Random Matrices, Boundaries and Branes

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This thesis is devoted to the application of random matrix theory to the study of random surfaces, both discrete and continuous; special emphasis is placed on surface boundaries and the associated boundary conditions in this formalism. In particular, using a multi-matrix integral with permutation symmetry, we are able to calculate the partition function of the Potts model on a random planar lattice with various boundary conditions imposed. We proceed to investigate the correspondence between the critical points in the phase diagram of this model and two-dimensional Liouville theory coupled to conformal field theories with global $\mathscr{W}$-symmetry. In this context, each boundary condition can be interpreted as the description of a brane in a family of bosonic string backgrounds. This investigation suggests that a spectrum of initially distinct boundary conditions of a given system may become degenerate when the latter is placed on a random surface of bounded genus, effectively leaving a smaller set of independent boundary conditions. This curious and much-debated feature is then further scrutinised by considering the double scaling limit of a two-matrix integral. For this model, we can show explicitly how this apparent degeneracy is in fact resolved by accounting for contributions invisible in string perturbation theory. Altogether, these developments provide novel descriptions of hitherto unexplored boundary conditions as well as new insights into the non-perturbative physics of boundaries and branes.

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Chapter 1

Introduction

This thesis concerns the application of the theory of random matrices to the foundations of string theory and the quantum mechanical description of gravitation, or quantum gravity for short. A major motivation is that the tools of random matrix theory afford us a uniquely detailed window into the quantum physics of strings and gravity beyond the perturbative expansion in the string coupling. In particular, geometric objects such as boundaries and branes acquire a simple interpretation in terms of averages of characteristic polynomials of random matrices. Below we briefly review the history of work that has intertwined these subjects, with a view towards the open problems addressed in this work, and subsequently provide an outline of the following chapters.

1.1 History of the subject

The application of the theory of random matrices to problems in physics was pioneered by Wigner in the fifities of the last century [1]. Since its inception, this field has expanded tremendously and today, the applications of random matrix theory include areas as diverse as signal processing, number theory in mathematics, RNA folding in biology, and portfolio optimisation in finance [2]. In the subsequent decade, Tutte initiated the enumeration of *planar maps* [3, 4], defined as graphs embeddable in the plane, modulo homeomorphisms.

When endowed with a statistical lattice model defined thereon, the detailed knowledge of the asymptotic properties of maps with a large number of vertices allows a rigorous definition of the path integral for two-dimensional gravity coupled to matter or equivalently, bosonic string theory in a non-critical target space dimension. The classification of boundary conditions that can consistently be imposed on the boundary of the graph then provides important insights into the spectrum of the theory.

The intimate relationship between the above two subjects first emerged when 't Hooft observed that averages of infinite matrices admit an expansion in planar diagrams [5]; the connection to the enumeration of planar maps was further fleshed out in the seminal works of Brezin et al. and Bessis et al. [6, 7]. In the eighties, David, Ambjørn et al. and Kazakov et al. exploited these insights to compute obervables in pure two-dimensional quantum gravity [8, 9, 10, 11] or equivalently, strings propagating in a zero-dimensional target space. These investigations revealed a powerful connection between the combinatorial problems at hand and algebraic geometry: for example, using random matrices, Boulatov and Kazakov discovered that the generating function for planar triangulations, weighted by the partition function of the Ising model defined thereon, can be obtained as the solution to a polynomial equation [12, 13]. As a result, upon analytic continuation, this generating function defines a Riemann surface called the *spectral curve* – see Figure 1.1 for a cartoon of this correspondence.

Shortly afterwards, Kazakov introduced a multi-matrix integral that describes the *Potts model* on a random lattice [14], which is a generalisation of the Ising model dating back to [15]. In the same year, a breakthrough by Distler and Kawai allowed for the development of a complementary description of two-dimensional quantum gravity using Liouville conformal field theory [16], leading to significant efforts to work out the correspondence to the random matrix description in the following decade: First, Lian and Zuckerman determined the physical Hilbert space of Liouville theory coupled to the so-called Virasoro minimal models [17], of which the critical Ising model is a special case. Good agreement with the

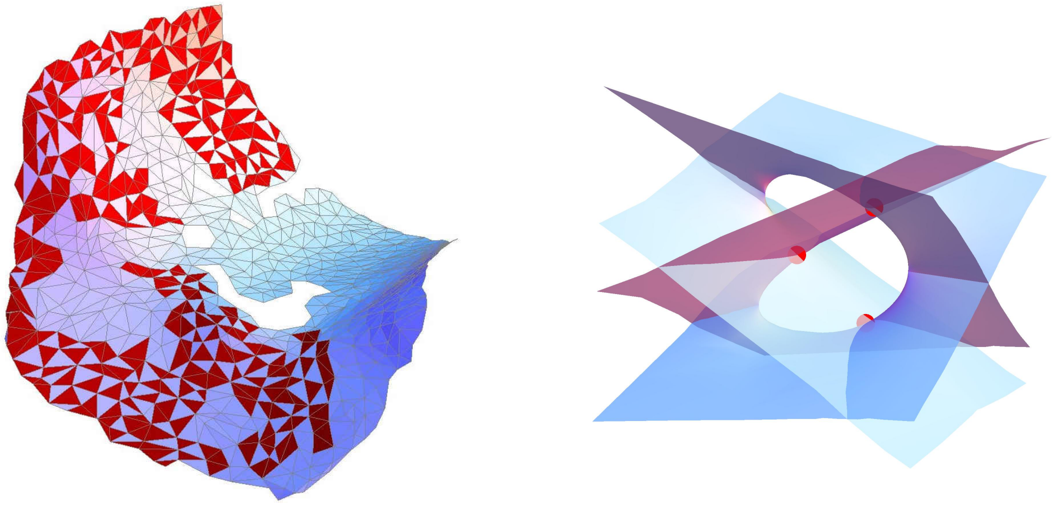


Figure 1.1: A configuration of the Ising model on a triangulated surface (left), and the Riemann surface arising from the analytic continuation at the critical point of the model (right).

random matrix description was found following the solution of the so-called two-matrix model by Daul, Kazakov and Kostov [18]. These developments were preceeded by the discovery of the *double scaling limit* of matrix models by Douglas and Shenker [19], in which large maps of arbitrary genus contribute to the asymptoic behaviour.

Then in the mid-nineties, Daul found exact solutions for Kazakov's random matrix description of the Potts model on a random lattice [20]. Around the same time, Speicher reported on the connection between sums of *independent* random matrices and Voiculescu's free probability theory [21, 22]. Only a few years later, Carroll, Ortiz and Taylor calculated the partition function of the Ising model on the randomly triangulated disk for all independent boundary conditions by considering the average of the sum of two *correlated* random matrices [23, 24], a task still to be completed for the Potts model on random planar maps. Indeed, for the Potts model on a *fixed* lattice, a complete set of boundary conditions was only described a year later by Affleck, Oshikawa and Saleur [25]. With the advent of the new millenium, a classification of boundary conditions for Liouville theory was achieved by Fateev, the Zamolodchikov brothers and Teschner [26, 27, 28]. Building on this, Seiberg and Shih subsequently developed the string theoretic interpretation of Liouville theory coupled to the Virasoro minimal model, in which the boundary conditions correspond to extended objects called *branes* [29, 30]. Curiously, a degeneracy in the states describing such boundary conditions was conjectured, implying that boundary conditions that are distinct for a matter system on a fixed background can be rendered indistinguishable when coupled to gravity. This conjecture was later challenged by matrix model calculations performed by Atkin, Wheater and Zohren [31, 32].

Despite this being a fairly mature research field with a well-developed literature, a complete understanding of the allowed set of boundary conditions and their relationships has not yet been achieved even for simple models; the work described herein is an attempt to make progress towards filling these gaps.

1.2 Outline of the thesis

This thesis is organised as follows: In Chapter 2, we set the stage by introducing the concept of a matrix model and reviewing the description of boundaries and branes, employing the connections of random matrices to combinatorics, conformal field theory and string theory. After this prelude, the prerequisites are at hand to present the author's original contributions in three main chapters:

We begin by introducing Kazakov's multi-matrix model with permutation symmetry that describes the Potts model on a random lattice in Chapter 3. In this context, we generalise the work of Voiculescu and Speicher [33, 21] to the addition of *correlated* random matrices. Using Affleck et al.'s classification [25] of boundary conditions for the Potts model as a guide, this enables us to compute the partition function of the model on the randomly triangulated disk for a whole family of boundary conditions exactly, thus extending the results [23, 24, 31, 32] for the Ising model obtained by Carroll and others. We deduce novel relationships between these boundary conditions and investigate the phase diagram to derive the scaling behaviour of the generating functions when the coupling constants approach a critical point. The results of this chapter have been reported in the publications [34, 35].

In Chapter 4, we take advantage of the scale-invariance arising at the above-mentioned critical points to develop a description of the scaling functions using conformal field theory. The permutation symmetry of the matrix model and the conformal symmetry get enhanced to a larger continuous symmetry whose generators satisfy the so-called $\mathscr{W}$-algebra. We investigate the space of physical states for string worldsheets of both spherical and disk topology. On the sphere, our treatment extends the work of Lian, Zuckerman and Bouwknegt [17, 36] on Liouville theory coupled to the aforementioned minimal models, whose symmetries are captured by the smaller Virasoro algebra. Moreover, on the disk, the degeneracy of boundary conditions of the latter system observed by Seiberg and Shih [29] is found to persist in the more general case under study.

In Chapter 5, employing the double scaling limit, we go beyond the conformal field theory description and study the description of branes non-perturbatively for specific cases. This will reveal novel and important differences that are not visible in the asymptotic expansion in the string coupling which follow from a careful counting of independent degrees of freedom. In particular, we find that the above-mentioned degeneracy is resolved upon inclusion of contributions from maps of unbounded genus and different boundary conditions capture truly independent degrees of freedom, thus potentially resolving the debate initiated in [31, 32]. The results of this chapter will be part of a forthcoming publication [37].

Finally in Chapter 6, we summarise the key results that follow from the above investigations and comment on possible further applications and future developments.

