

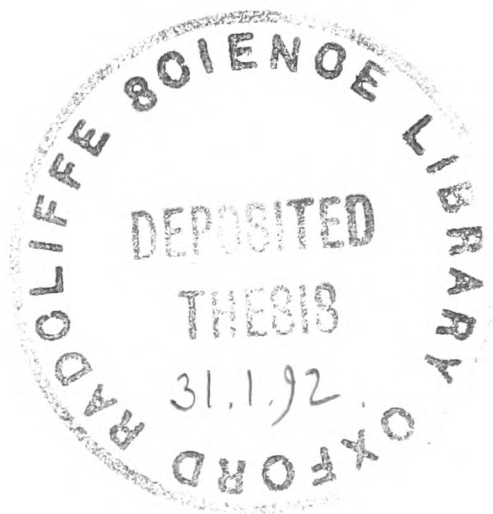
The Soliton of the Effective Chiral Action in the Two-Point Approximation

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Thesis submitted for the degree of Doctor of Philosophy
in the University of Oxford

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Abstract

In this thesis, we study the “two-point approximation” for highly non-local effective actions, in the particular case of the Chiral Soliton Model of the nucleon. The nucleon in this model is regarded as being made of three valence quarks bound together by a meson field in a soliton form. Mesons are treated in mean field theory and the vacuum energy due to one-quark loops is included. The theory is defined with a finite cut-off in momentum space, consistent with an effective theory for the low-energy description of the strong interactions.

We use the two-point approximation to calculate the vacuum correction to the chiral soliton energy for a variety of soliton profile functions, investigating the effect of different regularisation schemes. Results are little influenced by the choice of the cut-off, and are within 20% of exact calculations, done with the full inclusion of the Dirac sea.

We then perform a dynamical calculation of the chiral soliton by including sea-quark effects self-consistently in the two-point approximation. We find a typical 20% (or less) deviation in the soliton energy from exact calculations. We apply a further “pole” approximation which leads to a significant algebraic simplification in the self-consistent equations. We show, in particular, that a simple numerical fit of the pole form to the two-point cut-off function yields essentially indistinguishable results from the latter.

We finally calculate some static nucleon observables in the two-point approximation and find general agreement with exact calculations.

In view of the results obtained, we may hope that the pole form of the two-point approximation may prove to be a generally useful approach to similar problems involving highly non-local actions.

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To my parents

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Chapter 1

Introduction

Background

There is little doubt, with the beginning of this decade, that Quantum Chromodynamics (QCD) is the unique candidate for a theory of the strong interactions, supposedly responsible for most nuclear phenomena. The theory of QCD is formulated in terms of ‘coloured’ quarks, elementary matter fields, and gluons, self-interacting gauge fields mediating the strong interaction between the quarks. Nucleons were long believed (even before the advent of QCD in the early 1970’s) to be composite objects, the building blocks of which are called ‘quarks’. It is the discovery of ‘asymptotic freedom’, *i.e.* the coupling constant approaching zero at high energies or small distances, that led to the successful predictions of perturbative field theory (PFT). Calculations of high energy processes, using Feynman diagrams of quark and gluon lines, are in remarkable agreement with deep inelastic scattering phenomena. These experiments, which actually probe the inner structure of hadrons, constitute now an important experimental body supporting the existence of quarks, and the validity of QCD at high energies. Nature however prefers to hide the colour degrees of freedom at lower energies and only colourless bound states of quarks are seen at typical hadronic and nuclear energy scales.

This is known as ‘colour confinement’ and occurs in the non-perturbative regime of QCD. This regime is characterised by an increasing coupling constant as we go down in the energy scale or, equivalently, when the distance scale increases. Naturally, PFT breaks down at this scale since the perturbative series diverges. Unfortunately, there is – at the present time – no similar method to PFT which is applicable in the non-perturbative regime. Thus it is still an outstanding theoretical challenge to prove confinement or extract the asymptotic (hadronic) states from the perturbative (quark and gluon) degrees of freedom of QCD.

One way out is to use lattice computer calculations, solving the exact QCD equations numerically. Although they have shed some light on confinement and chiral symmetry breaking (also a property of low-energy QCD), lattice calculations are time-consuming and require a formidable computing power. Such direct QCD calculations may have to wait until substantially more powerful computers are built, before they can yield reliable results about the hadron spectrum.

Alternatively, many phenomenological models have been constructed for describing hadron physics and in particular nucleons, some of which even predate the discovery of QCD — for example, the early constituent quark models and the Skyrme model (invented when quarks were still unheard of). Despite the relatively large number of models found in the literature nowadays, a rough classification may be attempted, interpolating between two opposite views. At one end, we find the quark models which view hadrons as bound states of valence quarks (and anti-quarks) interacting through phenomenological potentials. These models, in their most recent form [Pat90], still offer the best way to account for meson and baryon spectroscopy. At the other end, we see the Skyrme model in which quarks are absent altogether and the baryon is a topological soliton (skyrmion) of a non-linear field theory of mesons. Skyrme-like models have had only partial success in explaining the spectrum of hadrons, and are also capable of describing

(with the same Lagrangian) soft meson scattering processes. All other models take an intermediate view where baryons are somehow ‘hybrid’ objects involving valence quarks in a meson cloud. The hybrid models seem to provide a conceptually economical way of interpolating between the high and the low energy regimes of the theory. The quarks are more adequate for describing the short distance properties and the mesons for the long range ones. All of these effective phenomenological models claim to be ‘inspired’ in one way or another from QCD.

The model we are concerned with in this thesis belongs to the ‘hybrid approach’ and is called the Chiral Soliton Model (CSM), where quarks are quantum fields and mesons are treated classically. The CSM possesses most known QCD properties including chiral symmetry breaking. It does not contain confinement though. It is believed, from lattice calculations, that the confinement energy scale is somewhat smaller than the chiral symmetry breaking scale, implying that the latter could be the determining factor in the low-energy spectrum and properties of hadrons. In the CSM, meson fields (in a soliton configuration) are background fields for the quarks, whose quantum effects are included by considering the polarisation of the Dirac sea due to the presence of the soliton. The effective action associated with the model arises from a fermion determinant and is highly non-local, since the soliton is an extended object. In the meson sector, an expansion in powers of the derivatives of the background fields turns out to be a good approximation to the non-local action (such an expansion provides corrections to “soft pion” results in π - π scattering, for example). However, the assumption of slowly varying background fields is not valid in the baryon sector. Hence an exact eigenvalue summation over the perturbed Dirac sea orbitals is required to calculate, for instance, the static soliton energy. As an alternative to such a direct but laborious evaluation of the effective action, this thesis is concerned with the study of a non-local approximation to the fermion determinant

in the soliton sector. Although we are only treating the special case of the chiral soliton model, we anticipate that this approximation may prove to be a useful approach to similar problems involving highly non-local actions.

Outline of the thesis

Besides the general introduction given above, chapter two starts with a short introduction to the Skyrme model, the qualitative features of which will be useful in understanding the chiral soliton model subsequently discussed in some detail.

In chapter three, we investigate a non-local (two-point) approximation to the static vacuum energy, due to Diakonov *et al.* [DPP89], for different regularisation schemes and soliton profile functions, in the baryon number-zero sector. We compare our results with similar work [MGG90] based on the exact evaluation of the Dirac sea. Our results are little influenced by the choice of the regularisation scheme, and within 20% of the exact calculations.

Chapter four presents a self-consistent calculation of the soliton profile function in the baryon number-one sector. We show that, using the two-point approximation, the problem is reduced to numerically solving a system of coupled integro-differential equations. Again, we compare our results with exact self-consistent calculations [MG91]. We find a typical 20% discrepancy in the soliton energy, for constituent quark masses between 300 and 500 MeV.

In chapter five, we use a further (pole) approximation to calculate explicitly the coordinate-space Green functions associated with the momentum space two-point function. This approximation leads to a significant simplification in the self-consistent equations. In particular, we show that a simple numerical fit of the pole two-point function to the proper-time two-point function, yields essentially indistinguishable results from the latter.

Finally in chapter six, we calculate some static nucleon observables, using the

two-point self-consistent solutions of chapter four, and compare them with the exact calculations of Refs. [MG91, WY91] to find general agreement. A ‘two-point’ expression for the vacuum contribution to the moment of inertia is seen to yield a considerable improvement over the derivative expansion, although it still only contributes 50% of the exact result. However, we find that the same ‘two-point’ expression approximates very well the exact result if the ‘exact’ self-consistent profile (taken from Ref. [Alk90]) is used.

Some results in chapter three are contained in Ref. [ZA91], submitted to *Int. J. Mod. Phys. A*. The main results in chapter four and five have appeared in *Phys. Lett. B* [AAZ91a] and a more comprehensive account is in preparation [AAZ91b] for submission to *Nucl. Phys. A*.

Chapter 2

The Effective Chiral Action

At low energies, the strong interactions are distinguished by both the broken chiral symmetry phase and the confinement of the colour degrees of freedom. However, we shall take the view that it is chiral symmetry breaking (CSB) which is the essential element in an attempt to try and understand the spectrum of the light strongly-interacting particles. Hence any model supposed to describe the low-energy properties of the underlying theory of the strong interactions, QCD, must incorporate at least chiral symmetry realised in the Goldstone mode. This fact was known to physicists for a long time and was used to construct phenomenological meson Lagrangians to describe soft pion processes, consistent with those predicted by current algebras (see Refs. [AD68, Lee72] for a general review). The Skyrme model, as a pre-QCD model for the nucleon based on the non-linear σ -model for low-energy pion physics, was the first attempt to unify in the same model both mesons and nucleons. Skyrme [Sky58, Sky61a, Sky61b] proposed to regard the nucleon as a soliton arising from a constrained pion Lagrangian. Then, after the advent of QCD, t'Hooft [t'H74] and Witten [Wit79] established a conceptual link between phenomenological meson Lagrangians and QCD. The basic idea is that, assuming confinement, theories of the QCD type with N_c colours

are equivalent in the large- N_c limit to non-interacting bosonic theories involving mesons (and glueballs) alone. Witten conjectured that this must be true for finite N_c and later, in two remarkable papers [Wit83a, Wit83b], went on to show how baryons could be accommodated into the same scheme by identifying them with solitons of the bosonic theory. Moreover he showed, by considering the topological Wess–Zumino action, that these solitons would be quantised as fermions for N_c odd and as bosons for N_c even (see [Ait87a] for a review). Witten’s conjecture made Skyrme’s original idea sound like a prophetic one, so much so that the early 1980’s saw numerous efforts to try and ‘derive’ the phenomenological meson Lagrangians, including Skyrme’s, directly from QCD. Current understanding however [Bal90] tends to disbelieve Witten’s conjecture for N_c finite (presumably not even large enough for QCD) and suggest that quarks may be an essential ingredient in any sensible model for the nucleon. There are comprehensive reviews on the Skyrme model (see for example [Ait88]) and we only give a brief account of its salient features in this chapter, after having highlighted the role of chiral symmetry breaking in the strong interactions. We will then go on to present the chiral soliton model we are concerned with in this thesis. An up-to-date general review on soliton models for the nucleon is given in [Bir90].

2.1 Chiral symmetry breaking

Experimentally, the lightest hadrons are a family of eight pseudo-scalar mesons forming an octet of the approximate $SU(3)$ flavour symmetry. These are the η , the kaons and the three light pions (π^0 , $\pi^\pm$). The pions form a triplet of the $SU(2)$ flavour subgroup. Restricting ourselves to the $SU(2)$ subgroup, these observations can be explained, from the point of view of QCD, by considering the

QCD Lagrangian density with two quark flavours: the up (u) and down (d):

$$\mathcal{L}_{QCD} = \bar{\psi}_u(i\gamma^\mu D_\mu - m_u)\psi_u + \bar{\psi}_d(i\gamma^\mu D_\mu - m_d)\psi_d \quad (2.1)$$

where D_μ is the covariant derivative and where colour indices and the pure gluon term are suppressed. If we make the approximation $m_u \simeq m_d$, then $\mathcal{L}_{QCD}$ is invariant under $SU_V(2)$ (isospin) transformations:

$$\Psi \longrightarrow \Psi' = e^{i\vec{\alpha} \cdot \frac{\vec{\tau}}{2}} \Psi. \quad (2.2)$$

If we further assume that $m_u \approx m_d \approx 0$, $\mathcal{L}_{QCD}$ is also invariant under $SU_A(2)$ (axial isospin) transformations:

$$\Psi \longrightarrow \Psi' = e^{i\vec{\beta} \cdot \frac{\vec{\tau}}{2} \gamma^5} \Psi. \quad (2.3)$$

The total symmetry group $SU_L(2) \times SU_R(2)$, where right (R) and left (L) subgroups are decoupled, is the chiral symmetry group. Experimentally, $SU_V(2)$ is manifest in the isospin multiplet structure of hadrons, while $SU_A(2)$ is *not*, since no parity degenerate states are observed. It follows that the axial symmetry must be realised in the Goldstone mode with the appearance of three Goldstone massless particles which, in the real world, are associated with the three light pions. Their non-vanishing mass is related to the non-zero, though small, current quark masses m_u and m_d .

The non-linear σ -model (NL σ M) is defined by the Lagrangian

$$\mathcal{L}_{NL\sigma M} = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2} \partial_\mu \vec{\pi} \partial^\mu \vec{\pi} \quad (2.4)$$

where σ is a scalar field and the chiral partner of the pseudo-scalar pions, subject to the constraint

$$\sigma^2 + \vec{\pi}^2 = f_\pi^2, \quad (2.5)$$

where f_π is the pion decay constant taken from experiment. $\mathcal{L}_{NL\sigma M}$ embodies chiral symmetry breaking and reproduces remarkably well all low-energy pion scattering processes [Lee72].

2.2 The Skyrme model

Skyrme [Sky58] remarked that the Lagrangian (2.4), together with the non-linear constraint (2.5), supported a non-trivial soliton solution characterised by a conserved topological winding number (defined below). Topological solitons are associated with finite energy field configurations divided into disconnected sectors [Raj84]. Such a field configuration is a map from the physical space into the field space of those configurations satisfying the constraint (2.5), which is the 3-D sphere S^3 . The finite energy requirement implies that the meson field approaches a unique value satisfying (2.5) at spatial infinity, hence compactifying the physical space R^3 into S^3 . Formally, the map $S^3 \rightarrow S^3$ leads to the homotopy classes $\pi_3(S^3) = Z$. Solitons are then characterised by a winding number $n \in Z$, defined as the spatial integral of the zeroth component W^0 of the conserved Skyrme topological current

$$W^\mu = -\frac{1}{12\pi^2 f_\pi^4} \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd} \phi_a \partial_\nu \phi_b \partial_\lambda \phi_c \partial_\rho \phi_d \quad (2.6)$$

with $\partial_\mu W^\mu = 0$ identically, and where the σ and pion fields are written in the $O(4)$ invariant form: $(\sigma, \vec{\pi}) = (\phi_0, \vec{\phi})$.

Skyrme associated the soliton of (2.4) with the nucleon by ingeniously identifying the winding number with baryon number, which is experimentally conserved. The so-called ‘hedgehog’ ansatz

$$\phi_0(\vec{r}) = f_\pi \cos \theta(r) \quad (2.7)$$

$$\vec{\phi}(\vec{r}) = f_\pi \frac{\vec{r}}{r} \sin \theta(r) \quad (2.8)$$

gives a soliton configuration with maximal symmetry under $SU_V(2)$, invariant under the combined spin and isospin $(\vec{J} + \vec{T})$ rotation. The winding number n is given by the difference of the values of the profile function $\theta(r)$ at $r \rightarrow \infty$ and

$r = 0$, through $\theta(\infty) - \theta(0) = n\pi$. The baryon density is given by the zeroth component of the topological current [Sky61b]:

$$\rho(r) = W^0(r) = -\frac{1}{2\pi r^2} \sin^2 \theta(r) \frac{d\theta(r)}{dr}. \quad (2.9)$$

The soliton we have just considered has however a major problem: left alone, it will inevitably collapse to zero size and zero energy. If R represents a characteristic size parameter of the soliton, the behaviour of the soliton energy will scale like $f_\pi^2 R$, which is minimised for $R = 0$. This result is in fact a consequence of Derrick's scaling theorem [Der64] for solitons in scalar field theories. To remedy this problem, Skyrme [Sky61b] added a further term to $\mathcal{L}_{NL\sigma M}$:

$$\mathcal{L}_{SK} = \frac{1}{4\varepsilon^2 f_\pi^2} \left[(\partial_\mu \phi_a \partial_\nu \phi_a)^2 - (\partial_\mu \phi_a)^4 \right]. \quad (2.10)$$

where ε is an arbitrary positive constant. $\mathcal{L}_{SK}$, which is fourth order in powers of derivatives, is known as the Skyrme term and scales like $1/(\varepsilon^2 R)$. The original Skyrme model Lagrangian is thus $\mathcal{L}_{NL\sigma M} + \mathcal{L}_{SK}$, supporting solitons with finite size and energy. These stable solitons are called skyrmions. Hence, the Skyrme Lagrangian can describe both baryons via its soliton solutions, and low-energy pion scattering by taking it up to second order in powers of derivatives, *i.e.* restricting it to the $NL\sigma M$ Lagrangian. Despite its attractiveness, Skyrme's idea was considered as no more than a mathematical curiosity for almost two decades, until Witten's conjecture gave it a theoretical foundation by relating it to QCD. The challenge was then to somehow justify the Skyrme model from first principles.

To this end, two approaches have been followed [Ait87b]. The first is based on Witten's conjecture and involves phenomenological meson (Skyrme-like) Lagrangians with a finite number of meson degrees of freedom, while the second makes use of the QCD 'inspired' quark-meson Lagrangians. The works of Pham and Truong [PT85]; Aitchison, Fraser and Miron [AFM86] and Tudor [Tud86] are

representative of the first approach. By considering various meson Lagrangians, these authors show how the decoupling of heavier mesons, like the vector-isovector ρ , leads to a purely pionic Lagrangian where the Skyrme term $\mathcal{L}_{SK}$ arises. They also find that, in general, other ‘non-Skyrme’ terms are present in such derivations which destabilise the skyrmion. Aside from the fact that phenomenological meson Lagrangians are non-renormalisable and should only be used at tree-level, they also lead to a nucleon mass considerably larger than the experimental value, if the parameters of the models are such as to fit the low-energy pion scattering data [PT85]. Thus it seems that more (heavier) mesons should be included, although the results obtained so far are not very encouraging [LLMC87, Che88]. As we will see in the next section, this fact is reminiscent of the failure of the derivative expansion in the soliton sector.

2.3 Quark-meson models and the derivative expansion

Various approaches [MM85, Sim86, PRC87, Bal87] in an attempt to integrate out the gluons from the QCD action, seem to arrive at a Nambu-Jona-Lasinio type Lagrangian [NJL61]. A purely non-perturbative approach, based on an instanton-liquid model due to Diakonov and Petrov [DP86], appears to agree with the functional methods. In its simplest form, the NJL Lagrangian reads:

$$\mathcal{L}_{NJL} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + \frac{G}{2} [(\bar{\psi} \psi)^2 + (\bar{\psi} i \vec{\tau} \gamma^5 \psi)^2]. \quad (2.11)$$

The corresponding (normalised) generating functional

$$\mathcal{Z} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^4x \mathcal{L}_{NJL}} \quad (2.12)$$

can equivalently be written as

$$\mathcal{Z}' = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\phi_0 \mathcal{D}\vec{\phi} e^{i \int d^4x \mathcal{L}'_{NJL}} \quad (2.13)$$

with

$$\mathcal{L}'_{NJL} = \bar{\psi} i \gamma^\mu \partial_\mu \psi - g \bar{\psi} (\phi_0 + i \gamma^5 \vec{\tau} \cdot \vec{\phi}) \psi - \frac{g^2}{2G} (\phi_0^2 + \vec{\phi}^2). \quad (2.14)$$

The ϕ fields are non-dynamical auxiliary spin-zero fields. The equivalence between (2.11) and (2.14) can easily be seen by performing the formal Gaussian integration over the boson fields in (2.13). It is the latter form of the model in the mean field approximation (ϕ treated classically), which is directly related to the ‘hybrid’ quark-meson models. The NJL Lagrangian is in fact chirally invariant, and the broken chiral symmetry phase is implemented phenomenologically by giving the ϕ_0 field a non-zero vacuum expectation value:

$$\langle \bar{\psi} \psi \rangle \equiv \langle \phi_0 \rangle = f_\pi, \quad (2.15)$$

while it is kept zero for the pion field. As a result, the quarks acquire a dynamically generated mass $m = g f_\pi$. Since f_π is fixed, m depends only on the arbitrary coupling constant g . All field configurations satisfying $\phi_0^2 + \vec{\phi}^2 = f_\pi^2$ are degenerate minima in the theory, which implies the possible existence of soliton solutions. We shall impose this condition (as in the NL σ M), and as a result the last term in (2.14) drops out as an irrelevant constant.

Let us now see how the non-linear version of (2.14) relates to the Skyrme Lagrangian. When the meson fields are treated classically, the corresponding effective action is trivially obtained by performing the formal integration over the Grassmann variables $\bar{\psi}$ and ψ in (2.13):

$$e^{i\Gamma(\phi)} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^4x \bar{\psi} (i \gamma^\mu \partial_\mu - g \phi_0 - i g \gamma^5 \vec{\tau} \cdot \vec{\phi}) \psi} \quad (2.16)$$

leading to

$$\Gamma(\phi) = -i N_c \text{Tr} \ln(i \gamma^\mu \partial_\mu - g \phi_0 - i g \gamma^5 \vec{\tau} \cdot \vec{\phi}), \quad (2.17)$$

where Tr means functional as well as Dirac and $\text{SU}(2)$ traces. The action $\Gamma(\phi)$ is a highly non-local object giving the first order quantum corrections to the classical effective action, which (in the non-linear case) is not present since the classical meson fields are non-dynamical. These corrections represent in fact the one quark-loop contribution (colour degrees of freedom contribute additively) or, in other words, the polarised vacuum in the presence of the background fields ϕ .

One much studied approach to effective actions given by fermion determinants, such as Γ , is the so-called derivative expansion. It was first applied in this context by Goldstone and Wilczek [GW81] as a way to evaluate the fermionic induced charge in the polarised vacuum. Later, Aitchison and Fraser (AF) [AF85] devised a general procedure based on functional methods by means of which a derivative expansion can be obtained. Another method was developed by Zuk [Zuk85a, Zuk85c] using proper-time techniques. Formally, one takes the adiabatic approximation in which the meson fields are slowly varying and considers the expansion

$$\Gamma(\phi) = \int d^4x \left[-V_{eff}(\phi) + \frac{1}{2}Z_{eff}(\phi)(\partial\phi)^2 + \dots \right] \quad (2.18)$$

where the coefficients V_{eff} (effective potential), $Z_{eff} \dots$ are to be determined. The effective action thus takes the form of an infinite sum of local terms with higher and higher powers of field derivatives. The series can then be truncated at some finite number of terms and one is left with a local effective action suitable for low energy physics. AF find that the coefficient of the quadratic term Z_{eff} is logarithmically divergent (see section 3.1 in next chapter). Its renormalisation requires however the addition (by hand) of a corresponding local counter-term in the effective Lagrangian. The divergent $0(\partial\phi)^2$ piece in $\Gamma(\phi)$ is thus absorbed into a bare pion kinetic term ($\mathcal{L}_{NL\sigma M}$), while the finite contribution is fixed by taking the pion wave-function coefficient f_π to be given by its phenomenological value

93 MeV. We note that the absence of this term (corresponding to the pion kinetic energy) from (2.14) reflects the non-renormalisability of the four-fermi interaction in the original NJL Lagrangian (2.11). The infinite renormalisation prescription yields the chiral quark-meson models of Kahana, Ripka and Soni [KRS84] and Birse and Banerjee [BB85].

Following the renormalisable approach, Aitchison and Fraser [AF84, AF85, Ait87c] calculated the fourth order terms in (2.18) using their functional method. They found not only the Skyrme term $\mathcal{L}_{SK}$ (given in (2.10)), but also a destabilising ‘non-Skyrme’ term with an overwhelming negative coefficient. The result is the inevitable collapse of the soliton. The same result was obtained by McKenzie *et al.* [MWZ84] using a graphical technique. With the hope that higher order derivative terms in (2.18) might provide the necessary stability, the sixth order terms were calculated in [Zuk85b, Zuk85c] and in [AFTZ85]. The result is that the sixth order contribution, although stabilising, is not large enough to prevent the soliton from collapsing to too small a size.

The derivative expansion was recognised as being inadequate for the evaluation of the soliton energy, when Ripka and Kahana [RK85, KPR85] computed the full contribution of the (renormalised) effective action $\Gamma(\phi)$ to the static soliton energy, thus including all terms in the derivative expansion (2.18). Their calculation showed that quantum loop corrections could effectively stabilise the soliton, while the derivative expansion was found to be at best asymptotic [BS88]. The approximation can only be justified in the case of large soliton sizes ($mR \gg 1$), where the meson fields have a large spatial extent and are slowly varying, and is certainly ill-conditioned for realistic soliton sizes for which the background fields are rather strongly varying. This may also explain why the phenomenological meson Lagrangians with a finite number of degrees of freedom could not do the job, since they are in effect equivalent to the derivative expansion of the effective

chiral action (2.17).

As it is practically impossible to construct a Lagrangian with an infinite number of meson degrees of freedom, the only possible avenue for reaching a low-energy effective theory for the strong interactions seems to consider the compact form of the fermion determinant (2.17), thus keeping its non-locality by retaining all terms in the derivative expansion. However, even if the soliton is obtained in this way, it is not absolutely stable. It has been shown [RK87, DPP88, DPP89, BS88, Zuk90] that the vacuum energy for a static soliton goes to zero when the soliton size parameter vanishes, indicating that the finite-energy soliton may tunnel to the vacuum. A hard core of valence quarks is therefore needed to prevent this, somehow putting the quarks back into the picture of the nucleon and departing from Skyrme's original idea. Hence, the simple idea of identifying the winding number with baryon number does not seem realistic, since we cannot get rid of the non-locality of the fermion determinant after all. Nor does Witten's conjecture seem plausible in fact (at least in the case of QCD) as far as the absolute stability of the soliton is concerned.

2.4 The chiral soliton model with finite cut-off

In the Skyrme model and Skyrme-like models (*i.e.* based on Witten's conjecture), nucleons are identified with solitons in an *ad hoc* way, which amounts to equating the winding number of the meson field with the baryon number. In 'hybrid' quark-meson models, the baryon number is due to the valence quarks, each one carrying one third of the total baryon charge. But as we will now see, the original Skyrme's identification in fact still holds in a particular regime of the theory, thereby keeping the link between winding number and baryon number.

Kahana and Ripka (KR) [KR84] computed the static vacuum soliton energy

arising from the one-quark loop quantum correction by evaluating the (renormalised) fermion determinant (2.17). For time-independent background fields we have $\Gamma(\phi) = -E_{vac}(\phi)T$ with

$$E_{vac}(\phi) = N_c \sum_{E_n < 0} (E_n - E_n^0) \quad (2.19)$$

calculated relative to the uniform background field configuration ($\phi_0 = f_\pi$, $\vec{\phi} = 0$). KR consider a non-trivial topological meson field configuration with winding number one, given by the linear profile function θ as shown in Fig. 2.1. The parameter $X = mR$ refers to the dimensionless soliton size. They then diagonalise the Dirac Hamiltonian appearing in (2.17) in a suitable basis and obtain both the positive and negative energy spectra. Besides the distortion of the Dirac sea orbitals (vacuum polarisation), Fig. 2.1 shows that as the soliton size is increased, a positive energy orbital emerges from the continuum and acquires negative energy for a critical soliton size ($X \simeq 3$). As X is increased further, this same orbital (labelled as 0^+) becomes more and more negative until it merges eventually with the Dirac sea orbitals. No other orbitals acquire negative energy and a mass gap appears for even larger soliton sizes. A system with baryon number one relative to the physical vacuum (where all Dirac sea orbitals are filled) is obtained by putting N_c coloured quarks in the valence orbital 0^+ . Then, for $X < 3$, the valence orbital is positive and accounts for the baryon number of the system. For $X > 3$, the Dirac sea increases its baryon number by one. In the limit of large soliton size, the valence quarks lose completely their identity and the nucleon becomes a pure skyrmion, thus joining the view of the Skyrme model. As there is no discontinuity in the baryon number when the valence orbital crosses the zero-energy axis, we conclude that the baryon charge comes from the balanced contributions of the valence quarks, which prevail for short distances, and the Dirac sea which takes over at large distances, where a Skyrme picture of the nucleon is recovered. The

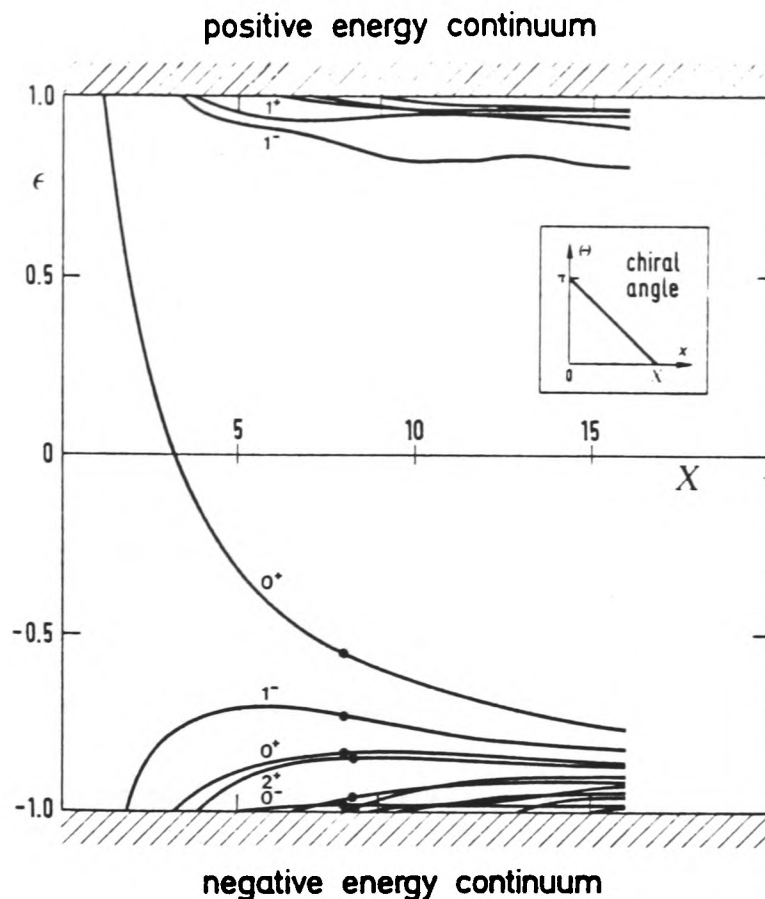


Figure 2.1: Energy spectrum of the Dirac Hamiltonian with a winding number-one meson field configuration. Energies are given in units of the constituent quark mass $m = g f_\pi$. This figure is taken from Ref. [KR84].

identification of winding number with baryon number is only true for the large soliton size limit. It is not an essential identification though, since in this limit the skyrmion is too large to be realistically interpreted as a nucleon. KR repeated their analysis for a winding number-two hedgehog field and found that, this time, two positive energy orbitals acquire negative energy, eventually becoming part of the Dirac sea for large solitons. These considerations seem to apply for any winding-number field configuration, where the lowest energy state happens to be the one for which the winding number is equal to the baryon number [Bal90].

The important point we would like to stress here however is the necessity to take the valence quarks into account in order to have an accurate picture of the nucleon. The dynamical stability between the opposed tendencies of the expanding valence quarks and the shrinking skyrmion occurs in fact in an intermediate region, where neither component can be neglected completely (see last part of chapter 4). This region turns out to be of the order of the nucleon size ($mR \simeq 1$).

However, Soni [Son87a] (see also Ripka and Kahana [RK85]) found a major defect in this programme. He noticed that the translationally invariant vacuum ($\phi_0 = f_\pi, \vec{\phi} = 0$) had higher energy than a vacuum in which the meson field had a non-trivial topological configuration. Thus, the translationally invariant vacuum would be unstable, invalidating the whole approach. The reason for this problem can be traced to the contribution from high ($\gg 1$ GeV) momenta in the loop integrations. When these integrations are cut off at a relatively low value (≤ 1 GeV) the vacuum instability does not occur [DPP88, Son87b]. Thus one should adopt the (physically sensible) point of view that the chiral quark-meson theory (though in fact renormalisable) should only be regarded as an effective theory, valid below some cut-off of order 1 GeV. If we take this view, then we no longer require a local counter-term for the meson kinetic energy in the effective Lagrangian, thus keeping within the spirit of the NJL approach.

Specialising to the non-linear case, the model Lagrangian we consider is therefore the NJL Lagrangian (2.14) without a pure meson term:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - g\phi_0 - ig\gamma^5\vec{\phi}\cdot\vec{\tau})\psi. \quad (2.20)$$

The absence of a kinetic energy (Skyrme-type) term for the meson fields ϕ in (2.20) is the main difference with the naive quark-meson models suggested in Refs. [KR84, DSW85, BB84]. The pions introduced in (2.20) are interpreted as composite (quark bilinear) fields, whose kinetic energy is radiatively induced when the quarks are integrated out. In contrast to the Skyrme model, pions are not point-like and have the same typical size as the nucleon, which emerges as a bound state of three valence quarks bound together by a meson field in a skyrmion form.

Before closing this chapter, we present here a more convenient expression of the fermion determinant (2.17). The Dirac operator appearing in (2.17) is linear,

while most field theoretic and proper-time techniques are easier to use in the case of quadratic operators such as those arising in scalar field theories. It turns out that a convenient form for the effective action (2.17) is given by [Zuk85c, DPP88]:

$$\Gamma(\phi) = -\frac{i N_c}{2} \text{Tr} \ln(D) - \frac{i N_c}{2} \text{Tr} \ln(D^+) = -\frac{i N_c}{2} \text{Tr} \ln(DD^+) \quad (2.21)$$

where D refers to the Dirac operator in (2.17). The r.h.s. of (2.21) represents in fact the real part of the effective action when it is rotated into Euclidian space. It so happens that in the SU(2) flavour symmetry and without spin-one fields, there is no imaginary (Wess–Zumino) action and the effective action is simply given by the real Euclidian action. Eq. (2.21) can be recast in the simplified form:

$$\Gamma(\phi) = -\frac{i N_c}{2} \text{Tr} \ln(\partial^2 + m^2 + V) \quad (2.22)$$

where the potential term is

$$V = ig\partial_\mu(\phi_0 - i\gamma^5\vec{\phi} \cdot \vec{\tau})\gamma^\mu. \quad (2.23)$$

The effective chiral action as given in (2.22) is now in a convenient form for us to start our study of the two-point approximation in the finite cut-off theory.

Chapter 3

The Two-Point Casimir Energy

A number of physicists have calculated the full effective action (or the vacuum contribution to the static soliton energy) in the cut-off case. This is rather an involved calculation and requires the computation of a very large number of sea orbitals following the method described by Ripka and Kahana [KR84]. Apart from the self-consistent calculation of the soliton energy, a very limited number of soliton profiles and regularisation schemes have been explored. It is therefore worth calculating the vacuum energy for various profiles and cut-offs, in order to explore the sensitivity of the theory to arbitrariness in these quantities. Since the computation of the full effective action is a major numerical task, it would be useful to have an approximation to it which would reduce the amount of numerical work involved. As we saw in the previous chapter, the derivative expansion — as an approximation scheme — proved to be inadequate for the study of the soliton sector.

