

# Études in Holography



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## Abstract

The holographic principle posits that a higher-dimensional system with gravity can be fully described by a lower-dimensional “boundary” theory without gravity. This remarkable relationship has profound implications across physics, influencing fields from quantum gravity and black hole thermodynamics to condensed matter physics and quantum information. This thesis explores two specific problems within this broad domain.

The first part addresses the identification and counting of black hole microstates. Specifically, we study the non-supersymmetric, asymptotically flat  $2d$  black hole through its dual  $c = 1$  matrix quantum mechanics. Our central result is a novel bound state that dominates the partition function in the double-scaling limit, providing an explicit dual description of the black hole microstates.

The second part investigates open effective field theories of fermions interacting with holographic baths. After establishing the case for using holography in studying open effective field theories, we introduce the *gravitational Schwinger–Keldysh saddle* (grSK saddle). We then analyze Dirac fermions propagating in these backgrounds, demonstrating how this leads to Fermi-Dirac statistics in the boundary theory. Our analysis reveals that the holographic Schwinger-Keldysh boundary conditions naturally select a unique combination of modes consistent with fermionic KMS relations. Through evaluation of the bulk on-shell action using these solutions, we derive an explicit expression for the probe fermion’s influence phase in a gradient expansion. We conclude by extending the holographic Schwinger-Keldysh prescription to charged black branes, enabling the study of systems at both finite temperature and chemical potential.

## Statement of Originality

This thesis is the work of the author unless otherwise stated. No part has been submitted for any other qualification. Chapter 0 contains an original review of the literature.

Chapter 1 is based on:

- Amr Ahmadain, Alexander Frenkel, Krishnendu Ray, and Ronak M. Soni. *Boundary description of microstates of the two-dimensional black hole*, Sci- Post Phys., 16(1):020, 2024.

Chapters 2 and 3 are based on:

- R. Loganayagam, Krishnendu Ray, and Akhil Sivakumar. *Fermionic Open EFT from Holography*. 11 2020 (*Submitted, Sci-Post*).

Chapter 4 is based on:

- R. Loganayagam, Krishnendu Ray, Shivam K. Sharma, and Akhil Sivakumar. *Holographic KMS relations at finite density*, JHEP, 03:233, 2021.

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## Chapter 0

# *Prelude: An introduction and a guide to this thesis*

### 0.1 Holography: An invitation

In the vast landscape of modern theoretical physics, few ideas have been as dramatic and striking as holography. Bridging the seemingly disparate realms of quantum gravity and strongly coupled quantum field theories, it has emerged as a powerful framework for understanding not only some the deepest, most fundamental questions in physics, but also in revealing its very intricate architecture.

The statement of the holographic principle is as elegant as it is profound: the complete physics of a higher-dimensional system with gravity can be described by a lower-dimensional “boundary” theory devoid of it. This principle finds its most concrete and celebrated manifestation in the AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence. Proposed by Juan Maldacena in 1997 [113], this duality establishes a precise equivalence between a gravitational theory in a  $(d + 1)$ -dimensional Anti-de Sitter spacetime and a conformal field theory residing on its  $d$ -dimensional boundary. The implications of this relationship are profound and far-reaching, influencing research across a wide spectrum of physics, from quantum gravity and black hole thermodynamics to condensed matter physics, quantum information and beyond.

This thesis explores two specific problems within this expansive domain of holography. The first, *Opus I*, addresses a long-standing challenge in quantum gravity: identifying the

microstates of black holes. This is accomplished through the application of the holographic principle using a toy model known as the  $c = 1$  matrix model. *Opus II* shifts our focus to open quantum systems with fermionic degrees of freedom, a crucial step towards understanding more general non-equilibrium and open systems in condensed matter physics. Here, we extend the standard holographic dictionary to encompass finite-temperature, real-time quantum field theories. Although these two problems may appear distinct, both are deeply rooted in holographic techniques, underscoring its remarkable ability to weave together different strands of physics.

Before delving into these specific quests, however, let's take a moment to appreciate the bigger picture. This chapter contains a whirlwind tour of the development of holography. From its humble beginnings in the study of black holes to its current status as a cornerstone of modern theoretical physics, we will see how this principle has not only revolutionized our approach to some of the most challenging problems in physics but also opened up entirely new avenues of inquiry. The holographic principle stands as a triumph of modern theoretical physics, a shining testament to the power of abstract ideas in yielding concrete insights about our universe's fundamental nature. This detour is indeed well worth our time.

### 0.1.1 Historical origins: Black hole thermodynamics

The story of holography begins not in the realm of string theory, but in the fascinating and enigmatic world of black holes. In the 1970s, a series of groundbreaking discoveries in black hole physics set the stage for a revolutionary new understanding of the relationship between gravity, quantum mechanics, and information.

#### Bekenstein–Hawking entropy

Our journey starts in 1972 with a bold proposal by Jacob Bekenstein, then a graduate student working with John Wheeler at Princeton. Bekenstein suggested that black holes should be attributed an entropy proportional to their horizon area [18]:

$$S_{\text{BH}} = \alpha \frac{A}{\ell_P^2}. \tag{1}$$

Here  $A$  is the horizon area,  $\ell_P$  is the Planck length (the fundamental length scale in quantum gravity), and  $\alpha$  is a constant<sup>1</sup>. This proposal was not merely a mathematical curiosity; it

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<sup>1</sup>In his seminal work [19] published the next year, he argued that  $\alpha$  is approximately  $\frac{\ln 2}{8\pi}$ .

was motivated by a deep physical principle — the need to preserve the second law of thermodynamics in processes involving black holes.

Bekenstein’s idea was revolutionary, but was unsurprisingly met with skepticism. Indeed, at the time, black holes were primarily understood through classical general relativity. In this framework, black holes were thought to be simple objects characterized by only a few parameters (mass, angular momentum, and electric charge). How then could this simple object, possess entropy, which is typically associated with a large number of microscopic degrees of freedom?

In 1974, Stephen Hawking, building on Bekenstein’s work, made a discovery that would shake the foundations of physics. Through a careful analysis, Hawking showed that black holes are not eternally black — they emit radiation [76]. This radiation, now known as Hawking radiation, has a thermal spectrum with a temperature given by:

$$T_{\text{H}} = \frac{\hbar c^3}{8\pi G M k_B}$$

Where  $M$  is the black hole mass. This result not only confirmed Bekenstein’s notion of black hole entropy but also fixed the constant  $\alpha$  to be exactly  $1/4$ , leading to the famous Bekenstein-Hawking entropy formula<sup>2</sup>:

$$S_{\text{BH}} = \frac{A}{4 \ell_P^2}. \tag{2}$$

While Hawking’s work provided a precise thermodynamic interpretation to the entropy of black holes, it shed no light on its statistical origins. However, this elegant equation carried a profound implication: the information content of a black hole is proportional not to its volume, as one might naively expect, but to its surface area. As we will see, this area-scaling of entropy would become a central tenet of the holographic principle.

## The information paradox

While Hawking’s discovery of black hole radiation was a triumph, it also led to a profound puzzle that would perplex physicists for decades: the information paradox. If a black hole can evaporate completely through Hawking radiation, what happens to the information about everything that fell into it? The laws of quantum mechanics insist that information

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<sup>2</sup>In one of the more pleasing coincidences in physics, the subscript “BH” can mean both “Bekenstein-Hawking” and “black hole”.

can never be truly lost - the evolution of quantum states is unitary, meaning that information is preserved over time. Yet Hawking's calculations suggested that the radiation emitted by a black hole was purely thermal, carrying no information about the black hole's history or contents.

This paradox struck at the heart of the compatibility between quantum mechanics and general relativity. It suggested that either quantum mechanics breaks down in the presence of black holes, or our understanding of black hole evolution was incomplete.

### **0.1.2 The birth of the holographic principle**

Black hole thermodynamics and the information paradox became a crucible for theoretical physics, challenging our most fundamental ideas about the nature of space, time, and information. These challenges were particularly acute because they seemed to pit quantum mechanics against general relativity — two pillars of modern physics that had each passed every experimental test with flying colors. In attempting to resolve these ideas, the holographic principle was born.

#### **'t Hooft's proposal**

Indeed, in 1993, Gerard 't Hooft, a Dutch physicist known for his foundational work in particle physics, proposed a radical solution to the information paradox [155]. 't Hooft suggested that the information about everything that falls into a black hole is not lost, but rather encoded on its event horizon. This proposal was more than just a clever solution to a thorny problem — it was a fundamental reimagining of how information is stored in our universe. In our everyday experience, the amount of information that can be stored in a region of space scales with its volume — a larger hard drive can store more data. However, 't Hooft argued that the maximum amount of information that could be stored in any region of space was proportional not to its volume, but to the area of its boundary. This deeply counterintuitive concept would come to be known as the holographic principle.

#### **Susskind's formalization**

Building on 't Hooft's insights, Leonard Susskind took the next crucial step. In his seminal 1994 paper [154], Susskind formalized the holographic principle, extending its implications beyond black holes to asymptotically flat spacetimes and other geometries where a suitable boundary could be defined. In particular, Susskind proposed that any theory of quantum gravity must be describable by a theory living on the boundary of the spacetime, with at

most one degree of freedom per Planck area. This formalization was both precise and far-reaching, offering a tantalizing path towards reconciling quantum mechanics and gravity. The holographic principle, as articulated by 't Hooft and Susskind, suggested that the true nature of reality might be found not in the bulk of spacetime, but on its boundaries.

### **The final run-up to AdS/CFT**

In the lead-up to the formal proposal of the AdS/CFT correspondence, a series of crucial developments emerged from studies of  $Dp$ -branes and their near-horizon geometries. Two complementary descriptions of branes had emerged: one in terms of the gravitational fields they produce, and another in terms of the quantum field theories living on their world-volumes.

One line of investigation focused on five-branes near their extremal limit. The work of Maldacena [112] and Strominger [116], provided concrete evidence that the world-volume theory could capture detailed aspects of the gravitational physics, including quantum effects like Hawking radiation. The fact that two very different descriptions — one gravitational, one field-theoretic — could yield identical results was profound.

Perhaps the most significant contribution during this period was made by Igor Klebanov in his 1997 paper, “World-volume approach to absorption by non-dilatonic branes” [97], studying how scalar fields were absorbed by D3-branes. This work was further developed with Steven Gubser [67], demonstrating an explicit connection between calculations done in the world-volume theory and those done in the supergravity description.

Ultimately, the D3-brane case would prove especially fruitful. Here, the near-horizon geometry was the product space  $AdS_5 \times S^5$ , while the world-volume theory was a conformal field theory. The mathematical consistency between these two descriptions hinted at something profound: a possible exact equivalence between gravitational physics in the bulk spacetime and quantum field theory on its boundary. These developments collectively built the conceptual framework needed for a revolutionary insight. The growing evidence for such “holographic” relationships between gravity and field theory would soon crystallize in Maldacena’s AdS/CFT correspondence, leading to one of the most significant breakthroughs in theoretical physics in the past half-century.

#### **0.1.3 Interlude: Matrix models and string theory**

While the holographic principle was taking shape, another line of research was developing that would prove crucial to our understanding of holography: the study of matrix models in

string theory. These models, which involve large  $N \times N$  matrices, were originally introduced in the context of two-dimensional quantum gravity and string theory.

Matrix models might seem far removed from the physics of black holes and holography, but they would turn out to provide important insights into how lower-dimensional theories can encode higher-dimensional physics. In these models, the path integral is expressed in terms of matrix variables, and the large  $N$  limit often leads to emergent continuum descriptions.

### The $c = 1$ matrix model

One of the most notable and influential examples of matrix models is the  $c = 1$  matrix model [117, 163, 91, 93]. This model provides a non-perturbative formulation of two-dimensional string theory, offering a simplified setting where many of the ideas of holography could be explored with greater mathematical control. In the  $c = 1$  model, a matrix quantum mechanics system describes a theory of non-critical strings — string theories in lower dimensions that do not require the full machinery of critical string theory. This model represents a string theory with a single space dimension and one time dimension, providing an ideal testing ground for exploring the principles of holography in simpler settings.

The  $c = 1$  matrix model is described by the action:

$$S = \int dt \operatorname{tr} \left( \frac{1}{2} (\partial_t M)^2 - V(M) \right), \quad (3)$$

where  $M$  is an  $N \times N$  Hermitian matrix and  $V(M)$  is typically a polynomial potential<sup>3</sup>. While simple, as we will see, this action contains a universe of complexity.

Indeed, this model has a *double-scaling limit*: by carefully tuning the coupling constants<sup>4</sup> in  $V(M)$  as  $N \rightarrow \infty$ , there is an emergent smooth spacetime that is dual to the discrete matrix degrees of freedom. The matrix eigenvalues in the model can be interpreted as the collective coordinates of the string degrees of freedom. The non-trivial dynamics of these eigenvalues, including their potential interactions and configurations, encode the bulk spacetime geometry. The  $c = 1$  matrix model is significant because it provides an example of a holographic duality in a low-dimensional setting. The matrix model captures both the perturbative and non-perturbative aspects of the string theory, and its relative simplicity allows for detailed analysis of questions related to spacetime, black holes, and quantum gravity.

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<sup>3</sup>More on this later (§1.2).

<sup>4</sup>This is described in more detail in §1.2.

Perhaps most importantly, in the context of holography, the  $c = 1$  matrix model helped lay the conceptual groundwork for the broader and more sophisticated developments that were to come. The ideas developed in the study of matrix models — the emergence of space and time from more fundamental degrees of freedom, the duality between different descriptions of the same physics, the importance and the subtlety of large  $N$  limits — were all previews of the central ideas that would find expression in the AdS/CFT correspondence, the most fully realized example of holography to date.

#### 0.1.4 The AdS/CFT correspondence: Holography realized

The most concrete and influential realization of the holographic principle came in 1997 with Juan Maldacena’s formulation of the AdS/CFT correspondence [113]. This breakthrough provided a precise mathematical framework for holography and opened up new avenues for studying strongly coupled quantum systems and quantum gravity.

##### Maldacena’s conjecture

Maldacena’s insight arose from considering a particular configuration of  $Dp$ -branes in string theory. While the precise details of this construction are not important for our discussion<sup>5</sup>, its upshot is. He was lead to conjecture a surprising equivalence between:

- Type IIB string theory on  $AdS_5 \times S^5$ .
- $\mathcal{N} = 4$  super Yang–Mills theory in  $3 + 1$  dimensions.

At first glance, these two theories could hardly seem more different. On one side, we have a theory of quantum gravity in a five-dimensional anti-de Sitter (AdS) space, combined with a five-dimensional sphere. On the other, we have a quantum field theory without gravity, living in four-dimensional spacetime. Yet Maldacena argued that these two theories were, in fact, different descriptions of the same underlying physics. More precisely, he conjectured that these two theories are exactly equivalent descriptions of the same physical system.

The implications of this conjecture were staggering. It provided a concrete realization of the holographic principle, showing how a theory of gravity in a higher-dimensional space could be entirely equivalent to a non-gravitational theory living on its boundary. This was the proverbial cosmic Rosetta Stone, allowing us to translate between the language of gravity and the language of quantum fields.

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<sup>5</sup>In the author’s opinion, the best review remains the canonical one: [2].

## The AdS/CFT Dictionary

Maldacena’s conjecture sparked a flurry of activity as physicists worked to flesh out the details of this correspondence. The result was the AdS/CFT dictionary, a set of precise relationships between quantities in the bulk AdS space and their counterparts in the boundary conformal field theory (CFT). This dictionary is a thing of beauty, revealing deep connections between seemingly unrelated fields, and it is worth detailing some of its salient features.

- **The GKPW formula:** Each field  $\varphi$  in the bulk AdS space corresponds to an operator  $\mathcal{O}$  in the boundary CFT. More specifically, the boundary value ( $\varphi_0$ ) of the bulk field  $\varphi$  acts as a source for the operator  $\mathcal{O}$ . This is summarized by the Gubser–Klebanov–Polyakov–Witten (GKPW)[164, 65] relation, whose importance cannot be overstated:

$$e^{W_{\text{CFT}}[\varphi_0]} = \left\langle \exp \left( \int \varphi_0 \mathcal{O} \right) \right\rangle_{\text{CFT}} = Z_{\text{gravity}}[\varphi|_{\partial \text{AdS}} = \varphi_0]. \quad (4)$$

- **Symmetries:** The isometries of  $AdS_5 \times S^5$  correspond to the conformal and  $R$ -symmetries of the  $\mathcal{N} = 4$  SYM theory. This correspondence of symmetries is a key check on the duality and provides insight into how the symmetries of spacetime emerge from the symmetries of the quantum field theory.
- **Parameters:** The parameters of the two theories are related in precise ways:  $\lambda = g_{\text{YM}}^2 N = (L/\ell_s)^4$ , where  $\lambda$  is the ’t Hooft coupling,  $L$  is the AdS radius, and  $\ell_s$  is the string length.  $G_N \sim L^8/N^2$ , where  $G_N$  is Newton’s constant in the bulk.

### 0.1.5 Highlights of holography

Since the discovery of the AdS/CFT correspondence, the field of holography has seen remarkable progress, with numerous groundbreaking discoveries and applications. Here, we highlight some of the most significant developments<sup>6</sup> that have shaped our understanding of holography and its implications for physics.

### Holographic entanglement entropy

One of the most profound connections to emerge from holography is the relationship between quantum entanglement in the boundary theory and geometry in the bulk. In 2006, Shinsei

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<sup>6</sup>Any list of this nature comes with significant bias. This one is no different.

Ryu and Tadashi Takayanagi proposed a formula that relates the entanglement entropy of a region in the boundary CFT to the area of a minimal surface in the bulk AdS space:

$$S(A) = \frac{\text{Area}(\gamma_A)}{4 G_N}, \quad (5)$$

where  $S(A)$  is the entanglement entropy of a region  $A$  in the CFT,  $\gamma_A$  is the minimal surface in the bulk whose boundary coincides with the boundary of  $A$ , and  $G_N$  is Newton’s constant in the bulk<sup>7</sup>.

This remarkable result suggests that spacetime itself might emerge from quantum entanglement, a concept that later led to the idea that “entanglement equals geometry” or the “ER = EPR” proposal, put forth by Juan Maldacena and Leonard Susskind.

The RT formula has been generalized to include quantum corrections, leading to the Quantum Extremal Surface (QES) formula [50, 46]:

$$S(A) = \min_{\gamma} \left[ \frac{\text{Area}(\gamma)}{4 G_N} + S_{\text{bulk}}(\gamma) \right] \quad (6)$$

where  $S_{\text{bulk}}(\gamma)$  is the entanglement entropy of bulk quantum fields in the region enclosed by  $\gamma$ . This quantum-corrected version plays a crucial role in recent discussions of the information paradox, hinting at how information can escape from black holes in a manner consistent with unitary evolution.

## Holographic Quantum Error Correction

Building on the ideas of holographic entanglement, in 2015, Ahmed Almheiri, Xi Dong, and Daniel Harlow [9] proposed that the bulk spacetime is encoded in the boundary CFT in a redundant way, similar to how information is protected in a quantum error correcting code. Specifically, they proposed that the mapping between bulk and boundary states can be understood as an isometry  $V : \mathcal{H}_B \rightarrow \mathcal{H}_C \otimes \mathcal{H}_{\psi}$ , where  $\mathcal{H}_B$  is the bulk Hilbert space,  $\mathcal{H}_C$  is the code subspace of the boundary Hilbert space, and  $\mathcal{H}_{\psi}$  represents the complementary degrees of freedom.

The error correction perspective provides a natural explanation for several puzzling features of AdS/CFT, such as the ability to reconstruct bulk operators from different boundary regions (complementary recovery) and the absence of bulk operators that commute with all local boundary operators (no bulk locality). Recent work has extended these ideas to

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<sup>7</sup>Notice that this immediately generalizes 2 for these spacetimes.

subregion duality, where a portion of the bulk can be reconstructed from a subset of the boundary. This has led to the concept of “entanglement wedges” and their role in bulk reconstruction, deepening our understanding of the structure of the bulk–boundary map.

## Applications to Condensed Matter Physics

The power of holography extends far beyond formal theory and has found surprising applications in condensed matter physics, particularly in the study of strongly correlated systems. The AdS/CFT correspondence also provides tools for studying quantum critical points and non-Fermi liquids, systems that are notoriously difficult to analyze using traditional methods. Key developments include:

1. **The KSS bound and fluid/gravity:** The universal ratio of shear viscosity to entropy density ( $\eta/s = \hbar/4\pi k_B$ ) for strongly coupled theories with gravity duals, conjectured in 2004 as a lower bound for all fluids [100]. This insight gained further significance with the subsequent development of the fluid–gravity correspondence, which demonstrated that Einstein equations in bulk AdS space reduce to Navier–Stokes equations on the boundary in the long–wavelength limit [25]. This correspondence provides a systematic method to compute transport coefficients of strongly coupled field theories using relatively straightforward gravitational techniques, offering a broader theoretical framework that helped explain the universality observed in the KSS bound.
2. **Strange metals:** Strange metals are a class of materials that exhibit unusual electronic properties that fall outside the paradigm of Fermi liquid theory [41, 106]. Constructing gravitational duals of these systems and considering their low–energy behaviour explains many of their unusual characteristics, such as linear–in–temperature resistivity and anomalous scaling of their spectral functions [74, 48].
3. **Quantum many-body systems:** Applying holographic techniques to study non-equilibrium phenomena in quantum many–body systems [139], including quantum phase transitions, quantum quenches, and thermalization processes [14, 107].
4. **High-temperature superconductors:** Introduced in [66, 72, 71], these models describe a superconducting phase transition in the boundary theory by the formation of a charged scalar hair on a black hole in the bulk. While these models may not capture all details of real–world superconductors, they have provided valuable insights into universal aspects of these systems, such as the ratio of the energy gap to the critical temperature, which are in surprising agreement with experimental results for certain classes of high– $T_c$  superconductors.

## The information paradox

Unsurprisingly, holographic approaches have offered crucial insights into the black hole information paradox. The AdS/CFT correspondence suggests that black hole evolution should be unitary when viewed from the perspective of boundary theory, implying information preservation. This has led to concrete proposals for how subtle correlations in Hawking radiation might encode information about the black hole’s interior.

Recent holographic calculations have successfully reproduced the Page curve for black hole evaporation, providing strong evidence for information preservation. These calculations involve concepts like “islands” in the gravitational bulk, offering new perspectives on how information escapes from black holes. The holographic quantum error correction framework has further elucidated how bulk information can be redundantly encoded on the boundary, suggesting mechanisms for how the black hole interior might be encoded on its horizon in a way that is robust against local perturbations.

Despite these advances, challenges remain. While holography strongly suggests that information is preserved, the precise details of how information is encoded in Hawking radiation and how it can be extracted are still not fully understood. Additionally, extending these insights to more realistic scenarios beyond Anti-de Sitter space remains an active area of research.

### 0.1.6 Open questions

Holography in string theory represents one of the most profound and far-reaching ideas in modern theoretical physics. From its origins in the study of black hole thermodynamics to its realization in the AdS/CFT correspondence and beyond, the holographic principle has continuously challenged and expanded our understanding of the most fundamental aspects of reality.

Yet, for all its successes, holography also reminds us of how much we still have to learn. The open questions in the field point to fundamental gaps in our understanding of quantum mechanics, gravity, and their marriage. Here are some of them:

- **Bulk reconstruction:** How exactly does local bulk physics emerge from the boundary theory? Can we reconstruct the experience of an observer falling into a black hole from the CFT description? Recent work on quantum error correction and tensor networks has provided some insights, but a complete understanding remains elusive.

- **Singularity resolution:** Can holography help us resolve the long-standing puzzles about spacetime singularities, both in cosmology and black hole physics? What happens at the center of a black hole in the holographic description?
- **Non-perturbative quantum gravity:** Can we find a complete, non-perturbative formulation of quantum gravity through holographic principles? What are the fundamental degrees of freedom from which spacetime emerges?
- **Entanglement and geometry:** While the connection between entanglement and geometry has been established, many questions remain. How exactly does geometry emerge from entanglement? Can all aspects of spacetime be understood in terms of quantum information?
- **Holography beyond AdS/CFT:** Are there other realizations of the holographic principle beyond AdS/CFT? How general is the holographic principle, and what are its limits? Can we extend holographic ideas beyond AdS to more realistic spacetimes, including our own universe? What would a holographic description of de Sitter space or asymptotically flat spacetimes look like?

As we continue to explore holography, we may find that the answers to these questions lead us to an even deeper understanding of the nature of space, time, and quantum mechanics.

The journey of holography is a testament to the power of theoretical physics to uncover deep truths about the universe. It shows how seemingly abstract mathematical ideas can lead to profound physical insights, and at the same time, how deeply unified all of physics is. It is for this reason, as we look to the future, beyond the ideas explored in this thesis, holography promises to remain at the forefront of theoretical physics. Its ideas and techniques will likely play a crucial role in our quest for a theory of quantum gravity and a deeper understanding of the universe at its most fundamental level.

The holographic principle suggests that the universe may be far stranger and more interconnected than we ever imagined. As we continue to unravel its implications, we may find that the true nature of reality is more beautiful and surprising than we ever dared to dream.

## 0.2 This thesis

Having embarked on this grand adventure exploring the development and history of holography, we now return to the central ideas of this thesis. Here, we provide a brief guide, and summarise our main results.

This thesis is divided into two parts.

### ***Opus I: Microstate counting and the $c = 1$ matrix model***

The first part of this thesis deals with counting and identifying the microstates of black holes. In §1, we identify the microstates of the non-supersymmetric, asymptotically flat  $2d$  black hole in the dual  $c = 1$  matrix quantum mechanics. Our central result is a novel bound state that can be considered an explicit dual of the black hole microstates that dominates the partition function in the double-scaling limit. Pleasingly, this bound state is long-lived and evaporates slowly, exactly like a black hole in asymptotically flat space would. This is based on work that appeared in [3], and was done in collaboration with Dr. Amr Ahmadain, Dr. Ronak Soni, and Alexander Frenkel.

### ***Opus II: Open EFTs from holography***

The second part of this thesis is dedicated to the study of open effective field theories of fermions interacting with holographic baths.

In §2, we begin by making the case for using holography to study open effective field theories. We then introduce the *gravitational Schwinger–Keldysh saddle* (grSK saddle), and explain why they are particularly well-suited to the study of open effective theories. We then detail the holographic dictionary for these geometries, with a focus on fermions.

In §3, we study Dirac fermions propagating in these backgrounds, solving for both their ingoing (quasi-normal) and outgoing (Hawking) modes in a boundary gradient expansion. Rather beautifully, we find that the holographic Schwinger–Keldysh boundary conditions automatically select a unique combination of these modes consistent with fermionic KMS relations. Evaluating the bulk on-shell action using these solutions, we obtain an explicit expression for the influence phase of the probe fermion in a gradient expansion. This is based on work that appeared in [110], and was done in collaboration with Dr. Akhil Sivakumar and Prof. R. Loganayagam.

We then extend the holographic Schwinger–Keldysh prescription to charged black branes in §4, with a view towards studying systems held at finite temperature and finite chemical potential. We check our prescription using charged Dirac probe fields, solving the Dirac equation in a boundary derivative expansion. Once again, we find that the correlators derived using this prescription automatically satisfy the appropriate KMS relations with Fermi–Dirac factors, including the chemical potential. This is based on work that appeared in [109] and was done in collaboration with Dr. Akhil Sivakumar, Shivam K Sharma and Prof. R. Loganayagam.

*Opus I*

# Chapter 1

## Microstates of the $2d$ black hole.

### 1.1 Motivation

The fact that black holes have entropy has perhaps been the most significant discovery in the study of quantum gravity. It is a piece of UV physics that we can observe in the IR. In the 40 years since this discovery, however, there have been remarkably few examples in which we understand the microstates that make up this entropy. Examples in which they are understood in the bulk description are extremal maximally supersymmetric black holes — see [153] and its many follow-ups. There have been other proposals for how the entropy might be carried by gravitational degrees of freedom, either through semi-classical “shell” configurations [15] or stringy replacements of the classical geometry [120], though explicitly constructing and counting these states remains challenging. There are also a larger class of examples where there is some handle in a holographic description, e.g. BTZ black holes (via the Cardy formula) [152], theories that admit quantum mechanical descriptions [12, 32, 43],  $4d \mathcal{N} = 4$  SYM [31, 39, 21],  $3d$  holographic theories [38], etc. Even in this second list, many examples are supersymmetric and extremal.

This chapter adds an example to the second list. This is a two-dimensional, asymptotically flat black hole [117] with no supersymmetry that is held at a finite temperature. The caveat, however, is this black hole is non-standard in that the physics in this background is ‘stringy’ and its temperature cannot be varied. A lot is known about this solution — this two-dimensional black hole is dual to a one-dimensional matrix quantum mechanics (MQM) system. The finite-temperature partition function of this MQM has been calculated [91, 93] in the path integral formulation, and two candidates for the entropy and energy

were found based on this path integral result in [92]. We review the salient features of this model in §1.2.

We aim to go a step further than reproducing the Euclidean partition function in this example — rather than simply reproducing the entropy (a count of the microstates), we explicitly *list* the black hole microstates. By this, we mean the following: The partition function can of course be calculated by a trace over a Hilbert space,<sup>1</sup>

$$Z_{\text{MQM}}(\beta) = \text{tr} e^{-\beta H} = \sum_{E \in \text{spec}(H)} e^{-\beta E}. \quad (1.1)$$

This partition sum, in the thermodynamic limit, is typically dominated by states of approximately equal energy and can be written in the thermodynamic form,

$$Z_{\text{MQM}}(\beta) \approx e^{S(E)} e^{-\beta E} = \text{tr}_{(H=E)} e^{-\beta E}, \quad (1.2)$$

where  $S(E)$  is the entropy of the black hole. We have also rewritten this as a trace over a restricted Hilbert space. ‘Listing’ the microstates of the black hole then, is to find another characterization of this Hilbert space that doesn’t boil down to simply saying “all states with the right energy and other charges.” There is also a subtlety that these are really microstates of black hole spacetimes rather than black holes themselves; as we will see, we must be careful to disambiguate the microstates of the black hole from states describing exterior degrees of freedom. It should be noted that by this standard, only in some extremal supersymmetric cases like [153] (and its follow-ups) have the microstates been listed; in the  $\mathcal{N} = 4$  case, this has been done in [30].

In what follows, we describe this restricted Hilbert space in the dual MQM system. The MQM has a  $PU(N)$  global symmetry, and the restricted Hilbert space turns out to be a particular set of non-singlet irreducible representations of this symmetry, as first pointed out in [63] (see [115, 55, 23] for other explicit investigations of the importance of these non-singlet irreps). The set of irreps is determined by the mass of the black hole which, as just mentioned, is related to the chemical potential. As we show in §1.3, the dimension of this restricted Hilbert space matches one of the two candidates for the entropy found in [92]. However, we also find that this entropy is the count of IR degrees of freedom that decouple from bulk dynamics in the double-scaling limit. We argue that this decoupling can be thought of as a ‘renormalisation’ in the double-scaling limit in §1.5.1, and provide

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<sup>1</sup>It should be noted that the relevant ensemble in our case is grand canonical rather than canonical, and so (1.1) and (1.2) are only schematic.

a (speculative) pictorial description in §1.6.3. We thus argue that the actual entropy of the black hole is a value much smaller than found in [92].

To actually describe the microstates of the black hole, we have to find the state of the degrees of freedom that do participate in the bulk dynamics. We analyse the Schrödinger equation to find this state, and find explicitly the state that dominates the grand canonical partition function. It is a novel bound state that we call the ‘*hole-in-the-world*’ solution, and we discuss this in §1.4.2. We argue that the properties of this state *within quantum mechanics* match properties expected from the bulk dual in §1.6.1.

The plan of this chapter is as follows:

1. In §1.2, we review the duality between the black hole and the MQM system and what is known about the free energy of the black hole.
2. In §1.3, we show that the degeneracy due to the  $PU(N)$  symmetry of the MQM exactly reproduces one of the two candidates for the entropy found in [92]. As we argue later in §1.5.2, while this points us the correct prescription, this prediction for the entropy is incorrect.
3. The main section, §1.4, reproduces the energetic term in the partition function.  
In §1.4.2, we identify an interesting state, which we call the ‘hole-in-the-world’ state, with the right energy. We argue that it dominates the fixed-charge partition function in the double-scaling limit in §1.4.3.
4. The contents of §1.5 deal with the grand canonical partition function calculated in [93, 91]. In §1.5.1, we address whether the  $PU(N)$  symmetry should be gauged or not, and argue that the passage from the ungauged to the gauged quantum mechanics can be thought of as renormalisation in the double-scaling limit. Finally, we comment on the entropy of the black hole in §1.5.2, and argue that its entropy is much smaller than that found in [92].
5. We conclude with some speculations and discussion in §1.6. After a broad overview of our results and their limitations, we discuss how the hole-in-the-world state matches expectations about bulk physics in §1.6.1. §1.6.2 talks about the implications on the phase structure of the theory. Finally, we provide a stringy picture for our analysis in terms of the stretched strings of [115] in §1.6.3 and end by discussing a surprising connection between our counting of states and Motzkin walks in §1.6.4.

**Some notation:** For this chapter alone, we use  $\iota$  for the imaginary unit. The indices  $i, j, k, l$ , etc. will be exclusively used to denote elements of  $\{1 \dots N\}$ .

## 1.2 A review of the duality

The system we work with is the two-dimensional black hole with asymptotically flat boundary conditions [117, 163]. It was argued in [91, 93] that this is dual to a quantum mechanical system with a single matrix degree of freedom. Some introductions to this set of dualities that we have found useful are [96, 59, 135, 8, 144, 141, 88]; a relatively short review is [91]. In this section, we outline the various models in this chain of dualities and summarise the main results of [92].

The cigar is the two-dimensional Euclidean black hole geometry specified by the metric and dilaton,

$$ds^2 = k \alpha' d\rho^2 + \tanh^2 \rho dT^2, \quad T \sim T + 2\pi\sqrt{k \alpha'}$$

$$\text{and} \quad \Phi = \Phi_0 - \log \cosh \rho. \quad (1.3)$$

As a manifold, this is the group coset  $\frac{SL(2, \mathbb{C})}{SU(2)} / U(1)$ . The inverse temperature and mass of this black hole are

$$\beta = 2\pi R \equiv 2\pi\sqrt{k \alpha'}, \quad M = \frac{1}{\sqrt{k \alpha'}} e^{-2\Phi_0}. \quad (1.4)$$

Henceforth, apart from some crucial equations, we will set  $\alpha' = 1$ .

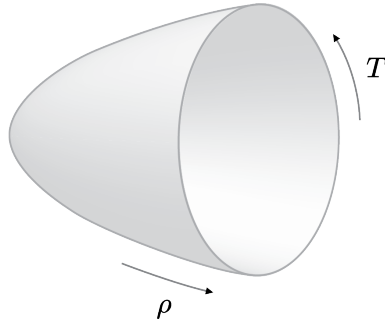


Figure 1.1: The cigar background

Consider a string propagating on this background,<sup>2</sup> with worldsheet action

$$I_{\text{ws}} = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma [G_{\mu\nu}(X) \partial X^{\mu} \bar{\partial} X^{\nu} + \mathcal{R}_{\Sigma}(X) \Phi(X)] + I_{\text{WZW}}. \quad (1.5)$$

Here,  $G_{\mu\nu}$  is the metric of the cigar, and  $I_{\text{WZW}}$  is a Wess–Zumino–Witten term.  $\mathcal{R}_{\Sigma}$  is the Ricci scalar of the worldsheet. The constant  $k$  is fixed by the requirement that the central charge of this CFT be 26,

$$c_{\text{cigar}} = \frac{3k}{k-2} - 1 = 26 \quad \Rightarrow \quad k = \frac{9}{4}. \quad (1.6)$$

This means that the  $2d$  string theory has a black hole with a fixed temperature given by

$$\beta = 3\pi\sqrt{\alpha'}, \quad R = \frac{3}{2}\sqrt{\alpha'}. \quad (1.7)$$

However, its mass can be varied freely.

### 1.2.1 From the cigar to the MQM

This black hole can be described by a matrix quantum mechanics (MQM) through a series of dualities.

1. By FZZ duality [91, 93, 115], the cigar CFT is dual to the sine–Liouville model.
2. The sine–Liouville model is the zero cosmological constant limit of the sine–Gordon model coupled to Liouville gravity.
3. The sine–Gordon model coupled to Liouville gravity is dual to an MQM with twisted boundary conditions in a double–scaling limit.

The sine–Liouville model is the string theory with worldsheet action

$$I_{\text{sL}} = \frac{1}{4\pi\alpha'} \int_{\tilde{\Sigma}} \left[ \left( \partial \tilde{T} \right)^2 + (\partial \phi)^2 + Q \mathcal{R}_{\tilde{\Sigma}} \phi + \mathfrak{z} e^{b\phi} \cos R\tilde{T} \right], \quad \tilde{T} \sim \tilde{T} + 2\pi \frac{\alpha'}{R} \quad (1.8)$$

The FZZ duality provides an identification between the fields and parameters in the sine–Liouville and cigar CFTs. The fields  $\Phi$  and  $\phi$  can be identified in the region  $\Phi, \phi \gg 1$

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<sup>2</sup>It is clear that the cigar and the Lorentzian black hole as metric manifolds are related by an analytic continuation. However, the relation between the string theories defined on the two backgrounds is rather more subtle, as explained for example in [61].

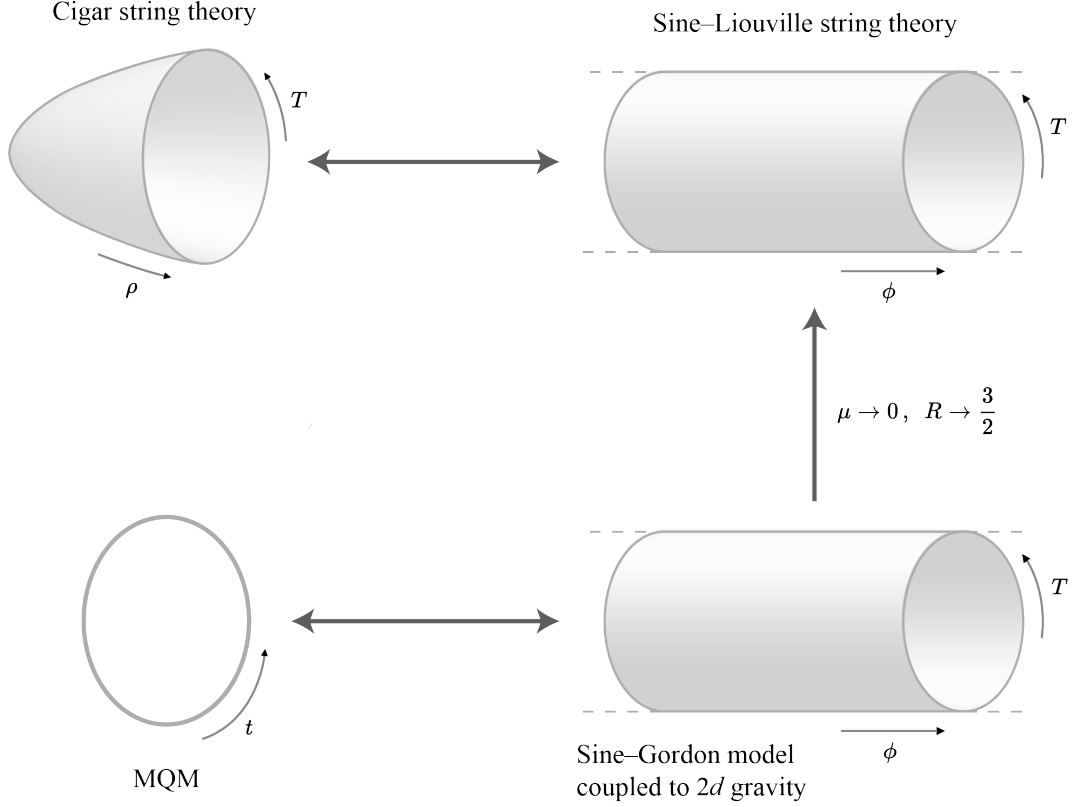


Figure 1.2: An illustration of the series of dualities that relate the cigar to the MQM.

(where the background is approximately just flat space with a linear dilaton, i.e. the weak-coupling region). The fields  $T$  and  $\tilde{T}$  are  $T$ -duals of each other;  $\mathfrak{z}$  is a chemical potential for momentum  $\pm 1$  excitations in the  $\tilde{T}$  direction, and therefore winding  $\pm 1$  excitations in the  $T$  direction — its value is set by the mass  $\mathfrak{z} \propto M^{(2-R)/4}$ . The central charge of this theory is [143]

$$c_{\text{sL}} = 2 + 6Q^2. \quad (1.9)$$

The Liouville-field dressing of any operator in the action with conformal dimension  $\Delta$  satisfies

$$b(Q - b) + \Delta = 1. \quad (1.10)$$

The undressed sine-Liouville operator  $\cos R\tilde{T}$  has conformal dimension  $\Delta = R^2$ . Matching central charges across the FZZ duality then gives

$$Q = -\frac{1}{b} = \frac{1}{\sqrt{k-2}} \xrightarrow{R=\frac{3}{2}} 2. \quad (1.11)$$

Unlike the cigar, the target manifold in this case is an infinite cylinder, implying an ER =

EPR duality [88].

An important and non-obvious point about the sine-Liouville theory is that it cannot be studied in perturbation theory in  $\mathfrak{z}$ . This is because any non-zero  $\mathfrak{z}$  can be rescaled to an  $\mathcal{O}(1)$  number by shifting the dilaton  $\phi$ ,

$$\mathfrak{z}e^{b(\phi-\phi_0)} = \tilde{\mathfrak{z}}e^{b\phi}, \quad \tilde{\mathfrak{z}} \equiv \mathfrak{z}e^{-b\phi_0}. \quad (1.12)$$

This shift also acts on the  $Q\mathcal{R}\phi$  term, and produces a topological term that can be thought of as the string coupling. Thus, the sine-Liouville CFT is not perturbatively related to Liouville theory, and it is precisely this fact that makes it difficult to study.

There are two important differences between the sine-Liouville and cigar CFTs. Firstly, the sine-Liouville CFT is on-shell (has  $c_{\text{sL}} = 26$ ) for all  $R$  as long as  $Q = 2$ . Secondly,  $k \rightarrow 2$  is a strong-coupling limit in the cigar CFT but a weak-coupling limit in the sine-Liouville (since the dressing parameter,  $b$  tends to 0 in this limit). Since  $k = 9/4$ , we henceforth stick to the sine-Liouville side of the duality.