Chapter 2

Review of the Hermitian Matrix Model

This chapter introduces the so-called Hermitian matrix model. This review will necessarily be incomplete and biased towards the applications in this thesis; more comprehensive reviews of these topics include [38, 39, 40]. We define the Hermitian matrix model as a probability measure for an $N \times N$ Hermitian matrix X,

$$\mathrm{d}\mu(X) = \frac{1}{Z_N} \mathrm{e}^{-N \mathrm{tr} V(X)} \mathrm{d}X, \tag{2.1}$$

for a polynomial $V(x) = \sum_{m=2}^{k+2} t_m x^m / m$, where $\mathrm{d}X$ denotes the integration over independent components,

$$\mathrm{d}X = \prod_{1 \le i \le j \le N} \mathrm{dRe} X^i{}_j \prod_{1 \le i < j \le N} \mathrm{dIm} X^i{}_j \ . \tag{2.2}$$

Here, the *partition function* Z_N normalises expectation values such that $\int \mathrm{d}\mu(X) = 1$. The above measure is invariant under the adjoint action of the unitary group, in components

$$X^i{}_j \longrightarrow \tilde{X}^i{}_j = U^i{}_k X^k{}_l U_j{}^{l*} \ , \quad U \in U(N) \ . \tag{2.3}$$

When $k = 0$, $\mathrm{d}\mu(X)$ defines the so-called Gaussian unitary ensemble (GUE) – one of

Wigner's three ensembles[1] [1]. All results in this thesis pertain to the generalisation of this to the following probability measure on q Hermitian matrices:

$$\mathrm{d}\mu(X_1,X_2,\dots X_q) = \frac{1}{Z_{N,q}} \prod_{\langle ij \rangle} \mathrm{e}^{N \mathrm{tr} X_i X_j} \times \prod_{i=1}^{q} \mathrm{e}^{-N \mathrm{tr} V_i(x)} \mathrm{d}X_i \;, \tag{2.4}$$

where $\langle ij \rangle$ denotes the product over distinct i, j. As the main Chapters 3, 4 and 5 will concern the so-called planar, scaling and double scaling limits of the model (2.4), we introduce these limits in Sections 2.1, 2.2 and 2.3 below for the simpler model (2.1). In doing so, we elucidate their connection to random planar maps, boundary conformal field theory and branes in string theory, respectively. Worked examples at the end of each section illustrate the relative ease with which results free of approximations can be obtained in these limits.

2.1 Planar Limit

In Chapter 3, we will be interested in the large-N spectral density of sums of random matrices of the form $X_1 + X_2 + \dots X_p$, $1 \leq p \leq q$, distributed according to (2.4). To this end, we shall discuss the application of the saddle point method in the large N limit in Subsection 2.1.1, first discussed in [6]. To pave the way for the interpretation of the measure (2.4) as a description of the Potts model on a random lattice, we proceed to review the connection to statistical physics on planar surfaces in Subsection 2.1.2, which will reveal the origin of the term "planar limit" for $N \to \infty$.

2.1.1 Saddle point equations

Given a Hermitian random matrix X, we woud like to compute large-N spectral density, defined as

$$\rho_X(x) = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta(x - x_i) \right\rangle \;, \tag{2.5}$$

[1] Besides the latter, these include the Gaussian orthogonal and symplectic ensembles, named analogously according to their respective symmetry groups.

where $\{x_i\}_{i=1}^N$ denote the eigenvalues of X. Introducing $U \in U(N)$ such that $UXU^\dagger$ is diagonal, we can integrate out the off-diagonal components by performing the integration over the unitary group, allowing us to write the partition function as

$$Z_N = \frac{\mathrm{vol}\, U(N)}{(\mathrm{vol}\, U(1))^N} \prod_{i=1}^N \int_\gamma \mathrm{d}x_i \; \mathrm{e}^{-NV(x_i)} \Delta^2(x) \; , \tag{2.6}$$

where $\Delta(x) = \det_{i,j} x_i^{j-1}$ is the Vandermonde determinant. To allow for odd values of k, we have generalised to normal matrices whose eigenvalues are supported on a one-dimensional cycle $\gamma \subset \mathbb{C}$. Note that when k is even and Re $t_{k+2} > 0$, we can always let X be Hermitian, i.e. choose $\gamma = \mathbb{R}$. The saddle points are then given by the eigenvalue configurations satisfying

$$V'(x_i) = \frac{2}{N} \sum_{j \neq i} \frac{1}{x_i - x_j} \; , \quad 1 \leq i \leq N \; . \tag{2.7}$$

These are N coupled algebraic equations, with a total of $N!\binom{N+k}{N}$ solutions, where the first factor arises from the invariance under permutations of eigenvalues, and the second from the dimensionality of the space of integration cycles. Since the left-hand side of (2.7) is holomorphic, we can anlytically continue this equation for any N. At this stage it is useful to introduce the *Stieltjes transform* of ρ_X,

$$W_X(z) = \int_{\mathrm{supp}\; \rho_X} \mathrm{d}x \; \frac{\rho_X(x)}{z - x} \; , \quad z \in \mathbb{C} \setminus \mathrm{supp}\; \rho_X \; . \tag{2.8}$$

By construction, $W_X(z)$ computes the average of the trace of the resolvent $(z - X)^{-1}$ for large N:

$$W_X(z) = \frac{1}{N} \left\langle \mathrm{tr} \frac{1}{z - X} \right\rangle + \mathscr{O}(1/N) \; . \tag{2.9}$$

The normalised spectral density is obtained by inverting (2.8),

$$\rho_X(x) = \frac{1}{\pi}\mathrm{Im}\, W_X(x)_+ \,. \tag{2.10}$$

Here and in what follows we use the notation $f(z)_\pm = \lim_{\varepsilon\searrow 0} f(z \pm \mathrm{i}\varepsilon)$. We can consequently rewrite the saddle point equation (2.7) as an equation for the Stieltjes transform of the spectral density,

$$V'(z) = W_X(z)_+ + W_X(z)_- \,, \quad z \in \mathbb{C} \,, \tag{2.11}$$

subject to the condition $\lim_{z\to\infty} zW_X(z) = 1$. Writing

$$W_X(z) = \frac{1}{2}V'(z) - y(z) \,, \tag{2.12}$$

we are left to determine a function $y(z)$ holomorphic on $\mathbb{C} \setminus \mathrm{supp}\, \rho_X$. Throughout this thesis, we focus exclusively on those saddle points for which the spectral density has connected support[2] as $N \to \infty$. Together with the requirement $y(z) = \frac{1}{2}V'(z) - z^{-1} + \mathscr{O}(z^{-2})$ for large z, this restriction fixes $y(z)$ entirely as the solution of a well-defined Riemann-Hilbert problem.

Example 2.1.1. *For the GUE ($k = 0$),* (2.11) *implies*

$$W_X(z) = \frac{t_2}{2}\left(z - \sqrt{z^2 - 4/t_2}\right) \,. \tag{2.13}$$

From (2.10) *it follows that the spectral density is given by the well-known semi-circle distribution*

$$\rho_X(x) = \begin{cases} \frac{t_2}{2\pi}\sqrt{4/t_2 - x^2} \,, & |x| < \frac{2}{\sqrt{t_2}} \,, \\ 0 \,, & |x| \geq \frac{2}{\sqrt{t_2}} \,. \end{cases} \tag{2.14}$$

Example 2.1.2. *When $V(z)$ is quartic ($k = 2$) and even, we expect the support of the eigenvalue density to be symmetric under reflections of the real axis. Then $W_X(z)$ can be written*

[2] This poses an implicit restriction on the range of t_m.

as

$$W_X(z) = \frac{1}{2}\left(t_4 z^3 + t_2 z - P(z)\sqrt{z^2 - z_c^2}\right) \ , \tag{2.15}$$

where $P(z)$ is a quadratic polynomial and z_c depends on t_2, t_4 only. Requiring $\lim_{z\to\infty} zW_X(z) = 1$ *fixes*

$$P(z) = t_4 z^2 + \frac{1}{3}\left(2t_2 + \sqrt{t_2^2 + 12t_4}\right) \ , \quad z_c^2 = \frac{2}{3t_4}\left(\sqrt{t_2^2 + 12t_4} - t_2\right) \ . \tag{2.16}$$

2.1.2 Statistical physics on planar lattices

The application of matrix integrals to the enumeration of random graphs was pioneered in [5, 6, 7]. These developments have since been extended significantly and we refer the reader to [39, 40] for a more comprehensive overview of these topics. In his seminal work [5], 't Hooft considered the new matrix integral obtained from Z_N by expanding the exponential in the integrand and reversing the order of integration of summation:

$$Z_N^{\text{formal}} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}X \ \mathrm{e}^{-\frac{t_2}{2} N \mathrm{tr} X^2} \left(-N\mathrm{tr}V(X)\right)^n \ . \tag{2.17}$$

Because each term is polynomial in $t_{m\geq 3}$, the above expression can be regarded as a formal power series in these parameters. The quantity (2.17) is consequently referred to as a *formal* matrix integral [41], an a priori different quantity than the convergent expression (2.6). Equally, throughout this section, we will regard averages $\langle\cdot\rangle$ as formal power series in the parameters t_m. An application of Wick's theorem tells us that (2.17) can be evaluated by a sum over closed fatgraphs, in which a given graph $\mathscr{G}$ with l internal lines and n_m vertices of coordination number m comes with a weight

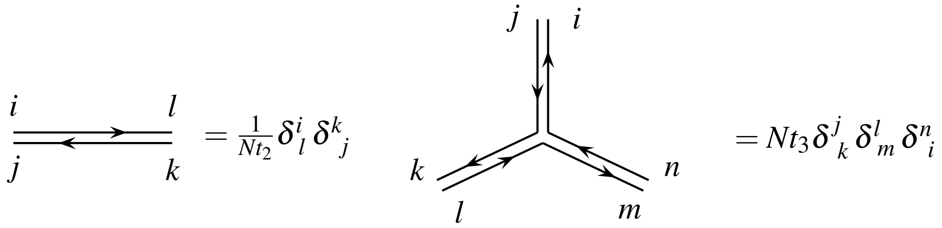


Figure 2.1: Feynman rules for the matrix model with cubic potential; lines are oriented according to index positions.

$$\frac{N^{2-2h}}{|\mathrm{Aut}(\mathscr{G})|}t_2^{-l}\prod_{m=3}^{k+2}t_m^{n_m}\ , \tag{2.18}$$

where $|\mathrm{Aut}(\mathscr{G})|$ is the order of the automorphism group of $\mathscr{G}$ and h its genus. The dual graph is obtained by associating m-gons to m-valent vertices, with sides identified when connected by a propagator. In this way, the logarithm of (2.17) enumerates maps, i.e. embeddings of connected graphs into surfaces; the parameters $\{t_m\}_{m=2}^{k+2}$ are the respective fugacities of m-gons in the map. Each series coefficient is polynomial in N and at leading order in $1/N$, only planar graphs contribute to average[3] – hence the name "planar limit" for taking $N\to\infty$. Because the number of planar maps is exponentially bounded, averages will have a convergent power series expansion as $N\to\infty$ for t_m of small enough modulus.

