Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 56-2 (2023) 8451–8456

Adaptive robust predictive control with sample-based persistent excitation

Xiaonan Lu * and **Mark Cannon** **

* *Bosch China Investment Ltd, Shanghai, China*

** *Department of Engineering Science, University of Oxford, UK*

Email: `xiaonan.lu@cn.bosch.com`, `mark.cannon@eng.ox.ac.uk`

Abstract: We propose a robust adaptive Model Predictive Control strategy with online set-based estimation for constrained linear systems with unknown parameters and bounded disturbances. A sample-based test applied to predicted trajectories is used to ensure convergence of parameter estimates by enforcing a persistence of excitation condition on the closed loop system. The control law robustly satisfies constraints and has guarantees of feasibility and input-to-state stability. Convergence of parameter set estimates to the actual system parameter vector is guaranteed under conditions on reachability and tightness of disturbance bounds.

Keywords: Predictive control, Recursive identification, randomised algorithms.

1. INTRODUCTION

Model Predictive Control (MPC) repeatedly solves a receding horizon optimisation problem to define a feedback control law. Controller performance relies on model accuracy, and mismatch between the controlled system and its model can degrade performance (Mayne et al., 2000). Various adaptive MPC algorithms have been proposed to improve model accuracy by estimating model parameters online. The control input of adaptive MPC has the dual purpose of regulating the controlled system and providing sufficient excitation to allow model identification.

Adaptive controllers rely on Persistent Excitation (PE) conditions to ensure asymptotic identification of model parameters (Narendra and Annaswamy, 1987), but combining PE conditions with state and control constraints is challenging (Mayne, 2014). Several MPC algorithms have been developed with the aim of achieving both objectives. A common approach includes additional constraints in the MPC receding horizon optimisation problem. For example, Genceli and Nikolaou (1996) impose an additional periodic input constraint to guarantee a periodic persistently exciting feedback law for FIR systems, but the proposed algorithm has no guarantees of closed loop stability or convergence. Marafioti et al. (2014) consider ARMA models and formulate a PE constraint using past and current control inputs. The proposed algorithm ensures a PE condition if a periodic solution is found, but does not guarantee robust constraint satisfaction.

An alternative approach uses controller performance measures to balance conflicting objectives of identifying model parameters and regulating the system state. Lu et al. (2021) propose an adaptive robust tube MPC formulation for linear systems that includes a PE measure in the predicted performance objective and suitably linearised PE conditions as constraints. This algorithm is proven to be recursively feasible and input-to-state stable, but it does not guarantee satisfaction of a PE condition in closed loop operation. Heirung et al. (2015) avoid a nonconvex PE constraint by including an estimate of the parameter error covariance matrix in the MPC cost function. The optimisation problem is reformulated into a quadratically-constrained quadratic programming (QCQP) problem but recursive feasibility and stability were not proven. Similarly, Weiss and Di Cairano (2014) include an information functional of the predicted parameter error covariance and enforce robust constraint satisfaction and stability by selecting the control input from the Robust Admissible Invariant (RAI) set, which results in a nonconvex optimisation problem. However, in both of these approaches there is no guarantee of persistently exciting control inputs.

Other methods of promoting persistence of excitation in MPC have been proposed. Brüggemann and Bitmead (2022) reformulate the PE condition as an algebraic condition on a reference trajectory around a nominal equilibrium point. The algorithm ensures local convergence of the state and the parameter estimates. However, the approach does not guarantee robust satisfaction of state and control constraints for all feasible initial conditions. Gonzalez et al. (2014) achieve stability and satisfaction of a PE condition using a switching control strategy. The algorithm drives the system state into a pre-defined invariant region for identification, within which an excitation signal is injected into the control law. The proposed algorithm is only applicable for open loop stable linear systems and the existence of the target region is example-dependent.

In this paper we ensure (similarly to Lu and Cannon, 2022) persistence of excitation and recursive feasibility of the MPC optimisation. However, unlike Lu and Cannon (2022), additional constraints to enforce persistent excitation are not included in the optimization because this would require nonconvex constraints on control inputs. Instead we propose a computationally tractable check on the solution by using postulated values (or samples) of unknown model parameters and disturbances to generate sets of likely future trajectories and hence estimate the amount of excitation in past and future trajectories. For this check, persistence of excitation is evaluated over a

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2023.10.1131

single time-window rather than multiple windows. Each of these innovations reduces computation relative to Lu and Cannon (2022). For linear systems with uncertain parameters and bounded disturbances, the algorithm satisfies constraints robustly and achieves input-to-state practical stability (ISpS) while retaining computational tractability.

The paper is organised as follows. Section 2 summarises the robust adaptive MPC formulation, explains set-based parameter estimation method, and states conditions for persistence of excitation. The sample-based check of a PE condition applied to predicted trajectories is described in Section 3, and Section 4 shows that the control strategy ensures recursive feasibility, input-to-state practical stability and asymptotic parameter convergence. Section 5 illustrates the approach with a numerical example.

Notation: $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{>0}$ are the sets of non-negative and positive integers, and $\mathbb{Z}_{[p,q]} = \{n \in \mathbb{Z} : p \leq n \leq q\}$. The Euclidean norm is $\|x\|$ and $\|x\|_Q = (x^\top Q x)^{1/2}$. The identity matrix is $\mathcal{I}$, and $A \succeq 0$ (or $A \succ 0$) indicates that A is a positive semidefinite (positive definite) matrix. For $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$, $A\mathcal{X} = \{Ax : x \in \mathcal{X}\}$, $\mathcal{X} \oplus \mathcal{Y} = \{x + y : x \in \mathcal{X},\, y \in \mathcal{Y}\}$ and $\mathrm{Co}(\mathcal{X})$ is the convex hull of $\mathcal{X}$. The value of a variable y at the kth discrete time instant is denoted y_k and $y_{k|t}$ denotes the value of y_{k+t} predicted at time t. Probabilities and expectations conditioned on the model state x_t are denoted $\mathbb{P}(\cdot|x_t) = \mathbb{P}_t(\cdot)$ and $\mathbb{E}(\cdot|x_t) = \mathbb{E}_t(\cdot)$, and $\mathbb{P}(\cdot)$, $\mathbb{E}(\cdot)$ are respectively equivalent to $\mathbb{P}_0(\cdot)$, $\mathbb{E}_0(\cdot)$.

2. ADAPTIVE ROBUST MPC

Consider systems described by the uncertain linear model

$$x_{t+1} = A(\theta^*)x_t + B(\theta^*)u_t + Fw_t \tag{1}$$

with state, control input and unknown disturbance $x_t \in \mathbb{R}^{n_x}$, $u_t \in \mathbb{R}^{n_u}$ and $w_t \in \mathbb{R}^{n_w}$ at discrete time $t \in \mathbb{Z}_{\geq 0}$. Here $\theta^* \in \mathbb{R}^p$ is an unknown parameter independent of t.

