

Effects of Surface Tension on the Richtmyer-Meshkov Instability in Fully Compressible and Inviscid Fluids

Kaitao Tang,^{1,2} Wouter Mostert,^{1,3} Daniel Fuster,⁴ and Luc Deike^{1,5}

¹*Department of Mechanical and Aerospace Engineering,
Princeton University, Princeton, NJ 08544, USA*

²*Center for Combustion Energy and School of Aerospace Engineering, Tsinghua University, Beijing, 100084, China*

³*Department of Mechanical and Aerospace Engineering,
Missouri University of Science and Technology, Rolla, MO 65401, USA*

⁴*Sorbonne Universités, UPMC Univ Paris 06, CNRS, UMR 7190,
Institut Jean Le Rond d'Alembert, F-75005 Paris, France.*

⁵*High Meadows Environmental Institute, Princeton University, Princeton, NJ 08544, USA*
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Novel numerical simulations investigating the Richtmyer-Meshkov instability (RMI) with surface tension are presented. We solve the two-phase compressible Euler equation with surface tension and interface reconstruction by a volume-of-fluid method. We validate and bridge existing theoretical models of effects of surface tension on the RMI in the linear, transitional and nonlinear post-shock growth regimes. After deriving a consistent non-dimensional formulation [from an existing linear incompressible theory predicting perturbation development under large surface tension](#), we find good agreement with theoretical prediction in the small-amplitude (linear) oscillatory regime for positive Atwood numbers, and we show that negative Atwood numbers can be accommodated by an appropriate modification to the theory. Next, we show good agreement with nonlinear theory for asymptotic interface growth in the limit of small surface tension. Finally, we [use the non-dimensional formulation to define a heuristic criterion which identifies the transition from the linear regime to the nonlinear regime at intermediate surface tension](#). These results highlight the utility of this numerical method for compressible problems featuring surface tension, and pave the way for a broader investigation into mixed compressible/incompressible problems.

I. INTRODUCTION

The Richtmyer-Meshkov instability (RMI) occurs when two regions of fluids with different densities and separated by a perturbed interface are subjected to the passage of a shock wave, which is often modeled as an impulsive acceleration. The misalignment of the pressure and density gradients incurred by the shock-interface interaction results in the baroclinic generation of vorticity deposited on the interface, and hence in the subsequent growth and development of perturbations. The RMI is relevant to many applications: it is well-known to inhibit the attainment of fusion in inertial confinement fusion contexts [1]; it can enhance mixing in high-speed airbreathing engines [2]; and may also drive mixing in certain supernovae [3] and magnetic field amplification in supernova remnants travelling through the interstellar medium [4].

If at least one of the fluids is a liquid, or if the interface between the fluids is a membrane [5], surface tension may play a role. This is the case for the familiar example of dropping a bucket filled with water on the ground [6]; the impact on the ground constitutes an impulsive acceleration which promotes the RMI on the water surface. Compressible effects in the gas phase are explicitly involved in the problem of high-speed droplet aerobreakup, which is challenging to investigate both in the laboratory (see Theofanous [7] and references therein) and numerically [8]. In this connection, the RMI may be relevant to the early-time behaviour of the shocked droplet. In any case, the effect of surface tension on the fine-scale structures at these early times is typically not accounted for and clarified by large-scale numerical simulations on grounds of insufficient resolution, even with adaptive mesh refinement (AMR) techniques (e.g. as noted by Meng and Colonius [8]).

As a fundamental problem, the effects of surface tension on the RMI, along with the closely-related Rayleigh-Taylor instability (RTI), have seen prior investigation. Mikaelian [5] performed a linear stability analysis, finding that the (sufficiently strong) surface tension induces a stable oscillation in the shocked interface whose amplitude and frequency depend on the magnitude of the surface tension. Bigdelou [9] recently conducted a series of numerical simulations on shock-driven multiphase flow problems using the level-set method [10], where Mikaelian's predictions [5] are used for a brief validation of the numerical model (see their Figure 4.49). If the surface tension is sufficiently weak, the interface enters a nonlinear growth regime featuring asymmetrically formed narrow "spike" and broad "bubble" structures interpenetrating between the fluids [11, 12]; in this regime, Sohn [13] provided an asymptotic analysis of the velocity of the bubble-structures, and included a

brief numerical verification using a phase-field interface model [14] in incompressible fluids. Matsuoka [15] also studied the interfacial behaviour in incompressible RMI with surface tension by numerically solving the Birkhoff-Rott Equation for vortex sheet motion, and demonstrated that large surface tension is able to stabilize the interface and suppress its late-time rollup behaviour. The weakly nonlinear RTI under the effect of surface tension has also been investigated in a variety of studies, see e.g. Garnier *et al.* [16], Guo *et al.* [17]. For a broad review of studies of the RTI and RMI including surface tension are covered, among other topics, see the review of Zhou [18].

The existing studies reveal only a limited understanding of the RMI with surface tension. First, the stabilizing effect of surface tension on the RMI suggests that it may inhibit transition to the nonlinear regime of bubble and spike development. However, for sufficiently large Weber numbers (defined as the ratio between inertial and surface tension forces), the interface perturbations may grow large enough to escape the linear regime before reaching the maximum amplitude of oscillation, so that Mikaelian's analysis [5] no longer applies. This introduces a *critical Weber number* for the RMI with surface tension, which to our knowledge has not been systematically studied besides a brief heuristic calculation by Bigdelou [9], whose numerical validation is conducted in the linear regime. Second, this critical value will not in general correspond with the vanishingly small surface tension required by Sohn's [13] asymptotic analysis; we expect that there exist cases with intermediately small surface tensions (that is, large Weber numbers) whose perturbation growth patterns are currently not well understood. Finally, besides Bigdelou [9], none of these studies, which rely on numerical support of theoretical results, consider the effects of compressible flow, which may appear for example in the case of shocked-membrane problems arising in shock tube environments [5], or in shock-bubble interactions [19] where to our knowledge the very early time-evolution of the shocked bubble under the influence of surface tension has not been investigated in detail. This may be partly due to the relative lack of compressible-flow solvers that include surface tension effects, although there has been recent progress in this direction (see e.g. [20], used as the basis for this study, and [9, 21]).

In this study we present fully nonlinear, compressible numerical simulations of the inviscid RMI with surface tension, using Fuster and Popinet's [20] recently developed and implemented numerical technique in the Basilisk package. In addition to the technical significance of this study, its purpose is, firstly, to provide numerical support for the studies of Mikaelian [5] and Sohn [13] in a compressible environment; secondly, to provide insight into the nonlinear development of the problem, considering especially the asymptotic large-time behaviour at small surface tensions (large Weber numbers); and finally, to find the critical Weber number required to suppress the RMI and, in particular, to restrict oscillations of the perturbation to the linear regime.

Our study is structured as follows. In §II we adapt the theoretical work of Mikaelian [5] [on the RMI perturbation development under large surface tension \(with typical post-shock Weber number \$We^+ < 10^2\$, as defined in \(7\)\)](#) to form a fully non-dimensional framework, based on which the following analyses are conducted. In §III we present a formulation of the problem and introduce the numerical method. Afterwards, with the post-shock parameters determined, we compare Mikaelian's [5] theory with the scaled numerical results in §IV A, and propose a modified theoretical model based on Vandenboomgaerde *et al.* [22] to better account for both positive and negative Atwood numbers. [For the development of nonlinear perturbation structures in the limit of small surface tension \(\$We^+ > 10^3\$ \)](#), we discuss the theory of Sohn [13] and compare it with the scaled numerical results in §IV B, [although a complete comparison of the relative large-time nonlinear behavior of bubbles and spikes at varying Weber numbers remains a topic for future study](#). Then in §IV C we [use the non-dimensional framework to identify the transitional nonlinear development of the interface at intermediate surface tension \(\$10^2 < We^+ < 10^3\$ \)](#), and hence to propose a heuristic criterion to delineate the transition to [nonlinear development](#). We eventually conclude the study in §V with some remarks on future work.

II. NON-DIMENSIONAL SCALING MODEL

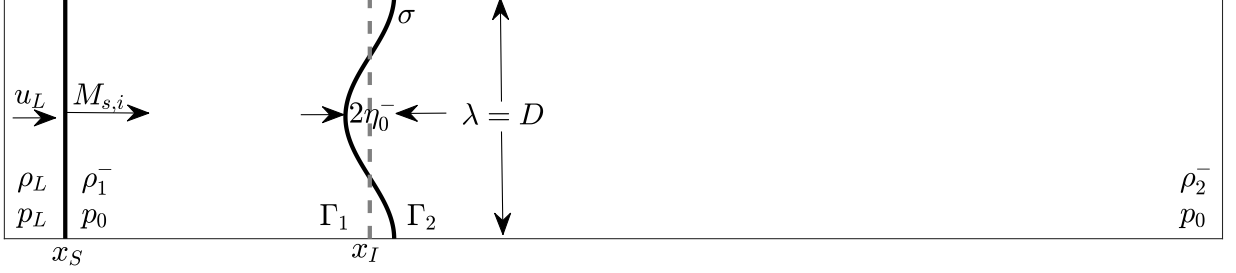
We first adapt the theoretical result from a linear stability analysis of the incompressible RMI with surface tension, due to Mikaelian [5], into a non-dimensional formulation which can then be interpreted in the compressible flow problem featuring a shock wave.

The flow configuration is illustrated in Figure 1, where the fluid to the left is labelled '1', and the fluid to the right labelled '2'. Initially, the velocities of the two inviscid fluids and the sinusoidally perturbed interface are all set to zero. The incident shock travels rightwards from the left boundary $x = x_S$ to hit the interface, whose passage will first bring the density and pressure in fluid 1 to intermediate values denoted by subscript 'L'. As is shown in Figure 1b, after the shock-interface interaction, the interface will be accelerated and acquire a velocity jump Δv ; in the meanwhile, part of the shock energy will be transferred into fluid 2 in the form of a transmitted shock, and the rest will reflect back into fluid 1 as a reflected wave, whose passage will bring the

density and pressure in fluid 1 to their post-shock values. The perturbed interface will then evolve under the influence of surface tension σ .

The non-dimensional groups governing the problem can be derived with respect to pre- and post-shock state of the interface. We first discuss the pre-shock state, which represents the *a priori* understanding of the system, before next discussing the post-shock state, to which Mikaelian's analysis [5] naturally applies. Afterwards, we will discuss how the post-shock state may be determined from knowledge of the pre-shock state. In the following, minus-sign superscripts denote the pre-shock state, and plus-sign superscripts denote post-shock state.

