

Quasi-ALE metrics with holonomy $SU(m)$ and $Sp(m)$

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This is the sequel to the author's paper [11] on *Asymptotically Locally Euclidean metrics*, or *ALE metrics* for short, with holonomy $SU(m)$. Let G be a finite subgroup of $U(m)$ and suppose (X, π) is a *resolution* of \mathbb{C}^m/G , that is, X is a normal nonsingular variety with a proper birational morphism $\pi : X \rightarrow \mathbb{C}^m/G$.

When G acts freely on $\mathbb{C}^m \setminus \{0\}$, so that \mathbb{C}^m/G has an *isolated quotient singularity* at 0, we defined in [11] a special class of Kähler metrics on X called *ALE Kähler metrics*, and proved an existence result for Ricci-flat ALE Kähler metrics on *crepant resolutions* of \mathbb{C}^m/G , using a version of the Calabi conjecture for ALE manifolds.

This paper will generalize the ideas of [11] to the case when G does not act freely on $\mathbb{C}^m \setminus \{0\}$, so that the singularities of \mathbb{C}^m/G are not isolated. The appropriate class of Kähler metrics on resolutions X of non-isolated quotient singularities \mathbb{C}^m/G will be called *Quasi-ALE*, or *QALE* for short. We are particularly interested in *Ricci-flat* QALE Kähler manifolds, and our main result is an existence theorem for Ricci-flat QALE Kähler metrics on crepant resolutions of \mathbb{C}^m/G .

The key to understanding the structure of non-isolated singularities \mathbb{C}^m/G is to observe that if s is a singular point of \mathbb{C}^m/G and $s \neq 0$, then an open neighbourhood of s in \mathbb{C}^m/G is isomorphic to an open neighbourhood of $(0, 0)$ in $\mathbb{C}^k \times \mathbb{C}^{m-k}/H$, where $0 < k < m$ and H is a finite subgroup of $U(m-k)$, and also of G . Thus, away from zero the singularities of \mathbb{C}^m/G look locally like *products* $\mathbb{C}^k \times \mathbb{C}^{m-k}/H$ for $k > 0$.

In §1 we will define a special class of resolutions X of \mathbb{C}^m/G called *local*

product resolutions, which have the property that if \mathbb{C}^m/G is locally modelled on $\mathbb{C}^k \times \mathbb{C}^{m-k}/H$ then X is locally modelled on $\mathbb{C}^k \times Y$, where Y is a resolution of \mathbb{C}^{m-k}/H . Crepant resolutions are automatically of this form. If X is a local product resolution, we want to impose some suitable asymptotic conditions ‘near infinity’ on Kähler metrics g on X .

In §2 we define g_X to be a *QALE Kähler metric* on X if g_X is asymptotic to $h_{\mathbb{C}^k} \times g_Y$ on the part of X modelled on $\mathbb{C}^k \times Y$, where $h_{\mathbb{C}^k}$ is the Euclidean metric on \mathbb{C}^k , and g_Y is a metric on Y , the resolution of \mathbb{C}^{m-k}/H . So g_X converges to $h_{\mathbb{C}^k} \times g_Y$ at infinity. Now the points of \mathbb{C}^m/G will be of several different kinds, modelled on $\mathbb{C}^k \times \mathbb{C}^{m-k}/H$ for different $0 \leq k \leq m$ and subgroups $H \subseteq G$. Thus we impose not one but many asymptotic conditions on g_X , which must all be satisfied for g_X to be QALE.

Section 3 discusses *Ricci-flat QALE Kähler manifolds*. We state the main result of the paper, Theorem 3.3, an existence result for Ricci-flat QALE metrics on crepant resolutions X of \mathbb{C}^m/G . Its proof follows that of Theorem 3.3 of [11], and takes up most of sections 4-7. We first develop the appropriate ideas of weighted Hölder spaces and elliptic regularity on QALE manifolds. Then we state (without proof) two versions of the Calabi conjecture on QALE manifolds, and apply them to construct Ricci-flat QALE Kähler metrics on X .

The original motivation for this paper and [11] is that ALE and QALE metrics with holonomy $SU(2)$, $SU(3)$, $SU(4)$ and $Sp(2)$ are essential ingredients in a new construction by the author of compact manifolds with the exceptional holonomy groups G_2 and $Spin(7)$, which generalizes that of [7, 8]. This construction will be described at length in the author’s forthcoming book [10].

For simplicity we restrict our attention in this paper to *resolutions* of \mathbb{C}^m/G . But in [10, §9.9] we will give a more general definition of QALE Kähler manifolds, in which the underlying complex manifold is allowed to be a *deformation* of \mathbb{C}^m/G , or a resolution of a deformation. We show that our main result, Theorem 3.3, also holds in this more general context.

The book will also contain the results of this paper and [11], and many other results on QALE manifolds, including the proofs of the QALE Calabi conjectures stated in §6, and existence results for QALE manifolds with holonomy G_2 and $Spin(7)$. The author does not know of any previous papers on QALE manifolds at all, at the time of writing; this may be the first paper on the subject.

1 Local product resolutions

In this section we will study the structure of *nonisolated quotient singularities* \mathbb{C}^m/G and their resolutions. Here a *resolution* (X, π) of \mathbb{C}^m/G is a normal nonsingular variety X with a proper birational morphism $\pi : X \rightarrow \mathbb{C}^m/G$. We shall define a special kind of resolution of \mathbb{C}^m/G called a *local product resolution*. Our main goal is to set up a lot of notation describing resolutions of \mathbb{C}^m/G , that we will use in the rest of the paper.

Let G be a finite subgroup of $U(m)$. If A is a subgroup of G and V a subspace of \mathbb{C}^m , define the *fixed point set* $\text{Fix}(A)$, the *centralizer* $C(V)$ and the *normalizer* $N(V)$ by

$$\begin{aligned} \text{Fix}(A) &= \{x \in \mathbb{C}^m : ax = x \text{ for all } a \in A\}, \\ C(V) &= \{g \in G : gv = v \text{ for all } v \in V\} \\ \text{and } N(V) &= \{g \in G : gV = V\}. \end{aligned} \tag{1}$$

Then $C(V)$ and $N(V)$ are subgroups of G , and $C(V)$ is a normal subgroup of $N(V)$.

Definition 1.1 Define a finite set \mathcal{L} of linear subspaces of \mathbb{C}^m by

$$\mathcal{L} = \{\text{Fix}(A) : A \text{ is a subgroup of } G\}. \tag{2}$$

Let I be an *indexing set* for \mathcal{L} , so that we may write $\mathcal{L} = \{V_i : i \in I\}$. Let the indices of $\text{Fix}(\{1\})$ and $\text{Fix}(G)$ be $0, \infty$ respectively, so that $0, \infty \in I$ and $V_0 = \mathbb{C}^m$, $V_\infty = \text{Fix}(G)$ by definition. Usually $V_\infty = \{0\}$.

Define a *partial order* \succeq on I by $i \succeq j$ if $V_i \subseteq V_j$. Then $\infty \succeq i \succeq 0$ for all $i \in I$. Let W_i be the perpendicular subspace to V_i in \mathbb{C}^m , so that $\mathbb{C}^m = V_i \oplus W_i$. Define $A_i = C(V_i)$. Then $V_i = \text{Fix}(A_i)$ and A_i acts on W_i , with $\mathbb{C}^m/A_i \cong V_i \times W_i/A_i$. If $i \succeq j$ then $W_i \supseteq W_j$ and $A_i \supseteq A_j$. Define B_i to be the quotient group $N(V_i)/C(V_i)$. Then B_i acts naturally on V_i and W_i/A_i , and $(V_i \times W_i/A_i)/B_i \cong \mathbb{C}^m/N(V_i)$. Hence, if $N(V_i) = G$ then $(V_i \times W_i/A_i)/B_i \cong \mathbb{C}^m/G$.

Let $V_i, V_j \in \mathcal{L}$, let A be the subgroup of G generated by A_i and A_j , and set $V = \text{Fix}(A)$. It is easy to show that $V = V_i \cap V_j$. Thus $V_i \cap V_j \in \mathcal{L}$, and \mathcal{L} is closed under intersection of subspaces. Also, if $g \in G$ and $V_i \in \mathcal{L}$ then $gV_i \in \mathcal{L}$, since $gV_i = \text{Fix}(gA_i g^{-1})$. For each $g \in G$ and $i \in I$, let $g \cdot i$ be the unique element of I such that $V_{g \cdot i} = gV_i$. This defines an action of G on I , which satisfies $W_{g \cdot i} = gW_i$ and $A_{g \cdot i} = gA_i g^{-1}$.

Let $v \in \mathbb{C}^m$. Then vG is a singular point of \mathbb{C}^m/G if and only if the subgroup of G fixing v is nontrivial, that is, if $v \in \text{Fix}(A)$ for some nontrivial subgroup $A \subset G$. Thus vG is a singular point if and only if $v \in V_i$ for some $i \in I$ with $i \neq 0$, and the *singular set* S of \mathbb{C}^m/G is

$$S = \bigcup_{i \in I \setminus \{0\}} V_i / G. \quad (3)$$

For generic points $v \in V_i$, the subgroup of G fixing v is A_i , and the singularity of \mathbb{C}^m/G at vG is locally modelled on the product $V_i \times W_i/A_i$.

Definition 1.2 Let (X, π) be a resolution of \mathbb{C}^m/G . We say that (X, π) is a *local product resolution* if for each $i \in I$ there exists a resolution (Y_i, π_i) of W_i/A_i such that the following conditions hold. Let $R > 0$, and define subsets S_i and T_i in $V_i \times W_i/A_i \cong \mathbb{C}^m/A_i$ by

$$S_i = \bigcup_{j \in I: i \neq j} V_j / A_i, \quad (4)$$

$$\begin{aligned} T_i &= \{x \in \mathbb{C}^m/A_i : d(x, S_i) \leq R\} \\ &= \{x \in \mathbb{C}^m : d(x, V_j) \leq R \text{ for some } j \in I \text{ with } i \neq j\} / A_i, \end{aligned} \quad (5)$$

where $d(\cdot, \cdot)$ is the distance in \mathbb{C}^m/A_i or \mathbb{C}^m . Let U_i be the pull-back of T_i to $V_i \times Y_i$ under $\text{id} \times \pi_i : V_i \times Y_i \rightarrow V_i \times W_i/A_i$. Let ϕ_i be the natural projection from $V_i \times W_i/A_i$ to \mathbb{C}^m/G . Then there should exist a map $\psi_i : V_i \times Y_i \setminus U_i \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} V_i \times Y_i \setminus U_i & \xrightarrow{\psi_i} & X \\ \downarrow \text{id} \times \pi_i & & \downarrow \pi \\ V_i \times W_i/A_i \setminus T_i & \xrightarrow{\phi_i} & \mathbb{C}^m/G. \end{array} \quad (6)$$

This ψ_i should be a local isomorphism of complex manifolds, and whenever $x \in X$ and $y \in V_i \times W_i/A_i \setminus T_i$ satisfy $\pi(x) = \phi_i(y)$, there should exist a unique $z \in V_i \times Y_i \setminus U_i$ with $x = \psi_i(z)$ and $y = (\text{id} \times \pi_i)(z)$.

Here is an equivalent way to write this. Define \tilde{X}_i by

$$\tilde{X}_i = \left\{ (x, y) \in X \times (V_i \times W_i/A_i \setminus T_i) : \pi(x) = \phi_i(y) \right\}.$$

For any resolution X of \mathbb{C}^m/G , one can show that \tilde{X}_i is a well-defined complex manifold, and the projection to X is a local isomorphism. We say that X is a local product resolution if \tilde{X}_i is isomorphic to $V_i \times Y_i \setminus U_i$ in such a way that the natural projections to $V_i \times W_i/A_i \setminus T_i$ agree.

The idea of the definition is that every point in \mathbb{C}^m/G has an open neighbourhood isomorphic to an open neighbourhood of 0 in $V_i \times W_i/A_i$ for some $i \in I$. (For nonsingular points we use $V_0 \times W_0/A_0 = \mathbb{C}^m$.) A resolution X of \mathbb{C}^m/G is a local product resolution if and only if the resolution of each singular point modelled on $V_i \times W_i/A_i$ is locally isomorphic to $V_i \times Y_i$.

Local product resolutions are a very large class of resolutions of \mathbb{C}^m/G . There do exist (rather artificial) examples of resolutions which are not local product resolutions, but in practice they include all interesting resolutions of \mathbb{C}^m/G , in particular crepant resolutions, as we shall see in §3.

Proposition 1.3 *Let (X, π) be a local product resolution of \mathbb{C}^m/G . Then for each $g \in G$ and $i \in I$ there is a unique isomorphism $\chi_{g,i} : Y_i \rightarrow Y_{g \cdot i}$ making a commutative diagram*

$$\begin{array}{ccc} Y_i & \xrightarrow{\chi_{g,i}} & Y_{g \cdot i} \\ \downarrow \pi_i & & \downarrow \pi_{g \cdot i} \\ W_i/A_i & \xrightarrow{g} & W_{g \cdot i}/A_{g \cdot i}, \end{array} \quad (7)$$

where the map $W_i/A_i \xrightarrow{g} W_{g \cdot i}/A_{g \cdot i}$ is given by $g(w A_i) = (g w) A_{g \cdot i}$.