Assumption 1. (Disturbance). The disturbance sequence $\{w_t \in \mathcal{W},\, t \in \mathbb{Z}_{\geq 0}\}$ is independent and identically distributed (i.i.d.), with $\mathbb{E}(w_t) = 0$, $\mathbb{E}(w_t w_t^\top) \succeq \epsilon_w \mathcal{I}$, $\epsilon_w > 0$, and $\mathcal{W}$ is a known compact and convex set.

Assumption 2. (Model parameters). (a). $(A(\theta), B(\theta)) = (A_0, B_0) + \sum_{i=1}^{p} (A_i, B_i)[\theta]_i$ for known (A_i, B_i) $i \in \mathbb{Z}_{[0,p]}$.
(b). $\Theta_0 \ni \theta^*$ is a polytopic initial parameter set estimate.
(c). $(A(\theta), B(\theta))$ is a reachable pair for all $\theta \in \Theta_0$.
(d). $(A(\theta^*), B(\theta^*)) = (A(\theta), B(\theta))$ if and only if $\theta = \theta^*$.

Starting from an initial set Θ_0, we recursively refine a polytopic parameter set estimate $\Theta_t = \{\theta : \Pi_\Theta \theta \leq \mu_t\}$ defined in terms of $\mu_t \in \mathbb{R}^{n_\Theta}$ and fixed $\Pi_\Theta \in \mathbb{R}^{n_\Theta \times p}$.

2.1 Robust MPC

Consider a predicted control law parameterised at time t in terms of decision variables $\mathbf{v}_t = \{v_{0|t}, \ldots, v_{N-1|t}\}$ as

$$u_{k|t} = \begin{cases} Kx_{k|t} + v_{k|t}, & k \in \mathbb{Z}_{[0,N-1]} \\ Kx_{k|t} + s_{k|t}, & k \geq N \end{cases} \tag{2}$$

where N is the MPC prediction horizon and the gain K is such that $z_{t+1} \in \mathrm{Co}\{A_K(\theta)z_t,\, \theta \in \Theta_0\}$ is quadratically stable with $A_K(\theta) = A(\theta) + B(\theta)K$. Here $\{s_{k|t},\, k \in \mathbb{Z}_{\geq 0}\}$ is a zero-mean i.i.d. random sequence that is introduced into the terminal control law to ensure persistent excitation, with $s_{k|t} \in \mathcal{S}$ for some known compact and convex set $\mathcal{S}$, and $\mathbb{E}(s_{k|t} s_{k|t}^\top) \succeq \epsilon_s \mathcal{I} \succ 0$ for all $k, t \in \mathbb{Z}_{\geq 0}$.

We assume linear state and control input constraints:

$$x_{k|t} \in \mathcal{X}, \quad u_{k|t} \in \mathcal{U}, \quad \forall k \geq 0, \tag{3}$$

where $\mathcal{X}$, $\mathcal{U}$ are given polytopic sets. To enforce these constraints we define a terminal set $\mathcal{X}_T$ satisfying

$$\mathcal{X}_T \subseteq \mathcal{X}, \quad K\mathcal{X}_T \oplus \mathcal{S} \subseteq \mathcal{U}, \tag{4}$$

$$A_K(\theta)\mathcal{X} \oplus B(\theta)\mathcal{S} \oplus F\mathcal{W} \subseteq \mathcal{X}_T \quad \forall \theta \in \Theta_0 \tag{5}$$

To ensure satisfaction of (3) we construct a tube $\mathbf{X}_t$,

$$\mathbf{X}_t = \{\mathcal{X}_{k|t},\, k \in \mathbb{Z}_{[0,N]}\},$$

where the sets $\mathcal{X}_{k|t}$ satisfy, for all $\theta \in \Theta_t$, $k \in \mathbb{Z}_{[0,N-1]}$,

$$\mathcal{X}_{k|t} \subseteq \mathcal{X}, \quad K\mathcal{X}_{k|t} \oplus \{v_{k|t}\} \subseteq \mathcal{U}, \tag{6}$$

$$A_K(\theta)\mathcal{X}_{k|t} \oplus \{B(\theta)v_{k|t}\} \oplus F\mathcal{W} \subseteq \mathcal{X}_{k+1|t}, \tag{7}$$

and the initial and terminal conditions

$$x_t \in \mathcal{X}_{0|t}, \quad \mathcal{X}_{N|t} \subseteq \mathcal{X}_T. \tag{8}$$

Remark 1. The sets comprising $\mathbf{X}_t$ may be defined, for example, as homothetic or fixed-complexity polytopic sets (Lorenzen et al., 2019; Lu and Cannon, 2019). However, in the example of Section 5, we use ellipsoidal sets $\mathcal{X}_{k|t} = \{x \in \mathbb{R}^{n_x} : (x - z_{k|t})^\top P (x - z_{k|t}) \leq \beta_{k|t}\}$, which ensure favourable scaling of computation with n_x when P is computed offline and $\boldsymbol{\beta}_t = \{\beta_{0|t}, \ldots, \beta_{N|t}\}$ and $\mathbf{z}_t = \{z_{0|t}, \ldots, z_{N|t}\}$ are online decision variables.

We consider a nominal performance objective defined for a given parameter vector $\hat{\theta}_t \in \Theta_t$ by the cost function

$$J(x_t, \hat{\theta}_t, \mathbf{v}_t) = \sum_{k=0}^{N-1} l(\hat{x}_{k|t}, K\hat{x}_{k|t} + v_{k|t}) + V_{N|t}(\hat{x}_{N|t}, \hat{\theta}_t)$$

with $\hat{x}_{0|t} = x_t$, $\hat{x}_{k+1|t} = A_K(\hat{\theta}_t)\hat{x}_{k|t} + B(\hat{\theta}_t)v_{k|t}$, $k \in \mathbb{Z}_{[0,N-1]}$. The stage cost $l(\cdot,\cdot)$ and terminal cost $V_{N|t}(\cdot, \hat{\theta}_t)$ are assumed to be positive definite quadratic functions satisfying, for given $\hat{\theta}_t \in \Theta_t$ and all $x \in \mathbb{R}^{n_x}$,

$$V_{N|t}(x, \hat{\theta}_t) = V_{N|t}\big(A_K(\hat{\theta}_t)x, \hat{\theta}_t\big) + l(x, Kx). \tag{9}$$

The nominal parameter vector $\hat{\theta}_t \in \Theta_t$ may be obtained by projecting a least squares parameter estimate onto Θ_t (Lorenzen et al., 2019) or by projecting $\hat{\theta}_{t-1}$ onto Θ_t:

$$\hat{\theta}_t = \arg\min_{\theta \in \Theta_t} \|\hat{\theta}_{t-1} - \theta\|. \tag{10}$$

A robust tube MPC law is determined by the solution, denoted $(\mathbf{v}_t^o, \mathbf{X}_t^o)$, of the problem of minimizing $J(x_t, \hat{\theta}_t, \mathbf{v}_t)$ over $\mathbf{v}_t$ and $\mathbf{X}_t$ subject to (6)-(8). This is the basis of the algorithm defined in Section 3.

