

# Moduli spaces of complexes of sheaves



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A thesis submitted for the degree of  
*Doctor of Philosophy*  
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# **Abstract**

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This thesis is on moduli spaces of complexes of sheaves and diagrams of such moduli spaces. The objects in these diagrams are constructed as geometric invariant theory quotients and the points in these quotients correspond to certain equivalence classes of complexes. The morphisms in these diagrams are constructed by taking direct sums with acyclic complexes. We then study the colimit of such a diagram and in particular are interested in studying the images of quasi-isomorphic complexes in the colimit.

As part of this thesis we construct categorical quotients of a group action on unstable strata appearing in a stratification associated to a complex projective scheme with a reductive group action linearised by an ample line bundle. We study this stratification for a closed subscheme of a quot scheme parametrising quotient sheaves over a complex projective scheme and relate the Harder–Narasimhan types of unstable sheaves with the unstable strata in the associated stratification. We also study the stratification of a parameter space for complexes with respect to a linearisation determined by certain stability parameters and show that a similar result holds in this case.

The objects in these diagrams are indexed by different Harder–Narasimhan types for complexes and are quotients of parameter schemes for complexes of this fixed Harder–Narasimhan type. This quotient is given by a choice of linearisation of the action and so the diagrams depend on these choices. We conjecture that these choices can be made so that for any quasi-isomorphism between complexes representing points in this diagram both complexes are identified in the colimit of this diagram.



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## Statement of Originality

This thesis contains no material that has already been accepted, or is concurrently being submitted, for any degree or diploma or certificate or other qualification in this University or elsewhere. To the best of my knowledge and belief this thesis contains no material previously published or written by another person, except where due reference is made in the text or commonly known.

Victoria Hoskins

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## 0.1 Introduction

The initial idea behind this thesis was to construct a diagram in the category of schemes whose objects are moduli spaces of  $S$ -equivalence classes of bounded complexes of coherent algebraic sheaves over a fixed base scheme and whose morphisms are given by taking direct sums with acyclic complexes. The idea was then to relate the notion of equivalence in the colimit of this diagram with quasi-isomorphisms.

The objects in this diagram are constructed using geometric invariant theory (GIT). We give only a brief summary of GIT now as greater details can be found in Chapter 1. To construct a quotient of a scheme  $Y$  by the action of a group scheme  $G$  in algebraic geometry is more complicated than just taking the topological quotient; we need to ensure that the quotient is a scheme and the quotient map is a morphism of schemes. Mumford's GIT provides a way to obtain such a quotient scheme and morphism under certain conditions. If the group is reductive and we have a linearisation of the  $G$ -action (i.e. a lift of the  $G$ -action to a line bundle  $L$  over  $Y$ ), then there is a GIT quotient scheme  $Y//_L G$  which is a categorical quotient of an open subscheme  $Y^{ss}(L)$  of semistable points in  $Y$ . The points in the complement of the semistable set are known as unstable points and must be removed to obtain a reasonable quotient. If the linearisation is ample, then topologically the GIT quotient is the set of semistable points modulo  $S$ -equivalence:  $y$  and  $y'$  are  $S$ -equivalent if their orbit closures meet in the semistable set.

Many moduli spaces in algebraic geometry are constructed as GIT quotients  $Y//G$  where  $Y$  is a parameter space for some collection of algebro-geometric objects and the  $G$ -orbits correspond to equivalence classes of these objects. For example, the moduli space of semistable sheaves (in the sense of Gieseker [13]) with fixed Hilbert polynomial is constructed as a GIT quotient of a closed subscheme  $\overline{Q}$  of a quot scheme by the action of a special linear group [13, 31, 40].

An important GIT construction for us is the moduli spaces of semistable quiver sheaves over a smooth integral complex projective scheme  $X$  due to Schmitt [39]. A quiver sheaf is a representation of a quiver in the category of coherent algebraic sheaves, although Schmitt only deals with representations in the subcategory of torsion free sheaves. A bounded complex of coherent algebraic sheaves is a representation of a quiver

$$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet$$

with relations imposed so that the composition of two consecutive morphisms is zero. From now on we use the term complex to mean a bounded complex of torsion free sheaves over  $X$ . As with any moduli problem we fix invariants  $\underline{P}$ , a Hilbert polynomial for each sheaf in the complex, and only parametrise semistable complexes. The notion of semistability for complexes due to Schmitt is determined by a collection of stability parameters which are used to define a weighted analogue of the reduced Hilbert polynomial. A complex is semistable if every nonzero proper subcomplex has weighted reduced Hilbert polynomial less than or equal to that of the complex. There is a parameter space  $\mathfrak{T} = \mathfrak{T}_{\underline{P}}$  for the semistable complexes and a reductive group  $G = G_{\underline{P}}$  acts on this parameter space such that the orbits correspond to isomorphism classes of complexes. Schmitt uses the stability parameters to determine an ample linearisation of the  $G$ -action such that GIT semistability corresponds to semistability as defined above and then takes the GIT quotient to construct moduli spaces of S-equivalence classes of complexes.

The moduli spaces of complexes constructed by Schmitt are not quite suitable for our purposes as complexes with wildly different cohomology can be identified in these moduli spaces. In an attempt to rectify this situation we instead take a quotient of a subscheme of the parameter scheme  $\mathfrak{T}$  such that all complexes parametrised by this subscheme have cohomology sheaves of a certain type. This statement will be made

more precise later on in this introduction, but first we discuss work on stratifications of schemes with group actions due to Hesselink [17], Kempf and Ness [22] and Kirwan [24].

Let  $Y$  be a nonsingular complex projective variety with an action by a reductive group  $G$  and  $L$  be an ample linearisation of this action. Associated to this action and linearisation there is a stratification

$$Y = \bigsqcup_{\beta \in \mathcal{B}} S_{\beta}$$

into  $G$ -invariant locally closed subvarieties such that the open stratum is the GIT semistable subvariety with respect to  $L$  (see [17, 22, 24]). The GIT quotient provides a categorical quotient of the  $G$ -action on the semistable subscheme and we consider the problem of constructing quotients of the  $G$ -action on the unstable strata. In this thesis we find a canonical linearisation on a projective completion of each stratum which can be used to obtain a categorical quotient of the  $G$ -action on that stratum. Unfortunately, the categorical quotient has the disadvantage that in general there are no stable points and so too many orbits are collapsed in this quotient. Instead we suggest a perturbation of the canonical linearisation should be used to construct a categorical quotient of an open subscheme of the unstable strata. We also note that this can all be extended to a projective scheme  $Y$  which is  $G$ -equivariantly embedded in an ambient projective space by intersecting the stratification of the projective space with  $Y$ .

We are ultimately interested in understanding the strata occurring in the stratification of the parameter space  $\mathfrak{T}$  for complexes associated to the linearisation for a suitable choice of Schmitt stability parameters. As a first step in this direction we study the stratification of the parameter scheme  $\overline{Q}$  for sheaves over a complex projective scheme.

Recall that every pure sheaf  $\mathcal{E}$  has a unique filtration by subsheaves

$$0 = \mathcal{E}^{(0)} \subsetneq \mathcal{E}^{(1)} \subsetneq \mathcal{E}^{(2)} \subsetneq \dots \subsetneq \mathcal{E}^{(s)} = \mathcal{E}$$

such that the successive quotients  $\mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$  are semistable with decreasing reduced Hilbert polynomial. This filtration is called the Harder–Narasimhan filtration of  $\mathcal{E}$  and the Harder–Narasimhan type of  $\mathcal{E}$  is the tuple of Hilbert polynomials of the successive quotients  $\mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$ .

Atiyah and Bott used Yang–Mills theory to stratify an infinite dimensional affine space of unitary connections on a fixed smooth vector bundle over a Riemann surface  $\Sigma$  and calculate the Betti numbers of moduli spaces of vector bundles over  $\Sigma$ . The stratification they obtained was a stratification by Harder–Narasimhan types of vector bundles [3].

This gives us reason to believe that the strata in the stratification of  $\overline{Q}$  should also be related to Harder–Narasimhan types. By using the explicit algebraic descriptions of the strata given in [24], we are able to show that this is the case. More precisely, we prove that for a fixed nontrivial Harder–Narasimhan type the construction of the parameter space for sheaves and its linearisation can be chosen so that all sheaves with this Harder–Narasimhan type are parametrised by a subscheme of  $\overline{Q}$  and that this subscheme is a union of connected components of an unstable stratum in the stratification of  $\overline{Q}$ .

Before we can start to discuss the stratification we will consider for complexes we need to choose a stability condition which is used to determine this stratification. For any choice of stability parameters a complex of torsion free sheaves has a Harder–Narasimhan filtration with respect to these parameters. We find a collection of stability parameters indexed by a small positive rational parameter  $\epsilon$  such that the Harder–Narasimhan filtration of a complex  $\mathcal{F}$  for  $\epsilon$  sufficiently small is given by

refining the filtration

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 \cdots & \rightarrow & \mathcal{F}_{i-1} & \rightarrow & \operatorname{Im} d_{i-1} & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \cdots \\
 & & \cap & & \cap & & \cap & & \cap & & \\
 \cdots & \rightarrow & \mathcal{F}_{i-1} & \rightarrow & \ker d_i & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \cdots \\
 & & \cap & & \cap & & \cap & & \cap & & \\
 \cdots & \rightarrow & \mathcal{F}_{i-1} & \rightarrow & \mathcal{F}_i & \rightarrow & \operatorname{Im} d_i & \rightarrow & 0 & \rightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & 
 \end{array}$$

by the Harder–Narasimhan filtrations of the cohomology sheaves  $\mathcal{H}^i(\mathcal{F})$  and the Harder–Narasimhan filtrations of the image sheaves  $\operatorname{Im} d_i$ . Therefore, the Harder–Narasimhan type is given by  $\tau = (\underline{H}, \underline{I})$  where  $\underline{H}$  denotes the Harder–Narasimhan types of the cohomology sheaves and  $\underline{I}$  denotes the Harder–Narasimhan types of the images. If two complexes have the same Harder–Narasimhan type with respect to these conditions for  $\epsilon$  sufficiently small, then their cohomology sheaves have the same Harder–Narasimhan types.

We use the stability conditions described above to obtain a stratification of a projective completion  $\overline{\mathfrak{T}}$  of the parameter space for complexes. For a nontrivial Harder–Narasimhan type  $\tau$  we show that the set up can be chosen so that there is a subscheme  $R_\tau$  of the parameter space  $\mathfrak{T}$  which parametrises complexes with this Harder–Narasimhan type  $\tau$  and that this subscheme is the intersection of the parameter space with a union of connected components of an unstable stratum. The subscheme  $R_\tau$  is invariant under the action of  $G_{\underline{P}(\tau)}$  where  $\underline{P}(\tau)$  is the tuple of Hilbert polynomials of the sheaves occurring in any complex with Harder–Narasimhan type  $\tau$ .

The objects in our diagram of schemes should be quotients of the  $G_{\underline{P}(\tau)}$ -action on the subscheme  $R_\tau$  for different Harder–Narasimhan types  $\tau$ ; such a quotient is

given by a choice of linearisation. Recall that there is a canonical linearisation on each unstable stratum which gives a categorical quotient of the  $G_{\underline{P}(\tau)}$ -action on that stratum. In this case the quotient of  $R_\tau$  obtained from the canonical linearisation is isomorphic to the product of moduli spaces for the successive quotients in the Harder–Narasimhan filtrations of the cohomology and image sheaves. In particular, any complex  $\mathcal{E}$  with Harder–Narasimhan filtration

$$0 = \mathcal{E}^{(0)} \subsetneq \mathcal{E}^{(1)} \subsetneq \mathcal{E}^{(2)} \subsetneq \dots \subsetneq \mathcal{E}^{(s)} = \mathcal{E}.$$

of type  $\tau$  is identified with the graded object  $\oplus \mathcal{E}^{(i)} / \mathcal{E}^{(i-1)}$  associated to this filtration in this quotient. In general this will not be desirable and so we will instead want to pick a perturbation  $\mathcal{L}_\tau^{\text{per}}$  of this canonical linearisation. We actually work with a slightly larger scheme  $\hat{R}_\tau$  than  $R_\tau$  whose points parametrise complexes with higher Harder–Narasimhan types than that specified by  $\tau$ .

Given some starting data (a choice of perturbed linearisation for each Harder–Narasimhan type) we construct a diagram in the category of schemes as follows. Firstly we fix the Harder–Narasimhan types  $\underline{H}$  of the cohomology sheaves. The objects in this diagram are indexed by Harder–Narasimhan types  $\tau = (\underline{H}, \underline{I})$  where  $\underline{I}$  is a collection of trivial Harder–Narasimhan types for the images. The object corresponding to  $\tau$  is the quotient  $M_\tau = \hat{R}_\tau //_{\mathcal{L}_\tau^{\text{per}}} G_{\underline{P}(\tau)}$ . For any complex  $\mathcal{A}$ , we can define a morphism of parameter spaces

$$\mathfrak{T}_{\underline{P}} \rightarrow \mathfrak{T}_{\underline{P} + \underline{P}(\mathcal{A})}$$

corresponding to taking the direct sum with this fixed complex. For a Harder–Narasimhan type  $\tau = (\underline{H}, \underline{I})$  this restricts to a morphism

$$\hat{R}_\tau \rightarrow \hat{R}_{\tau'}$$

where  $\tau' = \tau + \tau(\mathcal{A})$ . If  $\mathcal{A}$  is acyclic and this descends to a morphism

$$M_\tau = \hat{R}_\tau //_{\mathcal{L}_\tau^{\text{per}}} G_{\underline{P}(\tau)} \rightarrow M_{\tau'} = \hat{R}_{\tau'} //_{\mathcal{L}_{\tau'}^{\text{per}}} G_{\underline{P}(\tau')}$$

of the quotients, then this is a morphism in our diagram.

Finally, we consider the colimit of the above diagram. Suppose  $f : \mathcal{F} \rightarrow \mathcal{G}$  is a quasi-isomorphism between two complexes which represent points in this diagram. If the perturbed linearisations are chosen in a compatible way so that there are direct sum morphisms

$$M_{\tau(\mathcal{F})} \xrightarrow{\oplus \text{Cone}(f)} M'_\tau \quad \text{and} \quad M_{\tau(\mathcal{G})} \xrightarrow{\oplus \text{Cone}(\text{id}_{\mathcal{F}})} M'_\tau$$

in this diagram where  $\tau' = \tau(\mathcal{F}) + \tau(\text{Cone}(f)) = \tau(\mathcal{G}) + \tau(\text{Cone}(\text{id}_{\mathcal{F}}))$ , then we prove that  $\mathcal{F}$  and  $\mathcal{G}$  are identified in the colimit of this diagram. We end by conjecturing that one can choose the starting data so that this is possible for any such quasi-isomorphism.

The layout of this thesis is as follows:

In Chapter 1 we review some general facts about geometric invariant theory. We describe the standard constructions of affine and projective quotients, but also describe a way to generalise these approaches in Section 1.6.

In Chapter 2 we describe a stratification of a complex projective scheme with a reductive group action linearised by an ample line bundle (see [17, 22, 24]). We find a canonical linearisation on a compactification of each unstable stratum which gives a categorical quotient of the group action on that stratum (Proposition 2.2.8).

In Chapter 3 we apply the ideas of Chapter 2 to a closed subscheme  $\overline{Q}$  of a quotient scheme parametrisng quotient sheaves with fixed Hilbert polynomial over a complex projective scheme. For a nontrivial Harder–Narasimhan type we show that there is a scheme which parametrises all sheaves with this fixed Harder–Narasimhan type and that this scheme is a union of connected components of an unstable stratum in

the associated stratification of  $\overline{Q}$  (Theorem 3.5.7). An earlier version of some of the material in Chapters 2 and 3 was included in [18].

Chapter 4 is mostly a review of a construction due to Schmitt [39] for moduli spaces of S-equivalence classes of complexes of semistable complexes. In Section 4.2 we calculate the fixed points and weights of  $\mathbb{C}^*$ -actions on the parameter space from which this moduli space is constructed.

In Chapter 5 we study properties of the semistability conditions for complexes. We show that there is a collection of stability conditions indexed by a rational parameter  $0 < \epsilon \ll 1$  such that the Harder–Narasimhan filtration of a complex with respect to these conditions for  $\epsilon$  sufficiently small tells us about the cohomology sheaves of this complex (Proposition 5.3.7).

In Chapter 6 we study the stratifications of a projective completion  $\overline{\mathfrak{T}}$  of the parameter space for complexes with respect to the linearisations determined by the collection of stability parameters studied in Chapter 5. We show there is a scheme parametrising complexes of a fixed nontrivial Harder–Narasimhan type which is the intersection of the parameter scheme with a union of connected components of an unstable stratum (Theorem 6.4.9).

In Chapter 7 we consider diagrams in the category of schemes which are constructed from some starting data. The objects are moduli spaces of S-equivalence classes of complexes and the morphisms are given by taking direct sums with acyclic complexes. We conjecture that this starting data can be chosen so for any quasi-isomorphism between complexes representing points in this diagram both complexes represent the same point in the colimit of this diagram.

## 0.2 Notation and conventions

We use the word scheme (respectively variety) to mean algebraic scheme (respectively algebraic variety) and the word sheaf to mean coherent algebraic sheaf unless otherwise stated. The term point is used to mean geometric point unless otherwise stated.

For a locally free sheaf  $\mathcal{E}$  on a scheme  $X$ , we define the dual of  $\mathcal{E}$  to be the locally free sheaf

$$\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X).$$

If  $\mathcal{E}$  is a locally free sheaf on a scheme  $X$ , then we define the projective bundle associated to  $\mathcal{E}$  to be

$$\mathbb{P}(\mathcal{E}) := \text{Proj Sym}(\mathcal{E}^\vee).$$

If  $X$  is a scheme over a field  $k$ , then the points of  $\mathbb{P}(\mathcal{E})$  correspond to punctured lines through the origin in the fibres of  $\mathcal{E}$ . If we view a finite dimensional  $k$ -vector space  $V$  as a locally free sheaf over the point  $\text{Spec } k$  then

$$\mathbb{P}(V) = (V - \{0\})/\mathbb{G}_m$$

where  $\mathbb{G}_m := \text{Spec } k[t, t^{-1}]$  is the multiplicative group scheme.

If  $X \rightarrow Y$  and  $Z \rightarrow Y$  are morphisms of schemes, then  $X \times_Y Z$  denotes the fibre product of these schemes. Whereas if  $G$  is a group scheme and  $H \subset G$  is a subgroup acting on a scheme  $X$ , then we let  $G \times_H X$  denote the quotient of  $G \times X$  by the diagonal action of  $H$  on the right on  $G$  and on the left on  $X$ .

# Chapter 1

## Geometric invariant theory

In this chapter we review standard results and constructions in GIT and moduli theory to establish notation and conventions. The standard reference for GIT is [34] and for a more elementary introduction see [33, 36, 41].

### 1.1 Moduli problems

Given a set of algebro-geometric objects  $\mathcal{A}$  with an equivalence relation  $\sim$  on  $\mathcal{A}$  the aim of moduli theory is to provide a more geometric description of the set of equivalence classes  $\mathcal{A}/\sim$ . Recall that a moduli problem consists of more than just the set  $\mathcal{A}$  and equivalence relation  $\sim$ ; it requires the notion of families of objects in  $\mathcal{A}$  parametrised by a base scheme  $S$  and a lift of the equivalence relation to families. We also require that

- 1) A family parametrised by a point is an object in  $\mathcal{A}$  and the equivalence relation over a point is just  $\sim$ ,
- 2) Given a family  $X$  parametrised by a scheme  $S$  and a morphism of schemes  $\varphi : T \rightarrow S$ , there is a pullback family  $\varphi^*X$  parametrised by  $T$ .

This should all be functorial, so that a moduli problem is given by a contravariant functor

$$\mathcal{M} : \mathfrak{Sch} \longrightarrow \mathfrak{Sets}$$

$$S \mapsto \{\text{families parametrised by } S\} / \sim$$

which is often referred to as the moduli functor. Recall the following definitions of fine and coarse moduli spaces.

**Definition 1.1.1.** A scheme  $M$  is a fine moduli space if it represents a moduli functor  $\mathcal{M}$ ; that is,  $\mathcal{M}$  is isomorphic to the functor of points  $\underline{M} = \text{Hom}(-, M)$  associated to  $M$ .

This situation is ideal since we have a scheme structure on the set of equivalence classes  $\mathcal{A}/\sim$  and a universal family  $U$  parametrised by  $M$  which corresponds to the identity morphism on  $M$ . This family is universal in the sense that for any family  $X$  parametrised by  $S$  there is a unique map  $\varphi : S \rightarrow M$  such that  $\varphi^*U \sim X$ . Unfortunately moduli functors are not always representable and so we may instead look for a scheme which best approximates this.

**Definition 1.1.2.** A functor  $\mathcal{M} : \mathfrak{Sch} \rightarrow \mathfrak{Sets}$  is corepresented by a scheme  $M$  if there is a natural transformation  $\varphi : \mathcal{M} \rightarrow \underline{M}$  such that for all schemes  $Y$  and natural transformations  $h : \mathcal{M} \rightarrow \underline{Y}$  there is a unique morphism  $\phi : M \rightarrow Y$  such that  $h = \phi \circ \varphi$ . We say  $M$  is a coarse moduli space if it corepresents a moduli functor  $\mathcal{M}$  and for every algebraically closed field  $k$  the  $k$ -valued points of  $M$  are in bijection with the equivalence classes  $\mathcal{A}/\sim$ .

## 1.2 Construction of GIT quotients

We assume  $G$  is a linear algebraic group acting on a scheme  $X$  over an algebraically closed field  $k$  of characteristic zero.

**Definition 1.2.1.** A categorical quotient of  $X$  by  $G$  is a scheme  $Y$  and a  $G$ -invariant morphism  $\phi : X \rightarrow Y$  with the following universal property. For each scheme  $Z$  and  $G$ -invariant morphism  $\varphi : X \rightarrow Z$  there is a unique morphism  $\varphi' : Y \rightarrow Z$  such that  $\varphi = \varphi' \circ \phi$ . If in addition, for each point  $y \in Y$  the preimage  $\phi^{-1}(y)$  is a single orbit, then we call  $Y$  an orbit space.

Mumford's GIT allows us to construct categorical quotients of an open subset of  $X$  for actions by reductive groups. Recall that every linear algebraic group  $G$  has a unique maximal connected normal solvable subgroup called the radical of  $G$ .

**Definition 1.2.2.** A linear algebraic group  $G$  over  $k$  is reductive if its radical is an algebraic torus  $(\mathbb{G}_m)^n$ .

A useful property of reductive groups is the following theorem of Nagata:

**Theorem 1.2.3** (Nagata). *Let  $G$  be a reductive group acting on a finitely generated  $k$ -algebra  $R$ . Then the invariant subalgebra  $R^G$  is a finitely generated  $k$ -algebra.*

### 1.2.1 Affine quotients

Let  $G$  be a reductive group acting on an affine scheme  $X = \operatorname{Spec} R$  over  $k$ . This action induces an action of  $G$  on  $R$  and the inclusion  $R^G \rightarrow R$  induces a morphism  $\phi : X \rightarrow Y := \operatorname{Spec} R^G$ .

**Theorem 1.2.4.** ([34], Theorem 1.1) *Let  $G$  be a reductive group acting on an affine scheme  $X$ . Then the above morphism  $\phi : X \rightarrow Y$  is a categorical quotient.*

**Remark 1.2.5.** If  $X$  is an affine variety, then by Nagata's Theorem the invariant subalgebra  $R^G$  is a finitely generated  $k$ -algebra. Moreover, as  $R^G$  is a subalgebra of  $R$  it has no nilpotents and so the categorical quotient  $Y$  is also an affine variety.

### 1.2.2 Projective quotients

In order to construct a GIT quotient of a projective scheme  $X$  we need some extra data. This is given by a lift of the  $G$ -action to an ample invertible sheaf on  $X$ ; such a choice is called a linearisation of the action.

**Definition 1.2.6.** Let  $\pi : L \rightarrow X$  be a line bundle on  $X$ . A linearisation of the action of  $G$  with respect to  $L$  is an action of  $G$  on  $L$  such that

- i) For all  $g \in G$  and  $l \in L$ , we have  $\pi(g \cdot l) = g \cdot \pi(l)$ ,
- ii) For all  $x \in X$  and  $g \in G$  the map of fibres  $L_x \rightarrow L_{g \cdot x}$  is a linear map.

The linearisation is often also denoted by  $L$ . We say a linearisation  $L$  is (very) ample if the invertible sheaf associated to  $L$  is (very) ample.

**Remark 1.2.7.** Let  $\mathcal{L}$  denote the invertible sheaf associated to the line bundle  $L$ ; then the notion of a linearisation of an action  $\sigma : G \times X \rightarrow X$  can be stated in terms of sheaves as follows. Let  $\pi_X : G \times X \rightarrow X$  denote the projection onto the second factor and  $\mu : G \times G \rightarrow G$  denote the group action. A linearisation of the action with respect to  $\mathcal{L}$  is an isomorphism

$$\Phi : \sigma^* \mathcal{L} \rightarrow \pi_X^* \mathcal{L}$$

which satisfies the cocycle condition:

$$(\mu \times \text{id}_X)^* \Phi = \pi_{23}^* \Phi \circ (\text{id}_G \times \sigma)^* \Phi$$

where  $\pi_{23} : G \times G \times X \rightarrow G \times X$  is the projection onto the last two factors (see [34], Definition 1.6).

**Example 1.2.8.** Let  $L = X \times k$  be the trivial line bundle on a scheme  $X$  over  $k$ ; then a linearisation of a  $G$ -action on  $X$  with respect to  $L$  corresponds to a character

$\chi : G \rightarrow \mathbb{G}_m$ . The character  $\chi$  defines a lift of the action to  $L$  by

$$g \cdot (x, z) = (g \cdot x, \chi^{-1}(g)z)$$

where  $(x, z) \in X \times k$ . More generally, we can use a character  $\chi : G \rightarrow \mathbb{G}_m$  to modify any given linearisation  $L$  of a  $G$ -action on  $X$  by defining

$$g \cdot (x, z) = (g \cdot x, \chi(g)g \cdot z)$$

for  $x \in X$  and  $z \in L_x$ .

**Example 1.2.9.** There is a natural linearisation of the  $\mathrm{SL}(n+1, k)$  action on  $\mathbb{P}^n = \mathbb{P}(k^{n+1})$  with respect to  $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$  such that the induced action of  $G$  on

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = (k^{n+1})^*$$

is dual to the natural  $\mathrm{SL}(n+1, k)$ -action on  $k^{n+1}$ .

The tensor product of two linearisations has a natural linearisation as does the inverse of the invertible sheaf associated to a given linearisation; hence there is a group of isomorphism classes of  $G$ -linearisations over  $X$  which is denoted  $\mathrm{Pic}^G(X)$ .

**Remark 1.2.10.** Suppose that  $X$  is a projective scheme and  $\mathcal{L}$  is a very ample linearisation. There is an induced action of  $G$  on  $H^0(X, \mathcal{L})$  such that the natural evaluation map

$$H^0(X, \mathcal{L}) \otimes_k \mathcal{O}_X \rightarrow \mathcal{L}$$

is  $G$ -equivariant and so induces a  $G$ -equivariant embedding

$$X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L})^*).$$

In this case we have a linear representation of  $G$  on  $H^0(X, \mathcal{L})^*$  and the action of  $G$  on  $X$  is induced by the action of  $G$  on  $\mathbb{P}(H^0(X, \mathcal{L})^*)$ .

Now suppose  $X$  is projective and the linearisation  $\mathcal{L}$  is ample. The linearisation induces an action of  $G$  on the space of sections  $H^0(X, \mathcal{L}^{\otimes r})$  and the invariant sections

$$\bigoplus_{r \geq 0} H^0(X, \mathcal{L}^{\otimes r})^G \subset \bigoplus_{r \geq 0} H^0(X, \mathcal{L}^{\otimes r})$$

form a graded subalgebra. In this case the GIT quotient of  $X$  by  $G$  with respect to the linearisation  $\mathcal{L}$  is the projective scheme associated to the graded algebra of invariant sections,

$$X//_{\mathcal{L}} G := \text{Proj} \bigoplus_{r \geq 0} H^0(X, \mathcal{L}^{\otimes r})^G.$$

The inclusion of the subalgebra of invariant sections induces a rational map  $X \dashrightarrow X//_{\mathcal{L}} G$ . This map is in fact a categorical quotient of the open subset of  $X$  on which this map is a morphism. This motivates the definitions of (semi)stability with respect to  $\mathcal{L}$ :

**Definition 1.2.11.** Let  $G$  be a reductive group acting on a projective scheme  $X$  and suppose we have an ample linearisation  $\mathcal{L}$  of the action.

- 1) A point  $x \in X$  is semistable (with respect to  $\mathcal{L}$ ) if there is an invariant section  $s \in H^0(X, \mathcal{L}^{\otimes r})^G$  for some  $r > 0$  such that  $s(x) \neq 0$ . The open subset of semistable points with respect to  $\mathcal{L}$  is denoted  $X^{ss}(\mathcal{L})$ .
- 2) A point  $x \in X$  is stable (with respect to  $\mathcal{L}$ ) if  $\dim G \cdot x = \dim G$  and there is an invariant section  $s \in H^0(X, \mathcal{L}^{\otimes r})^G$  for some  $r > 0$  such that  $s(x) \neq 0$  and the action of  $G$  on  $X_s := \{x \in X : s(x) \neq 0\}$  is closed. The open subset of stable points with respect to  $\mathcal{L}$  is denoted  $X^s(\mathcal{L})$ .
- 3) The points which are not semistable are called unstable.

**Theorem 1.2.12.** ([34] Theorem 1.10) *Let  $G$  be a reductive group acting on a projective scheme  $X$  and  $\mathcal{L}$  be an ample linearisation of this action. Then there is a projective GIT quotient*

$$X//_{\mathcal{L}}G = \text{Proj} \bigoplus_{r \geq 0} H^0(X, \mathcal{L}^{\otimes r})^G$$

*and an affine morphism*

$$\phi : X^{ss}(\mathcal{L}) \rightarrow X//_{\mathcal{L}}G$$

*which is a categorical quotient of  $X^{ss}(\mathcal{L})$  by  $G$ . Furthermore, there is an open subset  $Y^s \subset X//_{\mathcal{L}}G$  such that  $\phi^{-1}(Y^s) = X^s(\mathcal{L})$  and  $\phi : X^s(\mathcal{L}) \rightarrow Y^s$  is an orbit space for the  $G$ -action on  $X^s(\mathcal{L})$ .*

We say two semistable points  $x$  and  $x'$  are S-equivalent if the orbit closures of  $x$  and  $x'$  meet in the semistable subset  $X^{ss}(\mathcal{L})$ . Then, topologically, the GIT quotient is the semistable set modulo S-equivalence:

**Proposition 1.2.13.** *Let  $x$  and  $x'$  be points in  $X^{ss}(\mathcal{L})$  such that  $\phi(x) = \phi(x')$  then  $x$  and  $x'$  are S-equivalent.*

### 1.3 The Hilbert–Mumford criterion

To determine from first principles whether a point is stable, semistable or unstable can be difficult, but Mumford developed a numerical criterion to determine (semi)stability known as the Hilbert–Mumford criterion. This criterion involves checking (semi)stability with respect to every one-parameter subgroup (1-PS) of  $G$ ; that is, every nontrivial homomorphism  $\lambda : \mathbb{G}_m \rightarrow G$ . In this section we follow the original conventions of Mumford.

Suppose that  $X$  is embedded in  $\mathbb{P}^n$  and the action is given by a representation  $\rho : G \rightarrow \text{GL}(n+1)$ . A 1-PS  $\lambda : \mathbb{G}_m \rightarrow G$  induces a linear action of  $\mathbb{G}_m$  on  $\mathbb{A}^{n+1}$  which

we can diagonalise by choosing a basis  $e_0, \dots, e_n$  of  $\mathbb{A}^{n+1}$  so that

$$\lambda(t) \cdot e_i = t^{r_i} e_i \quad \text{for } r_i \in \mathbb{Z}.$$

We call the  $r_i$  the  $\lambda$ -weights of this action on  $\mathbb{A}^{n+1}$ . For  $x \in X$  we can pick  $\hat{x} \in \mathbb{A}^{n+1} - \{0\}$  lying above this point and write  $\hat{x} = \sum_{i=0}^n \hat{x}_i e_i$ . The action of  $\lambda$  on  $x$  is given by

$$\lambda(t) \cdot \hat{x} = \sum_{i=0}^n t^{r_i} \hat{x}_i e_i.$$

We define the Hilbert–Mumford function  $\mu$  of  $x$  at  $\lambda$  by

$$\mu(x, \lambda) := \max\{-r_i : \hat{x}_i \neq 0\}.$$

More generally, suppose  $X$  is a projective scheme with a  $G$ -action and linearisation  $L$ . If  $\lambda$  is a 1-PS of  $G$  and  $x \in X$ , then the limit

$$\bar{x} = \lim_{t \rightarrow 0} \lambda(t) \cdot x$$

exists in  $X$  as  $X$  is proper. The limit  $\bar{x}$  is fixed by the  $\mathbb{G}_m$  action induced by  $\lambda$  and so  $\mathbb{G}_m$  acts on the fibre  $L_{\bar{x}}$  by a character  $t \mapsto t^r$ . In this case we define

$$\mu^L(x, \lambda) := r$$

**Remark 1.3.1.** These definitions are consistent when  $X$  is a projective scheme and  $L$  is a very ample linearisation. The linearisation gives rise to a  $G$ -equivariant embedding  $X \hookrightarrow \mathbb{P}(H^0(X, L)^*)$  and the action is given by a linear representation of  $G$  on  $H^0(X, L)^*$ . The action of a 1-PS  $\lambda$  on  $H^0(X, L)^*$  is diagonalisable; therefore, there is a basis  $s_0^*, \dots, s_n^*$  of  $H^0(X, L)^*$  and integers  $r_0, \dots, r_n$  such that  $\lambda(t) \cdot s_i^* = t^{r_i} s_i^*$ .

Then

$$\mu^L(x, \lambda) := \max\{-r_i : \hat{x}_i \neq 0\} = -\min\{r_i : \hat{x}_i \neq 0\}$$

is equal to the weight of the character acting on the fibre of  $L$  over the fixed point  $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ .

**Theorem 1.3.2.** (*Hilbert–Mumford Criterion, [34] Theorem 2.1*) *Let  $G$  be a reductive group scheme acting on a projective scheme  $X$  with respect to an ample linearisation  $L$ . Then*

$$\begin{aligned} x \in X^{ss}(L) &\iff \mu^L(x, \lambda) \geq 0 \text{ for all 1-PSs } \lambda \text{ of } G, \\ x \in X^s(X) &\iff \mu^L(x, \lambda) > 0 \text{ for all 1-PSs } \lambda \text{ of } G. \end{aligned}$$

If we are working over the complex numbers, then we can use the representation theory of  $G$  to define the quantity  $\mu^L(x, \lambda)$ . Suppose that  $L$  is very ample so that we have a  $G$ -equivariant embedding  $X \hookrightarrow \mathbb{P}(V)$  where  $V = H^0(X, L)^*$ . For a fixed maximal torus  $T$  of  $G$  we get a  $T$ -weight space decomposition of  $V$

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi$$

where  $V_\chi = \{v \in V : g \cdot v = \chi(g)v\}$  and  $X^*(T) = \text{Hom}(T, \mathbb{C}^*)$ . The characters  $\chi$  which index nonzero  $V_\chi$  are called the  $T$ -weights of  $V$ . For  $x \in X$  we can choose  $v \in V$  lying over this point and write  $v = \sum v_\chi$ . We define the  $T$ -state of  $x$  to be the set

$$\text{st}_T(x) := \{\chi : v_\chi \neq 0\}$$

and let  $\text{conv}_T(x)$  denote the convex hull of  $\text{st}_T(x)$  in  $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ . If we consider instead the action of  $T$  on  $X$  and the associated linearisation  $L_T$  then for any 1-PS  $\lambda$  of  $T$

$$\mu^{L_T}(x, \lambda) = \max\{-\langle \lambda, \chi \rangle : \chi \in \text{st}_T(x)\}$$

where  $\langle \lambda, \chi \rangle$  denotes the integer defined by  $\chi \circ \lambda(t) = t^{\langle \lambda, \chi \rangle}$ . Therefore we have a geometric condition for  $T$ -(semi)stability of  $x$ :

$$x \in X^{ss}(L_T) \iff 0 \in \text{conv}_T(x),$$

$$x \in X^s(L_T) \iff 0 \in \text{conv}_T(x)^\circ.$$

The (semi)stability of  $x$  under the maximal tori of  $G$  tells us the (semi)stability of  $x$  under  $G$  because the 1-PSs of  $G$  correspond to the union of the 1-PSs of all maximal tori of  $G$ . Therefore,

$$X^{ss}(L) = \bigcap_{T \subset G} X^{ss}(L_T).$$

As any two maximal tori are conjugate and

$$\mu^L(g \cdot x, \lambda) = \mu^L(x, g^{-1} \lambda g)$$

(see [34], §2.1) we have:

**Corollary 1.3.3.** (*Hilbert-Mumford criterion*)

$$x \in X^{ss}(L) \iff 0 \in \text{conv}_T(g \cdot x) \text{ for all } g \in G,$$

$$x \in X^s(L) \iff 0 \in \text{conv}_T(g \cdot x)^\circ \text{ for all } g \in G.$$

## 1.4 Instability in GIT

Let  $G$  be a reductive group scheme acting on a projective scheme  $X$  and suppose  $L$  is an ample linearisation. If  $x \in X - X^{ss}(L)$  is unstable, then, by the Hilbert–Mumford criterion, there exists a 1-PS  $\lambda$  such that  $\mu^L(x, \lambda) < 0$ . In this section we describe work of Kempf [21] on a natural class of one-parameter subgroups associated to such

an unstable point.

Let  $T$  be a maximal torus of  $G$  and  $W := N_G(T)/T$  be the Weyl group. We denote the sets of 1-PS and characters of  $G$  by

$$X_*(G) := \text{Hom}(\mathbb{G}_m, G)$$

$$X^*(G) := \text{Hom}(G, \mathbb{G}_m).$$

Let  $\langle, \rangle$  be a  $W$ -invariant inner product on  $X_*(T)_{\mathbb{R}} := X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ . For any 1-PS  $\lambda$  of  $G$  there is a  $g \in G$  such that  $g^{-1}\lambda g$  is a 1-PS of  $T$ . Therefore, we define the norm of the 1-PS  $\lambda$  to be the norm of  $g^{-1}\lambda g$  viewed as an element of  $X_*(T)_{\mathbb{R}}$

$$\|\lambda\| := \sqrt{\langle g^{-1}\lambda g, g^{-1}\lambda g \rangle}.$$

Consider the function on  $X$  defined by

$$M^L(x) := \inf_{\lambda \in X_*(G)} \frac{\mu^L(x, \lambda)}{\|\lambda\|}.$$

By [34], Proposition 2.17 this function takes finite values as  $L$  is ample. The Hilbert–Mumford criterion can be rephrased as

$$x \in X^{ss}(L) \iff M^L(x) \geq 0,$$

$$x \in X^s(L) \iff M^L(x) > 0.$$

**Definition 1.4.1.** If  $x$  is unstable with respect to  $L$ , then  $M^L(x) < 0$ . Any 1-PS  $\lambda$  which achieves this minimum

$$\frac{\mu^L(x, \lambda)}{\|\lambda\|} = M^L(x)$$

is said to be adapted to  $x$ . We let  $\wedge^L(x)$  denote the set of non-divisible 1-PSs which are adapted to  $x$ .

**Definition 1.4.2.** For any 1-PS  $\lambda$  of  $G$  we define a parabolic subgroup

$$P(\lambda) := \left\{ g \in G : \lim_{t \rightarrow 0} \lambda(t)g\lambda(t^{-1}) \text{ exists in } G \right\}$$

of  $G$ .

The relationship between adapted 1-PSs and their associated parabolic is given by the following theorem of Kempf.

**Theorem 1.4.3.** ([21] Theorem 6) *Let  $G$  be a reductive group scheme acting on a projective scheme  $X$  and suppose  $L$  is an ample linearisation. If  $x \in X - X^{ss}(L)$  is an unstable point, then*

- 1)  $\wedge^L(x)$  is nonempty.
- 2) There is a parabolic subgroup  $P_x^L$  such that  $P_x^L = P(\lambda)$  for all  $\lambda \in \wedge^L(x)$ .
- 3) All elements of  $\wedge^L(x)$  are conjugate to each other by elements of  $P_x^L$ .

## 1.5 GIT for non-ample linearisations

In the above sections we were working with an ample linearisation of a reductive group action on a scheme. In Mumford's book on GIT [34] the case when the line bundle is non-ample is discussed although most of the results use ample linearisations; for example, we need an ample linearisation for useful results like the Hilbert–Mumford criterion. In this section we give a quick survey of the construction of GIT quotients for arbitrary linearisations.

**Definition 1.5.1.** Let  $X$  be a scheme over  $k$  with an action by a reductive group  $G$  and  $\mathcal{L}$  be a linearisation of this action.

- 1) A point  $x \in X$  is semistable (with respect to  $\mathcal{L}$ ) if there is an invariant section  $s \in H^0(X, \mathcal{L}^{\otimes r})^G$  for some  $r \geq 0$  such that  $s(x) \neq 0$  and  $X_s = \{x \in X : s(x) \neq 0\}$  is affine. The open subset of semistable points with respect to  $\mathcal{L}$  is denoted  $X^{ss}(\mathcal{L})$ .
- 2) A point  $x \in X$  is stable (with respect to  $\mathcal{L}$ ) if  $\dim G \cdot x = \dim G$  and there is an invariant section  $s \in H^0(X, \mathcal{L}^{\otimes r})^G$  for some  $r \geq 0$  such that  $s(x) \neq 0$  and  $X_s$  is affine and the action of  $G$  on  $X_s$  is closed. The open subset of stable points with respect to  $\mathcal{L}$  is denoted  $X^s(\mathcal{L})$ .
- 3) The points which are not semistable are called unstable.

**Remark 1.5.2.** If  $X$  is projective and  $\mathcal{L}$  is ample, then this agrees with Definition 1.2.11 as  $X_s$  is affine for any non-constant section.

In this case the GIT quotient is constructed by covering  $X$  by affine schemes and gluing the GIT quotients of these affine schemes.

**Theorem 1.5.3.** ([34] Theorem 1.10) *Let  $G$  be a reductive group acting on a scheme  $X$  over  $k$  and  $\mathcal{L}$  be a linearisation of this action. Then there is a quasi-projective scheme  $X//_{\mathcal{L}}G$  and an affine morphism  $\phi : X^{ss}(\mathcal{L}) \rightarrow X//_{\mathcal{L}}G$  which is a categorical quotient of the  $G$ -action on  $X^{ss}(\mathcal{L})$ . Furthermore, there is an open subset  $Y^s \subset X//_{\mathcal{L}}G$  such that  $\phi^{-1}(Y^s) = X^s(\mathcal{L})$  and  $\phi : X^s(\mathcal{L}) \rightarrow Y^s$  is an orbit space for the  $G$ -action on  $X^s(\mathcal{L})$ .*

## 1.6 Generalising GIT

So far in this chapter we have mostly focused on reductive group actions on projective schemes with respect to ample linearisations. In this situation GIT works best, but sometimes we may want to do GIT when we have either a non-reductive group or a non-ample linearisation of the action. In Section 1.5 above we discussed the case

when the linearisation is non-ample. Work of Doran and Kirwan [10] and others gives constructions for non-reductive groups, but here we take a simplified point of view which suffices for this thesis.

In this section we describe a method for constructing quotients in the following set up. Suppose we have a quasi-projective scheme  $X$  with an action by a linear algebraic group  $G$  and a  $G$ -equivariant projective completion  $\overline{X}$ . Given a linearisation  $\mathcal{L}$  of the  $G$ -action on  $\overline{X}$  and a finitely generated graded subalgebra

$$R \subset \bigoplus_{r \geq 0} H^0(\overline{X}, \mathcal{L}^{\otimes r})^G$$

of the algebra of invariant sections we define a generalised GIT quotient with respect to  $(\mathcal{L}, R)$  as follows. There is a rational morphism

$$\overline{X} \dashrightarrow \text{Proj } R$$

which restricts to a morphism on the open subscheme  $\overline{X}^{nss}$  of naively semistable points; that is, the points  $x \in \overline{X}$  for which there is a section  $\sigma \in R_+ := \bigoplus_{r > 0} R_r$  such that  $\sigma(x) \neq 0$ . Let  $\overline{X} //_{(\mathcal{L}, R)} G$  denote the scheme theoretic image of the morphism

$$\overline{X}^{nss} \rightarrow \text{Proj } R.$$

The generalised GIT quotient of the  $G$ -action on  $X$  with respect to  $(\mathcal{L}, R)$  is given by

$$X^{nss} \rightarrow \overline{X} //_{(\mathcal{L}, R)} G.$$

**Remark 1.6.1.** Note that:

- 1) In the case when  $X$  is projective and  $G$  is reductive and  $\mathcal{L}$  is ample we can pick  $R$  to be the whole invariant subalgebra and then the generalised GIT quotient

is the standard GIT quotient in the sense of Mumford.

- 2) If  $G$  is reductive and  $\mathcal{L}$  is not ample the generalised GIT quotient may differ from Mumford's GIT quotient discussed in Section 1.5. For example, if  $X$  is projective and we take  $\mathcal{L} = \mathcal{O}_X$  to be the trivial line bundle then Mumford's quotient is empty whereas the generalised GIT quotient is a point.
- 3) Note that the morphism to the generalised GIT quotient is no longer necessarily surjective (this can be the case with non-reductive GIT when  $X$  is projective and  $L$  is ample and  $R$  is the whole invariant subalgebra cf. [10]).

Ideally we want to choose  $R$  to be reasonably large subject to the condition that it is finitely generated so that we have as many invariant sections as possible to distinguish between orbits.

**Remark 1.6.2.** Suppose  $G$  is a (not necessarily reductive) subgroup of a reductive group  $\tilde{G}$  which acts linearly on a projective scheme  $\tilde{X}$  with respect to an ample linearisation  $\tilde{\mathcal{L}}$ . If there is a  $G$ -equivariant morphism  $f : \overline{X} \rightarrow \tilde{X}$  such that  $f^*\tilde{\mathcal{L}}$  and  $\mathcal{L}$  are isomorphic linearisations, then the image  $R$  of the induced map

$$\bigoplus_{r \geq 0} H^0(\tilde{X}, \tilde{\mathcal{L}}^{\otimes r})^{\tilde{G}} \rightarrow \bigoplus_{r \geq 0} H^0(\overline{X}, \mathcal{L}^{\otimes r})^G$$

is finitely generated and so the pair  $(\mathcal{L}, R)$  defines a generalised linearisation on  $X$ . Moreover, the generalised GIT quotient  $\overline{X} //_{(\mathcal{L}, R)} G$  is the closure of  $\overline{X}^{nss}$  in the ordinary GIT quotient  $\tilde{X} //_{\tilde{\mathcal{L}}} \tilde{G}$ .

## 1.7 Constructing moduli spaces via GIT

Suppose we have a moduli problem for which we can express the equivalence classes as orbits of a reductive group action  $G$  on a scheme  $X$ . We want to know when

the categorical quotient really is a moduli space; for this we need the local universal property:

**Definition 1.7.1.** For a given moduli problem we say a family  $X$  parametrised by a scheme  $S$  has the local universal property if for any family  $X'$  parametrised by a scheme  $S'$  and point  $s \in S'$  there exists a neighbourhood  $U$  of  $s$  in  $S'$  and a morphism  $\phi : U \rightarrow S$  such that  $X'|_U \sim \phi^*X$ .

**Proposition 1.7.2.** ([36] Proposition 2.13) Suppose for a given moduli problem we have a family  $X$  parametrised by a scheme  $S$  which has the local universal property. Suppose also there is a reductive group  $G$  which acts on  $S$  so that  $s, t \in S$  lie in the same orbit if and only if  $X_s \sim X_t$ . Then:

- 1) Any coarse moduli space is a categorical quotient of  $S$  by  $G$ .
- 2) A categorical quotient of  $S$  by  $G$  is a coarse moduli space if and only if it is an orbit space.

## Chapter 2

# Quotients of unstable subschemes

Let  $X$  be a complex projective scheme and  $G$  be a complex reductive group acting on  $X$  with respect to an ample linearisation  $\mathcal{L}$ . The GIT quotient  $X//_{\mathcal{L}}G$  is a projective scheme which is a categorical quotient of an open subscheme  $X^{ss}$  of  $X$ , whose geometric points are semistable for the action of  $G$  with respect to  $\mathcal{L}$ . Associated to this action of  $G$  on  $X$  and linearisation there is a stratification  $\{S_{\beta} : \beta \in \mathcal{B}\}$  of  $X$  into disjoint  $G$ -invariant locally closed subschemes, such that the open stratum is  $X^{ss}(\mathcal{L})$  [17, 22, 24].

In this chapter we consider the problem of finding quotients for the  $G$ -actions on each unstable stratum  $S_{\beta}$  separately. For each  $\beta \in \mathcal{B}$  we find a canonical linearisation on a projective completion of the stratum  $S_{\beta}$  which in general is non-ample. In fact this linearisation will be used as a generalised linearisation (see Section 1.6). The subscheme of naively semistable points with respect to this canonical linearisation is the stratum  $S_{\beta}$  itself and so we can use this linearisation to take a quotient of the  $G$ -action on  $S_{\beta}$ . The generalised GIT quotient is a categorical quotient of the  $G$ -action on  $S_{\beta}$ , but has the disadvantage that in general it has no stable points. To rectify

this we suggest that a perturbation of the canonical linearisation should instead be used to take the quotient.

## 2.1 Stratifications of projective varieties

In this section we state the results needed from [24] in the case when  $X$  is a nonsingular complex projective variety. The stratification of  $X$  has both an algebraic and symplectic description; for more details on symplectic geometry see [8, 30, 37, 41]. We will also be using standard results from the representation theory of Lie algebras; good references for this are [7, 9, 11].