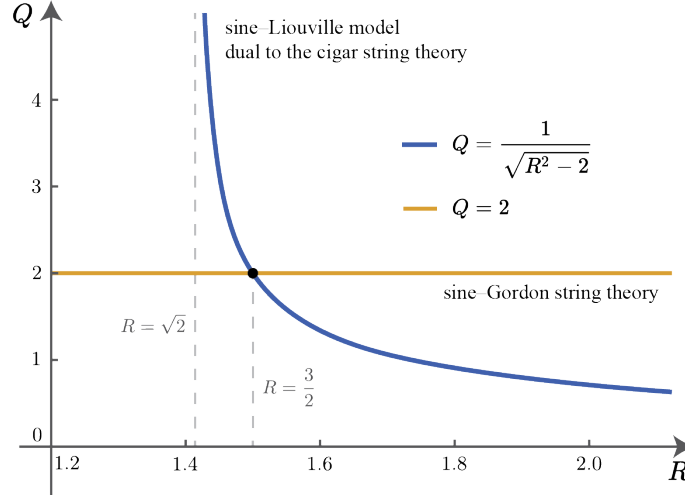


Figure 1.3: The sine-Liouville theories dual to the cigar and the sine-Gordon theory that can non-anomalously be a worldsheet theory only intersect at  $R = 3/2$ , when the cigar is on-shell.

The sine-Gordon model coupled to gravity defines a string theory given by the worldsheet action

$$I_{\text{sG}} = \frac{1}{4\pi\alpha'} \int_{\tilde{\Sigma}} \left[ (\partial T)^2 + (\partial\phi)^2 + Q\mathcal{R}_{\tilde{\Sigma}}\phi + \mu e^{-2\phi} + \mathfrak{z}e^{(R-2)\phi} \cos R\tilde{T} \right]. \quad (1.13)$$

Here, we have  $T$ -dualised again to the  $T$  field in the kinetic part of the action, and added

a cosmological constant for the Liouville sector. The  $\mu \rightarrow 0$  limit of this theory is the sine–Liouville model.<sup>3</sup> The  $\mathfrak{z} \rightarrow 0$  limit of this theory is known as  $c = 1$  string theory. Once again, because of the presence of the dilaton, the manner in which we take these perturbative limits is subtle. In sine–Gordon string theory, perturbation theory in  $\mathfrak{z}$  is valid where  $\mathfrak{z} \ll \mu$ , since the zero–mode of  $\phi$  can’t be shifted while keeping  $\mu$  constant. However, we are interested in the opposite limit, where perturbation theory in  $\mu$  is valid.

The sine–Gordon model is dual to a quantum mechanical model with one Hermitian matrix degree of freedom,

$$I_{\text{MQM}} = \int dt \operatorname{tr} \left[ \dot{X}^2 - V(X) \right], \quad V(X) = -\frac{1}{2}X^2 + \frac{g}{N}X^4. \quad (1.14)$$

A significant portion of the early literature on the subject uses a potential of the form,  $V(X) = -X^2/2 + g/\sqrt{N}X^3$ ; we use the quartic potential following the discussion in [141] (see [45, 156] for the supersymmetric versions).

This action has a  $PU(N) = U(N)/U(1) = SU(N)/\mathbb{Z}_N$  global symmetry given by

$$X(t) \rightarrow U^\dagger X(t) U. \quad (1.15)$$

Notice that the above transformation is the same for  $U$  and  $e^{i\phi}U$ , meaning that the symmetry group is not the unitary group  $U(N)$  but the *projective* unitary group  $PU(N)$  — as indicated, this is nothing but  $U(N)$  quotiented by the overall phase mode. In terms of  $SU(N)$ ,  $PU(N)$  is the quotient of  $SU(N)$  by its  $\mathbb{Z}_N$  centre, which exists because a phase rotation by  $e^{ij2\pi/N}$ ,  $j = 0, 1 \dots N-1$  has determinant 1. As an example,  $PU(2) = SU(2)/\mathbb{Z}_2 = SO(3)$ .

Some authors, like [115, 23, 24], gauge this symmetry while others, like [28, 91], don’t. We will keep this symmetry global for the bulk of this paper; in §1.5.1 and §1.6.3, we argue that there is nevertheless a sense in which the gauging ‘emerges.’

With these prerequisites in place, we are now in a position to precisely state the duality:

$$e^{-Z_{\text{sG}}(R, \mu, \mathfrak{z})} = \sum_{N=0}^{\infty} e^{2\pi R \mu_F N} \int_{PU(N)} d\Omega e^{N \mathfrak{z} b \operatorname{Re}[\operatorname{tr} \Omega]} \int_{X(\beta) = \Omega^\dagger X(0) \Omega} e^{-\int_0^\beta dt L(X, \dot{X})} \Big|_{\text{dsl}}. \quad (1.16)$$

The subscript “dsl” is short for double–scaling limit.  $Z_{\text{sG}}$  is the vacuum partition function of the sine–Gordon model and  $e^{-Z_{\text{sG}}}$  is the partition function of the corresponding string

<sup>3</sup>Up to the fact that the dilaton dressing for the vortex term is now  $\exp[(R-2)\phi]$  instead of  $\exp[-\sqrt{R^2-2}\phi]$ . It is easily checked that they agree at  $R = 3/2$ . The two different expressions correspond to plugging in either  $Q = 2$  or  $Q = 1/\sqrt{k-2}$  into (1.10).

field theory.  $\mu_F$  is a chemical potential for the size of the matrix, and  $\mathfrak{z}_b$  for the non-trivial representations of  $PU(N)$ . At  $\mathfrak{z}_b = 0$ , the  $\Omega$  integral localises the MQM to the singlet sector of this symmetry — this is dual to  $c = 1$  string theory, which is just the sine-Gordon theory at  $\mathfrak{z} = 0$ .  $\mu_F$  is related to the Liouville cosmological constant,  $\mu$ , in (1.13), and  $\mathfrak{z}_b$  to the fugacity for vortices,  $\mathfrak{z}$ ; the precise relations involve some renormalisations.

Importantly, the duality only emerges in the *double-scaling limit*, which can be thought of as a specific way of taking  $N \rightarrow \infty$ . There are two different but equivalent pictures for this limit, one based on Feynman diagrams and the other based on the Fermi statistics of the eigenvalues of these matrices. We outline the main points of how this is to be understood; more details can be found in the reviews listed at the beginning of this section.

In classic large  $N$  fashion, each Feynman diagram can be interpreted as a triangulation of a  $2d$  Riemann surface. Since the expansion in the coupling constant  $g$  has zero radius of convergence,<sup>4</sup> the series is asymptotic; near a critical value  $g_c$  diagrams with  $\mathcal{O}\left(\frac{1}{g_c - g}\right)$  vertices dominate the sum. The double-scaling limit is the limit  $g \rightarrow g_c, N \rightarrow \infty$ , with  $N(g_c - g)$  kept finite. In this limit, each diagram becomes a worldsheet and the sum over diagrams becomes a sum over worldsheets; the constant parameter (after renormalisation) becomes the Liouville cosmological constant  $\mu$ . This is illustrated in figure 1.4a.

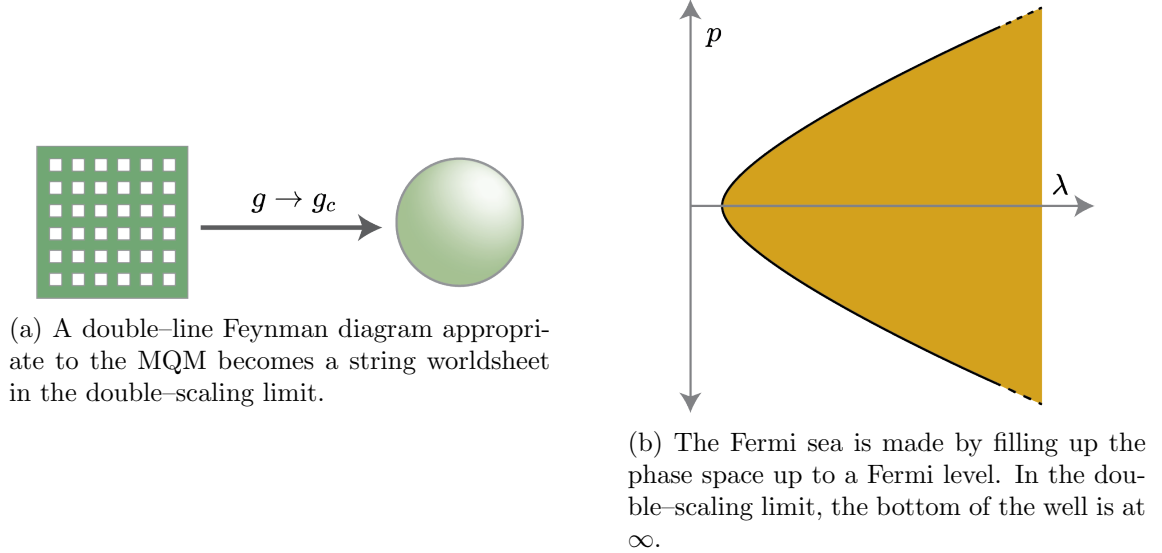


Figure 1.4: The double-scaling limit.

The other equivalent picture for the double-scaling limit is as follows. In the ground state, because of the  $PU(N)$  symmetry, we can focus on only the eigenvalues of the matrix  $X$ . As we explain in §1.4.1, because of a Vandermonde determinant, these eigenvalues are fermionic. In the vacuum, the  $N$  fermions fill up the potential up to a Fermi level, see figure

<sup>4</sup>This is most easily seen by considering the case  $g < 0$ .

1.4b. The Fermi level grows monotonically with  $g$  up to a value  $g_c$  where it becomes equal to the local maximum at 0. We take the  $g \rightarrow g_c, N \rightarrow \infty$  keeping the Fermi level at some finite value  $-\mu_F \sim \mathcal{O}(N^0)$ . This  $\mu_F$  is the same as the chemical potential in (1.16), by standard free fermion arguments. This free fermion theory becomes the dual string *field* theory, with the direction given by the log of the eigenvalue being identified with the target space spatial direction  $\phi$  (i.e. dilaton), up to a non-local transformation that won't be important to us [144].

The renormalisation is controlled by Khnizhnik–Polyakov–Zamolodchikov (KPZ) [98] dimensions; in particular, the sine–Liouville fugacity and the MQM fugacity are related as

$$\mathfrak{z} = \mathfrak{z}_b N^{\frac{2-R}{2}}. \quad (1.17)$$

The relation between  $\mu$  and  $\mu_F$ , which are both renormalised quantities, is a non-universal  $\mathcal{O}(1)$  factor. Since we will carry out an MQM calculation, an important consistency check for us will be that, along with this renormalisation, the sine–Liouville answer be independent of  $N$ .

### 1.2.2 The free energy and thermodynamics

The matrix partition function can be calculated explicitly [91, 93], and the piece in the free energy corresponding to the genus 0 partition function of the sine–Liouville model is

$$\mathcal{F}^{(0)} = -\frac{1}{4} (2-R)^2 \mathfrak{z}^{\frac{4}{2-R}}. \quad (1.18)$$

Here, the mass is  $M \sim \mathfrak{z}^{\frac{4}{2-R}}$  — it is taken to be large to suppress higher genus effects. On the face of it, this is surprising, since it is a non-zero sphere partition function in string theory. Usually, this vanishes because of the infinite volume of the conformal Killing group,  $SL(2, \mathbb{C})$ , on the sphere. In the cigar case, however, this result indicates that this volume cancels with the volume of the  $SL(2, \mathbb{C})$  factor in the quotient [93].

There are two different approaches using this result to understand the thermodynamics, as Kazakov and Tseytlin (henceforth, KT) argue [92]. Following the conventions laid out in their paper, we will refer to them as the first and second interpretations.

### The first interpretation

1. The MQM free energy is the free energy corresponding to a grand canonical ensemble for a dilaton current. The non-zero value of the free energy corresponds entirely to the charge term [57] and is proportional to  $(R - 3/2)$ .<sup>5</sup>
2.  $R$  is varied while keeping the mass constant (i.e. varying  $\mathfrak{z}$  as a function of  $R$ ).

This approach gives a free energy,

$$-\beta F_1(R, M) = 2\pi \left( \frac{3}{2} - R \right) M - \frac{R + R^{-1}}{24} \log M. \quad (1.19)$$

The entropy–energy relation is then,

$$S = 3\pi M. \quad (1.20)$$

The vanishing of the leading term in the free energy is attributed to a Hagedorn phase transition at this temperature, that exists for reasons similar to [133].

### The second interpretation

1. The free energy is given by (1.18), without any subtractions.
2. The parameter  $\mathfrak{z}$  in the sine–Liouville model is interpreted as a chemical potential for vortices. Thus  $R$  is varied while keeping  $\mathfrak{z}$  fixed, letting  $M$  vary as it wills. This is a Gibbons–Hawking–type variation.

Given this, if we now consider the partition function for an ensemble of fixed, but equal number of vortices and anti–vortices (required to keep the charge neutral), we see that this can be represented by the contour integral given by:

$$\exp \left( -\beta F_2(\mathbb{k}) \right) = \frac{1}{2\pi i} \oint \frac{d\mathfrak{z}}{\mathfrak{z}^{\mathbb{k}+1}} \mathcal{F}^{(0)}. \quad (1.21)$$

where  $\mathcal{F}^{(0)}$  is the sphere partition function that we previously defined, 1.18. Performing a saddle point computation, which is justified for large  $\mathbb{k}$ , we get,

$$-\beta F_2(R, \mathbb{k}) = \left( 2 - \frac{\beta}{2\pi} \right) \mathbb{k} \log \frac{\Lambda \alpha'}{\sqrt{\mathbb{k}}} + \mathcal{O}(\mathbb{k}). \quad (1.22)$$

---

<sup>5</sup>Equivalently, we can subtract off the vacuum free energy as in [92].

Here,  $\Lambda$  is a UV cutoff on the worldsheet with dimensions of inverse length, and  $\mathbb{k}$  is related to  $\mathfrak{z}$  and the mass (at leading order) as

$$\mathbb{k} \approx \mathfrak{z}^{\frac{4}{2-R}} \propto M. \quad (1.23)$$

This is finite but large in the sine-Liouville theory, and so we have  $1 \ll \mathbb{k} \ll N$  in the dual MQM. For energy and entropy (setting  $\alpha' = 1$  again), it is natural to interpret the  $\beta$ -independent term with the entropy and the  $\beta$ -dependent term with the mass. This gives,

$$M = \frac{1}{2\pi} \mathbb{k} \log \frac{\Lambda}{\sqrt{\mathbb{k}}}, \quad S = 4\pi M. \quad (1.24)$$

We also note that we can identify  $\Lambda\alpha' \sim N$  based on the arguments of [140, 99].<sup>6</sup>

The authors of [92] concluded with the remark that the best way to decide between the two interpretations would be to perform a microscopic calculation in the MQM. Note that the difference between the two interpretations is not just a matter of convention or ensemble; the relation between entropy and energy is a statement about the Hilbert space that is either true or not, and we can check which one agrees with the Hilbert space structure. This is what we do in the following sections. We find that the Hamiltonian analysis suggests that the second interpretation is correct. In our analysis, we find a slowly evaporating metastable bound state that looks like a black hole — while the energy of this state matches the KT result, as we detail later, their prediction for the entropy is much too large.

### 1.3 Entropy in the ungauged model

In this section, we reproduce the entropy in the KT result (1.22) by assuming that the  $PU(N)$  symmetry is a global symmetry rather than a gauge symmetry. This entropy was originally calculated in [64], but we clarify some points and fill in some details. We present two calculations, one in §1.3.1 and the other in §1.3.2, and find that they agree. The calculation in §1.3.1 is quick and direct but badly motivated; the calculation presented in §1.3.2, which is a version of the calculation in [64], is slightly longer but better motivated.

While we calculate the entropy of the ungauged MQM in this section, we will eventually argue (in §1.5.2 and §1.6.3) that this entropy should not be considered the entropy of the black hole but of some IR degrees of freedom that decouple in the double-scaling limit.

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<sup>6</sup>[140, 99] find an expression very similar to this for the free energy, with  $\Lambda$  being a worldsheet cutoff. In this case, the worldsheet cutoff is controlled by  $N$ .

In particular, we will argue in §1.5.1 that in the double-scaling limit this symmetry is effectively gauged and the large entropy we calculate here is therefore not relevant to bulk physics. An important intermediate result in this section that retains its importance in the gauged model is the fact that a single irrep, illustrated in figure 1.5, dominates the subspace with a large number of adjoints. This result has also been found in [24].

### 1.3.1 The direct calculation

The direct calculation proceeds based on the simple observation that  $\mathbb{k}$  is the charge conjugate to the fugacity for singlets and non-singlets,  $\mathfrak{z}_b$ . Since the basic representation of the symmetry in the model is an adjoint representation (1.15), the subspace with fixed  $\mathbb{k}$  is the one constructed out of  $\mathbb{k}$  copies of the adjoint representation. The dimension of this reducible representation, remembering to quotient by the symmetric group  $S_{\mathbb{k}}$  because the adjoints are indistinguishable, is

$$\log \dim \left( \mathcal{H}_{\text{adj}}^{\otimes \mathbb{k}} / S_{\mathbb{k}} \right) \approx \log \frac{N^{2\mathbb{k}}}{\mathbb{k}!} \approx 2\mathbb{k} \log \frac{N}{\sqrt{\mathbb{k}}} . \quad (1.25)$$

This is exactly the entropy in (1.22)!

### 1.3.2 Calculation using irreps

While we seem to have reproduced the KT entropy, we have no reason to believe that the subspace whose dimension we just counted above is even approximately degenerate. Exact degeneracy is, however, guaranteed within each irreducible representation of the  $PU(N)$  group. We now proceed break up the fixed  $\mathbb{k}$  subspace into irreps of  $PU(N)$  and compute their dimensions — we will see that we find the same answer. Further, we show that the mistake in [64] that was pointed out by [28] is subleading. Substantial parts of this section are a re-iteration of the work in [64] as well as [118, 28] — we present them here only so that the calculation may be presented as a unified whole.

The irreps of  $U(N)$  are usually given by Young diagrams (YDs) with at most  $N$  rows. Each box corresponds to a copy of the fundamental representation  $\mathcal{H}_f$ ; boxes in the same row are symmetrised and those in the same column are anti-symmetrised. We modify this prescription, following [94, 95, 64], to also allow for ‘*anti*-boxes,’ which correspond to copies of the *anti*-fundamental representation  $\mathcal{H}_{\bar{f}}$ . An example irrep, using green to denote the

anti-boxes, is

$$r_{\text{eg}} = \begin{array}{cccc} \blacksquare & \blacksquare & \blacksquare & \square & \square \\ & \blacksquare & \blacksquare & \square & \\ & & \blacksquare & & \end{array} . \quad (1.26)$$

With these anti-boxes included, the conjugate representation is constructed by mirroring the diagram along the horizontal axis. We can ensure that we are not over- or under-counting by restricting the maximum number of rows to  $N/2$ .<sup>7</sup>

The irreps of  $PU(N)$  are then given by Young diagrams with an *equal* number of boxes and anti-boxes. This equality ensures that the overall phase mode of  $U(N)$  acts trivially in this irrep. So, for example, (1.26) has 3 extra anti-boxes compared to boxes — meaning that it has charge  $(-3)$  under the phase mode, and it is therefore an invalid irrep of  $PU(N)$ . A valid irrep of  $PU(N)$  is

$$r_{\text{eg}}^{PU(N)} = \begin{array}{cccccc} \blacksquare & \blacksquare & \blacksquare & \square & \square & \square & \square \\ & \blacksquare & \blacksquare & \square & \square & & \\ & & \blacksquare & & & & \end{array} . \quad (1.27)$$

As pointed out by [28], the authors of [64] erroneously imposed the stronger condition that the irrep be self-conjugate, i.e. that the YD be symmetric under reflection.

Once again, the group  $PU(2) = SU(2)/\mathbb{Z}_2$  makes for a simple example. In this case, the fundamental is its own conjugate and so anti-boxes and boxes are identical. That there are an equal number of boxes and anti-boxes is simply the requirement that there are an even number of boxes; these are just the integer spin representations of  $SU(2)$ , which are well known to be the representations of  $SO(3)$ .

States in the irrep are given by *Young tableaux* (YTs) — this is a filling in of the boxes with numbers taken from  $\{1, \dots, N\}$  following a set of rules detailed in [94, 95]. Thus,

$$\mathcal{H}_{r_{\text{eg}}} = \text{span} \left\{ \left| \begin{array}{cccc} \blacksquare j_3 & \blacksquare j_2 & \blacksquare j_1 & \square i_1 & \square i_2 & \square i_3 & \square i_4 \\ & \blacksquare j_5 & \blacksquare j_4 & \square i_5 & \square i_6 & & \\ & & \blacksquare j_6 & & & & \end{array} \right. \right. \left. \left. \begin{array}{l} j_a, i_a \in \{1, \dots, N\} \\ \text{following the rules below.} \end{array} \right\} . \quad (1.28)$$

The number in the (anti-) box denotes the corresponding basis element in the (anti-) fundamental representation. The rules can be found in [94, 95]; they follow more or less directly from the fact that two boxes in the same row are symmetrised and two boxes in the

<sup>7</sup>More precisely, we have to restrict the sum of the number of rows and the number of anti-rows to be at most  $N$ . The precise specification then depends on whether  $N$  is even or odd in the obvious way.

same column are anti-symmetrised, along with the requirement that every basis element in the irrep have only one associated YT. The two most important rules are

- $R_1$ : The entries in both boxes and anti-boxes increase from top to bottom.
- $R_2$ : The entries for boxes (anti-boxes) increase from right to left (left to right).

These follow from the fact that different YTs, e.g.  $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$  and  $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}$ , would represent the same state due to the (anti-)symmetrisation rules. Note that these two rules apply separately to the boxes and the anti-boxes.

The new complication due to the presence of both boxes as well as anti-boxes is as follows. The state

$$\sum_{i=1}^N \left| \begin{array}{|c|c|}\hline i & i \\ \hline\end{array} \right\rangle \in \text{singlet}. \quad (1.29)$$

Any fundamental and any anti-fundamental can pair up into such a ‘trace.’ To define a true irrep then, we need to project out such possibilities; [94, 95] use a third rule stating that “If  $r(i)$  and  $r(\bar{i})$  are the lowest rows containing the indices  $i$  and  $\bar{i}$ ,  $r(i) + r(\bar{i}) \leq i$ ” to deal with this subtlety; they exclude enough possible YTs that a trace is never possible but keep enough so that the irrep space has the right dimension. We will not impose this rule, and instead deal with the possibility of traces in a more explicit fashion, aided by the fact that we are working in the limit of large  $N$ .

With all of this in place, let us return to the tensor product of  $\mathbb{k}$  adjoints that we considered in the previous section — this can be broken up into a direct sum of irreps of  $PU(N)$  with at most  $\mathbb{k}$  boxes and  $\mathbb{k}$  anti-boxes. For the moment, let us focus on the YDs with exactly  $\mathbb{k}$  boxes and anti-boxes. In the large  $N$  limit, each of these irreps, in turn, can be written as a product of an irrep  $r_1$  with  $\mathbb{k}$  boxes, and  $\bar{r}_2$  with  $\mathbb{k}$  anti-boxes. Let us first calculate the dimension, i.e. the number of YTs, corresponding to the YD  $r_1$ . A useful trick to do this counting follows from a simple observation [64]: since the number of potential labels is much larger than the number of boxes, repetitions are non-generic. Thus the number of ways to choose the labels is just  $\binom{N}{\mathbb{k}}$ . Such a Young tableau, constructed out of numbers with a total ordering, is called a *standard Young tableau* (sYT) and labels a state in an irrep of the symmetric group  $S_{\mathbb{k}}$ . Letting  $d_{r_1}^{S_{\mathbb{k}}}$  denote the dimension of this irrep of  $S_{\mathbb{k}}$ , we have

$$\dim \mathcal{H}_{r_1} = \binom{N}{\mathbb{k}} d_{r_1}^{S_{\mathbb{k}}}. \quad (1.30)$$

Of course, the dimension  $d_{r_1}^{S_{\mathbb{k}}}$  depends on the particular shape of the irrep of  $S_{\mathbb{k}}$ . An explicit verification of this trick can be found in [1]. Using the same argument for the anti-boxes,

we find

$$\dim \mathcal{H}_{r_1 \otimes \bar{r}_2} = \binom{N}{\mathbb{k}}^2 d_{r_1}^{\mathcal{S}_{\mathbb{k}}} d_{\bar{r}_2}^{\mathcal{S}_{\mathbb{k}}}. \quad (1.31)$$

Summing over all such irreps with  $\mathbb{k}$  boxes, we find

$$\sum_{r_1, \bar{r}_2 \mid \mathbb{k}} \dim \mathcal{H}_{r_1 \otimes \bar{r}_2} = \binom{N}{\mathbb{k}}^2 \left( \sum_{r \mid \mathbb{k}} d_r^{\mathcal{S}_{\mathbb{k}}} \right)^2. \quad (1.32)$$

The sum is a standard result in combinatorics (e.g. see Theorem 8.26b in [6]) and the result is called the  $\mathbb{k}^{th}$  telephone number. At large  $\mathbb{k}$ , it is approximated by [162]

$$\sum_{r \mid \mathbb{k}} d_r^{\mathcal{S}_{\mathbb{k}}} = \sqrt{\mathbb{k}!}. \quad (1.33)$$

The sum of dimensions of all  $PU(N)$  irreps with  $\mathbb{k}$  boxes and  $\mathbb{k}$  anti-boxes is thus

$$\binom{N}{\mathbb{k}}^2 \mathbb{k}! \approx \left( \frac{N}{\sqrt{\mathbb{k}}} \right)^{2\mathbb{k}}, \quad (1.34)$$

reproducing again the result of §1.3.1.

Of course, it is not correct to sum over all irreps — degeneracy is only guaranteed within an irrep of  $PU(N)$ , whereas we have counted the dimension of a *reducible* representation. This is in fact the same unjustified step we took in §1.3.1. However, let us note something simple:

$$\left( \sum_{r \mid \mathbb{k}} d_r^{\mathcal{S}_{\mathbb{k}}} \right)^2 \approx \mathbb{k}! = |\mathcal{S}_{\mathbb{k}}| = \sum_{r \mid \mathbb{k}} \left( d_r^{\mathcal{S}_{\mathbb{k}}} \right)^2. \quad (1.35)$$

In other words, at large  $\mathbb{k}$ , the dimension of the irrep of  $\mathcal{S}_{\mathbb{k}}$  has zero variance, indicating that the sum is dominated by a single YD. This is an old result [108]; they found that the dominant irrep is roughly a right isosceles triangle with somewhat concave sides i.e., the longest column, row and anti-row have approximately  $\sqrt{2\mathbb{k}}$  boxes each. This irrep is shown schematically in figure 1.5. This domination by a single diagram is why we were able to sum over all irreps and still reproduce the entropy of (1.22). Another interesting consequence is that our counting is actually valid in the regime  $1 \ll \mathbb{k} \ll N^2$  and not just  $1 \ll \mathbb{k} \ll N$  as originally claimed.

Finally, let us deal with the possibility of traces; naively, the count above includes states in smaller irreps since we didn't project out traces. Since a trace causes the number of

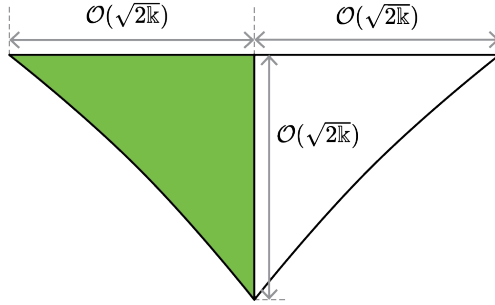


Figure 1.5: The shape of the YD that dominates the counting of states in the  $k \ll N$  phase.

(anti-)boxes in the YD to reduce by one, the counting above shows that the number of these states is approximately  $(N/\sqrt{k})^{2k-2}$  which is a factor of  $N^2$  smaller. Further, the leading corrections in the count just given (from corrections to Stirling's formula and the approximation we made for the telephone number) is a factor of  $e^{\#k}$ , which is a correction of  $\mathcal{O}(k)$  in the entropy. Thus, the uncertainty from the shape of the YD (comparable to the correction in the telephone number) dominates the uncertainty from the possibility of traces.

Since both the boxes and anti-boxes are dominated by the same irrep, the dominant irrep in the full counting is self-conjugate. This is why our result agrees with the counting in [64] even though they restricted their count to only self-conjugate representations.

## 1.4 Energy

Having found that the dominant degenerate space with  $k$  boxes reproduces the entropy of KT, we now calculate its energy. In §1.4.1, we decompose the Hilbert space in a useful way and set up the Schrodinger equation problem that we will attempt to solve. To reproduce the fixed-charge partition function (1.22), we proceed in two steps. First, in §1.4.2, assuming that within a fixed  $k$  sector the canonical and microcanonical ensembles are equivalent, we construct a state whose energy matches the KT energy; the construction of this state is our central result. In §1.4.3, we justify this assumption, arguing that the two ensembles are indeed equivalent within the sector.

Section 1.4.2 contains the central new result: the construction of a state when  $1 \ll k \ll N$  in the irrep that dominates the counting above whose energy matches that from (1.22) on the nose. This state is a new, interesting bound state that has all the characteristics of a black hole. We explore this facet, and argue that the dominance of such a bound state is exactly what is needed for the MQM to be dual to the sine-Liouville string theory, in §1.6.1.

### 1.4.1 Decomposition of the Hilbert space

The Hilbert space of the MQM is

$$\mathcal{H}_{\text{MQM}} = \text{span} \left\{ |X\rangle \mid X^\dagger = X \right\}. \quad (1.36)$$

We will first decompose this Hilbert space into its eigenvalue and angular directions and then “Fourier transform” the angular directions using the Peter–Weyl theorem. This has the nice benefit of isolating the action of the  $PU(N)$  symmetry.

We decompose the matrix  $X$  as

$$X(t) = U^\dagger(t) \Lambda(t) U(t) \\ \text{s.t.} \quad \Lambda_{ij} = \lambda_i \delta_{ij}, \quad \lambda_i \in \mathbb{R}, \quad i > j \Rightarrow \lambda_i \geq \lambda_j, \quad U \in U(1)^N \backslash U(N). \quad (1.37)$$

As we will shortly detail, the conditions on the diagonal and unitary matrices are required to make the decomposition uniquely defined — we explain each of these in turn. First, the eigenvalues need to be ordered since we can always rearrange them without changing  $X$  by absorbing an element  $P_\sigma \in \mathbf{S}_N \in U(N)$  into  $U$ ,

$$U^\dagger \Lambda U = (P_\sigma U)^\dagger (P_\sigma \Lambda P_\sigma^\dagger) (P_\sigma U). \quad (1.38)$$

Thus, denoting the space of  $N \times N$  diagonal matrices by  $D(N)$ , we require  $\Lambda \in D(N)/\mathbf{S}_N$ .<sup>8</sup> Similarly, the  $U(N)$  needs to be *left*-quotiented by  $U(1)^N$  to account for the gauge symmetry,

$$X = U^\dagger \Lambda U = U^\dagger e^{-i\Theta} \Lambda e^{i\Theta} U, \quad \Theta_{ij} = \theta_i \delta_{ij}. \quad (1.39)$$

The global symmetry (1.15) is given by a *right*-action on  $U$ ,  $U \rightarrow UV$ . There is also a left-action,  $U \rightarrow VU$ , which doesn’t commute with the Hamiltonian. The measure over the space decomposes into

$$d^{N \times N} X = \frac{d^N \lambda \, dU}{N! [\text{vol } U(1)]^N} \Delta(\lambda)^2, \quad \Delta(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j). \quad (1.40)$$

The two terms in the denominator are the size of  $\mathbf{S}_N$  and the volume of  $U(1)^N$  in the

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<sup>8</sup>As we will clarify below,  $D(N)/\mathbf{S}_N$  is not a strictly correct description of the space of  $\Lambda$ s.

Haar measure; we will henceforth absorb them into normalisation factors.  $\Delta(\lambda)$  is the Vandermonde determinant, whose square is the Jacobian for this change of basis. Thus, finally, we have

$$\mathcal{H}_{\text{MQM}} = \mathcal{H}_{D(N)/S_N} \bigotimes \mathcal{H}_{U(1)^N \setminus U(N)}, \quad (1.41)$$

where the two factors are spaces of normalisable functions over the respective spaces.

$\mathcal{H}_{D(N)/S_N}$  can be thought of as the space of wavefunctions on the subspace of  $\mathbb{R}^N$  satisfying the ordering conditions. To extend it to all of  $\mathbb{R}^N$ , we need to contend with the fact that permutation of eigenvalues also requires absorbing a left-action into the unitary, meaning that the extension is therefore not independent of the state in the angular directions. However, since we are interested in the case  $1 \ll k \ll N$ , we can forget about this subtlety and extend it to  $\mathbb{R}^N$  as a symmetric wavefunction [55]. Furthermore, because the inner product on these wavefunctions is

$$\langle \psi_1 | \psi_2 \rangle = \int d^N \lambda \, \Delta(\lambda)^2 \, \psi_1^*(\lambda) \, \psi_2(\lambda), \quad (1.42)$$

it is conventional to absorb a factor of  $\Delta(\lambda)$  into the wavefunction to get completely *anti*-symmetric wavefunctions (ignoring the angular directions) and a conventional measure for the inner product.

As we will see, a significant part of the analysis will concern the angular part of the Hilbert space,  $\mathcal{H}_{U(1)^N \setminus U(N)}$ . This Hilbert space can be thought of as simply  $\mathcal{H}_{U(N)}$  specified by a gauge constraint. To do so, we first parameterise the algebra  $\mathfrak{u}(N)$  in the weight basis  $\{H_i, T_{ij}, \tilde{T}_{ij} \mid i, j \in \{1, \dots, N\}, i < j\}$ . In the fundamental representation, these basis elements have the matrix representations

$$\begin{aligned} (H_i)_{kl} &= \delta_{kl} \delta_{ki}, \\ (T_{ij})_{kl} &= \delta_{ki} \delta_{lj} + \delta_{li} \delta_{kj}, \\ (\tilde{T}_{ij})_{kl} &= -i (\delta_{ki} \delta_{lj} - \delta_{li} \delta_{kj}). \end{aligned} \quad (1.43)$$

The generators  $T_{ij}$  and  $\tilde{T}_{ij}$  can respectively be thought of as the Pauli matrices  $\sigma_1$  and  $\sigma_2$  acting on the two-dimensional subspace corresponding to  $i, j$ . To each of these generators

correspond two operators on  $\mathcal{H}_{U(N)}$ , the left and right actions,

$$\begin{aligned} e^{\iota [\alpha^{ij} \hat{T}_{ij}^L + \beta^{ij} \hat{\tilde{T}}_{ij}^L + \gamma^i \hat{H}_i^L]} |U\rangle &= \left| e^{\iota [\alpha^{ij} T_{ij} + \beta^{ij} \tilde{T}_{ij} + \gamma^i H_i]} U \right\rangle, \\ e^{\iota [\alpha^{ij} \hat{T}_{ij}^R + \beta^{ij} \hat{\tilde{T}}_{ij}^R + \gamma^i \hat{H}_i^R]} |U\rangle &= \left| U e^{-\iota [\alpha^{ij} T_{ij} + \beta^{ij} \tilde{T}_{ij} + \gamma^i H_i]} \right\rangle. \end{aligned} \quad (1.44)$$

Now, the subspace  $\mathcal{H}_{U(1)^N \setminus U(N)}$  is given by

$$\begin{aligned} \mathcal{H}_{U(1)^N \setminus U(N)} &= \left\{ |v\rangle = \int dU \, v(U) |U\rangle \mid v(e^{\iota\Theta} U) = v(U) \right\} \\ &= \left\{ |v\rangle \in \mathcal{H}_{U(N)} \mid \hat{H}_i^L |v\rangle = 0 \right\}. \end{aligned} \quad (1.45)$$

The second equation arises from the fact that the change  $U \rightarrow e^{\iota\Theta} U$  is implemented by the exponential of  $\Theta^i \hat{H}_i$ . We will refer to the gauge constraint as the *zero-weight condition*.

To implement this constraint, we use the Peter–Weyl theorem to go to the “Fourier transformed” basis:

$$\begin{aligned} |r, \alpha, \beta\rangle &\equiv \sqrt{d_r} \int dU \, D_{\beta\alpha}^r(U^\dagger) |U\rangle, \\ |U\rangle &= \sum_r \sum_{\alpha, \beta} \sqrt{d_r} D_{\alpha\beta}^r(U) |r, \alpha, \beta\rangle. \end{aligned} \quad (1.46)$$

Here,  $r$  is an irrep of  $U(N)$ , usually represented by a YD.  $\alpha, \beta$  are two different, unrelated YTs filling in the YD corresponding to  $r$ .<sup>9</sup> In the  $N = 2$  case, with standard  $SU(2)$  notation, the states are  $|j, m, m'\rangle$  and the coefficients are matrix elements of Wigner  $D$ -matrices. The change of basis is unitary because of the orthogonality relations

$$\int dU \, D_{\beta\alpha}^r(U^\dagger) D_{\gamma\delta}^{r'}(U) = \frac{1}{d_r} \delta^{rr'} \delta_{\alpha\gamma} \delta_{\beta\delta}, \quad \sum_r \sum_{\alpha, \beta} d_r D_{\beta\alpha}^r(U^\dagger) D_{\alpha\beta}^r(U') = \delta(U^{-1} U'). \quad (1.47)$$

The most useful fact about these states is that they ‘decouple’ the right and left actions;

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<sup>9</sup>  $\beta$  should actually be considered a basis element in the conjugate irrep, but there is a canonical map and so it can also be considered a YT of the same shape.

denoting by  $\hat{V}^L, \hat{V}^R$  the left and right multiplications by  $V$  respectively, we have

$$\begin{aligned}\hat{V}^L |r, \alpha, \beta\rangle &= |r, \gamma, \beta\rangle D_{\gamma\alpha}^r(V) \\ \hat{V}^R |r, \alpha, \beta\rangle &= D_{\beta\delta}^r(V^\dagger) |r, \alpha, \delta\rangle.\end{aligned}\tag{1.48}$$

Indeed, we see that the left and right actions act on the two indices (i.e. YTs) separately. We remind the reader that the right action is a symmetry of the theory, and therefore the YTs denoted by the  $\beta$  index here are the ones we counted in §1.3 (more correctly, in that case  $r$  was an irrep of  $PU(N)$  — a restriction that we will impose presently).

In the Fourier-transformed basis, the zero-weight condition reads

$$\hat{H}_i^L |r, \alpha, \beta\rangle = |r, \gamma, \beta\rangle D_{\gamma\alpha}^r(H_i) = 0.\tag{1.49}$$

We can simultaneously diagonalise all the weights in  $\mathcal{H}_r$ , and then just pick the zero-weight subspace  $\mathcal{H}_r^{(0)}$  as the span of all  $\alpha^{(0)}$  that have all weights vanishing. Thus, we have

$$\mathcal{H}_{U(1)^N \setminus U(N)} = \bigoplus_r \mathcal{H}_r^{(0)} \otimes \mathcal{H}_{\bar{r}} = \text{span} \left\{ |r, \alpha^{(0)}, \beta\rangle \right\}.\tag{1.50}$$

We emphasise that the zero-weight condition is a condition on the irreps of  $U(N)$  as well as the states within the representations. In particular, there are irreps of  $U(N)$ , like the fundamental representation, that don't contain any zero-weight states.

The set of irreps that contain zero-weight states are exactly the irreps of  $PU(N)$ . We can see this as follows:  $H_i$  annihilates every basis vector of the fundamental (anti-fundamental) except the  $i^{th}$  one where it takes the value 1 (−1). Thus, the zero-weight condition is simply [94, 95]

$$\forall i, \quad \hat{H}_i^L |\text{YT}\rangle = \left\{ \# \boxed{i} - \# \textcolor{red}{\boxed{i}} \right\} |\text{YT}\rangle = 0.\tag{1.51}$$

This cannot vanish for all  $i$  unless the number of boxes and anti-boxes agree in the YD, as was pointed out in [28]. This is an alternative way of deriving the restriction to irreps of  $PU(N)$ . This condition is actually stronger, and says that the boxes and the anti-boxes have to carry the same set of labels.

To calculate the energy, we first decompose the classical Hamiltonian according to (1.37).

This gives,

$$H = \text{tr} \left( \frac{1}{2} \dot{\Lambda}^2 + V(\Lambda) + \frac{1}{2} [\dot{U}U^\dagger, \Lambda]^2 \right). \quad (1.52)$$

Notice that the Hamiltonian contains  $\dot{U}U^\dagger$  but not  $U^\dagger\dot{U}$ . The operator  $\dot{U}U^\dagger$  generates a change from the left —  $(1 + dt \dot{U}U^\dagger)U = U + dt \dot{U}$  — whereas  $U^\dagger\dot{U}$  generates a change from the right; the fact that the latter doesn't appear in the action but the former does is indicative of the fact that only the latter is a symmetry. We can now quantize this Hamiltonian following the discussion above, by implementing the Fourier transform for the angular part and explicitly working out the traces for the whole expression to produce:

$$H = \sum_i \left[ -\frac{1}{2\Delta(\lambda)} \frac{\partial^2}{\partial \lambda_i^2} \Delta(\lambda) + V(\lambda_i) \right] + \frac{1}{4} \sum_{i < j} P_{0\text{wt}} \frac{\hat{T}_{L,ij}^2 + \tilde{T}_{L,ij}^2}{(\lambda_i - \lambda_j)^2} P_{0\text{wt}}. \quad (1.53)$$

Once again there are only left-action generators in the Hamiltonian, and the weight operators don't appear at all, since the weights are fixed by the gauge condition.  $P_{0\text{wt}}$  is an explicit projector onto the zero-weight subspace; this is necessary because  $\hat{T}_{L,ij}^2$  doesn't commute with it in arbitrary representations. We will drop the  $L$  subscript henceforth since the right-acting operators don't play a role at all. We will also drop the hats since only the quantum operators appear in the following discussion. In what follows, we will find it useful to define

$$H_{\text{ns}} \equiv \sum_{i < j} H_{ij}, \quad H_{ij} \equiv \frac{1}{4} \frac{T_{ij}^2 + \tilde{T}_{ij}^2}{(\lambda_i - \lambda_j)^2}, \quad (1.54)$$

where  $H_{\text{ns}}$  is the Hamiltonian of the *non-singlet* sector.

The singlet sector, which is annihilated by  $H_{\text{ns}}$ , is just  $N$  non-interacting fermions. Denoting the ground state of this sector by  $|0\rangle_{\text{sing}}$ , its energy is obtained by filling up  $N$  levels of a Fermi sea of eigenvalues,<sup>10</sup>

$$E_0 \sim \mathcal{O}(N^2). \quad (1.55)$$

In comparison,  $H_{\text{ns}}$  scales with  $\mathbb{k} \ll N^2$  and can thus be treated as a perturbation [118]. With  $H_{\text{ns}}$  turned off, the Hamiltonian does not act on the angular directions at all, and

---

<sup>10</sup>We subtract off this zero-point energy in the double-scaling limit.

there is a huge degenerate subspace, given (after dropping the right-action index) by

$$\mathcal{H}_{\text{degen}} = |0\rangle_{\text{sing}} \otimes \text{span} \left\{ \left| r, \alpha^{(0)} \right\rangle \right\}. \quad (1.56)$$

$H_{\text{ns}}$  breaks this degeneracy, and we have to use the techniques of degenerate perturbation theory. The perturbed energy levels are the eigenvalues of the operator  $P_{\text{degen}} H_{\text{ns}} P_{\text{degen}}$ , where  $P_{\text{degen}}$  is the projector onto the degenerate subspace (1.56). Below, we will find that this will not be enough to reproduce the KT partition function. However, this is nonetheless a useful starting point.

