



DEPARTMENT OF ECONOMICS

DISCUSSION PAPER SERIES

**LARGE DEVIATIONS AND EQUILIBRIUM SELECTION IN LARGE
POPULATIONS**

Alan Beggs

Number 129

November 2002

Manor Road Building, Oxford OX1 3UQ

Large Deviations and Equilibrium Selection in Large Populations

A. W. Beggs
Wadham College
Oxford
OX1 3PN
UK

Current Version: July 2002

Abstract

This paper uses the theory of large deviations to analyse equilibrium selection in one-dimensional games with large populations where the system evolves according to a jump Markov process. The equilibria selected maximise a quasi-potential function which can be determined by solving a polynomial equation. Estimates of waiting times are also given. It shows that equilibria about which there is more noise are less likely to be selected and clarifies the role of the limiting deterministic dynamic in selection.

JEL Classification Numbers: C72, C73

Keywords: equilibrium selection, large deviations, large populations, games

I am grateful to seminar participants at Oxford for helpful comments.

1. Introduction

This paper examines the problem of equilibrium selection in games played by large populations. Since the pioneering work of Foster and Young (1990), Kandori et al. (1993) and Young (1993) there has been much interest in the idea that introducing noise in players' actions may help to select between equilibria in games where there are multiple equilibria. Most of this literature has proceeded by considering a fixed finite population of players who play a game repeatedly using some myopic but noisy learning rule. It then examines whether if the noise is small the resulting system spends most of its time near one of the equilibria of the underlying unperturbed game.

This paper also looks at the problem of equilibrium selection but rather than introducing exogenous noise in players' choices and examining what happens as this noise becomes small, it asks what happens if players' choices are persistently noisy but the population size is large, so that in the aggregate noise is small. This paper therefore belongs to the line of research pioneered by Binmore and Samuelson (1997), Blume (1994) and Amir and Berninghaus (1996). These authors have previously considered the problem of learning in large populations, but under quite special assumptions.

This paper presents a more general analysis. It considers the evolution of strategies in a population with noisy choices which evolves according to a one-dimensional jump Markov chain. It shows that the theory of large deviations allows one to determine which equilibrium is selected in a large population simply by solving a polynomial equation, even though the exact equilibrium distribution may be intractable. It also provides estimates of waiting times for the selected equilibrium to be reached. It shows that an increase in the noise about an equilibrium, in the sense of first-order stochastic dominance, makes it less likely to be selected. It also clarifies the role of the form of the limiting deterministic dynamic in equilibrium selection.

Although much of the recent literature has been interested in equilibrium selection in simple coordination games, the relevance of equilibrium selection in large population games is much wider. Models of search or congestion, for example, often have multiple equilibria and are studied when populations are large. The results of this paper can be used to study these more general dynamic games.

In a large population one might expect that noise will average out and that the motion of the system can be well approximated by that of an ordinary differential equation. This is indeed so if one is interested in behaviour over a finite horizon — see for example Binmore et al. (1995) and Sandholm (1999) for proofs in an economic context or more generally Ethier and Kurtz (1986). Nevertheless differential equations are not satisfactory if one is interested in the long-run behaviour of the system, which involves behaviour over an infinite one. As Binmore et al. (1995) point out it is quite possible for the system to have a unique stationary distribution for each finite population size but the differential equation approximation to have multiple stable steady-states. The differential equation approximation is therefore inadequate as a guide to long-run finite population behaviour.

One reaction to this problem is to find models where the finite population stationary distributions can be computed explicitly and then examine their behaviour as the population size becomes large. This strategy is followed by Binmore and Samuelson (1997), Blume (1994) and Amir and Berninghaus (1996). They look at the case of birth-death processes where the number of players playing each strategy can change by at most one at a given time. A motivating example given by Binmore and Samuelson (1997) is that every so often a player becomes discontented and chooses to change strategy.

Although the assumption of unit changes in the number of players playing a strategy may seem innocuous, it does rule out a number of plausible learning stories. For example consider a large population where pairs of players are randomly matched at different times. Suppose that when a match occurs, each player in the pair reviews their strategy. If both players were playing the same strategy and then choose to switch, the number of players playing that strategy will change by two.

Once one relaxes the assumptions of unit jumps and leaves the world of birth-death processes, however, it becomes very hard to solve for the stationary distributions. Kandori (1999) analyses a finite version of the Diamond (1982) search model. Here if a pair of traders meet, they exchange goods and become inactive. Inactive agents become active with a certain probability. This model violates the birth-death assumption as pairs of traders leave the market, although agents enter only singly. The finite distribution is intractable but Kandori obtains some equilibrium selection results by considering an approximation to the stationary distribution.

This paper aims to provide a more general analysis of equilibrium selection in large population models. To do so it goes back to ideas introduced by Freidlin and Wentzell (1998). Suppose that the differential equation approximation has a finite number of equilibria. If the population size is large the system will spend most of its time in the neighbourhood of these equilibria, with occasional escapes between them. One can therefore think of the system as describing a Markov chain between these states. To estimate the transition probabilities one needs to find the most likely way for the system to leave a given equilibrium — which involves the theory of rare events, that is of large deviations.

This paper shows that this approach leads to a clean approach to the problem of equilibrium selection. It considers a general class of models whose evolution in a finite population is described a Markov process with jumps whose state is one-dimensional (say the fraction of players playing a given strategy in a 2×2 game). It shows that in a large population almost all the weight of the stationary distribution will be put on the equilibrium which has the highest value of a certain quasi-potential function. To find this function simply requires solving a polynomial equation (and integrating), which is considerably easier than trying to solve for the equilibrium distribution directly.

The paper also provides general results on the nature of equilibrium selection. It makes precise the idea that an equilibrium which is noisier is less likely to be selected, as escape is easier. It also clarifies the role of the deterministic dynamic in equilib-

rium selection. In the literature on equilibrium selection there has been interest in the observation that equilibrium selection is often independent of the underlying deterministic dynamic (see for example Kandori et al. (1993), though not if mutations can be strongly state-dependent (see Bergin and Lipman (1996))). In contrast, with a continuous-state space the underlying dynamic seems to matter — see for example Fudenberg and Harris (1992) and Binmore and Samuelson (1997). The paper observes that many deterministic dynamics are in fact simply time changes of one another and therefore have equivalent selection properties. Apparent dependence on the deterministic dynamic can often be thought of as instead resulting from applying different perturbations to the same dynamic.

The paper also investigates when the conclusions of models based on birth-death processes generalise. It shows that if players take actions which do not depend on the players with whom they are currently matched, for example take a perturbed best reply to the aggregate distribution of play, then the conclusions generalise. In particular it shows that Blume (1994)'s result that the risk-dominant equilibrium is selected in 2×2 coordination games is robust to allowing simultaneous revisions.

The theory of large deviations is quite technical and for simple models such as birth-death processes it brings in unnecessarily heavy armoury. Nevertheless it has several advantages apart from mere generality. In the first place although justification of the theory is involved, the actual application is straightforward. Secondly, the formulae it yields are simple and intuitive and yield insight into the nature of the problem. Thirdly, it allows ready calculation of other quantities beyond those needed for equilibrium selection results.

If the population size is large, so aggregate noise is small, one may be sceptical about the relevance of the equilibrium selection results: if one starts near a stable equilibrium then it will take a very long time for the system to be perturbed away from it, even if it is not the one selected in the long run. For this reason, much of the literature on large deviations (see for example Shwartz and Weiss (1995) and the references therein) concentrates on the exit problem: to calculate how long (and in what way) the process is likely to take to exit from a stable equilibrium. Mean exit times are easy to calculate using the apparatus of large deviations and may be more relevant in practice.

Ellison (2000) and Binmore and Samuelson (1997) have stressed the importance of estimating the waiting time until a selected equilibrium is reached in order to gain some idea of its relevance. Exit times calculations can be used to do this and the paper does so.

Foster and Young (1990) first introduced the ideas of Freidlin and Wentzell into the economics literature. They considered equilibrium selection in the case of a differential equation perturbed by a small Brownian motion. Most of the work since then has concentrated on fixed finite populations or cases where equilibrium distributions can be calculated explicitly. Sargent (1999) uses exit time calculations via large deviations in another context.

Freidlin and Wentzell develop their theory in detail for processes perturbed by

Brownian motion. It is, however, straightforward to adapt it to the case of jump Markov processes as is done in this paper. The models considered are essentially one-dimensional. The same ideas can be used in higher dimensions but are less useful as the equations obtained, though still much simpler than the full equilibrium conditions, are much harder to solve than in the one-dimensional case.

The paper is organised as follows. Section 2 gives examples to motivate the framework and sketches the main ideas behind the analysis. Section 3 lays out the main assumptions. Section 4 gives the basic results on large deviations and equilibrium selection. It also provides results on exit times from stable equilibria and waiting times to hit the selected equilibrium. Section 5 applies the results to some examples. Section 6 shows that if choices of players do not depend upon whom they are matched with, then results obtained with birth-death processes generalise. Section 7 gives general results on the effect of noise on and the role of deterministic dynamic in equilibrium selection. Section 8 concludes.

2. Examples and Informal Sketch

The paper considers a finite population of N players. Each player is assumed to have two available actions 1 and 2 and at each time plans to play one of these. The fraction of the population playing action 1 is denoted x . Time is continuous and players switch occasionally between strategies.