Let  $G$  be a complex reductive group acting linearly on a nonsingular complex projective variety  $X$  with respect to an ample linearisation  $\mathcal{L}$ . For the purposes of GIT we can assume without loss of generality that  $\mathcal{L}$  is very ample, so that  $X$  is embedded in projective space  $\mathbb{P}^n = \mathbb{P}(H^0(X, \mathcal{L})^*)$  and the action of  $G$  is given by a homomorphism  $\rho : G \rightarrow \mathrm{GL}(n+1)$ . As  $G$  is a complex reductive group, it is the complexification of a maximal compact subgroup  $K$ . We can assume without loss of generality that  $K$  acts unitarily on  $\mathbb{P}^n$  via a representation  $\rho : K \rightarrow \mathrm{U}(n+1)$ .

Alternatively we can view  $X$  as a symplectic manifold with symplectic form  $\omega$  inherited from the Kähler structure on  $\mathbb{P}^n$  defined by the Fubini-Study metric. The action of  $K$  on the symplectic manifold  $(X, \omega)$  is Hamiltonian and so if  $\mathfrak{K} = \mathrm{Lie}K$ , then there is an associated moment map  $\mu : X \rightarrow \mathfrak{K}^*$  given by

$$\mu(x) := \rho^* \left( \frac{x^* \bar{x}^{*t}}{2\pi i \|x^*\|^2} \right)$$

where  $x^* \in \mathbb{C}^{n+1}$  lies over  $x \in \mathbb{P}^n$  and  $\rho^* : \mathfrak{u}(n+1)^* \rightarrow \mathfrak{K}^*$  is dual to  $\mathrm{Lie}\rho$ . The symplectic analogue of the GIT quotient is the symplectic reduction  $\mu^{-1}(0)/K$  of Marsden and Weinstein [30]. A point  $x \in X$  is semistable if and only if the closure of

its  $G$ -orbit meets  $\mu^{-1}(0)$ . The inclusion of  $\mu^{-1}(0)$  in  $X^{ss}$  induces a homeomorphism

$$\mu^{-1}(0)/K \cong X//G.$$

We fix an inner product on  $\mathfrak{K}$  which is invariant under the adjoint action of  $K$ . The norm square of the moment map  $\|\mu\|^2 : X \rightarrow \mathbb{R}$  with respect to this inner product induces a Morse-type stratification of  $X$  into  $G$ -invariant locally closed nonsingular subvarieties

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

where the indexing set  $\mathcal{B}$  is a finite set of adjoint orbits in the Lie algebra  $\mathfrak{K}$  (or equivalently a finite set of points in a fixed positive Weyl chamber  $\mathfrak{t}_+$  in  $\mathfrak{K}$ ). In particular  $0 \in \mathcal{B}$  indexes the open stratum  $S_0$  which is equal to the semistable subset  $X^{ss}$ .

**Remark 2.1.1.** It is important to note that this stratification depends on the choice of linearisation and the choice of invariant inner product on  $\mathfrak{K}$ .

The strata indexed by nonzero  $\beta \in \mathcal{B}$  have both algebraic and symplectic descriptions given in [24]. In this thesis we concentrate on the algebraic description but add some comments on the symplectic description. We fix a maximal torus  $T$  of  $K$  and positive Weyl chamber  $\mathfrak{t}_+$  in the Lie algebra  $\mathfrak{t}$  of  $T$ . The restriction  $\rho|_T : T \rightarrow \mathrm{U}(n+1)$  is diagonalisable with weights

$$\alpha_0, \dots, \alpha_n : T \rightarrow S^1.$$

If we identify the tangent space of  $S^1$  at the identity with the line  $2\pi i\mathbb{R}$  in the complex plane, and identify  $2\pi i\mathbb{R}$  with  $\mathbb{R}$ , then by taking the derivative of the weights  $\alpha_i$  at the identity we get elements of  $\mathfrak{t}^* \cong \mathfrak{t}$  which we also denote by  $\alpha_i$ . The index set  $\mathcal{B}$  is defined in [24] to be the set of  $\beta \in \mathfrak{t}_+$  such that  $\beta$  is the closest point to zero of the

convex hull in  $\mathfrak{t}$  of some nonempty subset of the set of weights  $\{\alpha_0, \dots, \alpha_n\}$ .

**Remark 2.1.2.** Since the set of weights  $\{\alpha_0, \dots, \alpha_n\}$  is invariant under the Weyl group, the index set  $\mathcal{B}$  can also be identified with the set of  $K$ -orbits in  $\mathfrak{K}$  of closest points to 0 of convex hulls of subsets of  $\{\alpha_0, \dots, \alpha_n\}$ .

**Definition 2.1.3.** For  $\beta \in \mathcal{B}$  we define subvarieties

$$Z_\beta := X \cap \{[x_0 : \dots : x_n] \in \mathbb{P}^n : x_i = 0 \text{ if } \alpha_i \cdot \beta \neq \|\beta\|^2\}.$$

and

$$Y_\beta := X \cap \left\{ [x_0 : \dots : x_n] \in \mathbb{P}^n : \begin{array}{l} x_i = 0 \text{ if } \alpha_i \cdot \beta < \|\beta\|^2 \text{ and } x_i \neq 0 \\ \text{for some } i \text{ such that } \alpha_i \cdot \beta = \|\beta\|^2 \end{array} \right\}$$

of  $X$ .

**Remark 2.1.4.** These subvarieties have the following symplectic description. The closed subvariety  $Z_\beta$  is the set of critical points for the function  $\mu_\beta(x) := \mu(x) \cdot \beta$  on which  $\mu_\beta$  takes the value  $\|\beta\|^2$  and  $Y_\beta$  is the set of points in  $X$  whose path of steepest descent under  $\mu_\beta$  has limit in  $Z_\beta$ .

There is an obvious surjection  $p_\beta : Y_\beta \rightarrow Z_\beta$  which is a retraction onto  $Z_\beta$ . Let  $\text{Stab}\beta$  be the stabiliser of  $\beta$  under the adjoint action of  $G$  on its Lie algebra  $\mathfrak{g}$ . The closed subvariety  $Z_\beta$  is  $\text{Stab}\beta$ -invariant (see [24] §4.8) and the original linearisation  $\mathcal{L}$  restricts to a  $\text{Stab}\beta$ -linearisation on  $Z_\beta$ . The rational weight  $\beta \in \mathfrak{t} \cong \mathfrak{t}^*$  defines a character of  $\text{Stab}\beta$  which can be used to twist this linearisation.

**Definition 2.1.5.** Let  $Z_\beta^{ss}$  be the GIT semistable subset for the  $\text{Stab}\beta$  action on  $Z_\beta$  linearised by  $\mathcal{L}$  twisted by the character defined by  $-\beta$ . Then define  $Y_\beta^{ss} := p_\beta^{-1}(Z_\beta^{ss})$  and  $S_\beta := GY_\beta^{ss}$ .

**Remark 2.1.6.** The stratum  $S_\beta$  depends only on the adjoint orbit of  $\beta$ , but in order to define  $Z_\beta$  we need to fix an element in that adjoint orbit.

**Remark 2.1.7.** We can restrict the moment map  $\mu$  to  $Z_\beta$  and  $\mu - \beta$  is a moment map for the  $\text{Stab}\beta$  action on  $Z_\beta$ . The symplectic description of  $Z_\beta^{ss}$  is that it is the open stratum for the Morse stratification of the function  $\|\mu - \beta\|^2$  on  $Z_\beta$ . The critical point set of  $\|\mu\|^2$  is the disjoint union over  $\beta \in \mathcal{B}$  of the closed subsets  $C_\beta := K(Z_\beta \cap \mu^{-1}(\beta))$  by [24], Lemma 3.15. Moreover, the stratum  $S_\beta$  is the set of points in  $X$  whose path of steepest descent under  $\|\mu\|^2$  has a limit point in  $C_\beta$ .

For  $\beta \in \mathfrak{t}_+$  we define a parabolic subgroup

$$P_\beta := \{g \in G : \lim_{t \rightarrow -\infty} \exp(it\beta) g \exp(it\beta)^{-1} \text{ exists in } G\}$$

of  $G$  as in [24] §6. The subvarieties  $Y_\beta^{ss}$  and  $Y_\beta$  are  $P_\beta$ -invariant (see [24] Lemma 6.10) and by [24] Theorem 6.18 there is an isomorphism

$$S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}.$$

**Remark 2.1.8.** This stratification can also be described in terms of the work of Hesselink [17] and Kempf and Ness [22]. Recall from Section 1.4 that for each unstable point  $x \in X$  there is a collection  $\wedge^\mathcal{L}(x)$  of adapted 1-PSs to  $x$  and a parabolic subgroup  $P_x$  of  $G$  which is equal to

$$P(\lambda) := \{g \in G : \lim_{t \rightarrow 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G\}$$

for any  $\lambda \in \wedge^\mathcal{L}(x)$ . The alternative stratification [17, 22] uses conjugacy classes of 1-PSs to stratify  $X$  and agrees with the stratification of  $X$  described above.

To relate these stratifications we can view  $\beta$  as a rational weight in the sense that there is a natural number  $m > 0$  such that  $m\beta$  defines a 1-PS  $\mathbb{C}^* \rightarrow T_\mathbb{C}$  whose

restriction to  $S^1 \rightarrow T$  has derivative at the identity

$$\mathbb{R} \cong 2\pi i \mathbb{R} \cong \mathrm{Lie} S^1 \rightarrow \mathfrak{t}$$

sending 1 to  $m\beta$ . For any rational  $\beta \in \mathfrak{t}$  let  $\lambda_\beta : \mathbb{C}^* \rightarrow T_{\mathbb{C}}$  be the unique non-divisible 1-PS which is defined by  $q\beta$  for some positive rational number  $q$ . Then if  $\beta$  is a nonzero index the 1-PS  $\lambda_\beta$  is adapted to every  $x \in Y_\beta^{ss}$ . Moreover, we have that

$$P_\beta = P(\lambda_\beta).$$

## 2.2 Quotients of the unstable subvarieties

Let  $\beta$  be a nonzero index for the stratification  $\{S_\beta : \beta \in \mathcal{B}\}$ . In this section we find a canonical linearisation of the  $G$ -action on  $S_\beta$ . From the description  $S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$  we see that a quotient of the  $G$ -action on  $S_\beta$  is equivalent to a quotient of the  $P_\beta$ -action on  $Y_\beta^{ss}$ . We note at this point that the parabolic group  $P_\beta$  is in general non-reductive.

We take a  $G$ -equivariant projective completion  $\hat{S}_\beta$  of  $S_\beta$  given by

$$\hat{S}_\beta := G \times_{P_\beta} \overline{Y_\beta} \subset G \times_{P_\beta} X$$

where  $\overline{Y_\beta}$  is the closure of  $Y_\beta^{ss}$  in  $X$ . We construct a  $G$ -linearisation on this projective completion which we call the canonical linearisation for reasons which become clear towards the end of this chapter (cf. Proposition 2.2.8)

**Remark 2.2.1.** It is always the case that

$$\overline{Y_\beta} \subseteq X \cap \{[x_0 : \cdots : x_n] : x_j = 0 \text{ if } \alpha_j \cdot \beta < \|\beta\|^2\}.$$

We often have equality here (for example when  $X = \mathbb{P}^n$ ) but it might be the case, for

example, that  $X \cap \{[x_0 : \cdots x_n] : x_j = 0 \text{ if } \alpha_j \cdot \beta < ||\beta||^2\}$  has connected components which do not meet  $Y_\beta$ .

**Remark 2.2.2.** It may be the case that  $Z_\beta, Y_\beta$  and  $S_\beta$  are disconnected (cf. [24] §5), in which case we can, if we wish, refine the stratification by replacing  $Z_\beta$  with its connected components  $Z_{\beta,j}$ , say, and setting  $S_{\beta,j} = GY_{\beta,j}^{ss}$  where  $Y_{\beta,j}^{ss} = p_\beta^{-1}(Z_{\beta,j}^{ss})$ . Then each  $S_{\beta,j}$  will be a connected component of  $S_\beta$  (so long as  $Z_{\beta,j}^{ss}$  is nonempty). In what follows we will work with  $S_\beta$  for simplicity of notation, but we could equally well work with its connected components separately.

There is a natural isomorphism

$$G \times_{P_\beta} X \cong G/P_\beta \times X$$

$$(g, x) \mapsto (gP_\beta, g \cdot x)$$

and so we can construct new linearisations on  $\hat{S}_\beta$  by restricting linearisations on  $G \times_{P_\beta} X \cong G/P_\beta \times X$  to  $\hat{S}_\beta$ . The quotient  $G/P_\beta$  is a partial flag variety and linearisations of the  $G$ -actions on such varieties are well understood.

### 2.2.1 Line bundles on partial flag varieties $G/P$

In this section we review the construction of line bundles on partial flag varieties; for more detailed information see [26, 27, 28, 29]. We assume for the moment that  $G$  is semisimple and simply connected. We fix sets of positive roots  $\Phi_+ \subset \Phi$  and simple roots  $\Pi$ . Let  $\omega_i$  denote the fundamental dominant weight associated to a simple root  $\alpha_i$ . If  $\lambda = \sum a_i \omega_i$  is a dominant weight, then define

$$\Pi_\lambda := \{\alpha_i \in \Pi : a_i = 0\} \subset \Pi.$$

As we have fixed an invariant inner product on the Lie algebra  $\mathfrak{K}$  of the maximal compact subgroup  $K$  of  $G$ , we can identify the Lie algebra  $\mathfrak{t}$  of our maximal torus  $T$  with its dual  $\mathfrak{t}^*$ . In particular this defines a one-to-one correspondence between the characters of  $T$  and the one-parameter subgroups of  $T$ . By abuse of notation we let  $\lambda$  also denote the corresponding one-parameter subgroup; then the parabolic subgroup  $P(\lambda)$  associated to the 1-PS  $\lambda$  has associated simple roots

$$\Pi_{P(\lambda)} := \{\alpha_i \in \Pi : -\alpha_i \text{ is a root of } P(\lambda)\} \subset \Pi$$

and these sets agree, so that  $\Pi_\lambda = \Pi_{P(\lambda)}$ .

A character  $\chi : T_{\mathbb{C}} \rightarrow \mathbb{C}^*$  extends to  $P(\lambda)$  if and only if  $\chi \cdot \alpha^\vee = 0$  for all coroots  $\alpha^\vee$  such that  $\alpha \in \Pi_{P(\lambda)}$ . The weights naturally correspond to characters and the character defined by  $\lambda$  extends to  $P(\lambda)$  since

$$\lambda \cdot \alpha_i^\vee = a_i = 0 \text{ for all } \alpha_i \in \Pi_{P(\lambda)}$$

by the definition of this set. We let  $\lambda$  also denote the associated character of  $P(\lambda)$  and define a line bundle  $\mathcal{L}(\lambda)$  on  $G/P(\lambda)$  to be the line bundle associated to the character  $\lambda^{-1}$ ; that is,

$$\begin{array}{c} \mathcal{L}(\lambda) := G \times_{P(\lambda)} \mathbb{C} \\ \downarrow \\ G/P(\lambda) \end{array}$$

where  $(g, z)$  and  $(gp, \lambda(p)z)$  are identified for all  $p \in P(\lambda)$ . The sections of  $\mathcal{L}(\lambda)$  are given by

$$H^0(G/P(\lambda), \mathcal{L}(\lambda)) = \{f : G \rightarrow \mathbb{C} : f(gp) = \lambda(p)f(g) \text{ for all } g \in G, p \in P\}$$

and the natural left  $G$ -action gives this vector space a  $G$ -module structure. Let  $V(\lambda)$

denote the representation of  $G$  of highest weight  $\lambda$ . By the Borel–Weil–Bott theorem [5], there is an isomorphism of  $G$ -modules

$$H^0(G/P(\lambda), \mathcal{L}(\lambda)) \cong V(\lambda)^*.$$

The line bundle  $\mathcal{L}(\lambda)$  is very ample if and only if

$$\lambda \cdot \alpha_i^\vee = a_i > 0 \text{ for all } \alpha_i \notin \Pi_{P(\lambda)}$$

which is clearly the case by definition of  $\Pi_\lambda = \Pi_{P(\lambda)}$ . Thus there is an embedding

$$G/P(\lambda) \hookrightarrow \mathbb{P}(H^0(G/P(\lambda), \mathcal{L}(\lambda))^*) \cong \mathbb{P}(V(\lambda))$$

which is the natural projective embedding of the partial flag variety  $G/P(\lambda)$ . More concretely, let  $v_{\max}$  denote the highest weight vector in  $V(\lambda)$ , so that  $v_{\max}$  is an eigenvector for the action of  $T$  with eigenvalue  $\lambda$ ; then the embedding is given by the inclusion of the orbit  $G \cdot v_{\max}$ ,

$$\begin{aligned} G/P(\lambda) &\hookrightarrow \mathbb{P}(V(\lambda)) \\ gP(\lambda) &\mapsto [g \cdot v_{\max}]. \end{aligned}$$

**Remark 2.2.3.** If  $G$  is a subgroup of  $\mathrm{GL}(n)$  and the weight  $\lambda$  is restricted from  $\mathrm{GL}(n)$  we can view the weight  $\lambda$  as an element of the dual of the Lie algebra of both  $\mathrm{GL}(n)$  and  $\mathrm{PGL}(n)$  or equivalently  $\mathrm{SL}(n)$ . There are associated parabolics  $P(\lambda_{\mathrm{GL}})$  and  $P(\lambda_{\mathrm{SL}})$ , and the partial flag varieties for these two parabolics agree

$$\mathrm{GL}(n)/P(\lambda_{\mathrm{GL}}) = \mathrm{SL}(n)/P(\lambda_{\mathrm{SL}}).$$

Since  $\mathrm{SL}(n)$  is semisimple and simply connected there is a projective embedding of

this partial flag variety into  $\mathbb{P}(V(\lambda_{\text{SL}}))$  where  $V(\lambda_{\text{SL}})$  is the representation of  $\text{SL}(n)$  with highest weight  $\lambda_{\text{SL}}$ . As  $G \subset \text{GL}(n)$  and  $P(\lambda) = G \cap P(\lambda_{\text{GL}})$  we have

$$G/P(\lambda) \subseteq \text{GL}(n)/P(\lambda_{\text{GL}}) = \text{SL}(n)/P(\lambda_{\text{SL}}).$$

Hence we can use this inclusion and the embedding of  $\text{SL}(n)/P(\lambda_{\text{SL}})$  described above to obtain a projective embedding of  $G/P(\lambda)$ .

### 2.2.2 The canonical linearisation on $\hat{S}_\beta$

We can apply the above to the case when the parabolic subgroup is  $P_\beta = P(\lambda_\beta)$  and  $\lambda_\beta$  is the 1-PS associated to  $\beta \in \mathfrak{t}$ . The natural embedding of the partial flag variety  $G/P_\beta$  is thus given by the very ample line bundle  $\mathcal{L}(\lambda_\beta)$

$$G/P_\beta \hookrightarrow \mathbb{P}(H^0(G/P_\beta, \mathcal{L}(\lambda_\beta))^*) \cong \mathbb{P}(V(\beta)).$$

Unfortunately we actually need to twist by  $-\beta$  as we did in Definition 2.1.5 to construct  $Z_\beta^{ss}$ . Therefore we consider the non-ample linearisation  $\mathcal{L}(\lambda_{-\beta})$  on  $G/P_\beta$ .

**Definition 2.2.4.** Let  $\mathcal{L}_\beta$  be the  $G$ -linearisation on  $G/P_\beta \times X$  given by the tensor product of the pullbacks of  $\mathcal{L}(\lambda_{-\beta})$  on  $G/P_\beta$  and  $\mathcal{L}$  on  $X$  to  $G/P_\beta \times X$ . We let  $\mathcal{L}_\beta$  also denote the restriction of this linearisation to  $\hat{S}_\beta$  and call this the canonical linearisation on  $\hat{S}_\beta$ .

**Example 2.2.5.** Consider  $G = \text{SL}(2, \mathbb{C})$  acting on the complex projective line  $X = \mathbb{P}^1$  with respect to  $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1)$ . The semistable set is empty and the action is transitive so there will be one nonzero index in the stratification of  $X$ . We choose a maximal torus  $T = \{\text{diag}(t, t^{-1}) : t \in S^1\}$ ; then the weights of  $T$  acting on  $\mathbb{C}^2$  are  $\alpha_0 = \alpha, \alpha_1 = \alpha^{-1}$  where

$$\alpha : T \longrightarrow S^1$$

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto t.$$

The Lie algebra of  $T$  is  $\mathfrak{t} \cong \mathbb{R}$  and we pick the positive Weyl chamber  $\mathfrak{t}_+$  which contains  $\alpha$ . Then  $\beta = \alpha$  is an index for the stratification of  $X$  and we have that

$$Z_\beta = Z_\beta^{ss} = Y_\beta = Y_\beta^{ss} = \{[1 : 0]\}$$

and  $S_\beta = X$ . The parabolic subgroup  $P_\beta$  is the Borel subgroup of upper triangular matrices and we have an isomorphism

$$G/P_\beta \cong \mathbb{P}^1$$

$$gP_\beta \mapsto g \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The very ample line bundle on  $G/P_\beta$  defined by  $\beta$  is  $\mathcal{O}_{\mathbb{P}^1}(1)$  and the line bundle defined by  $-\beta$  is  $\mathcal{O}_{\mathbb{P}^1}(-1)$ . The canonical linearisation is given by restricting  $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$  on  $\mathbb{P}^1 \times \mathbb{P}^1 \cong G/P_\beta \times X$  to  $\hat{S}_\beta \cong S_\beta = X$ . The morphism  $S_\beta \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is the diagonal morphism and so the canonical linearisation on  $S_\beta = X$  is  $\mathcal{L}_\beta = \mathcal{O}_{\mathbb{P}^1}$ . Note that although  $\mathcal{L}_\beta$  is not ample, it is globally generated.

As we saw in the example above, the canonical linearisation  $\mathcal{L}_\beta$  on  $\hat{S}_\beta$  is not necessarily ample. However it is still possible to use this linearisation to produce GIT quotients, we just need to use the generalised notion of GIT of Section 1.6.

Instead of using the canonical linearisation  $\mathcal{L}_\beta$  of the  $G$ -action on  $\hat{S}_\beta$  we can instead describe this canonical linearisation in terms of an ample linearisation of the  $P_\beta$  action on  $\bar{Y}_\beta$ . Let  $\chi_{-\beta} : P_\beta \rightarrow \mathbb{C}^*$  denote the character associated to  $-\beta$ . Then  $\mathcal{L}^{\chi_{-\beta}}$  is a  $P_\beta$ -linearisation on  $\bar{Y}_\beta$  such that  $G \times_{P_\beta} \mathcal{L}^{\chi_{-\beta}}$  is the canonical linearisation on  $\hat{S}_\beta$ . Therefore we will also refer to the  $P_\beta$ -linearisation  $\mathcal{L}^{\chi_{-\beta}}$  on  $\bar{Y}_\beta$  as the canonical

linearisation and use  $\mathcal{L}_\beta$  to denote this linearisation.

**Lemma 2.2.6.** *We have an isomorphism of graded algebras*

$$\bigoplus_{r \geq 0} H^0(\hat{S}_\beta, \mathcal{L}_\beta^{\otimes r})^G \cong \bigoplus_{r \geq 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{P_\beta}.$$

*Proof.* This follows immediately from the fact that  $\hat{S}_\beta = G \times_{P_\beta} \bar{Y}_\beta$  and the canonical  $G$ -linearisation on  $\hat{S}_\beta$  is equal to  $G \times_{P_\beta} \mathcal{L}_\beta$  where here  $\mathcal{L}_\beta$  is the canonical  $P_\beta$ -linearisation on  $\bar{Y}_\beta$ .  $\square$

Although the canonical linearisation on  $\bar{Y}_\beta$  is ample, the group  $P_\beta$  is not reductive and so it is not immediate that the ring of invariant sections

$$\bigoplus_{r \geq 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{P_\beta}$$

is finitely generated. Luckily we can relate this canonical linearisation with an ample linearisation of a reductive group action on  $Z_\beta$  to show the ring of invariant sections is finitely generated:

**Lemma 2.2.7.** *Let  $\mathcal{L}_\beta = \mathcal{L}^{X-\beta}$  be the canonical  $P_\beta$ -linearisation on  $\bar{Y}_\beta$ . Then the algebra of invariant sections*

$$R := \bigoplus_{r \geq 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{P_\beta} \cong \bigoplus_{r \geq 0} H^0(\hat{S}_\beta, \mathcal{L}_\beta^{\otimes r})^G$$

*is finitely generated.*

*Proof.* Let  $\lambda_\beta : \mathbb{C}^* \rightarrow G$  be the 1-PS determined by the rational weight  $\beta$ . Then  $\lambda_\beta(\mathbb{C}^*) \subset P_\beta$  and so

$$\bigoplus_{r \geq 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{P_\beta} \subset \bigoplus_{r \geq 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{\lambda_\beta(\mathbb{C}^*)}.$$

The torus  $\lambda_\beta(\mathbb{C}^*)$  acts on  $\overline{Y}_\beta$  with respect to the canonical linearisation  $\mathcal{L}_\beta$  with non-negative weights and zero weights exactly on  $Z_\beta$ . Hence

$$\bigoplus_{r \geq 0} H^0(\overline{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{\lambda_\beta(\mathbb{C}^*)} \cong \bigoplus_{r \geq 0} H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\lambda_\beta(\mathbb{C}^*)} \quad \text{and}$$

$$\bigoplus_{r \geq 0} H^0(\overline{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{P_\beta} \subset \bigoplus_{r \geq 0} H^0(\overline{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{\text{Stab}\beta} \cong \bigoplus_{r \geq 0} H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\text{Stab}\beta}.$$

Let  $\sigma \in H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\text{Stab}\beta}$  and consider  $p_\beta^* \sigma \in H^0(Y_\beta, \mathcal{L}_\beta^{\otimes r})^{\text{Stab}\beta}$  where  $p_\beta : Y_\beta \rightarrow Z_\beta$  is the retraction defined by  $\beta$ . We have that  $P_\beta = \text{Stab}\beta U_\beta$  where  $U_\beta$  is the unipotent radical of  $P_\beta$  and there is a retraction  $q_\beta : P_\beta \rightarrow \text{Stab}\beta$  such that

$$p_\beta(p \cdot y) = q_\beta(p) \cdot p_\beta(y) \tag{2.1}$$

for all  $y \in Y_\beta$  and  $p \in P_\beta$ . For  $p \in P_\beta$  we can view the action by  $p$  as a morphism from  $Y_\beta$  to itself and let  $p^* : H^0(Y_\beta, \mathcal{L}_\beta^{\otimes r}) \rightarrow H^0(Y_\beta, \mathcal{L}_\beta^{\otimes r})$  be the induced morphism on cohomology. Then

$$p \cdot p_\beta^* \sigma := (p_\beta \circ p^{-1})^* \sigma = (q_\beta(p^{-1}) \circ p_\beta)^* \sigma = p_\beta^*(q_\beta(p) \cdot \sigma) = p_\beta^* \sigma$$

as  $\sigma$  is  $\text{Stab}\beta$  invariant. Therefore,

$$\bigoplus_{r \geq 0} H^0(\overline{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{P_\beta} \cong \bigoplus_{r \geq 0} H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\text{Stab}\beta}.$$

As  $\text{Stab}\beta$  is reductive, the algebra of invariant sections

$$\bigoplus_{r \geq 0} H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\text{Stab}\beta}$$

is finitely generated. □

We can now use the pair  $(\mathcal{L}_\beta, R)$  to give a generalised GIT quotient of  $\overline{Y}_\beta$  which provides a quotient for the  $G$ -action on  $\hat{S}_\beta$ .

**Proposition 2.2.8.** *The set of naively semistable points for the  $P_\beta$ -action on  $\overline{Y}_\beta$  with respect to  $(\mathcal{L}_\beta, R)$  is equal to  $Y_\beta^{ss}$ . Moreover, the generalised GIT quotient is isomorphic to the GIT quotient of the  $\text{Stab}\beta$  action on  $Z_\beta$  with respect to  $\mathcal{L}_\beta := \mathcal{L}^{X-\beta}$ :*

$$\hat{S}_\beta //_{(\mathcal{L}_\beta, R)} G = \overline{Y}_\beta //_{(\mathcal{L}_\beta, R)} P_\beta := \text{Proj}(R) \cong Z_\beta //_{\mathcal{L}_\beta} \text{Stab}\beta.$$

*Proof.* By definition  $Z_\beta^{ss}$  is the semistable subset for the  $\text{Stab}\beta$  action on  $Z_\beta$  with respect to  $\mathcal{L}_\beta$  and so  $Z_\beta^{ss}$  is contained in the naively semistable set for the  $P_\beta$  action on  $\overline{Y}_\beta$  by the proof of Lemma 2.2.7 above. The retraction

$$p_\beta : Y_\beta \rightarrow Z_\beta$$

sends a point  $y$  in  $Y_\beta$  to its limit  $\overline{y}$  under a 1-PS of  $\text{Stab}\beta$ , and so  $\overline{y}$  is in the  $P_\beta$ -orbit closure of  $y$ . Therefore, all points of  $Y_\beta^{ss}$  are naively semistable as they contain naively semistable points in their orbit closures and the naively semistable subscheme is open. Conversely, if  $y \in \overline{Y}_\beta - Y_\beta$  then  $y$  is not semistable for the  $\lambda_\beta(\mathbb{C}^*)$  action and hence is not naively semistable for the action of  $P_\beta$ . By the proof of Lemma 2.2.7 above

$$R \cong \bigoplus_{r \geq 0} H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\text{Stab}\beta},$$

so that  $y \in Y_\beta$  is naively semistable if and only if  $\overline{y} \in Z_\beta$  is semistable for the  $\text{Stab}\beta$  action; that is,  $\overline{y} \in Z_\beta^{ss}$ . Moreover, we have an isomorphism

$$\text{Proj} R \cong Z_\beta //_{\mathcal{L}_\beta} \text{Stab}\beta$$

with the GIT quotient for the  $\text{Stab}\beta$ -action on  $Z_\beta$ . □

**Remark 2.2.9.** Since the whole of the invariant subalgebra is finitely generated, we drop the indexing by  $R$  and replace the notation  $\hat{S}_\beta //_{(\mathcal{L}_\beta, R)} G = \bar{Y}_\beta //_{(\mathcal{L}_\beta, R)} P_\beta$  with  $\hat{S}_\beta //_{\mathcal{L}_\beta} G = \bar{Y}_\beta //_{\mathcal{L}_\beta} P_\beta$ .

**Proposition 2.2.10.** *The morphism  $S_\beta \rightarrow Z_\beta //_{\mathcal{L}_\beta} \text{Stab}\beta$  is a categorical quotient of the  $G$ -action on  $S_\beta$ .*

*Proof.* As  $S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$ , this is equivalent to showing that  $Y_\beta^{ss} \rightarrow Z_\beta //_{\mathcal{L}_\beta} \text{Stab}\beta$  is a categorical quotient of the  $P_\beta$ -action on  $Y_\beta^{ss}$ . The  $P_\beta$ -invariance of the morphism  $Y_\beta^{ss} \rightarrow Z_\beta //_{\mathcal{L}_\beta} \text{Stab}\beta$  follows from (2.1) and the  $\text{Stab}\beta$ -invariance of  $Z_\beta^{ss} \rightarrow Z_\beta //_{\mathcal{L}_\beta} \text{Stab}\beta$ . If we have a  $P_\beta$ -invariant morphism  $f : Y_\beta^{ss} \rightarrow Y$ , then we need to show this morphism factors through  $Z_\beta //_{\mathcal{L}_\beta} \text{Stab}\beta$ . As  $f$  is  $P_\beta$ -invariant it is constant on orbit closures and so  $f = f|_{Z_\beta^{ss}} \circ p_\beta$ . Since  $Z_\beta^{ss} \rightarrow Z_\beta //_{\mathcal{L}_\beta} \text{Stab}\beta$  is a categorical quotient of the  $\text{Stab}\beta$ -action on  $Z_\beta^{ss}$  and  $f|_{Z_\beta^{ss}} : Z_\beta^{ss} \rightarrow Y$  is  $\text{Stab}\beta$ -invariant, there is a morphism  $h : Z_\beta //_{\mathcal{L}_\beta} \text{Stab}\beta \rightarrow Y$  such that  $f|_{Z_\beta^{ss}}$  is the composition of the categorical quotient and  $h$ . Then we have a commutative diagram

$$\begin{array}{ccccc}
 Y_\beta^{ss} & \xrightarrow{p_\beta} & Z_\beta^{ss} & \longrightarrow & Z_\beta //_{\mathcal{L}_\beta} \text{Stab}\beta \\
 & \searrow f & \downarrow f| & \swarrow h & \\
 & & Y & & 
 \end{array}$$

and so the morphism  $f$  factors through  $Z_\beta //_{\mathcal{L}_\beta} \text{Stab}\beta$ .  $\square$

**Remark 2.2.11.** The generalised GIT quotient produced by the canonical linearisation is a suitable quotient for the action of  $\text{Stab}\beta$  on  $Z_\beta$  although it does not provide a satisfactory quotient of the  $G$  action on  $S_\beta$  unless  $Z_\beta^{ss} = Y_\beta^{ss}$ . As we saw above every point  $y \in Y_\beta^{ss}$  is identified in this quotient with its projection to  $Z_\beta^{ss}$  via  $p_\beta : Y_\beta^{ss} \rightarrow Z_\beta^{ss}$ . This means that the generalised GIT quotient will in general have lower dimension than expected. In fact for any generalised linearisation such that  $Z_\beta$  is naively semistable we will have this problem. Therefore, in order to get a better

quotient of  $S_\beta$  we need to find a  $G$  linearisation for which  $Z_\beta^{ss}$  is unstable. We suggest that using a perturbation of the canonical linearisation will give a better quotient.

**Example 2.2.12.** Consider  $G = \mathrm{SL}(2, \mathbb{C})$  acting on the complex projective line  $X = \mathbb{P}^1$  with respect to  $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1)$  as in Example 2.2.5. Then there is only one index  $\beta$  and  $S_\beta = \mathbb{P}^1$ . The subvarieties  $Z_\beta$ ,  $Z_\beta^{ss}$ ,  $Y_\beta$  and  $Y_\beta^{ss}$  all coincide and are equal to the point  $[1 : 0]$  in  $\mathbb{P}^1$ . In this case all sections are invariant

$$H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\mathrm{Stab} \beta} = \mathbb{C}$$

and so the generalised GIT quotient of the  $G$ -action on  $\hat{S}_\beta = S_\beta$  with respect to the canonical linearisation is a point as expected.

## 2.3 Extending to projective schemes

In this section we extend the above constructions for nonsingular projective varieties to the case when  $X$  is a closed subscheme of  $\mathbb{P}^n$  and the action of  $G$  on  $X$  is given by a linear representation  $G \rightarrow \mathrm{GL}(n+1)$ . For the  $G$ -action on the ambient projective space we can define the subvarieties  $Z_\beta^{ss}$  and  $Y_\beta^{ss}$  as before. We can also define the closed subvariety  $\overline{Y}_\beta$  of  $\mathbb{P}^n$  and use the scheme structure on  $\mathbb{P}^n$  to give this the reduced induced closed scheme structure as in [16], II Example 3.2.6. This gives  $\hat{S}_\beta := G \times_{P_\beta} \overline{Y}_\beta$  its scheme structure. Then the open subsets  $S_\beta \subset \hat{S}_\beta$  and  $Y_\beta^{ss} \subset \overline{Y}_\beta$  get an induced scheme structure as open subsets of schemes. We have a stratification

$$\mathbb{P}^n = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

into  $G$ -invariant locally closed subschemes and the morphism

$$G \times_{P_\beta} Y_\beta^{ss} \rightarrow \mathbb{P}^n$$

induced by the group action

$$G \times_{P_\beta} \mathbb{P}^n \rightarrow \mathbb{P}^n$$

is an isomorphism onto  $S_\beta$ .

To go from the stratification of the ambient projective space  $\mathbb{P}^n$  to a stratification of  $X$  we intersect the above stratification by taking fibre products. For any subscheme  $S$  of  $\mathbb{P}^n$  we let

$$S(X) := S \times_{\mathbb{P}^n} X$$

be the fibre product of  $X$  and  $S$  over  $\mathbb{P}^n$ . Then  $\bar{Y}_\beta(X)$  is a closed subscheme of  $X$  and  $\hat{S}_\beta(X) = G \times_{P_\beta} \bar{Y}_\beta(X)$  is a projective completion of  $S_\beta(X)$ . The morphism

$$G \times_{P_\beta} Y_\beta^{ss}(X) \rightarrow X$$

is an isomorphism onto  $S_\beta(X)$  by using the universal property of the fibre product  $S_\beta(X)$  and the fact that  $G \times_{P_\beta} Y_\beta^{ss} \cong S_\beta$  for the ambient projective space. We have a stratification

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta(X)$$

into  $G$ -invariant locally closed subschemes, but for some indices  $\beta$  we may have  $S_\beta(X)$  is the empty stratum. We note at this point that this stratification can be refined by taking connected components of  $Z_\beta^{ss}(X)$  in the same way as it can for varieties.

We can also define the canonical linearisation on  $\hat{S}_\beta$  in exactly the same way as we do for varieties and this can be restricted to  $\hat{S}_\beta(X)$ . In this situation we also have that the generalised GIT quotient with respect to the canonical linearisation is a categorical quotient of the  $G$ -action on  $S_\beta(X)$  and is isomorphic to the GIT quotient

$$Z_\beta(X) //_{\mathcal{L}_\beta} \text{Stab} \beta$$

of the  $\text{Stab}_\beta$ -action on  $Z_\beta(X)$  with respect to the linearisation  $\mathcal{L}_\beta = \mathcal{L}^{X-\beta}$ .

## Chapter 3

# The stratification for sheaves

We apply the ideas of the previous chapter to moduli spaces of sheaves on a complex projective scheme  $W$ . There is a well-known construction due to Simpson [40] of the moduli space of semistable pure sheaves on  $W$  of fixed Hilbert polynomial as the GIT quotient of a linear action of a special linear group  $G$  on a closed scheme  $\overline{Q}$  of a quotient scheme which is  $G$ -equivariantly embedded in a projective space.

In this chapter we study the stratification of  $\overline{Q}$  as in Section 2.1 and give a description of the unstable strata. We show that the above construction can be chosen so that elements of  $\overline{Q}$  which parametrise sheaves of a fixed Harder–Narasimhan type are a union of connected components of a stratum in this stratification. This generalises the Yang–Mills stratification of Atiyah and Bott [3] for vector bundles over a nonsingular projective curve (see also [25, 35]).

### 3.1 Simpson’s construction

Let  $W$  be a complex projective scheme with ample invertible sheaf  $\mathcal{O}(1)$ . We consider the moduli problem of classifying pure coherent algebraic sheaves on  $W$  up to isomorphism. Gieseker introduced a notion of semistability for sheaves in [13] and

constructed coarse moduli spaces of semistable torsion free sheaves in the case when  $W$  is a smooth projective surface. Maruyama generalised this to torsion free sheaves over integral smooth projective schemes in [31, 32]. Later Simpson [40] constructed coarse moduli spaces of semistable pure sheaves on complex projective schemes. We follow the more general construction of Simpson where the moduli space of semistable pure sheaves of fixed Hilbert polynomial is constructed as a GIT quotient of a subscheme  $\overline{Q}$  of a quot scheme by the action of a special linear group  $G$ . The linearisation is given by using Grothendieck's embedding of the quot scheme into a Grassmannian and then using the Plücker embedding of the Grassmannian into a projective space.

We fix a rational polynomial  $P \in \mathbb{Q}[x]$  of degree  $e$  which takes integer values when  $x$  is integral. Then any sheaf which has Hilbert polynomial  $P$  will have support of dimension  $e$ . Recall that a sheaf is pure of dimension  $e$  if its support has dimension  $e$  and all its nonzero subsheaves have support of dimension  $e$ . We consider pure sheaves of dimension  $e$  with Hilbert polynomial  $P$  calculated with respect to  $\mathcal{O}(1)$ .

**Definition 3.1.1.** Let  $\mathcal{F}$  be a pure sheaf of dimension  $e$  over  $W$ . We define the multiplicity of  $\mathcal{F}$  to be  $r(\mathcal{F}) = e!a_e(\mathcal{F})$  where  $a_e(\mathcal{F})$  is the leading coefficient in the Hilbert polynomial of  $\mathcal{F}$ . If  $\mathcal{F}$  is torsion free this is just the rank of  $\mathcal{F}$ . The reduced Hilbert polynomial of  $\mathcal{F}$  is defined to be the quotient  $P(\mathcal{F})/r(\mathcal{F})$ .

**Definition 3.1.2.** A sheaf  $\mathcal{F}$  is semistable if it is pure and every nonzero subsheaf  $\mathcal{F}' \subset \mathcal{F}$  satisfies

$$\frac{P(\mathcal{F}')}{r(\mathcal{F}')} \leq \frac{P(\mathcal{F})}{r(\mathcal{F})}$$

where the ordering on polynomials is given by lexicographic ordering of their coefficients. The sheaf is stable if the above inequality is strict for every proper nonzero subsheaf. A semistable sheaf  $\mathcal{F}$  has a Jordan–Hölder filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s = \mathcal{F}$$

where  $\mathcal{F}_i/\mathcal{F}_{i-1}$  is stable with reduced Hilbert polynomial  $P(\mathcal{F})/r(\mathcal{F})$  for  $1 \leq i \leq s$ . This filtration is not in general canonical but the associated graded sheaf

$$Gr^{JH}(\mathcal{F}) = \bigoplus_{i=1}^s \mathcal{F}_i/\mathcal{F}_{i-1}$$

is canonically associated to  $\mathcal{F}$  (up to isomorphism). Two semistable sheaves  $\mathcal{F}$  and  $\mathcal{G}$  over  $W$  are S-equivalent if  $Gr^{JH}(\mathcal{F})$  and  $Gr^{JH}(\mathcal{G})$  are isomorphic.

Simpson shows that the semistable sheaves with Hilbert polynomial  $P$  are bounded (see [40], Theorem 1.1), and hence we can choose  $n \gg 0$  so that all such sheaves are  $n$ -regular. In particular this means that for any such sheaf  $\mathcal{F}$  the evaluation map  $H^0(\mathcal{F}(n)) \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  is surjective and the higher cohomology of  $\mathcal{F}(n)$  vanishes; i.e.,

$$H^i(\mathcal{F}(n)) = 0 \text{ for } i > 0,$$

so that  $P(n) = P(\mathcal{F}, n) = \dim H^0(\mathcal{F}(n))$ .

Let  $V$  be a vector space of dimension  $P(n)$ . The evaluation map for  $\mathcal{F}$  and a choice of isomorphism  $H^0(\mathcal{F}(n)) \cong V$  determine a point  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  in the quot scheme

$$\text{Quot}(V \otimes \mathcal{O}(-n), P)$$

of quotients with Hilbert polynomial  $P$  of the sheaf  $V \otimes \mathcal{O}(-n)$  on  $W$ . We consider the open subscheme  $Q \subset \text{Quot}(V \otimes \mathcal{O}(-n), P)$  consisting of quotients  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  such that  $\mathcal{F}$  is pure of dimension  $e$  and  $H^0(\rho(n)) : V \rightarrow H^0(\mathcal{F}(n))$  is an isomorphism. The group  $G := \text{SL}(V)$  acts on this quot scheme by acting on the vector space  $V$ , so that  $g \cdot \rho$  is the composition

$$g \cdot \rho : V \otimes \mathcal{O}(-n) \xrightarrow{g^{-1}} V \otimes \mathcal{O}(-n) \xrightarrow{\rho} \mathcal{F}$$

for  $g \in G$  and  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  in the quot scheme. The subscheme  $Q$  is preserved by this action and the  $G$ -orbits correspond to isomorphism classes of sheaves.

Simpson considers a linearisation of this action given by an equivariant embedding of the quot scheme  $\text{Quot}(V \otimes \mathcal{O}(-n), P)$  into a Grassmannian. For  $m > n$  we let  $H := H^0(\mathcal{O}(m-n))$  and  $\text{Gr}(V \otimes H, P(m))$  be the Grassmannian of  $P(m)$ -dimensional quotients of the vector space  $V \otimes H$ . Grothendieck showed that for  $m \gg n$  the morphism

$$\text{Quot}(V \otimes \mathcal{O}(-n), P) \longrightarrow \text{Gr}(V \otimes H, P(m))$$

$$\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F} \longmapsto H^0(\rho(m)) : V \otimes H \rightarrow H^0(\mathcal{F}(m))$$

is an embedding. The Plücker embedding

$$\text{Gr}(V \otimes H, P(m)) \hookrightarrow \mathbb{P}((\wedge^{P(m)}(V \otimes H))^*)$$

$$H^0(\rho(m)) \mapsto [\wedge^{P(m)} H^0(\rho(m))]$$

then gives an embedding of  $\text{Quot}(V \otimes \mathcal{O}(-n), P)$  in the projective space  $\mathbb{P}((\wedge^{P(m)}(V \otimes H))^*)$ . Let  $\overline{Q}$  denote the closure of  $Q$  in the quot scheme  $\text{Quot}(V \otimes \mathcal{O}(-n), P)$  and let  $\mathcal{U}$  be the restriction to  $\overline{Q} \times W$  of the universal quotient sheaf on the quot scheme. We let  $\pi_{\overline{Q}}$  and  $\pi_W$  denote the projections from  $\overline{Q} \times W$  to  $\overline{Q}$  and  $W$ . Then if  $m \gg n$  the higher cohomology groups  $H^i(\mathcal{F}(m))$  for  $i > 0$  all vanish for  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  in  $\text{Quot}(V \otimes \mathcal{O}(-n), P)$  and

$$\mathcal{L} = \det(\pi_{\overline{Q}*}(\mathcal{U} \otimes \pi_W^* \mathcal{O}(m))) \tag{3.1}$$

is the ample invertible sheaf corresponding to the embedding of  $\overline{Q}$  into the projective space  $\mathbb{P}((\wedge^{P(m)}(V \otimes H))^*)$  above. There is a natural lift of the  $G$ -action on  $\overline{Q}$  to the universal quotient  $\mathcal{U}$  and this gives an action of  $G$  on  $\mathcal{L}$ ; we let  $\mathcal{L}$  denote this linearisation as well as the line bundle underlying it.

**Theorem 3.1.3.** ([40], Theorem 1.21) *Let  $W$  be a projective scheme,  $e$  a positive*

integer and  $P$  a Hilbert polynomial of degree  $e$ . If  $m \gg n \gg 0$  the GIT quotient  $\overline{Q} //_{\mathcal{L}G} G$  is a coarse moduli space for  $S$ -equivalence classes of semistable sheaves of pure dimension  $e$  with Hilbert polynomial  $P$ .

**Remark 3.1.4.** If  $W$  is a smooth integral projective complex scheme and we look at pure sheaves of maximal dimension, i.e., torsion free sheaves, then we can instead use the original linearisation due to Gieseker [13] of the  $\mathrm{SL}(V)$ -action on  $Q$ . This linearisation is given by an embedding of  $Q$  into a projective bundle over part of the Picard scheme  $\mathrm{Pic}(W)$ . Let  $R$  be the components of  $\mathrm{Pic}(W)$  which contain  $\det \mathcal{F}$  for a quotient sheaf  $V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  in  $\overline{Q}$ . The determinant of the universal quotient sheaf  $\mathcal{U}$  on  $\overline{Q}$  is equivalent to a homomorphism

$$D : \overline{Q} \rightarrow R.$$

Let  $\mathcal{P}$  be a Poincaré line bundle on  $R \times W$ ; then there is an isomorphism

$$\det \mathcal{U} \cong (D \times \mathrm{id}_W)^* \mathcal{P} \otimes \pi_Q^* \mathcal{A} \tag{3.2}$$

where  $\mathcal{A}$  is an invertible sheaf on  $\overline{Q}$ .

Let  $\pi_1$  and  $\pi_2$  be the projections from  $R \times W$  to the first and second factors. We can assume that  $n$  is sufficiently large so that the invertible sheaves in the image of  $D : \overline{Q} \rightarrow R$  are all  $n$ -regular. Our assumptions on  $n$  ensure that the sheaf

$$\mathcal{W} := (\wedge^r V)^\vee \otimes (\pi_1)_* (\mathcal{P} \otimes (\pi_2)^* \mathcal{O}(rn))$$

over  $R$  is locally free, where  $r$  is the rank determined by the top coefficient of  $P$ . Let  $B := \mathbb{P}(\mathcal{W})$  denote the projective bundle over  $R$  associated to  $\mathcal{W}$ . We construct a morphism  $h : \overline{Q} \rightarrow B$  by showing there is an injective homomorphism  $\mathcal{A}^\vee \rightarrow D^* \mathcal{W}$  of sheaves over  $\overline{Q}$  (as in [16], II Proposition 7.12 where the opposite convention for

projective bundles is used). Let

$$\wedge^r V \rightarrow \pi_{\overline{Q}*} \left( \det(\mathcal{U}) \otimes (\pi_{\overline{W}}^{\overline{Q} \times W})^* \mathcal{O}(rn) \right)$$

be the  $r$ th exterior product of the universal quotient morphism on  $\overline{Q}$ . This homomorphism and the isomorphism

$$p_* \left( \det(\mathcal{U}) \otimes (\pi_{\overline{W}}^{\overline{Q} \times W})^* \mathcal{O}(rn) \right) \cong D^*(\pi_1)_*(\mathcal{P} \otimes (\pi_2)^* \mathcal{O}(rn))$$

(defined by the isomorphism (3.2)) gives the injective homomorphism  $\mathcal{A}^\vee \rightarrow D^* \mathcal{W}$  defining the morphism  $h : \overline{Q} \rightarrow B$ .