2.2 Set-based parameter estimate

Set membership identification uses information on u_{t-1}, x_t and x_{t-1} to define a set Δ_t of unfalsified model parameters at time t. This is combined with the parameter set estimate Θ_{t-1} to construct an updated estimate Θ_t as follows. The linear dependence of the system model (1) on θ^* implies

$$x_{t+1} = \Phi(x_t, u_t)\theta^* + \phi(x_t, u_t) + Fw_t,$$

where $\Phi_t = \Phi(x_t, u_t)$ and $\phi_t = \phi(x_t, u_t)$ are defined

$$\Phi_t = \Phi(x_t, u_t) = [A_1 x_t + B_1 u_t \ \cdots \ A_p x_t + B_p u_t] \tag{11}$$

$$\phi_t = \phi(x_t, u_t) = A_0 x_t + B_0 u_t. \tag{12}$$

A set of unfalsified parameters at time t given x_t, x_{t-1}, u_{t-1}, and the disturbance set $\mathcal{W}$, is given by

$$\Delta_t = \{\theta : x_t - \Phi_{t-1}\theta - \phi_{t-1} \in F\mathcal{W}\}.$$

Using N_μ unfalsified sets, the minimum volume Θ_t satisfying $\Theta_t \supseteq \bigcap_{\tau=t-N_\mu+1}^{t} \Delta_\tau \cap \Theta_{t-1}$ is found by solving n_Θ convex programs to compute the update μ_t. This ensures that $\theta^* \in \Theta_t \subseteq \Theta_{t-1}$ for all $t \in \mathbb{Z}_{>0}$ (Lu et al., 2021).

2.3 Persistent excitation

The regressor Φ_t in (11) is persistently exciting (Narendra and Annaswamy, 1987) if there exists a horizon N_u and scalar $\epsilon_\Phi > 0$ such that $\sum_{k=t}^{t+N_u-1} \Phi_k^\top \Phi_k \succeq \epsilon_\Phi \mathcal{I}$ for all $t \in \mathbb{Z}_{\geq 0}$. In the current work, we define persistent excitation using the requirement that there exists $\epsilon_\Phi > 0$ and an infinite sequence of time instants $t_0, t_1, \ldots$ such that

$$\sum_{k=t_i}^{t_i+N_u-1} \Phi_k^\top \Phi_k \succeq \epsilon_\Phi \mathcal{I} \quad \forall\, i \in \mathbb{Z}_{\geq 0}. \tag{13}$$

We refer to the interval $\mathbb{Z}_{[t_i, t_i+N_u-1]}$ as a *PE window.*

Assumption 3. (Tight disturbance bound). For all $w^0 \in \partial\mathcal{W}$ and any $\epsilon > 0$ the disturbance sequence $\{w_0, w_1, \ldots\}$ satisfies $\mathbb{P}\{\|w_t - w^0\| < \epsilon\} \geq p_w(\epsilon)$, for all $t \in \mathbb{Z}_{\geq 0}$, where $p_w(\epsilon) > 0$ whenever $\epsilon > 0$.

The PE condition (13) ensures convergence of the parameter set Θ_t (see e.g. Lu and Cannon (2022) Lemma 1).

Lemma 2. Under Assumptions 1 and 3, if (13) holds and $N_u \leq N_\mu$, then $\lim_{t\to\infty} \Theta_t = \{\theta^*\}$ with probability 1 (w.p. 1).

3. PERSISTENCE OF EXCITATION CHECK

The PE condition (13) is not suitable to be included directly as a constraint in the optimization of predicted trajectories because it involves nonconvex quadratic matrix inequalities. Instead, we impose (13) with a certain level of statistical confidence using a sample-based check applied to the solution of the MPC optimization. This approach performs a comparison with a reference solution that is known to satisfy the PE condition on average, and adopts the reference as a fallback solution if the check fails.

Given the solution, $(\mathbf{v}_t^o, \mathbf{X}_t^o)$, of the MPC optimization at time t, we compare a measure of PE for $\mathbf{v}_t^o$ and a reference solution $\hat{\mathbf{v}}_t$ over a set of samples of model parameters $\theta_t \in \Theta_t$ and disturbance sequences $\mathbf{w}_t = \{w_{0|t}, \ldots, w_{N-1|t}\} \in \mathcal{W}^N$. The reference $\hat{\mathbf{v}}_t = \{\hat{v}_{0|t}, \ldots, \hat{v}_{N-1|t}\}$, is defined using the solution $\mathbf{v}_{t-1}^o$ adopted at time $t-1$ and the random input $s_{N-1|t}$:

$$\hat{v}_{k|t} = \begin{cases} v_{k+1|t-1}^o, & k \in \mathbb{Z}_{[0,N-2]} \\ s_{N-1|t}, & k = N-1. \end{cases}$$

Let $\zeta_t^{(i)} = (\theta_t^{(i)}, w_{0|t}^{(i)}, \ldots, w_{N-1|t}^{(i)})$ be a sample from the uniform distribution on $\Theta_t \times \mathcal{W}^N$, and let $\{\zeta_t^{(1)}, \ldots, \zeta_t^{(N_s)}\}$ denote a set of N_s independent samples. Corresponding predicted state and control sequences $(\tilde{x}_{k|t}^{(i)}, \tilde{u}_{k|t}^{(i)})$, for $k \in \mathbb{Z}_{[0,N-1]}$, are generated using

$$\begin{aligned} \tilde{x}_{k+1|t}^{(i)} &= A(\theta_t^{(i)})\tilde{x}_{k|t}^{(i)} + B(\theta_t^{(i)})\tilde{u}_{k|t}^{(i)} + Fw_{k|t}^{(i)}, \\ \tilde{u}_{k|t}^{(i)} &= K\tilde{x}_{k|t}^{(i)} + v_{k|t}, \end{aligned} \tag{14}$$

and we set $\tilde{x}_{k|t}^{(i)} = x_{k+t}$ if $k \leq 0$ and $\tilde{u}_{k|t}^{(i)} = u_{k+t}$ if $k < 0$. For a PE window starting at $t+\kappa$ with $\kappa \in \mathbb{Z}_{[-N_u+1, N-N_u]}$, the associated *PE matrix* $\Psi_{\kappa|t}(\mathbf{v}_t, \theta_t^{(i)}, \mathbf{w}_t^{(j)})$ is defined as

$$\Psi_{\kappa|t}(\mathbf{v}_t, \zeta_t^{(i)}) = \sum_{k=\kappa}^{\kappa+N_u-1} \Phi(\tilde{x}_{k|t}^{(i)}, \tilde{u}_{k|t}^{(i)})^\top \Phi(\tilde{x}_{k|t}^{(i)}, \tilde{u}_{k|t}^{(i)}).$$