The eigenvalue ground state is characterised by a phase space density function,

$$\rho(\lambda, p_\lambda) = \frac{1}{2\pi} \theta \left( E_F - \frac{p_\lambda^2}{2} - V(\lambda) \right). \quad (1.57)$$

Integrating over momenta gives the eigenvalue density function,

$$\rho(\lambda) = \frac{1}{\pi} \sqrt{2E_F - 2V(\lambda)} \approx \frac{1}{\pi} \sqrt{-2\mu_F + \lambda^2} \xrightarrow{\lambda \gg \sqrt{2\mu_F}} \frac{\lambda}{\pi}, \quad (1.58)$$

where  $E_F = -\mu_F$  is the Fermi energy we mentioned in §1.2; the double scaling limit of interest involves keeping  $\mu_F \sim \mathcal{O}(1)$  as  $N \rightarrow \infty$ . Here, we have replaced the quartic part of the potential by a wall at  $\lambda_c \sim \mathcal{O}(\sqrt{N})$ , since the precise form of the cutoff doesn't matter in the large  $N$  limit. The number of eigenvalues between  $\lambda_1$  and  $\lambda_2$  is  $\int_{\lambda_1}^{\lambda_2} \rho(\lambda) d\lambda$  — the eigenvalue density is then normalised as,

$$\int_{\sqrt{2\mu_F}}^{\lambda_c} \rho(\lambda) d\lambda = N. \quad (1.59)$$

We will for many purposes drop the  $\mu_F$ -dependence in what follows, reinstating it only when necessary. Any expectation value in the ground state  $|0\rangle_{\text{sing}}$  is controlled by this eigenvalue density — thus, the projector  $P_{\text{degen}}$  ensures that the part of  $H_{\text{ns}}$  acting on the eigenvalues is determined by this eigenvalue density.

Clearly then, the eigenvectors of  $P_{\text{degen}} H_{\text{ns}} P_{\text{degen}}$  have the form [118] (dropping the right-action indices)

$$|0, r, \{w\}\rangle = |0\rangle_{\text{sing}} \otimes \sum w_{\alpha^{(0)}} \left| r, \alpha^{(0)} \right\rangle. \quad (1.60)$$

The distribution over zero-weight states has to be found by solving the eigenvalue equation

$$H_{\text{ns}} |0, r, \{w\}\rangle = \Delta E |0, r, \{w\}\rangle. \quad (1.61)$$

We proceed to do (approximately) this.

### 1.4.2 A state with the KT energy

We construct the state of interest in three steps. First, in §1.4.2 we review the calculation of the energy in the adjoint sector [118, 64, 28, 115]. We then argue that, at finite  $N$ , a straightforward extension of these methods to a state with a gas of  $1 \ll \mathbb{k} \ll N$  weakly-interacting adjoints gives an energy much higher than the KT value (1.22). This failure to reproduce the KT energy arises from a trade-off between keeping the momenta of each adjoint low and minimising the interactions between them; we find that reducing one in a straightforward way increases the other. An important distinction in the approach we take is that we keep  $N$  finite for much of the discussion, taking it to  $\infty$  only at the end; the state of interest is more clearly isolated at finite  $N$ .

We then show that a ‘liquid’ state in which we ‘clump’ these  $\mathbb{k}$  adjoints together has a much lower energy in §1.4.2; however, its energy is still higher than the KT value. We describe this as a liquid state because although the adjoints have condensed together, there is no appreciable barrier for one or more adjoints to ‘spill’ out and leave the clump.

To get the KT energy, it turns out that we need to ‘solidify’ this clump, i.e. make it so that there is a large barrier for an adjoint to leave. We solidify it by modifying the eigenvalue distribution in a radical way by creating a large  $\mathcal{O}(\mathbb{k}^{1/4})$  hole in it; since the Fermi surface of the eigenvalue distribution is the spatial direction in the string theory, and since we are positing breaking it up into two disconnected pieces, we call this a “hole-in-the-world state.” This is a solid because any adjoint that wants to leave has to tunnel across the hole in the world — this has a large cost and thus prevents adjoints from spilling out. This is the content of §1.4.2.

## A gas of adjoints

### A single adjoint

We begin by reviewing the calculation in the adjoint ( $\mathbb{k} = 1$ ) sector [118, 64, 28, 115]. The zero-weight states are given by  $\sum_k w_k \left| \begin{array}{|c|} \hline k \\ \hline \end{array} \right\rangle$ , with  $\sum_k w_k = 0$ . The action of the

algebra generators on this state are ( $i < j$ )

$$\begin{aligned}
T_{ij} \left| \begin{array}{|c|c|} \hline k & k \\ \hline \end{array} \right\rangle &= (\delta_{ik} - \delta_{jk}) \left| \begin{array}{|c|c|} \hline i & j \\ \hline \end{array} \right\rangle + (i \leftrightarrow j), \\
\tilde{T}_{ij} \left| \begin{array}{|c|c|} \hline k & k \\ \hline \end{array} \right\rangle &= \iota(\delta_{ik} - \delta_{jk}) \left| \begin{array}{|c|c|} \hline i & j \\ \hline \end{array} \right\rangle - (i \leftrightarrow j), \\
T_{ij}^2 \left| \begin{array}{|c|c|} \hline k & k \\ \hline \end{array} \right\rangle &= \tilde{T}_{ij}^2 \left| \begin{array}{|c|c|} \hline k & k \\ \hline \end{array} \right\rangle = 2(\delta_{ik} - \delta_{jk}) \left| \begin{array}{|c|c|} \hline i & i \\ \hline \end{array} \right\rangle + (i \leftrightarrow j).
\end{aligned} \tag{1.62}$$

Thus, the eigenvalue equation becomes

$$\sum_{i \neq j} \frac{w_i - w_j}{(\lambda_i - \lambda_j)^2} |0\rangle_{\text{sing}} \otimes \left| \begin{array}{|c|c|} \hline i & i \\ \hline \end{array} \right\rangle = \Delta E |0\rangle_{\text{sing}} \otimes \sum_k w_k \left| \begin{array}{|c|c|} \hline k & k \\ \hline \end{array} \right\rangle. \tag{1.63}$$

This can be rewritten as the integral equation

$$\Delta E w(\lambda) = \mathcal{P} \int_{\sqrt{2\mu_F}}^{\lambda_c} \rho(\lambda') \frac{w(\lambda) - w(\lambda')}{(\lambda - \lambda')^2} d\lambda' \quad \text{s.t.} \quad \int_{\sqrt{2\mu_F}}^{\lambda_c} d\lambda \rho(\lambda) w(\lambda) = 0, \tag{1.64}$$

where the  $\mathcal{P}$  means that the integral is evaluated using a principal value prescription (which follows from the  $i \neq j$  in the sum). As pointed out in [115], it is useful to shift to the following variables,

$$h(\lambda) \equiv \rho(\lambda) w(\lambda) \quad \text{and} \quad \tau \equiv \cosh^{-1} \frac{\lambda}{\sqrt{2\mu_F}} \xrightarrow{\lambda \gg \sqrt{2\mu_F}} \log \frac{\lambda}{\sqrt{\mu_F}} + \dots \tag{1.65}$$

Matching the normalization condition (1.59) then gives,  $\tau_c \sim \frac{1}{2} \log(N/\mu_F)$ . The eigenvalue equation becomes [115]

$$\Delta E h(\tau) = \frac{1}{2} \left( \log \frac{N}{\mu_F} - 2\tau \coth \tau \right) h(\tau) - \frac{1}{4\pi} \mathcal{P} \int_{-\frac{1}{2} \log(N/\mu_F)}^{\frac{1}{2} \log(N/\mu_F)} d\tau' \frac{h(\tau')}{\sinh^2 \frac{\tau - \tau'}{2}}. \tag{1.66}$$

This last term can be thought of as a kinetic term, whereas the rest can be thought of as a potential term. The kinetic term can be Fourier transformed to give

$\frac{1}{\sqrt{2\pi^3}} (1 + \pi p \coth \pi p) h(p)$ . This eigenvalue equation was explicitly solved in [53]; we will not use the details of this solution.

Taking  $\lambda \gg \sqrt{\mu_F}$ , all the coths can be approximated by 1 and this Hamiltonian becomes

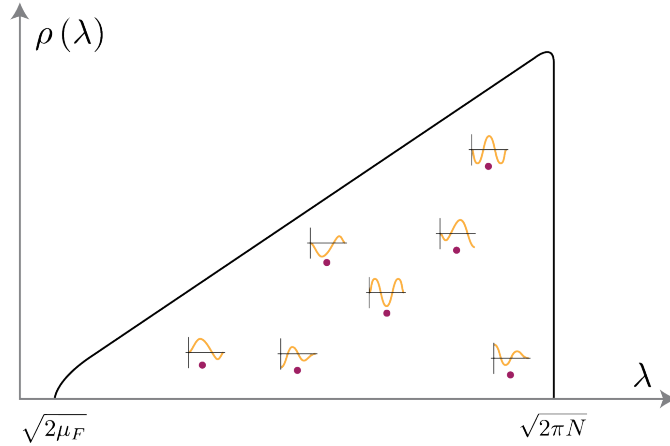


Figure 1.6: A gas of adjoints. We have drawn an adjoint at eigenvalue  $\lambda$  as a purple dot in the position in the Fermi sea where the corresponding eigenvalue is. By a gas, we mean that the wavefunction approximately factorises into a product of wavefunctions of individual adjoints. The wavefunction above each dot is placed to remind the reader that the adjoints are not localised.

just

$$H_{\text{1pt}} = \frac{1}{2\pi} \log \frac{N}{\lambda^2} + \frac{1}{\sqrt{2\pi}} |\hat{p}|, \quad (1.67)$$

where  $\hat{p}$  is the momentum operator that we defined implicitly by the Fourier transform of the last term in (1.66), and we have gone back from  $\tau$  to  $\lambda$ .

### A gas of adjoints

We first consider states where the distribution of matrix eigenvalues is perturbatively close to the singlet sector ground state, and where the adjoints are well-spread out and weakly interacting, as in figure 1.6. More explicitly, consider the effect of  $T_{ij}^2 + \tilde{T}_{ij}^2$  on a state with multiple fundamentals and anti-fundamentals. Because the Hamiltonian comes with projectors onto the zero-weight subspace, we only consider the branches of the action that result in terms within the zero-weight subspace.

We take YTs of size 2 as an example. Suppose we take the YT

$$\begin{array}{|c|c|c|} \hline i & i & j \\ \hline j & & \\ \hline \end{array}. \quad (1.68)$$

Defining

$$|T_{ij}|^2 = \frac{T_{ij}^2 + \tilde{T}_{ij}^2}{4}, \quad (1.69)$$

when  $k \neq i, j$  we have

$$\begin{aligned}
|T_{ik}|^2 \left| \begin{array}{|c|c|c|} \hline i & i & j \\ \hline j & & \\ \hline \end{array} \right\rangle &= \left| \begin{array}{|c|c|c|} \hline i & i & j \\ \hline j & & \\ \hline \end{array} \right\rangle - \left| \begin{array}{|c|c|c|} \hline k & k & j \\ \hline j & & \\ \hline \end{array} \right\rangle, \\
|T_{jk}|^2 \left| \begin{array}{|c|c|c|} \hline i & i & j \\ \hline j & & \\ \hline \end{array} \right\rangle &= \left| \begin{array}{|c|c|c|} \hline i & i & j \\ \hline j & & \\ \hline \end{array} \right\rangle - \left| \begin{array}{|c|c|c|} \hline i & i & k \\ \hline k & & \\ \hline \end{array} \right\rangle.
\end{aligned} \tag{1.70}$$

In other words, we notice that the action of any generator that rotates an index appearing within the YT to one not appearing within the YT decouples into independent action on each box–antibox pair that share an index. We can therefore treat these terms as  $\mathbb{k}$  copies of  $H_{1\text{pt}}$  in (1.67).

We now analyze the action of  $|T_{ij}|^2$  where both indices  $i, j$  appear in the YT. For simplicity, we first consider a YT where the boxes labeled  $i$  and  $j$  share neither a row nor a column.

$$\begin{aligned}
|T_{ij}|^2 \left| \begin{array}{|c|c|c|c|} \hline i & k & k & i \\ \hline & j & j & \\ \hline \end{array} \right\rangle &= \left| \begin{array}{|c|c|c|c|} \hline i & k & k & i \\ \hline & j & j & \\ \hline \end{array} \right\rangle - \left| \begin{array}{|c|c|c|c|} \hline j & k & k & j \\ \hline & j & j & \\ \hline \end{array} \right\rangle \\
&+ \left| \begin{array}{|c|c|c|c|} \hline i & k & k & i \\ \hline & j & j & \\ \hline \end{array} \right\rangle - \left| \begin{array}{|c|c|c|c|} \hline i & k & k & i \\ \hline & i & i & \\ \hline \end{array} \right\rangle \\
&+ \left| \begin{array}{|c|c|c|c|} \hline j & k & k & i \\ \hline & i & j & \\ \hline \end{array} \right\rangle + \left| \begin{array}{|c|c|c|c|} \hline i & k & k & j \\ \hline & j & i & \\ \hline \end{array} \right\rangle.
\end{aligned} \tag{1.71}$$

Taking apart this equation, we see that the first two lines on the RHS correspond to the  $H_{1\text{pt}}$  action we would expect from  $|T_{ij}|^2$  on the  $i$  boxes and the  $j$  boxes respectively. The third line contains a new action — one that exchanges the  $i$  and  $j$  anti-boxes in the first term and the  $i$  and  $j$  boxes in the second term. Thus, the last line is generated by an exchange operator, which we denote  $\mathfrak{s}_{ij}$ , and suggests that the interaction term between boxes is given by the ‘cross’ term

$$H_{\times} \equiv \sum_{i,j} \frac{\mathfrak{s}_{ij}}{(\lambda_i - \lambda_j)^2} \tag{1.72}$$

We now check that this works for states where  $i$  and  $j$  share either a row or a column.

Returning to two-box representations for simplicity, we find

$$\begin{aligned}
|T_{ij}|^2 \left| \begin{array}{|c|c|} \hline i & i \\ \hline j & j \\ \hline \end{array} \right\rangle &= 0 = \left| \begin{array}{|c|c|} \hline i & i \\ \hline j & j \\ \hline \end{array} \right\rangle - \left| \begin{array}{|c|c|} \hline j & j \\ \hline i & i \\ \hline \end{array} \right\rangle + \\
&+ \left| \begin{array}{|c|c|} \hline i & i \\ \hline j & j \\ \hline \end{array} \right\rangle - \left| \begin{array}{|c|c|} \hline i & i \\ \hline i & i \\ \hline \end{array} \right\rangle + \\
&+ \left| \begin{array}{|c|c|} \hline j & i \\ \hline i & j \\ \hline \end{array} \right\rangle + \left| \begin{array}{|c|c|} \hline i & j \\ \hline j & i \\ \hline \end{array} \right\rangle.
\end{aligned} \tag{1.73}$$

The LHS is equal to zero because the columns comprising the boxes and antiboxes are singlets under rotations from  $i$  to  $j$ . The RHS is equal to zero due to the antisymmetry properties of the YT states. States corresponding to YTs with two  $i$ 's/ $j$ 's in the same column vanish, and the states appearing in the third row on the RHS cancel the first two states of the first two lines due to the anti-symmetrisation. We may again decompose this action into the sum of the  $H_{1\text{pt}}$  action and an exchange operator, neither of which vanishes separately. A similar analysis can be done for the other possible combinations we can have of boxes sharing rows and columns, and we find that we may decompose the non-singlet Hamiltonian into

$$H_{\text{ns}} = H_{\text{dir}} + H_{\times}, \tag{1.74}$$

where  $H_{\text{dir}}$  is the sum of  $\mathbb{k}$  copies of  $H_{1\text{pt}}$ , and  $H_{\times}$  is given by (1.72).

With this decomposition of the Hamiltonian in hand, we now search for wavefunctions of the labels of the dominant representation (figure 1.5) that match the KT prediction for the energy. We first present several general types of non-singlet excitation configurations on the singlet saddle point eigenvalue distribution and demonstrate why they fail. Then, we will resolve these confusions by showing that a certain class of states introduces a violent deformation of the eigenvalue configuration which precisely lowers the energy enough to match the KT result.

We first consider wavefunctions where the labels are taken to be well-separated, i.e., the maximum label that is allowed to appear in the YT is  $i_{\text{max}} \gg \mathbb{k}$  — this allows us to neglect the cross-term to leading order. This state can be likened to a gas of adjoints. Due to the antisymmetrisation rules of the YT, we need at least  $\mathcal{O}(\sqrt{\mathbb{k}})$  different labels for its construction. Since  $\tau$  takes values in a range of width  $\log N$ , the spectrum of the momentum operator  $\hat{p}$  has a spacing of  $\mathcal{O}(1/\log N)$ . Pauli exclusion then pushes labels up into states with a typical momentum of at least  $p \sim \frac{\mathbb{k}^{1/2}}{\log N}$ . The energy of such a configuration then

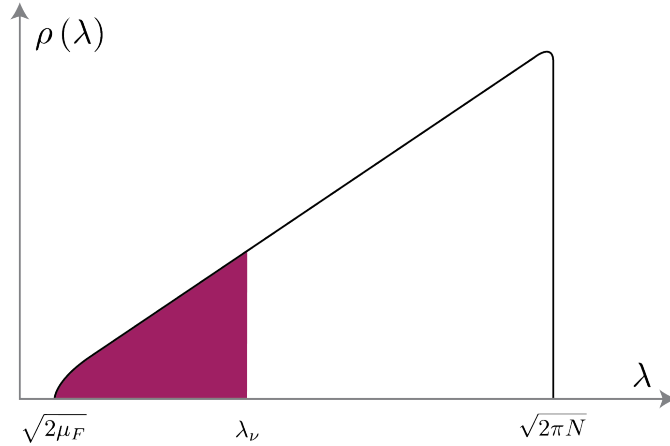


Figure 1.7: We can reduce the energy. at finite  $N$ , by clumping the adjoints together inside the Fermi sea as shown here.

scales as

$$E_{\mathbb{k}} = \frac{\mathbb{k}}{2\pi} \log N + \mathcal{O}\left(\frac{\mathbb{k}^{3/2}}{\log N}\right). \quad (1.75)$$

The leading term in the above expression comes from the  $\propto \log N$  term in  $H_{\text{dir}}$  (1.67). The  $\propto \log \lambda^2$  term can be ignored because of a subtlety in the order with which we take limits—for a fixed  $N$ , we populate the gas by choosing  $\mathbb{k}$  ( $\ll N^2$ ) eigenvalues to label the adjoints. We then take the large  $N$  limit, whilst keeping these labels fixed. Thus the contribution from the  $\propto \log \lambda^2$  term does not scale with  $N$  and can be ignored to leading order.

If we assume  $\mathbb{k}^{1/2} \leq \mathcal{O}(\log N)$  we can certainly find states with the KT energy where the non-singlet excitations are spread out as described above.<sup>11</sup> However, once  $\mathbb{k}$  crosses this bound, the kinetic energy on its own is much higher than the KT energy. Indeed, in the double-scaling limit of interest,  $\mathbb{k} = \mathcal{O}(N^0)$  and so such a state likely exists. However, there is nothing special about the state with energy  $(2\pi)^{-1} \mathbb{k} \log N / \sqrt{\mathbb{k}}$  and we have no reason to believe it will dominate the partition function.

As a result, we are forced to look for other classes of states where the momentum and interaction cancel each other (the potential energy is strictly positive and can't cancel the kinetic energy). This is what we turn to now.

### Condensing the adjoints into a liquid

We now show how to use the interactions to cancel the large momentum contributions and achieve a state with a much lower energy. Our strategy to do this is to kill significant subsets

<sup>11</sup>The minimum spreading required goes to the entire eigenvalue distribution as  $\sqrt{\mathbb{k}} \rightarrow \log N$ .

of the  $U(N)$  generators by considering states that are singlets under subgroups of  $U(N)$ . In analogy with the ‘gas’ of adjoints that we described earlier, we think of this as a ‘liquid’ solution, where the adjoints are clumped together as in Figure 1.7 but there is no significant barrier against them ‘spilling out’ of the clump.<sup>12</sup> This is an intermediate step in our analysis — as we will see, this reduces the energy but still gives a larger value than the KT energy (1.22); this ‘liquid’ state never dominates the partition function and should not be regarded as an intermediate phase.

The interaction term between a box–antibox pair with index  $i$  and one with index  $j$  is the positive operator  $|T_{ij}|^2/(\lambda_i - \lambda_j)$ . Thus the way we minimise its contributions is to construct a state that is annihilated by as many of the operators  $T, \tilde{T}$  as possible. As a start, we first consider YTs with all indices taken from the set  $\{1, \dots, \nu\}$  — this ensures that all the generators with *both* indices larger than  $\nu$  kill the state. Further, as we explain below, we can ensure this state is an  $U(\nu)$  singlet — this results in all the generators with *both* indices in  $\{1, \dots, \nu\}$  also annihilating it. Thus, the state that we are after is a singlet of the subgroup  $U(\nu) \times U(N - \nu)$ . We note that if we are interested in the approximately isosceles YD that dominates the entropy, we need

$$\nu \geq \sqrt{2\mathbb{k}}, \quad (1.76)$$

since this is the size of the largest (anti–)column (see figure 1.5) and different boxes in a column must have distinct indices (otherwise they can’t be anti–symmetrised).

Now we turn to constructing a  $U(\nu)$  singlet. Such a singlet in the adjoint sector is simply a trace over the subset of indices<sup>13</sup>,

$$|\nu\rangle \equiv \frac{1}{\sqrt{\nu}} \sum_{i=1}^{\nu} \left| \begin{array}{|c|c|} \hline i & i \\ \hline \end{array} \right\rangle. \quad (1.77)$$

The state we are interested in is  $\mathbb{k}$  copies of this  $U(\nu)$  singlet (1.77), projected into the irrep of interest.

That the state remains a  $U(\nu)$  singlet after the projection is clear by the fact that the generators act irreducibly within each irrep, and so the state can only be annihilated by

<sup>12</sup>Only a few adjoints can spill out without a high energy cost.

<sup>13</sup>This state isn’t exactly in the adjoint irrep — for instance, it has a non-zero inner product with the state  $\mathbb{1}_{N \times N}/\sqrt{N}$ , which is a  $U(N)$  singlet. However this inner product is  $\sqrt{\nu/N} \ll 1$ , and so we can safely ignore this subtlety.

any generator if each projection is; schematically, for any generator  $T$  we have

$$(e^{\iota T_{\text{adj}}})^{\otimes \mathbb{k}} |\psi\rangle = \bigoplus_{r \in \text{adj}^{\otimes \mathbb{k}}} e^{\iota T_r} |\psi\rangle = 1 \quad \Rightarrow \quad T_r P_r |\psi\rangle = 0. \quad (1.78)$$

Further, the projection is unnecessary, since we know from the discussion in §1.3.2 that a single irrep dominates the tensor product of  $\mathbb{k}$  adjoints. One can check using the same techniques that the dimension of the zero-weight subspace of the dominant irrep is  $\binom{N}{\mathbb{k}} \cdot (\sqrt{\mathbb{k}}!)^2 = N^{\mathbb{k}}$ , which is the dimension of  $\mathbb{k}$  zero-weight adjoints. Thus, the the  $U(\nu) \times U(N - \nu)$  singlet of interest is merely,

$$|\nu, \mathbb{k}\rangle \equiv |\nu\rangle^{\otimes \mathbb{k}} = \left( \frac{1}{\sqrt{\nu}} \sum_{i=1}^{\nu} \left| \begin{array}{|c|c|} \hline \textcolor{red}{i} & i \\ \hline \end{array} \right\rangle \right)^{\otimes \mathbb{k}}, \quad (1.79)$$

and the liquid state is therefore given by,

$$|0_l\rangle \equiv |\rho(\lambda)\rangle \otimes |\nu, \mathbb{k}\rangle. \quad (1.80)$$

Written in terms of YTs in the dominant irrep, this state is a uniform superposition over all reflection-symmetric (i.e. self-conjugate) YTs with indices in  $\{1, \dots, \nu\}$ .<sup>14</sup>

We can also check this last point explicitly by calculating the value of the second Casimir and its standard deviation in the state. The second Casimir is

$$C_2 = \sum_{i < j} \frac{T_{ij}^2 + \tilde{T}_{ij}^2}{2} + \sum_i H_i^2. \quad (1.81)$$

The only generators that don't annihilate the state are those with  $i < \nu, j > \nu$ ; and each of these can act on any of the  $2\mathbb{k}$  (anti-)boxes. Its expectation value in the state above is

$$\langle 0_l | C_2 | 0_l \rangle = \frac{2\mathbb{k}}{\nu} \sum_{i \leq \nu, j > \nu} 1 \approx 2\mathbb{k}N + \mathcal{O}(\mathbb{k}\nu). \quad (1.82)$$

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<sup>14</sup> A zero-weight state in any fixed irrep can also be written as a tensor  $T_{i_{\sigma(1)} \dots i_{\sigma(\mathbb{k})}}^{i_1 \dots i_{\mathbb{k}}}$  where  $\sigma$  is a permutation in  $S_{\mathbb{k}}$  and the symmetry properties of the upper (lower) indices are dictated by the shape of the YD (anti-YD). This tensor is the wavefunction on the space of YTs. The wavefunction of  $|\nu, \mathbb{k}\rangle$  is zero if any of the indices  $i_1 \dots i_{\mathbb{k}}$  is bigger than  $\nu$  or if the permutation  $\sigma$  is not the identity element of  $S_{\mathbb{k}}$ , and 1 (up to a normalisation) otherwise.

which can easily be checked to be the value of the Casimir in the dominant irrep of Figure 1.5, using standard formulae found in e.g. [85]. However, this is not enough; we must also check the variance to make sure that it has a small spread in irreps. By a similar logic as above, we find

$$\langle C_2^2 \rangle_l \approx \langle C_2 \rangle_l^2 \quad (1.83)$$

and the variance is therefore subleading. All of this is concomitant with our result in §1.3 that one irrep dominates the product of many adjoints.

The expectation value of the non-singlet Hamiltonian in this state is

$$\langle H_{\text{ns}} \rangle_l \equiv \langle 0_l | H_{\text{ns}} | 0_l \rangle = \frac{1}{\nu^{\mathbb{k}}} \left( \sum_{i=1}^{\nu} \left\langle \begin{array}{|c|} \hline i \\ \hline \end{array} \right| \right)^{\otimes \mathbb{k}} \left( \frac{1}{4} \sum_{j < k} \frac{T_{jk}^2 + \tilde{T}_{jk}^2}{(\lambda_j - \lambda_k)^2} \right) \left( \sum_{i'=1}^{\nu} \left| \begin{array}{|c|} \hline i' \\ \hline \end{array} \right| \right)^{\otimes \mathbb{k}}. \quad (1.84)$$

We need only consider the case when  $j < \nu$ ,  $k > \nu$ , because the state is annihilated by all the other generators. The action that contributes to this expectation value is the one where both operators in each  $T^2$  act on the same (anti-)box. This gives

$$\langle H_{\text{ns}} \rangle_l = \frac{\mathbb{k}}{\nu} \sum_{j=1}^{\nu} \sum_{k=\nu+1}^N \frac{1}{(\lambda_j - \lambda_k)^2} = \frac{\mathbb{k}}{\nu} \int_0^{\lambda_{\nu}} d\lambda' \int_{\lambda_{\nu+1}}^{\lambda_N} d\lambda \frac{\rho(\lambda') \rho(\lambda)}{(\lambda - \lambda')^2}. \quad (1.85)$$

Here,  $\lambda_i$  is the value of the  $i^{\text{th}}$  eigenvalue, which is the solution of the equation

$$\int_0^{\lambda_i} \rho(\lambda) d\lambda = i. \quad (1.86)$$

Let us plug in now the vacuum eigenvalue density  $\rho(\lambda) = \lambda/\pi$ . We find

$$\begin{aligned} \langle H_{\text{ns}} \rangle_l &= \frac{\mathbb{k}}{2\pi^2\nu} \left[ [\lambda \lambda' + (\lambda^2 + \lambda'^2) \log |\lambda - \lambda'|]_0^{\lambda_{\nu}} \right]_{\lambda_{\nu+1}}^{\lambda_c} \\ &\approx \frac{\mathbb{k}}{2\pi^2\nu} \left\{ -\lambda_{\nu} \lambda_{\nu+1} + \lambda_{\nu}^2 \log \lambda_c - (\lambda_{\nu}^2 + \lambda_{\nu+1}^2) \log(\lambda_{\nu+1} - \lambda_{\nu}) + \lambda_{\nu+1}^2 \log \lambda_{\nu+1} \right\}. \end{aligned} \quad (1.87)$$

To simplify this further, we have to use the facts that  $\lambda_i = \sqrt{2\pi i}$  and  $\lambda_{\nu+1} - \lambda_{\nu} \sim \sqrt{\frac{\pi}{\nu}}$  in

the vacuum eigenvalue density. Thus we find

$$\langle H_{\text{ns}} \rangle_l = \frac{\mathbb{k}}{2\pi} \left( \log N + \frac{3}{2} \log \mathbb{k} \right) + \mathcal{O}(\mathbb{k}). \quad (1.88)$$

The UV divergence coming from  $\lambda_{\nu+1}$  approaching  $\lambda_\nu$ , while of the same magnitude as the subleading term in the KT energy, causes the subleading term to have the wrong sign. The UV divergent term is the penultimate one  $-(\lambda_\nu^2 + \lambda_{\nu+1}^2) \log(\lambda_{\nu+1} - \lambda_\nu)$ ; this is a positive quantity because the argument of logarithm is small. This divergence is the last remnant of the large kinetic energy discussed in the previous section, and it seems we cannot push this contribution lower without drastically modifying the structure of the UV cutoff in the emergent geometry.

In summary, in our efforts to use the interaction between the non-singlet excitations to eliminate as many of the  $|T_{ij}|^2$ 's in the Hamiltonian as possible, we were forced to introduce a sharp cutoff in the non-singlet distribution. This sharp cutoff functions as a domain wall between a region dense with boxes and a region with no boxes, and the UV divergence can then be thought of as a domain wall energy.

We have not been able to conclusively prove that there is no state in the unperturbed eigenvalue distribution that matches the KT energy in the regime where  $\mathbb{k} \gg \log N$ . There may still be room for careful interplay between the cross term and direct term in the non-singlet Hamiltonian to cancel the large kinetic energies seemingly forced on us by Pauli exclusion. However, the methods we have considered have not yielded a way past this obstruction. In the next section we show how allowing the domain wall described above to violently backreact on the background eigenvalue distribution, ripping a hole in the emergent geometry, eliminates the UV divergences and lowers the energy to precisely match the KT result.

### Freezing the liquid by blowing up a hole in the world

We now show that there is always a bound state for  $1 \ll \mathbb{k} \ll N^2$  — the same regime where the entropy calculation is valid — whose energy matches the KT value (1.22). This state is obtained by allowing the non-singlet excitations to backreact strongly on the eigenvalues, creating a large gap in the eigenvalue distribution. Since the bulk spacetime on which the string theory lives is formed by the eigenvalue Fermi sea (up to a non-local integral transform), this configuration is like a hole in the world; we call it the hole-in-the-world

state.<sup>15</sup> Blowing up a hole in the world naively seems to be in contradiction with the string theory, but we argue that this hole is exactly what is needed for a duality with sine–Liouville string theory in §1.6.1.

The hole-in-the-world solution can be motivated by the following rearrangement of (1.87)

$$E_{\text{ns}} = \frac{\mathbb{k}}{2\pi^2\nu} \left\{ -\lambda_\nu \lambda_{\nu+1} + \lambda_\nu^2 \log \lambda_c - \lambda_\nu^2 \log (\lambda_{\nu+1} - \lambda_\nu) - \lambda_{\nu+1}^2 \log \left( 1 - \frac{\lambda_\nu}{\lambda_{\nu+1}} \right) \right\}. \quad (1.89)$$

The last term here gave a large positive contribution to (1.88); to make it small and to have the penultimate term yield the KT energy we only need

$$\lambda_{\nu+1} - \lambda_\nu = \lambda_\nu \cdot \mathcal{O}(1) = \mathbb{k}^{1/4} \cdot \mathcal{O}(1) \quad \implies \quad \langle H_{\text{ns}} \rangle_l \longmapsto \frac{\mathbb{k}}{2\pi} \log \frac{N}{\sqrt{\mathbb{k}}} + \mathcal{O}(\mathbb{k}). \quad (1.90)$$

Here we’ve again taken  $\nu = \mathcal{O}(\sqrt{\mathbb{k}})$  and  $\lambda_\nu, \lambda_{\nu+1} = \mathcal{O}(\mathbb{k}^{1/4})$ . The only problem is that this is a radical change in the eigenvalue distribution, demanding that there are no eigenvalues for an  $\mathcal{O}(\mathbb{k}^{1/4})$  region of eigenvalue space, as shown in figure 1.8. Denoting the new eigenvalue distribution by  $\rho_n(\lambda)$ , the hole-in-the-world state is given by,

$$|0\rangle \equiv |\rho_n(\lambda)\rangle \otimes |\nu, \mathbb{k}\rangle. \quad (1.91)$$

Of course, such a radical change to the state needs to be treated with care, checking for other large contributions to the energy and carefully dealing with any new parameters that may arise; this is what we turn to now. The remainder of this subsection is a technical exercise in showing that all corrections to the energy in (1.90) are  $\mathcal{O}(\mathbb{k})$ . Readers interested only in the main physics points may safely skip to §1.4.3.

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<sup>15</sup>This might not be precisely a hole in the string theory, due to the non-local integral transformation relating the eigenvalue and dilaton directions. However, as we discuss in §1.6.1, there are indications of hole-like behaviour in the string side as well. We argue that the scattering and bound states of sine–Liouville theory are excitations to the right and left of the hole respectively.

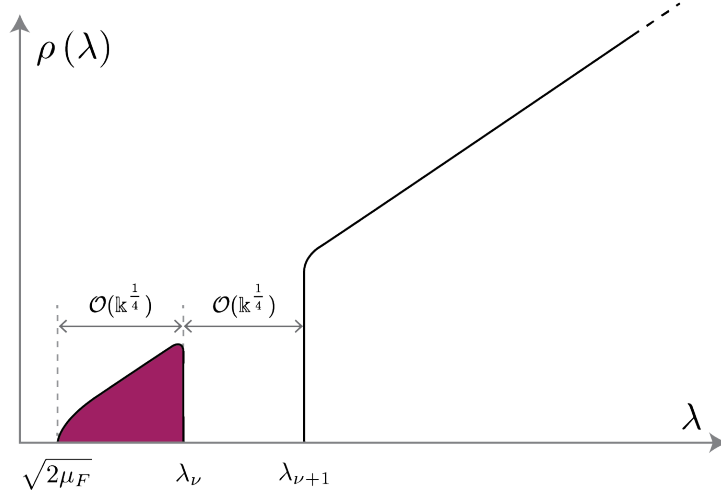


Figure 1.8: A visual representation of the hole-in-the-world state, showing the new eigenvalue density (1.92). The coloured eigenvalue are host to non-singlet excitations and the uncoloured ones are unoccupied.  $\lambda_\nu$ ,  $\lambda_{\nu+1}$ , and  $\lambda_{\nu+1} - \lambda_\nu$  are all  $\mathcal{O}(\ell^{1/4})$ . Here, we have taken the double-scaling limit  $N \rightarrow \infty$ .

We now modify the eigenvalue density to

$$\rho_n(\lambda) = \frac{1}{\pi} \begin{cases} \lambda + n'(\lambda) & 0 < \lambda < \lambda_1 \\ 0 & \lambda_1 < \lambda < \lambda_2 \\ \lambda + n(\lambda) & \lambda > \lambda_2 \end{cases} . \quad (1.92)$$

The functions  $n, n'$  are constrained by

$$\int_0^{\lambda_c} \rho_n(\lambda) d\lambda = N \quad \implies \quad \frac{\lambda_2^2 - \lambda_1^2}{2} = \int_0^{\lambda_1} n'(\lambda) d\lambda + \int_{\lambda_2}^{\lambda_c} n(\lambda) d\lambda . \quad (1.93)$$

We will minimise the energy over the parameters  $\lambda_1, \lambda_2$  and the functions  $n, n'$ . We assume that the maximum value of the eigenvalue  $\lambda_c = \sqrt{2\pi N}$  is not modified.

To calculate the energy of these contributions, we first review the calculation of the energy in the singlet vacuum, using the phase space density as in (1.57). The eigenvalue density is given by integrating over the momentum, and the energy by integrating the Hamiltonian against this phase space density. Setting  $\mu_F = 0$ , which we are now allowed to do, the integral is over  $|p| \leq \lambda$ .

The energy difference between the new and the old eigenvalue distributions is

$$\Delta E_{\text{eig}} = \frac{1}{2\pi} \int_0^{\lambda_c} d\lambda \cdot 2 \int_{\pi\rho(\lambda)}^{\pi\rho_n(\lambda)} dp \frac{p^2 - \lambda^2}{2}. \quad (1.94)$$

The factor of 2 comes from the fact that we have to sum over both positive as well as negative momenta. We find

$$\Delta E_{\text{eig}} = \frac{1}{2\pi} \int_0^{\lambda_1} d\lambda \, n'^2 \left( \frac{n'}{3} + \lambda \right) + \frac{\lambda_2^4 - \lambda_1^4}{12\pi} + \frac{1}{2\pi} \int_{\lambda_2}^{\lambda_c} d\lambda \, n^2 \left( \frac{n}{3} + \lambda \right). \quad (1.95)$$

The term in the middle is the cost of scooping out the eigenvalues to create the hole — it is a positive term because the original energy of these eigenvalues was negative.

The non-singlet energy with this new density of states is found by plugging in the new density of states into (1.85). This gives

$$E_{\text{ns}} \equiv \langle 0 | H_{\text{ns}} | 0 \rangle = \frac{\mathbb{k}}{\pi^2 \nu} \int_0^{\lambda_1} d\lambda' \int_{\lambda_2}^{\lambda_c} d\lambda \frac{(\lambda' + n')(\lambda + n)}{(\lambda - \lambda')^2}, \quad \nu = \int_0^{\lambda_1} \rho_n(\lambda) d\lambda. \quad (1.96)$$

The  $n, n'$ -independent part was discussed in (1.90), modulo the fact that the scalings taken there have not yet been justified. The  $n$ -dependent part of this is

$$E_{\text{ns}}|_n = \frac{\mathbb{k}}{\pi^2 \nu} \int_{\lambda_2}^{\lambda_c} d\lambda \, n(\lambda) \left[ \log \left( 1 - \frac{\lambda_1}{\lambda} \right) + \frac{\lambda_1}{\lambda - \lambda_1} \right] + \frac{\mathbb{k}}{\pi^2 \nu} \int_0^{\lambda_1} d\lambda' \int_{\lambda_2}^{\lambda_c} d\lambda \frac{n(\lambda) n'(\lambda')}{(\lambda - \lambda')^2}, \quad (1.97)$$

and its  $n'$ -dependent part is

$$E_{\text{ns}}|_{n'} = \frac{\mathbb{k}}{\pi^2 \nu} \int_0^{\lambda_1} d\lambda' \, n'(\lambda') \left[ \log \frac{\lambda_c}{\lambda_2 - \lambda'} + \frac{\lambda'}{\lambda_2 - \lambda'} \right] + \frac{\mathbb{k}}{\pi^2 \nu} \int_0^{\lambda_1} d\lambda' \int_{\lambda_2}^{\lambda_c} d\lambda \frac{n(\lambda) n'(\lambda')}{(\lambda - \lambda')^2}. \quad (1.98)$$

Because of the large  $\log \lambda_c \sim \log N$  term in (1.98), minimisation requires that we set

$$\int_0^{\lambda_1} n'(\lambda') d\lambda' = 0. \quad (1.99)$$

In other words, all the eigenvalues we scooped out between  $\lambda_1$  and  $\lambda_2$  get sent to larger eigenvalues — this makes sense since the repulsion wants to send them far away and there's a lot more space at larger eigenvalues. The ones to the left of the hole can move around

but stay to the left of the hole. In particular, this means that

$$\lambda_1 = \lambda_\nu = \sqrt{2\pi\nu} \quad (1.100)$$

as was the case even in the old eigenvalue distribution.