A motivating example would be a population playing a 2×2 co-ordination game, as shown in Figure 1. Players are assumed to revise their strategies according to some boundedly rational model of learning. Some models consistent with the framework of the paper might be:

Example 1: Binmore and Samuelson (1997) In this model all players play against each other simultaneously at each moment. Each player's expected payoff is therefore determined by the strategy he plays and the fraction of players playing strategy 1 (x). Players become discontented according to a Poisson process at a rate which depends on their current payoff (and thus x and their type). A discontented player picks another player at random and imitates him with some probability. Since, under a Poisson process, players are exceedingly unlikely to become discontented at the same time, this yields a model where per unit time the number of players picking strategy 1 increases by 1 with probability $N\lambda_1^N$ and decreases by 1 with probability $N\lambda_{-1}^N$. The factor N reflects the fact that as population size increases revisions happen more often. The superscripts denote any further dependence on population size. This yields a birth-death process.

Example 2 Suppose instead that pairs of players are selected at random (perhaps meet) according to a Poisson process. When they meet they know the distribution of strategies in the population, x , but not their opponent's action in advance. Assume that they take a best response to this distribution, but their payoffs may be subject to random noise, so the best response is noisy (cf. stochastic fictitious play, see for example

Fudenberg and Kreps (1993)). The number of players playing strategy 1 therefore may change at any integer between -2 and 2, with rates $N\lambda_{-2}^N(x)$, $N\lambda_{-1}^N(x)$, $N\lambda_1^N(x)$ and $N\lambda_2^N(x)$ (in obvious notation). This is a Markov jump process but not a birth-death process.

Example 3 A variation on Example 2 would be K players selected at random to play a $K \times 2$ game (with K independent of N). One might also imagine K players selected to play a 2×2 game in round-robin fashion against one another (cf. Young (1993)). After the round is completed they revise their strategies. These models yield a jump process with jumps between K and $-K$ possible.

Example 4 A rather different dynamic is obtained if one supposes as above that pairs of players are selected to play the game (perhaps meet) according to a Poisson process but only revise strategies after the game is played. After playing the game, they revise their strategies in the light of their payoffs attained (but have no memory of the past payoffs before this). The payoffs they achieve depend on their own strategy and whom they are matched against, the probability of which depends on x . This yields a jump Markov process with jumps between -2 and 2 possible. One could generalise this along the lines of Example 3 and allow groups of K players to meet and revise strategies after playing the game. This would again yield a jump process with jumps between $-K$ and K .

Example 5 One could also consider a more symmetric version of the learning process considered in Example 1. Suppose that pairs of players occasionally meet and compare the performance of their strategies, perhaps only observed with noise (compare for example Weibull (1995) Section 4.4). If one assumes that the player with the worse strategy switches to the better one with a probability increasing in the payoff difference, this yields a birth-death process. If one adds the feature that players may randomly experiment even if the person they meet is using the same strategy, however, this yields a jump process with jumps between -2 and 2. If one allows large groups of players to meet and compare payoffs (cf. Schlag (1999)), this yields a general jump process also.

More generally one can consider dynamic games whose state is one-dimensional. For example

Example 6: Kandori (1999) Kandori considers a version of the Diamond (1982) search model where players are either inactive or active. Players become active at a rate, $\lambda_1(x)$, which depends on the fraction of players active, x . Pairs of active players meet at random according to a Poisson process and if they meet become inactive. The rate at which pairs meet depends on the fraction of the population active. The number of active players therefore increases by 1 at rate $N\lambda_1^N(x)$ and decreases by 2 at rate $N\lambda_{-2}^N(x)$. This is not a birth-death process.

These models can all be described as jump Markov processes. Now the total

number of players playing strategy 1 changes by

$$N \sum_{-K \leq i \leq K} i \lambda_i^N(x) h + o(h) \quad (1)$$

where $o(h)$ denotes a term of order smaller than h . The change in the fraction of players playing strategy 1 is therefore

$$\Delta x = \sum_i i \lambda_i^N(x) h + o(h) \quad (2)$$

If N is large and the λ 's converge, one would expect, by a law of large numbers, the motion of the system to be well-described by the differential equation:

$$\frac{dx}{dt} = \sum_i i \lambda_i(x) \quad (3)$$

This is indeed so if one is interested in behaviour over a finite horizon. Nevertheless this is not satisfactory if one is interested in equilibrium selection. It is quite possible for (3) to have multiple steady states and yet the model to have a unique stationary distribution for each N .

One might be tempted to improve the approximation by appealing to the central limit theorem and to obtain a diffusion approximation. While this is possible, this is not accurate enough for equilibrium selection — Binmore et al. (1995) give examples in an economic context. Kushner (1982) discusses this issue more generally.¹

Freidlin and Wentzell (1998) suggest approaching the problem by noting that for large N the system will spend most of its time in the neighbourhood of the stable points of (3). The system will occasionally escape, on account of the finite population randomness, to another equilibrium. One can therefore think of the system as moving according a Markov chain between equilibria and one needs to estimate the probability of these rare events in order to determine its transition matrix. This leads to the theory of large deviations.

To gain some intuition, consider for a moment an *i.i.d.* sequence of random variables X_i with mean 0. Let X denote a random variable with this distribution. By the law of large numbers, $\bar{X}_n = \sum_i X_i/n$ converges. Suppose one wishes to estimate the probability that $\bar{X}_n > u$, where $u > 0$. Now by Chebyshev's inequality for any $\alpha > 0$,

$$P(\bar{X}_n > u) < E \exp n(-\alpha u + \alpha X) \quad (4)$$

Equivalently,

$$\frac{\ln P(\bar{X}_n > u)}{n} < -(\alpha u - \ln M(\alpha)) \quad (5)$$

¹Sandholm (1999) suggests diffusion approximations may be useful for local stability analysis. Beggs (2002a) shows that they may be useful for characterising long-run behaviour if selection is weak in the sense that λ_N is small for large N .

where $M(t)$ is the moment generating function of X . To obtain as tight a bound as possible, one optimises over α , so²

$$\lim_{n \rightarrow \infty} \frac{\ln P(\bar{X}_n > u)}{n} < -\Lambda^*(u) \quad (6)$$

where

$$\Lambda^*(u) = \sup_{\alpha} \alpha u - C(\alpha) \quad (7)$$

is the Fenchel-Legendre transform of $C(\alpha) = \ln M(\alpha)$, that is of the log of the moment-generating function (the cumulant-generating function). The limit in (6) is redundant but is added for conformity with other results.

(6) gives an upper bound on the probability that \bar{X} exceeds u in terms of $\Lambda^*(u)$. A slightly more intricate argument shows that $\Lambda^*(u)$ also yields a lower bound. In more general cases, the lower and upper large deviations bounds do not coincide (see Section 4).

In the more complicated setting of this paper, one wishes to estimate the probability that the process finite population process deviates from its large sample average, (3). One might hope to do this by looking for analogues of the moment-generating function.

The function

$$H(x, \alpha) = \sum_i \lambda_i(x) (e^{i\alpha} - 1) \quad (8)$$

can be thought of as the logarithm of the ‘instantaneous’ moment-generating for the finite process. Recall that the moment-generating function of a Poisson random variable with parameter λ is $\exp(\lambda(e^\alpha - 1))$.

The Fenchel-Legendre transform of H is

$$L(x, \beta) = \sup_{\alpha} \beta \alpha - H(x, \alpha) \quad (9)$$

and one might hope this measures the likelihood of an instantaneous jump of size β . Now the cost of a path should be the sum of the cost of its jumps so the (log of the) likelihood of a path of duration T should be measured by

$$I(x) = \int_{t=0}^T L(x, \dot{x}) dt \quad (10)$$

where I is defined to be ∞ if the path x is not absolutely continuous. This turns out to be the case. A precise statement is given in the next section.

For the mean path, $L(x, \dot{x}) = 0$, that is following it is costless. If the system is at a stable equilibrium then, for large N , it is likely to stay near it. If the system escapes from it, then it seems plausible that it will do so by the route which makes (10)

²(5) was derived under the assumption that $\alpha > 0$, but it turns out this can be ignored in the optimisation.

smallest. Other escape routes are possible but for large N their relative probability goes to zero at an exponential rate.

L does not have a closed form expression in general. For the case of the Poisson process with constant rate λ , $H(x, \alpha) = \lambda(\exp(\lambda\alpha) - 1)$ and it is easy to check that for $\beta > 0$.

$$L(x, \beta) = \beta \ln \frac{\beta}{\lambda} - \beta + \lambda \quad (11)$$

As λ tends to zero, L is ill-behaved. This will cause some technical problems later as jumps in some directions must go to zero near the boundaries if the process is to remain in the unit interval.

To implement the Freidlin-Wentzell theory, and estimate the transition probabilities of the approximate Markov chain between equilibria, it seems one needs to solve a variational problem involving (10) to find the cheapest escape route from an equilibrium. In the one-dimensional case this turns out to be remarkable easy and simply involves solving a polynomial, as is shown in Section 4. Given this it is easy to determine which equilibrium is selected in the one-dimensional case, since the induced Markov chain has a rather simple structure (one can only exit to adjacent equilibria).

The following sections make these ideas precise.