To see how the description of boundaries in the random graph can be achieved, note that applying Wick's theorem to the expansion of the first derivative of the planar free energy per degree of freedom

$$\frac{1}{N}\langle \mathrm{tr}X^j\rangle = \frac{1}{N^2}\frac{\partial}{\partial t_j}F_0 + \mathscr{O}(1/N^2)\ , \qquad F_0 = -\lim_{N\to\infty}\ln Z_N^{\mathrm{formal}}\ , \tag{2.19}$$

gives a sum over all maps with n_m m-gons and one marked j-gon called the *root*, whose links define the boundary of $\mathscr{G}$. Since the Stieltjes transform $W_X(z)$ of the spectral density is the generating function for the moments

[3]Note that this is inequivalent to the genuine $1/N$ expansion of convergent integrals as encountered in the next section.

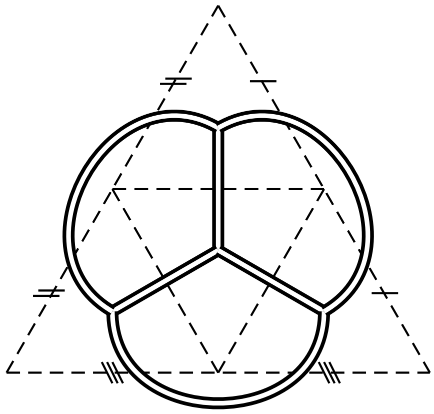


Figure 2.2: Example of a Feynman graph dual to a triangulation of the sphere.

$$\lim_{N\to\infty}\frac{1}{N}\langle \mathrm{tr}X^j\rangle = \frac{1}{2\pi \mathrm{i}}\oint_C \mathrm{d}z\, z^j W_X(z)\ , \tag{2.20}$$

where the contour C encloses supp ρ_X counter-clockwise, we see that $W_X(z)$ may be understood as the generating function for planar maps with connected boundary, i.e. maps of disk topology, where z is the fugacity of a boundary link. For this reason, $W_X(z)$ is also referred to as the *disk function*. The partition function of the model on the disk is defined by dividing by the order of the automorphism group of the boundary – which is simply the number of boundary links – at each order in z. Equivalently, $W_X(z)$ is the first derivative of the disk partition function:

$$\frac{1}{N}\frac{\partial}{\partial z}\langle \mathrm{tr}\ln(z-X)\rangle = W_X(z)+\mathscr{O}(1/N)\ . \tag{2.21}$$

Example 2.1.3. *When V is cubic ($k=1$), Z_N^{formal} is a power series in the single variable t_3,*

$$Z_N^{\mathrm{formal}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{Nt_3}{3}\right)^n\int \mathrm{d}X \mathrm{e}^{-\frac{t_2}{2}N\mathrm{tr}X^2}\left(\mathrm{tr}X^3\right)^n\ . \tag{2.22}$$

The resulting Feynman rules are depicted in Figure 2.1; an example of a term proportional to $t_3^4N^2$ arising from Wick's theorem applied to the above series in Figure 2.2.

Example 2.1.4. *When V is quartic ($k=2$) and even, Z_N^{formal} is a power series in the single* *variable* t_4. *Using* (2.20) *for the explicit result* (2.15) *and setting* $t_2 = 1$ *without loss of generality, one recovers Tutte's famous result [4] for the generating function of rooted planar quadrangulations:*

$$\lim_{N\to\infty} \frac{1}{N}\langle \mathrm{tr} X^4 \rangle = \sum_{n=0}^{\infty} (-t_4)^{n-1} \frac{2}{n!} \frac{3^n (2n)!}{(n+2)!} \,. \tag{2.23}$$

2.2 Scaling limit

In Chapter 3, we will derive scaling limit of averages computed with (2.4); in Chapter 4, we will study the same system using conformal field theory. To set the stage, Subsections 2.2.1 and 2.2.2 therefore introduce this limit and discuss its connection to conformal field theory, respectively.

2.2.1 Phase diagram and critical points

The description of continuous surfaces via the critical behaviour of large random matrices dates back to the seminal works of David, Ambjørn, Kazakov and collaborators [8, 9, 10, 11]. The planar limit may be understood as the thermodynamic limit of the eigenvalue statistics: at infinite N, averages are non-analytic in the fugacities $\{t_m\}_{m=2}^{k+2}$, which span a k-dimensional phase diagram. Consequently, power series like (2.19) generally have a finite radius of convergence. Tuning a single fugacity t_j towards a critical value $t_{j,c}$ such that we approach a $(k-1)$-dimensional critical submanifold, the resulting universal behaviour can be characterised by the scaling exponent γ_j of the second derivative of the planar free energy,

$$\frac{\partial^2}{\partial t_j^2} F_0 = \text{const.} \times (t_j - t_{j,c})^{-\gamma_j} + \text{ terms analytic in } t_j \,. \tag{2.24}$$

Generally, we see from (2.19) that then

$$\frac{1}{N}\langle \mathrm{tr}X^j\rangle = \text{const.} \times (t_j - t_{j,c})^{1-\gamma_j} + \text{ terms analytic in } t_j \ , \tag{2.25}$$

which implies that the number of rooted planar maps with n_j j-gons is exponentially bounded by a constant multiple of $t_{j,c}^{-n_j}(n_j)^{\gamma_j - 2}$ for large n_j. In particular, when $\gamma_j > 0$, the terms analytic in t_j are subleading and the average number of j-gons contributing to the average in (2.24) diverges linearly with the distance from the critical point[4]:

$$\langle n_j\rangle = \frac{\partial}{\partial t_j}\ln\left(\frac{\partial^2}{\partial t_j^2}F_0\right) \propto \frac{\gamma_j}{t_j - t_{j,c}} \quad \text{as } t_j \to t_{j,c} \ . \tag{2.26}$$

Assigning a fixed length ε to each edge in the dual of $\mathscr{G}$ thus yields an expectation value of the dimensionful surface area proportional $\varepsilon^2\langle n_j\rangle$. Then the *scaling limit* is obtained by sending $t_j \to t_{j,c}$, $\varepsilon \to 0$, keeping $\mu = (t_j - t_{j,c})/\varepsilon^2$ and hence the dimensionful area fixed. For this reason, μ is also referred to as the renormalised *cosmological constant*. More generally, for higher-order critical behaviour near lower-dimensional critical submanifolds, the exact relationship between μ and the fugacities $\{t_m\}_{m=2}^{k+2}$ depends more intrically on the direction in which we approach the critical submanifold in question and was worked out by Moore et al. [42], who coined the term *conformal background* for the scaling limit in which no other couplings besides μ are nonzero. A widely-used diagnostic discriminating between different universality classes is the scaling exponent γ_s of the susceptibility

$$\frac{\partial^2}{\partial \mu^2}F_0 = \text{const.} \times \mu^{-\gamma_s} + \text{ terms analytic in } \mu \ . \tag{2.27}$$

For a single random matrix with the measure (2.1), we can at most arrange for an algebraic singularity of the form $\gamma_s = 1/2 - k$; a more general algebraic singularity $\gamma_s = -(k+2)^{-1}$ can arise when we choose the measure (2.4) with $q = 2$ [18].

For graphs with boundaries, we can similarly let the average number of boundary links

[4]When $\gamma_j < 0$, the analogous conclusions follow after taking sufficiently many derivatives with respect to t_j, that is, for maps with a sufficient number of marked j-gons.

diverge: as $N \to \infty$, the disk function $W_X(z)$ develops a branch cut located at the support of the eigenvalue density and the corresonding power series in z displays a finite radius of convergence z_c. Recalling that z is the fugacity of a boundary link, we introduce by analogy the *boundary cosmological constant* via $z = z_c(1-\varepsilon\mu_B)$. For $0 > \gamma_s > -1$, the expansion of $W_X(z)$ near the critical point is then of the form

$$W_X(z) = W_X(z_c) + \varepsilon c_1 \mu_B + \varepsilon^{1-\gamma_s} c_2 \left.\frac{\partial D(\mu,\mu_B)}{\partial \mu_B}\right|_{\mu} + \dots, \tag{2.28}$$

where the leading non-analytic term is the universal first derivative of the *continuum* disk partition function, and c_1 and c_2 are non-universal constants. It will turn out convenient to express the above in the dimensionless variables

$$\zeta = \frac{\mu_B}{\sqrt{\mu}}\ , \quad Q = \mu^{(\gamma_s-1)/2} \left.\frac{\partial D(\mu,\mu_B)}{\partial \mu_B}\right|_{\mu}\ . \tag{2.29}$$

Example 2.2.1. *Consider again the case $k=2$ with even V. Upon setting $t_2 = 1$ and applying Stirling's approximation to the coefficients in* (2.23)*, one finds the number of rooted planar quadrangulations grows asymptotically as $2/\sqrt{\pi} \times 12^n n^{-5/2}$, which implies $t_{4,c} = -1/12$ and $\gamma_s = -1/2$; this universality class describes a random planar surface which has been called the "Brownian map" in the mathematics literature [43]. At the critical point, the spectral density* (2.5) *is proportional to $(8-x^2)^{3/2}$, so $z_c = \sqrt{8}$. Approaching this point via the parametrisation*

$$t_4 = t_{4,c}(1-\varepsilon^2\mu)\ , \quad z = z_c(1+\varepsilon\mu_B/2)\ , \quad \varepsilon \searrow 0\ , \tag{2.30}$$

we find the following expansion of the disk function:

$$W_X(z) = \frac{\sqrt{2}}{3} - \varepsilon\frac{1}{\sqrt{2}}\mu_B + \varepsilon^{3/2}\frac{\sqrt{2}}{3}(2\mu_B - \sqrt{\mu})\sqrt{\mu_B + \sqrt{\mu}} + \mathcal{O}(\varepsilon^2)\ . \tag{2.31}$$

In the dimensionless variables (2.29)*, the above can be succinctly summarised as the solution to* $T_3(\zeta) - T_2(Q) = 0$*, where* $T_n(x)$ *denotes the* n^{th} *Chebyshev polynomial of the first kind.*

Remark 2.2.2. Had we considered cubic V ($k = 1$), $W_X(z)$ would have become the generating function for rooted planar triangulations. The scaling limit would have given different values of z_c, c_1 and c_2, but the same value of γ_s, and the same form ζ and Q due to the universality of the Brownian map.

2.2.2 Conformal field theory description

Generally, the existence of critical points suggests that the universal properties of the scaling limit can be captured by a scale-invariant field theory. To see how this expectation is borne out in Chapter 4, we will first need to recall some crucial results and widespread terminology. A general introduction to two-dimensional conformal field theory is [44], the foundations of which will be assumed to be familiar to the reader. Generalities of BRST cohomology in the context of string theory are reviewed in [45]; for an overview on $\mathscr{W}$-algebras in conformal field theory, see [46].