(a)



(b)

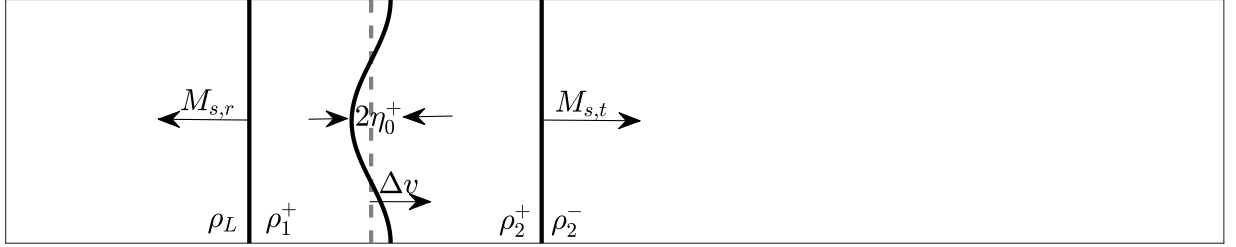


FIG. 1: Sketches for the pre- (a) and post-shock (b) states of the compressible RMI problem with surface tension. The domain is separated by a sinusoidal interface featuring initial amplitude η_0^- and wavelength λ . The fluids to the left and right of the interface are labelled '1' and '2' respectively. The incident shock with Mach number $M_{s,i}$ changes the state of fluid 1 from pre-shock (with superscript '-') to intermediate (with subscript 'L') before its interaction with the interface, which compresses the interfacial perturbation to η_0^+ . The transmitted shock and reflected wave bring the two fluids to their final post-shock states (with superscript '+'), while the interface perturbation develops under the influence of surface tension σ .

A. Pre-shock dimensionless parameters

We discuss the state of the problem prior to the shock-interface interaction. The densities of the pre-shock fluids are ρ_i^- ($i = 1, 2$), the unperturbed pressure in each fluid is p_0 and the specific heat ratio of each fluid is Γ_i . The fluids are considered inviscid in this model. The interface which separates the fluids has a monochromatic, sinusoidal perturbation of wavelength λ and amplitude η_0^- , and corresponding wavenumber $k = 2\pi/\lambda$.

The surface tension is given by σ . The corrugated interface slightly perturbs the pressure in the two fluid by the action of surface tension; which may be approximated as $\Delta p = 2\sigma/R_c = 2\sigma\eta_0^- k^2$, where we use the Young-Laplace Equation and take the characteristic radius of curvature R_c as that at the extrema of the sinusoidal interface. In our work, the maximum magnitude of the ratio $\Delta p/p_0$ is 10^{-2} ; thus we conclude that this pressure-perturbation Δp has negligible effect on the subsequent evolution of the system.

The incident shock is planar with a speed $u_{s,i}$ and induces a velocity u_L in fluid 1, which is known analytically from the Rankine-Hugoniot shock relations. Applying Buckingham's theorem, we form the following four dimensionless groups apart from Γ_i :

$$s^- \equiv \eta_0^- k = 2\pi \frac{\eta_0^-}{\lambda}, \quad A^- \equiv \frac{\rho_2^- - \rho_1^-}{\rho_2^- + \rho_1^-}, \quad We^- \equiv \frac{\rho_1^- + \rho_2^-}{\sigma k} A^{-2} u_L^2, \quad M_{s,i} = \frac{u_{s,i}}{\sqrt{\Gamma_1 p_0 / \rho_1^-}}, \quad (1)$$

where s^- characterizes the slope of the initial perturbed interface; A^- is the Atwood number, which represents

the initial density setup; We^- is the Weber number, which measures the strength of surface tension, and $M_{s,I}$ is the Mach number of incident shock, which satisfies $M_{s,I} > 1$. It should be noted that the following incident shock strength $\epsilon \in (0, 1)$, defined as the relative pressure change in Fluid 1 after the passage of the incident shock, is sometimes used instead of $M_{s,I}$ within literature (e.g. [22, 23]) and may be calculated from $M_{s,I}$ and Γ_1 using the Rankine-Hugoniot shock relation,

$$\epsilon \equiv 1 - \frac{p_0}{p_L} = \frac{2\Gamma_1(M_{s,I}^2 - 1)}{\Gamma_1(2M_{s,I}^2 - 1) + 1}. \quad (2)$$

B. Post-shock dimensionless parameters

We will now discuss the application of Mikaelian's incompressible, impulsive model [5] to the compressible case, and develop the appropriate dimensionless parameters. Mikaelian predicts that, if the perturbation development is in the linear regime (i.e. when the slope s remains small), and the effect of compressibility is negligible after the shock-interface interaction, surface tension will act as a restoring force and cause the post-shock perturbation amplitude $\eta(t)$ to oscillate sinusoidally,

$$\eta(t) = \eta_0^+ \cos(\omega t) + \frac{\dot{\eta}_0^+}{\omega} \sin(\omega t) = \eta_0^+ \sqrt{1 + \left(\frac{\dot{\eta}_0^+}{\eta_0^+ \omega}\right)^2} \sin\left(\omega t + \arctan \frac{\eta_0^+ \omega}{\dot{\eta}_0^+}\right), \quad (3)$$

where η_0^+ is the post-shock initial perturbation amplitude, and the capillary angular frequency ω is defined in the following form, with post-shock fluid densities ρ_1^+ and ρ_2^+ :

$$\omega = \sqrt{\frac{k^3 \sigma}{\rho_1^+ + \rho_2^+}}. \quad (4)$$

$\dot{\eta}_0^+$ is the post-shock initial perturbation growth rate, which Mikaelian [5] gives as,

$$\dot{\eta}_0^+ = \Delta v k A^+ \eta_0^+, \quad (5)$$

where the post-shock Atwood number $A^+ \equiv (\rho_2^+ - \rho_1^+)/(\rho_2^+ + \rho_1^+)$, and Δv is the post-shock velocity jump of the interface. Equation (5) is identical to Richtmyer's [24] prescription for growth rate in the RMI without surface tension. Equations (4) and (5) combined yield the following result:

$$\left(\frac{\dot{\eta}_0^+}{\eta_0^+ \omega}\right)^2 = \frac{(\rho_1^+ + \rho_2^+) A^{+2} \Delta v^2}{\sigma k}. \quad (6)$$

The term $(\dot{\eta}_0^+/\eta_0^+ \omega)^2 = We^+$ is a post-shock Weber number of the perturbed interface, and measures the relative importance of the inertia of the flow compared to the surface tension. A set of dimensionless groups to describe the growth characteristics of the RMI can be further defined in terms of post-shock variables to be,

$$s^+ = 2\pi \frac{\eta_0^+}{\lambda}, \quad A^+ \equiv \frac{\rho_2^+ - \rho_1^+}{\rho_2^+ + \rho_1^+}, \quad We^+ \equiv \left(\frac{\dot{\eta}_0^+}{\eta_0^+ \omega}\right)^2 = \frac{\rho_1^+ + \rho_2^+}{\sigma k} A^{+2} \Delta v^2. \quad (7)$$

These are of the same form as the pre-shock parameters, and both will be used throughout this work. According to Equation (3), the time-dependent amplitude $\eta(t) \equiv \eta$ depends on $\sqrt{1 + We^+}$. Further, combining (4) with (7) yields $\omega = A^+ \Delta v k / \sqrt{We^+}$. These can be used to non-dimensionalize amplitude and time t by,

$$\tilde{\eta} \equiv \frac{\eta}{\eta_0^+ \sqrt{1 + We^+}}, \quad \tilde{t} \equiv \omega t = \frac{k A^+ \Delta v}{\sqrt{We^+}} t, \quad (8)$$

where the tildes indicate nondimensional variables. With these parameters introduced, Mikaelian's model (3)

can be summarized in the following normalized form,

$$\tilde{\eta} = \sin \left(\tilde{t} + \arctan \frac{1}{\sqrt{We^+}} \right). \quad (9)$$

We note that saturation at $\tilde{t} = \pi/2 - \arctan(1/\sqrt{We^+})$, in the regime discussed by Mikaelian [5] and in which our Equation (9) is presented, is governed by linear processes dependent on strong surface tension. On the other hand, for small surface tensions nonlinear effects in underlying instability come to dominate, which cannot be predicted by Equation (9); [but the post-shock normalization scheme \(8\) may still be used for comparison with strong surface tension cases, especially for the transition to nonlinear regime with intermediate surface tension, as will be discussed in §IV C.](#)

It is of interest to understand the relation between pre-shock and post-shock parameters. It has been concluded in Velikovich [25] that in the weak-shock limit ([incident shock strength \$\epsilon \rightarrow 0\$](#)), the pre- and post-shock values of perturbation slope, Weber and Atwood number are close and there is no need to distinguish between them. It is also in the same weak-shock limit that Richtmyer's prescription is reported to give most accurate results (see for example Figure 1 in Velikovich *et al.* [26].) In general, however, the post-shock parameters may deviate significantly from the pre-shock values. For the present study, the post-shock parameters are determined numerically. For a discussion on its effectiveness in comparison with alternative methods, see the appendix §B.

Finally, a change in sign of the pre-shock Atwood number A^- may introduce a qualitative change in the shock interaction process [22, 27, 28]. When the shock wave interacts with the interface, it undergoes a refraction process which results in a transmitted shock and a reflected wave, which may be a shock or a rarefaction wave. The reflected wave type depends on A^- and Γ_i . Drake [27] identifies the following critical pre-shock Atwood number A_c^- for an unperturbed interface,

$$A_c^- = \frac{\Gamma_1 - \Gamma_2}{\Gamma_1 + \Gamma_2 + 2}. \quad (10)$$

When $A^- < A_c^-$, a reflected rarefaction is expected to form in fluid 1; otherwise a reflected shock is expected. Within our work, we set the specific heat ratios as $\Gamma_1 = \Gamma_2 = 1.4$, hence $A_c^- = 0$, i.e. negative pre-shock Atwood numbers A^- correspond to reflected rarefactions, and positive ones to reflected shocks. This effect is not accounted for in Mikaelian's model [5]; its consequences will be examined in §IV A.

III. FORMULATION AND METHODOLOGY

A. Governing equations

We solve the two-phase compressible Euler equations with surface tension, which are written in their averaged dimensional form,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (11)$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla p + \sigma \kappa \delta_s \mathbf{n}, \quad (12)$$

$$\frac{\partial}{\partial t} \left(\rho e + \frac{1}{2} \rho \mathbf{u}^2 \right) + \nabla \cdot \left[\mathbf{u} \left(\rho e + \frac{1}{2} \rho \mathbf{u}^2 \right) \right] = -\nabla \cdot (\mathbf{u} p) + \sigma \kappa \delta_s \mathbf{n} \cdot \mathbf{u}. \quad (13)$$

Equations (11) and (12) are respectively the continuity and momentum equation, where p is the fluid pressure. The influence of surface tension is incorporated as the volumetric term $\sigma \kappa \delta_s \mathbf{n}$ within Equation (12), where σ is the surface-tension coefficient, κ and \mathbf{n} the local curvature and normal vector on the interface. The Dirac delta δ_s is non-zero only on the interface, indicating the local concentration of surface tension effects [29, 30]. The energy equation (13) is included owing to the presence of compressibility, where e denotes the specific internal energy. Following [20], influences of thermal diffusion and mass transfer are neglected. Equations of state are still required to close the equation system. While both incompressible and compressible fluids can be modeled simultaneously by Mie-Grüneisen equation of state in the numerical solver, for this study we restrict

our attention to entirely compressible flow by applying the ideal gas law as a special case:

$$\rho_i e_i = \frac{p_i}{\Gamma_i - 1}, \quad (14)$$

where ρ_i , e_i , p_i ($i = 1, 2$) are respectively the density, specific internal energy and pressure of each fluid.