3. The Model

The paper considers a family of stochastic processes indexed by N . For each N the state of the system is the number of agents, L , taking a certain action, say strategy 1, or equivalently the fraction of agents taking that action, $x = L/N$. Time is continuous.

Assumption 1 *For each N , L evolves according to a jump Markov process with parameters $N\lambda_i^N(x)$ with i an integer between $-K_1$ and K_2 , independent of N .*

This assumption implies that the number of agents changing strategy is bounded independently of N . It is satisfied in the examples of the previous section. It guarantees that, for large population sizes, the change in the fraction of agents playing strategy 1 in a small length of time is negligible. It could be relaxed to allow the possible jumps to grow with N , provided they do so sufficiently slowly. Without some such assumption, the process could make large jumps in small time periods and so would not yield a differential equation in the limit.

Assumption 2 *For each i , $\lambda_i^N(x)$ converges to $\lambda_i(x)$ uniformly in x .*

For finite N , only multiples of $1/N$ are possible values of x . This is unimportant in the statements of the assumptions.

This assumption implies that for large N , the rate of change of x only depends on x , not N . This cannot be exactly true near the boundaries. For example in Kandori (1999)'s model (Example 6) the number of active agents decreases by 2 when a pair meets. This cannot, however, be true if only 1 agent is playing that strategy. One therefore needs to modify the jump rates near the boundary for finite N . This is allowed for in Assumption 2.

Assumption 3 For each i , $\lambda_i(x)$ is a Lipschitz-continuous function of x on $[0, 1]$.

This is a standard technical assumption. It guarantees, amongst other things, that (3) has a unique solution.

The slightly complicated phrasing of the next assumption is to allow for the fact that some jumps between $-K_1$ and K_2 may not be possible. Let $\Phi = \{i : \lambda_i(x) = 0 \text{ for all } x\}$.

Assumption 4 (a) For some $i < 0$, $i \notin \Phi$. For all $i < 0$ with $i \notin \Phi$, $\lambda_i(x) = 0$ if and only if $x = 0$. (ii) For some $i > 0$, $i \notin \Phi$. For all $i > 0$ with $i \notin \Phi$, $\lambda_i(x) = 0$ if and only if $x = 1$.

This assumption implies that the differential equation (3) does not have any equilibria on the boundary. It is satisfied in the applications above. It is stronger than this, however. As discussed in Section 2, points where transition rates vanish cause some technical difficulties. This cannot be avoided at the boundaries, since the process cannot leave the unit interval, but this rules out any other points where this occurs. It could be relaxed somewhat (see the discussion in Schwartz and Weiss (1995) Chapter 6), but in the applications it is quite natural as away from the boundaries there is usually some minimal rate of switching (for example due to error or mutation). It is satisfied in the examples in Section 2.

Assumption 5 For $i < 0$, $i \notin \Phi$, $x \ln \lambda_i(x)$ tends to 0 as x tends to zero. For $i > 0$, $i \notin \Phi$, $(1 - x) \ln \lambda_i(x)$ tends to zero as x tends to 1.

This assumption requires that rates not go to zero too fast near the boundary and is needed for technical reasons. It is fairly natural as, in most economic examples, the transition rates away from a strategy (considering the case $-i < 0$) can usually be written in the form $\lambda_{-i}(x) = x^i \mu_i(x)$ with $\mu_i(x)$ bounded away from zero. This is because to have i people switching away from strategy 1, there must be at least i people playing it amongst the group revising strategies, and the probability of such a group is usually proportional to x^i . So long as there is some minimal rate of switching of individual players, for example due to error or mutation, then $\mu_i(x)$ is bounded away from zero. It is satisfied in the examples in Section 2.³

Assumption 6 For each N , the process is an ergodic Markov chain.

This assumption implies that for each N , there is a unique stationary distribution. It is evident in the examples in Section 4, but at this level of generality needs to be imposed.

Assumption 7 The differential equation (3) has finitely many equilibrium points.

This assumption is natural in most applications, where in games of interest, for example those of Section 2, there are usually a finite number of equilibria. It implies that for large N , the process essentially executes a finite Markov chain with states

³Kandori (1999)'s assumptions imply it in the case of Example 6.

corresponding to these equilibria. The analysis would go through unchanged if one allowed intervals of equilibria, provided there are only finitely many (essentially all points in an interval are treated as equivalent). In higher dimensions, this would be re-drafted as a finite number of limit sets, but equilibria are the only possible limit sets in one dimension.

4. Formal Results

This section presents the main technical results. They are applied in later sections. It uses the ideas of Freidlin and Wentzell (1998) together with results for Markov jump processes. An accessible reference for the latter is Shwartz and Weiss (1995). More advanced treatments can be found in Dupuis and Ellis (1997) and Wentzell (1990) (see also Freidlin and Wentzell (1998) Chapter 5.2 for an introduction to the latter's results).

4.1 Preliminary Results

The first result confirms that for large N , the model of Section 3 yields an ordinary differential equation limit. Let $x^N(t)$ denote a sample path of the random process and $x(t)$ that of the differential equation (3). Let P_x denote probability conditional on starting at x at time 0.

Theorem 1 *Under Assumptions 1–3, for all finite $T > 0$, there exists a constant C_1 and a function $C_2(\epsilon)$, with $\lim_{\epsilon \rightarrow 0} \frac{C_2(\epsilon)}{\epsilon^2} \in (0, \infty)$ and $\lim_{\epsilon \rightarrow \infty} \frac{C_2(\epsilon)}{\epsilon} = \infty$, such that for all $N \geq 1$ and $\epsilon > 0$,*

$$P_x \left(\sup_{0 \leq t \leq T} |x_N(t) - x(t)| \geq \epsilon \right) \leq C_1 e^{-NC_2(\epsilon)} \quad (12)$$

where C_1 and C_2 can be chosen independently of x .

This follows from Theorem 5.3 of Shwartz and Weiss (1995) (Kurtz's Theorem). The proof given there is for the case of $\lambda_i^N(x)$ independent of N but it is easy to check that it goes through under Assumption 2. (12) states that the probability that the sample path is ever more than ϵ away from that of the ordinary differential equation goes to zero at an exponential rate in N . The conditions on C_2 give information on the dependence of the bound on ϵ .

The theorem is essentially a law of large numbers result. The next result estimates the probabilities of fluctuations away from the mean path.

The sample paths of $x_N(t)$ on $[0, T]$ belong to $D[0, T]$, the set of real-valued functions on $[0, T]$ which are right-continuous and have limits to the left at every point. $D[0, T]$ can be made into a topological space under the Skorokhod topology. On the set of continuous functions this coincides with the uniform metric but it allows processes which have jumps at nearby points to be close, which the uniform metric does not. A formal definition can be found in Shwartz and Weiss.

Theorem 2 *Let G and F be open and closed sets of paths respectively in $D[0, T]$ under the Skorokhod topology which do not hit $x = 0$ or $x = 1$. Under Assumptions 1–4, if $I(z)$ is the function defined in (10) then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln P_x(x_N(t) \in G) \geq -\inf\{I(z) : z \in G, \quad z(0) = x\} \quad (13)$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln P_x(x_N(t) \in F) \leq -\inf\{I(z) : z \in F, \quad z(0) = x\} \quad (14)$$

The bound in (13), for given G , holds uniformly over x in compact subsets of the interior of $[0, 1]$. The bound in (14) also holds uniformly if, for given F , the right-hand side is continuous in x .

This result is implied by Shwartz and Weiss (1995) Theorems 5.51 and 5.54 and Corollary 5.65—they prove the result for the case when $\lambda_i^N(x)$ is independent of N but it is easy to check that the result goes through. Alternatively, see Wentzell (1990) Theorems 3.2.2 and 3.2.3 (with the use of Theorem 4.1.1 to verify the hypotheses), which allow for dependence on N .

To interpret this result, note that for the mean path $I(x) = 0$. If the set of paths does not include the mean path, then loosely the theorem says that its probability goes to zero exponentially with N , at a rate which depends on the cost of the least costly path in the set. The different bounds for open and closed sets arise for similar reasons to those occurring in the theory of weak convergence. Indeed one can develop some analogies between the theory of large deviations and that of weak convergence — see for example Dupuis and Ellis (1997).

T is finite in the Theorem. Nevertheless the Theorem can be used to estimate, for example, escape probabilities from a neighbourhood of an equilibrium point, which may take place over an unbounded horizon, as these events will with high probability take place quickly if at all.

The restriction to paths not hitting the boundaries, and the need for Assumption 4, arises because as discussed in Section 2, the rate function is not well-behaved as rates hit zero. It is not immediate that one can extend Theorem 2, though one can in some cases. For a detailed discussion of the problem of boundaries, see Shwartz and Weiss (1995) (especially chapters 6, 8 and 11).

Here this is not a problem. Firstly, the fact that the boundaries are not absorbing implies the process spends little time there for large N . Secondly, in order to estimate the probability of moving between equilibria, one needs to estimate the probability of paths that start at one equilibrium and end at the other without returning to the original or hitting any other equilibrium. In the one-dimensional case such paths cannot touch the boundary (see Figure 2) and so one does need a large deviations principle that applies there. In higher dimensions, however, such paths may touch the boundary and so one requires a more delicate argument.

4.2 Equilibrium Selection

Theorem 2 allows the derivation of the equilibrium selection results. The key ideas are outlined in the text. Rigorous proofs are in the Appendix.