To minimise the energy, we have to minimise the ‘action’

$$E = \Delta E_{\text{eig}} + E_{\text{ns}} - \frac{\alpha}{2\pi} \left( \int_{\lambda_2}^{\lambda_c} n(\lambda) d\lambda - \frac{\lambda_2^2 - \lambda_1^2}{2} \right) - \frac{\alpha'}{2\pi} \int_0^{\lambda_\nu} n'(\lambda') d\lambda', \quad (1.101)$$

where  $\alpha, \alpha'$  are Lagrange multipliers. Setting  $\delta E/\delta n, \delta E/\delta n'$  to 0 gives

$$\begin{aligned} n(\lambda) &= \sqrt{\lambda^2 + \alpha - \beta(\lambda)} - \lambda, \\ n'(\lambda') &= \sqrt{\lambda'^2 + \alpha' - \beta'(\lambda')} - \lambda', \end{aligned} \quad (1.102)$$

where

$$\begin{aligned} \frac{\beta(\lambda)}{2\pi} &= \frac{\mathbb{k}}{\pi^2\nu} \left\{ \log \left( 1 - \frac{\lambda_1}{\lambda} \right) + \frac{\lambda_1}{\lambda - \lambda_1} + \int_0^{\lambda_1} d\lambda' \frac{n'(\lambda')}{(\lambda - \lambda')^2} \right\}, \\ \frac{\beta'(\lambda')}{2\pi} &= \frac{\mathbb{k}}{\pi^2\nu} \left\{ -\log(\lambda_2 - \lambda') + \frac{\lambda'}{\lambda_2 - \lambda'} + \int_{\lambda_2}^{\lambda_c} d\lambda \frac{n(\lambda)}{(\lambda - \lambda')^2} \right\}. \end{aligned} \quad (1.103)$$

Now we have to plug in these solutions into the constraints to solve for the Lagrange multipliers. The constraint for  $n$  is easier to deal with, because the integral has a large IR region  $\lambda \gg \lambda_{1,2}$ . At large  $\lambda$ , the  $n'$ -dependent term in  $\beta$  is effectively proportional to  $\int d\lambda' n'(\lambda') = 0$ , up to corrections that scale with  $\lambda_1^2/\lambda^2$ , and the  $1/\lambda$  contributions from the first two terms cancel each other, giving  $\beta(\lambda) \sim \mathbb{k}/\nu \cdot \lambda_1^2/\lambda^2$ . Thus, at large  $\lambda$ , we find that  $n \sim \alpha/2\lambda$ . Plugging this back into the constraint to solve for  $\alpha$  gives<sup>16</sup>

$$\alpha = \frac{\lambda_2^2 - \lambda_1^2}{\log \lambda_c} \implies n \approx \lambda \left\{ \sqrt{1 - \frac{\beta(\lambda)}{\lambda^2}} - 1 \right\}. \quad (1.104)$$

Close to the hole, this quantity is negative and we have thus found that the non-singlet

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<sup>16</sup>Here, we are assuming that  $\log N \gg \sqrt{\mathbb{k}}$ , consistent with our interest in the double-scaling limit. It can be checked, by not dropping the  $\alpha$  term in this equation, that this is not necessary for the main result of this section to hold. We will use this fact in the discussion to follow.

interaction pushes the eigenvalues away from the hole. Far away from the hole, it vanishes. Since  $\lambda_1/\lambda_2 < 1$ , we expand in  $\lambda_1/\lambda$  to find

$$\frac{\beta(\lambda)}{2\pi} \approx \frac{\mathbb{k}}{2\pi^2\nu} \frac{\lambda_1^2}{\lambda^2}, \quad n \approx -\frac{\mathbb{k}}{4\pi^2\nu} \frac{\lambda_1^2}{\lambda^3} = -\frac{\mathbb{k}}{2\pi\lambda^3}. \quad (1.105)$$

Plugging this into (1.103) and expanding in  $\lambda'/\lambda_2$  gives

$$\frac{\beta'}{2\pi} \approx -\frac{\mathbb{k}}{\pi^2\nu} \log \lambda_2, \quad n' \approx \sqrt{\lambda'^2 + \alpha' + \frac{\mathbb{k} \log \lambda_2}{\pi^2\nu}} - \lambda'. \quad (1.106)$$

The constraint (1.99) can only be satisfied if

$$n' = 0. \quad (1.107)$$

It is straightforward to plug (1.104) and (1.107) back into the energy (1.101) to find

$$E = \frac{\lambda_2^4 - \lambda_1^4}{12\pi} + a \frac{\mathbb{k}^2 \lambda_1^4}{\nu^2 \lambda_2^4} + \frac{\mathbb{k}}{2\pi^2\nu} \left[ -\lambda_1 \lambda_2 + \lambda_1^2 \log \frac{\lambda_c}{\lambda_2 - \lambda_1} - \lambda_2^2 \log \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \right]. \quad (1.108)$$

Here,  $a$  is an  $\mathcal{O}(1)$  factor. To minimise with respect to  $\lambda_2$ , we set the derivative with respect to  $\lambda_2$  to 0, giving the equation (setting  $\lambda_1^2 = 2\pi\nu$  and not keeping track of coefficients)<sup>17</sup>

$$0 = a_1 \frac{\mathbb{k}^2}{\lambda_2^5} + a_2 \lambda_2^3 + a_3 \frac{\mathbb{k}}{\sqrt{\nu}} + a_4 \frac{\mathbb{k}}{\lambda_2 - \lambda_1} = \mathcal{O}(\mathbb{k}^{1/4}) \quad \implies \quad \sqrt{\nu}, \lambda_2 = \mathcal{O}(\mathbb{k}^{1/4}). \quad (1.109)$$

With this scaling the last three terms are all  $\mathcal{O}(\mathbb{k}^{3/4})$  and the first term is smaller. We can now check with this scaling that the energy is

$$E = \frac{\mathbb{k}}{2\pi} \log \frac{N}{\sqrt{\mathbb{k}}} + \mathcal{O}(\mathbb{k}). \quad (1.110)$$

### Fluctuations in the energy

Since this is not an exact eigenstate, we should also check that it is an approximate eigenstate of the Hamiltonian by calculating the standard deviation of the energy. Since the eigenvalue sector is in a classical limit (which is why we could use the language of eigenvalue densities),

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<sup>17</sup>We have dropped terms arising from the derivatives of the logarithms for brevity. Because we are only interested in the scaling of  $\lambda_2$ , it is easily checked that the addition of these terms don't change the result.

it doesn't contribute to the variance at leading order.

To calculate the variance due to the non-singlets, we first consider the action of the non-singlet Hamiltonian on one copy of the  $U(\nu)$  singlet,  $|\nu\rangle$ ,

$$\begin{aligned}
H_{\text{ns}} |\nu\rangle \equiv |\nu'\rangle &= \frac{1}{\sqrt{\nu}} \sum_{j \leq \nu, k > \nu} \frac{|\boxed{j} \ j\rangle - |\boxed{k} \ k\rangle}{(\lambda_j - \lambda_k)^2} \\
&= \frac{1}{\sqrt{\nu}} \left\{ \sum_{j \leq \nu} \underbrace{\sum_{k > \nu} \frac{1}{(\lambda_j - \lambda_k)^2}}_{\equiv \alpha_j} |\boxed{j} \ j\rangle - \sum_{k > \nu} \underbrace{\sum_{j \leq \nu} \frac{1}{(\lambda_j - \lambda_k)^2}}_{\equiv \beta_k} |\boxed{k} \ k\rangle \right\}.
\end{aligned} \tag{1.111}$$

Converting the  $\alpha_j$  and  $\beta_k$  to integrals give,

$$\begin{aligned}
\alpha_j &= \int_{\lambda_{\nu+1}}^{\lambda_c} d\lambda \frac{\rho_n(\lambda)}{(\lambda - \lambda_j)^2} = \frac{1}{\pi} \left\{ \log \frac{\lambda_c}{\lambda_{\nu+1} - \lambda_j} + \frac{\lambda_j}{\lambda_{\nu+1} - \lambda_j} \right\}, \\
\beta_k &= \int_0^{\lambda_\nu} d\lambda \frac{\rho_n(\lambda)}{(\lambda - \lambda_k)^2} = \frac{1}{\pi} \left\{ \log \left( 1 - \frac{\lambda_\nu}{\lambda_k} \right) + \frac{1}{1 - \frac{\lambda_\nu}{\lambda_k}} - 1 \right\}.
\end{aligned} \tag{1.112}$$

Acting on the hole-in-the-world state  $|0\rangle$ , the Hamiltonian gives

$$H_{\text{ns}} |0\rangle = |\rho_n(\lambda)\rangle \otimes \left( \sum_{r=1}^{\mathbb{k}} |\nu\rangle^{\otimes(r-1)} \otimes |\nu'\rangle \otimes |\nu\rangle^{\otimes(\mathbb{k}-r)} \right). \tag{1.113}$$

The expectation value of the Hamiltonian is

$$\langle H_{\text{ns}} \rangle = \frac{\mathbb{k}}{\nu} \sum_j \alpha_j \equiv \mathbb{k} \bar{\alpha}. \tag{1.114}$$

Now we can calculate the two-point function of  $H_{\text{ns}}$

$$\langle 0 | H_{\text{ns}}^2 | 0 \rangle = \|H_{\text{ns}} |0\rangle\|^2 = \frac{\mathbb{k}}{\nu} \left( \sum_j \alpha_j^2 + \sum_k \beta_k^2 \right) + 2 \binom{\mathbb{k}}{2} \bar{\alpha}^2. \tag{1.115}$$

The first term proportional to  $\mathbb{k}/\nu$  is from the terms where both copies of  $H_{\text{ns}}$  act on the same adjoint, and the term proportional to  $\binom{k}{2}$  comes from the case when the two  $H_{\text{ns}}$ 's act on different adjoints. The variance is

$$\langle H_{\text{ns}}^2 \rangle - \langle H_{\text{ns}} \rangle^2 = \overline{\mathbb{k}(\alpha - \bar{\alpha})^2} + \frac{\mathbb{k}}{\nu} \sum_{k=\nu+1}^N \beta_k^2. \quad (1.116)$$

where the overline means the average over the first  $\nu$  eigenvalues. The two terms can be calculated, as always, by converting the sum into an integral

$$\overline{\mathbb{k}(\alpha - \bar{\alpha})^2} \sim \mathbb{k} \frac{\lambda_\nu^2}{\nu} = \mathcal{O}(\mathbb{k}), \quad (1.117)$$

$$\frac{\mathbb{k}}{\nu} \sum_{k=\nu+1}^N \beta_k^2 = \mathcal{O}(\mathbb{k}), \quad (1.118)$$

$$\implies \langle H_{\text{ns}}^2 \rangle - \langle H_{\text{ns}} \rangle^2 = \mathcal{O}(\mathbb{k}). \quad (1.119)$$

Thus, the standard deviation is  $\mathcal{O}(\sqrt{\mathbb{k}})$ , which is significantly smaller than both the energy and the corrections to it that we have not kept track of. We can then safely conclude that the hole-in-the-world state is an approximate eigenstate whose energy matches the energy found by KT (1.22).

### 1.4.3 The fixed-charge partition function

We have isolated a particular bound-state within the dominant  $PU(N)$  irrep, so we have picked one state out of a possible  $N^{\mathbb{k}}$ .<sup>18</sup> Further the considerations in §1.4.2 and §1.4.2 together provide some (inconclusive) evidence that this is more or less the lowest energy state in the sector. However, our Hamiltonian derivation of the KT partition function is not yet complete. This is because this state may not dominate the partition function, since in general the interplay of energy and entropy can cause a higher energy state to dominate the partition function.

The hole-in-the-world state has two free parameters that the analysis in §1.4.2 did not fix, the position of the hole's edges  $\lambda_1, \lambda_2$  and the modification of the eigenvalue distribution to its right  $n(\lambda)$ . All we found was that the former is  $\mathcal{O}(\mathbb{k}^{1/4})$  and the latter has the asymptotic form (1.104). Thus, the number of orthogonal hole-in-the-world states grows

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<sup>18</sup>We have factored out the degeneracy discussed in §1.3. The number  $N^{\mathbb{k}}$  can be derived by the argument above (1.79).

polynomially with  $\mathbb{k}$ , giving an entropy  $\sim \log \mathbb{k}$  (apart from the large entropy discussed in §1.3). Thus, we have to check if there are any other states such that

$$S - 2\pi RE > -R\mathbb{k} \log \frac{N}{\sqrt{\mathbb{k}}} + \mathcal{O}(\mathbb{k}), \quad 1 < R < 2. \quad (1.120)$$

The condition on  $R$  comes from the string theory [91, 93]. Since  $R$  is an  $\mathcal{O}(1)$  number, all we need to check is that the energy cost of excitations outweighs the entropy cost. There are two sorts of states to check, excitations of the solid itself and states where we let all or part of the solid evaporate into the large eigenvalue region.

There are roughly  $\sqrt{\mathbb{k}}^{\mathbb{k}}$  states that are excitations of the solid. The count of states is the dimension of  $\mathbb{k}$  zero-weight  $U(\nu)$  adjoints. Thus the entropic advantage here is  $\sim \mathbb{k} \log \mathbb{k}$ . To calculate the energy cost, we use the result (1.107) that the eigenvalue distribution doesn't get modified within the solid. We start with the observation that compressing  $\mathbb{k}$  vortices into a region of size  $\frac{1}{4} \log \mathbb{k}$  in  $\tau$ -space will lead to kinetic energies of order  $\frac{\mathbb{k}^{3/2}}{\log \mathbb{k}}$ . We now evaluate the variance due to the interactions around this large kinetic energy by noting that the exchange operators appearing in the numerators in the cross-term have precisely one positive and one negative eigenvalue of unit magnitude. We therefore model a generic contribution from the cross term as a random walk — each pair of boxes located at eigenvalues  $i, j$  will randomly contribute either  $(\lambda_i - \lambda_j)^{-2}$  or  $-(\lambda_i - \lambda_j)^{-2}$  to the energy. The number of steps in the random walk will be the number of pairs of boxes —  $\mathbb{k}^2/2$ . The mean square of the step size is evaluated as

$$\frac{1}{\mathbb{k}} \int_0^{\mathbb{k}^{1/4}} \lambda d\lambda \int_0^{\mathbb{k}^{1/4}} \lambda' d\lambda' \frac{1}{(\lambda - \lambda')^4} \approx \mathcal{O}(\mathbb{k}^{1/4}). \quad (1.121)$$

Here the  $1/\mathbb{k}$  factor comes from averaging over  $\mathbb{k}$  pairs of eigenvalues inside the solid. The random walk approximation therefore implies that the standard deviation in the cross-term energy goes like

$$\Delta E_{\times} \sim \mathbb{k} \cdot \mathbb{k}^{1/8} \sim \mathcal{O}(\mathbb{k}^{9/8}). \quad (1.122)$$

Thus, we have determined that the typical energy in this class of states is  $\mathcal{O}\left(\frac{\mathbb{k}^{3/2}}{\log \mathbb{k}}\right) \pm \mathcal{O}(\mathbb{k}^{9/8})$ . This dominates the entropic contribution to the free energy of  $\mathcal{O}(\mathbb{k} \log \mathbb{k})$ , showing the ground state we have picked out indeed dominates the partition function.

We can perform a similar analysis if we allow the box labels to range over the entirety of the eigenvalue range, from  $\lambda = 0$  to  $\lambda = \sqrt{2\pi N}$ . There will be  $\left(\frac{N}{\sqrt{\mathbb{k}}}\right)^{\mathbb{k}}$  such states, with a typical energy of order  $\frac{\mathbb{k}^{3/2}}{\log N}$ . The energy will again suppress the contribution from these

states provided  $\mathbb{k} \log N \ll \frac{\mathbb{k}^{3/2}}{\log N}$ , i.e. when

$$\mathbb{k} \gg \log N. \quad (1.123)$$

We saw a similar but stronger condition in §1.4.2 as the condition that there are no gas of adjoints states with the KT value of the energy. Thus, we have found that the bound state above genuinely dominates the partition function at  $\mathbb{k} \gg \log N$ .

Unfortunately, however, (1.123) is not satisfied in the double-scaling limit, where we expect  $\mathbb{k} \sim \mathfrak{z}^{\frac{4}{2-R}} = \mathcal{O}(N^0)$ . This is natural, since in the double-scaling limit the black hole is not in a finite-volume box and so should evaporate, whereas when (1.123) is satisfied there is a box with finite size relative to the black hole. However, as we argue in the following section, the hole-in-the-world state is nevertheless metastable when  $\mathbb{k} = \mathcal{O}(N^0)$ , with a lifetime that grows polynomially in  $\mathbb{k}$ .

### Evaporation rate and the Page time of the black hole

Let us now calculate the lifetime of the state above in the double-scaling limit  $1 \ll \mathbb{k} \sim N^0$ . We will find that it grows polynomially in  $\mathbb{k}$ , which, as we discuss in §1.5, we may consider to be a renormalised mass .

The action of an infinitesimal amount of time-evolution on the state yields just a phase with high probability, since it is an eigenstate, and with a small probability, a state where one adjoint escapes, i.e.,

$$e^{\iota H_{\text{ns}} \epsilon} |\nu, \mathbb{k}\rangle = e^{\iota E_{\text{ns}} \epsilon} \sqrt{1-r} |\nu, \mathbb{k}\rangle + \iota \epsilon \sqrt{r} |\nu, \mathbb{k}-1\rangle \otimes |\text{free adjoint}\rangle. \quad (1.124)$$

There is also a slight rearranging of the adjoints within the solid, as given by the  $\alpha_j$  term in (1.113); the branch where the adjoint escapes is the one proportional to the  $\beta_k$ s in the same equation. We ignore the  $\alpha_j$  branch for simplicity of exposition, but the following discussion is not modified by its inclusion.

The second branch in (1.124), where a single adjoint escapes, is illustrated in Figure 1.9. Because the hole-in-the-world state is an approximate eigenstate, the second branch has a much smaller amplitude that we have denoted by  $\sqrt{r}$ .  $r$ , which we may call an escape

rate, can be estimated straightforwardly to be<sup>19</sup>

$$r = \langle H_{\text{ns}}^2 \rangle - \langle H_{\text{ns}} \rangle^2 \approx \mathbb{k}. \quad (1.125)$$

The log of this escape rate can also be thought of as the imaginary part of the energy of the state, and the lifetime of the state can be calculated using formulas for that of a resonance.

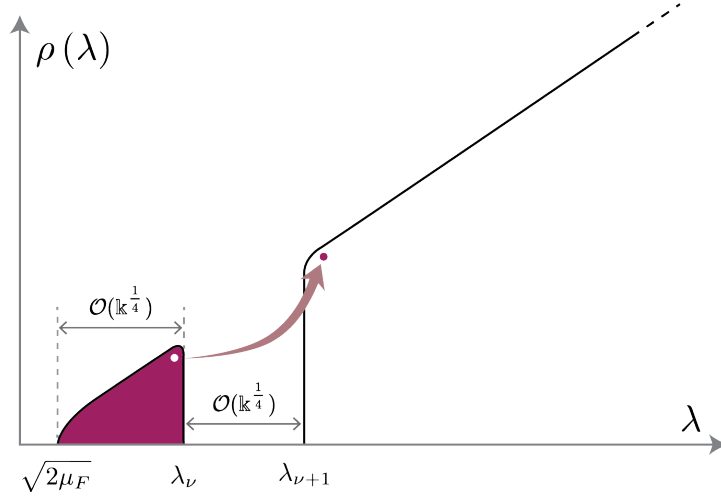


Figure 1.9: The evaporation process of the solid — an adjoint jumps out of the solid and into the rest of the Fermi sea.

We adopt a more direct method here. Upon  $n$  steps of time-evolution, we find

$$(e^{\iota H_{\text{ns}} \epsilon})^n |\nu, \mathbb{k}\rangle = e^{\iota E_{\text{ns}} n \epsilon} (1 - r)^{n/2} |\nu, \mathbb{k}\rangle + \iota n \epsilon \sqrt{r} |\nu, \mathbb{k} - 1\rangle \otimes |\text{free adjoint}\rangle + \mathcal{O}(\epsilon^2). \quad (1.126)$$

Thus, we see that the branch where a single adjoint has escaped has an  $\mathcal{O}(1)$  amplitude after time

$$t_{\text{esc}} \sim n \epsilon \sim \frac{1}{\sqrt{r}} \sim \frac{1}{\sqrt{\mathbb{k}}}. \quad (1.127)$$

The inverse of this quantity is the evaporation rate of the black hole. Notice that this calculation also agrees with the expectation that one may have from the energy-time uncertainty relation, recalling that the energy width of this state is  $\mathcal{O}(\sqrt{\mathbb{k}})$ .

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<sup>19</sup>Consider a Hamiltonian  $H = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$ . After time-evolution by an infinitesimal time  $\epsilon$ , the state  $|0\rangle$  evolves to  $e^{\iota H \epsilon} |0\rangle = e^{\iota a \epsilon} \left(1 - \frac{\epsilon^2}{2} |c|^2\right) |0\rangle + \iota \epsilon c |1\rangle$ . Comparing with (1.124), we see that  $r = |c|^2$ . It is also easy to verify that  $\langle 0 | H^2 | 0 \rangle - \langle 0 | H | 0 \rangle^2 = |c|^2$  in this two-state system.

Since there are  $\mathbb{k}$  adjoints, the time for most of them to evaporate is

$$t_{\text{Page}} \approx \int_1^{\mathbb{k}} d\mathbb{k}' t_{\text{esc}}(\mathbb{k}') \sim \sqrt{\mathbb{k}}. \quad (1.128)$$

We may consider this the Page time of this black hole.

The inclusion of a third branch in (1.124) corresponding to the  $\alpha_j$ s in (1.113) doesn't modify this discussion because it contributes to the variance at the same order, and so that rearrangement of the adjoint distribution becomes important on the same time-scale.

## 1.5 From the fixed-charge to the grand canonical ensemble

The partition function that [91] calculate isn't in the fixed-charge ensemble but in a grand canonical ensemble,

$$Z(\mathfrak{z}_b) = \int_{U(N)} d\Omega e^{N\mathfrak{z}_b[\text{tr}\Omega + \text{tr}\Omega^\dagger]} Z(\Omega), \quad (1.129)$$

where  $Z(\Omega)$  is the partition function with twisted boundary conditions  $X(\beta) = \Omega^\dagger X(0) \Omega$ . In this section, we verify that our results are consistent with theirs and attempt to shed some light on the question of whether the  $PU(N)$  symmetry is gauged or not.<sup>20</sup>

Let us first assume that the  $PU(N)$  is a global symmetry and show that our results are consistent with those of [91, 93]. Since the twisted partition function is a class function [28], we can decompose it into fixed  $PU(N)$  irrep sectors as

$$Z(\Omega) = \sum_r \frac{\chi_r(\Omega)}{d_r} Z_r. \quad (1.130)$$

Here, the characters are normalised so that

$$\chi_r(\mathbb{1}) = d_r, \quad \int d\Omega \chi_{r_1}(\Omega)^* \chi_{r_2}(\Omega) = \delta_{r_1, r_2}. \quad (1.131)$$

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<sup>20</sup>We thank Juan Maldacena for sharing some unpublished notes which directly inspired this section.

The factors  $d_r$  of the dimension of the irreps ensures that

$$Z(\mathbb{1}) = \sum_r Z_r = \sum_r \text{tr}_{\mathcal{H}_r} e^{-\beta H} = \text{tr}_{\mathcal{H}} \left( \bigoplus_{\mathcal{H}_r} e^{-\beta H} \right) = \text{tr}_{\mathcal{H}} e^{-\beta H}, \quad (1.132)$$

meaning that  $Z(\mathbb{1})$  is the trace over the entire Hilbert space and  $Z_r$  is the trace over the fixed-irrep Hilbert space  $\mathcal{H}_r$ . Here, unlike in some earlier section,  $\mathcal{H}_r$  is not the irrep Hilbert space but the space of all states that transform in the irrep  $r$  — in a hydrogen atom where the irreps are labelled by  $l$ ,  $\mathcal{H}_l$  is spanned by states with all  $n, m$  indices consistent with  $l$ . This justification for the factor of  $d_r$  is only sensible in the case that  $PU(N)$  is a global symmetry. We now show that the fixed irrep partition function we have found results in the partition function of [91] after the convolution (1.129) and a crucial renormalisation.

The first step is to expand the exponent

$$e^{N\mathfrak{z}_b \{\text{tr } \Omega + \text{tr } \Omega^\dagger\}} = \sum_{\mathbb{k}, \mathbb{k}'=0}^{\infty} \frac{(N\mathfrak{z}_b)^{\mathbb{k}+\mathbb{k}'}}{(\mathbb{k}+\mathbb{k}')!} \binom{\mathbb{k}+\mathbb{k}'}{\mathbb{k}} (\text{tr } \Omega)^{\mathbb{k}} (\text{tr } \Omega^\dagger)^{\mathbb{k}'}. \quad (1.133)$$

Using the fact that  $Z(\Omega)$  is independent of the global phase mode of  $\Omega$  (by definition), we can integrate over it to find

$$\int_{U(1)_{\text{centre}}} e^{N\mathfrak{z}_b \{\text{tr } \Omega + \text{tr } \Omega^\dagger\}} = \sum_{\mathbb{k}=0}^{\infty} \frac{(N\mathfrak{z}_b)^{2\mathbb{k}}}{(\mathbb{k}!)^2} (\text{tr } \Omega')^{\mathbb{k}} (\text{tr } \Omega'^\dagger)^{\mathbb{k}}, \quad (1.134)$$

where  $\Omega'$  is valued in  $PU(N)$ . The integral over the phase mode is proportional to  $\delta_{\mathbb{k}, \mathbb{k}'}$ . So we have, renaming  $\Omega' \rightarrow \Omega$ ,

$$Z(\mathfrak{z}_b) = \sum_{\mathbb{k}} \frac{(N\mathfrak{z}_b)^{2\mathbb{k}}}{\mathbb{k}!^2} \int_{PU(N)} (\text{tr } \Omega)^{\mathbb{k}} (\text{tr } \Omega^\dagger)^{\mathbb{k}} \sum_r \frac{\chi_r(\Omega)}{d_r} Z_r. \quad (1.135)$$

The next step is to do the  $\Omega$  integral

$$\frac{1}{d_{r_1} d_{\bar{r}_2}} \int_{U(N)} \chi_{f \otimes \mathbb{k} \otimes \bar{f} \otimes \mathbb{k}}(\Omega) \chi_{r_1 \bar{r}_2}(\Omega). \quad (1.136)$$

Notice that we are integrating over all of  $U(N)$  and not just  $PU(N)$  — because the integrand is invariant and we aren't keeping track of normalisations, this makes no difference. We have also written the irrep  $r$  as the symmetrisation of an irrep  $r_1$  of the boxes and an irrep

$\bar{r}_2$  of the anti-boxes. At leading order in large  $N$ , we can forget about the symmetrisation and take the full irrep to be simply  $r_1 \otimes \bar{r}_2$ .<sup>21</sup> Now, Schur–Weyl duality states that  $\mathbb{k}$  copies of the fundamental decomposes into irreps with  $\mathbb{k}$  boxes in the YD with multiplicity  $d_r^{(\mathbb{S}_{\mathbb{k}})}$  equal to the dimension of the corresponding irrep of  $\mathbb{S}_{\mathbb{k}}$ . For the characters, this implies<sup>22</sup>

$$f^{\otimes \mathbb{k}} = \bigoplus_{r \mid \mathbb{k}} r_{U(N)} \otimes r_{\mathbb{S}_{\mathbb{k}}} \quad \Rightarrow \quad \chi_{f^{\otimes \mathbb{k}}} = \sum d_r^{\mathbb{S}_{\mathbb{k}}} \chi_r. \quad (1.137)$$

The symbol  $r \mid \mathbb{k}$  denotes the restriction to representations with exactly  $\mathbb{k}$  boxes. Thus, for  $r_1, r_2$  with  $\mathbb{k}$  boxes each, we find

$$\frac{1}{d_{r_1} d_{\bar{r}_2}} \int_{U(N)} \chi_{f^{\otimes \mathbb{k}} \otimes \bar{f}^{\otimes \mathbb{k}}}(\Omega) \chi_{r_1 \bar{r}_2}(\Omega) \approx \frac{d_{r_1}^{\mathbb{S}_{\mathbb{k}}} d_{\bar{r}_2}^{\mathbb{S}_{\mathbb{k}}}}{d_{r_1} d_{\bar{r}_2}} \approx \binom{N}{\mathbb{k}}^{-2} \approx \frac{(\mathbb{k}!)^2}{N^{2\mathbb{k}}}. \quad (1.138)$$

Here, we’ve used (1.131) and (1.137) in the first equality and (1.30) in the second equality. The first equality is only approximately true because we’re ignoring the possibility that some pairs of fundamentals and anti-fundamentals could fuse to a singlet state.

The final partition function, plugging (1.138) into (1.135) and using the KT free energy, is

$$Z(\mathfrak{z}) = \sum_{\mathbb{k}} \mathfrak{z}_b^{2\mathbb{k}} \sum_{r_1, r_2 \mid \mathbb{k}} Z_{r_1 \bar{r}_2} = \sum_{\mathbb{k}} \mathfrak{z}_b^{2\mathbb{k}} \left( \frac{N}{\sqrt{\mathbb{k}}} \right)^{(2-R)\mathbb{k}} = \sum_{\mathbb{k}} \left( \mathfrak{z}_b N^{\frac{2-R}{2}} \right)^{2\mathbb{k}} \left( \sqrt{\mathbb{k}} \right)^{-(2-R)\mathbb{k}}. \quad (1.139)$$

The quantity within the brackets is exactly the renormalised fugacity (1.17).<sup>23</sup> Evaluating the sum by saddle-point, we find exactly the answer (1.18) from [91] with  $\mathfrak{z}$  being the *renormalised* fugacity.

Let us observe that we can define a renormalised fixed-charge partition function<sup>24</sup>

$$Z_{\text{ren}, \mathbb{k}} = \left( \sqrt{\mathbb{k}} \right)^{(2+R)\mathbb{k}}, \quad S_{\text{ren}} = 2\mathbb{k} \log \sqrt{\mathbb{k}}, \quad E_{\text{ren}} = -\frac{\mathbb{k}}{2\pi} \log \sqrt{\mathbb{k}}, \quad (1.140)$$

<sup>21</sup>At finite  $N$ ,  $r_1 \otimes \bar{r}_2$  contains irreps with fewer than  $\mathbb{k}$  boxes as well.

<sup>22</sup>The fact that each irrep appears with such a multiplicity can be rephrased as the statement that we are summing over irreps with a ‘Poissonised–Plancherel’ measure [24].

<sup>23</sup>We thank Juan Maldacena for pointing this out.

<sup>24</sup>This quantity was originally defined by Juan Maldacena in unpublished notes.

which is dual to the renormalised fugacity  $\mathfrak{z}$  rather than the bare fugacity  $\mathfrak{z}_b$ ,

$$Z(\mathfrak{z}) = \sum_{\mathbb{k}} \frac{\mathfrak{z}^{2\mathbb{k}}}{\mathbb{k}!^2} Z_{\text{ren},\mathbb{k}}. \quad (1.141)$$

Comparing this with (1.135), it seems that  $Z_{\text{ren},\mathbb{k}}$  is a renormalised version of the fixed  $\mathbb{k}$  partition function  $\int_{PU(N)} (\text{tr } \Omega)^{\mathbb{k}} (\text{tr } \Omega^\dagger)^{\mathbb{k}} Z(\Omega)$ . We will return to this point in §1.5.1 and §1.6.4.

### 1.5.1 Gauging as renormalisation

We finally address the difference between our approach and that of [115], which is the question of whether the  $PU(N)$  symmetry should be gauged or not. We find that, even if we start with the ungauged theory, a gauged description might still emerge after renormalisation in the double-scaling limit.

To begin, we note that we can rewrite the twisted partition function  $Z(\Omega)$  as the path integral of a gauged theory,<sup>25</sup>

$$Z(\Omega) = \int_{Pe^\iota \oint A = \Omega} \frac{[\mathcal{D}X][\mathcal{D}A]'}{\text{vol}(\text{gauge})} e^{-S[X,A]}, \quad (1.142)$$

The integral over  $A$  excludes the zero-mode, which is fixed. This is the same as the ungauged twisted partition function, since for every  $A$  degree of freedom, i.e., the non-zero modes of  $A$ , that we include in  $[\mathcal{D}A]$ , we introduce a corresponding constraint. The zero-mode of the gauge field is just the twist.

Where the gauged and ungauged partition function differ is in the definition of the fixed-irrep partition function. In the ungauged theory, it can naturally be defined as the trace over a Hilbert space as in (1.132). In the gauged theory, however, the most sensible definition is

$$Z_r^{(\text{g})} = \int \frac{[\mathcal{D}X][\mathcal{D}A]}{\text{vol}(\text{gauge})} \text{tr}_r \left( Pe^\iota \oint A \right) e^{-S[X,A]}. \quad (1.143)$$

We have reintroduced the integral over the zero-mode of  $A$  here. This was how the adjoint partition function was defined in [115]. Comparing this with (1.130), we find that

$$Z_r^{(\text{g})} = \frac{Z_r}{d_r} \xrightarrow{\text{dominant YD}} e^{-R\mathbb{k} \log \frac{N}{\sqrt{\mathbb{k}}}}. \quad (1.144)$$

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<sup>25</sup>We thank Shiraz Minwalla for emphasising this fact to us.

The partition function of [91, 93] is then

$$Z(\mathfrak{z}_b) = \sum_{\mathbb{k}=0}^{\infty} \frac{(N\mathfrak{z}_b)^{2\mathbb{k}}}{\mathbb{k}!^2} \int d\Omega \chi_{f^{\otimes \mathbb{k}} \otimes \bar{f}^{\otimes \mathbb{k}}}(\Omega) \sum_{r_1, r_2} \chi_{r_1 \bar{r}_2}(\Omega) e^{-R\mathbb{k} \log \frac{N}{\sqrt{\mathbb{k}}}}. \quad (1.145)$$

The  $\Omega$  integral gives, as in (1.138),

$$\sum_{r_1, r_2} \int d\Omega \chi_{f^{\otimes \mathbb{k}} \otimes \bar{f}^{\otimes \mathbb{k}}}(\Omega) \chi_{r_1 \bar{r}_2}(\Omega) \approx \sum_{r_1} d_{r_1}^{(S_{\mathbb{k}})} \sum_{r_2} d_{r_2}^{(S_{\mathbb{k}})} \approx \mathbb{k}!. \quad (1.146)$$

The sums are done as in §1.3.2, and we know from there that they are dominated by one irrep. Thus, we see that the fixed  $\mathbb{k}$  partition function has extra entropy of  $2\mathbb{k} \log \sqrt{\mathbb{k}}$  compared to the fixed-irrep partition function, and that this entropy is entirely from the multiplicity of the dominant irrep in the product of  $\mathbb{k}$  adjoints.

Absorbing the powers of  $N$  to renormalise  $\mathfrak{z}$ , we see that the fixed- $\mathbb{k}$  partition function in the gauged interpretation is

$$Z_{\mathbb{k}}^{(\text{g})} = \left(\sqrt{\mathbb{k}}\right)^{(2+R)\mathbb{k}} = Z_{\text{ren}, \mathbb{k}}, \quad (1.147)$$

i.e. we get exactly the renormalised partition function defined in (1.140).

So we see that we can reproduce the result of [91, 93] from both the gauged as well as the ungauged interpretations. Since the gauged entropy agrees with the renormalised entropy defined in (1.140), we speculate that the passage from the ungauged theory to the gauged one corresponds to a renormalisation in the double-scaling limit. We give pictorial evidence for this speculation in §1.6.3. We leave further analysis of this difference to future work.

### 1.5.2 Entropy of the black hole

Now, for the sake of clarity, we collect our results on the entropy. There are three important partition functions: the fixed-irrep partition function, the fixed-charge partition function, and the grand canonical partition function.

There are two versions of the fixed-irrep partition function: the gauged and the ungauged. The ungauged model has an entropy given by the log of the dimension of the irrep dimension, as we found in §1.3; for the dominant irrep at a given  $\mathbb{k}$  this entropy is  $2\mathbb{k} \log N/\sqrt{\mathbb{k}} + \mathcal{O}(\mathbb{k})$ . The gauged fixed-irrep partition function (1.143), which is really the

expectation value of a Wilson line wrapping the thermal circle, is the number of excitations of the hole-in-the-world state that change the energy only at  $\mathcal{O}(\mathbb{k})$ . As discussed above (1.120), this entropy is  $\mathcal{O}(\log \mathbb{k})$ .<sup>26</sup>

For the fixed-charge partition function, there is a new contribution to the entropy from the dominant irrep appearing with a certain multiplicity in the product of  $\mathbb{k}$  adjoints. This contribution is  $\mathbb{k} \log \mathbb{k}$ , as we discuss around (1.146). The grand canonical ensemble is at leading order equivalent to the fixed- $\mathbb{k}$  ensemble and so there is nothing new in this case.

As we have argued in sections 1.5.1 and 1.6.3, the ungauged entropy is dominated by IR degrees of freedom that decouple from the bulk theory and thus shouldn't count towards the entropy of the black hole. The actual entropy of the black hole should be the log of the number of states in the gauged theory in the sector with  $\mathbb{k}$  adjoints,

$$S_{\text{BH}} = \mathbb{k} \log \mathbb{k} + \mathcal{O}(\mathbb{k}). \quad (1.148)$$

We prefer this answer over the number of states in the dominant irrep because to actually see this state we would need to collapse  $\mathbb{k}$  adjoints; we would see  $e^S$  distinct bound states, corresponding to the different permutations of the  $\mathbb{k}$  adjoints that we collapsed.

## 1.6 Discussion

In this work, we have found a Hamiltonian derivation of the free energy of the  $2d$  black hole. The microstates of the black hole are dual to the subspace of  $\mathbb{k}$  adjoints of the  $PU(N)$  symmetry. The entropy can be calculated either directly in terms of  $\mathbb{k}$  indistinguishable adjoints or by counting the dimension of the irrep that dominates this tensor product. The energy was calculated by a direct analysis of the Schrödinger equation, and finding a non-trivial bound state that dominates the fixed-charge partition function. Since [92] had two interpretations that resulted in different free energies, the final outcome is not only that we have *reproduced* the free energy but that we have *picked out* one of these two prescriptions.

Our main result, however, is the existence of a novel, long-lived bound state — the hole-in-the-world state — that has properties very reminiscent of a black hole. We will discuss further the ways in which the properties of this state match the two-dimensional bulk description in §1.6.1, but we may also wonder about higher-dimensional black holes. The purely two-dimensional black hole has the unsatisfactory property that, because  $\mathbb{k}-2$  is

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<sup>26</sup>This is in agreement with unpublished results of Juan Maldacena.

small, strings on this background are strongly coupled, because of which its black hole-like nature is unclear. However, as discussed recently in [36] among others, the  $SL(2, \mathbb{R})_k/U(1)$  CFT when considered as a two-dimensional sector of a higher-dimensional string theory can have  $k > 3$  where it is weakly coupled. Further, the WZW model is believed to be exactly dual to a sine-Liouville CFT at arbitrary  $k$  and so sine-Liouville theory is relevant for the description of higher-dimensional black holes. While we have not verified that the MQM description is relevant in this higher-dimensional setting, our result indicates that the Lorentzian description of the microstates in the sine-Liouville CFT might be a condensate of stretched strings similar to our hole-in-the-world state. In other words, we speculate that this hole-in-the-world state points the way to a more general Lorentzian description of the winding condensate that is believed to carry all the entropy of a black hole, see e.g. [29].

Another important avenue to explore in the future is to obtain the stringy counterparts of the black hole microstates we found in this paper on the *bulk* side, specifically on the cigar, using the off-shell formulation of string theory presented in [4, 5].

In the rest of this section, we speculate on what our results might imply for the string-theoretic interpretation of this system. First, in §1.6.1 we discuss aspects of our central result — the hole-in-the-world state. We heuristically sketch out ways in which it behaves like a black hole and the sine-Liouville string theory. We go on in §1.6.2 with a short summary of the thermal phase structure of string theory, and what our results seem to imply for the phase structure of the  $c = 1$  theory. In §1.6.3, we present a natural extension of the string theory construction of [115] that seems consistent with our results; it also provides some motivation for thinking of the passage from the ungauged MQM to the gauged one as renormalisation in the double-scaling limit. Finally, in §1.6.4, we point out that our count of states is related to a random walk model called the Motzkin walk model; we speculate on the interpretation of this in string-theoretic terms.

### 1.6.1 Bulk properties of the ‘hole-in-the-world’ state

Since the spatial direction of the string theory target space is identified with the Fermi surface, one might worry that a hole in the eigenvalue distribution shouldn’t be allowed because it would cause a radical change on the string theory side. In this section, we argue heuristically that the formation of the hole is dual to the passage from sine-Gordon theory to sine-Liouville theory in the bulk.

In the usual duality between the singlet sector of the MQM and  $c = 1$  string theory, a probe closed string is dual to a particle-hole pair in the Fermi sea. In the bulk, such a

string scatters off the tachyon wall — the  $\mu e^{-2\phi}$  term in the worldsheet action — and in the MQM the particle-hole pair scatters off the edge of the Fermi sea at  $\lambda = \sqrt{2\mu_F}$ . Thus, the reflection amplitudes in both the bulk and the boundary are non-analytic in  $\mu$ .

In the hole-in-the-world state, a particle-hole pair now scatters off the new edge of the Fermi sea at  $\mathbb{k}^{1/4} \sim \mathfrak{z}^{\frac{1}{2-R}}$  — note that the  $\mathfrak{z}$  scaling is that found in (1.23). In sine-Gordon theory, the worldsheet action contains a term  $\left(\mathfrak{z}^{\frac{2}{2-R}} e^{-2\phi}\right)^\#$ , much as it contains a term  $\mu e^{-2\phi}$ . We see that the coefficient of  $e^{-2\phi}$  is exactly the square of the position of the edge of the Fermi sea in both cases. Thus, it is plausible that scattering of closed strings at large  $\mathfrak{z}$  and the scattering of particle-hole pairs in the hole state are non-analytic in the same parameter  $\mathfrak{z}^{\frac{2}{2-R}}$  and are further exactly dual. For example, the leading non-analyticity of the susceptibility at large  $\mathfrak{z}$  calculated in [93] is precisely  $\log \mathfrak{z}^{\frac{2}{2-R}}$ . Such similarities between  $\mu$  and  $\mathfrak{z}^{\frac{2}{2-R}}$  can also be seen in the reflection coefficients, see e.g. [130, 132]. We leave a detailed check of such a duality to future work.

More evidence of the duality with sine-Liouville can be seen in the excitations of the solid itself. Since it is a compact region in the double-scaling limit, we expect that the spectrum of these excitations is quantised. Further, we should not be able to excite the eigenvalues without ‘carrying around’ the adjoints, since the adjoint index is associated with the corresponding eigenvalue.

Sine-Liouville theory *also* has a spectrum of bound states in which momentum and winding modes are excited together,

$$j_N = \frac{\mathbb{k}|w| - |n|}{2} - N \in \left(\frac{1}{2}, \frac{\mathbb{k} - 1}{2}\right), \quad N \in \mathbb{N},$$

$$h_{jnw} = -\frac{j(j-1)}{\mathbb{k}-2} + \frac{(n-kw)^2}{4\mathbb{k}} \quad \bar{h}_{jnw} = -\frac{j(j-1)}{\mathbb{k}-2} + \frac{(n+kw)^2}{4\mathbb{k}}. \quad (1.149)$$

Here,  $n$  counts momentum modes and  $w$  winding modes. The constraint on  $j_N$  ensures that momentum modes can’t be excited without winding modes. We can associate a momentum mode with the eigenvalue and the winding mode with an adjoint. Thus, the constraint is entirely analogous to what we expect from the solid.

Checking this duality by making the preceding statements precise is a promising avenue of future work.

## The Hole-in-the-World and black holes

The hole-in-the-world state bears several qualitative resemblances to black holes. The state radiates into free particles — because the interaction between adjoints falls off like

$1/(\Delta\lambda)^2$ , an adjoint far away from the solid will behave to first order as a free particle.