Distler and Kawai pioneered the definition of the measure on the space of physically distinct configurations of the continuum surface taking advantage of the fact that any two-dimensional metric g may be written in the *conformal gauge* $g = f^*(\mathrm{e}^{\varphi}\hat{g})$, where the f^* denotes the action of a diffeomorphism and the *background metric* $\hat{g}$ is specified by a unique point in *moduli space* – the finite-dimensional compact space of two-dimensional metrics modulo diffeomorphisms and local Weyl transformations [16]. This change of variables contributes a Jacobian to the measure which is the product of a contribution from the non-invariance of the field measures under Weyl transformations, leading to the appearance of the Liouville action for the scalar field φ and a determinant which as usual can be written as a functional integral over Grassmann-valued "ghost" fields b, c of spin -1 and 2; the measure then displays a residual gauge invariance under the subset of diffeomorphisms which preserve g up to a local Weyl transformation. In complex coordinates $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, the algebra of the latter is enlarged by the infinitely many additional generators $\ell_n = -z^{n+1}\partial_z$, $n \in \mathbb{Z}$ of two copies of the *Witt algebra*

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} \ , \tag{2.32}$$

whose subalgebra with $|n| < 2$ exponentiates to the *conformal group* $SL(2,\mathbb{C})/\mathbb{Z}_2$. One way to compute observables is via the *conformal bootstrap*, taking advantage of the fact that invariance under (2.32) implies differential equations for correlation functions. Another route represents the algebra (2.32) explicitly on a Fock space, which is the procedure we shall employ. On the latter, (2.32) is represented only up to a phase, that is by operators L_n generating the *Virasoro algebra*, which includes a central term

$$\frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{2.33}$$

in addition to the right-hand side of (2.32). In particular, when the statistical system defined on the random surface approaches its critical point, the Liouville field φ interacts with another conformal field theory, frequently referred to as the *matter* theory and the total *central charge* is given by $c = c_L + c_M - 26$, where c_L and c_M denote the central charges of the two systems, offset by the negative contribution of the ghost fields. However, since here (2.32) describes an algebra of residual gauge transformations arising from partial gauge fixing, it must be respected exactly so that c must vanish, thus fixing c_L given c_M. This allows for an interpretation of this theory as a description of bosonic strings propagating in a $c_M + 1$-dimensional target space. For this reason, we will sometimes also refer to the random surface as the *worldsheet* of the string. As we shall see below, for $c_M \notin [1,25]$, the susceptibility exponent is determined by the celebrated Knizhnik-Polyakov-Zamolodchikov (KPZ) relation [47]:

$$\gamma_s = \frac{1}{12}\left(c_M - 1 - \sqrt{(c_M-1)(c_M-25)}\right) . \tag{2.34}$$

The subspace of physically distinct states – i.e. those states in the Fock space that do not differ by just a gauge transformation – is then obtained from the cohomology of the nilpotent Becchi-Rouet-Stora-Tyutin (BRST) operator, whose holomorphic component is expressed in terms of the respective stress-energy tensors and the anticommuting ghost field as[5]

$$\mathrm{d} = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} J(z) , \qquad J(z) = :\left(T_M(z) + T_L(z) + \frac{1}{2}T_{gh}(z)\right) c(z): , \tag{2.35}$$

where we introduced standard notation for the normal ordering of free quantum fields [44],

$$:\mathscr{O}_1(z)\mathscr{O}_2(z): = \lim_{w\to z}\left(\mathscr{O}_1(z)\mathscr{O}_2(w) - \langle \mathscr{O}_1(z)\mathscr{O}_2(w)\rangle\right) . \tag{2.36}$$

Finite representations of (2.32) with $c_M \leq 1$ have been classified and are labelled by pairs of simply laced Dynkin diagrams [48], constituing the Hilbert space of the so-called *minimal models*. In particular, the values of γ_s mentioned in the previous section identify the universality classes describing highest critical point of a single random matrix with the (A_1, A_{2k}) minimal model with $c_M = 10 - 6[k + 2/(2k+1)]$ and those of (2.4) with $q = 2$ with the (A_{k+1}, A_{k+2}) minimal model with with $c_M = 1 - 6/[(k+1)(k+2)]$, respectively coupled to Liouville theory; these will arise as special cases of the construction outlined below. For an explicit expression for the BRST operator and the space it acts on, let us fix some definitions for the matter, Liouville and ghost systems. Because in Chapter 4 we are looking for a conformal field theory description of the scaling limit for (2.4), we will consider the $\mathscr{W}_q$ minimal model as defining the matter theory, which reduces to the aforementioned minimal models for $q = 1$ and $q = 2$, respectively. For definiteness, we begin by discussing each system on the Riemann sphere, allowing us to choose the flat background

[5]We omit a total derivative $3\partial^2 c/2$ which has to be added to ensure J transforms as a tensor.

metric $\hat{g} = \mathrm{d}z\mathrm{d}\bar{z}$ and drop the integration over moduli, before moving on to the inclusion of a single boundary.

Matter sector. The $\mathscr{W}_q$ minimal models are a family of rational conformal field theories dating back to [49] that can be specified by a positive integer q and a pair of coprime integers (p, p'); here we describe their free-field realisation paralleling the presentation in [50, 51]. Their action functional can be represented as

$$S_M[\phi, \hat{g}] = \frac{1}{8\pi} \int_{\mathbb{CP}^1} \mathrm{d}^2 x \sqrt{\det \hat{g}} \, \left(\hat{g}^{\alpha\beta} \partial_\alpha \phi \cdot \partial_\beta \phi + 2\mathrm{i} Q_0 R[\hat{g}] \, \rho \cdot \phi \right) , \tag{2.37}$$

where $\phi = (\phi^1, \dots \phi^{q-1})$ is a vector in the root space of $SU(q)$, ρ is the Weyl vector, R denotes the Ricci scalar and $Q_0 = (p' - p)/\sqrt{pp'}$. Variation of (2.37) with respect to the background metric yields the stress-energy tensor, whose holomorphic component reads

$$T_M(z) = -\frac{1}{2} : \partial\phi \cdot \partial\phi : + \mathrm{i} Q_0 \rho \cdot \partial^2 \phi \, . \tag{2.38}$$

We shall group zero modes with the holomorphic field components, expanding the fields $\phi^i(z, \bar{z}) = \phi^i(z) + \bar{\phi}^i(\bar{z})$ as

$$\phi^i(z) = \phi^i_0 - \mathrm{i} a^i_0 \ln z + \mathrm{i} \sum_{n \neq 0} \frac{a^i_n}{n} z^{-n} \, , \quad \bar{\phi}^i(\bar{z}) = -\mathrm{i} a^i_0 \ln \bar{z} + \mathrm{i} \sum_{n \neq 0} \frac{\bar{a}^i_n}{n} \bar{z}^{-n} \, . \tag{2.39}$$

Adopting an orthogonal cartesian basis in root space, the commutation relations read

$$[\phi^i_0, a^j_n] = \mathrm{i} \delta_{n,0} \delta^{ij} \, , \qquad [a^i_n, a^j_m] = n \delta_{n+m,0} \delta^{ij} \, . \tag{2.40}$$

These imply that the generators

$$L_n^M = \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_m \cdot a_{n-m} : -(n+1) Q_0 \rho \cdot a_n \tag{2.41}$$

obey the Virasoro algebra[6] with central charge

$$c_M = (q-1)\left(1 - \frac{(p'-p)^2}{pp'} q(q+1)\right) , \tag{2.42}$$

where we used the Freudenthal-de-Vries 'strange formula' $\rho \cdot \rho = (q^3 - q)/12$. The allowed highest-weight states are obtained by the action of the chiral vertex operators $V_\alpha^M(z) = \,:\exp(\mathrm{i}\alpha \cdot \phi(z)):$ on the $SL(2,\mathbb{C})$-invariant vacuum,

$$|\lambda\rangle_M = \lim_{z \to 0} V^M_{Q_0 \rho - \frac{1}{\sqrt{pp'}}\lambda}(z)|0\rangle_M , \qquad a_n^i |\lambda\rangle_M = 0 \ \forall \ n > 0 . \tag{2.43}$$

In the above λ is short-hand for the $SU(q)$ representation weight $\lambda = (p'r^i - ps^i)\omega_i$, where ω_i denotes the dual weight corresponding to the simple root e_i, satisfying

$$e_i \cdot \omega_j = \delta_{ij} , \quad \omega_i \cdot \omega_j = \frac{i(q-j)}{q} , \quad i \le j . \tag{2.44}$$

From the operator product expansion (OPE) with $T_M(z)$, we find that the primary field corresponding to λ has conformal weight

$$L_0^M |\lambda\rangle_M = \left(\frac{\lambda^2}{pp'} - Q_0^2 \rho^2\right) |\lambda\rangle_M . \tag{2.45}$$

Consistency of the fusion rules requires that r^i and s^i are positive integers satisfying

$$\sum_{i=1}^{q-1} r^i < p , \qquad \sum_{i=1}^{q-1} s^i < p' . \tag{2.46}$$

These restrictions identify the sets $[r^1, \dots, r^{q-1}]$ and $[s^1, \dots, s^{q-1}]$ as Dynkin labels for rep-

[6] The Virasoro algebra generated by (2.41) is merely a subalgebra of the larger, non-linear $\mathscr{W}_q$-algebra generated by the chiral spin-s currents, $s = 2 \dots q$. These currents are primary with respect to Virasoro, but arise as descendants of the $\mathscr{W}_q$ vacuum module.

resentations of $\widehat{\mathfrak{su}}(q)_k$ and $\widehat{\mathfrak{su}}(q)_{k+1}$, respectively, with the additional node of the affine diagram omitted and the level k given by

$$k = \frac{p}{p'-p} - q \ . \tag{2.47}$$

In particular, when k is a positive integer and $q \geq 2$, all primary fields come with a positive weight and the model is unitary. Since the action of the Weyl group of $SU(q)$ leaves the root space inner product invariant, the weight (2.45) is invariant under $\lambda \to w\lambda$, for w an element of the symmetric group S_q of order $q!$. Consequently the conditions (2.46) still leave degeneracy in the spectrum: we need to restrict λ further to a fundamental domain $\mathscr{B}^{(q)}_{p,p'}$ to avoid overcounting[7]. For each such $\lambda \in \mathscr{B}^{(q)}_{p,p'}$, we then define the Fock space

$$\mathscr{F}_M(\lambda) = \text{span}\left\{ \prod_{i=1}^{q-1} \prod_{j=1}^{k_i} a^i_{-n^{(i)}_j} |\lambda\rangle_M \mid k_i \geq 0,\ 0 < n^{(i)}_1 \leq \cdots \leq n^{(i)}_{k_i} \right\} \ , \tag{2.48}$$

which is a reducible $\mathscr{W}_q$-module. To obtain the irreducible $\mathscr{W}_q$-module $\mathscr{M}(\lambda)$ defining the subspace of physical states, we introduce the so-called Felder complex $(\mathscr{C}(\lambda), \mathrm{d}')$, where d' is a nilpotent operator assembled from integrals of products of field exponentials acting on

$$\mathscr{C}(\lambda) = \bigoplus_{N^i \in \mathbb{Z}} \bigoplus_{w \in S_q} \mathscr{F}_M\left(\lambda^w - pp'N^i e_i\right) \ , \tag{2.49}$$

with $\lambda^w = p'r^i w\omega_i - ps^i\omega_i$. The complex is called a *resolution* of $\mathscr{M}(\lambda)$; that is, the n^{th} cohomology cohomology group reads $H^n(\mathscr{C}_M(\lambda), \mathrm{d}') = \delta_{n0}\mathscr{M}(\lambda)$, where n denotes the grading with respect to d' [46]. The partition function thus obtained agrees with that obtained from other definitions of $\mathscr{M}(\lambda)$ [52]. This construction has first been described in detail for $q = 2$ in [53] and for $q = 3$ in [54]. The main feature of importance for Chapter

[7] See [51] for an explicit derivation of $\mathscr{B}^{(q)}_{p,p'}$.