Proof. If v is a generic point of V_i then \mathbb{C}^m/G is locally isomorphic to $V_i \times W_i/A_i$ near vG , and X is locally isomorphic to $V_i \times Y_i$ near $\pi^{-1}(vG)$, as X is a local product resolution. But if $g \in G$ then $vG = (gv)G$, and gv is a generic point of $V_{g \cdot i}$, so that \mathbb{C}^m/G is also locally isomorphic to $V_{g \cdot i} \times W_{g \cdot i}/A_{g \cdot i}$ near vG , and X is locally isomorphic to $V_{g \cdot i} \times Y_{g \cdot i}$ near $\pi^{-1}(vG)$. Hence $V_i \times Y_i$ and $V_{g \cdot i} \times Y_{g \cdot i}$ are locally isomorphic. In fact they are globally isomorphic, and this gives an isomorphism $\chi_{g,i} : Y_i \rightarrow Y_{g \cdot i}$, which makes (7) commutative. \square

If $g \in N(V_i)$ then $g \cdot i = i$, so that $\chi_{g,i}$ is an automorphism of Y_i . Also, if $g \in A_i$ then $\chi_{g,i}$ is the identity on Y_i . Thus $N(V_i)$ acts on Y_i , and the normal subgroup A_i of $N(V_i)$ acts trivially on Y_i . Therefore the action descends to an action of the quotient group $B_i = N(V_i)/A_i$ on Y_i . That is, the action of

B_i on W_i/A_i must lift to an action of B_i on the resolution Y_i on W_i/A_i . So each resolution Y_i in a local product resolution must be B_i -equivariant.

We can use this to give a more thorough explanation of the idea of local product resolution, under a simplifying assumption. Suppose that $N(V_i) = G$ for each $i \in I$. This happens if G is abelian, and in other cases too. Then we have $(V_i \times W_i/A_i)/B_i \cong \mathbb{C}^m/G$ for $i \in I$, from Definition 1.1. Since B_i acts on V_i and Y_i we can take the quotient $(V_i \times Y_i)/B_i$, which is a complex orbifold with a natural projection to $(V_i \times W_i/A_i)/B_i \cong \mathbb{C}^m/G$.

Effectively, $(V_i \times Y_i)/B_i$ is a *partial resolution* of \mathbb{C}^m/G , which resolves the singularities of \mathbb{C}^m/G due to fixed points of elements of A_i , but leaves unresolved those caused by fixed points of elements of $G \setminus A_i$. Now S_i is exactly the subset of $V_i \times W_i/A_i$ fixed by some $b \neq 1$ in B_i , and therefore S_i/B_i is the set of singularities of \mathbb{C}^m/G due to fixed points of $G \setminus A_i$. Also T_i is the subset of $V_i \times W_i/A_i$ within distance R of S_i , so T_i/B_i is the subset of \mathbb{C}^m/G within distance R of singularities due to fixed points of $G \setminus A_i$.

All the fixed points of the B_i -action on $V_i \times Y_i$ lie in $(\text{id} \times \pi_i)^{-1}(S_i)$, which is in the interior of U_i . Thus B_i acts freely on $V_i \times Y_i \setminus U_i$, and $(V_i \times Y_i \setminus U_i)/B_i$ is nonsingular. In fact $\psi_i : (V_i \times Y_i \setminus U_i)/B_i \rightarrow X$ is an *isomorphism* between $(V_i \times Y_i \setminus U_i)/B_i$ and $X \setminus \pi^{-1}(T_i/B_i)$. Thus, the definition requires that the resolution X of \mathbb{C}^m/G has to coincide with the partial resolution $(V_i \times Y_i)/B_i$ of \mathbb{C}^m/G outside the set U_i/B_i which contains the singularities of $(V_i \times Y_i)/B_i$.

Our next result shows that local product resolutions are built out of local product resolutions of smaller dimension.

Proposition 1.4 *Suppose X is a local product resolution of \mathbb{C}^m/G . Then each of the resolutions Y_i of W_i/A_i in Definition 1.2 is also a local product resolution.*

Proof. Let $i \in I$ be fixed throughout the proof. Above we defined a lot of notation such as \mathcal{L}, I, V_j, W_j and so on, associated to \mathbb{C}^m/G . The corresponding data associated to W_i/A_i will be written $\mathcal{L}', I', V'_j, W'_j$, etc., in the obvious way. We will express the data for W_i/A_i in terms of that for \mathbb{C}^m/G . Define the indexing set I' by $I' = \{j \in I : i \succeq j\}$, and for each $j \in I'$ set $V'_j = V_j \cap W_i$. Then $\mathcal{L}' = \{V'_j : j \in I'\} = \{\text{Fix}(A) : A \text{ is a subgroup of } A_i\}$ is a finite set of subspaces of W_i . The two special elements of I' are $0' = 0$ and $\infty' = i$. Also W'_j, A'_j and Y'_j are the same as W_j, A_j and Y_j for each $j \in I'$.

Let ϕ'_j, T'_j and U'_j be as in Definition 1.2. It can be shown that there is a unique map $\psi'_j : V'_j \times Y'_j \setminus U'_j \rightarrow Y_i$, such that the product $\text{id} \times \psi'_j$ of ψ'_j with

the identity on V_i makes the following picture into a commutative diagram:

$$\begin{array}{ccc}
V_i \times V'_j \times Y'_j \setminus U_j & \xrightarrow{\text{id} \times \psi'_j} & V_i \times Y_i \setminus U_i \\
\downarrow \cong & & \downarrow \psi_i \\
V_j \times Y_j \setminus U_j & \xrightarrow{\psi_j} & X.
\end{array} \tag{8}$$

It easily follows that Y_i is a local product resolution of W_i/A_i . \square

2 Quasi-ALE Kähler metrics

We will now define a class of Kähler metrics on local product resolutions of \mathbb{C}^m/G called *Quasi Asymptotically Locally Euclidean*, or *Quasi-ALE* or *QALE* for short, which generalize the ALE Kähler metrics considered in [11]. Let G be a finite subgroup of $U(m)$, let X be a local product resolution of \mathbb{C}^m/G , and let all notation be as in §1.

Definition 2.1 For each pair $i, j \in I$, define $\mu_{i,j} : V_i \times Y_i \rightarrow [0, \infty)$ by $\mu_{i,j}(z) = d((\text{id} \times \pi_i)(z), V_j A_i / A_i)$, where $V_j A_i / A_i = \{v A_i : v \in V_j\}$, as a subset of \mathbb{C}^m / A_i , and $d(y, T)$ is the shortest distance between the point y and the subset T in \mathbb{C}^m / A_i . For each $i \in I$, define $\nu_i : V_i \times Y_i \rightarrow [1, \infty)$ by $\nu_i(z) = 1 + \min\{\mu_{i,j}(z) : j \in I, j \neq i\}$. Then $\mu_{i,j}$ and ν_i are both continuous functions on $V_i \times Y_i$. For each $i \in I$ define $d_i = 2 - 2 \dim W_i = 2 - 2m + 2 \dim V_i$, and let h_i be the Euclidean metric on V_i .

Let g be a Kähler metric on X . We say g is *Quasi-ALE* if the complex codimension n of the singular set S of \mathbb{C}^m/G , given by $n = \min\{\dim W_i : i \in I, i \neq 0\}$ satisfies $n \geq 2$, and for each $i \in I$ there is a Kähler metric g_i on Y_i , such that the metric $h_i \times g_i$ on $V_i \times Y_i$ satisfies

$$\nabla^l(\psi_i^*(g) - h_i \times g_i) = \sum_{j \in I: i \neq j} O(\mu_{i,j}^{d_j} \nu_i^{-2-l}) \tag{9}$$

on $V_i \times Y_i \setminus U_i$, for all $l \geq 0$. If X is a local product resolution of \mathbb{C}^m/G with complex structure J , and g is a QALE metric on X , then we say that (X, J, g) is a *QALE Kähler manifold asymptotic to \mathbb{C}^m/G* .

We now discuss this definition. The pull-back to $V_i \times Y_i$ of the singular set S of \mathbb{C}^m/G splits up into a number of pieces parametrized by $j \in I \setminus \{0\}$, and

$\mu_{i,j}$ is a measure of the distance to piece j . Similarly, ν_i measures the distance in $V_i \times Y_i$ to the pull-back of S , but we add 1 to this to avoid problems when ν_i is small. Also, by definition U_i is the subset of $V_i \times Y_i$ on which $\mu_{i,j} \leq R$ for some $j \in I$ with $i \neq j$. Thus $\mu_{i,j} > R > 0$ in (9), so there are no problems when $\mu_{i,j}$ is small.

Thus eqn (9) says that at large distances from U_i , the pull-back $\psi_i^*(g)$ of g to $V_i \times Y_i$ must approximate the product metric $h_i \times g_i$ on $V_i \times Y_i$. As in §1, we can explain this more clearly if we assume that $N(V_i) = G$. In this case $h_i \times g_i$ pushes down to an orbifold metric on $(V_i \times Y_i)/B_i$, and X is isomorphic to $(V_i \times Y_i)/B_i$ outside U_i/B_i . So (9) says that the metrics g and $h_i \times g_i$ on the isomorphic subsets of X and $(V_i \times Y_i)/B_i$ must agree asymptotically at large distances from U_i/B_i .

We assume that the singular set S of \mathbb{C}^m/G has codimension $n \geq 2$ because many of the results in the rest of the paper are false when $n = 1$. One reason for this is that if $\dim W_j = 1$ then $d_j = 2 - 2 \dim W_j = 0$, and so the ‘error term’ $O(\mu_{i,j}^{d_j} \nu_i^{-2-l})$ in (9) may not be small when $\mu_{i,j}$ is large. But some of our results rely on the errors being small at large distances.

Another reason is that the equation $\Delta u = f$ on \mathbb{C}^k behaves differently in the cases $k = 1$ and $k > 1$. When $k > 1$ and f is a smooth function on \mathbb{C}^k that decays rapidly at infinity, there is a unique smooth function u on \mathbb{C}^k with $\Delta u = f$ and $u = O(r^{2-2k})$ for large r . But when $k = 1$ this is false, and instead $u = O(\log r)$ and is not unique. Because of this, some of our results about the Laplacian on QALE manifolds are false when $n = 1$.

Consider what (9) means when $i = 0$ and $i = \infty$. Now $V_0 = \mathbb{C}^m$ and W_0 and Y_0 are both a single point, so $V_0 \times W_0 \cong \mathbb{C}^m$. The metric $h_0 \times g_0$ on $V_0 \times W_0$ is just the Euclidean metric h_0 on \mathbb{C}^m , and (9) says that the metrics g on X and h_0 on \mathbb{C}^m/G must be asymptotic at large distances from the singular set of \mathbb{C}^m/G .

When $i = \infty$ we have $U_\infty = \emptyset$, the map $\psi_\infty : V_\infty \times Y_\infty \rightarrow X$ is an isomorphism, and the r.h.s. of (9) is zero, which forces $\psi_\infty^*(g) = h_\infty \times g_\infty$. Using ψ_∞ to identify X and $V_\infty \times Y_\infty$, we see that $g = h_\infty \times g_\infty$, the product of a Euclidean metric on V_∞ and a metric g_∞ on Y_∞ . If $\text{Fix}(G) = \{0\}$ then $V_\infty = \{0\}$ and $Y_\infty = X$, and (9) holds trivially by taking $g_\infty = g$. It is often convenient to assume that $\text{Fix}(G) = \{0\}$.

In particular, suppose \mathbb{C}^m/G has an *isolated* singularity at 0. Then $I = \{0, \infty\}$ and $\text{Fix}(G) = \{0\}$, so (9) is trivial for $i = \infty$. We have $d_\infty = 2 - 2m$ and $\mu_{0,\infty} = r$, the radius function on \mathbb{C}^m , and $\nu_0 = 1 + r$, so when $i = 0$ eqn

(9) becomes

$$\nabla^l(\psi_0^*(g) - h_0) = O(r^{2-2m}(1+r)^{-2-l}) \quad (10)$$

wherever $r > R$ on \mathbb{C}^m , for all $l \geq 0$. But this is equivalent to the equation [11, eqn (3)] defining ALE Kähler metrics. So we have proved:

Lemma 2.2 *Suppose \mathbb{C}^m/G has an isolated singularity at 0. Then QALE Kähler metrics on X are the same thing as ALE Kähler metrics on X , in the sense of [11, Def. 2.3].*

The idea used to prove Proposition 1.3 also yields the following result.

Proposition 2.3 *In the situation of Definition 2.1, the metrics g_i satisfy $\chi_{\gamma,i}^*(g_{\gamma \cdot i}) = g_i$ for each $\gamma \in G$ and $i \in I$. Thus g_i is invariant under the natural action of B_i on Y_i .*

Next we show that QALE metrics are made out of other QALE metrics of lower dimension.

