP poles and/or zeros, and can be used to reduce conservatism. It was shown that the assumption of arcwise connectivity is not an unreasonable one in practice because it is closely tied to the genericity of feedback stability. Obviously, if the stability property is not generic, it cannot possibly be robust.

After a brief introductory chapter, the technical discussion begins in chapter 2 where necessary preliminaries are presented. The problem of representing uncertainty is then critically examined in chapter 3. Confining ourselves to the consideration of square, lumped and linear time-invariant systems, we introduce and present some of our main robust stability results in chapter 4. The assumptions of squareness, and finite dimensionality are then relaxed in chapter 5 where we extend our robust stability results to non-square as well as distributed systems. The reprentations of uncertainties are then refined in chapter 6, where we consider more structured perturbations. Our discussion on robustness

analysis ends in chapter 7 where we study the effect of nonlinear perturbations.

In chapter 8, based on our results of the earlier chapters, a robustness optimization problem is posed as a non-square H^{∞} - optimization : find a stabilizing controller K(s) which minimizes the cost function

$$||Z(s)||_{\infty} = \left|\left|\begin{bmatrix} W_1(s)S(s) \\ W_2(s)T(s) \end{bmatrix}\right|\right|_{\infty}$$

where S and T are the sensitivity and complementary sensitivity function matrices respectively, and W_1 and W_2 are weighting matrices. All solutions to the above optimization problem are then derived, and a 'best' solution is proposed (in appendix VIII-1) based on the following nested optimization problem : find a stabilizing K(s) which solves

$\min \sigma_m[Z]$

subject to $\sigma_{m-1}[Z]$ is minimized

subject to $\sigma_{m-2}[Z]$ is minimized

.

.

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subject to $\sigma_1[Z]$ is minimized.

9.2 Suggestions for Future Reaearch

We suggest a few directions which we think deserve further attention.

9.2-1 Non-square plants

Although almost all the analysis and design results reported in this thesis are applicable to non-square plants as well as square plants, the problem of robust design for non-square plants appears to be much more difficult in practice. For example, if a plant has more outputs than inputs, then we have

$$\bar{\sigma}[(I+G(j\omega)K(j\omega))^{-1}] \geq 1 \quad , \quad \text{all } \omega.$$

This implies that worst-case disturbance attenuation, [DAN3], and sensitivity reduction cannot be achieved. It is therefore both important and interesting to investigate the practical implications of these limitations. In many existing linear multivariable design techniques, one usually first attempts to "square-down", [KOU5], the plant in a way so that all the zeros of the "squared-down" plant are in the open LHP. High-gain feedback may then be applied to reduce the sensitivity, and improve the performance (disturbance rejection) of the system. However, a closer examination reveals that this approach could break down because:

(i) Good responses given by the "squared-down" outputs do not necessarily imply good responses in the actual outputs. For example, let $y = x_1 + x_2$, where y is the "squared-down" output, and x_1 and x_2 are the actual outputs to be controlled. Assuming that regulation is a design objective, then it is clear that $y \approx 0$ does not necessarily imply that $x_1 \approx 0$ and $x_2 \approx 0$.

(ii) The zeros of a "squared-down" plant can be extremely sensitive (see examples 3.3 and 3.4). If a zero has the possibility of "moving" into the CRHP, then obviously high-gain feedback cannot be used near that

frequency, thus severely restricting the achievable performance of the feedback system.

The problem of robust design for non-square plants therefore requires further attention. It would be desirable to know how one can square a plant down so that the minimum-phase property is robust, or, indeed, whether it can be achieved at all. It is also important to know if good performance and robustness cannot be achieved, what is the next best alternative. For example, one might consider controlling fewer outputs, or introducing more actuators and inputs (if possible), or introducing nonlinear control.

9.2-2 Capturing the structure of uncertainty

To-date, much remains to be done on how one can best capture the structure of the plant uncertainty in a usable and practical way for robustness analysis and design. Although the structured singular value of Doyle et. al., [DOY6], and the optimal non-similarity scaling technique of Kouvaritakis and Latchman, [KOU3], represent some significant work in this direction, much remains to be done especially in using the available phase information, [LOH1].

9.2-3 Numerical implementation of the H^∞ optimization

Although solutions and algorithms for deriving the solutions to H^∞ - optimization problem have been given in chapter 8, we have so far evaded the important problem of numerical efficiency and stability. Future efforts should be devoted to the development of efficient and reliable software for solving the optimization problem. In this direction, the recent and outstanding work of Glover, [GLO1], on optimal Hankel-norm model reduction could prove useful.

APPENDIX A

SINGULAR VALUES OF A MATRIX

For any complex mxn matrix A, there exists mxm and nxn unitary matrices U and V such that

$$A = U\Sigma V^H \tag{A.1}$$

$$= \sum_{i=1}^{r} \sigma_i u_i v_i^H \tag{A.2}$$

where Σ is an mxn matrix in the form

$$\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} ,$$

$\Sigma_r = \text{diag}[\sigma_1, \ldots, \sigma_r]$, $r = \text{rank}(A)$, and $u = \text{col}[u_1, \ldots, u_m]$, $v = \text{col}[v_1, \ldots, v_n]$. Note that σ_i's are real and positive. The elements of the n-tuple $[\sigma_1, \ldots, \sigma_r, 0, .., 0]^T$ are called the singular values of A, and the vectors u_i and v_i the left and right singular vectors of A, respectively. (A.1) is called a singular value decomposition (SVD) of A, [STR1].

It may be shown that the singular values of A are the positive square roots of the eigenvalues of the matrix $A^H A$, or equivalently, AA^H. The reader who is interested in the computational aspects of the SVD is referred to [DON1].

The following results on the singular values of a matrix is due to Fan, [FAN1].

<u>Theorem A</u> Let $A, B \in C^{mxm}$, then

(i) $\sigma_{i+j+1}(A + B) \leq \sigma_{i+1}(A) + \sigma_{j+1}(B)$ (A.3)

(ii) $\sigma_{i+j+1}(AB) \leq \sigma_{i+1}(A)\,\sigma_{j+1}(B)$ (A.4)

The maximum and minimum singular values of a matrix satisfy the following properties in which A and B may be non-square:

(i) $\underline{\sigma}(A) = \min_{\bar{\sigma}(x)=1} \bar{\sigma}(Ax)$; x a complex column vector (A.5)

(ii) $\bar{\sigma}(A) = \max_{\bar{\sigma}(x)=1} \bar{\sigma}(Ax)$; x a complex column vector (A.6)

(iii) $\bar{\sigma}(A^{-1}) = \frac{1}{\underline{\sigma}(A)}$ (A.7)

(iv) $\bar{\sigma}(A+B) \leq \bar{\sigma}(A) + \bar{\sigma}(B)$ (A.8)

(v) $\bar{\sigma}(AB) \leq \bar{\sigma}(A)\bar{\sigma}(B)$ (A.9)

(vi) $\sigma(A) - \bar{\sigma}(B) \leq \sigma(A+B) \leq \sigma(A) + \bar{\sigma}(B)$ (A.10)

(vii) $\underline{\sigma}(A)\underline{\sigma}(B) \leq \underline{\sigma}(AB) \leq \bar{\sigma}(A)\underline{\sigma}(B)$ (A.11)

(viii) If $m \geq n$ then $\underline{\sigma}(A) = 0 \Leftrightarrow$ A is rank deficient (A.12)

(ix) If A is square, then $\underline{\sigma}(A) \leq \rho(A) \leq \bar{\sigma}(A)$ (A.13)

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