4 in this construction is that the above-mentioned degeneracy in the definition of λ implies that there exists more than one resolution for given $\mathscr{M}(\lambda)$.

Liouville sector. Liouville theory is governed by the action functional[8]

$$S_L[\varphi,\hat{g}] = \frac{1}{8\pi}\int_{\mathbb{CP}^1} \mathrm{d}^2x\sqrt{\det\,\hat{g}}\left(\hat{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + 2Q_L R[\hat{g}]\varphi + 8\pi\mu\mathrm{e}^{\sqrt{2}b\varphi}\right)\,. \tag{2.50}$$

Requiring invariance under Weyl transformations fixes the background charge in terms of the Liouville coupling b as $Q_L = (b+b^{-1})/\sqrt{2}$; the cosmological constant μ then corresponds to a marginal deformation. The relationship (2.34) can now be inferred by inspecting the change of the action under translations in field space: according to the Gauss-Bonnet theorem, upon shifting $\varphi \to \varphi + \sigma$, the second term in (2.50) contributes $2Q_L\sigma$ to the change of the action. Moreover, choosing $\sigma = -\ln\mu/(\sqrt{2}b)$ renders the third term in (2.50) independent of μ. Together with the invariance of the remaining contributions to the measure, this implies that the partition function on the sphere obeys $F_0(\mu) = \mu^{2-\gamma_s}F_0(1)$, where $\gamma_s = 1-b^{-2}$; expressing b as a function of c_M then yields (2.34). Since the short-distance behaviour of fields is controlled by large negative values of φ, we do not expect the exponential interaction term to affect the expression for the central charge and conformal weights. This allows us to set $\mu = 0$ throughout the remainder of this section. The holomorphic component of the stress-energy tensor is then

$$T_L(z) = -\frac{1}{2}\,:\partial\varphi\partial\varphi:\, + Q_L\partial^2\varphi\,. \tag{2.51}$$

Expanding the holomorphic component of φ as

[8]Our normalisation of the Liouville field differs by a factor of $\sqrt{2}$ from [38] and keeps our conventions close to the free-field treatments in [44, 36].

$$\varphi(z) = \varphi_0 - \mathrm{i}\alpha_0 \ln z + \mathrm{i}\sum_{n\neq 0}\frac{\alpha_n}{n}z^{-n} \tag{2.52}$$

results in the mode commutation relations

$$[\varphi_0, \alpha_n] = \mathrm{i}\delta_{n,0}\,, \qquad [\alpha_n, \alpha_m] = n\delta_{n+m,0}\,. \tag{2.53}$$

As a result, the generators

$$L^L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} :\alpha_m\alpha_{n-m}: \; + \mathrm{i}Q_L(n+1)\alpha_n \tag{2.54}$$

obey the Virasoro algebra with central charge

$$c_L = 1 + 12Q_L^2\,. \tag{2.55}$$

From the $SL(2,\mathbb{C})$-invariant vacuum we obtain a continuous family of highest-weight states $|P\rangle_L$ by acting with vertex operators $V^L_\alpha(z) = :\exp(\alpha\varphi(z)):$

$$|P\rangle_L = \lim_{z\to 0} V^L_{Q_L+\mathrm{i}P}(z)|0\rangle_L\,, \qquad \alpha_n|P\rangle_L = 0\;\forall\; n>0\,. \tag{2.56}$$

The OPE with $T_L(z)$ reveals that the corresponding bulk fields are spinless primaries of weight

$$L^L_0|P\rangle_L = \frac{1}{2}(Q_L^2 + P^2)|P\rangle_L\,, \tag{2.57}$$

which is evidently invariant under the reflection $P \to -P$. The states (2.56) are normalisable iff $P \in \mathbb{R}$ - the corresponding operators then create macroscopic loops in the worldsheet as we will discuss below. On the other hand, operators with $\mathrm{i}P \in \mathbb{R}$ create non-normalisable states; on the latter, one must impose the Seiberg bound $\mathrm{i}P \geq 0$ when $\mu > 0$ to avoid double-counting [55]. For given P, we define the Feigin-Fuchs module

$$\mathscr{F}_L(P) = \mathrm{span}\left\{ \prod_{i=1}^{k} \alpha_{-n_i} |P\rangle_L \mid k \geq 0,\ 0 < n_1 \leq \cdots \leq n_k \right\} . \tag{2.58}$$

Ghost sector. The action for the ghost system arising form the partial gauge-fixing of worldsheet diffeomorphisms is

$$S_{gh}[b,c]\big|_{\hat{g}=\mathrm{d}z\mathrm{d}\bar{z}} = \frac{1}{2\pi}\int_{\mathbb{CP}^1} \mathrm{d}^2 z (b\bar{\partial}c + \bar{b}\partial\bar{c}) , \tag{2.59}$$

with corresponding holomorphic stress-energy tensor

$$T_{gh}(z) = :(2b\partial c + c\partial b):\ . \tag{2.60}$$

The mode expansion of the holomorphic fields and commutation relations read

$$c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1} , \qquad b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2} , \tag{2.61}$$

$$\{b_n, c_m\} = \delta_{n+m,0} . \tag{2.62}$$

so that the generators[9]

$$L_n^{gh} = \sum_{m\in\mathbb{Z}} (m-n)\ :c_{-m}b_{m+n}: - \delta_{n,0} \tag{2.63}$$

obey the Virasoro algebra with central charge $c_{gh} = -26$. Of the two possible ground states, we shall define the ghost vacuum by $b_{n-1}|0\rangle_{gh} = c_n|0\rangle_{gh} = 0$ for $n > 0$ and normalise the ghost number $\sum_{n\in\mathbb{Z}} :c_{-n}b_n:$ such that $|0\rangle_{gh}$ has ghost number zero. By repeated action of the creation operators on this state we generate the Fock space

[9]The normal ordering constant in L_n^{gh} is determined by $2L_0|0\rangle_{gh} = [L_1, L_{-1}]|0\rangle_{gh} = -2|0\rangle_{gh}$.

$$\mathscr{F}_{gh} = \text{span}\left\{ \prod_{i=1}^{k} c_{-n_i} \prod_{j=1}^{l} b_{-m_j} |0\rangle_{gh} \mid k,l \geq 0 \ , \ 0 < n_1 < \cdots < n_k, \ 0 < m_1 \cdots < m_l \right\} . \tag{2.64}$$

Let us now turn to the description of boundaries in the present context. Conformally invariant boundary conditions were determined in [26, 28, 56] for Liouville theory and in [50, 51] for the $\mathscr{W}_q$ minimal model. Let us consider worldsheets with disk topology, and for concreteness we shall map the disk interior to the upper half plane $\{z \in \mathbb{CP}^1 | \text{Im}\, z > 0\}$ such that the boundary is located at $z = \bar{z}$.

Matter sector. We briefly summarise the free field construction of the matter boundary states given in [51]. Let us define the coherent states

$$|B(\lambda)\rangle_\Lambda = \exp\left(\sum_{m>0} \frac{1}{m} a_{-m}^T \cdot \Lambda \cdot \bar{a}_{-m} \right) \lim_{z,\bar{z}\to 0} V^M_{Q_0\rho - \frac{1}{\sqrt{pp'}}\lambda}(z) \bar{V}^M_{Q_0\rho + \frac{1}{\sqrt{pp'}}\lambda}(\bar{z}) |0\rangle_M \, . \tag{2.65}$$

The two possible forms of the $(q-1)\times(q-1)$ matrix Λ compatible with conformal symmetry were determined in [50, 51]: either $\Lambda = -\mathbb{I}$ or $\Lambda = w_\rho$, where w_ρ is the longest element of the Weyl group [10]. The matter Ishibashi states corresponding to a bulk primary λ are given by a sum of such coherent states over the Felder complex,

$$|\lambda;\Lambda\rangle\rangle_M = \sum_{w\in S_q} \sum_{N^i \in \mathbb{Z}} \kappa_N^w \left| B(\lambda^w - pp' N^i e_i) \right\rangle_\Lambda \, , \tag{2.66}$$

with λ^w defined as in (2.49) and κ_N^w a pure phase which will be unimportant for our discussion. For each $\lambda \in \mathscr{B}^{(q)}_{p,p'}$, we obtain two physical boundary states that additionally obey

[10] Only the former choice of Λ additionally preserves the higher-spin symmetries.

the Cardy condition [57], one for each of the allowed choices of Λ:

$$
\begin{aligned}
|\lambda\rangle_{\mathrm{C}} &= \sum_{\lambda'\in\mathscr{B}} \Psi^*_\lambda(\lambda')|\lambda';-\mathbb{I}\rangle\rangle_M \;, \\
|\tilde{\lambda}\rangle_{\mathrm{C}} &= \sum_{\lambda'\in\mathscr{B}} \Psi^*_\lambda(\lambda')|\lambda';w_\rho\rangle\rangle_M \;,
\end{aligned}
\tag{2.67}
$$

where the one-point function the primary field λ' on the disk with boundary condition λ is given in terms of the modular S-matrix

$$
S_{\lambda\lambda'} = \frac{Q_0^{q-1}}{\sqrt{\det A}} \sum_{w\in S_q}\sum_{w'\in S_q} \det w\; \exp\left(2\pi \mathrm{i} Q_0^2 \lambda'\cdot w'(p'r_0 w - p s_0)\lambda\right) \;, \tag{2.68}
$$

as $\Psi_\lambda(\lambda') = S_{\lambda\lambda'}/\sqrt{S_{\rho\lambda'}}$, with $\lambda=\rho$ the identity field and A the Cartan matrix of $SU(q)$. In the above, r_0 and s_0 denote the unique[11] pair of integers integers satisfying $1\leq r_0\leq p-1$, $1\leq s_0\leq p'-1$ and $p'r_0-ps_0=1$. Note that states with $\Lambda=w_\rho$ may decouple so that in general we do not obtain two boundary states per primary field as (2.67) might suggest. For example, for $(q,k)=(3,1)$, there are 6 primary fields, but only 8 independent boundary states [50].