As discussed in §1.4.3, the state radiates like a black hole at a slow rate, with a lifetime polynomially large in  $\mathbb{k}$ . When considering the bound state in  $\tau$ -space, which is perhaps more natural from the perspective of the free adjoints, the size of the hole is always  $\mathcal{O}(\lambda_{\nu+1}/\lambda_\nu) = \mathcal{O}(1)$ , while the size of the bound solid itself is  $\mathcal{O}(\log \mathbb{k})$ . We can see from the discussion in §1.4.2 that a configuration of free adjoints will only be energetically favored when they are spread over a region  $\Delta\lambda$  of size  $\log \Delta\lambda \sim \Delta\tau \sim \mathcal{O}(\mathbb{k}^{1/2})$ , which we can interpret as a Schwarzschild radius. We note that both the evaporation time and this effective radius are polynomial in the mass of the black hole. This distribution of scales in  $\tau$ -space is shown in Fig. 1.10:

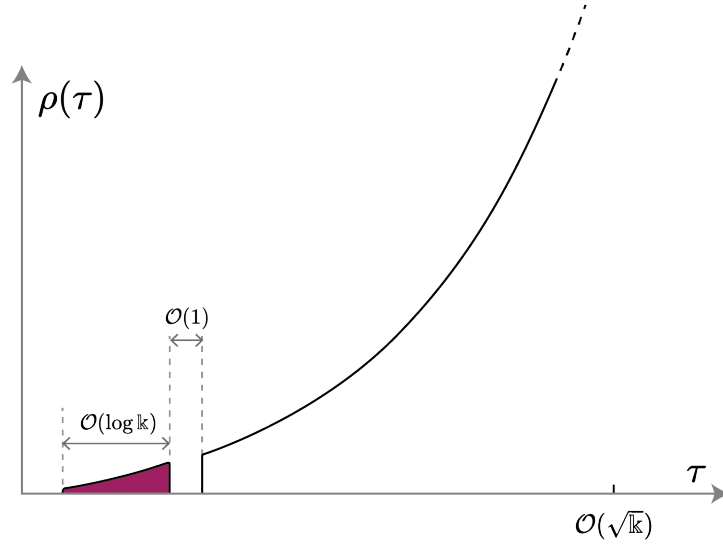


Figure 1.10: A depiction of the eigenvalue distribution shown in Fig. 1.8, this time with respect to  $\tau \sim \log \lambda$ . We note that in  $\tau$ -space, the hole-in-the-world's size does not scale with  $\mathbb{k}$ . The disconnected solid of non-singlets is of size  $\mathcal{O}(\mathbb{k})$ , and its Schwarzschild radius is of size  $\mathcal{O}(\mathbb{k})$ .

This radiation can furthermore be mined as the black hole radiates entirely. It would be interesting to compare the resulting entanglement curve between the solid and the collected radiation with the semiclassical Liouville calculation [26].

The bound state of eigenvalues further bears resemblance to bound  $D0$ -brane models of black holes in BFSS, whose evaporation processes were considered in [22]. However, in such models the bound states radiate due to a supersymmetry-induced cancellation of strong attractive forces due to open strings stretching between the branes. Despite being non-supersymmetric, this model reproduces similar behavior.

### 1.6.2 Thermal phase transitions in string theory

In this subsection, we begin with a review of some of the classic results about the canonical ensemble in string theory [140, 99, 13] and try to put the story reviewed in §1.2 into the context of this literature. Finally, we attempt to physically interpret the fact that our calculation picked out the second prescription of [92].

One of the first conclusions one can reach from the spectrum of closed string states is that there's a Hagedorn divergence in the one-loop partition function. This is because the log of the density of states has the same asymptotic behaviour as the energy spectrum at large dimension — both grow as  $\sqrt{\Delta}$  where  $\Delta$  is the conformal dimension of the oscillators (that of the full vertex operator must be 2). As a result, the partition function looks like

$$Z(\beta) \sim \int dE e^{\beta_H E} e^{-\beta E}, \quad (1.150)$$

where the first piece is the density of states and  $\beta_H = 4\pi\sqrt{\alpha'}$  is the reciprocal of the Hagedorn temperature  $T_H$ ; at  $T > T_H$  the exponent grows at large energies and the integral diverges. Naïvely then, it appears as if string theory has a limiting temperature.

Further progress was made by [140, 99], who made a remarkable observation about this temperature. The observation was that  $T_H$  is exactly the temperature at which two things happen:

1. The lowest *winding mode* of the string becomes massless. In the expansion of the worldsheet field  $T(\tau, \sigma)$  where  $T$  is the target space Euclidean time and  $\tau, \sigma$  are worldsheet coordinates,

$$T = T_0 + \frac{n}{R}\tau + mR\sigma + \text{oscillators}, \quad (1.151)$$

the mode with no oscillators excited and  $n = 0, m = 1$  is the lowest winding mode, since it is the lowest mass-squared mode where a spatial circle of the string winds around the cylinder. The corresponding vertex operator is  $e^{\iota R(T_L - T_R)}$ . Its mass depends on  $R$ ; at lower temperatures, it has  $m^2 > 0$  and at higher temperatures it has  $m^2 < 0$ .

This means that this mode becomes a *winding tachyon* at  $T > T_H$  and so one should think of the Hagedorn temperature as the location of a phase transition where this tachyon condenses.

Of course, there is the question of whether the winding tachyon potential has a stable minimum.

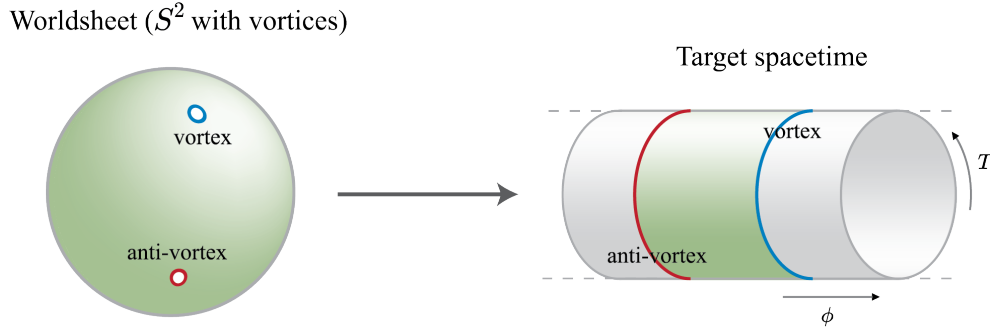


Figure 1.11: A vortex is a hole that winds around the thermal circle.

2. The worldsheet undergoes a Berezinskii–Kosterlitz–Thouless (BKT) transition, see e.g. [10] for an introduction, in which vortices get deconfined and proliferate.

From the stringy point-of-view, a vortex is just a hole that winds around the thermal circle, see figure 1.11. Interestingly enough, since a vortex needs to have a core where the derivatives diverge, this proliferation also introduces a *worldsheet UV cutoff* — meaning that conformal invariance is broken on the worldsheet. Since conformal invariance was the main reason that the  $S^2$  partition function of string theory vanished, this phase transition also causes the genus 0 partition function<sup>27</sup> to become non-zero.

The fact that these happen at the same temperature is not terribly surprising, since the insertion of a winding mode operator can equivalently be thought of as that of a vortex.<sup>28</sup> The condensation of the winding tachyon, then, is just the proliferation of vortices.

Despite their equivalence, the BKT point of view has an important advantage: the question of whether the winding tachyon potential has a stable minimum becomes the question of whether the worldsheet theory above the BKT transition flows to a non-trivial CFT when you add a fugacity for the winding mode operator, see figure 1 and subsequent discussion in [140] for an exceptionally clear exposition. In  $\mathbb{R}^{25} \times S^1$ , the answer is no; there is a Jeans instability where, because of the infinite volume of space, the matter starts “clumping” together.

In the  $c = 1$  case, however, the answer is yes — the sine-Liouville theory is a non-trivial CFT at a non-zero value of the fugacity  $\mathfrak{z}$  for  $R < 2$ . An important point here is that the sine-Liouville theory cannot be studied perturbatively in  $\mathfrak{z}$ , see the discussion around (1.12), which can be thought of as the fact that adding this term triggers a non-trivial RG

<sup>27</sup>As a pedantic point, we note that introducing holes causes the Euler characteristic to change but not the genus.

<sup>28</sup>To see this, consider a small disc around a winding mode operator. Using the state-operator correspondence and the fact that the state looks like (1.151) with  $n = 0, m = 1$ , the boundary of this disc winds around the thermal circle. This is the same property that characterises a vortex.

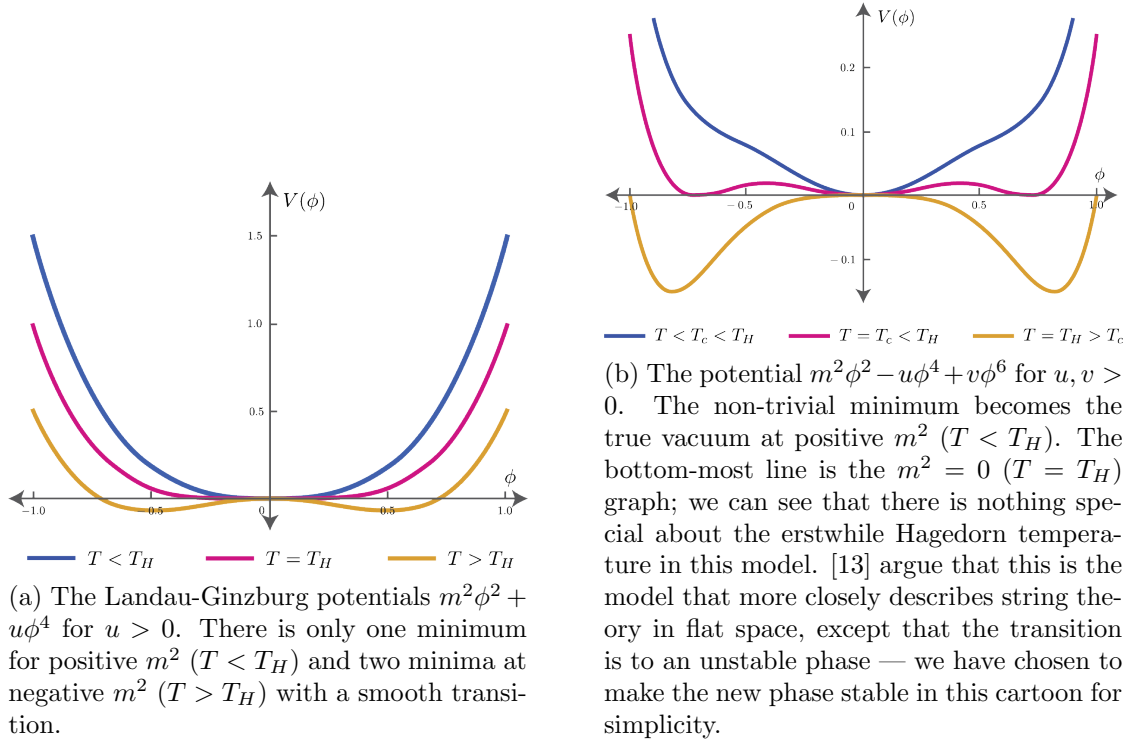


Figure 1.12: A cartoon of how a first-order transition happens at a lower than expected temperature. In both cases, increasing the temperature takes  $m^2$  from positive to negative.

flow to a new fixed point. This new fixed point, in turn, can be thought of as being dual to our hole-in-the-world state. The fact that the arguments of [140, 99] predict the location of the transition at  $R = 2\sqrt{\alpha'}$  in the  $c = 1$  model was noted in [64].

An important point added by [13] was about the order of the transition. Since the mass-squared of the winding mode goes smoothly to 0 and then negative values as temperature is increased, the above discussion might lead one to conclude that the Hagedorn transition is second or higher order.<sup>29</sup> However, it turns out that the coupling between the winding mode and the dilaton modifies the effective Landau-Ginzburg potential and the transition is first order and, more importantly, *it occurs at a lower temperature  $T_c < T_H$* . A simple cartoon for this can be seen in figure 1.12.

How do the two interpretations mentioned in §1.2 tie in to this discussion? Firstly, both interpretations agree on the fact that there is an actual transition of  $c = 1$  string theory at the Hagedorn value  $R = 2$ .<sup>30</sup> This is similar to figure 1.12a. The first interpretation, however, posits a *second* Hagedorn transition at the black hole temperature  $R = 3/2$  and that is why the sphere free energy vanishes [91]. The free energy in the second interpretation

<sup>29</sup>It should be noted that BKT transitions are generically of infinite order.

<sup>30</sup>See [91] for a discussion of this transition in the context of the first interpretation.

is completely analytic at the black hole temperature and so under this interpretation there is only one phase transition for  $R \geq 3/2$ .

Thus, our results imply that two-dimensional string theory has only one Hagedorn-type phase transition, at the naïve Hagedorn temperature. Various arguments suggest another phase transition at  $R = 1$ , and our results have no bearing on this question.

### 1.6.3 A stringy picture for the entropy

In this section, we attempt to understand the relation between the MQM and the stretched strings of [115]. We will also find a natural picture for the renormalisation from the ungauged MQM to the gauged one we noted in §1.5.1. This section is entirely speculative and we hope to return to some of these ideas in more detail in future work.

We begin by going to the deep UV of the worldsheet, which is a Feynman diagram for the MQM, as emphasised in §1.2. A Feynman diagram that corresponds to the genus 0 partition function in the adjoint sector is one with the topology of  $S^2$  and with two faces that wind around the thermal circle in opposite ways. These two faces are the vortex and the anti-vortex respectively, see figure 1.14a. For simplicity, we consider the case where the vortices are the only non-trivial faces. The vortex and the anti-vortex have an index each, which we call  $i, j$  respectively.

This is a cylinder diagram corresponding to the propagation of the matrix element  $X_{ij}$  around the circle. Using the eigenvalue decomposition, this matrix element can be written as  $(U^\dagger)_{ik} \lambda_k U_{kj}$ . The  $i, j$  indices are the right-action indices that carry the  $PU(N)$  symmetry and the  $k$  index is the left-action index whose wavefunction is given by the eigenvalue equation. In the double-scaling limit,  $k$  is kept at  $\mathcal{O}(1)$  inside the solid, while  $i, j$  are unconstrained and therefore typically of  $\mathcal{O}(N)$ .

Now, to give this Feynman diagram a Hamiltonian interpretation, we cut it at  $t = 0$ . Remembering that the eigenvalue corresponds to the spatial direction in the string theory, we can associate the cut loop at  $t = 0$  to a string that goes from  $\lambda_i$  to  $\lambda_k$  and comes back to  $\lambda_j$ , see figure 1.13c. In the double-scaling limit, this is an open string that comes in from  $\infty$  to  $\lambda_k$  and goes back out — similar to the stretched string of [115].

In [115], the stretched string is anchored to an FZZT brane at  $\infty$ . However, in our picture, the open string has two endpoints at different eigenvalues  $\lambda_i, \lambda_j$ , which one might naively associate to two different branes. One way to make our picture consistent with that of [115] would be to remember that  $i, j$  are summed over all  $\mathcal{O}(N)$  values, which could be consistent with the fact that an FZZT brane ranges over all values of the dilaton from some  $\phi_{\min}$  to  $\infty$ . We caution the reader that this is a departure from the precise solution in [115],

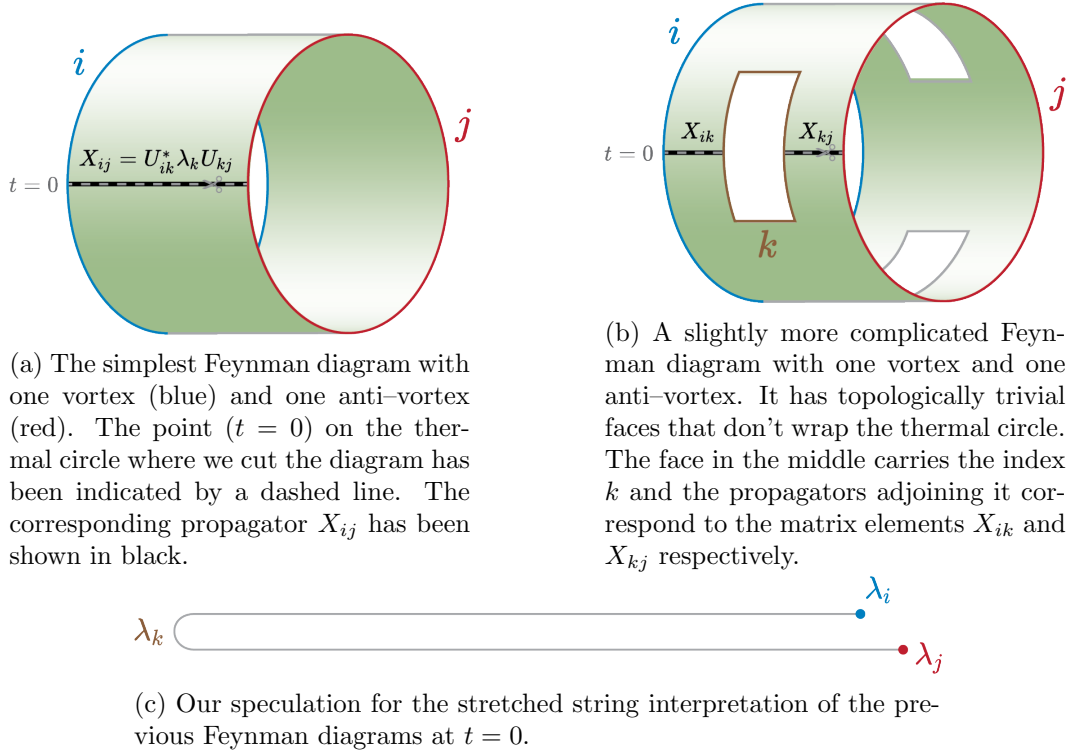


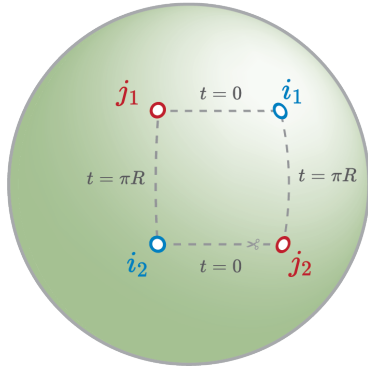
Figure 1.13: Feynman diagrams with one vortex and one anti-vortex.

were the two ends of the stretched string were at the same dilaton value (which is the same eigenvalue, up to an integral transformation). Since the entropy count was a sum over the different values that  $i, j$  could take, this suggests that the stringy picture for the entropy is related to the length of the FZZT brane. The fact that the latter diverges is nothing but the fact that the entropy depends on  $N$  and also diverges in the double-scaling limit.

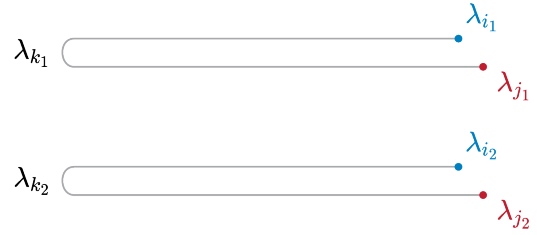
Before moving on to the  $k = 2$  case with two vortices, let us address how to think about this when there are  $n_F$  topologically trivial faces between the vortex and the anti-vortex, as in figure 1.13b. Here, the matrix we have to diagonalise is  $(X_1)_{ii_1}(X_2)_{i_2i_3}\dots(X_{n_F+1})_{i_{n_F}i}$ . The rest of the discussion carries through at the low level of precision we have maintained in this section.

Let us now look at the case of two vortices, which we can draw as in figure 1.14a. At  $t = 0$  we see two stretched strings; and two corresponding stretched strings at  $t = \pi R$ . The entire discussion carries through for each string, with the small caveat that we can exchange the two strings by shuffling the indices on the vortices. Therefore the number of states passing through  $t = 0$  is a factor of 2 less than what one might expect from two stretched strings.

More generally, at arbitrary  $k$ , each stretched string should be thought of as an adjoint, and the different stretched strings as indistinguishable. The number of states is therefore



(a) A simple Feynman diagram with two vortices and two anti-vortices. It has two faces with no winding (one not visible), two with winding 1 and two with winding  $-1$ .



(b) A stretched string interpretation of the Feynman diagram to the left, at  $t = 0$ . There is a similar situation on the other side of the thermofield double, but the strings stretch between different pairs of eigenvalues there as can be read off from the dotted lines in figure 1.14a.

Figure 1.14: Feynman diagrams with two vortices and two anti-vortices.

counted by the number of states in  $\mathbb{k}$  indistinguishable copies of the adjoint — which was exactly the counting we did in §1.3.1. Thus, the sine-Liouville model seems to directly realise one of the two countings we have presented. We hope to return to these ideas and make them precise in future work.

Further, the renormalised partition function (1.140), which we saw in §1.5.1 to be natural for the gauged theory, has a natural interpretation here. This is because the entropy we calculated in §1.3 corresponds to the number of places the open string can end, and the indices at the edge of the open strings are infinitely far away in the double-scaling limit. These indices thus shouldn't matter very much for dynamics at finite distance, and so the renormalisation associated to the double-scaling limit could be regarded as decoupling the  $PU(N)$  global symmetry from the bulk dynamics by sending the objects charged under the symmetry (the ends of the open strings) infinitely far away.

Finally, the major caveat about the pictures in this section is that, in this interpretation, the stretched string seems to ‘jump’ across the hole in the world. This suggests that there might be a more useful picture in the black hole phase. This is what we turn to now.

#### 1.6.4 Motzkin walks

We end our discussion by describing another equivalent way of performing the entropy calculation in 1.3.2 using a class of discrete random walks called *Motzkin walks*. As we will explain, this allows us to draw a speculative but tantalizing connection between the solutions of the cigar string theory and these random walks.

First, some definitions. Let  $\{\mathbf{e}_1, \mathbf{e}_2\}$  be the standard basis of  $\mathbb{R}^2$ . A Motzkin walk (or path) of length  $n$  is a continuous lattice path from  $(0, 0)$  to  $(n, 0)$  that never dips below the  $x$ -axis created out of the following unit steps:

$$\{\mathbf{e}_1\} \cup \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2\}. \quad (1.152)$$

For obvious reasons, the above are often referred to as ‘flat’ steps, ‘up’ steps and ‘down’ steps respectively. An example of a valid and an invalid Motzkin walks are illustrated in figure 1.15. The number of Motzkin walks of length  $n$  are counted by the  $n^{\text{th}}$  Motzkin number,  $\mathcal{T}_n$ . These numbers appear while enumerating a host of other combinatorial objects — amongst other things,  $\mathcal{T}_n$  counts the number of non-intersecting chords that can be drawn between  $n$  points on a circle, the number of grammatically allowed ways of placing left and right parentheses, and spaces in a sentence and the number of configurations of a spin-1 chain satisfying  $\sum_{i=1}^m S_i^z \geq 0$  for  $m < n$  and  $\sum_{i=1}^n S_i^z = 0$ . Given their prevalence then, it is no surprise that these numbers have been quite extensively studied in the combinatorics literature. For a more complete list of the different manifestations of Motzkin numbers and a thorough discussion of their properties and relations, see [42].

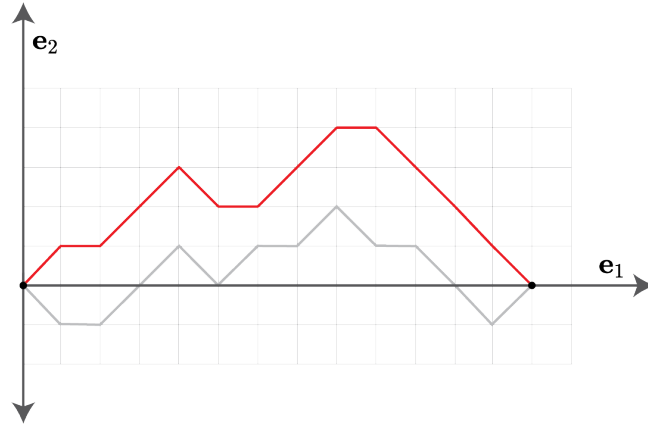


Figure 1.15: A valid example (red) and an invalid example (gray) of an uncoloured Motzkin walk.

In what follows, we will be interested in a higher dimensional generalization of the Motzkin walk — for historical reasons, these extra dimensions are often referred to as *colours*, and these generalized walks are called *coloured* Motzkin walks [47].<sup>31</sup> Once again,

<sup>31</sup>Rather unhelpfully, there are two separate notions of a Motzkin walk with colour degrees of freedom that appear in the literature. Unfortunately, despite sharing the same name, these two generalizations of the Motzkin walk have very different properties and it is therefore important that a distinction is made for our discussion. We will not describe the other generalization of the walk or the differences between the two

let  $\{\mathbf{e}_1, \dots, \mathbf{e}_s, \mathbf{e}_{s+1}\}$  be the standard basis of  $\mathbb{R}^{s+1}$ . An  $s$ -coloured Motzkin walk is a lattice path from  $(0, \dots, 0)$  to  $(n, 0, \dots, 0)$  that always remains in the positive ‘quadrant’ of  $\mathbb{R}^{s+1}$ , created out of the following unit steps,

$$\{\mathbf{e}_1\} \cup \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2\} \cup \left( \bigcup_{i=2}^s \{\mathbf{e}_1 - \mathbf{e}_i + \mathbf{e}_{i+1}, \mathbf{e}_1 + \mathbf{e}_i - \mathbf{e}_{i+1}\} \right). \quad (1.153)$$

An example of a valid and an invalid 2-coloured Motzkin walks with ten steps are illustrated in figure 1.16. We will denote the number of Motzkin walks of length  $n$  with *at most*  $s$  colours by  $\mathcal{T}_n^s$ . The constraint that the walker stay in the positive quadrant bounds the maximum number of colours  $s$  by half the number of steps  $n/2$ .

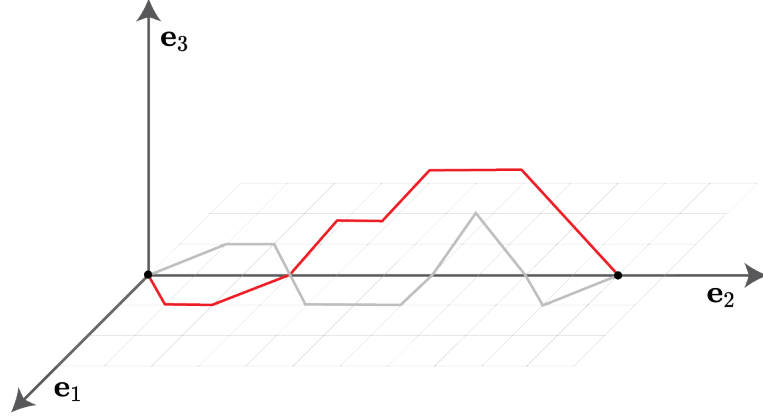


Figure 1.16: A valid (red) and an invalid (gray) example of a 2-coloured Motzkin walk.

The relevance of these walks to our story begins with the highly non-trivial observation that the number of sYTs with  $n$  boxes and at most  $2s + 1$  rows is also given by  $\mathcal{T}_n^s$ .<sup>32</sup> Indeed, the authors of [47] take this a step further and construct a bijection between the two sets, assigning each Motzkin walk to a specific sYT. The complete bijection is rather involved and will not be necessary for our present discussion. However, the mere existence of such a map implies that we can reproduce everything we that have derived about the string theory from the counting of the sYTs using these walks. In particular, the following picture emerges:

1. Every state in the Hilbert space is assigned to two independent Motzkin walks — one corresponding to the sYT for the boxes, and the other for the anti-boxes. Because the symmetry group is  $PU(N)$ , these two Motzkin walks have the same number of steps.

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here, but only provide references for the interested reader. The generalization of the Motzkin walk that we use is defined in [47]. The other generalisation of the coloured Motzkin walk appears in [126, 7].

<sup>32</sup>A slightly smaller set of walks can be associated to sYTs with  $2s$  rows. As elsewhere in this paper, we ignore all even-odd effects.

In the gauged model, we may think of these two Motzkin walks as corresponding to the state in the irrep of the  $S_k$  that permutes the adjoints.

2. Given a state with  $k$  steps, each of the two Motzkin walkers can explore any number of the available dimensions. However, in strict analogy with what happens with the sYTs, the dominant contribution to the entropy arises from the walks that explore  $\mathcal{O}(\sqrt{k})$  dimensions.
3. If we now calculate the entropy of all the doubled Motzkin walks traversing  $k$  steps, we see that this is exactly equal to the renormalized entropy (1.140), i.e.

$$S_{\text{ren}, k} = 2k \log \sqrt{k}. \quad (1.154)$$

In other words, the Motzkin walks appear to directly compute quantities associated with the fixed-charge partition function in the gauged model,  $Z_{\text{ren}, k}$ .

Bolstered by this last observation, it is extremely tempting to ‘forget’ about the sYTs entirely and attempt to draw a deeper and more direct connection between these Motzkin walks and the string theory on the cigar background. This idea is not new — beginning with the work of Horowitz and Polchinski [83], a common motif in various investigations of string theory near the Hagedorn transition have involved drawing parallels with various random walk models (see [101, 122, 123] and references therein). The Horowitz–Polchinski solution (see [37] for a recent review) describes a self-gravitating gas of strings that then condense into a single stretched string near the Hagedorn transition. The Euclidean picture of this computation involves a condensate that winds around the thermal circle that grows massless at the Hagedorn temperature — in a first quantized description, the path integral for this field is just a sum over random walks.

A similar cartoon emerges while studying the Euclidean cigar. Near the Hagedorn transition there is a winding condensate that is concentrated at the tip of the geometry. More precisely, the profile of this winding mode is described by the Nambu–Goto action of a string stretching from the tip of the cigar up to a particular radial distance. As the work of [29] emphasize, it is this condensate that carries the entropy of this system. We suspect that the Motzkin walks can provide a direct Hamiltonian realization of this entropy.

If this picture is true, it has two neat consequences. First, [60] draws a connection between the cigar random walks and ‘string bits’. These are fundamental point-like objects first introduced in [157] in order to make the causality and stability of string theory manifest — strings are then regarded as composite ‘multi-bit’ objects with a particular linear

arrangement [158]. This would allow us to interpret each step of the Motzkin walk as an individual bit. Secondly, the picture that we have drawn describes a repackaging of the dynamics of a gas of strings (akin to dealing with multiple adjoints) into Motzkin walks (which, like the sYTs are simply irreps with the correct number of boxes). In other words, Schur–Weyl duality reorganises the physics of the system in a manner very reminiscent of ER=EPR duality!

Of course, there is much work to be done before we can make this picture precise. To begin with, here are some simple questions:

1. Why are there two Motzkin walks for each state?
2. What does the restriction of the walk to the positive quadrant of the  $\mathbb{R}^{s+1}$  mean from the perspective of the string theory?
3. What do the dimensions (colours) correspond to in the string theory?

We don't have satisfactory answers to these questions yet — we leave them here as launching pads for future exploration.

## *Opus II*

## Chapter 2

# The gravitational Schwinger–Keldysh saddle and fermionic open EFTs

### 2.1 Fermionic open EFTs and holography

Several strongly correlated electronic systems have been reported to show novel metallic behaviour [68, 56, 139]. Their low energy physics cannot be captured by quasiparticle excitations. These systems thus fall outside the standard Fermi–liquid paradigm and are hence termed non–Fermi liquids. A distinct feature of these novel metals is that many of them seem to undergo critical phase transitions at low temperatures, and are thus governed by the fermionic cousins of the bosonic quantum phase transitions [138]. This motivates a plausible origin of their unusual properties — the fermionic excitations in these systems are thought to be interacting with a conformal sector leading to a quick decay of quasiparticles and a breakdown of Fermi–liquid structure [160, 128, 104]. While many examples of this type are known and well-studied, an overarching theoretical framework is not yet available.

Many fundamental questions about non–Fermi liquids remain unanswered. What is the most efficient description of their long distance dynamics? What is the best characterisation of their low energy transport [151, 150]? We propose that a useful point of view to characterise strongly correlated metallic phases of matter is via the effective theory of probe

fermions propagating in these systems. After all, a universal experimental feature uniting materials with or without quasiparticles is the fact that they can be probed by injecting electrons into them (say in STM/ARPES type measurements). Since such probe electrons often suffers inelastic scattering, decoherence and dissipation as they move through the material, the effective theories for such probes should be understood from the perspective of open effective field theories (open EFTs). Open EFTs describe the effective dynamics of a collection of fields of interest (the system) interacting with a set of inaccessible fields (the bath), after the latter fields are integrated out. Developing such a systematic framework is especially necessary if we have to go beyond the lower point correlations (which characterise conductivity etc.) and to include the physics of out-of-time ordered correlations.

In most cases, deriving open effective actions from a unitary microscopic theory is a tricky exercise. This is because one usually lacks a good way to separate out the bath and system degrees of freedom. To have a local effective action at some time scale  $\tau_{\text{eff}}$ , it is necessary that the correlators of the bath relax at a time scale  $\tau_{\text{bath}} < \tau_{\text{eff}}$ . This accords with the intuition that the bath should ‘forget’ fast enough to let the system evolve in a Markovian or local fashion. For generic baths,  $\tau_{\text{bath}}$  decreases as the interaction coupling increases; stronger coupling implies faster mixing of information and thus faster decay of correlations. This indicates that non-Fermi liquids can be thought of as Markovian baths – they lack long-lived quasiparticles and have a characteristic relaxation time of the order of the inverse temperature,  $\beta$  [138, 165]. Therefore at time scales larger than  $\beta$ , the probe fields interacting with them are described by local effective actions. When long-lived excitations are present they show up as memory effects in the open EFT of the probe.

As mentioned above, the fermionic fields in novel metals are thought to be interacting with some critical sector of the metallic system, leading to the the break down of the conventional Fermi liquid description. A simple way to model such interactions is to couple free fermions to a critical theory, like gapless bosons [79, 124, 111]. A major difficulty with such models is that one has to carefully devise an expansion parameter giving a controlled approximation of the strongly coupled dynamics, without losing its essential features giving rise to the non-Fermi liquid behaviour.

Another class of toy models widely used to study strongly coupled critical systems is based on holographic CFTs. Holographic models offer a great advantage over the field theoretical ones as the former come with a natural perturbative description in terms of the dual gravitational dynamics. The finite temperature states of these models are dual to black brane backgrounds which dissipate and relax at a time scale of order  $\beta$  [82]. This in turn fixes the relaxation time scale of the field theory, making them qualitatively similar to the novel metals discussed above. The application of holographic techniques to fermionic

systems have uncovered a rich variety of phases of Fermi liquid and non-Fermi liquid kinds [75, 166]. Recent developments in real time holography (which we review below) make these holographic models a good avenue to develop the open EFT perspective described above.

With these motivations, we initiate a study of fermionic open EFTs using holography. The setup we consider is similar to the semi-holographic model pioneered by [51]: the microscopic action consists of an external probe fermionic field  $\psi$  which couples to a holographic CFT at a finite temperature via a fermionic, single trace primary  $\mathcal{O}$ . The open effective action for  $\psi$  fields is then derived by integrating out the CFT bath in the real time Schwinger–Keldysh path integral [52, 142] (see [136, 20, 90] for a textbook description). The effect of bath fields are then encoded in the *influence phase* [52] added to the Schwinger–Keldysh effective action of  $\psi$ . Our goal is to compute this influence phase holographically. The fermionic single trace primary  $\mathcal{O}$  is dual to a bulk Dirac field  $\Psi$ , and the influence phase comes from evaluating the Dirac action on a field configuration of  $\Psi$  probing a ‘gravitational Schwinger–Keldysh’ (grSK) geometry [146, 147, 159, 62, 34, 89].

## The gravitational Schwinger–Keldysh saddle

We begin by reminding the reader that we still lack a systematic formalism to compute real time SK path integrals in AdS/CFT. Over the last two decades, many prescriptions have been proposed, gradually expanding the class of real time observables that can be computed holographically — complex contours played a foundational role in the early development of these calculations. Perhaps the first seminal contribution came from Son and Starinets [148], who showed that retarded Green’s functions in equilibrium CFTs could be computed from ingoing quasi-normal modes on an AdS black brane. Their work established a precise dictionary between bulk modes and boundary thermal correlators, laying the groundwork for systematic real-time calculations in AdS/CFT. This framework was further illuminated by Motl’s studies of quasi-normal modes in AdS black holes [125], which deepened our understanding of black hole response functions through complex analytic methods.

This prescription was successfully generalized to near-equilibrium hydrodynamic configurations in the fluid-gravity correspondence (see [137, 84] and references therein). However, extending beyond causal Green’s functions to compute other real time observables (encoding the physics of fluctuations) proved challenging. The core difficulty on the gravitational side lies in accounting for Hawking radiation and its interactions with the infalling modes. This led researchers to approach the problem through analytical continuation from Euclidean or eternal black brane computations [80, 149]. A particularly relevant example is the work of Giecold [58], who studied a Dirac field probing an eternal black brane background.

While these analytic continuation techniques have proven successful for computing certain classes of out-of-time ordered correlation functions in equilibrium [145], they fall short of capturing the vast landscape of non-equilibrium, real time correlations computed by generalized SK path integrals. These developments collectively motivated a new direction: developing prescriptions that directly compute real time correlations in holography without relying on analytic continuation gymnastics.

A pioneering development came from Skenderis and van Rees [146, 147, 159], who proposed a doubled geometry that fills out the SK contour in the CFT (See Fig.2.3). Their construction for thermal CFT states consists of two Schwarzschild black branes glued together along a common future horizon, providing a geometric realization of the SK contour. The construction was further developed in subsequent works [105, 17, 16, 27], which clarified the role of horizon gluing conditions in maintaining unitarity.

A parallel development came from Hartman and Maldacena [69], who employed a doubled Lorentzian AdS geometry connected by a Euclidean section to study the evolution of entanglement entropy. Their construction used two copies of the exterior black hole geometry joined by an interior region and connected to a Euclidean cap, demonstrating how such doubled geometries could be combined with complex extremal surfaces to probe time-dependent quantities. This work showed both the power of complex contour methods and the utility of doubled geometric constructions in capturing real-time dynamics in holography.

A significant improvement to the Skenderis–van Rees prescription came from Glorioso, Crossley and Liu [62], who provided greater clarity to the horizon structure of these geometries. In the updated prescription, the horizon region is replaced by a “horizon cap” interpolating between the two black brane exterior spacetimes. We discuss this geometry more elaborately in §2.3. Following [89], we refer to this geometry as *gravitational Schwinger-Keldysh saddle* (grSK). For the interested reader, we have included a detailed comparison between our adopted methods and previous approaches to real-time holography in §2.1.1.

This Schwinger–Keldysh saddle has demonstrated its utility in deriving interacting open effective actions for Brownian particles and scalar fields probing the CFT [34, 89], passing various non-trivial checks at the interacting level. Our goal in this work is to extend these ideas to fermionic fields to construct fermionic open EFTs.

### 2.1.1 A comparison with previous methods

In this section, we provide a detailed comparison between the approach to real-time holography presented in this chapter and previous attempts at such prescriptions. We will argue that the grSK approach advocated here is better suited to study interacting systems and questions involving out-of-equilibrium physics.

We begin by reminding the reader of the construction of thermofield double (TFD) states in the (doubled) Hilbert space of our field theory. The TFD state is a particular purification of the thermal state, with correlators generated via the the partition function,

$$\mathcal{Z}_{\text{TFD}} [\psi, \bar{\psi}] \equiv \text{Tr} \left\{ U[\psi_R, \bar{\psi}_R] \rho_{\beta}^{\frac{1}{2}} U[\psi_L, \bar{\psi}_L]^{\dagger} \rho_{\beta}^{\frac{1}{2}} \right\}. \quad (2.1)$$

The above partition function should be contrasted with the expression for the Schwinger–Keldysh partition function, (2.3). A quick way to appreciate the distinction between the two constructions is to compare the time contours on which these two path integrals are defined — this is illustrated in (Fig.2.1). An early work of Maldacena [114] identified TFD correlators with semi-classical gravity in the background of a maximally Kruskal extended Schwarzschild–AdS<sub>d+1</sub> black brane. Soon after, Herzog and Son [80] provided a precise prescription for calculating TFD Green’s functions using this eternal black brane dual. Giecold [58] later used this same prescription to derive 2-pt fermionic TFD correlation functions of single trace primaries in the strongly couple  $\mathcal{N} = 4$  SYM plasma.

As we mentioned, the TFD construction is extremely useful if one is interested in the properties of the field theory at thermal equilibrium. In order to obtain real-time response, we have to then analytically continue these results to the Schwinger–Keldysh time contour. Indeed, this is precisely the approach that authors of [80, 149, 58] take in order to derive Schwinger–Keldysh correlators. In principle, this analytical continuation is possible when there are no sources deforming the system away from equilibrium. However, when one is interested in understanding systems in local equilibrium, or more generally out-of-equilibrium, this analytical continuation proves to be much harder. This issue is particularly relevant when we have to compute higher point real-time correlation functions, and is the reason for the paucity of such calculations in the literature. Further, it is often the case that analytical results from holographic computations are available only as boundary derivative expansions — in these situations, such analytic continuations are inherently ambiguous.

Aside from the issue with analytical continuation, there is a more basic issue with calculating higher point TFD correlators from holography. To appreciate this, recall how we compute Euclidean higher point functions from AdS/CFT. We use the boundary-to-bulk

propagators to evolve fields into the bulk from the sources set on the boundary. Then, we use bulk local interaction vertices to convolve these fields — to get the relevant correlator, one then has to integrate the position of this vertex over the entire Euclidean bulk manifold. However, an analogous discussion for holographically computing higher point TFD correlators is complicated by the presence of horizons in Lorentzian spacetimes — *a priori*, it is not clear over what region(s) of the blackhole geometry one has to integrate these bulk interaction vertices.

The authors of [17] provided a partial answer to this question, demonstrating that integrating the bulk vertices over only the exterior regions of the eternal black brane correctly reproduces the 3-pt functions of a scalar primary. Though appealing, note that this result does not completely resolve the issue. A computation of 4-pt correlation functions and higher would receive contributions from additional (Lorentzian) ‘Witten’ diagrams where virtual excitations fall into the horizon. Such diagrams do not appear in the calculation of 3-pt functions. To the best of our knowledge, there has been no consistent proposal to deal with these additional contributions.

Indeed, correctly treating infalling virtual excitations is also important when studying loop corrections to bulk propagators<sup>1</sup>. As an example relevant to condensed matter theory, consider [49], where the authors use holography to calculate the contribution to charge conductivity from Fermi surfaces. As they detail in their paper, this requires computing the fermionic one-loop corrections to the bulk photon propagator, the dominant contribution to which arises when the fermionic loop goes into the horizon. Thus, correctly accounting for virtual infalling excitations is at the heart of this calculation. However, the authors of [49] circumvent this problem by adopting a refined analytical continuation to the real-time domain. While this trick proves ample for their purposes, it is unsatisfying for two reasons. Firstly, from the gravitational perspective, there is a loss of clarity in which blackhole processes correspond to the real-time processes that occur in the field theory. Secondly, while sufficient to determine the corrections to the 2-pt function they are interested in, it is not clear how these calculations generalize to higher point functions.

For these reasons, it is useful and important to have a prescription that directly computes Schwinger–Keldysh correlation functions from holography. As mentioned in §2.1, the pioneering work of Skenderis and van Rees [146, 147, 159] was the first attempt at such a dictionary. They argued that the Schwinger–Keldysh contour should be piece-wise extended in to the bulk, augmenting the real-time segments of the boundary contour with Lorentzian geometries and the imaginary-time segments with Euclidean geometries. These geometries are glued along codimension-1 surfaces. The authors then developed a robust dictionary

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<sup>1</sup>This would be necessary if we were interested in obtaining the  $1/N$  corrections to the CFT correlators.