The following advective equation (15) is applied to determine the interface position, featuring a tracer function f that distinguishes between fluids 1 and 2 [20]:

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = 0. \quad (15)$$

Finally, in Figure 1, the wavelength λ , wave number k and post-shock perturbation growth rate $\dot{\eta}_0^+$ suffice to provide natural reference scales for the length, mass and time quantities that appear in the system (15)-(13). Thus, following the work of Dimonte [31, 32], the non-dimensional variables,

$$\frac{\eta}{\lambda}, \quad \frac{u}{|\dot{\eta}_0^+|}, \quad k|\dot{\eta}_0^+|t, \quad (16)$$

will be used in the remainder of the study as an alternative to the non-dimensional parameters presented in §II.

B. Numerical method

The simulations in this work are conducted within the open-source solver Basilisk using the all-Mach scheme proposed by Fuster and Popinet [20] for multiphase flows, which is capable of handling mixed compressible-incompressible fluids. This is a second-order accurate finite volume numerical scheme with hyperbolic upwinding suitable for shock capturing. Within the solver, the mass and energy conservation equations (11), (13) are solved separately for each phase, while the geometric volume-of-fluid (VOF) reconstruction method guarantees a sharp representation of the fluid interface and reduces the parasitic currents induced by surface tension. For modelling of surface tension effects, $\delta_s \mathbf{n}$ in Equation (12) is approximated as the gradient of the VOF tracer ∇f following Brackbill's method [33], and the curvature κ is calculated by taking the finite-difference approximation of the derivatives of interface height functions [29].

The problem is initialized within a rectangular simulation domain of size $nD \times D$, where D is the width of the domain and $n = 5, 7$ or 11 , depending on the particular case. The shock is initialized at the left boundary, so that the domain is defined by $\Omega = [x_S, nD + x_S] \times [-D, 0]$. The boundary conditions on the top and bottom of the domain are periodic. We use a zero-gradient boundary condition at the right boundary, while at the left side Dirichlet conditions are applied according to the post-shock conditions for a incident shock of Mach number M_s ; these are discussed further below.

The fluid interface is initially a cosine function of the form $x(y)/\lambda = x_I/\lambda + (\eta_0^-/\lambda) \cos ky$ where λ is set to the domain size D . The average position of the interface x_I/λ is set to 0. The incident shock is initialized at the left boundary of the simulation domain, at $x = x_S$ by assigning the values of ρ_L , q_L , E_L to the conservative variables via the aforementioned Dirichlet boundary conditions. We set the left boundary x_S immediately next to the initial interface at $x_I = 0$, so that the surface-tension-induced self-oscillation behavior does not have sufficient time to influence the pre-shock perturbation amplitude before its interaction with the incident shock.

The discretized grid size, being the same in x and y direction, is defined as $h = D/2^L$, where L is the resolution level. Most of our simulations are conducted on $L = 9$, while for convergence studies we also run certain cases on levels $L = 8$ and $L = 10$ for comparison. The discretized timestep is characterized and controlled by a non-dimensional constant $CFL_{ac} = (c_m + u_m)\Delta t/h$, where c_m and u_m are the magnitudes of the expected maximum local speed of sound and the fluid velocity respectively. For simulations conducted in this work, we set $CFL_{ac} = 0.5$ and 0.25 for strong-shock and weak-shock cases respectively.

We now present the pre-shock non-dimensional parameter space to be investigated in this work. Namely, we conduct the simulations with pre-shock slope s^- values going from low (0.02π), medium (0.03π) to high (0.04π), all of which are sufficiently small to satisfy the linear regime prerequisite $s^- \ll 1$. We thus allocate approximately 5-10 grid cells across the initial perturbation profile at resolution level $L = 9$, which is sufficient to capture the growth characteristics of the RMI independent of the numerical resolution, as shown in §A. We set $M_{s,I} = 2$ (strong-shock, corresponding to incident shock strength $\epsilon = 0.78$) in most situations, where influence of compressibility is already nontrivial but has not yet caused significant deviations from the impulsive

model (see Figure 8 in [34]); while also setting $M_{s,I} = 1.2$ (weak-shock, corresponding to $\epsilon = 0.40$) in certain cases to allow for discussions on its influence. Pre-shock Atwood numbers are set as $A^- = 9/11$ and $-9/11$. The magnitude of pre-shock Weber numbers We^- we investigate ranges from 10^1 to 10^3 , which covers all three of the linear, transitional and nonlinear post-shock growth-rate regime.

IV. RESULTS

A. Linear regime

Mikaelian's existing theory is written in terms of the post-shock state, which is not known *a priori* in this study. We therefore determine the post-shock state, characterized by ρ_1^+ , ρ_2^+ , Δv , and η^+ numerically. This sets post-shock dimensionless parameters s^+ , A^+ and We^+ , as discussed in §B, where we find the post-shock Atwood numbers A^+ to be 0.793 and -0.819 , corresponding to the light-heavy and heavy-light density configurations.

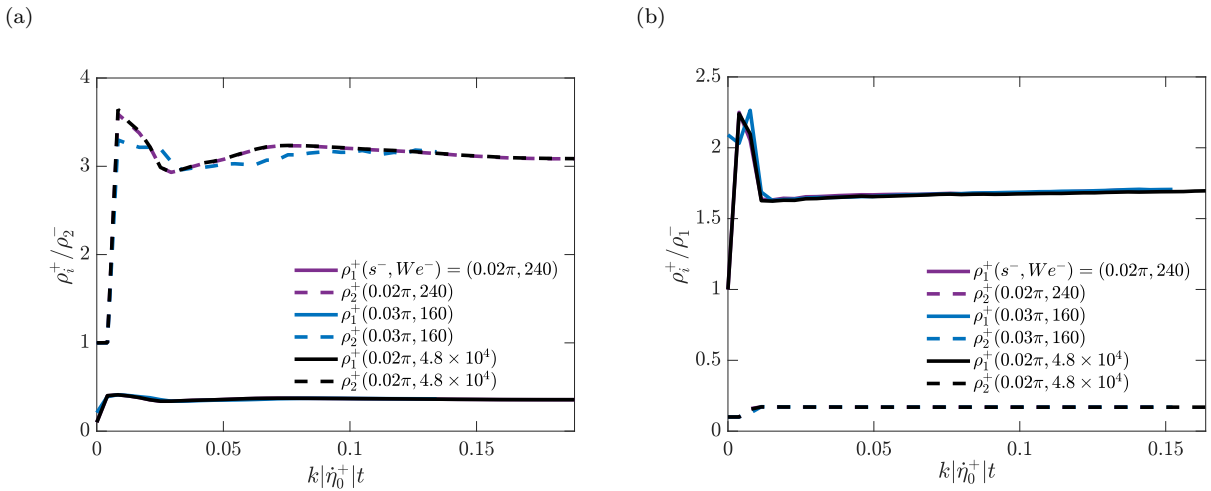


FIG. 2: The development of post-shock density values ρ_i^+ in cases with pre-shock Atwood number $A^- = 9/11$ (a) and $A^- = -9/11$ (b) measured from numerical diagnostics. The post-shock densities of the two fluids are observed to eventually settle down at steady-state values, which are chosen as the post-shock values determined by numerical diagnostics.

In the diagnostics, the post-shock state variables are extracted close to the interface, and averaged over y to remove variations due to the interface perturbation. Then we non-dimensionalize the post-shock variables by the natural units introduced in §III A. Figure 2 shows the numerical outputs for sample cases with $A^- = \pm 9/11$, $We^- = 160, 240$, and $s^- = 0.02\pi, 0.03\pi$.

All measured variables behave similarly in the diagnostics: there is a short transient period at early times associated with shock-interface interaction, before the measured variables stabilize as the transmitted and reflected wavefronts move away from the post-shock interface. The steady-state values are not affected by the value of s^- , as expected, and are taken as the post-shock state. As discussed in the appendix B, these numerical diagnostics are more reliable for the present study than theoretical alternatives. Note that the density remaining constant after the initial transients supports the assumption that compressible effects are limited to the time during and immediately after the shock-interface interaction, and that the flow is approximately incompressible afterwards. This is consistent with conclusions from other numerical studies [6, 35].

We now test the normalized model of Mikaelian [5] given by Equation (9), which predicts a relationship between the non-dimensional amplitude $\tilde{\eta}$ and the non-dimensional time \tilde{t} independent of the other non-dimensional parameters (provided that the post-shock Weber numbers We^+ are larger than 200, which causes an error less than 4.5% at the first peak if the phase shift term $\arctan 1/\sqrt{We^+}$ is neglected).

The results are presented in Figure 3, where they are organised into three major categories: strong shock ($M_{s,I} = 2$) with positive Atwood number ($A^- = 9/11$) in the first row, strong shock with negative Atwood number ($A^- = -9/11$) in the second and weak shock ($M_{s,I} = 1.2$) with positive Atwood number in the third row. For each Atwood/Mach number category, a sweeping of Weber number We^- is conducted. In the left column (Figures 3a, 3c, 3e), the amplitude and time are normalized naturally (see §III); in the right column,

the scaled amplitude and time based on Mikaelian's model (see §II) are used (Figures 3b, 3d, 3f). To facilitate comparison of simulation results with Mikaelian's model [5], we also plot in dashed lines the theoretical $\tilde{\eta} - \tilde{t}$ curves for Weber numbers larger than 200, at which the phase shift term $\arctan 1/\sqrt{We^+}$ becomes negligible.

In the following parts, we will first discuss the influence of Atwood number A^- on post-shock perturbation growth in §IV A 1 by comparing the first and second row in Figure 3, and then the influence of incident shock Mach number $M_{s,I}$ in §IV A 2 by comparing the first and third row of the same figure.

1. Effect of initial fluid density configuration

As is shown in Figures 3a, 3c, when normalized using the natural scheme introduced in §III A, the post-shock perturbation amplitude η/λ will first increase with diminishing growth rate. At small post-shock Weber number We^+ , η/λ will reach a peak and then decrease, whereas for large We^+ values the peak will not be reached in the limited simulation domain. These findings agree with the numerical results of [21].

When the time developments of perturbation are normalized using the scheme developed in §II B (see Figures 3d, 3f), they overlap very well for both positive and negative Atwood number cases with $M_{s,i} = 2$. However, discrepancy exists between the two Atwood number classes: the positive Atwood number cases have their first normalized peaks around $\tilde{\eta} = 1.1$ (Figure 3b), while those of the negative Atwood number cases are much higher, being around $\tilde{\eta} = 7.3$ (Figure 3d). Consequently, the positive Atwood number cases conform to Mikaelian's model (9) more closely, whereas the negative Atwood number cases show a nontrivial deviation from the same model.