Liouville sector. In presence of a boundary, the boundary cosmological constant μ_B arises as an additional marginal coupling as the Liouville action has to be augmented by a boundary term, which in our coordinates is simply

$$
\mu_B \int_{\mathbb{R}} \mathrm{d}s\; \mathrm{e}^{b\varphi} \;. \tag{2.69}
$$

The parameter μ_B labels a family of Neumann boundary conditions on φ along the real axis,

$$
\mathrm{i}(\partial-\bar{\partial})\varphi = 4\pi\mu_B\, \mathrm{e}^{b\varphi} \;. \tag{2.70}
$$

[11]The uniqueness of r_0 and s_0 is a consequence of Bézout's idenity.

At sufficiently strong coupling b, we need to account for the presence of the semiclassically invisible, marginal counterterms [27]

$$\tilde{\mu}\int_{\mathrm{Re}\ z,\bar{z}>0} \mathrm{d}^2 z\ \mathrm{e}^{2\varphi/b}\ , \qquad \tilde{\mu}_B\int_{\mathbb{R}} \mathrm{d}s\ \mathrm{e}^{\varphi/b}\ , \tag{2.71}$$

rendering the action (2.50) invariant under the *strong-weak duality* transformation $(b,\mu,\mu_B)\to(b^{-1},\tilde{\mu},\tilde{\mu_B})$. The boundary conditions thus defined by the dimensionless ratios

$$\zeta^2=\frac{\mu_B^2}{\mu}\sin(\pi b^2)\ , \quad \eta^2=\frac{\tilde{\mu}_B^2}{\tilde{\mu}}\sin(\pi/b^2)\ , \tag{2.72}$$

can be parametrised by a single variable σ, which we define as

$$\zeta=\cosh(\pi b\sigma)\ , \quad \eta=\cosh(\pi\sigma/b)\ . \tag{2.73}$$

The Liouville Ishibashi states are given by

$$|P\rangle\rangle_L=\ \exp\left(-\sum_{k>0}\frac{1}{k}\alpha_{-k}\bar{\alpha}_{-k}\right)\lim_{z,\bar{z}\to 0}V^L_{Q_L+\mathrm{i}P}(z)\bar{V}^L_{Q_L-\mathrm{i}P}(\bar{z})|0\rangle_L\ , \tag{2.74}$$

where now P is real. The physical boundary state corresponding to the boundary condition (2.73) was worked out in [26, 56]

$$|\sigma\rangle_{\mathrm{FZZT}}=\int_0^\infty \mathrm{d}P\ \Psi^*_\sigma(P)|P\rangle\rangle_L\ , \tag{2.75}$$

where the disk one-point function with boundary condition σ was found to be

$$\Psi_\sigma(P)=\mu^{-\mathrm{i}P/b}\frac{\cos(2\pi\sigma P)}{\mathrm{i}P}\Gamma(1+2\mathrm{i}P/b)\Gamma(1+2\mathrm{i}Pb)\ , \tag{2.76}$$

which is manifestly invariant under the strong-weak duality. In the context of string theory, this defines the so-called Fateev-Zamolodchikov-Zamolodchikov-Teschner (FZZT) brane. An infinite discrete family of Dirichlet boundary conditions has been found in [28] – the

Zamolodchikov-Zamolodchikov (ZZ) brane – but will not be of concern in this thesis.

Ghost sector. For completeness, we finally spell out the conformally invariant boundary state for the ghost system [58, p.226]

$$|B\rangle_{gh} = (c_0 + \bar{c}_0) \exp\left(- \sum_{k>0} (b_{-k}\bar{c}_{-k} + \bar{b}_{-k}c_{-k}) \right) |0\rangle_{gh} \, . \tag{2.77}$$

2.3 Double scaling limit

In Chapter 5, we study averages of products of characteristic polynomials in the double-scaling limit, employing the operator approach developed by Douglas [19] and applied to the Hermitian two-matrix model by Daul, Kazakov and Kostov [18]. Below we shall define the double scaling limit in the context of this formalism. The application of the theory of monodromy preserving deformations of linear ordinary differential equations [59, 60, 61] to the present context was developed by Moore [62, 63] – see also [64] for a more recent discussion. Further details on this formalism can be found in the reviews [38, 39]; the connection to the theory of integrable systems is reviewed in [65].

In the previous section, we saw that planar maps are an exponentially bounded family, yielding convergent expressions for series expansions in $\{t_m\}_{m=3}^{k+2}$ for generic values of the latter. Resting on the fact that generating functions for maps of fixed positive genus display the same radius of convergence, the topological recursion algorithm [66] yields finite answers for the free energy F_h for maps of arbitrary genus h from the initial data at $h = 0$. For large N, the free energy to all orders can then be estimated by an asymptotic series[12], in the following denoted by '$\simeq$':

$$Z_N \simeq \exp\left(\sum_{h=0}^{\infty} N^{2-2h} F_h \right) , \quad N \to \infty \, . \tag{2.78}$$

[12] Though finite at each order in $1/N$, this estimate is formal because this series, neglecting exponentially small corrections, has vanishing radius of convergence.

At the boundary of the domain of analyticity, the scaling relation (2.27) generalises for the genus-h free energy as

$$F_h = \varepsilon^{(2-\gamma_s)(2-2h)} \mathscr{F}_h(\mu) \,, \tag{2.79}$$

where μ is the renormalised cosmological constant as defined in the previous section. Introducing the *string coupling* $g_s = \varepsilon^{\gamma_s - 2}/N$, we recast (2.78) as an asymptotic expansion in g_s,

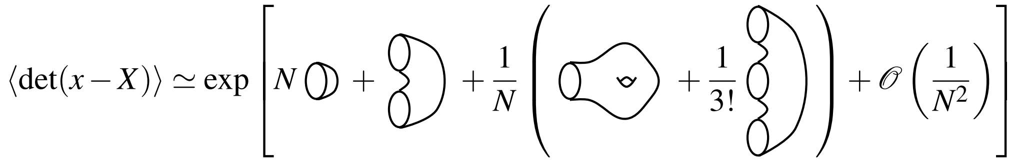
$$Z_N \simeq \exp\left(\sum_{h=0}^{\infty} g_s^{2h-2} \mathscr{F}_h(\mu) \right) \,, \quad g_s \to 0 \,. \tag{2.80}$$

We can thus retain significant contributions from all topologies even at large N by taking the double scaling limit $N \to \infty$, $\varepsilon \to 0$, keeping g_s fixed.

To study this limit effectively, we need a handle on contributions from all worldsheet topologies that does not rely on the asymptotic expansion (2.80) about zero string coupling. One way to achieve this is to employ the operator approach [19] for the measure (2.4) with $q = 2$ and $\deg V_1 = p$, $\deg V_2 = p'$ for coprime integers p and p', whose scaling limit about the highest critical point is described by the $(A_{p-1}, A_{p'-1})$ minimal model coupled to Liouville theory [18]. In this context, it is useful to study the exponentiation of the operator $N^{-1}\mathrm{tr}\ln(x-X)$ encountered in equation (2.21) of Subection 2.1.2: denoting the contribution from connected surfaces by $\langle\cdot\rangle_c$, the latter can be written as

$$\begin{aligned} \langle \det(x-X) \rangle &= \exp\left(\left\langle \mathrm{e}^{\mathrm{tr}\ln(x-X)} - 1 \right\rangle_c \right) \\ &\simeq \exp\left(\sum_{n=1}^{\infty} \frac{1}{n!} \langle (\mathrm{tr}\ln(x-X))^n \rangle_c \right) \,. \end{aligned} \tag{2.81}$$

The large-N expansion of the above can be represented graphically as



$$, \quad (2.82)$$

thus accounting for an arbitrary number of boundaries with the same boundary condition labelled by x. In the context of string theory, this expectation value is therefore interpreted as the partition function of a brane at target space position x. More generally, the principal objects of interest are averages of characteristic polynomials labelled by $1 \leq n \leq N$,

$$\alpha_n(x) = \langle \det(x - X_1) \rangle_{n \times n} \ , \qquad \beta_n(y) \ = \langle \det(y - X_2) \rangle_{n \times n} \ , \tag{2.83}$$

where $\langle \cdot \rangle_{n \times n}$ denotes the average with respect to the measure on $n \times n$ minors,

$$\begin{aligned} \mathrm{d}\mu_{n \times n}(X_1, X_2) &= \frac{1}{Z_n} \mathrm{e}^{-N \mathrm{tr}[V_1(X_1) + V_2(X_2) - X_1 X_2]} \mathrm{d}^{n^2} X_1 \ \mathrm{d}^{n^2} X_2 \ , \\ \mathrm{d}^{n^2} X &= \prod_{1 \leq i \leq j \leq n} \mathrm{dRe} X^i_{\ j} \prod_{1 \leq i < j \leq n} \mathrm{dIm} X^i_{\ j} \ . \end{aligned} \tag{2.84}$$

Then the *Baker-Akhiezer functions*

$$\psi_n(x) = \frac{\mathrm{e}^{-N V_1(x)}}{\sqrt{h_n}} \alpha_n(x) \ , \quad \chi_n(y) = \frac{\mathrm{e}^{-N V_2(y)}}{\sqrt{h_n}} \beta_n(y) \ , \tag{2.85}$$

are bi-orthonormal, i.e. $\int \mathrm{d}x \ \mathrm{d}y \ \psi_n(x) \chi_m(y) \mathrm{e}^{xy} = \delta_{mn}$ for a suitable choice of the normalisation constant h_n, and obey recursion relations of the form

$$\begin{aligned} x \psi_n(x) &= P_n(z) \psi_n(x) \ , & \frac{1}{N} \frac{\partial}{\partial x} \psi_n(x) &= -Q_n(z)^T \psi_n(x) \ , \\ y \chi_n(y) &= Q_n(z) \chi_n(y) \ , & \frac{1}{N} \frac{\partial}{\partial y} \chi_n(y) &= -P_n(z)^T \chi_n(y) \ , \end{aligned} \tag{2.86}$$

where the difference operators P_n and Q_n have an expansion in powers of $z = \exp(-\partial_n)$ and satsify $[P_n, Q_n] = 1/N$. Here, the transpose is defined by $(f(n)\mathrm{e}^{s\partial_n})^T = \mathrm{e}^{-s\partial_n} f(n)$. Introducing the scaling variables

$$g_s = \varepsilon^{-\frac{p+p'}{p}}/N \;, \quad t = \varepsilon^{-\frac{p+p'-1}{p}}(N-n)/N \;, \tag{2.87}$$

so that $\varepsilon^{-1/p}\partial_n = -g_s\partial_t$ and taking the double scaling limit $N \to \infty$, $\varepsilon \to 0$, keeping g_s and t finite, the difference operators become