The action of  $\mathrm{SL}(V)$  on  $\wedge^r V$  defines a  $\mathrm{SL}(V)$ -action on  $B$  such that the morphism  $h : \overline{Q} \rightarrow B$  is  $\mathrm{SL}(V)$ -equivariant with respect to this action. The scheme  $Q$  only parametrises torsion free sheaves and so the class of a quotient sheaf  $V \otimes \mathcal{O}(-n) \rightarrow \mathcal{E}$  in  $Q$  is completely determined by the homomorphism  $\wedge^r V \rightarrow H^0(\det(\mathcal{E})(rn))$ . Hence  $h$  is injective on  $Q$  and  $\mathcal{A}|_Q$  is ample relative to  $\mathrm{Pic}(W)$ .

The morphism  $h$  maps semistable sheaves in  $Q$  to the semistable subscheme  $B^{ss}$  for the  $\mathrm{SL}(V)$ -action on  $B$  and the moduli space of semistable torsion free sheaves over  $W$  is constructed as the GIT quotient of this action.

### 3.1.1 The Hilbert–Mumford function

In this section we calculate the Hilbert–Mumford function  $\mu^{\mathcal{L}}(\rho, \lambda)$  for a quotient sheaf  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  in  $\overline{Q}$  and a 1-parameter subgroup  $\lambda$  of  $G = \mathrm{SL}(V)$ . Recall this is equal to the weight of the  $\mathbb{C}^*$ -action induced by  $\lambda$  on the fibre of  $\mathcal{L}$  over the fixed point  $\lim_{t \rightarrow 0} \lambda(t) \cdot \rho$ . Moreover, by the Hilbert–Mumford criterion (see Section 1.3 for more details) we know that  $\rho$  is a GIT semistable point if and only if  $\mu^{\mathcal{L}}(\rho, \lambda) \geq 0$  for all 1-PSs  $\lambda$  of  $G$ .

First of all we make use of the fact that any 1-PS induces a decomposition of  $V$

as a direct sum of weight spaces:

$$\left\{ \begin{array}{c} \text{1-PSs of } \mathrm{SL}(V) \\ \lambda \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{decompositions } V = \bigoplus_{k \in \mathbb{Z}} V_k \\ \text{such that } \sum k \dim V_k = 0 \end{array} \right\}$$

$$\lambda \mapsto V_k := \{v \in V : \lambda(t) \cdot v = t^k v\}.$$

The relation  $\sum k \dim V_k = 0$  is required so that we get a 1-PS of the special linear group as opposed to the general linear group. Such a decomposition determines a filtration of  $V$  given by

$$\cdots \subseteq V_{\geq k+1} \subseteq V_{\geq k} \subseteq V_{\geq k-1} \subseteq \cdots$$

where  $V_{\geq k} := \bigoplus_{l \geq k} V_l$ . There are only finitely many integers  $k$  such that  $V_k \neq 0$ , say

$$k_1 > \cdots > k_s;$$

then if we let  $V^{(i)} = V_{\geq k_i}$  we get a finite filtration  $0 \subset V^{(1)} \subset \cdots \subset V^{(s)} = V$ . In particular we obtain a map

$$\left\{ \begin{array}{c} \text{1-PSs of } \mathrm{SL}(V) \\ \lambda \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} 0 = V^{(0)} \subset V^{(1)} \subset \cdots \subset V^{(s)} = V \\ \text{filtrations of } V \text{ and integers } k_1 > \cdots > k_s \\ \text{such that } \sum k_i \dim V^{(i)} / V^{(i-1)} = 0 \end{array} \right\}.$$

Let  $\lambda$  be a 1-PS of  $G = \mathrm{SL}(V)$  and  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  be a point in  $\overline{Q}$ . Then the filtration of  $V$  determined by  $\lambda$  induces a filtration of  $\mathcal{F}$  given by

$$0 = \mathcal{F}^{(0)} \subset \mathcal{F}^{(1)} \subset \cdots \subset \mathcal{F}^{(s)} = \mathcal{F}$$

where  $\mathcal{F}^{(i)} = \rho(V^{(i)} \otimes \mathcal{O}(-n))$ . Let  $\mathcal{F}_{\geq k}$  denote the image of  $V_{\geq k} \otimes \mathcal{O}(-n)$  under  $\rho$

for any integer  $k$ . Then  $\rho$  induces surjective morphisms

$$\rho_k : V_k \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}_k = \mathcal{F}_{\geq k} / \mathcal{F}_{\geq k+1}$$

for each integer  $k$ ; note that  $\mathcal{F}_k$  and  $\rho_k$  can only be nonzero if  $k = k_i$  for some  $i$  with  $1 \leq i \leq s$ . We define

$$\bar{\rho} = \bigoplus_{k \in \mathbb{Z}} \rho_k : \bigoplus_{k \in \mathbb{Z}} V_k \otimes \mathcal{O}(-n) \rightarrow \bar{\mathcal{F}} = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}_k \quad (3.3)$$

which by [19], Lemma 4.4.3 is the fixed point  $\lim_{t \rightarrow 0} \lambda(t) \cdot \rho$ . We now have a formula for the Hilbert–Mumford function.

**Lemma 3.1.5.** *The Hilbert–Mumford function evaluated at a one-parameter subgroup  $\lambda$  of  $G = \mathrm{SL}(V)$  for a point  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  in  $\bar{Q}$  is given by*

$$\mu^{\mathcal{L}}(\rho, \lambda) = \sum_{i=1}^{s-1} (k_i - k_{i+1}) \left( P(\mathcal{F}^{(i)}, m) - \dim V^{(i)} \frac{P(m)}{P(n)} \right)$$

where  $V^{(i)}$  and  $\mathcal{F}^{(i)}$  are defined as above.

*Proof.* For this we follow the argument of [19], Lemma 4.4.4, but use a left action as opposed to a right action. To calculate the value of the Hilbert–Mumford function we need to calculate the weight of the  $\mathbb{C}^*$ -action on the fibre at  $\bar{\rho}$  of the line bundle  $\mathcal{L}$  defined at (3.1). Since  $m \gg n \gg 0$  we have  $H^i(\bar{\mathcal{F}}(m)) = 0$  for  $i > 0$  and the line bundle

$$\mathcal{L} = \det(\pi_{\bar{Q}*}(\mathcal{U} \otimes \pi_W^* \mathcal{O}(m)))$$

has fibre

$$\det(H^0(\bar{\mathcal{F}}(m)))^* = \wedge^{P(m)} H^0(\bar{\mathcal{F}}(m))^*$$

at  $\bar{\rho}$ . The  $\mathbb{C}^*$ -action induced by  $\lambda$  on  $\rho_k$  has weight  $-k$  because  $\lambda(t) \cdot \rho_k$  is the

composition

$$V_k \otimes \mathcal{O}(-n) \xrightarrow{\lambda^{-1}(t)} V_k \otimes \mathcal{O}(-n) \xrightarrow{\rho_k} \mathcal{F}_k$$

and

$$\lambda^{-1}(t) \cdot v_k = t^{-k} v_k \text{ for all } v_k \in V_k.$$

Therefore the weight of the  $\mathbb{C}^*$ -action on  $\det H^0(\mathcal{F}_k(m))$  is equal to  $-k$  times the dimension of  $H^0(\mathcal{F}_k(m))$ , which is the value  $P(\mathcal{F}_k, m)$  of the Hilbert polynomial  $P(\mathcal{F}_k)$  at  $m$ . The weight of the  $\mathbb{C}^*$ -action on the fibre of  $\mathcal{L}$  over  $\bar{\rho}$  is equal to minus the weight of the  $\mathbb{C}^*$ -action on a point in  $(\wedge^{P(m)}(V \otimes H))^*$  which lies over  $\bar{\rho}$  and so is equal to minus the sum of the weights of the  $\mathbb{C}^*$ -action on  $\det H^0(\rho_k(m))$ :

$$\mu^{\mathcal{L}}(\rho, \lambda) = \sum_{k \in \mathbb{Z}} k P(\mathcal{F}_k, m) = \sum_{i=1}^s k_i P(\mathcal{F}_{k_i}, m).$$

As  $\lambda$  is a 1-PS of the special linear group, we have  $\sum_{i=1}^s k_i \dim V_{k_i} = 0$ . Therefore,

$$\begin{aligned} \mu^{\mathcal{L}}(\rho, \lambda) &= \sum_{i=1}^s k_i \left( P(\mathcal{F}_{k_i}, m) - \dim V_{k_i} \frac{P(m)}{P(n)} \right) \\ &= \sum_{i=1}^s k_i \left( P(\mathcal{F}^{(i)}, m) - P(\mathcal{F}^{(i+1)}, m) - \dim V^{(i)} \frac{P(m)}{P(n)} + \dim V^{(i+1)} \frac{P(m)}{P(n)} \right) \\ &= k_s \left( P(m) - \dim V \frac{P(m)}{P(n)} \right) + \sum_{i=1}^{s-1} (k_i - k_{i+1}) \left( P(\mathcal{F}^{(i)}, m) - \dim V^{(i)} \frac{P(m)}{P(n)} \right) \end{aligned}$$

which gives the required result since  $\dim V = P(n)$ .  $\square$

**Remark 3.1.6.** In Remark 3.1.4 we saw that when  $W$  is smooth and integral and  $e = \dim W$  we can use the linearisation  $\mathcal{A}$  due to Gieseker of the  $\mathrm{SL}(V)$ -action on  $G$ . The Hilbert–Mumford function with respect to Gieseker’s linearisation is given by

$$\mu^{\mathcal{A}}(\rho, \lambda) = \sum_{i=1}^s k_i \mathrm{rk} \mathcal{F}_{k_i} = \sum_{i=1}^{s-1} (k_i - k_{i+1}) \left( \mathrm{rk} \mathcal{F}^{(i)} - \dim V^{(i)} \frac{r}{P(n)} \right)$$

where  $r$  denotes the rank determined by the leading coefficient of  $P$ . This calculation is very similar to the above calculation and can be found in [13]. The main difference is that under Gieseker's linearisation the quotient sheaf  $\bar{\rho}$  is embedded as  $H^0(\wedge^r \bar{\rho}(rn))$  and the weight of the  $\mathbb{C}^*$ -action on  $H^0(\wedge^{r_k} \rho_k(r_k n))$  is equal to  $-kr_k$  where  $r_k$  denotes the rank of the quotient sheaf  $\mathcal{F}_k$ .

## 3.2 The stratification and index set

We consider the group  $G = \mathrm{SL}(V)$  acting on the closed subscheme  $\bar{Q}$  of the quotient scheme  $\mathrm{Quot}(V \otimes \mathcal{O}(-n), P)$  with respect to the linearisation  $\mathcal{L}$  defined at (3.1), for which the GIT quotient  $\bar{Q} //_{\mathcal{L}} G$  is a coarse moduli space for S-equivalence classes of semistable sheaves on  $W$  with Hilbert polynomial  $P$ . The linearisation  $\mathcal{L}$  defines a  $G$ -equivariant embedding of  $\bar{Q}$  in the projective space  $\mathbb{P}((\wedge^{P(m)}(V \otimes H))^*) = \mathbb{P}^N$  and we can choose a Kähler structure on  $\mathbb{P}^N$  which is invariant under the maximal compact subgroup  $SU(V)$  of  $G = \mathrm{SL}(V)$ . There is a moment map  $\mu : \mathbb{P}^N \rightarrow \mathfrak{su}(V)^*$  associated to this action and we use the Killing form on  $\mathfrak{su}(V)$  as an invariant inner product to identify the Lie algebra  $\mathfrak{su}(V)$  with its dual  $\mathfrak{su}(V)^*$ . Then applying Morse theory to  $\|\mu\|^2$  gives us a stratification of  $\mathbb{P}^N$  which we can intersect with  $\bar{Q}$  to get a stratification

$$\bar{Q} = \bigsqcup_{\beta \in \mathcal{B}} S_{\beta}$$

into  $G$ -invariant locally closed subschemes as in Section 2.3. We note at this point that the strata  $S_{\beta}$  may not be connected and so may be stratified further into their connected components (cf. Remark 2.2.2).

We can choose a basis of the fixed vector space  $V$  of dimension  $P(n)$  and pick the maximal torus  $T \subset \mathrm{SU}(V)$  consisting of matrices which are diagonal with respect to this basis. The Lie algebra  $\mathfrak{t} = \mathrm{Lie} T$  consists of purely imaginary tracefree diagonal matrices with respect to this basis. We choose a positive Weyl chamber  $\mathfrak{t}_+ \subset \mathfrak{t}$  given

by

$$\mathfrak{t}_+ = \left\{ \text{id}\text{diag}(a_1, \dots, a_{\dim(V)}) : \begin{array}{l} a_i \in \mathbb{R} \text{ such that } \sum a_i = 0 \\ \text{and } a_1 \geq a_2 \geq \dots \geq a_{\dim(V)} \end{array} \right\}.$$

The indexing set  $\mathcal{B}$  for the stratification  $\{S_\beta : \beta \in \mathcal{B}\}$  can then be identified with a finite set of points in  $\mathfrak{t}_+$ .

### 3.3 Harder–Narasimhan filtrations

Any sheaf of pure dimension  $e$  over  $W$  has a canonical filtration by subsheaves whose successive quotients are semistable with decreasing reduced Hilbert polynomials. This filtration is known as the Harder–Narasimhan filtration due to the original existence and uniqueness results of Harder and Narasimhan [15] for such filtrations of vector bundles over smooth projective curves.

**Definition 3.3.1.** Let  $\mathcal{F}$  be a pure sheaf; then its Harder–Narasimhan filtration is a filtration

$$0 = \mathcal{F}^{(0)} \subsetneq \mathcal{F}^{(1)} \subsetneq \dots \subsetneq \mathcal{F}^{(s)} = \mathcal{F}$$

such that the successive quotients  $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$  are semistable with decreasing reduced Hilbert polynomials

$$\frac{P(\mathcal{F}^{(1)})}{r(\mathcal{F}^{(1)})} > \frac{P(\mathcal{F}^{(2)}/\mathcal{F}^{(1)})}{r(\mathcal{F}^{(2)}/\mathcal{F}^{(1)})} > \dots > \frac{P(\mathcal{F}^{(s)}/\mathcal{F}^{(s-1)})}{r(\mathcal{F}^{(s)}/\mathcal{F}^{(s-1)})}.$$

We will denote by  $\text{Gr}^{HN}(\mathcal{F})$  the associated graded sheaf

$$\text{Gr}^{HN}(\mathcal{F}) = \bigoplus_{i=1}^s \mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}.$$

We call the first sheaf  $\mathcal{F}^{(1)}$  appearing in the Harder–Narasimhan filtration the maximal destabilising subsheaf. The Harder–Narasimhan type of  $\mathcal{F}$  is the tuple of Hilbert

polynomials of the successive quotients:

$$HN(\mathcal{F}) := (P(\mathcal{F}^{(1)}), P(\mathcal{F}^{(2)}/\mathcal{F}^{(1)}), \dots, P(\mathcal{F}^{(s)}/\mathcal{F}^{(s-1)})).$$

For each point  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  in  $Q$  we define the Harder–Narasimhan type of  $\rho$  to be the Harder–Narasimhan type of  $\mathcal{F}$ .

The different types of Harder–Narasimhan filtrations allow us to decompose the points of  $Q$  into subsets of fixed Harder–Narasimhan type.

**Definition 3.3.2.** If  $\tau$  is a Harder–Narasimhan type, let  $R_\tau \subseteq Q$  be the set of points  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  in  $Q$  such that  $\mathcal{F}$  has Harder–Narasimhan type  $\tau$ . Then we can write  $Q$  as

$$Q = \bigsqcup_{\tau} R_\tau.$$

Let  $\tau_0 = (P)$  denote the trivial Harder–Narasimhan type; then  $R_{\tau_0}$  contains only semistable sheaves and so is equal to the set of points in the stratum  $S_0 = \overline{Q}^{ss}$  by [40] Theorem 1.21 (cf. Theorem 3.1.3). We want to give the subsets  $R_\tau$  associated to higher Harder–Narasimhan types  $\tau$  a geometric structure and relate this to the higher strata  $S_\beta$ .

For the rest of this chapter we fix a nontrivial Harder–Narasimhan type  $\tau = (P_1, \dots, P_s)$ , where  $P_1, \dots, P_s$  are polynomials of degree  $e$  such that  $P_1 + \dots + P_s = P$ . We assume that there is a sheaf of pure dimension  $e$  over  $W$  with this Harder–Narasimhan type so for  $n \gg 0$  the subset  $R_\tau$  is nonempty.

**Lemma 3.3.3.** *The set of sheaves of pure dimension  $e$  with Hilbert polynomial  $P$  and Harder–Narasimhan type  $\tau$  is bounded.*

*Proof.* This follows from a result of Simpson (see [40] Theorem 1.1) that a set of sheaves on  $W$  of pure dimension  $e$  and Hilbert polynomial  $P$  is bounded if the slopes of their subsheaves are bounded above by a fixed constant, where the slope of a sheaf

is (up to multiplication by a positive constant) the second to top coefficient of its reduced Hilbert polynomial. Any sheaf  $\mathcal{F}$  with Harder–Narasimhan type  $\tau$  has a maximal destabilising subsheaf  $\mathcal{F}^{(1)}$  with Hilbert polynomial  $P_1$ , and all subsheaves of  $\mathcal{F}$  have reduced Hilbert polynomial less than or equal to the reduced Hilbert polynomial of  $\mathcal{F}^{(1)}$ . Let  $\mu_1$  denote the slope of  $\mathcal{F}^{(1)}$ , which depends only on the polynomial  $P_1$ ; then any subsheaf  $\mathcal{F}'$  of  $\mathcal{F}$  has slope less than or equal to  $\mu_1$  and this proves the result.  $\square$

This boundedness result means that we may assume  $n$  is chosen so that all pure sheaves with Hilbert polynomial  $P$  and Harder–Narasimhan type  $\tau$  are  $n$ -regular, and therefore are parametrised by  $Q$ . We may also assume that  $n$  is sufficiently large so that all sheaves with Harder–Narasimhan type  $(P_{i_1}, \dots, P_{i_k})$  for any  $1 \leq i_1 < i_2 < \dots < i_k \leq s$  are  $n$ -regular; in particular, the sheaves  $\mathcal{F}^{(i)}$  occurring in the Harder–Narasimhan filtration of any sheaf  $\mathcal{F}$  of Harder–Narasimhan type  $\tau$  are  $n$ -regular.

### 3.4 The index associated to a Harder–Narasimhan type

We want to show for a fixed Harder–Narasimhan type  $\tau = (P_1, \dots, P_s)$  that  $R_\tau$  is contained in a stratum  $S_{\beta(\tau)}$  occurring in the stratification  $\{S_\beta : \beta \in \mathcal{B}\}$ . In fact we will actually show that  $R_\tau$  is a union of connected components of a stratum in this stratification provided  $n$  and  $m$  are sufficiently large (for details see the proof of Theorem 3.5.7). Firstly we need to find a candidate for the index  $\beta = \beta(\tau)$  depending on the information coming from the Harder–Narasimhan type  $\tau$ . The definitions of  $Z_\beta$  and  $Y_\beta$ ,  $Z_\beta^{ss}$  and  $Y_\beta^{ss}$  are valid for any  $\beta \in \mathfrak{t}$  and do not require  $\beta$  to belong to the indexing set  $\mathcal{B}$ , but  $Y_\beta^{ss}$  is nonempty if and only if  $\beta \in \mathcal{B}$ . Therefore we can look for a candidate  $\beta \in \mathfrak{t}_+$ , and if  $Y_\beta^{ss}$  is nonempty then this will imply that  $\beta$  is an index in the stratification.

We assume  $n$  is sufficiently large so that all sheaves with Harder–Narasimhan

type  $\tau$  are parametrised by  $Q$ . We choose a point  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  in  $Q$  which represents a sheaf  $\mathcal{F}$  with Harder–Narasimhan type  $\tau$  and let

$$0 = \mathcal{F}^{(0)} \subsetneq \mathcal{F}^{(1)} \subsetneq \dots \subsetneq \mathcal{F}^{(s)} = \mathcal{F}$$

denote the Harder–Narasimhan filtration of  $\mathcal{F}$ . By Remark 2.1.8 rather than finding  $\beta \in \mathfrak{t}_+$  such that  $\rho$  belongs to  $Y_\beta^{ss}$  we can look for a 1-PS  $\lambda_\beta$  of  $G = \mathrm{SL}(V)$  which is adapted to  $\rho$ . We have seen that all 1-PSs give rise to filtrations of  $\mathcal{F}$  and it is reasonable to expect that a 1-PS which is adapted to  $\rho$  will give rise to the filtration of  $\mathcal{F}$  which is most responsible for its instability, namely its Harder–Narasimhan filtration. With this in mind we let

$$V^{(i)} := H^0(\rho(n))^{-1}(H^0(\mathcal{F}^{(i)}(n)))$$

and choose a basis of  $V$  (and corresponding maximal torus of  $G = \mathrm{SL}(V)$ ) by first taking a basis of  $V^{(1)}$ , then extending to  $V^{(2)}$  and so on. This gives us a decomposition

$$V = V_1 \oplus \dots \oplus V_s$$

of  $V$  where  $V_i \cong V^{(i)}/V^{(i-1)}$ . We consider 1-PSs in  $G = \mathrm{SL}(V)$  of the form

$$\lambda(t) = \begin{pmatrix} t^{\beta_1} I_{V_1} & & & \\ & t^{\beta_2} I_{V_2} & & \\ & & \ddots & \\ & & & t^{\beta_s} I_{V_s} \end{pmatrix}$$

where the weights  $\beta_i$  are decreasing  $\beta_1 > \dots > \beta_s$  and satisfy the relation

$$\sum_{i=1}^s \beta_i P(\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}, n) = 0.$$

**Remark 3.4.1.** We may assume  $n$  is sufficiently large so that  $\mathcal{F}^{(i)}$  and  $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$  are  $n$ -regular for  $i = 1, \dots, s$ . Then  $\mathcal{F}^{(i)}(n)$  and  $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}(n)$  have no higher cohomology and so we have that

$$P_i(n) = P(\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}, n) = \dim V^{(i)}/V^{(i-1)} = \dim V_i$$

for  $i = 1, \dots, s$ .

A nontrivial 1-PS  $\lambda$  of  $G$  is adapted to  $\rho$  if

$$\frac{\mu^{\mathcal{L}}(\rho, \lambda)}{\|\lambda\|}$$

is minimal among nontrivial 1-PSs of  $G$ . Therefore, we choose the weights  $(\beta_1, \dots, \beta_s)$  to minimise the function

$$f(\beta_1, \dots, \beta_s) := \frac{\sum_{i=1}^{s-1} (\beta_i - \beta_{i+1}) \left( P(\mathcal{F}^{(i)}, m) - P(\mathcal{F}^{(i)}, n) \frac{P(m)}{P(n)} \right)}{(\sum_{i=1}^s \beta_i^2 P_i(n))^{1/2}}$$

subject to the condition  $g(\beta_1, \dots, \beta_s) := \sum_{i=1}^s \beta_i P_i(n) = 0$ .

We introduce a Lagrangian multiplier  $\eta$  and define

$$\Lambda(\beta_1, \dots, \beta_s, \eta) := f(\beta_1, \dots, \beta_s) - \eta g(\beta_1, \dots, \beta_s);$$

then we look for solutions to

$$\begin{aligned} \frac{\partial}{\partial \beta_j} \Lambda(\beta_1, \dots, \beta_s, \eta) &= 0 \quad \text{for } j = 1, \dots, s \\ \frac{\partial}{\partial \eta} \Lambda(\beta_1, \dots, \beta_s, \eta) &= 0. \end{aligned} \tag{3.4}$$

Note that for any  $a \in \mathbb{R}_{>0}$  we have that  $f(a\beta_1, \dots, a\beta_s) = f(\beta_1, \dots, \beta_s)$  and that

$$g(a\beta_1, \dots, a\beta_s) = 0 \iff g(\beta_1, \dots, \beta_s) = 0.$$

It is easy to check that

$$(\beta_1, \dots, \beta_s, \eta) = \left( \frac{P(m)}{P(n)} - \frac{P_1(m)}{P_1(n)}, \dots, \frac{P(m)}{P(n)} - \frac{P_s(m)}{P_s(n)}, 0 \right)$$

provides a solution to the equations (3.4). Note that for  $m \gg n \gg 0$  the weights are decreasing

$$\beta_1 > \dots > \beta_s$$

because the reduced Hilbert polynomial of  $P_i$  is strictly greater than that of  $P_{i+1}$  by definition of the Harder–Narasimhan filtration.

**Definition 3.4.2.** Let  $\tau = (P_1, \dots, P_s)$  be a Harder–Narasimhan type such that  $P = \sum_{i=1}^s P_i$ . We define

$$\beta_i(\tau) := \beta_i = \frac{P(m)}{P(n)} - \frac{P_i(m)}{P_i(n)}$$

and

$$\beta = \beta(\tau) := \text{idia}(\beta_1, \dots, \beta_1, \beta_2, \dots, \beta_2, \dots, \beta_s, \dots, \beta_s) \in \mathfrak{t}_+ \quad (3.5)$$

where  $\beta_i$  appears  $P_i(n)$  times.

**Remark 3.4.3.** This rational weight  $\beta(\tau)$  depends on  $n$  and  $m$  and will be written as  $\beta(\tau, n, m)$  if it is necessary to make this dependence explicit. Note that if we have two different Harder–Narasimhan types  $\tau = (P_1, \dots, P_s)$  and  $\tau' = (P'_1, \dots, P'_s)$  for pure sheaves with Hilbert Polynomial  $P$ , then for certain values of  $n$  and  $m$  we may have  $\beta(\tau, n, m) = \beta(\tau', n, m)$  (for example, if  $P_i(n) = P'_i(n)$  and  $P_i(m) = P'_i(m)$  for  $1 \leq i \leq s$ ). However, if  $m \gg n \gg 0$  then the rational weights associated to two different Harder–Narasimhan types will be distinct.

**Remark 3.4.4.** In Remarks 3.1.4 and 3.1.6 we discussed the linearisation due to Gieseker when  $W$  is smooth and integral and we are looking at torsion free sheaves.

In exactly the same way as above we can calculate the rational weight  $\beta = \beta(\tau)$  corresponding to a Harder–Narasimhan type  $\tau$  for the linearisation used by Gieseker.

In this case we have

$$\beta_i = \beta_i(\tau) := \frac{r}{P(n)} - \frac{r_i}{P_i(n)}$$

and

$$\beta(\tau) := \beta = \text{idiat}(\beta_1, \dots, \beta_1, \beta_2, \dots, \beta_2, \dots, \beta_s \cdots \beta_s) \in \mathfrak{t}_+$$

where  $\beta_i$  appears  $P_i(n)$  times.

### 3.5 Describing components of $S_\beta$

As in the previous sections we fix a Harder–Narasimhan type  $\tau = (P_1, \dots, P_s)$  and define  $\beta = \beta(\tau) \in \mathfrak{t}_+$  as above. In this section we prove that  $\beta$  is an index for the stratification of  $\overline{Q}$  and also relate certain connected components of  $S_\beta$  to the set  $R_\tau \subset Q$  of points with Harder–Narasimhan type  $\tau$ .

We have a decomposition  $V = V_1 \oplus \dots \oplus V_s$  of  $V$  into weight spaces for the 1-PS  $\lambda_\beta$ . By definition  $Z_\beta$  is the connected components of the fixed point locus of  $\lambda_\beta(\mathbb{C}^*)$  acting on  $\overline{Q}$  on which this 1-PS acts with weight  $\|\beta\|^2$ . The fixed point locus of  $\lambda_\beta$  is well-known:

**Lemma 3.5.1.** *If  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  is a point in  $\overline{Q}$ , then  $\rho$  is fixed by the 1-PS  $\lambda_\beta$  if and only if  $\mathcal{F}$  has a decomposition*

$$\mathcal{F} = \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_s$$

*and we also have a decomposition of the sheaf homomorphism*

$$\rho = \rho_1 \oplus \dots \oplus \rho_s$$

where  $\rho_i : V_i \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}_i$  lies in the quot scheme  $\text{Quot}(V_i \otimes \mathcal{O}(-n), P(\mathcal{F}_i))$ .

In particular, the fixed point locus of  $\lambda_\beta(\mathbb{C}^*)$  acting on  $\overline{Q}$  decomposes into components indexed by the tuple of Hilbert polynomials of the direct summands. Let  $Q_i := \text{Quot}(V_i \otimes \mathcal{O}(-n), P_i)$  and consider

$$F = \{q \in \text{Quot}(V \otimes \mathcal{O}(-n), P) : q = \oplus_{i=1}^s q_i \text{ such that } q_i \in Q_i\} \cong Q_1 \times \cdots \times Q_s.$$

**Corollary 3.5.2.** *The scheme  $F \cap \overline{Q}$  is a union of connected components of  $Z_\beta$ .*

*Proof.* Clearly  $F$  is a union of connected components of the fixed point locus of the one-parameter subgroup  $\lambda_\beta$ . By definition  $Z_\beta$  is the connected components of the fixed point locus in  $\overline{Q}$  such that  $\lambda_\beta$  acts on points in the affine cone  $\mathbb{C}^{N+1}$  lying over these components with weight  $\|\beta\|^2$ . Let  $q = \oplus_{i=1}^s q_i$  be a point in  $F$  where  $q_i : V_i \otimes \mathcal{O}(-n) \rightarrow \mathcal{E}_i$  is a quotient sheaf in  $Q_i$ . The Hilbert-Mumford function  $\mu^\mathcal{L}(q, \lambda_\beta)$  is equal to the weight of the action of  $\lambda_\beta$  on the fibre of  $\mathcal{L}$  over  $q$ ; this is equal to minus the weight of the  $\lambda_\beta$  action on a point in  $\mathbb{C}^{N+1}$  lying over  $q$ . By direct calculation we have

$$\|\beta\|^2 = \sum_{i=1}^s \beta_i^2 P_i(n) = \sum_{i=1}^s \frac{P_i(m)^2}{P_i(n)} - \frac{P(m)^2}{P(n)}$$

and

$$\mu^\mathcal{L}(q, \lambda_\beta) = \sum_{i=1}^s \beta_i P(\mathcal{E}_i, m) = \sum_{i=1}^s \beta_i P_i(m) = \frac{P(m)^2}{P(n)} - \sum_{i=1}^s \frac{P_i(m)^2}{P_i(n)} = -\|\beta\|^2$$

which shows that  $F \cap \overline{Q}$  is a union of connected components of  $Z_\beta$ .  $\square$

The subscheme  $Z_\beta$  is invariant under the subgroup of  $\text{SL}(V)$  which stabilises  $\beta$ ,

$$\text{Stab}\beta = \left( \prod_{i=1}^s \text{GL}(V_i) \right) \cap \text{SL}(V).$$

The original linearisation  $\mathcal{L}$  restricts to a  $\text{Stab}\beta$  linearisation on  $Z_\beta$  which we also denote by  $\mathcal{L}$ . Associated to  $-\beta$  is a character

$$\chi_{-\beta} : \text{Stab}\beta \rightarrow \mathbb{C}^*$$

$$(g_1, \dots, g_s) \mapsto \prod_{i=1}^s \det g_i^{-\beta_i}$$

which we can use to twist the linearisation  $\mathcal{L}$ ; we let  $\mathcal{L}^{\chi_{-\beta}}$  denote this twisted linearisation on  $Z_\beta$ . By definition

$$Z_\beta^{ss} := Z_\beta^{\text{Stab}\beta-ss}(\mathcal{L}^{\chi_{-\beta}})$$

is the open subscheme of  $Z_\beta$  whose geometric points are semistable for this  $\text{Stab}\beta$  action.

**Lemma 3.5.3.** *The central subgroup*

$$Z(\text{Stab}\beta) = \{(t_1, \dots, t_s) \in (\mathbb{C}^*)^s : \prod_{i=1}^s t_i^{P_i(n)} = 1\}$$

fixes every point of  $F$  and so acts by a character  $\chi_F : Z(\text{Stab}\beta) \rightarrow \mathbb{C}^*$  on the fibres of  $\mathcal{L}$  over points of  $F$  where

$$\chi_F(t_1, \dots, t_s) = \prod_{i=1}^s t_i^{-\beta_i P_i(n)}.$$

*Proof.* There is a copy of  $\mathbb{C}^* \subset \text{GL}(V)$  acting trivially on each quot scheme  $\text{Quot}(V \otimes \mathcal{O}(-n), P)$ . Since  $F$  is isomorphic to a product of  $s$  quot schemes  $Q_i$  the centre  $Z(\text{Stab}\beta)$  fixes each point  $q = \oplus q_i$  in  $F$ . Therefore,  $Z(\text{Stab}\beta)$  acts on the fibre of  $\mathcal{L}$  at  $q \in F$  by a character  $\chi_F$ . To calculate this character we revisit the calculations of Section 3.1.1 and generalise these calculations to larger torus actions. Let  $(t_1, \dots, t_s)$  be an element of this central subgroup then we consider its action on  $q$ . The di-

rect summand  $q_i$  is multiplied by  $t_i^{-1}$  and so  $\det H^0(q_i(m))$  is multiplied by  $t_i^{-P_i(m)}$ .

Therefore

$$\chi_F(t_1, \dots, t_s) = \prod_{i=1}^s t_i^{P_i(m)}$$

and since  $\prod t_i^{P_i(n)} = 1$  we may rewrite this as

$$\chi_F(t_1, \dots, t_s) = \prod_{i=1}^s t_i^{-\left(P_i(n) \frac{P(n)}{P(m)} - P_i(m)\right)} = \prod_{i=1}^s t_i^{-\beta_i P_i(n)}$$

which completes the proof.  $\square$

Consider the subgroup

$$G' = \prod_{i=1}^s \mathrm{SL}(V_i)$$

of  $\mathrm{Stab}\beta$ .

**Lemma 3.5.4.** *The semistable subscheme*

$$F^{ss} := F^{\mathrm{Stab}\beta - ss}(\mathcal{L}^{\chi - \beta})$$

for the  $\mathrm{Stab}\beta$  action on  $F$  with respect to  $\mathcal{L}^{\chi - \beta}$  is equal to the semistable subscheme for the  $G'$  action on  $F$  with respect to  $\mathcal{L}$ .

*Proof.* There is a surjective group homomorphism

$$G' \times Z(\mathrm{Stab}\beta) \rightarrow \mathrm{Stab}\beta$$

$$((g'_1, \dots, g'_m), (t_1, \dots, t_s)) \mapsto (t_1 g'_1, \dots, t_s g'_s)$$

whose kernel is a product of finite cyclic groups  $\prod_{i=1}^s \mathbb{Z}/P_i(n)\mathbb{Z}$ . Therefore the reductive group  $\mathrm{Stab}\beta$  is the quotient of the product  $G' \times Z(\mathrm{Stab}\beta)$  by this product of the finite cyclic groups of order  $P_i(n)$ . Since finite groups do not alter GIT semistability

we have that

$$F^{ss} := F^{G' \times Z(\text{Stab}\beta) - ss}(\mathcal{L}^{\chi_{-\beta}}).$$

By Lemma 3.5.3 the centre  $Z(\text{Stab}\beta)$  fixes  $F$  and acts on the fibres of  $\mathcal{L}$  over  $F$  by multiplication by the character  $\chi_F$ . This character given in Lemma 3.5.3 is equal to the character  $\chi_{-\beta} : \text{Stab}\beta \rightarrow \mathbb{C}^*$  defined by the weight  $\beta$ . Since the linearisation  $\mathcal{L}^{\chi_{-\beta}}$  is the original linearisation  $\mathcal{L}$  twisted by the character  $\chi_{-\beta}$  (see Example 1.2.8 for our conventions for such twists) the centre  $Z(\text{Stab}\beta)$  acts trivially on the twisted linearisation. Therefore the semistable set for the action of  $\text{Stab}\beta = G'Z(\text{Stab}\beta)$  with respect to  $\mathcal{L}^{\chi_{-\beta}}$  is equal to the semistable set for the  $G'$  action with respect to  $\mathcal{L}$ .  $\square$

Recall that we have an isomorphism of schemes

$$F \cong Q_1 \times \cdots \times Q_s$$

where  $Q_i = \text{Quot}(V_i \otimes \mathcal{O}(-n), P_i)$ . Let  $\mathcal{L}_i = \det(\pi_{Q_i*}(\mathcal{U}_i \otimes \pi_W^* \mathcal{O}(m)))$  denote Simpson's linearisation of the  $\text{SL}(V_i)$ -action on  $Q_i$  where  $\mathcal{U}_i$  is the universal quotient sheaf on this quot scheme. By [40], Theorem 1.19 provided  $n$  and  $m$  are sufficiently large the points of the semistable subscheme

$$Q_i^{ss} := Q_i^{\text{SL}(V_i) - ss}(\mathcal{L}_i)$$

are quotient sheaves  $q_i : V_i \otimes \mathcal{O}(-n) \rightarrow \mathcal{E}_i$  where  $\mathcal{E}_i$  is Gieseker semistable.

**Proposition 3.5.5.** *Under the isomorphism  $F \cong Q_1 \times \cdots \times Q_s$ , the semistable set  $F^{ss}$  for the  $\text{Stab}\beta$  action on  $F$  with respect to  $\mathcal{L}^{\chi_{-\beta}}$  is isomorphic to the product of the GIT semistable subschemes  $Q_i^{ss}$ :*

$$F^{ss} \cong Q_1^{ss} \times \cdots \times Q_s^{ss}.$$

*Proof.* Let  $q = \oplus_{i=1}^m q_i$  be a point in  $F$  where  $q_i : V_i \otimes \mathcal{O}(-n) \rightarrow \mathcal{E}_i$  represents a quotient sheaf in  $Q_i$ . By Lemma 3.5.4 we have that

$$F^{ss} = F^{G' - ss}(\mathcal{L})$$

where  $G' = \prod_{i=1}^s \mathrm{SL}(V_i)$  and so, by the Hilbert-Mumford criterion,  $q$  is in  $F^{ss}$  if and only  $\mu^{\mathcal{L}}(q, \lambda) \geq 0$  for all 1-PSs  $\lambda$  of  $G'$ ,

Firstly suppose each  $q_i$  is semistable for the  $\mathrm{SL}(V_i)$ -action. Then for any 1-PS  $\lambda_i : \mathbb{C}^* \rightarrow \mathrm{SL}(V_i)$  we have

$$\mu^{\mathcal{L}_i}(q_i, \lambda_i) \geq 0.$$

A 1-PS  $\lambda$  of  $G'$  corresponds to 1-PSs  $\lambda_i$  of  $\mathrm{SL}(V_i)$  for  $i = 1, \dots, s$ . By diagonalising these 1-PS simultaneously we see that

$$\mu^{\mathcal{L}}(q, \lambda) = \sum_{i=1}^s \mu^{\mathcal{L}_i}(q_i, \lambda_i) \geq 0,$$

and so  $q \in F^{ss}$ .

Conversely suppose there is an  $1 \leq i \leq s$  such that  $q_i$  is unstable. Then there is a 1-PS  $\lambda_i$  of  $\mathrm{SL}(V_i)$  such that

$$\mu^{\mathcal{L}_i}(q_i, \lambda_i) < 0.$$

Let  $\lambda = (\lambda_1, \dots, \lambda_s)$  be a 1-PS of  $G'$  where for  $j \neq i$  the 1-PS  $\lambda_j(t) = I_{V_j}$  is trivial and  $\lambda_i$  is the 1-PS above which destabilises  $q_i$ ; then this 1-PS destabilises  $q$  as

$$\mu^{\mathcal{L}}(q, \lambda) = \mu^{\mathcal{L}_i}(q_i, \lambda_i) < 0.$$

Therefore under the isomorphism

$$F \cong Q_1 \times \dots \times Q_s$$

the semistable subscheme  $F^{ss}$  corresponds to the product of the semistable subschemes  $Q_i^{ss}$ .  $\square$

We can now relate the set  $R_\tau$  consisting of points with Harder–Narasimhan type  $\tau$  to the stratum  $S_\beta$  where  $\beta = \beta(\tau)$  is the rational weight given in Definition 3.5. Recall that  $S_\beta = GY_\beta^{ss}$  where  $Y_\beta^{ss} = p_\beta^{-1}(Z_\beta^{ss})$  and  $p_\beta : Y_\beta \rightarrow Z_\beta$  is a retraction onto  $Z_\beta$ . We can also refine the strata by taking connected components:

**Remark 3.5.6.** Recall from Remark 3.1 that from the decomposition  $Z_\beta^{ss} = \sqcup Z_{\beta,j}$  into disjoint closed subsets we get similar decompositions  $Y_\beta^{ss} = \sqcup Y_{\beta,j}^{ss}$  and

$$S_\beta = \bigsqcup GY_{\beta,j}^{ss}$$

where  $Y_{\beta,j}^{ss} = p_\beta^{-1}(Z_{\beta,j}^{ss})$ .

Let  $R_i$  be the open subscheme of  $Q_i$  consisting of quotient sheaves  $q_i : V_i \otimes \mathcal{O}(-n) \rightarrow \mathcal{E}_i$  which are pure of dimension  $e$  and such that  $H^0(q_i(n))$  is an isomorphism. Let  $R_i^{ss}$  denote the semistable subscheme for the  $\mathrm{SL}(V_i)$ -action on  $R_i$ . Then consider the subschemes

$$Z_{(\tau)}^{ss} = \{q = \oplus_{i=1}^s q_i : (q_i : V_i \otimes \mathcal{O}(-n) \rightarrow \mathcal{E}_i) \in R_i^{ss}\}$$

of  $Z_\beta^{ss}$  and  $Y_{(\tau)}^{ss} := p_\beta^{-1}(Z_{(\tau)}^{ss})$  of  $Y_\beta^{ss}$ .

**Theorem 3.5.7.** *Let  $\tau = (P_1, \dots, P_s)$  be the Harder–Narasimhan type of a pure sheaf of dimension  $e$  on  $W$  such that  $P = P_1 + \dots + P_s$ . Let  $\beta(\tau) = \beta(\tau, n, m) \in \mathfrak{t}_+$  be the associated rational weight. Then if  $n$  and  $m$  are sufficiently large we have:*

*i) The rational weight  $\beta(\tau)$  is an index in the stratification  $\{S_\beta : \beta \in \mathcal{B}\}$  of  $\overline{Q}$  associated to the action of  $\mathrm{SL}(V)$  with respect to the linearisation  $\mathcal{L}$ .*

*ii)  $R_\tau = GY_{(\tau)}^{ss}$ ,*

iii)  $R_\tau$  is a union of connected components of  $S_{\beta(\tau)}$ .

*Proof.* By Lemma 3.3.3 we can choose  $n$  sufficiently large so all sheaves with Harder–Narasimhan type  $\tau$  are parametrised by  $Q$ . We assume  $n$  and  $m$  are sufficiently large so for  $i = 1, \dots, s$  the points in  $Q_i^{ss}$  correspond to semistable sheaves. Let  $\beta = \beta(\tau)$  then for  $i$ ) it suffices to show that  $Y_\beta^{ss}$  is nonempty. Let  $\rho : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  be the quotient sheaf with Harder–Narasimhan type  $\tau$  which was used to define  $\beta$ ; then we claim  $\rho \in Y_\beta^{ss}$ , or equivalently,  $p_\beta(\rho) \in Z_\beta^{ss}$ . Since

$$\bar{\rho} := p_\beta(\rho) = \lim_{t \rightarrow 0} \lambda_\beta(t) \cdot \rho$$

we have that  $\bar{\rho} := \oplus_{i=1}^s \rho_i$  represents the graded sheaf  $\oplus_{i=1}^s \mathcal{F}_i$  associated to the Harder–Narasimhan filtration of  $\mathcal{F}$ . By definition of the Harder–Narasimhan filtration the successive quotients  $\mathcal{F}_i$  are semistable and so each  $\rho_i$  is a point in  $R_i^{ss}$ . Then by Proposition 3.5.5,  $\bar{\rho} \in Z_{(\tau)}^{ss} \subset F^{ss} \cap \bar{Q} \subset Z_\beta^{ss}$ .

Now suppose  $q : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{E}$  is a point in  $R_\tau$ ; then the Harder–Narasimhan filtration of  $\mathcal{E}$  gives rise to a filtration of  $V$  by subspaces

$$W^{(i)} = H^0(q(n))^{-1}(H^0(\mathcal{E}^{(i)}(n))).$$

We can choose  $g \in G = \mathrm{SL}(V)$  to be a change of basis matrix sending  $W^{(i)}$  to  $V^{(i)}$  for each  $i$ , which is possible since  $\dim W^{(i)} = \dim V^{(i)} = \sum_{j \leq i} P_i(n)$ . Then, by the same argument as above,  $g \cdot q \in Y_{(\tau)}^{ss}$  and so  $q \in GY_{(\tau)}^{ss}$ .

Since  $\bar{R}_i^{ss} = R_i^{ss}$  (cf. [40] Theorem 1.19) the subscheme  $Z_{(\tau)}^{ss}$  is closed in  $F^{ss} \cap \bar{Q}$  and so, by Corollary 3.5.2,  $Z_{(\tau)}^{ss}$  is a union of connected components of  $Z_\beta^{ss}$ . The points of  $Y_{(\tau)}^{ss} := p_\beta^{-1}(Z_{(\tau)}^{ss})$  are quotient sheaves  $q : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  such that the filtration  $0 \subset V^{(1)} \subset \dots \subset V^{(s)} = V$  used to define  $\beta$  induces the Harder–Narasimhan filtration of  $\mathcal{F}$ . In particular,  $Y_{(\tau)}^{ss}$  is contained in  $R_\tau$  and since  $R_\tau$  is  $G$ -invariant we

have  $GY_{(\tau)}^{ss} \subset R_\tau$ . Therefore, we have  $R_\tau = GY_{(\tau)}^{ss}$ . Since  $Z_{(\tau)}^{ss}$  is a union of connected components of  $Z_\beta^{ss}$  we have that  $R_\tau = GY_{(\tau)}^{ss}$  is a union of connected components of  $S_\beta$  by Remark 3.5.6.  $\square$

As  $\overline{Q}$  and  $Q$  both depend on  $n$  and the strata  $S_\beta$  depend on both  $n$  and  $m$ , we should really write  $\overline{Q}(n)$  and  $Q(n)$  and  $S_\beta(n, m)$ . We can intersect the stratification  $\{S_\beta(n, m) : \beta \in \mathcal{B}(n, m)\}$  of  $\overline{Q}(n)$  with  $Q(n)$  to obtain a stratification

$$Q(n) = \bigsqcup_{\beta \in \mathcal{B}'(n, m)} S_\beta(n, m) \cap Q(n)$$

where  $\mathcal{B}'(n, m) \subset \mathcal{B}(n, m)$  is the set of rational weights  $\beta$  whose associated stratum  $S_\beta(n, m)$  meets  $Q(n)$ . We are really interested in comparing this stratification of  $Q(n)$  with the stratification of  $Q(n)$  by Harder–Narasimhan type. By Remark 3.4.3 we have that for two Harder–Narasimhan types  $\tau$  and  $\tau'$  of pure sheaves with Hilbert polynomial  $P$ , the associated rational weights  $\beta(\tau, n, m)$  and  $\beta(\tau', n, m)$  will be distinct for  $m \gg n \gg 0$ . This leads us to consider the limit of the stratifications  $\{S_\beta(n, m) \cap Q(n) : \beta \in \mathcal{B}'\}$  of  $Q(n)$  as  $n$  and  $m$  tend to infinity which we write as

$$Q(\infty) = \bigsqcup_{\beta \in \mathcal{B}'(\infty)} S_\beta(\infty) \cap Q(\infty).$$

In particular, we see:

**Corollary 3.5.8.** *The limit of these stratifications as  $n$  and  $m$  tend to infinity*

$$Q(\infty) = \bigsqcup_{\beta \in \mathcal{B}'(\infty)} S_\beta(\infty) \cap Q(\infty)$$

*agrees with the stratification by Harder–Narasimhan type.*

**Remark 3.5.9.** In Remark 3.4.4 we defined a rational weight  $\beta(\tau)$  for the linearisation used by Gieseker when  $W$  is smooth and integral and  $e = \dim W$ . In this case an

analogue of the above theorem also holds although this time we only have one integer  $n$  to choose sufficiently large.

### 3.6 Some remarks on taking a quotient of the $G$ -action on $R_\tau$

Since  $R_\tau = GY_{(\tau)}^{ss} \cong G \times_{P_\beta} Y_{(\tau)}^{ss}$  to take a quotient of the  $G$ -action on  $R_\tau$  is equivalent to taking a quotient of the  $P_\beta$ -action on  $Y_{(\tau)}^{ss}$ . In Section 2.2.2 we studied a canonical linearisation  $\mathcal{L}_\beta$  of the  $P_\beta$ -action on a compactification  $\bar{Y}_\beta$  of  $Y_\beta^{ss}$ . This can be restricted to the closure  $\bar{Y}_{(\tau)}$  of  $Y_{(\tau)}^{ss}$  and the generalised GIT quotient with respect to this canonical linearisation is

$$Y_{(\tau)}^{ss} //_{\mathcal{L}_\beta} P_\beta = Z_{(\tau)}^{ss} //_{\mathcal{L}_\beta} P_\beta = Z_{(\tau)}^{ss} //_{\mathcal{L}_\beta} \text{Stab} \beta.$$

The canonical linearisation  $\mathcal{L}_\beta$  on  $Z_\beta$  is equal to  $\mathcal{L}^{\chi-\beta}$  and so by Lemma 3.5.4

$$Z_{(\tau)}^{ss} //_{\mathcal{L}_\beta} \text{Stab} \beta = Z_{(\tau)}^{ss} //_{\mathcal{L}} G'$$

where  $G' = \Pi_{i=1}^s \text{SL}(V_i)$ . Therefore, two sheaves  $\mathcal{F}$  and  $\mathcal{E}$  with Harder–Narasimhan type  $\tau$  are identified in this quotient if and only if the successive quotients in their Harder–Narasimhan filtrations are S-equivalent in the usual sense; that is, the graded objects associated to their Jordan–Hölder filtrations are isomorphic. This is the problem with the canonical quotient: the notion of S-equivalence is too weak. The map

$$p_\beta : Y_{(\tau)}^{ss} \rightarrow Z_{(\tau)}^{ss}$$

sends a quotient sheaf  $q : V \otimes \mathcal{O}(-n) \rightarrow \mathcal{F}$  to the graded quotient sheaf  $\oplus_{i=1}^s q_i$  associated to its Harder–Narasimhan filtration. Since this map corresponds to taking

the limit under a 1-PS, the graded quotient sheaf  $p_\beta(q) = \oplus_{i=1}^s q_i$  is contained in the orbit closure of  $q$ . So if  $p_\beta(q) \in Z_{(\tau)}^{ss}$  is semistable for any linearisation of this action then  $q$  will also be semistable (as the GIT semistable subscheme is open and  $p_\beta(q)$  is in the orbit closure of  $q$ ) and both these points will be identified in the generalised GIT quotient. As suggested in the previous chapter, we need to perturb the canonical linearisation so that  $Z_{(\tau)}^{ss}$  is no longer semistable.