Proposition 2.4 *Let g_i be the Kähler metric on the resolution Y_i of W_i/A_i in Definition 2.1. Then g_i is also a QALE metric.*

Proof. We shall use the notation defined in the proof of Proposition 1.4. In addition, let h'_j be the Euclidean metric on $V'_j = V_j \cap W_i$. Let $j \in I'$, so that $i \succeq j$ and $V_j = V_i \times V'_j$. Writing the Euclidean metric h_j on V_j as $h_i \times h'_j$, eqn (9) with i replaced by j becomes

$$\nabla^l(\psi_j^*(g) - h_i \times h'_j \times g_j) = \sum_{k \in I: j \not\succeq k} O(\mu_{j,k}^{d_k} \nu_j^{-2-l}) \quad (11)$$

on $V_i \times V'_j \times Y_j \setminus U_j$. Using $\text{id} \times \psi'_j$ to pull eqn (9) from $V_i \times Y_i \setminus U_i$ back to $V_i \times V'_j \times Y_j \setminus U_j$, and substituting $(\text{id} \times \psi'_j)^*(\psi_i^*(g)) = \psi_j^*(g)$ since $\psi_i \circ (\text{id} \times \psi'_j) = \psi_j$ by (8), gives

$$\nabla^l(\psi_j^*(g) - h_i \times (\psi'_j)^*(g_i)) = \sum_{k \in I: i \not\succeq k} O(\mu_{j,k}^{d_k} \nu_j^{-2-l}) \quad (12)$$

on $V_i \times V'_j \times Y_j \setminus U_j$. Hence, subtracting (11) and (12) gives

$$\nabla^l(h_i \times (\psi'_j)^*(g_j) - h_i \times h'_j \times g_j) = \sum_{k \in I: j \nprec k} O(\mu_{j,k}^{d_k} \nu_j^{-2-l}).$$

By restricting to $\{v\} \times V'_j \times Y_j$ for $v \in V_i$ and taking v to be large, we show that

$$\nabla^l((\psi'_j)^*(g_i) - h'_j \times g_j) = \sum_{k \in I': j \nprec k} O((\mu'_{j,k})^{d_k} (\nu'_j)^{-2-l})$$

on $V'_j \times Y_j \setminus U'_j$, for all $l \geq 0$. Thus g_i is a QALE Kähler metric on Y_i . \square

So local product resolutions and QALE Kähler metrics are built up by a kind of *induction on dimension*. If X is a local product resolution with a QALE Kähler metric, then the Y_i that appear as limits of X are also local product resolutions with QALE metrics, but in a lower dimension. This suggests a method of proving results about QALE metrics: we assume the result is true for all Y_i with $\dim Y_i < \dim X$, and prove it for X . Then the result holds for all QALE metrics, by induction on $\dim X$. We will use this idea several times later on.

Here is the definition of Kähler classes and the Kähler cone on QALE manifolds.

Definition 2.5 Let X be a local product resolution of \mathbb{C}^m/G , and g a QALE Kähler metric on X with Kähler form ω . The de Rham cohomology class $[\omega] \in H^2(X, \mathbb{R})$ is called the *Kähler class* of g . Define the *Kähler cone* \mathcal{K} of X to be the set of Kähler classes $[\omega] \in H^2(X, \mathbb{R})$ of QALE Kähler metrics on X .

By studying the de Rham cohomology of a QALE Kähler manifold one can show that \mathcal{K} is an open convex cone in $H^2(X, \mathbb{R})$, not containing zero.

3 Ricci-flat QALE Kähler manifolds

A resolution (X, π) of \mathbb{C}^m/G with first Chern class $c_1(X) = 0$ is called a *crepant resolution*, as in Reid [13]. A great deal is known about the algebraic geometry of crepant resolutions, especially in dimensions 2 and 3. In particular, if \mathbb{C}^m/G has a crepant resolution then $G \subset \mathrm{SU}(m)$, and conversely when $m = 2, 3$ a crepant resolution of \mathbb{C}^m/G exists for every finite subgroup

G of $\mathrm{SU}(m)$. If G is abelian then all the crepant resolutions of \mathbb{C}^m/G can be understood explicitly using toric geometry.

Since any Ricci-flat Kähler manifold X has $c_1(X) = 0$, we deduce:

Proposition 3.1 *Suppose (X, J, g) is a Ricci-flat QALE Kähler manifold asymptotic to \mathbb{C}^m/G . Then $G \subset \mathrm{SU}(m)$ and X is a crepant resolution of \mathbb{C}^m/G .*

We shall construct Ricci-flat QALE Kähler manifolds by starting with a crepant resolution X of \mathbb{C}^m/G , and constructing a suitable Kähler metric on X . This is why we are interested in crepant resolutions. The proof of the following proposition is fairly straightforward if you know enough algebraic geometry, and we omit it.

Proposition 3.2 *Suppose X is a crepant resolution of \mathbb{C}^m/G . Then X is a local product resolution. Moreover, each resolution Y_i of W_i/A_i is also a crepant resolution.*

Here is our main result, which generalizes Theorem 3.3 of [11] to the QALE case. Its proof works by first proving a version of the Calabi conjecture for QALE manifolds.

Theorem 3.3 *Let G be a finite subgroup of $\mathrm{SU}(m)$, and X a crepant resolution of \mathbb{C}^m/G . Then each Kähler class of QALE metrics on X contains a unique Ricci-flat QALE Kähler metric.*

If X is a crepant resolution of \mathbb{C}^m/G then it is a local product resolution, by the previous proposition. Also, since $G \subset \mathrm{SU}(m)$ the singular set of \mathbb{C}^m/G has codimension $n \geq 2$, which was one of the conditions for X to admit QALE metrics in Definition 2.1. In practice this means that any Kähler crepant resolution of \mathbb{C}^m/G also admits QALE Kähler metrics, and so has a family of Ricci-flat QALE Kähler metrics by Theorem 3.3.

We call a quotient singularity \mathbb{C}^m/G *reducible* if we can write $\mathbb{C}^m/G = \mathbb{C}^{m_1}/G_1 \times \mathbb{C}^{m_2}/G_2$, where $G_j \subset \mathrm{U}(m_j)$ and $m_1, m_2 > 0$ satisfy $m_1 + m_2 = m$. Otherwise we say \mathbb{C}^m/G is *irreducible*. If \mathbb{C}^m/G is reducible and (X, J, g) is a Ricci-flat QALE Kähler manifold asymptotic to \mathbb{C}^m/G , then it is easy to see that (X, J, g) is a product of lower-dimensional Ricci-flat QALE Kähler manifolds. So we shall restrict our attention to irreducible \mathbb{C}^m/G . In this case we can show:

Theorem 3.4 *Let (X, J, g) be a Ricci-flat QALE Kähler manifold asymptotic to \mathbb{C}^m/G , where \mathbb{C}^m/G is irreducible. If $m \geq 4$ is even and G is conjugate to a subgroup of $\mathrm{Sp}(m/2)$ then $\mathrm{Hol}(g) = \mathrm{Sp}(m/2)$, and otherwise $\mathrm{Hol}(g) = \mathrm{SU}(m)$.*

The proofs of Theorems 3.3 and 3.4 will be given in §7.

3.1 Examples of QALE manifolds with holonomy $\mathrm{SU}(m)$

Here are three examples of QALE manifolds with holonomy $\mathrm{SU}(m)$, for $m = 3$ and 4.

Example 3.5 Define $\alpha : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ by

$$\alpha : (z_1, z_2, z_3) \mapsto (-z_1, iz_2, iz_3).$$

Then $\langle \alpha \rangle$ is a subgroup of $\mathrm{SU}(3)$ isomorphic to \mathbb{Z}_4 . Let I be $\{0, 1, \infty\}$, and set

$$V_0 = \mathbb{C}^3, \quad V_1 = \{(z_1, 0, 0) : z_1 \in \mathbb{C}\} \quad \text{and} \quad V_\infty = \{0\}.$$

Then $\mathcal{L} = \{V_i : i \in I\}$, and the groups A_i are

$$A_0 = \{1\}, \quad A_1 = \{1, \alpha^2\}, \quad \text{and} \quad A_\infty = \{1, \alpha, \alpha^2, \alpha^3\} = \mathbb{Z}_4.$$

The quotient $\mathbb{C}^3/\mathbb{Z}_4$ has a unique crepant resolution X , with $b^2(X) = 2$. It can be constructed using toric geometry, and admits QALE Kähler metrics. Thus X has a 2-parameter family of Ricci-flat QALE Kähler metrics g by Theorem 3.3, which have holonomy $\mathrm{SU}(3)$ by Theorem 3.4.

We can describe the asymptotic behaviour of these Ricci-flat metrics g quite simply. The vector space W_1 is $\{(0, z_2, z_3) : z_2, z_3 \in \mathbb{C}\}$, and $\alpha^2 \in A_1$ acts as -1 on W_1 . Thus $W_1/A_1 \cong \mathbb{C}^2/\{\pm 1\}$. Let Y_1 be the blow-up of $\mathbb{C}^2/\{\pm 1\}$ at 0. Then Y_1 carries a 1-parameter family of ALE metrics with holonomy $\mathrm{SU}(2)$, given explicitly by Eguchi and Hanson [5]. Let g_1 be an Eguchi-Hanson metric on Y_1 , and h_1 the Euclidean metric on $V_1 \cong \mathbb{C}$. Then $h_1 \times g_1$ is a Ricci-flat metric on $V_1 \times Y_1$.

Now α acts on $V_1 \times Y_1$ with $\alpha^2 = 1$, and the fixed points of α are a copy of \mathbb{CP}^1 in $V_1 \times Y_1$. Thus $(V_1 \times Y_1)/\langle \alpha \rangle$ is a complex orbifold, and X is its unique crepant resolution. The metric $h_1 \times g_1$ is preserved by α and

descends to $(V_1 \times Y_1)/\langle \alpha \rangle$. Identifying X with $(V_1 \times Y_1)/\langle \alpha \rangle$ outside $\pi^{-1}(0)$, the asymptotic conditions (9) on our QALE metric g become

$$\nabla^l(g - h_1 \times g_1) = O(\pi^*(r)^{-4}(1 + \pi^*(s))^{-2-l}),$$

where $r, s : \mathbb{C}^3/\mathbb{Z}_4 \rightarrow [0, \infty)$ are the distances in $\mathbb{C}^3/\mathbb{Z}_4$ to 0 and to the singular set $S = \{\pm(z_1, 0, 0) : z_1 \in \mathbb{C}\}$ respectively. Thus at infinity g is asymptotic to $h_1 \times g_1$, a metric we can write down explicitly in coordinates.

Example 3.6 Define $\alpha, \beta : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ by

$$\alpha : (z_1, z_2, z_3) \mapsto (z_1, -z_2, -z_3), \quad \beta : (z_1, z_2, z_3) \mapsto (-z_1, z_2, -z_3).$$

Then $\langle \alpha, \beta \rangle$ is a subgroup of $SU(3)$ isomorphic to \mathbb{Z}_2^2 . Let I be $\{0, 1, 2, 3, \infty\}$, and set

$$\begin{aligned} V_0 &= \mathbb{C}^3, & V_\infty &= \{0\}, & V_1 &= \{(z_1, 0, 0) : z_1 \in \mathbb{C}\}, \\ V_2 &= \{(0, z_2, 0) : z_2 \in \mathbb{C}\} & \text{and} & & V_3 &= \{(0, 0, z_3) : z_3 \in \mathbb{C}\}. \end{aligned}$$

Then $\mathcal{L} = \{V_i : i \in I\}$, and the groups A_i are

$$A_0 = \{1\}, \quad A_1 = \{1, \alpha\}, \quad A_2 = \{1, \beta\}, \quad A_3 = \{1, \alpha\beta\}, \quad A_\infty = \mathbb{Z}_2^2.$$

The quotient $\mathbb{C}^3/\mathbb{Z}_2^2$ has four distinct crepant resolutions X_1, \dots, X_4 , with $b^2(X_j) = 3$. They can be constructed explicitly using toric geometry, and all admit QALE Kähler metrics. Thus X_1, \dots, X_4 carry 3-parameter families of Ricci-flat QALE Kähler metrics by Theorem 3.3, which have holonomy $SU(3)$ by Theorem 3.4.

Example 3.7 Define $\alpha, \beta, \gamma : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ by

$$\begin{aligned} \alpha : (z_1, \dots, z_4) &\mapsto (-z_1, -z_2, z_3, z_4), \\ \beta : (z_1, \dots, z_4) &\mapsto (z_1, -z_2, -z_3, z_4) \\ \text{and} \quad \gamma : (z_1, \dots, z_4) &\mapsto (z_1, z_2, -z_3, -z_4). \end{aligned}$$

Then $\langle \alpha, \beta, \gamma \rangle \cong \mathbb{Z}_2^3$ is a subgroup of $SU(4)$. Let I be $\{0, \infty\} \cup \{jk : j, k = 1, \dots, 4, j < k\}$, an 8-element set. Then $\mathcal{L} = \{V_i : i \in I\}$, where

$$V_0 = \mathbb{C}^4, \quad V_\infty = \{1\}, \quad V_{jk} = \{(z_1, \dots, z_4) \in \mathbb{C}^4 : z_j = z_k = 0\}.$$

The quotient $\mathbb{C}^4/\mathbb{Z}_2^3$ has 48 distinct crepant resolutions X_1, \dots, X_{48} , with $b^2(X_j) = 6$. They can be constructed explicitly using toric geometry, and all admit QALE Kähler metrics. Thus X_1, \dots, X_{48} carry 6-parameter families of Ricci-flat QALE Kähler metrics by Theorem 3.3, which have holonomy $SU(4)$ by Theorem 3.4.

In each of these examples we know by Theorem 3.3 that QALE metrics with holonomy $SU(m)$ exist on the resolutions X of \mathbb{C}^m/G , but we are unable to write these metrics down explicitly in coordinates. The author believes that QALE metrics with holonomy $SU(m)$ for $m \geq 3$ are not algebraic objects, and it is not possible to give an explicit formula for these metrics in coordinates.

3.2 Examples of QALE manifolds with holonomy $Sp(m)$

It can be shown [10, §9.3] that if X is a QALE manifold with holonomy $Sp(m)$ for $m \geq 2$ asymptotic to \mathbb{C}^{2m}/G , then G is nonabelian. So to find examples we must consider nonabelian groups.