This negative-Atwood discrepancy is rooted in the incompressible nature of Mikaelian's model [5], and might be traced further back to the situations where surface tension σ is absent. As the post-shock Weber number We^+ asymptotically approaches infinity, Mikaelian's model [5] will reduce to Richtmyer's impulsive prescription [24] in the form of Equation (5). Vandenboomgaerde *et al.* [22] and Velikovich [25] observe that this prescription usually gives good results for positive Atwood number cases, but fails for negative Atwood number cases where $\dot{\eta}$ is not proportional to A^+ [23] and other alternatives are available (e.g. [22, 28, 36]). In particular, the negative Atwood number is qualitatively different from the positive case, as it generates a reflected rarefaction wave rather than a reflected shock, as noted in §II B. This is the physical reason for this discrepancy.

We will now seek to develop a correction to the linear theory of [5] that can effectively reduce the discrepancy caused by opposite signs of Atwood numbers. An alternative to Equation (5) is given in [22] as:

$$\eta = \eta_0^+ + k\Delta v \left[\frac{1}{2} (A^+ \eta_0^+ + A^- \eta_0^-) - \frac{1}{6} (A^+ - A^-) (\eta_0^+ - \eta_0^-) \right] t. \quad (17)$$

In this prescription, the post-shock perturbation growth rate now depends on both pre- and post-shock states, which is different from Mikaelian's model [5], as the latter is only related to the post-shock state. According to [22], this prescription takes into account the shock-induced compression of the perturbation and variation of Atwood number in a simplified way; and while Dimonte [32] notes that this prescription might be problematic when A^+ and A^- are very different, Table I suggests that the two Atwood numbers are very close in our numerical calculations, especially for the negative Atwood number cases where $A^- = -9/11$, in which case Equation (17) reduces to the following prescription of Meyer and Blewett [28]:

$$\eta = \eta_0^+ + k\Delta v \left[\frac{1}{2} A^+ (\eta_0^+ + \eta_0^-) \right] t, \quad (18)$$

which has been reported to match well with the RMI perturbation growth trend in cases with negative A^- [32, 37]. Consequently, we expect good performance of Vandenboomgaerde's prescription [22] in the parameter space we explore. We then seek to compare Vandenboomgaerde's prescription [22] with Richtmyer's [24] in the zero-surface-tension cases with different Atwood number setups.

As is shown in Figures 4a and 4b, we normalize time t and the perturbation amplitude η by the natural units introduced in §III A. It is found that Vandenboomgaerde's prescription (17) matches much better with simulation results for the negative Atwood number cases compared with Richtmyer's prescription (5) [24], and is also close to the numerical results in the positive Atwood number cases. The failure of Richtmyer's prescription [24] for the negative Atwood number case corresponds back to the deviation patterns of our simulation results from the scaling model in the previous part, where the $A^- = 9/11$ cases slightly overshoots the theoretical maximum $\tilde{\eta}_{max} = 1$ under the scaling, and those with $A^- = -9/11$ significantly overestimates

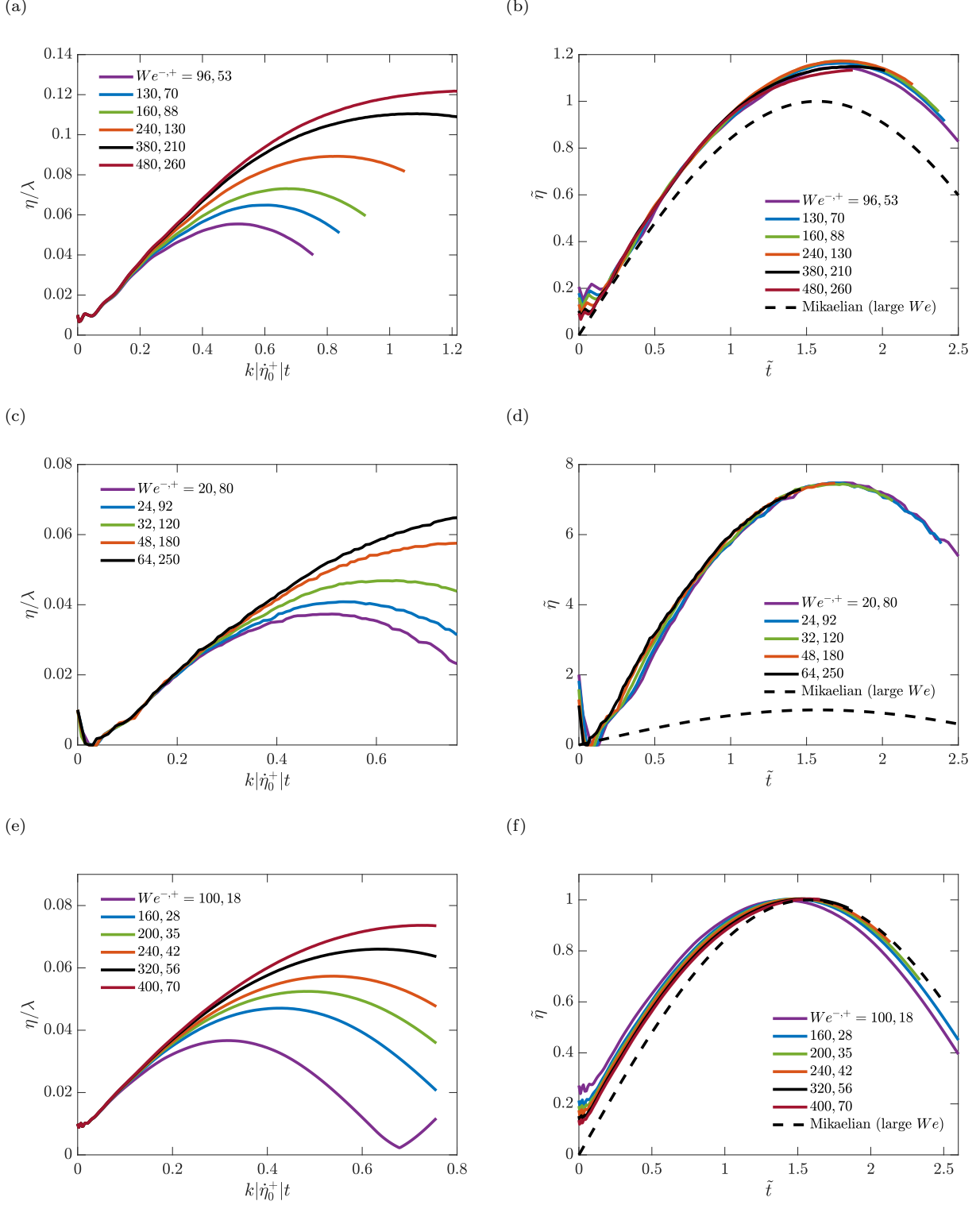


FIG. 3: Numerical results compared with Mikaelian's theory [5]. Upper row: strong shock ($M_{s,I} = 2$) with negative Atwood number ($A^- = -9/11$); middle row: strong shock ($M_{s,I} = 2$) with positive Atwood number ($A^- = 9/11$); lower row: weak shock ($M_{s,I} = 1.2$) with positive Atwood number ($A^- = 9/11$). Left column: perturbation growth scaled with the natural units. Right column: results scaled using the post-shock dimensionless parameters. Good collapsing patterns are observed for strong-shock cases ($M_{s,I} = 2$), among which those with positive Atwood number $A^- = 9/11$ show a good qualitative agreement with Mikaelian's theory [5]; whereas the weak-shock cases display poor collapsing under the normalization.

the same maximum.

Consequently, in order to modify Mikaelian's model [5] for a better performance, we tentatively replace the Richtmyer's prescription embodied in (3) with Vandenboomgaerde's [22]. To this end, we introduce the modified Atwood number \tilde{A} based on Vandenboomgaerde's work [22], which may be viewed as an average of pre- and post-shock Atwood numbers A^- and A^+ involving compression ratio $r \equiv \eta_0^+/\eta_0^-$:

$$\tilde{A} \equiv \frac{1}{2} \left(A^+ + \frac{A^-}{r} \right) - \frac{1}{6} (A^+ - A^-) \left(1 - \frac{1}{r} \right). \quad (19)$$

With this modified Atwood number defined, the post-shock dimensionless parameters (7), (8) and normalized perturbation-growth model (9) proposed in §II B can be formally retained by replacing all A^+ with \tilde{A} , which is then used to non-dimensionalize the results for Figures 4c and 4d.