## Chapter 4

# Moduli spaces of complexes

Let  $X$  be a smooth integral projective scheme over the complex numbers and  $\mathcal{O}_X(1)$  be an ample invertible sheaf on  $X$ . From now on we use the term complex to mean a bounded cochain complex of coherent sheaves and all complexes will be over  $X$  unless otherwise stated.

In this chapter we describe a GIT construction due to Schmitt [39] of moduli spaces of S-equivalence classes of semistable complexes. The notion of semistability is determined by a choice of stability parameters and essentially comes from physics; it is closely related to a notion of semistability coming from a Hitchin–Kobayashi correspondence for quiver bundles due to Álvarez-Cónsul and García-Prada [2]. The notion of S-equivalence is weaker than isomorphism and arises from the GIT construction of these moduli spaces.

The Hilbert–Mumford criterion (see Section 1.3) allows us to study semistability by studying one dimensional torus actions. In this chapter we calculate the weights of such actions as this will be used later on to describe a Harder–Narasimhan type stratification for complexes. We also construct a morphism corresponding to taking the direct sum of two complexes which will also be used later in this thesis.

**Notation.** For notational convenience we index the complexes at the bottom:

$$\cdots \rightarrow \mathcal{E}_{k-1} \rightarrow \mathcal{E}_k \rightarrow \mathcal{E}_{k+1} \rightarrow \cdots.$$

## 4.1 Moduli spaces of semistable complexes

We can view a complex of fixed length as a representation of a certain quiver with relations in the category  $\text{Coh } X$  of coherent sheaves on  $X$ . To clarify what is meant by a representation of a quiver in the category of coherent sheaves we give the following definition:

**Definition 4.1.1.** A representation of a quiver  $Q = (V, A, h, t)$  with relations  $R$  in a category  $\mathcal{A}$  is:

- i) For each vertex  $v \in V$ , an object  $A_v \in \text{Ob}(\mathcal{A})$ ,
- ii) For each arrow  $a : u \rightarrow v \in A$ , a morphism  $f_a : A_u \rightarrow A_v$ ,

such that the morphisms  $f_a$  satisfy the relations in  $R$ .

The classical case is when  $\mathcal{A}$  is the category of vector spaces over a field  $k$  and representations of a quiver in this category are simply called quiver representations. Moduli spaces of semistable quiver representations have been constructed by King [23] as GIT quotients where the notion of semistability is determined by a collection of integers which are used to define the linearisation.

We are interested in the case when  $\mathcal{A} = \text{Coh } X$  is the category of coherent sheaves on  $X$  since studying moduli of complexes (up to isomorphism) is equivalent to studying moduli of representations of the following quiver

$$Q = \bullet \xrightarrow{a_1} \bullet \longrightarrow \cdots \xrightarrow{a_{m-1}} \bullet$$

with the relations  $a_i \circ a_{i-1} = 0$  in the category of coherent sheaves. A representation of a quiver  $Q$  in the subcategory of locally free sheaves is referred to as a  $Q$ -bundle and is often used to describe a vector bundle with additional structure.

There is a so-called vortex version of the Hitchin–Kobayashi correspondence for quiver bundles due to Álvarez-Cónsul and García-Prada [2]. The original Hitchin–Kobayashi correspondence due to Uhlenbeck and Yau [42] relates the stability of a vector bundle on a Riemann surface to the existence of a Hermitian-Einstein metric on the bundle. The vortex version also relates stability of a quiver bundle to the existence of Hermitian metrics satisfying certain gauge theoretic equations determined by a collection of stability parameters. Moduli spaces of semistable quiver bundles and torsion free quiver sheaves have been studied by Álvarez-Cónsul [1] and Schmitt [38, 39] respectively. In this section we will describe and use a construction of moduli of torsion free quiver sheaves due to Schmitt [39].

As with any moduli problem we start by fixing any invariants of isomorphism classes of complexes. This is done by fixing the Hilbert polynomials of each of the sheaves occurring in the complex with respect to  $\mathcal{O}_X(1)$  by fixing a collection of Hilbert polynomials  $\underline{P} = (P_1, \dots, P_m)$  of degree  $\dim X$ . Note this fixes the length of the complex and we have assumed for simplicity that the complex is zero outside of the degrees 1 to  $m$ .

#### 4.1.1 Semistability for complexes

The moduli spaces of complexes can only parametrise a certain collection of complexes with invariants  $\underline{P}$  and this collection is determined by a notion of (semi)stability. Schmitt introduces a notion of (semi)stability for complexes which depends on a collection of stability parameters  $(\underline{\sigma}, \underline{\chi})$  where

- $\underline{\sigma} = (\sigma_1, \dots, \sigma_m) \in \mathbb{Z}_{>0}^m$ ,

- $\underline{\eta} = (\eta_1, \dots, \eta_m) \in \mathbb{Q}^m$ ,
- $\delta$  is a positive rational polynomial such that  $\deg \delta = \max(\dim X - 1, 0)$

and we define  $\underline{\chi} := \delta \underline{\eta}$ .

**Remark 4.1.2.** There will be two points of view which we will take when it comes to thinking about these stability parameters. Firstly, when we are thinking about semistability of complexes with fixed invariants  $\underline{P}$ , we will assume that the  $\eta_i$  satisfy  $\sum \eta_i r_i = 0$  where  $r_i$  is the rank determined by the top coefficient of the Hilbert polynomial  $P_i$ . This condition is needed so we can define the linearisation of the group action on the parameter space for complexes with invariants  $\underline{P}$ . In fact  $\underline{\eta}$  is used to define a character of a product of general linear groups and this condition ensures this character is trivial on the copy of  $\mathbb{C}^*$  which acts trivially on the parameter space. Secondly, when we are thinking about semistability of complexes with varying invariants (for example, when we want to study Harder–Narasimhan filtrations), we will not impose this condition on  $\underline{\eta}$ . When we wish to emphasise the fact that we are thinking of stability parameters in the first sense we will refer to these as stability parameters for  $\underline{P}$ . In Remark 4.1.5 we will see how to go between these points of view.

**Definition 4.1.3.** The reduced Hilbert polynomial of a complex of torsion free sheaves  $\mathcal{E}$  with respect to the parameters  $(\underline{\sigma}, \underline{\chi})$  is defined as

$$P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}) := \frac{\sum_{i=1}^m \sigma_i P(\mathcal{E}_i) - \chi_i \text{rk } \mathcal{E}_i}{\sum_{i=1}^m \sigma_i \text{rk } \mathcal{E}_i}$$

where  $P(\mathcal{E}_i)$  and  $\text{rk } \mathcal{E}_i$  are the Hilbert polynomial and rank of the sheaf  $\mathcal{E}_i$  calculated with respect to the ample line bundle  $\mathcal{O}_X(1)$ . We call this weighted analogue of the reduced Hilbert polynomial the reduced Hilbert polynomial with respect to  $(\underline{\sigma}, \underline{\chi})$  or simply the reduced Hilbert polynomial if the choice of stability parameters is clear.

The following definition is due to Schmitt [39] (see also [2, 6, 38]).

**Definition 4.1.4.** A nonzero complex  $\mathcal{E}$  is  $(\underline{\sigma}, \underline{\chi})$ -semistable if each sheaf  $\mathcal{E}_i$  is torsion free and for any nonzero proper subcomplex  $\mathcal{F} \subset \mathcal{E}$ , we have an inequality of polynomials

$$P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{F}) \leq P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}).$$

The ordering ‘ $\leq$ ’ on polynomials is given by lexicographic ordering of the coefficients of the polynomials; therefore  $P \leq Q$  if and only if  $P(x) \leq Q(x)$  for all  $x \gg 0$ . We say the complex is  $(\underline{\sigma}, \underline{\chi})$ -stable if this inequality is strict for all such subcomplexes.

**Remark 4.1.5.** Note that for any rational number  $C$  if we let  $\underline{\eta}' = \underline{\eta} - C\underline{\sigma}$  and  $\underline{\chi}' = \delta\underline{\eta}'$ , then the notions of  $(\underline{\sigma}, \underline{\chi})$ -semistability and  $(\underline{\sigma}, \underline{\chi}')$ -semistability are equivalent. In particular, for any invariants  $\underline{P}$  if we let

$$C = \frac{\sum_{i=1}^m \eta_i r_i}{\sum_{i=1}^m \sigma_i r_i},$$

then the stability parameters  $(\underline{\sigma}, \underline{\chi}')$  are stability parameters for  $\underline{P}$ ; i.e.,  $\sum_{i=1}^m \eta'_i r_i = 0$ . We will call  $(\underline{\sigma}, \underline{\chi}')$  the stability parameters for  $\underline{P}$  associated to  $(\underline{\sigma}, \underline{\chi})$ .

**Remark 4.1.6.** These stability conditions have many useful properties similar to Mumford’s slope stability for vector bundles on curves and Gieseker’s reduced Hilbert polynomial stability for coherent sheaves on projective schemes. As usual, these notions of stability satisfy the see-saw property; that is, if we have a short exact sequence of complexes

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$$

then either

- 1)  $P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}) < P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{F}) < P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{G})$ ,
- 2)  $P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}) > P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{F}) > P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{G})$ , or

$$3) \ P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}) = P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{F}) = P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{G}).$$

A useful consequence of this property is that if  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  is a nonzero morphism of  $(\underline{\sigma}, \underline{\chi})$ -semistable complexes then we have an inequality

$$P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}) \leq P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{F}).$$

In particular we can use this property to construct Harder–Narasimhan filtrations (see Definition 5.1.1) and Jordan–Hölder filtrations for semistable complexes (see Definition 4.1.15).

The moduli space of  $(\underline{\sigma}, \underline{\chi})$ -semistable complexes with Hilbert polynomials  $\underline{P}$  is constructed as a GIT quotient of a parameter scheme  $\mathfrak{T} = \mathfrak{T}_{\underline{P}}$  by a reductive group. The linearisation of this action is determined by the stability parameters. In this section we carefully describe this GIT construction of Schmitt [39].

#### 4.1.2 The parameter space

As we have fixed invariants  $\underline{P}$ , we assume we have picked stability parameters  $(\underline{\sigma}, \underline{\chi})$  for  $\underline{P}$  so that  $\sum_{i=1}^m \eta_i r_i = 0$ . The set of sheaves occurring in a  $(\underline{\sigma}, \underline{\chi})$ -semistable complex  $\mathcal{E}$  with invariants  $\underline{P}$  is bounded by the usual arguments (see [40], Theorem 1.1) and so we may choose  $n \gg 0$  so that all these sheaves are  $n$ -regular. Fix a complex vector space  $V_i$  of dimension  $P_i(n)$  and let  $Q_i$  be the open subscheme of  $\text{Quot}(V_i \otimes \mathcal{O}_X(-n), P_i)$  parametrising torsion free quotient sheaves  $q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i$  such that  $H^0(q_i(n))$  is an isomorphism. The parameter scheme  $\mathfrak{T}$  is constructed as a locally closed subscheme of a projective bundle  $\mathfrak{D}$  over the product  $Q := Q_1 \times \cdots \times Q_m$ .

Given a  $(\underline{\sigma}, \underline{\chi})$ -semistable complex  $\mathcal{E}$  with Hilbert polynomials  $\underline{P}$  we can use the evaluation maps

$$H^0(\mathcal{E}_i(n)) \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i$$

along with a choice of isomorphism  $V_i \cong H^0(\mathcal{E}_i(n))$  to parametrise the  $i$ th sheaf  $\mathcal{E}_i$  by a point  $q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i$  in  $Q_i$ . The boundary maps of the complex  $d_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$  can be used to define a homomorphism

$$d : \oplus_{i=1}^m \mathcal{E}_i \rightarrow \oplus_{i=1}^m \mathcal{E}_i.$$

Let  $V = \oplus_{i=1}^m V_i$  and

$$\psi := d(n) \circ (\oplus_{i=1}^m q_i(n)) : V \otimes \mathcal{O}_X \rightarrow \oplus_{i=1}^m \mathcal{E}_i(n).$$

We construct a projective bundle  $\mathfrak{D}$  which parametrises these homomorphisms  $\psi$  and a complex number  $\zeta$ . We are only interested in studying the locus where  $\zeta = 1$  so that the points correspond to such homomorphisms; however, by introducing  $\zeta$  we compactify this set up.

Let  $V_i \otimes (\pi_X^{Q_i \times X})^* \mathcal{O}_X(-n) \rightarrow \mathcal{U}_i$  denote the universal quotient on  $Q_i \times X$ ; then if we pull this back to  $Q \times X$  via  $\pi_i : Q \times X \rightarrow Q_i \times X$  and take the direct sum for  $i = 1, \dots, m$ , we get a universal quotient

$$\Gamma : V \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(-n) \rightarrow \mathcal{U} = \oplus_{i=1}^m \pi_i^* \mathcal{U}_i$$

where  $V = \oplus_{i=1}^m V_i$ . Let  $p : Q \times X \rightarrow Q$  be the projection morphism and define  $\mathfrak{D} := \mathbb{P}(\mathcal{R})$  to be the projective bundle over  $Q$  associated to the locally free sheaf

$$\mathcal{R} := V^\vee \otimes p_* \left( \mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n) \right) \oplus \mathcal{O}_Q$$

on  $Q$ . Note that here we need to assume  $n \gg 0$  so that  $\mathcal{R}$  is locally free. Recall that our convention is that the points of a projective bundle correspond to lines in the fibres of the corresponding vector bundle. Let  $\pi_{\mathfrak{D}} : \mathfrak{D} \rightarrow Q$  denote the projection

map to  $Q$ ; then the tautological inclusion  $\mathcal{O}_{\mathfrak{D}}(-1) \hookrightarrow \pi_{\mathfrak{D}}^* \mathcal{R}$  over  $\mathfrak{D}$  splits into two tautological homomorphisms:

$$\psi_{\mathfrak{D}} : V \otimes \mathcal{O}_{\mathfrak{D}} \rightarrow \pi_{\mathfrak{D}}^* p_* \left( \mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n) \right) \otimes \mathcal{O}_{\mathfrak{D}}(1)$$

and

$$\zeta_{\mathfrak{D}} : \mathcal{O}_{\mathfrak{D}} \rightarrow \mathcal{O}_{\mathfrak{D}}(1).$$

A point of  $\mathfrak{D}$  over a point  $(q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i)_{i=1}^m$  in  $Q$  is given (up to scalar multiplication) by a homomorphism  $\psi : V \rightarrow \oplus_{i=1}^m H^0(\mathcal{E}_i(n))$  and a complex number  $\zeta$ .

We can construct a closed subscheme  $\tilde{\mathfrak{D}}$  of the projective bundle  $\mathfrak{D}$  consisting of points  $(q_1, \dots, q_m, [\psi : \zeta])$  where  $\psi$  factors through the quotient map  $\oplus_{i=1}^m q_i(n)$ , so that  $\psi$  is constructed from a morphism  $d : \oplus \mathcal{E}_i \rightarrow \oplus \mathcal{E}_i$ . More precisely, let  $\mathcal{K}$  be the kernel of  $\Gamma(n) : V \otimes \mathcal{O}_{Q \times X} \rightarrow \mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n)$  and consider the composition

$$\pi_{\mathfrak{D}}^* p_* \mathcal{K} \rightarrow V \otimes \mathcal{O}_{\mathfrak{D}} \rightarrow \pi_{\mathfrak{D}}^* p_* \left( \mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n) \right) \otimes \mathcal{O}_{\mathfrak{D}}$$

where the second map is the tautological map  $\psi_{\mathfrak{D}}$ . Then for  $n \gg 0$  this is a homomorphism of locally free sheaves and we define  $\tilde{\mathfrak{D}}$  to be the vanishing locus of this homomorphism.

Let  $\mathcal{S} = p_* \mathcal{H}om(\mathcal{U}, \mathcal{U})$ ; then the homomorphism  $p_* \Gamma(n) : V \otimes \mathcal{O}_Q \rightarrow p_*(\mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n))$  defines a homomorphism of sheaves

$$\mathcal{S} \rightarrow V^{\vee} \otimes p_*(\mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n))$$

over  $Q$ . Let  $\mathcal{S} \oplus \mathcal{O}_Q \rightarrow \mathcal{R}$  be the homomorphism obtained from this homomorphism and the identity homomorphism on  $\mathcal{O}_Q$ . The following lemma follows immediately from the construction of  $\tilde{\mathfrak{D}}$ .

**Lemma 4.1.7.** *The restriction to  $\tilde{\mathfrak{D}}$  of the tautological inclusion  $\mathcal{O}_{\mathfrak{D}}(-1) \hookrightarrow \pi_{\mathfrak{D}}^* \mathcal{R}$  factors uniquely through the restriction to  $\tilde{\mathfrak{D}}$  of the homomorphism  $\pi_{\mathfrak{D}}^*(\mathcal{S} \oplus \mathcal{O}_Q) \rightarrow \pi_{\mathfrak{D}}^* \mathcal{R}$ . We call  $\mathcal{O}_{\mathfrak{D}}(-1)|_{\tilde{\mathfrak{D}}} \hookrightarrow \pi_{\mathfrak{D}}^*(\mathcal{S} \oplus \mathcal{O}_Q)|_{\tilde{\mathfrak{D}}}$  the tautological inclusion on  $\tilde{\mathfrak{D}}$ .*

We may interpret the tautological inclusion on  $\tilde{\mathfrak{D}}$  as two tautological morphisms on  $\tilde{\mathfrak{D}} \times X$ :

$$d_{\tilde{\mathfrak{D}} \times X} : \mathcal{U}_{\tilde{\mathfrak{D}} \times X} \rightarrow \mathcal{U}_{\tilde{\mathfrak{D}} \times X} \otimes (\pi_{\tilde{\mathfrak{D}}}^{\mathfrak{D} \times X})^* \mathcal{O}_{\mathfrak{D}}(1)|_{\tilde{\mathfrak{D}} \times X}$$

and

$$\zeta_{\tilde{\mathfrak{D}} \times X} : \mathcal{O}_{\tilde{\mathfrak{D}} \times X} \rightarrow (\pi_{\tilde{\mathfrak{D}}}^{\mathfrak{D} \times X})^* \mathcal{O}_{\mathfrak{D}}(1)|_{\tilde{\mathfrak{D}} \times X}$$

where  $\mathcal{U}_{\tilde{\mathfrak{D}} \times X} := (\pi_{Q \times X}^{\mathfrak{D} \times X})^*(\mathcal{U})|_{\tilde{\mathfrak{D}} \times X}$ . Since  $\mathcal{U} = \bigoplus_{i=1}^m \pi^* \mathcal{U}_i$ , we may write  $d_{\tilde{\mathfrak{D}} \times X} = (d_{i,j})_{\tilde{\mathfrak{D}} \times X}$  where

$$(d_{i,j})_{\tilde{\mathfrak{D}} \times X} : (\mathcal{U}_i)_{\tilde{\mathfrak{D}} \times X} \rightarrow (\mathcal{U}_j)_{\tilde{\mathfrak{D}} \times X} \otimes (\pi_{\tilde{\mathfrak{D}}}^{\mathfrak{D} \times X})^* \mathcal{O}_{\mathfrak{D}}(1)|_{\tilde{\mathfrak{D}} \times X}$$

and  $(\mathcal{U}_i)_{\tilde{\mathfrak{D}} \times X} = (\pi_{Q \times X}^{\mathfrak{D} \times X})^*(\mathcal{U}_i)|_{\tilde{\mathfrak{D}} \times X}$ .

**Definition 4.1.8.** The parameter scheme  $\mathfrak{T}$  is the subscheme of  $\tilde{\mathfrak{D}}$  defined by

- i)  $(d_{i,j})|_{\mathfrak{T} \times X} = 0$  if  $j \neq i + 1$ ,
- ii)  $(d_{i,i+1})|_{\mathfrak{T} \times X} \circ (d_{i-1,i})|_{\mathfrak{T} \times X} = 0$ ,
- iii)  $\zeta|_{\mathfrak{T} \times X} \neq 0$ .

The first two conditions are closed conditions whereas the third condition is open and so  $\mathfrak{T}$  is a locally closed subscheme of  $\mathfrak{D}$ .

**Remark 4.1.9.** Let  $\iota : \mathfrak{T} \hookrightarrow \mathfrak{D}$  denote the inclusion morphism; then the tautological inclusion on  $\mathfrak{T}$  is given by  $\iota^* \mathcal{O}_{\mathfrak{D}}(-1) \hookrightarrow \iota^* \pi_{\mathfrak{D}}^* \mathcal{S}$  where  $\mathcal{S}$  is the sheaf  $p_* \mathcal{H}om(\mathcal{U}, \mathcal{U})$  over  $Q$ .

We will write points of  $\mathfrak{T}$  as  $(q_1, \dots, q_m, d)$  where  $q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i$  represents a point in  $Q_i$  and  $d$  is given by  $d_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$  which satisfy  $d_i \circ d_{i-1} = 0$ . If we wish to view  $z$  as a point of the projective bundle  $\mathfrak{D}$  we may instead write it as  $(q_1, \dots, q_m, [d(n) \circ \oplus_{i=1}^m q_i(n) : 1])$ .

**Remark 4.1.10.** The construction of the parameter scheme  $\mathfrak{T}$  depends on the choice of  $n$  and the Hilbert polynomials  $\underline{P}$ . We write  $\mathfrak{T}_{\underline{P}}$  if we wish to emphasise its dependence on the Hilbert polynomials.

### 4.1.3 The group action

For  $i = 1, \dots, m$  we have fixed vector spaces  $V_i$  of dimension  $P_i(n)$ . The group  $\mathrm{GL}(V_i)$  acts on the scheme  $Q_i$  by

$$g \cdot q_i : V_i \otimes \mathcal{O}_X(-n) \xrightarrow{g^{-1}} V_i \otimes \mathcal{O}_X(-n) \xrightarrow{q_i} \mathcal{E}_i$$

where  $g \in \mathrm{GL}(V_i)$  and  $q_i$  represents a quotient sheaf in  $Q_i$ .

The product  $\prod_{i=1}^m \mathrm{GL}(V_i) \subset \mathrm{GL}(\oplus_{i=1}^m V_i)$  acts on the parameter scheme  $\mathfrak{T}$  by acting on  $V := \oplus_{i=1}^m V_i$ : if  $\psi : V \rightarrow \oplus_{i=1}^m H^0(\mathcal{E}_i(n))$  is a point in the parameter scheme  $\mathfrak{T}$  over  $(q_1, \dots, q_m) \in Q$  and  $g = (g_1, \dots, g_m) \in \prod_{i=1}^m \mathrm{GL}(V_i)$ , then

$$g \cdot \psi := \psi \circ g^{-1}$$

which sits over  $(g_1 \cdot q_1, \dots, g_m \cdot q_m) \in Q$ . If instead we consider  $\tilde{\psi} := \oplus H^0(q_i(n))^{-1} \circ \psi : V \rightarrow V$  then this action corresponds to conjugating  $\tilde{\psi}$  by  $g$ ; that is,

$$g \circ \tilde{\psi} \circ g^{-1} = \widetilde{g \cdot \psi}.$$

The orbits of this action on the parameter scheme  $\mathfrak{T}$  are in one-to-one correspondence with isomorphism classes of complexes. Since a copy of  $\mathbb{C}^* = \mathbb{C}^*(I_{V_1}, \dots, I_{V_m})$  acts trivially we can look at the action of  $(\prod_i \mathrm{GL}(V_i))/\mathbb{C}^*$  on the parameter scheme. We

take a GIT quotient by a closely related group

$$G(\underline{\sigma}, \underline{P}(n)) = \prod_{i=1}^m \mathrm{GL}(V_i) \cap \mathrm{SL}(\oplus_i V_i^{\oplus \sigma_i}) = \left\{ \begin{array}{l} (h_1, \dots, h_m) \in \prod \mathrm{GL}(V_i) : \\ \det(h_1)^{\sigma_1} \cdots \det(h_m)^{\sigma_m} = 1 \end{array} \right\}$$

which maps with finite kernel onto  $(\prod_i \mathrm{GL}(V_i))/\mathbb{C}^*$ .

#### 4.1.4 The associated decoration

The first part of the linearisation of this action on the parameter scheme  $\mathfrak{T}$  is given by embedding  $\mathfrak{T}$  in another projective bundle  $\mathfrak{B}_{\underline{\sigma}}$  over  $Q$  which depends on the stability parameters  $\underline{\sigma}$ . This embedding is given by associating to the boundary maps of a given complex a decoration; that is, another homomorphism of sheaves on  $X$  which depends on the stability parameters  $\underline{\sigma}$  and a complex number  $\zeta$ .

Let  $\mathcal{E}.$  be a complex of torsion free sheaves with Hilbert polynomials  $\underline{P}$ ; then Schmitt associates to the boundary maps  $d_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$  of the complex  $\mathcal{E}.$  and a complex number  $\zeta$  a decoration

$$\varphi = \varphi(\underline{\sigma}) : (\mathcal{E}^{\otimes r})^{\oplus 2} \rightarrow \det(\mathcal{E})$$

where  $\mathcal{E} := \oplus_i \mathcal{E}_i^{\oplus \sigma_i}$  and  $r = \sum \sigma_i r_i$ . In this section we explain how to associate such a decoration to a given complex.

The boundary maps  $d_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$  define a homomorphism  $D = (D_{i,j}) : \mathcal{E} \rightarrow \mathcal{E}$  by

$$D_{i,j} = \begin{cases} d_i^{\oplus \sigma_i \sigma_j} : \mathcal{E}_i^{\oplus \sigma_i} \rightarrow \mathcal{E}_j^{\oplus \sigma_j} & \text{if } j = i + 1 \\ 0 & \text{if } j \neq i + 1. \end{cases}$$

Let  $U \subset X$  be the largest open set on which all of the sheaves  $\mathcal{E}_i$  are locally free. Since these sheaves are torsion free and  $X$  is smooth the complement of  $U$  has codimension at least two.

The homomorphism  $D|_U$  and complex number  $\zeta$  define a homomorphism

$$\varphi_0 : \mathcal{H}om(\mathcal{E}|_U, \mathcal{E}|_U) \oplus \mathcal{O}_U \rightarrow \mathcal{O}_U$$

$$(H, w) \rightarrow \text{tr}(HD|_U) + \zeta w$$

where  $\text{tr} : \mathcal{H}om(\mathcal{E}|_U, \mathcal{E}|_U) \rightarrow \mathcal{O}_U$  is the trace homomorphism.

Let  $R$  be a complex vector space of dimension  $r = \sum \sigma_i r_i$  where the positive integers  $r_i$  are the ranks determined by the top coefficients of the Hilbert polynomials  $P_i$ . The wedge product  $\wedge^{r-1} R \otimes R \rightarrow \wedge^r R$  is a perfect pairing and so we have an isomorphism

$$(\wedge^r R)^\vee \otimes \wedge^{r-1} R \cong R^\vee.$$

The vector space  $\wedge^{r-1} R \otimes R$  is a direct summand of  $R^{\otimes r}$ : the inclusion and projection maps are given by

$$\wedge^{r-1} R \otimes R \longleftrightarrow R^{\otimes r}$$

$$a_1 \wedge \cdots \wedge a_{r-1} \otimes b \rightarrow \frac{1}{(r-1)!} \sum_{\sigma \in \Sigma_{r-1}} \text{sign}(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(r-1)} \otimes b$$

$$a_1 \wedge \cdots \wedge a_{r-1} \otimes a_r \leftarrow a_1 \otimes \cdots \otimes a_r.$$

We have isomorphisms

$$\text{Hom}(R, R) \cong R^\vee \otimes R \cong (\wedge^r R)^\vee \otimes \wedge^{r-1} R \otimes R$$

and so  $\text{Hom}(R, R)$  is a direct summand of  $(\wedge^r R)^\vee \otimes R^{\otimes r}$  and similarly the diagonal  $\mathbb{C} \subset \text{Hom}(R, R)$  is also a direct summand of  $(\wedge^r R)^\vee \otimes R^{\otimes r}$ . Therefore, we have a decomposition

$$(\wedge^r R)^\vee \otimes (R^{\otimes r})^{\oplus 2} = \text{Hom}(R, R) \oplus \mathbb{C} \oplus Z.$$

The sheaf  $\mathcal{E}|_U$  is locally free of rank  $r = \sum \sigma_i r_i$  and so similarly there is a direct

sum decomposition of locally free sheaves

$$\det(\mathcal{E}|_U)^\vee \otimes (\mathcal{E}|_U^{\otimes r})^{\oplus 2} \cong \mathcal{H}om(\mathcal{E}|_U, \mathcal{E}|_U) \oplus \mathcal{O}_U \oplus \mathcal{Z}.$$

Let

$$\varphi_1 : \det(\mathcal{E}|_U)^\vee \otimes (\mathcal{E}|_U^{\otimes r})^{\oplus 2} \rightarrow \mathcal{O}_U$$

be the composition of the projection  $\det(\mathcal{E}|_U)^\vee \otimes (\mathcal{E}|_U^{\otimes r})^{\oplus 2} \twoheadrightarrow \mathcal{H}om(\mathcal{E}|_U, \mathcal{E}|_U) \oplus \mathcal{O}_U$  with  $\varphi_0$  and let  $j : U \hookrightarrow X$  denote the inclusion of the open subscheme  $U$ . The decoration associated to the complex  $\mathcal{E}$  is the composition

$$\varphi : (\mathcal{E}^{\otimes r})^{\oplus 2} \otimes \det(\mathcal{E})^\vee \rightarrow j_*((\mathcal{E}|_U^{\otimes r})^{\oplus 2} \otimes \det(\mathcal{E}|_U)^\vee) \rightarrow j_*\mathcal{O}_U \cong \mathcal{O}_X$$

where the first map is the natural map and the second map is  $j_*\varphi_1$ . The isomorphism  $j_*\mathcal{O}_U \cong \mathcal{O}_X$  follows from the fact that  $X$  is smooth and  $\text{codim}(X - U, X) \geq 2$  (cf. [31] Lemma 4.2).

**Definition 4.1.11.** Let  $z = (q_1, \dots, q_m, d)$  be a point in  $\mathfrak{T}$  where  $q_i$  denotes a quotient sheaf  $q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i$ . Then the morphism  $d : \oplus \mathcal{E}_i \rightarrow \oplus \mathcal{E}_i$  and complex number  $\zeta = 1$  define a decoration  $\varphi(d, 1) : (\mathcal{E}^{\otimes r})^{\oplus 2} \rightarrow \det(\mathcal{E})$  as above where  $\mathcal{E} = \oplus \mathcal{E}_i^{\oplus \sigma_i}$ . The decoration associated to  $z$  is the homomorphism

$$\varphi(z) = \varphi(d, 1) \circ ((\oplus q_i^{\oplus \sigma_i})^{\otimes r})^{\oplus 2} : ((\oplus V_i^{\oplus \sigma_i})^{\otimes r})^{\oplus 2} \otimes \mathcal{O}_X(-rn) \rightarrow \det(\mathcal{E}).$$

**Remark 4.1.12.** More generally given a point  $z = (q_1, \dots, q_m, [d : \zeta]) \in \tilde{\mathfrak{T}}$  we can associate to this point a decoration which is defined up to scalar multiplication,

$$\varphi(z) = \varphi(d, \zeta) \circ ((\oplus q_i^{\oplus \sigma_i})^{\otimes r})^{\oplus 2} : ((\oplus V_i^{\oplus \sigma_i})^{\otimes r})^{\oplus 2} \otimes \mathcal{O}_X(-rn) \rightarrow \det(\mathcal{E})$$

where  $\varphi(d, \zeta)$  is the decoration associated to the homomorphism  $d : \oplus \mathcal{E}_i \rightarrow \oplus \mathcal{E}_i$  and

complex number  $\zeta$ . Note that for a nonzero complex number  $c$  we have  $\varphi(cd, c\zeta) = c\varphi(d, \zeta)$  and so the decoration is only defined up to scalar multiplication.

Let  $M = \oplus_{i=1}^m V_i^{\oplus \sigma_i}$ ; then

$$\mathcal{C}_{\underline{\sigma}} := ((M^{\otimes r})^{\oplus 2})^\vee \otimes p_* \left( \det(\mathcal{U}_{\underline{\sigma}}) \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(rn) \right)$$

is a locally free sheaf over  $Q$  provided  $n$  is sufficiently large. Let  $\pi_{\mathfrak{B}_{\underline{\sigma}}} : \mathfrak{B}_{\underline{\sigma}} = \mathbb{P}(\mathcal{C}_{\underline{\sigma}}) \rightarrow Q$  be the associated projective bundle over  $Q$ . There is an action of

$$G = G(\underline{\sigma}, \underline{P}(n)) = \prod_{i=1}^m \mathrm{GL}(V_i) \cap \mathrm{SL}(M)$$

on  $\mathfrak{B}_{\underline{\sigma}}$ : if  $g \in G$  and  $\varphi$  lies over  $(q_1, \dots, q_m)$  in the projective bundle  $\mathfrak{B}_{\underline{\sigma}}$ , then we define  $g \cdot \varphi$  to be the composition

$$(M^{\otimes r})^{\oplus 2} \xrightarrow{g^{-1}} (M^{\otimes r})^{\oplus 2} \xrightarrow{\varphi} \det(\mathcal{E})(rn).$$

The embedding  $h : \mathfrak{T} \rightarrow \mathfrak{B}_{\underline{\sigma}}$  is defined on points by

$$z = (q_1, \dots, q_m, d) \mapsto (q_1, \dots, q_m, \varphi(z))$$

where  $\varphi(z)$  denotes the decoration associated this point. More precisely, by [16] II Proposition 7.12, constructing a morphism  $h : \mathfrak{T} \rightarrow \mathfrak{B}_{\underline{\sigma}}$  is equivalent to giving an invertible sheaf  $L$  over  $\mathfrak{T}$ , a morphism of schemes  $\tilde{h} : \mathfrak{T} \rightarrow Q$  and an injective homomorphism  $L \rightarrow \tilde{h}^* \mathcal{C}_{\underline{\sigma}}$  of sheaves over  $\mathfrak{T}$  (note that the convention in [16] for projective bundles is opposite to our convention).

The morphism  $h : \mathfrak{T} \rightarrow \mathfrak{B}_{\underline{\sigma}}$  is constructed by taking  $L = \iota^* \mathcal{O}_{\mathfrak{D}}(-1)$  where  $\iota : \mathfrak{T} \hookrightarrow \mathfrak{D}$  is the inclusion and letting  $\tilde{h} = \pi_{\mathfrak{D}} \circ \iota : \mathfrak{T} \rightarrow Q$  where  $\pi_{\mathfrak{D}} : \mathfrak{D} \rightarrow Q$ . The inclusion defining  $h$  is given by the composition

$$\iota^* \mathcal{O}_{\mathfrak{D}}(-1) \rightarrow \iota^* \pi_{\mathfrak{D}}^* p_* \mathcal{H}om(\mathcal{U}, \mathcal{U}) \rightarrow \iota^* \pi_{\mathfrak{D}}^* \mathcal{C}_{\underline{\sigma}}$$

where the first homomorphism is the tautological inclusion on  $\mathfrak{T}$  and the second homomorphism corresponds to associating to a homomorphism  $d : \oplus \mathcal{E}_i \rightarrow \oplus \mathcal{E}_i$  and the complex number  $\zeta = 1$  the decoration as defined above. Schmitt shows the associated morphism  $h : \mathfrak{T} \rightarrow \mathfrak{B}_{\underline{\sigma}}$  is  $G$ -equivariant, injective and proper (see [39] Section 2.2). Therefore we can pullback a linearisation of the  $G$ -action on  $\mathfrak{B}_{\underline{\sigma}}$  via  $h$  to get a linearisation of the  $G$ -action on the parameter scheme  $\mathfrak{T}$ .

#### 4.1.5 The linearisation

The linearisation of the  $G$ -action on  $\mathfrak{B}_{\underline{\sigma}}$  used by Schmitt [39] is given by taking another equivariant embedding of  $\mathfrak{B}_{\underline{\sigma}}$  into a projective  $G$ -scheme  $B_{\underline{\sigma}}$  and pulling back a linearisation on  $B_{\underline{\sigma}}$  to  $\mathfrak{T}$ .

For each quot scheme  $Q_i$  we use a Gieseker style embedding (cf. [13] and Remark 3.1.4) into a projective bundle  $B_i$  over a union of connected components  $\text{Pic}(X)_i$  of the Picard scheme of  $X$ . Let  $\text{Pic}(X)_i \subset \text{Pic}(X)$  be the union of the components of the Picard scheme which contain the determinant of a sheaf  $\mathcal{E}_i$  parametrised by  $Q_i$  and  $\mathcal{P}_i$  be a Poincaré line bundle on  $\text{Pic}(X)_i \times X$ . Then there is an isomorphism

$$\det \mathcal{U}_i \cong (D_i \times \text{id}_X)^* \mathcal{P}_i \otimes p_i^* \mathcal{A}_i \quad (4.1)$$

where  $\mathcal{A}_i$  is an invertible sheaf on  $Q_i$  and  $p_i : Q_i \times X \rightarrow Q_i$  is the projection morphism and  $D_i : Q_i \rightarrow \text{Pic}(X)_i$  is the morphism defined by the family of line bundles  $\det \mathcal{U}_i$  over  $Q_i$ . Let

$$\mathcal{W}_i := (\wedge^{r_i} V_i)^\vee \otimes (\pi_i^1)_* (\mathcal{P}_i \otimes (\pi_i^2)^* \mathcal{O}_X(r_i n))$$

where  $\pi_i^1$  and  $\pi_i^2$  be the projections from  $\text{Pic}(X)_i \times X$  to the first and second factors. For  $n$  sufficiently large this sheaf is locally free and we let  $B_i = \mathbb{P}(\mathcal{W}_i)$  denote the associated projective bundle. As we saw in Remark 3.1.4, there is an injective homomorphism  $\mathcal{A}_i^\vee \rightarrow D_i^* \mathcal{W}_i$  of sheaves over  $Q_i$  which defines an  $\text{SL}(V_i)$ -equivariant and

injective morphism  $h_i : Q_i \rightarrow B_i$ . Moreover, by modifying the choice of Poincaré bundle  $\mathcal{P}_i$  we can assume that  $\mathcal{A}_i$  is very ample relative to  $\text{Pic}(X)_i$ . We let  $\mathcal{L}_i := \mathcal{O}_{B_i}(1)$  denote the natural  $\text{SL}(V_i)$ -linearisation where the action on  $B_i$  is described in Remark 3.1.4.

Let  $R := \prod_{i=1}^m \text{Pic}(X)_i$  and  $\mathcal{P}_{\underline{\sigma}} := \otimes (\pi_i^R)^* \mathcal{P}_i^{\otimes \sigma_i}$  where  $\pi_i^R : R \times X \rightarrow \text{Pic}(X)_i \times X$  is the projection. The sheaf

$$\mathcal{W}'_{\underline{\sigma}} = ((M^{\otimes r})^{\oplus 2})^{\vee} \otimes (\pi_R^{R \times X})_*(\mathcal{P}_{\underline{\sigma}} \otimes (\pi_X^{R \times X})^* \mathcal{O}_X(rn))$$

over  $R$  is locally free for  $n$  sufficiently large and so let  $B'_{\underline{\sigma}} = \mathbb{P}(\mathcal{W}'_{\underline{\sigma}})$  denote the associated projective bundle over  $R$ . A morphism  $h' : \mathfrak{B}_{\underline{\sigma}} \rightarrow B'_{\underline{\sigma}}$  is constructed by giving an invertible sheaf  $\Lambda$  on  $\mathfrak{B}_{\underline{\sigma}}$  and an injective homomorphism  $\Lambda \rightarrow \pi_{\mathfrak{B}_{\underline{\sigma}}}^* D^* \mathcal{W}'_{\underline{\sigma}}$  where

$$D = D_1 \times \cdots \times D_m : Q \rightarrow R.$$

We can use the isomorphisms (4.1) and the invertible sheaves  $\mathcal{A}_i$  to construct an invertible sheaf

$$\mathcal{A}_{\underline{\sigma}} = \otimes_{i=1}^m (\pi_{Q_i}^Q)^* \mathcal{A}_i^{\otimes \sigma_i} \quad (4.2)$$

over  $Q$  such that

$$\det(\mathcal{U}_{\underline{\sigma}}) \cong (D \times \text{id}_X)^* \mathcal{P}_{\underline{\sigma}} \otimes p^* \mathcal{A}_{\underline{\sigma}}. \quad (4.3)$$

We choose the invertible sheaf  $\Lambda$  to be  $\pi_{\mathfrak{B}_{\underline{\sigma}}}^* \mathcal{A}_{\underline{\sigma}}^{\vee}$  and the injection  $\Lambda \rightarrow \pi_{\mathfrak{B}_{\underline{\sigma}}}^* D^* \mathcal{W}'_{\underline{\sigma}}$  is given by the tautological inclusion  $\mathcal{O}_{\mathfrak{B}_{\underline{\sigma}}}(-1) \rightarrow \pi_{\mathfrak{B}_{\underline{\sigma}}}^* \mathcal{C}_{\underline{\sigma}}$  and the isomorphism

$$p_* \left( \det(\mathcal{U}_{\underline{\sigma}}) \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(rn) \right) \cong D^* (\pi_R^{R \times X})_* \left( \mathcal{P}_{\underline{\sigma}} \otimes (\pi_X^{R \times X})^* \mathcal{O}_X(rn) \right) \otimes \mathcal{A}_{\underline{\sigma}}$$

defined by (4.3). The resulting morphism  $h' : \mathfrak{B}_{\underline{\sigma}} \rightarrow B'_{\underline{\sigma}}$  is  $G$ -equivariant and as before we can ensure that  $\mathcal{L}' = \mathcal{O}_{B'}(1)$  is very ample by choosing  $\mathcal{P}_{\underline{\sigma}}$  appropriately.

We let  $\mathcal{L}'$  also denote the natural linearisation of the  $G$ -action on  $B'_{\underline{\sigma}}$  arising from the  $G$ -action on  $\mathcal{W}'_{\underline{\sigma}}$ .

The morphisms  $h_i : Q_i \rightarrow \text{Pic}(X)_i$  and  $h' : \mathfrak{B}_{\underline{\sigma}} \rightarrow B'_{\underline{\sigma}}$  constructed above define a morphism

$$\mathfrak{B}_{\underline{\sigma}} \rightarrow B_{\underline{\sigma}} := B_1 \times \cdots \times B_m \times B'_{\underline{\sigma}}.$$

The composition of this morphism with the embedding  $h : \mathfrak{T} \rightarrow \mathfrak{B}_{\underline{\sigma}}$  of Section 4.1.4 defines a morphism  $H : \mathfrak{T} \rightarrow B_{\underline{\sigma}}$  which is  $G$ -equivariant and injective (cf. [39], Section 2.1). The linearisation on  $\mathfrak{T}$  is given by pulling back a linearisation on  $B_{\underline{\sigma}}$  via the equivariant and injective morphism  $H$ . If  $\underline{a} = (a', a_1, \dots, a_m)$  is a tuple of positive integers, then

$$\mathcal{O}_B(\underline{a}) := \pi_{B'_{\underline{\sigma}}}^* \mathcal{L}'(a') \otimes \pi_{B_1}^* \mathcal{L}_1(a_1) \otimes \cdots \otimes \pi_{B_m}^* \mathcal{L}_m(a_m)$$

is a very ample linearisation on  $B_{\underline{\sigma}}$  which can be modified by a character  $\rho : G \rightarrow \mathbb{C}^*$ . Schmitt uses the stability parameters  $\underline{\sigma}$  and  $\underline{\chi}$  and the invariants  $\underline{P}$  to define the positive integers  $\underline{a}$  and the character  $\rho$  as follows. Let  $a_i$  and  $a'$  to be positive integers such that

$$\frac{a_i}{a'} = \sigma_i \left( \frac{P(n)}{r\delta(n)} - 1 \right) + \eta_i$$

where  $P = \sum \sigma_i P_i$  and  $r = \sum \sigma_i r_i$ . This is possible as for sufficiently large  $n$  the right hand side is a positive rational number.

The rational numbers

$$c_i := a' \left[ \sigma_i \left( \frac{P(n)}{r\delta(n)} - 1 \right) \left( \frac{r}{P(n)} - \frac{r_i}{P_i(n)} \right) - \frac{r_i \eta_i}{P_i(n)} \right]$$

can be used to define a character  $\rho : G \rightarrow \mathbb{C}^*$  as follows. If the  $c_i$  are all integral we define

$$\rho((g_1, \dots, g_m)) = \det g_1^{c_1} \cdots \det g_m^{c_m}$$

whose restriction to the centre

$$Z(G) \cong \{(t_1, \dots, t_m) \in (\mathbb{C}^*)^m : \prod_{i=1}^m t_i^{P_i(n)\sigma_i} = 1\}$$

is  $\rho((t_1, \dots, t_m)) = \prod t_i^{P_i(n)c_i}$ . If the  $c_i$  are rational, then we can find the minimal positive integer  $N$  such that  $c_i N$  are all integers and then scale everything used in the linearisation (that is, the positive integers  $a$  and  $a_i$ ) by  $N$  and define  $\rho$  using the integers  $c_i N$ .

**Definition 4.1.13.** The linearisation  $\mathcal{L} = \mathcal{L}^{(\underline{\sigma}, \underline{\chi})}$  of the  $G$ -action on  $\mathfrak{T}$  determined by the stability parameters  $(\underline{\sigma}, \underline{\chi})$  is equal to the pullback via  $H : \mathfrak{T} \rightarrow B$  of the linearisation  $\mathcal{O}_B(\underline{a})$  on  $B$  modified by  $\rho$ .

**Remark 4.1.14.** For simplicity in calculations we will often assume  $a' = 1$  since  $a'$  is only introduced to ensure this tuple defining the linearisation is a tuple of integers rather than a tuple of rational numbers. We will also ignore the scaling by  $N$  needed if  $c_i$  are not integral.

#### 4.1.6 S-equivalence and moduli of $(\underline{\sigma}, \underline{\chi})$ -semistable complexes

The moduli space of  $(\underline{\sigma}, \underline{\chi})$ -semistable complexes is constructed as a GIT quotient. A GIT quotient identifies S-equivalent orbits; that is, those orbits whose closures meet in the semistable locus. Therefore, we have to weaken the notion of equivalence from isomorphism to S-equivalence:

**Definition 4.1.15.** A Jordan–Hölder filtration of a  $(\underline{\sigma}, \underline{\chi})$ -semistable complex  $\mathcal{E}$  is a filtration by subcomplexes

$$0. = \mathcal{E}^0 \subsetneq \mathcal{E}^1 \subsetneq \dots \subsetneq \mathcal{E}^k = \mathcal{E}.$$

such that the quotients  $\mathcal{E}^i / \mathcal{E}^{i-1}$  are  $(\underline{\sigma}, \underline{\chi})$ -stable with reduced Hilbert polynomials

which satisfy

$$P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}^i / \mathcal{E}^{i-1}) = P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}).$$

This filtration is in general not canonical but the associated graded object

$$\text{gr}_{(\underline{\sigma}, \underline{\chi})}(\mathcal{E}) := \bigoplus_{i=1}^k \mathcal{E}^i / \mathcal{E}^{i-1}$$

is canonically associated to  $\mathcal{E}$  up to isomorphism. We say two  $(\underline{\sigma}, \underline{\chi})$ -semistable complexes are S-equivalent if their graded objects associated to their Jordan–Hölder filtrations with respect to  $(\underline{\sigma}, \underline{\chi})$  are isomorphic.

The proof of the existence of Jordan–Hölder filtrations of  $(\underline{\sigma}, \underline{\chi})$ -semistable complexes is given by modifying the proof of the existence statement for semistable sheaves (for example, see [13]) in the appropriate way.

We are now able to state one of the main results of [39] for us: the existence of moduli spaces of  $(\underline{\sigma}, \underline{\chi})$ -semistable complexes with invariants  $\underline{P}$ . Recall there is a parameter scheme  $\mathfrak{T}$  for  $(\underline{\sigma}, \underline{\chi})$ -semistable complexes with invariants  $\underline{P}$  which is quasi-projective. There is an action of a reductive group  $G$  on  $\mathfrak{T}$  such that the  $G$ -orbits correspond to isomorphism classes of complexes. The stability parameters  $(\underline{\sigma}, \underline{\chi})$  are used to determine a linearisation  $\mathcal{L}$  of this action.

**Theorem 4.1.16.** ([39], p3) *Let  $X$  be a smooth integral projective scheme over the complex numbers and let  $\underline{P} = (P_1, \dots, P_m)$  be a tuple of Hilbert polynomials of degree  $\dim X$ . Let  $(\underline{\sigma}, \underline{\chi})$  be stability parameters for  $\underline{P}$ ; then there is a quasi-projective coarse moduli space*

$$M^{(\underline{\sigma}, \underline{\chi})-ss}(X, \underline{P}) = \mathfrak{T} //_{\mathcal{L}} G$$

*for S-equivalence classes of  $(\underline{\sigma}, \underline{\chi})$ -semistable complexes over  $X$  with Hilbert polynomials specified by  $\underline{P}$ .*

## 4.2 Calculating the Hilbert–Mumford function

In the above section we described a construction of the moduli space of  $(\underline{\sigma}, \underline{\chi})$ -semistable complexes of fixed invariants due to Schmitt [39]. The construction is as a GIT quotient of a parameter scheme  $\mathfrak{T} = \mathfrak{T}_{\underline{P}}$  by the action of

$$G := \prod_{i=1}^m \mathrm{GL}(V_i) \cap \mathrm{SL}(\oplus_{i=1}^m V_i^{\oplus \sigma_i})$$

with respect to a linearisation  $\mathcal{L}$  determined by the stability parameters.