We also allow the PE window to extend beyond the first N time steps of the prediction horizon by defining $v_{k|t} = s_{k|t}$ and $w_{k|t}^{(i)} = w_{k+t}$ for all $i \in \mathbb{Z}_{[1,N_s]}$ and all $k \geq N$ in (14), and defining

$$\begin{aligned} \Psi_{\kappa|t}(\mathbf{v}_t, \zeta_t^{(i)}) = &\sum_{k=\kappa}^{N-1} \Phi(\tilde{x}_{k|t}^{(i)}, \tilde{u}_{k|t}^{(i)})^\top \Phi(\tilde{x}_{k|t}^{(i)}, \tilde{u}_{k|t}^{(i)}) \\ &+ \sum_{k=N}^{\kappa+N_u-1} \mathbb{E}_{N|t}\big[\Phi(\tilde{x}_{k|t}^{(i)}, \tilde{u}_{k|t}^{(i)})^\top \Phi(\tilde{x}_{k|t}^{(i)}, \tilde{u}_{k|t}^{(i)})\big] \end{aligned}$$

for $\kappa \in \mathbb{Z}_{[N-N_u+1, N-1]}$. Here $\mathbb{E}_{N|t}(\cdot)$ denotes the expectation (conditioned on $\tilde{x}_{N|t}$) that is obtained by marginalising over the distributions of $s_{k|t}$ and w_{k+t} for $k \geq N$.

To perform the check, we compare $\delta_{\kappa|t}$ and $\hat{\delta}_{\kappa|t}$ defined by

$$\begin{aligned} \delta_{\kappa|t} &= \max_{\delta \in \mathbb{R}} \ \delta \text{ subject to } \Psi_{\kappa|t}(\mathbf{v}_t^o, \zeta_t^{(i)}) \succeq \delta\mathcal{I} \ \forall i \in \mathbb{Z}_{[1,N_s]} \\ \hat{\delta}_{\kappa|t} &= \max_{\delta \in \mathbb{R}} \ \delta \text{ subject to } \Psi_{\kappa|t}(\hat{\mathbf{v}}_t, \zeta_t^{(i)}) \succeq \delta\mathcal{I} \ \forall i \in \mathbb{Z}_{[1,N_s]} \end{aligned} \tag{15}$$

The check is said to have failed if $\delta_{\kappa|t} < \hat{\delta}_{\kappa|t}$ and in this case we redefine $\mathbf{v}_t^o$ by setting its value to $\hat{\mathbf{v}}_t$.

At each time t we propose to check a single PE window with start time $t+\kappa$, and to decrease κ by 1 at successive time steps until $\kappa = -N_u+1$, when it is reset to $\kappa = N-1$. The overall strategy of Algorithm 1 is therefore to track a PE window with a fixed start time as time advances until the window contains only the past and current time instants from $t - N_u + 1$ to t. We show in Sections 4 and 5 that this strategy ensures that the closed loop system has a non-zero probability of satisfying the PE condition (13) at least once every $N + N_u - 1$ time steps, and therefore (13) necessarily holds at an infinite number of time instants.

Algorithm 1 Adaptive MPC with sampled PE check

Offline: Compute $\mathcal{X}_T$ satisfying (4)-(5) and set $\kappa = N-1$.
Online: At times $t = 0, 1, \ldots$:
(a). Obtain the current state x_t.
(b). Update Θ_t and $\hat{\theta}_t$, and determine $V_{N|t}$ satisfying (9).
(c). Compute the solution $(\mathbf{v}_t^o, \mathbf{X}_t^o)$ of the convex program:

$$\mathcal{P}(x_t, \Theta_t, \hat{\theta}_t): \ \underset{\mathbf{v}_t, \mathbf{X}_t}{\text{minimise}} \quad J(x_t, \hat{\theta}_t, \mathbf{v}_t) \text{ s.t. (6)-(8).}$$

(d). Draw samples $\{\zeta^{(1)}, \ldots, \zeta^{(N_s)}\}$ and compute $\delta_{\kappa|t}, \hat{\delta}_{\kappa|t}$. Set $\mathbf{v}_t^o \leftarrow \hat{\mathbf{v}}_t$ if $\delta_{\kappa|t} < \hat{\delta}_{\kappa|t} + \varepsilon$ for a given threshold ε.
(e). Apply the control input $u_t = Kx_t + v_{0|t}^o$. If $\kappa > -N_u+1$ set $\kappa \leftarrow \kappa-1$, otherwise set $\kappa \leftarrow N-1$.

Remark 3. The threshold ϵ in step (d) is a positive value used to distinguish non-zero minimum eigenvalues in (15) from those that are zero to numerical precision.

Remark 4. The computational effort to find $\delta_{\kappa|t}$ and $\hat{\delta}_{\kappa|t}$ in (15) is equivalent to determining the minimum eigenvalues of $2N_s$ positive semidefinite $p \times p$ matrices.

Remark 5. If the disturbance bound $\mathcal{W}$ is ellipsoidal and the sets $\mathcal{X}_{k|t}$ are ellipsoidal (as described in Remark 1), then the update of Θ_t in step (b) and problem $\mathcal{P}$ in step (c) become second-order conic programs (SOCPs).

4. CLOSED LOOP SYSTEM PROPERTIES

The MPC optimisation (problem $\mathcal{P}$ in Algorithm 1) is by construction feasible at all times $t > 0$ if it is feasible at $t = 0$ since $\Theta_t \subseteq \Theta_{t-1}$ for all t. In addition, the closed loop system has the following input-to-state stability property (see e.g. Theorem 1 and Corollary 1 of Lu et al. (2021)).

Proposition 6. (Stability). Under Assumptions 1 and 2, the system (1) with Algorithm 1 is input-to-state practically stable (ISpS) (Limon et al., 2009, Def. 6) with respect to an input defined by $Fw_t + B(\theta^*)s_t$, where $s_t = s_{N|t-N}$ for $t \geq N$ and $s_t = 0$ for $t < N$, in the set of initial conditions x_0 for which $\mathcal{P}(x_0, \Theta_0, \hat{\theta}_0)$ is feasible.