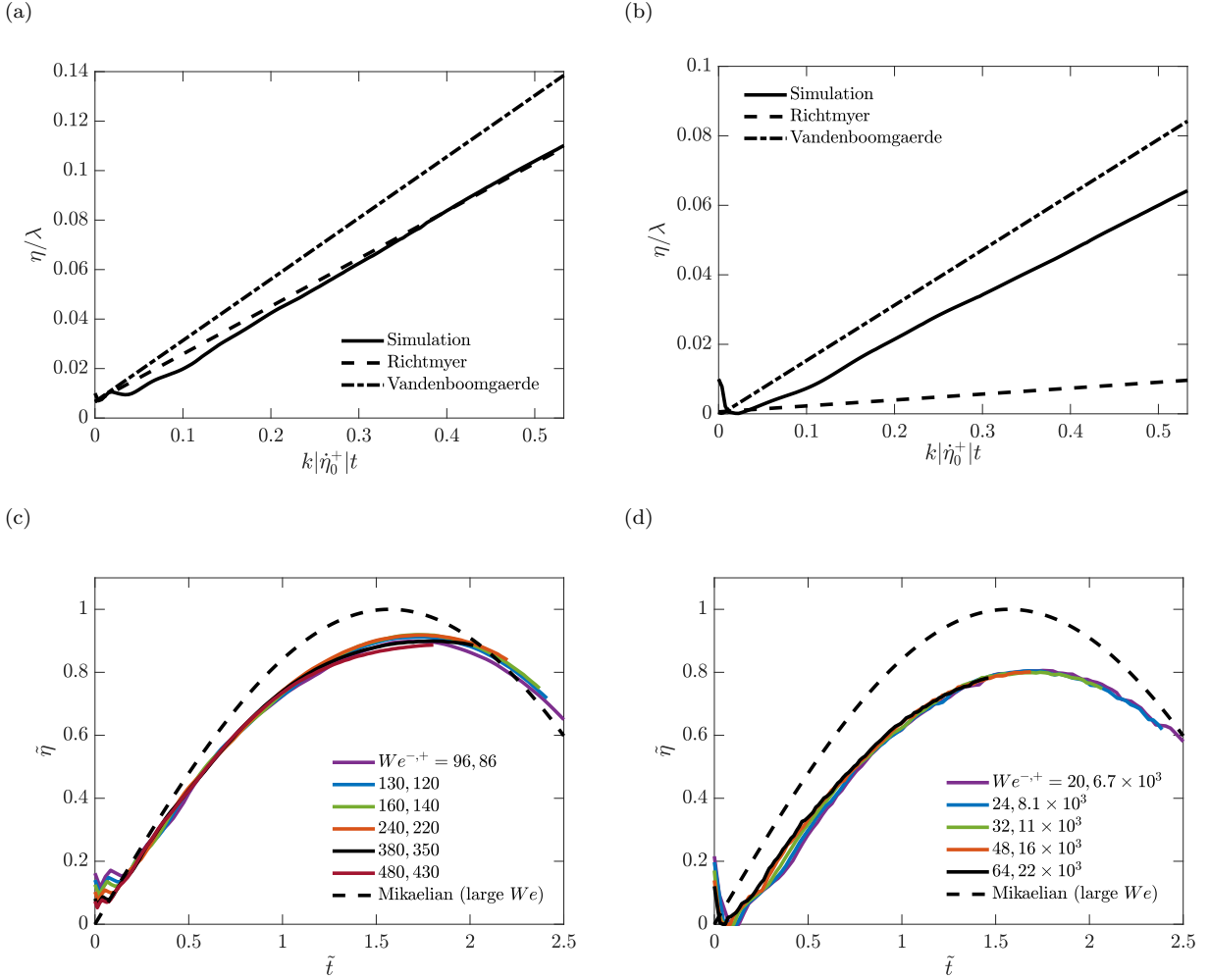


FIG. 4: Upper row: Comparison of the simulation results with the theoretical predictions of Richtmyer [24] and Vandenboomgaerde *et al.* [22]. Lower row: simulation results scaled using the modified model in cases with surface tension. (a, c): $A^- = 9/11$, (b, d): $A^- = -9/11$. The theory of Vandenboomgaerde [22] shows a better match with simulation results in both negative and Atwood number cases without surface tension, leading to improved performance of the modified theoretical model.

When the modified model is compared with simulation results, as is shown in Figures 4c and 4d, within each Atwood number category the normalized curves still show good collapsing, and good agreement is found between the results of the two Atwood numbers with opposite signs, as the first peaks of all normalized curves are now around $(\tilde{t}, \tilde{\eta}) = (1.65, 0.8)$. This maximum $\tilde{\eta}$ value of 0.8 is less than unity as expected by the modified model, which is likely caused by the aforementioned overestimation of η_0^+ by Vandenboomgaerde's prescription [22] in zero-surface-tension cases; but it is still a considerable improvement compared with the original model of Mikaelian [5], especially for negative Atwood number cases since their discrepancy in peak values with

positive Atwood ones is now greatly reduced.

2. Effect of incident shock strength

Apart from surface tension and initial density setups, Mach number of the incident shock $M_{s,i}$ also plays a significant role in the post-shock perturbation growth, as it is directly connected with shock strength and effects of compressibility via Equation (2) [22]. Since Mikaelian's model [5] is essentially impulsive and incorporates Richtmyer's prescription [24], it is natural to expect better matching of the simulation results with our scaled model in the weak shock limit. We find that this is indeed the case for $M_{s,i} = 1.2$, as the perturbation growth patterns shown in Figure 3f are closer to Mikaelian's prediction compared with Figure 3b, with maximum normalized perturbation amplitude values almost exactly equaling 1. The recent numerical work by Bigdelou [9] has also confirmed a good match of linear-regime perturbation growth with Mikaelian's prediction for RMI cases with $M_{s,i} = 1.2$, where a different pre-shock Atwood number $A^- = 2/3$ is used.

We further discuss the applicability of our modification of Mikaelian's model (9) in §IV A 1. Mikaelian approximates the impact of the incident shock on the interface as an impulsive acceleration, which is compatible with the prescriptions of both Richtmyer [24] and Vandenboomgaerde [22]. More accurate models for perturbation growth under the influence of surface tension can be derived using the same approach in §IV A 1, i.e. swapping Vandenboomgaerde *et al.*'s prescription (19) for more precise ones, preferably those accounting for nonlinear perturbation development (e.g. [38]), and combining them with Mikaelian's model, as long as the prescriptions introduced are compatible with the impulsive base of Mikaelian. However, it should be noted that the impulsive model will become inaccurate at higher values of $M_{s,I}$ (see e.g. the limit of $\epsilon \rightarrow 1$ in Figure 5 of [11]), where compressible theories predicting perturbation growth have to be proposed. This is a fundamental problem for RMI studies and requires detailed investigation of the shock-interface interaction period, which is out of the scope of this work.

Consequently, we conclude that in the linear oscillation regime with strong surface tension, the scaled model (9) based on Mikaelian [5] match very well with weak-shock cases, whereas the strong-shock positive Atwood number cases show maximum perturbation values slightly larger than those predicted by Mikaelian due to stronger influence of compressibility. The strong-shock negative Atwood number cases show maximum $\tilde{\eta}$ values that are several times larger than Mikaelian's prediction, but still keep good collapsing patterns under the proposed non-dimensionalization scheme (8). This non-trivial negative-Atwood deviation is due to the well-attested failure of Richtmyer's prescription [5] incorporated in Mikaelian's model, and may be reduced by introducing more accurate prescriptions for post-shock initial perturbation growth rate $\dot{\eta}_0^+$.

B. Nonlinear regime

As the simulation cases with incident shock Mach number $M_{s,I} = 1.2$ feature relatively small Weber numbers and weak influence of compressibility, and their perturbation growth patterns are thus close to the prediction of Mikaelian [5], in the following parts we mainly focus on the analysis of the simulation results with $M_{s,I} = 2$.

When surface tension becomes small enough, it can no longer curb the perturbation growth and the development of asymmetric spikes and bubbles on the interface. Figure 5 shows the evolution of a bubble (the broad structure straddling the periodic boundary of the domain) and a spike (the narrow structure near the center of the domain) for $We^- = 3800$. As the surface tension is very small, the asymmetry between bubbles and spikes becomes evident as the non-dimensional time $k|\dot{\eta}_0^+|t$ approaches unity, as reported by Dimonte [31] for the onset of nonlinear effects in RMI without surface tension. Sizes of the bubble and the spike can be calculated by measuring the difference between the local and average interface positions, whose time derivatives yield bubble velocity U_B and spike velocity U_S .

The development of calculated bubble velocities U_B and spike velocities U_S at different Weber numbers are first non-dimensionalized using the natural units discussed in §III A, and then shown in Figure 6. The development patterns of bubble velocity are almost the same initially for different Weber numbers, displaying early-time damped oscillating behavior followed by a constantly decreasing period, which agree well in trend with [39] where front-tracking simulations are used. Similar early-time oscillations of bubble and spike velocities have been observed in [34], where compressibility is involved; but are lacking in works focusing on incompressible flows (e.g. [13, 40]), indicating that this is an effect of compressibility. Mikaelian [23] reported the oscillations of perturbation growth rate $\dot{\eta}$, and ascribed it to the 'rippling' behavior of the transmitted and reflected shocks; i.e. the profile of the two shocks are gradually flattened after they travel a distance of magnitude $\lambda = D$ away from the interface, as is also noted in §II.

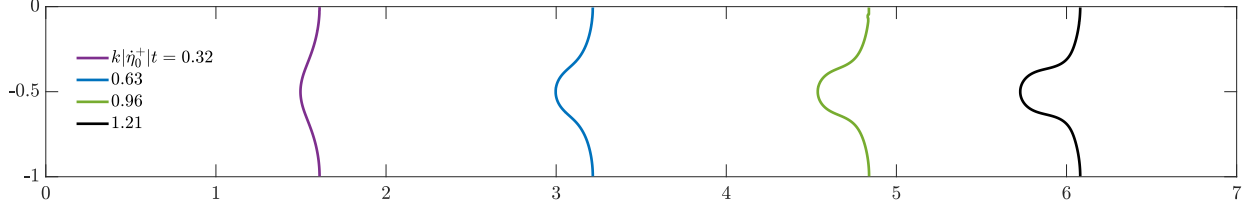


FIG. 5: A series of $We^- = 3800$ simulation snapshots showing the evolution of the perturbed interface from the initially sinusoidal shape to one with a bubble (across the periodic boundaries) and a spike (at the center).

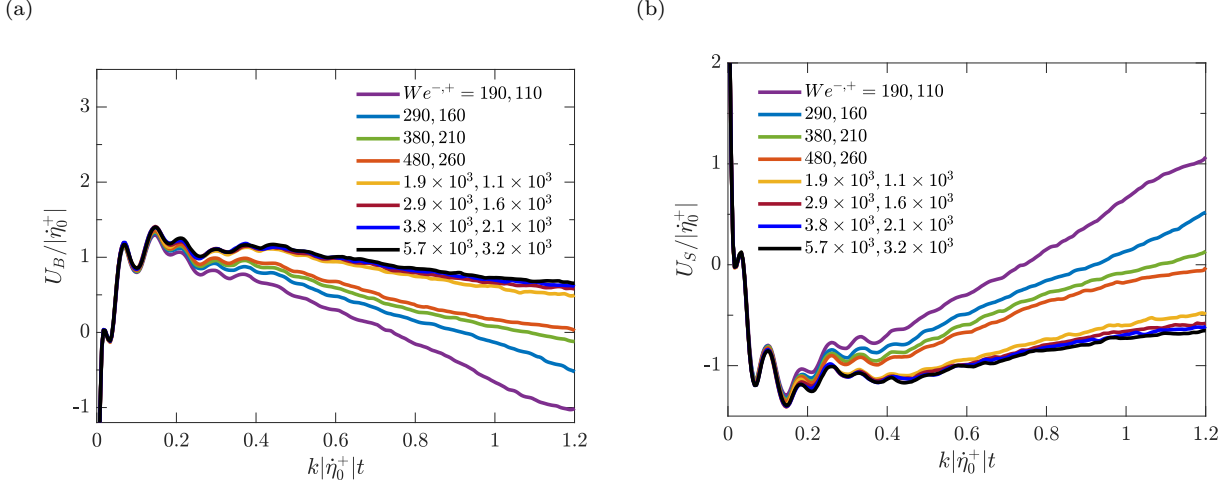


FIG. 6: Development of bubble (a) and spike (b) velocity for transitional and nonlinear cases with $A^- = 9/11$. As Weber number increases, the curbing effect of surface tension on the post-shock perturbation growth diminishes, and the time development patterns of bubble and spike velocities remain roughly the same within a large range of high Weber numbers (greater than 10^3).

Within the decreasing phase of bubble growth rate, for the cases with pre-shock Weber numbers less than ~ 500 , the ‘bubble velocity’ (more precisely, the growth rate of the sinusoidal crests as bubbles have not yet formed in these cases) eventually decreases below 0 due to the long-term restoring effect of surface tension. As the Weber number is further increased, the curves remain positive for the entire simulated time and become less sensitive to pre-shock Weber number beyond ~ 1900 (i.e. post-shock Weber number We^+ beyond 1100, which we consider as the lower bound of the nonlinear regime, as noted in §I). This indicates that within this range of Weber numbers, the development of bubbles velocity departs from an oscillatory regime and approaches asymptotically zero velocity. The absolute values of spike velocities $|U_S|$ in the high Weber number cases are also decreasing after the oscillation period, similar to that of the bubble velocity.