We propose in this chapter to adopt a non-local approximation to the full determinant, suggested by Diakonov and co-workers [DPP88]. Our aim will be mainly to see to what extent the vacuum energy depends on different profiles and cut-offs, and check the validity of the approximation whenever a direct comparison

can be made with work done using the full determinant. This chapter specifically will provide a preliminary study of the approximation before proceeding to a self-consistent calculation of the soliton, which we relegate to the following two chapters.

3.1 The two-point effective action

Let us rewrite the fermion effective action as given in the previous chapter (2.22):

$$\Gamma(\phi) = -\frac{i N_c}{2} \text{Tr} \ln \left(\frac{P^2 - m^2 - V}{P^2 - m^2} \right) \quad (3.1)$$

where the effective action corresponding to the uniform background field configuration (of winding number zero) has been subtracted. In (3.1), $P_\mu = i\partial_\mu$ is the momentum operator in coordinate space and N_c the number of colours. Tr means functional as well as Dirac and $SU(2)$ traces. The potential

$$V = ig \partial_\mu (\phi_0 - i\vec{\phi} \cdot \vec{\tau} \gamma^5) \gamma^\mu \quad (3.2)$$

is a function of x and does not commute with the momentum operator, making the action (3.1) highly non-local.

Diakonov, Petrov and Pobilitza (DPP) [DPP88] proposed an approximation to this action based on a perturbative expansion in powers of the potential V . Expanding the logarithm in (3.1) we find

$$\Gamma(\phi) = \frac{i N_c}{2} \text{Tr} \left(V \frac{1}{P^2 - m^2} \right) + \frac{i N_c}{4} \text{Tr} \left(V \frac{1}{P^2 - m^2} V \frac{1}{P^2 - m^2} \right) + \dots \quad (3.3)$$

Since the trace of an odd number of γ matrices is zero, the first-order term vanishes and it turns out that the lowest-order non-vanishing term in the expansion is quadratic in V . The approximation

$$\Gamma(\phi) \simeq \Gamma^{[2]}(\phi) = \frac{i N_c}{4} \text{Tr} \left(V \frac{1}{P^2 - m^2} V \frac{1}{P^2 - m^2} \right) \quad (3.4)$$

was named the interpolation formula by DPP, who claimed that it becomes exact both for small and for large soliton sizes and that it is accurate to within 10% for all intermediate ranges. The exact limits are due to the fact that the effective expansion parameter in (3.3) is $mp/(m^2 + p^2)$ [DPP88], where m is the constituent quark mass and p the momentum of the pions*. Zuk [Zuk85c] had previously arrived at the same expression by summing up all diagrams with two external legs and called it therefore the two-point effective action. One can also make use of the functional method of Aitchison and Fraser [AF85] to obtain $\Gamma^{[2]}$.

In the static case V is independent of time and the two-point effective action is written

$$\Gamma^{[2]}(\phi) = -E^{[2]} \int dt, \quad (3.5)$$

where $E^{[2]}$ is the two-point Casimir or vacuum energy.

To perform the functional trace which we take to be in momentum space $\int \frac{d^4 p}{(2\pi)^4}$, we insert complete sets of eigenvectors of the coordinate and momentum operators in the form of identity operators:

$$\begin{aligned} \Gamma^{[2]} = & \frac{i N_c}{4} \text{tr} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} d^4 x d^4 x' \langle p | \frac{1}{P^2 - m^2} | p \rangle \\ & \langle p | x \rangle \langle x | V | x \rangle \langle x | p' \rangle \langle p' | \frac{1}{P^2 - m^2} | p' \rangle \langle p' | x' \rangle \langle x' | V | x' \rangle \langle x' | p \rangle \end{aligned} \quad (3.6)$$

where now tr refers only to the Dirac and $SU(2)$ traces. Using the plane wave representation $\langle p | x \rangle = e^{ixp}$ and introducing a new integration variable $q = p - p'$ we arrive at the following expression:

$$\begin{aligned} \Gamma^{[2]} = & \frac{i N_c}{4} \text{tr} \int \frac{d^4 q}{(2\pi)^4} \int d^4 x e^{ixq} V(x) \int d^4 x' e^{-ix'q} V(x') \\ & \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2} \frac{1}{(p - q)^2 - m^2}. \end{aligned} \quad (3.7)$$

For a static field configuration, integration over x^0 yields a delta function $2\pi\delta(q^0)$ and, after performing the q^0 integral, an infinite term $\int dx^0$ remains which

*The large momentum limit corresponding to the small soliton size limit and vice versa.

is identified with the coefficient $\int dt$ in eq. (3.5). We thus obtain the expression for the two-point Casimir energy

$$E^{[2]} = -\frac{i N_c}{4} \int \frac{d^3 q}{(2\pi)^3} \text{tr} \tilde{V}(\vec{q}) \tilde{V}(-\vec{q}) \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2} \frac{1}{(p - q)^2 - m^2} \quad (3.8)$$

where $\tilde{V}$ denotes the spatial Fourier transform of V as defined in eq. (3.2). Performing the Dirac trace we get

$$\text{tr} \tilde{V}(\vec{q}) \tilde{V}(-\vec{q}) = 4 g^2 q^2 \text{tr}_{\text{SU}(2)} \tilde{\phi}_a(\vec{q}) \tilde{\phi}_a(-\vec{q}) \quad (3.9)$$

where $\tilde{\phi}_a \tilde{\phi}_a$ ($a = 0, \dots, 3$) denotes an internal space dot-product and where now $q = |\vec{q}|$. Eq. (3.9) yields the following expression for $E^{[2]}$:

$$E^{[2]}(\phi) = N_c g^2 \int \frac{d^3 q}{(2\pi)^3} q^2 \text{tr}_{\text{SU}(2)} \tilde{\phi}_a(\vec{q}) \tilde{\phi}_a(-\vec{q}) F(q^2) \quad (3.10)$$

where $F(q^2)$ represents the two-point function:

$$\equiv i \int_0^\infty \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m^2)((p - q)^2 - m^2)}.$$

The function $F(q^2)$ does in fact include the ultraviolet divergence of the full determinant. To see this, we perform a Wick rotation and write the two-point function in the proper-time representation:

$$F(q^2) = \frac{1}{16\pi^2} \int_0^\infty \frac{d\tau}{\tau} e^{-m^2 \tau} \int_0^1 ds e^{-\tau q^2 s(1-s)} \quad (3.11)$$

and, making use of the identity

$$\int_0^1 ds e^{s(1-s)A} = \sum_{n=0}^\infty \frac{n!}{(2n+1)!} A^n, \quad (3.12)$$

we obtain the series representation

$$F(q^2) = \frac{1}{16\pi^2} \sum_{n=0}^\infty \frac{n!}{(2n+1)!} \int_0^\infty d\tau \tau^{n-1} e^{-m^2 \tau} (q^2)^n. \quad (3.13)$$

The leading term ($n = 0$) is divergent and corresponds to the pion kinetic energy term (quadratic in q) in eq. (3.13). All higher order ($n \geq 2$) terms are convergent. If an infinite renormalisation of this divergence is performed (by adding a counter-term to the Lagrangian (2.20) proportional to the pion kinetic term), the vacuum instability problem is found to be present. Thus, the two-point approximation to the effective action is a faithful one in this respect. We shall calculate the vacuum contribution to the static soliton energy using the two-point approximation with finite cut-off, which will avoid the vacuum instability problem.

3.2 Soliton profile functions

Restricting our study to the non-linear σ model, we can parametrise the meson fields ϕ with a single scalar field $\theta(r)$ through the well-known hedgehog ansatz* which satisfies automatically the non-linear constraint $\phi \cdot \phi = f_\pi$. This ansatz is believed to be a true minimum of the classical action [KR84]. The winding number n of the soliton configuration is related to the boundary conditions of $\theta(r/R)$

$$\theta(0) = n\pi \quad \text{and} \quad \lim_{r \rightarrow \infty} \theta(r/R) \rightarrow 0. \quad (3.14)$$

In this thesis, we confine our attention to the unit winding number sector ($n = 1$). A non-linear representation of the pions is conveniently provided by the unitary matrix:

$$U = e^{i\theta(r)\frac{\vec{\tau} \cdot \vec{\tau}}{f_\pi}} = \frac{1}{f_\pi}(\phi_0 + i\vec{\phi} \cdot \vec{\tau}) \quad (3.15)$$

where τ_i , $i = 1, 2, 3$, are Pauli matrices. The profile-dependent part of the two-point Casimir energy (eq. (3.10)) can then have the form (with $\tilde{U}$ denoting the

*given in (2.7)–(2.8) with $\theta(r)$ replaced by $\theta(r/R)$, where R is a scale parameter characterising the spatial extent of the hedgehog field.

Fourier transform of U):

$$\text{tr}_{\text{SU}(2)} q^2 \widetilde{\phi}_a(\vec{q}) \widetilde{\phi}_a(-\vec{q}) = \Omega(q^2, R) = f_\pi^2 \text{tr}_{\text{SU}(2)} q^2 \tilde{U}(\vec{q}) \tilde{U}^\dagger(\vec{q}) \quad (3.16)$$

and after performing the $SU(2)$ trace,

$$\Omega(q^2, R) = 2q^2 \left[\widetilde{\phi}_0(\vec{q}) \widetilde{\phi}_0(-\vec{q}) + \vec{\tilde{\phi}}(\vec{q}) \cdot \vec{\tilde{\phi}}(-\vec{q}) \right], \quad (3.17)$$

where (using the hedgehog configuration),

$$\widetilde{\phi}_0(\vec{q}) = f_\pi \int d^3r e^{i\vec{q}\cdot\vec{r}} \cos \theta(r/R) = 4\pi f_\pi \int_0^\infty dr r^2 j_0(qr) \cos \theta(r/R) \quad (3.18)$$

$$\vec{\tilde{\phi}}(\vec{q}) = f_\pi \int d^3r \hat{r} e^{i\vec{q}\cdot\vec{r}} \sin \theta(r/R) = 4\pi f_\pi i\hat{q} \int_0^\infty dr r^2 j_1(qr) \sin \theta(r/R) \quad (3.19)$$

where j_0 and j_1 are the spherical Bessel functions of order 0 and 1:

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = -\frac{d}{dx} j_0(x). \quad (3.20)$$

The integral appearing in the Fourier transform of $\phi_0(\vec{r})$ yields a spurious delta function $\delta(q)$, arising from the non-vanishing large- r limit of $\cos \theta$. This divergence does not however affect the two-point Casimir energy as there are enough positive powers of q to damp the δ function when the q -integral is performed. Hence, we find it convenient to eliminate the δ function by subtracting from the Fourier transform the uniform background field ($\theta = 0$) contribution. With this prescription, which does not affect the final result but nevertheless has a practical purpose, the profile-dependent part of the two-point Casimir energy takes the form:

$$\Omega(q^2, R) = 32\pi^2 f_\pi^2 q^2 \left[\left(\int_0^\infty dr r^2 j_0(qr) [\cos \theta(r/R) - 1] \right)^2 + \left(\int_0^\infty dr r^2 j_1(qr) \sin \theta(r/R) \right)^2 \right]. \quad (3.21)$$

Now the algebraic profile functions commonly used in the literature are the one-parameter exponential and linear profiles:

$$\theta(r/R) = \pi e^{-r/R}, \quad \theta(r/R) = \begin{cases} \pi(1 - \frac{r}{R}) & \text{if } r \leq R, \\ 0 & \text{if } r > R. \end{cases} \quad (3.22)$$

Another family of profile functions is provided by the so-called generalised stereographic profile [Zuk90] given by:

$$\theta(r/R) = 2 \arctan \left(\frac{R}{r} \right)^\alpha, \quad \alpha \geq 1, \quad (3.23)$$

which has two interesting features:

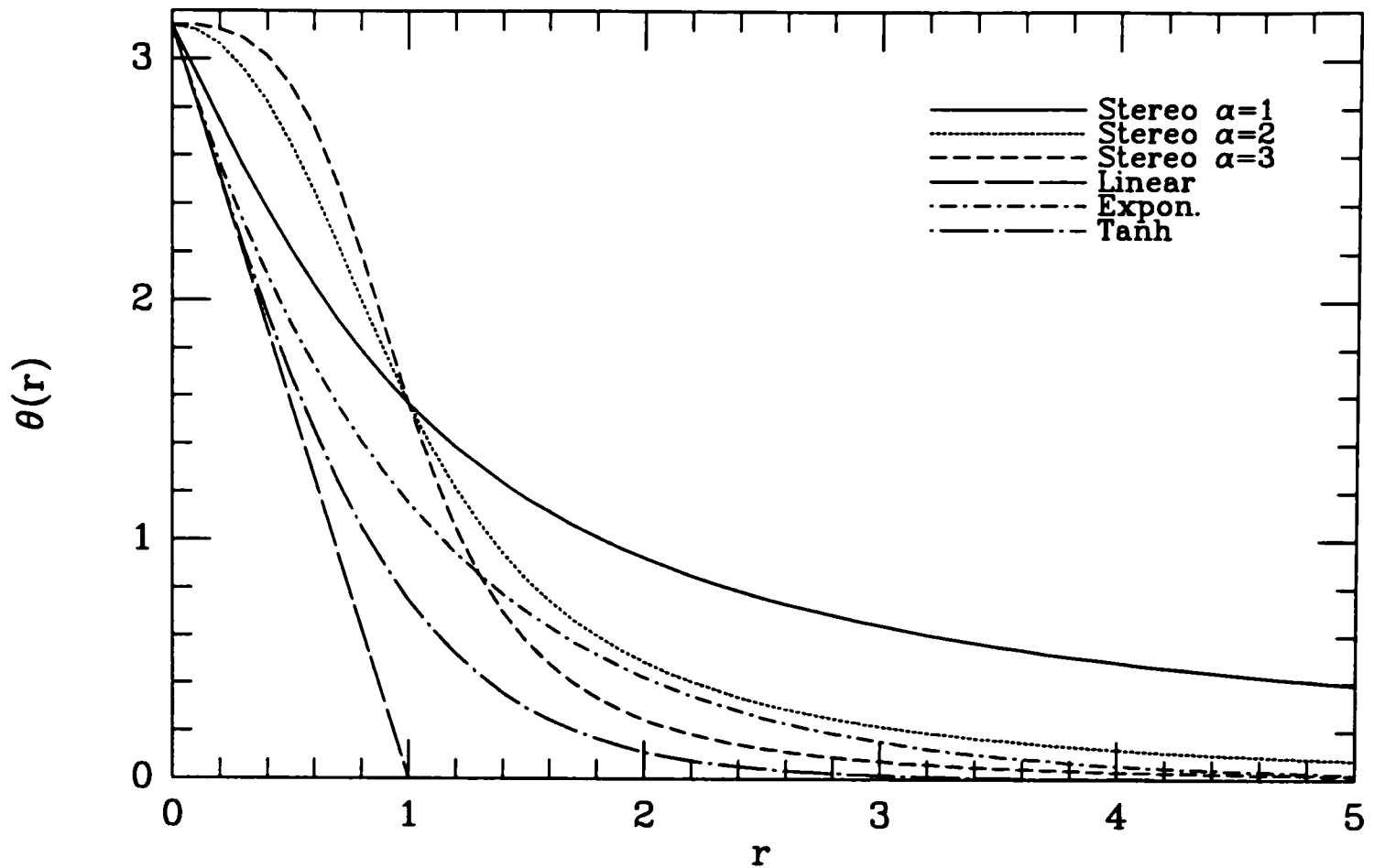
1. The Fourier transforms (3.18) and (3.19) have a tractable form and yield algebraic expressions in the cases where α is an odd integer.
2. The asymptotic behaviour of θ for $r \rightarrow \infty$ is $\theta(r/R) \simeq 2R^\alpha r^{-\alpha}$. The two-point Casimir energy can therefore be studied as a function of α for an arbitrary inverse power decay law in θ . Note that both the exponential and linear profiles fall to zero faster than any power of r^{-1} .

For our calculations, we shall specialise to the $\alpha = 1, 2, 3$ stereographic profiles and use also, besides the exponential and linear profiles, another version of the exponential profile, namely the tanh profile:

$$\theta(r/R) = \pi[1 - \tanh(r/R)]. \quad (3.24)$$

This profile has the advantage of yielding exponentially decaying Fourier transforms [ZA91], a desirable property for numerical integrations. Fig. 3.1 shows the above-mentioned profiles for a soliton size $R = 1$.

Let us now present the expressions of $\Omega(q^2, R)$ for the linear as well as the $\alpha = 1$ and $\alpha = 3$ stereographic profiles, for which the integrals in eq. (3.21) can be done explicitly.

Figure 3.1: Profile functions described in the text for a soliton size $R = 1$.

The linear profile yields:

$$\begin{aligned} \Omega(q^2, R) = 16\pi^2 f_\pi^2 R^4 & \left\{ \left[\left(\frac{qR}{(qR)^2 - \pi^2} - \frac{1}{qR} \right) \cos qR - \left(\frac{(qR)^2 + \pi^2}{((qR)^2 - \pi^2)^2} - \frac{1}{(qR)^2} \right) \sin qR \right]^2 \right. \\ & \left. + \left[\left(\frac{-\pi}{(qR)^2 - \pi^2} \right) \cos qR + \left(\frac{2\pi qR}{((qR)^2 - \pi^2)^2} + \frac{\pi/qR}{(qR)^2 - \pi^2} \right) \sin qR \right]^2 \right\} \end{aligned} \quad (3.25)$$

while the stereographic profile (for arbitrary α) yields the general form:

$$\begin{aligned} \Omega(q^2, R, \alpha) = 64\pi^2 f_\pi^2 R^2 & \left\{ \left(\int_0^\infty dx \frac{x^\alpha}{x^{2\alpha} + 1} \sin qRx \right)^2 \right. \\ & - 2qR \left(\int_0^\infty dx \frac{x^{\alpha+1}}{x^{2\alpha} + 1} \cos qRx \right) \left(\int_0^\infty dx \frac{x^\alpha}{x^{2\alpha} + 1} \sin qRx \right) \\ & \left. + (qR)^2 \left[\left(\int_0^\infty dx \frac{x^{\alpha+1}}{x^{2\alpha} + 1} \cos qRx \right)^2 + \left(\int_0^\infty dx \frac{x}{x^{2\alpha} + 1} \sin qRx \right)^2 \right] \right\} \end{aligned} \quad (3.26)$$

where $x = r/R$ is a dimensionless integration variable. For the cases $\alpha = 1$ and $\alpha = 3$ the integrals are easily done in the complex plane, giving:

$$\Omega(q^2, R, \alpha=1) = 16\pi^4 f_\pi^2 R^2 (1 + 2qR + 2(qR)^2) e^{-2qR} \quad (3.27)$$

and

$$\Omega(q^2, R, \alpha=3) = \frac{16}{9}\pi^4 f_\pi^2 R^2 (A_+^2 + 2qR A_+ A_- + 2(qR)^2 A_-^2) e^{-qR}, \quad (3.28)$$

with

$$A_\pm = e^{-qR/2} \pm 2 \sin\left(\frac{\sqrt{3}}{2}qR \mp \frac{\pi}{6}\right). \quad (3.29)$$

One can easily derive the large-momentum behaviour of $\Omega(q^2, R)$ in these analytic cases; for the linear profile Ω goes like q^{-2} while for the stereographic profile $\Omega(\alpha=1)$ and $\Omega(\alpha=3)$ decay exponentially with q .

3.3 The regularised two-point function

As we already mentioned in the previous chapter, the two-point function $F(q^2)$ is ultraviolet divergent and needs therefore to be provided with some sort of cut-off. If we were to perform an infinite renormalisation of the theory (by removing the divergent bit of the pion kinetic term and replacing it by a corresponding phenomenologically determined quantity — the pion decay constant f_π), the choice of the regularisation used would hardly matter since the cut-off would be taken to infinity in the end. This however may not be the case for the problem at hand where a finite cut-off is needed in order to avoid the vacuum instability problem. In this section, we review the regularisation schemes which we will use in calculating the two-point vacuum energy. They include the covariant momentum cut-off (MC), the Pauli–Villars cut-off (PV), the sharp proper-time cut-off (PT) and the momentum-dependent quark mass cut-off (MDQM). We first start by explaining how the cut-off is fixed in the CSM.

Cut-off determination

Let $F(q^2, \varepsilon)$ denote the regularised two-point function, where ε refers to the cut-off parameter in some given regularisation scheme. In Euclidean space,

$$F(q^2, \varepsilon) = \int_{\varepsilon} \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 + m^2)((p - q)^2 + m^2)}. \quad (3.30)$$

The cut-off parameter will be conveniently defined, for all regularisation schemes considered, such that setting it to zero means removing the cut-off. Accordingly, ε will be proportional to the inverse of the cut-off Λ and dimensionless if expressed in units of m^{-1} . We saw in the leading section of this chapter that the lowest-order term (divergent) in a series expansion of $E^{[2]}$ in powers of q^2 reproduces the kinetic energy term for the pions. In coordinate space, it corresponds to the Non-Linear σ Model Lagrangian (2.4) arising as the lowest-order term in a derivative expansion of the effective action $\Gamma^{[2]}$ (or the full action Γ). For time-independent meson fields, the meson kinetic energy is:

$$E_{kin} = \frac{1}{2} \int d^3 r \, \vec{\partial} \phi \cdot \vec{\partial} \phi = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} q^2 \tilde{\phi}(\vec{q}) \cdot \tilde{\phi}(-\vec{q}). \quad (3.31)$$

On the other hand, by keeping the leading term in the q^2 -expansion of $F(q^2, \varepsilon)$ in eq. (3.10), we also have:

$$E_{kin} = 2 N_c g^2 \int \frac{d^3 q}{(2\pi)^3} q^2 \tilde{\phi}(\vec{q}) \cdot \tilde{\phi}(-\vec{q}) F(q^2 = 0, \varepsilon). \quad (3.32)$$

By identifying eqs. (3.31) and (3.32) we immediately obtain a constraint equation on ε :

$$4 N_c g^2 F(0, \varepsilon) = 1. \quad (3.33)$$

Equation (3.33) fixes the value of the cut-off parameter ε once the value of the coupling constant g (or the dynamical quark mass through $m = g f_\pi$) has been fixed. Fig. 3.2 shows the dependence of the cut-off Λ on the quark mass m for

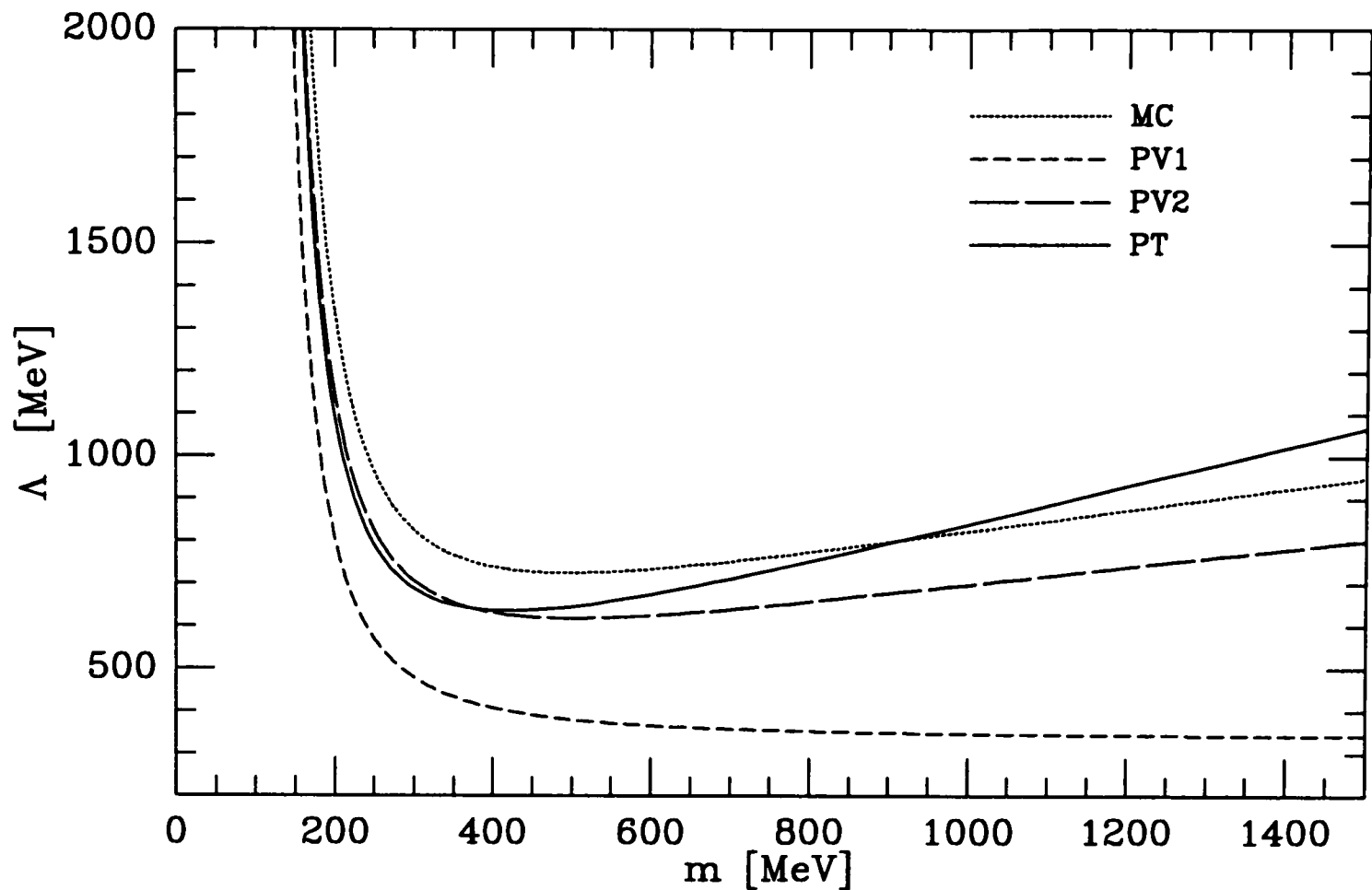


Figure 3.2: The cut-off Λ as a function of the quark mass m for the regularisations considered in the text.

the regularisations considered. We therefore see that the cut-off theory has still one free parameter just like in the case of the renormalised theory.

Before going any further, let us note that the two-point function F is a dimensionless quantity, and that making the change of variable $q \rightarrow t = q/m$, we can write:

$$F(q^2, \varepsilon) = F(t^2, \varepsilon) = \int_{\varepsilon} \frac{d^4 u}{(2\pi)^4} \frac{1}{(u^2 + 1)((u - t)^2 + 1)} \quad (3.34)$$

where $u = p/m$ is the dimensionless internal momentum. From now on, we shall find it convenient to work with these dimensionless quantities in anticipation to the numerical calculations.

Covariant momentum cut-off

This regularisation is implemented by cutting off the radial momentum integration ($p \leq \Lambda$) in the loop integral of the Euclidean two-point function;

$$F_{MC}(t^2, \varepsilon) = \int d\Omega_4 \int_0^{1/\varepsilon} du \frac{u^3}{(u^2 + 1)((t - u)^2 + 1)} \quad (3.35)$$

Evaluating the two-point function at $t^2 = 0$

$$F(0, \varepsilon) = \int d\Omega_4 \int_0^{1/\varepsilon^2} du \frac{u}{(u + 1)^2}, \quad (3.36)$$

is easily done to yield the cut-off condition (from eq. (3.33)):

$$\frac{N_c g^2}{4\pi^2} \left[\ln \left(1 + \frac{1}{\varepsilon^2} \right) - \frac{1}{1 + \varepsilon^2} \right] = 1. \quad (3.37)$$

Lengthy but otherwise straightforward calculations yield the expression of the two-point function for arbitrary t^2 :

$$\begin{aligned} F_{MC}(t^2, \varepsilon) = & \frac{1}{32\pi^2} \left\{ 1 - \ln 2 + \ln \left(1 + \frac{1}{\varepsilon^2} \right) + \frac{1}{4t^2} \left(1 + \frac{1}{\varepsilon^2} \right) \right. \\ & - 2\sqrt{1 + \frac{1}{t^2}} \ln \frac{\sqrt{1 + 1/t^2} + 1}{\sqrt{1 + 1/t^2} - 1} - \frac{1}{4t^2} \sqrt{16t^4 + 8 \left(1 - \frac{1}{\varepsilon^2} \right) t^2 + \left(1 + \frac{1}{\varepsilon^2} \right)^2} \\ & + \sqrt{1 + \frac{1}{t^2}} \ln \frac{\left(1 + \frac{1}{\varepsilon^2} \right) + \sqrt{16t^4 + 8 \left(1 - \frac{1}{\varepsilon^2} \right) t^2 + \left(1 + \frac{1}{\varepsilon^2} \right)^2} + 4t^2 \sqrt{1 + 1/t^2}}{\left(1 + \frac{1}{\varepsilon^2} \right) + \sqrt{16t^4 + 8 \left(1 - \frac{1}{\varepsilon^2} \right) t^2 + \left(1 + \frac{1}{\varepsilon^2} \right)^2} - 4t^2 \sqrt{1 + 1/t^2}} \\ & \left. + \ln \left[\left(1 + \frac{1}{\varepsilon^2} \right) - 4t^2 + \sqrt{16t^4 + 8 \left(1 - \frac{1}{\varepsilon^2} \right) t^2 + \left(1 + \frac{1}{\varepsilon^2} \right)^2} \right] \right\}. \quad (3.38) \end{aligned}$$

We also find that $F_{MC}(t^2, \varepsilon)$ falls to zero like t^{-2} :

$$\lim_{t \rightarrow \infty} F_{MC}(t^2, \varepsilon) = \frac{1}{16\pi^2} \left(\frac{1}{\varepsilon^2} - \ln \left(1 + \frac{1}{\varepsilon^2} \right) \right) \frac{1}{t^2}. \quad (3.39)$$

Pauli-Villars cut-off

Contrasting with the rather naive covariant momentum cut-off scheme which breaks gauge invariance in the presence of gauge fields, a more generally useful

regularisation is that of Pauli-Villars (PV). This scheme consists in substituting for the diverging quantity (the two-point effective action here) an expression of the form [IZ80]:

$$\Gamma^{[2]}(m) \longrightarrow \Gamma_{PV}^{[2]}(m, \varepsilon) = \Gamma^{[2]}(m) - \sum_{i=1}^n c_i \Gamma^{[2]}(M_i), \quad (3.40)$$

which amounts to adding to the fermion loop (of mass m) the contribution of n fermions (the i th having a mass $M_i^2 = \left(1 + \frac{k_i}{\varepsilon^2}\right) m^2$). The coefficients c_i and k_i are judiciously chosen so as to remove any divergence present in $\Gamma^{[2]}$. They must specifically satisfy the equations

$$\sum_{i=1}^n c_i = 1 \quad \text{and} \quad \sum_{i=1}^n c_i k_i^s = 0; \quad s = 1, 2, \dots, (n-1) \quad (3.41)$$

which admit the following solutions:

$$c_i = (-1)^{i+1} \binom{n}{i} \quad \text{and} \quad k_i = i. \quad (3.42)$$

It turns out that for the one-loop effective action Γ , one Pauli-Villars field is sufficient to regularise the ultraviolet divergence. Let $F_{PV1}(t^2, \varepsilon)$ denote the ($n = 1$) regularised two-point function. Using standard techniques in the computation of scalar loop diagrams, it is relatively straightforward to arrive at the formula:

$$\begin{aligned} F_{PV1}(t^2, \varepsilon) = & \frac{1}{16\pi^2} \left\{ \ln \left(1 + \frac{1}{\varepsilon^2} \right) \right. \\ & \left. + \sqrt{1 + \frac{4}{t^2} \left(1 + \frac{1}{\varepsilon^2} \right)} \ln \frac{\sqrt{1 + 4 \left(1 + \frac{1}{\varepsilon^2} \right) / t^2} + 1}{\sqrt{1 + 4 \left(1 + \frac{1}{\varepsilon^2} \right) / t^2} - 1} - \sqrt{1 + \frac{4}{t^2}} \ln \frac{\sqrt{1 + 4/t^2} + 1}{\sqrt{1 + 4/t^2} - 1} \right\}. \end{aligned} \quad (3.43)$$

The cut-off condition (3.33) in this case leads to:

$$\frac{N_c g^2}{4\pi^2} \ln \left(1 + \frac{1}{\varepsilon^2} \right) = 1, \quad (3.44)$$

while $F_{PV1}(t^2, \varepsilon)$ is found to exhibit a $\ln t/t^2$ fall-off:

$$\lim_{t \rightarrow \infty} F_{PV1}(t^2, \varepsilon) = \frac{1}{4\pi^2 \varepsilon^2} \frac{\ln t}{t^2}. \quad (3.45)$$

Using (3.40) and (3.42), we can easily generalise to any number of PV regulators and express $F_{PVn}(t^2, \varepsilon)$ in terms of $F_{PV1}(t^2, \varepsilon)$;

$$F_{PVn}(t^2, \varepsilon) = \sum_{i=1}^n (-1)^{i+1} \binom{n}{i} F_{PV1}(t^2, \varepsilon/\sqrt{i}) \quad (3.46)$$

Let us quote the results for the case $n = 2$ which we will use in our numerical calculations:

$$F_{PV2}(t^2, \varepsilon) = 2F_{PV1}(t^2, \varepsilon) - F_{PV1}(t^2, \varepsilon/\sqrt{2}) \quad (3.47)$$

from which follow the PV2 cut-off condition

$$\frac{N_c g^2}{4\pi^2} \left[2 \ln \left(1 + \frac{1}{\varepsilon^2} \right) - \ln \left(1 + \frac{2}{\varepsilon^2} \right) \right] = 1 \quad (3.48)$$

and the large-momentum limit:

$$\lim_{t \rightarrow \infty} F_{PV2}(t^2, \varepsilon) = \frac{1}{8\pi^2} \left[\left(1 + \frac{2}{\varepsilon^2} \right) \ln \left(1 + \frac{2}{\varepsilon^2} \right) - 2 \left(1 + \frac{1}{\varepsilon^2} \right) \ln \left(1 + \frac{1}{\varepsilon^2} \right) \right] \frac{1}{t^2} \quad (3.49)$$

exhibiting a t^{-2} fall-off unlike the PV1 two-point function. In fact, it can be shown that this is a property of the PVn two-point function for any $n \neq 1$.

Let us finally mention that the PV regularisation belongs to the general class of proper-time regularisations [Zuk90]. We can write

$$F_{PVn}(t^2, \varepsilon) = \frac{1}{16\pi^2} \int_0^\infty \frac{d\tau}{\tau} C(\tau, \varepsilon) e^{-\tau} \int_0^1 ds e^{-\tau s(1-s)t^2}, \quad (3.50)$$

where $C(\tau, \varepsilon) = (1 - e^{-\tau/\varepsilon^2})^n$ is the regulating function for n PV regulators.

Sharp proper-time cut-off

In this scheme, an infra-red cut-off is applied to the proper-time parameter in the proper-time (PT) regularised two-point function (see eq. (3.11))

$$F_{PT}(t^2, \varepsilon) = \frac{1}{16\pi^2} \int_{\varepsilon^2}^\infty \frac{d\tau}{\tau} e^{-\tau} \int_0^1 ds e^{-\tau s(1-s)t^2}. \quad (3.51)$$

The PT cut-off condition is

$$\frac{N_c g^2}{4\pi^2} \int_{\epsilon^2}^{\infty} \frac{d\tau}{\tau} e^{-\tau} = 1 \quad (3.52)$$

and the asymptotic limit of F_{PT} is

$$\lim_{t \rightarrow \infty} F_{PT}(t^2, \epsilon) = \frac{1}{8\pi^2} \left(\int_{\epsilon^2}^{\infty} \frac{d\tau}{\tau^2} e^{-\tau} \right) \frac{1}{t^2}, \quad (3.53)$$

exhibiting a t^{-2} fall-off as in the cases of the MC and PV2 regularisations. Despite the fact that these expressions cannot be evaluated in closed form, we have considered this regularisation for the sake of completeness as it enjoys a wide use in the literature relevant to the work presented in this thesis.