A consequence of Proposition 6 (Limon et al., 2009; Lu et al., 2021) is that a $\mathcal{KL}$-function η and $\mathcal{K}$-functions ψ, ξ exist such that, for all $t \in \mathbb{Z}_{\geq 0}$ and all feasible x_0,

$$\|x_t\| \leq \eta(\|x_0\|, t) + \psi\big(\max_{\tau \in \mathbb{Z}_{[0,t-1]}} \|Fw_\tau + B(\theta^*)s_\tau\|\big) + \xi\big(\max_{\tau \in \mathbb{Z}_{[0,t-1]}} \|\hat{\theta}_t - \theta^*\|\big).$$

4.1 Closed loop system persistence of excitation

This section considers the randomised check performed in step (d) of Algorithm 1. We show that this procedure ensures that the closed loop system satisfies the PE condition (13) using an approach that does not require knowledge of the probability distribution of the disturbance w_t or the prior distribution of the unknown model parameters θ^*. To simplify notation, let $d_{\kappa|t}(\mathbf{v})$ denote the minimum over $\zeta \in \Theta_t \times \mathcal{W}^N$ of the smallest eigenvalue of $\Psi_{\kappa|t}(\mathbf{v}, \zeta)$.

Lemma 7. If $\delta_{\kappa|t} \geq \hat{\delta}_{\kappa|t} + \varepsilon$, then, for some $\rho \in [0, 1)$,

$$\mathbb{P}_t\big\{d_{\kappa|t}(\mathbf{v}_t^o) \geq d_{\kappa|t}(\hat{\mathbf{v}}_t)\big\} \geq 1 - \rho^{N_s}. \tag{16}$$

Proof. The minimum eigenvalue of $\Psi_{\kappa|t}(\mathbf{v}_t^o, \zeta)$ is a Lipschitz continuous function of ζ since $\{(A_i, B_i),\, i \in \mathbb{Z}_{[1,p]}\}$, $\mathcal{X}$, $\mathcal{U}$, Θ_0 and N_u are by assumption bounded. Therefore $\delta_{\kappa|t} \geq \hat{\delta}_{\kappa|t} + \varepsilon$ implies $d_{\kappa|t}(\mathbf{v}_t^o) \geq \hat{\delta}_{\kappa|t}$ whenever $\zeta_t^{(i)}$ for some $i \in \mathbb{Z}_{[1,N_s]}$ lies in a certain neighbourhood, $\mathcal{N}$, of a minimizing argument ζ_t^* such that $d_{\kappa|t}(\mathbf{v}_t^o) = \Psi_{\kappa|t}(\mathbf{v}_t^o, \zeta_t^*)$ (where $\mathcal{N}$ depends on $\{(A_i, B_i),\, i \in \mathbb{Z}_{[1,p]}\}$, $\mathcal{X}$, $\mathcal{U}$, Θ_0, N_u, and ε, and $\mathcal{N}$ necessarily has a non-empty interior). But the samples $\zeta_t^{(i)}$, $i \in \mathbb{Z}_{[1,N_s]}$ are uniformly distributed over $\Theta_t \times \mathcal{W}^N$. Denoting the probability that $\zeta_t^{(i)}$ lies outside $\mathcal{N}$ as ρ, where $\rho \leq 1 - \mathrm{vol}\{\mathcal{N} \cap (\Theta_t \times \mathcal{W}^N)\}/\mathrm{vol}\{\Theta_t \times \mathcal{W}^N\} \in [0, 1)$, it follows that $\zeta_t^{(i)} \in \mathcal{N}$ for one or more index $i \in \mathbb{Z}_{[1,N_s]}$ with probability $1 - \rho^{N_s}$ Therefore (16) must hold because $\hat{\delta}_{\kappa|t} \geq d_{\kappa|t}(\hat{\mathbf{v}}_t)$. □

The following result demonstrates that the PE condition (13) holds under Algorithm 1, and therefore that Θ_t converges to $\{\theta^*\}$ with probability 1 by Lemma 2.

Theorem 8. Suppose Assumptions 1 and 2 are satisfied and $N_u > n_x$, then Algorithm 1 ensures that (13) holds at an infinite number of instants $t_0, t_1, \ldots$, for some $\epsilon_\Phi > 0$.

Proof. For all $\theta \in \mathbb{R}^p$ and any $\zeta^{(i)} \in \Theta_t \times \mathcal{W}^N$ we have

$$\mathbb{E}_{N|t}\big[\theta^\top \Psi_{N-1|t}(\hat{\mathbf{v}}_t, \zeta^{(i)})\,\theta\big] = \sum_{k=N-1}^{N+N_u-2} \mathbb{E}_{N|t}\big[\|\Phi(\tilde{x}_{k|t}^{(i)}, \tilde{u}_{k|t}^{(i)})\theta\|^2\big],$$

where $\tilde{x}_{k|t}^{(i)}$ and $\tilde{u}_{k|t}^{(i)}$ are generated by (14) with $v_{k|t} = s_{k|t}$. Therefore, from Assumptions 2(c) and 2(d), and $\Theta_t \subseteq \Theta_0$, $\epsilon_s > 0$, and $\hat{v}_{N-1|t} = s_{N-1|t}$, it follows that $\epsilon_\Phi > 0$ exists so that (e.g. see Lu and Cannon (2022), Thm. 3):

$$\mathbb{E}_{N|t}\big[\theta^\top \Psi_{N-1|t}(\hat{\mathbf{v}}_t, \zeta^{(i)})\,\theta\big] \geq \epsilon_\Phi \|\theta\|^2. \tag{17}$$

Hence the probability that the minimum eigenvalue of $\Psi_{N-1|t}(\hat{\mathbf{v}}_t, \zeta)$ for all $\zeta \in \Theta_t \times \mathcal{W}^N$ is no less than ϵ_Φ satisfies

$$\mathbb{P}_t\big[d_{N-1|t}(\hat{\mathbf{v}}_t) \geq \epsilon_\Phi\big] \geq \alpha \tag{18}$$

for some $\alpha \in (0, 1]$. This property of predicted trajectories ensures the PE condition (13) because Algorithm 1 implies

$$\mathbb{P}_t\big[d_{\kappa|t}(\mathbf{v}_t^o) \geq d_{\kappa|t}(\hat{\mathbf{v}}_t)\big] \geq 1 - \rho^{N_s} \in (0, 1] \tag{19}$$

(this follows from Lemma 7 if $\delta_{\kappa|t} \geq \hat{\delta}_{\kappa|t} + \epsilon$ and from $\mathbf{v}_t^o = \hat{\mathbf{v}}_t$ if $\delta_{\kappa|t} < \hat{\delta}_{\kappa|t} + \epsilon$), and hence, for any $\beta \in (0, 1]$,

$$\mathbb{P}_t\big[d_{\kappa|t}(\hat{\mathbf{v}}_t) \geq \epsilon_\Phi\big] \geq \beta \;\Rightarrow\; \mathbb{P}_t\big[d_{\kappa|t}(\mathbf{v}_t^o) \geq \epsilon_\Phi\big] \geq \beta(1 - \rho^{N_s}). \tag{20}$$