Based on potential flow methods, Sohn derived a nonlinear and incompressible model in [13] for the late-time bubble development, which accounts for effects of both viscosity and surface tension in RMI at the same time. A similar model is also proposed in [41]. Specifically, when the two fluids are inviscid, and the non-dimensional expression of Sohn’s model [13] reads:

$$\hat{U}_B = \cot \hat{t}, \quad (20)$$

where

$$\hat{U}_B \equiv \frac{3}{A^+} \sqrt{\frac{(1+A^+) We^+}{2}} \frac{U_B}{\Delta v}, \quad \hat{t} \equiv \frac{\sqrt{2(1+A^+)}}{3+A^+} \hat{t} = \frac{\sqrt{2(1+A^+)}}{3+A^+} \frac{k A^+ \Delta v}{\sqrt{We^+}} t. \quad (21)$$

Note that the non-dimensional formulation (21) is not derived from the model of Mikaelian (Equations (8) and (9)), which is not applicable in the regime with asymptotically small surface tension discussed here. Rather, the post-shock non-dimensional parameters defined in (7) are retained to re-formulate Sohn’s nonlinear theory for consistency.

We now seek to compare our measured bubble velocities with Sohn's model [13]. For this purpose, we normalize bubble velocity and time extracted from simulation cases with $We^- = 480, 1900, 3800$ and 5700 according to the definitions of \hat{U}_B and \hat{t} . The normalized curves are plotted in Figure 7 and compared with Sohn's model [13] in the form of (20).

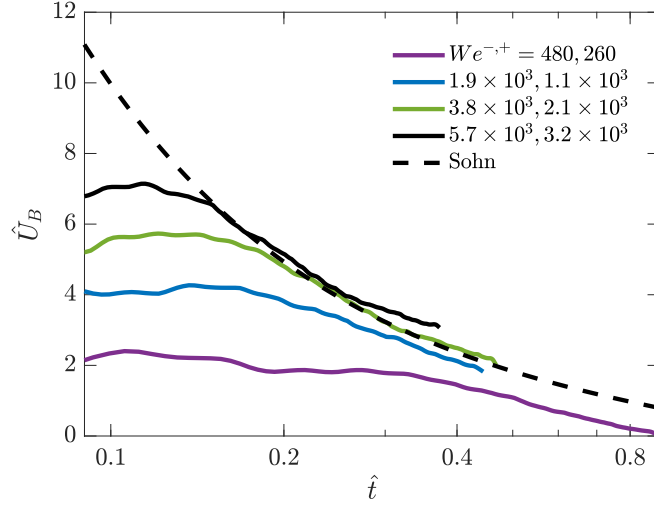


FIG. 7: Comparison between simulated (solid lines) and theoretical (dashed line) bubble velocity developments in cases with $We^- = 480, 1900, 3800$ and 5700 . Good agreements with Sohn's theory are found at asymptotically large Weber numbers.

As is shown in Figure 7, for the transitional Weber number case with $We^- = 480$, Sohn's model [13] significantly overestimates the U_B development within the simulated time range, despite correctly capturing the late-time decreasing trend. As for the cases with higher Weber numbers exceeding 10^3 , where surface tension is weak enough to give way to formation of bubbles and spikes, the normalized simulation results asymptotically converges to Sohn's cotangent model [13] as normalized time \hat{t} increases. Also, as the Weber number increases, convergence to Sohn's model [13] will occur at earlier normalized time \hat{t} . This verifies that Sohn's model [13] applies for asymptotically high Weber numbers, since, for lower Weber numbers, surface tension still has some curbing effects on the development of bubbles and spikes.

Within Sohn's text [13], Equation (20) is defined for asymptotically large We^+ so that the singularity at $\hat{t} = \pi/2$ is not reached for any physical time t . However, our simulations feature large but finite We^+ , so that for sufficiently large physical time t , the $\hat{t} = \pi/2$ singularity may be reached in simulations. This may correspond with the bubble velocity becoming negative and the interface thus exhibiting nascent oscillatory behavior, as speculated by Sohn [13].

Therefore, we conclude that within the time range investigated, our results agree well with Sohn's model [13] in the bubble-development period of very high Weber number cases, besides the inability of Sohn's model to capture the early oscillatory behaviour of the bubble and spike velocities, due to compressible effects, observed in this study. The development of spikes and bubbles at later time remains to be investigated, where Dimonte [32] reports that bubble amplitudes will eventually saturate due to nonlinear effects; and in the meantime spikes will continually grow and eventually break off from the interface, as has been predicted by Sohn [13] and Matsuoka [15, 42], and recently observed in simulations by Corot *et al.* [21].

C. Transition to the nonlinear regime

In §IV A, we discussed the linear regime of interface evolution, which occurs for small Weber numbers (strong surface tension), for which the interface perturbations oscillate with small amplitude. Then, in §IV B, we discussed the late-time development of the highly nonlinear bubbles and spikes, which appear for large Weber numbers (small surface tension). However, for intermediate Weber numbers the surface tension may curb but not prevent transition into a nonlinear evolution regime, which still exhibits oscillatory behavior. The pre-shock initial perturbation slope s^- also plays a significant role in the later transition process, as larger s^- values result in more rapid depositions of baroclinic vorticity at the interface, which causes it to evolve from the initially sinusoidal shape into the complex late-time structures [40]. Therefore, both slope and Weber

number will determine the nonlinear transition. Since the non-dimensional formulation (7) and (8) is based on the linear incompressible theory of Mikaelian [5], it is used in this section to diagnose the departure from the linear regime, and to develop a heuristic criterion for transition to nonlinear behaviour.

Firstly, we seek a quantitative indicator of nonlinear transition. The earliest such signature is the departure from the sinusoidal oscillation predicted by the linear theory. Therefore, as Weber number and slope increase, nonlinear deviations in the shape of the $\tilde{\eta} - \tilde{t}$ curves are expected to be found first near the normalized peaks, as the onset of nonlinearity should be relatively subtle and achieved most easily at maximum amplitudes. Figures 8a,b,c show the first peak of the perturbation oscillation as a function of Weber number, each at a different slope s^- . In these figures, the axes are normalized according to the linear theory [5].

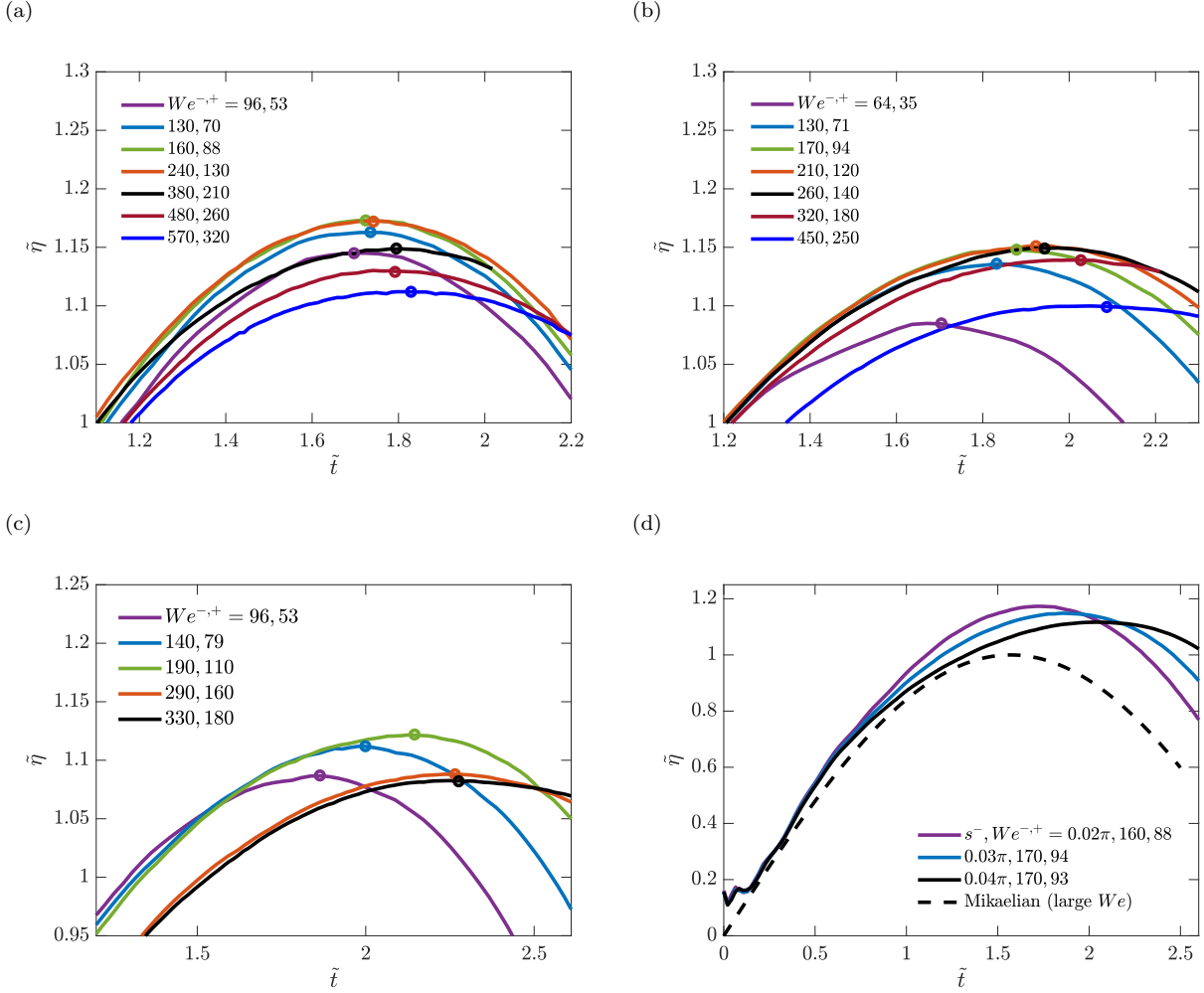


FIG. 8: Normalized perturbation development curves for $A^- = 9/11$. (a): $s^- = 0.02\pi$; (b): $s^- = 0.03\pi$; (c): $s^- = 0.04\pi$; (d): comparisons of cases with $s^- = 0.02\pi, 0.03\pi$ and 0.04π , at We^+ values around 90. For (a)(b)(c), the zoom-in views near the first peaks of the curves are displayed, and the peak points of each curve are marked with circles. Rightward shift of the peak points is found as the nonlinear indicator.