Example 3.8 Define $\alpha, \beta : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ by

$$\begin{aligned}\alpha : (z_1, \dots, z_4) &\mapsto (e^{2\pi i/3} z_1, e^{4\pi i/3} z_2, e^{4\pi i/3} z_3, e^{2\pi i/3} z_4), \\ \beta : (z_1, \dots, z_4) &\mapsto (z_3, z_4, z_1, z_2).\end{aligned}$$

Then $G = \langle \alpha, \beta \rangle$ is a nonabelian subgroup of $Sp(2)$ of order 6 isomorphic to the symmetric group S_3 , that preserves the complex symplectic form $dz_1 \wedge dz_2 + dz_3 \wedge dz_4$.

Let S be the singular set of \mathbb{C}^4/G , and define a map $\phi : \mathbb{C}^4/G \setminus S \rightarrow \mathbb{CP}^4$ by

$$\begin{aligned}\phi((z_1, \dots, z_4)G) &= [z_1 z_2 - z_3 z_4, z_1^3 - z_3^3, \\ &\quad z_1^2 z_4 - z_2 z_3^2, z_1 z_4^2 - z_2^2 z_3, z_2^3 - z_4^3].\end{aligned}\tag{13}$$

The five polynomials in z_1, \dots, z_4 given here are invariant under α and change sign under β , and they are all zero if and only if $(z_1, z_2, z_3, z_4)G$ lies in S . Let X be the closure of the graph of ϕ in $\mathbb{C}^4/G \times \mathbb{CP}^4$, and $\pi : X \rightarrow \mathbb{C}^4/G$ the natural projection. Then a careful analysis shows that X is actually nonsingular, and (X, π) is a *crepant resolution* of \mathbb{C}^4/G .

Let $[x_0, x_1, x_2]$ be homogeneous coordinates on the *weighted projective space* $\mathbb{CP}_{3,1,1}^2$. Define a map $\psi : \mathbb{CP}_{3,1,1}^2 \rightarrow \mathbb{CP}^4$ by

$$\psi([x_0, x_1, x_2]) = [x_0, x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3].\tag{14}$$

Then ψ is injective, and it can be shown that $\pi^{-1}(0)$ is the subset $\{0\} \times \text{Im } \psi$ in $\mathbb{C}^4/G \times \mathbb{CP}^4$. Thus $\pi^{-1}(0)$ is isomorphic to the (singular) weighted projective space $\mathbb{CP}_{3,1,1}^2$. Since X retracts onto $\pi^{-1}(0)$, it follows that $b^2(X) = 1$, $b^4(X) = 1$ and $b^6(X) = 0$. Clearly X is a quasi-projective variety, and therefore Kähler. Theorem 3.3 shows that X has a 1-parameter family of Ricci-flat QALE Kähler metrics, which have holonomy $\text{Sp}(2)$ by Theorem 3.4.

In fact Example 3.8 generalizes to give an action of the symmetric group S_{m+1} on \mathbb{C}^{2m} , and a crepant resolution X_m of \mathbb{C}^{2m}/S_{m+1} carrying QALE Kähler metrics with holonomy $\text{Sp}(m)$. To prove this we adapt an idea of Beauville, and regard the *Hilbert scheme* or *Douady space* $\text{Hilb}^{m+1} \mathbb{C}^2$ of $m+1$ points in \mathbb{C}^2 as a crepant resolution of $\mathbb{C}^{2m+2}/S_{m+1}$. Further details are given in [3, §6-7].

Example 3.9 Let \mathbb{C}^4 have coordinates (z_1, z_2, z_3, z_4) . Let H be a subgroup of $\text{SU}(2)$. Then $H \times H$ acts on \mathbb{C}^4 , with the first H acting only on the coordinates (z_1, z_2) and the second H acting only on (z_3, z_4) . Define $\alpha : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ by $\alpha : (z_1, \dots, z_4) \mapsto (z_3, z_4, z_1, z_2)$. Let G be the subgroup of $\text{Sp}(2)$ generated by $H \times H$ and α . Then G is a semidirect product $\mathbb{Z}_2 \ltimes (H \times H)$. We can define a crepant resolution X of \mathbb{C}^4/G as follows.

Let Y be the unique crepant resolution of \mathbb{C}^2/H . Then $Y \times Y$ is a crepant resolution of $\mathbb{C}^4/(H \times H)$, and the action of α lifts to $Y \times Y$ by $\alpha : (y_1, y_2) \mapsto (y_2, y_1)$ in the obvious way. The singular set of the quotient $(Y \times Y)/\langle \alpha \rangle$ is the ‘diagonal’ $\Delta_Y = \{(y, y) : y \in Y\}$, and each singular point is modelled locally on $\mathbb{C}^2 \times (\mathbb{C}^2/\{\pm 1\})$. Let X be the blow-up of $(Y \times Y)/\langle \alpha \rangle$ along the diagonal Δ_Y . Then X is nonsingular, and is a crepant resolution of \mathbb{C}^4/G .

Since we understand Y very well, it is easy to compute the Betti numbers of X . For example, in the case $H = \mathbb{Z}_k$ we have $b^2(X) = k$, $b^4(X) = \frac{1}{2}(k+2)(k-1)$ and $b^6(X) = 0$. Again, each such resolution X has a family of Ricci-flat QALE Kähler metrics by Theorem 3.3, which have holonomy $\text{Sp}(2)$ by Theorem 3.4.

We claimed in §3.1 that QALE metrics with holonomy $\text{SU}(m)$ for $m \geq 3$ are nonalgebraic objects, and cannot be explicitly written down in coordinates. However, for QALE metrics with holonomy $\text{Sp}(m)$, the reverse is true. The author has a proof that *every* QALE metric with holonomy $\text{Sp}(m)$ has an algebraic description. This proof uses the theory of *hypercomplex algebraic geometry*, which is described in Joyce [9].

It seems likely that QALE manifolds with holonomy $\mathrm{Sp}(m)$ can be explicitly constructed using the hyperkähler quotient, as in Kronheimer's construction of ALE manifolds with holonomy $\mathrm{Sp}(1)$, [12]. However, at present the author has no proof of this, nor any explicit examples.

4 Kähler potentials on QALE Kähler manifolds

If g, g' are two QALE Kähler metrics in the same Kähler class on X , with Kähler forms ω, ω' , then we expect that $\omega' = \omega + \mathrm{dd}^c \phi$ for some function ϕ on X . (Here we use the notation that $\mathrm{d}^c \phi = i(\bar{\partial} - \partial)\phi$, so that d^c is a real operator with $\mathrm{dd}^c = 2i\partial\bar{\partial}$.) Conversely, if g is a QALE Kähler metric on X with Kähler form ω , then we can try to define other QALE metrics g' on X with Kähler forms $\omega' = \omega + \mathrm{dd}^c \phi$ for suitable functions ϕ on X . In this section we will study the properties of such functions ϕ in detail.

Definition 4.1 Let (X, J, g) be a QALE Kähler manifold asymptotic to \mathbb{C}^m/G , and use the notation of Definition 2.1. We say that a smooth real function ϕ on X is of *Kähler potential type* if for each $i \in I$ there exists a smooth real function ϕ_i on Y_i , with $\phi_0 = 0$, such that

$$\nabla^l(\psi_i^*(\phi) - \phi_i) = \sum_{j \in I: i \not\leq j} O(\mu_{i,j}^{d_j} \nu_i^{-l}) \quad (15)$$

on $V_i \times Y_i \setminus U_i$, for all $l \geq 0$. Here we identify ϕ_i with its pull-back to $V_i \times Y_i$.

By comparing (15) with (9), we immediately deduce:

Proposition 4.2 *Let (X, J, g) be a QALE Kähler manifold, ω the Kähler form of g , and ϕ a function of Kähler potential type on X . Suppose $\omega' = \omega + \mathrm{dd}^c \phi$ is a positive $(1, 1)$ -form on X . Then the Kähler metric g' on X with Kähler form ω' is QALE.*

Here is a converse to this proposition. It can be proved by combining the method of [11, Th. 5.4] with the results on the Laplacian on QALE manifolds that we will prove in §5, extended along the lines sketched in [10, §9.7]. We will not give a proof here.

Theorem 4.3 *Let X be a local product resolution of \mathbb{C}^m/G , and suppose that g, g' are QALE metrics on X in the same Kähler class, with Kähler forms ω, ω' . Then there is a unique function ϕ of Kähler potential type on X such that $\omega' = \omega + \text{dd}^c \phi$.*

Proposition 4.2 and Theorem 4.3 show that functions of Kähler potential type are the natural class of functions to use as Kähler potentials on QALE Kähler manifolds. The next result can be proved by following the proofs of Propositions 2.3 and 2.4.

Proposition 4.4 *Let (X, J, g) be a QALE Kähler manifold, and ϕ a function of Kähler potential type on X . Then the functions ϕ_i on Y_i introduced in Definition 4.1 satisfy the following conditions:*

- (i) *For each $\gamma \in G$ and $i \in I$ we have $\chi_{\gamma,i}^*(\phi_{\gamma \cdot i}) = \phi_i$. Hence ϕ_i is invariant under the action of B_i on Y_i .*
- (ii) *Whenever $i, j \in I$ with $i \succeq j$, then on $V_j' \times Y_j' \setminus U_j'$ we have*

$$\nabla^l((\psi_j')^*(\phi_i) - \phi_j) = \sum_{\substack{k \in I: \\ i \succeq k, j \not\succeq k}} O((\mu'_{j,k})^{d_k} (\nu_j')^{-l})$$

for all $l \geq 0$, using the notation of Propositions 1.4 and 2.4. Thus the functions ϕ_i on Y_i are also of Kähler potential type.

The proposition shows that (i) and (ii) are *necessary* conditions for a set of functions ϕ_i to be associated to a function ϕ of Kähler potential type. In fact they are also *sufficient*.

Theorem 4.5 *Let (X, J, g) be a QALE Kähler manifold, and suppose that for each $i \in I \setminus \{\infty\}$ there is a smooth function ϕ_i on Y_i , with $\phi_0 = 0$, such that the ϕ_i satisfy conditions (i) and (ii) of Proposition 4.4. Then there exists a function ϕ of Kähler potential type on X asymptotic to these ϕ_i for all $i \in I \setminus \{\infty\}$.*

Proof. Let $\eta : [0, \infty) \rightarrow [0, 1]$ be smooth with $\eta(x) = 0$ for $x \leq R$ and $\eta(x) = 1$ for $x \geq 2R$, where $R > 0$ is the constant in Definition 1.2. For each $i \in I \setminus \{\infty\}$, define a smooth function Φ_i on $V_i \times Y_i$ by

$$\Phi_i(v, y) = \phi_i(y) \cdot \prod_{j \in I: i \not\succeq j} \eta(\mu_{i,j}(v, y)). \quad (16)$$

The idea here is that $\Phi_i = \phi_i$ at distance at least $2R$ from the pull-back of S_i in $V_i \times Y_i$, that $\Phi_i = 0$ at distance no more than R from the pull-back of S_i , and that between distances R and $2R$ we join the two possibilities $\Phi_i = \phi_i$ and $\Phi_i = 0$ smoothly together using a partition of unity. Note that $\Phi_i \equiv 0$ in U_i .

For $i \in I \setminus \{\infty\}$, let k_i be integers satisfying

$$\sum_{i \in I \setminus \{\infty\}: i \succeq j} k_i = 1 \quad \text{for each } j \in I \setminus \{\infty\}. \quad (17)$$

It can be shown that these equations have a unique solution $\{k_i\}$. Now define

$$\phi(x) = \sum_{i \in I \setminus \{\infty\}} \frac{k_i |A_i|}{|G|} \sum_{\substack{(v,y) \in V_i \times Y_i \setminus U_i: \\ \psi_i(v,y)=x}} \Phi_i(v,y). \quad (18)$$

As $\Phi_i \equiv 0$ in U_i , we see that ϕ is smooth. It turns out that this ϕ is of Kähler potential type on X and asymptotic to ϕ_i for all $i \in I \setminus \{\infty\}$, so that it satisfies the conditions of the theorem. The proof of this is rather complicated, and we will not give it in full. Instead, we will explain the important points under the simplifying assumption that $N(V_i) = G$ for all $i \in I$.

In this case $B_i = G/A_i$, so $|B_i| = |G|/|A_i|$. Thus we may rewrite (18) as

$$\phi(x) = \sum_{i \in I \setminus \{\infty\}} k_i \Phi'_i(x), \quad \text{where} \quad \Phi'_i(x) = \frac{1}{|B_i|} \sum_{\substack{(v,y) \in V_i \times Y_i \setminus U_i: \\ \psi_i(v,y)=x}} \Phi_i(v,y). \quad (19)$$

Now B_i acts on Y_i and $V_i \times Y_i$, and ϕ_i is B_i -invariant by condition (i). Thus Φ_i is also B_i -invariant, and pushes down to a function on $(V_i \times Y_i)/B_i$. From §1, ψ_i induces an isomorphism between $(V_i \times Y_i \setminus U_i)/B_i$ and $X \setminus \pi^{-1}(T_i/B_i)$. In fact Φ'_i is the push-forward of Φ_i under this isomorphism. The factor $1/|B_i|$ in (19) ensures this, because each point of $X \setminus \pi^{-1}(T_i/B_i)$ pulls back to an orbit of B_i in $V_i \times Y_i$, consisting of $|B_i|$ points.