In this section we calculate the Hilbert–Mumford function  $\mu^{\mathcal{L}}(z, \lambda)$  for  $z \in \mathfrak{T}$  and  $\lambda : \mathbb{C}^* \rightarrow G$  a one-parameter subgroup (1-PS) of  $G$ . In fact we will actually work with a projective completion  $\overline{\mathfrak{T}}$  of the parameter scheme (see Definition 4.2.1 below) so that we can study the limits of points  $z$  under such one-parameter subgroups. Some details of these calculations can be found in [39] Section 2.1, but we give some additional details which will be needed for this thesis.

### 4.2.1 Limits of points under one-parameter subgroups

Let  $\overline{Q}_i$  denote the closure of  $Q_i$  in the quot scheme  $\mathrm{Quot}(V_i \otimes \mathcal{O}_X(-n), P_i)$  and  $\overline{Q} = \overline{Q}_1 \times \cdots \times \overline{Q}_m$ . Then the projective bundle  $\mathfrak{D}$  over  $Q$  extends to a projective bundle over  $\overline{Q}$  which we denote by  $\overline{\mathfrak{D}}$ . A point of  $\overline{\mathfrak{D}}$  is given by  $z = (q_1, \dots, q_m, [\psi : \zeta])$  where  $q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i$  is in  $\overline{Q}_i$  and  $[\psi : \zeta]$  is a nonzero class given by  $\zeta \in \mathbb{C}$  and  $\psi : V = \oplus_{i=1}^m V_i \rightarrow \oplus_{i=1}^m \mathcal{E}_i(n)$  defined up to scalar multiplication.

**Definition 4.2.1.** We define  $\overline{\mathfrak{T}}$  to be the closed subscheme of  $\overline{\mathfrak{D}}$  consisting of points  $z = (q_1, \dots, q_m, [\psi : \zeta])$  as above such that

- i) The morphism  $\psi$  factors through  $\oplus_{i=1}^m q_i(n) : V \rightarrow \oplus_{i=1}^m \mathcal{E}_i(n)$ ; i.e.,  $\psi$  is defined by a morphism  $d : \oplus_{i=1}^m \mathcal{E}_i \rightarrow \oplus_{i=1}^m \mathcal{E}_i$ .
- ii) The morphism  $d = (d_{i,j} : \mathcal{E}_i \rightarrow \mathcal{E}_j)$  satisfies  $d_{i,j} = 0$  unless  $j = i + 1$  and

$$d_{i,i+1} \circ d_{i-1,i} = 0 \text{ for } i = 2, \dots, m-1.$$

These conditions are both closed as they are cut out by the vanishing locus of a homomorphism of locally free sheaves (cf. the construction of the closed subscheme  $\tilde{\mathfrak{D}}$  of  $\mathfrak{D}$  in Section 4.1.2). The scheme  $\overline{\mathfrak{T}}$  is a projective completion of the parameter scheme  $\mathfrak{T}$  and the group action and linearisation  $\mathcal{L}$  extend to  $\overline{\mathfrak{T}}$ ; therefore, we can calculate the Hilbert–Mumford function for points in  $\overline{\mathfrak{T}}$ .

Let  $\lambda = (\lambda_1, \dots, \lambda_m)$  be a one-parameter subgroup of  $G$  where  $\lambda_i : \mathbb{C}^* \rightarrow \mathrm{GL}(V_i)$  satisfy

$$\det(\lambda_1)^{\sigma_1} \times \dots \times \det(\lambda_m)^{\sigma_m} = 1.$$

We have seen that a 1-PS of  $\mathrm{GL}(V_i)$  gives rise to a weight space decomposition of  $V_i$  indexed by a finite collection of integers  $k_1 > \dots > k_s$  which we can use to filter  $V_i$ . We diagonalise each  $\lambda_i$  simultaneously so for each  $i = 1, \dots, m$  there is a decomposition  $V_i = \bigoplus_{j=1}^s V_i^j$  (where we may have  $V_i^j = 0$ ), and a filtration

$$0 \subset V_i^{(1)} \subset V_i^{(2)} \subset \dots \subset V_i^{(s)} = V_i$$

where  $V_i^{(j)} = V_i^1 \oplus \dots \oplus V_i^j$  such that

$$\lambda_i(t) = \begin{pmatrix} t^{k_1} I_{V_i^1} & & \\ & \ddots & \\ & & t^{k_s} I_{V_i^s} \end{pmatrix}.$$

Let  $z = (q_1, \dots, q_m, [d : \zeta])$  be a point in  $\overline{\mathfrak{T}}$  and let

$$\bar{z} := \lim_{t \rightarrow 0} \lambda(t) \cdot z$$

denote the limit of  $z$  under the action of  $\lambda(t)$  as  $t$  tends to zero. We may write this point as  $\bar{z} = (\bar{q}_1, \dots, \bar{q}_m, [\bar{d} : \bar{\zeta}])$ . Then by [19] Lemma 4.4.3, the quotient sheaves are

direct sums:

$$\overline{q_i} = \lim_{t \rightarrow 0} \lambda_i(t) \cdot q_i = \oplus_{j=1}^s q_i^j : \oplus_{j=1}^s V_i^j \otimes \mathcal{O}_X(-n) \rightarrow \oplus_{j=1}^s \mathcal{E}_i^j$$

where  $\mathcal{E}_i^j$  are the successive quotients in the filtration

$$0 \subset \mathcal{E}_i^{(1)} \subset \cdots \subset \mathcal{E}_i^{(j)} := q_i(V_i^{(j)} \otimes \mathcal{O}_X(-n)) \subset \cdots \subset \mathcal{E}_i^{(s)} = \mathcal{E}_i$$

induced by  $\lambda_i$ .

In the case when  $\zeta = 1$ , there are two possibilities: either the boundary maps respect these filtrations (see case *i*) of Lemma 4.2.2 below) or they do not respect these filtrations (see case *iii*) of Lemma 4.2.2 below).

**Lemma 4.2.2.** *Let  $\lambda$  be a 1-PS of  $G$  as above with weights  $k_1 > \cdots > k_s$ . Let  $z = (q_1, \dots, q_m, [d : \zeta])$  be a point in  $\overline{\mathfrak{T}}$  and*

$$\overline{z} := \lim_{t \rightarrow 0} \lambda(t) \cdot z = (\overline{q_1}, \dots, \overline{q_m}, [\overline{d} : \overline{\zeta}])$$

where  $\overline{q_i}$  are as above and  $\overline{d}$  is given by  $\overline{d}_i : \oplus_{j=1}^s \mathcal{E}_i^j \rightarrow \oplus_{j=1}^s \mathcal{E}_{i+1}^j$  for  $i = 1, \dots, m-1$  which satisfy  $\overline{d}_i \circ \overline{d}_{i-1} = 0$ . Then:

- i) (Diagonal) If  $d_i(\mathcal{E}_i^{(j)}) \subset \mathcal{E}_{i+1}^{(j)}$  for  $i = 1, \dots, m-1$  and  $j = 1, \dots, s$  and if either  $\zeta \neq 0$  or there is an  $i$  and  $j$  such that  $d_i(\mathcal{E}_i^{(j)}) \not\subset \mathcal{E}_{i+1}^{(j-1)}$ , then  $\overline{\zeta} = \zeta$  and  $\overline{d}_i = \oplus_{j=1}^s d_i^j$  where  $d_i^j : \mathcal{E}_i^j \rightarrow \mathcal{E}_{i+1}^j$  for  $i = 1, \dots, m$ .*
- ii) (Strictly upper triangular) If  $\zeta = 0$  and  $d_i(\mathcal{E}_i^{(j)}) \subset \mathcal{E}_{i+1}^{(j-1)}$  for  $i = 1, \dots, m-1$  and  $j = 1, \dots, s$ , then  $\overline{\zeta} = 0$  and there is a  $N \in \mathbb{Z}_{>0}$  such that  $\overline{d}_i(\mathcal{E}_i^j) \cap \mathcal{E}_{i+1}^l = 0$  unless  $k_l - k_j = N$ .*
- iii) (Strictly lower triangular) If there is an  $i$  and  $j$  such that  $d_i(\mathcal{E}_i^{(j)}) \not\subset \mathcal{E}_{i+1}^{(j)}$ , then  $\overline{\zeta} = 0$  and there is a  $N \in \mathbb{Z}_{<0}$  such that  $\overline{d}_i(\mathcal{E}_i^j) \cap \mathcal{E}_{i+1}^l = 0$  unless  $k_l - k_j = N$ .*

*Proof.* For the purposes of this proof we assume that  $H^0(q_i(n))$  is an isomorphism for each  $i$  so that rather than working with  $d_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$  we can work with

$$\tilde{\psi}_i := H^0(q_{i+1}(n))^{-1} \circ d_i(n) \circ H^0(q_i(n)) : V_i \rightarrow V_{i+1}.$$

If this were not the case we can use  $\varphi_i = d_i \circ q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_{i+1}$  and techniques similar to that of [19] Lemma 4.4.3 to prove the result.

For  $i = 1, \dots, m$  pick a basis of  $V_i$  which is compatible with the filtration  $0 \subset V_i^{(1)} \subset \dots \subset V_i^{(s)} = V_i$  and write  $\tilde{\psi}_i$  as a block matrix  $A^i = (a_{jl}^i)$  with respect to this basis where  $a_{jl}^i : V_i^l \rightarrow V_{i+1}^j$ . Then  $\lambda$  acts by conjugation on  $A^i$  and

$$\lambda(t) \cdot A^i = \begin{pmatrix} a_{11}^i & t^{k_1-k_2}a_{12}^i & \dots & t^{k_1-k_s}a_{1s}^i \\ t^{k_2-k_1}a_{21}^i & a_{22}^i & \dots & t^{k_2-k_s}a_{2s}^i \\ \vdots & \vdots & \ddots & \vdots \\ t^{k_s-k_1}a_{s1}^i & t^{k_s-k_2}a_{s2}^i & \dots & a_{ss}^i \end{pmatrix}. \quad (4.4)$$

In case *i*) we have that for each  $i$  the corresponding matrix  $A^i$  is block upper triangular and either  $\zeta \neq 0$  or there is an  $i$  such that the matrix  $A^i$  has a nonzero diagonal entry  $a_{jj}^i \neq 0$ . Then as  $k_1 > \dots > k_s$  the limit as  $t \rightarrow 0$  is

$$\begin{pmatrix} a_{11}^i & 0 & 0 & 0 \\ 0 & a_{22}^i & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{ss}^i \end{pmatrix}.$$

Therefore,  $\bar{\zeta} = \zeta$  and  $\bar{d}_i = \oplus_{j=1}^s d_i^j$  where  $d_i^j : \mathcal{E}_i^j \rightarrow \mathcal{E}_{i+1}^j$  is the quotient of  $\mathcal{E}_i^{(j)} \rightarrow \mathcal{E}_{i+1}^{(j)}$  by  $\mathcal{E}_i^{(j-1)} \rightarrow \mathcal{E}_{i+1}^{(j-1)}$ .

In case *ii*) we have that  $\zeta = 0$  and for each  $i$  the corresponding matrix  $A^i$  is

strictly upper triangular; i.e.,  $a_{jl}^i = 0$  if  $j \geq l$ . Therefore,  $t$  only appears with positive exponents in the versions of (4.4) for  $i = 1, \dots, m-1$ . Let  $N \in \mathbb{Z}_{>0}$  be the smallest exponent of  $t$  appearing in these matrices; then since we are working projectively we can multiply  $[\lambda(t) \cdot d : \zeta = 0]$  by  $t^{-N}$ . As  $t \rightarrow 0$ , the limit is  $[\bar{d} : 0]$  where  $\bar{d}_i$  are all strictly upper triangular and  $\bar{d}_i(\mathcal{E}_i^j) \cap \mathcal{E}_{i+1}^l = 0$  unless  $k_l - k_j = N$ .

In case *iii*) we know that there is an  $i$  such that the corresponding matrix  $A^i$  has a nonzero entry  $a_{jl}^i$  for  $j > l$ . Therefore,  $t$  appears with a negative exponent in (4.4). Let  $N \in \mathbb{Z}_{<0}$  be the smallest exponent of  $t$  appearing in the versions of (4.4) for  $i = 1, \dots, m-1$ ; then we can multiply  $[\lambda(t) \cdot d : \zeta]$  by  $t^{-N}$  to remove the negative exponents. As  $t \rightarrow 0$ , the limit is  $[\bar{d} : 0]$  where  $\bar{d}_i$  are all strictly lower triangular and  $\bar{d}_i(\mathcal{E}_i^j) \cap \mathcal{E}_{i+1}^l = 0$  unless  $k_l - k_j = N$ .  $\square$

**Definition 4.2.3.** Let  $\lambda$  be a 1-PS of  $G$  and  $z = (q_1, \dots, q_m, [d : 1])$  be a point in  $\mathfrak{T}$ . We say  $\lambda$  induces a filtration of  $z$  (or the corresponding complex  $\mathcal{E}$ .) by subcomplexes if the filtrations  $0 \subset \mathcal{E}_i^{(1)} \subset \dots \subset \mathcal{E}_i^{(s)} = \mathcal{E}_i$  induced by  $\lambda_i$  are compatible with the boundary maps  $d_i$ ; that is,  $d_i(\mathcal{E}_i^{(j)}) \subset \mathcal{E}_{i+1}^{(j)}$  for all  $i$  and  $j$ .

An important consequence of Lemma 4.2.2 is the following corollary:

**Corollary 4.2.4.** *Let  $z = (q_1, \dots, q_m, [d : 1])$  be a point of  $\mathfrak{T}$  parametrising a complex  $\mathcal{E}$ . and  $\lambda$  be a 1-PS of  $G$ . Then  $\lambda$  induces a filtration of  $z$  by subcomplexes if and only if  $\lim_{t \rightarrow 0} \lambda(t) \cdot z \in \mathfrak{T}$ .*

### 4.2.2 The weight of $\mathbb{C}^*$ -actions over fixed points

Let  $z = (q_1, \dots, q_m, [d : \zeta])$  be a point in  $\overline{\mathfrak{T}}$  and  $\lambda = (\lambda_1, \dots, \lambda_m)$  be a 1-PS of  $G$ ; then in this section we calculate the weight of this copy of  $\mathbb{C}^*$  acting on the fibre of the linearisation  $\mathcal{L}$  over the fixed point  $\bar{z} := \lim_{t \rightarrow 0} \lambda(t) \cdot z$  which is by definition the value  $\mu^{\mathcal{L}}(z, \lambda)$  of the Hilbert–Mumford function  $\mu^{\mathcal{L}}$  evaluated at  $(z, \lambda)$ . Let  $\varphi = \varphi(z)$  denote the decoration associated to  $z$  (cf. Remark 4.1.12) then by construction of  $\mathcal{L}$

we have

$$\mu^{\mathcal{L}}(z, \lambda) = \mu^{\mathcal{L}'}(\varphi, \lambda) + \sum_{i=1}^m a_i \mu^{\mathcal{L}_i}(q_i, \lambda_i) - \rho \cdot \lambda \quad (4.5)$$

where the rational numbers  $a_i$  and the character  $\rho$  are defined in section 4.1.5. We now calculate each of these quantities separately.

**Lemma 4.2.5.** *Let  $\lambda_i$  be a 1-PS of  $\mathrm{GL}(V_i)$  corresponding to a decomposition  $V_i = \oplus_{j=1}^s V_i^j$  and sequence of integers  $k_1 > \dots > k_s$  as above. Let  $\mathcal{L}_i$  be the linearisation of the  $\mathrm{SL}(V_i)$ -action on  $\overline{Q_i}$  defined in Section 4.1.5. Suppose  $q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i$  is a point in  $\overline{Q_i}$ , then*

$$\mu^{\mathcal{L}_i}(q_i, \lambda_i) = \sum_{j=1}^s k_j \left( \mathrm{rk} \mathcal{E}_i^j - \dim V_i^j \frac{r_i}{P_i(n)} \right)$$

where  $\mathcal{E}_i^{(j)} = q_i(V_i^{(j)} \otimes \mathcal{O}_X(-n))$  and  $\mathcal{E}_i^j = \mathcal{E}_i^{(j)} / \mathcal{E}_i^{(j-1)}$ .

*Proof.* By [19], Lemma 4.4.3 we know that

$$\overline{q_i} := \lim_{t \rightarrow 0} \lambda'_i(t) \cdot q_i = \oplus_{j=1}^s q_i^j$$

where  $q_i^j : V_i^j \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i^j$  is the quotient of  $q_i|_{V_i^{(j)}} : V_i^{(j)} \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i^{(j)}$  by  $q_i|_{V_i^{(j-1)}} : V_i^{(j-1)} \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i^{(j-1)}$ .

The linearisation  $\mathcal{L}_i$  is a lift of the  $\mathrm{SL}(V_i)$ -action and so we should view  $\lambda_i(t)$  as elements of  $\mathrm{SL}(V_i)$  as follows. We can associate to the 1-PS  $\lambda_i$  of  $\mathrm{GL}(V_i)$  a rational 1-PS  $\lambda'_i$  of  $\mathrm{SL}(V_i)$  by

$$\lambda'_i(t) = \begin{pmatrix} t^{k'_1} I_{V_i^1} & & \\ & \ddots & \\ & & t^{k'_s} I_{V_i^s} \end{pmatrix}$$

where  $k'_j := k_j - (\sum_{l=1}^s k_l \dim V_i^l) / P_i(n)$ .

The weight of the  $\mathbb{C}^*$ -action induced by  $\lambda'_i$  on the fibre of  $\mathcal{L}_i$  over  $\overline{q_i}$  is calculated

in [14]:

$$\mu^{\mathcal{L}_i}(q_i, \lambda'_i) = \sum_{j=1}^s k'_j \operatorname{rk} \mathcal{E}_i^j = \sum_{j=1}^s k_j \left( \operatorname{rk} \mathcal{E}_i^j - \dim V_i^j \frac{r_i}{P_i(n)} \right)$$

(cf. Remark 3.1.6). □

As above we let  $\lambda = (\lambda_1, \dots, \lambda_m)$  be a 1-PS of  $G$  which corresponds to integers  $k_1 > \dots > k_s$  and decompositions  $V_i = \bigoplus_{j=1}^s V_i^j$ . This induces a decomposition and filtration of the vector space  $M = \bigoplus_{i=1}^m V_i^{\oplus \sigma_i}$ . Let  $M^j = \bigoplus_{i=1}^m (V_i^j)^{\oplus \sigma_i}$ ; then we get a decomposition  $M = \bigoplus_{j=1}^s M^j$ . The associated filtration is given by

$$0 \subset M^{(1)} \subset \dots \subset M^{(s)} = M$$

where  $M^{(j)} = \bigoplus_{i=1}^m (V_i^{(j)})^{\oplus \sigma_i}$ .

Given an index  $I = (i_1, \dots, i_r) \in \{1, \dots, s\}^{\times r}$ , we let  $M^I = M^{i_1} \otimes \dots \otimes M^{i_r} \subset M^{\otimes r}$ . Then  $\lambda(t)$  acts on  $M^I$  by multiplication by  $t$  to the power of  $\sum k_j n_j(I)$  where

$$n_j(I) = \#\{i_l : i_l = j \text{ for } 1 \leq l \leq r\}.$$

**Lemma 4.2.6.** *Let  $\lambda$  denote a 1-PS of  $G$  and  $z = (q_1, \dots, q_m, [d : \zeta]) \in \overline{\mathfrak{T}}$  as above.*

*If  $\varphi$  denotes the decoration associated to  $z$ , then*

$$\mu^{\mathcal{L}'}(\varphi, \lambda) = - \min \left\{ - \sum_{j=1}^s k_j n_j(I) : \begin{array}{l} I \in \{1, \dots, s\}^{\times r} \text{ and there is a } c \in M^I \\ \text{such that } \varphi(c, 0) \text{ or } \varphi(0, c) \neq 0 \end{array} \right\}.$$

*Furthermore, if  $z$  is a point in  $\mathfrak{T}$  such that  $\lambda$  induces a filtration of  $z$  by subcomplexes then*

$$\mu^{\mathcal{L}'}(\varphi, \lambda) = \sum_{j=1}^s \sum_{i=1}^m k_j \sigma_i \operatorname{rk} \mathcal{E}_i^j$$

*where  $\mathcal{E}_i^j$  are the successive quotients in the filtrations  $0 \subset \mathcal{E}_i^{(1)} \subset \dots \subset \mathcal{E}_i^{(s)} = \mathcal{E}_i$  induced by  $\lambda$ .*

*Proof.* The first statement is simply the definition of the Hilbert–Mumford function. Suppose that  $z = (q_1, \dots, q_m, [d : 1])$  is a point in  $\mathfrak{T}$  representing a complex  $\mathcal{E}$  such that  $\lambda$  induces a filtration of  $\mathcal{E}$  by subcomplexes. By Lemma 4.2.2 and Corollary 4.2.4 the fixed point is

$$\bar{z} = (\bar{q}_1, \dots, \bar{q}_m, [\bar{d}, \bar{\zeta}]) = \lim_{t \rightarrow 0} \lambda(t) \cdot z \in \mathfrak{T}$$

where  $\bar{q}_i = \oplus_{j=1}^s (q_i^j : V_i^j \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i^j)$  and  $[\bar{d}, \bar{\zeta}] = [\oplus_{j=1}^s d^j, 1]$ . By definition of the Hilbert–Mumford function we have that

$$\mu^{\mathcal{L}'}(\varphi, \lambda) = \mu^{\mathcal{L}'}(\bar{\varphi}, \lambda)$$

where  $\bar{\varphi}$  denote the decoration associated to the fixed point  $\bar{z}$ . Since  $\bar{\varphi}$  is fixed by  $\lambda(\mathbb{C}^*)$  we have

$$\mu^{\mathcal{L}'}(\bar{\varphi}, \lambda) = \sum_{j=1}^s k_j n_j(I)$$

for any  $I \in \{1, \dots, s\}^{\times r}$  such that there is an  $c \in M^I$  such that  $\bar{\varphi}(c, 0) \neq 0$  or  $\bar{\varphi}(0, c) \neq 0$ . In order to find such an  $I$  we need to recall the construction of  $\bar{\varphi}$  given in Section 4.1.4. Let  $\mathcal{E} = \oplus_{i=1}^m \oplus_{j=1}^s (\mathcal{E}_i^j)^{\oplus \sigma_i}$ ; then  $\bar{\varphi} : (M^{\otimes r})^{\oplus 2} \rightarrow H^0(\det(\mathcal{E})(rn))$  is given by two components. The first component  $\bar{\varphi}_1 : M^{\otimes r} \rightarrow H^0(\det(\mathcal{E})(rn))$  is defined using the morphism  $\bar{d}$  and the second component  $\bar{\varphi}_2 : M^{\otimes r} \rightarrow H^0(\det(\mathcal{E})(rn))$  is defined by the complex number  $\bar{\zeta} = 1$  viewed as a global section of  $\mathcal{O}_X$ . The second component is needed so that  $\bar{\varphi} \neq 0$  and so we know there is a  $c \in M^{\otimes r}$  such that  $\bar{\varphi}_2(c) \neq 0$ .

As  $\mathcal{E}_i^j(n)$  is generated by global sections we know that there are sections  $s_{i,j}^l \in H^0(\mathcal{E}_i^j(n))$  for  $l = 1, \dots, r_i^j := \text{rk } \mathcal{E}_i^j$  such that

$$s_{i,j}^1 \wedge \dots \wedge s_{i,j}^{r_i^j} \neq 0 \text{ in } H^0(\det(\mathcal{E}_i^j)(r_i^j n)).$$

These sections for  $i = 1, \dots, m$  and  $j = 1, \dots, s$  define an element  $c \in M^{\otimes r} \cong H^0(\mathcal{E}(n))^{\otimes r}$  (where  $s_{i,j}^l$  appears  $\sigma_i$  times) and by reviewing the construction of this decoration we see that  $\bar{\varphi}_2(c) \neq 0$ . Then  $c \in M^I$  where  $n_j(I) = \sum_i \sigma_i r_i^j$  and

$$\mu^{\mathcal{L}'}(\bar{\varphi}, \lambda) = \sum_{j=1}^s \sum_{i=1}^m k_j \sigma_i \operatorname{rk} \mathcal{E}_i^j$$

as required.  $\square$

**Lemma 4.2.7.** *Let  $\lambda$  denote a 1-PS of  $G$  as above and  $z = (q_1, \dots, q_m, [d : 1])$  be a point in  $\mathfrak{T}$  with associated decoration  $\varphi$ . If  $\lambda$  does not induces a filtration of  $z$  by subcomplexes, then there is a negative integer  $N = k_{j_2} - k_{j_1}$  such that*

$$\mu^{\mathcal{L}'}(\varphi, \lambda) = \sum_{j=1}^s \sum_{i=1}^m k_j \sigma_i \operatorname{rk} \mathcal{E}_i^j - N$$

where  $\mathcal{E}_i^j$  are the successive quotients in the filtrations  $0 \subset \mathcal{E}_i^{(1)} \subset \dots \subset \mathcal{E}_i^{(s)} = \mathcal{E}_i$  induced by  $\lambda$ .

*Proof.* As  $\lambda$  does not induce a filtration by subcomplexes, we are in case *iii*) of Lemma 4.2.2 and the limit  $\bar{z} = \lim_{t \rightarrow 0} \lambda(t) \cdot z \notin \mathfrak{T}$ . In fact there is  $N \in \mathbb{Z}_{<0}$  such that  $\bar{d}_i(\mathcal{E}_i^j) \cap \mathcal{E}_{i+1}^l = 0$  unless  $k_l - k_j = N$ . As  $\bar{d}$  is nonzero, there are  $1 \leq i_0 \leq m-1$  and  $1 \leq j_1 < j_2 \leq s$  such that  $\bar{d}_{i_0}(\mathcal{E}_{i_0}^{j_1}) \cap \mathcal{E}_{i_0+1}^{j_2} \neq 0$ . As in Lemma 4.2.6 above

$$\mu^{\mathcal{L}'}(\varphi, \lambda) = \mu^{\mathcal{L}'}(\bar{\varphi}, \lambda) = \sum_{j=1}^s k_j n_j(I)$$

for any  $I \in \{1, \dots, s\}^{\times r}$  for which there is an  $c \in M^I$  such that  $\bar{\varphi}(c, 0) \neq 0$  or  $\bar{\varphi}(0, c) \neq 0$ .

Since  $\bar{\zeta} = 0$  the second component  $\bar{\varphi}_2 = 0$  and so cannot be used to calculate the Hilbert–Mumford function. Instead we use the fact that  $\bar{d}_{i_0}(\mathcal{E}_{i_0}^{j_1}) \cap \mathcal{E}_{i_0+1}^{j_2} \neq 0$ . The sheaf  $\mathcal{E}_i^j(n)$  is generated by global sections and so there are sections  $s_{i,j}^l \in H^0(\mathcal{E}_i^j(n))$

for  $l = 1, \dots, r_i^j := \text{rk } \mathcal{E}_i^j$  such that

$$s_{i,j}^1 \wedge \dots \wedge s_{i,j}^{r_i^j} \neq 0 \text{ in } H^0(\det(\mathcal{E}_i^j)(r_i^j n)).$$

We can pick these sections so that  $H^0(\bar{d}_{i_0}(n))(s_{i_0,j_1}^1) = s_{i_0+1,j_2}^1$ . Let  $c \in M^{\otimes r} \cong (\oplus_{i,j} H^0(\mathcal{E}_i^j(n))^{\oplus \sigma_i})^{\otimes r}$  to be the tensor product of the  $s_{i,j}^l$  where  $s_{i_0,j_1}^1$  appears  $\sigma_i + 1$  times and  $s_{i_0+1,j_2}^1$  appears  $\sigma_{i+1} - 1$  times and all other  $s_{i,j}^l$  appear  $\sigma_i$  times. By carefully studying the construction of this decoration we see that  $\bar{\varphi}_1(c) \neq 0$  and  $c \in M^I$  where

$$n_j(I) = \begin{cases} 1 + \sum_i \sigma_i r_i^j & \text{if } j = j_1 \\ -1 + \sum_i \sigma_i r_i^j & \text{if } j = j_2 \\ \sum_i \sigma_i r_i^j & \text{otherwise.} \end{cases}$$

Therefore

$$\mu^{\mathcal{L}'}(\bar{\varphi}, \lambda) = \sum_{j=1}^s \sum_{i=1}^m k_j \sigma_i \text{rk } \mathcal{E}_i^j + k_{j_1} - k_{j_2} = \sum_{j=1}^s \sum_{i=1}^m k_j \sigma_i \text{rk } \mathcal{E}_i^j - N$$

as  $k_{j_2} - k_{j_1} = N$ . □

**Proposition 4.2.8.** *Let  $\lambda = (\lambda_1, \dots, \lambda_m)$  be a 1-PS of  $G$  which corresponds to integers  $k_1 > \dots > k_s$  and decompositions  $V_i = \oplus_{j=1}^s V_i^j$  for  $i = 1, \dots, m$  as above. Let  $z = (q_1, \dots, q_m, [d : 1])$  be a point in  $\mathfrak{T}$  where  $q_i$  represents a quotient sheaf  $q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i$  and  $d$  represents the boundary maps. Then we have:*

i) *If  $\lambda$  induces a filtration of  $z$  by subcomplexes, then*

$$\mu^{\mathcal{L}}(z, \lambda) = \sum_{j=1}^s \sum_{i=1}^m k_j \left( \sigma_i \frac{P(n)}{r\delta(n)} + \eta_i \right) \text{rk } \mathcal{E}_i^j$$

where  $\mathcal{E}_i^j = \mathcal{E}_i^{(j)} / \mathcal{E}_i^{(j-1)}$  and  $\mathcal{E}_i^{(j)} = q_i(V_i^{(j)} \otimes \mathcal{O}_X(-n))$ .

ii) If  $\lambda$  does not induce a filtration of  $z$  by subcomplexes, then

$$\mu^{\mathcal{L}}(z, \lambda) = \sum_{j=1}^s \sum_{i=1}^m k_j \left( \sigma_i \frac{P(n)}{r\delta(n)} + \eta_i \right) \text{rk } \mathcal{E}_i^j - N$$

where  $N$  is the negative integer given by Lemma 4.2.7.

*Proof.* By the formula (4.5) and the definition of  $\rho$ , we have

$$\mu^{\mathcal{L}}(z, \lambda) = \mu^{\mathcal{L}'}(\varphi, \lambda) + \sum_{i=1}^m a_i \mu^{\mathcal{L}'}(q_i, \lambda'_i) - \sum_{j=1}^s \sum_{i=1}^m k_j c_i \dim V_i^j$$

where  $a_i$  and  $c_i$  are rational numbers defined in Section 4.1.5. By Lemma 4.2.5 and the definition of these rational numbers, we have

$$\begin{aligned} \mu = & \mu^{\mathcal{L}'}(\varphi, \lambda) + \sum_{j=1}^s \sum_{i=1}^m k_j \left( \sigma_i \frac{P(n)}{r\delta(n)} - \sigma_i + \eta_i \right) \left( \text{rk } \mathcal{E}_i^j - \dim V_i^j \frac{r_i}{P_i(n)} \right) \\ & - \sum_{j=1}^s \sum_{i=1}^m k_j \left[ \sigma_i \left( \frac{P(n)}{r\delta(n)} - 1 \right) \left( \frac{r}{P(n)} - \frac{r_i}{P_i(n)} \right) - \frac{r_i \eta_i}{P_i(n)} \right] \dim V_i^j. \end{aligned}$$

As  $\lambda$  is a 1-PS of  $\text{SL}(\oplus_i V_i^{\oplus \sigma_i})$ ,

$$\sum_{j=1}^s \sum_{i=1}^m k_j \sigma_i \dim V_i^j = 0$$

so that

$$\mu = \mu^{\mathcal{L}'}(\varphi, \lambda) + \sum_{j=1}^s \sum_{i=1}^m k_j \left( \sigma_i \frac{P(n)}{r\delta(n)} - \sigma_i + \eta_i \right) \text{rk } \mathcal{E}_i^j.$$

Then *i)* and *ii)* follow from Lemma 4.2.6 and Lemma 4.2.7 respectively.  $\square$

### 4.3 Direct sum constructions

Given  $\underline{P} = (P_1, \dots, P_m)$  and  $\underline{P}' = (P'_1, \dots, P'_m)$  two tuples of Hilbert polynomials, we may construct the associated parameter schemes  $\mathfrak{T}_{\underline{P}}$  and  $\mathfrak{T}_{\underline{P}'}$  as in Section 4.1.2. In

this section we describe a morphism

$$\mathfrak{T}_{\underline{P}} \times \mathfrak{T}_{\underline{P}'} \rightarrow \mathfrak{T}_{\underline{P}''}$$

where  $\underline{P}'' = \underline{P} + \underline{P}'$  which corresponds to taking the direct sum of two complexes. If  $(q_1, \dots, q_m, d)$  (respectively  $(q'_1, \dots, q'_m, d')$ ) denotes a point in  $\mathfrak{T}_{\underline{P}}$  (respectively  $\mathfrak{T}_{\underline{P}'}$ ), then their image under this morphism is the direct sum

$$(q_1 \oplus q'_1, \dots, q_m \oplus q'_m, d \oplus d').$$

Suppose  $n \gg 0$ ; then for  $i = 1, \dots, m$  fix vector spaces  $V_i$  (resp.  $V'_i$ ) of dimension  $P_i(n)$  (resp.  $P'_i(n)$ ) and define  $V''_i = V_i \oplus V'_i$  which has dimension  $P''_i(n)$ . We define  $Q_i$  to be the locally closed subscheme of the quot scheme  $\text{Quot}(V_i \otimes \mathcal{O}_X(-n), P_i)$  parametrising torsion free quotient sheaves  $q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i$  such that  $H^0(q_i(n))$  is an isomorphism. Analogously we define subschemes  $Q'_i$  of  $\text{Quot}(V'_i \otimes \mathcal{O}_X(-n), P'_i)$  and  $Q''_i$  of  $\text{Quot}(V''_i \otimes \mathcal{O}_X(-n), P''_i)$ . Let  $Q := Q_1 \times \dots \times Q_m$  and  $Q' := Q'_1 \times \dots \times Q'_m$  and  $Q'' := Q''_1 \times \dots \times Q''_m$ .

**Lemma 4.3.1.** *There is a  $\prod_{i=1}^m \text{GL}(V_i) \times \prod_{i=1}^m \text{GL}(V'_i)$ -equivariant injective morphism of schemes*

$$\phi : Q \times Q' \rightarrow Q''$$

given by

$$((q_i)_{i=1}^m, (q'_i)_{i=1}^m) \mapsto (q_i \oplus q'_i)_{i=1}^m$$

*Proof.* It is sufficient to construct for  $i = 1, \dots, m$  a morphism  $\phi_i : Q_i \times Q'_i \rightarrow Q''_i$  which is  $\text{GL}(V_i) \times \text{GL}(V'_i)$ -equivariant and injective. As the quot scheme is a fine moduli space, giving a morphism  $\phi_i : Q_i \times Q'_i \rightarrow \text{Quot}(V''_i \otimes \mathcal{O}_X(-n), P''_i)$  is equivalent to giving a family  $\mathcal{V}_i$  of quotient sheaves of  $V''_i \otimes \mathcal{O}_X(-n)$  with Hilbert polynomial  $P''_i$  parametrised by  $Q_i \times Q'_i$ . Let  $\mathcal{U}_i$  (respectively  $\mathcal{U}'_i$ ) denote the universal quotient sheaf

on  $Q_i \times X$  (respectively  $Q'_i \times X$ ) and let  $\pi_{jk}^i$  be the projection from  $Q_i \times Q'_i \times X$  to the  $j^{\text{th}}$  and  $k^{\text{th}}$  factors. Then

$$\mathcal{V}_i = (\pi_{13}^i)^* \mathcal{U}_i \oplus (\pi_{23}^i)^* \mathcal{U}'_i$$

is a family of quotient sheaves parametrised by  $Q_i \times Q'_i$  and so defines a morphism  $\phi_i : Q_i \times Q'_i \rightarrow \text{Quot}(V_i'' \otimes \mathcal{O}_X(-n), P_i'')$ . From the construction of this family  $\mathcal{V}_i$  we see that this morphism corresponds to taking the direct sum of two quotient sheaves and that the image of  $\phi_i$  is contained in the open subscheme  $Q_i''$ .

Two quotient sheaves represent the same point in a quot scheme if and only if their kernels are equal; thus, if  $\phi_i(q_i, q'_i) = \phi_i(\rho_i, \rho'_i)$  then  $\ker q_i = \ker \rho_i$  and  $\ker q'_i = \ker \rho'_i$ . Hence, the morphism  $\phi_i$  is injective. There is an inclusion of  $\text{GL}(V_i) \times \text{GL}(V'_i)$  into  $\text{GL}(V_i'')$  and it is easy to verify that  $\phi_i$  is equivariant with respect to the action of  $\text{GL}(V_i) \times \text{GL}(V'_i)$ .  $\square$

We recall the construction of the parameter scheme  $\mathfrak{T} = \mathfrak{T}_P$  from Section 4.1.2. Let  $V = \oplus_{i=1}^m V_i$  and consider the projective bundle  $\mathfrak{D} = \mathbb{P}(\mathcal{R})$  over  $Q$  where

$$\mathcal{R} = V^\vee \otimes p_* \left( \mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n) \right) \oplus \mathcal{O}_Q,$$

$p : Q \times X \rightarrow Q$  and  $\mathcal{U}$  is the universal quotient sheaf on  $Q \times X$ . Then  $\mathfrak{T}$  is the locally closed subscheme of  $\mathfrak{D}$  defined in Section 4.1.2 consisting of points  $(q_1, \dots, q_m, d)$  where  $q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i$  represents a point in  $Q_i$  and  $d$  is given by  $d_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$  satisfying  $d_i \circ d_{i-1} = 0$ .

Let  $\iota : \mathfrak{T} \hookrightarrow \mathfrak{D}$  denote the inclusion; then there is a tautological inclusion

$$\iota^* \mathcal{O}_{\mathfrak{D}}(-1) \hookrightarrow \iota^* \pi_{\mathfrak{D}}^* \mathcal{S}$$

where  $\pi_{\mathfrak{D}} : \mathfrak{D} \rightarrow Q$  denotes the projection map and  $\mathcal{S} = p_* \mathcal{H}om(\mathcal{U}, \mathcal{U})$  (cf. Remark

4.1.9). Analogously we define  $V' = \bigoplus_{i=1}^m V'_i$  and  $\mathfrak{T}' = \mathfrak{T}_{P'}$  as a subscheme of the projective bundle  $\mathfrak{D}' = \mathbb{P}(\mathcal{R}')$  over  $Q'$  which comes with a tautological inclusion  $\iota'^* \mathcal{O}_{\mathfrak{D}'}(-1) \hookrightarrow \iota'^* \pi_{\mathfrak{D}'}^* \mathcal{S}'$ . We also define  $V'' = \bigoplus_{i=1}^m V''_i$  and  $\mathfrak{T}'' = \mathfrak{T}_{P''}$  as a subscheme of the projective bundle  $\mathfrak{D}'' = \mathbb{P}(\mathcal{R}'')$  over  $Q''$  where

$$\mathcal{R}'' = (V'')^\vee \otimes p''_* \left( \mathcal{U}'' \otimes (\pi_X^{Q'' \times X})^* \mathcal{O}_X(n) \right) \oplus \mathcal{O}_{Q''}''$$

and  $p'' : Q'' \times X \rightarrow Q''$  and  $\mathcal{U}''$  is the universal quotient sheaf on  $Q'' \times X$ .

Consider the sheaves

$$\mathcal{T} := p_*(\mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n)) \quad \text{and} \quad \mathcal{T}' := p'_*(\mathcal{U}' \otimes (\pi_X^{Q' \times X})^* \mathcal{O}_X(n))$$

over  $Q$  and  $Q'$ .

**Lemma 4.3.2.** *Let  $\phi : Q \times Q' \rightarrow Q''$  be the direct sum morphism of Lemma 4.3.1. There are invertible sheaves  $\mathcal{C}$  over  $Q$  and  $\mathcal{C}'$  over  $Q'$  such that there is an isomorphism*

$$\phi^* \mathcal{R}'' \cong (V'')^\vee \otimes \left( (\pi_Q^{Q \times Q'})^* (\mathcal{T} \otimes \mathcal{C}) \oplus (\pi_{Q'}^{Q' \times Q'})^* (\mathcal{T}' \otimes \mathcal{C}') \right) \oplus \mathcal{O}_{Q \times Q'}.$$

of sheaves on  $Q \times Q'$ . Moreover, the constant section  $1 \in H^0(\mathcal{O}_{Q \times Q'})$  and the decomposition  $V'' = V \oplus V'$  define an injective homomorphism

$$\left( V^\vee \otimes (\pi_Q^{Q \times Q'})^* (\mathcal{T} \otimes \mathcal{C}) \right) \oplus \left( (V')^\vee \otimes (\pi_{Q'}^{Q' \times Q'})^* (\mathcal{T}' \otimes \mathcal{C}') \right) \hookrightarrow \phi^* \mathcal{R}'' \quad (4.6)$$

of sheaves over  $Q \times Q'$ .

*Proof.* Recall from the proof of Lemma 4.3.1 that the morphism  $\phi$  is equivalent to the family

$$\mathcal{V} = \pi_{13}^* \mathcal{U} \oplus \pi_{23}^* \mathcal{U}'$$

parametrised by  $Q \times Q'$  where  $\pi_{jk}$  is the projection from  $Q \times Q' \times X$  to the  $j^{\text{th}}$  and  $k^{\text{th}}$

factors and  $\mathcal{U}$  (respectively  $\mathcal{U}'$ ) is the universal quotient sheaf on  $Q \times X$  (respectively  $Q' \times X$ ). Since the families of quotient sheaves  $\mathcal{V}$  and  $(\phi \times \text{id}_X)^*\mathcal{U}''$  are equivalent, there are invertible sheaves  $\mathcal{C}$  over  $Q$  and  $\mathcal{C}'$  over  $Q'$  such that

$$(\phi \times \text{id}_X)^*\mathcal{U}'' \cong \pi_{13}^*(\mathcal{U} \otimes p^*\mathcal{C}) \oplus \pi_{23}^*(\mathcal{U}' \otimes p'^*\mathcal{C}'). \quad (4.7)$$

By flat base change and (4.7) we have isomorphisms

$$\begin{aligned} \phi^*p''_* \left( \mathcal{U}'' \otimes (\pi_X^{Q'' \times X})^* \mathcal{O}_X(n) \right) &\cong \pi_{12*} \left( (\phi \times \text{id}_X)^*\mathcal{U}'' \otimes (\pi_X^{Q \times Q' \times X})^* \mathcal{O}_X(n) \right) \\ &\cong \pi_{12*} \left( (\pi_{13}^*\mathcal{U} \oplus \pi_{23}^*\mathcal{U}') \otimes (\pi_X^{Q \times Q' \times X})^* \mathcal{O}_X(n) \right) \otimes (\pi_Q^{Q \times Q'})^*\mathcal{C} \otimes (\pi_{Q'}^{Q' \times Q'})^*\mathcal{C}'. \end{aligned}$$

Similarly we have isomorphisms

$$\pi_{12*}\pi_{13}^* \left( \mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n) \right) \cong (\pi_Q^{Q \times Q'})^*p_* \left( \mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n) \right)$$

and

$$\pi_{12*}\pi_{23}^* \left( \mathcal{U}' \otimes (\pi_X^{Q' \times X})^* \mathcal{O}_X(n) \right) \cong (\pi_{Q'}^{Q' \times Q'})^*p'_* \left( \mathcal{U}' \otimes (\pi_X^{Q' \times X})^* \mathcal{O}_X(n) \right).$$

Hence,

$$\phi^*\mathcal{R}'' \cong (V'')^\vee \otimes \left( (\pi_Q^{Q \times Q'})^*(\mathcal{T} \otimes \mathcal{C}) \oplus (\pi_{Q'}^{Q' \times Q'})^*(\mathcal{T}' \otimes \mathcal{C}') \right) \oplus \mathcal{O}_{Q \times Q'}.$$

Note that  $V^\vee \otimes (\pi_Q^{Q \times X})^*(\mathcal{T} \otimes \mathcal{C})$  and  $(V')^\vee \otimes (\pi_{Q'}^{Q' \times X})^*(\mathcal{T}' \otimes \mathcal{C}')$  are direct summands of

$$(V'')^\vee \otimes \left( (\pi_Q^{Q \times Q'})^*(\mathcal{T} \otimes \mathcal{C}) \oplus (\pi_{Q'}^{Q' \times Q'})^*(\mathcal{T}' \otimes \mathcal{C}') \right)$$

and so the injective homomorphism of (4.6) is constructed by using the inclusion of these direct summands.  $\square$

**Proposition 4.3.3.** *Let  $\underline{P} = (P_1, \dots, P_m)$  and  $\underline{P}' = (P'_1, \dots, P'_m)$  be tuples of Hilbert polynomials. Then there is a morphism of schemes*

$$\Phi : \mathfrak{T}_{\underline{P}} \times \mathfrak{T}_{\underline{P}'} \rightarrow \mathfrak{T}_{\underline{P}''}$$

where  $\underline{P}'' = \underline{P} + \underline{P}'$ , given by

$$\Phi((q_1, \dots, q_m, d), (q'_1, \dots, q'_m, d')) = (q_1 \oplus q'_1, \dots, q_m \oplus q'_m, d \oplus d').$$

*Proof.* We use the notation as above: let  $\iota : \mathfrak{T} = \mathfrak{T}_{\underline{P}} \hookrightarrow \mathfrak{D}$  denote the inclusion into the projective bundle and  $\pi_{\mathfrak{D}} : \mathfrak{D} = \mathbb{P}(\mathcal{R}) \rightarrow Q$  the projection map (and use the analogous notation for  $\underline{P}'$  and  $\underline{P}''$ ). Since  $\mathfrak{T}''$  is a subscheme of the projective bundle  $\mathfrak{D}'' = \mathbb{P}(\mathcal{R}'')$  over  $Q''$ , we first construct a morphism

$$\Phi : \mathfrak{T} \times \mathfrak{T}' \rightarrow \mathfrak{D}''$$

which is, by [16] II Proposition 7.12, equivalent to giving an invertible sheaf  $\Lambda$  over  $\mathfrak{T} \times \mathfrak{T}'$  and an injective homomorphism  $\Lambda \rightarrow j^* \phi^* \mathcal{R}''$  of sheaves over  $\mathfrak{T} \times \mathfrak{T}'$  where  $\phi : Q \times Q' \rightarrow Q''$  is the morphism defined in Lemma 4.3.1 and  $j : \mathfrak{T} \times \mathfrak{T}' \rightarrow Q \times Q'$  is defined to be  $(\pi_{\mathfrak{D}} \circ \iota) \times (\pi_{\mathfrak{D}'} \circ \iota')$ . Let

$$\Lambda := (\pi_{\mathfrak{T} \times \mathfrak{T}'}^*)^* \iota^* (\mathcal{O}_{\mathfrak{D}}(-1) \otimes \pi_{\mathfrak{D}}^* \mathcal{C}) \otimes (\pi_{\mathfrak{T}' \times \mathfrak{T}'}^*)^* \iota'^* (\mathcal{O}_{\mathfrak{D}'}(-1) \otimes \pi_{\mathfrak{D}'}^* \mathcal{C}')$$

where  $\mathcal{C}$  and  $\mathcal{C}'$  are the invertible sheaves defined by Lemma 4.3.2. By Remark 4.1.9, there is a tautological inclusion

$$\iota^* \mathcal{O}_{\mathfrak{D}}(-1) \hookrightarrow \iota^* \pi_{\mathfrak{D}}^* \mathcal{S}$$

over  $\mathfrak{T}$  where  $\mathcal{S} = p_* \mathcal{H}om(\mathcal{U}, \mathcal{U})$  and a similar tautological inclusion over  $\mathfrak{T}'$ . Therefore,

there is an injective homomorphism

$$\Lambda \rightarrow j^* \left( (\pi_Q^{Q \times Q'})^* (\mathcal{S} \otimes \mathcal{C}) \oplus (\pi_{Q'}^{Q \times Q'})^* (\mathcal{S}' \otimes \mathcal{C}') \right). \quad (4.8)$$

The pushforward of the universal quotient

$$p_* \Gamma(n) : V \otimes \mathcal{O}_Q \rightarrow p_*(\mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n))$$

defines an injective homomorphism

$$\mathcal{S} \rightarrow V^\vee \otimes \mathcal{T}$$

over  $Q$  where  $\mathcal{T} := p_*(\mathcal{U} \otimes (\pi_X^{Q \times X})^* \mathcal{O}_X(n))$ . Similarly using

$$p'_* \Gamma'(n) : V' \otimes \mathcal{O}_Q \rightarrow p'_*(\mathcal{U}' \otimes (\pi_X^{Q' \times X})^* \mathcal{O}_X(n))$$

we have an injective homomorphism

$$\mathcal{S}' \rightarrow (V')^\vee \otimes \mathcal{T}'$$

over  $Q'$  where  $\mathcal{T}' := p'_*(\mathcal{U}' \otimes (\pi_X^{Q' \times X})^* \mathcal{O}_X(n))$ . By combining these homomorphisms with (4.8) we have a homomorphism

$$\Lambda \rightarrow j^* \left( V^\vee \otimes (\pi_Q^{Q \times Q'})^* (\mathcal{T} \otimes \mathcal{C}) \oplus (V')^\vee \otimes (\pi_{Q'}^{Q \times Q'})^* (\mathcal{T}' \otimes \mathcal{C}') \right). \quad (4.9)$$

We take the composition of (4.9) with the pullback via  $j$  of (4.6) to get the injective homomorphism

$$\Lambda \rightarrow j^* \phi^* \mathcal{R}''$$

which defines the morphism  $\Phi : \mathfrak{T} \times \mathfrak{T}' \rightarrow \mathfrak{D}''$ . From the construction of this homo-

morphism (see also Lemma 4.3.2) we see that this morphism corresponds to taking the direct sum and that its image is contained in  $\mathfrak{T}''$ .  $\square$

**Remark 4.3.4.** It is straight forward to check that  $\Phi$  is injective. Consider the groups  $G = \Pi_{i=1}^m \mathrm{GL}(V_i)/\mathbb{C}^*$  and  $G' = \Pi_{i=1}^m \mathrm{GL}(V'_i)/\mathbb{C}^*$  which act on  $\mathfrak{T}$  and  $\mathfrak{T}'$ ; then the morphism  $\Phi : \mathfrak{T} \times \mathfrak{T}' \rightarrow \mathfrak{T}''$  constructed above is  $G \times G'$ -equivariant.