Momentum dependent quark mass cut-off

In a paper published in 1986, Diakonov and Petrov [DP86] derive from their instanton-liquid picture for the non-perturbative regime of QCD, an expression for the dynamically generated quark mass. We quote their main result in a form convenient to us:

$$M(q^2) = m \left(2 \int_0^{\infty} dx \frac{x^2 J_2(x)}{(x^2 + (\rho q)^2)^{3/2}} \right)^2, \quad (3.54)$$

where m is the value of $M(q^2)$ at $q^2 = 0$, ρ is the average instanton size which they take to be of the order of $(600 \text{ MeV})^{-1}$, and finally $J_2(x)$ is the Bessel function of order 2. The asymptotic limits of $M(q^2)$ are found to be

$$\begin{cases} M(q^2) = m = \frac{2\pi^2 \epsilon N \rho^2}{V N_c} & \rho q = 0 \\ M(q^2) \simeq 36m \frac{1}{(\rho q)^6} & \rho q \gg 1 \end{cases} \quad (3.55)$$

N is the average number of instantons in the 4-volume V occupied by the system and the quantity ϵ is determined self-consistently from the gap equation

$$\int \frac{d^4 q}{(2\pi)^4} \frac{M^2(q^2)}{q^2 + M^2(q^2)} = \frac{N}{4 N_c V} (1 - m_b \epsilon). \quad (3.56)$$

It is worth noting that, given all numerical values of these parameters (including the bare quark mass $m_b = 0$), they find the dynamical mass m to be 345 MeV.

To implement the momentum dependent quark mass regularisation (MDQM) in our case, we replace the quark mass m in the two-point function by the running mass $M(q^2)$. This cuts off the contribution from the high momentum region which is responsible for the logarithmic divergence. We have

$$F_{MDQM}(q^2) = \frac{1}{m^2} \int \frac{d^4 p}{(2\pi)^4} \frac{M(p^2)}{p^2 + M^2(p^2)} \frac{M((p-q)^2)}{(p-q)^2 + M^2((p-q)^2)}. \quad (3.57)$$

Changing to dimensionless variables as in previous cases, with $q = mt$, $p = mu$, $M = mA$ and $m\rho' = \rho$; and expressing the momentum integral in polar coordinates, we obtain:

$$F_{MDQM}(t^2) = \frac{1}{4\pi^3} \int_0^\infty du \frac{u^3 A(u^2)}{u^2 + A^2(u^2)} \int_0^\pi d\theta \frac{\sin^2 \theta A((t-u)^2)}{t^2 + u^2 - 2tu \cos \theta + A^2((t-u)^2)} \quad (3.58)$$

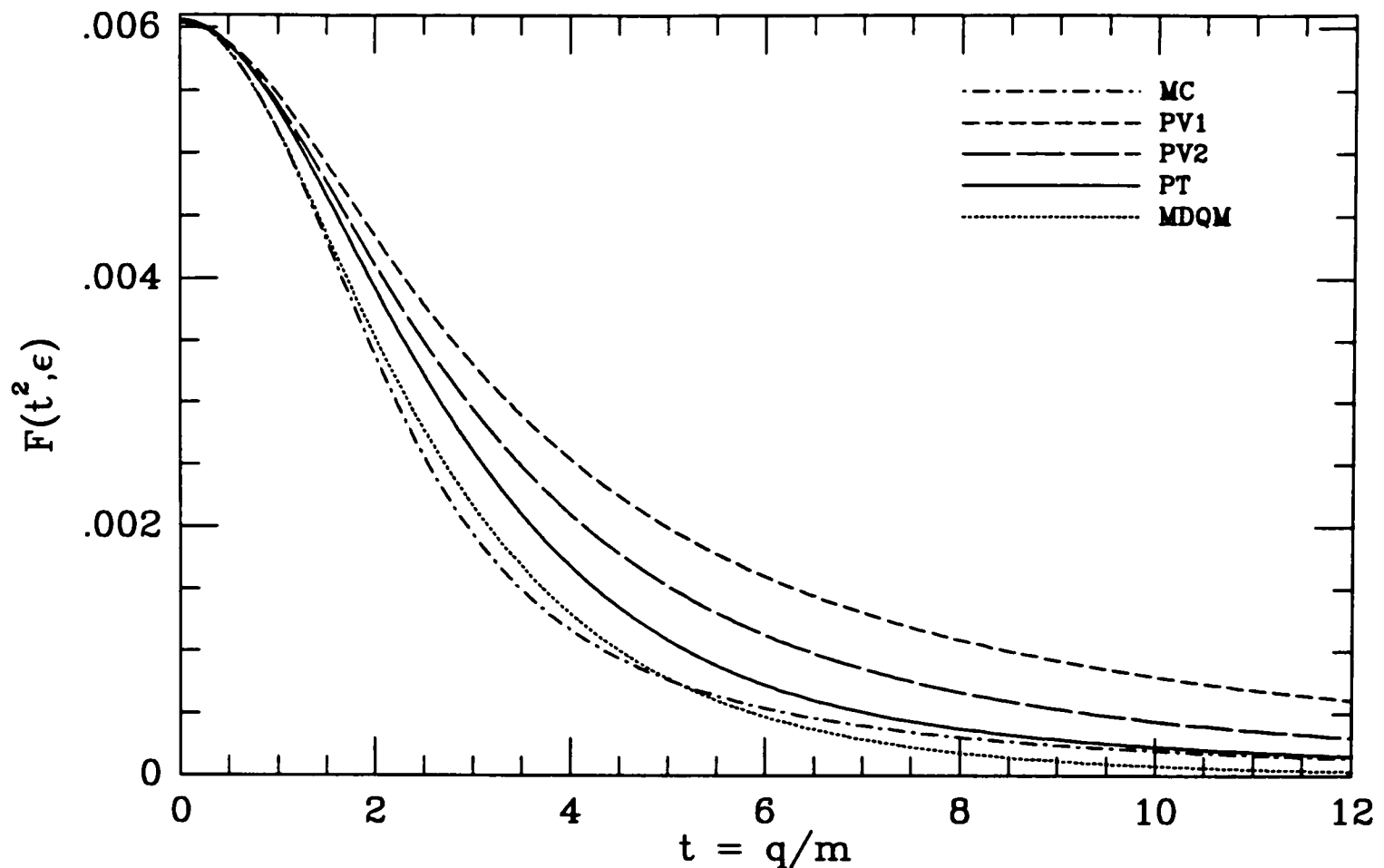
and for the cut-off condition:

$$\frac{N_c g^2}{2\pi^2} \int_0^\infty du \frac{u^3 A^2(u^2)}{[u^2 + A^2(u^2)]^2} = 1. \quad (3.59)$$

The dimensionless instanton size ρ' plays here the role (through the dependence of $A(u^2)$ on ρ) of the cut-off ε in the foregoing regularisations. It is not difficult to establish from eq. (3.58) that $F_{MDQM}(t^2)$ behaves like t^{-5} in the large t limit.

As no explicit sharp cut-off ε is required in this case, the MDQM regularisation seems a more natural and less *ad hoc* scheme to use should one be concerned with giving a physical understanding to the finite cut-off approach to the effective chiral action.

Figure 3.3 contains the curves of $F(t^2, \varepsilon)$ for the four regularisations we have described with the quark mass $m = 345$ MeV, corresponding to a coupling constant $g = 3.71$.

Figure 3.3: Two-point functions for $m = 345$ MeV.

3.4 Results and discussion

In order to perform numerical calculations, it is convenient to deal with dimensionless expressions. To this end, we scale the quark mass m out of all dimensionful variables in the expression of $E^{[2]}$. In particular $q = mt$ and $mR = X$, and the profile dependent part may then be written:

$$g^2 \Omega(q^2, R) = \frac{1}{m^2} \Omega(t^2, X) \quad (3.60)$$

and recalling that $F(q^2, \epsilon) = F(t^2, \epsilon)$ from eq. (3.34) we obtain for the two-point vacuum energy

$$E^{[2]}(X) = \frac{N_c m}{2\pi^2} \int_0^\infty dt t^2 \Omega(t^2, X) F(t^2, \epsilon). \quad (3.61)$$

We shall instead work with a simpler quantity, namely $\mathcal{E}^{[2]}$ (the dimensionless two-point Casimir energy per unit colour) defined by:

$$E^{[2]}(X) = N_c m \mathcal{E}^{[2]}(X). \quad (3.62)$$

We further split $\mathcal{E}^{[2]}$ into the kinetic piece (quadratic in powers of derivatives) and the vacuum energy with the kinetic term subtracted, representing all terms of fourth and higher order in a derivative expansion;

$$\mathcal{E}^{[2]}(X) = \mathcal{E}_{kin}(X) + \mathcal{E}_{sub}^{[2]}(X). \quad (3.63)$$

$\mathcal{E}_{kin}$ is, by construction, independent of the regularisation scheme and can be evaluated in closed form for some of the profile functions we have considered. Using the appropriate soliton profile functions in eq. (3.32) $\mathcal{E}_{kin}$ is found to depend linearly on X^* :

$$\mathcal{E}_{kin}(X) = C_{pf} \frac{\pi^2}{N_c g^2} X, \quad (3.64)$$

where C_{pf} is given for different profiles in Table 3.1.

Table 3.1: Slopes C_{pf} of the kinetic energy for different profiles.

Profile	C_{pf}
Linear	$\left(\frac{2\pi}{3} + \frac{2}{\pi}\right)$
Exponential	$\left(\frac{\pi}{2} + \frac{4}{\pi} \int_0^\infty dx \sin^2 \pi e^{-x}\right)$
Tanh	$\frac{2}{\pi} \int_0^\infty dx \left[\frac{4x^2}{(1 + e^{-2x})^2} \theta^2(x) + 2 \sin^2 \theta(x) \right]$
Stereographic	$\frac{4 + 2\alpha^2}{\alpha^2 \sin \frac{\pi}{2\alpha}}$

We have found that a Simpson quadrature rule is generally adequate for the evaluation of most of the integrals appearing in the expression of the two-point vacuum energy $\mathcal{E}^{[2]}$. However, integrals involving the linear profile as well as evaluation of the Fourier transforms of the exponential, the ‘tanh’ and the stereographic ($\alpha = 2$) profiles required a variable stepsize algorithm, due to the presence of highly oscillating functions in the integrands. We have used a Gauss

*It may be shown on dimensional grounds that E_{kin} is proportional to R , by scaling R out of the integrand in (3.31).

quadrature rule for this purpose. For the numerical values of the parameters, we have taken $g = 3.71$, corresponding to $m = 345$ MeV which is the value Diakonov and Petrov [DP86] extracted from instanton physics. We have also used another value of the coupling constant: $g = 4$ ($m = 372$ MeV), in order to compare our results with those of Meissner and coworkers [MGG90], who are the only group to our knowledge to have published calculations of the vacuum energy for fixed profiles in the cut-off theory. Table 3.2 gives the numerical values of both the dimensionful cut-off Λ and the dimensionless cut-off parameter $\varepsilon = m/\Lambda$ for the different regularisation schemes and for the two values of m considered. Since the MDQM regularisation does not have any free parameter, it will only be applied for $m = 345$ MeV which is fixed by the set of parameters Diakonov and Petrov use in their instanton liquid model. As can be seen from Tab. 3.2, the cut-off Λ is found numerically to be in agreement with the interaction scale in the instanton vacuum ($1/\rho \sim 600$ MeV), apart from the PV1 cut-off which is somewhat smaller. It should be noted however, that only the MC cut-off has a direct physical meaning amongst the regularisation schemes we have used. We note also a slight discrepancy (about 5 MeV) in the PT cut-off for $g = 4$ with the value quoted in ref. [MGG90]: 637 MeV. This discrepancy may be due to numerical inaccuracies.

Table 3.2: Cut-offs for different regularisations.

	$m = 345$ MeV		$m = 372$ MeV	
	ε	Λ [MeV]	ε	Λ [MeV]
MC	0.448	770.0	0.494	753.7
PV1	0.792	435.6	0.885	420.3
PV2	0.525	657.1	0.579	642.5
PT	0.529	652.2	0.580	641.8

In Figs. 3.4 and 3.5, we have plotted the two-point Casimir energy $E_{vac}^{[2]}$ and

the meson kinetic energy E_{kin} as functions of the dimensionless soliton size X for the six profiles discussed in section 3.2, in the case of the PT cut-off and with $m = 345$ MeV. The straight lines, as expected, refer to the kinetic energy. The graphs show that in all cases the two-point vacuum energy is positive for all X and, since $E_{vac}^{[2]}$ is in fact calculated relatively to the uniform background field ($\theta = 0$) energy point, we deduce that vacuum instability is not encountered in the cut-off theory. This is in agreement with Ref. [MGG90] where the full Dirac sea E_{vac} is calculated. This conclusion is supported by analytical calculations of the small- R limit of $E_{vac}^{[2]}$ [Zuk90]*.

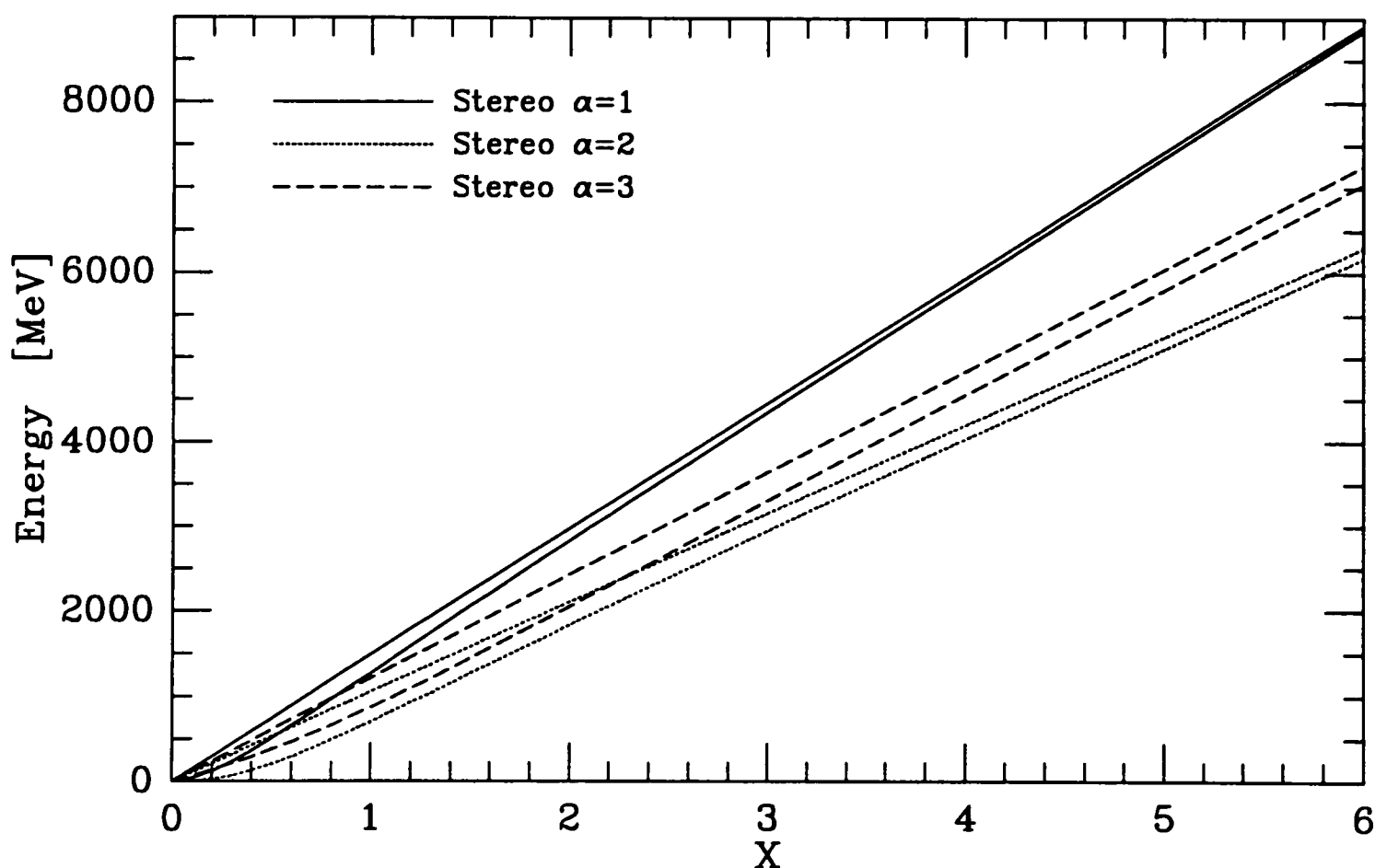


Figure 3.4: $E_{kin}(X)$ (straight lines) and $E_{vac}^{[2]}(X)$ (curved lines) using a PT cut-off and $m = 345$ MeV for 3 stereographic profiles.

We can further see from Figs. 3.4 and 3.5 that, with the exception of the

*Broniowski and Kutschera argued in a recent paper [BK90] that vacuum instability could still occur in the finite cut-off theory for some regularisation schemes. It turns out that this problem can be avoided [Zuk91] if one requires that the proper-time regulating function be everywhere positive.

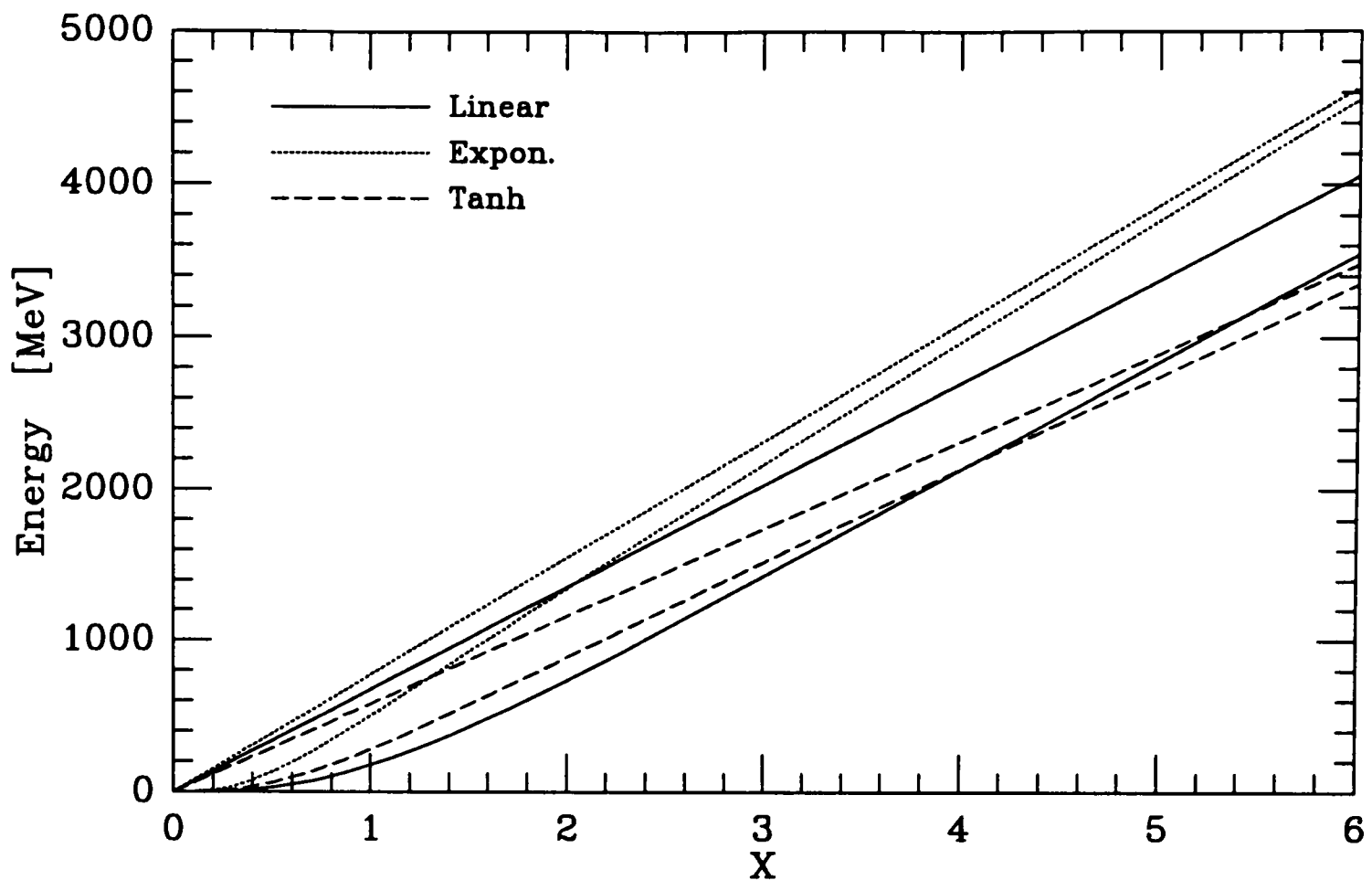


Figure 3.5: $E_{kin}(X)$ (straight lines) and $E_{vac}^{[2]}(X)$ (curved lines) using a PT cut-off and $m = 345$ MeV for the linear, exponential and tanh profiles.

linear profile (see below), $E_{vac}^{[2]}$ is well approximated for large X by the kinetic term E_{kin} alone. This is a manifestation of the fact that the derivative expansion is a good approximation for large solitons and in the limit of large- X when the leading term (E_{kin}) is expected to dominate.

The difference between $E_{vac}^{[2]}$ and E_{kin} can be seen to be relatively small, certainly for larger X . In order to appreciate the dependence of this difference on X , we have plotted in Fig. 3.6 the vacuum energy with the linear term subtracted (referred to as $E_{sub}^{[2]}$) for the six different profiles considered. The difference varies appreciably for the different profiles, but it is in all cases maximum at roughly $X = 1$ (or $R = m^{-1}$), which happens to be the relevant region for nucleon sizes. This fact rules out the use of the derivative expansion for the evaluation of $E_{vac}^{[2]}$, since all terms in the series will be of order 1 in this region.

In Fig. 3.6, all curves except that for the linear profile seem to go to zero as

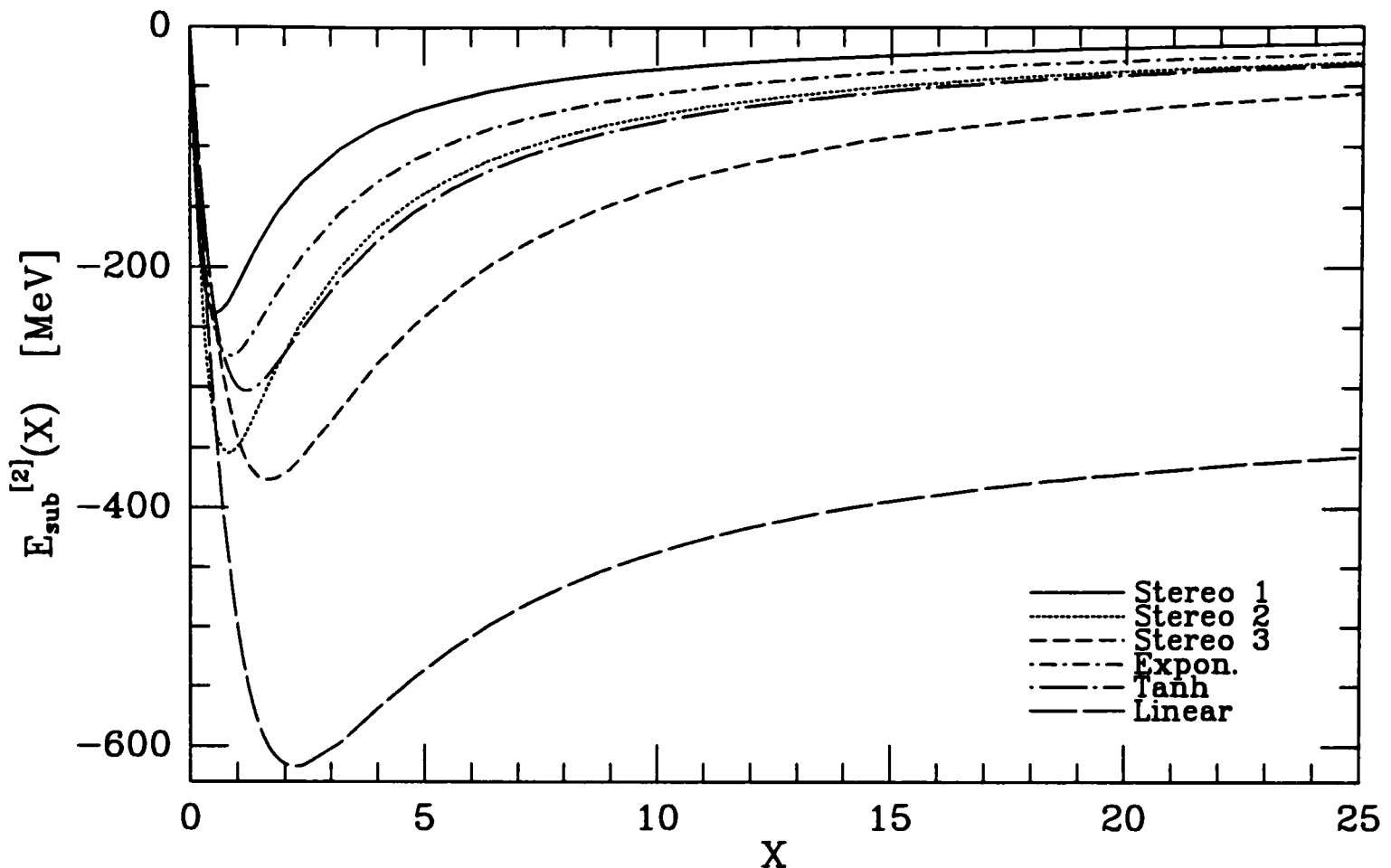


Figure 3.6: $E_{sub}^{[2]}(X)$ for different profiles with a PT cut-off and $m = 345$ MeV.

the soliton size X tends to infinity. The derivative expansion would dictate on dimensional grounds that, in this limit, the next to leading order will go like X^{-1} since it is fourth order in powers of derivatives. To see this, we have plotted in Fig. 3.7 the same curves as in Fig. 3.6 but multiplied this time by the soliton size X . As expected, the graphs (except for the linear) show that the decay of $E_{sub}^{[2]}$ is indeed of order X^{-1} . The case of the linear profile is a special one. In fact it has been shown [ZA91] that, for this profile, all terms in the derivative expansion beyond the leading term E_{kin} are formally divergent, but they can be summed to yield a negative constant for large X as the leading correction to E_{kin} . In other words, the two-point Casimir energy for the linear profile will not converge towards the kinetic term with increasing X but will rather run parallel to it, shifted by a negative constant. Fig. 3.6 does not however show that $E_{vac}^{[2]}$ clearly goes to a constant. We have calculated $E_{vac}^{[2]}$ for even larger X 's and it turns out

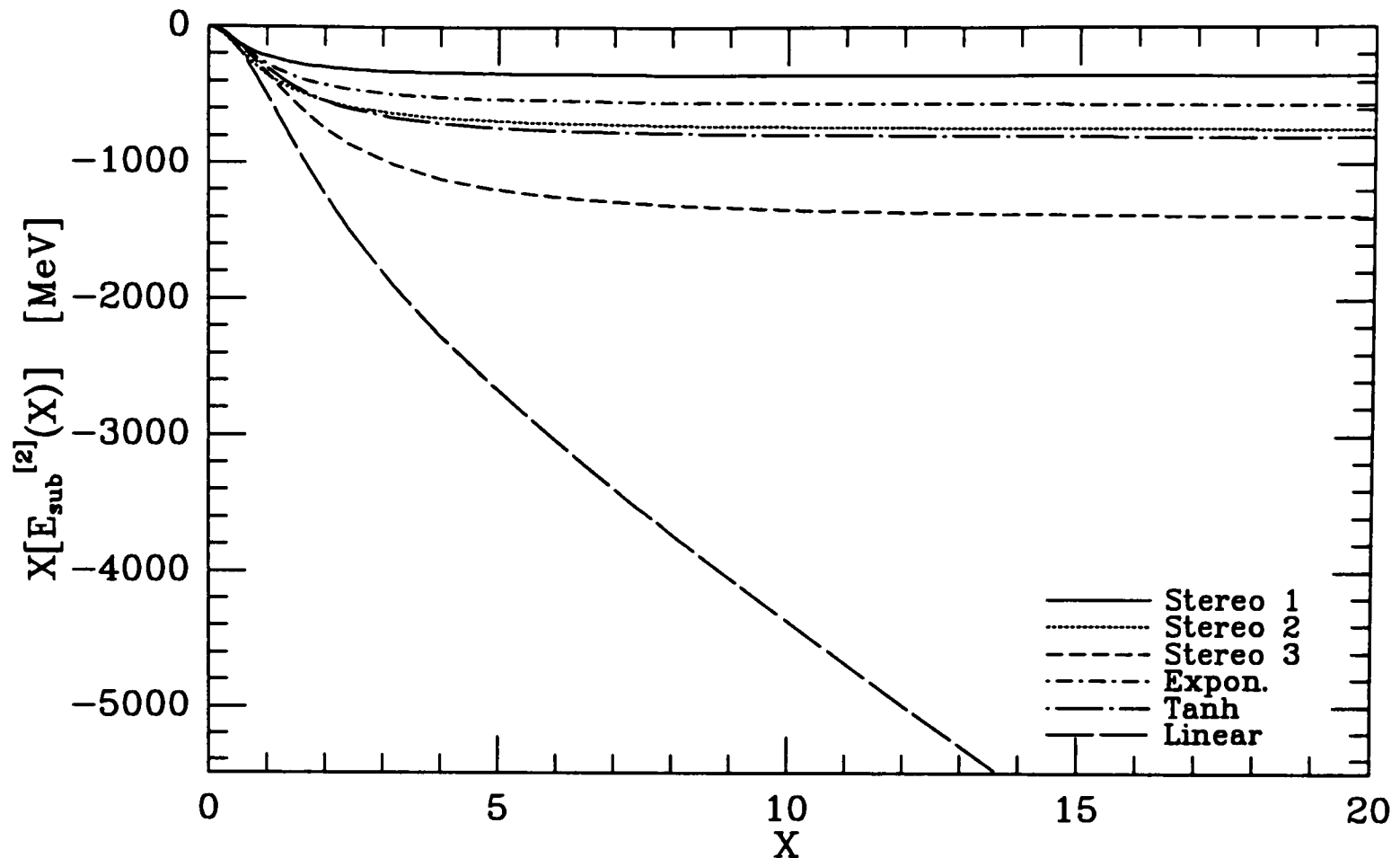


Figure 3.7: $X E_{sub}^{[2]}(X)$ for different profiles with a PT cut-off and $m = 345$ MeV.

that the asymptotic limit is not attained until a soliton size of order $X = 100$ is reached. The full Casimir energy is expected to behave similarly for all the profiles considered, since the two-point approximation is exact in both limits of small and large soliton sizes. The difference between E_{vac} and $E_{vac}^{[2]}$ may be in how fast the asymptotic limits are reached. It turns out that for the linear profile the full vacuum energy is much faster in reaching its limit $X \simeq 20$ [ZA91].

We come now to the question of comparing different regularisation schemes and their effect on the two-point Casimir energy. As the linear contribution E_{kin} is independent of the scheme used, it is sufficient to consider here the subtracted vacuum energy $E_{sub}^{[2]}$. In Fig. 3.8 we have plotted the curves $E_{sub}^{[2]}(X)$ for the regularisations reviewed in section 3.3, and using an exponential profile with $m = 345$ MeV. The renormalised case (first computed for ‘stereo 1’ by Tudor [Tud86]) when the cut-off Λ is taken to infinity (or ε set to zero) irrespective

of the regularisation, is also shown along with the other cases by way of comparison. $E_{sub}^{[2]}$ is in fact a finite quantity which, if evaluated with an infinite cut-off, exhibits an infinite negative slope at the origin proportional to $X \ln X$ [Zuk85c]; this is responsible for the vacuum instability problem.

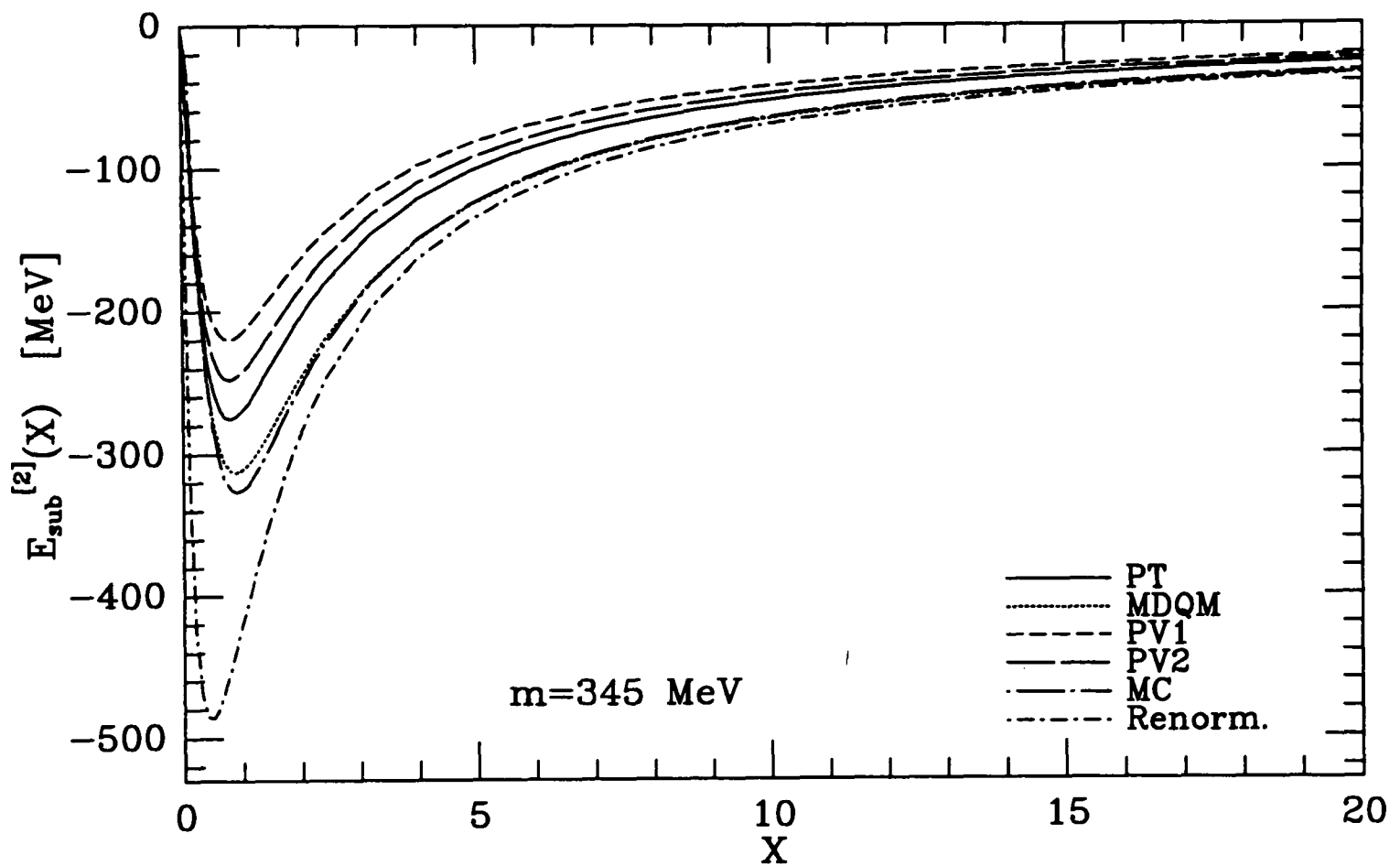


Figure 3.8: $E_{sub}^{[2]}(X)$ for different regularisations and infinite renormalisation, using an exponential profile and $m = 345$ MeV.