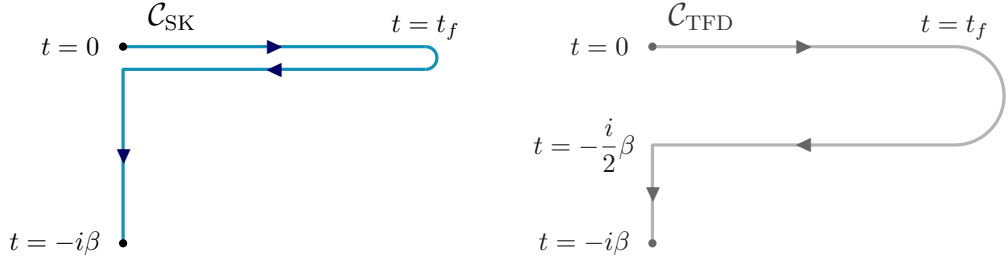


Figure 2.1: A comparison of the Schwinger–Keldysh and the thermofield double time contours. The endpoints of each of the two contours are identified, with the periodicity in the imaginary time (Euclidean) direction being set by the inverse temperature,  $\beta$ .

that computes real-time correlation functions of operators in the probe limit, that passed several checks. However, if we were to move past the probe limit and consider scenarios when we have dynamically evolving geometries in the bulk, piecewise-smooth geometries glued along their horizons pose a potential issue. It is in this regard that the modified prescription put forth by Glorioso, Crossley and Liu [62] is significant. As we detailed in §2.3.2, the authors propose a smooth geometry as the gravitational dual to the asymptotic Schwinger–Keldysh contour, replacing the horizon region of the geometry by a ‘horizon cap’ that interpolates between the two copies of the AdS spacetime. It is this geometry that we use to derive fermionic Schwinger–Keldysh correlation functions in this chapter.

### 2.1.2 A summary and an outline

We will pause here to quote the central physical result of this chapter followed by an outline.

A crucial check that our prescription does indeed reproduce the correct real time dynamics of fermionic radiation is the derivation of the Fermi–Dirac function from the bulk. More precisely, our answers should be consistent with the fermionic Kubo–Martin–Schwinger [102, 119] conditions for SK correlators. Note that this is extremely non-trivial: after all in the bulk we work completely classically and it is *a priori* not obvious at all how one then derives Bose–Einstein or Fermi–Dirac statistics without any quantisation whatsoever. As explained in [34, 89] this works in the bosonic examples as follows: the structure of grSK saddles is such that the outgoing Hawking radiated modes behave as evanescent waves near the horizon cap undergoing a Boltzmann-like suppression *classically* when compared across the two exteriors. The horizon cap region is however completely transparent to the ingoing quasi-normal modes. An interference between these two then results in the appearance of the Bose–Einstein function in the classical calculation.

As we will argue later in this work, for a field of spin  $s$ , the classical transmission amplitude for the outgoing modes across the horizon cap is given by<sup>2</sup>,

$$e^{-\beta\omega+2\pi is}. \quad (2.2)$$

For integer  $s$  this reduces to the simple Boltzmann weight, whereas for half integer  $s$  we get an additional phase factor of  $-1$ . This then leads to a destructive interference of the ingoing and the outgoing modes, perfectly reproducing the Pauli exclusion felt by half-integer fields. The classical physics of grSK saddles hence automatically reproduces the spin-statistics relation implementing the Fermi-Dirac statistics of the boundary correlators!

Here is an outline of the rest of this work: we begin in §2.2 with a review of the Schwinger-Keldysh formalism as applied to fermions, followed by a review of necessary ideas from real time holography in §2.3. In particular, we introduce the grSK geometry in detail §2.3.2. The main results of this work are then summarised in two sections: in §3.1, we explain in detail the classical origin of Fermi-Dirac statistics when solving Dirac equations in the grSK saddle. In §3.2, we give, for the first time, an explicit solution of the Dirac equation in the black-brane background in a derivative expansion. We also describe the structure of the ingoing quasi-normal modes and the outgoing Hawking modes as well as the superposition of the two that satisfies the appropriate boundary conditions on the grSK saddle.

In §4 we extend this geometry and prescription to compute real time correlators at finite chemical potential in the dual CFT. We do this via a geometry made by stitching two copies of RN-AdS black brane exteriors (referred to as the RN-SK saddle/geometry henceforth) that generalises the grSK construction. This is detailed in §4.2. In §4.2, we devise a method to write down the outgoing Hawking modes in this geometry when the ingoing modes are known. We test this method in §4.3 using a probe Dirac field and once again show that it indeed yields the correct Fermi-Dirac physics (as well as the correct Kubo-Martin-Schwinger [102, 119] relations) at finite chemical potential. Finally, in §4.4 we derive an explicit solution to Dirac equation upto second order in boundary derivative expansion and the corresponding influence functional.

Various technical computations are relegated to the appendices including the details about spinors and gamma matrices in Appendix A.

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<sup>2</sup>In the case of the charged black brane, there is an extra term proportional to the chemical potential in the exponent.

## 2.2 Fermions in the Schwinger–Keldysh formalism

In this section, we will review the Schwinger–Keldysh formalism below with an emphasis on fermionic fields (see [136, 20, 90] for a textbook description).

Consider a unitary field theory which contains fermionic operators collectively denoted as  $\mathcal{O}$ . The SK generating functional is given by

$$\mathcal{Z}_{\text{SK}}[\psi, \bar{\psi}] \equiv \text{Tr} \left\{ U[\psi_R, \bar{\psi}_R] \rho_{\text{init}} U[\psi_L, \bar{\psi}_L]^\dagger \right\}, \quad (2.3)$$

where  $\rho_{\text{init}}$  is the initial density matrix and  $\psi_{R,L}$  represent the sources for the fermionic operator  $\mathcal{O}$  on the right/left contour. Equivalently, we would like to think of  $\psi_{R,L}$  as probe fermion fields coupling to the field theory under question. This provides an equivalent interpretation of the above generating function as the *influence function* for this probe field which encodes the effect of the environment.

The operator  $U$  is the unitary evolution operator deformed by the sources  $\psi_{R,L}$ . When  $\rho_{\text{init}}$  is thermal, the above generating function can be regarded as the path integral of the field theory living on a complex time contour  $\mathcal{C}$ , as shown in Fig.2.2. The part of the contour indicating forward (backward) evolution in time is labeled  $\mathcal{C}_R$  ( $\mathcal{C}_L$ ).  $\mathcal{C}_E$  corresponds to an Euclidean evolution of length  $\beta$  and is a surrogate for the thermal circle. This part of the contour prepares the thermal initial state.

When a Lagrangian description is available, we can write down a SK action defined by,

$$\mathbb{S}_{\text{SK}} \equiv \oint_{\mathcal{C}} dt_{\mathcal{C}} d^{d-1}\mathbf{x} \mathcal{L}[\mathcal{O}], \quad (2.4)$$

where  $t_{\mathcal{C}}$  stands for the contour time on  $\mathcal{C}$  and  $\mathcal{L}[\mathcal{O}]$  is the Lagrangian density of the field theory. The corresponding SK generating functional can be schematically written as,

$$\mathcal{Z}_{\text{SK}}[\psi, \bar{\psi}] = \int \mathcal{D}[\mathcal{O}, \bar{\mathcal{O}}] \exp \left\{ i \mathbb{S}_{\text{SK}} + i \int_{\mathcal{C}} dt_{\mathcal{C}} d^{d-1}\mathbf{x} \left( \bar{\psi} \mathcal{O} + \bar{\mathcal{O}} \psi \right) \right\}. \quad (2.5)$$

Here  $\mathcal{D}[\mathcal{O}, \bar{\mathcal{O}}]$  schematically represents the path integral over the elementary fields out of which the operators  $\mathcal{O}, \bar{\mathcal{O}}$  are constructed.

It is convenient to distinguish between the operators supported on the segments  $\mathcal{C}_R$  and

$\mathcal{C}_L$  of the contour. We define,

$$\begin{aligned}
\mathcal{O}(t_{\mathcal{C}}, \mathbf{x}) &\equiv \mathcal{O}_R(t, \mathbf{x}), & \psi(t_{\mathcal{C}}, \mathbf{x}) &\equiv \psi_R(t, \mathbf{x}), & t = t_{\mathcal{C}}; & t_{\mathcal{C}} \in \mathcal{M}_R \\
\mathcal{O}(t_{\mathcal{C}}, \mathbf{x}) &\equiv \mathcal{O}_L(t, \mathbf{x}), & \psi(t_{\mathcal{C}}, \mathbf{x}) &\equiv \psi_L(t, \mathbf{x}), & t = 2t_f - t_{\mathcal{C}}; & t_{\mathcal{C}} \in \mathcal{M}_L \\
\mathcal{O}_R(t_f, \mathbf{x}) &\equiv \mathcal{O}_L(t_f, \mathbf{x}), & \psi_R(t_f, \mathbf{x}) &\equiv \psi_L(t_f, \mathbf{x}),
\end{aligned} \tag{2.6}$$

where  $t_f$  is the future most or turning point on the contour. Once we adopt these new variables, the time variable  $t$  can be identified with the physical time  $x^0$ , and fields  $\mathcal{O}_R$  and  $\mathcal{O}_L$  are field degrees of freedom corresponding to forward and backward temporal evolutions respectively; they are identified at  $t_f$ . The initial time configurations of these fields are weighted by an Euclidean path integral on the segment  $\mathcal{C}_E$  which corresponds to the preparation of the thermal initial state. This segment of the contour will play a largely passive role in our story as we do not intend to source the fields there<sup>3</sup>.

The SK two-point functions are defined by,

$$\mathcal{S}^{IJ}(x, y) \equiv \frac{1}{i} \frac{\delta}{\delta \bar{\psi}_I(x)} i \frac{\delta}{\delta \psi_J(y)} \ln \mathcal{Z} \Big|_{\psi=\bar{\psi}=0} = \begin{pmatrix} \mathcal{S}^{RR} & \mathcal{S}^{RL} \\ \mathcal{S}^{LR} & \mathcal{S}^{LL} \end{pmatrix}, \tag{2.7}$$

where we follow the convention that Latin letters  $I, J = R, L$  denote contour indices and Greek letters  $\alpha, \beta$  are fermionic indices<sup>4</sup>. Since path integrals generate contour ordered correlators, we have

$$\begin{aligned}
\mathcal{S}_{\alpha\beta}^{RR}(x, y) &= \langle \mathbb{T} \mathcal{O}_{\alpha}(x) \bar{\mathcal{O}}_{\beta}(y) \rangle, & \mathcal{S}_{\alpha\beta}^{RL}(x, y) &= - \langle \bar{\mathcal{O}}_{\beta}(y) \mathcal{O}_{\alpha}(x) \rangle, \\
\mathcal{S}_{\alpha\beta}^{LR}(x, y) &= \langle \mathcal{O}_{\alpha}(x) \bar{\mathcal{O}}_{\beta}(y) \rangle, & \mathcal{S}_{\alpha\beta}^{LL}(x, y) &= \langle \hat{\mathbb{T}} \mathcal{O}_{\alpha}(x) \bar{\mathcal{O}}_{\beta}(y) \rangle.
\end{aligned} \tag{2.8}$$

Here  $\mathbb{T}$  and  $\hat{\mathbb{T}}$  denote the time ordering and anti-time ordering of the operators. The additional negative sign on  $\mathcal{S}^{RL}$  comes from the Grassmann odd property of fermionic operators.

<sup>3</sup>We will consider only the thermal initial state. Field insertions in the Euclidean segment correspond to driving the thermal state out of equilibrium.

<sup>4</sup>We adopt these conventions for this section only. Further, we do not distinguish between upper and lower indices.

The SK Green's functions given above simplify further in the so called 'Keldysh-rotated' field basis, given by the average and difference of the  $R$  and  $L$  field variables<sup>5</sup>.

$$\begin{aligned}\mathcal{O}_a &= \frac{1}{2}(\mathcal{O}_R + \mathcal{O}_L) , & \mathcal{O}_d &= \mathcal{O}_R - \mathcal{O}_L . \\ \overline{\mathcal{O}}_a &= \frac{1}{2}(\overline{\mathcal{O}}_R + \overline{\mathcal{O}}_L) , & \overline{\mathcal{O}}_d &= \overline{\mathcal{O}}_R - \overline{\mathcal{O}}_L .\end{aligned}\tag{2.9}$$

Similarly, we define the average/difference sources,

$$\begin{aligned}\psi_a &= \frac{1}{2}(\psi_R + \psi_L) , & \psi_d &= \psi_R - \psi_L . \\ \overline{\psi}_a &= \frac{1}{2}(\overline{\psi}_R + \overline{\psi}_L) , & \overline{\psi}_d &= \overline{\psi}_R - \overline{\psi}_L .\end{aligned}\tag{2.10}$$

The Green's function matrix, when expressed in the Keldysh basis becomes,

$$\begin{aligned}\mathcal{S}_{\alpha\beta}^{aa}(x, y) &= \frac{1}{2} \left\langle \left[ \mathcal{O}_\alpha(x), \overline{\mathcal{O}}_\beta(y) \right] \right\rangle , & \mathcal{S}_{\alpha\beta}^{ad}(x, y) &= \theta(x^0 > y^0) \left\langle \left\{ \mathcal{O}_\alpha(x), \overline{\mathcal{O}}_\beta(y) \right\} \right\rangle , \\ \mathcal{S}_{\alpha\beta}^{dd}(x, y) &= 0 , & \mathcal{S}_{\alpha\beta}^{da}(x, y) &= -\theta(x^0 < y^0) \left\langle \left\{ \mathcal{O}_\alpha(x), \overline{\mathcal{O}}_\beta(y) \right\} \right\rangle .\end{aligned}\tag{2.11}$$

where,

$$\mathcal{S}^{ij}(x, y) \equiv \langle \mathcal{O}_i(x) \overline{\mathcal{O}}_j(y) \rangle , \quad i, j = a, d .\tag{2.12}$$

From the above we recognise the retarded and advanced Green's functions as,

$$\mathcal{S}_{\alpha\beta}^{\text{ret}}(x, y) \equiv \mathcal{S}_{\alpha\beta}^{ad}(x, y) , \quad \mathcal{S}_{\alpha\beta}^{\text{adv}}(x, y) \equiv \mathcal{S}_{\alpha\beta}^{da}(x, y) .\tag{2.13}$$

An important advantage of the Keldysh basis is the statement that the two point function  $\mathcal{S}_{\alpha\beta}^{dd}(x, y) = 0$ . This can be traced to the fact that the correlations of difference operators  $\mathcal{O}_d$  are generated by setting  $\psi_R = \psi_L$  in the original SK generating function (2.3). The underlying unitarity and the cyclicity of trace then implies that the SK generating function reduces to unity, demonstrating that any  $n$ -point correlation function of only difference operators identically vanishes. This is the *Schwinger-Keldysh collapse rule* obeyed by SK

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<sup>5</sup>Our convention here differs from that of [90] where the definitions of  $\overline{\mathcal{O}}_a$  and  $\overline{\mathcal{O}}_d$  are interchanged.

correlators. Since,

$$\bar{\psi}_R \mathcal{O}_R - \bar{\psi}_L \mathcal{O}_L = \bar{\psi}_a \mathcal{O}_d + \bar{\psi}_d \mathcal{O}_a, \quad (2.14)$$

this implies that the generating function of correlations never has a term with only  $\psi_a$ s.

The third and last basis set we introduce is the *retarded-advanced (RA)* basis,

$$\psi_{\bar{F}} \equiv n_{\omega}^{\text{FD}} (\psi_R - \psi_L) - \psi_R, \quad \psi_{\bar{P}} \equiv n_{\omega}^{\text{FD}} (\psi_R - \psi_L), \quad (2.15)$$

where we have defined the Fermi-Dirac factor,

$$n_{\omega}^{\text{FD}} = \frac{1}{1 + e^{\beta\omega}}. \quad (2.16)$$

The combination above (which differs slightly from the RA basis defined in say [40]) is motivated by the analysis presented in [35]. This combination is chosen to make manifest the Kubo-Martin-Schwinger [102, 119] relations arising from thermality and cyclicity of trace. We will refer the reader to [35] for a complete description of such constraints on the real time-correlators. For our purposes, it is sufficient to simply state the final result: the SK collapse rules and the KMS conditions together imply that *the generating function of SK correlations can never have term with only  $\psi_{\bar{P}}$ s or terms with only  $\psi_{\bar{F}}$ s. All non-zero terms have at least one  $\psi_{\bar{P}}$  and at least one  $\psi_{\bar{F}}$* . As we will see, these conditions arise naturally in our analysis in the bulk. For two point functions in Keldysh basis, these two conditions imply  $\mathcal{S}^{dd} = 0$ , and in addition,

$$\mathcal{S}^{aa} = \left( \frac{1}{2} - n_{\omega}^{\text{FD}} \right) (\mathcal{S}^{ad} - \mathcal{S}^{da}). \quad (2.17)$$

## 2.3 The grSK geometry and open EFTs

As mentioned in §2.1, our goal is to construct an open effective field theory for a fermionic probe field coupled to a thermal environment with a gravitational dual — we present the framework for this computation in this section.

We first focus on the holographic thermal bath / environment, detailing a holographic recipe for calculating its real-time correlators. This is done in three steps: in §2.3.1, we quickly review the standard prescription for computing response functions for spinors from

the AdS/CFT correspondence. We then introduce a complex two-sheeted geometry that is the relevant gravitational dual of the asymptotic Schwinger–Keldysh contour in §2.3.2. In §2.3.3, we describe how the usual AdS/CFT dictionary can be adapted to this new geometry, allowing us to compute the SK generating function (which as we explained before is equivalent to computing the open effective action for the fermionic probe).

### 2.3.1 Fermions in AdS/CFT

The discussion of the AdS/CFT correspondence for spinors that we present here closely follows the original references on the subject [78, 77, 129, 86]. A large part of this short review will be carried out in Euclidean space. At the very end, we will describe how to move all our expressions back to the real-time domain, which is the starting point for the discussion in §2.3.3.

We begin with the metric of an asymptotically Euclidean  $AdS_{d+1}$  spacetime,  $\mathcal{M}_E$ , written out in the (Euclidean) Eddington–Finkelstein coordinates. We are interested in studying a single free fermion living in this spacetime.

$$ds^2 = g_{AB} dx^A dx^B = g_{v_E v_E} dv_E^2 + 2i dv_E dr + g_{ij} dx^i dx^j \quad (2.18)$$

$$g_{v_E v_E} \xrightarrow[r \rightarrow \infty]{} r^2, \quad g_{ij} \xrightarrow[r \rightarrow \infty]{} r^2 \delta_{ij}$$

Here  $v_E$  is the advanced Euclidean Eddington–Finkelstein time and the  $AdS$  radius has been set to unity. In addition to the coordinates and metric, we need to align the Gamma matrices and spinors that we use to write down the Dirac action in this geometry. This is done using a tetrad (tangent) frame, i.e., a set of  $(d+1)$  linearly independent, smooth 1-form fields<sup>6</sup>  $E_A^a$  that satisfy,

$$\delta_{ab} E_A^a E_B^b = g_{AB}, \quad (2.19)$$

where  $\delta_{ab}$  is the  $(d+1)$  dimensional Euclidean metric. The Gamma matrices and spinor fields are now defined with respect to this tangent frame. Collectively, these tetrads fix

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<sup>6</sup>A word on notation: we use the uppercase alphabet  $(A, B, \dots)$  to denote abstract spacetime indices, and the lowercase alphabet  $(a, b, \dots)$  to denote abstract tetrad frame indices. Field theory indices will be denoted by  $(\mu, \nu, \dots)$ . Whenever a particular value of a tetrad frame index is referred to, we will enclose it within a parenthesis to distinguish it from the corresponding spacetime index.

the orientations of these objects in the ambient spacetime — as an important example, the spacetime Gamma matrices are defined as  $\Gamma^A = \Gamma^a E_a^A$ .

From (2.19) is clear that the choice of tangent frame is hardly unique, with different frames leading to different equations of motion. For now, we leave our choice of tetrads unspecified as our current discussion is independent of its details.

With these structures defined, the Euclidean action for a minimally coupled Dirac fermion  $\Psi$  propagating in this spacetime is,

$$\mathbb{S}_E = - \int_{\mathcal{M}_E} d^{d+1}x \sqrt{g} \bar{\Psi} \left( \Gamma^A \nabla_A - m \right) \Psi + \mathbb{S}_{\partial\mathcal{M}_E}, \quad (2.20)$$

where  $\bar{\Psi} = \Psi^\dagger \Gamma^{(v_E)}$ ,

$$\nabla_A = \partial_A + \frac{1}{4} \omega_{abA} \Gamma^{ab}. \quad (2.21)$$

where the  $\omega_{abA}$  are, in turn, the spin connection 1-forms. The action (2.20) is the sum of two terms. The first is a bulk term that determines the equations of motion. The second is a boundary term that does not alter the equations of motion. It is fixed by demanding a well-posed variational principle for this system, and will play a crucial role in what is to follow. More will be said about it soon.

As we mentioned in §0, the AdS/CFT correspondence posits an equivalence between a dynamical theory of gravity on  $AdS_{d+1}$  (the bulk) and a  $d$ -dimensional CFT that lives on the boundary of this spacetime. In this instance, the bulk fermionic field  $\Psi$  is mapped to a fermionic operator  $\mathcal{O}$  in the conformal field theory. The precise statement of the duality relates the on-shell bulk action for the field  $\Psi$  with the generating functional for connected correlation functions of  $\mathcal{O}$ .

$$\left\langle \exp \left[ \int d^d x \left( \bar{\psi} \mathcal{O} + \bar{\mathcal{O}} \psi \right) \right] \right\rangle_{\text{CFT}} = e^{-\mathbb{S}_E^{\text{cl}}[\psi, \bar{\psi}]} \quad (2.22)$$

Here  $\mathbb{S}_E^{\text{cl}}[\psi, \bar{\psi}]$  is the on-shell action, which is computed by evaluating (2.20) on a classical solution  $\Psi$ . The arguments of the on-shell action lets us know how this solution is selected: in addition to being regular in the interior,  $\Psi$  must satisfy a suitable Dirichlet boundary condition set by the boundary spinor  $\psi$  at  $r \rightarrow \infty$ . It is the interpretation of this boundary condition that introduces two subtleties for fermions. First, the spinors  $\Psi$  and  $\psi$  belong

to spacetimes with different dimensions, and therefore may have a different number of components. Secondly, the Dirac equation that (2.20) gives rise to is a first order differential equation; naïvely imposing a Dirichlet boundary condition on a solution to this equation would fix it everywhere without guaranteeing its regularity in the interior. Of course, these two issues are closely related to one another and have been carefully examined and elucidated before, most notably in [86]. Here, we quickly summarize their prescription, directing the reader to the extremely readable original work for further details.

As a preliminary step, it is useful to first split the spinor  $\Psi$  in terms of its eigenvalues under the  $\Gamma^{(r)}$  matrix.

$$\Psi = \Psi_+ + \Psi_-, \quad \Psi_{\pm} = \mathcal{P}_{\pm} \Psi. \quad (2.23)$$

Using this decomposition in (2.20), one immediately sees that  $\Psi_{\pm}$  are conjugate to one another. The resolution is to demand the regularity of the solution  $\Psi$  in the interior of the bulk geometry — this naturally allows us to apply Dirichlet boundary conditions on only half the components of  $\Psi$ , i.e., either on  $\Psi_+$  or  $\Psi_-$ .

This prescription neatly resolves the issue of  $\Psi$  and  $\psi$  having (in general) a different number of components. Recall that the bulk spinor  $\Psi$  is a Dirac spinor (2.20). When the dimension of the bulk geometry is odd ( $d + 1 = 2n + 1$ ), we are free to decompose the *boundary* Dirac spinors into Weyl spinors, which transform under the  $(\Sigma, \bar{\Sigma})$  matrices. Correspondingly, we choose the bulk Gamma matrices<sup>7</sup> as follows,

$$\Gamma^{(\mu)} = \begin{pmatrix} 0 & \Sigma^{(\mu)} \\ -\bar{\Sigma}^{(\mu)} & 0 \end{pmatrix}, \quad \Gamma^{(r)} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \quad (2.24)$$

Importantly, the radial Gamma matrix  $\Gamma^{(r)}$  and the chirality matrix of the boundary theory coincide; with this identification the probe spinor  $\psi$  transforms as a  $d$ -dimensional Weyl spinor. Thus when  $d + 1 = 2n + 1$ , the bulk Dirac spinor  $\Psi$  is mapped to boundary Weyl spinor primary  $\mathcal{O}$ .

On the other hand, when the dimension of the bulk is even  $d + 1 = 2n$ , we make the

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<sup>7</sup>The conventions that we adopt differ slightly from those used in [86]. Appendix A contains a detailed discussion.

following choice for our Gamma matrices,

$$\Gamma^{(\mu)} = \begin{pmatrix} 0 & \gamma^{(\mu)} \\ \gamma^{(\mu)} & 0 \end{pmatrix}, \quad \Gamma^{(r)} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \quad (2.25)$$

In this case, the boundary spinor  $\psi$  transforms as a Dirac spinor. Thus, when  $d + 1 = 2n$ , the bulk Dirac spinor  $\Psi$  is mapped to a boundary Dirac spinor  $\mathcal{O}$ . In all cases then, the number of components of the bulk Dirac spinor  $\Psi$  is always double that of the the boundary spinor  $\psi$  and the operator  $\mathcal{O}$ .

To figure out which to half of the spinor  $\Psi$  to provide Dirichlet boundary conditions, we have to look at its asymptotic fall-offs near the *AdS* boundary. Solving the Dirac equation arising from (2.20), we have,

$$\begin{aligned} \lim_{r \rightarrow \infty} \Psi_+(r, k) &= A_+(k) r^{-\frac{d}{2}+m} + B_+(k) r^{-\frac{d}{2}-m-1} \\ \lim_{r \rightarrow \infty} \Psi_-(r, k) &= A_-(k) r^{-\frac{d}{2}-m} + B_-(k) r^{-\frac{d}{2}+m-1} \end{aligned} \quad (2.26)$$

Clearly, the transformation  $m \longleftrightarrow -m$  exchanges  $\Psi_+$  and  $\Psi_-$ , with  $A_+ \longleftrightarrow A_-$  and  $B_+ \longleftrightarrow B_-$ . In what follows, we will choose  $m \geq 0$ . This choice selects the fall-off with coefficient  $A_+$  as the dominant contribution across all terms in (2.26) and should therefore be set equal to the source  $\psi(k)$ . This fixes the Dirichlet condition for  $\Psi$  that we seek, to wit,

$$A_+(k) = \psi(k), \quad \lim_{r \rightarrow \infty} r^{\frac{d}{2}-m} \Psi_+ = \psi. \quad (2.27)$$

Ergo, the conformal dimension  $\Delta$  of the boundary operator  $\mathcal{O}$  is,

$$\Delta = \frac{d}{2} + m. \quad (2.28)$$

Knowing the Dirichlet condition for the bulk fermion  $\Psi$ , in turn, determines the boundary term in the action (2.20). Having chosen to fix  $\Psi_+$  at the *AdS* boundary, we are no longer free to vary  $\Psi_-$ . As a result, the variation of  $S_E$  shouldn't contain any terms

proportional to  $\delta\Psi_-$ . This can be arranged by choosing  $\mathbb{S}_{\partial\mathcal{M}_E}$  to be,

$$\mathbb{S}_{\partial\mathcal{M}_E} = - \int_{\partial\mathcal{M}_E} d^d x \, r^d \, \bar{\Psi} \, \mathcal{P}_- \Psi. \quad (2.29)$$

We can now place the Dirac action (2.20) on-shell. Notice that the entire contribution arises from the boundary term (2.29); the first term in (2.20) vanishes when evaluated on a solution of the EOM. However, as can be seen from (2.26), this quantity has divergent contributions if  $m > 1/2$ . The standard holographic prescription asks us to select only the finite pieces from the on-shell action, which is the end result of a careful holographic renormalization [11], i.e.,

$$\mathbb{S}_E^{\text{ren}}[\psi, \bar{\psi}] = \left\{ \mathbb{S}_E^{\text{cl}}[\psi, \bar{\psi}] \right\}_{r_0} = - \int_{\partial\mathcal{M}_E} d^d x \, r^d \, \bar{\mathbf{A}}_+ \mathbf{A}_-. \quad (2.30)$$

Using (2.22) we can now compute *renormalized* Euclidean correlation functions for  $\mathcal{O}$  and  $\bar{\mathcal{O}}$ . Of course, in order to generate retarded and advanced correlators, we would have to analytically continue these results to real-time.

In order to generate Schwinger–Keldysh correlators we will upgrade the above discussion to an appropriately ‘doubled’ version of an asymptotically *AdS* spacetime. While detailing this prescription in §2.3.3, we will work directly in a Lorentzian spacetime, using the real-time versions of the expressions presented here. These can be obtained by the following analytic continuation,

$$v_E \rightarrow i \, v, \quad \delta_{ab} \rightarrow \eta_{ab}, \quad \Gamma^{(v_E)} \rightarrow i \, \Gamma^{(v)}, \quad \mathbb{S}_E \rightarrow -i \, \mathbb{S}. \quad (2.31)$$

### 2.3.2 The gravitational Schwinger–Keldysh saddle

As we had described in §2.2, the Schwinger–Keldysh contour for a thermal state is a *complex* time path, travelling from  $t = 0$  to  $t = T$ , looping back to  $t = 0$  and proceeding to  $t = -i\beta$ , as illustrated in Fig.2.2. In order to have a holographic interpretation of this, we would like to supplement the contour by a suitable bulk geometry, in a manner that the contour forms a time-like codimension-1 boundary of the larger spacetime.

We begin by writing the asymptotically *AdS* <sub>$d+1$</sub>  black brane in ingoing Eddington–

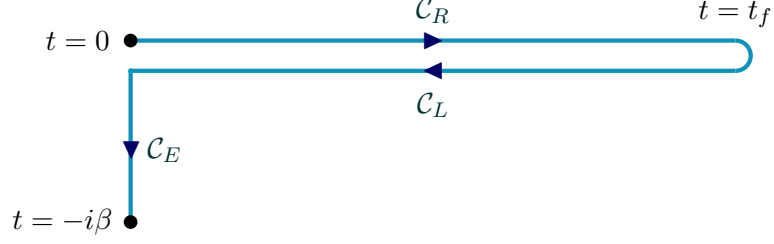


Figure 2.2: The Schwinger–Keldysh complex time contour. The direction of the contour is as indicated, with the forward and backward evolutions occurring along the sections  $\mathcal{C}_R$  and  $\mathcal{C}_L$ , respectively. The length of the Euclidean time segment,  $\mathcal{C}_E$ , is determined by the inverse temperature  $\beta$ .

Finkelstein coordinates.

$$ds^2 = -r^2 f(r) dv^2 + 2 dv dr + r^2 d\mathbf{x}_{d-1}^2, \quad f(r) = 1 - \frac{r_h^d}{r^d} \quad (2.32)$$

Here  $r_h$  is the radius of the horizon and the  $AdS$  radius has been set to unity. The temperature of this black brane is given by,

$$\beta = \frac{4\pi}{dr_h} \quad (2.33)$$

In order to realize the Schwinger–Keldysh contour on the boundary theory, we promote the coordinate  $r$  to a complex variable and select a suitable line in the complex plane as our radial coordinate. The particular curve we choose is the following: we select two copies of a line extending from the black hole horizon to infinity, each infinitesimally displaced in the imaginary direction on either side of the real line. That is, one line extends from  $r = r_h + i\epsilon$  to  $r = \infty + i\epsilon$  and the other extends from  $r = r_h - i\epsilon$  to  $r = \infty - i\epsilon$ . We then join these two legs by a *horizon cap* – a smooth curve connecting the two points  $r = r_h \pm i\epsilon$ , encircling the point  $r = r_h$ . For computational ease, it will prove convenient to terminate both these legs at finite cut-offs,  $r = r_c \pm i\epsilon$ . We will eventually take  $r_c \rightarrow \infty$  at the end of our calculations. This is the radial contour for the full spacetime and is illustrated in Fig.2.3.

We will find it useful to utilise a different parametrization for the radial direction, dubbed the *mock tortoise* coordinate  $\zeta$ . This is defined by the differential equation,

$$\frac{dr}{d\zeta} = \frac{i\beta}{2} r^2 f(r) \quad (2.34)$$

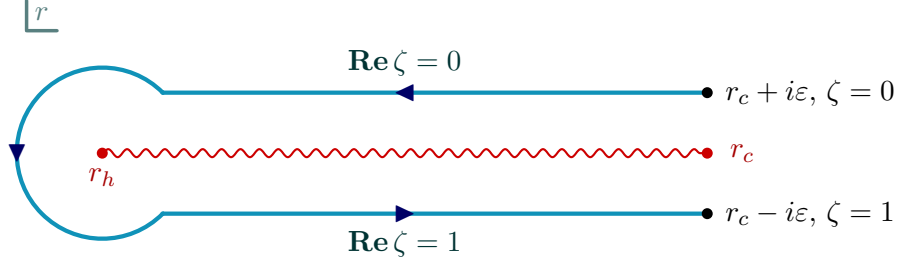


Figure 2.3: The radial contour drawn on the complex  $r$  plane, at fixed  $v$ . The locations of the two boundaries and the horizon have been indicated, along with the direction of the contour. The red line denotes the branch cut beginning at the horizon.

This coordinate is not an analytic function of  $r$  on the entire complex plane, and instead has a logarithmic branch point at  $r = r_h$ , the zero of the emblackening factor  $f(r)$  — we choose the corresponding branch cut to traverse from  $r = r_h$  to  $r = \infty$ . With our choice of normalization, the monodromy that  $\zeta$  picks up around this branch is unity.

Evaluated on each leg of the radial contour described above, the mock tortoise coordinate is a complex parameter, with an imaginary part running from 0 at the *AdS* boundary to  $\infty$  at the horizon. The real parts of the two legs differ by the monodromy around the horizon, viz., unity. As a matter of convention, we choose its real part to vanish on the upper leg of the contour, and set it to unity on the lower leg — this can now be used to distinguish between the two. Thus, the mock tortoise coordinate asymptotes to,

$$\zeta(r_c + i\epsilon) = 0, \quad \zeta(r_c - i\epsilon) = 1 \quad (2.35)$$

Indeed, the mock tortoise coordinate  $\zeta$  can be immediately integrated in terms of  $r$  to give,

$$\zeta + \zeta_c = \frac{i}{2\pi} \frac{d}{d-1} \left( \frac{r_h}{r} \right)^{d-1} {}_2F_1 \left( 1, 1 - \frac{1}{d}; 2 - \frac{1}{d}; \frac{r^d}{r_h^d} \right) \quad (2.36)$$

Here  $\zeta_c$  is a constant chosen to set  $\zeta(r_c + i\epsilon) = 0$ . A constant time slice of the mock tortoise co-ordinate in the complex plane, along with our choice of branch cut is depicted in Fig.2.3.

When expressed in these coordinates, the metric reads,

$$ds^2 = -r^2 f dv^2 + i \beta r^2 f d\zeta dv + r^2 d\mathbf{x}_{d-1}^2. \quad (2.37)$$

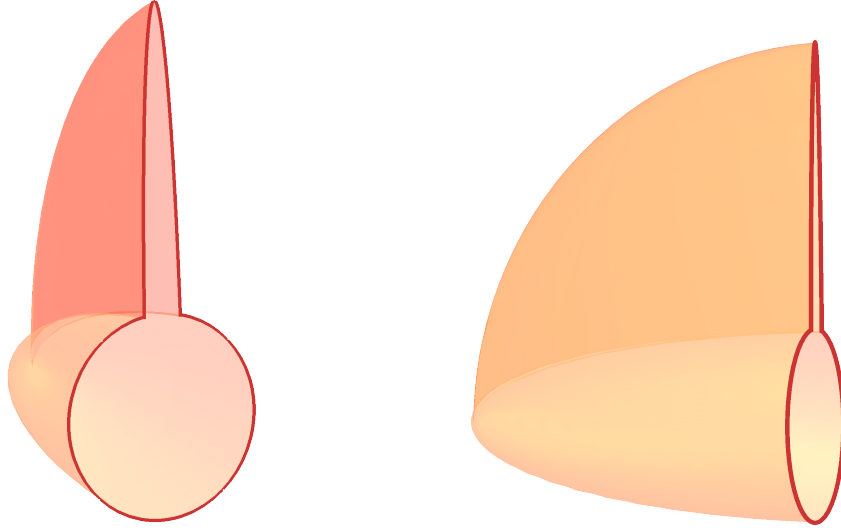


Figure 2.4: Two perspectives of the two-sheeted complex grSK geometry. The left panel displays the boundary thermal SK contour which is filled in the Euclidean portion by the Euclidean black hole (cigar) geometry and in the Lorentzian section by two copies of the exterior of a Lorentzian black hole spacetime. The right panel emphasize the smooth join of the two sheets of the Lorentzian section.

The spacetime thus consists of two manifolds  $\mathcal{M}_L$  and  $\mathcal{M}_R$  (with  $\mathbf{Re} \zeta = 0, 1$  respectively), smoothly joined along their radial directions by the horizon cap. The advanced time  $v$  is now identified with the time coordinate  $t$  on the boundary of this spacetime at  $r \rightarrow \infty \pm i\epsilon$ . In order to make contact with the Schwinger–Keldysh contour, these two manifolds require one further identification – each radially constant slice of  $\mathcal{M}_L$  and  $\mathcal{M}_R$  meets at the future turning point,  $v \rightarrow \infty$ . Thus, the sheets  $\mathcal{M}_L$  and  $\mathcal{M}_R$  are the bulk extensions of the  $\mathcal{C}_L$  and  $\mathcal{C}_R$  regions of the Schwinger–Keldysh contour, respectively. This resulting geometry is called the *gravitational Schwinger–Keldysh saddle* [89], and is illustrated in Fig.2.4.

### A grSK involution isometry

To conclude our discussion of the geometry, we record a feature that will play an important role in the analysis to follow. The gravitational Schwinger–Keldysh saddle is invariant under the following involutive diffeomorphism,

$$v \longmapsto i\beta\zeta - v, \quad \mathbf{x} \longmapsto -\mathbf{x}. \quad (2.38)$$

It is immediately clear that the metric (2.37) is unchanged by this transformation. While it is tempting to interpret this isometry as a  $PT$  reversal of the coordinates on the boundary

of the spacetime, this isn't quite correct: the transformation of the advanced Eddington–Finkelstein time  $v$  depends on the radial coordinate,  $\zeta$ . Indeed, these geometries do not possess a time reversal symmetry, as we may have anticipated for a gravitational dual to a non-equilibrium field theory. As noted by the authors of [34, 89], the above symmetry<sup>8</sup> is extremely useful for obtaining propagating solutions. In the following sections, we will exploit this involution extensively to construct Green's functions that solve the Dirac equation in the grSK geometry.

### 2.3.3 Open EFTs from holography

Finally, in order to derive the Schwinger–Keldysh generating functional for the operator  $\mathcal{O}$ , all that is left to do now is to lift the prescription in §2.3.1 to the grSK saddle. This is easily done, switching coordinates, and writing the Lorentzian version of the action (2.20) as a contour integral over the doubled geometry.

$$\mathbb{S}_\Psi = \oint d\zeta \int d^d x \sqrt{-g} i \bar{\Psi} \left( \Gamma^A D_A - m \right) \Psi + \int d^d x r^d i \bar{\Psi} \mathcal{P}_-^\zeta \Psi \Bigg|_{\zeta=0}^{\zeta=1} \quad (2.39)$$

Notice that the boundary term is now evaluated at both the right and the left boundaries of the grSK spacetime. The rest of the recipe proceeds analogously: we first need to solve for the fermion  $\Psi$  in this spacetime. Crucially, we will now impose Dirichlet boundary conditions at *both* the (right and left) boundaries of the grSK saddle,

$$\lim_{\zeta \rightarrow 0} r^{\frac{d}{2}-m} \mathcal{P}_+^\zeta \Psi = S_0 \psi_L, \quad \lim_{\zeta \rightarrow 1} r^{\frac{d}{2}-m} \mathcal{P}_+^\zeta \Psi = S_0 \psi_R. \quad (2.40)$$

In contrast to the situation in §2.3.1, such a solution will (in general) *not* be regular in the interior of the grSK geometry, with branch-cuts at the horizon of the  $AdS$  black brane. As we will see, these branch-cuts encode a lot of the physics of these systems and have to be dealt with carefully. Having obtained a solution with these specifications, we can now evaluate the action (2.39) on-shell, again selecting only its finite part to give us the generating functional for renormalized Schwinger–Keldysh correlators of the boundary CFT,  $\mathcal{Z}_{\text{SK}}[\psi_R, \bar{\psi}_R, \psi_L, \bar{\psi}_L]$ . Deriving the above generating functional is equivalent to deriving the open effective field theory that we seek.

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<sup>8</sup>In [34, 89], the authors do not flip the spatial co-ordinates  $\mathbf{x}$ .

## Chapter 3

# An open EFT for fermions from holography

### 3.1 Classical origin of Fermi–Dirac statistics

Having outlined how to derive an open effective field theory using holographic methods, the next task at hand is to study the dynamics of a fermionic field propagating in the bulk geometry. As emphasized in the previous section the influence phase for the probe field is obtained by taking the Dirac action on-shell — to do so, we have to solve the Dirac equation in this spacetime, subject to the boundary conditions (2.40). This is what we turn to now.

We organize this chapter as follows: we begin by introducing the Dirac equation for the bulk fermion in §3.1.1. In §3.1.2, we then detail how to use the involutive diffeomorphism (2.38) to ‘time–reverse’ solutions to this wave equation, allowing us to generate a full solution once the solution satisfying the infalling boundary conditions is known. As an important consequence, in §3.1.3 we will see how this leads to the correct statistics for this system. Once these constructions have been laid out, we will move on to computing the infalling boundary–to–bulk Green’s function. In general  $d$ , the Dirac wave equation is not known to admit closed form solutions in these spacetimes. However, we can solve them in a gradient expansion, i.e., determining them order by order in increasing powers of frequency and momenta. This is carried out in §3.2.

### 3.1.1 The Dirac equation.

In order to set up the Dirac equation in this geometry, we use the following tangent frame [33],

$$E^{(v)} = \frac{r}{2} dv - \frac{f}{2} r (i\beta d\zeta - dv), \quad E^{(\zeta)} = \frac{r}{2} dv + \frac{f}{2} r (i\beta d\zeta - dv), \quad E^{(i)} = r dx^i. \quad (3.1)$$

These tetrads have two advantages. First, none of the components of the frame or their derivatives are badly behaved at the blackhole horizon. Second, there are no square roots of the the emblackening factor  $f$  that appear. These features are of particular importance to the grSK saddle because it involves an analytical continuation of the radial direction across the future horizon — the above tangent frame is thus smooth on the whole spacetime. The spin connection 1-forms corresponding to these tetrads are,

$$\omega_{(v)(\zeta)} = \left( rf - \frac{1}{i\beta f} \frac{df}{d\zeta} \right) dv - \frac{i\beta}{2} rf d\zeta, \quad \omega_{(i)(v)} = -\frac{1-f}{2} r dx^i, \quad \omega_{(i)(\zeta)} = \frac{1+f}{2} r dx^i, \quad (3.2)$$

with all other components, not related by symmetry, vanishing. A little algebra then leads us to the following Dirac equation,

$$\left[ \mathbb{F} (\partial_\zeta + i\beta \partial_v) + \mathbb{F}^\dagger f^{-1} \partial_\zeta + \frac{i\beta}{2} (\Gamma^{(i)} \partial_i - mr) + \frac{1}{2} \mathbb{F} \frac{d}{d\zeta} \log(r^d f) + \frac{1}{2} \mathbb{F}^\dagger f^{-1} \frac{d}{d\zeta} \log(r^d) \right] \Psi = 0, \quad (3.3)$$

where we have introduced the matrices  $\mathbb{F}$  and  $\mathbb{F}^\dagger$  to keep expressions compact.

$$\mathbb{F} = \frac{1}{2} (\Gamma^{(\zeta)} + \Gamma^{(v)}) \implies \mathbb{F}^\dagger = \frac{1}{2} (\Gamma^{(\zeta)} - \Gamma^{(v)}). \quad (3.4)$$

We emphasize that by definition (2.34), both  $\partial_\zeta$  and  $f^{-1} \partial_\zeta$  are regular vector fields at the future horizon. Consequently, despite initial appearances, both the spin connection 1-forms and the Dirac equation are well defined along the holographic contour.