For each s^- , corresponding with each of Figure 8a,b,c, the peaks of the perturbation amplitude curves are marked out. In each case, the peak perturbation amplitude first increases, then decreases with increasing We^- . The time of peak amplitude, however, increases with We^- in all cases. We also note that this phenomenon becomes apparent at lower Weber numbers for the cases $s^- = 0.03\pi, 0.04\pi$ (Figures 8b,c) than at $s^- = 0.02\pi$ (8a). The phase shift that occurs with increasing We^- cannot be explained by the change of $\Delta\varphi = \arctan 1/\sqrt{We^+}$ alone in (9); for example, the phase-shift value predicted by this term for the $s^- = 0.02\pi$ cases with We^+ between 130 and 260 is only 0.025, much less than the measured value of 0.05 in Figure 8a. Since the phase shift we observed in Figures 8a,b,c occurs around post-shock Weber number $We^+ \sim 10^2$, we select 10^2 as the typical Weber number for the lower bound of the transitional regime and the upper bound of the linear oscillatory regime, as also noted in §I; while a more rigorous criterion incorporating the influence of

the slope s^- will be proposed below.

We also varied s^- at constant $We^+ \simeq 90$ to study the effect of amplitude alone (Figure 8d). Compared with the nonlinear effects of increasing Weber number, the rightward phase shift caused by increasing s^- occurs at a global scale, as the normalized curves start to diverge at $\tilde{t} \approx 0.65$, while the peak values decrease slightly as s^- increases. Again, this rightward shift cannot be explained by the change of $\Delta\varphi = \arctan 1/\sqrt{We^+}$ in (9), as the term does not explicitly contain s^- , and are almost the same for the three cases as the post-shock Weber numbers We^+ are nearly fixed. The initial amplitude of perturbation therefore has a material effect on the critical Weber number required for nonlinear transition.

Therefore, nonlinear transition appears to manifest most clearly as a shift in (normalized) time of the peak of the first oscillation. We quantify this directly as the relative error between the detected peak time (normalized, denoted \tilde{t}_m), and that predicted by the linear theory,

$$\Delta \equiv \left| \frac{\tilde{t}_m}{\frac{\pi}{2} - \arctan \frac{1}{\sqrt{We^+}}} - 1 \right|. \quad (22)$$

Note that this is defined according to the post-shock Weber number, We^+ . For properly linear evolution, $\Delta = 0$, corresponding to exact matching with the linear theory of [5], but this is not attained for any cases in this study for even the smallest We^+, s^- with incident shock number $M_{s,I} = 2$, primarily due to strong effects of compressibility as discussed in §IV A. Of course, the transition to nonlinearity is also gradual, so that a critical value Δ_c for nonlinear transition can only be heuristically chosen. Here we first choose $\Delta_c = 0.255$, and plot in Figure 9a a phase diagram that identifies linear cases ($\Delta < \Delta_c$, in yellow squares) and transitional cases ($\Delta \geq \Delta_c$, in blue pluses).

We now seek a simple predictive model for Δ and Δ_c , in order to identify the presence of nonlinear effects due to the effect of Weber number We^+ and post-shock perturbation slope s^+ . The criterion $k|\dot{\eta}_0^+|t \sim 1$ proposed by Dimonte [31] is not applicable in this section as surface tension may significantly influence the transition mechanism. We therefore assume heuristically that nonlinear effects become apparent when the peak perturbation amplitude-to-wavelength ratio reaches ~ 0.1 , or equivalently when $s_{max} \equiv k\eta_{max} \sim 0.6$ (Bigdelou [9] proposed a similar criterion of $k\eta_{max} \sim 1$). From (7), (8) and for $We^+ \gg 1$, this suggests that transition begins to occur for values of the parameter $\chi_1 \simeq 0.6$ where

$$\chi_\alpha \equiv (s^+)^{\alpha} \sqrt{We^+} \quad (23)$$

and $\alpha = 1$ as a first estimate.

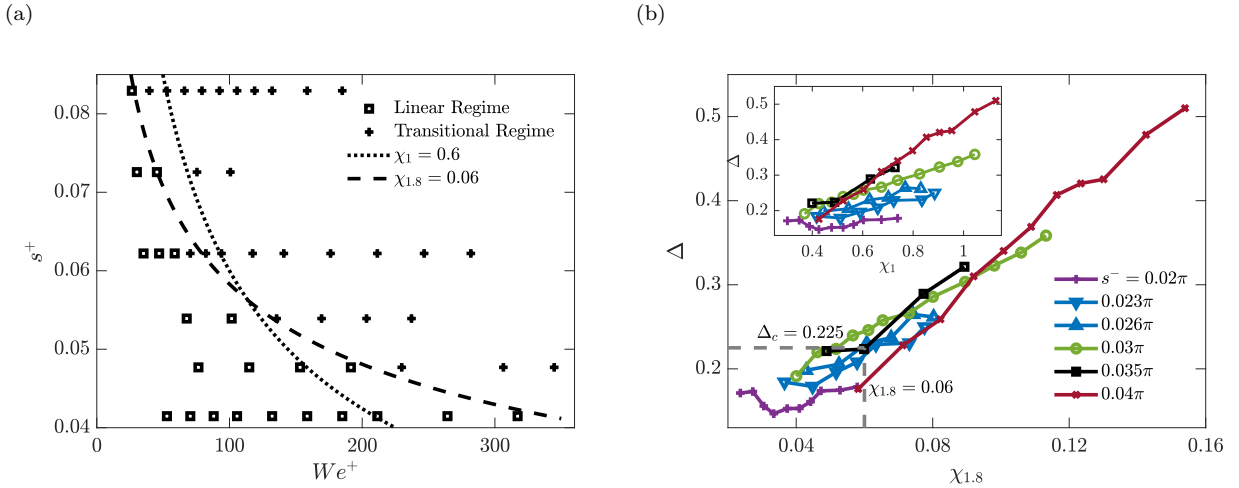


FIG. 9: Left: phase diagram showing simulation cases in linear growth-rate regime (squares) and transitional regime (crosses). The boundary between the two is selected as $\chi_{1.8} = 0.06$ and plotted in dashed line, while $\chi_1 = 0.6$ is also plotted in dotted line for comparison. Right: relationship between Δ and $\chi_{1.8}$ with different s^- setups. A common growth stage of Δ is reached for different s^- setups as $\chi_{1.8}$ increases, and the critical Δ_c and $\chi_{1.8}$ values are plotted.

We plot $\chi_1 = 0.6$ in Figure 9a as a dotted line, but it does not correctly delineate linear and transitional cases for the given choice of Δ_c . Moreover, as shown in the inset of Figure 9b, χ_1 does not fully explain the

variation in Δ . In that plot, while Δ increases with χ_1 , the rate of increase is clearly dependent on s^- (hence s^+). There is therefore a further dependence of the transition to nonlinearity on s^+ , suggesting a better choice of α in (23), which may be related to the local competition of surface tension and baroclinic vorticity on the interface, and awaits a more detailed investigation. Figure 9b shows the resulting scaling with a modified $\chi_{1.8}$ where $\alpha = 1.8$, which better collapses the data. Plotting the dashed line $\chi_{1.8} = 0.06$ on Figure 9a also more clearly delineates the linear and nonlinear cases, especially for those with larger perturbation slope s^+ . The measure is not perfect, as it does not fully delineate all linear and transitional cases. We attribute this to the curves not fully collapsing in Figure 9b. Nevertheless, for any choice of Δ_c a critical value for $\chi_{1.8}$ can always be found that reasonably separates those conditions that will remain linear from those that transit to nonlinear behaviour.

V. CONCLUSIONS

We have presented results of nonlinear and compressible numerical simulations of the Richtmyer-Meshkov instability with surface tension in the linear, nonlinear and transitional regime of perturbation development. Our core results are summarized below corresponding to each perturbation development regime, with the Weber number range of each regime highlighted:

First, in the *linear regime* where $We^+ < 10^2$, using appropriate dimensional analysis, we find that the existing theoretical impulsive model due to Mikaelian [5] predicts well the evolution of the shocked interface in this regime, with an appropriate modification based on the theory of Vandenboomgaerde [22] to accommodate Atwood numbers of either sign.

Next, in the *nonlinear regime* where $We^+ > 10^3$, we show agreement with theoretical predictions of Sohn [13] on the asymptotic (large time) bubble velocity in the limit of large Weber number, while a complete quantitative comparison of bubble and spike behavior at late times is left for future work.

Finally, in the *transitional regime* where $10^2 \leq We^+ \leq 10^3$, we use Eq. (8) to diagnose the onset of this regime to allow for comparison with results in the linear oscillatory regime, and develop a heuristic criterion based on non-dimensional parameters s^+ and We^+ for transition to nonlinear development.

These results indicate the utility of this numerical model for problems involving multiphase compressible flows, and constitute a further validation of its surface tension model and implementation. This study sheds light on influence of surface tension on the compressible RMI, which is important for related multiphase compressible flow problems where surface tension effects have not yet been systematically investigated, for example shock-bubble interactions [19]; while also providing a stepping stone towards the mixed compressible-incompressible problem which may influence the early-time development of the shocked-droplet or aerobreakup problem [43].

VI. ACKNOWLEDGMENT

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Appendix A: Numerical convergence and maximum resolvable Weber number

Since each fluid is inviscid, the smallest length scale in the bulk is set by numerical dissipation, so that pointwise grid convergence is not expected. Nevertheless, we require that the primary characteristics of the RMI growth, both with and without surface tension, be independent of grid resolution at our chosen resolution of $L = 9$. Here, four groups of convergence tests are conducted in total for four different categories of initial setups; namely, light-heavy ($A^- > 0$) and heavy-light ($A^- < 0$) density setups with and without surface tension. Specifically, we set $A^- = \pm 9/11$, while $We^- = +\infty$ or 160.

The raw outputs of the tests are provided in Figure 10, where time t and perturbation η are normalized by the natural units introduced in §III A. Absolute values are taken to facilitate the comparison between the results of light-heavy and heavy-light initial density setups, as in the latter case there will be a phase reversal of the perturbation profile at early time. When relatively weak surface tension is introduced to the heavy-light density setup case or the resolution level L is low, noise may appear in the neighborhood of the interface during the simulation, causing spurious high-frequency oscillations on the curve. Despite these problems, we