The renormalised $E_{sub}^{[2]}$ put aside, all regularisations yield comparable results. In particular, MDQM and MC are closer to each other than any other two regularisations. This can be understood as the MC regularisation, which involves a sharp cut-off in momentum space, is expected to mimic the smooth cut-off version of the MDQM regularisation. PV1 yields the smallest difference between $E_{vac}^{[2]}$ and E_{kin} , MC the largest one (discarding the limiting case of the renormalised $E_{sub}^{[2]}$). PT lies somewhere in between. All these observations still hold when a different profile or a different quark mass is used.

Going back to Figs. 3.4 and 3.5, we notice that the slope of the kinetic energy

varies significantly from one profile to the other. For example, the difference is a factor of roughly 3 between the stereo 1 and the tanh profiles (as can be checked by numerical evaluation of the constants C_{pf} in Table 3.2). However X (or the dimensionful R) is a dummy parameter which can be rescaled independently for each profile. It is interesting to see whether adopting the same scaling requirement for all profiles will make the slopes C_{pf} comparable. We propose to define a mean (dimensionless) soliton radius $\bar{X}$ as the value for which the pion field has maximum amplitude. This is obtained for $\theta(\bar{X}/X) = \frac{\pi}{2}$. The new slope $\bar{C}_{pf}$ will be given (in analogy to (3.64)) through

$$\mathcal{E}_{kin}(\bar{X}) = \bar{C}_{pf} \frac{\pi^2}{N_c g^2} \bar{X}. \quad (3.65)$$

Table 3.3 gives the slopes C_{pf} for different profiles for both the unscaled and the

Table 3.3: Scaled and unscaled slopes C_{pf} of the kinetic energy for different profiles.

Profile	$\bar{X}/X$	C_{pf} (unscaled)	$\bar{C}_{pf}$ (scaled)
linear	1/2	2.731	5.462
exponential	$\ln 2$	3.120	4.500
tanh	$\ln 3/2$	2.342	4.264
stereo $\alpha = 1$	1	6.000	6.000
stereo $\alpha = 2$	1	4.243	4.243
stereo $\alpha = 3$	1	4.889	4.889

scaled soliton radii. The new slopes are found to be fairly comparable.

Finally, the question of comparing the two-point approximation of the Casimir energy with the exact evaluation is in order. In Figs. 3.9 and 3.10, the graphs of the exact Casimir energy taken from Ref. [MGG90], for the exponential and the linear profile respectively, are plotted as dotted lines. The full lines represent the two-point Casimir energy and the dashed ones the linear contribution, E_{kin} . The value of the coupling constant is $g = 4$ ($m = 372$ MeV) and a PT cut-off was used in both cases. Let us consider the exponential profile first.

Fig. 3.9 shows that the general form for both E_{vac} and $E_{vac}^{[2]}$ is the same, leading to a reasonable overall agreement. There is however a systematic discrepancy of about 10-20%. As mentioned in the leading section of this chapter, the behaviour of E_{vac} for both the large- R and small- R limits is dominated by $E_{vac}^{[2]}$ and the two curves will necessarily agree in these regions. Examination of the graph of E_{vac} for

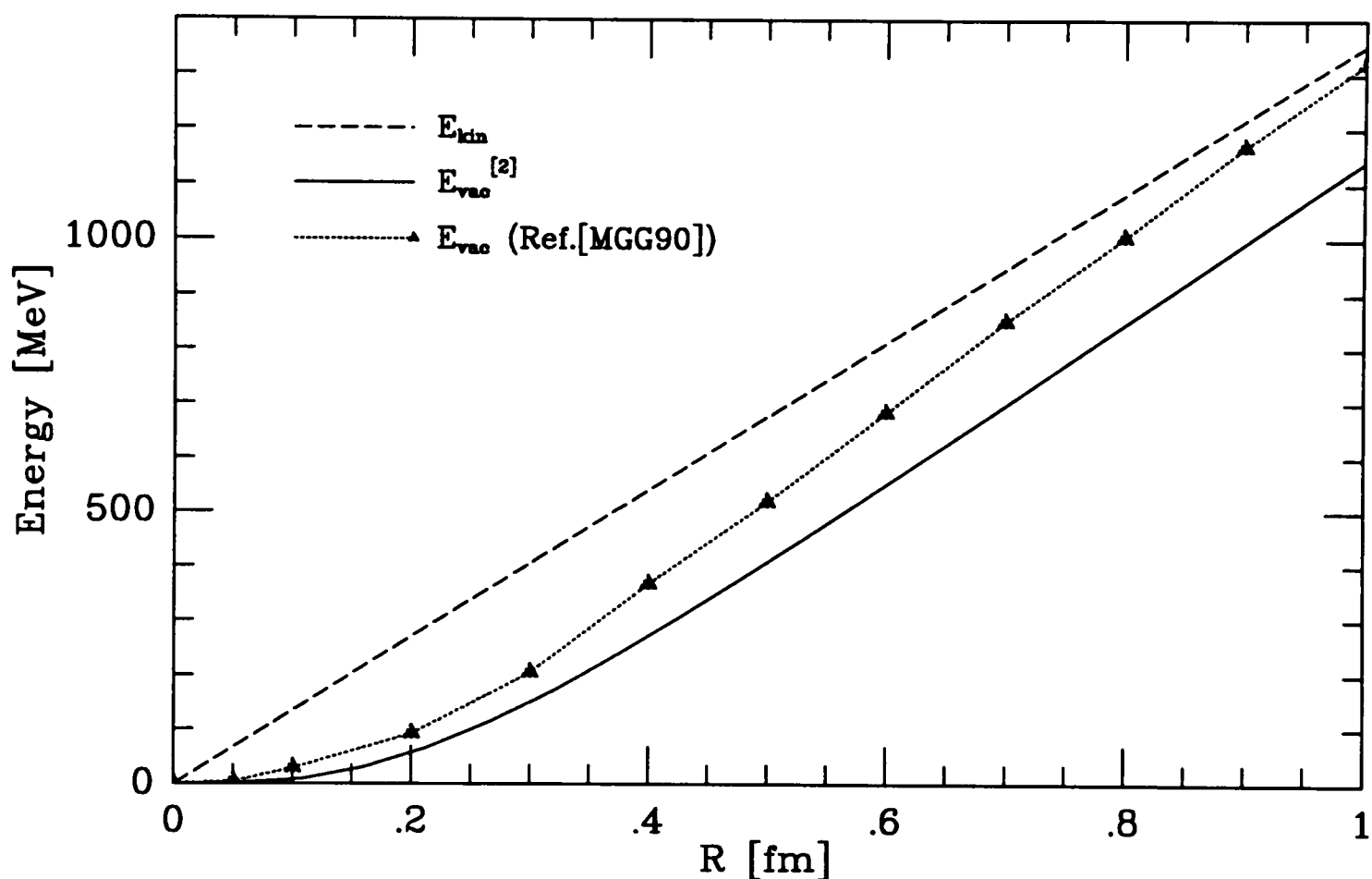


Figure 3.9: $E_{vac}(X)$ (Ref.[MGG90]), $E_{vac}^{[2]}(X)$ and $E_{kin}(X)$ for the exponential profile, using a PT cut-off and $m = 372$ MeV.

the largest R values available from [MGG90] (which is only about 1 fm or 2.5 in dimensionless units) shows that it converges considerably faster to E_{kin} than $E_{vac}^{[2]}$. As we saw, the difference $E_{sub}^{[2]}$ is dominated by a R^{-1} contribution arising from the fourth order term in powers of derivatives, part of which is contained in $E_{vac}^{[2]}$. However, unlike the second order term (the linear contribution) which is fully contained in $E_{vac}^{[2]}$, there is a further fourth order contribution (proportional to R^{-1}) arising from the ‘four-point’ Casimir energy $E_{vac}^{[4]}$ and this is not contained in $E_{vac}^{[2]}$. It is therefore tempting to think of including the fourth order term

($\sim R^{-1}$) exactly. This is however impossible since a R^{-1} correction would make $E_{vac}^{[2]}$ diverge for small R , a manifestation of the failure of the derivative expansion. The only consistent way to do this would be to incorporate the full (non-local) four-point approximation $E_{vac}^{[4]}$. Unfortunately, we have not been able to find a simple way to calculate $E_{vac}^{[4]}$.

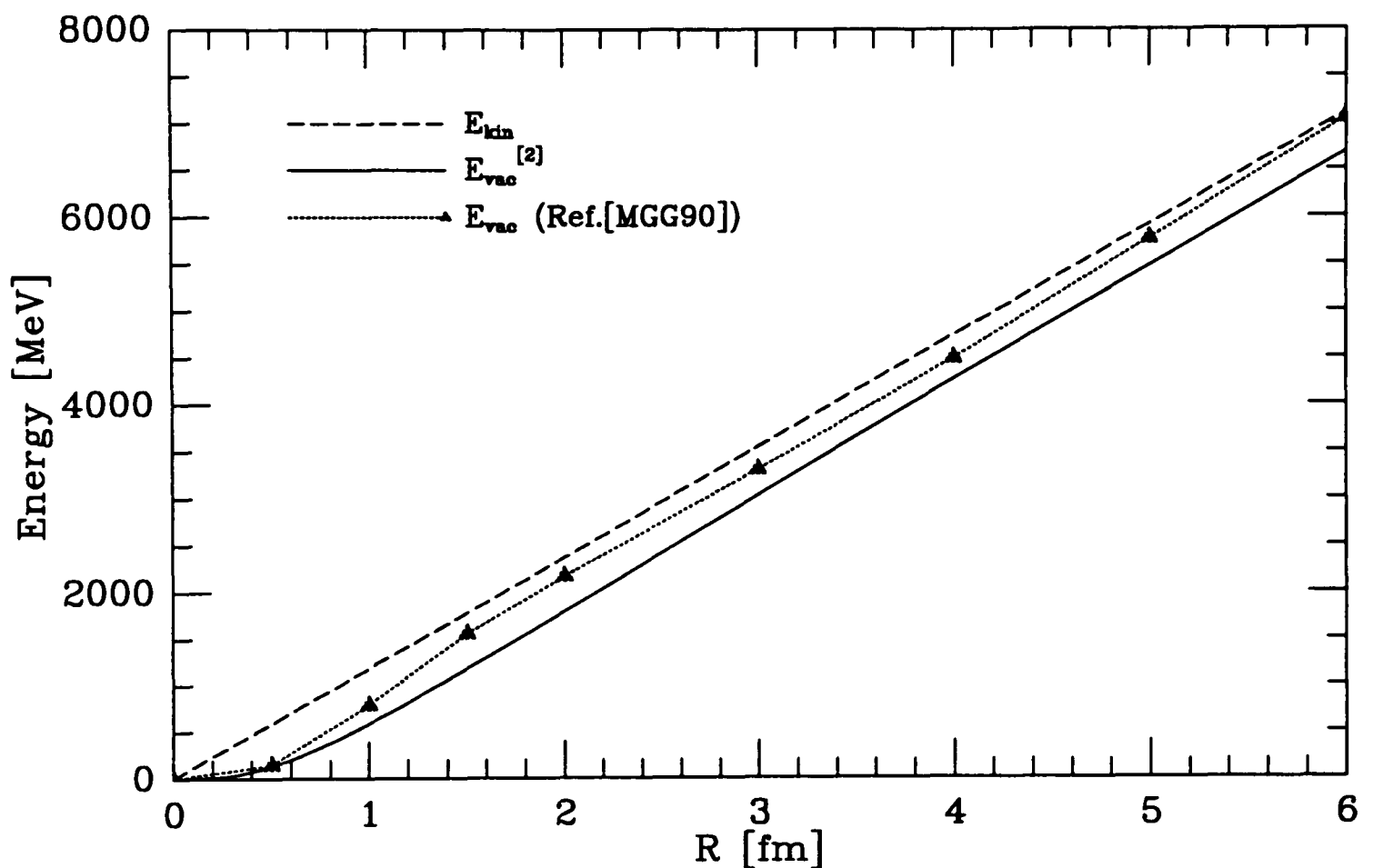


Figure 3.10: $E_{vac}(X)$ (Ref.[MGG90]), $E_{vac}^{[2]}(X)$ and $E_{kin}(X)$ for the linear profile, using a PT cut-off and $m = 372$ MeV.

Examination of the graphs for the linear profile in Fig. 3.10, however, leads to a quite different conclusion. The foregoing analysis about the exponential profile does not hold here for high values of R . For R between 2.5 fm and 4.5 fm E_{vac} runs parallel to E_{kin} but, at about $R = 5.5 - 6$ fm, it suddenly converges with it. Since $E_{vac}^{[2]}$ (and E_{vac} for this matter) for the linear profile is expected to approach E_{kin} up to a negative constant, we conclude that the discrepancy must be due to a numerical problem which Meissner *et al.* have had in calculating the Casimir energy for large R .

3.5 Conclusion

In this chapter, we have investigated the two-point approximation to the static energy of the polarised vacuum in the presence of a soliton field. We have considered six different one-parameter soliton profiles and regularised the logarithmic divergence of the two-point vacuum energy in four different ways. The cut-off is kept finite and is fixed so as to reproduce the pion decay constant $f_\pi = 93$ MeV. In view of the results obtained, we can conclude that the dreaded vacuum instability problem is not encountered in the cut-off theory, which it is sensible to consider since we are dealing here with a low-energy effective theory. The two-point vacuum energy is found to be well approximated for large soliton sizes by its leading contribution in a derivative expansion, corresponding to the radiatively generated pion kinetic energy. The difference goes to zero like the inverse soliton size for all profiles, except in the special case of the linear profile which leads to a constant negative shift from the leading contribution. Comparison with exact calculation of the vacuum energy is conclusive in only one case out of the two we have considered. It shows that the two-point approximation is in reasonable agreement for all soliton sizes with the full Dirac sea, although the discrepancy amounts to up to 20% in intermediate regions, contradicting Diakonov's claim of only 10%. The two-point Casimir energy seems to be stable with respect to all regularisation schemes used, with all cut-offs comparable. Variation with respect to profile functions used is however appreciable, indicating the need to work with a self-consistently determined soliton.

Chapter 4

The Self-Consistent Static Soliton

Much of the work on the soliton of the ECA reported in the literature has involved fixed profiles with a single parameter, the soliton size R , of the kind we used in the last chapter, with the exponential and the linear profiles being the most popular ones. In the last two or three years however, several physicists have reported work on a fully self-consistent calculation of the static soliton energy which includes the contribution of the polarised vacuum in the finite cut-off approach to the theory [RW88, RW89, MGG89, Alk90, MG91, WY91]. All of these workers use the numerical method of Kahana and Ripka [KR84] to compute the exact vacuum contribution. Wakamatsu and Yoshiki [WY91], in particular, have used an improved version of the KR method to carry out an extensive study in which they obtained many nucleon properties. They have also addressed in the same paper the controversial question of the spin of the proton. Pushing on with our programme on the study of the two-point approximation to the ECA, we tackle in this chapter the question of the self-consistent static soliton within this approximation. We hope by doing so to be able to further assess its validity

by comparison with some of the works mentioned above.

We will first write the soliton energy functional in the more general framework of the linear σ -model. We will later restrict our study to the non-linear case in order to derive the self-consistent equation for the meson field, parametrised by the soliton profile function $\theta(r)$. We will use three of the regularisations reviewed in the last chapter to see to what extent the self-consistent soliton depend on the choice of the scheme. Relevant to this point is Zuk's result that the soliton energy does not depend significantly on the regularisation scheme used, as long as the latter involves 'well-behaved' regulating functions [Zuk91].

4.1 The static soliton energy

As was emphasised in chapter two, the nucleon (a baryon-number one system in the CSM) is constructed by giving a winding-number one configuration to the meson field and by filling the lowest positive energy orbital with three quarks. If a one-parameter soliton profile is used, then this and only this valence orbital is seen to dive in the Dirac sea for a critical size, eventually becoming indistinguishable from the rest of the negative energy orbitals in the large soliton size limit. A Skyrme picture of the nucleon is obtained in this limit.

Correspondingly, the total energy of the system will be the sum of the energy of the three valence quarks and that of the polarised vacuum, both of which are determined by a non-trivial meson field configuration minimising the functional

$$E_{tot}(\phi) = N_c E_{val}(\phi) + E_{vac}(\phi) + E_{pot}(\phi), \quad (4.1)$$

where 'val' and 'vac' stand for valence and vacuum respectively. $E_{pot}(\phi) = \frac{g^2}{2\mu} \int d^3r (\phi_0^2 + \vec{\phi}^2)$ corresponds to the last term in the NJL Lagrangian (2.14)*.

*Note that we have changed the initial notation of G in (2.14) to μ . This will avoid unnecessary confusion with the Dirac wave-function, to be discussed below.

We then approximate the vacuum energy with the two-point Casimir energy $E_{tot} \simeq E_{tot}^{[2]} = N_c E_{val} + E_{vac}^{[2]} + E_{pot}$ and search for the self-consistent meson field which satisfies the extremum condition:

$$\frac{\delta E_{tot}(\phi)}{\delta \phi^i} = 0, \quad (4.2)$$

where $i = 0, 1, 2, 3$ denotes the components of the meson field. If we let the latter assume the familiar hedgehog ansatz in the linear σ -model:

$$\begin{cases} \phi_0(\vec{r}) &= f_\pi \sigma(r) \\ \vec{\phi}(\vec{r}) &= f_\pi \frac{\vec{r}}{r} \pi(r) \end{cases} \quad (4.3)$$

then eq. (4.2) leads to two variational (coupled) equations, one for $\sigma(r)$ and one for $\pi(r)$. If we further impose the non-linear condition $\sigma^2 + \pi^2 = 1$, E_{pot} , being now a constant, will not contribute in eq. (4.2) and we get one single equation for the profile function $\theta(r)$. On the other hand, one has to solve the Dirac equation associated with the valence quark orbital simultaneously with eq. (4.2) since it depends directly on the valence eigenvalue E_{val} .

The valence quark orbital

To obtain the valence quark energy E_{val} as a functional of the meson fields σ and π , we make use of the Dirac equation

$$H_D |\psi\rangle = E |\psi\rangle \quad (4.4)$$

and, assuming the unit normalisation of the Dirac spinor $\langle \psi | \psi \rangle = 1$, we write:

$$E_{val}(\sigma, \pi) = \langle \psi_{val} | H_D(\sigma, \pi) | \psi_{val} \rangle, \quad (4.5)$$

where the one-particle Dirac Hamiltonian is given by:

$$H_D(\sigma, \pi) = \vec{\alpha} \cdot \vec{P} + \beta g f_\pi [\sigma - i \vec{\tau} \cdot \hat{r} \gamma^5 \pi]. \quad (4.6)$$

The non-trivial coupling to the isospin which appears in the potential energy term involving π breaks both angular momentum and isospin symmetries. If $\vec{J} = \vec{L} + \vec{S}$ and $\vec{T}$ denote respectively the total angular momentum and the isospin of the quark, then it can be shown that, although the Hamiltonian H_D does not commute with either $\vec{J}$ or $\vec{T}$, their sum $\vec{G} = \vec{J} + \vec{T}$ commutes with the Hamiltonian. The observable G is therefore a conserved quantity, called the grand spin. It follows that any rotation in real space is accompanied by a corresponding reverse rotation in internal (isospin) space. This reflects the same property of the hedgehog ansatz (4.3) which, by mapping internal space onto real space, leads to a topologically non-trivial meson field configuration. Since the parity operator Π also leaves the Hamiltonian invariant, one can use the simultaneous eigenstates of H_D , $\vec{G}$ and Π as a Dirac spinor basis $|\psi\rangle$ for the corresponding Hilbert space. Quark orbitals may therefore be labelled by G^π where G and π represent the quantum numbers of the grand spin and parity operators. The quark orbital we are interested in here, is the lowest positive energy one which becomes negative in the presence of an extended winding-number one meson field. As we saw in Chapter two, it corresponds to the 0^+ (positive parity and zero grand spin) orbital. In the coordinate space representation $\psi_{val}(\vec{r}) = \langle \vec{r} | \psi_{val} \rangle$ and using the γ matrices $\gamma^0 = \beta$ and $\gamma^i = \beta\sigma^i$, equation (4.5) becomes

$$E_{val} = -i \int d^3r \bar{\psi}_{val}(\vec{r}) \vec{\gamma} \cdot \vec{\nabla} \psi_{val}(\vec{r}) + m \int d^3r \bar{\psi}_{val}(\vec{r}) \sigma(r) \psi_{val}(\vec{r}) - m \int d^3r \bar{\psi}_{val}(\vec{r}) i\gamma^5 \vec{\tau} \cdot \hat{r} \pi(r) \psi_{val}(\vec{r}), \quad (4.7)$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ and the normalisation condition $\int d^3r \psi_{val}^\dagger(\vec{r}) \psi_{val}(\vec{r}) = 1$ is assumed.

Since the Dirac equation (4.4) reduces to a central potential problem resulting from the spherical symmetry of the hedgehog ansatz, the parity of the Dirac spinor will be given by $(-1)^\ell$, where ℓ is the orbital momentum quantum number. The

state 0^+ , which is of positive parity, requires therefore an even ℓ . For $G = 0$ the rules of composition of angular momenta will allow only the value $\ell = 0$. Thus the form of the 0^+ orbital is given by:

$$\psi_{val}(\vec{r}) = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} F(r) & |\xi\rangle \\ iG(r) & \vec{\sigma} \cdot \hat{r} |\xi\rangle \end{pmatrix} \quad (4.8)$$

where $F(r)$ and $G(r)$ are the radial functions of respectively the upper and lower components of the Dirac spinor. The Pauli spinor $|\xi\rangle$ satisfies the relation

$$(\vec{S} + \vec{T}) |\xi\rangle = 0, \quad (4.9)$$

which couples the spin operator $\vec{S} = \frac{1}{2}\vec{\sigma}$ and the isospin operator $\vec{T} = \frac{1}{2}\vec{\tau}$ to zero, for zero orbital angular momentum and grand spin. As a result of the spherical symmetry of the state 0^+ , the normalisation of the Dirac spinor $\psi_{val}(\vec{r})$ will involve radial and angular degrees of freedom separately:

$$\int_0^\infty dr r^2 [F^2(r) + G^2(r)] = 1 \text{ and } \langle \xi | \xi \rangle = 1 \quad (4.10)$$

Furthermore the energy functional (4.7) can be reduced to a form involving a single integral over the radial variable r . We will therefore integrate out the angular part $\int d\Omega$ by substituting the Dirac spinor (4.8) in eq. (4.7). The expressions we need to calculate are $-i\bar{\psi}_{val}\vec{\gamma} \cdot \vec{\nabla}\psi_{val}$, $\bar{\psi}_{val}\psi_{val}$ and $\bar{\psi}_{val}i\gamma^5\vec{\tau} \cdot \hat{r}\psi_{val}$. We shall evaluate the first one by way of example, the two others can be calculated in the same way.

In the Pauli-Dirac representation of the γ matrices,

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (4.11)$$

we have:

$$i\bar{\psi}_{val}\gamma^i\nabla_i\psi_{val} = \frac{1}{4\pi} \langle \xi | \begin{pmatrix} -F(r) & iG(r)\vec{\sigma} \cdot \hat{r} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -i\vec{\sigma} \cdot \vec{\nabla}F(r) \\ \vec{\sigma} \cdot \vec{\nabla}G(r)\vec{\sigma} \cdot \hat{r} \end{pmatrix} |\xi\rangle. \quad (4.12)$$

Writing $\vec{\sigma} \cdot \vec{\nabla}$ in spherical coordinates,

$$\vec{\sigma} \cdot \vec{\nabla} = \sigma_r \frac{\partial}{\partial r} + \sigma_\theta \frac{\partial}{r \partial \theta} + \sigma_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}, \quad (4.13)$$

and noting that $\vec{\sigma} \cdot \hat{\vec{r}} = \sigma_r$ corresponds to the radial component of the spin vector, we get:

$$\begin{aligned} \vec{\sigma} \cdot \vec{\nabla} F(r) &= \sigma_r \frac{\partial}{\partial r} F(r) \\ \vec{\sigma} \cdot \vec{\nabla} (G(r) \sigma_r) &= \left(\frac{\partial}{\partial r} + \frac{2}{r} \right) G(r) \end{aligned} \quad (4.14)$$

which leads to:

$$-i \bar{\psi}_{val}(\vec{r}) \vec{\gamma} \cdot \vec{\nabla} \psi_{val}(\vec{r}) = \frac{1}{4\pi} \left[F(r) \left(\frac{\partial}{\partial r} + \frac{2}{r} \right) G(r) - G(r) \frac{\partial}{\partial r} F(r) \right], \quad (4.15)$$

where we used the property of Pauli matrices that $\sigma_i^2 = 1$, i denoting here the spherical coordinates. We obtain in a similar fashion the expressions:

$$\bar{\psi}_{val}(\vec{r}) \psi_{val}(\vec{r}) = \frac{1}{4\pi} [F^2(r) - G^2(r)] \quad (4.16)$$

and

$$\bar{\psi}_{val}(\vec{r}) i \gamma^5 \vec{\tau} \cdot \hat{\vec{r}} \psi_{val}(\vec{r}) = \frac{1}{4\pi} 2F(r)G(r). \quad (4.17)$$

With the integration over the angular part in eq. (4.7) done trivially ($\int d\Omega = 4\pi$), we can write the functional E_{val} in its final form:

$$\begin{aligned} E_{val}(\sigma, \pi) &= \int_0^\infty r^2 dr \left[m\sigma(r) (F^2(r) - G^2(r)) \right. \\ &\quad \left. - 2m\pi(r)F(r)G(r) + F(r) \left(\frac{\partial}{\partial r} + \frac{2}{r} \right) G(r) - G(r) \frac{\partial}{\partial r} F(r) \right]. \end{aligned} \quad (4.18)$$

Multiplying the left-hand side by the constraint $\int_0^\infty r^2 dr (F^2 + G^2) = 1$ and varying with respect to the wavefunctions $F(r)$ and $G(r)$, we arrive at a set of two coupled differential equations:

$$\begin{cases} F'(r) &= -m\pi(r) F(r) - (E_{val} + m\sigma(r))G(r) \\ G'(r) &= -\left(\frac{2}{r} - m\pi(r)\right) G(r) + (E_{val} - m\sigma(r))F(r) \end{cases} \quad (4.19)$$

which corresponds to the radial part of the Dirac equation (4.4).

The vacuum energy functional

We now turn to the vacuum contribution to the static soliton energy in the two-point approximation: $E_{vac}^{[2]}$. Let us write $E_{vac}^{[2]}$ in the form

$$E_{vac}^{[2]} = E_{kin} + E_{sub}^{[2]}, \quad (4.20)$$

where we have separated the radiatively generated kinetic energy for the mesons:

$$E_{kin} = N_c g^2 \int \frac{d^3 q}{(2\pi)^3} \Omega(\phi) F(0, \varepsilon) \quad (4.21)$$

and the rest of the quantum correction in the two-point approximation corresponding to the fourth and higher order terms in an expansion of powers of the derivatives of the field ϕ :

$$E_{sub}^{[2]} = N_c g^2 \int \frac{d^3 q}{(2\pi)^3} \Omega(\phi) [F(q^2, \varepsilon) - F(0, \varepsilon)], \quad (4.22)$$

with the profile-dependent function Ω as given in eq. (3.21) in the last chapter. The constraint equation (3.33) fixes $F(0, \varepsilon) = 1/4 N_c g^2$ and by inserting the hedgehog ansatz (4.3) in Ω we find that

$$E_{kin} = \frac{f_\pi^2}{4} \text{tr}_{SU(2)} \int d^3 r \left[\left(\vec{\nabla} \sigma(r) \right)^2 + \left(\vec{\nabla} [\vec{\tau} \cdot \hat{r} \pi(r)] \right)^2 \right], \quad (4.23)$$

and after performing the $SU(2)$ trace and the angular integral, we get

$$E_{kin} = 2\pi f_\pi^2 \int_0^\infty dr \left[r^2 \left(\frac{\partial \sigma}{\partial r} \right)^2 + r^2 \left(\frac{\partial \pi}{\partial r} \right)^2 + 2\pi^2 \right]. \quad (4.24)$$

Now, recalling the general form of the profile-dependent function Ω from eq. (3.21) in the preceding chapter, we can write

$$\Omega(\sigma, \pi) = 32\pi^2 f_\pi^2 q^2 \left[\left(\int_0^\infty dr r^2 j_0(qr) [\sigma(r) - 1] \right)^2 + \left(\int_0^\infty dr r^2 j_1(qr) \pi(r) \right)^2 \right], \quad (4.25)$$

leading to the explicit form:

$$E_{sub}^{[2]} = 16 N_c m^2 \int_0^\infty dq q^4 F_{sub}(q^2, \varepsilon) \left[\left(\int_0^\infty dr r^2 j_0(qr) [\sigma(r) - 1] \right)^2 + \left(\int_0^\infty dr r^2 j_1(qr) \pi(r) \right)^2 \right], \quad (4.26)$$

where $F_{sub}(q^2, \varepsilon) = F(q^2, \varepsilon) - F(0, \varepsilon)$.

Putting the different contributions to the total energy E_{tot} together, we get the final result:

$$\begin{aligned} E_{tot}(\sigma, \pi) = & N_c \int_0^\infty r^2 dr \left[m\pi(r) (F^2(r) - G^2(r)) \right. \\ & \left. - 2m\pi(r)F(r)G(r) + F(r) \left(\frac{\partial}{\partial r} + \frac{2}{r} \right) G(r) - G(r) \frac{\partial}{\partial r} F(r) \right] \\ & + 8\pi N_c m^2 F(0, \varepsilon) \int_0^\infty dr \left[r^2 \left(\frac{\partial \sigma}{\partial r} \right)^2 + r^2 \left(\frac{\partial \pi}{\partial r} \right)^2 + 2\pi^2 \right] + 16 N_c m^2 \int_0^\infty dq q^4 \\ & F_{sub}(q^2, \varepsilon) \left[\left(\int_0^\infty dr r^2 j_0(qr) [\sigma(r) - 1] \right)^2 + \left(\int_0^\infty dr r^2 j_1(qr) \pi(r) \right)^2 \right] \\ & + \frac{2\pi m^2}{\mu} \int_0^\infty dr r^2 (\sigma^2 + \pi^2). \end{aligned} \quad (4.27)$$

4.2 The meson equation of motion

The linear σ -model

The extremisation equation (4.2) is equivalent to the Euler-Lagrange equations:

$$\frac{\partial E_{tot}}{\partial \sigma} - \frac{d}{dr} \left(\frac{\partial E_{tot}}{\partial \sigma'} \right) = 0 \quad \text{and} \quad \frac{\partial E_{tot}}{\partial \pi} - \frac{d}{dr} \left(\frac{\partial E_{tot}}{\partial \pi'} \right) = 0, \quad (4.28)$$

where $\sigma' = d\sigma/dr$ and $\pi' = d\pi/dr$. Eqs. (4.28) lead to the following second order coupled integro-differential equations:

$$\begin{aligned} \frac{d^2 \sigma(r)}{dr^2} = & -\frac{2}{r} \frac{d\sigma(r)}{dr} + \frac{m^2}{8\mu F(0, \varepsilon)} \sigma(r) + \frac{1}{16m\pi F(0, \varepsilon)} [F^2(r) - G^2(r)] \\ & + \frac{2}{\pi} \int_0^\infty dq q^4 \frac{F_{sub}(q^2, \varepsilon)}{F(0, \varepsilon)} j_0(qr) \int_0^\infty dr' r'^2 j_0(qr') (\sigma(r') - 1) \end{aligned} \quad (4.29)$$

and

$$\begin{aligned} \frac{d^2\pi(r)}{dr^2} = & -\frac{2}{r}\frac{d\pi(r)}{dr} + \left(\frac{m^2}{8\mu F(0,\varepsilon)} + \frac{2}{r^2}\right)\pi(r) - \frac{1}{16m\pi F(0,\varepsilon)}[2F(r)G(r)] \\ & - \frac{2}{\pi}\int_0^\infty dq q^4 \frac{F_{sub}(q^2,\varepsilon)}{F(0,\varepsilon)} j_1(qr) \int_0^\infty dr' r'^2 j_1(qr') \pi(r'). \end{aligned} \quad (4.30)$$

We note that if we drop the integrals in (4.29) and (4.30), we recover the equations of motion of σ and π as found by Birse and Banerjee (eq. (2.12) in Ref. [BB85]) for the linear σ -model at the classical level (without inclusion of the fermion loop correction), with the difference that we have a quadratic potential whereas they used a quartic one. It should also be noted that their conventions relate to ours by making the transformation $\sigma \rightarrow -\sigma$ in the Dirac Hamiltonian and taking the lower Dirac component with the opposite sign $G(r) \rightarrow -G(r)$.

The non-linear σ -model

Let us now take the non-linear constraint $\sigma^2 + \pi^2 = 1$ into account and find the appropriate expressions for the static soliton energy and the meson equation of motion. The meson field is parameterised with the soliton profile function $\theta(r)$ such that $\sigma(r) = \cos \theta(r)$ and $\pi(r) = \sin \theta(r)$. As a result of this, the potential energy $E_{pot}(\theta)$ becomes equal to an infinite constant which can be taken as the reference energy point. This is equivalent to renormalising E_{pot} to zero and hence there is no contribution from this term to $E_{tot}(\theta)$, which reads in this case:

$$\begin{aligned} E_{tot}(\theta) = & N_c \int_0^\infty r^2 dr \left[m \cos \theta(r) (F^2(r) - G^2(r)) \right. \\ & \left. - 2m \sin \theta(r) F(r) G(r) + F(r) \left(\frac{\partial}{\partial r} + \frac{2}{r} \right) G(r) - G(r) \frac{\partial}{\partial r} F(r) \right] \\ & + 8\pi N_c m^2 F(0,\varepsilon) \int_0^\infty dr \left[r^2 \left(\frac{\partial \theta}{\partial r} \right)^2 + 2 \sin^2 \theta \right] + 16 N_c m^2 \int_0^\infty dq q^4 F_{sub}(q^2,\varepsilon) \\ & \left[\left(\int_0^\infty dr r^2 j_0(qr) [\cos \theta(r) - 1] \right)^2 + \left(\int_0^\infty dr r^2 j_1(qr) \sin \theta(r) \right)^2 \right] \end{aligned} \quad (4.31)$$

and, applying the Euler-Lagrange equation $\frac{\partial E_{tot}}{\partial \theta} - \frac{d}{dr} \left(\frac{\partial E_{tot}}{\partial \theta'} \right) = 0$, we obtain the equation of motion for the field $\theta(r)$:

$$\begin{aligned} \frac{d^2 \theta(r)}{dr^2} = & -\frac{2}{r} \frac{d\theta(r)}{dr} + \frac{\sin 2\theta(r)}{r^2} - \frac{F^2(r) - G^2(r)}{16m\pi F(0, \varepsilon)} \sin \theta(r) - \frac{2F(r)G(r)}{16m\pi F(0, \varepsilon)} \cos \theta(r) \\ & - \frac{2}{\pi} \sin \theta(r) \int_0^\infty dq q^4 \frac{F_{sub}(q^2, \varepsilon)}{F(0, \varepsilon)} j_0(qr) \int_0^\infty dr' r'^2 j_0(qr') (\cos \theta(r') - 1) \\ & + \frac{2}{\pi} \cos \theta(r) \int_0^\infty dq q^4 \frac{F_{sub}(q^2, \varepsilon)}{F(0, \varepsilon)} j_1(qr) \int_0^\infty dr' r'^2 j_1(qr') \sin \theta(r'). \end{aligned} \quad (4.32)$$

In this case also, our formulation allows to recover, by discarding the integrals in (4.32), the equation for θ found by Kahana *et al.* (eq. (4.5b) in Ref. [KRS84]) for their model. We see that our respective conventions agree by making the transformations $\pi \longrightarrow -\pi$ in the Dirac Hamiltonian, $\theta \longrightarrow -\theta$ and finally $G(r) \longrightarrow -G(r)$.