$$P_n(z) = x_c + \varepsilon\mathbb{P}(t;\partial_t) \;, \quad Q_n(z)^T = y_c - \varepsilon^{p'/p}\mathbb{Q}(t;\partial_t) \;, \quad \partial_t \equiv g_s\frac{\partial}{\partial t} \tag{2.88}$$

where the p^{th} and $(p')^{\text{th}}$ order differential operators can be brought to the form

$$\mathbb{P}(t;\partial_t) = 2^{p-1}\partial_t^p + \sum_{n=2}^{p} u_n(t)\partial_t^{p-n} \;, \tag{2.89a}$$

$$\mathbb{Q}(t;\partial_t) = \beta_{p,p'}\left(2^{p'-1}\partial_t^{p'} + \sum_{n=2}^{p'} v_n(t)\partial_t^{p'-n}\right) \;, \tag{2.89b}$$

where $\beta_{p,p'} = (-1)^{p+p'}\beta_{p',p}$ is a real constant and the coefficients $\{u_n(t)\}_{n=2}^{p}$, $\{v_n(t)\}_{n=2}^{p'}$ solve the *string equation* [19]

$$[\mathbb{P},\mathbb{Q}] = g_s \;. \tag{2.90}$$

Upon a suitable rescaling of $\mu_B = \varepsilon^{-1}(x-x_c)$, $\widetilde{\mu}_B = \varepsilon^{-p'/p}(-1)^{p+1}(y-y_c)$ and introducing the dimensionless variables $\zeta = \mu_B/\sqrt{\mu}$, $\eta = \beta_{p,p'}^{-1}\widetilde{\mu}_B/\sqrt{\widetilde{\mu}}$, the functions (2.85) then satisfy the overdetermined couples of differential equations

$$\zeta\psi(t;\zeta) = \mathbb{P}(t;\partial_t)\psi(t;\zeta) \;, \tag{2.91a}$$

$$\partial_\zeta\psi(t;\zeta) = \mathbb{Q}(t;\partial_t)\psi(t;\zeta) \;, \quad \partial_\zeta \equiv g_s\frac{\partial}{\partial\zeta} \tag{2.91b}$$

and similarly, with $(f(t)\partial^n)^T = (-\partial)^n f(t)$,

$$\eta\chi(t;\eta) = \frac{(-1)^{p'}}{\beta_{p,p'}}\mathbb{Q}^T(t;\partial_t)\chi(t;\eta) \ , \tag{2.92a}$$

$$\partial_\eta\chi(t;\eta) = (-1)^{p'}\beta_{p,p'}\mathbb{P}^T(t;\partial_t)\chi(t;\eta) \ , \quad \partial_\eta \equiv g_s\frac{\partial}{\partial\eta} \tag{2.92b}$$

whose compatibility is expressed by (2.90). In the theory of integrable systems, the operators $(\mathbb{Q},\mathbb{P})$ are often called the *Lax pair* and the set of differential equations for $u_n(t)$ and $v_n(t)$ summarised by (2.90) is referred to as the p^{th} *reduction of the Kadomtsev-Petviashvili (KP) hierarchy*. They are invariant under

1. the $SL(2,\mathbb{C})$-family of linear canonical transformations

$$(\mathbb{P},\mathbb{Q}) \longmapsto (a\mathbb{P}-c\mathbb{Q}, d\mathbb{Q}-b\mathbb{P}) \ , \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \ , \tag{2.93}$$

2. the "charge conjugation"

$$(\mathbb{P},\mathbb{Q}) \longmapsto (\mathbb{P}^T, -\mathbb{Q}^T) \ . \tag{2.94}$$

Each $\gamma \in SL(2,\mathbb{C})$ with $b \neq 0$ can be represented by an integral transform acting on functions of ζ,

$$\gamma[f](\eta) = \mathrm{e}^{-\frac{d}{2b}\eta^2}\int \mathrm{d}\zeta \ f(\zeta)\mathrm{e}^{-\frac{1}{b}(a\zeta^2/2-\eta\zeta)} \ . \tag{2.95}$$

The particular case $a = d = 0$, $c = -b$ yields the Laplace transform, for which we reserve the notation

$$\mathscr{L}_b[f](\eta) = \int \mathrm{d}\zeta \mathrm{e}^{\eta\zeta/b} f(\zeta) \ , \tag{2.96}$$

and drop the subscript when $b = -1$. A symmetry of particular importance in Chapter 5 is the *duality* transformation $(p,p') \to (p',p)$ as first introduced in this context in [67]. From (2.84) we see this amounts to the interchange of the matrices X_1 and X_2 – a definition that extends to finite N and has been considered in [68, 69, 70, 71] under the name of *spectral duality*. We observe that the effect of this transformation on the differential equations (2.91) and (2.92) is, up to a coefficient, the composition of the charge conjugation (2.94) and a Laplace transform (2.96), namely

$$(\mathbb{P},\mathbb{Q}) \longmapsto \left((-1)^{p'}\beta_{p,p'}^{-1}\mathbb{Q}^T, (-1)^{p'}\beta_{p,p'}\mathbb{P}^T\right) , \tag{2.97}$$

and thus preserves the string equation (2.90).

We close this section by introducing a few concepts originating from the application of the inverse monodromy problem to (2.91) and (2.92), which will be useful for the definition of the spectral curve and the analysis of the semiclassical limit $g_s \to 0$. In brief, defining the p- and p'-vectors

$$\vec{\psi}(t;\zeta) = \left(\psi(t;\zeta), \partial_\zeta \psi(t;\zeta), \ldots, \partial_\zeta^{p-1}\psi(t;\zeta)\right)^T , \tag{2.98a}$$

$$\vec{\chi}(t;\eta) = \left(\chi(t;\eta), \partial_\eta \chi(t;\eta), \ldots, \partial_\eta^{p'-1}\chi(t;\eta)\right)^T , \tag{2.98b}$$

the relations (2.91) can be expressed as linear differential systems with $p \times p$ resp. $p' \times p'$ matrix-valued coefficients that are rational functions[13],

$$\partial_\zeta \vec{\psi}(t;\zeta) = \mathscr{Q}(t;\zeta)\vec{\psi}(t;\zeta) , \qquad \partial_t \vec{\psi}(t;\zeta) = \mathscr{B}(t;\zeta)\vec{\psi}(t;\zeta) ; \tag{2.99a}$$

$$\partial_\eta \vec{\chi}(t;\eta) = \widetilde{\mathscr{Q}}(t;\eta)\vec{\chi}(t;\eta) , \qquad \partial_t \vec{\chi}(t;\eta) = \widetilde{\mathscr{B}}(t;\eta)\vec{\chi}(t;\eta) . \tag{2.99b}$$

Accordingly, the spectrum of the Lax operators $\mathbb{P}$, $\mathbb{Q}$ and their dual images under (2.97)

[13]These relations are also frequently expressed in other variables such as $\lambda = (2\zeta)^{1/p}/2$ [64].

can conveniently be encoded in the zero locus of the polynomials

$$G(t;\zeta,Q) = \det(Q\mathbb{I}_{p\times p} - \mathscr{Q}(t;\zeta)) \,, \tag{2.100a}$$

$$\widetilde{G}(t;\eta,P) = \det(P\mathbb{I}_{p'\times p'} - \widetilde{\mathscr{Q}}(t;\eta)) \,, \tag{2.100b}$$

which defines a Riemann surface $\mathscr{C}_{p,p'}(t) = \{(\zeta,Q)\in\mathbb{C}^2 | G(t;\zeta,Q)=0\}$, called the *spectral curve*[14]. In the semiclassical limit $g_s \to 0$, the coefficients $v_n(t)$ and $u_n(t)$ are approximately constant. To compare this limit to the results of conformal field theory, we need to choose the conformal background [42] described in Subsection 2.2.1. This limit was first computed in [18], with the result[15]

$$\lim_{g_s\to 0} G(t;\zeta,Q) = \frac{1}{2^{p-1}}\left(T_p(Q) - T_{p'}(\zeta)\right) \,, \tag{2.101}$$

where $T_n(x)$ denotes the n^{th} Chebyshev polynomial. This corresponds to the algebraic equation satisfied by the dimensionless disk amplitude (2.29); as will become evident in later chapters, the duality transformation $(p,p') \to (p',p)$ then reduces to the strong-weak duality $b \to b^{-1}$ of the Liouville part of the conformal field theory.

Example 2.3.1. $(p,p') = (2,1)$. *This describes the double scaling limit of the GUE. The recursion relations* (2.86) *are solved by the* n^{th} *Hermite polynomial* H_n*,*

$$\langle \det(x-X)\rangle_{n\times n} = \left(\frac{1}{2t_2}\right)^{n/2} H_n\left(x\sqrt{\frac{t_2}{2}}\right) \,. \tag{2.102}$$

Using the integral representation of the latter [73] to write

$$\langle \det(x-X)\rangle_{n\times n} = \sqrt{\frac{t_2}{2\pi}}\int_{\mathbb{R}} (x+\mathrm{i}z)^n \mathrm{e}^{-t_2 z^2/2}\mathrm{d}z \tag{2.103}$$

[14] This is to be contrasted with the proposal to define a ‘quantum’ spectral curve by generalising the notion of a Riemann surface by allowing non-commuting coordinates in accordance with (2.90), see e.g. [62, 63, 72].