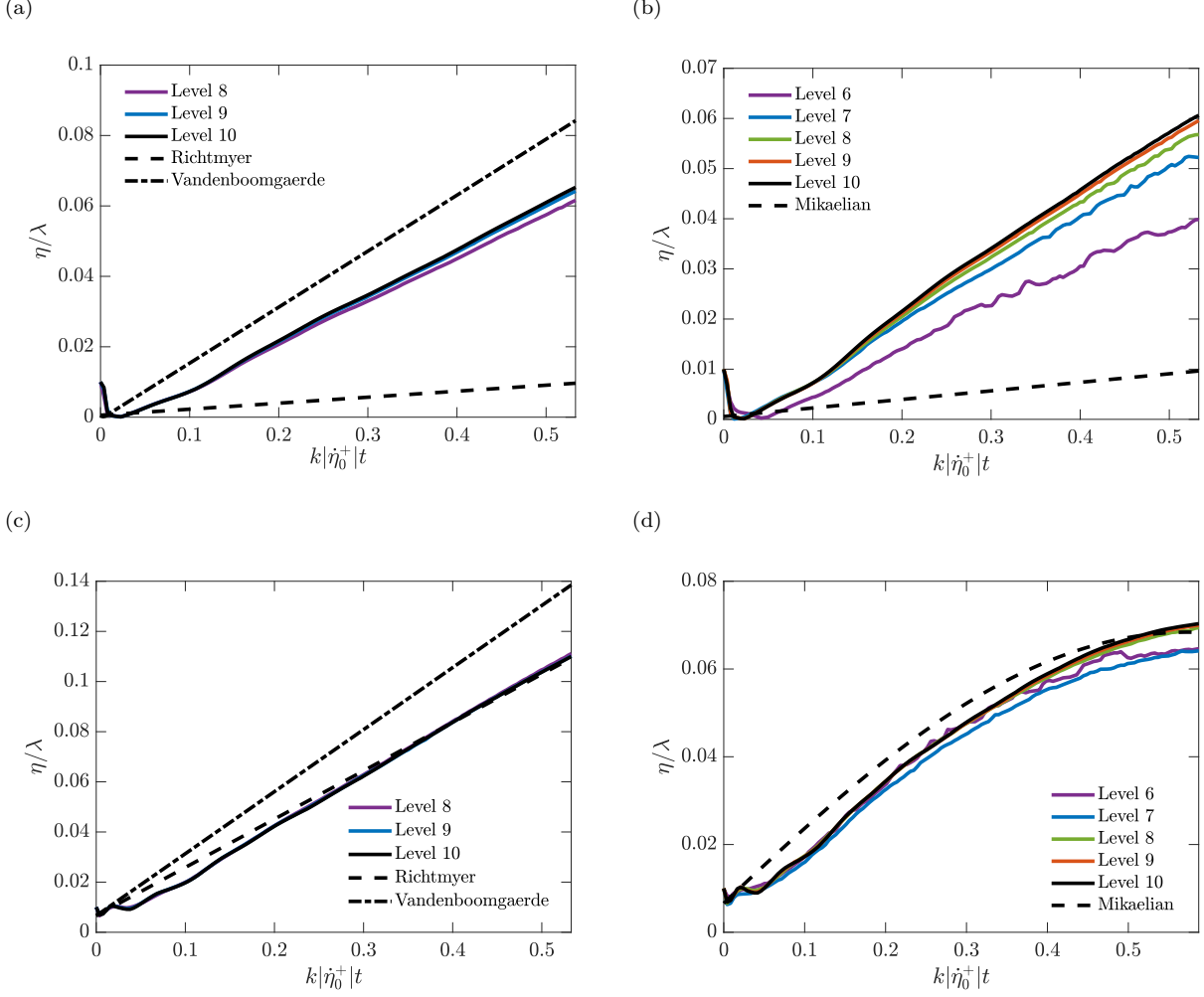


FIG. 10: Convergence test results (Upper row: heavy-light density setup without (a) and with (b) surface tension; lower row: light-heavy density setup without (c) and with (d) surface tension). Good numerical convergence is observed for all the inviscid test cases.

still observe good converging trend at resolution level $L = 9$ for all four groups of convergence tests.

As §IV B investigates RMI with asymptotically weak surface tension, it is also of interest to determine the maximum Weber number We_m resolvable at a certain resolution level. This is achieved by setting the following grid Weber number We_{grid} to 1:

$$We_{grid} \equiv \frac{(\rho_1^+ + \rho_2^+)(\dot{\eta}_0^+)^2 h}{\sigma_c}, \quad (\text{A1})$$

and the critical surface tension σ_c is then plugged into the definition of post-shock Weber number We^+ as given by (7). We then find that for the positive Atwood number cases at resolution level $L = 9$, the maximum resolvable Weber number is

$$We_m = \frac{1}{k^3(\eta_0^+)^2 h} = 4.7 \times 10^4, \quad (\text{A2})$$

which is much larger than the maximum post-shock Weber number investigated in this work (3.2×10^3), and we therefore expect that our numerical results effectively capture surface tension at the Weber numbers and resolutions that were tested.

Appendix B: Determination of post-shock state

While in this study, the post-shock state is determined through numerical diagnostics, it is instructive to compare them with theoretical predictions.

If surface tension and interface perturbation are absent, the flow state constitutes a Riemann problem when the incident shock arrives at the interface, whose solution yields the post-shock state. Mikaelian [23] proposes a set of equations for this problem. Lying at the core of this equation system are two alternative transcendental equations (Equations A4 and A16 in the appendix of [23]) for which there are generally no analytical solutions:

$$\frac{\xi - \frac{p_0}{p_L}}{\sqrt{\xi + \frac{\gamma-1}{\gamma+1} \frac{p_0}{p_L}}} = \begin{cases} \sqrt{\frac{\rho_2^-}{\rho_1^-}} \cdot \frac{1 - \frac{p_0}{p_L} - (\xi - 1) \sqrt{\frac{\gamma-1+(\gamma+1)\frac{p_0}{p_L}}{(\gamma+1)\xi + \gamma - 1}}}{\sqrt{1 + \frac{\gamma-1}{\gamma+1} \frac{p_0}{p_L}}}, & \xi \geq 1 \\ \sqrt{\frac{\rho_2^-}{\rho_1^-}} \cdot \frac{1 - \frac{p_0}{p_L} + \left(1 - \xi^{\frac{\gamma-1}{2\gamma}}\right) \sqrt{\frac{2\gamma}{\gamma-1} + \frac{\gamma+1}{(\gamma-1)^2} \frac{p_0}{p_L}}}{\sqrt{1 + \frac{\gamma-1}{\gamma+1} \frac{p_0}{p_L}}}, & \xi < 1 \end{cases}. \quad (B1)$$

The first equation is physically valid when its root satisfies $\xi \geq 1$, which indicates a reflected shock; otherwise, the second one will produce a root satisfying $\xi \leq 1$, which indicates a reflected rarefaction instead. Once the value of ξ has been determined, the post-shock quantities ρ_i^+ , Δv can be determined via the following equations,

$$\begin{aligned} \frac{\rho_1^+}{\rho_1^-} &= \begin{cases} \frac{(\gamma+1) + (\gamma-1)\frac{p_0}{p_L}}{(\gamma-1) + (\gamma+1)\frac{p_0}{p_L}} \cdot \frac{(\gamma+1)\xi + \gamma - 1}{(\gamma-1)\xi + \gamma + 1}, & \xi \geq 1 \\ \frac{(\gamma+1) + (\gamma-1)\frac{p_0}{p_L}}{(\gamma-1) + (\gamma+1)\frac{p_0}{p_L}} \cdot \xi^{\frac{1}{\gamma}}, & \xi < 1 \end{cases}, \\ \frac{\rho_2^+}{\rho_2^-} &= \frac{(\gamma+1)\xi + (\gamma-1)\frac{p_0}{p_L}}{(\gamma-1)\xi + (\gamma+1)\frac{p_0}{p_L}}, \\ (\Delta v)^2 &= \frac{2\xi p_L}{\rho_2^-} \cdot \frac{\left(1 - \frac{p_0}{\xi p_L}\right)^2}{(\gamma+1) + (\gamma-1)\frac{p_0}{\xi p_L}}. \end{aligned} \quad (B2)$$

It should be noted that equation sets (B1, B2) do not include effects of surface tension or the perturbed interface profile investigated in our work. Both factors could potentially cause the post-shock state to deviate from the solution of the Riemann problem. In particular, the perturbed interface profile gives rise to the RMI, and causes the transmitted and reflected wavefronts to have corrugated shapes initially, which are similar to the sinusoidal shape of the perturbed interface itself. These wavefront corrugations will oscillate and die out after the waves travel a distance of several wavelengths away from the post-shock interface [44], as is the situation shown in Figure 1(b). This rippling behavior has been observed in the experiments of [45], and reproduced afterwards in many simulation works (e.g. [23, 39, 46]).

We compare now the predictions of theory and numerical diagnostics for the cases presented in Figure 2. The transcendental equation sets ((B1), (B2)) yields $\xi = 1.8982$ for the cases with $A^- = 9/11$, and $\xi = 0.4668$ for those with $A^- = -9/11$. These solutions agree with the categorization of reflected wave (Equation (10)) by Drake [27], which is based on the sign of pre-shock Atwood number A^- .

The post-shock parameters are then derived using the solution ξ , and their comparisons with the values of numerical diagnostics measured at around $t = 0.4$ are shown in the following Table I. Here $r \equiv \eta_0^+/\eta_0^-$ is the compression ratio of perturbation amplitude, which may be calculated by $r = 1 - \Delta v/u_{s,I}$ according to [22, 23].

Within each pre-shock Atwood number category, the results gained via numerical diagnostics and equation solving for the same post-shock parameter are roughly on the same level of magnitude. Particularly good agreements are found for ρ_2^+ values in cases with $A^- = -9/11$, and also A^+ and r in those with $A^- = 9/11$. However, generally speaking, nontrivial discrepancies do exist between the numerical and analytical results.

The discrepancies are most likely caused by the equation system (B1) 's not accounting for the influence of the 'rippling' behavior of the post-shock wavefronts due to the initially perturbed interface profile (see §II). In cases without surface tension, such behavior is reported in [23] to cause the post-shock perturbation growth rate $\dot{\eta}$ to reach an asymptotic value, for which no simple analytic solution exists [23, 47], after going through

A^- (Method)	9/11 (ND)	9/11 (ES)	-9/11 (ND)	-9/11 (ES)
ρ_1^+	0.356	0.418	1.700	1.591
ρ_2^+	3.084	3.593	0.169	0.168
A^+	0.793	0.792	-0.819	-0.809
Δv	2.030	2.333	2.231	2.109
r	0.66	0.688	0.0565	0.109

TABLE I: Comparison between post-shock values gained from numerical diagnostics ("ND") and equation solving ("ES").

a damped oscillation period, which also matches the trend of our results in Figure 2.

The ascription of discrepancies above is further consolidated by the following observation. As is shown in Figure 2, for the case with $A^- = 9/11$, our state diagnostic case captures $\rho_1^+ = 0.411$ and $\rho_2^+ = 3.590$ at a very early time $t = 0.02$ after the shock-interface interaction, which matches very well with the solution of Mikaelian's equation sets [23] (see column 3 of Table I). However, the two densities eventually settle down at the steady-state values, as discussed in §IV A.

The relationship between pre- and post-shock Weber numbers We^- and We^+ may now be derived using the post-shock states determined by numerical diagnostics, which enables direct prediction of We^+ from We^- . Based on the definitions of We^- and We^+ (Equations (1) and (7)), we find that

$$\frac{We^+}{We^-} = \frac{\rho_1^+ + \rho_2^+}{\rho_1^- + \rho_2^-} \left(\frac{A^+ \Delta v}{A^- u_L} \right)^2. \quad (B3)$$

That is, the postshock Weber number depends only on the ratio of post-shock to pre-shock densities, and the Mach number of the incident shock, and is independent of the (small) pre-shock perturbation amplitude η_0^- . As a result, for simulation cases with pre-shock Atwood numbers $A^- = 9/11$ and $-9/11$, we have $We^+/We^- = 0.553$ and 3.875 respectively.

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