Under Assumption 7, the limiting ordinary differential equation has finitely many equilibria. The latter will be indexed by consecutive positive integers $i = 1, 2, \dots$, with the i th equilibrium occurring at $x = x_i$, in increasing order of x (so $x_i > x_j$ if and only if $i > j$).

For large N , the process spends most of its time near these equilibria with occasional rare exits between them. Movements between them follow an approximate Markov chain. When N is large, one cannot pass between two equilibria which are not adjacent without passing close to an equilibrium — close because of the discreteness of x for finite N — so only nearest neighbour transitions are possible (see Figure 2).

Let j and k be adjacent equilibria at x_j and x_k respectively with, j stable (at least from the direction of k), k unstable. Note that an equilibrium will be called stable if the flow of (3) converges from to it from an open set of initial points containing it, unstable otherwise. Stable on the left and right have the obvious definitions. Note that if two equilibria are adjacent, one of them must be unstable in the direction of the other (since the equilibria are finite in number).

Let

$$V_{jk} = \inf_{z, T} \left\{ \int_0^T L(z, \dot{z}) dt : z(0) = x_j, \quad z(T) = x_k \right\} \quad (15)$$

Intuitively, V_{jk} measures the cost of moving between j and k . For large N the process is approximately a Markov chain with transition probabilities $p_{jk} = \exp(-NV_{jk})$ for adjacent j and k . V_{jk} and V_{kj} therefore need to be determined.

Now $V_{kj} = 0$ as travel in the direction of the flow of the differential equation is free. To determine V_{jk} consider the more general problem of moving from a point x to x_k , and replace inf by sup for convenience:

$$V(x) = \sup_{z, T} \left\{ \int_0^T -L(z, \dot{z}) dt : z(0) = x, \quad z(T) = x_k \right\} \quad (16)$$

Clearly $V(x_k) = 0$. Now the Bellman equation for this problem is

$$\max_u -L(x, u) + uV' = 0 \quad (17)$$

For if one moves at rate u for a length of time dt , this has instantaneous payoff $-Ldt$ and one moves a distance udt closer to the desired destination. Yet, (17) is simply the Fenchel-Legendre transform of $L(x, u)$. Now L was itself the Fenchel-Legendre transform of H , and, since H is continuous and convex, taking the transform again gets back to H (see for example Rockafellar (1970) Theorem 12.2, p. 104). Hence (17) is equivalent to

$$H(x, V') = 0 \quad (18)$$

Equivalently

$$\sum_i \lambda_i(x) \left(e^{iV'(x)} - 1 \right) = 0 \quad (19)$$

or putting $V'(x) = \ln y$,

$$\sum_i \lambda_i(x) (y^i - 1) = 0 \quad (20)$$

For fixed x , this is a polynomial equation in y . It always has a trivial root $y = 1$ or $V' = 0$: this corresponds to the fact that moving in the direction of the flow is costless. Only roots with $y > 0$ are meaningful, and it is easy to show (see Appendix) that

Lemma 1 *For each x in the interior of $[0, 1]$, (20) has a root $y = 1$ and one other positive root, $U(x)$. $U(x)$ is continuous in x and $U(x) - 1$ has the opposite sign to $\sum_i i\lambda_i(x)$.*

The condition on the sign simply reflects the fact that it is costly to move against the flow ($L \geq 0$) and that the total cost falls as one moves closer to one's destination ($V' = \ln y$ and recall that in (16) one is maximising minus the costs).

Now

$$V(x) = \int_{x_k}^x \ln U(\xi) d\xi \quad (21)$$

or since costs are the negative of this and $V_{kj} = 0$,

Lemma 2 *For adjacent j and k*

$$V_{jk} - V_{kj} = \int_{x_j}^{x_k} \ln U(\xi) d\xi \quad (22)$$

For this kind of chain — one-dimensional with only nearest neighbour transitions possible — one must have

$$\pi_j p_{jk} = \pi_k p_{kj} \quad \text{for all } j, k \quad (23)$$

where π_j and π_k are its stationary probabilities, that is it is reversible — see for example Kelly (1979). Note that the original process is not itself reversible in general, except in the birth-death case. Hence if j and k are adjacent and $V_{jk} - V_{kj} > 0$, π_k is negligible in comparison to π_j for large N .

For non-adjacent j and k with $k > j$ one has for large N approximately, using (23),

$$\frac{\pi_k}{\pi_j} = \exp \left\{ -N \sum_{j \leq l < k} (V_{l,l+1} - V_{l+1,l}) \right\} \quad (24)$$

where the sum is over equilibria l that lie between j and k .

To determine which equilibrium is selected one therefore needs to calculate $\sum_l V_{l,l+1} - V_{l+1,l}$. To do this one simply patches together the formulae found above. Let

$$W(x) = \int_x^{x_0} \ln U(\xi) d\xi \quad (25)$$

where x_0 is an arbitrary point in the interior of $[0, 1]$. Then it follows from the above that:

Lemma 3 $\sum_{j \leq l < k} V_{l,l+1} - V_{l+1,l} = W(x_j) - W(x_k)$.

The argument above suggests that the equilibrium for which W is largest will be selected. (Note that no non-equilibrium point can have $W(x)$ maximal as one can move for free from it to a non-equilibrium point). This is indeed the case:

Theorem 3 *Under Assumptions 1–7, any limit points of the sequence of stationary distributions as $N \rightarrow \infty$ in the weak topology are contained in the set of distributions which have support contained in the set of x for which $W(x)$ is maximal. In particular, if $W(x)$ is maximised at a unique point x^* , the stationary distributions converge to a point mass at x^* .*

The proof is in the Appendix. It is a straightforward adaptation of the Freidlin and Wentzell's result for differential equations perturbed by Brownian motion to the case of jump processes.

$W(x)$ is usually referred to as a quasi-potential function (quasi because except in special cases the transition rates are not simply related to it).⁴ Note that since one can leave an unstable equilibrium for free:

Corollary *No unstable equilibrium can belong to the support of any limit point of the finite stationary distributions.*

4.3 Waiting Times

For large N , if one starts near a stable equilibrium, then even if it is not selected it will take a long time to move away from it, so one might doubt the relevance of the stationary distribution. One might be interested in estimating how long it will take to leave a small neighbourhood of the equilibrium. Let τ_x^F the time of first escape from F starting at x .

Theorem 4 *Let x^* be a stable equilibrium and let $F = [a, b]$ be a closed interval, containing it and contained in the closure of its domain of attraction. If $a = 0$ or $b = 1$, set $V_a = \infty$ or $V_b = \infty$ respectively. Otherwise, let $V_a = W(x^*) - W(a)$ and $V_b = W(x^*) - W(b)$. and $V^* = \min\{V_a, V_b\}$. Then under Assumptions 1–7 for x in any compact subset of the interior of F and any $\epsilon > 0$,*

$$(i) \quad \lim_{N \rightarrow \infty} E \frac{\ln \tau_x^F}{N} = V^* \quad (26)$$

$$(ii) \quad \lim_{N \rightarrow \infty} P_x \left(\frac{\ln \tau_x^F}{N} \in (V^* - \epsilon, V^* + \epsilon) \right) = 1 \quad (27)$$

In other words, mean escape times go to infinity at an exponential rate determined by V^* and realisations of escape times are strongly concentrated around V^* . The proof

⁴Freidlin and Wentzell's results are phrased in terms of minimising the quasi-potential, which corresponds to taking the opposite sign for W .

is in the Appendix. It can also be shown that with probability tending to 1, escape occurs by the side where exit is cheapest.

The above result restricts attention to escape from a domain containing a single equilibrium (except possibly on its boundary). One might be interested in estimating how long the system will take to reach the equilibrium selected in Theorem 3. Ellison (2000) and Binmore and Samuelson (1997), for example, have stressed the importance of estimating waiting times to determine the relevance of selection results. This problem may involve estimating the exit time from a domain containing several equilibria. This can also be estimated using the Freidlin and Wentzell theory.

To simplify the statement of the result suppose that W has a unique maximum, x^* . In order to determine the relevance of the selected equilibrium it is desirable to estimate the expected time to reach a neighbourhood of x^* from another point.

Theorem 5 *Let $F = [a, b]$ be a closed neighbourhood containing x^* and no other equilibria. Let σ_x^F be the first time the process enters F starting from x . Let*

$$H^* = \max_{0 \leq x \leq y \leq a} W(x) - W(y) \quad (28)$$

then if $0 \leq x < a$,

$$\lim_{N \rightarrow \infty} E \frac{\ln \sigma_x^F}{N} \leq H^* \quad (29)$$

with equality for the x where the maximum in (28) is attained. A symmetric formula holds for $b < x \leq 1$.

A formal proof is in the Appendix. Intuitively, H^* is the greatest difference in potential occurring between points between 0 and a and so measures the difficulty of escaping to a . Note that if there is an equilibrium in $[0, a)$, then the maximum is attained at an equilibrium point as the Assumptions imply the left-most equilibrium is stable from the left. If there are no equilibria, the maximum is zero — in this case the system is rapidly attracted to the nearest equilibrium, at x^* .

It is straightforward to generalise Theorem 5 to the case of multiple global maxima of W . In the general case, one needs to consider intervals from which exit is possible to the left or the right. The exit time from one of these intervals is bounded by the lesser of the greatest potential difference incurred moving to the left and the greatest incurred moving to the right.