Thus, abusing notation a little, we can write $\phi = \sum_{i \neq \infty} k_i \Phi_i$, because the factor $|A_i|/|G|$ in (18) compensates for the fact that each generic point in X pulls back to $|G|/|A_i|$ points in $V_i \times Y_i$. But $\Phi_i = \phi_i$ away from U_i , and so away from $\pi^{-1}(S)$ in X we have $\phi = \sum_{i \neq \infty} k_i \phi_i$, again by an abuse of notation.

We must prove that ϕ satisfies (15). One way to interpret this is to say that X is divided roughly into overlapping regions corresponding to $j \in I$,

where in the j^{th} region X is locally isomorphic to $V_j \times Y_j$ and $\phi \approx \phi_j$. Now on the j^{th} region we have $\phi_i \approx \phi_j$ if $i \succeq j$ and $\phi_i \approx 0$ if $i \not\succeq j$, by condition (ii) of Proposition 4.2. Therefore on the j^{th} region we have

$$\phi \approx \sum_{i \in I \setminus \{\infty\}: i \succeq j} k_i \phi_j = \phi_j,$$

by (17), as we want. This idea can be used to show that (15) holds away from $\pi^{-1}(S)$ in X .

It remains to consider the parts of X near $\pi^{-1}(S)$. Within distance $2R$ of $\pi^{-1}(S)$ in X , the functions $\eta(\mu_{i,j})$ in (16) are not all equal to 1, and so we do not have $\Phi_i = \phi_i$ for all i . The reason for introducing the $\eta(\mu_{i,j})$ is that the push-forward of ϕ_i to $X \setminus \pi^{-1}(S_i/B_i)$ may not extend smoothly to X . So we modify ϕ_i to get Φ_i , whose push-forward is zero near $\pi^{-1}(S_i/B_i)$ and does extend smoothly to X . But we then have to make sure that the ‘errors’ due to the $\eta(\mu_{i,j})$ are within the bounds allowed by (15). One can show using (17) that this is always so, and the proof is complete. \square

The reason for excluding $i = \infty$ in this theorem is that in general $X = V_\infty \times Y_\infty$ and $\phi = \phi_\infty$, and so if we assumed that ϕ_∞ existed we could just take $\phi = \phi_\infty$ and there would be nothing to prove. Although the proof works by writing down ϕ explicitly, there are in fact many suitable ϕ , since if we add to ϕ any smooth function on Y_∞ with sufficiently fast decay at infinity, the resulting function also satisfies the theorem. The proof above will play an important part in the construction of Ricci-flat QALE metrics in §7, which is why we covered it in some detail.

Proposition 4.6 *In the situation of Theorem 4.5, let ω be the Kähler form of g and ω_i the Kähler form of g_i , and suppose $\omega_i + \text{dd}^c \phi_i$ is a positive $(1,1)$ -form on Y_i for each $i \in I \setminus \{\infty\}$. Then we can choose the function ϕ such that $\omega' = \omega + \text{dd}^c \phi$ is a positive $(1,1)$ -form on X . Thus the Kähler metric g' on X with Kähler form ω' is QALE, by Proposition 4.2.*

Proof. Let ϕ be as in the proof of Theorem 4.5. Then $\omega + \text{dd}^c \phi$ is positive outside a compact set in X , because at large distances from $\pi^{-1}(0)$ we have $\phi \approx \phi_i$ for some $i \neq \infty$, and so $\omega + \text{dd}^c \phi$ is positive because $\omega_i + \text{dd}^c \phi_i$ is positive. Let $\hat{R} > 0$, and choose a smooth function $\eta : X \rightarrow [0, 1]$ such that $\eta = 0$ at distance less than \hat{R} from $\pi^{-1}(0)$ in X , and $\eta = 1$ at distance more than $2\hat{R}$ from $\pi^{-1}(0)$ in X , and $\nabla \eta = O(\hat{R}^{-1})$, $\nabla^2 \eta = O(\hat{R}^{-2})$. For large \hat{R} it turns out that $\eta\phi$ satisfies the conditions of the proposition. \square

One moral of this proposition is that QALE Kähler metrics on X are actually very abundant. That is, the asymptotic conditions on QALE metrics are not so restrictive that they admit few solutions. Rather, given any set of Kähler metrics g_i on Y_i for $i \in I \setminus \{\infty\}$ satisfying the obvious consistency conditions, we expect to find many QALE Kähler metrics g on X asymptotic to these g_i .

Finally, we give an analogue of [11, Prop. 5.5] for QALE manifolds.

Theorem 4.7 *Let X be a local product resolution of \mathbb{C}^m/G that admits QALE Kähler metrics. Then in each Kähler class there exists a QALE Kähler metric g' on X such that for each $i \in I$ we have $\psi_i^*(g') = h_i \times g'_i$ on the set $V_i \times Y_i \setminus U_i$, where g'_i is a Kähler metric on Y_i , and U_i is defined as in Definition 1.2 using a constant $R > 0$, which may depend on the Kähler class.*

Taking $i = 0$ gives $g' = \pi^*(h_0)$ on $\{x \in X : d(\pi(x), S) > R\}$, where h_0 is the Euclidean metric, S the singular set and $d(\cdot, \cdot)$ the distance on \mathbb{C}^m/G . Thus g' is flat outside a fixed distance from the exceptional set of the resolution X . Also, g' is a product metric $h_i \times g'_i$ on the parts of X modelled on $V_i \times Y_i$, except within a fixed distance of some other component of the exceptional set.

Here is a sketch of the proof of this theorem; we leave the details as an exercise for the reader. The basic idea is to start with a QALE Kähler manifold (X, J, g) with Kähler form ω , and to find a function ϕ of Kähler potential type on X such that $\omega' = \omega + dd^c\phi$ is a positive (1,1)-form, and the QALE Kähler metric g' with Kähler form ω' satisfies the conditions of the theorem. We construct ϕ by induction on $m = \dim X$.

The inductive step of the proof works as follows. Assume by induction that such a ϕ exists when $\dim X < m$, and let (X, J, g) be QALE Kähler manifold asymptotic to \mathbb{C}^m/G . The inductive hypothesis shows that for each $i \neq \infty$ in I there is a function ϕ_i on Y_i such that $\omega'_i = \omega_i + dd^c\phi_i$ is positive on Y_i , and the corresponding Kähler metric g'_i on Y_i satisfies the theorem.

We can also arrange that these functions ϕ_i for $i \neq \infty$ satisfy conditions (i) and (ii) of Proposition 4.4. Then using the ideas of Theorem 4.5 and Proposition 4.6, we construct a function ϕ on X with the properties we need, that is asymptotic to ϕ_i on Y_i for each $i \neq \infty$. To prove such a ϕ exists requires the analytical results of §5 and also some discussion of the de Rham cohomology $H^2(X, \mathbb{R})$, along the lines of the proof of [11, Prop. 5.5].

5 Analysis on QALE Kähler manifolds

In our previous paper [11, §4] we described the *weighted Hölder spaces* $C_{\beta}^{k,\alpha}(X)$ over an ALE manifold X . These are Banach spaces of functions on X defined using powers of a *radius function* ρ on X . They are important tools in solving analysis problems on ALE manifolds, because elliptic operators such as the Laplacian Δ have good regularity properties on these spaces.

In this section we develop a similar theory of analysis on QALE manifolds. We shall define weighted Hölder spaces of functions on QALE Kähler manifolds X , and study the action of the Laplacian on them. We begin by defining functions ρ, σ on X which are analogues of the radius functions used in [11].

Definition 5.1 Let G be a finite subgroup of $U(m)$, and S the singular set of \mathbb{C}^m/G . Define continuous functions $r : \mathbb{C}^m/G \rightarrow [0, \infty)$ and $s : \mathbb{C}^m/G \rightarrow [0, \infty)$ by $r(x) = d(x, 0)$ and $s(x) = d(x, S)$, where $d(\cdot, \cdot)$ is the distance on \mathbb{C}^m/G . Let (X, π) be a local product resolution of \mathbb{C}^m/G , and g a QALE Kähler metric on X . We say that (ρ, σ) is a *pair of radius functions on X* if $\rho, \sigma : X \rightarrow [1, \infty)$ are smooth functions such that $\rho \geq \sigma$ and for some $K > 0$ we have

$$\pi^*(r) + 1 \leq \rho \leq \pi^*(r) + 2, \quad |\nabla \rho| \leq K, \quad \text{and} \quad |\nabla^2 \rho| \leq K\rho^{-1}, \quad (20)$$

$$\frac{1}{2}\pi^*(s) + 1 \leq \sigma \leq \pi^*(s) + 2, \quad |\nabla \sigma| \leq K, \quad \text{and} \quad |\nabla^2 \sigma| \leq K\sigma^{-1}. \quad (21)$$

A pair of radius functions (ρ, σ) exists for every QALE Kähler manifold (X, J, g) .

Here ρ and σ are smoothed versions of $\pi^*(r)$ and $\pi^*(s)$, adjusted to ensure that $\rho \geq 1$ and $\sigma \geq 1$. The reason why we have $\pi^*(r) + 1 \leq \rho$ but $\frac{1}{2}\pi^*(s) + 1 \leq \sigma$ is that s is a rather less smooth function on \mathbb{C}^m/G than r , and so to make a smoothed version σ with $|\nabla^2 \sigma|$ small we must allow a greater difference between σ and $\pi^*(s)$ than between ρ and $\pi^*(r)$.

The author believes that the following is a natural definition of weighted Hölder spaces on QALE manifolds.

Definition 5.2 Let (X, J, g) be a QALE Kähler manifold and (ρ, σ) a pair of radius functions on X . For $\beta, \gamma \in \mathbb{R}$ and k a nonnegative integer, define $C_{\beta, \gamma}^k(X)$ to be the space of continuous functions f on X with k continuous derivatives, such that $\rho^{-\beta} \sigma^{\gamma-j} |\nabla^j f|$ is bounded on X for $j = 0, \dots, k$. Define

the norm $\|\cdot\|_{C_{\beta,\gamma}^k}$ on $C_{\beta,\gamma}^k(X)$ by

$$\|f\|_{C_{\beta,\gamma}^k} = \sum_{j=0}^k \sup_X |\rho^{-\beta} \sigma^{j-\gamma} \nabla^j f|.$$

Let $\delta(g)$ be the injectivity radius of g , and write $d(x, y)$ for the distance between x, y in X . For T a tensor field on X and $\alpha, \beta, \gamma \in \mathbb{R}$, define

$$[T]_{\alpha,\beta,\gamma} = \sup_{\substack{x \neq y \in X \\ d(x,y) < \delta(g)}} \left[\min(\rho(x), \rho(y))^{-\beta} \cdot \min(\sigma(x), \sigma(y))^{-\gamma} \cdot \frac{|T(x) - T(y)|}{d(x, y)^\alpha} \right]. \quad (22)$$

Here we interpret $|T(x) - T(y)|$ using parallel translation along the unique geodesic of length $d(x, y)$ joining x and y . For $\beta, \gamma \in \mathbb{R}$, k a nonnegative integer, and $\alpha \in (0, 1)$, define the *weighted Hölder space* $C_{\beta,\gamma}^{k,\alpha}(X)$ to be the set of $f \in C_{\beta,\gamma}^k(X)$ for which the norm

$$\|f\|_{C_{\beta,\gamma}^{k,\alpha}} = \|f\|_{C_{\beta,\gamma}^k} + [\nabla^k f]_{\alpha,\beta,\gamma-k-\alpha} \quad (23)$$

is finite. Define $C_{\beta,\gamma}^\infty(X)$ to be the intersection of the $C_{\beta,\gamma}^k(X)$ for all $k \geq 0$. Both $C_{\beta,\gamma}^k(X)$ and $C_{\beta,\gamma}^{k,\alpha}(X)$ are Banach spaces, but $C_{\beta,\gamma}^\infty(X)$ is not a Banach space.

This generalizes the definition of weighted Hölder spaces $C_\beta^{k,\alpha}(X)$ on ALE manifolds, [11, Def. 4.2]. The reasoning leading up to the definition is discussed in [10, §9.5], where we also define weighted Sobolev spaces on QALE manifolds. The basic idea is that if $C_{\beta,\gamma}^k(X)$ then $\nabla^j f = O(\rho^\beta \sigma^{\gamma-j})$ for $j \leq k$. If \mathbb{C}^m/G has an isolated singularity, so that X is an ALE manifold, then $\rho = \sigma$ and $C_{\beta,\gamma}^{k,\alpha}(X)$ agrees with $C_{\beta+\gamma}^{k,\alpha}(X)$, the usual weighted Hölder space on an ALE manifold.

In [11, Th. 4.5(a)] we showed that if X is an n -dimensional ALE manifold and $\beta \in (-n, -2)$, then $\Delta : C_{\beta+2}^{k+2,\alpha}(X) \rightarrow C_\beta^{k,\alpha}(X)$ is an isomorphism. Our next few results generalize this to a QALE manifold X . We show that $\Delta : C_{\beta,\gamma}^{k+2,\alpha}(X) \rightarrow C_{\beta,\gamma-2}^{k,\alpha}(X)$ is an isomorphism, for (β, γ) in a certain nonempty open set \mathcal{I}_X in \mathbb{R}^2 . This is a useful analytic tool, and the principal justification for Definition 5.2.

We begin with a fairly weak existence result for solutions of the equation $\Delta u = f$. It can be proved by adapting known methods for ALE manifolds.