4.3 Numerical solution

Similarly to section 3.4 of the preceding chapter, we prefer to work with dimensionless quantities when carrying out the numerical part of the calculations. The constituent quark mass $m = gf_\pi$ in particular is scaled out of all dimensionful variables so that the dependence on m will come in through the sole dependence on the cut-off parameter $\varepsilon = m/\Lambda$ (or, equivalently, the coupling constant g , since m , ε and g are all inter-dependent by virtue of eq. (3.33)). This implies that we shall work with the new variables; $x = mr$, $t = q/m$, $u(x) = m^{-3/2}F(r)$, $v(x) = m^{-3/2}G(r)$. Another numerically useful quantity is the dimensionless energy per unit colour; $\mathcal{E}_{str} = E_{str}/N_c m$ where the subscript *str* can be *tot*, *kin* or *vac*.

Non-linear integro-differential equations are known to be highly unstable under iteration and any hope of reaching self-consistency must rely on starting with

an approximate solution lying in the neighbourhood of the true solution. To help develop a workable algorithm for tackling equation (4.32), we recall that in the last chapter the vacuum energy was found to be well approximated by the meson kinetic energy alone. As this conclusion is definite for several different fixed profiles, it is reasonable to expect it to hold for the self-consistently determined one. The integrals appearing in eq. (4.32) may therefore be regarded as small perturbations with respect to other terms in the same equation. We shall now use this fact to construct the following two-stage iterative procedure:

A) In the first stage, we neglect the integrals in (4.32) and solve the remaining “mean-field” equation (similarly to Birse and Banerjee [BB85] for the linear σ model, or Kahana, Ripka and Soni [KRS84] for the non-linear case) in its dimensionless form:

$$\begin{aligned} \frac{d^2\theta^{(0)}(x)}{dx^2} = & -\frac{2}{x} \frac{d\theta^{(0)}(x)}{dx} + \frac{\sin 2\theta^{(0)}(x)}{x^2} \\ & - \frac{u^{(0)2}(x) - v^{(0)2}(x)}{16\pi F(0, \varepsilon)} \sin \theta^{(0)}(x) - \frac{2u^{(0)}(x)v^{(0)}(x)}{16\pi F(0, \varepsilon)} \cos \theta^{(0)}(x) \end{aligned} \quad (4.33)$$

along with the dimensionless form of (4.19) in the non-linear case:

$$\begin{cases} \frac{d}{dx}u^{(0)}(x) = -\sin \theta^{(0)}(x) u^{(0)}(x) - (\mathcal{E}_{val}^{(0)} + \cos \theta^{(0)}(x))v^{(0)}(x), \\ \frac{d}{dx}v^{(0)}(x) = -\left(\frac{2}{x} - \sin \theta^{(0)}(x)\right)v^{(0)}(x) + (\mathcal{E}_{val}^{(0)} - \cos \theta^{(0)}(x))u^{(0)}(x), \end{cases} \quad (4.34)$$

where the superscript (0) (‘zeroth’ order) refers to self-consistent quantities calculated without the subtracted sea contribution (integrals in (4.32)). We note that, in its dimensionless form, the radial Dirac equation is explicitly independent of the coupling constant g . The system of simultaneous equations (4.33) and (4.34) is still non-linear and requires itself an iterative procedure. First, an initial analytic profile $\theta_0^{(0)}$ is substituted in (4.34) which is solved to yield an eigenvalue $\mathcal{E}_0^{(0)}$ and eigenfunctions $u_0^{(0)}$ and $v_0^{(0)}$. After being normalised to unity, $u_0^{(0)}$ and $v_0^{(0)}$ are then inserted in eq. (4.33) which is solved to give a new profile $\theta_1^{(0)}$. This

completes one iteration and the cycle is repeated (replacing the subscripts 0 by 1 and 1 by 2) as many times as is required to obtain the profile $\theta^{(0)}$ to the desired accuracy. About 5 iterations were necessary for most starting functions used. We also checked independently the convergence of the solution by calculating the soliton energy $\mathcal{E}_{tot,i}^{(0)} = \mathcal{E}_{val,i}^{(0)} + \mathcal{E}_{kin,i}^{(0)}$, where $\mathcal{E}_{kin,i}^{(0)}$ is given by the first term in the third line of (4.31) in its dimensionless form.

As the differential equations to be solved present a two-point ($x = 0$ and $x = \infty$) boundary-value problem, knowledge of the behaviour of the functions θ , $u(x)$ and $v(x)$ for both the limits $x \rightarrow 0$ and $x \rightarrow \infty$ is required. In appendix A, we derive the appropriate limits which we summarise here:

$$\begin{array}{ll}
 x \longrightarrow 0 & x \longrightarrow \infty \\
 \theta(x) \simeq \pi - bx & \theta(x) \simeq \frac{c}{x^2} \\
 u(x) \simeq a & u(x) \simeq \sqrt{1-\lambda^2} \frac{1}{x} e^{\sqrt{1-\lambda^2} x} \\
 v(x) \simeq \frac{1+\lambda}{3} a x & v(x) \simeq \frac{1}{x} e^{\sqrt{1-\lambda^2} x}
 \end{array} \tag{4.35}$$

In (4.35) $\lambda = \mathcal{E}_{val}$ is the dimensionless valence-quark eigenvalue, which is such that $\lambda^2 \leq 1$ for a bound state. A shooting (Runge-Kutta) method (adequate for boundary-value problems) was used for integrating eqs. (4.34) and (4.33). The unknown parameters a , b , c and λ were determined numerically by matching the outward (from small x) and the inward (from large x) integrations of the differential equations. We note that in our conventions, which are the same as those Dodd and Lohe [DL85] use, all functions $\theta(x)$, $u(x)$ and $v(x)$ are positive*.

B) Once the solution $\theta^{(0)}(x)$ is obtained, we move to the second stage of the iterative solution of eq. (4.32), in which we consider the next ‘first’ order where

*As can be deduced from their asymptotic behaviour (see appendix A).

$\theta^{(1)}(x)$ is defined to be solution of

$$\begin{aligned} \frac{d^2\theta^{(1)}}{dx^2} = & -\frac{2}{x} \frac{d\theta^{(1)}}{dx} + \frac{\sin 2\theta^{(1)}}{x^2} - \frac{u^{(0)^2} - v^{(0)^2}}{16\pi F(0, \varepsilon)} \sin \theta^{(1)} \\ & - \frac{2u^{(0)}v^{(0)}}{16\pi F(0, \varepsilon)} \cos \theta^{(1)} - \frac{2}{\pi} \sin \theta^{(1)} I_0[\theta^{(0)}] + \frac{2}{\pi} \cos \theta^{(1)} I_1[\theta^{(0)}] \end{aligned} \quad (4.36)$$

where I_0 and I_1 denote the corresponding integrals of (4.32) involving j_0 and j_1 . As indicated by the superscripts in (4.36), the ‘zeroth’ order solutions serve as coefficients on the right-hand side of the θ equation which is solved to yield the ‘first’ order solution $\theta^{(1)}$. This in turn will serve as input to the Dirac system (4.34) to yield $\mathcal{E}_{val}^{(1)}$, $u^{(1)}$ and $v^{(1)}$ which now include some sea effects through eq. (4.36). The ‘second’ order of this iterative procedure proceeds by incrementing the superscripts in (4.36) by one, *i.e.* $(0) \rightarrow (1)$, $(1) \rightarrow (2)$. Between 15 and 20 iterations were necessary to reach the desired accuracy; that is the cycle was repeated up to the ‘fifteenth’ or ‘twentieth’ order of the (stage B) procedure we have just described. Here again, an independent check on the convergence was carried out by calculating $\mathcal{E}_{val} + \mathcal{E}_{vac}^{[2]}$, where $\mathcal{E}_{vac}^{[2]}$ given in its dimensionless form, corresponding to the third and fourth lines of eq. (4.31) when $\theta = \theta^{(k)}$.

As a final remark in this section, we emphasise that the iterative procedure in stage B would *only* converge when the ‘zeroth’ order solution was used as the initial guess. This indicates that the inclusion of the Dirac sea (in our approximation) yields a relatively small change in the self-consistent soliton with respect to the ‘mean-field’ soliton $\theta^{(0)}$, obtained in stage A above. Of course, once a self-consistent $\theta(x)$ is obtained for a given value of the constituent quark mass m , it can itself be used as an input profile for eq. (4.36) with a neighbouring value of m , thus speeding up the convergence.

4.4 Results and discussion

We present in this section some numerical results, following the iterative procedure discussed in the previous section. We first solved the ‘zeroth’ order problem (eq. (4.33)) for a few values of the quark mass m , namely for $m = 320, 345, 372$ and 400 MeV. These solutions were then used as initial trial solutions for the full integro-differential equation (4.36) for the same values of m and subsequently for a wider domain ranging from $m = 300$ MeV to $m = 500$ MeV. The lower limit is slightly larger than a critical quark mass m_c (which we have not been able to fix exactly) below which there is no stable baryon number-one system. This is due to the fact that smaller quark mass means smaller coupling g which implies more weakly bound quarks. We shall come back to this point later on when discussing Fig. 4.7. The upper limit of the quark mass domain ($m = 500$ MeV) marks a region beyond which the cut-off Λ becomes smaller than the quark mass m (see Fig. 3.2). The theory is questionable in this regime. Moreover, it is believed that, phenomenologically, the constituent quark mass is roughly a third of the nucleon mass, implying that it should lie within the range 300 – 400 MeV.

Fig. 4.1 shows the self-consistent solutions of eqs. (4.33) and (4.36) in dotted and full line respectively, as functions of the dimensionless radial coordinate $x = mr$. A PT cut-off was used with a quark mass $m = 345$ MeV. The two profiles are fairly close to each other, while the shape of θ is seen to wiggle slightly on both sides of $\theta^{(0)}$. θ also seems to vanish faster than $\theta^{(0)}$ with increasing x . The spatial extent can be seen to be of the order of m^{-1} .

In Fig. 4.2 we have plotted the self-consistent profile $\theta(x)$ obtained using three different cut-off two-point functions, for $m = 345$ MeV. As can be seen from the graphs, the form of the cut-off has little influence on θ . Physical quantities (nucleon mass and nucleon observables) which are presumably sensitive to

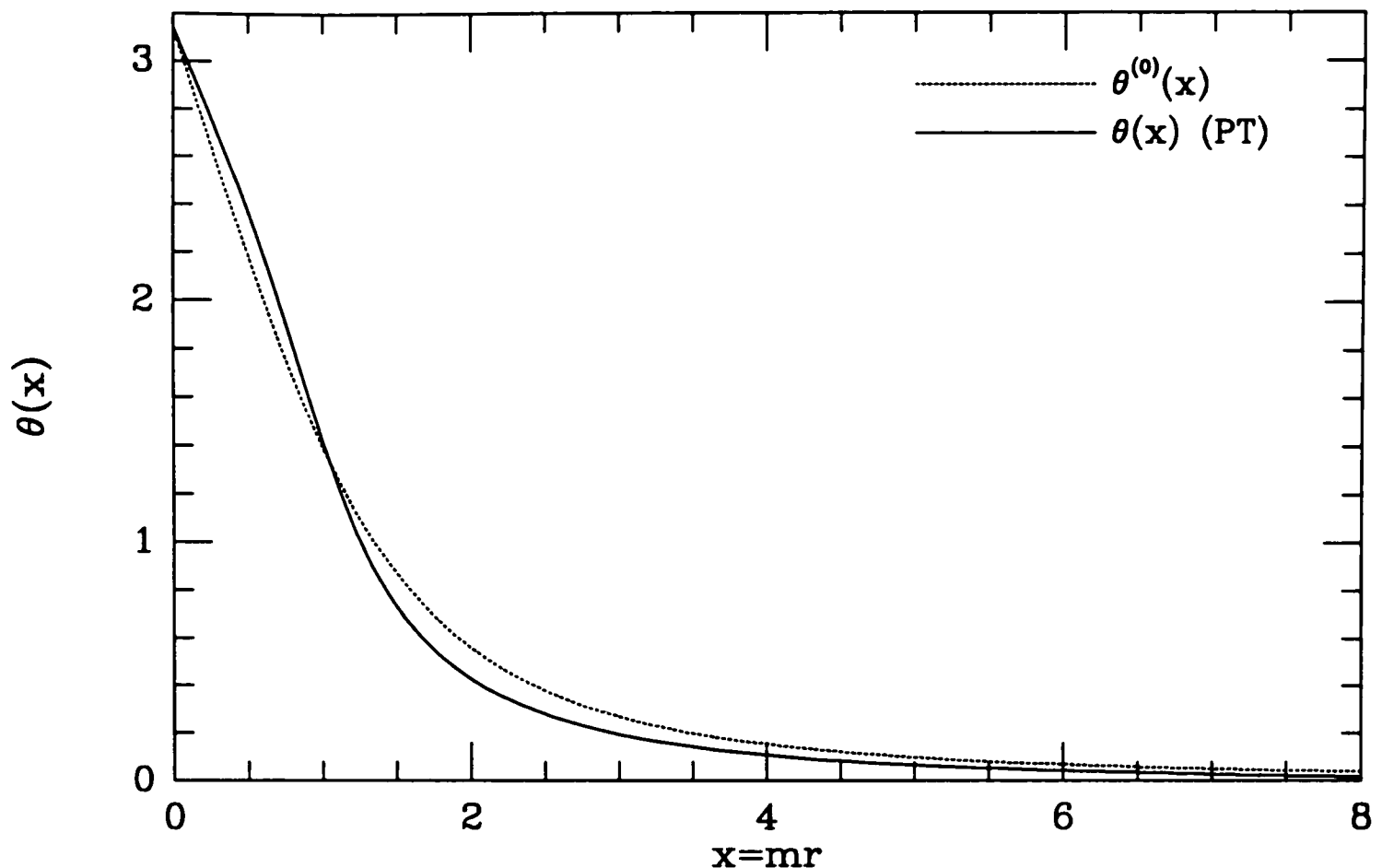


Figure 4.1: The 0th order solution $\theta^{(0)}$ and the full self-consistent profile θ with a PT cut-off and $m = 345$ MeV.

the details of the shape of the profile will be more useful in comparing different regularisations. We can nevertheless state at this point that the two-point approximation stabilises the soliton at a reasonable physical size $\sim m^{-1}$, whereas the derivative expansion loses the soliton by collapse. Next, in Figs. 4.3 and 4.4, the self-consistent profile and the meson fields in units of f_π are shown for three values of the parameter m . The use of the radial variable $x = mr$, which is in units of m^{-1} , is not appropriate here since we are comparing quantities for different quark masses. We have therefore plotted all the curves as functions of r in fermis.

We see from Fig. 4.3 that increasing the value of the parameter m spreads the spatial extent of the self-consistent profile θ but has little effect on its overall shape. In other words the self-consistent soliton becomes larger with tighter binding. This same trend is noticeable in the pion field in Fig. 4.4; its maximum

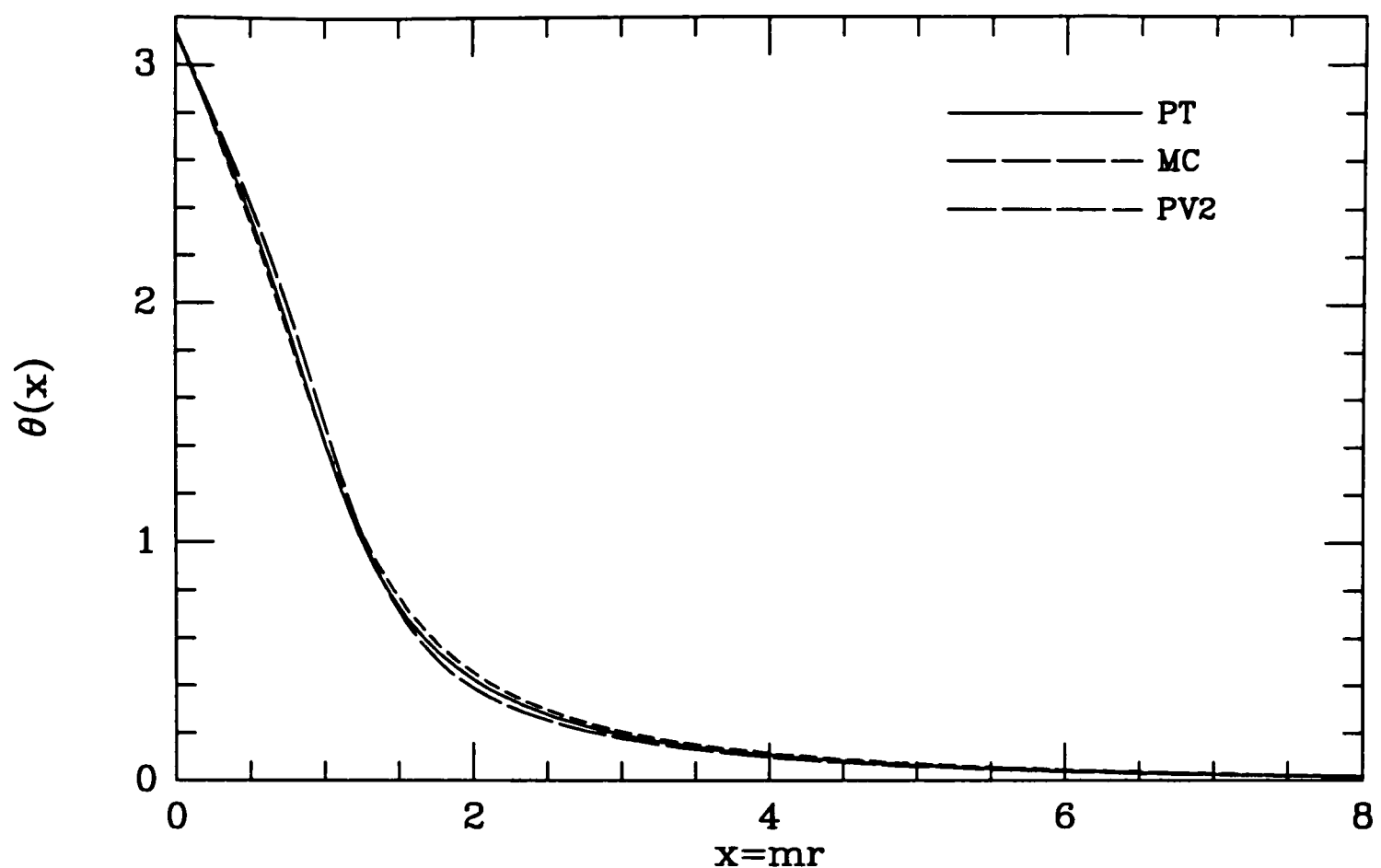
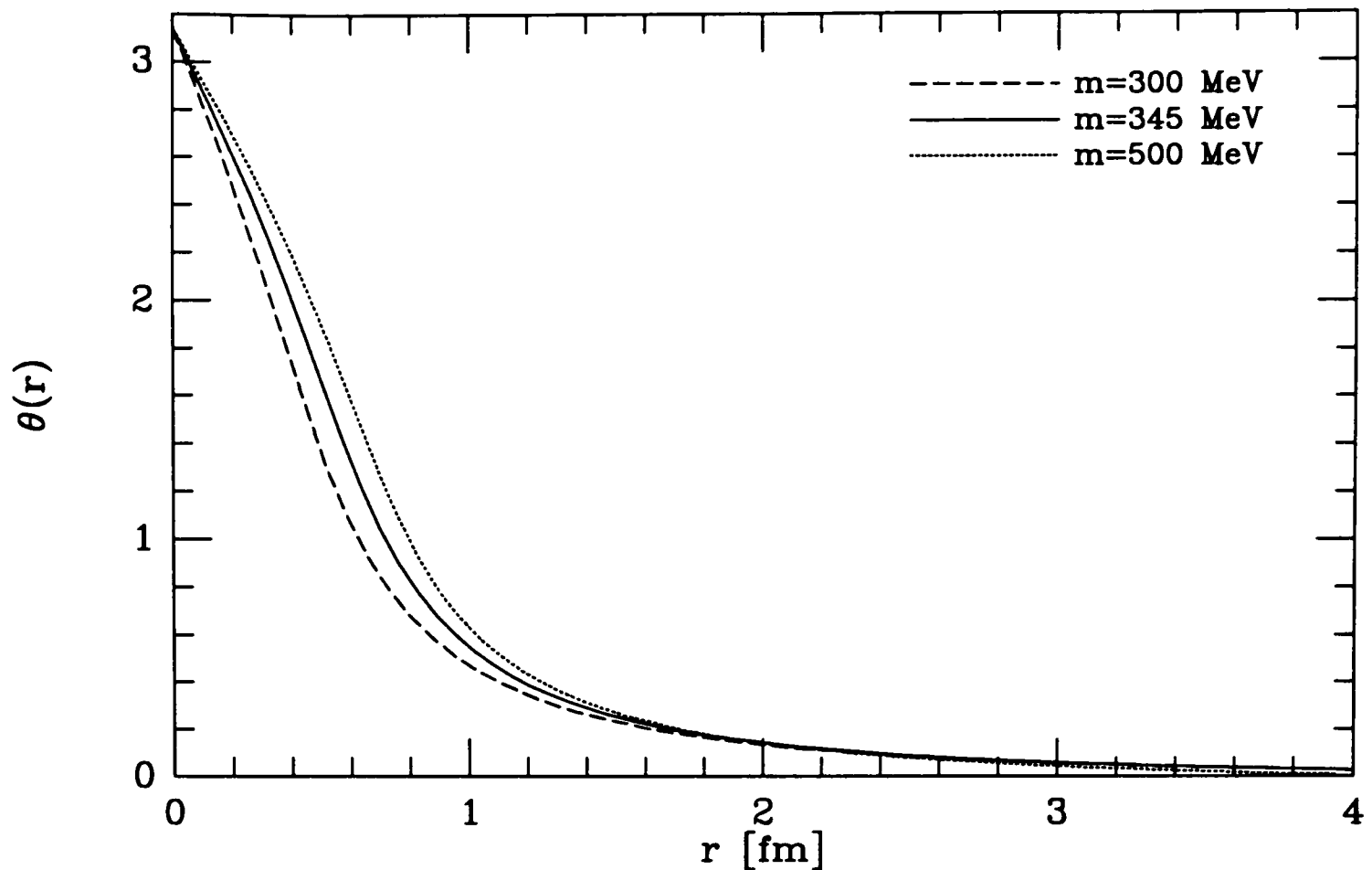


Figure 4.2: The self-consistent profile θ for various cut-offs and $m = 345$ MeV.

value is reached at $r = 0.45$ fm for $m = 300$ MeV and $r = 0.6$ fm (a typical nucleon size) for $m = 500$ MeV. It is interesting to mention at this point that Meissner *et al.* [MG91] find that their self-consistent pion field (with inclusion of the full Dirac sea) hardly changes with variation of the constituent quark mass m . The maximum amplitude, they find, is stationary at around $r = 0.41$ fm.

Fig. 4.5 shows the valence quark wave-functions for different values of m . The wave-functions $F(r)$ and $G(r)$ are normalised according to (4.10). The valence quarks seem to become more localised towards the origin with increasing m . We shall have more to say about this point when we discuss in chapter six baryon observables in the two-point approximation to the ECA.

Table 4.1 compares the values of the different contributions to the soliton energy E_{tot} for the three cut-offs as well as with the corresponding results from Ref. [MG91], where E_{tot} was calculated using the proper-time regularised eigen-

Figure 4.3: The self-consistent profile θ with a PT cut-off and various quark masses.Table 4.1: Comparison of E_{tot} and its contributions as well as g_A for different cut-offs and $m = 372$ MeV.

Cut-off	E_{val} [MeV]	E_{vac} [MeV]	E_{tot} [MeV]	g_A
MC 1	370.75	595.00	965.75	0.963
PT	381.16	661.02	1042.18	1.066
PV2	376.88	700.22	1077.10	1.098
Ref.[MG91]	595.00	617.00	1213.00	0.810

value summation of the Dirac sea for $m = 372$ MeV. The rightmost column gives the axial vector coupling constant g_A , which is evaluated in a very simple way. We defer the explanation of how to calculate it to the proper context of chapter six where we discuss some nucleon observables. The figures for E_{tot} and g_A show a 10% discrepancy between the MC (lowest) and the PV2 (highest) values, while the PT case is roughly half-way in between. Comparing our PT results with Ref. [MG91], we note that the two-point approximation to E_{vac} gives a soliton

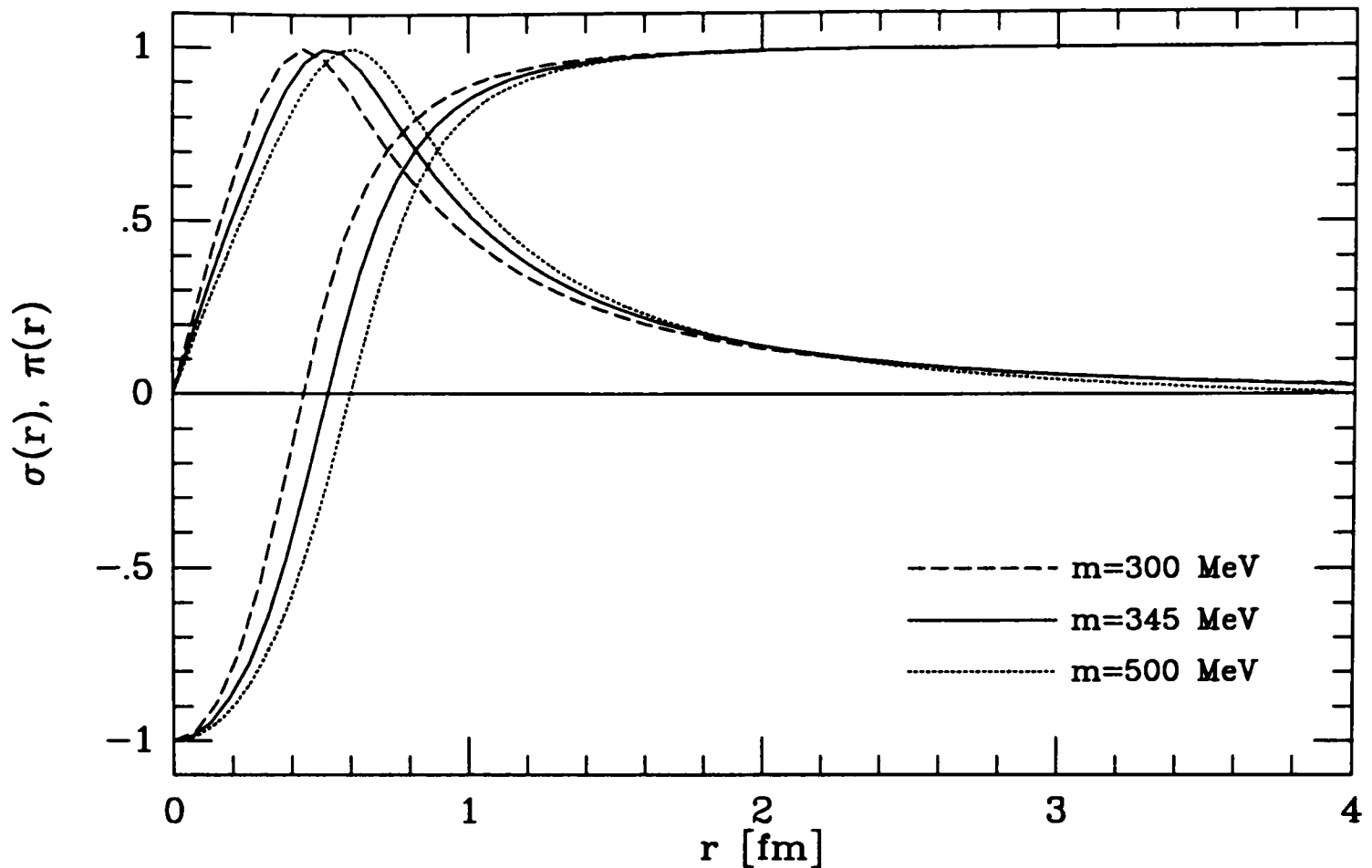


Figure 4.4: The self-consistent meson fields σ and π (vanishing at the origin) with a PT cut-off and various quark masses.

energy some 20% below the exact value while our g_A is some 30% above. As far as the soliton energy is concerned, this is due to a considerable reduction in the valence quark energy and a moderate increase in the two-point vacuum energy. Although our figures are somewhat less accurate than what was to be expected according to Diakonov [Dia87], they are sufficiently close to the exact results to make the two-point approximation well worth considering. It is interesting to notice that the two-point approximation gives results nearer the experimental figures of less than 1000 MeV* for the chiral soliton and 1.23 for g_A . This should not however be taken seriously since we are only concerned here with testing the two-point approximation.

From Fig. 4.6 we see that the self-consistent energy in our approximation de-

*The chiral soliton considered is in fact a classical object with no definite quantum numbers. Its mass can not be therefore identified with that of a nucleon before the soliton has been quantised. See chapter six for more details.

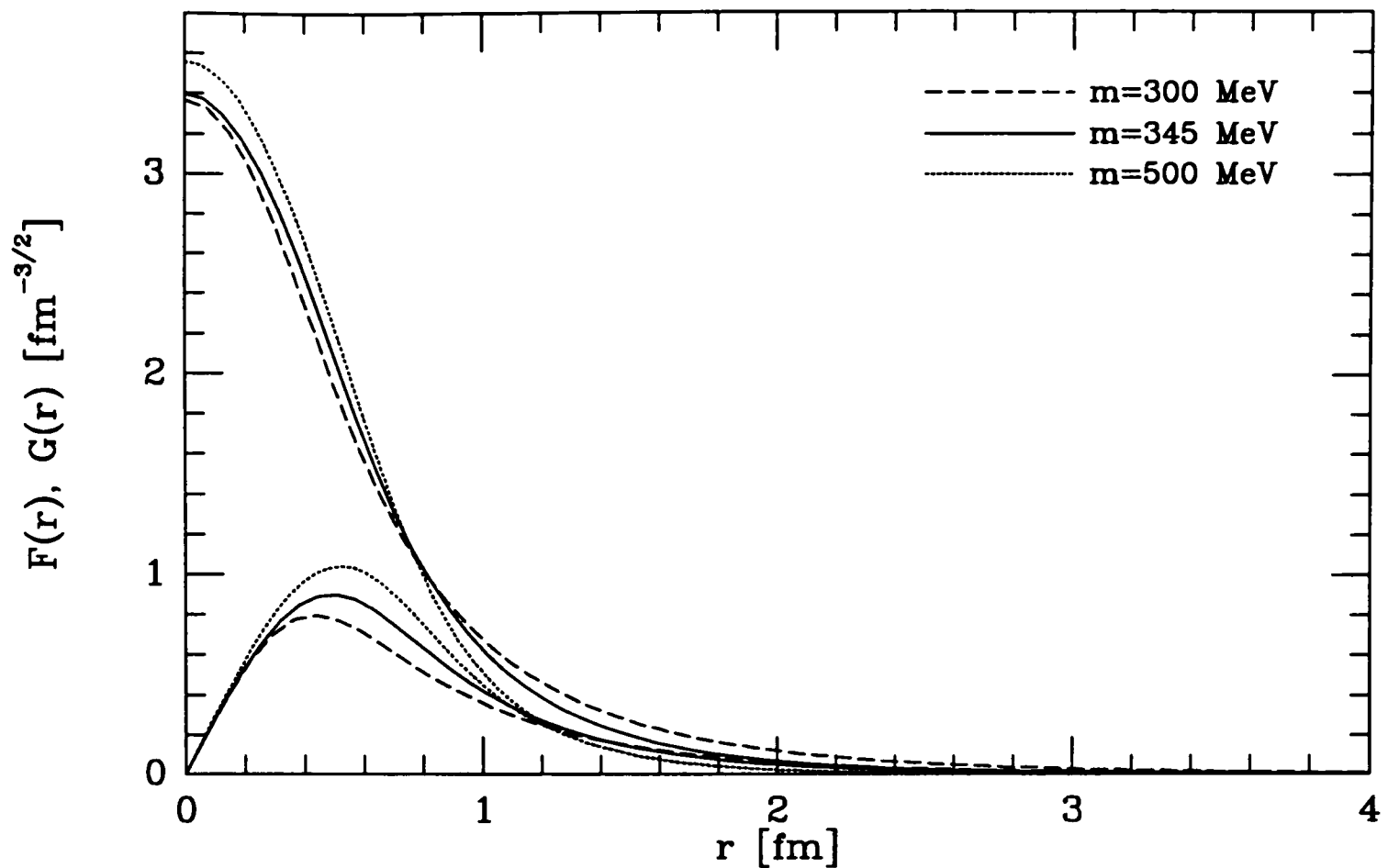


Figure 4.5: The self-consistent valence quark wave-functions (the smaller component $G(r)$ vanishing at the origin) with a PT cut-off and various quark masses.

creases noticeably with increasing m , whereas the curve taken from Ref. [MG91] shows only a very gradual variation. The exact calculation seems to give a stable soliton energy at around 1200 MeV with respect to variation of m (in agreement also with Refs. [Alk90, WY91]), implying that the two-point approximation worsens as the constituent quark mass increases. Recalling that m is proportional to the coupling constant g , it may be that the perturbative expansion on which our approximation is based deteriorates with stronger coupling, thus requiring the addition of higher order terms in the expansion. From the results in Tab. 4.1 and Fig. 4.6 however, it can be concluded that the approximation works reasonably well if one restricts the range of m to the phenomenologically preferred values (300–400 MeV) and gets better for the smaller values.