As a final observation, we note that the metric components (2.37) are independent of  $v$  and  $\mathbf{x}$ . Thus, it is useful to perform a Fourier transformation with respect to these coordinates, allowing these modes to decouple. We adopt the following shorthand for the Fourier transform,

$$\Psi(v, \zeta, \mathbf{x}) = \int \frac{d\omega d^{d-1}\mathbf{k}}{(2\pi)^d} \Psi(\omega, \zeta, \mathbf{k}) e^{-i\omega v + i\mathbf{k}\cdot\mathbf{x}} \equiv \int_{\mathbf{k}} \Psi(\omega, \zeta, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (3.5)$$

The field  $\Psi(\omega, \zeta, \mathbf{k})$  is then easily seen to satisfy,

$$\left[ \mathbb{F} (\partial_\zeta + \beta\omega) + \mathbb{F}^\dagger f^{-1} \partial_\zeta - \frac{\beta}{2} (\Gamma^{(i)} k_i + imr) + \frac{1}{2} \mathbb{F} \frac{d}{d\zeta} \log(r^d f) + \frac{1}{2} \mathbb{F}^\dagger f^{-1} \frac{d}{d\zeta} \log(r^d) \right] \Psi = 0. \quad (3.6)$$

### 3.1.2 ‘Time-reversing’ solutions.

In the following sections, we will solve the above Dirac equation (3.6) for the boundary-to-bulk Green’s function. As we will see, these propagators come in two types: an *infalling* (*retarded*) propagator and an *outgoing* (*advanced*) one<sup>1</sup>, with the full boundary-to-bulk Green’s function being a linear combination of the two. The involutive diffeomorphism described at the end of §2.3.2 allows us to exploit this structure: using (2.38), we can construct a map between infalling and outgoing solutions. Thus, once the infalling Green’s function is known, the full boundary-to-bulk Green’s function can be immediately written down.

This construction isn’t new — the utility of this symmetry for this very purpose was first recognized and exploited by the authors of [34] in the context of a classical string probing the grSK geometry, and later in [89] for a probe scalar field. In this section, we describe how this construction works for the Dirac field. As we will see, this discussion contains subtleties not present in the other two cases.

To begin with, let us consider the effect of the global coordinate transformation (GCT) (2.38) on the Dirac operator (3.3) — it simply exchanges the derivatives  $\partial_\zeta \longleftrightarrow \partial_\zeta + i\beta \partial_v$ , and therefore does *not* leave the operator unchanged. This is to be expected — crucially, the form of the Dirac equation depends both on our choice of coordinates and our choice of tangent frame. While the isometry leaves the metric invariant, it has a non-trivial action

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<sup>1</sup>The terms infalling and outgoing are used in reference to the radial direction of the geometry.

on the space of 1-forms induced by its Jacobian  $\mathcal{J}_B^A$ ,

$$E_A^a \longrightarrow \mathcal{J}_A^B E_B^a, \quad \text{where} \quad \mathcal{J}_A^B = \begin{pmatrix} -1 & 0 & 0 \\ i\beta & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (3.7)$$

In order to realize the involution (2.38) as a discrete symmetry of the Dirac equation, we have to compensate for the above transformation of the tetrads by an appropriate linear transformation of the tangent frame. In other words, we seek a matrix  $\mathcal{T}^a_b$  that satisfies,

$$\mathcal{T}^a_b \mathcal{J}_A^B E_B^b = E_A^a. \quad (3.8)$$

This matrix equation is easily solved, giving,

$$\mathcal{T}^a_b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh \vartheta & \sinh \vartheta & 0 \\ \sinh \vartheta & \cosh \vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \vartheta = \log f. \quad (3.9)$$

Thus, the linear transformation  $\mathcal{T}$  is actually an idempotent local Lorentz transformation (LLT) composed of a boost along  $(\zeta)$  with rapidity parameter  $\vartheta$  followed by a reflection of the  $(v)$  and  $(x)$  axes. This LLT is an improper transformation when the dimension of the bulk is even, and proper when its dimension is odd.

With the metric and the tetrads left invariant, the involution (2.38) can now be regarded as a symmetry of the Dirac operator (3.3) — the lesson from the above calculation is that this requires both the GCT as well a compensatory LLT.

### Constructing outgoing solutions.

We are now in a position to construct the map between infalling and outgoing solutions. Consider an infalling solution of (3.6),  $\Psi^{\text{in}}(\omega, \zeta, \mathbf{k})$ . Drawing from our previous discussion, in order to generate an outgoing solution from this, simply performing the coordinate transformation (2.38) is not enough — the involution induces a linear transformation on spinor

space that we have to undo by a compensatory transformation,  $\mathfrak{T}$ . Let us deal with these two steps one by one. The effect of the GCT is given by,

$$\Psi^{\text{in}}(\omega, \zeta, \mathbf{k}) \longmapsto \Psi^{\text{in}}(-\omega, \zeta, -\mathbf{k}) e^{-\beta\omega\zeta} \equiv \Psi^{\text{rev}}(\omega, \zeta, \mathbf{k}) e^{-\beta\omega\zeta}, \quad (3.10)$$

where we have defined the *frequency reversed* spinor,  $\Psi^{\text{rev}}(\omega, \zeta, \mathbf{k})$ . The relevant linear transformation in spinor space,  $\mathfrak{T}$ , is simply the LLT in spinor space that corresponds to the Lorentz transformation  $\mathcal{T}$ , to wit,

$$\mathfrak{T} = \Gamma^{(\zeta)} \cdot \exp\left(\frac{\vartheta}{2} \Gamma^{(\zeta)} \Gamma^{(v)}\right) = \sqrt{f} \mathbb{F} + \sqrt{f}^{-1} \mathbb{F}^\dagger. \quad (3.11)$$

Indeed, exactly like (3.9), this matrix is constructed out of two parts. The first factor,  $\exp\left(\frac{\vartheta}{2} \Gamma^{(\zeta)} \Gamma^{(v)}\right)$ , is a boost in spinor space along  $(\zeta)$  with rapidity  $\vartheta$ . This is then multiplied by  $\Gamma^{(\zeta)}$ , a matrix that *anti-commutes* with  $\Gamma^{(v)}$  and all the  $\Gamma^{(i)}$ , accounting for the reflection of the  $(v)$  and  $(\mathbf{x})$  directions. It is easily checked that the matrix  $\mathfrak{T}$  is idempotent.

Thus, combining the above, we are led to the following recipe: given an ingoing solution  $\Psi^{\text{in}}(\omega, \zeta, \mathbf{k})$ , we can construct an outgoing solution  $\Psi^{\text{out}}(\omega, \zeta, \mathbf{k})$  as follows,

$$\Psi^{\text{out}}(\omega, \zeta, \mathbf{k}) = \mathfrak{T} \cdot \Psi^{\text{rev}}(\omega, \zeta, \mathbf{k}) e^{-\beta\omega\zeta} = \mathfrak{T} \cdot \Psi^{\text{in}}(-\omega, \zeta, -\mathbf{k}) e^{-\beta\omega\zeta}. \quad (3.12)$$

The infalling solution  $\Psi^{\text{in}}(\omega, \zeta, \mathbf{k})$  is (by definition) regular at the future horizon of the black brane. The above construction then implies that the outgoing solution is *not* smooth in this region, with branch-cuts arising from the matrix  $\mathfrak{T}$  and the factor  $e^{-\beta\omega\zeta}$ . This ties in with our previous discussion in §2.3.3 — the full solution, which is an appropriate linear combination of the infalling and outgoing solutions is not regular in the interior.

The validity of the above prescription for outgoing solutions is easily checked. Evaluating  $\Psi^{\text{out}}(\omega, \zeta, \mathbf{k})$  on the Dirac operator (3.6), a short calculation demonstrates that this spinor solves the wave equation if and only if  $\Psi^{\text{rev}}(\omega, \zeta, \mathbf{k})$  satisfies,

$$\left[ \mathbb{F} (\partial_\zeta - \beta\omega) + \mathbb{F}^\dagger f^{-1} \partial_\zeta + \frac{\beta}{2} (\Gamma^{(i)} k_i - imr) + \frac{1}{2} \mathbb{F} \frac{d}{d\zeta} \log(r^d f) + \frac{1}{2} \mathbb{F}^\dagger f^{-1} \frac{d}{d\zeta} \log(r^d) \right] \Psi^{\text{rev}} = 0. \quad (3.13)$$

This is just the Dirac operator (3.6) with the signs of its frequency and momenta flipped. By definition,  $\Psi^{\text{in}}(-\omega, \zeta, -\mathbf{k}) = \Psi^{\text{rev}}(\omega, \zeta, \mathbf{k})$  is a solution to this equation.

### 3.1.3 The emergence of Fermi–Dirac statistics

Following the prescription described in Section 2.3.2, we can write the full solution of the Dirac equation on the Schwinger Keldysh contour as,

$$\Psi(\zeta, \omega, \mathbf{k}) = S^{\text{in}}(\zeta, \omega, \mathbf{k}) c(\omega, \mathbf{k}) + \left[ \sqrt{f} \mathbb{I} + \frac{1}{\sqrt{f}} \mathbb{I}^\dagger \right] S^{\text{rev}}(\zeta, \omega, \mathbf{k}) h(\omega, \mathbf{k}) e^{-\beta\omega\zeta}. \quad (3.14)$$

Here,  $c(\omega, \mathbf{k})$  and  $h(\omega, \mathbf{k})$  are (*causal* and *Hawking*) boundary spinors.  $S^{\text{in/rev}}(\zeta, \omega, \mathbf{k})$  are the boundary to bulk Green’s functions, and,

$$S^{\text{rev}}(\zeta, \omega, \mathbf{k}) \equiv S^{\text{in}}(\zeta, -\omega, -\mathbf{k}). \quad (3.15)$$

We now impose boundary conditions on this solution using the sources  $\psi_L(\omega, \mathbf{k})$  and  $\psi_R(\omega, \mathbf{k})$  at the two conformal boundaries ( $r \rightarrow r_c \pm i\epsilon$ ) of the holographic SK contour.

$$\begin{aligned} \lim_{r \rightarrow r_c + i\epsilon} r^{\frac{d}{2}-m} \mathcal{P}_+^\zeta \Psi &= \mathcal{P}_+^\zeta S_0 \psi_L \\ \lim_{r \rightarrow r_c - i\epsilon} r^{\frac{d}{2}-m} \mathcal{P}_+^\zeta \Psi &= \mathcal{P}_+^\zeta S_0 \psi_R \end{aligned} \quad (3.16)$$

Here,  $S_0$  is a constant matrix related to  $S^{\text{in/rev}}$  via,

$$\lim_{r \rightarrow r_c \pm i\epsilon} r^{\frac{d}{2}-m} S^{\text{in}} = \lim_{r \rightarrow r_c \pm i\epsilon} r^{\frac{d}{2}-m} S^{\text{rev}} = S_0. \quad (3.17)$$

Further, at these two boundaries,

$$\lim_{r \rightarrow r_c + i\epsilon} \sqrt{f} = 1, \quad \lim_{r \rightarrow r_c - i\epsilon} \sqrt{f} = -1. \quad (3.18)$$

These boundary conditions immediately yield the following equations,

$$\begin{aligned}\psi_L(\omega, \mathbf{k}) &= c(\omega, \mathbf{k}) + h(\omega, \mathbf{k}) \\ \psi_R(\omega, \mathbf{k}) &= c(\omega, \mathbf{k}) - h(\omega, \mathbf{k}) e^{-\beta\omega}\end{aligned}\tag{3.19}$$

These, in turn, are readily solved to give the full solution,

$$\begin{aligned}\Psi(\zeta, \omega, \mathbf{k}) &= S^{\text{in}}(\zeta, \omega, \mathbf{k}) \left[ n_{\omega}^{\text{FD}} \psi_L + (1 - n_{\omega}^{\text{FD}}) \psi_R \right] \\ &+ \left[ \sqrt{f} \mathbb{F} + \frac{1}{\sqrt{f}} \mathbb{F}^{\dagger} \right] S^{\text{rev}}(\zeta, \omega, \mathbf{k}) \left[ (1 - n_{\omega}^{\text{FD}}) (\psi_L - \psi_R) \right] e^{-\beta\omega\zeta}.\end{aligned}\tag{3.20}$$

Where  $n_{\omega}^{\text{FD}}$  is the familiar Fermi–Dirac factor (2.16). The solution can be alternately written as,

$$\Psi(\zeta, \omega, \mathbf{k}) = -S^{\text{in}}(\zeta, \omega, \mathbf{k}) \psi_{\bar{F}} - \left[ \sqrt{f} \mathbb{F} + \frac{1}{\sqrt{f}} \mathbb{F}^{\dagger} \right] S^{\text{rev}}(\zeta, \omega, \mathbf{k}) \psi_{\bar{P}} e^{\beta\omega(1-\zeta)},\tag{3.21}$$

where we have again introduced the retarded–advanced basis,

$$\psi_{\bar{F}} = n_{\omega}^{\text{FD}} (\psi_R - \psi_L) - \psi_R, \quad \psi_{\bar{P}} = n_{\omega}^{\text{FD}} (\psi_R - \psi_L).\tag{3.22}$$

We see that when the solution is written in terms of sources in the RA basis, the two combinations of sources precisely multiply the ingoing/quasi–normal bulk–to–boundary propagator and the outgoing bulk–to–boundary propagator respectively. This fact can be exploited to argue why the generating function of correlations computed from the bulk necessarily satisfy the SK collapse rules as well as KMS conditions, i.e., the statement that the generating function cannot contain terms with only  $\psi_{\bar{P}}$  or terms with only  $\psi_{\bar{F}}$ .

The argument proceeds as follows: if we think of the contributions to the generation function as arising from Witten diagrams on grSK contour, the terms under question are either made entirely out of the ingoing propagators or made entirely out of the outgoing propagators. A diagram made completely out of ingoing propagators is analytic on the grSK contour and hence evaluates to zero, when the vertices are integrated all over the grSK contour, viz., the left and right contributions cancel out. A diagram made completely out of outgoing propagators is also analytic on the grSK contour: the branch cut due to

$\zeta$  cancels by frequency conservation, while the square root branch cut cancels because any non-zero fermionic correlator has an even number of operator insertions. Hence, by a similar argument, terms with only  $\psi_{\bar{P}}$  cannot occur in the generating function. This argument is a direct generalisation of a similar argument in the bosonic examples [34, 89].

Finally, in the Keldysh or average-difference basis (2.10) the solution above takes the form,

$$\Psi(v, \zeta, \mathbf{k}) = S^{\text{in}} \psi_a - \left\{ \left( n_{\omega}^{\text{FD}} - \frac{1}{2} \right) S^{\text{in}} + n_{\omega}^{\text{FD}} e^{\beta\omega(1-\zeta)} \left[ \sqrt{f} \mathbb{I} + \frac{1}{\sqrt{f}} \mathbb{I}^{\dagger} \right] S^{\text{rev}} \right\} \psi_d. \quad (3.23)$$

Having provided a derivation of the statistical factor for Dirac fermions, it is easy to see how this argument generalises to arbitrary spins. The crucial fact in this regard is the statement that the outgoing Hawking modes suffer a Boltzmann suppression as well as a complex boost of rapidity  $2\pi i$  when they pass through the horizon cap region. The rapidity  $2\pi i$  arises by comparing the Lorentz boost factor  $\mathcal{T}$  (3.9) across the horizon cap — the factor  $2\pi i$  is essentially the jump in the rapidity  $\vartheta = \ln f$  appearing in  $\mathcal{T}$ . It is then clear that for a field of spin  $s$ , the classical transmission amplitude for the outgoing modes across the horizon cap is given by,

$$e^{-\beta\omega + 2\pi i s}, \quad (3.24)$$

As we argued in the introductory chapter to this *Opus*, for integer  $s$  this reduces to a simple Boltzmann weight. For half integer  $s$ , on the other hand, we get an additional phase factor of  $-1$ , which then leads to a destructive interference between the ingoing and the outgoing modes, perfectly reproducing the Pauli exclusion felt by half-integer fields. The classical physics of grSK saddles thus automatically reproduces the correct spin-statistics relation for the boundary correlators.

## 3.2 Solution in gradient expansion

In this section, we will describe how the Dirac equation can be solved explicitly in a derivative expansion. We will begin by first discussing the massless case where the functions involved are simpler and then generalise our discussion to the massive case.

### 3.2.1 Massless gradient expansion

In the massless case, the most general ingoing solution compatible with rotational invariance can be expanded in a derivative expansion as,

$$\begin{aligned} \Psi = \frac{1}{r^{d/2}} \Big\{ \mathbb{1} + C_a^{(1)} M^a \partial_v + D_a^{(1)} M^a \Gamma^{(i)} \partial_i + C_a^{(2)} M^a \partial_v^2 \\ + D_a^{(2)} M^a \partial_i^2 + F_a^{(2)} M^a \Gamma^{(i)} \partial_i \partial_v + \dots \Big\} S_0 \psi . \end{aligned} \quad (3.25)$$

Here  $\psi = \psi(v, \mathbf{x})$  is a (boundary) spinor that acts as the source at the boundary.  $S_0$  is the leading order ingoing propagator (up to factor of  $r^{-\frac{d}{2}}$ ) and satisfies,

$$\mathbb{T} S_0 = 0 . \quad (3.26)$$

In the above ansatz,  $\{C_a^{(n)}, D_a^{(n)}, F_a^{(n)}, \dots\}$  are functions of the radial coordinate  $\zeta$  and  $M^a \in \{\mathbb{1}, \Gamma^{(\zeta)}\}$ . Terms with  $M^a = \Gamma^{(v)}$  can be eliminated using (3.26).

The boundary conditions at infinity and the regularity condition at the horizon fix the ingoing solution.

$$\begin{aligned} \Psi_{\text{in}} = \frac{1}{r^{d/2}} \Big\{ 1 - \frac{i\beta}{2} \left( H \Gamma^{(\zeta)} - H_{(0)} \right) \Gamma^{(i)} \partial_i - \frac{\beta^2}{2} \left[ \tilde{H} \Gamma^{(\zeta)} - \tilde{H}_{(0)} \right] \Gamma^{(i)} \partial_i \partial_v \\ + \frac{\beta^2}{8} \left( f H^2 - f_c H_{(0)}^2 \right) \partial_i^2 + \frac{\beta^2}{4} H_{(0)} \left( H \Gamma^{(\zeta)} - H_{(0)} \right) \partial_i^2 \Big\} S_0 \psi . \end{aligned} \quad (3.27)$$

Here,  $H$  and  $\tilde{H}$  are functions defined by the differential equations and boundary conditions,

$$\frac{d}{d\zeta} (H \sqrt{f}) = \sqrt{f} , \quad \frac{d}{d\zeta} (\tilde{H} \sqrt{f}) = H \sqrt{f} , \quad H_{(0)} = H(\zeta_c) , \quad \tilde{H}_{(0)} = \tilde{H}(\zeta_c) . \quad (3.28)$$

All these functions are regular in ingoing coordinates. The function  $H$  can be explicitly computed and is given by,

$$H(\zeta) = \frac{1}{i\pi} {}_2F_1 \left( \frac{1}{2}, \frac{d-1}{d}; \frac{3}{2}; f(\zeta) \right) \quad \Rightarrow \quad H_{(0)} = -\frac{i}{2\sqrt{\pi}} \frac{\Gamma(1/d)}{\Gamma(\frac{d+2}{2d})} . \quad (3.29)$$

We also obtain the value for  $\tilde{H}_{(0)}$  as,

$$\begin{aligned} \tilde{H}_{(0)} = & -\frac{d \cos \frac{\pi}{d}}{(d-2)\pi^{\frac{5}{2}}} \Gamma\left(\frac{1}{d}\right) \Gamma\left(\frac{3d-2}{2d}\right) {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{d-1}{d}; \frac{d+2}{2d}, \frac{3}{2}; 1\right) \\ & \stackrel{d \rightarrow 2}{=} -\frac{2}{\pi^2} G, \end{aligned} \quad (3.30)$$

where  $G$  is the Catalan's constant.

Given this ingoing solution in derivative expansion, the solution on the full grSK contour can be obtained from the expressions derived in the last section. In the a-d basis the solution, up to second order in gradients, can be expressed as,

$$\begin{aligned} \Psi = & \frac{1}{r^{d/2}} \left\{ \left[ \mathbb{1} + \frac{\beta}{2} \left( H \Gamma^{(\zeta)} - H_{(0)} \right) \Gamma^{(i)} k_i - \frac{\beta^2}{2} \left( \tilde{H} \Gamma^{(\zeta)} - \tilde{H}_{(0)} \right) \Gamma^{(i)} \omega k_i \right. \right. \\ & \left. \left. - \frac{\beta^2}{4} \left( H_{(0)} \left( H \Gamma^{(\zeta)} - H_{(0)} \right) + \frac{1}{2} \left( f H^2 - H_{(0)}^2 \right) \right) k^2 \right] S_0 \psi_a \right. \\ & + \frac{1}{2} \left[ -\frac{\Gamma^{(\zeta)}}{\sqrt{f}} + \frac{\beta}{2} \left( \mathbb{1} + (2\zeta - 1) \frac{\Gamma^{(\zeta)}}{\sqrt{f}} \right) \omega \right. \\ & + \frac{\beta}{2} \left( H \sqrt{f} - H_{(0)} \frac{\Gamma^{(\zeta)}}{\sqrt{f}} \right) \Gamma^{(i)} k_i + \frac{\beta^2}{2} \zeta (1 - \zeta) \frac{\Gamma^{(\zeta)}}{\sqrt{f}} \omega^2 \\ & + \frac{\beta^2}{8} \left( 2H_{(0)} H \sqrt{f} + \left( f H^2 - 3H_{(0)}^2 \right) \frac{\Gamma^{(\zeta)}}{\sqrt{f}} \right) k^2 \\ & + \frac{\beta^2}{4} \left( 2 \left( \tilde{H} \sqrt{f} - \tilde{H}_{(0)} \frac{\Gamma^{(\zeta)}}{\sqrt{f}} \right) \right. \\ & + (1 - 2\zeta) \left( H \sqrt{f} - H_{(0)} \frac{\Gamma^{(\zeta)}}{\sqrt{f}} \right) \\ & \left. \left. + \left( H \Gamma^{(\zeta)} - H_{(0)} \right) \right) \Gamma^{(i)} \omega k_i \right] S_0 \psi_d \left. \right\}. \end{aligned} \quad (3.31)$$

We shall now calculate the on-shell action using the above solution. It is given by a

boundary term that evaluates to

$$\begin{aligned}
r^d \bar{\Psi} \mathcal{P}_-^\zeta \Psi \Big|_{\zeta=0}^{\zeta=1} &= -\bar{\psi}_a \left( \gamma^0 + \beta H_{(0)} \gamma^i k_i + \frac{\beta^2}{2} H_{(0)}^2 \gamma^0 k^2 + \beta^2 \tilde{H}_{(0)} \gamma^i \omega k_i \right) \psi_d \\
&\quad + \bar{\psi}_d \left( \gamma^0 - \beta H_{(0)} \gamma^i k_i + \frac{\beta^2}{2} H_{(0)}^2 \gamma^0 k^2 + \beta^2 \tilde{H}_{(0)} \gamma^i \omega k_i \right) \psi_a \\
&\quad + \frac{1}{2} \bar{\psi}_d (\beta \gamma^0 \omega) \psi_d , \\
&= \bar{\psi}_d \gamma^0 \psi_a - \bar{\psi}_a \gamma^0 \psi_d + \frac{\beta \omega}{2} \bar{\psi}_d \gamma^0 \psi_d - \beta H_{(0)} k_i (\bar{\psi}_d \gamma^i \psi_a + \bar{\psi}_a \gamma^i \psi_d) \\
&\quad - \frac{\beta^2}{2} H_{(0)}^2 k^2 (\bar{\psi}_a \gamma^0 \psi_d - \bar{\psi}_d \gamma^0 \psi_a) - \beta^2 \tilde{H}_{(0)} \omega k_i (\bar{\psi}_a \gamma^i \psi_d - \bar{\psi}_d \gamma^i \psi_a) .
\end{aligned} \tag{3.32}$$

In the RL basis we have,

$$\begin{aligned}
r^d \bar{\Psi} \mathcal{P}_-^\zeta \Psi \Big|_{\zeta=0}^{\zeta=1} &= \left( 1 - \frac{\beta}{2} \omega \right) \bar{\psi}_R \gamma^0 \psi_L - \left( 1 + \frac{\beta}{2} \omega \right) \bar{\psi}_L \gamma^0 \psi_R \\
&\quad + \frac{\beta}{2} \bar{\psi}_R (\omega \gamma^0 - 2 H_{(0)} \gamma^i k_i) \psi_R + \frac{\beta}{2} \bar{\psi}_L (\omega \gamma^0 + 2 H_{(0)} \gamma^i k_i) \psi_L \\
&\quad + \frac{\beta^2}{2} \bar{\psi}_R \left( H_{(0)}^2 \gamma^0 k^2 + 2 \tilde{H}_{(0)} \gamma^i \omega k_i \right) \psi_L \\
&\quad - \frac{\beta^2}{2} \bar{\psi}_L \left( H_{(0)}^2 \gamma^0 k^2 + 2 \tilde{H}_{(0)} \gamma^i \omega k_i \right) \psi_R .
\end{aligned} \tag{3.33}$$

We quote the corresponding expression in even boundary dimensions  $d = 2n$  is (check Appendix A for Clifford algebra conventions)

$$\begin{aligned}
r^d \bar{\Psi} \mathcal{P}_-^\zeta \Psi \Big|_{\zeta=0}^{\zeta=1} &= \psi_a^\dagger \psi_d - \psi_d^\dagger \psi_a - \frac{\beta \omega}{2} \psi_d^\dagger \psi_d + \beta H_{(0)} \left( \psi_a^\dagger \Sigma^{(i)} k_i \psi_d + \psi_d^\dagger \Sigma^{(i)} k_i \psi_a \right) \\
&\quad + \frac{\beta^2}{2} \psi_a^\dagger \left( H_{(0)}^2 k^2 + 2 \omega k_i \tilde{H}_{(0)} \Sigma^{(i)} \right) \psi_d - \frac{\beta^2}{2} \psi_d^\dagger \left( H_{(0)}^2 k^2 + 2 \omega k_i \tilde{H}_{(0)} \Sigma^{(i)} \right) \psi_a , \\
&= \psi_L^\dagger \left( 1 + \frac{\beta \omega}{2} \right) \psi_R - \psi_R^\dagger \left( 1 - \frac{\beta \omega}{2} \right) \psi_L \\
&\quad + \frac{\beta}{2} \psi_R^\dagger \left( 2 H_{(0)} \Sigma^{(i)} k_i - \omega \right) \psi_R - \frac{\beta}{2} \psi_L^\dagger \left( 2 H_{(0)} \Sigma^{(i)} k_i + \omega \right) \psi_L \\
&\quad + \frac{\beta^2}{2} \psi_L^\dagger \left( H_{(0)} k^2 + 2 \tilde{H}_{(0)} \omega k_i \Sigma^{(i)} \right) \psi_R - \frac{\beta^2}{2} \psi_R^\dagger \left( H_{(0)} k^2 + 2 \tilde{H}_{(0)} \omega k_i \Sigma^{(i)} \right) \psi_L
\end{aligned} \tag{3.34}$$

### 3.2.2 Massive gradient expansion

In this section we will discuss the massive solution up to first order in derivative expansion. Note that while writing the following solution, we have chosen the function  $\sqrt{f}$  to assume positive values at the left boundary (the segment of contour with positive imaginary radius). In contrast to the massless solution, the structure of the ingoing solution in this case is more involved. While a direct derivative expansion results in somewhat intricate differential equations even at first order in derivative expansion, we will show below that simpler equations can be obtained by a judicious parametrisation of the relevant functions.

We will begin by writing the zeroth order ingoing solution for the massive case in the form,

$$\Psi = \frac{1}{r^{\frac{d}{2}}} \frac{1}{\mathcal{N}_m} \left\{ \mathcal{F}_+ + \frac{\mathcal{F}_-}{\sqrt{f}} \Gamma^{(\zeta)} \right\} S_0 \psi. \quad (3.35)$$

The functions  $\mathcal{F}_+$  and  $\mathcal{F}_-$  are in turn defined via a function  $\mathcal{F}$  as,

$$\mathcal{F}_+ \equiv \frac{1}{2} \left( \mathcal{F} + \frac{1}{\mathcal{F}} \right), \quad \mathcal{F}_- \equiv \frac{1}{2} \left( \mathcal{F} - \frac{1}{\mathcal{F}} \right), \quad (3.36)$$

where the function  $\mathcal{F}$  satisfies,

$$\frac{d}{d\zeta} \mathcal{F} = m \frac{i\beta}{2} r \sqrt{f} \mathcal{F}, \quad \mathcal{F}(\zeta_h) = 1. \quad (3.37)$$

The above conditions fix  $\mathcal{F}$  to be,

$$\mathcal{F} = \left( \frac{1 + \sqrt{f}}{1 - \sqrt{f}} \right)^{m/d}. \quad (3.38)$$

$\mathcal{N}_m$  is a normalisation constant given by,

$$\mathcal{N}_m \equiv \lim_{r \rightarrow \infty} r^{-m} \mathcal{F} = \frac{4^{\frac{m}{d}}}{r_h^m}. \quad (3.39)$$

Notice that across the horizon,  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  are exchanged because the sign of the function  $\sqrt{f}$  reverses. This in turn reverses the sign of  $\mathcal{F}_-$  while the sign of  $\mathcal{F}_+$  remains invariant. Nevertheless, in the full zeroth order solution to the massive Dirac equation given above,

no such square root branch cuts appear, thanks to the extra factor of  $\sqrt{f}$  multiplying  $\mathcal{F}_-$ .

At first order in derivatives we can write an ingoing solution of the form,

$$\begin{aligned} \Psi = \frac{1}{r^{\frac{d}{2}}} \frac{1}{\mathcal{N}_m} \left\{ \left( \mathcal{F}_+ + \mathcal{F}_- \frac{\Gamma(\zeta)}{\sqrt{f}} \right) \left( 1 + \frac{i\beta}{2} \mathcal{W}_+ \partial_v + \frac{i\beta}{2} \mathcal{K}_+ \Gamma^{(i)} \partial_i \right) \right. \\ \left. - \left( \mathcal{F}_- + \mathcal{F}_+ \frac{\Gamma(\zeta)}{\sqrt{f}} \right) \left( \frac{i\beta}{2} \mathcal{W}_- \partial_v + \frac{i\beta}{2} \mathcal{K}_- \Gamma^{(i)} \partial_i \right) \right\} S_0 \psi. \end{aligned} \quad (3.40)$$

Here the functions  $\mathcal{W}_\pm$  and  $\mathcal{K}_\pm$  are solutions to the differential equations,

$$\begin{aligned} \frac{d}{d\zeta} \mathcal{W}_+ &= 2 (\mathcal{F}_-)^2, & \frac{d}{d\zeta} \mathcal{W}_- &= 2 \mathcal{F}_+ \mathcal{F}_-, \\ \frac{d}{d\zeta} \mathcal{K}_+ &= 2 \sqrt{f} \mathcal{F}_+ \mathcal{F}_-, & \frac{d}{d\zeta} \mathcal{K}_- &= \sqrt{f} ((\mathcal{F}_+)^2 + (\mathcal{F}_-)^2). \end{aligned} \quad (3.41)$$

They are uniquely fixed by the boundary conditions

$$\begin{aligned} \lim_{\zeta \rightarrow 0} (\mathcal{W}_+ - \mathcal{W}_-) &= 0, & \lim_{\zeta \rightarrow 0} (\mathcal{K}_+ - \mathcal{K}_-) &= 0 \\ \lim_{\zeta \rightarrow \zeta_h} \mathcal{W}_- &= 0, & \lim_{\zeta \rightarrow \zeta_h} \mathcal{K}_- &= 0. \end{aligned} \quad (3.42)$$

In the above, exactly like the functions at zeroth order, the functions with subscript ‘+’ remain the same as we traverse the horizon cap. The functions with subscript ‘-’, on the other hand, pick up a minus sign because of a square root branch cut. We can write an integral representation for  $\mathcal{W}_\pm$  and  $\mathcal{K}_\pm$  as,

$$\begin{aligned} \mathcal{W}_+(\zeta) &= 2 \int_0^\zeta (\mathcal{F}_-)^2 d\hat{\zeta} - 2 \int_0^{\zeta_h} \mathcal{F}_+ \mathcal{F}_- d\hat{\zeta}, \\ \mathcal{W}_-(\zeta) &= 2 \int_{\zeta_h}^\zeta \mathcal{F}_+ \mathcal{F}_- d\hat{\zeta}, \\ \mathcal{K}_+(\zeta) &= 2 \int_0^\zeta \sqrt{f} \mathcal{F}_+ \mathcal{F}_- d\hat{\zeta} - 2 \int_0^{\zeta_h} \sqrt{f} ((\mathcal{F}_+)^2 + (\mathcal{F}_-)^2) d\hat{\zeta}, \\ \mathcal{K}_-(\zeta) &= \int_{\zeta_h}^\zeta \sqrt{f} ((\mathcal{F}_+)^2 + (\mathcal{F}_-)^2) d\hat{\zeta}. \end{aligned} \quad (3.43)$$

The reader can check that these functions satisfy the differential equations and the horizon jump conditions mentioned above.

The  $\mathcal{W}_-$  and  $\mathcal{K}_-$  functions vanish at the horizon, as can be checked by computing the following limits.

$$\begin{aligned}\lim_{\zeta \rightarrow \zeta_h} \frac{1}{\sqrt{f}} \mathcal{W}_-(\zeta) &= \lim_{\zeta \rightarrow \zeta_h} \frac{1}{\sqrt{f}} \int_{\zeta_h}^{\zeta} 2 \mathcal{F}_+ \mathcal{F}_- d\hat{\zeta} = \frac{1}{i\pi} \frac{4}{d} m, \\ \lim_{\zeta \rightarrow \zeta_h} \frac{1}{\sqrt{f}} \mathcal{K}_-(\zeta) &= \lim_{\zeta \rightarrow \zeta_h} \frac{1}{\sqrt{f}} \int_{\zeta_h}^{\zeta} \sqrt{f} ((\mathcal{F}_+)^2 + (\mathcal{F}_-)^2) d\hat{\zeta} = \frac{1}{i\pi}.\end{aligned}\tag{3.44}$$

As in the massless case, the solution on the grSK saddle can be obtained by using the ingoing solution. In the a-d basis, we get (upto first order in derivatives),

$$\begin{aligned}\Psi = \frac{1}{r^{\frac{d}{2}}} \frac{1}{\mathcal{N}_m} \Bigg\{ & \left( \left[ \mathcal{F}_+ + \frac{\mathcal{F}_-}{\sqrt{f}} \Gamma(\zeta) \right] \left( 1 + \frac{\beta\omega}{2} \mathcal{W}_+ - \frac{\beta k_i}{2} \mathcal{K}_+ \Gamma^{(i)} \right) \right. \\ & - \left[ \mathcal{F}_- + \frac{\mathcal{F}_+}{\sqrt{f}} \Gamma(\zeta) \right] \left( \frac{\beta\omega}{2} \mathcal{W}_- - \frac{\beta k_i}{2} \mathcal{K}_- \Gamma^{(i)} \right) \Bigg) S_0 \psi_a \\ & - \frac{1}{2} \left( \left[ \mathcal{F}_- + \frac{\mathcal{F}_+}{\sqrt{f}} \Gamma(\zeta) \right] \left( 1 + \frac{\beta\omega}{2} (1 - 2\zeta - \mathcal{W}_+) + \frac{\beta k_i}{2} \mathcal{K}_+ \Gamma^{(i)} \right) \right. \\ & \left. \left. + \left[ \mathcal{F}_+ + \frac{\mathcal{F}_-}{\sqrt{f}} \Gamma(\zeta) \right] \left( \frac{\beta\omega}{2} (\mathcal{W}_- - 1) - \frac{\beta k_i}{2} \mathcal{K}_- \Gamma^{(i)} \right) \right) S_0 \psi_d \right\}\end{aligned}\tag{3.45}$$

In order to calculate the influence phase, we need to evaluate on-shell action with appropriate counterterms. In order to do this, we need the boundary limit of the above solution. We start by defining the following boundary limits of the functions,

$$\begin{aligned}\lim_{r \rightarrow r_c + i\epsilon} \mathcal{F} &= \mathcal{F}_c, & \lim_{r \rightarrow r_c + i\epsilon} \mathcal{W}_{\pm} &= \mathcal{W}_c, & \lim_{r \rightarrow r_c + i\epsilon} \mathcal{K}_{\pm} &= \mathcal{K}_c, \\ \lim_{r \rightarrow r_c - i\epsilon} \mathcal{F} &= \frac{1}{\mathcal{F}_c}, & \lim_{r \rightarrow r_c - i\epsilon} \mathcal{W}_{\pm} &= \pm \mathcal{W}_c, & \lim_{r \rightarrow r_c - i\epsilon} \mathcal{K}_{\pm} &= \pm \mathcal{K}_c,\end{aligned}\tag{3.46}$$

More explicitly, we have,

$$\mathcal{F}_c = \left( 4 \frac{r_c^d}{r_h^d} \right)^{|m|/d} \implies \mathcal{N}_m r_c^m = \mathcal{F}_c,\tag{3.47}$$

In terms of this data, the boundary influence phase can be computed by evaluating the

on-shell Dirac action using our solution. In the a-d basis, we obtain,

$$\begin{aligned} \left[ r^d \bar{\Psi} \mathcal{P}_-^\zeta \Psi \right]_{\zeta=0}^{\zeta=1} &= \left( \frac{r_h^{2m}}{16^{(m/d)}} \right) \left\{ -\bar{\psi}_a \left( (1 - \beta\omega \mathcal{W}_c) \gamma^0 + \beta k_i \mathcal{K}_c \gamma^i \right) \psi_d \right. \\ &\quad \left. + \bar{\psi}_d \left( (1 + \beta\omega \mathcal{W}_c) \gamma^0 - \beta k_i \mathcal{K}_c \gamma^i \right) \psi_a + \frac{\beta\omega}{2} \bar{\psi}_d \gamma^0 \psi_d \right\}. \end{aligned} \quad (3.48)$$

The relevant finite coefficients are given by the series expansions involving incomplete Beta functions (or digamma functions).

$$\begin{aligned} \left\{ \frac{4\pi i}{16^{\frac{m}{d}} d} \mathcal{W}_c \right\}_{r^0} &= -\frac{1}{16^{\frac{m}{d}}} \left[ \frac{4^{\frac{1}{d}+1} m}{4m^2 - 1} + \frac{2}{d} \left\{ \psi \left( \frac{d+1-2m}{d} \right) - \psi \left( \frac{d+1+2m}{d} \right) \right\} \right. \\ &\quad \left. - \sum_{j=1}^{\infty} \left[ B \left( j, \frac{2}{d}, -1 \right) - 2^{\frac{2}{d}-1} \frac{(-)^j}{j} \right] \frac{1}{B \left( j, \frac{2}{d}, 1 \right)} \frac{8m}{4m^2 - (1+dj)^2} \right] \\ \left\{ \frac{4\pi i}{16^{\frac{m}{d}} d} \mathcal{K}_c \right\}_{r^0} &= 4^{\frac{1-2m}{d}} \frac{\Gamma \left( \frac{1+2m}{d} \right) \Gamma \left( \frac{1-2m}{d} \right)}{d \Gamma \left( \frac{2}{d} \right)}. \end{aligned} \quad (3.49)$$

This gives an explicit expression for the influence phase for any value of the mass of the bulk Dirac fermion.

## Chapter 4

# An open EFT for fermions at finite density

### 4.1 Introduction

In the previous chapter, we showed that for probe Dirac fermions, the grSK geometry reproduces the correct Fermi–Dirac statistical factors in real time correlations. This setup can be used to compute the influence phase of an external fermion probing the dual CFT at finite temperature. A natural generalisation of this work is to ask what happens if we consider near–equilibrium states with finite density/chemical potential. Such a generalisation is necessary to make contact with more realistic models of holographic condensed matter [87, 74, 70, 121, 81, 44, 103, 127, 73, 166]. This is what we turn to now. As we will see, on the gravitational side, the relevant physics involves a generalization of the grSK geometry to the RN–AdS black brane, and to consider its quasi–normal modes and its Hawking radiation.

### 4.2 Hawking radiation in RN AdS backgrounds

In this section, we will describe how to construct Hawking modes in charged black branes when the ingoing quasi–normal modes are known. We will assume that the ingoing solution is naturally specified in ingoing Eddington–Finkelstein coordinates as an analytic function

(possibly in a boundary derivative expansion as is the case, for example, in the fluid–gravity correspondence [137, 84]). We will construct these Hawking modes using a bulk  $\mathbb{Z}_2$  action which is dual to a  $CPT$  transformation on the boundary CFT.