**Corollary 4.3.5.** *Given a finite number of tuples of Hilbert polynomials  $\underline{P}^j$  for  $j = 1, \dots, s$  there is a direct sum map*

$$\Phi : \mathfrak{T}_{\underline{P}^1} \times \cdots \times \mathfrak{T}_{\underline{P}^s} \rightarrow \mathfrak{T}_{\sum_{j=1}^s \underline{P}^j}$$

*extending the above construction which is equivariant and injective.*

## Chapter 5

# A study of stability conditions for complexes

Let  $X$  be a smooth integral projective scheme over the complex numbers and  $\mathcal{O}_X(1)$  be an ample invertible sheaf on  $X$  as in the previous chapter. We continue to use the term complex to mean bounded cochain complex of coherent sheaves over  $X$ . In the last chapter we gave a notion of semistability for complexes due to Schmitt [39] which depends on stability parameters  $(\underline{\sigma}, \underline{\chi})$  where  $\underline{\chi} = \delta \underline{\eta}$  and

- $\underline{\sigma} = (\sigma_1, \dots, \sigma_m) \in \mathbb{Z}_{>0}^m$ ,
- $\underline{\eta} = (\eta_1, \dots, \eta_m) \in \mathbb{Q}^m$ ,
- $\delta$  is a positive rational polynomial such that  $\deg \delta = \max(\dim X - 1, 0)$ .

In this chapter we study these notions of stability for complexes in greater depth. As we are now studying complexes with varying invariants we do not impose any condition on  $\underline{\eta}$ . An important property of these stability conditions is that we can describe any complex of torsion free sheaves as a finite sequence of extensions of semistable complexes by studying its Harder–Narasimhan filtration. Note that the

Harder–Narasimhan filtration of a complex depends on the stability parameters we are using. We describe the notion of semistability when the complex is a sheaf and also when we are working over the point  $X = \text{Spec } \mathbb{C}$ .

We also describe a collection of stability conditions indexed by a small positive rational number  $\epsilon$  which can be used to study the cohomology sheaves of a given complex. More precisely, the Harder–Narasimhan filtration of a complex of torsion free sheaves with torsion free cohomology with respect to these notions of stability (for  $\epsilon$  sufficiently small) tells us the Harder–Narasimhan filtration of the cohomology sheaves (see Proposition 5.3.7 below).

## 5.1 Basic properties

In this section we study the basic properties of these stability conditions for complexes such as the existence of Harder–Narasimhan filtrations.

### 5.1.1 Harder–Narasimhan filtrations of complexes

Given a choice of stability parameters  $(\underline{\sigma}, \underline{\chi})$  every complex of torsion free sheaves has a maximal destabilising filtration known as the Harder–Narasimhan filtration of the complex with respect to  $(\underline{\sigma}, \underline{\chi})$ .

**Definition 5.1.1.** Let  $\mathcal{E}.$  be a complex of torsion free sheaves and  $(\underline{\sigma}, \underline{\chi})$  be a fixed stability condition. A Harder–Narasimhan filtration for  $\mathcal{E}.$  with respect to  $(\underline{\sigma}, \underline{\chi})$  is a filtration by subcomplexes

$$0. = \mathcal{E}^{(0)} \subsetneq \mathcal{E}^{(1)} \subsetneq \cdots \subsetneq \mathcal{E}^{(s)} = \mathcal{E}.$$

such that the successive quotients  $\mathcal{E}^{(j)} / \mathcal{E}^{(j-1)}$  are  $(\underline{\sigma}, \underline{\chi})$ -semistable and have decreas-

ing reduced Hilbert polynomials with respect to  $(\underline{\sigma}, \underline{\chi})$

$$P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}^{(1)}) > P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}^{(2)}/\mathcal{E}^{(1)}) > \dots > P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}^{(s)}/\mathcal{E}^{(s-1)}).$$

The Harder–Narasimhan filtration can be constructed inductively once we know how to construct a maximal destabilising subcomplex:

**Definition 5.1.2.** Let  $\mathcal{E}$  be a complex of torsion free sheaves and  $(\underline{\sigma}, \underline{\chi})$  be a fixed stability condition. A subcomplex  $\mathcal{E}' \subset \mathcal{E}$  is a maximal destabilising subcomplex for  $\mathcal{E}$  with respect to  $(\underline{\sigma}, \underline{\chi})$  if

- i) The complex  $\mathcal{E}'$  is  $(\underline{\sigma}, \underline{\chi})$ -semistable and,
- ii) For every subcomplex  $\mathcal{F}$  of  $\mathcal{E}$  such that  $\mathcal{E}' \subsetneq \mathcal{F}$  we have

$$P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}') > P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{F}).$$

Following the original existence statement of Harder and Narasimhan [15] we have:

**Lemma 5.1.3.** *A complex of torsion free sheaves has a unique maximal destabilising subcomplex with respect to  $(\underline{\sigma}, \underline{\chi})$ .*

The first term  $\mathcal{E}^{(1)}$  in the Harder–Narasimhan filtration is given by the maximal destabilising subcomplex. Then the second term is given by taking the preimage of the maximal destabilising subcomplex in  $\mathcal{E}/\mathcal{E}^{(1)}$  via the quotient map  $\mathcal{E} \rightarrow \mathcal{E}/\mathcal{E}^{(1)}$ . In this manner the Harder–Narasimhan filtration is constructed inductively and so we have:

**Lemma 5.1.4.** *A complex of torsion free sheaves has a unique Harder–Narasimhan filtration with respect to  $(\underline{\sigma}, \underline{\chi})$ .*

**Definition 5.1.5.** Let  $\mathcal{E}.$  be a complex of torsion free sheaves and  $(\underline{\sigma}, \underline{\chi})$  be stability parameters. Let

$$0. = \mathcal{E}^{(0)} \subsetneq \mathcal{E}^{(1)} \subsetneq \dots \subsetneq \mathcal{E}^{(s)} = \mathcal{E}.$$

denote the Harder–Narasimhan filtration of  $\mathcal{E}.$  with respect to  $(\underline{\sigma}, \underline{\chi})$ . Then the Harder–Narasimhan type of  $\mathcal{E}.$  with respect to  $(\underline{\sigma}, \underline{\chi})$  is given by a  $s$ -tuple  $\tau = (\underline{P}^1, \dots, \underline{P}^s)$  where  $\underline{P}^j$  is the tuple of Hilbert polynomials of the complex  $\mathcal{E}^{(j)}/\mathcal{E}^{(j-1)}$  so that

$$P_i^j := P(\mathcal{E}_i^{(j)}/\mathcal{E}_i^{(j-1)}).$$

### 5.1.2 Relation to Gieseker semistability of sheaves

Suppose  $\mathcal{E}.$  is a complex concentrated in only one degree so that  $\mathcal{E}_i = 0$  for all but one  $i$ . Then  $\mathcal{E}.$  is a shift of a sheaf  $\mathcal{E}$  over  $X$  and we can relate  $(\underline{\sigma}, \underline{\chi})$ -semistability of this complex to Gieseker semistability of the corresponding sheaf.

**Lemma 5.1.6.** *Let  $\mathcal{E}. = \mathcal{E}[-k]$  be the shift of a torsion free sheaf. Then  $\mathcal{E}.$  is a  $(\underline{\sigma}, \underline{\chi})$ -semistable complex if and only if  $\mathcal{E}$  is a Gieseker semistable sheaf.*

*Proof.* A subcomplex  $\mathcal{F}.$  of  $\mathcal{E}.$  is equivalent to a subsheaf  $\mathcal{F} \subset \mathcal{E}$ . The complex  $\mathcal{E}.$  is  $(\underline{\sigma}, \underline{\chi})$ -semistable if and only if for all  $\mathcal{F} \subset \mathcal{E}$  we have an inequality

$$P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{F}.) := \frac{\sigma_k P(\mathcal{F}) - \chi_k \text{rk } \mathcal{F}}{\sigma_k \text{rk } \mathcal{F}} \leq P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}.) := \frac{\sigma_k P(\mathcal{E}) - \chi_k \text{rk } \mathcal{E}}{\sigma_k \text{rk } \mathcal{E}},$$

or equivalently,

$$\frac{P(\mathcal{F})}{\text{rk } \mathcal{F}} \leq \frac{P(\mathcal{E})}{\text{rk } \mathcal{E}};$$

that is, the sheaf  $\mathcal{E}$  is Gieseker semistable. □

### 5.1.3 Relation to quiver representations

If  $X = \operatorname{Spec} \mathbb{C}$  is a point, then a sheaf over  $X$  is a complex vector space and its Hilbert polynomial and rank both coincide and are equal to the dimension of the vector space. A complex  $\mathcal{E}$ . over  $X$  is a representation of an  $A_m$ -quiver with the obvious orientation and relations in the category of complex vector spaces.

We fix a dimension vector  $\underline{d} = (d_1, \dots, d_m)$  for the vector spaces rather than a tuple of polynomials and suppose that  $(\underline{\sigma}, \underline{\chi})$  be stability parameters for  $\underline{d}$  so that  $\sum_{i=1}^m \eta_i d_i = 0$ . Then if  $V$ . is a complex of vector spaces with dimension vector  $\underline{d}$ ,

$$P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(V) = \frac{\sum_{i=1}^m \sigma_i \dim V_i}{\sum_{i=1}^m \sigma_i \dim V_i} = 1$$

and for a subcomplex  $W \subset V$ .

$$P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(W) = \frac{\sum_{i=1}^m \sigma_i \dim W_i - \delta \sum_{i=1}^m \eta_i \dim W_i}{\sum_{i=1}^m \sigma_i \dim W_i} = 1 - \delta \frac{\sum_{i=1}^m \eta_i \dim W_i}{\sum_{i=1}^m \sigma_i \dim W_i}.$$

In particular

$$P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(W) \leq P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(V) \iff \sum_{i=1}^m \eta_i \dim W_i \geq 0.$$

As the  $\eta_i$  are rational numbers, there is a positive integer  $N$  such that  $N\eta_i \in \mathbb{Z}$  for  $i = 1, \dots, m$ . Let  $\theta = N(\eta_1, \dots, \eta_m) \in \mathbb{Z}^m$ ; then this is a stability parameter for quiver representations in the sense of King [23].

**Definition 5.1.7.** A quiver representation  $V$ . is  $\theta$ -semistable if for all subrepresentations  $W$ . of  $V$ . we have  $\sum_{i=1}^m \theta_i \dim W_i \geq 0$ .

**Lemma 5.1.8.** Fix a dimension vector  $\underline{d} = (d_1, \dots, d_m)$  and let  $(\underline{\sigma}, \underline{\chi})$  be stability parameters for  $\underline{d}$ . A complex  $V$ . over  $X = \operatorname{Spec} \mathbb{C}$  with dimension vector  $\underline{d}$  is  $(\underline{\sigma}, \underline{\chi})$ -semistable if and only if it is  $\theta$ -semistable as a quiver representation.

## 5.2 Natural choices of stability

In this section we study a few natural choices of the stability parameters  $(\underline{\sigma}, \underline{\chi})$ .

**Lemma 5.2.1.** *A complex of torsion free sheaves  $\mathcal{E}$ . is  $(\underline{\sigma}, \underline{0})$ -semistable if and only if  $\oplus \mathcal{E}_i$  is a Gieseker semistable sheaf.*

*Proof.* Firstly suppose  $\oplus \mathcal{E}_i$  is Gieseker semistable with reduced Hilbert polynomial  $P$ ; then each  $\mathcal{E}_i$  must be semistable with reduced Hilbert polynomial  $P$ . Let  $\mathcal{F}$ . be a subcomplex of  $\mathcal{E}$ .; then by semistability of each  $\mathcal{E}_i$  we have

$$\sum_i \sigma_i P(\mathcal{F}_i) \leq \sum_i \sigma_i \operatorname{rk} \mathcal{F}_i P = \sum_i \sigma_i \operatorname{rk} \mathcal{F}_i \frac{\sum_i \sigma_i P(\mathcal{E}_i)}{\sum_i \sigma_i \operatorname{rk} \mathcal{E}_i}.$$

Conversely suppose  $\mathcal{E}$ . is  $(\underline{\sigma}, \underline{0})$ -semistable. We first show inductively that the sheaves  $\mathcal{E}_i$  have reduced Hilbert polynomial equal to

$$P := P_{\underline{\sigma}, \underline{0}}^{\operatorname{red}}(\mathcal{E}.) = \frac{\sum_{i=1}^m \sigma_i P(\mathcal{E}_i)}{\sum_{i=1}^m \sigma_i \operatorname{rk} \mathcal{E}_i}.$$

The image of each boundary map  $d_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$  can be viewed as a subcomplex and quotient complex of  $\mathcal{E}$ . and so the nonzero images have reduced Hilbert polynomial  $P$ . Similarly the nonzero kernels of the boundary maps have reduced Hilbert polynomial less than or equal to  $P$ . By inductively using subcomplexes of the form

$$\mathcal{E}_1 \rightarrow \cdots \rightarrow \mathcal{E}_{i-1} \rightarrow \mathcal{E}_i \rightarrow \operatorname{Im} d_i \rightarrow 0 \cdots$$

$$\mathcal{E}_1 \rightarrow \cdots \rightarrow \mathcal{E}_{i-1} \rightarrow \ker d_i \rightarrow 0 \rightarrow 0 \cdots$$

we can show that the reduced Hilbert polynomial of  $\mathcal{E}_i$  is less than or equal to  $P$  and then by using the quotient complex

$$\mathcal{E}_1 \rightarrow \cdots \rightarrow \mathcal{E}_{i-1} \rightarrow \mathcal{E}_i \rightarrow 0 \rightarrow 0 \cdots$$

we see that  $\mathcal{E}_i$  has reduced Hilbert polynomial  $P$ .

As  $\mathcal{E}_i$  and  $\text{Im } d_i$  both have reduced Hilbert polynomial  $P$ , the kernel of  $d_i$  must also have reduced Hilbert polynomial  $P$  by the see-saw property. Since  $\ker d_i$  and  $\text{Im } d_i$  are both subcomplexes of  $\mathcal{E}$ , with the same reduced Hilbert polynomial as  $\mathcal{E}$ , with respect to  $(\underline{\sigma}, \underline{0})$  these subcomplexes are also  $(\underline{\sigma}, \underline{0})$ -semistable. By Lemma 5.1.6 the nonzero kernels and images of the boundary maps are Gieseker semistable. There is a short exact sequence

$$0 \rightarrow \ker d_i \rightarrow \mathcal{E}_i \rightarrow \text{Im } d_i \rightarrow 0$$

where the outer sheaves are semistable with reduced Hilbert polynomial  $P$ ; therefore, each  $\mathcal{E}_i$  is Gieseker semistable with reduced Hilbert polynomial  $P$  and so is their direct sum  $\oplus \mathcal{E}_i$ .  $\square$

**Remark 5.2.2.** In the case when  $\underline{\chi} = 0$  the boundary maps of the complex have no impact on semistability. In fact to write down the Harder–Narasimhan filtration of a complex with respect to such stability parameters we need only study the Harder–Narasimhan filtrations of each sheaf in this complex. Given the maximal destabilising subsheaf  $\mathcal{E}'_i$  of each sheaf  $\mathcal{E}_i$  we construct the maximal destabilising subcomplex  $\mathcal{E}^{(1)}$  of  $\mathcal{E}$ , as follows. Let  $P$  be the maximal reduced Hilbert polynomial of the  $\mathcal{E}'_i$  and define

$$\mathcal{E}_i^{(1)} := \begin{cases} \mathcal{E}'_i & \text{if } P(\mathcal{E}'_i) = P \text{ rk } \mathcal{E}'_i \\ 0 & \text{otherwise.} \end{cases}$$

This is actually a subcomplex of  $\mathcal{E}$ , as there are no morphisms between two semistable sheaves when the domain sheaf has strictly larger reduced Hilbert polynomial than the codomain sheaf.

**Lemma 5.2.3.** *Let  $\mathcal{E}$  be a  $(\underline{1}, (-1)^i \delta)$ -semistable complex. Then for every even integer  $i$  the boundary map  $d_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$  is zero.*

*Proof.* Let  $d_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$  denote the boundary maps of the complex. If  $d_i$  is nonzero, then  $\text{Im } d_i$  is a subcomplex and quotient complex of  $\mathcal{E}$ . and so by  $(\underline{1}, (-1)^i \delta)$ -semistability of  $\mathcal{E}$ . we have

$$\frac{P(\text{Im } d_i)}{\text{rk Im } d_i} - \delta(-1)^{i+1} \leq P_{\underline{1}, (-1)^i \delta}^{\text{red}}(\mathcal{E}).$$

and

$$\frac{P(\text{Im } d_i)}{\text{rk Im } d_i} - \delta(-1)^i \geq P_{\underline{1}, (-1)^i \delta}^{\text{red}}(\mathcal{E}).$$

Therefore, if  $d_i$  is nonzero  $\delta(-1)^i \leq 0$ . As  $\delta$  is positive, the even boundary maps must all be zero.  $\square$

### 5.3 Notions of stability relating to cohomology

In this section we describe a collection of stability conditions indexed by a small positive rational number  $\epsilon$  which can be used to study the cohomology sheaves of a given complex. The stability parameters we are interested in are of the form  $(\underline{\epsilon}, \underline{\chi})$  where  $\underline{\epsilon}$  is the constant vector and  $\underline{\chi} = \delta \underline{\eta}$  where  $\eta_i$  are strictly increasing rational numbers. We see for a given complex with torsion free cohomology that for  $\epsilon > 0$  sufficiently small the Harder–Narasimhan filtration of this complex encodes the Harder–Narasimhan filtration of each cohomology sheaf in this complex.

#### 5.3.1 Stability with respect to $(\underline{0}, \underline{\chi})$

In this section we take the limit as  $\epsilon$  tends to zero and study semistability with respect to the parameters  $(\underline{0}, \underline{\chi})$  where  $\underline{\chi} = \delta \underline{\eta}$ . This is a slight generalisation of the parameters consider by Schmitt [39] where we now allow  $\sigma_i$  to be zero. Although these generalised stability parameters will no longer define an ample linearisation on the parameter space (cf. Definition 4.1.13), we can still study a corresponding notion

of semistability. Rather than requiring an inequality of reduced Hilbert polynomials we say that  $\mathcal{F}$  is  $(0, \underline{\chi})$ -semistable if it is a complex of torsion free sheaves and for all nonzero proper subcomplexes  $\mathcal{E}$  we have

$$\frac{\sum \eta_i \operatorname{rk} \mathcal{E}_i}{\sum \operatorname{rk} \mathcal{E}_i} \geq \frac{\sum \eta_i \operatorname{rk} \mathcal{F}_i}{\sum \operatorname{rk} \mathcal{F}_i}.$$

If we look at  $(\epsilon, \underline{\chi})$ -semistability of  $\mathcal{F}$ , then for subcomplexes  $\mathcal{E}$  we want an inequality

$$\left( \epsilon \sum P(\mathcal{E}_i) - \delta \sum \eta_i \operatorname{rk} \mathcal{E}_i \right) \left( \sum \operatorname{rk} \mathcal{F}_i \right) \leq \left( \epsilon \sum P(\mathcal{F}_i) - \delta \sum \eta_i \operatorname{rk} \mathcal{F}_i \right) \left( \sum \operatorname{rk} \mathcal{E}_i \right)$$

and as  $\epsilon$  tends to zero we get the inequality required for  $(0, \underline{\chi})$ -semistability.

**Lemma 5.3.1.** *Let  $\eta_i < \eta_{i+1}$  for all integers  $i$ . A complex  $\mathcal{E}$  of torsion free sheaves is  $(0, \underline{\chi})$ -semistable if and only if it is a shift of a torsion free sheaf or isomorphic to a shift of the cone of the identity morphism on a torsion free sheaf.*

*Proof.* Suppose  $\mathcal{E}$  is  $(0, \underline{\chi})$ -semistable. If for each  $i$  the boundary map  $d_i$  is zero, then each nonzero sheaf  $\mathcal{E}_k$  is a subcomplex and quotient complex simultaneously so by semistability

$$\eta_k = \frac{\sum \eta_i \operatorname{rk} \mathcal{E}_i}{\sum \operatorname{rk} \mathcal{E}_i}.$$

However as  $\eta_i < \eta_{i+1}$  this can only happen for one  $k$  so that  $\mathcal{E}$  is the shift of a torsion free sheaf.

If there is a nonzero boundary map  $d_k$  then the image of this boundary map can be viewed as a quotient complex and a subcomplex so that

$$\eta_k \leq \frac{\sum \eta_i \operatorname{rk} \mathcal{E}_i}{\sum \operatorname{rk} \mathcal{E}_i} \leq \eta_{k+1}. \quad (5.1)$$

As the  $\eta_i$  are strictly increasing this can only happen for one  $k$ , so this is the only nonzero boundary map. If  $i \neq k, k+1$  then  $\eta_i \neq (\sum \eta_i \operatorname{rk} \mathcal{E}_i) / (\sum \operatorname{rk} \mathcal{E}_i)$  and so by the

first part we see that  $\mathcal{E}_i = 0$ . If

$$\eta_k = \frac{\sum \eta_i \operatorname{rk} \mathcal{E}_i}{\sum \operatorname{rk} \mathcal{E}_i} = \frac{\eta_k \operatorname{rk} \mathcal{E}_k + \eta_{k+1} \operatorname{rk} \mathcal{E}_{k+1}}{\operatorname{rk} \mathcal{E}_k + \operatorname{rk} \mathcal{E}_{k+1}},$$

then  $\eta_k = \eta_{k+1}$  and so the inequalities of (5.1) are strict. If  $d_k$  is not injective, then the kernel is a subcomplex and so by semistability

$$\eta_k \geq \frac{\sum \eta_i \operatorname{rk} \mathcal{E}_i}{\sum \operatorname{rk} \mathcal{E}_i}.$$

Hence  $d_k$  is injective. Similarly  $d_k$  must be surjective and so  $\mathcal{E}$  is isomorphic to a shift of the cone on the identity morphism of a torsion free sheaf.

If  $\mathcal{E}$  is a shift of a torsion free sheaf  $\mathcal{E}_k$ , then a subcomplex  $\mathcal{E}'$  of  $\mathcal{E}$  is just a subsheaf and

$$\frac{\sum \eta_i \operatorname{rk} \mathcal{E}'_i}{\sum \operatorname{rk} \mathcal{E}'_i} = \eta_k = \frac{\sum \eta_i \operatorname{rk} \mathcal{E}_i}{\sum \operatorname{rk} \mathcal{E}_i}.$$

If  $\mathcal{E}$  is isomorphic to a shift of the cone on the identity morphism of a torsion free sheaf, then there is an integer  $k$  such that  $d_k$  is an isomorphism and  $\mathcal{E}_i = 0$  unless  $i = k$  or  $k + 1$ . A subcomplex  $\mathcal{E}'$  of  $\mathcal{E}$  is either concentrated in degree  $k + 1$  or concentrated in degrees  $k$  and  $k + 1$ . In the first case we have

$$\frac{\sum \eta_i \operatorname{rk} \mathcal{E}'_i}{\sum \operatorname{rk} \mathcal{E}'_i} = \eta_{k+1} \geq \frac{\eta_k \operatorname{rk} \mathcal{E}_k + \eta_{k+1} \operatorname{rk} \mathcal{E}_{k+1}}{\operatorname{rk} \mathcal{E}_k + \operatorname{rk} \mathcal{E}_{k+1}}.$$

In the second case we have  $\operatorname{rk} \mathcal{E}'_k \leq \operatorname{rk} \mathcal{E}'_{k+1}$  and so

$$\frac{\eta_k \operatorname{rk} \mathcal{E}'_k + \eta_{k+1} \operatorname{rk} \mathcal{E}'_{k+1}}{\operatorname{rk} \mathcal{E}'_k + \operatorname{rk} \mathcal{E}'_{k+1}} \geq \frac{\eta_k \operatorname{rk} \mathcal{E}_k + \eta_{k+1} \operatorname{rk} \mathcal{E}_{k+1}}{\operatorname{rk} \mathcal{E}_k + \operatorname{rk} \mathcal{E}_{k+1}}.$$

In particular,  $\mathcal{E}$  is  $(\underline{0}, \underline{\chi})$ -semistable. □

**Lemma 5.3.2.** *Let  $\eta_i < \eta_{i+1}$  for all integers  $i$  and  $\mathcal{E}$  be a complex of torsion free sheaves. Let  $k$  be the minimal integer for which  $\mathcal{E}_k$  is nonzero. Then the maximal*

destabilising subcomplex of  $\mathcal{E}$ . with respect to  $(\underline{0}, \underline{\chi})$  is

$$i) \mathcal{E}^{(1)} : \quad \cdots \rightarrow 0 \rightarrow \ker d_k \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \quad \text{if } \ker d_k \text{ is nonzero,}$$

$$ii) \mathcal{E}^{(1)} : \quad \cdots \rightarrow 0 \rightarrow \mathcal{E}_k \rightarrow \operatorname{Im} d_k \rightarrow 0 \rightarrow 0 \rightarrow \cdots \quad \text{if } \ker d_k \text{ is zero.}$$

*Proof.* By Lemma 5.3.1 both of these complexes are  $(\underline{0}, \underline{\chi})$ -semistable. Therefore in order to prove that this gives the maximal destabilising subcomplex, we need to show that if  $\mathcal{E}^{(1)} \subsetneq \mathcal{F} \subset \mathcal{E}$ . then

$$\frac{\sum \eta_i \operatorname{rk} \mathcal{F}_i}{\sum \operatorname{rk} \mathcal{F}_i} > \frac{\sum \eta_i \operatorname{rk} \mathcal{E}_i^{(1)}}{\sum \operatorname{rk} \mathcal{E}_i^{(1)}}.$$

As  $\mathcal{F}$ . is not equal to  $\mathcal{E}^{(1)}$ , the set  $I := \{i \in \mathbb{Z} : \mathcal{F}_i \neq \mathcal{E}_i^{(1)}\}$  is nonempty. Furthermore, for all  $i \in I$  as  $\mathcal{F}_i$  contains  $\mathcal{E}_i^{(1)}$  we must have that  $\mathcal{F}_i$  is nonzero.

Suppose that we are in the situation of case *i*). If  $k$  is a member of  $I$ , then  $\ker d_k \subsetneq \mathcal{F}_k$  and as  $\mathcal{F}$ . is a subcomplex we must have  $\mathcal{F}_{k+1} \neq 0$  so that  $k+1 \in I$ . As the  $\eta_i$  are strictly increasing we have

$$\eta_k \operatorname{rk} \mathcal{F}_k + \eta_{k+1} \operatorname{rk} \mathcal{F}_{k+1} > \eta_k (\operatorname{rk} \mathcal{F}_k + \operatorname{rk} \mathcal{F}_{k+1}).$$

If  $i > k+1$  and belongs to  $I$  we have  $\eta_i \operatorname{rk} \mathcal{F}_i > \eta_k \operatorname{rk} \mathcal{F}_i$  and so

$$\sum_{i \in I} \eta_i \operatorname{rk} \mathcal{F}_i > \sum_{i \in I} \eta_k \operatorname{rk} \mathcal{F}_i.$$

Then since  $\sum_{i \notin I} \eta_i \operatorname{rk} \mathcal{F}_i = \sum_{i \notin I} \eta_k \operatorname{rk} \mathcal{F}_i$  we see

$$\frac{\sum \eta_i \operatorname{rk} \mathcal{F}_i}{\sum \operatorname{rk} \mathcal{F}_i} > \eta_k = \frac{\sum \eta_i \operatorname{rk} \mathcal{E}_i^{(1)}}{\sum \operatorname{rk} \mathcal{E}_i^{(1)}}.$$

Suppose that we are in the situation of case *ii*) and so the first boundary morphism

is injective. Then  $k \notin I$  and so for all  $i \in I$  we have

$$\eta_i \operatorname{rk} \mathcal{F}_i > \frac{\eta_k + \eta_{k+1}}{2} \operatorname{rk} \mathcal{F}_i.$$

Since

$$\sum_{i \in I} \eta_i \operatorname{rk} \mathcal{F}_i > \sum_{i \in I} \frac{\eta_k + \eta_{k+1}}{2} \operatorname{rk} \mathcal{F}_i \quad \text{and} \quad \sum_{i \notin I} \eta_i \operatorname{rk} \mathcal{F}_i \geq \sum_{i \notin I} \frac{\eta_k + \eta_{k+1}}{2} \operatorname{rk} \mathcal{F}_i,$$

we have

$$\frac{\sum \eta_i \operatorname{rk} \mathcal{F}_i}{\sum \operatorname{rk} \mathcal{F}_i} > \frac{\eta_k + \eta_{k+1}}{2} = \frac{\sum \eta_i \operatorname{rk} \mathcal{E}_i^{(1)}}{\sum \operatorname{rk} \mathcal{E}_i^{(1)}}.$$

□

**Remark 5.3.3.** If  $\mathcal{E}$ . has torsion free cohomology, then its Harder–Narasimhan filtration with respect to these stability parameters picks out the kernels and images of the boundary maps successively:

$$\begin{array}{cccccccc} \mathcal{E}^{(1)} : & \cdots \rightarrow 0 & \rightarrow & \ker d_k & \rightarrow & 0 & \rightarrow & 0 \cdots \\ & & & \cap & & \cap & & \cap \\ \mathcal{E}^{(2)} : & \cdots \rightarrow 0 & \rightarrow & \mathcal{E}_k & \rightarrow & \operatorname{Im} d_k & \rightarrow & 0 \cdots \\ & & & \cap & & \cap & & \cap \\ \mathcal{E}^{(3)} : & \cdots \rightarrow 0 & \rightarrow & \mathcal{E}_k & \rightarrow & \ker d_{k+1} & \rightarrow & 0 \cdots \rightarrow 0 \\ & & & \cap & & \cap & & \cap \\ \mathcal{E}^{(4)} : & \cdots \rightarrow 0 & \rightarrow & \mathcal{E}_k & \rightarrow & \mathcal{E}_{k+1} & \rightarrow & \operatorname{Im} d_{k+1} \rightarrow 0 \cdots \\ & & & \vdots & & \vdots & & \vdots \end{array}$$

In particular every other successive quotient is a cohomology sheaf

$$\mathcal{H}^i(\mathcal{E}) = \ker d_i / \operatorname{Im} d_{i-1}.$$

### 5.3.2 Semistability with respect to $(\underline{\epsilon}, \underline{\chi})$

The stability parameters  $(\underline{0}, \underline{\chi})$  used above are not Schmitt stability parameters and so we consider a small perturbation of these parameters which are Schmitt stability parameters. Let  $\epsilon$  be a small positive rational number and consider the stability parameters  $(\underline{\epsilon}, \underline{\chi})$  where  $\underline{\chi} = \delta \underline{\eta}$  and  $\underline{\eta}$  are strictly increasing. Strictly speaking, Schmitt only allows integral values for  $\sigma_i$  and so we should instead think of the stability parameters  $(\underline{1}, \underline{\chi}/\epsilon)$  as the notions of  $(\underline{\epsilon}, \underline{\chi})$ -semistability and  $(\underline{1}, \underline{\chi}/\epsilon)$ -semistability agree.

**Lemma 5.3.4.** *Suppose  $\mathcal{E}$ . is a complex for which there is an  $\epsilon_0 > 0$  such that for all positive rational  $\epsilon < \epsilon_0$  this complex is  $(\underline{\epsilon}, \underline{\chi})$ -semistable. Then  $\mathcal{E}$ . is either a shift of a semistable torsion free sheaf or isomorphic to a shift of the cone on the identity morphism of a semistable torsion free sheaf.*

*Proof.* By studying the limit as  $\epsilon$  tends to zero we see that  $\mathcal{E}$ . is  $(\underline{0}, \underline{\chi})$ -semistable. By Lemma 5.3.1 the complex  $\mathcal{E}$ . is either a shift of a torsion free sheaf or isomorphic to a shift of the cone on the identity morphism of a torsion free sheaf.

In the first case  $\mathcal{E}$ . must be the shift of a torsion free semistable sheaf by Lemma 5.1.6. If  $\mathcal{E}$ . is isomorphic to the cone on the identity morphism of a torsion free sheaf  $\mathcal{F}$ , then for any subsheaf  $\mathcal{F}' \subset \mathcal{F}$  we can take the cone on the identity morphism of this subsheaf as a subcomplex. By semistability we have for any  $0 < \epsilon < \epsilon_0$

$$\epsilon \frac{P(\mathcal{F}')}{\text{rk } \mathcal{F}'} - \delta \frac{\eta_k + \eta_{k+1}}{2} \leq \epsilon \frac{P(\mathcal{F})}{\text{rk } \mathcal{F}} - \delta \frac{\eta_k + \eta_{k+1}}{2}$$

and so  $\mathcal{F}$  is Gieseker semistable. □

**Remark 5.3.5.** Conversely, a shift of a semistable torsion free sheaf or a shift of a cone on the identity morphism of a semistable torsion free sheaf is  $(\underline{\epsilon}, \underline{\chi})$ -semistable for any  $\epsilon > 0$ .

We now consider Harder–Narasimhan filtrations with respect to  $(\underline{\epsilon}, \underline{\chi})$ . Firstly suppose we have a  $(\underline{0}, \underline{\chi})$ -semistable complex then it is easy to determine the Harder–Narasimhan filtration of this complex with respect to  $(\underline{\epsilon}, \underline{\chi})$ :

**Lemma 5.3.6.** *Let  $\mathcal{E}$  be a  $(\underline{0}, \underline{\chi})$ -semistable complex.*

- i) If  $\mathcal{E}$  is a shift of a torsion free sheaf  $\mathcal{E}$ , then the Harder–Narasimhan filtration of  $\mathcal{E}$  with respect to  $(\underline{\epsilon}, \underline{\chi})$  is the Harder–Narasimhan filtration of the sheaf  $\mathcal{E}$ .*
- ii) If  $\mathcal{E}$  is isomorphic to the cone on the identity morphism of a torsion free sheaf  $\mathcal{E}$ , then the Harder–Narasimhan filtration of  $\mathcal{E}$  with respect to  $(\underline{\epsilon}, \underline{\chi})$  is given by taking cones on the identity morphism of each term in the Harder–Narasimhan filtration of the sheaf  $\mathcal{E}$ .*

Let  $\mathcal{E}$  be a complex of torsion free sheaves with torsion free cohomology. We have seen the Harder–Narasimhan filtration of  $\mathcal{E}$  with respect to  $(\underline{0}, \underline{\chi})$  picks out the successive kernels and images of each boundary map. The successive quotient complexes in this filtration are either shifts of the cohomology sheaves  $\mathcal{H}^i(\mathcal{E})$  or isomorphic to shifts of cones on the identity morphism of the images of the boundary morphisms  $d_i$ .

In the following proposition we show that, provided  $\epsilon$  is sufficiently small, the Harder–Narasimhan filtration of  $\mathcal{E}$  with respect to  $(\underline{\epsilon}, \underline{\chi})$  is given by refining the Harder–Narasimhan filtration of  $\mathcal{E}$  with respect to  $(\underline{0}, \underline{\chi})$  by using the sheaves occurring in the Harder–Narasimhan filtrations of the cohomology sheaves and image sheaves.

We assume for simplicity that  $\mathcal{E}$  is a complex concentrated in degrees  $1, \dots, m$ . Let  $0 \subsetneq \mathcal{H}^i(\mathcal{E})^{(1)} \subsetneq \dots \subsetneq \mathcal{H}^i(\mathcal{E})^{(s_i)} = \mathcal{H}^i(\mathcal{E})$  be the Harder–Narasimhan filtration of the  $i$ th cohomology sheaf and  $0 \subsetneq \text{Im } d_i^{(1)} \subsetneq \dots \subsetneq \text{Im } d_i^{(t_i)} = \text{Im } d_i$  be the Harder–Narasimhan filtration of the image sheaf  $\text{Im } d_i$ .

For  $i = 1, \dots, m$  and  $j = 1, \dots, s_i$  we define a subcomplex  $\mathcal{A}^{i,(j)} \subset \mathcal{E}$  by

$$\mathcal{A}_k^{i,(j)} = \begin{cases} \mathcal{E}_k & \text{if } k < i, \\ \pi_i^{-1}(\mathcal{H}^i(\mathcal{E})^{(j)}) & \text{if } k = i, \\ 0 & \text{if } k > i, \end{cases}$$

where  $\pi_i : \ker d_i \rightarrow \mathcal{H}^i(\mathcal{E})$ .

For  $i = 1, \dots, m-1$  and  $j = 1, \dots, t_i$  we define a subcomplex  $\mathcal{B}^{i,(j)} \subset \mathcal{E}$  by

$$\mathcal{B}_k^{i,(j)} = \begin{cases} \mathcal{E}_k & \text{if } k < i, \\ p_i^{-1}(\text{Im } d_i^{(j)}) & \text{if } k = i, \\ \text{Im } d_i^{(j)} & \text{if } k = i+1, \\ 0 & \text{if } k > i+1, \end{cases}$$

where  $p_i : \mathcal{E}_i \rightarrow \text{Im } d_i$ .

**Proposition 5.3.7.** *Let  $\underline{\eta}$  be strictly increasing and  $\mathcal{E}$  be a complex of torsion free sheaves with torsion free cohomology. Then there is an  $\epsilon_0 > 0$  such that for all positive rational  $\epsilon < \epsilon_0$  the Harder–Narasimhan filtration of  $\mathcal{E}$  with respect to  $(\underline{\epsilon}, \underline{\chi})$  is the filtration*

$$0 \subset \mathcal{A}^{1,(1)} \subset \dots \subset \mathcal{A}^{1,(s_1)} \subset \mathcal{B}^{1,(1)} \subset \dots \subset \mathcal{B}^{1,(t_1)} \subset \mathcal{A}^{2,(1)} \subset \dots \subset \mathcal{A}^{m,(s_m)}.$$

*Proof.* Let  $\mathcal{H}^i(\mathcal{E})^j$  (respectively  $\text{Im } d_i^j$ ) denote the successive quotient sheaves in the Harder–Narasimhan filtration of  $\mathcal{H}^i(\mathcal{E})$  (respectively  $\text{Im } d_i$ ). Then the successive quotients in the above filtration are either shifts of  $\mathcal{H}^i(\mathcal{E})^j$  or isomorphic to shifts of the cone on the identity morphism of  $\text{Im } d_i^j$ . In particular, the successive quotients are  $(\underline{\epsilon}, \underline{\chi})$ -semistable by Remark 5.3.5. Therefore we just need to verify that the reduced Hilbert polynomials of the successive quotients are strictly decreasing provided  $\epsilon$  is sufficiently small. More precisely, we need to show that there is an  $\epsilon_0$  such that for

all  $0 < \epsilon < \epsilon_0$  we have

$$\begin{aligned} \epsilon \frac{P(\mathcal{H}^1(\mathcal{E})^1)}{\text{rk } \mathcal{H}^1(\mathcal{E})^1} - \delta \eta_1 &> \epsilon \frac{P(\mathcal{H}^1(\mathcal{E})^2)}{\text{rk } \mathcal{H}^1(\mathcal{E})^2} - \delta \eta_1 > \dots > \epsilon \frac{P(\mathcal{H}^1(\mathcal{E})^{s_1})}{\text{rk } \mathcal{H}^1(\mathcal{E})^{s_1}} - \delta \eta_1 > \\ \epsilon \frac{P(\text{Im } d_1^1)}{\text{rk Im } d_1^1} - \delta \frac{\eta_1 + \eta_2}{2} &> \dots > \epsilon \frac{P(\text{Im } d_1^{t_1})}{\text{rk Im } d_1^{t_1}} - \delta \frac{\eta_1 + \eta_2}{2} > \epsilon \frac{P(\mathcal{H}^2(\mathcal{E})^1)}{\text{rk } \mathcal{H}^1(\mathcal{E})^1} - \delta \eta_2 \end{aligned}$$

and so on. Since we know the reduced Hilbert polynomials of the successive quotients in the Harder–Narasimhan filtrations of the cohomology and image sheaves are decreasing, it suffices to show for  $i = 1, \dots, m-1$  that:

$$\begin{aligned} 1) \quad & \epsilon \frac{P(\mathcal{H}^i(\mathcal{E})^{s_i})}{\text{rk } \mathcal{H}^i(\mathcal{E})^{s_i}} - \delta \eta_i > \epsilon \frac{P(\text{Im } d_i^1)}{\text{rk Im } d_i^1} - \delta \frac{\eta_i + \eta_{i+1}}{2} \quad \text{and} \\ 2) \quad & \epsilon \frac{P(\text{Im } d_i^{t_i})}{\text{rk Im } d_i^{t_i}} - \delta \frac{\eta_i + \eta_{i+1}}{2} > \epsilon \frac{P(\mathcal{H}^{i+1}(\mathcal{E})^1)}{\text{rk } \mathcal{H}^{i+1}(\mathcal{E})^1} - \delta \eta_{i+1}. \end{aligned} \tag{5.2}$$

As these inequalities involve polynomials we start by looking at the highest coefficients. We have equality for the top coefficients and so we next look at the second to top coefficients. Let  $\mu(\mathcal{A})$  denote the second to top coefficient of the reduced Hilbert polynomial of  $\mathcal{A}$  which is, up to multiplication by a positive constant, the slope of  $\mathcal{A}$ . We assume for simplicity that  $\delta$  is monic. For  $i = 1, \dots, m-1$  let

$$M_i := \max \left\{ \mu(\text{Im } d_i^1) - \mu(\mathcal{H}^i(\mathcal{E})^{s_i}), \mu(\mathcal{H}^{i+1}(\mathcal{E})^1) - \mu(\text{Im } d_i^{t_i}) \right\}.$$

We can pick  $\epsilon_0 > 0$  so that if  $M_i > 0$  then  $\epsilon_0 < (\eta_{i+1} - \eta_i)/2M_i$ . Then if  $\epsilon < \epsilon_0$  we get strict inequalities in the second to top coefficients of (5.2).  $\square$

## Chapter 6

# Stratifications for complexes by Harder–Narasimhan type

In the previous chapter we saw that every complex of torsion free sheaves has a unique Harder–Narasimhan filtration with respect to a fixed notion of stability determined by parameters  $(\underline{\sigma}, \underline{\chi})$ . In particular we studied semistability and Harder–Narasimhan filtrations with respect to  $(\underline{\epsilon}, \delta \underline{\eta})$  (or equivalently the associated Schmitt stability parameters  $(\underline{1}, \delta \underline{\eta}/\epsilon)$ ) where  $\epsilon > 0$  is an arbitrarily small rational number,  $\eta_i$  are strictly increasing rational numbers and  $\delta$  is a positive rational polynomial such that  $\deg \delta = \max(\dim X - 1, 0)$ . The Harder–Narasimhan filtration of a complex with torsion free cohomology with respect to  $(\underline{\epsilon}, \delta \underline{\eta})$  tells us about the Harder–Narasimhan filtration of the cohomology sheaves of this complex provided  $\epsilon > 0$  is chosen sufficient small (see Proposition 5.3.7).

In Section 4.1 (see also [39]) we saw that the moduli space of  $(\underline{1}, \delta \underline{\eta}/\epsilon)$ -semistable complexes with fixed Hilbert polynomials  $\underline{P} = (P_1, \dots, P_m)$  is given by a GIT quotient of a parameter space  $\mathfrak{T} := \mathfrak{T}_{\underline{P}}$  by a reductive group

$$G = \prod_{i=1}^m \mathrm{GL}(P_i(n)) \cap \mathrm{SL}(\sum_{i=1}^m P_i(n)).$$

This quotient is taken respect to a linearisation  $\mathcal{L}$  determined by the stability parameters  $(1, \delta\tilde{\eta}/\epsilon)$  where  $\tilde{\eta} := \underline{\eta} - (\sum \eta_i r_i / \sum r_i)$  are the stability parameters for  $\underline{P}$  associated to  $\underline{\eta}$ . In this chapter we consider an open subscheme  $\mathfrak{T}^{tf}$  of  $\mathfrak{T}$  which parametrises complexes with torsion free cohomology. There is a stratification of the closure of  $\mathfrak{T}^{tf}$  associated to the linearisation of this action as in Sections 2.1 and 2.3 which we can intersect with  $\mathfrak{T}^{tf}$  to get a stratification of this scheme. We fix a Harder–Narasimhan type with respect to these parameters and show there is a scheme which is a union of connected components of a stratum in this stratification of  $\mathfrak{T}^{tf}$  which parametrises all complexes with this fixed Harder–Narasimhan type.

## 6.1 The stratification of the parameter scheme

As we are interested in points of  $\mathfrak{T}$  which parametrise complexes with torsion free cohomology, we consider an open subscheme  $\mathfrak{T}^{tf}$  which parametrises such complexes. We continue to use the notation of Section 4.1.2. Let  $\iota : \mathfrak{T} \hookrightarrow \mathfrak{D}$  be the inclusion of  $\mathfrak{T}$  into the projective bundle  $\mathfrak{D}$  and  $(\mathcal{U}_i)_{\mathfrak{T} \times X} = (\iota \circ \text{id}_X)^*(\pi_{Q_i \times X}^{\mathfrak{D} \times X})^* \mathcal{U}_i$  be the pullback of the universal quotient sheaf on the quot scheme  $Q_i$ . For  $i = 1, \dots, m-1$  let

$$(d_i)_{\mathfrak{T} \times X} : (\mathcal{U}_i)_{\mathfrak{T} \times X} \rightarrow (\mathcal{U}_{i+1})_{\mathfrak{T} \times X} \otimes (\iota \times \text{id}_X)^*(\pi_{\mathfrak{D}}^{\mathfrak{D} \times X})^* \mathcal{O}_{\mathfrak{D}}(1)$$

be the tautological homomorphisms over  $\mathfrak{T} \times X$  which describe the boundary maps of the complexes parametrised by  $\mathfrak{T}$ . Then

$$\mathcal{H}^i := \ker d_i / \text{Im } d_{i-1}$$

is a family of sheaves over  $X$  parametrised by  $\mathfrak{T}$  which describe the cohomology sheaves of the complexes parametrised by  $\mathfrak{T}$ . By performing a flattening stratification (see [19], Theorem 2.1.5) we can construct an open subscheme  $\mathfrak{T}^i \subset \mathfrak{T}$  parametrisng

complexes whose  $i$ th cohomology sheaf is torsion free. Then we define  $\mathfrak{T}^{tf}$  to be the intersection of these subschemes.

Let  $\overline{\mathfrak{T}}$  be the projective completion of the parameter scheme as in Definition 4.2.1 and let  $\overline{\mathfrak{T}}^{tf}$  denote the closure of  $\mathfrak{T}^{tf}$  in  $\overline{\mathfrak{T}}$ . We shall work with the stratification of the projective completion  $\overline{\mathfrak{T}}^{tf}$  as the stratification as described in Section 2.3 is for projective schemes. The group action and linearisation  $\mathcal{L}$  extend to  $\overline{\mathfrak{T}}^{tf}$  and this linearisation defines a  $G$ -equivariant embedding of this scheme into an ambient projective space. By intersecting the associated stratification of the ambient projective space with  $\overline{\mathfrak{T}}^{tf}$  we get a stratification

$$\overline{\mathfrak{T}}^{tf} = \bigsqcup_{\beta \in \mathcal{B}} S_{\beta} \quad (6.1)$$

into  $G$ -invariant locally closed subschemes as in Section 2.3. Of course, we can intersect this stratification with the parameter scheme  $\mathfrak{T}^{tf}$ . However, the strata occurring in the stratification of projective schemes have nice descriptions (see Section 2.1) and so we work with the projective completion.

By choosing a basis of the fixed vector space  $V_i$  of dimension  $P_i(n)$  we determine a compact maximal torus  $T_i$  of  $\mathrm{GL}(V_i)$  which consists of the matrices which are diagonal with respect to this basis. We choose the positive Weyl chamber

$$(\mathfrak{t}_i)_+ := \{i\mathrm{diag}(a_1, \dots, a_{\dim V_i}) : a_1 \geq \dots \geq a_{\dim V_i}\}$$

in  $\mathfrak{t}_i = \mathrm{Lie} T_i$ . The maximal tori  $T_i$  of  $\mathrm{GL}(V_i)$  determine a maximal torus  $T$  of  $G$  and the choices  $(\mathfrak{t}_i)_+$  of positive Weyl chambers determine a positive Weyl chamber  $\mathfrak{t}_+$  in  $\mathfrak{t} = \mathrm{Lie} T$ . The index set  $\mathcal{B}$  for the stratification  $\{S_{\beta} : \beta \in \mathcal{B}\}$  is a finite set of points in  $\mathfrak{t}_+$ .

## 6.2 Boundedness

In this section we give some boundedness results for complexes of fixed Harder–Narasimhan type. The set of sheaves occurring in  $(\underline{\sigma}, \underline{\chi})$ -unstable complexes with Hilbert polynomials  $\underline{P}$  and fixed Harder–Narasimhan type  $\tau$  is bounded by the following lemma.