Furthermore, at time t, $\Psi_{\kappa-1|t+1}(\hat{\mathbf{v}}_{t+1}, \zeta)$ and $\Psi_{\kappa|t}(\mathbf{v}_t^o, \zeta)$ are identically distributed random variables for all $\kappa \in \mathbb{Z}_{[-N_u+1,N-1]}$, so that, for any $\beta \in (0, 1]$,

$$\mathbb{P}_t\big[d_{\kappa|t}(\mathbf{v}_t^o) \geq \epsilon_\Phi\big] \geq \beta \;\Rightarrow\; \mathbb{P}_t\big[d_{\kappa-1|t+1}(\hat{\mathbf{v}}_{t+1}) \geq \epsilon_\Phi\big] \geq \beta. \tag{21}$$

Let $N_{pe} := N + N_u - 1$, starting from (18) and applying (20)-(21) N_{pe} times, we obtain

$$\mathbb{P}_t\big[d_{-N_u+1|t+N_{pe}}(\mathbf{v}_{t+N_{pe}}^o, \zeta) \geq \epsilon_\Phi\big] \geq (1 - \rho^{N_s})^{N_{pe}}\alpha.$$

But $\theta^\top \Psi_{-N_u+1|t+N_{pe}}(\mathbf{v}_t^o, \zeta)\,\theta = \sum_{k=N-1}^{N+N_u-2} \|\Phi_{t+k}\theta\|^2$ for all $\theta \in \mathbb{R}^p$, where $\Phi_{t+k} = \Phi(x_{t+k}, u_{t+k})$ is the regressor defined in (11) evaluated along the closed loop trajectories of (1) under Algorithm 1. Therefore, for all $\theta \in \mathbb{R}^p$,

$$\mathbb{P}_t\Big(\sum_{k=N-1}^{N+N_u-2} \|\Phi_{t+k}\theta\|^2 \geq \epsilon_\Phi \|\theta\|^2\Big) \geq (1 - \rho^{N_s})^{N_{pe}}\alpha > 0.$$

Since this holds at $t = nN_{pe}$, $n \in \mathbb{Z}_{>0}$, the PE condition (13) must be satisfied at an infinite number of instants. □

5. NUMERICAL EXAMPLES

We consider a system with 3 unknown parameters, defined by the following matrices in (1) and Assumption 2(a):

$$(A_0, B_0, F) = \begin{bmatrix} 1.08 & -0.12 & 0 & 0 \\ -0.12 & -1.18 & 0 & 0 \\ 0.36 & 0.99 & 0.70 & 0.04 \\ -2.19 & -0.04 & 0.04 & 0.17 \end{bmatrix}, \begin{bmatrix} 0 & -0.97 \\ 0 & -0.78 \\ 0 & -0.44 \\ -1.12 & 0.24 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1.57 & 0 \\ 0 & -0.49 \end{bmatrix}$$

$$(A_1, B_1) = \begin{bmatrix} -3 & 1 & 0 & 0 \\ -2 & -9 & 0 & 0 \\ -8 & 6 & -6 & -8 \\ -3 & 1 & -8 & -8 \end{bmatrix} \times 10^{-2}, \begin{bmatrix} -4 & -7 \\ 3 & -7 \\ -4 & -9 \\ 5 & 8 \end{bmatrix} \times 10^{-2}$$

$$(A_2, B_2) = \begin{bmatrix} -7 & -4 & 0 & 0 \\ 7 & 1 & 0 & 0 \\ 5 & -8 & -10 & -7 \\ -4 & 0 & 2 & 1 \end{bmatrix} \times 10^{-2}, \begin{bmatrix} 5 & 8 \\ -5 & 7 \\ -7 & 0 \\ 8 & 2 \end{bmatrix} \times 10^{-2}$$

$$(A_3, B_3) = \begin{bmatrix} 5 & 6 & 0 & 0 \\ -8 & 10 & 0 & 0 \\ 0 & -5 & -3 & -7 \\ -6 & 8 & 8 & 1 \end{bmatrix} \times 10^{-2} \begin{bmatrix} -7 & -6 \\ 7 & -5 \\ -3 & -2 \\ 10 & 5 \end{bmatrix} \times 10^{-2}$$

and true parameter vector $\theta^* = [-0.5\ \ -0.152\ \ 0.44]^\top$. The system constraints are $\|x_t\|_\infty \leq 1$, $\|u_t\|_\infty \leq 1$, the cost matrices are $Q = \mathcal{I}_{4\times 4}$, $R = \mathcal{I}_{2\times 2}$, and the disturbance sequence $\{w_0, w_1, \ldots\}$ is uniformly distributed on $\mathcal{W} = \{w \in \mathbb{R}^2 : w^\top P_w w \leq 1\}$ with $P_w = \left[\begin{smallmatrix} 2.18 & 0.07 \\ 0.07 & 2.19 \end{smallmatrix}\right] \times 10^4$.

This system satisfies Assumptions 1 and 2, but the structure of F and the block-lower-triangular structure of A_0, A_1, A_2 and A_3 ensures that the system is not reachable from the disturbance input w_t. Therefore persistence of excitation cannot be achieved through the action of the random disturbance w_t alone, whereas u_t may be made persistently exciting with a suitably chosen control law.

A robust MPC law was used with ellipsoidal tubes bounding predicted states. The terminal set is defined as a robustly invariant ellipsoidal set $\mathcal{X}_T = \{x : x^\top P x \le 1\}$, and the tube cross-sections $\mathcal{X}_{k|t} = \{x : (x - z_{k|t})^\top P(x - z_{k|t}) \le \beta_{k|t}\}$ are defined using the same matrix P, with $\{\beta_{0|t}, \ldots, \beta_{N|t}\}$ and $\{z_{0|t}, \ldots, z_{N|t}\}$ as decision variables in problem $\mathcal{P}$, where

$$z_{k+1|t} = A_K(\bar{\theta}_t) z_k + B(\bar{\theta}_t) v_{k|t},\ k \in \mathbb{Z}_{[0,N-1]},$$

and $\bar{\theta}_t = \arg\min_{\bar{\theta}} \max_{\theta \in \Theta_t} \|\bar{\theta} - \theta\|$. Simulations were performed in Matlab on a 2.9 GHz Intel Core i7 processor with 16 GB RAM. The offline MPC design steps and the online optimisation in step (c) of Algorithm 1 were formulated using Yalmip (Löfberg, 2019) and solved using Mosek (MOSEK ApS, 2022). Identical sequences of random disturbances were used in parallel tests of different control strategies. By omitting step (d) from Algorithm 1 we obtain an adaptive robust MPC law without PE guarantees, and Algorithm 1 reduces to a robust MPC law with a fixed parameter set if steps (b) and (d) are omitted.