The result obtained is weaker than that in Theorem 4: here only the expectation of the exit time is estimated. Other features of its distribution, for example quantiles such as the median, may go to infinity at a much slower rate. To understand this, note that in order to estimate the expectation one must take into account the possibility that an unlikely transition results in the system moving from the current equilibrium to another equilibrium with long escape times. Such a transition may be rare enough that it does not affect probabilities much, but may make a large contribution to expected waiting times. A further discussion of this point can be found in Freidlin and Wentzell (1998) Chapter 6, Section 5. For this reason, Theorem 5 is a relatively weak

statement about the behaviour of waiting times, though it a weakness shared by the results in Ellison (2000), who also only estimates expected waiting times.

The results of this Section can in principle be extended to higher dimensions. The obstacles are first that as noted the problem of behaviour near boundaries poses more technical obstacles. Secondly, the equation corresponding to (18) is harder to solve - it is an example of the Hamilton-Jacobi equation. Numerical solution is necessary in general.

These results are applied to the examples of Section 2 in the next two Sections. They are used to give general characterisations in Section 7.

Section 5. Examples

This Section applies the results to the examples of Section 2.

Example 1 (Binmore and Samuelson) Here only unit jumps to the left and right are possible. (20) therefore becomes

$$\lambda_{-1}(x)(y^{-1} - 1) + \lambda_1(x)(y - 1) = 0 \quad (30)$$

or equivalently

$$(y - 1)(-\lambda_{-1}(x) + y\lambda_1(x)) = 0 \quad (31)$$

The root $y = 1$ is, as noted, irrelevant and so taking the other root, one obtains,

$$W(x) = \int_x^{x_0} \ln \frac{\lambda_{-1}(\xi)}{\lambda_1(\xi)} d\xi \quad (32)$$

Theorem 3 implies that the equilibrium with the greatest value of W is selected as $N \rightarrow \infty$, which is exactly the result obtained by Binmore and Samuelson (1997) by direct calculation.

Example 6 (Kandori) Here jumps to the left by two units and to the right by one unit are the only ones possible. (20) therefore becomes

$$\lambda_{-2}(x)(y^{-2} - 1) + \lambda_1(x)(y - 1) = 0 \quad (33)$$

or equivalently

$$(y - 1)(-\lambda_{-2}(x)(1 + y) + \lambda_1(x)y^2) = 0 \quad (34)$$

The positive root not equal to one is again the relevant one, so one obtains

$$W(x) = \int_x^{x_0} \ln \frac{\lambda_{-2}(\xi) + \sqrt{\lambda_{-2}(\xi)^2 + 4\lambda_1(\xi)\lambda_{-2}(\xi)}}{2\lambda_1(\xi)} d\xi \quad (35)$$

Again Theorem 3 implies that the equilibrium with the largest value of W is selected as $N \rightarrow \infty$, as obtained by Kandori (1999) by approximating the stationary distribution.

Waiting times can be calculated in this and the previous example using Theorems 4 and 5.

Examples 2 to 5 If jumps may take place by up to K_1 to the left and K_2 to the right, then one obtains, after removing the factor $y - 1$, a polynomial of degree $K_1 + K_2 - 1$:

$$(1 - y) \left(\sum_{i=0}^{K_1-1} y^i \left(\sum_{j=K_1-i}^{K_1} \lambda_{-j}(x) \right) - \sum_{i=K_1}^{K_1+K_2-1} y^i \left(\sum_{j=i-K_1+1}^{K_2} \lambda_j(x) \right) \right) = 0 \quad (36)$$

For the case of pairs meeting at random and revising strategies before or after an encounter, $K_1 = K_2 = 2$, so this yields a cubic equation, which is soluble analytically. In general of course one would need to resort to numerical solution, but these equations are more tractable than the difference equations required to find the steady-state distributions directly.

In the case of Examples 2 and 3 (36) is soluble analytically and this case is considered in detail in the next section. Section 7 considers general properties of equilibrium selection obtainable even when analytical solutions are not available.

6. Match-Independent Choices

This section shows that the case of Examples 2 and 3 can be analysed rather simply. These dynamics share the feature that when a player changes strategy, the action he takes, there a perturbed best-reply to the the current distribution of strategies, is independent of the strategies of the players with whom he is actually matched. This contrasts with Examples 4 and 5. The analysis will apply to any dynamic with the property of match-independent choices, not just ones where players take perturbed best replies.

The general framework considered is that a group of K players revises strategy at a time. Revisions take place at rate 1 per unit time (as will be seen in the next section, this is without loss of generality, since it is assumed that players are selected purely randomly from the population). If a player is currently playing strategy 2, he plays strategy 1 with probability $p(x)$, if he is playing strategy 1 he switches to strategy 2 with probability $q(x)$.⁵ In the case of perturbed best-reply dynamics, $q(x) = 1 - p(x)$, but the current formulation allows for some inertia in behaviour.

If $K = 1$, this yields a birth-death process with rates $\lambda_1(x) = (1 - x)p(x)$ and $\lambda_{-1} = xq(x)$. $p(x)$ and $q(x)$ are assumed strictly positive and Lipschitz-continuous on $[0, 1]$, so that the Assumptions of Section 3 are satisfied. These restrictions rule out pure best-reply dynamics and the Kandori et al. (1993) mistakes model, but are consistent with the models of perturbed best-replies considered by Blume (1994) and in another context Fudenberg and Kreps (1993). In this case of course, one finds just

⁵For convenience it is assumed that a player includes himself in the population average. The Lipschitz-continuity assumptions on p and q imply this does not affect the limiting results.

as in (32)

$$W(x) = \int_x^{x_0} \ln \frac{\xi q(\xi)}{(1-\xi)p(\xi)} d\xi \quad (37)$$

Blume (1994) has considered this dynamic in the case of 2×2 symmetric co-ordination games with $p = 1 - q$ and

$$\ln \frac{p}{1-p} = \beta g(\Delta\pi) \quad (38)$$

where $\Delta\pi$ is the difference in payoffs between the strategies 1 and 2 and β is a parameter. The larger β the closer to pure best-replies choices are.

In a 2×2 symmetric co-ordination game there are three equilibria: $x = 0$, $x = 1$, and a mixed equilibrium x^* . If $x^* < 1/2$, then $x = 1$ is the risk-dominant equilibrium since it has the larger basin of attraction (see Figure 3). Blume (1994) shows that if only the difference in payoffs affects choices, so g is an odd function $g(y) = -g(-y)$, then for large β the risk-dominant equilibrium is selected. This is easy to show in the current framework:

Lemma 4 *Assume that g is a C^1 increasing function of $\Delta\pi$ with $g' > 0$ in a neighbourhood of 0. For large β , (3) has three zeroes $x_0(\beta)$, $x_1(\beta)$ and $x_3(\beta)$ with $\lim_{\beta \rightarrow \infty} x_0 = 0$, $\lim_{\beta \rightarrow \infty} x_1 = 1$, and $\lim_{\beta \rightarrow \infty} x_3 = x^*$. If $x^* < 1/2$, then for large β , $W(x)$ is maximised at x_1 , so x_1 is the selected equilibrium.*

The proof is in the Appendix. The first part of the statement states for large β , the equilibria of the system are close to those of the original game.

To analyse the general case, note that (20) can be re-written as

$$\sum_{i>0} \lambda_i(x) y^i + \sum_{i<0} \lambda_i(x) y^i = \sum_{i>0} \lambda_i(x) + \sum_{i<0} \lambda_i(x) \quad (39)$$

Explicit expressions for $\lambda_i(x)$ rapidly become very messy, so a more elegant approach will be used. (39) can be written more symmetrically by introducing dummy jumps of size 0. If $\lambda_0(x) = 1 - \sum_{i \neq 0} \lambda_i(x)$, then (39) is equivalent to

$$G(y) = 1 \quad (40)$$

where

$$G(y) = \sum_i \lambda_i(x) y^i \quad (41)$$

is the probability generating function of the distribution defined by the λ_i .

In the cases under consideration, by assumption each player's revision of strategy is independent of the choices of the other players revising strategy at the same time. If a single player playing strategy 2 is chosen to revise his strategy and switches to 1, the number of players playing 1 increases by 1. This event has probability $p(1-x)$. With probability qx a player playing strategy 1 switches to 2, so the number playing 1 decreases by 1. In the remaining cases, the change is zero. Now the K players revise

independently, so the distribution of total change in numbers playing 1 is that of the sum of K independent random variables, each taking value 1 with probability $p(1-x)$, -1 with probability qx and 0 with probability $1 - p(1-x) - qx$.

(40) can therefore be re-written as

$$(G_1(y))^K = 1 \tag{42}$$

where

$$G_1(y) = p(x)(1-x)y + xq(x)y^{-1} + (1 - p(x)(1-x) - xq(x)) \tag{43}$$

is the probability generating function when a single player is chosen to revise strategy.

It is easy to see that the only roots of (42) with $y > 0$ are those of $G_1(y) = 0$, namely $y = 1$ and $y = (1-x)p/qx$. It follows that

Theorem 6 *With match-independent choices, the quasi-potential function for any K is the same as for the $K = 1$, the birth-death case. The equilibrium selection properties are therefore exactly the same.*

In particular, Blume (1994)'s result in Lemma 4 on the selection of the risk-dominant equilibrium is robust to allowing simultaneous revisions.