Comparison with the ‘zeroth’ order curve (dotted line) shows that quantum corrections (additional to the radiatively induced meson kinetic energy) lower the

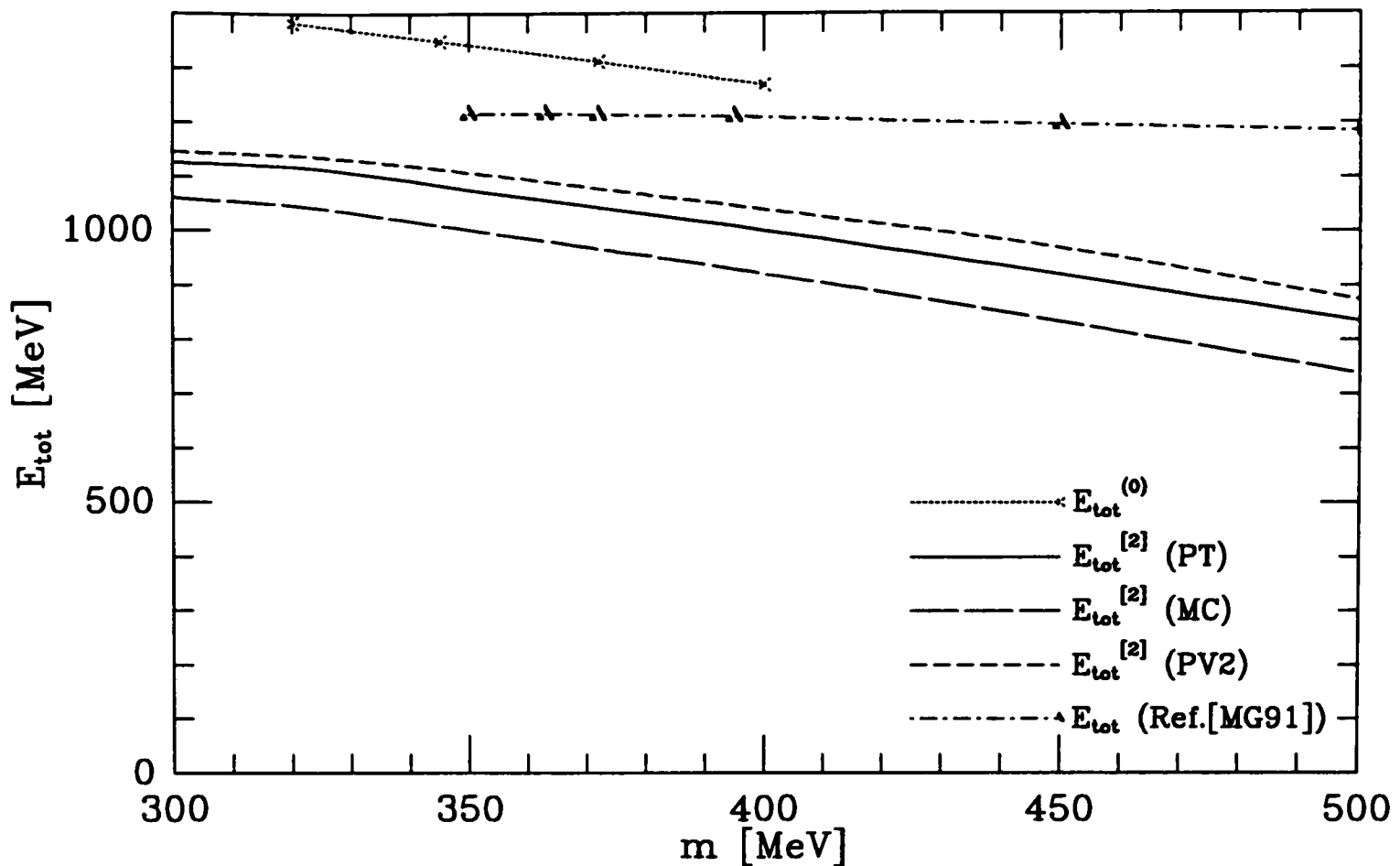


Figure 4.6: Variation with m of the self-consistent energy in the two-point approximation, $E_{tot}^{[2]}$, for the 3 stated cut-offs, compared with that of the exact energy as given in Ref. [MG91]; also shown is the 0^{th} order energy $E_{tot}^{(0)}$.

soliton energy significantly in the two-point approximation and only moderately (up to $m = 400$ MeV) with the full inclusion of the Dirac sea.

Fig. 4.7 shows the variation of the self-consistent soliton energy and its different contributions with m in the case of a PT cut-off. The total soliton energy is above the three-quark threshold for $m < 350$ MeV and below for $m > 350$ MeV, to be compared with the value $m = 400$ MeV found in Refs. [Alk90, MG91, WY91]. The valence energy starts at 640 MeV for $m = 300$ MeV, where the quarks are weakly bound. Below this critical value of m the Dirac wave-function becomes that of a scattering state and the baryon number-one solution is lost. The valence energy decreases gradually to zero for $m \sim 500$ MeV and becomes negative for higher values. The corresponding values in Refs. [Alk90, MG91, WY91] are about 325–360 MeV for the critical quark mass and about 750 MeV for the zero

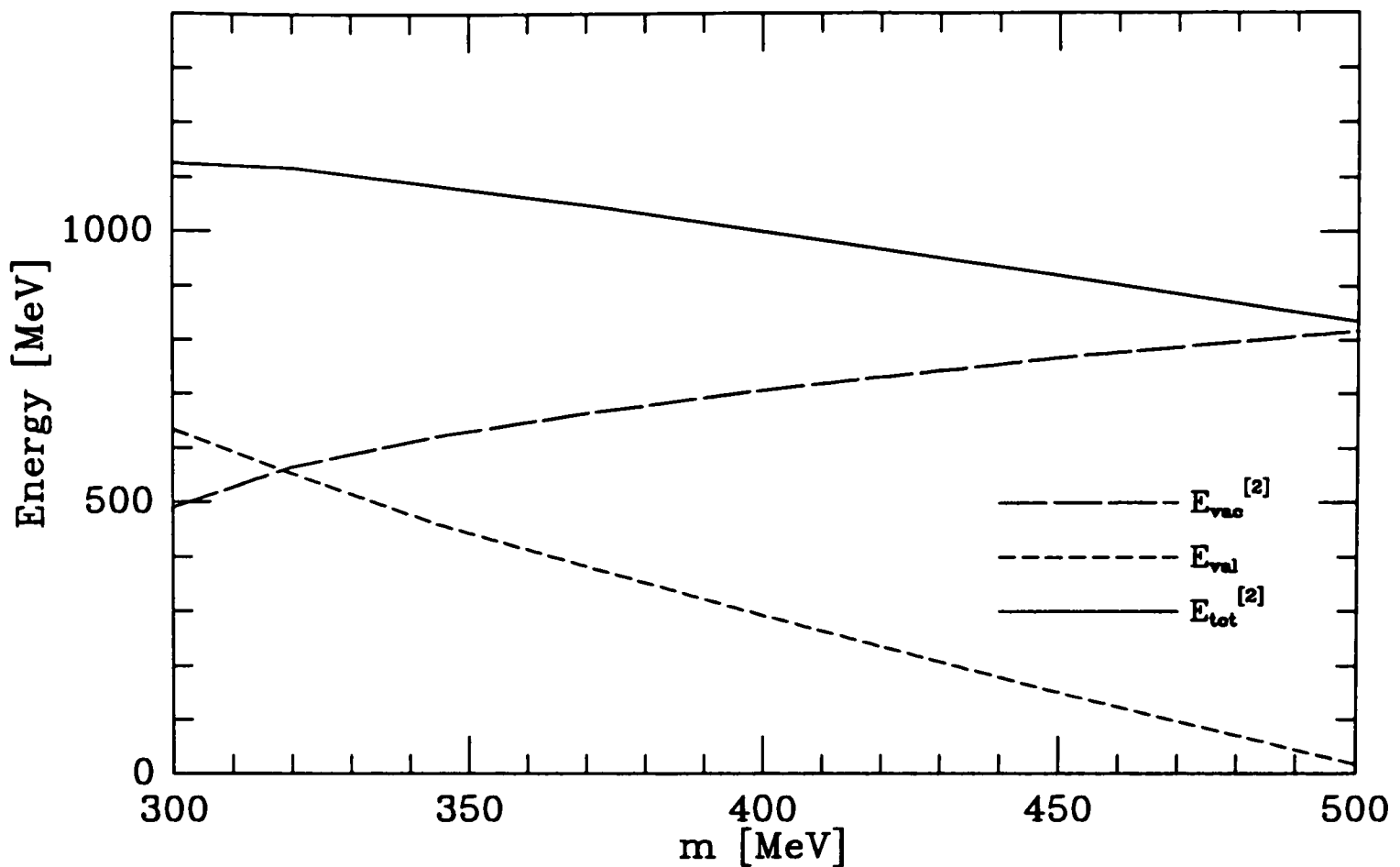


Figure 4.7: Variation with m of the different contributions to $E_{tot}^{[2]}$, using a PT cut-off.

valence-energy point. In any case, the two-point approximation seems to favour a slight domination of the sea over the valence quarks whereas this picture is exactly reversed with the sea fully included [MG91].

The question of how much energy is gained by the self-consistent profile over the fixed profiles we considered in the previous chapter deserves some attention. Table 4.2 summarises the results for $E_{tot}^{[2]}$ obtained for two values of m by minimising the soliton energy with respect to the soliton-size parameter R . The seventh row gives the self-consistent energy. The stereo 2, exponential and tanh profiles are the closest to the self-consistent, with the stereo 2 profile retaining a mere 15 MeV over the dynamically determined one. Perhaps this is due to stereo 2 having the right long range behaviour $\sim r^{-2}$, to which the soliton energy may be sensitive.

Finally, it is usually an interesting question to ask whether a solution to a

Table 4.2: Comparison of the total soliton energy $E_{tot}^{[2]}$ for the self-consistent profile and various fixed profiles, using a PT cut-off.

Profile	$E_{tot}^{[2]}$ [MeV]	
	$m = 345$ MeV	$m = 372$ MeV
stereo 1	1411.40	1243.02
stereo 2	1096.24	1059.63
stereo 3	1354.13	1309.52
linear	1194.30	1155.05
expon.	1103.54	1082.23
tanh	1099.53	1067.90
θ (self-cons.)	1081.60	1042.18

Euler-Lagrange equation is an absolute minimum. Recently, Moussallam [Mou89] argued that any two-parameter soliton profile only yields saddle point solutions which are local minima. For the problem at hand, we may regard our self-consistent solution as a function with an arbitrary number of parameters, for which a functional is extremised in parameter space, but the nature of the stationary point has not been investigated yet. Following the one-parameter profile discussion of the preceding chapter, we propose here to extract a size parameter λ by extending and contracting the self-consistent soliton profile;

$$\theta(r) \longrightarrow \theta(\lambda r), \quad (4.37)$$

such that setting $\lambda = 1$ recovers the self-consistent profile obtained. The stability of the soliton energy with respect to the scaling transformation (4.37) can then be checked. Results from Fig. 4.8 show that the overall minimum with respect to variation of λ occurs at $\lambda = 1$, indicating that the soliton energy is indeed stable with respect to rescaling of the soliton profile. The minima for $m = 300$ MeV and $m = 345$ MeV are only local minima as should be expected, since the soliton energy in this region is above the three-quark threshold.

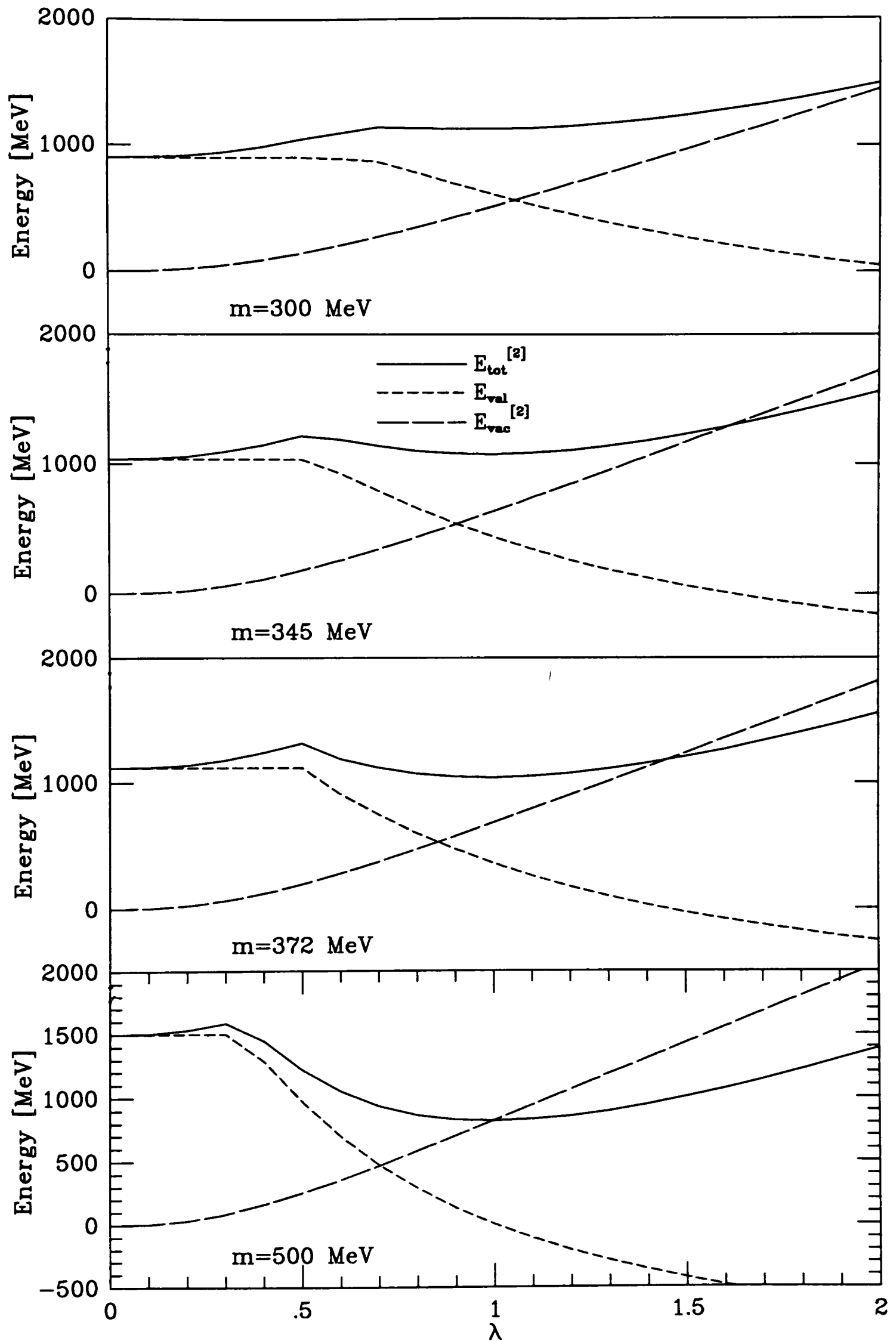


Figure 4.8: Variation of the different contributions to the soliton energy $E_{\text{tot}}^{[2]}$ with the scaling parameter λ , using a PT cut-off, for four values of m .

All we have done so far is checking the stability of our solution with respect to one parameter and so we have not yet checked Moussallam's claim. Being unable to extract a second variational parameter from our self-consistent profile, we have alternatively made use of a two-parameter version of the stereo 2 profile invented by Diakonov *et al.* [DPP89]:

$$\theta(r) = \begin{cases} \arctan \left[\left(\frac{r}{R} \right)^2 \frac{(r/R)^4 + 1 + \delta}{(r/R)^4 + 1 - \delta} \right] & \text{for } 0 \leq \delta < 1, \\ \arctan \left[\left(\frac{r}{R} \right)^2 \frac{1 - \delta}{1 + \delta} \right] \cdot \frac{(r/R)^4 + 1 - \delta}{(r/R)^4 + 1 + \delta} \frac{1 + \delta}{1 - \delta} & \text{for } -1 < \delta < 0. \end{cases} \quad (4.38)$$

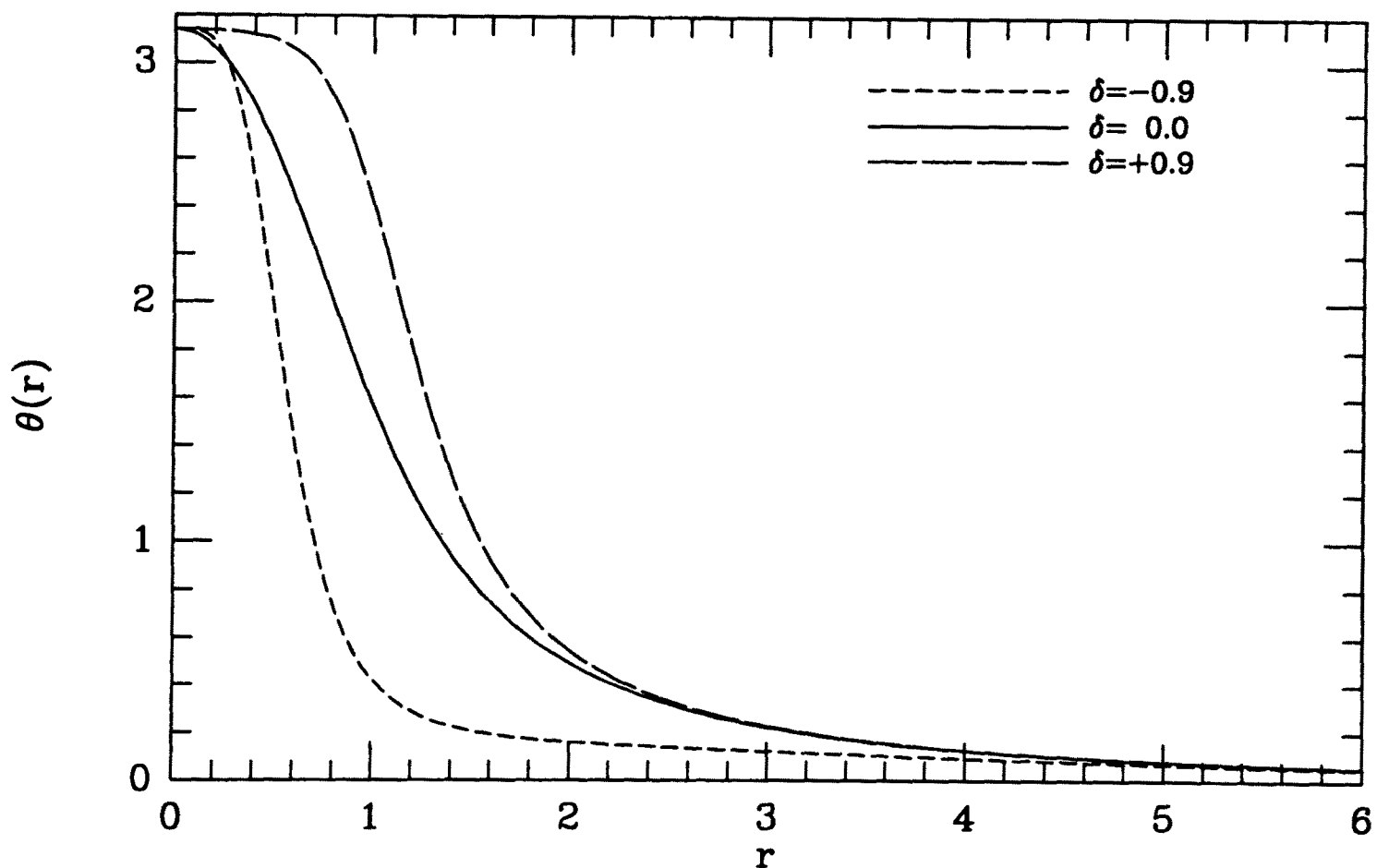


Figure 4.9: The two-parameter profile defined in (4.38) for 3 values of δ , with $R = 1$ in all cases.

In (4.38) R is the usual size parameter, while δ changes the shape of the profile as shown in Fig. 4.9. By setting $\delta = 0$ we obtain the stereo 2 profile. As can be seen from Fig. 4.10, where the soliton energy surface $E_{tot}(R, \delta)$ is

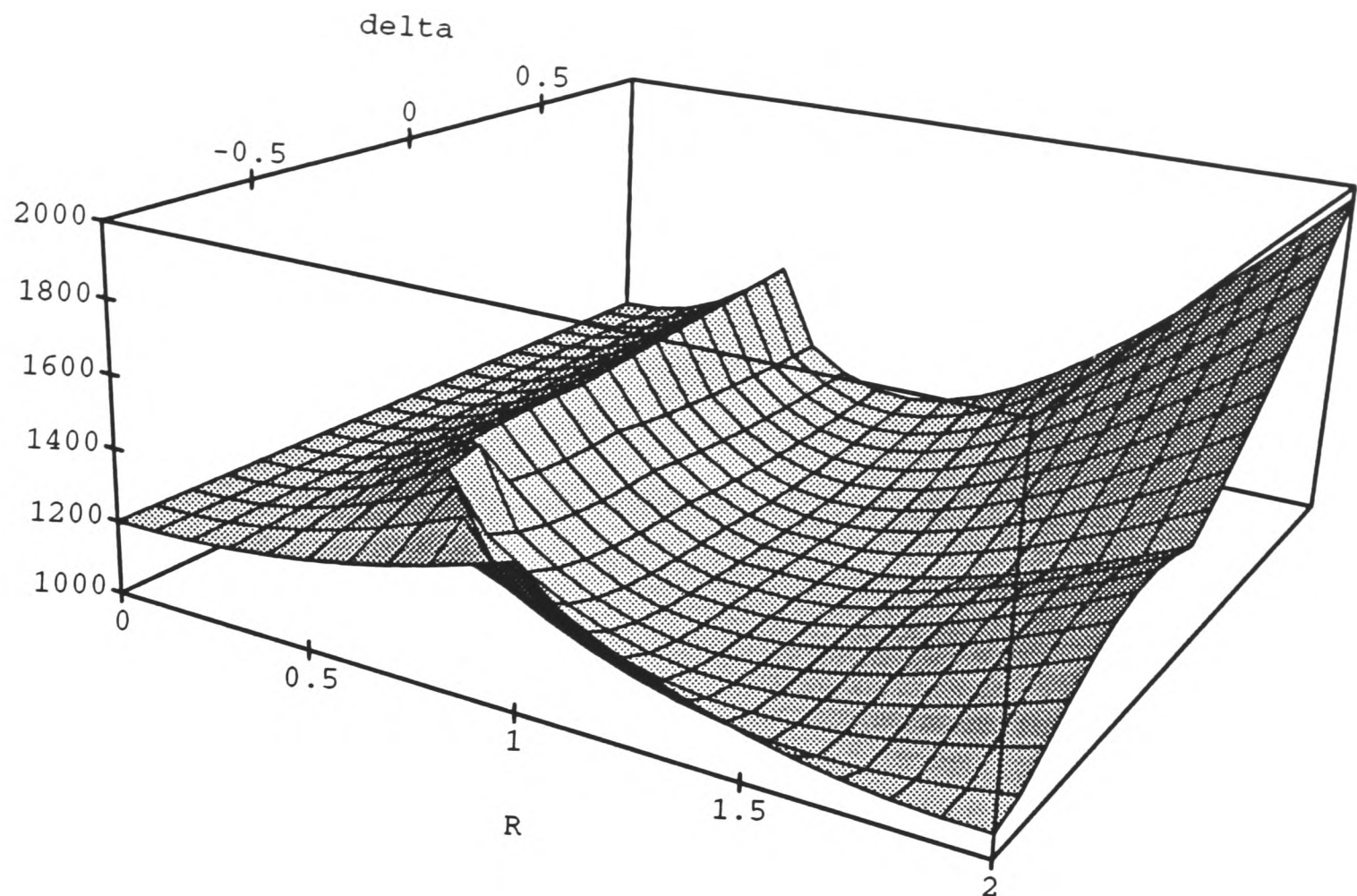


Figure 4.10: soliton energy as a function of the two-parameter profile defined in (4.38), using a PT cut-off with $m = 372$ MeV.

plotted, the overall minimum occurs for $\delta = 0$ while the δ -dependence turns out to be rather weak, confirming Diakonov's conclusion who calculated the soliton energy including the full Dirac sea. Besides providing a further check for the two-point approximation, our two-parameter calculation disagrees with Moussallam's statement.

4.5 Conclusion

Using the two-point approximation to the polarised vacuum energy, we have carried out a self-consistent calculation of the soliton profile. The numerical procedure we followed simplifies the complicated integro-differential equation problem by first solving the ‘mean-field’ equations, where only the meson kinetic energy contribution (from the sea) is retained, and then including the rest of the (two-point) quantum corrections as perturbation in the main equation. We were able to find self-consistent solutions for a wide range of constituent quark masses 300–500 MeV and for three different forms of the cut-off. No solutions seem to exist for masses below 300 MeV. The soliton energy changes by up to 10% as the form of the cut-off is changed. Comparing our results with exact calculations, we found that the two-point approximation leads to soliton masses about 20% too low and an axial coupling g_A 30% too high for a constituent quark mass $m = 372$ MeV, disagreeing with initial claims by Diakonov of only 10% in the energy. Furthermore, the approximation is better for low values of m and systematically worsens for high values. No substantial gain in energy was obtained by preferring the self-consistent soliton over some of the fixed profiles of the last chapter. The soliton energy was found to be stable against rescaling of the self-consistent profile. The fact that the ρ -approximation leads to stable soliton solutions and reasonable agreement with exact results (for the phenomenologically preferred constituent quark masses) makes it well worth considering as a useful approximation for similar problems involving non-local effective actions.

Chapter 5

The Pole Approximation

Using the two-point approximation to the effective chiral action, we were able to find a simpler expression, though still non-local, for the vacuum contribution to the static soliton energy. Numerically, it simplified the problem of summing up the eigenvalues of a very large number of negative energy orbitals to a more manageable two-dimensional integral involving the two-point function and the Fourier transforms of the meson field. However, in chapter four we encountered difficulties in solving the coupled equations for quarks and mesons, due to the inherent complexity of strongly non-linear integro-differential equations. There is so far no known systematic method (whether analytical or numerical) for solving this type of equations, apart from iterative procedures which are – in any case – not very reliable for non-linear problems. It would clearly be desirable if there existed a simplification which would lead to an alternative way for solving equation (4.32).

In this chapter, we shall use an approximation to the two-point function which will change the integro-differential equation (4.32) into a self-consistent equation for the profile θ , involving no derivatives and only a single integral. It represents a significant improvement both analytically and numerically. This

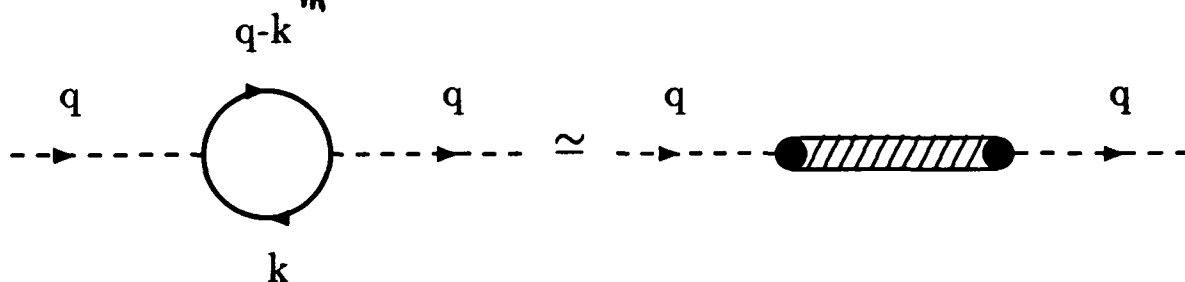
approximation was first mentioned by Diakonov in Ref. [Dia87] and is called the pole approximation, a name which is explained in the next section. As we shall see, this approximation could be regarded as ‘just’ another regularisation scheme, and therefore it is effectively no further approximation in the context of the two-point effective action with finite cut-off.

5.1 The pole approximation to the two-point function

We saw in chapter three that, with the exception of the PV1 cut-off and the phenomenological MDQM cut-off, all the regularisation schemes we considered yielded cut-off functions with a q^{-2} behaviour for large momenta. An approximation to the two-point function in these cases is provided by the simple pole form:

$$F(q^2, \varepsilon) \simeq F_{pole}(q^2, \alpha, \beta) = \frac{\alpha}{q^2/m^2 + \beta^2} \quad (5.1)$$

which, using a diagrammatic language, is equivalent to:



meaning that a scalar propagator of mass βm is substituted for the two-point function. Like the MC, PV2 and PT cut-off functions, the pole two-point function has an explicit q^{-2} fall-off. Whereas the cases treated in chapter three depended on a single parameter ε , F_{pole} introduces two free parameters α and β . One of these parameters is fixed using the cut-off condition – eq. (3.33) – which reads in this case:

$$4 N_c g^2 \alpha = \beta^2. \quad (5.2)$$

The other parameter, which we will take for convenience to be β , may be obtained by fitting the large- q behaviour of $F(q^2, \varepsilon)$. This prescription fits $F_{pole}(q^2, \alpha, \beta)$ to $F(q^2, \varepsilon)$ by making the two corresponding curves coincide at $q = 0$ and $q = \infty$.

Alternatively, one can fix β by fitting F_{pole} to the ‘bulk’ of $F(q^2, \varepsilon)$, which turns out to be the important region. In Fig. 5.1 we have plotted the two-point functions MC, PV2 and PT versus the dimensionless momentum $t = q/m$ for four values of the constituent quark mass. We notice that for all four values the cut-off functions delineate a ‘corridor’ of possible two-point functions, where MC represents the ‘hardest’ cut-off, PV2 the ‘softest’ one and PT lies in between. The dotted curves represent $F_{pole}(t^2, \alpha, \beta)$ which have been fitted to the $t \leq 3$ region of the PT curves. Figure 5.1 indicates that the pole approximation F_{pole} could be considered on the same footing as cut-off functions belonging to the family of functions $F(t^2, \varepsilon)$ having a q^{-2} fall-off. Table 5.1 shows values of α and β for the four values of the quark mass considered. We note that β is of order 2 in all cases, implying a pole of a mass roughly twice the mass of a quark (or a quark-antiquark pair). This seems reasonable, since – in terms of singularity structure – the pole is presumably “mocking up” the two-particle normal threshold in the spacelike region.

Table 5.1: Numerical values of α and β for F_{pole} fitted to F_{PT} .

g	$m = gf_\pi$ [MeV]	fitted to tail region		fitted to $t \leq 3$ region	
		α	β	α	β
3.23	300	0.0393	2.216	0.0714	2.986
3.71	345	0.0221	1.910	0.0425	2.649
4.00	372	0.0165	1.781	0.0328	2.511
5.38	500	0.0059	1.435	0.0129	2.116

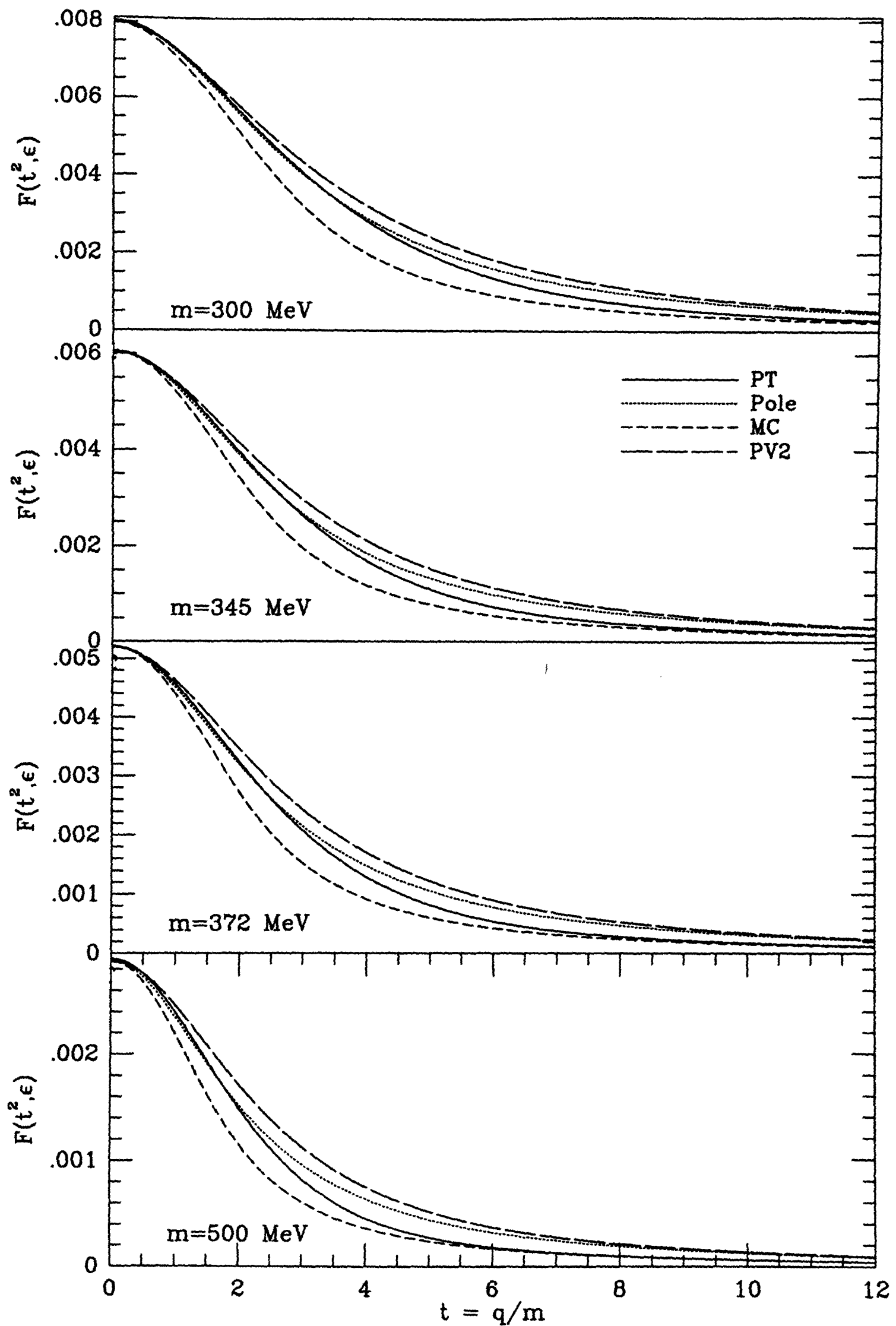


Figure 5.1: The two-point functions $F(t^2, \epsilon)$ for three covariant cut-offs and the pole approximation versus t , for four values of m .