Theorem 5.3 *Let (X, J, g) be a QALE Kähler manifold, $\alpha \in (0, 1)$ and $\beta, \gamma < 0$. Then for each $f \in C_{\beta, \gamma-2}^{0, \alpha}(X)$ there exists a unique $u \in C^{2, \alpha}(X)$ such that $\Delta u = f$ and $u(x) \rightarrow 0$ as $x \rightarrow \infty$ in X .*

We quote a result about Schauder estimates on balls in \mathbb{R}^n , taken from [6, Th. 6.17], that we will need next.

Theorem 5.4 *Let B_1, B_2 be the balls of radius 1, 2 in \mathbb{R}^n . Let P be a linear elliptic operator of order 2 on functions on B_2 , defined by*

$$Pu(x) = a^{ij}(x) \frac{\partial^2 u}{\partial x^i \partial x^j}(x) + b^i(x) \frac{\partial u}{\partial x^i}(x) + c(x)u(x).$$

Let $k \geq 0$ be an integer and $\alpha \in (0, 1)$. Suppose the coefficients a^{ij}, b^i and c lie in $C^{k, \alpha}(B_2)$ and there are constants $\lambda, \Lambda > 0$ such that $|a^{ij}(x)\xi_i\xi_j| \geq \lambda|\xi|^2$ for all $x \in B_2$ and $\xi \in \mathbb{R}^n$, and $\|a^{ij}\|_{C^{k, \alpha}} \leq \Lambda$, $\|b^i\|_{C^{k, \alpha}} \leq \Lambda$, and $\|c\|_{C^{k, \alpha}} \leq \Lambda$ on B_2 for all $i, j = 1, \dots, n$. Then there exists a constant C depending on n, k, α, λ and Λ such that whenever $u \in C^2(B_2)$ and $f \in C^{k, \alpha}(B_2)$ with $Pu = f$, we have $u|_{B_1} \in C^{k+2, \alpha}(B_1)$ and

$$\|u|_{B_1}\|_{C^{k+2, \alpha}} \leq C(\|f\|_{C^{k, \alpha}} + \|u\|_{C^0}).$$

Using Theorems 5.3 and 5.4 we derive a sufficient condition for $\Delta : C_{\beta, \gamma}^{k+2, \alpha}(X) \rightarrow C_{\beta, \gamma-2}^{k, \alpha}(X)$ to be an isomorphism.

Theorem 5.5 *Let (X, J, g) be a QALE Kähler manifold, (ρ, σ) a pair of radius functions on X , and $\beta, \gamma < 0$. Suppose there exists a smooth function $F : X \rightarrow (0, \infty)$ satisfying*

$$\Delta F \geq \rho^\beta \sigma^{\gamma-2} \quad \text{and} \quad K_1 \rho^\beta \sigma^\gamma \leq F \leq K_2 \rho^\beta \sigma^\gamma \quad (24)$$

for some $K_1, K_2 > 0$. Then whenever $k \geq 0$ and $\alpha \in (0, 1)$, there exists $C > 0$ such that for each $f \in C_{\beta, \gamma-2}^{k, \alpha}(X)$ there is a unique $u \in C_{\beta, \gamma}^{k+2, \alpha}(X)$ with $\Delta u = f$, which satisfies $\|u\|_{C_{\beta, \gamma}^{k+2, \alpha}} \leq C\|f\|_{C_{\beta, \gamma-2}^{k, \alpha}}$. In other words, $\Delta : C_{\beta, \gamma}^{k+2, \alpha}(X) \rightarrow C_{\beta, \gamma-2}^{k, \alpha}(X)$ is an isomorphism.

Proof. Let $f \in C_{\beta, \gamma-2}^{k, \alpha}(X)$, and suppose for simplicity that $\|f\|_{C_{\beta, \gamma-2}^{k, \alpha}} \leq 1$. Then $f \in C_{\beta, \gamma-2}^{0, \alpha}(X)$, so by Theorem 5.3 there exists a unique $u \in C^{2, \alpha}(X)$

such that $\Delta u = f$ and $u(x) \rightarrow 0$ as $x \rightarrow \infty$ in X . Since $\Delta u = f \leq \rho^\beta \sigma^{\gamma-2} \leq \Delta F$, we see that $\Delta(u - F) \leq 0$ on X . Suppose that $u - F > 0$ at some point of X . Then $u - F$ is non-constant and has a maximum in X , since $F > 0$ and $u(x) \rightarrow 0$ as $x \rightarrow \infty$ in X . But this contradicts the maximum principle, as $\Delta(u - F) \leq 0$. Therefore $u - F \leq 0$, and $u \leq F$. Similarly we show that $u \geq -F$, and so $|u| \leq F \leq K_2 \rho^\beta \sigma^\gamma$.

To complete the proof, it is enough to show that $u \in C_{\beta,\gamma}^{k+2,\alpha}(X)$ and $\|u\|_{C_{\beta,\gamma}^{k+2,\alpha}} \leq C$ for some $C > 0$ independent of f . We shall do this by applying Theorem 5.4 to balls of radius $L\sigma(x)$ about x , for small $L \in (0, \frac{1}{2})$.

Let B_1, B_2 be the balls of radius 1 and 2 about 0 in \mathbb{R}^{2m} , where $m = \dim X$. Fix $x \in X$, and choose a vector space isometry $T_x X \cong \mathbb{R}^{2m}$. Let $L \in (0, \frac{1}{2})$ be small, and define a map $\Psi_x : B_2 \rightarrow B_{2L\sigma(x)}(x)$ by $\Psi_x(y) = \exp_x(L\sigma(x)y)$, where $\exp_x : T_x X \rightarrow X$ is the exponential map, and we have identified \mathbb{R}^{2m} and $T_x X$.

Using the definition of QALE metric, we can show that if $L > 0$ is small, then

- Ψ_x is a diffeomorphism between B_2 and $B_{2L\sigma(x)}(x)$,
- $\Psi_x|_{B_1}$ is a diffeomorphism between B_1 and $B_{L\sigma(x)}(x)$, and
- the metric $g_x = L^{-2}\sigma(x)^{-2}\Psi_x^*(g)$ on B_2 is close to the Euclidean metric h_0 on B_2 in $C^{k+2,\alpha}$, and $\|g_x - h_0\|_{C^{k+2,\alpha}}$ is bounded independently of $x \in X$.

The idea here is that $\sigma(x)$ is the approximate length-scale at which the metric on X near x differs from a Euclidean metric. Thus, balls of radius $L\sigma(x)$ and $2L\sigma(x)$ should resemble Euclidean balls of the same radius provided L is sufficiently small.

Define an operator P and functions u', f' on B_2 by

$$P = (L\sigma(x))^2 \Psi_x^*(\Delta), \quad u' = (L\sigma(x))^{-2} \Psi_x^*(u) \quad \text{and} \quad f' = \Psi_x^*(f).$$

Then P is the Laplacian of g_x on B_2 , which is close to h_0 in $C^{k+2,\alpha}$, and therefore P is close in $C^{k,\alpha}$ to the Euclidean Laplacian Δ_0 on B_2 . Also $Pu' = f'$, as $\Delta u = f$.

On B_2 we use the Euclidean metric h_0 , and on $B_{2L\sigma(x)}(x)$ we use the metric g . Since $h_0 \approx L^{-2}\sigma(x)^{-2}\Psi_x^*(g)$, it follows that

$$|\nabla^j f'| \approx (L\sigma(x))^j \Psi_x^*(|\nabla^j f|) \quad \text{and} \quad |\nabla^j u'| \approx (L\sigma(x))^{j-2} \Psi_x^*(|\nabla^j u|).$$

Thus, as $\nabla^j f = O(\rho(x)^\beta \sigma(x)^{\gamma-2-j})$ on $B_{2L\sigma(x)}$ we have $\nabla^j f' = O(\rho(x)^\beta \sigma(x)^{\gamma-2})$ on B_2 , and as $u = O(\rho(x)^\beta \sigma(x)^\gamma)$ on $B_{2L\sigma(x)}$ we have $u' = O(\rho(x)^\beta \sigma(x)^{\gamma-2})$ on $B_{2L\sigma(x)}$.

Therefore Theorem 5.4 shows that $\nabla^j u' = O(\rho(x)^\beta \sigma(x)^{\gamma-2})$ on B_1 for $0 \leq j \leq k+2$, together with the appropriate Hölder estimate. So $\nabla^j u = O(\rho(x)^\beta \sigma(x)^{\gamma-j})$ on $B_{L\sigma(x)}(x)$ for $0 \leq j \leq k+2$, together with the appropriate Hölder estimate. We have shown that u is bounded in $C_{\beta,\gamma}^{k+2,\alpha}(X)$, which completes the proof. \square

Here when we say that Δ is an isomorphism between two Banach spaces of functions on X , we mean that it is an *isomorphism of topological vector spaces*. That is, Δ is an isomorphism of vector spaces and is bounded and has bounded inverse, but it does not necessarily identify the norms on the two Banach spaces. In fact (24) is also necessary for $\Delta : C_{\beta,\gamma}^{k+2,\alpha}(X) \rightarrow C_{\beta,\gamma-2}^{k,\alpha}(X)$ to be an isomorphism.

Proposition 5.6 *Let (X, J, g) be a QALE Kähler manifold, let (ρ, σ) be a pair of radius functions on X , and let $\beta, \gamma < 0$. Suppose that $\Delta : C_{\beta,\gamma}^{2,\alpha}(X) \rightarrow C_{\beta,\gamma-2}^{0,\alpha}(X)$ is an isomorphism. Then there exists a smooth function $F : X \rightarrow (0, \infty)$ satisfying (24).*

Proof. From (20) and (21) we see that $\rho^\beta \sigma^{\gamma-2}$ lies in $C_{\beta,\gamma-2}^1(X)$, and hence in $C_{\beta,\gamma-2}^{0,\alpha}(X)$. So there exists a unique $F \in C_{\beta,\gamma}^{2,\alpha}(X)$ with $\Delta F = \rho^\beta \sigma^{\gamma-2}$. Clearly F is smooth, $\Delta F \geq \rho^\beta \sigma^{\gamma-2}$ and $F \leq K_2 \rho^\beta \sigma^\gamma$ for some $K_2 > 0$. Also, using (20) and (21) we can show that $\Delta(K_1 \rho^\beta \sigma^\gamma) \leq \rho^\beta \sigma^{\gamma-2}$ for some small $K_1 > 0$, and thus by the proof of the previous theorem we have $K_1 \rho^\beta \sigma^\gamma \leq F$. Thus F satisfies (24), as we want. \square

Definition 5.7 Let (X, J, g) be a QALE Kähler manifold, and define \mathcal{I}_X to be the set of pairs $(\beta, \gamma) \in \mathbb{R}^2$ such that $\beta < 0$, $\gamma < 0$ and $\Delta : C_{\beta,\gamma}^{k+2,\alpha}(X) \rightarrow C_{\beta,\gamma-2}^{k,\alpha}(X)$ is an isomorphism for $k \geq 0$ and $\alpha \in (0, 1)$. Theorem 5.5 and Proposition 5.6 prove that this condition is independent of k, α , and that $(\beta, \gamma) \in \mathcal{I}_X$ if and only if there exists a smooth function F on X satisfying (24). We can also show that \mathcal{I}_X is an open set in \mathbb{R}^2 .

Of course these ideas are of no use at all if \mathcal{I}_X is empty. In the following three results we show that it is not. The proof of the next proposition is elementary, and we omit it.

Proposition 5.8 *Let \mathbb{C}^{m-n} be a subspace of \mathbb{C}^m for $n > 0$, and define $r, s : \mathbb{C}^m \rightarrow [0, \infty)$ by $r(x) = d(x, 0)$ and $s(x) = d(x, \mathbb{C}^{m-n})$. Let $\beta, \gamma \in \mathbb{R}$. Then on $\mathbb{C}^m \setminus \mathbb{C}^{m-n}$ we have*

$$\Delta(r^\beta s^\gamma) = -\frac{1}{2}r^{\beta-2}s^{\gamma-2}[\gamma(\gamma + 2n - 2)r^2 + \beta(2m - 2 + \beta + 2\gamma)s^2]. \quad (25)$$

Since $0 \leq s \leq r$, there exists $C > 0$ such that $\Delta(Cr^\beta s^\gamma) \geq r^\beta s^{\gamma-2}$ if and only if

$$\gamma(\gamma + 2n - 2) < 0 \quad \text{and} \quad \gamma(\gamma + 2n - 2) + \beta(2m - 2 + \beta + 2\gamma) < 0. \quad (26)$$

The pair of inequalities (26) are equivalent to $2 - 2n < \gamma < 0$ and

$$|\beta + \gamma + m - 1| < \sqrt{(m - 1)^2 + 2\gamma(m - n)}.$$

For these to have a solution we must have $n > 1$. Note also that

$$(m - 1)^2 + 2\gamma(m - n) = (m + 1 - 2n)^2 + 2(\gamma + 2n - 2)(m - n).$$

Thus if $2 - 2n < \gamma < 0$ the square root $\sqrt{(m - 1)^2 + 2\gamma(m - n)}$ exists, and

$$|m + 1 - 2n| \leq \sqrt{(m - 1)^2 + 2\gamma(m - n)} \leq m - 1.$$

Hence, any solutions β, γ to (26) must satisfy $2 - 2m < \beta + \gamma < 0$. Also, if $2 - 2n < \gamma < 0$ and $\beta + \gamma$ lies between $2 - 2n$ and $2(n - m)$ then β, γ satisfy (26). To get the factor $\frac{1}{2}$ in (25), remember that Δ on Kähler manifolds is by convention half that on Riemannian manifolds.