One reaction might be that this is not very surprising. If N is large, x changes rather slowly so K players changing action simultaneously should not be very different from K changing in quick succession. Nevertheless, it is not at all obvious if one writes down the explicit equilibrium distributions. A virtue of the current approach is that it gives a clean and simple proof of this fact.

It follows that with match-independent choices the conclusions of the birth-death model are robust. For other models, for example those of Examples 4 and 5, this need not be so. If it is assumed that large groups sample one another or play one another in round-robin fashion, then this should not be very different from sampling independently from the whole population, so the birth-death conclusions should also be robust, though this will depend on the particular model. For small samples, however, a more detailed analysis is required.

7. General Characterisation Results

This section explores how equilibrium selection can be characterised when the quasi-potential function cannot be found explicitly. It makes precise the idea that an equilibrium which is noisier is less likely to be selected, as escape is easier. It also clarifies the role of the deterministic dynamic in equilibrium selection. In the literature of equilibrium selection there has been interest in the observation that equilibrium selection is often independent of the underlying deterministic dynamic (see for example Kandori et al. (1993), though not if mutations can be strongly state-dependent (see Bergin and Lipman (1996)). In contrast, with a continuous-state space the underlying dynamic seems to matter — see for example Fudenberg and Harris (1992) and Binmore

and Samuelson (1997). This section observes that many deterministic dynamics are in fact simply time changes of one another and therefore have equivalent selection properties. Apparent dependence on the deterministic dynamic can often be thought of as instead resulting from applying different perturbations to the same dynamic.

7.1 Noise

It is convenient to introduce a change of variables. For each x , let $\lambda(x) = \sum_{-K_1}^{K_2} \lambda_i(x)$ and $p_i(x) = \lambda_i(x)/\lambda(x)$, $i = -K_1, \dots, K_2$. $\lambda(x)$ can be thought of as the rate at which events take place at x . When an event occurs, the process jumps to another state with probabilities given by $P(x) = (p_i(x))$. Note that, in contrast to the previous section, the device of dummy jumps of size zero is not used here. Dividing through by $\lambda(x)$, (19) becomes

$$\sum_i ip_i(x) \exp(ih) = 1 \quad (44)$$

where h is the variable to be determined.

An immediate observation is that equilibrium selection depends only on the p_i , the jump probabilities, not λ . This will be interpreted further below. For the moment hold λ fixed and consider variations in p_i . Suppose $\sum_i ip_i(x)$ is also held fixed for each x . This holds the mean drift constant and so from (3), the deterministic drift is left constant.

Consider comparing a stable equilibrium x^* with another equilibrium x' . Now

$$W(x^*) - W(x') = \int_{x^*}^{x'} \ln U(x) dx \quad (45)$$

Suppose without loss of generality that $x' > x^*$. For $x > x^*$ in the basin of attraction of x^* , the desired root $h = \ln U(x)$ of (44) is positive (escape is costly—see Lemma 1). \exp is a convex function, so a mean-preserving spread (see for example Diamond and Rothschild (1978)) of the p 's raises the right-hand side for each h , and the desired root falls (see Appendix for proof). Hence the integrand in (25) is reduced at x . It follows that $W(x^*) - W(x')$ falls.

Hence

Theorem 7 *A mean-preserving increase in noise at points in the basin of attraction of stable equilibrium reduces the value of its quasi-potential relative to that of other equilibria and so, other things equal, make it less likely to be selected.*

The term ‘increase in noise’ seems reasonable as the mean jump size is preserved but it becomes more variable. The intuition is clear: there is more noise, so jumps away from the mean path become easier.

Note that with a birth-death model, such as Binmore and Samuelson (1997), or any model with only two jump sizes, for example Kandori (1999), if one preserves the

mean the p 's are determined, so this result is vacuous. The more general framework here allows these features to be separated.

The alert reader may have noted that any transform of the p 's which preserves the left-hand side of (44) will have the same equilibrium selection properties, regardless of whether the drift is preserved. It is hard to give interpretable conditions for such a utility or exponential-moment preserving transform, however, as which moment needs to be preserved is not known in advance. This line of enquiry will therefore not be pursued.

7.2 Deterministic Dynamic

As noted above,

Theorem 8 *The equilibrium selected depends only on $P(x)$, not $\lambda(x)$.*

This is at first sight surprising. If one slows down events at a particular point a jump Markov chain will spend more time there and so the stationary distribution will put more weight there. The equilibrium distribution of a jump chain does not therefore depend solely on $P(x)$ — or the embedded Markov chain as it is often referred to as (see Wolff (1989) for example). The difference here is that Assumption 3 restricts $\lambda(x)$ to be a (Lipschitz-)continuous function of x . If one slows down the process at a particular point x , so makes departures less frequent, it also slows it down at neighbouring points, so making returns less frequent. For large N these effects exactly offset one another.

The waiting time estimates of Theorem 4 are also unaffected by changes in λ . Of course, actual waiting times cannot be unaffected. The point is that the estimates of Theorem 4 only give an estimate of their rate of growth as N becomes large and not an estimate of their levels.

Although changing $\lambda(x)$ does not affect the equilibrium selection results it does affect the form the limiting differential equation. This can be written as

$$\frac{dx}{dt} = \lambda(x) \sum_i ip_i(x) \quad (46)$$

Nevertheless, since $\lambda(x) > 0$ everywhere, this has exactly the same orbits as

$$\frac{dx}{dt} = \sum_i ip_i(x) \quad (47)$$

The factor $\lambda(x)$ simply corresponds to changing the time scale — possibly at a rate depending on x . At points where $\lambda(x)$ is large, time is measured in smaller units than at points where it is small. See for example Chicone (1999) p. 14–17 for further details.

From the point of view of equilibrium selection, (46) and (47) are equivalent. They both can be generated by the same noise process $P(x)$. They simply differ according to the time scale used. Any difference in equilibrium selection must therefore result from using different noise processes. This however stems from the fact that different

noise process can give different equilibrium selection results from the same equation, not from the form of the deterministic dynamic.

It is therefore natural to ask when two deterministic dynamics are equivalent in this sense. Now consider

$$\frac{dx}{dt} = f(x) \quad (48)$$

$$\frac{dx}{dt} = g(x) \quad (49)$$

Definition 1 (48) is **equivalent** to (49) if $f(x) = \lambda(x)g(x)$ for some Lipschitz-continuous $\lambda(x)$ with $\lambda(x) > 0$ all x .

This definition of equivalence implies that (48) can be obtained from (49). The Lipschitz continuity hypothesis guarantees that if Assumption 3 is satisfied for a finite model yielding (49), it is also satisfied if rates are rescaled by $\lambda(x)$ to obtain (48). The other Assumptions also carry over.

The only candidate for $\lambda(x)$ is $f(x)/g(x)$. For $\lambda(x)$ to be always positive f and g must always have the same sign and vanish at the same points. That is, in the terminology of Kandori et al. (1993), they must be sign-preserving transformations of one another. To obey the continuity conditions, however, it must be possible to extend the definition of $f(x)/g(x)$ in a Lipschitz-continuous way to the common zeroes of f and g .

Definition 2 f and g have the same order at a common zero x_0 , if $f(x)/g(x)$ can be extended to a strictly positive Lipschitz continuous function in a neighbourhood of x_0 .

Essentially this condition says that f and g must go to zero at the same rate. This puts some restriction on which sign-preserving dynamics are equivalent. One obvious remark is:

Lemma 5 If f and g are C^2 and their derivatives are non-zero, that is f and g are hyperbolic at zeroes, and have the same sign, then they have the same order at common zeroes.

Hyperbolic equilibria are in a sense generic. On the other hand, economic restrictions may place restrictions on the nature of f and g , so the assumption of genericity may sometimes be unreasonable — see for example Binmore and Samuelson (1999) for further discussion.

If the derivatives of both f and g both vanish then one can look at higher-order derivatives to settle the matter (by L'Hôpital's rule). The condition amounts to the requirement that the order of the first non-zero higher order derivative, if any, be the same (and have the same sign). When f and g are analytic, this is equivalent to requiring that the first non-zero term of their Taylor series expansions be of the same degree and have the same sign.

In any case, note that the order condition is purely a local condition. To see if f and g are equivalent one needs know the details of f and g only around equilibria.

To summarise

Theorem 8 *If two deterministic dynamics are sign-preserving transforms of one another and have the same order at zeroes, then they have the same equilibrium selection properties.*

This is an analogous result to that of Kandori et al. (1993) who show that the same equilibrium is selected in their model for any sign-preserving transform of the Darwinian dynamic. It is weaker in that one needs to restrict behaviour around zeroes and there is no need (or natural analogue) for this in the discrete space model they analyse. On the other hand it suggests, that the dependence on the deterministic dynamic is less in one-dimensional continuous space models than has been supposed.

In discrete space models the result does not generalise to higher dimensions as the sign of the dynamic is not enough to determine basins of attraction (see for example Ellison (2000), Hahn (1995)). Similarly here, even if every component of two vector fields have the same sign, they cannot in general be written as scalar multiples of one another. A sign-preserving transform of a given dynamic need not therefore be a time transform of the original.