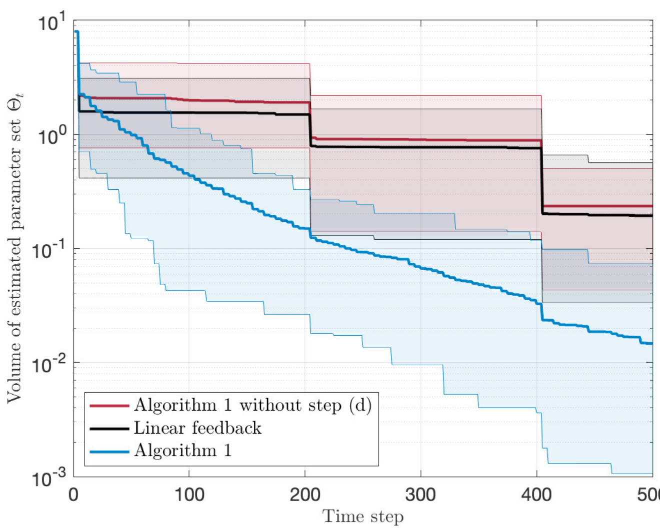

Fig. 1. Parameter set volume vol(Θ_t) for Algorithm 1 with and without step (d), and linear feedback $u_t = Kx_t$.

Figure 1 shows the volume of the estimated parameter set with Algorithm 1 (with and without step (d)) and with linear feedback. The model state switches to new initial conditions at $t = 200$ and $t = 400$ in order to illustrate the effect of transients, which could arise, for example, after reference changes in tracking problems. Each simulation was repeated with 30 disturbance sequences; mean parameter set volumes are shown by solid lines and ranges are shown by shaded regions. In agreement with Lemma 2 and Theorem 8, Alg. 1 (blue) results in parameter convergence, while the adaptive MPC law of Alg. 1 without step (d) (red) and linear feedback (black) are unable to improve their parameter estimates except during the transient responses around $t = 200$ and $t = 400$.

The convergence rates of the estimated parameter sets in Figure 1 are explained by the measures of persistence of excitation shown in Figure 2. Here the closed loop PE coefficient at time t is the largest scalar ϵ_t satisfying

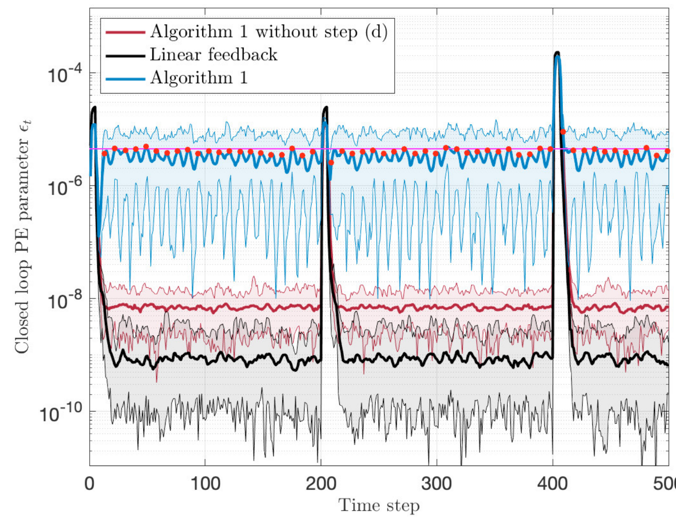

Fig. 2. Closed loop PE coefficients ϵ_t for Algorithm 1 with and without step (d), and linear feedback $u_t = Kx_t$.

$$\sum_{k=t-N_u+1}^{t} \Phi_k^\top \Phi_k \succeq \epsilon_t \mathcal{I}.$$

The solid lines show the values of ϵ_t averaged over 30 uncertainty realisations, and the shaded regions show the ranges of values observed for ϵ_t. With the exception of short periods following the times when the system switches to a new initial condition, ϵ_t is typically orders of magnitude greater when step (d) is included in Algorithm 1, and ϵ_t for linear feedback is effectively zero (within limits on numerical precision). The threshold employed in step (d) was $\epsilon = 10^{-6}$, the number samples used to compute $\delta_{\kappa|t}$ and $\hat{\delta}_{\kappa|t}$ in step (d) was $N_s = 20$, the PE window length was $N_u = 5$, and the MPC prediction horizon was $N = 10$.

The pink line in Figure 2 shows the value of ϵ_Φ, which was defined in (17) as a lower bound on the minimum eigenvalue of $\mathbb{E}_{N|t}[\Psi_{N-1|t}(\hat{\mathbf{v}}_t, \zeta)]$ for all $\zeta \in \Theta_t \times \mathcal{W}^N$. For convenience we define ϵ_Φ here by minimizing over Θ_0, considering the contribution from $s_{k|t}$, $k \ge N$, and neglecting the contribution from the free response starting from $x_{N|t}$. The sequence $\{s_{0|t}, s_{1|t}, \ldots\}$ is uniformly distributed on $\mathcal{S} = \{s : s^\top P_s s \le 1\}$ with $P_s = \left[\begin{smallmatrix} 1.45 & 0.16 \\ 0.16 & 1.3 \end{smallmatrix}\right] \times 10^4$. The values of ϵ_t for Algorithm 1 at times when $\kappa = -N_u + 1$ are shown by red circles in Figure 2. Clearly the empirical expected value of ϵ_t at these times has a non-zero probability of exceeding ϵ_Φ, in agreement with Theorem 8.

To investigate the effectiveness of the PE check in Algorithm 1 during transient responses, we now consider switching between randomly chosen initial conditions at intervals of 20 time steps. Figure 3 shows that Algorithm 1 with step (d) (in blue) ensures that the PE condition (13) is met more reliably than either Algorithm 1 without step (d) (red) or linear feedback (black). Here the PE check in step (d) failed at 32% of time steps on average (reduced from 84% in Figure 1) and linear feedback violated input or state constraints at 25% of time steps.