**Lemma 6.2.1.** *Fix a choice of stability parameters  $(\underline{\sigma}, \underline{\chi})$  and a Harder–Narasimhan type with respect to these parameters  $\tau = (\underline{P}^1, \dots, \underline{P}^s)$ . Then the set of sheaves occurring in a complex of torsion free sheaves with Harder–Narasimhan type  $\tau$  with respect to  $(\underline{\sigma}, \underline{\chi})$  is bounded.*

*Proof.* This follows from a result of Simpson (see [40] Theorem 1.1) which states that a set of torsion free sheaves on  $X$  of fixed Hilbert polynomial is bounded if the slopes of their subsheaves are bounded above by a fixed constant.

Recall that the slope of a sheaf is (up to multiplication by a positive constant) the second to top coefficient in its reduced Hilbert polynomial. Let  $\mathcal{E}$  be a complex with Harder–Narasimhan type  $\tau$ ; then for any subcomplex  $\mathcal{F}$  of  $\mathcal{E}$  we have

$$P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{F}) \leq P_{\underline{\sigma}, \underline{\chi}}^{\text{red}}(\mathcal{E}^{(1)}) = \frac{\sum_{i=1}^m \sigma_i P_i^1 - \delta \sum_{i=1}^m \eta_i r_i^1}{\sum_{i=1}^m \sigma_i r_i^1} =: R$$

where  $\mathcal{E}^{(1)}$  is the maximal destabilising subcomplex of  $\mathcal{E}$  which has Hilbert polynomials specified by  $\underline{P}^1$ . Suppose  $\mathcal{F}$  is a subsheaf of  $\mathcal{E}_i$  and consider the subcomplex

$$\mathcal{F} : 0 \rightarrow \dots \rightarrow 0 \rightarrow \mathcal{F} \rightarrow \mathcal{E}_{i+1} \rightarrow \dots \rightarrow \mathcal{E}_m;$$

then we have an inequality of polynomials

$$\frac{\sigma_i P(\mathcal{F}) - \delta \eta_i \text{rk } \mathcal{F} + \sum_{j=i+1}^m \sigma_j P_j^1 - \delta \sum_{j=i+1}^m \eta_j r_j^1}{\sigma_i \text{rk } \mathcal{F} + \sum_{j=i+1}^m \sigma_j r_j^1} \leq R.$$

The top coefficients agree and so looking at the second to top coefficients we have

$$\begin{aligned} \frac{\deg(\mathcal{F})}{\operatorname{rk} \mathcal{F}} &\leq \frac{\delta^{\operatorname{top}}}{\sigma_i} \left( \eta_i + \frac{\sum_{j=i+1}^m \eta_j r_j^1}{\operatorname{rk} \mathcal{F}} \right) - \frac{\sum_{j=i+1}^m \sigma_j d_j^1}{\sigma_i \operatorname{rk} \mathcal{F}} \\ &\quad + \left( 1 + \frac{\sum_{j=i+1}^m \sigma_j r_j^1}{\sigma_i \operatorname{rk} \mathcal{F}} \right) \frac{\sum_{j=1}^m \sigma_j d_j^1 - \delta^{\operatorname{top}} \sum_{j=1}^m \eta_j r_j^1}{\sum_{j=1}^m \sigma_j r_j^1} \end{aligned} \quad (6.2)$$

where  $d_j^1$  is (up to multiplication by a positive constant) the second to top coefficient of  $P_j^1$  and  $\delta^{\operatorname{top}}$  is (up to multiplication by a positive constant) the leading coefficient of  $\delta$ . Since the rank of  $\mathcal{F}$  is bounded, it follows that the slope of subsheaves  $\mathcal{F} \subset \mathcal{E}_i$  are bounded.  $\square$

**Corollary 6.2.2.** *Let  $\tau = (\underline{P}^1, \dots, \underline{P}^s)$  be a Harder–Narasimhan type with respect to  $(\underline{\sigma}, \underline{\chi})$ . Then we can choose  $n$  sufficiently large so that for  $1 \leq i_1 < \dots < i_k \leq s$  all the sheaves occurring in a complex of torsion free sheaves with Harder–Narasimhan type  $(\underline{P}^{i_1}, \dots, \underline{P}^{i_k})$  are  $n$ -regular.*

### 6.3 The associated index

Let  $\mathcal{F}.$  be a complex of torsion free sheaves with Hilbert polynomials given by  $\underline{P} = (P_1, \dots, P_m)$ . We assume that the cohomology sheaves of  $\mathcal{F}.$  are torsion free and that  $\mathcal{F}.$  is not the shift of a semistable sheaf or the cone on the identity morphism of a semistable sheaf. By Lemma 5.3.2 and Proposition 5.3.7, the Harder–Narasimhan filtration of  $\mathcal{F}.$  with respect to  $(\underline{0}, \delta\underline{\eta})$  picks out the cohomology and image sheaves of this complex successively and if the rational parameter  $\epsilon > 0$  is sufficiently small then the Harder–Narasimhan filtration of  $\mathcal{F}.$  with respect to  $(\underline{\epsilon}, \delta\underline{\eta})$  refines this filtration by using the Harder–Narasimhan filtrations of the cohomology and image sheaves.

Let  $H^{k,j}$  denote the Hilbert polynomial of the  $j$ th successive quotient in the Harder–Narasimhan filtration  $0 \subset \mathcal{H}^k(\mathcal{F}.)^{(1)} \subset \dots \subset \mathcal{H}^k(\mathcal{F}.)^{(s_k)} = \mathcal{H}^k(\mathcal{F}.)$  of the  $k$ th cohomology sheaf and similarly let  $I^{k,j}$  denote the Hilbert polynomial of the

$j$ th successive quotient in the Harder–Narasimhan filtration  $0 \subset \operatorname{Im} d_k^{(1)} \subset \cdots \subset \operatorname{Im} d_k^{(t_k)} = \operatorname{Im} d_k$  of the  $k$ th image sheaf. We define  $\underline{H}^{k,j}$  to be the vector of Hilbert polynomials which is zero except in degree  $k$  where it is  $H^{k,j}$  and  $\underline{I}^{k,j}$  to be the vector of Hilbert polynomials which is zero except in degrees  $k$  and  $k+1$  where it is  $I^{k,j}$ . For  $\epsilon$  sufficiently small the Harder–Narasimhan type of  $\mathcal{F}$  with respect to  $(\epsilon, \delta\eta)$  is

$$\tau = (\underline{H}^{1,1}, \dots, \underline{H}^{1,s_1}, \underline{I}^{1,1}, \dots, \underline{I}^{1,t_1}, \underline{H}^{2,1}, \dots, \underline{H}^{n,s_m})$$

which we abbreviate to  $\tau = (\underline{H}, \underline{I})$  where  $\underline{H}$  records all the information of the vectors  $\underline{H}^{k,j}$  and  $\underline{I}$  records all the information of the vectors  $\underline{I}^{k,j}$ .

We assume  $\epsilon > 0$  is sufficiently small so that Proposition 5.3.7 holds for the above complex  $\mathcal{F}$  and that  $n$  is sufficiently large so that Lemma 6.2.1 and Corollary 6.2.2 hold for this Harder–Narasimhan type  $\tau$ . In particular this means all complexes with Harder–Narasimhan type  $\tau$  are parametrised by  $\mathfrak{T}^{tf}$ .

**Definition 6.3.1.** Let  $R_\tau$  denote the set of points in  $\mathfrak{T}^{tf}$  which parametrise complexes with Harder–Narasimhan type  $\tau$ . Then we may write

$$\mathfrak{T}^{tf} = \bigsqcup_{\tau} R_\tau$$

where the union is over all Harder–Narasimhan types.

In this section we associate to this nontrivial Harder–Narasimhan type  $\tau$  an index  $\beta(\tau)$  and in the next section we show for  $n$  sufficiently large (depending on  $\tau$ ) that  $R_\tau$  is a union of connected components of  $S_{\beta(\tau)} \cap \mathfrak{T}^{tf}$  where  $S_{\beta(\tau)}$  is a stratum appearing in the stratification defined at (6.1).

Let  $z = (q_1, \dots, q_m, d)$  be a point in  $\mathfrak{T}^{tf}$  which parametrises the fixed complex  $\mathcal{F}$  with Harder–Narasimhan type  $\tau$ . By Remark 2.1.8, rather than looking for the rational weight  $\beta = \beta(\tau)$  such that  $z \in Y_\beta^{ss}$  we can equivalently look for a 1-PS  $\lambda_\beta$  of

$G$  which is adapted to  $z$ . As in the case of sheaves, we expect a 1-PS which is adapted to  $z$  to induce the Harder–Narasimhan filtration of  $\mathcal{F}$  with respect to  $(\underline{\epsilon}, \underline{\delta\eta})$ . Let

$$0 \subsetneq \mathcal{A}^{1,(1)} \subsetneq \dots \subsetneq \mathcal{A}^{1,(s_1)} \subsetneq \mathcal{B}^{1,(1)} \dots \mathcal{B}^{1,(t_1)} \subsetneq \mathcal{A}^{2,(1)} \subsetneq \dots \subsetneq \mathcal{A}^{m,(s_m)} = \mathcal{F}.$$

denote the Harder–Narasimhan filtration of  $\mathcal{F}$  with respect to  $(\underline{\epsilon}, \underline{\delta\eta})$ . For the construction of the parameter scheme (see Section 4.1.2) we fixed vector spaces  $V_i$  of dimension  $P_i(n)$ . For  $i = 1, \dots, m$  the Harder–Narasimhan filtration of  $\mathcal{F}$  induces filtrations

$$0 \subset V_i^{1,(1)} \subset \dots \subset V_i^{1,(s_1)} \subset W_i^{1,(1)} \subset \dots \subset W_i^{1,(t_1)} \subset \dots \subset V_i^{m,(s_m)} = V_i \quad (6.3)$$

of these vector spaces where

$$V_i^{k,(j)} := H^0(q_i(n))^{-1} H^0(\mathcal{A}_i^{k,(j)}(n))$$

and

$$W_i^{k,(j)} := H^0(q_i(n))^{-1} H^0(\mathcal{B}_i^{k,(j)}(n)).$$

Let  $V_i^{k,j}$  (respectively  $W_i^{k,j}$ ) denote the quotient of  $V_i^{k,(j)}$  (respectively  $W_i^{k,(j)}$ ) by the previous vector space in this filtration. Note that by the construction of the Harder–Narasimhan filtration (see Proposition 5.3.7) we have that  $V_i^{k,j} = 0$  unless  $k = i$  and  $W_i^{k,j} = 0$  unless  $k = i$  or  $k = i - 1$ .

Given integers  $a_{k,j}$  for  $k = 1, \dots, m$  and  $j = 1, \dots, s_k$  and integers  $b_{k,j}$  for  $k = 1, \dots, m - 1$  and  $j = 1, \dots, t_k$  which satisfy

$$\begin{aligned} i) \quad & a_{1,1} > \dots > a_{1,s_1} > b_{1,1} > \dots > b_{1,t_1} > a_{2,1} > \dots > a_{m,s_m} \\ ii) \quad & \sum_{i=1}^m \sum_{j=1}^{s_i} a_{i,j} \dim V_i^{i,j} + 2 \sum_{i=1}^{m-1} \sum_{j=1}^{t_i} b_{i,j} \dim W_i^{i,j} = 0, \end{aligned} \quad (6.4)$$

we can define a 1-PS  $\lambda(\underline{a}, \underline{b})$  of  $G$  as follows. Let  $\lambda(H)_i^k : \mathbb{C}^* \rightarrow \mathrm{GL}(V_i^k)$  be given by

$$\lambda(H)_i^k(t) = \begin{pmatrix} t^{a_{k,1}} I_{V_i^{k,1}} & & \\ & \ddots & \\ & & t^{a_{k,s_k}} I_{V_i^{k,s_k}} \end{pmatrix}$$

where  $V_i^k = \bigoplus_{j=1}^{s_k} V_i^{k,j}$  and let  $\lambda(I)_i^k : \mathbb{C}^* \rightarrow \mathrm{GL}(W_i^k)$  be given by

$$\lambda(I)_i^k(t) = \begin{pmatrix} t^{b_{k,1}} I_{W_i^{k,1}} & & \\ & \ddots & \\ & & t^{b_{k,t_k}} I_{W_i^{k,t_k}} \end{pmatrix}$$

where  $W_i^k = \bigoplus_{j=1}^{t_k} W_i^{k,j}$  and  $b_{0,j} = b_{m,j} = 0$ . Then  $(\underline{a}, \underline{b})$  define a 1-PS  $\lambda(\underline{a}, \underline{b}) := (\lambda_1, \dots, \lambda_m)$  of  $G$  where

$$\lambda_i(t) := \begin{pmatrix} \lambda(I)_i^{i-1}(t) & & \\ & \lambda(H)_i^i(t) & \\ & & \lambda(I)_i^i(t) \end{pmatrix}. \quad (6.5)$$

We want to pick  $(\underline{a}, \underline{b})$  to minimise the (normalised) Hilbert–Mumford function

$$\frac{\mu^{\mathcal{L}}(z, \lambda(\underline{a}, \underline{b}))}{\|\lambda(\underline{a}, \underline{b})\|}$$

subject to condition *ii*) of (6.4). For all pairs  $(\underline{a}, \underline{b})$  the associated 1-PS  $\lambda(\underline{a}, \underline{b})$  of  $G$  induces the Harder–Narasimhan filtration of  $\mathcal{F}$ . and so by Proposition 4.2.8

$$\begin{aligned} \mu^{\mathcal{L}}(z, \lambda(\underline{a}, \underline{b})) &= \sum_{i=1}^m \sum_{j=1}^{s_i} a_{i,j} \left( \frac{P(n)}{r\delta(n)} + \frac{\tilde{\eta}_i}{\epsilon} \right) \mathrm{rk} \mathcal{H}^i(\mathcal{F})^j \\ &\quad + \sum_{i=1}^{m-1} \sum_{j=1}^{t_i} b_{i,j} \left( 2 \frac{P(n)}{r\delta(n)} + \frac{\tilde{\eta}_i + \tilde{\eta}_{i+1}}{\epsilon} \right) \mathrm{rk} \mathrm{Im} d_i^j \end{aligned} \quad (6.6)$$

where  $P = \sum_{i=1}^m P_i$  and  $r = \sum_{i=1}^m r_i$ .

**Definition 6.3.2.** For a Harder–Narasimhan type  $\tau = (\underline{H}, \underline{I})$  as above we define rational numbers  $(\underline{a}, \underline{b}) = (\underline{a}, \underline{b})(\tau)$  by

$$a_{i,j} := \frac{1}{\delta(n)} - \left( \frac{P(n)}{r\delta(n)} + \frac{\tilde{\eta}_i}{\epsilon} \right) \frac{\text{rk}(H^{i,j})}{H^{i,j}(n)}$$

and

$$b_{i,j} := \frac{1}{\delta(n)} - \left( \frac{P(n)}{r\delta(n)} + \frac{\tilde{\eta}_i + \tilde{\eta}_{i+1}}{2\epsilon} \right) \frac{\text{rk}(I^{i,j})}{I^{i,j}(n)}$$

where  $\text{rk}(H^{i,j})$  and  $\text{rk}(I^{i,j})$  are the ranks determined by the leading coefficients of the polynomials  $H^{i,j}$  and  $I^{i,j}$ .

We have chosen  $(\underline{a}, \underline{b})$  to be the rational numbers which minimise the normalised Hilbert–Mumford function

$$\frac{\mu^{\mathcal{L}}(z, \lambda(\underline{a}, \underline{b}))}{\|\lambda(\underline{a}, \underline{b})\|}$$

subject to condition *ii*) of (6.4) where  $z \in \mathfrak{T}$  is the point which represents the complex  $\mathcal{F}$  with Harder–Narasimhan type  $\tau$ . The choice of  $\epsilon$  given by Proposition 5.3.7 ensures that  $(\underline{a}, \underline{b})$  also satisfy the inequalities *i*) of (6.4) for all sufficiently large  $n$ .

We choose a maximal torus  $T_i$  of the maximal compact subgroup  $U(V_i)$  of  $GL(V_i)$  as follows. Take a basis of  $V_i$  which is compatible with the filtration of  $V_i$  defined at (6.3) and define  $T_i$  to be the maximal torus of  $U(V_i)$  given by taking diagonal matrices with respect to this basis. We pick the positive Weyl chamber

$$\mathfrak{t}_{i+} := \{i\text{diag}(a_1, \dots, a_{\dim V_i}) \in \mathfrak{t}_i : a_1 \geq \dots \geq a_{\dim V_i}\}.$$

Let  $T$  be the maximal torus of the maximal compact subgroup of  $G$  determined by the maximal tori  $T_i$  and let  $\mathfrak{t}_+$  be the positive Weyl chamber associated to the  $\mathfrak{t}_{i+}$ .

**Definition 6.3.3.** For a Harder–Narasimhan type  $\tau$  with respect to  $(\underline{\epsilon}, \delta\underline{\eta})$  we define

$\beta = \beta(\tau) \in \mathfrak{t}_+$  to be the point defined by the rational weights

$$\beta_i = \text{idia}(b_{i-1,1}, \dots, b_{i-1,t_{i-1}}, a_{i,1}, \dots, a_{i,s_i}, b_{i,1}, \dots, b_{i,t_i}) \in \mathfrak{t}_+$$

where  $a_{i,j}$  appears  $H^{i,j}(n)$  times and  $b_{k,j}$  appears  $I^{k,j}(n)$  times. This rational weight defines a rational 1-PS  $\lambda_\beta$  of  $G$  by  $\lambda_\beta = \lambda(\underline{a}, \underline{b})$ .

## 6.4 The stratum containing $R_\tau$

In this section we show that for  $\epsilon$  sufficiently small and  $n$  sufficiently large the rational weight  $\beta = \beta(\tau)$  is an index for the stratification of  $\overline{\mathfrak{Z}}^{tf}$  and that  $R_\tau$  is a union of connected components of  $S_\beta \cap \mathfrak{Z}$ .

Let  $Z_\beta$  and  $Y_\beta$  denote the subschemes of the stratum  $S_\beta$  given in Definition 2.1.3. By definition  $Z_\beta$  is equal to the union of the connected components of the fixed point locus for the action of  $\lambda_\beta$  on  $\overline{\mathfrak{Z}}^{tf}$  such that  $\lambda_\beta$  acts with weight  $||\beta||^2$  on these components. By Lemma 4.2.2 the fixed point locus decomposes into three components: a diagonal component, a strictly upper triangular component and a strictly lower triangular component. Each of these components decomposes further in terms of the Hilbert polynomials of the direct summands of each sheaf in this complex. We are interested in the component(s) of the diagonal part which may contain the graded object

$$\bigoplus_{i=1}^m \bigoplus_{j=1}^{s_i} \mathcal{H}^i(\mathcal{F})^j[-i] \oplus \bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^{s_i} \text{Cone}(\text{id} : \text{Im } d_i^j \rightarrow \text{Im } d_i^j)[-i-1]$$

associated to the Harder–Narasimhan filtration of the complex  $\mathcal{F}$  with respect to  $(\epsilon, \delta\eta)$ . Here  $[-]$  denotes the shift operator and  $\text{Cone}(\text{id} : \mathcal{A} \rightarrow \mathcal{A})$  is the acyclic complex which is  $\mathcal{A}$  in degrees -1 and 0 and zero elsewhere.

We consider a closed subscheme  $F_{(\underline{H}, \underline{I})}$  of  $\overline{\mathfrak{Z}}^{tf}$  consisting of  $z = (q_1, \dots, q_m, [d : \zeta])$

such that

- we have a decomposition

$$q_i = \bigoplus_{j=1}^{t_{i-1}} p_i^{i-1,j} \oplus \bigoplus_{j=1}^{s_i} q_i^{i,j} \oplus \bigoplus_{j=1}^{t_i} p_i^{i,j}$$

where  $q_i^{i,j} : V_i^{i,j} \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{E}_i^{i,j}$  is a point in  $\text{Quot}(V_i^{i,j} \otimes \mathcal{O}_X(-n), H_i^{i,j})$  and  $p_i^{k,j} : W_i^{k,j} \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{G}_i^{k,j}$  is a point in  $\text{Quot}(W_i^{k,j} \otimes \mathcal{O}_X(-n), I_i^{k,j})$ ,

- $d$  is given by  $d_i = \bigoplus_{j=1}^{t_i} d_i^j$  where  $d_i^j : \mathcal{G}_i^{i,j} \rightarrow \mathcal{G}_{i+1}^{i,j}$ .

Following the discussion above we have:

**Lemma 6.4.1.**  *$F_{(\underline{H}, \underline{I})}$  is a union of connected components of the fixed point locus of  $\lambda_\beta$  acting on  $\overline{\mathfrak{T}}^{tf}$  which is contained completely in the diagonal part of this fixed point locus.*

**Remark 6.4.2.** In Section 4.3 we described an injective and equivariant morphism

$$\Gamma : \mathfrak{T}_{\underline{H}^{1,1}} \times \cdots \times \mathfrak{T}_{\underline{H}^{1,s_1}} \times \mathfrak{T}_{\underline{I}^{1,1}} \times \cdots \times \mathfrak{T}_{\underline{I}^{1,t_1}} \times \mathfrak{T}_{\underline{H}^{2,1}} \times \cdots \times \mathfrak{T}_{\underline{H}^{m,s_m}} \rightarrow \mathfrak{T}$$

which corresponds to taking the direct sum of complexes. When we restrict this morphism to the open subschemes of these parameter schemes which parametrise complexes with torsion free cohomology

$$\Gamma : \mathfrak{T}_{\underline{H}^{1,1}} \times \cdots \times \mathfrak{T}_{\underline{H}^{1,s_1}} \times \mathfrak{T}_{\underline{I}^{1,1}}^{tf} \times \cdots \times \mathfrak{T}_{\underline{I}^{1,t_1}}^{tf} \times \mathfrak{T}_{\underline{H}^{2,1}} \times \cdots \times \mathfrak{T}_{\underline{H}^{m,s_m}} \rightarrow \mathfrak{T}^{tf}$$

we see that the image of this restricted morphism is  $F_{(\underline{H}, \underline{I})} \cap \mathfrak{T}^{tf}$ .

**Lemma 6.4.3.** *The scheme  $\mathfrak{T}_{(\tau)} := F_{(\underline{H}, \underline{I})} \cap \mathfrak{T}^{tf}$  is contained in  $Z_\beta$ .*

*Proof.* Let  $z = (q_1, \dots, q_m, [d : 1])$  be a point of  $\mathfrak{T}_{(\tau)}$ ; then  $q_i$  and  $d_i$  are direct sums

as above. By Proposition 4.2.8

$$\begin{aligned} \mu^{\mathcal{L}}(z, \lambda_\beta) &= \sum_{i=1}^m \sum_{j=1}^{s_i} a_{i,j} \left( \frac{P(n)}{r\delta(n)} + \frac{\tilde{\eta}_i}{\epsilon} \right) \text{rk } \mathcal{H}_i^{i,j} \\ &\quad + \sum_{i=1}^{m-1} \sum_{j=1}^{t_i} b_{i,j} \left( 2 \frac{P(n)}{r\delta(n)} + \frac{\tilde{\eta}_i + \tilde{\eta}_{i+1}}{\epsilon} \right) \text{rk } I_i^{i,j} \end{aligned} \quad (6.7)$$

where the weights  $(\underline{a}, \underline{b})$  are given in Definition 6.3.2. The weights  $(\underline{a}, \underline{b})$  are scaled so that  $\|\beta\|^2 = -\mu^{\mathcal{L}}(z, \lambda_\beta)$  and so this proves the lemma as  $-\mu^{\mathcal{L}}(z, \lambda_\beta)$  is equal to the weight of the action of  $\lambda_\beta$  on a point in the affine cone which lies over  $z$ .  $\square$

**Corollary 6.4.4.** *The scheme  $\mathfrak{T}_{(\tau)}$  is a union of connected components of  $Z_\beta \cap \mathfrak{T}^{tf}$ .*

Let  $F$  be the union of connected components of  $Z_\beta$  meeting  $\mathfrak{T}_{(\tau)}$ ; then  $F$  is a closed subscheme of  $\overline{\mathfrak{T}}^{tf}$  which is completely contained in the diagonal part of  $Z_\beta$ . Consider the subgroup

$$\text{Stab}\beta = \left( \prod_{i=1}^m \prod_{j=1}^{s_i} \text{GL}(V_i^{i,j}) \times \prod_{i=1}^{m-1} \prod_{j=1}^{t_i} \text{GL}(W_i^{i,j}) \times \text{GL}(W_{i+1}^{i,j}) \right) \cap \text{SL}(\oplus_{i=1}^m V_i)$$

of  $G$  which is the stabiliser of  $\beta$  under the adjoint action.

**Lemma 6.4.5.** *Stab $\beta$  has a central subgroup*

$$\hat{G} = \left\{ \begin{aligned} &(u^{i,j}, w^{i,j}) \in (\mathbb{C}^*)^{\sum_{i=1}^m s_i} \times (\mathbb{C}^*)^{\sum_{i=1}^{m-1} t_i} : \\ &\prod_{i=1}^m \prod_{j=1}^{s_i} (u^{i,j})^{H^{i,j}(n)} \times \prod_{i=1}^{m-1} \prod_{j=1}^{t_i} (w^{i,j})^{2I^{i,j}(n)} = 1 \end{aligned} \right\} \quad (6.8)$$

which fixes every point of  $F$ . This subgroup acts on the fibre of  $\mathcal{L}$  over any point of  $F$  by a character  $\chi_F : \hat{G} \rightarrow \mathbb{C}^*$  where  $\chi_F(u^{i,j}, w^{i,j})$  is the product

$$\prod_{i=1}^m \prod_{j=1}^{s_i} (u^{i,j})^{\text{rk } H^{i,j} \left( \frac{P(n)}{r\delta(n)} + \frac{\eta_i}{\epsilon} \right)} \prod_{i=1}^{m-1} \prod_{j=1}^{t_i} (w^{i,j})^{\text{rk } I^{i,j} \left( \frac{2P(n)}{r\delta(n)} + \frac{\eta_i + \eta_{i+1}}{\epsilon} \right)}.$$

*Proof.* The inclusion of  $\hat{G}$  into  $\text{Stab}\beta$  is given by

$$(u^{i,j}, w^{i,j}) \mapsto (u^{i,j} I_{V_i^{i,j}}, w^{i,j} I_{W_i^{i,j}}, w^{i,j} I_{W_{i+1}^{i,j}}).$$

Let  $z = (q_1, \dots, q_m, [d : \zeta])$  be a point of  $F$  then we have a decomposition

$$q_i = \bigoplus_{j=1}^{s_i} q_i^{i,j} \oplus \bigoplus_{j=1}^{t_i} p_i^{i,j} \oplus \bigoplus_{j=1}^{t_{i-1}} p_i^{i-1,j}.$$

A copy of  $\mathbb{C}^*$  acts trivially on each quot scheme and so the central subgroup  $\hat{G}$  fixes this quotient sheaf. The boundary maps determined by  $d$  also have a decomposition  $d_i = \bigoplus_{j=1}^{t_i} d_i^j$  where  $d_i^j : \mathcal{G}_i^{i,j} \rightarrow \mathcal{G}_{i+1}^{i,j}$ . As  $(u^{i,j}, w^{i,j}) \in \hat{G}$  acts on both  $\mathcal{G}_i^{i,j}$  and  $\mathcal{G}_{i+1}^{i,j}$  by multiplication by  $w^{i,j}$ , the boundary maps are also fixed by the action of  $\hat{G}$ .

To calculate the character  $\chi_F : \hat{G} \rightarrow \mathbb{C}^*$  with which this torus acts we fix  $(u^{i,j}, w^{i,j}) \in \hat{G}$  and calculate the weight of the action of this element on the fibre over a point  $z \in F$ . This is done by carefully modifying the calculations about weights of  $\mathbb{C}^*$ -actions in Section 4.2 to general torus actions.  $\square$

Let  $\mathcal{L}^{\chi_{-\beta}}$  denote the linearisation of the  $\text{Stab}\beta$  action on  $Z_\beta$  given by twisting the original linearisation  $\mathcal{L}$  by the character  $\chi_{-\beta}$  associated to  $-\beta$ . Recall that  $Z_\beta^{ss}$  is the GIT semistable set with respect to this linearisation. Consider the subgroup

$$\begin{aligned} G' &:= \prod_{i=1}^m \prod_{j=1}^{s_i} \text{SL}(V_i^{i,j}) \times \prod_{i=1}^{m-1} \prod_{j=1}^{t_i} (\text{GL}(W_i^{i,j}) \times \text{GL}(W_{i+1}^{i,j})) \cap \text{SL}(W_i^{i,j} \oplus W_{i+1}^{i,j}) \\ &= \left\{ (g^{i,j}, h_i^{i,j}, h_{i+1}^{i,j}) \in \text{Stab}\beta : \det g^{i,j} = 1 \text{ and } \det h_i^{i,j} \det h_{i+1}^{i,j} = 1 \right\}. \end{aligned}$$

**Proposition 6.4.6.** *Let  $F$  be the components of  $Z_\beta$  which meet  $\mathfrak{T}_{(\tau)}$ ; then we have an equality of GIT semistable subschemes*

$$F^{\text{Stab}\beta-ss}(\mathcal{L}^{\chi_{-\beta}}) = F^{G'-ss}(\mathcal{L}).$$

*Proof.* Let  $\hat{G}$  be the central subgroup defined in Lemma 6.4.5; then there is a surjective homomorphism  $\Phi$  from  $\text{Stab}\beta$  to  $\hat{G}$  given by

$$\Phi(g_{i,j}, h_i^{i,j}, h_{i+1}^{i,j}) = ((\det g^{i,j})^{D/H^{i,j}(n)}, (\det h_i^{i,j} \det h_{i+1}^{i,j})^{D/2I^{i,j}(n)})$$

where  $D = \prod_{i=1}^m \prod_{j=1}^{s_i} H^{i,j}(n) \times \prod_{i=1}^{m-1} \prod_{j=1}^{t_i} 2I^{i,j}(n)$ . The composition of the inclusion of  $\hat{G}$  into  $\text{Stab}\beta$  with  $\Phi$  is a homomorphism  $\hat{G} \rightarrow \hat{G}$  given by

$$(u^{i,j}, w^{i,j}) \mapsto ((u^{i,j})^D, (w^{i,j})^D).$$

Therefore,  $\ker \Phi \times \hat{G}$  surjects onto  $\text{Stab}\beta$  with finite kernel and so

$$F^{\text{Stab}\beta-ss}(\mathcal{L}^{\chi-\beta}) = F^{\ker \Phi \times \hat{G}-ss}(\mathcal{L}^{\chi-\beta}).$$

The restriction of the character  $\chi_{-\beta}$  to the central subgroup  $\hat{G}$

$$\chi_\beta(u^{i,j}, v^{i,j}) = \prod_{i=1}^m \prod_{j=1}^{s_i} (u^{i,j})^{-a_{i,j}H^{i,j}(n)} \times \prod_{i=1}^{m-1} \prod_{j=1}^{t_i} (w^{i,j})^{-b_{i,j}2I^{i,j}(n)}$$

is equal to the character  $\chi_F : \hat{G} \rightarrow \mathbb{C}^*$  defined in Lemma 6.4.5. As we are considering the action of  $\ker \Phi \times \hat{G}$  on  $F$  linearised by  $\mathcal{L}^{\chi-\beta}$ , the effects of the action of  $\hat{G}$  and the modification by the character corresponding to  $-\beta$  cancel so that

$$F^{\ker \Phi \times \hat{G}-ss}(\mathcal{L}^{\chi-\beta}) = F^{\ker \Phi-ss}(\mathcal{L}).$$

Then since  $G'$  injects into  $\ker \Phi$  with finite cokernel,

$$F^{\ker \Phi-ss}(\mathcal{L}) = F^{G'-ss}(\mathcal{L})$$

which proves the proposition. □

We are now ready to describe the semistable subscheme  $\mathfrak{T}_{(\tau)}^{ss} := \mathfrak{T}_{(\tau)}^{\text{Stab}^{\beta-ss}}(\mathcal{L}^{\chi-\beta})$  for the  $\text{Stab}^{\beta}$ -action on  $\mathfrak{T}_{(\tau)}$  with respect to  $\mathcal{L}^{\chi-\beta}$ . Recall from Remark 6.4.2 that we have an isomorphism

$$\mathfrak{T}_{(\tau)} \cong \mathfrak{T}_{\underline{H}^{1,1}} \times \cdots \times \mathfrak{T}_{\underline{H}^{1,s_1}} \times \mathfrak{T}_{\underline{I}^{1,1}}^{tf} \times \cdots \times \mathfrak{T}_{\underline{I}^{1,t_1}}^{tf} \times \mathfrak{T}_{\underline{H}^{2,1}} \times \cdots \times \mathfrak{T}_{\underline{H}^{m,s_m}}.$$

**Proposition 6.4.7.** *Let  $\tau$  be a fixed Harder–Narasimhan type with respect to  $(\underline{\epsilon}, \underline{\delta\eta})$  where  $\underline{\eta}$  are strictly increasing rational numbers and let  $\beta = \beta(\tau)$  be the associated rational weight. Then for  $\epsilon > 0$  sufficiently small and for  $n$  sufficiently large we have*

$$\mathfrak{T}_{(\tau)}^{ss} \cong \mathfrak{T}_{\underline{H}^{1,1}}^{ss} \times \cdots \times \mathfrak{T}_{\underline{H}^{1,s_1}}^{ss} \times \mathfrak{T}_{\underline{I}^{1,1}}^{ss} \times \cdots \times \mathfrak{T}_{\underline{I}^{1,t_1}}^{ss} \times \mathfrak{T}_{\underline{H}^{2,1}}^{ss} \times \cdots \times \mathfrak{T}_{\underline{H}^{m,s_m}}^{ss}$$

where  $\mathfrak{T}_{\underline{H}^{i,j}}^{ss}$  (respectively  $\mathfrak{T}_{\underline{I}^{i,j}}^{ss}$ ) is the subscheme of  $\mathfrak{T}_{\underline{H}^{i,j}}$  (respectively  $\mathfrak{T}_{\underline{I}^{i,j}}^{tf}$ ) which parametrises  $(\underline{\epsilon}, \underline{\delta\eta})$ -semistable complexes with Hilbert polynomials  $\underline{H}^{i,j}$  (respectively  $\underline{I}^{i,j}$ ).

*Proof.* Let  $\epsilon$  be sufficiently small so that the only  $(\underline{\epsilon}, \underline{\delta\eta})$ -semistable complexes with Hilbert polynomials  $\underline{I}^{i,j}$  are isomorphic to cones on the identity morphism of a torsion free semistable sheaf. Let  $n$  be sufficiently large so:

- i) Gieseker semistability of a torsion free sheaf with Hilbert polynomial  $H^{i,j}$  (respectively  $I^{i,j}$ ) is equivalent to GIT-semistability of a point in the relevant quot scheme representing this sheaf with respect to the linearisation given by Gieseker (see Section 4.1.5 and also [13]).
- ii) For  $1 \leq i \leq m-1$  and  $1 \leq j \leq t_i$ , we have that  $(\underline{\epsilon}, \underline{\delta\eta})$ -semistability of a complex with Hilbert polynomials  $\underline{I}^{i,j}$  is equivalent to GIT semistability of a point in  $\mathfrak{T}_{\underline{I}^{i,j}}$  for the linearisation defined by these stability parameters.
- iii) For  $i = 1, \dots, m$ , we have  $\tilde{\eta}_i + \epsilon P(n)/r\delta(n) > 0$  (this is possible as  $P$  and  $\delta$  are both positive for large  $n$  and  $P$  is of greater degree than  $\delta$ ).

Let

$$z = \bigoplus_{i=1}^m \bigoplus_{j=1}^{s_i} z^{i,j} \oplus \bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^{t_i} y^{i,j}$$

be a point in  $\mathfrak{T}_{(\tau)}$ ; that is,  $z^{i,j} = (q_i^{i,j}, [0, 1])$  is a point in  $\mathfrak{T}_{\underline{H}^{i,j}}$  corresponding to a complex  $\mathcal{H}^{i,j}$  concentrated in degree  $i$  and  $y^{i,j} = (p_i^{i,j}, p_{i+1}^{i,j}[d_i^j, 1])$  is a point in  $\mathfrak{T}_{\underline{I}^{i,j}}$  corresponding to a complex  $\mathcal{I}^{i,j}$  concentrated in degrees  $i$  and  $i+1$ . By Proposition 6.4.6, we have

$$\mathfrak{T}_{(\tau)}^{ss} = \mathfrak{T}_{(\tau)}^{G'-ss}(\mathcal{L}|_{\mathfrak{T}_{(\tau)}});$$

therefore,  $z$  is in  $\mathfrak{T}_{(\tau)}^{ss}$  if and only if  $\mu^{\mathcal{L}}(z, \lambda) \geq 0$  for every 1-PS  $\lambda$  of  $G'$ .

Suppose that  $\mathcal{H}^{i,j}$  and  $\mathcal{I}^{i,j}$  are all  $(\epsilon, \delta\eta)$ -semistable, so  $\mathcal{H}_i^{i,j}$  is a Gieseker semistable sheaf and  $\mathcal{I}^{i,j}$  is isomorphic to the cone on a semistable sheaf. A 1-PS  $\lambda$  of  $G'$  is given by

- 1-PSs  $\lambda_H^{i,j}$  of  $\mathrm{SL}(V_i^{i,j})$  and
- 1-PSs  $\lambda_I^{i,j} = (\lambda_{I,i}^{i,j}, \lambda_{I,i+1}^{i,j})$  of  $(\mathrm{GL}(W_i^{i,j}) \times \mathrm{GL}(W_{i+1}^{i,j})) \cap \mathrm{SL}(W_i^{i,j} \oplus W_{i+1}^{i,j})$ .

We diagonalise these 1-PSs simultaneously to get decreasing integers

$$\gamma_1 > \cdots > \gamma_u$$

and decompositions  $V_i^{i,j} = V_{i,1}^{i,j} \oplus \cdots \oplus V_{i,u}^{i,j}$  and similarly  $W_i^{i,j} = W_{i,1}^{i,j} \oplus \cdots \oplus W_{i,u}^{i,j}$  and  $W_{i+1}^{i,j} = W_{i+1,1}^{i,j} \oplus \cdots \oplus W_{i+1,u}^{i,j}$  such that

$$\lambda_H^{i,j}(t) = \begin{pmatrix} t^{\gamma_1} I_{V_{i,1}^{i,j}} & & \\ & \ddots & \\ & & t^{\gamma_u} I_{V_{i,u}^{i,j}} \end{pmatrix}$$

and similarly for  $\lambda_I^{i,j}$ . The corresponding filtrations of these vector spaces gives rise to filtrations of the sheaves  $\mathcal{H}_i^{i,j}$ ,  $\mathcal{I}_i^{i,j}$  and  $\mathcal{I}_{i+1}^{i,j}$  and we let  $\mathcal{H}_{i,k}^{i,j}$ ,  $\mathcal{I}_{i,k}^{i,j}$  and  $\mathcal{I}_{i+1,k}^{i,j}$  denote

the successive quotients. By Proposition 4.2.8 (see also Lemma 4.2.7)

$$\begin{aligned} \mu^{\mathcal{L}}(z, \lambda) = & \sum_{i=1}^{m-1} \sum_{j=1}^{t_i} \sum_{k=1}^u \gamma_k \left[ \left( \frac{P(n)}{r\delta(n)} + \frac{\tilde{\eta}_i}{\epsilon} \right) \operatorname{rk} \mathcal{I}_{i,k}^{i,j} + \left( \frac{P(n)}{r\delta(n)} + \frac{\tilde{\eta}_{i+1}}{\epsilon} \right) \operatorname{rk} \mathcal{I}_{i+1,k}^{i,j} \right] \\ & + \sum_{i=1}^m \sum_{j=1}^{s_i} \sum_{k=1}^u \gamma_k \left( \frac{P(n)}{r\delta(n)} + \frac{\tilde{\eta}_i}{\epsilon} \right) \operatorname{rk} \mathcal{H}_{i,k}^{i,j} - N \end{aligned} \quad (6.9)$$

where  $N = 0$  if  $\lambda$  induces a filtration of each  $\mathcal{I}^{i,j}$  by subcomplexes and otherwise  $N = \gamma_{l_2} - \gamma_{l_1} < 0$  for some  $1 \leq l_1 < l_2 \leq u$ . As  $\mathcal{H}_i^{i,j}$ ,  $\mathcal{I}_i^{i,j}$  and  $\mathcal{I}_{i+1}^{i,j}$  are Gieseker semistable, we have the following inequalities:

$$\begin{aligned} 1) \quad & \sum_{k=1}^u \gamma_k \operatorname{rk} \mathcal{H}_{i,k}^{i,j} \geq 0, \\ 2) \quad & \sum_{k=1}^u \gamma_k \left( \operatorname{rk} \mathcal{I}_{i,k}^{i,j} - \frac{\operatorname{rk} I^{i,j}}{I^{i,j}(n)} \dim W_{i,k}^{i,j} \right) \geq 0, \\ 3) \quad & \sum_{k=1}^u \gamma_k \left( \operatorname{rk} \mathcal{I}_{i+1,k}^{i,j} - \frac{\operatorname{rk} I^{i,j}}{I^{i,j}(n)} \dim W_{i+1,k}^{i,j} \right) \geq 0. \end{aligned} \quad (6.10)$$

Firstly suppose that  $\lambda$  induces a filtration of  $z$  by subcomplexes. The choice of  $n$  and the inequalities of (6.10) imply that

$$\begin{aligned} \mu^{\mathcal{L}}(z, \lambda) \geq & \frac{\operatorname{rk} I^{i,j}}{I^{i,j}(n)} \sum_{i=1}^{m-1} \sum_{j=1}^{t_i} \sum_{k=1}^u \gamma_k \left[ \frac{P(n)}{r\delta(n)} \left( \dim W_{i,k}^{i,j} + \dim W_{i+1,k}^{i,j} \right) \right. \\ & \left. + \frac{\tilde{\eta}_i}{\epsilon} \dim W_{i,k}^{i,j} + \frac{\tilde{\eta}_{i+1}}{\epsilon} \dim W_{i+1,k}^{i,j} \right]. \end{aligned}$$

As  $\lambda$  induces a filtration by subcomplexes, we know that

$$\dim(W_{i,1}^{i,j} \oplus \cdots \oplus W_{i,k}^{i,j}) \leq \dim(W_{i+1,1}^{i,j} \oplus \cdots \oplus W_{i+1,k}^{i,j}).$$

Since  $\lambda_I^{i,j}$  is a 1-PS of  $(\operatorname{GL}(W_i^{i,j}) \times \operatorname{GL}(W_{i+1}^{i,j})) \cap \operatorname{SL}(W_i^{i,j} \oplus W_{i+1}^{i,j})$ , we know that the weights satisfy a relation  $\sum_{k=1}^u \gamma_k (\dim W_{i,k}^{i,j} + \dim W_{i+1,k}^{i,j}) = 0$ . It follows that

$\sum_{k=1}^u \gamma_k \dim W_{i+1,k}^{i,j} \geq 0$  and

$$\mu^{\mathcal{L}}(z, \lambda) \geq \frac{\mathrm{rk} I^{i,j}}{I^{i,j}(n)} \sum_{i=1}^{m-1} \sum_{j=1}^{t_i} \frac{(\tilde{\eta}_{i+1} - \tilde{\eta}_i)}{\epsilon} \sum_{k=1}^u \gamma_k \dim W_{i+1,k}^{i,j} \geq 0.$$

Now suppose that  $\lambda$  does not induce a filtration of  $z$  by subcomplexes. Then we still have

$$\mu^{\mathcal{L}}(z, \lambda) \geq \frac{1}{\epsilon} \sum_{i=1}^{m-1} \sum_{j=1}^{t_i} \sum_{k=1}^u \gamma_k (\tilde{\eta}_i \mathrm{rk} \mathcal{I}_{i,k}^{i,j} + \tilde{\eta}_{i+1} \mathrm{rk} \mathcal{I}_{i+1,k}^{i,j}) - N$$

where  $N$  is a negative number depending on the  $\gamma_k$ . In this situation we can use the ideas of [39], Theorem 1.7.1. As the set of sheaves occurring in these expressions is bounded (cf. [39] Section 2.3), there is a positive constant  $K$  such that for all such pairs  $(z, \lambda)$  we have

$$\frac{1}{\epsilon} \sum_{i=1}^{m-1} \sum_{j=1}^{t_i} \sum_{k=1}^u \gamma_k (\tilde{\eta}_i \mathrm{rk} \mathcal{I}_{i,k}^{i,j} + \tilde{\eta}_{i+1} \mathrm{rk} \mathcal{I}_{i+1,k}^{i,j}) \geq NK.$$

We pick  $C > K$  and replace  $\tilde{\eta}/\epsilon$  and  $\delta$  with  $\tilde{\eta}/\epsilon C$  and  $C\delta$  in the definition of the linearisation. Then

$$\mu^{\mathcal{L}}(z, \lambda) \geq NK/C - N > 0. \quad (6.11)$$

Conversely suppose there is a direct summand which is unstable. We claim that there is a 1-PS  $\lambda = (\lambda_H^{i,j}, \lambda_I^{i,j})$  of  $G'$  which destabilises  $z$  so that  $\mu^{\mathcal{L}}(z, \lambda) < 0$ . Firstly suppose  $\mathcal{H}_i^{i,j}$  is unstable for some  $(i, j)$ ; then there exists a subsheaf  $\mathcal{H}_{i,1}^{i,j} \subset \mathcal{H}_i^{i,j}$  such that

$$\frac{H^0(\mathcal{H}_{i,1}^{i,j}(n))}{\mathrm{rk} \mathcal{H}_{i,1}^{i,j}} > \frac{H^{i,j}(n)}{\mathrm{rk} H^{i,j}}. \quad (6.12)$$

We construct a 1-PS  $\lambda$  with three weights  $\gamma_1 > \gamma_2 = 0 > \gamma_3$ . Let

$$V_{i,1}^{i,j} = H^0(q_i^{i,j}(n))^{-1} H^0(\mathcal{H}_{i,1}^{i,j}(n))$$

and let  $V_{i,3}^{i,j}$  be an orthogonal complement to  $V_{i,1}^{i,j} \subset V_i^{i,j}$ . Define

$$\lambda_H^{i,j} = \begin{pmatrix} t^{\gamma_1} I_{V_{i,1}^{i,j}} & \\ & t^{\gamma_3} I_{V_{i,3}^{i,j}} \end{pmatrix}$$

and define all the other parts of  $\lambda$  to be trivial (the weights  $\gamma_i$  should be chosen so  $\lambda_H^{i,j}$  has determinant 1). Then

$$\mu^{\mathcal{L}}(z, \lambda) = \left( \frac{P(n)}{r\delta(n)} + \frac{\tilde{\eta}_i}{\epsilon} \right) \left[ H^{i,j}(n) \operatorname{rk} \mathcal{H}_{i,1}^{i,j} - H^0(\mathcal{H}_{i,1}^{i,j}(n)) \operatorname{rk} H^{i,j} \right] < 0.$$

Secondly suppose  $\mathcal{I}^{i,j}$  is unstable with respect to  $(\underline{\epsilon}, \delta\underline{\eta})$ ; i.e., it is not isomorphic to the cone on the identity map of a semistable sheaf. Let  $d: \mathcal{I}_i^{i,j} \rightarrow \mathcal{I}_{i+1}^{i,j}$  denote the boundary morphism of this complex. If  $d = 0$ , then choose the 1-PS  $\lambda$  to pick out the subcomplex  $I_i^{i,j} \rightarrow 0$ . For example, let  $\lambda$  have three weights  $1 > 0 > -1$  and let the only nontrivial part be

$$\lambda_I^{i,j} = (tI_{W_i^{i,j}}, t^{-1}I_{W_{i+1}^{i,j}});$$

then  $\mu^{\mathcal{L}}(z, \lambda) < 0$ . If  $d \neq 0$  but has nonzero kernel, then consider the reduced Hilbert polynomial of this kernel. If the kernel has reduced Hilbert polynomial strictly larger than  $\mathcal{I}_i^{i,j}$ , then choose  $\lambda$  to pick out the subcomplex  $\ker d \rightarrow 0$ . If the kernel has reduced Hilbert polynomial strictly smaller than  $\mathcal{I}_i^{i,j}$ , then choose  $\lambda$  to pick out the subcomplex  $0 \rightarrow \operatorname{Im} d$ . If the kernel has reduced Hilbert polynomial equal to  $I^{i,j}/\operatorname{rk} I^{i,j}$ , then choose  $\lambda$  to pick out the subcomplex  $\mathcal{I}_i^{i,j} \rightarrow \operatorname{Im} d$ . In all three cases we see that  $\mu^{\mathcal{L}}(z, \lambda) < 0$ . If  $d$  is an isomorphism but  $\mathcal{I}_i^{i,j}$  is not Gieseker semistable, then let  $\mathcal{I}_{i,1}^{i,j}$  be its maximal destabilising subsheaf. A 1-PS which picks out the subcomplex  $\mathcal{I}_{i,1}^{i,j} \rightarrow d(\mathcal{I}_{i,1}^{i,j})$  will destabilise  $z$ .  $\square$

Recall from Definition 2.1.3 that there is a subscheme  $Y_\beta \subset S_\beta$  and a retraction  $p_\beta : Y_\beta \rightarrow Z_\beta$  where

$$p_\beta(y) = \lim_{t \rightarrow 0} \lambda_\beta(t) \cdot y.$$

**Lemma 6.4.8.** *Let  $F^{ss}$  denote the connected components of  $Z_\beta^{ss}$  meeting  $\mathfrak{T}_{(\tau)}^{ss}$ ; then for  $\epsilon$  sufficiently small and  $n$  sufficiently large*

$$p_\beta^{-1}(F^{ss}) \cap \mathfrak{T}^{tf} = p_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss}).$$

*Proof.* Let  $n$  and  $\epsilon$  be chosen as in Proposition 6.4.7. Let  $y \in p_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss})$  so that  $p_\beta(y) = \lim_{t \rightarrow 0} \lambda_\beta(t) \cdot y \in \mathfrak{T}^{tf} \cap F^{ss}$ . If  $y \notin \mathfrak{T}^{tf}$ , then for all  $t \neq 0$  we have  $\lambda_\beta(t) \cdot y \notin \mathfrak{T}^{tf}$ . Then this would contradict the openness of  $\mathfrak{T}^{tf} \cap F^{ss}$  in  $F^{ss}$  as  $\lim_{t \rightarrow 0} \lambda_\beta(t) \cdot y \in \mathfrak{T}^{tf}$ .