Even for a single dynamic, different noise processes $P(x)$, may yield different selection results. The continuous framework yields state-dependent noise in the sense of Bergin and Lipman (1996) rather naturally. The point is simply that apparent differences between deterministic dynamics may actually be equivalent to applying different noise processes to the same dynamic.

8. Conclusion

This paper has presented a general approach to equilibrium selection in one-dimensional games with large populations. The birth-death approach has many virtues, including simplicity. It leaves open, however, the question of the generality of the results obtained. The current approach allows one to assess this. It requires more technical apparatus but the end results are simple and transparent.

Appendix

Proof of Lemma 1

The assertion on the number of positive roots follows easily from Descartes' rule of signs applied to the equivalent form (36). Let $y^* = U(x)$. Call the left-hand side of (20) h . The derivative of h at $y = 1$ equals $\sum_i i\lambda_i(x)$ and the assertion on the sign of $y^* - 1$ follows from this.

At a point where $\sum_i i\lambda_i(x) \neq 0$, the derivative of h at y^* must be non-zero, since the root is simple. Continuity in the coefficients, and so x , follows from the implicit function theorem. Consider a point x^* where $\sum_i i\lambda_i(x^*) = 0$. Since $\lambda_i(x)$ are continuous by Assumption 3, it follows from Assumption 4 that one can find a neighbourhood of x^* , \mathcal{N} , such that any coefficient of h which is non-zero at x^* is uniformly bounded away from zero in \mathcal{N} . Any coefficients zero at x^* are zero for all x in \mathcal{N} . It follows that one can find positive M and k such that if $y > M$, then $h > k$ and if $y < 1/M$, $h < -k$ in \mathcal{N} . It follows that $U(x) \in [1/M, M]$ for $x \in \mathcal{N}$. Considering convergent subsequences as x tends to x^* shows that $U(x)$ must tend to $U(x^*) = 1$.

Proof of Lemma 2

Lemmas C.9 and C.11 of Appendix C of Shwartz and Weiss (1995) justify the variational formula. The remainder is immediate.

Proof of Theorem 3

The proof of the Theorem follows that of Theorems 4.1 and 4.2 of Chapter 6 of Freidlin and Wentzell (1998). The necessary changes are sketched below.

The proof relies on a series of auxiliary Lemmas in Section 1 of Chapter 6 of Freidlin and Wentzell (1998). Lemma 1.1 follows from Lemmas 6.21 and 6.23 of Shwartz and Weiss (1995) using Assumptions 1–5. Lemma 1.2 follows immediately. Lemmas 1.3 and 1.4 are not needed here (and in case are true away from the boundary of $[0, 1]$). Lemmas 1.5 and 1.6 are immediate.

Lemmas 1.7 and 1.8, replacing ϵ^{-2} with n here and subsequently, hold for compacta away from the boundary. This is adequate as they are only needed in Theorem 4.1 for equilibrium compacta and by Assumption 6 these lie away from the boundary. Theorem 2 of this paper supplies the necessary large deviations estimates.

Lemma 1.9 and its Corollary also hold. The necessary estimate $P_x^n(\tau_K > kT) \leq \exp^{-n C k}$, for some C , follows from the argument of Lemma 6.32 of Shwartz and Weiss (1995) for odd integers. Note that this argument requires only Kurtz's Theorem (Theorem 1 of this paper), which holds even near the boundary.

Theorem 4.1 also requires estimate (2.3) from Lemma 2.1 of Chapter 6 of Freidlin and Wentzell (1998). This follows here with Theorem 2 providing the necessary large deviations estimates — note that only paths joining equilibria directly are considered, and these do not pass near the boundary. The necessary uniform continuity of the upper bound holds as the right-hand side of (14) is continuous in x for the closed sets required. This can be shown using Lemma 6.21 (extended in exercises 6.22 and 6.23)

and Lemma 6.25 of Shwartz and Weiss (1995) (cf. the proof of Lemma 6.36 of Shwartz and Weiss (1995)) since Assumptions 3–5 hold. The δ -perturbation of the boundary is irrelevant here. The boundaries of the sets indicated need to be replaced by thin bands owing to the discreteness of the state space, but this does not affect the proof.

Theorem 4.1 now follows. The tree-characterisation there is easily seen to be equivalent to the characterisation here, on account of the one-dimensional nature of the state. Theorem 4.2 is immediate.

Proof of Theorem 4

This follows from Theorems 6.15 and 6.17 of Shwartz and Weiss (1995), together with exercise 6.68 for the cases when $a = 0$ or $b = 1$. These results extend to allow (unstable) equilibrium points at a or b by a similar argument to that in Dembo and Zeitouni (1993) Corollary 5.7.16.

Proof of Theorem 5

The proof of the Theorem follows that of Theorem 5.3 of Chapter 6 in Freidlin and Wentzell (1998). Most of the necessary changes are given in the proof of Theorem 2, the remainder are sketched below. Let $G = [0, a)$.

Condition A of Freidlin and Wentzell (1998) is satisfied since any trajectory of (3) starting in the domain either leaves it and does not return or is attracted to one of the equilibrium points.

The necessary results from Lemmas 1.1 to 1.9 of Chapter 6 of Freidlin and Wentzell (1998) follow as in the proof of Theorem 3. Note that Lemma 1.4 is now required but holds as the boundary of G relative to $[0, 1]$ is a , $\neq 0, 1$.

Estimates (2.3) and (2.4) of Lemma 2.1 of Chapter 6 of Freidlin and Wentzell (1998) follow as in the proof of Theorem 3 (here using a δ -perturbation of the boundary where necessary). Estimates (2.8) and (2.9) of Lemma 2.2 also follow similarly. Note that for $x = 0$ or $x = 1$, the latter follow vacuously as the chain considered in the lemma will hit the nearest equilibrium point with probability 1 and cannot reach the boundary (unless there are no equilibria within G , in which case it hits the boundary with probability 1).

Theorem 5.3 now follows. The formula given by Freidlin and Wentzell (1998) implies the one given in this particular case, as can be seen as follows. A more detailed discussion can be found in Beggs (2002b).

The case when G contains no equilibria is immediate, so assume G contains equilibria $\{x_1, \dots, x_m\}$ with $x_1 < x_2 < \dots < x_m$. It is easy to see that the worst case for exit times in the formula in Freidlin and Wentzell (1998), and here, occurs when the process starts at an equilibrium, so concentrate on estimating the exit time starting from an equilibrium.

The formula in Freidlin and Wentzell (1998) implies that for any point x_i the logarithm of the expected time to exit G divided by N converges to

$$W_G - M_G(x_i) \tag{50}$$

where

$$W_G = \sum_{i=1}^{m+1} C(x_i, x_{i+1}) \quad (51)$$

with

$$C(x_i, x_j) = \begin{cases} (W(x_i) - W(x_j))^+ & j = i - 1, i + 1 \\ = +\infty & \text{otherwise} \end{cases} \quad (52)$$

and $x_{m+1} = a$. $()^+$ denotes the positive part. In other words W_G gives the cost of connecting all points in G to a .

$M_G(x_i)$ is the minimum cost of a graph without loops which connects x_i to some point $x_j \neq a$ (possibly itself), to be chosen, and all other points either to x_j or to a . All points to the left of x_j must be connected to x_j . Suppose that all points from $x_k > x_j$ are connected to a (if $x_k = a$ this means all points in G are connected to x_j).

For fixed x_j and x_k , (50) becomes

$$\sum_{l=j}^{k-2} C(x_l, x_{l+1}) - C(x_{l+1}, x_l) + C(x_{k-1}, x_k) \quad (53)$$

$$= W(x_j) - W(x_{k-1}) + (W(x_{k-1}) - W(x_k))^+ \quad (54)$$

using (52).

For given x_i , calculating $M_G(x_i)$ is equivalent to choosing x_j and $x_k > x_i$ to maximise this expression. An upper bound for all x_i is obtained by maximising it only subject to the constraint $x_k > x_j$. The bound is exact when x_i equals one of the maximising values of x_j . This yields the bounds in the text.

Proof of Lemma 4

The fact that g is C^1 guarantees the Lipschitz continuity assumptions. Assumption 7 will follow from the proof below and the other assumptions are immediate. A rest point of (3) is a point where $p(x) = x$.

$$p(x) = \frac{\exp(\beta g(\Delta\pi))}{1 + \exp(\beta g(\Delta\pi))} \quad (55)$$

and

$$\frac{dp}{dx} = A\beta g' p(1 - p) \quad (56)$$

where A is a constant.

Since $g(0) = 0$ and $g' > 0$ in a neighbourhood of 0, as $\beta \rightarrow \infty$, $p(0) \rightarrow 0$ and $p(1) \rightarrow 1$, while $p(x^*) = 1/2$. The same assumptions imply one can find a neighbourhood, \mathcal{N} , of x^* such that $g'(\Delta(\pi(x)))$ is bounded below by $k > 0$ and so $|g'(\Delta(\pi(x)))| > \epsilon$ for $x \notin \mathcal{N}$, some $\epsilon > 0$. Given $\eta > 0$ small enough, one can therefore find β^* such that for all $\beta \geq \beta^*$, for all $x \notin \mathcal{N}$, $\min\{p(x), 1 - p(x)\} < \eta$ and so, using (56), $dp/dx < 1$. Enlarging β^* if necessary, from (56), $dp/dx > 1$ for all $x \in \mathcal{N}$ for which $p = x$. The assertions on the number and properties of the equilibria follow easily from this.

$$W(x_1) - W(x_0) = - \int_{x_0}^{x_1} \ln \frac{\xi}{1-\xi} d\xi + \beta \int_{x_0}^{x_1} g(\Delta\pi(\xi)) d\xi \quad (57)$$

The first term is bounded and can be neglected for large β . For large enough β , x_0 and x_1 are close to 0 and 1 respectively. $g(\Delta\pi(x))$ is an odd function of $x - x^*$, since payoffs are linear in x (see Figure 3), and so for large β the second integral is positive if $x^* < 1/2$. It follows that for large enough β , $W(x_1) > W(x_0)$ and so x_1 is selected.