Coordinate space Green functions

Rather than substitute the pole form F_{pole} in place of F in the θ equation of motion (4.32), let us go back to the original dimensionless form of the two-point Casimir energy without the meson kinetic term subtracted, *i.e.*

$$\mathcal{E}_{vac}^{[2]} = 16 \int_0^\infty dt t^4 F(t^2, \varepsilon) \left[\left(\int_0^\infty dx x^2 j_0(tx) [\cos \theta(x) - 1] \right)^2 + \left(\int_0^\infty dx x^2 j_1(tx) \sin \theta(x) \right)^2 \right] \quad (5.3)$$

and rewrite it in the form:

$$\mathcal{E}_{vac}^{[2]} = 16 \int_0^\infty dx \int_0^\infty dx' xx' \left[K_0(x, x') (\cos \theta(x) - 1) (\cos \theta(x') - 1) + K_1(x, x') \sin \theta(x) \sin \theta(x') \right] \quad (5.4)$$

where we have defined

$$K_n(x, x') = \int_0^\infty dt t^2 F(t^2, \varepsilon) tx j_n(tx) tx' j_n(tx'), \quad n = 0, 1. \quad (5.5)$$

By replacing $F(t^2, \varepsilon)$ by the pole form F_{pole} eq. (5.5) becomes

$$K_n(x, x') = \int_0^\infty \frac{\alpha t^2}{t^2 + \beta^2} tx j_n(tx) tx' j_n(tx'). \quad (5.6)$$

Now recalling the orthogonality relations of the spherical Bessel functions:

$$\int_0^\infty dt tx j_n(tx) tx' j_n(tx') = \frac{\pi}{2} \delta(x - x'), \quad n \in N, \quad (5.7)$$

we note that, writing $\frac{\alpha t^2}{t^2 + \beta^2} = \alpha - \frac{\alpha \beta^2}{t^2 + \beta^2}$, the integral in eq. (5.6) includes a contribution from eq. (5.7). We then isolate the infinite part of $K_n(x, x')$:

$$K_n(x, x') = \frac{\alpha \pi}{2} \delta(x - x') - \alpha \beta^2 \int_0^\infty dt \frac{1}{t^2 + \beta^2} tx j_n(tx) tx' j_n(tx'). \quad (5.8)$$

The remaining integral, which can be done in the complex plane, is readily available in tabulated form:

$$\int_0^\infty dt \frac{1}{t^2 + \beta^2} tx j_n(tx) tx' j_n(tx') = \begin{cases} \frac{\pi}{2} xx' i_n(\beta x) k_n(\beta x') & x < x' \\ \frac{\pi}{2} xx' i_n(\beta x') k_n(\beta x) & x > x' \end{cases} \quad (5.9)$$

where i_n and k_n are the modified spherical Bessel functions of the first and second kind respectively and given by:

$$\begin{aligned} i_n(x) &= x^n \left(\frac{d}{dx} \right)^n \frac{\sinh x}{x}, \\ i_n(x) &= (-1)^n x^n \left(\frac{1}{x} \frac{d}{dx} \right)^n \frac{e^{-x}}{x}. \end{aligned} \quad (5.10)$$

Defining $G_n(x, x')$ by

$$K_n(x, x') = \frac{\alpha\pi}{2} \delta(x, x') - \frac{\alpha\beta^2\pi}{2} G_n(x, x') \quad (5.11)$$

and substituting (5.11) for $n = 0$ and $n = 1$ in (5.4) we get the expression of the two-point vacuum energy in the pole approximation:

$$\begin{aligned} \mathcal{E}_{vac}^{[2]} &= 32\alpha\pi \int_0^\infty dx x^2 \sin^2 \frac{\theta(x)}{2} - 8\alpha\beta^2\pi \int_0^\infty \int_0^\infty dx' x x' \\ &\quad [G_0(x, x') (\cos \theta(x) - 1) (\cos \theta(x') - 1) + G_1(x, x') \sin \theta(x) \sin \theta(x')] \end{aligned} \quad (5.12)$$

where a little algebra gives $G_0(x, x')$ and $G_1(x, x')$ explicitly:

$$\beta G_0(x, x') = \begin{cases} \sinh \beta x e^{-\beta x'} & x < x' \\ \sinh \beta x' e^{-\beta x} & x > x' \end{cases} \quad (5.13)$$

and

$$\beta G_1(x, x') = \begin{cases} \left(\cosh \beta x - \frac{1}{\beta x} \sinh \beta x \right) \left(1 + \frac{1}{\beta x'} \right) e^{-\beta x'} & x < x' \\ \left(\cosh \beta x' - \frac{1}{\beta x'} \sinh \beta x' \right) \left(1 + \frac{1}{\beta x} \right) e^{-\beta x} & x > x' \end{cases} \quad (5.14)$$

An alternative approach to the derivation presented in this section is outlined in appendix B. There, we show how the functions $G_0(x, x')$ and $G_1(x, x')$ can be identified with the coordinate space Green functions in one dimension, which are solutions of [AAZ91b]:

$$\left[-\partial_x^2 + \frac{\ell(\ell+1)}{x^2} + \beta^2 \right] G_\ell(x, x') = \delta(x - x') \quad (5.15)$$

and obey the boundary conditions

$$\lim_{x \rightarrow 0} x^{-\ell} G_\ell(x, x') = 0, \text{ for } \ell = 0, 1. \quad (5.16)$$

5.2 The self-consistent equation for θ

Similarly to section 4.2 in chapter four, the equation of motion for the field $\theta(x)$ in the present context is obtained by varying the total energy $\mathcal{E}_{tot} = \mathcal{E}_{val} + \mathcal{E}_{vac}^{[2]}$ with respect to $\theta(x)$. Since there are no terms involving derivatives of $\theta(x)$ in the present form of $\mathcal{E}_{vac}^{[2]}$ (eq. (5.12)), the variational equation $\delta\mathcal{E}_{tot}/\delta\theta = 0$ leads to a relatively simple self-consistent equation, reading:

$$A(x) \cos \theta(x) = B(x) \sin \theta(x), \quad (5.17)$$

where

$$A(x) = 2x u(x)v(x) + 16\pi\alpha\beta^2 \int_0^\infty dx' x' G_1(x, x') \sin \theta(x') \quad (5.18)$$

and

$$B(x) = -x (u^2(x) - v^2(x)) + 16\pi\alpha\beta^2 \int_0^\infty dx' x' G_0(x, x') \cos \theta(x'). \quad (5.19)$$

Eq. (5.17) has still to be solved along with the system of differential equations (4.19) for $u(x)$ and $v(x)$, which are unaltered in this formalism since they are not sensitive to regularisation of the Dirac sea. Whereas eq. (4.32) necessitated the (numerically awkward) evaluation of a double integral with highly oscillating integrands, eq. (5.17) involves no derivatives of $\theta(x)$ and requires only a one-dimensional integration over smooth functions. In view of this, eq. (5.17) presents a remarkable improvement over eq. (4.32).

5.3 Numerical solution

The numerical procedure we have adopted in this chapter follows the same lines as the ‘stage B’ calculation discussed in section 4.3. First, an initial fixed profile $\theta^{(0)}$ is substituted into the Dirac system (4.19) and an eigenvalue $\mathcal{E}_{val}^{(0)}$ and eigenfunctions $(u^{(0)}, v^{(0)})$ are extracted. Then, after being normalised, $u^{(0)}$ and $v^{(0)}$ are

used (along with $\theta^{(0)}$) to compute $A(x)$ and $B(x)$. The self-consistent equation (5.17) then yields a new profile function $\theta^{(1)}(x)$:

$$\tan \theta^{(1)}(x) = \frac{A^{(0)}(x)}{B^{(0)}(x)}. \quad (5.20)$$

One iteration has now been completed and the cycle is repeated until we have $|\theta^{(k+1)}(x) - \theta^{(k)}(x)| \leq \varepsilon$ for arbitrary x , and where ε is the required absolute error on the solution. Similarly to the numerical method used in chapter four, we carry out an independent check on the convergence of the solution by calculating and comparing the total soliton energy for the last two iterations. Although eq. (5.20) is non-local and still strongly non-linear, we find that convergence is quickly reached (within 5 or 6 iterations*) and requires in average about 20 times less computing time than the integro-differential algorithm of the last chapter. Of course, the same accuracy requirement in both cases is assumed.

Before closing this section, let us mention an interesting feature of the pole approximation. This concerns the possibility of analytically calculating the small- x limit of the exact $\theta(x)$, which obeys eq. (5.17). We recall from appendix A that the Dirac wavefunctions $u(x)$ and $v(x)$ are independent of the value of θ at the origin and, for small x , go like a and bx respectively (a and b being positive constants). Taking the limit $x \rightarrow 0$ of eq. (5.20), we arrive at the following asymptotic form:

$$\lim_{x \rightarrow 0} \tan \theta(x) = \frac{-(2ab + 16\pi\alpha\beta^2 C_1/3)x^3}{(a^2 + 16\pi\alpha\beta^2 C_0)x^2}, \quad (5.21)$$

where

$$C_0 = \int_0^\infty dx' x' e^{-\beta x'} \cos \theta(x') \quad \text{and} \quad C_1 = \int_0^\infty dx' (1 + \beta x') e^{-\beta x'} \sin \theta(x'). \quad (5.22)$$

*using any reasonable initial (fixed) profile and not necessarily lying close to the true solution.

Since C_0 and C_1 are both independent of x , $\tan \theta(x)$ is proportional to x up to a constant *i.e.* $\tan \theta(x) \simeq -Cx$, which implies that

$$\theta(x) \simeq n\pi - Cx \quad \text{for } x \rightarrow 0. \quad (5.23)$$

We see therefore that the solution which is of interest to us, *i.e.* the unit baryon number system ($n = 1$), effectively extremises the static soliton energy.

5.4 Results and discussion

We have implemented the self-consistent numerical procedure described in section 5.3 with the pole two-point function fitted to the covariant regularisations used in chapter four, for several values of the constituent quark mass m in the range 300 MeV to 500 MeV. In this section, we present results in the case of the pole form adjusted to model the low-momentum region of the PT two-point function (see Tab. 5.1 and Fig. 5.1 in section 5.1). Fig 5.2 shows the self-consistent solution of equation (5.17) in dotted line for $m = 345$ MeV, along with the solutions with a PT, MC and PV2 cut-off obtained in the preceding chapter. Like the graphs of the two-point functions in Fig. 5.1, the profile θ obtained with the pole approximation lies within the ‘corridor’ delimited by the MC and the PV2 profiles. In fact, it is quite indistinguishable from the PT curve, as can also be deduced from the graphs of the meson fields and the valence quark wave-functions, in Figs. 5.3 and 5.4 respectively. The maximum deviation is found to be no larger than 1% in all cases.

In Table 5.2 the soliton energy and its different contributions are given, calculated with a PT cut-off (chapter four) in the first row for each value of m and with a pole form in the second row. The rightmost column compares the values for the nucleon axial coupling constant g_A obtained as will be explained in chapter six.

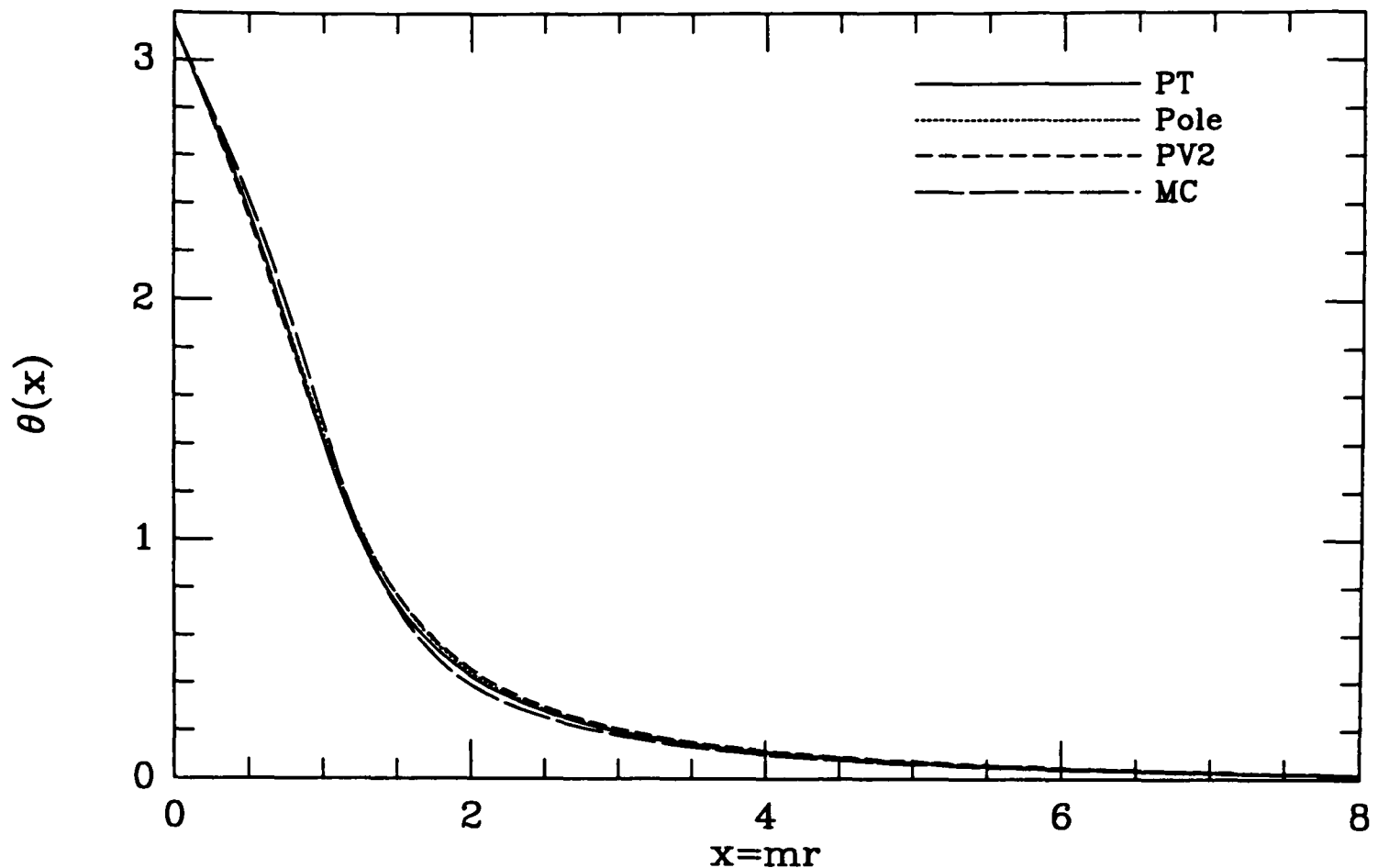


Figure 5.2: The self-consistent profile $\theta(x)$ in the pole approximation for $m = 345$ MeV, compared with the profiles determined via the full two-point approximation for the MC, PT and PV2 cut-offs.

Table 5.2: Different contributions to the self-consistent soliton energy as well as g_A , calculated with a PT cut-off and its pole approximation for 2 values of the constituent quark mass.

m [MeV]	cut-off	E_{val} [MeV]	$E_{vac}^{[2]}$ [MeV]	E_{tot} [MeV]	g_A
345	PT	459.19	622.69	1081.88	1.065
	Pole	440.15	637.35	1077.50	1.089
372	PT	381.16	661.02	1042.18	1.066
	Pole	357.96	683.85	1041.81	1.110

The figures (in Tab. 5.2) for the total energy show a discrepancy of up to 4% for $m = 345$ MeV between the PT and the pole calculations, and less than 1% for $m = 372$ MeV. The discrepancy in the values of g_A is less than 4% for both cases. The deviation in the cases of E_{val} and E_{vac} is somewhat higher. This should however not be a matter of concern since, supposedly, neither quantity

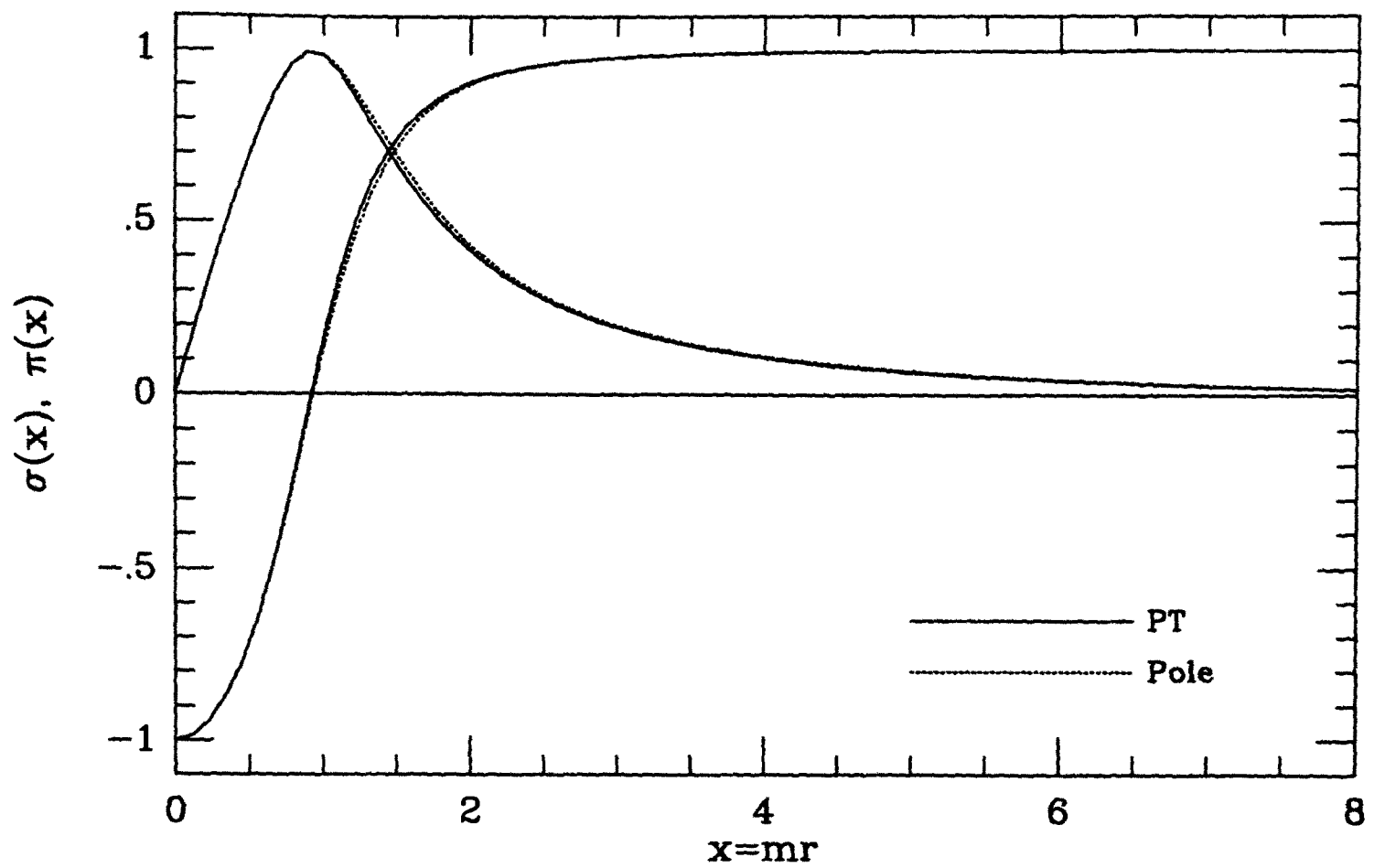


Figure 5.3: The self-consistent meson fields $\sigma(x)$ and $\pi(x)$ for the PT cut-off and the pole approximation, with $m = 345$ MeV.

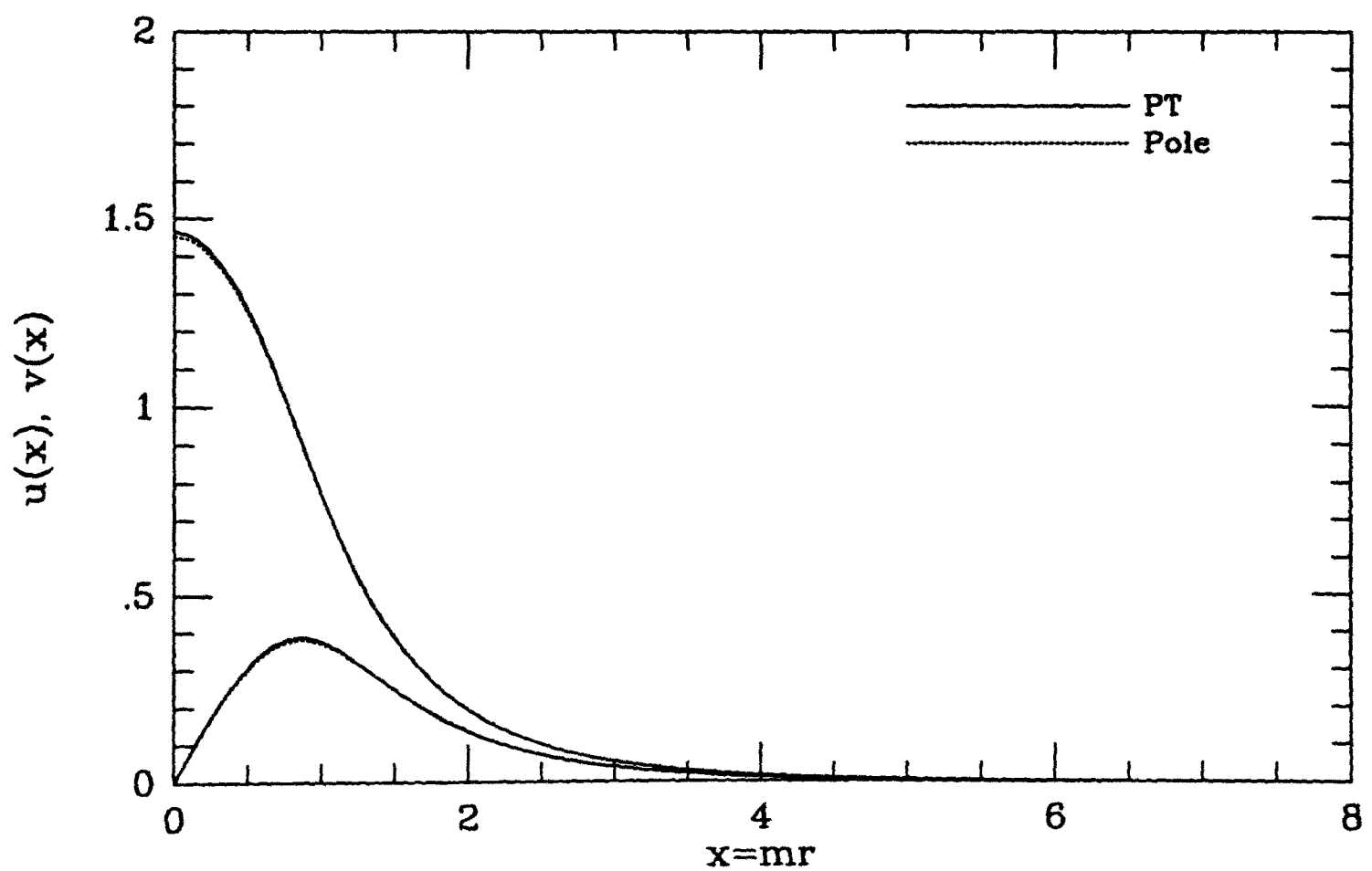


Figure 5.4: The normalised self-consistent valence wave-functions $u(x)$ and $v(x)$ for the PT cut-off and the pole approximation, with $m = 345$ MeV.

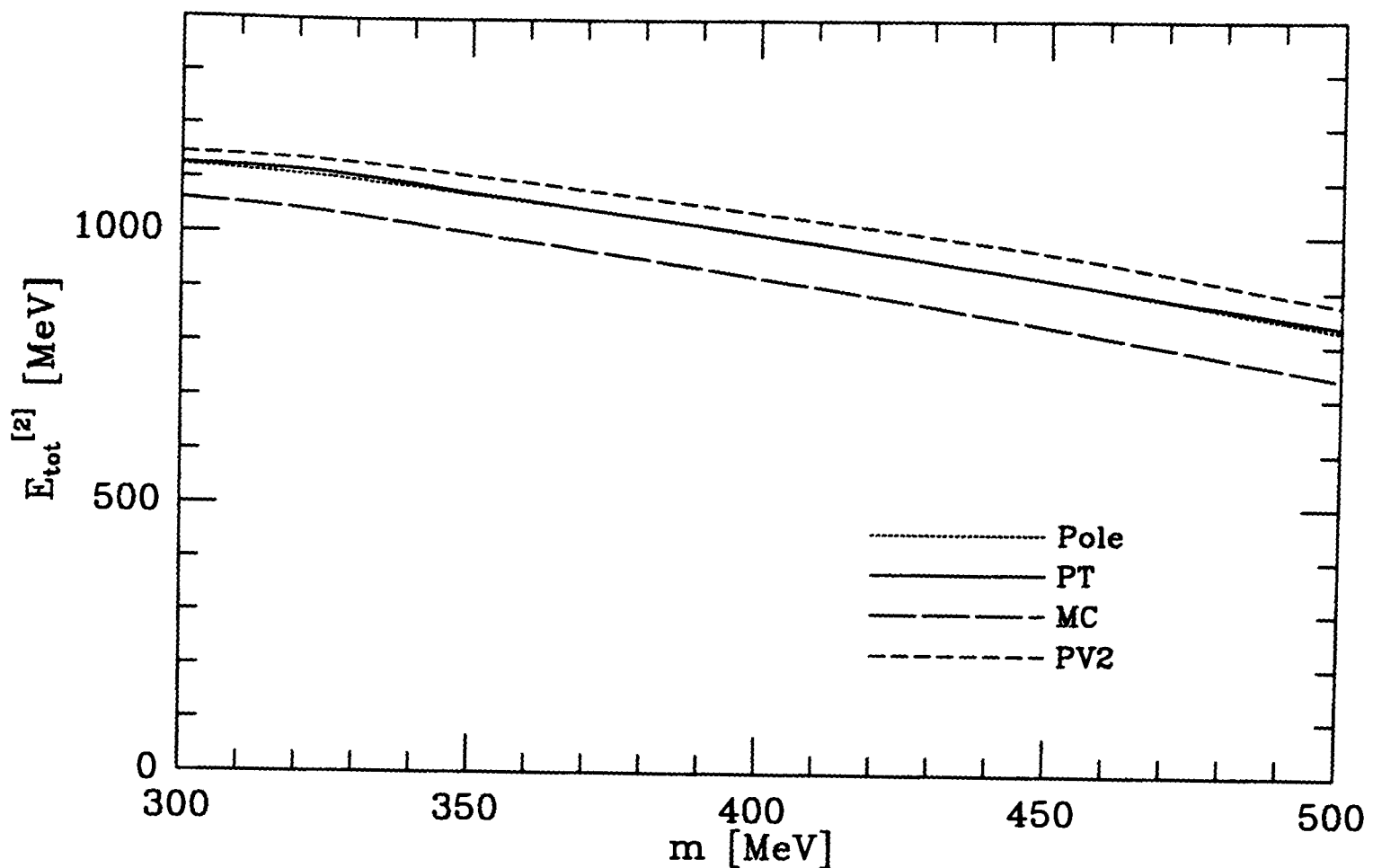


Figure 5.5: Variation of the soliton energy $E_{tot}^{[2]}$ with the constituent quark mass m , for the three cut-offs and the pole approximation, as indicated in the key.

is separately observable. Results for other values of the constituent quark mass m confirm these observations. We have especially plotted in Fig. 5.5 the soliton energy $E_{tot}^{[2]}$ for the three cut-offs and the pole approximation, as a function of m . It is clear from the graphs that the curve for the pole approximation follows closely that for the PT cut-off and never strays away from it by more than a few percent. We therefore conclude that our prescription of fitting the pole two-point function to the $q \leq 3m$ region of the PT cut-off, has led to a genuine simplification without effectively introducing any further approximation above the basic one: the two-point approximation. Results for the other two covariant cut-offs (MC and PV2) confirm these observations.

5.5 Conclusion

In this chapter, we considered an approximation (known as the pole approximation) to the two-point function introduced in chapter three. Implementing this approximation for the calculation of the two-point Casimir energy, we showed how it leads to a very significant simplification when deriving the self-consistent equation for the profile function $\theta(r)$. The numerical algorithm for solving this self-consistent equation requires a simple iterative procedure with fast convergence, unlike the corresponding equation of chapter four. We furthermore showed that the simple prescription of fitting the pole approximation to the low-momentum region of the proper-time cut-off function, yields results essentially indistinguishable from the latter. We may particularly conclude that the results obtained in this chapter confirm those obtained in the previous one, since the pole approximation provides an entirely independent check on the numerical procedure of chapter four. Results for different cut-off schemes and several values of the constituent quark mass allow us to conclude that the pole approximation can be legitimately employed, being effectively no further approximation to the basic two-point approximation in this context. Recalling from the previous chapter that the latter approximation seems to be within 10–20% of the exact results, we may hope that a judicious combination of the two-point approximation with the pole approximation may provide an efficient and amenable approach to similar problems involving highly non-local effective actions.

Chapter 6

Some Static Nucleon Properties

The static soliton we have so far considered in this thesis is a classical object with no definite quantum numbers such as angular momentum, spin or isospin. In fact, the static soliton energy we have calculated in chapters four and five is not the mass of a nucleon but rather that of a mixture of states with different spin and isospin. The identification of the chiral soliton with the nucleon will only be possible after quantising the classical soliton and projecting onto good quantum states. In this chapter, we explain very briefly what is meant by semi-classical quantisation of the soliton and how states with the correct quantum numbers of the nucleon can be obtained. We go on to define some of the nucleon observables calculated by Diakonov *et al.* [DPP88], Meissner *et al.* [MG91] and Wakamatsu and Yoshiki [WY91]. These authors show that some observables are valence-quark dominated, and that the valence quark contribution alone can often be a good approximation. This is particularly the case for the baryon density and related properties. These quantities are in fact of zeroth order (in Ω , see below) in the semi-classical expansion and hence, involve merely the 0^+ valence orbital without interference from other orbitals. A two-point approximation to the vacuum contribution for two nucleon properties is known: the moment of

inertia [Zuk] and the isovector magnetic moment [DPP88]. As we shall see, the two quantities turn out to be related to each other. We shall be interested in calculating these and other quantities using our self-consistent solutions of chapter four and comparing them (wherever possible) with the works mentioned above. We finally conclude with some remarks about the validity of the two-point approximation as far as nucleon observables (other than the static soliton energy) are concerned.

6.1 Semi-classical quantisation

In order to discuss the quantum mechanical states representing baryons, we must first somehow quantise the classical soliton. There is, however, no general procedure for quantising the kind of extended object that a soliton is*, which is after all non-perturbative in nature. The alternative is to consider the semi-classical expansion up to the next order in $\hbar$, representing the leading quantum corrections. Several methods have been developed for this purpose, such as the stationary phase approximation for the quantisation of static solitons, or the WKB method for more general time-dependent systems [Raj84]. In the course of following this scheme however, one has to deal with the zero modes of the theory to be quantised. These are associated with the continuous symmetries that are present in the theory and lead to nasty divergences when quantum corrections are calculated. In the case of the static soliton of the ECA, a displacement of the centre of the soliton or a rotation round one axis of symmetry does not change the total energy. Consequently, there are three translational and three rotational zero modes, corresponding to the three directions in space. The method of collective coordinate quantisation in the semi-classical expansion allows one to deal effec-

*Creation and annihilation operators for solitons have been obtained in some low-dimensional cases, such as the (1+1) dimensional Sine-Gordon system.

tively with the zero modes. It consists essentially in replacing the zero modes with the same number of ‘collective’ coordinates, which are then integrated out. For example, if this method is applied to the translation modes, non-zero modes may be calculated and identified with small boson (quantum) fluctuations of the soliton allowing, in particular, the study of meson-nucleon scattering. Likewise, treatment of the rotation modes will lead to quantum states with definite quantum numbers such as angular momentum or isospin, thereby providing the first step towards identifying the chiral soliton with the nucleon. Many of these techniques are performed in the path integral formalism and are quite involved in many cases of physical interest.

6.2 The moment of inertia

An important physical quantity to know when quantising the rotation modes of the soliton is the moment of inertia. One useful approximation scheme in this case is the so-called ‘cranking’ technique, which assumes that the system is rigidly rotating with uniform angular velocity. The rotation speed is considered low enough so that the system remains undistorted. This relatively simple technique is in fact borrowed from Nuclear Physics where similar situations arise. It was first applied to the Skyrme model by Adkins *et al.* [ANW83] and later to chiral soliton models in both their linear and non-linear versions by, amongst others, McGovern and Birse [MB88] (for the SU(3) flavour group in the zero-loop approximation), Diakonov *et al.* (DPP) [DPP88], Reinhardt [Rei89], Meissner and Goeke (MG) [MG91], Goeke *et al.* [GGG⁺91], Wakamatsu [Wak90] and Wakamatsu and Yoshiki (WY) [WY91]. The simplest way to implement the cranking procedure is to consider a ‘rotating hedgehog’ field [ANW83, DPP88]:

$$\tilde{U}(r, t) = A(t) U(r) A^\dagger(t) \quad (6.1)$$

where U is a unitary matrix characterising the static meson field and $\tilde{A}$ is an $SU(2)$ time-dependent matrix representing a global rotation in space or isospace. As was already noted in section 4.1, the two kinds of rotation are equivalent by virtue of the symmetries of the hedgehog ansatz. Then, DPP [DPP88] introduce the collective angular velocity matrix (in isospin space) $\Omega = \frac{1}{2}\Omega_a\tau_a = -iA^+\partial_t A$ and find the corresponding effective action $\Gamma[\tilde{U}(r, t)]$ for the rotating hedgehog. The assumption of slow rotation allows one to expand the effective action in powers of Ω and, from symmetry requirements, the lowest order contribution is found to be quadratic in Ω . After considering the effect of both the vacuum and the valence quarks, DPP show that the new effective action is:

$$\Gamma[\tilde{U}(r, t)] = \int dt \left(-E_{tot}[U(r)] + \frac{1}{2}\mathcal{I}_{tot}\Omega_a^2(t) \right) \quad (6.2)$$

where $\mathcal{I}_{tot}$ is interpreted as the moment of inertia which receives contributions from both the valence and the sea quarks. Integrating over all rotation matrices in the path integral $\int \mathcal{D}A(t) \exp[\frac{i}{2}\mathcal{I}_{tot} \int dt \Omega_a^2(t)]$ gives the rotational Hamiltonian which, after being diagonalised, yields the soliton energy with an additional rotational kinetic energy piece

$$E_{tot,J} = E_{tot} + \frac{J(J+1)}{2\mathcal{I}_{tot}} \quad (6.3)$$

where J is the total angular momentum number. Recalling that for the hedgehog configuration we have $J = T$ (section 4.1), we see that the energy spectrum implied by (6.3) describes a series of resonances with $(J, T) = (\frac{1}{2}, \frac{1}{2}), (\frac{3}{2}, \frac{3}{2}), (\frac{5}{2}, \frac{5}{2}), \dots$ where the two lowest lying states are the nucleon N (spin 1/2) and the Δ (spin 3/2). As higher spin states have not been experimentally observed, the cranking method can only be taken seriously for the nucleon and Δ . Eq. (6.3) may then be used to predict the N - Δ mass difference:

$$E_{tot,3/2} - E_{tot,1/2} = \frac{3}{2\mathcal{I}}. \quad (6.4)$$

We recall that, using the Kahana-Ripka method, the vacuum contribution to the static soliton energy is calculated by performing an eigenvalue summation over all negative energy orbitals of the single particle Hamiltonian. The case of the vacuum contribution to the moment of inertia is considerably more complicated; the expression DPP find for $\mathcal{I}_{vac}$ (unregularised) involves a double summation, reading

$$\mathcal{I}_{vac} = \frac{N_c}{2} \sum_{m,n} \frac{\langle n | \tau_3 | m \rangle \langle m | \tau_3 | n \rangle}{E_m - E_n}, \quad (6.5)$$

where m and n represent respectively all positive and negative energy orbitals. A question of interest to us here is whether a two-point approximation to $\mathcal{I}_{vac}$ can be found which, similarly to the case of the vacuum energy, would considerably simplify the numerical effort needed to calculate (6.5). It turns out that, using the pole approximation defined in chapter five, such a ‘two-point vacuum’ moment of inertia can be calculated, yielding the simple expression [Zuk]:

$$\mathcal{I}_{vac}^{[2]} = \frac{8\pi f_\pi^2}{3} \int_0^\infty dr \int_0^\infty dr' K(r, r') \sin \theta(r/R) \sin \theta(r'/R) \quad (6.6)$$

where

$$\begin{aligned} K(r, r') = & \frac{1}{4\beta m} e^{-\beta m(r+r')} [(1 + \beta m(r+r') + \beta^2 m r r')(1 + \beta m(r+r')) + \beta m(r+r') + 2] \\ & - \frac{1}{4\beta m} e^{-\beta m|r-r'|} [(1 + \beta m|r-r'| - \beta^2 m r r')(1 + \beta m|r-r'|) + \beta m|r-r'| + 2]. \end{aligned} \quad (6.7)$$

$\theta(r/R)$ is the soliton profile and R is the soliton size parameter. R is viewed as a scaling parameter in case θ represents the (numerical) self-consistent profile (see section 4.4). One interesting feature of (6.6) is that its large soliton size limit can easily be computed, giving

$$\lim_{R \rightarrow \infty} \mathcal{I}_{vac}^{[2]} = \frac{8\pi f_\pi^2}{3} \int_0^\infty dr r^2 \sin^2 \theta(r/R) \quad (6.8)$$

which goes like the cubic power of R as can be seen by changing to the dimensionless variable $x = r/R$ in (6.8). DPP in [DPP88] argue that the moment of inertia

is sensitive to the long range behaviour of θ and therefore a derivative expansion of $\mathcal{I}_{vac}$ may be justified. Their result for the leading term agrees with (6.8), indicating that the two-point approximation $\mathcal{I}_{vac}^{[2]}$ is exact for large soliton sizes. In section 6.4, we shall discuss the validity of the approximation in an intermediate range (physically relevant to nucleons) in the light of the numerical results obtained. In the next section, we briefly define the rest of the nucleon observables which we shall be interested in calculating.