Motivated by Proposition 5.8, we can prove:

Theorem 5.9 *Let (X, J, g) be a QALE Kähler manifold asymptotic to \mathbb{C}^m/G , and let n be the complex codimension of the singular set of \mathbb{C}^m/G . Suppose $\beta, \gamma \in \mathbb{R}$ satisfy (26). Then there exists a smooth function $F : X \rightarrow (0, \infty)$ satisfying (24).*

The basic idea of the proof is to model F on $\rho^\beta \sigma^\gamma$. The details are complicated, and we will not give them. But here is a sketch of the case when X is an ALE manifold. For $1 - m < \delta < 0$ we want to find a function

F such that $\Delta F \geq \rho^{2\delta-2}$ and $F \leq C\rho^{2\delta}$. On \mathbb{C}^m/G we have $\Delta(r^2) = -2m$ and $|\nabla(r^2)|^2 = 4r^2$. We show that there exists a unique smooth function u on X such that $\Delta u = -2m$ and $u = \rho^2 + O(\rho^{2-2m})$ for large ρ , and $4u - |\nabla u|^2$ is bounded on X .

Choose $K \in \mathbb{R}$ such that $u + K \geq 1$ and $|\nabla u|^2 \leq 4(u + K)$ on X . Then

$$\begin{aligned} \Delta[(u + K)^\delta] &= \delta(u + K)^{\delta-1} \Delta u - \frac{1}{2} \delta(\delta - 1)(u + K)^{\delta-2} |\nabla u|^2 \\ &= -2\delta(\delta + m - 1)(u + K)^{\delta-1} \\ &\quad + \frac{1}{2} \delta(\delta - 1)(u + K)^{\delta-2} (4(u + K) - |\nabla u|^2) \\ &\geq -2\delta(\delta + m - 1)(u + K)^{\delta-1}, \end{aligned}$$

since $\delta(\delta - 1) > 0$ and $|\nabla u|^2 \leq 4(u + K)$. But there exist $C_1, C_2 > 0$ such that

$$-2\delta(\delta + m - 1)C_1(u + K)^{\delta-1} \geq \rho^{2\delta-2} \quad \text{and} \quad (u + K)^\delta \leq C_2\rho^{2\delta}. \quad (27)$$

So putting $F = C_1(u + K)^\delta$ we see that $\Delta F \geq \rho^{2\delta-2}$ and $F \leq C_1C_2\rho^{2\delta}$, and the proof is finished. To generalize this proof to the case that X is a QALE manifold, we use similar functions u on X and on each Y_i in the decomposition of X .

Corollary 5.10 *Let (X, J, g) be a QALE Kähler manifold asymptotic to \mathbb{C}^m/G , and let n be the complex codimension of the singular set of \mathbb{C}^m/G . Suppose that $\beta, \gamma \in \mathbb{R}$ satisfy*

$$\beta < 0, \quad 2 - 2n < \gamma < 0, \quad |\beta + \gamma + m - 1| < \sqrt{(m - 1)^2 + 2\gamma(m - n)}. \quad (28)$$

Then $(\beta, \gamma) \in \mathcal{I}_X$. Hence if $2 \leq n \leq m$ then \mathcal{I}_X is nonempty.

To prove the last part, observe that if $2 \leq n \leq m$ then (28) holds for $\beta = 1 - m$ and $\gamma < 0$ small. Note that $n \geq 2$ by definition, since g is a QALE metric. If we allowed $n = 1$ then the equation $2 - 2n < \gamma < 0$ in (28) would have no solutions, which illustrates the problems when S has codimension one.

By definition, if $(\beta, \gamma) \in \mathcal{I}_X$ then $\beta < 0$ and $\gamma < 0$. However, (26) admits solutions with $\gamma < 0$ but $\beta \geq 0$. This suggests that $\Delta : C_{\beta, \gamma}^{k+2, \alpha}(X) \rightarrow$

$C_{\beta, \gamma-2}^{k, \alpha}(X)$ can be an isomorphism when $\beta \geq 0$ as well. In fact this is the case, but we will not explore the idea because we shall need $\beta < 0$ in our applications anyway.

Using similar methods to those in the proof of Theorem 5.9, we can show:

Theorem 5.11 *Let (X, J, g) be a QALE Kähler manifold asymptotic to \mathbb{C}^m/G , and let $\gamma < 0$. Then there exists a smooth function $F : X \rightarrow (0, \infty)$ satisfying $\Delta F \geq \rho^{2-2m} \sigma^{\gamma-2}$ and $K_1 \rho^{2-2m} \leq F \leq K_2 \rho^{2-2m}$ for some $K_1, K_2 > 0$.*

From Theorem 5.11 and the proof of Theorem 5.5 we deduce

Corollary 5.12 *Let (X, J, g) be a QALE Kähler manifold of dimension m , and let $\gamma < 0$. Then whenever $k \geq 0$ and $\alpha \in (0, 1)$, there exists $C > 0$ such that for each $f \in C_{2-2m, \gamma-2}^{k, \alpha}(X)$ there is a unique $u \in C_{2-2m, 0}^{k+2, \alpha}(X)$ with $\Delta u = f$, which satisfies $\|u\|_{C_{2-2m, 0}^{k+2, \alpha}} \leq C \|f\|_{C_{2-2m, \gamma-2}^{k, \alpha}}$.*

The author believes that the following results are true.

Conjecture 5.13 *Let (X, J, g) be a QALE Kähler manifold asymptotic to \mathbb{C}^m/G , and let n be the complex codimension of the singular set of \mathbb{C}^m/G . Then*

$$\mathcal{I}_X = \{(\beta, \gamma) \in \mathbb{R}^2 : \beta < 0, \quad 2 - 2n < \gamma < 0, \quad \beta + \gamma > 2 - 2m\}. \quad (29)$$

Also, for generic $\beta, \gamma \in \mathbb{R}$ the map $\Delta : C_{\beta, \gamma}^{k+2, \alpha}(X) \rightarrow C_{\beta, \gamma-2}^{k, \alpha}(X)$ is Fredholm, with finite-dimensional kernel and cokernel.

6 The Calabi conjecture for QALE manifolds

The *Calabi conjecture* [4] describes the possible Ricci curvatures of Kähler metrics on a fixed compact complex manifold M , in terms of the first Chern class $c_1(M)$. It was proved by Yau [17] in 1976. Since then several authors such as Tian and Yau [15, 16] and Bando and Kobayashi [1, 2] have proved versions of the Calabi conjecture for *noncompact* manifolds.

We now state two versions of the Calabi conjecture for QALE manifolds, both of which will be proved in the author's book [10]. So far as the author

knows they do not follow from any published noncompact version of the Calabi conjecture, though there will obviously be similarities. The first, simpler version is based on the Calabi conjecture for ALE manifolds given in [11, §6], and is proved in [10, §9.6].

The Calabi conjecture for QALE manifolds (first version) *Let (X, J, g) be a QALE Kähler manifold of dimension m , with Kähler form ω . Then*

- (a) *Suppose that $(\beta, \gamma) \in \mathcal{I}_X$, as in Definition 5.7. Then for each $f \in C_{\beta, \gamma-2}^\infty(X)$ there is a unique $\phi \in C_{\beta, \gamma}^\infty(X)$ such that $\omega + \text{dd}^c \phi$ is a positive $(1, 1)$ -form and $(\omega + \text{dd}^c \phi)^m = e^f \omega^m$ on X .*
- (b) *Suppose $\gamma < 0$. Then for each $f \in C_{2-2m, \gamma-2}^\infty(X)$ there is a unique $\phi \in C_{2-2m, 0}^\infty(X)$ such that $\omega + \text{dd}^c \phi$ is a positive $(1, 1)$ -form and $(\omega + \text{dd}^c \phi)^m = e^f \omega^m$ on X .*

We prove the conjecture by combining the method of Yau's proof in the compact case [17] with the ideas in §5 on analysis on QALE manifolds. The proof is similar to that of the Calabi conjecture for ALE manifolds given in [10, §8.6-8.7], and discussed in [11, §6].

Here is a way to understand parts (a) and (b). Now $(\omega + \text{dd}^c \phi)^m = e^f \omega^m$ is really a nonlinear version of $\Delta \phi = -f$. Near infinity in X , where ϕ and f are small, the two equations become very close. Therefore, if we can solve the equation $\Delta \phi = -f$ uniquely for ϕ, f in some given Banach spaces, we would naïvely expect to be able to solve the Calabi conjecture in the same Banach spaces.

By Definition 5.7, if $(\beta, \gamma) \in \mathcal{I}_X$ then $\Delta : C_{\beta, \gamma}^{k+2, \alpha}(X) \rightarrow C_{\beta, \gamma-2}^{k, \alpha}(X)$ is an isomorphism. Thus we can solve the equation $\Delta \phi = -f$ uniquely with $\phi \in C_{\beta, \gamma}^{k+2, \alpha}(X)$ and $f \in C_{\beta, \gamma-2}^{k, \alpha}(X)$, and similarly with $\phi \in C_{\beta, \gamma}^\infty(X)$ and $f \in C_{\beta, \gamma-2}^\infty(X)$.

So part (a) above, which says we can solve the Calabi conjecture with $\phi \in C_{\beta, \gamma}^\infty(X)$ and $f \in C_{\beta, \gamma-2}^\infty(X)$, is as we would expect. Similarly, part (b) above corresponds to Corollary 5.12, which says that we can solve $\Delta \phi = -f$ uniquely with $\phi \in C_{2-2m, 0}^{k+2, \alpha}(X)$ and $f \in C_{2-2m, \gamma-2}^{k, \alpha}(X)$, and so with $\phi \in C_{2-2m, 0}^\infty(X)$ and $f \in C_{2-2m, \gamma-2}^\infty(X)$.

So far we have estimated the decay of functions ϕ and metrics g on X in two different ways. In §2-§4 we pulled g and ϕ back to $V_i \times Y_i$ and bounded them using powers of functions $\mu_{i,j}$ and ν_i on $V_i \times Y_i$. In §5, and

in the conjecture above and its proof, we estimated ϕ in terms of powers of functions ρ, σ on X .

There are good reasons for using these two approaches in the way we have — attempting to define QALE metrics directly on X , and trying to solve the Calabi conjecture by pulling back to $V_i \times Y_i$, both seem to lead to disaster. However, in order to construct QALE Kähler metrics with prescribed Ricci curvature on X we need to integrate these two approaches, because in solving the Calabi conjecture on X we need ϕ to be a function of *Kähler potential type*, as in §4.

Here is a version of the Calabi conjecture for QALE manifolds which achieves this, which is proved in [10, §9.7].

The Calabi conjecture for QALE manifolds (second version) *Let (X, J, g) be a QALE Kähler manifold asymptotic to \mathbb{C}^m/G with $\text{Fix}(G) = \{0\}$, let ω be the Kähler form and ξ the Ricci form of g , and let $\epsilon < -2$. Suppose f is a smooth function on X with*

$$\nabla^l \psi_i^*(f) = \sum_{j \in I: i \not\leq j} O(\mu_{i,j}^{d_j} \nu_i^{\epsilon-l}) \quad (30)$$

on $V_i \times Y_i \setminus U_i$, for all $i \in I \setminus \{\infty\}$ and $l \geq 0$. Then there is a unique smooth function ϕ on X such that $\omega' = \omega + \text{dd}^c \phi$ is a positive $(1,1)$ -form and $(\omega')^m = e^f \omega^m$ on X , and

$$\nabla^l \psi_i^*(\phi) = \sum_{j \in I: i \not\leq j} O(\mu_{i,j}^{d_j} \nu_i^{-l}) \quad (31)$$

on $V_i \times Y_i \setminus U_i$, for all $i \in I \setminus \{\infty\}$ and $l \geq 0$. Furthermore, ϕ is of Kähler potential type on X , and the Kähler metric g' on X with Kähler form ω' is QALE and has Ricci form $\xi' = \xi - \frac{1}{2} \text{dd}^c f$.

We prove this by translating (30) and (31) into equations directly on X , prescribing the asymptotic behaviour of ϕ and f in terms of ρ, σ and some similar functions ρ_i for $i \in I \setminus \{0\}$. Then we apply the method used to prove the first version above — this is really a more complicated version of part (b) of the first conjecture.

The point of assuming that $\text{Fix}(G) = V_\infty = \{0\}$ here is that taken over all $i \in I \setminus \{\infty\}$, eqns (30) and (31) prescribe the asymptotic behaviour of f and ϕ everywhere on X , except within a fixed distance of $\pi^{-1}(V_\infty)$. If $\text{Fix}(G) = \{0\}$ this is all of X except a compact subset. But if $\text{Fix}(G) \neq \{0\}$

then $\pi^{-1}(V_\infty)$ extends to infinity in X , and (30) and (31) only prescribe the behaviour on a part of the ‘boundary’ of X , so they are not good boundary conditions.

7 The proofs of Theorems 3.3 and 3.4

Before proving the theorems, we give two preliminary propositions.