Proof of Theorem 7

As noted in Lemma 1, the derivative of the left-hand side of (20) at $y = 1$ is negative if $\sum_i i\lambda_i(x)$ is negative, as it is for $x > x^*$ in the basin of attraction of x^* . Since $y^* = U(x) > 1$, it follows that at y^* , the slope of the left-hand side of (20) is positive. The result follows from the remarks in the text.

References

- Amir, M. and Berninghaus, S. (1996). Another Approach to Learning and Mutation in Games. *Games and Economic Behavior*, 14:19–43.
- Beggs, A. (2002a). Stochastic evolution with slow learning. *Economic Theory*, 19:379–405.
- Beggs, A. (2002b). Waiting times and equilibrium selection. mimeo, Wadham College, Oxford.
- Bergin, J. and Lipman, B. (1996). Evolution with state-dependent mutations. *Econometrica*, 64:943–956.
- Binmore, K. and Samuelson, L. (1997). Muddling through: Noisy equilibrium selection. *Journal of Economic Theory*, 74:235–65.
- Binmore, K. and Samuelson, L. (1999). Evolutionary drift and equilibrium selection. *Review of Economic Studies*, 66:363–393.
- Binmore, K., Samuelson, L., and Vaughan, R. (1995). Musical chairs: Modelling noisy evolution. *Games and Economic Behavior*, 11:1–35.
- Blume, L. (1994). How noise matters. mimeo, Cornell University.
- Chicone, C. (1999). *Ordinary Differential Equations with Applications*. Springer-Verlag, New York.
- Dembo, A. and Zeitouni, O. (1993). *Large Deviations Techniques and Applications*. Jones and Bartlett, Boston, MA.
- Diamond, P. (1982). Aggregate demand in search equilibrium. *Journal of Political Economy*, 90:881–894.
- Diamond, P. and Rothschild, M. (1978). *Uncertainty in Economics*. Academic Press, New York.
- Dupuis, P. and Ellis, R. (1997). *A Weak Convergence Approach to the Theory of Large Deviations*. John Wiley, London.
- Ellison, G. (2000). Basins of attraction, long run equilibria and the speed of step-by-step evolution. *Review of Economic Studies*, 67:17–45.
- Ethier, S. and Kurtz, T. (1986). *Markov Processes: Characterization and Convergence*. John Wiley and Sons, New York.
- Foster, D. and Young, P. (1990). Stochastic evolutionary game dynamics. *Journal of Theoretical Biology*, 38:219–232.
- Freidlin, M. and Wentzell, A. (1998). *Random Perturbations of Dynamical Systems*. Springer Verlag, New York, second edition.

- Fudenberg, D. and Harris, C. (1992). Evolutionary dynamics with aggregate shocks. *Journal of Economic Theory*, 57:420–441.
- Fudenberg, D. and Kreps, D. (1993). Learning mixed equilibria. *Games and Economic Behavior*, 5:320–367.
- Hahn, S. (1995). The long run equilibrium in an asymmetric coordination game. mimeo, Harvard University.
- Kandori, M. (1999). Fluctuations and potential in search equilibrium. mimeo, University of Tokyo.
- Kandori, M., Mailath, G., and Rob, R. (1993). Learning, mutation and long-run equilibria in games. *Econometrica*, 61:29–56.
- Kelly, F. (1979). *Reversibility and Stochastic Networks*. John Wiley, New York.
- Kushner, H. (1982). A cautionary note on the use of singular perturbation methods for “small noise” models. *Stochastics*, 6:117–120.
- Rockafellar, R. (1970). *Convex Analysis*. Princeton University Press, Princeton, NJ.
- Sandholm, W. (1999). Evolution and equilibrium under inexact information. mimeo, University of Wisconsin.
- Sargent, T. (1999). *The Conquest of American Inflation*. Princeton University Press, Princeton, NJ.
- Schlag, K. (1999). Which one should i imitate? *Journal of Mathematical Economics*, 31:493–522.
- Shwartz, A. and Weiss, A. (1995). *Large Deviations for Performance Analysis*. Chapman and Hall, London.
- Weibull, J. (1995). *Evolutionary Game Theory*. MIT Press, Cambridge, MA.
- Wentzell, A. (1990). *Limit Theorems on Large Deviations for Markov Stochastic Processes*. Kluwer, Dordrecht, Holland.
- Wolff, R. (1989). *Stochastic Modeling and the Theory of Queues*. Prentice Hall, Englewood Cliffs, NJ.
- Young, P. (1993). The evolution of conventions. *Econometrica*, 61:57–84.

Figures

	1	2
1	8, 8	7, 0
2	0, 7	9, 9

Figure 1

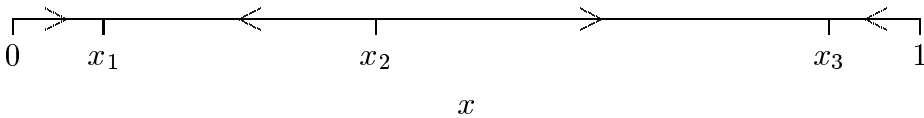


Figure 2

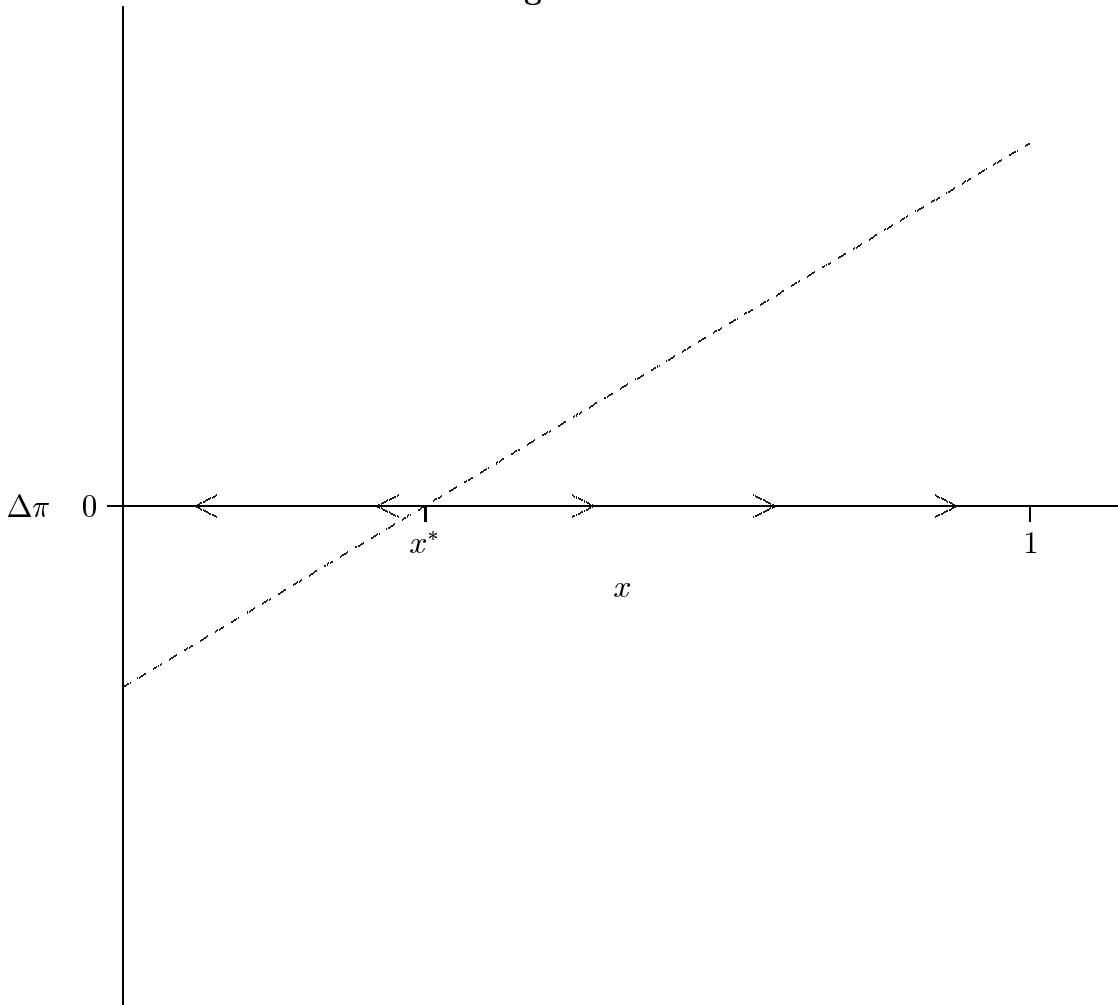


Figure 3