To compare performance, we consider the cumulative cost at time t, $J_t := \sum_{k=0}^{t} (\|x_k\|_Q^2 + \|u_k\|_R^2)$. Figure 4 shows the evolution of J_t in the first 20 times steps of Figure 3 while

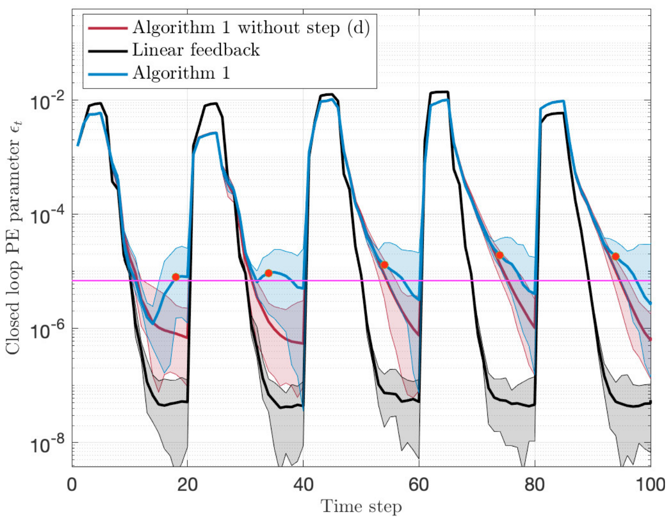

Fig. 3. Closed loop PE coefficients ϵ_t for Algorithm 1 with and without step (d), and linear feedback $u_t = Kx_t$.

the parameter set is updated (dashed lines), and when Θ_t is frozen at the value of Θ_{200} (solid lines). Comparing the blue and red dashed lines, the cumulative cost rises slightly faster when step (d) is included, indicating that the guarantee of parameter convergence provided by step (d) is obtained at the expense of slightly worse performance. However, the solid lines show that the more accurate parameter estimates obtained using Algorithm 1 with step (d) provide improved cumulative costs since the range of costs (shaded regions in Fig. 4), is smaller and the worst case cost is reduced by 5%.

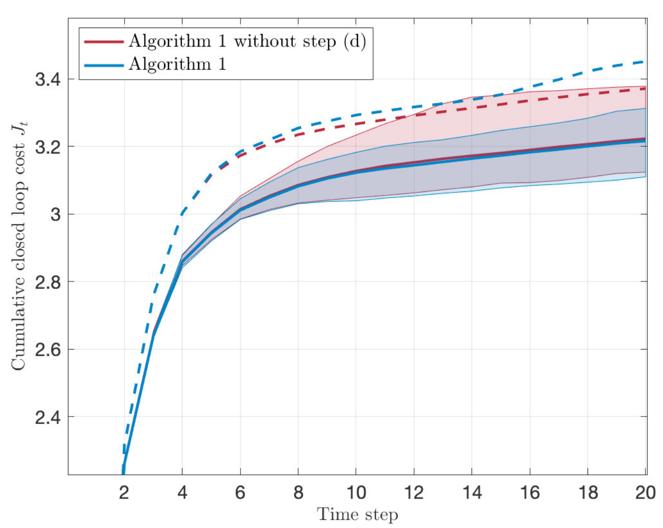

Fig. 4. Closed loop cumulative cost J_t for Algorithm 1 with and without step (d). Dashed lines: during parameter set estimation, Solid lines: constant parameter set.

Table 1. Mean {max} computation times (ms)

Model sizes and Alg. 1 parameters	step (b)	step (c)	step (d)
$(n_x, n_u, p) = (4, 2, 3)$	12.0	12.9	4.7
$(N, N_u, N_s) = (10, 5, 20)$	{14.3}	{17.2}	{8.9}
$(n_x, n_u, p) = (4, 2, \mathbf{5})$	22.0	43.2	17.7
$(N, N_u, N_s) = (10, 5, 20)$	{25.1}	{51.8}	{33.0}
$(n_x, n_u, p) = (\mathbf{8}, \mathbf{3}, 5)$	26.4	89.5	23.1
$(N, N_u, N_s) = (\mathbf{20}, \mathbf{10}, 20)$	{33.8}	{107.0}	{42.5}

The dependence of computation times on plant dimensions and algorithm parameters for a randomly chosen set of plant models is shown in Table 1.

6. CONCLUSIONS

An adaptive robust MPC algorithm is proposed, providing a persistently exciting control law while robustly satisfying input and state constraints. Persistence of excitation is ensured by a sample-based check on the solution of the MPC optimisation problem, and by using a fallback control law if this check fails. The approach provides is computationally tractable and robustly stable.

REFERENCES

Brüggemann, S. and Bitmead, R.R. (2022). Forward-looking persistent excitation in model predictive control. *Automatica*, 136, 110033.

Genceli, H. and Nikolaou, M. (1996). New approach to constrained predictive control with simultaneous model identification. *AIChE Journal*, 42(10), 2857–2868.

Gonzalez, A., Ferramosca, A., Bustos, G., Marchetti, J., Fiacchini, M., and Odloak, D. (2014). Model predictive control suitable for closed-loop re-identification. *Systems & Control Letters*, 69(1), 23–33.

Heirung, T., Foss, B., and Ydstie, B. (2015). MPC-based dual control with online experiment design. *Journal of Process Control*, 32, 64–76.

Limon, D., Alamo, T., Raimondo, D., Muñoz de la Peña, D., Bravo, J., Ferramosca, A., and Camacho, E. (2009). Input-to-state stability: A unifying framework for robust model predictive control. In *Nonlinear Model Predictive Control*, LNCIS vol. 384. Springer.

Löfberg, J. (2019). Yalmip. https://yalmip.github.io.

Lorenzen, M., Cannon, M., and Allgöwer, F. (2019). Robust MPC with recursive model update. *Automatica*, 103, 467–471.

Lu, X. and Cannon, M. (2022). Robust adaptive model predictive control with persistent excitation conditions. https://arxiv.org/abs/arxiv:2211.09275 (submitted to Automatica).

Lu, X., Cannon, M., and Koksal-Rivet, D. (2021). Robust adaptive model predictive control: performance and parameter estimation. *Int. J. Robust and Nonlinear Control*, 31, 8703–8724.

Lu, X. and Cannon, M. (2019). Robust adaptive tube model predictive control. *American Control Conference*, 3695–3701.

Marafioti, G., Bitmead, R.R., and Hovd, M. (2014). Persistently exciting model predictive control. *Int. J. Adaptive Control and Signal Processing*, 28(6), 536–552.

Mayne, D.Q., Rawlings, J., Rao, C., and Scokaert, P. (2000). Constrained model predictive control: stability and optimality. *Automatica*, 36, 789–814.

Mayne, D.Q. (2014). Model predictive control: Recent developments and future promise. *Automatica*, 50(12), 2967–2986.

MOSEK ApS (2022). *The MOSEK optimization toolbox V10.0.* https://www.mosek.com/documentation.

Narendra, K.S. and Annaswamy, A.M. (1987). Persistent excitation in adaptive systems. *Int. J. Control*, 45, 127.

Weiss, A. and Di Cairano, S. (2014). Robust dual control MPC with guaranteed constraint satisfaction. *53rd IEEE Conference on Decision and Control*, 6713–6718.