6.3 Other nucleon observables

Within models of the nucleon involving explicit quark degrees of freedom, nucleon observables are obtained by considering expectation values of the corresponding quark bilinear operators, between baryon states of definite spin and isospin

$$\langle J_N \bar{\psi} O^\mu \psi J_N^+ \rangle = \int \mathcal{D}\sigma \mathcal{D}\pi \mathcal{D}\psi \mathcal{D}\bar{\psi} J_N \bar{\psi} O^\mu \psi J_N^+ e^{i\Gamma[\tilde{U}]}. \quad (6.9)$$

As discussed for the moment of inertia, the classical hedgehog is considered rotating with uniform angular velocity (characterised by Ω). Then, an evaluation of (6.9) can be attempted in the ‘cranking’ approximation; one expands (6.9) in powers of Ω keeping the lowest order non-vanishing terms. As WY pointed out [WY91], observables can be classified in two categories. The first one corresponds to quantities for which the dominant contribution is of zeroth order in the Ω expansion. The second comprises those quantities for which the zeroth order term vanishes and one must therefore take into account the first order term, linear in Ω . The latter group includes the spin, isospin and the isoscalar magnetic moment for example and are rather tricky to calculate because of off-diagonal matrix elements (in the formalism of (6.5)) between valence and sea orbitals. However, like the static soliton energy (which is of zeroth order in Ω), the valence

quark contribution to observables in the first category depends solely on the filled valence quark orbital 0^+ and, likewise, the vacuum part on the negative energy orbitals only. We may then calculate these contributions using our self-consistent 0^+ wavefunctions (obtained in the two-point approximation) and compare them with the corresponding values in the above-mentioned works.

Baryon density and related quantities

The baryon density is a first-category quantity and is given by:

$$\rho(r) = \int d\Omega_3 \sum_n \bar{\psi}_n \gamma^0 \psi_n \quad (6.10)$$

where the sum over n extends over all occupied levels of the Dirac spectrum, including the valence orbital. The valence part $\rho_{val}(r)$ in terms of the 0^+ orbital is simply

$$\rho_{val}(r) = F^2(r) + G^2(r) \quad (6.11)$$

while we find that there is no two-point contribution to the sea part $\rho_{vac}^{[2]}(r)$. This comes as no surprise since the leading order term in a derivative expansion of ρ_{vac} is cubic in the potential $V(x)$ (see for example [AF84, KR84]). One would have therefore to consider the ‘four-point’ approximation to obtain a non-zero contribution. Meissner *et al.* [MG91] observe however that for those values of the constituent quark mass where the valence orbital has positive energy, the baryon density is strongly dominated (to 90%) by the valence quarks which account for the baryon number in this region. We may therefore consider it a good approximation to take $\rho(r) \simeq \rho(r)_{val}$ for solutions with a positive top-most orbital. We also calculate the isoscalar quadratic radius

$$\langle R^2 \rangle = \int_0^\infty dr r^4 \rho(r) \quad (6.12)$$

and the isoscalar electric form factor

$$G_E^{T=0}(q^2) = \int d^3r e^{i\vec{q}\cdot\vec{r}} \rho(r) = \int_0^\infty dr r^2 j_0(qr) \rho(r), \quad (6.13)$$

in the valence quark approximation.

The isovector magnetic moment and g_A

We present here two more zeroth order quantities in the cranking approximation.

The isovector magnetic moment is defined through the relation

$$\mu_V = G_M^{T=1}(q^2 = 0) \quad (6.14)$$

where $G_M^{T=1}$ is the isovector magnetic form factor. The valence part is given by [CB86, WY91]:

$$\mu_V^{val} = \frac{N_c E_{tot}}{3} \int d^3r \bar{\psi}_{val}(r) \varepsilon_{ijk} \gamma^0 \gamma^i r_j \tau^k \psi_{val}(r) = -\frac{2 N_c E_{tot}}{9} \int_0^\infty dr r^3 F(r) G(r), \quad (6.15)$$

in units of nuclear magnetons. Diakonov *et al.* [DPP88] give a two-point approximation to μ_V^{vac} which they claim is exact for both small and large solitons. In fact, when we replace the two-point function in their expression by the pole approximation (5.1) in chapter five, we find that it can be written as a function of the two-point moment of inertia (6.6):

$$\mu_V^{vac,[2]} = \frac{2}{3} E_{tot}^{[2]} \mathcal{I}_{vac}^{[2]}. \quad (6.16)$$

Note that if $\mu_V^{vac,[2]}$ is expressed in MeV^{-1} (1 nucl. magneton $\equiv 1/2 M_N$, where M_N is approximated by $E_{tot}^{[2]}$), it is found to be exactly one third of the two-point moment of inertia $\mathcal{I}_{vac}^{[2]}$. This observation was also made by Cohen and Bronjowski [CB86] in the Birse-Banerjee model regarding the meson contribution.

Finally, the axial-vector coupling constant g_A due to the valence quarks is given by [CB86]

$$g_A^{val} = \frac{N_c}{3} \langle \psi_{val} | \gamma^3 \gamma^5 \frac{\tau_3}{2} | \psi_{val} \rangle = \frac{N_c}{3} \int_0^\infty dr r^2 [F^2(r) - \frac{1}{3} G^2(r)]. \quad (6.17)$$

The case of g_A is a rather peculiar one; the (full) axial vector coupling constant can in fact be extracted from the long-range behaviour of the pion field. In the Skyrme model, Adkins *et al.* [ANW83] obtain (using the Goldberger–Treiman relation)

$$g_A = \frac{8}{3}\pi f_\pi^2 A. \quad (6.18)$$

The constant A is taken from the large- r limit of the pion field $\sim A/r^2$ and is related to the numerical parameter c in (4.35) by $A = \frac{m^2 c}{f_\pi}$. This result can still be (and has been) used in the CSM since the presence of the valence quark orbital, being a bound state, has an exponential fall-off and does not perturb the large- r behaviour of the pion field.

6.4 Results and discussion

For all calculated quantities introduced in this section, we have used the (PT) self-consistent solutions obtained in chapter four. Let us first consider the baryon density $\rho(r)$ in the valence-quark approximation. Fig. 6.1 shows $\rho_{val}(r)$ for three values of the constituent quark mass. The densities are comparable in the three cases, but the valence quarks seem to be more concentrated towards the origin for increasing m . The same trend can be observed in the isoscalar mean square radius $\langle R^2 \rangle$ shown in Fig. 6.2. A gradual decrease is observed with increasing m in $\langle R^2 \rangle$ (full line). The dashed and dotted lines show $\langle R^2 \rangle$ obtained by WY and MG respectively, as calculated with the full Dirac sea. $\langle R^2 \rangle$ are seen to fall noticeably faster with m in both cases. There is however some 15% discrepancy between WY and MG and our curve lies between the two ‘exact’ results in the range $m = 363 - 450$ MeV. Since the experimental value of $\langle R^2 \rangle$ is 0.62 fm, we deduce from the graphs that a quark mass between 363 and 400 MeV is favoured. We therefore conclude that our $\langle R^2 \rangle$ in the valence-quark approximation is in

good agreement with the exact results for the physically relevant values of m .

Table 6.1: Comparison for some nucleon properties with Ref.[WY91] and experimental data wherever available, for $m = 400$ MeV (Part 1).

Quantity	E_{tot} [MeV]	$\mathcal{I}_{vac}$ [MeV $^{-1}$]	$\langle R^2 \rangle$ [fm 2]
Two-point	1001	0.00337	0.578
Ref.[WY91]	1200	0.00192	0.617
Experiment	≤ 1000 (?)	–	0.620

Table 6.2: Comparison for some nucleon properties with Ref.[WY91] and experimental data wherever available, for $m = 400$ MeV (Part 2).

Quantity	g_A^{val}	g_A	μ_v^{val} (nucl. magn.)	μ_v^{vac}	μ_v^{tot}
Two-point	0.689	1.114	0.995	2.246	3.239
Ref.[WY91]	0.734	0.820	1.640	1.010	2.650
Experiment	–	1.230	–	–	4.710

Tables 6.1 and 6.2 summarise the results we obtained for various nucleon observables for $m = 400$ MeV in the first row. We have included the static soliton energy for completeness. Comparison with the corresponding values obtained by using the full sea by WY (second row) shows reasonable agreement for most quantities. In particular, besides $\langle R^2 \rangle$ already discussed above, the values of g_A^{val} agree within 5%. The biggest discrepancy is found in the vacuum contributions to $\mathcal{I}$ and μ_v . It amounts roughly to 50% in both cases. This is not a coincidence since we saw that the two quantities are related to each other (see eq. (6.16)). The two-point approximation does not seem particularly good in these two cases.

Curiously, with the exception of $\langle R^2 \rangle$, we observe that all results obtained in the two-point approximation seem to be closer to the experimental values (third row), as was already noticed in chapter four concerning the static soliton energy.

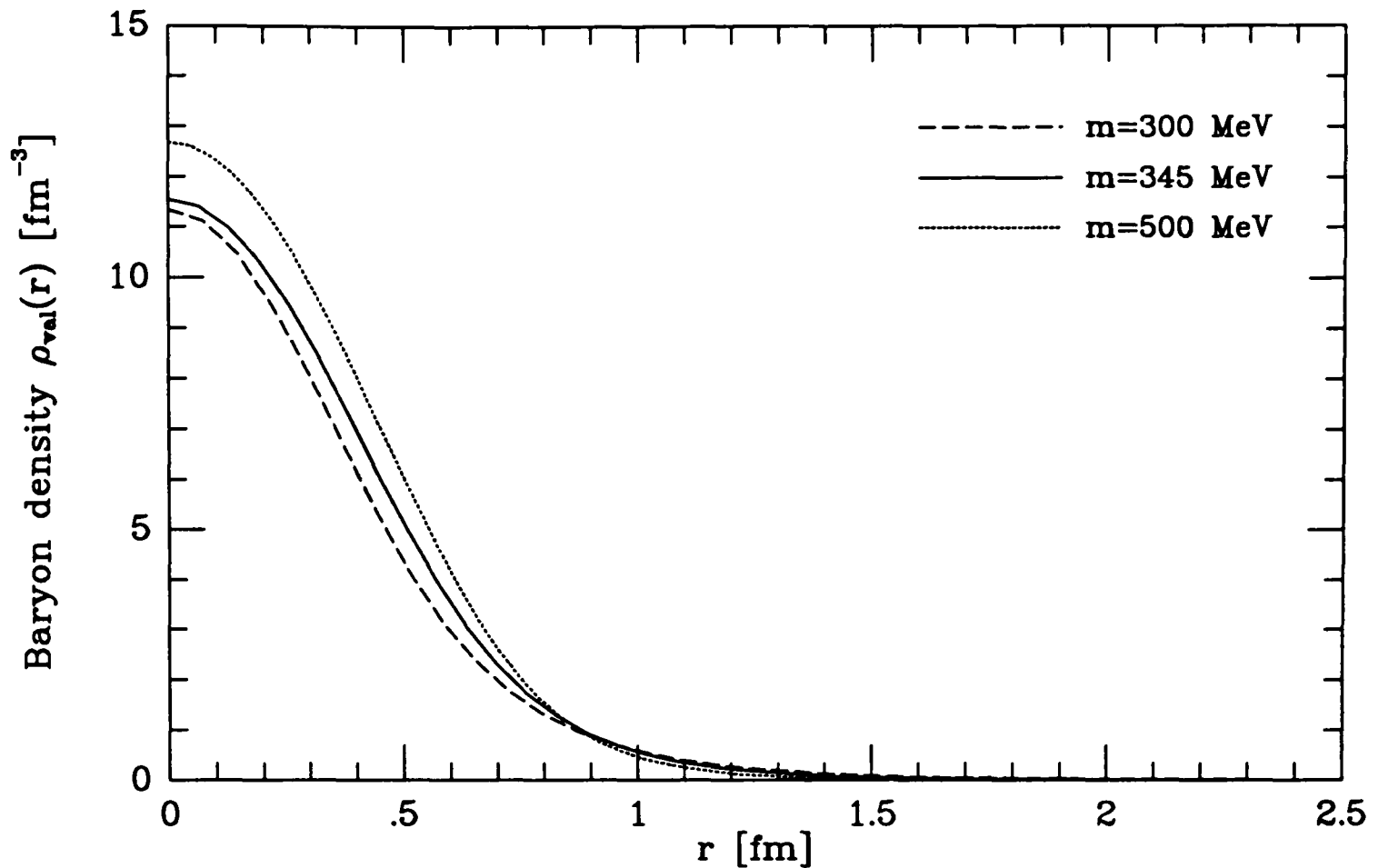


Figure 6.1: Normalised baryon density in the valence-quark approximation for three constituent quark masses.

Fig. 6.3 shows the two-point approximation to $\mathcal{I}_{vac}$ (full line), the leading term in the derivative expansion (dotted line) and the exact $\mathcal{I}_{vac}$ from Ref. [WY91] in dashed line. We see that, although $\mathcal{I}_{vac}^{[2]}$ presents an improvement over the derivative expansion, there is still a discrepancy of 30 to 50% with the exact result of WY. Moreover, the approximation seems to worsen slightly with increasing m due to the opposite sign in the slopes of the two curves (up to 450 MeV).

Comparison of the isoscalar electric form factor with the exact result of Ref. [MG91] shows good agreement in Fig. 6.4. The two-point moment of inertia $\mathcal{I}_{vac}^{[2]}$ in Tab. 6.1 and Fig. 6.3 was calculated using the ‘two-point’ self-consistent profile of chapter four. In fact everything we have been calculating so far in this thesis has been within the ‘two-point’ approximation philosophy. It is interesting to ask whether using the ‘exact’ self-consistent profile would lead to different results. In particular a direct comparison of $\mathcal{I}_{vac}^{[2]}$ with $\mathcal{I}_{vac}$ would be possible,

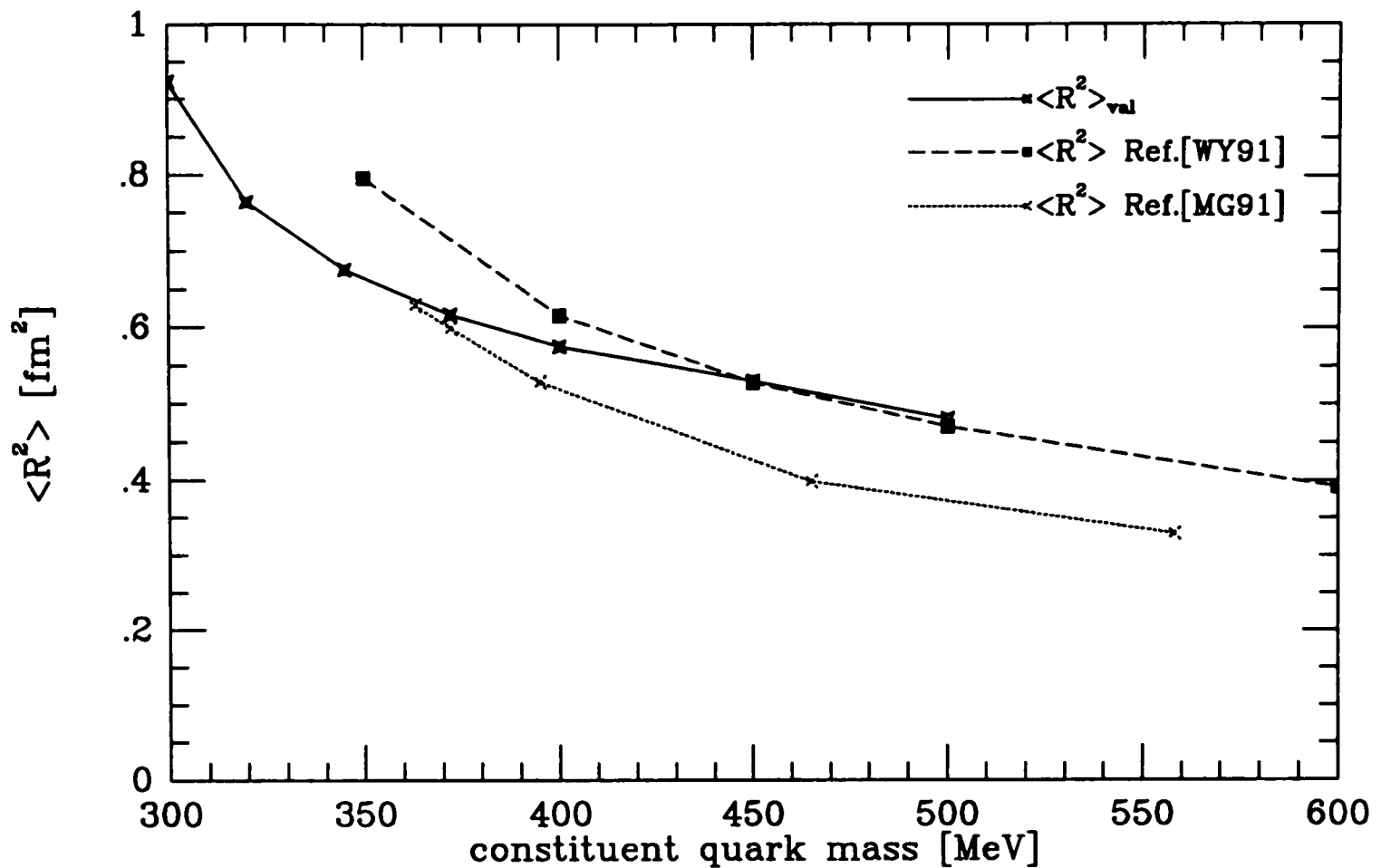


Figure 6.2: Isoscalar mean square radius versus constituent quark mass. Comparison is made with two independent references.

since both quantities would be calculated with the same soliton profile. To this end, we used the ‘exact’ self-consistent profile obtained by Alkofer [Alk90] for the two values of m : 300 and 500 MeV, to compute $\mathcal{I}_{vac}^{[2]}$ and its leading term in the derivative expansion $\mathcal{I}_{vac,0}^{[2]}$. We then compared (see Tab. 6.3) our results with WY after having checked that both authors agree on the self-consistent calculation.

When the ‘exact’ θ is used, we see consistency (to within 5%) between the ‘exact’ and the ‘two-point’ $\mathcal{I}_{vac}$. Furthermore, we note that the derivative expansion approximates rather well the exact result, which is also what can be inferred from Fig. 1 of Ref. [GGG⁺91], although the authors say otherwise in the text. We may then conclude that the two-point expression for the moment of inertia is a good approximation *if* the ‘exact’ self-consistent soliton profile is used.

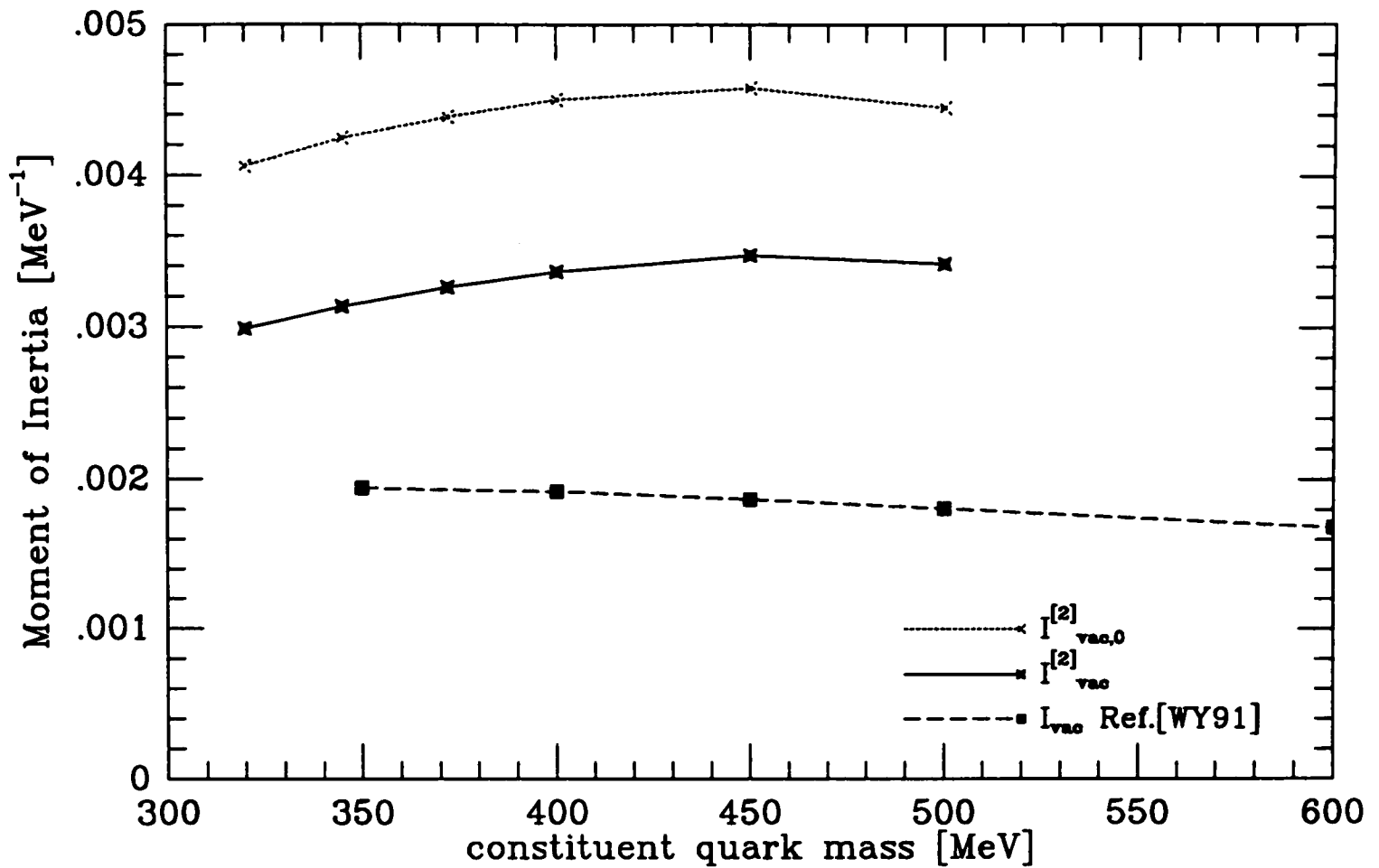


Figure 6.3: Comparison of the two-point vacuum contribution to the moment of inertia and the leading term in the derivative expansion, with the exact result of Ref. [WY91].

Table 6.3: $\mathcal{I}_{vac}$ [MeV⁻¹] in the derivative expansion and in the two-point approximation using the ‘two-point’ θ and the ‘exact’ θ (from Ref.[Alk90]), and compared with Ref.[WY91].

	‘two-point’ θ		‘exact’ θ		
m [MeV]	$\mathcal{I}_{vac,0}^{[2]}$	$\mathcal{I}_{vac}^{[2]}$	$\mathcal{I}_{vac,0}^{[2]}$	$\mathcal{I}_{vac}^{[2]}$	Ref.[WY91]
350	0.00425	0.0327	0.00227	0.00201	0.00195
500	0.00445	0.00314	0.00264	0.00198	0.00181

6.5 Conclusion

In this chapter, we used the self-consistent solutions obtained in chapter four to calculate some static nucleon observables. In particular, there is good agreement with the ‘exact’ results in the case of quantities which depend weakly on the Dirac sea (for constituent quark masses between 350 and 450 MeV) and hence, a valence-quark approximation may be justified. This is especially the case of

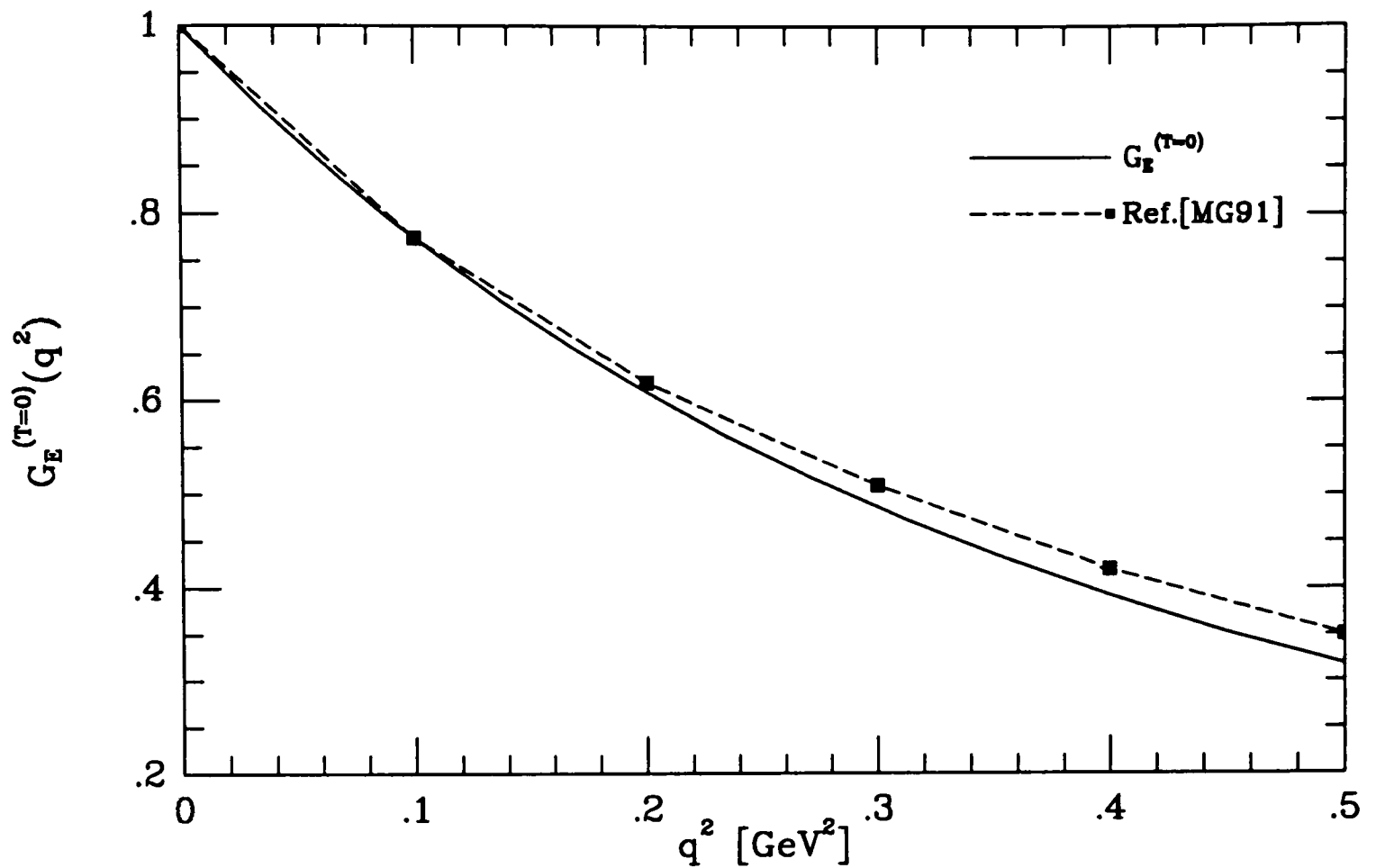


Figure 6.4: Comparison of the isoscalar electric form factor in the valence quark approximation with the exact result of Ref. [MG91].

the baryon density and related properties. A vacuum two-point approximation to the moment of inertia was presented and shown to give a poor estimate of the full vacuum moment of inertia. However, when the ‘exact’ self-consistent soliton profile is used, the same two-point expression for the moment of inertia yields results very near the full sea evaluations. We may therefore hope that similar ‘two-point’ expressions to other nucleon observables may prove to be reliable approximations, once an ‘exact’ self-consistent soliton profile is obtained.

Appendix A

Asymptotic behaviour of θ , u and v

In this appendix, we investigate the small and large distance behaviour of the soliton profile $\theta(x)$ and the Dirac components $u(x)$ and $v(x)$, which are solutions of the simultaneous equations (4.33) and (4.34).

Small- x behaviour

Let us suppose that $\theta(x)$ goes to ϑ for $x \rightarrow 0$, where ϑ is *a priori* an arbitrary function of x , regular at the origin. We require from the wavefunctions u and v to be also regular at $x = 0$, and expand them in a Taylor series round the origin:

$$u(x) \simeq a_0 + a_1 x + 0(x^2) \quad \text{and} \quad v(x) = a'_0 + a'_1 x + 0(x^2). \quad (A.1)$$

We then insert this form of u and v in (4.34) and, identifying terms of same degree in x , we obtain the following relations

$$\begin{aligned} a_0 & \text{ arbitrary constant} > 0, & a'_0 &= 0, \\ a_1 &= 0, & a'_1 &= a_0 \frac{\lambda - \cos \vartheta}{3}. \end{aligned} \quad (A.2)$$

where λ is the Dirac eigenvalue. We therefore see that for sufficiently small x , $u(x)$ is a constant function of x and $v(x)$ a linear one. As λ is a constant, these asymptotic limits are independent of any (regular) small- x behaviour of ϑ .

Next, for a winding number one configuration, the soliton profile θ will go like $\pi + f(x)$ for small x where we demand $f(0) = 0$. We then expand $f(x)$ in a Taylor series round the origin, substitute for $\theta(x)$ in eq. (4.33) and use the limits (A.2) for u and v to find (retaining the leading contribution to f):

$$\theta(x) \simeq \pi - bx \quad (\text{A.3})$$

where b is an undetermined positive constant.

Large- x behaviour

A finite energy configuration requires that $\lim_{x \rightarrow \infty} \theta(x) = n\pi$ and the radial Dirac equation reduces in this limit to the system

$$\begin{cases} u' = -(1 + \lambda)v \\ v' = -(1 - \lambda)u \end{cases} \quad (\text{A.4})$$

which is easily solved to yield (up to a multiplicative constant):

$$u(x) \simeq \sqrt{1 - \lambda^2} \frac{1}{x} e^{\sqrt{1 - \lambda^2} x}, \quad v(x) \simeq \frac{1}{x} e^{\sqrt{1 - \lambda^2} x}. \quad (\text{A.5})$$

The large- x behaviour of θ is then readily obtained by assuming that $\theta(x) \sim x^{-k}$ ($k > 0$) and making use of (A.5) in the θ equation (4.33). It turns out that

$$\theta(x) \simeq c x^{-2}, \quad (\text{A.6})$$

which is appropriate to a source of massless quanta (c being an undetermined positive constant).

Appendix B

Alternative derivation of $G_\ell(x, x')$

In this appendix we present an alternative approach [AAZ91b], which is of more general interest, to the evaluation of the functions $G_\ell(x, x')$ in section 5.1. Let us go back to the original expression of the two-point effective action (3.6) in chapter three. This time, we interpret the functional trace as $\int d^4x$, giving

$$\Gamma^{[2]} = \frac{i N_c}{4} \text{tr} \int d^4x d^4x' V(x) \langle x | \frac{1}{P^2 - m^2} | x' \rangle V(x') \langle x' | \frac{1}{P^2 - m^2} | x \rangle. \quad (B.1)$$

Defining the function $K(x, x')$ by

$$\begin{aligned} K(x, x') &= -i \int \frac{d^4q}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{e^{-iq(x-x')}}{(p^2 - m^2)((p - q)^2 - m^2)} \\ &= -i \int \frac{d^4q}{(2\pi)^4} F_M(q^2/m^2) e^{-iq(x-x')}, \end{aligned} \quad (B.2)$$

where F_M is the two-point function in Minkowski space, we then rewrite eq. (B.1) in the following form:

$$\Gamma^{[2]} = -\frac{N_c}{4} \text{tr} \int d^4x d^4x' V(x) K(x, x') V(x'). \quad (B.3)$$

Now, rotating to Euclidean space and inserting the pole form (5.1) for the Euclidean two-point function F we obtain

$$\Gamma^{[2]} = -\frac{N_c \alpha}{4} \text{tr} \int d^4x V(x) \frac{1}{P^2/m^2 + \beta^2} V(x) \quad (B.4)$$

and using the form (3.2) for the coordinate-space potential $V(x)$, we arrive at the following expression (having performed both Dirac and $SU(2)$ traces):

$$\Gamma^{[2]} = 2g^2 N_c \alpha \text{tr} \int d^4x \partial_\mu \phi_a(x) \frac{1}{P^2/m^2 + \beta^2} \partial_\mu \phi_a(x) \quad (B.5)$$

where the subscript a refers to the dot product in ϕ -space. The lowest order term in a derivative expansion of (B.5) (neglecting P^2) is the expected meson kinetic energy term. It is correctly normalised when $2g^2 N_c \alpha/\beta^2 = 1/2$ in agreement with (5.2). For time-independent field configurations we obtain the corresponding static energy $E^{[2]}$:

$$E^{[2]} = 2g^2 N_c \alpha \int d^3r \phi_a(\vec{r}) \left[\frac{-\vec{\partial}^2}{-\vec{\partial}^2/m^2 + \beta^2} \right] \phi_a(\vec{r}). \quad (B.6)$$

If we let the meson field assume the hedgehog ansatz (4.3), the system acquires spherical symmetry and it is more convenient to use the spherical form of the Laplacian:

$$\vec{\partial}^2 = \partial_r^2 - \frac{1}{r^2} \vec{L}^2 \quad (B.7)$$

where $\partial_r^2 f(r) = \frac{1}{r} \frac{d^2}{dr^2} (r f(r))$ and $\vec{L}^2 \hat{r} = 2$, leading to

$$\begin{aligned} E^{[2]} = & 32\pi\alpha N_c m \int_0^\infty dx x^2 \sin^2 \frac{\theta(x)}{2} \\ & - 8\pi\alpha\beta^2 N_c \int_0^\infty dx x (\cos \theta(x) - 1) \frac{1}{-\partial_x^2 + \beta^2} x (\cos \theta(x) - 1) \\ & - 8\pi\alpha\beta^2 N_c \int_0^\infty dx x \sin \theta(x) \frac{1}{-\partial_x^2 + \frac{2}{x^2} + \beta^2} x \sin \theta(x) \end{aligned} \quad (B.8)$$

where x is the dimensionless variable $x = mr$.

The corresponding coordinate-space Green functions are obtained by solving eq. (5.15), subject to the boundary conditions (5.16). The full solutions $G_\ell(x, x')$, $\ell = 0, 1$, are found to be those we obtained in (5.13) and (5.14) in chapter five.

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