Proposition 7.1 *Suppose (X, J, g) is a Ricci-flat QALE Kähler manifold. Then g is the only Ricci-flat QALE Kähler metric in its Kähler class.*

Proof. Suppose for a contradiction that g, g' are distinct Ricci-flat QALE Kähler metrics on X in the same Kähler class, and let X be of the smallest dimension in which this can happen. Clearly $\text{Fix}(G) = \{0\}$, since otherwise we can replace X by Y_∞ , which has smaller dimension. Let g, g' be asymptotic to metrics g_i, g'_i on Y_i for $i \in I$, as in Definition 2.1. Then g_i, g'_i are Ricci-flat, and in the same Kähler class. Thus $g_i = g'_i$ for all $i \neq \infty$ in I , since $\dim Y_i < \dim X$ when $i \neq \infty$.

Let ω, ω' be the Kähler forms of g, g' . Then by Theorem 4.3 we have $\omega' = \omega + \text{dd}^c \phi$, where ϕ is a function of Kähler potential type on X . Since $g_i = g'_i$ for $i \neq \infty$, the functions ϕ_i of Definition 4.1 are zero for $i \neq \infty$, and ϕ satisfies (31). But g, g' have the same Ricci curvature, so that $(\omega')^m = \omega^m$. Thus ϕ and $f \equiv 0$ satisfy the second version of the Calabi conjecture for QALE manifolds. By uniqueness in the conjecture we see that $\phi \equiv 0$, so that $g = g'$, a contradiction. Thus g is unique in its Kähler class. \square

Proposition 7.2 *Let (X, π) be a crepant resolution of \mathbb{C}^m/G with $\text{Fix}(G) = \{0\}$, and let g be a QALE Kähler metric on X asymptotic to metrics g_i on Y_i for $i \in I$, such that $\psi_i^*(g) = h_i \times g_i$ on $V_i \times Y_i \setminus U_i$. Suppose that for each $i \in I \setminus \{\infty\}$ there is a Ricci-flat Kähler metric g'_i on Y_i in the same Kähler class as g_i . Then there exists a QALE Kähler metric g' on X in the same Kähler class as g with Ricci form $\frac{1}{2}\text{dd}^c f$, where f is a smooth function on X satisfying*

$$\nabla^l \psi_i^*(f) = \sum_{\substack{j \neq k \in I \setminus \{\infty\}: \\ V_j \cap V_k = \{0\}}} O(\mu_{i,j}^{d_j} \mu_{i,k}^{d_k} \nu_i^{-4-l}) \quad \text{on } V_i \times Y_i \setminus U_i, \quad (32)$$

for $i \in I$ and $l \geq 0$. However, if there are no $j, k \in I \setminus \{\infty\}$ with $V_j \cap V_k = \{0\}$ then (32) does not hold, but instead f is compactly-supported on X .

Proof. Let g_i, g'_i have Kähler forms ω_i, ω'_i . Then Theorem 4.3 shows that $\omega'_i = \omega_i + \text{dd}^c \phi_i$, where ϕ_i is a unique function of Kähler potential type on Y_i . As g'_i is the only Ricci-flat QALE metric in its Kähler class by Proposition 7.1, we see that $\chi_{\gamma, i}^*(g'_{\gamma, i}) = g'_i$, and g'_i is asymptotic to g'_j when $i, j \neq \infty$ and $i \succeq j$. Using these facts one can show that the ϕ_i satisfy parts (i) and (ii) of Proposition 4.4.

Therefore we may apply Theorem 4.5 and Proposition 4.6 to find a function ϕ of Kähler potential type on X such that $\omega' = \omega + \text{dd}^c \phi$ is the Kähler form of a QALE Kähler metric g' on X , which is asymptotic to the Ricci-flat metrics g'_i on Y_i . Also, as $\text{Fix}(G) = \{0\}$, by construction ϕ satisfies (18) outside a compact subset T of X .

Let ω_0 be the Kähler form of the Euclidean metric h_0 on \mathbb{C}^m/G , and let $\Omega_0 = dz^1 \wedge \cdots \wedge dz^m$ be the holomorphic volume form on \mathbb{C}^m/G , which is well-defined as $G \subset \text{SU}(m)$, since X is a crepant resolution. Calculation shows that $\omega_0^m = C_m \Omega_0 \wedge \overline{\Omega}_0$ on \mathbb{C}^m/G , where $C_m = 2^{-m} i^m m! (-1)^{m(m-1)/2}$. Let $\Omega = \pi^*(\Omega_0)$. Then Ω is a nonsingular holomorphic volume form on X . Define a smooth real function f on X by $e^f (\omega')^m = C_m \Omega \wedge \overline{\Omega}$. So g' has Ricci form $\frac{1}{2} \text{dd}^c f$, as we want.

We must show that f satisfies (32). For simplicity we will restrict our attention to the case $N(V_i) = G$ for all $i \in I$. Then (19) gives

$$\phi = \sum_{i \in I \setminus \{\infty\}} k_i \Phi'_i \quad \text{on } X \setminus T. \quad (33)$$

Let $x \in X$ satisfy $d(\pi(x), S) > 2R$. Then putting $i = 0$ in $\psi_i^*(g) = h_i \times g_i$ on $V_i \times Y_i \setminus U_i$ shows that $g = \pi^*(h_0)$ near x . Thus $\omega = \pi^*(\omega_0)$ and $\Omega = \pi^*(\Omega_0)$, and so $\omega^m = C_m \Omega \wedge \overline{\Omega}$ near x .

If $i \neq \infty$ in I and $(v, y) \in V_i \times Y_i \setminus U_i$ with $\psi_i(v, y) = x$, then $\psi_i^*(g) = h_i \times g_i$ near (v, y) . Also the function Φ_i defined in (16) satisfies $\Phi_i = \phi_i$ near (v, y) , since $\mu_{i, j} > 2R$ near (v, y) for all $j \neq 0$. But $N(V_i) = G$, and so $\Phi_i = \psi_i^*(\Phi'_i)$, where Φ'_i is defined by (19). Therefore the metric $h_i \times g'_i$ on $V_i \times Y_i$ has Kähler form $\psi_i^*(\omega + \text{dd}^c \Phi'_i)$ near (v, y) .

But $h_i \times g'_i$ is Ricci-flat. So $\omega + \text{dd}^c \Phi_i$ is the Kähler form of a Ricci-flat metric near x . It is then not difficult to show that $(\omega + \text{dd}^c \Phi'_i)^m = C_m \Omega \wedge \overline{\Omega}$ near x . Thus we have $e^f (\omega + \text{dd}^c \phi)^m = \omega^m = (\omega + \text{dd}^c \Phi'_i)^m$ wherever $d(\pi(x), S) > 2R$ on X , for all $i \in I \setminus \{\infty\}$. Multiplying out $(\omega + \text{dd}^c \Phi'_i)^m = \omega^m$ and rearranging gives

$$m \text{dd}^c \Phi'_i \wedge \omega^{m-1} = -\frac{1}{2} m(m-1) (\text{dd}^c \Phi'_i)^2 \wedge \omega^{m-2} + \dots, \quad (34)$$

where ‘ \dots ’ are terms of order at least 3 in $\text{dd}^c \Phi'_i$. Substituting (33) into the equation $e^f(\omega + \text{dd}^c \phi)^m = \omega^m$, rearranging and using (34) shows that

$$\begin{aligned} \frac{-2f}{m(m-1)} \omega^m &= \sum_{i \neq j \in I \setminus \{\infty\}} k_i k_j \text{dd}^c \Phi'_i \wedge \text{dd}^c \Phi'_j \wedge \omega^{m-2} \\ &+ \sum_{i \in I \setminus \{\infty\}} k_i (k_i - 1) (\text{dd}^c \Phi'_i)^2 \wedge \omega^{m-2} + \dots, \end{aligned} \quad (35)$$

where ‘ \dots ’ are terms of order at least 3 in the $\text{dd}^c \Phi'_i$ or at least 2 in f , and the equation holds for $x \in X \setminus T$ with $d(\pi(x), S) > 2R$. Thus f is roughly quadratic in the $\text{dd}^c \Phi'_i$, to highest order.

Using (35) and the special properties of the k_i and Φ'_i , we can show that (32) holds. The proof of this is complicated, and we will not give it. The basic idea is that g' is made by combining the Ricci-flat metrics $h_i \times g'_i$. The dominant terms in $\text{Ric}(g')$ result from interference between $h_j \times g'_j$ and $h_k \times g'_k$ for $j \neq k$, and contribute $O(\mu_{i,j}^{d_j} \nu_i^{-2} \cdot \mu_{i,k}^{d_k} \nu_i^{-2})$ to $\psi_i^*(f)$ on $V_i \times Y_i \setminus U_i$, and corresponding terms to $\nabla^l \psi_i^*(f)$.

However, if $V_j \cap V_k = V_i$ with $i \neq \infty$ then $h_i \times g'_i$ is Ricci-flat and asymptotic to both $h_j \times g'_j$ and $h_k \times g'_k$. So we introduce no extra Ricci curvature by combining $h_j \times g'_j$ and $h_k \times g'_k$ in this case, which is why the sum in (32) is restricted to j, k with $V_j \cap V_k = \{0\}$. More details of this argument are given in [10, §9.8]. \square

7.1 Proof of Theorem 3.3

We work by induction on $m = \dim X$. The result is trivial for $m = 0, 1$, giving the first step. For the inductive step, suppose that X is a crepant resolution of \mathbb{C}^m/G for some $m \geq 2$, that κ is a Kähler class on X containing QALE Kähler metrics, and that the theorem is true in dimensions $0, 1, \dots, m-1$. We shall show that there exists a Ricci-flat QALE Kähler metric \hat{g} on X in κ . This \hat{g} is unique by Proposition 7.1, and so each Kähler class on X contains a unique Ricci-flat QALE Kähler metric. Thus by induction the theorem is true for all m .

If $\text{Fix}(G) \neq \{0\}$ then $X = V_\infty \times Y_\infty$, and as $\dim Y_\infty < m$ there is a unique Ricci-flat QALE Kähler metric \hat{g}_∞ on Y_∞ in $\kappa|_{Y_\infty}$. Then $\hat{g} = h_\infty \times \hat{g}_\infty$ is the metric on X that we seek. So suppose that $\text{Fix}(G) = \{0\}$.

By Theorem 4.7 we can choose QALE Kähler metrics g on X and g_i on Y_i such that g has Kähler class κ and $\psi_i^*(g) = h_i \times g_i$ on $V_i \times Y_i \setminus U_i$, where U_i

is defined using some $R > 0$ depending on κ . Now Y_i is a crepant resolution of W_i/A_i , and $\dim Y_i < m = \dim X$ if $i \neq \infty$. Thus for each $i \neq \infty$ in I there is a unique Ricci-flat Kähler metric g'_i on Y_i in the Kähler class of g_i , by the inductive hypothesis.

Proposition 7.2 applies, and gives a QALE Kähler metric g' on X in the Kähler class κ with Ricci form $\frac{1}{2}\text{dd}^c f$, where either f satisfies (32) if there exist $j, k \in I \setminus \{\infty\}$ with $V_j \cap V_k = \{0\}$, or else f is compactly-supported on X . Now it is easy to show from (32) that f satisfies (30) with $\epsilon = -4$.

Therefore by the second version of the Calabi conjecture for QALE manifolds, which holds by [10, §9.7], there is a unique function ϕ' on X such that $\hat{\omega} = \omega' + \text{dd}^c \phi'$ satisfies $\hat{\omega}^m = e^f (\omega')^m$ and (31) holds for ϕ' . The Kähler metric \hat{g} on X with Kähler form $\hat{\omega}$ is QALE, as ϕ' is of Kähler potential type, and has Ricci form $\xi - \frac{1}{2}\text{dd}^c f$, where ξ is the Ricci form of g' . But $\xi = \frac{1}{2}\text{dd}^c f$ from above, and so \hat{g} is Ricci-flat. Thus we have found a Ricci-flat QALE Kähler metric \hat{g} in the Kähler class κ , and the proof is complete.

7.2 Proof of Theorem 3.4

Let (X, J, g) be a Ricci-flat QALE Kähler manifold asymptotic to \mathbb{C}^m/G , where \mathbb{C}^m/G is irreducible. As X is simply-connected and g is Ricci-flat Kähler we have $\text{Hol}(g) \subseteq \text{SU}(m)$. But g is nonsymmetric as it is Ricci-flat, and since \mathbb{C}^m/G is irreducible one can show that g is irreducible. So Berger's classification of Riemannian holonomy groups [14, §10] shows that $\text{Hol}(g)$ is $\text{SU}(m)$ or $\text{Sp}(m/2)$. The Euclidean metric h_0 on \mathbb{C}^m/G has $\text{Hol}(h_0) = G$, and as g is asymptotic to h_0 we see that $G \subset \text{Hol}(g)$. Thus, if G is not conjugate to a subgroup of $\text{Sp}(m/2)$ then $\text{Hol}(g) \neq \text{Sp}(m/2)$, which forces $\text{Hol}(g) = \text{SU}(m)$.

So suppose $m \geq 4$ is even and G is conjugate to a subgroup of $\text{Sp}(m/2)$. Then there exists a G -invariant, constant complex symplectic form $\omega_{\mathbb{C}}$ on \mathbb{C}^m , which pushes down to \mathbb{C}^m/G . The pull-back $\pi^*(\omega_{\mathbb{C}})$ is a nonsingular complex symplectic form on X , and using a Bochner argument we can prove that $\nabla \pi^*(\omega_{\mathbb{C}}) = 0$. Therefore $\text{Hol}(g) \subset \text{Sp}(m/2)$, and so $\text{Hol}(g) = \text{Sp}(m/2)$.

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