Conversely suppose  $y = (q_1, \dots, q_m, [d : 1]) \in \mathfrak{T}^{tf}$  and  $z = p_\beta(y) \in F^{ss}$  where  $q_i : V_i \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{F}_i$  and  $d$  is given by  $d_i : \mathcal{F}_i \rightarrow \mathcal{F}_{i+1}$ . The scheme  $F^{ss}$  is contained in the diagonal components of  $Z_\beta^{ss}$ ; therefore the 1-PS  $\lambda_\beta$  induces a filtration of  $y$  by subcomplexes and the associated graded point is

$$z = (\oplus_{i,j} z^{i,j}) \oplus (\oplus_{i,j} y^{i,j})$$

where  $z^{i,j} = (q_i^{i,j})$  and  $y^{i,j} = (p_i^{i,j}, p_{i+1}^{i,j}, [d_i^j : 1])$  both represent complexes (cf. Lemma 4.2.2). By Proposition 6.4.6, the limit  $z$  is in the GIT semistable set for the action of  $G'$  on  $Z_\beta$  with respect to  $\mathcal{L}$ . We can apply the arguments used in the proof of Proposition 6.4.7 to show that  $z^{i,j} \in \mathfrak{T}_{\underline{H}^{i,j}}^{ss}$  and  $y^{i,j} \in \mathfrak{T}_{\underline{I}^{i,j}}^{ss}$ .  $\square$

We are now able to prove the main result of this chapter. Recall that  $X$  is a smooth integral projective scheme over the complex numbers. We choose Schmitt stability parameters  $(\underline{1}, \delta \underline{\eta}/\epsilon)$  for complexes over  $X$  where  $\epsilon > 0$  is a rational number,  $\eta_i$  are strictly increasing rational numbers indexed by the integers and  $\delta$  is a positive rational polynomial such that  $\deg \delta = \max(\dim X - 1, 0)$ . Let  $\tau$  be a fixed Harder–

Narasimhan type with respect to  $(\underline{1}, \delta\underline{\eta}/\epsilon)$  and  $\beta = \beta(\tau)$  be the rational weight given in Definition 6.3.3. Provided  $n$  is sufficiently large all complexes with this Harder–Narasimhan type can be represented by points in the scheme  $\mathfrak{T}^{tf}$ . We defined  $R_\tau$  to be the set of points in  $\mathfrak{T}^{tf}$  which parametrise complexes with this Harder–Narasimhan type  $\tau$ . There is an action of

$$G = \Pi_{i=1}^m \mathrm{GL}(P_i(n)) \cap \mathrm{SL}(\Sigma_{i=1}^m P_i(n))$$

on this scheme and the stability parameters determine a linearisation  $\mathcal{L}$  of this action. Associated to this action there is a stratification  $\{S_\beta : \beta \in \mathcal{B}\}$  of a projective completion  $\overline{\mathfrak{T}}^{tf}$  indexed by a finite set  $\mathcal{B}$  of rational weights.

**Theorem 6.4.9.** *Let  $\tau$  be a fixed Harder–Narasimhan type with respect to  $(\underline{1}, \delta\underline{\eta}/\epsilon)$ . Then for  $\epsilon$  sufficiently small and  $n$  sufficiently large we have:*

- i)  $\beta = \beta(\tau)$  belongs to the index set  $\mathcal{B}$  for the stratification  $\{S_\beta : \beta \in \mathcal{B}\}$  of  $\overline{\mathfrak{T}}^{tf}$ ,*
- ii)  $R_\tau = Gp_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss})$  and,*
- iii) The subscheme  $R_\tau = Gp_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss})$  of the parameter scheme  $\mathfrak{T}^{tf}$  parametrising complexes with Harder–Narasimhan type  $\tau$  is a union of connected components of  $S_\beta \cap \mathfrak{T}^{tf}$ .*

*Proof.* Suppose  $\epsilon$  is sufficiently small and  $n$  is sufficiently large as in Proposition 6.4.7. We defined  $\beta$  by fixing a point  $z = (q_1, \dots, q_m, [d : 1]) \in R_\tau$  corresponding to a complex  $\mathcal{F}$  with Harder–Narasimhan type  $\tau$ . We claim that  $\bar{z} := p_\beta(z) \in Z_\beta^{ss}$  which implies *i)*. The 1-PS  $\lambda_\beta$  induces the Harder–Narasimhan filtration of  $\mathcal{F}$  and  $\bar{z} = \lim_{t \rightarrow 0} \lambda_\beta(t) \cdot z$  is the graded object associated to this filtration. By Proposition 6.4.7 it suffices to show that each direct summand in the associated graded object is  $(\underline{1}, \delta\underline{\eta}/\epsilon)$ -semistable which follows by definition of the Harder–Narasimhan filtration.

In fact the above argument shows that  $p_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss}) \subset R_\tau$  and since  $R_\tau$  is  $G$ -invariant we have  $Gp_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss}) \subset R_\tau$ . To show *ii*) suppose  $y = (q_1, \dots, q_m, [d : 1]) \in R_\tau$  corresponds to a complex  $\mathcal{E}$  with Harder–Narasimhan filtration

$$0 \subsetneq \mathcal{H}^{1,(1)} \subsetneq \dots \subsetneq \mathcal{H}^{1,(s_1)} \subsetneq \mathcal{I}^{1,(1)} \dots \mathcal{I}^{1,(t_1)} \subsetneq \mathcal{H}^{2,(1)} \subsetneq \dots \subsetneq \mathcal{H}^{m,(s_m)} = \mathcal{E}.$$

of type  $\tau$ . Then this filtration induces a filtration of each vector space  $V_i$  and we can choose a change of basis matrix  $g$  which switches this filtration with the filtration of  $V_i$  given at (6.3) used to define  $\beta$ . Then  $g \cdot y \in p_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss})$  which proves *ii*).

Since  $F^{ss}$  is a union of connected components of  $Z_\beta^{ss}$ , the scheme  $Gp_\beta^{-1}(F^{ss})$  is a union of connected components of  $S_\beta$ . Therefore,  $Gp_\beta^{-1}(F^{ss}) \cap \mathfrak{T}^{tf}$  is a union of connected components of  $S_\beta \cap \mathfrak{T}^{tf}$ . By *ii*) and Lemma 6.4.8

$$R_\tau = Gp_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss}) = Gp_\beta^{-1}(F^{ss}) \cap \mathfrak{T}^{tf}$$

which proves *iii*). □

## 6.5 Constructing a quotient of the $G$ -action on $R_\tau$

Following Theorem 6.4.9 above, there is a scheme  $R_\tau = Gp_{\beta(\tau)}^{-1}(\mathfrak{T}_{(\tau)}^{ss})$  parametrising complexes of Harder–Narasimhan type  $\tau = (\underline{H}, \underline{I})$  with respect to  $(\underline{\epsilon}, \delta\underline{\eta})$  (provided  $\epsilon$  and  $n$  are chosen appropriately). Moreover,  $R_\tau = S_\tau \cap \mathfrak{T}^{tf}$  where  $S_\tau = Gp_\beta^{-1}(F^{ss})$  is a union of connected components of a stratum  $S_\beta$  occurring in the stratification given at (6.1) and  $\beta = \beta(\tau)$  is the associated rational weight.

In Section 2.2.2 we constructed a projective completion  $\hat{S}_\beta$  of this stratum and a canonical linearisation  $\mathcal{L}_\beta$  of the  $G$ -action on this completion. Recall that the projective completion was equal to  $\hat{S}_\beta = G \times_{P_\beta} \overline{Y}_\beta$  where  $\overline{Y}_\beta$  is the closure of  $Y_\beta^{ss}$  in  $\overline{\mathfrak{T}}^{tf}$ . Then the canonical linearisation is the pullback of the exterior tensor product

of the linearisation  $\mathcal{L}$  on  $\overline{\mathfrak{T}}^{tf}$  and  $\mathcal{L}(\lambda_{-\beta})$  on  $G/P_\beta$  via

$$\hat{S}_\beta \rightarrow G/P_\beta \times \overline{\mathfrak{T}}^{tf}$$

$$[g, y] \mapsto (gP_\beta, g \cdot y).$$

We also saw that the generalised GIT quotient of  $\hat{S}_\beta$  with respect to the canonical linearisation is a categorical quotient of the  $G$ -action on  $S_\beta$  and is isomorphic to the GIT quotient

$$Z_\beta //_{\mathcal{L}_\beta} \text{Stab} \beta$$

of the  $\text{Stab} \beta$ -action on  $Z_\beta$  linearised by  $\mathcal{L}_\beta = \mathcal{L}^{\chi_{-\beta}}$ .

We can do all of this with connected components of  $S_\beta$ . Let  $\hat{S}_\tau$  denote the closure of  $S_\tau$  in  $\hat{S}_\beta$ . We can restrict the canonical linearisation to  $\hat{S}_\tau \subset \hat{S}_\beta$  and take the generalised GIT quotient of  $\hat{S}_\tau$  with respect to  $\mathcal{L}_\beta$ . This is a categorical quotient of the  $G$ -action on  $S_\tau$  and is isomorphic to the GIT quotient of  $\text{Stab} \beta$  acting on the scheme  $F$  with respect to  $\mathcal{L}_\beta$ . As  $\hat{R}_\tau := \hat{S}_\tau \cap \overline{\mathfrak{T}}^{tf} \subset \hat{S}_\tau$  is the non-vanishing locus of an invariant section of the canonical linearisation, there is an open subscheme of this GIT quotient which we denote by  $\hat{R}_\tau //_{\mathcal{L}_\beta} G$  which is a categorical quotient of the  $G$ -action on  $R_\tau$ . By Proposition 2.2.8 and Proposition 6.4.6,

$$\hat{R}_\tau //_{\mathcal{L}_\beta} G \cong \mathfrak{T}_{(\tau)} //_{\mathcal{L}_\beta} \text{Stab} \beta = \mathfrak{T}_{(\tau)} //_{\mathcal{L}} G'.$$

Moreover, this is isomorphic to the product of GIT quotients

$$\mathfrak{T}_{\underline{H}^{1,1}} // G_{\underline{H}^{1,1}} \times \cdots \times \mathfrak{T}_{\underline{H}^{1,s_1}} // G_{\underline{H}^{1,s_1}} \times \mathfrak{T}_{\underline{I}^{1,1}}^{tf} // G_{\underline{I}^{1,1}} \times \cdots \times \mathfrak{T}_{\underline{H}^{m,s_m}} // G_{\underline{H}^{m,s_m}}.$$

Unfortunately, a complex  $\mathcal{F}$  with Harder–Narasimhan type  $\tau$  with respect to  $(\underline{\epsilon}, \underline{\delta\eta})$  is identified with the graded object associated to its Harder–Narasimhan filtration in

this quotient (cf. Sections 2.2.2 and 3.6). Therefore, this quotient does not provide a satisfactory quotient of the  $G$ -action on  $\hat{R}_\tau$ . Instead we want to find a perturbed linearisation  $\mathcal{L}_\tau^{\text{per}}$  on  $\hat{S}_\tau$  such that  $\mathfrak{T}_{(\tau)} \subset Z_\beta$  is unstable with respect to this linearisation and then take the GIT quotient with respect to this perturbed linearisation.

There are several possible ways to perturb the canonical linearisation. Since the canonical linearisation is defined by taking a tensor product of  $\mathcal{L}(\lambda_{-\beta})$  and  $\mathcal{L}$ , we can perturb either of these linearisations. For example, we can perturb the linearisation  $\mathcal{L}(\lambda_{-\beta})$  on the partial flag variety by taking  $\mathcal{L}(\lambda_{-(\beta+\epsilon'\beta')})$  where  $\beta' \in \mathfrak{t}_+$  is a rational weight and  $\epsilon' > 0$  is a small rational number. Alternatively, we can use a new Schmitt linearisation on the parameter scheme which is a perturbation of the original linearisation  $\mathcal{L}$ .

## Chapter 7

# Diagrams of moduli spaces of complexes

Let  $X$  be a smooth integral projective scheme over the complex numbers. In the previous chapters we have studied a collection of stability parameters  $(\underline{\epsilon}, \delta \underline{\eta})$  where  $0 < \epsilon \ll 1$  is a rational parameter,  $\underline{\eta}$  are strictly increasing rational numbers and  $\delta$  is a positive rational polynomial such that  $\deg \delta = \max(\dim X - 1, 0)$ . We have seen that for a fixed Harder–Narasimhan type  $\tau = (\underline{H}, \underline{I})$  there is a subscheme  $R_\tau$  of the parameter scheme which parametrises complexes of this Harder–Narasimhan type and there is a canonical linearisation which gives a categorical quotient of  $R_\tau$ .

In this chapter we construct a diagram in the category of schemes from some starting data. The starting data is given by choosing a perturbation of the canonical linearisation for each Harder–Narasimhan type. The objects, indexed by Harder–Narasimhan types  $\tau$ , are quotients of the schemes  $\hat{R}_\tau$  defined at the end of Section 6.5 and the morphisms are given by taking direct sums with acyclic complexes. We conjecture that this starting data can be chosen so for any quasi-isomorphism between complexes representing points in this diagram both complexes represent the same point in the colimit of this diagram.

## 7.1 Diagrams and colimits

Since we are interested in studying quasi-isomorphism classes of complexes, we can fix any invariants of complexes as objects in the derived category  $D^b(X)$  of coherent sheaves on  $X$ . For two complexes  $\mathcal{E}$  and  $\mathcal{F}$  to be quasi-isomorphic it is necessary that their cohomology sheaves are isomorphic, so that they must have the same Harder–Narasimhan type as sheaves. For each integer  $i$  we fix a Harder–Narasimhan type  $\tau(\mathcal{H}^i) = (H^{i,1}, \dots, H^{i,s_i})$  of a torsion free sheaf and we suppose that this type is nonzero for only finitely many  $i$ . As in the previous section we fix strictly increasing rational numbers  $\underline{\eta}$  and a positive polynomial  $\delta$  of degree  $\dim X - 1$ .

We can now describe the index category  $\mathcal{J}$  for our diagram which also depends on the starting data (a choice of perturbed linearisation for each Harder–Narasimhan type). The objects of  $\mathcal{J}$  are Harder–Narasimhan types  $\tau$  of complexes with torsion free cohomology such that the image sheaf of each boundary morphism is semistable (and thus has trivial Harder–Narasimhan type). More precisely, an object of  $\mathcal{J}$  is a tuple of polynomials  $\tau = (\underline{H}, \underline{I})$  where  $\underline{H} = (H^i)_{i \in \mathbb{Z}}$  and  $\underline{I} = (I^i)_{i \in \mathbb{Z}}$  such that for all  $i$  we have

$$\underline{H}^i = (H^{i,1}, \dots, H^{i,s_i})$$

is the fixed polynomials above and for all but finitely many  $i$  we have  $I^i = 0$ , otherwise  $I^i$  is a nonzero trivial Harder–Narasimhan type of a torsion free sheaf; i.e., a single Hilbert polynomial.

In the previous chapter we saw that associated to such a Harder–Narasimhan type  $\tau$  there is a parameter scheme  $\mathfrak{T}_{\underline{P}(\tau)}^{tf}$  and a subscheme  $R_\tau \subset \mathfrak{T}_{\underline{P}(\tau)}^{tf}$  which parametrises complexes of this Harder–Narasimhan type. In Section 6.5 we constructed a subscheme  $\hat{R}_\tau$  of  $\mathfrak{T}^{tf}$  which contains  $R_\tau$  and this scheme is invariant under the reductive group  $G = G_{\underline{P}(\tau)}$ . This scheme also parametrises complexes with Harder–Narasimhan types higher than  $\tau$ . Then if the perturbed linearisation is a nontrivial perturbation

of the canonical linearisation we consider the GIT quotient

$$\hat{R}_\tau^{ss}(\mathcal{L}_\tau^{\text{per}}) \rightarrow M_\tau := \hat{R}_\tau //_{\mathcal{L}_\tau^{\text{per}}} G_{\underline{P}(\tau)}$$

and if the perturbed linearisation is equal to the canonical linearisation we consider the generalised GIT quotient (see Section 1.6)

$$R_\tau \rightarrow M_\tau := \hat{R}_\tau //_{\mathcal{L}_\tau^{\text{can}}} G_{\underline{P}(\tau)}.$$

**Remark 7.1.1.** We may really need a more sophisticated notion of GIT quotient for nontrivial perturbations if they are not ample, but this is beyond the scope of this thesis.

**Definition 7.1.2.** For a complex  $\mathcal{E}_\cdot$  with torsion free cohomology we let  $\underline{H}(\mathcal{E}_\cdot)$  denote the Harder–Narasimhan types of the cohomology sheaves  $\mathcal{H}^i(\mathcal{E}_\cdot)$  and  $\underline{I}(\mathcal{E}_\cdot)$  denote the Hilbert polynomials of the images of the boundary maps in this complex. Then we define  $\tau(\mathcal{E}_\cdot) := (\underline{H}(\mathcal{E}_\cdot), \underline{I}(\mathcal{E}_\cdot))$ .

**Remark 7.1.3.** If  $\underline{H}(\mathcal{E}_\cdot) = \underline{H}$ , then  $\tau(\mathcal{E}_\cdot)$  is an object in  $\mathcal{J}$ .

The morphisms in  $\mathcal{J}$  are determined by the perturbed linearisations we fixed and so this category depends on the choice of these perturbed linearisations. Let  $\mathcal{A}_\cdot$  be an acyclic complex of torsion free sheaves. Then for any object  $\tau$  in  $\mathcal{J}$  if we choose  $n$  and  $\epsilon$  appropriately, there is a morphism of parameter schemes

$$\mathfrak{T}_{\underline{P}(\tau)} \rightarrow \mathfrak{T}_{\underline{P}(\tau + \tau(\mathcal{A}_\cdot))}$$

corresponding to taking the direct sum with  $\mathcal{A}_\cdot$  (see Section 4.3) and this restricts to a morphism

$$\varphi_\tau^{\mathcal{A}_\cdot} : \hat{R}_\tau \rightarrow \hat{R}_{\tau + \tau(\mathcal{A}_\cdot)}.$$

**Definition 7.1.4.** Let  $\mathcal{A}_\bullet$  be an acyclic complex of torsion free sheaves and  $\tau$  be an object in  $\mathcal{J}$ . Then there is a morphism

$$\phi_\tau^{\mathcal{A}_\bullet} : \tau \rightarrow \tau + \tau(\mathcal{A}_\bullet)$$

in  $\mathcal{J}$  if the image of the semistable scheme  $\hat{R}_\tau^{ss}(\mathcal{L}_\tau^{\text{per}})$  under  $\phi_\tau^{\mathcal{A}_\bullet}$  is contained in the semistable scheme  $\hat{R}_{\tau+\tau(\mathcal{A}_\bullet)}^{ss}(\mathcal{L}_{\tau+\tau(\mathcal{A}_\bullet)}^{\text{per}})$ .

**Lemma 7.1.5.** Let  $\mathcal{A}_\bullet$  be an acyclic complex of torsion free sheaves and  $\tau = (\underline{H}, \underline{I})$  be an object in  $\mathcal{J}$ . If this pair  $(\tau, \mathcal{A}_\bullet)$  defines a morphism

$$\phi_\tau^{\mathcal{A}_\bullet} : \tau \rightarrow \tau + \tau(\mathcal{A}_\bullet)$$

in  $\mathcal{J}$ , then there is a morphism of schemes

$$\Phi_\tau^{\mathcal{A}_\bullet} : M_\tau \rightarrow M_{\tau+\tau(\mathcal{A}_\bullet)}$$

given by taking the direct sum with this acyclic complex.

*Proof.* By Proposition 4.3.3, there is an equivariant and injective morphism

$$\mathfrak{T}_{\underline{P}(\tau)} \rightarrow \mathfrak{T}_{\underline{P}(\tau+\tau(\mathcal{A}_\bullet))}$$

corresponding to taking the direct sum with this acyclic complex. By construction of  $\hat{R}_\tau$  this restricts to a morphism

$$\varphi_\tau^{\mathcal{A}_\bullet} : \hat{R}_\tau \rightarrow \hat{R}_{\tau+\tau(\mathcal{A}_\bullet)}.$$

The assumption that the pair  $(\tau, \mathcal{A}_\bullet)$  defines a morphism in the diagram  $\mathcal{J}$  means that this morphism sends semistable points to semistable points; that is, this morphism

preserves the semistable subschemes

$$\varphi_\tau^{\mathcal{A}}(\hat{R}_\tau^{ss}(\mathcal{L}_\tau^{\text{per}})) \subset \hat{R}_{\tau+\tau(\mathcal{A})}^{ss}(\mathcal{L}_{\tau+\tau(\mathcal{A})}^{\text{per}}).$$

Therefore, the composition

$$\hat{R}_\tau^{ss}(\mathcal{L}_\tau^{\text{per}}) \rightarrow M_{\tau+\tau(\mathcal{A})}$$

is  $G_{P(\tau)}$ -invariant and so by the universal property of the categorical quotient there is an induced morphism

$$\Phi_\tau^{\mathcal{A}} : M_\tau \rightarrow M_{\tau+\tau(\mathcal{A})}.$$

□

**Definition 7.1.6.** Suppose that for each Harder–Narasimhan type  $\tau$  we have fixed a choice of perturbed linearisation  $\mathcal{L}_\tau^{\text{per}}$ . Let  $D := D(\underline{H}, \mathcal{L}_\tau^{\text{per}}) : \mathcal{J} \rightarrow \mathfrak{Sch}$  be the diagram given by

$$\begin{aligned} D(\tau) &:= M_\tau \quad \text{for } \tau \in \text{Ob}(\mathcal{J}) \\ D(\phi_\tau^{\mathcal{A}}) &:= \Phi_\tau^{\mathcal{A}} : M_\tau \rightarrow M_{\tau+\tau(\mathcal{A})}. \end{aligned}$$

**Definition 7.1.7.** A colimit of  $D$  is an object  $D_{\text{lim}}$  and a family of morphisms

$$\theta_\tau : M_\tau \rightarrow D_{\text{lim}}$$

such that for any morphism  $\phi_\tau^{\mathcal{A}} : \tau \rightarrow \tau + \tau(\mathcal{A})$  we have a commutative triangle

$$\begin{array}{ccc} M_\tau & \xrightarrow{\Phi_\tau^{\mathcal{A}}} & M_{\tau+\tau(\mathcal{A})} \\ & \searrow \theta_\tau & \swarrow \theta_{\tau+\tau(\mathcal{A})} \\ & D_{\text{lim}} & \end{array}$$

and this object is universal with respect to these properties.

**Remark 7.1.8.** A colimit may not exist in the category of schemes but will exist in an extension of this category which is closed under such colimits, namely the category of ind-schemes. A thorough treatment of colimits and ind-schemes is given by Kashiwara and Schapira in [20].

Consider two complexes  $\mathcal{F}$  and  $\mathcal{G}$  which correspond to points

$$[\mathcal{F}] \in M_\tau \quad \text{and} \quad [\mathcal{G}] \in M_{\tau'}$$

in the diagram  $D$  where  $\tau = \tau(\mathcal{F})$  and  $\tau' = \tau(\mathcal{G})$ . These points are identified in the colimit if and only if there is another object  $\tau''$  in  $\mathcal{J}$  and morphisms  $\phi_\tau^{\mathcal{A}} : \tau \rightarrow \tau''$  and  $\phi_{\tau'}^{\mathcal{A}'} : \tau' \rightarrow \tau''$  in  $\mathcal{J}$  such that

$$\Phi_\tau^{\mathcal{A}}([\mathcal{F}]) = \Phi_{\tau'}^{\mathcal{A}'}([\mathcal{G}])$$

or, equivalently  $\mathcal{F} \oplus \mathcal{A}$  and  $\mathcal{G} \oplus \mathcal{A}'$  represent the same point in  $M_{\tau''}$ .

## 7.2 Quasi-isomorphism and equivalence in the colimit

Suppose  $f : \mathcal{F} \rightarrow \mathcal{G}$  is a quasi-isomorphism and both these complexes correspond to points

$$[\mathcal{F}] \in M_{\tau(\mathcal{F})} \quad \text{and} \quad [\mathcal{G}] \in M_{\tau(\mathcal{G})}$$

in the diagram  $D$  defined above. Then we want to compare the images of  $\mathcal{F}$  and  $\mathcal{G}$  in the colimit  $D_{\text{lim}}$  described above.

**Definition 7.2.1.** Let  $f : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of complexes. Recall that the cone of  $f$  is a complex  $C(f)$  whose sheaves are given by

$$C(f)_i := \mathcal{F}_{i+1} \oplus \mathcal{G}_i$$

and whose boundary maps are given by

$$d_{C(f)} := \begin{pmatrix} d_{\mathcal{F}[1]} & 0 \\ f[1] & d_{\mathcal{G}} \end{pmatrix}.$$

There is an exact sequence of complexes

$$0 \rightarrow \mathcal{G} \rightarrow C(f) \rightarrow \mathcal{F}[1] \rightarrow 0$$

where  $\mathcal{F}[1]$  is the shift of  $\mathcal{F}$  defined by  $\mathcal{F}_i[1] = \mathcal{F}_{i+1}$  and  $d_{\mathcal{F}[1]} = -d_{\mathcal{F}}$ . Furthermore if  $f : \mathcal{F} \rightarrow \mathcal{G}$  is a quasi-isomorphism, the cone  $C(f)$  is acyclic.

Let  $C(\text{id}_{\mathcal{F}})$  denote the cone on the identity morphism of  $\mathcal{F}$ ; then both  $C(f)$  and  $C(\text{id}_{\mathcal{F}})$  are acyclic. If the perturbed linearisations are chosen in a compatible way so that they preserve semistability, then we will have morphisms

$$\Phi_{\tau(\mathcal{F})}^{C(f)} : M_{\tau(\mathcal{F})} \rightarrow M_{\tau(\mathcal{F}) + \tau(C(f))}$$

and

$$\Phi_{\tau(\mathcal{G})}^{C(\text{id}_{\mathcal{F}})} : M_{\tau(\mathcal{G})} \rightarrow M_{\tau(\mathcal{G}) + \tau(C(\text{id}_{\mathcal{F}}))}$$

in  $D$  corresponding to taking the direct sum with  $C(f)$  and  $C(\text{id}_{\mathcal{F}})$  respectively.

**Lemma 7.2.2.** *The morphisms  $\Phi_{\tau(\mathcal{F})}^{C(f)}$  and  $\Phi_{\tau(\mathcal{G})}^{C(\text{id}_{\mathcal{F}})}$  described above have the same codomain.*

*Proof.* Since  $\mathcal{F} \oplus C(f)$  and  $\mathcal{G} \oplus C(\text{id}_{\mathcal{F}})$  are both quasi-isomorphic to  $\mathcal{F}$ , their cohomology sheaves have the prescribed Harder–Narasimhan type  $\tau(\mathcal{H}^i)$ . For these morphisms to have the same codomain we need to check that the image sheaves of the boundary morphisms of  $\mathcal{F} \oplus C(f)$  and  $\mathcal{G} \oplus C(\text{id}_{\mathcal{F}})$  have the same Hilbert polynomial. Note that these complexes both consist of the same sheaves  $\mathcal{F}_i \oplus \mathcal{F}_{i+1} \oplus \mathcal{G}_i$  but different boundary maps. We also know that the cohomology sheaves of these

complexes have the same Hilbert polynomials. Then, by looking at the short exact sequences

$$0 \rightarrow \ker d_{\mathcal{F} \oplus C(f)}^i \rightarrow \mathcal{F}_i \oplus \mathcal{F}_{i+1} \oplus \mathcal{G}_i \rightarrow \operatorname{Im} d_{\mathcal{F} \oplus C(f)}^i \rightarrow 0$$

$$0 \rightarrow \operatorname{Im} d_{\mathcal{F} \oplus C(f)}^{i-1} \rightarrow \ker d_{\mathcal{F} \oplus C(f)}^i \rightarrow \mathcal{H}^i(\mathcal{F}) \rightarrow 0$$

and similar short exact sequences for  $\mathcal{G} \oplus C(\operatorname{id}_{\mathcal{F}})$ , we can prove inductively that the image sheaves of the boundary morphisms of these complexes also have the same Hilbert polynomials.  $\square$

Ideally we want to show that these complexes represent the same point in

$$M_{\tau(\mathcal{G}) + \tau(C(\operatorname{id}_{\mathcal{F}}))} = M_{\tau(\mathcal{F}) + \tau(C(f))}.$$

Note that even when a GIT quotient is taken with respect to a non-ample linearisation, two orbits whose closures meet in the GIT semistable set are identified in this GIT quotient.

We consider a complex which is, up to sign, the cylinder on  $f$ . Let  $\mathcal{C}$  be the complex with sheaves  $\mathcal{C}_i := \mathcal{F}_i \oplus \mathcal{F}_{i+1} \oplus \mathcal{G}_i$  and boundary maps given by

$$d_{\mathcal{C}} := \begin{pmatrix} d_{\mathcal{F}} & \operatorname{id}_{\mathcal{F}[1]} & 0 \\ 0 & d_{\mathcal{F}[1]} & 0 \\ 0 & f[1] & d_{\mathcal{G}} \end{pmatrix}.$$

The inclusion  $\mathcal{G} \hookrightarrow \mathcal{C}$  is a quasi-isomorphism (see [12] Lemma 2.2.1) and so the cohomology sheaves of this complex have the specified Harder–Narasimhan types.

**Proposition 7.2.3.** *The orbit closure of  $\mathcal{C}$  contains both  $\mathcal{F} \oplus C(f)$  and  $\mathcal{G} \oplus C(\operatorname{id}_{\mathcal{F}})$ .*

*Proof.* We define one-parameter subgroups  $\lambda$  and  $\lambda'$  whose limits at infinity when acting on the complex  $\mathcal{C}$  are  $C(\operatorname{id}_{\mathcal{F}}) \oplus \mathcal{G}$  and  $\mathcal{F} \oplus C(f)$  respectively.

Before we define these 1-PSs we recall the group action on the parameter space. Let  $n$  be sufficiently large so that everything of interest is  $n$ -regular. Let  $w_i := \dim H^0(\mathcal{F}_i(n))$  and  $v_i := \dim H^0(\mathcal{G}_i(n))$ ; then  $p_i := \dim H^0((\mathcal{F}_i \oplus \mathcal{F}_{i+1} \oplus \mathcal{G}_i)(n)) = w_i + w_{i+1} + v_i$ . The groups  $\mathrm{GL}(p_i)$  act on the sheaves by acting on the quot schemes  $\mathrm{Quot}(\mathbb{C}^{p_i} \otimes \mathcal{O}_X(-n), P(\mathcal{F}_i) + P(\mathcal{F}_{i+1}) + P(\mathcal{G}_i))$ . The subgroup  $\mathbb{C}^* I_{p_i} \subset \mathrm{GL}(p_i)$  acts trivially on the quot scheme. As the complexes  $\mathcal{F} \oplus C(f)$ ,  $C(\mathrm{id}_{\mathcal{F}}) \oplus \mathcal{G}$ , and  $\mathcal{C}$  all have the same sheaves  $\mathcal{F}_i \oplus \mathcal{F}_{i+1} \oplus \mathcal{G}_i$ , we want to look at elements of  $\mathrm{GL}(p_i)$  which fix these sheaves. These elements look like

$$\begin{pmatrix} aI_{w_i} & 0 & 0 \\ 0 & bI_{w_{i+1}} & 0 \\ 0 & 0 & cI_{v_i} \end{pmatrix} \in \mathrm{GL}(p_i).$$

The group  $G := \prod \mathrm{GL}(p_i)$  acts on the boundary maps  $d_i$  of a complex by acting by conjugation on the maps  $H^0(d_i(n))$ .

We can define one-parameter subgroups  $\lambda, \lambda' : \mathbb{C}^* \rightarrow G$  by instead defining a collection of 1-PSs  $\lambda_i, \lambda'_i$  of  $\mathrm{GL}(p_i)$ . We choose positive integers  $r$  and  $s$  and then define

$$\lambda_i(t) := \begin{pmatrix} t^r I_{w_i} & 0 & 0 \\ 0 & t^r I_{w_{i+1}} & 0 \\ 0 & 0 & t^{-s} I_{v_i} \end{pmatrix} \quad \lambda'_i(t) := \begin{pmatrix} t^{-s} I_{w_i} & 0 & 0 \\ 0 & t^r I_{w_{i+1}} & 0 \\ 0 & 0 & t^r I_{v_i} \end{pmatrix}$$

Then the action is given by

$$\lambda(t) \cdot H^0(d_{\mathcal{C}}^i(n)) = \begin{pmatrix} H^0(d_{\mathcal{F},i}(n)) & \mathrm{id}_{\mathcal{F}} & 0 \\ 0 & H^0(d_{\mathcal{F},[1],i}(n)) & 0 \\ 0 & t^{-(r+s)} H^0(f_{i+1}(n)) & H^0(d_{\mathcal{G},i}(n)) \end{pmatrix}$$

and

$$\lim_{t \rightarrow \infty} \lambda(t) \cdot H^0(d_{\mathcal{C}}^i(n)) = \begin{pmatrix} H^0(d_{\mathcal{F},i}(n)) & \text{id}_{\mathcal{F}} & 0 \\ 0 & H^0(d_{\mathcal{F},[1],i}(n)) & 0 \\ 0 & 0 & H^0(d_{\mathcal{G},i}(n)) \end{pmatrix},$$

so the limit is  $\mathcal{G} \oplus C(\text{id}_{\mathcal{F}})$ .

Similarly

$$\lambda'(t) \cdot H^0(d_{\mathcal{C}}^i(n)) = \begin{pmatrix} H^0(d_{\mathcal{F},i}(n)) & t^{-(r+s)} H^0(\text{id}_{\mathcal{F}_{i+1}}(n)) & 0 \\ 0 & H^0(d_{\mathcal{F},[1],i}(n)) & 0 \\ 0 & H^0(f_{i+1}(n)) & H^0(d_{\mathcal{G},i}(n)) \end{pmatrix}$$

and

$$\lim_{t \rightarrow \infty} \lambda'(t) \cdot H^0(d_{\mathcal{C}}^i(n)) = \begin{pmatrix} H^0(d_{\mathcal{F},i}(n)) & 0 & 0 \\ 0 & H^0(d_{\mathcal{F},[1],i}(n)) & 0 \\ 0 & H^0(f_{i+1}(n)) & H^0(d_{\mathcal{G},i}(n)) \end{pmatrix}$$

which corresponds to  $\mathcal{F} \oplus C(f)$ . □

**Corollary 7.2.4.** *Let  $f : \mathcal{F} \rightarrow \mathcal{G}$  be a quasi-isomorphism of complexes representing points in  $D$ . If we can pick the perturbed linearisations in a compatible way so that there are morphisms  $\Phi_{\tau(\mathcal{F})}^{C(f)}$  and  $\Phi_{\tau(\mathcal{G})}^{C(\text{id}_{\mathcal{F}})}$  in  $D$ , then these complexes are identified in the colimit of  $D$ .*

*Proof.* Assuming the existence of these perturbed linearisations, to show that  $\mathcal{F}$  and  $\mathcal{G}$  are identified in the colimit it suffices to show that  $\mathcal{F} \oplus C(f)$  and  $\mathcal{G} \oplus C(\text{id}_{\mathcal{F}})$  are identified in the colimit. Since these complexes are both points in

$$M_{\tau(\mathcal{G})+\tau(C(\text{id}_{\mathcal{F}}))} = M_{\tau(\mathcal{F})+\tau(C(f))},$$

it suffices to show that they both represent the same point. By Proposition 7.2.3, there is a complex  $\mathcal{C}$ . whose orbit closure contains both of these complexes. Since the GIT semistable set is open,  $\mathcal{C}$ . must also be semistable. Therefore the orbits of  $\mathcal{F} \oplus C(f)$  and  $\mathcal{G} \oplus C(\text{id}_{\mathcal{F}})$  and  $\mathcal{C}$ . are all identified and so these complexes represent the same point.  $\square$

Therefore, to ensure  $\mathcal{F}$ . and  $\mathcal{G}$ . are identified in the quotient we need to be able to pick the perturbed linearisations in a compatible way. Working forwards from the initial object in this diagram corresponding to  $\tau_0 := (\underline{H}, \underline{0})$  we hope to be able to choose these linearisations in a compatible way and so make the following conjecture.

**Conjecture 7.2.5.** *There is a collection of perturbed linearisations  $\mathcal{L}_\tau^{\text{per}}$  which define a diagram  $D(\underline{H}, \mathcal{L}_\tau^{\text{per}})$  such that:*

- 1) *For each perturbed linearisation the GIT semistable set  $\hat{R}_\tau^{ss}(\mathcal{L}_\tau^{\text{per}})$  is nonempty.*
- 2) *If  $f : \mathcal{F} \rightarrow \mathcal{G}$ . is a quasi-isomorphism between complexes representing points in this diagram, then these complexes both represent the same point in the colimit.*

**Remark 7.2.6.** Suppose for each Harder–Narasimhan type  $\tau$  we choose the trivial perturbation; i.e., the canonical linearisation  $\mathcal{L}_\tau^{\text{can}} := \mathcal{L}_{\beta(\tau)}$  and take the generalised GIT quotient which is a categorical quotient of  $\hat{R}_\tau^{nss}(\mathcal{L}_\tau^{\text{can}}) = R_\tau$ . By definition of  $D = D(\underline{H}, \mathcal{L}_\tau^{\text{can}})$ , a Harder–Narasimhan type  $\tau = (\underline{H}, \underline{I})$  indexes an object in  $D$  if and only if  $\underline{I}$  is a collection of trivial Harder–Narasimhan types for the image sheaves of the boundary morphisms. Suppose  $\tau$  is a Harder–Narasimhan type which indexes an object in  $D$ ; then if  $\mathcal{E}$ . has Harder–Narasimhan type  $\tau$  the image sheaves  $\text{Im } d_{\mathcal{E}}^i$  are Gieseker semistable with reduced Hilbert polynomial  $I^i / \text{rk } I^i$ . Let  $\mathcal{A}$ . be an acyclic complex of torsion free sheaves. By definition of  $D$ , there is a morphism  $\tau = (\underline{H}, \underline{I}) \rightarrow \tau + \tau(\mathcal{A})$  in  $D$  if and only if for every complex  $\mathcal{E}$ . with Harder–Narasimhan type  $\tau$  the direct sum  $\mathcal{E} \oplus \mathcal{A}$ . has Harder–Narasimhan type  $\tau + \tau(\mathcal{A})$ ; that is, if and only if

for all such  $\mathcal{E}$ . the sheaf  $\mathrm{Im} d_{\mathcal{E}}^i \oplus \mathrm{Im} d_{\mathcal{A}}^i$  is Gieseker semistable or equivalently  $\mathrm{Im} d_{\mathcal{A}}^i$  is Gieseker semistable with reduced Hilbert polynomial  $I^i / \mathrm{rk} I^i$ . In this case the colimit of the diagram is the product of moduli spaces for the successive quotients in the Harder–Narasimhan filtration of each cohomology sheaf.

**Example 7.2.7.** We consider the most basic example when  $X = \mathrm{Spec} \mathbb{C}$  is a point. Then a sheaf over  $X$  is just a complex vector space and its Hilbert polynomial and rank are both equal to the dimension of this vector space. Therefore, the parameter spaces are indexed by a tuple  $\underline{d}$  of dimension vectors.

As every vector space has the same reduced Hilbert polynomial, every vector space is automatically semistable and has trivial Harder–Narasimhan type. Therefore the Harder–Narasimhan type  $\tau = (\underline{H}, \underline{I})$  of any complex of vector spaces with respect to  $(\epsilon, \delta\eta)$  is given by two collections of trivial Harder–Narasimhan types  $\underline{H} = (H^i)_{i \in \mathbb{Z}}$  and  $\underline{I} = (I^i)_{i \in \mathbb{Z}}$  which give the dimension vectors of the cohomology sheaves and image sheaves respectively. In particular this Harder–Narasimhan type indexes an object in the diagram  $D(\underline{H}, \mathcal{L}_{\tau}^{\mathrm{per}})$ .

If  $V$  and  $W$  are complexes of vector spaces which have the same dimension vector  $\underline{d}$ ; then these complexes are isomorphic if and only if their cohomology sheaves are isomorphic. Therefore, the scheme  $R_{\tau}$  parametrising complexes with Harder–Narasimhan type  $\tau$  is a single  $G_{\underline{d}(\tau)}$ -orbit in the parameter scheme  $\mathfrak{T}_{\underline{d}(\tau)}$ . Moreover, the categorical quotient  $M_{\tau}$  of the  $G_{\underline{d}(\tau)}$ -action on  $R_{\tau} = \hat{R}_{\tau}^{nss}(\mathcal{L}_{\tau}^{\mathrm{can}})$  is just a point. In this case we do not have the problem of too many orbits being collapsed (see Section 6.5) and so there is no need to perturb the canonical linearisation.

Fix a dimension vector  $\underline{H}$  for the cohomology and consider the associated diagram  $D(\underline{H}, \mathcal{L}_{\tau}^{\mathrm{can}}) : \mathcal{J} \rightarrow \mathfrak{Sch}$ . The objects in this diagram are points  $M_{\tau} = \mathrm{Spec} \mathbb{C}$  indexed by all Harder–Narasimhan types  $\tau = (\underline{H}, \underline{I})$ . In fact, every complex  $V$  with cohomology of dimension  $\underline{H}$  is represented by a point  $M_{\tau(V)}$  in this diagram. For any acyclic complex  $A$  there is a morphism in this diagram  $M_{\tau} \rightarrow M_{\tau+\tau(A)}$  as every

vector space is semistable with the same reduced Hilbert polynomial (see Remark 7.2.6). If  $f : V \rightarrow W$  is a quasi-isomorphism between complexes with cohomology type  $\underline{H}$ , then these complexes are both identified in the colimit (see Corollary 7.2.4). Conversely if  $V$  and  $W$  are identified in the colimit then there are acyclic complexes  $A$  and  $A'$  such that  $V \oplus A \cong W \oplus A'$ ; i.e., these complexes are quasi-isomorphic. Therefore, the colimit of  $D$  is just the point  $\text{Spec } \mathbb{C}$  and corresponds to the one quasi-isomorphism class for complexes with cohomology of dimension vector  $\underline{H}$ .

### 7.3 Concluding remarks

We end this thesis with some concluding remarks about the diagrams we were considering above. The starting data for these diagrams is a choice of perturbed linearisation for each Harder–Narasimhan type  $\tau$ . As we mentioned in Section 6.5, one possible way to perturb the canonical linearisation on  $\hat{R}_\tau \subset G_{\underline{P}(\tau)} \times_{P_{\beta(\tau)}} \mathfrak{T}^{tf} \cong G_{\underline{P}(\tau)} / P_{\beta(\tau)} \times \mathfrak{T}^{tf}$  is by choosing a Schmitt linearisation  $\mathcal{L}'$  which is a perturbation of the original Schmitt linearisation  $\mathcal{L}$  and taking the exterior tensor product of the linearisation  $\mathcal{L}(\lambda_{-\beta(\tau)})$  on  $G_{\underline{P}(\tau)} / P_{\beta(\tau)}$  with  $\mathcal{L}'$ .

In fact we could extend the categories  $\mathcal{J}$  which indexed the diagrams we were considering to allow more than one choice of perturbed linearisation for each Harder–Narasimhan type  $\tau$  and construct several different quotients of  $\hat{R}_\tau$  using these perturbed linearisations. The advantage of this approach would mean that there is a greater chance of making the direct sum morphisms work and so there is a greater chance of showing that quasi-isomorphic complexes are identified in the colimit.

Ideally we would like to find a choice of perturbed linearisation for each  $\tau$  such that almost all complexes in  $\hat{R}_\tau^{ss}(\mathcal{L}_\tau^{\text{per}})$  are  $D(X)$ -indecomposable in the following sense:

**Definition 7.3.1.** We say a complex  $\mathcal{F}$  is  $D(X)$ -decomposable if it is a direct sum of

two non-acyclic complexes. Otherwise, we say the complex is  $D(X)$ -indecomposable.

Let  $D_\tau \subset M_\tau = \hat{R}_\tau //_{\mathcal{L}_\tau^{\text{per}}} G_{\underline{P}(\tau)}$  be the set of points corresponding to  $D(X)$ -decomposable complexes and let  $U_\tau$  be the complement in  $\hat{R}_\tau^{ss}(\mathcal{L}_\tau^{\text{per}})$  of the preimage of  $D_\tau$  under the quotient morphism

$$\hat{R}_\tau^{ss}(\mathcal{L}_\tau^{\text{per}}) \rightarrow M_\tau.$$

The following definition is motivated by [4], Theorem 4.2.

**Definition 7.3.2.** We define an equivalence relation  $\sim$  on  $U_\tau$  called S-equivalence:  $x$  and  $y$  are S-equivalent ( $x \sim y$ ) if there exists  $g$  and  $g' \in G_\tau$ , 1-PSs  $\lambda$  and  $\lambda'$  of  $G_\tau$  such that

$$\lim_{t \rightarrow 0} \lambda(t)gx = \lim_{t \rightarrow 0} \lambda'(t)g'y \in U_\tau.$$

**Lemma 7.3.3.** Assume that  $\tau$  is the Harder–Narasimhan type for a non-acyclic complex. If  $x$  and  $y$  are S-equivalent points in  $U_\tau$  which represent complexes  $\mathcal{E}$ . and  $\mathcal{F}$ . then these complexes are quasi-isomorphic.

*Proof.* By definition of S-equivalence there are elements  $g, g' \in G_{\underline{P}(\tau)}$  and 1-PSs  $\lambda, \lambda'$  of  $G_{\underline{P}(\tau)}$  such that

$$z := \lim_{t \rightarrow 0} \lambda(t)gx = \lim_{t \rightarrow 0} \lambda(t)g'y \in U_\tau.$$

Let  $\mathcal{G}. \cong \mathcal{E}$ . be the complex corresponding to  $gx$  and  $\mathcal{H}. \cong \mathcal{F}$ . be the complex corresponding to  $g'y$ . Then it suffices to show  $\mathcal{G}.$  and  $\mathcal{H}.$  are quasi-isomorphic.

As  $z \in U_\tau \subset \hat{R}_\tau$ , it corresponds to a complex too. Moreover, the 1-PS  $\lambda$  induces a filtration of  $\mathcal{G}.$  by subcomplexes  $0 \subset \mathcal{G}.^{[1]} \subset \dots \subset \mathcal{G}.^{[t]} = \mathcal{G}.$  and  $z$  represents the graded object  $\overline{\mathcal{G}.} := \bigoplus_{j=1}^t \mathcal{G}.^j$  associated to this filtration. Therefore by symmetry, it suffices to prove that  $\mathcal{G}.$  is quasi-isomorphic to the complex  $\overline{\mathcal{G}.}$  representing this limit point  $z$ .

By definition of  $U_\tau$  we know that  $\overline{\mathcal{G}}_\cdot$  is  $D(X)$ -indecomposable and so there is  $1 \leq j_o \leq t$  such that  $\mathcal{G}_\cdot^{j_o}$  is not acyclic and all other direct summands are acyclic. Then clearly  $\overline{\mathcal{G}}_\cdot$  is quasi-isomorphic to  $\mathcal{G}_\cdot^{j_o}$  and so it remains to prove that  $\mathcal{G}_\cdot^{j_o}$  is quasi-isomorphic to  $\mathcal{G}_\cdot$ . Note that  $\mathcal{G}_\cdot^{[j_o-1]}$  is acyclic as it is built up by taking successive extensions of acyclic complexes. We have a short exact sequence

$$0 \rightarrow \mathcal{G}_\cdot^{[j_o-1]} \rightarrow \mathcal{G}_\cdot^{[j_o]} \rightarrow \mathcal{G}_\cdot^{j_o} \rightarrow 0$$

and so  $\mathcal{G}_\cdot^{[j_o]} \rightarrow \mathcal{G}_\cdot^{j_o}$  is a quasi-isomorphism. Similarly  $\mathcal{G}_\cdot / \mathcal{G}_\cdot^{[j_o]}$  is acyclic and since we have a short exact sequence

$$0 \rightarrow \mathcal{G}_\cdot^{[j_o]} \rightarrow \mathcal{G}_\cdot \rightarrow \mathcal{G}_\cdot / \mathcal{G}_\cdot^{[j_o]} \rightarrow 0,$$

$\mathcal{G}_\cdot^{[j_o]} \rightarrow \mathcal{G}_\cdot$  is a quasi-isomorphism. Therefore,  $\mathcal{G}_\cdot$  is quasi-isomorphic to  $\mathcal{G}_\cdot^{j_o}$  which completes the proof.  $\square$

The ideal situation would be if  $U_\tau = \hat{R}_\tau^{ss}(\mathcal{L}_\tau^{\text{per}})$  and the perturbed linearisation was ample as then topologically  $M_\tau$  would just be  $U_\tau$  modulo S-equivalence. Then by the above lemma two S-equivalent complexes are actually quasi-isomorphic. However if  $U_\tau \neq \hat{R}_\tau^{ss}(\mathcal{L}_\tau^{\text{per}})$  then more points will be identified as two points in  $U_\tau$  may have orbit closures which meet in the complement of  $U_\tau$  in  $\hat{R}_\tau^{ss}(\mathcal{L}_\tau^{\text{per}})$ .

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