Let us begin with the Reissner–Nordström black brane solution in  $\text{AdS}_{d+1}$  written out in ingoing Eddington–Finkelstein time and a *mock tortoise coordinate*,  $\zeta$ :

$$\begin{aligned} ds^2 &= -r^2 f dv^2 + i \beta r^2 f dv d\zeta + r^2 d\mathbf{x}_{d-1}^2, \\ \mathcal{A}_M dx^M &= \mathcal{A}_v dv = -\mu \left( \frac{r_h}{r} \right)^{d-2} dv. \end{aligned} \quad (4.1)$$

which is the solution of the Einstein–Maxwell bulk action, given by,

$$\frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \left[ \mathcal{R} + d(d-1) - \frac{1}{g_F^2} \mathcal{F}_{MN} \mathcal{F}^{MN} \right], \quad (4.2)$$

where  $g_F$  is the gauge coupling constant and we have set the AdS radius to unity. Equivalently, we can replace the description in terms of the metric to one in terms of a set of orthonormal 1-forms [33, 110],

$$E^{(v)} = \frac{r}{2} dv - \frac{f}{2} r (i\beta d\zeta - dv), \quad E^{(\zeta)} = \frac{r}{2} dv + \frac{f}{2} r (i\beta d\zeta - dv), \quad E^{(i)} = r dx^i. \quad (4.3)$$

In the above equations,  $f$  is the emblackening factor of the RN–AdS black brane,  $\beta$  is its inverse Hawking temperature,  $r_h$  its outer horizon radius, and  $\mu$  its chemical potential. These quantities are, in turn, given in terms of the mass parameter  $M$  and the charge parameter  $Q$  of the black brane as,

$$f(r) \equiv 1 + \frac{Q^2}{r^{2d-2}} - \frac{M}{r^d}, \quad \frac{1}{\beta} \equiv \frac{d r_h}{4\pi} \left( 1 - \frac{(d-2)Q^2}{d r_h^{2d-2}} \right), \quad \mu \equiv \frac{g_F Q}{r_h^{d-2}} \left[ \frac{d-1}{2(d-2)} \right]^{\frac{1}{2}}. \quad (4.4)$$

The ADM mass density  $\mathcal{M}$  and the charge density  $\mathcal{Q}$  are given by,

$$\mathcal{M} = \frac{(d-1)M}{2\kappa^2}, \quad \mathcal{Q} = \frac{\sqrt{2(d-1)(d-2)}}{g_F \kappa^2} Q. \quad (4.5)$$

The mock tortoise coordinate  $\zeta$  is related to the standard radial coordinate  $r$  via the

following differential equation,

$$\frac{dr}{d\zeta} = \frac{i\beta}{2} r^2 f(r). \quad (4.6)$$

Since the RHS has simple zeroes at the inner and outer horizons, the coordinate  $\zeta$ , when thought of as a complex function on the complex  $r$  plane, has these points (along with infinity and other complex simple zeroes of  $f(r)$ ) as branch points. We will define  $\zeta$  by taking a branch cut to extend from the outer horizon to the asymptotic boundary along the real axis. We choose the inner horizon branch cut such that  $\zeta$  is analytic in the real interval between the two horizons (see Fig.2.3). The differential equation above then normalises  $\zeta$  such that it has a unit jump across the logarithmic branch cut. Without loss of generality, we will take  $\zeta(r = \infty + i\varepsilon) = 0$ . This condition along with the above differential equation then uniquely defines  $\zeta$  everywhere in the neighbourhood of  $r \in [r_h, \infty)$ .

We define the RN–SK geometry as one constructed by taking the RN–AdS exterior and replacing the radial interval extending from the outer horizon to infinity by a doubled contour, as indicated in Fig.4.1. We then obtain a geometry with two copies of RN–AdS exteriors smoothly stitched together by a ‘horizon cap’ region. This spacetime requires one further identification — each radially constant slice must meet at the future turning point,  $v \rightarrow \infty$ .

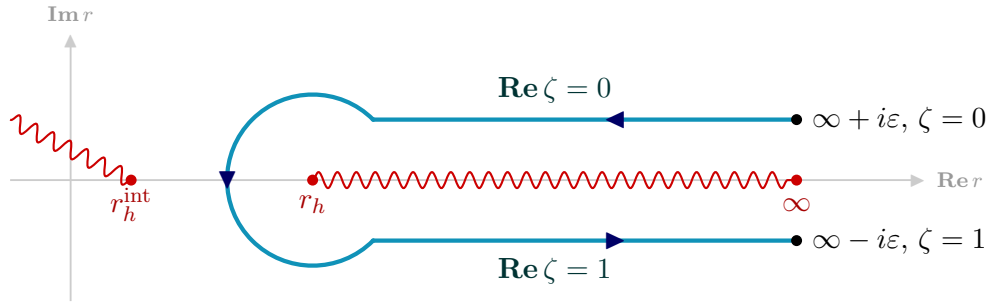


Figure 4.1: The radial contour drawn on the complex  $r$  plane, at fixed  $v$ . The locations of the two boundaries and the two horizons have been indicated.

Now that we have elaborated on the RN–SK geometry and its associated notation, let us return to the question posed at the beginning of this section: given an ingoing solution written in ingoing EF coordinates, how do we construct the outgoing Hawking modes? We

will do this by exploiting the *CPT* symmetry of the boundary CFT which acts as,

$$v \longmapsto -v, \quad \boldsymbol{x} \longmapsto -\boldsymbol{x}, \quad (4.7)$$

in addition to a charge conjugation. The reader can quickly verify that the transformations as written above do not preserve the RN–AdS solution given in (4.1) and (4.3). So how then do we implement this symmetry on the RN–AdS solution?

The answer to this conundrum lies in the fact that, by the standard rules of AdS/CFT, the boundary symmetries can always be composed with bulk gauge symmetries (since they act trivially on the Hilbert space of states of the CFT). In this case, we combine the boundary *CPT* transformation with bulk diffeomorphisms, local Lorentz transformations and gauge transformations. In fact, the reader can check that the following transformations preserve the RN–AdS solution: we combine a charge conjugation  $\mathcal{A}_M \mapsto -\mathcal{A}_M$ , a diffeomorphism,

$$v \longmapsto i\beta\zeta - v, \quad \boldsymbol{x} \longmapsto -\boldsymbol{x}, \quad (4.8)$$

a gauge transformation,

$$\mathbb{A} \equiv -i\beta \int_0^\zeta d\zeta' \mathcal{A}_v(\zeta'), \quad (4.9)$$

and a non-orthochronous local Lorentz transformation,

$$\mathcal{T}^a_b \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh \vartheta & \sinh \vartheta & 0 \\ \sinh \vartheta & \cosh \vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \vartheta \equiv \log f. \quad (4.10)$$

The linear transformation  $\mathcal{T}$  is an idempotent local Lorentz transformation (LLT) composed of a boost along  $\zeta$  with rapidity  $\vartheta$  followed by a reflection of the  $v$  and  $\boldsymbol{x}$  axes.

To check the invariance of the geometry, we will write the diffeomorphism in Eq.(4.8)

in the form  $X^A \mapsto X^B \mathcal{J}_B^A$  where,

$$\mathcal{J}_A^B \equiv \begin{pmatrix} -1 & 0 & 0 \\ i\beta & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (4.11)$$

The invariance is then the statement that

$$\mathcal{T}^a_b \mathcal{J}_A^B E_B^b = E_A^a, \quad -(\mathcal{J}_N^M \mathcal{A}_M + \partial_N \mathbb{A}) = \mathcal{A}_N, \quad (4.12)$$

as can be readily checked.

It is instructive to unify these transformations by imagining the gauge field as descending from a Kaluza–Klein reduction over a circle. Say there was an extra circle direction denoted by  $\varphi$ , along which we have a 1-form,  $d\varphi + \mathcal{A}_N dX^N$ , that is orthonormal to the set of 1-forms,  $E_M^a dX^M$ . The action of charge conjugation and gauge transformation in this picture can then be thought of as the KK diffeomorphism  $\varphi \mapsto \mathbb{A} - \varphi$  (which extends  $\mathcal{J}_A^B$  to the  $\varphi$  direction) along with a KK LLT that acts as a reflection along the  $\varphi$  axis (which extends  $\mathcal{T}^a_b$  to the  $\varphi$  direction). The mathematically inclined reader would recognise that we have essentially given a description of the boundary *CPT* transformation as an automorphism of the RN–AdS principle bundle.

Having implemented the boundary *CPT* in the bulk, we will now use this symmetry to generate new solutions to the bulk field equations linearised about the RN–AdS background. To see how this is done, let us begin with the simplest example of a complex scalar field with charge  $q$  probing the RN–AdS black brane. In the Fourier domain, we can expand a field configuration in terms of planar waves:

$$\int \frac{d\omega \, d^{d-1}\mathbf{k}}{(2\pi)^d} \Phi(q, \omega, \zeta, \mathbf{k}) e^{-i\omega v + i\mathbf{k} \cdot \mathbf{x}}. \quad (4.13)$$

Say this configuration solves the diffeomorphism/gauge covariant linear PDE in the bulk. How do we use the  $\mathbb{Z}_2$  we constructed above to get new solutions? The quickest way to answer this is to use the KK (or the principal bundle) picture described above. To this end, we think of the above field as drawn from a KK decomposition of a higher dimensional

scalar field of the form,

$$\sum_{q \in \mathbb{Z}} e^{iq\varphi} \int \frac{d\omega \, d^{d-1}\mathbf{k}}{(2\pi)^d} \Phi(q, \omega, \zeta, \mathbf{k}) e^{-i\omega v + i\mathbf{k} \cdot \mathbf{x}}. \quad (4.14)$$

The diffeomorphism/gauge covariant linear PDE can then be lifted to a PDE covariant under higher dimensional diffeomorphisms. We perform a higher dimensional diffeomorphism  $\{v \mapsto i\beta\zeta - v, \mathbf{x} \mapsto -\mathbf{x}, \varphi \mapsto \mathbb{A} - \varphi\}$  on the Fourier domain expression given above. After relabelling the sum/integrals, we obtain,

$$\sum_{q \in \mathbb{Z}} e^{iq\varphi} \int \frac{d\omega \, d^{d-1}\mathbf{k}}{(2\pi)^d} e^{-\beta\omega\zeta - iq\mathbb{A}} \Phi(-q, -\omega, \zeta, -\mathbf{k}) e^{-i\omega v + i\mathbf{k} \cdot \mathbf{x}}. \quad (4.15)$$

The higher dimensional diffeomorphism covariance of the linear PDE then guarantees that the map

$$\Phi(q, \omega, \zeta, \mathbf{k}) \mapsto e^{-\beta\omega\zeta - iq\mathbb{A}} \Phi(-q, -\omega, \zeta, -\mathbf{k}) \quad (4.16)$$

generates new solutions. In fact, the pre-factor in the exponent shows that this map takes ingoing solutions that are analytic in  $r$  to outgoing solutions which exhibit branch cuts. We can thus generate the Hawking modes from quasi-normal modes via the map,

$$\Phi^{\text{in}}(q, \omega, \zeta, \mathbf{k}) \mapsto e^{-\beta\omega\zeta - iq\mathbb{A}} \Phi^{\text{in}}(-q, -\omega, \zeta, -\mathbf{k}) \equiv \Phi^{\text{out}}(q, \omega, \zeta, \mathbf{k}). \quad (4.17)$$

The extension to other fields is straightforward. As a pertinent example, let us consider how this works for charged Dirac spinors. The main novelty in this case is the additional action of LLTs on spinor indices. The complex boost given in (4.10) acts on spinor indices via

$$\mathfrak{T} \equiv \Gamma^{(\zeta)} \cdot \exp\left(\frac{\vartheta}{2} \Gamma^{(\zeta)} \Gamma^{(v)}\right) = \sqrt{f} \, \mathbb{F} + \frac{1}{\sqrt{f}} \, \mathbb{F}^\dagger, \quad (4.18)$$

where we have defined,

$$\mathbb{F} \equiv \frac{1}{2} \left( \Gamma^{(v)} + \Gamma^{(\zeta)} \right). \quad (4.19)$$

The Gamma matrix notation here is standard and our convention for the Clifford algebra is  $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\mathbb{1}$ , with a mostly positive signature  $\eta^{ab}$ . The map from ingoing to outgoing solutions, in this case, is given by,

$$\Psi^{\text{in}}(q, \omega, \zeta, \mathbf{k}) \mapsto e^{-\beta\omega\zeta - iq\Lambda} \mathfrak{T} \cdot \Psi^{\text{in}}(-q, -\omega, \zeta, -\mathbf{k}) \equiv \Psi^{\text{out}}(q, \omega, \zeta, \mathbf{k}). \quad (4.20)$$

In order to streamline the analysis that is to follow, we end this section with a discussion of how the quantities that we have introduced behave as they traverse the horizon cap. As noted before, the mock tortoise coordinate has a unit jump when it encircles  $r = r_h$ . Noting that the gauge field  $\mathcal{A}_v$  has the value  $-\mu$  at the horizon, we see that the jump in  $e^{-iq\Lambda}$  is the fugacity of the boundary theory,

$$\lim_{\zeta \rightarrow 0} e^{-iq\Lambda} = 1, \quad \lim_{\zeta \rightarrow 1} e^{-iq\Lambda} = e^{q\beta\mu} \equiv z. \quad (4.21)$$

The rapidity parameter  $\vartheta$ , on the other hand, has a jump of  $2\pi i$  when it encircles the outer horizon — this can be traced to the phase  $e^{2\pi i}$  that the function  $f$  picks up around the same path. As a result, every spinor index receives an additional ‘fermionic’ minus sign as it crosses the horizon cap. For a field of spin  $s$ , one gains the statistical factor  $(-1)^{2s} e^{-\beta\omega + q\beta\mu}$  which is identical to the factor gained in Euclidean path integrals as we traverse the thermal circle. As we will argue below, this factor is crucial to how the RN–SK geometry gets the physics of Hawking radiation right.

### 4.3 The Dirac equation

We will now apply the preceding slightly abstract discussion to the explicit example of a Dirac equation in the RN–SK background. The action for a bulk Dirac spinor  $\Psi$  of charge  $q$  and mass  $m$  living on the RN–SK geometry is given by,

$$\mathbb{S}_\Psi = \oint d\zeta \int d^d x \sqrt{-g} \, i \bar{\Psi} (\Gamma^M \mathbb{D}_M - m) \Psi + \int d^d x \, i \bar{\Psi} \mathcal{P}_-^\zeta \Psi \Big|_{\zeta=0}^{\zeta=1}. \quad (4.22)$$

This is an immediate generalization of the discussion in the previous chapter. Nonetheless, let us take a moment to explain each of the above terms. The first is the standard bulk Dirac action, that fixes the equations of motion to be the Dirac equation in this curved

spacetime. We have introduced above the covariant Dirac derivative,

$$\mathbb{D}_M \equiv \partial_M - iq\mathcal{A}_M + \frac{1}{4}\omega_{abM}\Gamma^a\Gamma^b, \quad (4.23)$$

where  $\mathcal{A}_M$  is the background gauge potential and  $\omega_{abM}$  are the spin-connection 1-forms derived from Eq.(4.3).

The second term is a variational boundary term, added in to ensure that this action admits a well-posed variational principle when the spinors satisfy semi-Dirichlet boundary conditions [78, 129, 77, 86] (more will be said about this below). When the above action is evaluated on-shell, the bulk term vanishes and only the boundary term contributes to the final answer. Notice that the boundary term receives contributions from both the left and right boundaries of the RN-SK saddle. As before, the operator  $\mathcal{P}_-^\zeta$  involved in the definition of this term is one of two projection operators, defined as,

$$\mathcal{P}_\pm^\zeta \equiv \frac{1}{2} \left( \mathbb{1} \pm \Gamma^{(\zeta)} \right). \quad (4.24)$$

The Dirac equation  $(\Gamma^M \mathbb{D}_M - m) \Psi = 0$ , written out in the Fourier domain, takes the following form:

$$\left\{ \mathbb{F} \left( \partial_\zeta + \beta\omega + \beta q\mathcal{A}_v + \frac{1}{2}\partial_\zeta \ln f \right) + \mathbb{F}^\dagger f^{-1} \partial_\zeta - \frac{\beta}{2} \left( \Gamma^{(i)} k_i + imr \right) \right\} \left( r^{\frac{d}{2}} \Psi \right) = 0. \quad (4.25)$$

It can be explicitly checked that if  $\Psi(q, \omega, \zeta, \mathbf{k})$  solves this equation, applying the transformation described in (4.20) to it generates another solution of the same equation.

The relevant AdS/CFT boundary conditions are,

$$\lim_{\zeta \rightarrow 0} r^{\frac{d}{2}-m} \mathcal{P}_+^\zeta \Psi = \mathcal{P}_+^\zeta S_0 \psi_L, \quad \lim_{\zeta \rightarrow 1} r^{\frac{d}{2}-m} \mathcal{P}_+^\zeta \Psi = \mathcal{P}_+^\zeta S_0 \psi_R. \quad (4.26)$$

Here  $S_0$  is a constant boundary-to-bulk matrix [110]. It is defined in terms of the ingoing boundary-to-bulk propagator  $S^{\text{in}}(\omega, \zeta, \mathbf{k})$  as,

$$S_0 \equiv \lim_{\zeta \rightarrow 0} r^{\frac{d}{2}-m} S^{\text{in}}(\omega, \zeta, \mathbf{k}) = \lim_{\zeta \rightarrow 1} r^{\frac{d}{2}-m} S^{\text{in}}(\omega, \zeta, \mathbf{k}). \quad (4.27)$$

The above boundary conditions are a ‘doubled’ version of the standard semi-Dirichlet conditions in AdS/CFT. These boundary conditions uniquely determine the solution on the

RN–SK geometry.

To see this, we begin with the most general combination of ingoing and outgoing solutions:

$$\begin{aligned}
\Psi(q, \omega, \zeta, \mathbf{k}) &= -S^{\text{in}}(q, \omega, \zeta, \mathbf{k}) \psi_{\bar{F}}(q, \omega, \mathbf{k}) - S^{\text{out}}(q, \omega, \zeta, \mathbf{k}) \psi_{\bar{P}}(q, \omega, \mathbf{k}) e^{\beta(\omega - q\mu)} , \\
&= -S^{\text{in}}(q, \omega, \zeta, \mathbf{k}) \psi_{\bar{F}}(q, \omega, \mathbf{k}) \\
&\quad - S^{\text{rev}}(q, \omega, \zeta, \mathbf{k}) \psi_{\bar{P}}(q, \omega, \mathbf{k}) e^{\beta\omega(1-\zeta) + iq(i\beta\mu - \Lambda)} .
\end{aligned} \tag{4.28}$$

Here, we have defined,

$$S^{\text{rev}}(q, \omega, \zeta, \mathbf{k}) \equiv S^{\text{in}}(-q, -\omega, \zeta, -\mathbf{k}) , \tag{4.29}$$

and we have used the  $\mathbb{Z}_2$  action detailed in the last section to get the outgoing Hawking solution.

The boundary conditions described above fix

$$\psi_{\bar{F}} \equiv n_{\omega}^{\text{FD}} (\psi_R - \psi_L) - \psi_R , \quad \psi_{\bar{P}} \equiv n_{\omega}^{\text{FD}} (\psi_R - \psi_L) \tag{4.30}$$

with

$$n_{\omega}^{\text{FD}} \equiv \frac{1}{e^{\beta(\omega - q\mu)} + 1} \tag{4.31}$$

being the familiar Fermi–Dirac factor at finite chemical potential. We recognise here the *retarded–advanced basis* (RA) for the boundary spinor sources [40, 35]. Thus, when written in the RA basis, the two combinations of sources precisely multiply the ingoing/quasi-normal bulk-to-boundary propagator and the outgoing bulk-to-boundary propagator respectively. This is analogous to the corresponding statements at zero chemical potential [149, 34, 89, 110].

As pointed out in the previous chapter, the above fact can be used to argue why the generating function of correlations computed using our holographic prescription satisfy both the Schwinger–Keldysh *collapse rules* as well as the KMS conditions, i.e., the generating function cannot contain terms with only  $\psi_{\bar{P}}$  or terms with only  $\psi_{\bar{F}}$ . Exactly the same argument is valid for the case of finite chemical potential presented here.

Finally, in the *Keldysh-rotated* or *average-difference* basis, one gets,

$$\Psi(v, \zeta, \mathbf{k}) = S^{\text{in}} \psi_a - \left\{ \left( n_\omega^{\text{FD}} - \frac{1}{2} \right) S^{\text{in}} + n_\omega^{\text{FD}} e^{\beta\omega(1-\zeta) + iq(i\beta\mu - \Lambda)} \mathfrak{T} \cdot S^{\text{rev}} \right\} \psi_d. \quad (4.32)$$

This basis makes the Schwinger–Keldysh collapse rules manifest, i.e., there are no terms in the generating function for boundary correlators that have only  $\psi_a$  terms. Equivalently, correlation functions composed of only the *difference* operator  $\mathcal{O}_d$  vanish. As we will see in §4.4, the influence phase that we derive for the open EFT precisely has this property.

## 4.4 Gradient expansion

In this section, we will find the solution to the Dirac equation in gradient expansion, generalizing the results §3 to *charged* black branes. We only consider the case of a massless spinor field for simplicity.

We begin by writing the most general solution  $\Psi \equiv \Psi(v, \zeta, \mathbf{x})$  in a gradient expansion, compatible with rotational invariance as,

$$\Psi = \frac{1}{r^{d/2}} \left\{ \mathbb{1} + C_a^{(1)} M^a \partial_v + D_a^{(1)} M^a \Gamma^{(i)} \partial_i + C_a^{(2)} M^a \partial_v^2 + D_a^{(2)} M^a \partial_i^2 + \dots \right\} S_0 \psi \quad (4.33)$$

where  $\psi \equiv \psi(v, \mathbf{x})$  is the boundary spinor acting as a source at the boundary. The functions  $\{C^{(i)}, D^{(j)}, \dots\}$  are some unknown functions of the radial direction  $\zeta$ . The matrices  $M^a$  that appear in the above equation belong to the set  $\{\mathbb{1}, \Gamma^{(\zeta)}\}$  and in order to satisfy the Dirac equation (4.25) at zero derivative order, we will fix  $S_0$  to be a constant matrix annihilated by  $\mathbb{T}$ , i.e.,  $\mathbb{T} S_0 = 0$ .

The bulk-to-boundary Green’s function satisfying ingoing boundary conditions can be computed by substituting the above ansatz into the Dirac equation and solving it order by order in derivative expansion. In the Fourier domain we get an expression of the form,

$$S^{\text{in}} = \frac{1}{r^{d/2}} \left\{ 1 + \frac{\beta}{2} \left( H \Gamma^{(\zeta)} - H_{(0)} \right) \Gamma^{(i)} k_i - \frac{\beta^2 \omega}{2} \left( \tilde{H} \Gamma^{(\zeta)} - \tilde{H}_{(0)} \right) \Gamma^{(i)} k_i - \frac{\beta^2 \mathbf{k}^2}{8} \left( f G^2 - f_{(0)} G_{(0)}^2 \right) - \frac{\beta^2 \mathbf{k}^2 H_c}{4} \left( H \Gamma^{(\zeta)} - H_{(0)} \right) + \dots \right\} S_0. \quad (4.34)$$

where  $\mathbf{k}^2 = k_i k^i$  and the subscript ‘(0)’ denotes the value of the functions at the conformal boundary,  $\zeta = 0$ . The unknown functions  $H$ ,  $\tilde{H}$ , and  $G$  must satisfy the following radial differential equations:

$$\frac{d}{d\zeta} \left( e^{iq\Lambda} \sqrt{f} H \right) = e^{iq\Lambda} \sqrt{f} , \quad \frac{d}{d\zeta} \left( e^{iq\Lambda} \sqrt{f} \tilde{H} \right) = e^{iq\Lambda} \sqrt{f} H , \quad \frac{d}{d\zeta} \left( \sqrt{f} G \right) = \sqrt{f} . \quad (4.35)$$

Since we want an ingoing solution, we take  $H$ ,  $\tilde{H}$ , and  $G$  to be analytic at the outer horizon which then implies that the functions  $\sqrt{f}H$ ,  $\sqrt{f}\tilde{H}$  and  $\sqrt{f}G$  should vanish at the outer horizon. With this boundary condition specified, the above ODEs have a unique solution.

Given the above ingoing solution, the corresponding solution on the RN–SK geometry follows by using the formulae derived in §4.3. We will end this section by quoting the result for the influence phase in *average–difference* basis:

$$\mathbb{S}_{\text{IF}}[\psi_a, \psi_d] = \int d^d x r^d i \bar{\Psi} \mathcal{P}^\zeta \Psi \Big|_{\zeta=0}^{\zeta=1} = \int d^d x i \left\{ \bar{\psi}_a \mathcal{S}^{da} \psi_d + \bar{\psi}_d \mathcal{S}^{ad} \psi_a + \bar{\psi}_d \mathcal{S}^{aa} \psi_d \right\} \quad (4.36)$$

where  $\mathcal{S}^{ad}$ ,  $\mathcal{S}^{da}$ , and  $\mathcal{S}^{aa}$  are the retarded, advanced and Keldysh Green’s functions respectively. When the dimension of the boundary theory,  $d$ , is odd, their explicit forms are given by,

$$\begin{aligned} \mathcal{S}^{ad} &= \gamma^{(v)} - \beta H_{(0)} \gamma^{(i)} k_i + \beta^2 \omega \gamma^{(i)} k_i \tilde{H}_{(0)} + \frac{\beta^2 \mathbf{k}^2 (H_{(0)})^2}{2} \gamma^{(v)} + \dots \\ \mathcal{S}^{da} &= - \left( \gamma^{(v)} + \beta H_{(0)}^* \gamma^{(i)} k_i + \beta^2 \omega \gamma^{(i)} k_i \tilde{H}_{(0)}^* + \frac{\beta^2 \mathbf{k}^2 (H_{(0)}^*)^2}{2} \gamma^{(v)} + \dots \right) \\ \mathcal{S}^{aa} &= \mathfrak{e} \gamma^{(v)} + \frac{\beta \omega (1 - \mathfrak{e}^2)}{2} \gamma^{(v)} - \frac{\beta^2 \omega^2 [\mathfrak{e} (1 - \mathfrak{e}^2)]}{4} \gamma^{(v)} + \frac{\mathfrak{e} \beta (H_{(0)}^* - H_{(0)})}{2} \gamma^{(i)} k_i \\ &\quad + \frac{\beta^2 \omega}{4} \gamma^{(i)} k_i \left( (1 - \mathfrak{e}^2) (H_{(0)}^* - H_{(0)}) + 2 \mathfrak{e} (\tilde{H}_{(0)}^* + \tilde{H}_{(0)}) \right) \\ &\quad + \frac{\mathfrak{e} \beta^2 \mathbf{k}^2 [(H_{(0)}^*)^2 + H_{(0)}^2]}{4} \gamma^{(v)} + \dots \end{aligned} \quad (4.37)$$

where we have expanded the Fermi–Dirac factors in powers of  $\omega$  and the superscript ‘ $\star$ ’ denotes the replacement  $q \rightarrow -q$  in the arguments of the functions. We use the letter  $\mathfrak{e}$  to

denote the ground state charge

$$\mathfrak{e} \equiv \frac{1}{1 + e^{-q\beta\mu}} - \frac{1}{1 + e^{q\beta\mu}} , \quad (4.38)$$

which is the difference between the ground-state occupation of fermionic particles and anti-particles. A similar set of expressions can be obtained  $d$  is even, as described in [110]. This influence phase obeys the 2-point KMS relation which takes the following form in the a-d basis,

$$\mathcal{S}^{aa} = \frac{1}{2} \frac{e^{\beta\omega} - z}{e^{\beta\omega} + z} \left( \mathcal{S}^{ad} - \mathcal{S}^{da} \right). \quad (4.39)$$

In the limit  $\omega, \mathbf{k} \rightarrow 0$ , the influence phase has a very simple form,

$$\mathbb{S}_{\text{IF}} = \int d^d x \, i \left( \psi_a^\dagger \psi_d - \psi_d^\dagger \psi_a - \mathfrak{e} \psi_d^\dagger \psi_d \right). \quad (4.40)$$

Finally, in the limit of uncharged black branes or vanishing chemical potential ( $q, \mu \rightarrow 0$ ), all these Green's functions and the influence phase match with the results derived in §3.2.

## Conclusion

Holography ranks among physics’ most captivating ideas, offering us a radical new lens through which to view reality. What began as an elegant bridge between quantum gravity and strongly coupled field theories has blossomed into a framework whose insights illuminate corners of physics we never imagined it could reach. This thesis explores two distinct applications that showcase not just holography’s power, but its remarkable ability to unveil unexpected connections.

Our journey begins with one of quantum gravity’s most enduring mysteries: the nature of black hole microstates. In §1, we examine the non-supersymmetric, asymptotically flat  $2d$  black hole through its dual description in  $c = 1$  matrix quantum mechanics. Our central discovery is a novel bound state that serves as the explicit dual of black hole microstates, dominating the partition function in the double scaling limit. Indeed, this state exhibits precisely the behaviour that we would expect of a black hole in asymptotically flat space — it is a metastable entity and undergoes a gradual, steady evaporation.

The second part of our thesis explores holography’s remarkable utility in condensed matter physics. Here, we extend the AdS/CFT correspondence into new territory, developing tools to study (fermionic) open effective field theories interacting with holographic baths.

In §2, we lay the groundwork for understanding open effective field theories through holography. Central to this framework is the introduction of the *gravitational Schwinger–Keldysh saddle* (grSK saddle), a bulk spacetime that proves remarkably well-suited for probing open effective theories. Through careful development of the holographic dictionary for these geometries, we build a bridge between bulk and boundary physics, with special attention to fermionic systems.

In §3, we study Dirac fermions as propagating through these backgrounds and carefully demonstrate how they lead to Fermi–Dirac statistics on the boundary theory. Indeed, the holographic Schwinger–Keldysh boundary conditions naturally select exactly those mode combinations that satisfy fermionic KMS relations. Finally, we derive an explicit expression for the probe fermion’s influence phase through the evaluation of the bulk on-shell action.

Our investigation culminates in §4, where we extend all of our previous results to charged black branes — a crucial step toward modeling realistic condensed matter systems at finite temperature and chemical potential.

These explorations, rich as they are, represent just two threads in holography’s vast tapestry. The principle continues to reveal new connections and insights, exemplifying theoretical physics at its most profound — where mathematical elegance leads to deep physical truth. As we look to the future, holography stands poised to play an ever more

crucial role in our quest to understand quantum gravity and the fundamental nature of reality itself. The insights it continues to yield suggest that the universe's deepest structures may be even more beautiful and surprising than we dare to imagine.

## Appendix A

# Spinors and Clifford algebras: conventions and representations

In this appendix, we delineate our conventions for the spinors and Clifford algebras used in the main text. We organize our discussion from the perspective of the boundary conformal field theory, first describing the conventions for the boundary spinors and boundary Gamma / Sigma matrices, and then explaining how the corresponding objects in the bulk are defined. The exposition presented here and the choice of basis that we use differ slightly from the standard references on the subject [86, 134, 161, 54, 131], but it is one that we find useful for our purposes. Of course, none of the results quoted in the text depend on a particular representation of these matrices.

### A.1 Spinor and Clifford algebra conventions

The Gamma matrices that we use satisfy the Clifford algebra,

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \mathbb{1}, \tag{A.1}$$

where  $\eta_{ab} = \text{diag}(-1, +1, \dots, +1)$ . In particular, the lone time-like Gamma matrix  $\Gamma^{(v)}$  squares to  $-\mathbb{1}$ , while the space-like Gamma matrices are idempotent. We will always choose  $\Gamma^{(v)}$  to be *anti*-Hermitian and all the rest to be Hermitian. Recall that in a  $(d+1)$ -

dimensional spacetime, these are  $2^k \times 2^k$  dimensional matrices, with  $k$  given by,

$$k = \left\lfloor \frac{d+1}{2} \right\rfloor. \quad (\text{A.2})$$

Thus, the Dirac spinors in this spacetime also have  $2^k$  components. As we had noted before, the bulk and boundary spinors can have a different number of components, depending on the dimension of the spacetime we are in. In this regard, we adopt the following conventions: the spinors in the bulk will always be represented as Dirac spinors. When the dimension of the boundary is odd, i.e.,  $d = 2n + 1$ , the spinors of the CFT are Dirac spinors of dimension  $2^n$ . When the dimension of the boundary spacetime is even, i.e.,  $d = 2n$ , we will use Weyl spinors for the CFT, viz., chiral or anti-chiral spinors of dimension  $2^{n-1}$ . With these choices, in every dimension the bulk spinors always have twice as many components as the boundary spinors do.

We choose a basis for our Gamma matrices that plays well with the above conventions. When  $d = 2n$ , the boundary spinors are Weyl spinors that transform under the covariant Pauli (or Sigma) matrices  $(\Sigma, \bar{\Sigma})$  given by

$$\Sigma^{(\mu)} = (\mathbb{1}, \sigma^{(i)}) ; \quad \bar{\Sigma}^{(\mu)} = (\mathbb{1}, -\sigma^{(i)}) . \quad (\text{A.3})$$

In turn,  $\sigma^i$  are the generalized Pauli matrices that satisfy,

$$\{\sigma^{(i)}, \sigma^{(j)}\} = 2 \delta^{(i)(j)} \mathbb{1}. \quad (\text{A.4})$$

Correspondingly, we choose the the Gamma matrices in the bulk to be,

$$\Gamma^{(\zeta)} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \Gamma^{(\mu)} = \begin{pmatrix} 0 & \Sigma^{(\mu)} \\ -\bar{\Sigma}^{(\mu)} & 0 \end{pmatrix}. \quad (\text{A.5})$$

It is easily checked that these Gamma matrices satisfy the Clifford algebra. Furthermore, because the Sigma matrices are all Hermitian,  $\Gamma^{(v)}$  is an anti-Hermitian matrix, while all the rest are Hermitian. Finally, the ‘lightcone’ combinations of Gamma matrices  $\mathbb{F}$  and  $\mathbb{F}^\dagger$  (3.4) and the constant bulk to boundary matrix  $S_0$  (3.17) that we introduced in the main

text, read,

$$\begin{aligned}
\mathbb{F} &= \frac{1}{2} \left( \Gamma^{(v)} + \Gamma^{(\zeta)} \right) = \frac{1}{2} \begin{pmatrix} \mathbb{1} & \Sigma^{(v)} \\ -\bar{\Sigma}^{(v)} & -\mathbb{1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & -\mathbb{1} \end{pmatrix}, \\
\mathbb{F}^\dagger &= \frac{1}{2} \left( \Gamma^{(v)} - \Gamma^{(\zeta)} \right) = \frac{1}{2} \begin{pmatrix} \mathbb{1} & -\Sigma^{(v)} \\ \bar{\Sigma}^{(v)} & -\mathbb{1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix}, \\
S_0 &= \begin{pmatrix} \mathbb{1} \\ -\Sigma^{(v)} \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ -\mathbb{1} \end{pmatrix}.
\end{aligned} \tag{A.6}$$

When  $d = 2n - 1$ , the boundary spinors are Dirac spinors that transform under the boundary Gamma matrices, which we will denote by  $\gamma^{(\mu)}$ . In this case, the Gamma matrices for the bulk spinor can be chosen to be,

$$\Gamma^{(\zeta)} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \Gamma^{(\mu)} = \begin{pmatrix} 0 & \gamma^{(\mu)} \\ \gamma^{(\mu)} & 0 \end{pmatrix}. \tag{A.7}$$

Once again, it is easy to see that the above Gamma matrices satisfy the Clifford algebra, and that they inherit the correct Hermiticity properties from the boundary  $\gamma$ 's, i.e.,  $\Gamma^{(v)}$  is an anti-Hermitian matrix, and the rest of the  $\Gamma^{(a)}$  are Hermitian. In these dimensions, the matrices  $\mathbb{F}$ ,  $\mathbb{F}^\dagger$  and  $S_0$  then take the form,

$$\begin{aligned}
\mathbb{F} &= \frac{1}{2} \left( \Gamma^{(v)} + \Gamma^{(\zeta)} \right) = \frac{1}{2} \begin{pmatrix} \mathbb{1} & \gamma^{(v)} \\ \gamma^{(v)} & -\mathbb{1} \end{pmatrix}, \\
\mathbb{F}^\dagger &= \frac{1}{2} \left( \Gamma^{(v)} - \Gamma^{(\zeta)} \right) = \frac{1}{2} \begin{pmatrix} \mathbb{1} & -\gamma^{(v)} \\ -\gamma^{(v)} & -\mathbb{1} \end{pmatrix}, \\
S_0 &= \begin{pmatrix} \mathbb{1} \\ \gamma^{(v)} \end{pmatrix}.
\end{aligned} \tag{A.8}$$

## A.2 Pauli matrices and explicit representations

Having described how to construct the bulk Gamma matrices from the  $(\Sigma, \bar{\Sigma})$  and  $\gamma$  matrices, we now turn to providing explicit representations for these boundary matrices. We will do this using the generalized Pauli matrices.

As the name suggest, the generalized Pauli matrices are set of Hermitian matrices that generalize the standard  $2 \times 2$  Pauli matrices  $(\tau_1, \tau_2, \tau_3)$  to all odd dimensions. As we will soon see, we can always choose them to be real and symmetric, or imaginary and anti-symmetric. In  $p = 2n - 1$  dimensions, they are  $2^{n-1} \times 2^{n-1}$  matrices that satisfy the Euclidean Clifford algebra,

$$\{\sigma_p^{(i)}, \sigma_p^{(j)}\} = 2 \delta^{(i)(j)} \mathbb{1}. \quad (\text{A.9})$$

We organize these matrices into four different classes depending on their dimension:  $p = 8k + 1$ ,  $p = 8k + 3$ ,  $p = 8k + 5$ ,  $p = 8k + 7$ , and describe each of their explicit constructions separately. In each case, we will build them out of the standard Pauli matrices  $(\tau_1, \tau_2, \tau_3)$ .

$p = 8k + 1$

The very simplest case is of course  $p = 1$ , where the Pauli ‘matrix’  $\sigma_1^{(1)}$  is just the number 1. The next member of this family is  $p = 9$ , where *all* the Pauli matrices can be chosen to be real, symmetric matrices as follows,

$$\begin{aligned} \sigma_9^{(1)} &= \tau_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\ \sigma_9^{(2)} &= \varepsilon \otimes \mathbb{1} \otimes \tau_3 \otimes \varepsilon, \quad \sigma_9^{(3)} = \varepsilon \otimes \mathbb{1} \otimes \tau_1 \otimes \varepsilon, \\ \sigma_9^{(4)} &= \varepsilon \otimes \tau_3 \otimes \varepsilon \otimes \mathbb{1}, \quad \sigma_9^{(5)} = \varepsilon \otimes \tau_1 \otimes \varepsilon \otimes \mathbb{1}, \\ \sigma_9^{(6)} &= \varepsilon \otimes \varepsilon \otimes \mathbb{1} \otimes \tau_3, \quad \sigma_9^{(7)} = \varepsilon \otimes \varepsilon \otimes \mathbb{1} \otimes \tau_1, \\ \sigma_9^{(8)} &= \epsilon \otimes \epsilon \otimes \epsilon \otimes \epsilon, \quad \sigma_9^{(9)} = \tau_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}. \end{aligned} \quad (\text{A.10})$$

We have adopted the shorthand  $\varepsilon = i\tau_2$  to make the reality of these matrices manifest. With the above two cases defined, we can now recursively define the rest of the members of

this family.

$$\begin{aligned}
\sigma_{8k+9}^{(i)} &= \sigma_9^{(i)} \otimes^{4k} \mathbb{1}, \quad i \in \{1, \dots, 8\}; \\
\sigma_{8k+9}^{(8+j)} &= \sigma_9^{(9)} \otimes \sigma_{8k+1}^{(j)}, \quad j \in \{1, \dots, 8k+1\}.
\end{aligned} \tag{A.11}$$

$p = 8k + 3$

The matrices in these dimensions are defined as follows.

$$\begin{aligned}
\sigma_{8k+3}^{(1)} &= \tau_1 \otimes^{4k} \mathbb{1}, \\
\sigma_{8k+3}^{(2)} &= \tau_2 \otimes^{4k} \mathbb{1}, \\
\sigma_{8k+3}^{(2+j)} &= \tau_3 \otimes \sigma_{8k+1}^{(j)}, \quad j \in \{1, \dots, 8k+1\}.
\end{aligned} \tag{A.12}$$

Unsurprisingly, when  $p = 3$ , these are just the standard Pauli matrices.

$p = 8k + 5$

These matrices can be written down as follows.

$$\begin{aligned}
\sigma_{8k+5}^{(1)} &= i \tau_1 \otimes \varepsilon \otimes^{4k} \mathbb{1}, \\
\sigma_{8k+5}^{(2)} &= i \tau_3 \otimes \varepsilon \otimes^{4k} \mathbb{1}, \\
\sigma_{8k+5}^{(3)} &= \varepsilon \otimes \varepsilon \otimes^{4k} \mathbb{1}, \\
\sigma_{8k+5}^{(4)} &= \mathbb{1} \otimes \tau_3 \otimes^{4k} \mathbb{1}, \\
\sigma_{8k+5}^{(4+j)} &= \mathbb{1} \otimes \tau_1 \otimes \sigma_{8k+1}^{(j)}, \quad j \in \{1, \dots, 8k+1\}.
\end{aligned} \tag{A.13}$$

$p = 8k + 7$

All the matrices in these dimensions can be chosen to be purely imaginary and anti-

symmetric.

$$\begin{aligned}
\sigma_{8k+7}^{(1)} &= i \mathbb{1} \otimes \tau_3 \otimes \varepsilon \otimes^{4k} \mathbb{1} , \\
\sigma_{8k+7}^{(2)} &= i \mathbb{1} \otimes \tau_1 \otimes \varepsilon \otimes^{4k} \mathbb{1} , \\
\sigma_{8k+7}^{(3)} &= i \tau_3 \otimes \varepsilon \otimes \mathbb{1} \otimes^{4k} \mathbb{1} , \\
\sigma_{8k+7}^{(4)} &= i \tau_1 \otimes \varepsilon \otimes \mathbb{1} \otimes^{4k} \mathbb{1} , \\
\sigma_{8k+7}^{(5)} &= i \varepsilon \otimes \mathbb{1} \otimes \tau_3 \otimes^{4k} \mathbb{1} , \\
\sigma_{8k+7}^{(6)} &= i \varepsilon \otimes \mathbb{1} \otimes \tau_1 \otimes^{4k} \mathbb{1} , \\
\sigma_{8k+7}^{(6+j)} &= i \varepsilon \otimes \varepsilon \otimes \varepsilon \otimes \sigma_{8k+1}^{(j)} , \quad j \in \{1, \dots, 8k+1\}.
\end{aligned} \tag{A.14}$$

With these matrices explicitly laid out, we can now construct the boundary matrices in the AdS/CFT correspondence. For an even-dimensional boundary,  $d = 2n = p + 1$ , the Sigma matrices are constructed as,

$$\begin{aligned}
\Sigma_d^{(v)} &= \overline{\Sigma}_d^{(v)} = \mathbb{1} \\
\Sigma_d^{(i)} &= -\overline{\Sigma}_d^{(i)} = \sigma_{d-1}^{(i)} , \quad i \in \{1, \dots, d-1\}
\end{aligned} \tag{A.15}$$

For an odd-dimensional boundary,  $d = 2n + 1 = p + 2$ , the boundary Gamma matrices can be chosen as follows,

$$\begin{aligned}
\gamma_d^{(v)} &= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \\
\gamma_d^{(\mu)} &= \begin{pmatrix} 0 & \sigma_{d-2}^{(\mu)} \\ \sigma_{d-2}^{(\mu)} & 0 \end{pmatrix} , \quad \mu \in \{v, 1, \dots, d-2\} \\
\gamma_d^{(d-1)} &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}
\end{aligned} \tag{A.16}$$

The corresponding matrices in the bulk can now be immediately constructed from our previous discussion.

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