[15] Here and often in what follows, we suppressed the unimportant real constant $\beta_{p,p'}$.

and taking the double scaling limit with $z = t_2^{-1/2}(\mathrm{i} - \varepsilon^{1/2} g_s^{1/3} s)$ *and* g_s*,* t *as in* (2.87)*, this becomes the so-called Airy function,*

$$\psi(t;\zeta) = \mathrm{Ai}\left(g_s^{-2/3}(\zeta + t)\right) , \quad \mathrm{Ai}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}(s^3/3+xs)}\mathrm{d}s \; . \tag{2.104}$$

Upon rescaling $g_s \to g_s/\sqrt{2}$*, this indeed solves* (2.91) *with*

$$\mathbb{P} = 2\partial_t^2 + u_2(t) , \quad \mathbb{Q} = \partial_t , \tag{2.105}$$

where the string equation (2.90) *demands* $\dot{u}_2(t) = -1$*. The spectral curve is then given by the zeroes of*

$$G(t;\zeta,Q) = Q^2 - \frac{1}{2}(\zeta + t) \; . \tag{2.106}$$

Example 2.3.2. $(p,p') = (3,2)$*. The universality class of the critical cubic matrix model is controlled by* (2.91) *with*

$$\mathbb{P} = 4\partial_t^3 + u_2(t)\partial_t + u_3(t) , \quad \mathbb{Q} = \beta_{3,2}\left(2\partial_t^2 + v_2(t)\right) \; . \tag{2.107}$$

The string equation (2.90) *requires* $u_2(t) = 3v_2(t)$ *and* $u_3(t) = 3\dot{v}_2(t)/2$*, where* $v_2(t)$ *solves the first Painlevé equation:* $\ddot{v}_2(t) = 6v_2(t) + t$*. The spectral curve is given by the zeroes of*

$$G(t;\zeta,Q) = Q^3 - \frac{\zeta^2}{2} - Q\left(\frac{3v_2^2}{4} + \frac{\ddot{v}_2}{2}\right) - \frac{v_2^3}{4} - \frac{v_2\ddot{v}_2}{2} + \frac{\dot{v}_2^2}{2} , \tag{2.108}$$

see also Example 5.2.9 in Subsection 5.2.2 of Chapter 5. In the limit $g_s \searrow 0$*,* $v_2(t) = -1 + \mathscr{O}(g_s)$ *so that to leading order, the above becomes* $(T_3(Q) - T_2(\zeta))/4$*. After a duality transformation* $(p,p') \to (p',p)$*, this describes the universality classs encountered in Example 2.2.1 of the previous section.*

Chapter 3

Sums of Random Matrices and the Potts Model on Planar Maps

3.1 Overview

We compute the partition function $W_{(p)}(z)$ of the q-states Potts model [15] on a random planar lattice with $1 \leq p \leq q$ allowed, equally weighted colors on a connected boundary, where z denotes the fugacity of a boundary link. In the particular cases $q = 2$ and $q = 3$, all of these correspond to boundary conditions of the Ising and Potts lattice models that were found to be integrable on a fixed lattice by Behrend and Pearce [74]. To this end, we employ its matrix model formulation (2.4) as proposed long ago by Kazakov [14], who used it to solve the limits $q \to 0$ and $q \to 1$.

In the particular case of random *triangulations*, $W_{(1)}(z)$ was first found by Daul [20] and later Zinn-Justin [75] in the saddle point approximation for integer $0 \leq q \leq 4$, and by Bonnet and Eynard [76, 77] using the method of loop equations, who found an algebraic equation for $W_{(1)}(z)$ when $\arccos((q-2)/2)/\pi$ is rational; see also [78, 79] for related results. More recently, the authors of [80] considered a combinatorial approach using the so-called "loop-gas" representation of the Potts model on planar maps without reference to a matrix integral, from which a pair of coupled functional equations for $W_{(1)}(z)$ and a function related to $W_{(q)}(z)$ was obtained and solved. For $q = 2$, the relationship between $W_{(1)}(z)$ and $W_{(2)}(z)$ has been expressed succinctly from the perspective of the boundary renormalisation group [23, 24], a picture which later was extended to non-planar geometries [31] and arbitrary face degrees [32]. Indeed, these investigations revealed that different boundary conditions yield inequivalent algebraic equations satisfied by the corresponding generating functions. However, a systematic understanding of the relationship between different boundary conditions for more general values of q and p appears to be lacking and herein we report on some progress on this matter.

As will be discussed in Section 3.2, $W_{(p)}(z)$ is given by the Stieltjes transform of the spectral density of the sum p Hermitian random matrices of infinite size. A more mathematically inclined characterisation of the problem solved in this chapter thus goes as follows: given a set of Hermitian random matrices $\{X_i\}_{i=1}^{q}$ distributed according to (2.4) and a positive integer $p \leq q$, what is the spectral density of the sum $X_1 + X_2 + \ldots X_p$ as $N \to \infty$? For the simpler case of uncorrelated matrices, the answer has been neatly summarised in the context of free probability [21, 22], going back to Voiculescu's observation of the asymptotic freeness of Gaussian independent random matrices [33]: given a spectral density $\rho_X(z)$, define the *R-transform* via the functional inverse of its Stieltjes transform $W_X(z)$

$$R^X(z) = W_X^{-1}(z) - \frac{1}{z} \ . \tag{3.1}$$

Now assume Y is freely independent from X. Then the *free (additive) convolution* $\rho_X \boxplus \rho_Y$ is defined by ρ_{X+Y}. The results of free probability theory [33] state that the latter is obtained from ρ_X and ρ_Y by

1. computing R^{X+Y} by adding the respective R-transforms,

$$R^{X+Y}(z) = R^X(z) + R^Y(z) \ , \tag{3.2}$$

2. inverting the relationship (3.1),

$$W_{X+Y}^{-1}(z) = R^{X+Y}(z) + \frac{1}{z} \,. \tag{3.3}$$

The spectral density for the sum $X+Y$ can then be read off from the imaginary part of the inverse function,

$$\rho_{X+Y}(x) = \frac{1}{\pi}\mathrm{Im}\, W_{X+Y}(x)_{+} \,. \tag{3.4}$$

We follow [81] in referring to the key relationship (3.3) as Voiculescu's formula. Clearly the matrices $\{X_i\}_{i=1}^{q}$ distributed according to (2.4) are *not* freely independent – their correlations prevent us from applying Voiculescu's formula to compute the spectral densities for sums like $X_1 + X_2 + \ldots X_p$. Our strategy to obtain the disk partition function of the Potts model involves a suitable generalisation of the R-transform and using it to evaluate the spectral density and hence $W_{(p)}(z)$.

This chapter is organised as follows: Section 3.2 reviews the matrix model formulation and defines the observables of interest. In Section 3.3, we state the main results in Propositions 3.3.1 and 3.3.6 and discuss how our results reduce to Voiculescu's formula when the interactions of the Potts model are turned off. In Section 3.4, we study hard dimers, the Ising model and the 3-states Potts model on planar triangulations as simple examples in greater detail. We derive explicit expressions for the spectral curve for given p and compare our results to the literature where available. In Section 3.5, we proceed to investigate the phase diagram of the model when $0 < q < 4$ and comment on the conformal field theory description of the scaling behaviour associated with the critical points. Finally, we discuss the implications of our results in Section 3.6.

3.2 Definition of the model

Following [14, 20, 76, 77], we use the measure (2.4) to compute observables of the q-states Potts model on a random planar lattice. A distinguishing feature are the exponentials of $\mathrm{tr}X_iX_j$ in (2.4), breaking the overall $U(N)\times O(q)$-invariance of the remaining factors. Here we confine our study to the case $V_i(z) = U(z) + z^2/2\ \forall i$ for a fixed polynomial $U(z) = \sum_{m=2}^{k+2} t_m z^m/m$, rendering the q states of the statistical system indistinguishable. In this case, the measure (2.4) remains invariant under the overall symmetries

$$X_i \to U^\dagger X_i U \ , \quad U \in U(N) \ , \quad \text{and} \quad X_i \to X_{\sigma(i)} \ , \quad \sigma \in S_q \ , \tag{3.5}$$

where S_q dentoes the symmetric group of order $q!$. This is to be contrasted with the "multi-matrix chain" studied for example in [82, 83], for which $\mathbb{Z}_2$ is preserved in place of S_q. Our definition includes a subset of the statistical RSOS models on a random lattice, which are indexed by simply laced Dynkin diagrams [84] and have been described using matrix integrals by Kostov [85]. In particular, for $(q,k) = (2,1)$, (2.4) describes the A_3 model and for $(q,k) = (3,1)$ the D_4 model on random triangulations, respectively.

The desired quantities $W_{(p)}(z)$ can now be defined along the lines of our discussion in Subection 2.1.2 of the previous chapter: Given $\sigma \in S_q/(S_p \times S_{q-p})$, we define the partition function of the model on a random lattice with p allowed, equally weighted colors on a single connected boundary containing a marked point as[1]

$$W_{(p|\sigma)}(z) = \frac{1}{N}\left\langle \mathrm{tr}\frac{1}{z - X_{(p|\sigma)}} \right\rangle \ , \quad X_{(p|\sigma)} = \sum_{i=1}^{p} X_{\sigma(i)} \ , \quad 1 \leq p \leq q \ , \tag{3.6}$$

where here and in what follows $\langle\cdot\rangle$ denotes the average with respect to (2.4) and z denotes the fugacity of a boundary link. As a result of the permutation symmetry, for given p, all $|S_q/(S_p \times S_{q-p})| = \binom{q}{p}$ partition functions $W_{(p|\sigma)}(z)$ are described by the same function, so that we henceforth abbreviate $W_{(p)}(z) := W_{(p|\sigma_0)}(z)$ for a representative σ_0 and denote the

[1] Note that the $X_{(p|\sigma)}$ inherit a covariant transformation behaviour under $S_q \subset O(q)$.

spectral density of the sum $X_{(p|\sigma)}$ by $\rho_{(p)}(z)$. Note that for $p=1$, our definition of $W_{(p)}(z)$ reduces to the one studied in [14, 20, 75, 76, 77].

We conclude this section with a helpful lemma which expresses the partition function as single integral over effective matrix variables X_0 and $P_\pm$ by a series of integral transformations[2]. This circumvents a notorious difficulty presented by the first factor in (2.4), which leads to a complicated integral over the unitary group when the latter is written as a function of the eigenvalues of the matrices X_i with $i>0$ [14, 20].

Lemma 3.2.1. *Let $h>0$ and abbreviate the integral transformations*

$$\gamma_\pm(X) = \int_{\mathbb{R}} \mathrm{d}P_\pm f(P) \mathrm{e}^{-\frac{N}{2}\mathrm{tr}P_\pm^2} \mathrm{e}^{N\mathrm{tr}P_\pm X/\sqrt{\mathrm{e}^{\pm 2h}-1}}\,, \tag{3.7a}$$

$$\gamma'_\pm(P) = \int_\Gamma \mathrm{d}X f(X) \mathrm{e}^{N\mathrm{tr}PX\sqrt{1-\mathrm{e}^{\mp 2h}}}\,, \tag{3.7b}$$

where the subscripts below the integrals indicate the integration cycle for the corresponding eigenvalues. Then up to an overall constant, the partition function in (2.4) *can be written as*

$$Z_{N,q} = \int_{\mathbb{R}} \mathrm{d}P_+ \ \mathrm{e}^{-\frac{N}{2}(1-\mathrm{e}^{-2h})\mathrm{tr}P_+^2} \left(\gamma'_+\left[\mathrm{e}^{-N\mathrm{tr}U}\right](P_+)\right)^q \tag{3.8a}$$

$$= \int_{\mathbb{R}} \mathrm{d}X_0 \ \gamma_+\left[(\gamma'_+[\mathrm{e}^{-N\mathrm{tr}U}])^p\right](X_0)\ \gamma_-\left[(\gamma'_-[\mathrm{e}^{-N\mathrm{tr}U}])^{q-p}\right](X_0) \tag{3.8b}$$

$$= \int_{\mathbb{R}} \mathrm{d}X_0 \left(\prod_{i=1}^q \int_\Gamma \mathrm{d}X_i \ \mathrm{e}^{-N\mathrm{tr}U(X_i)}\right) \times \gamma_+[1]\left(X_0 + 2\sinh(h)\sum_{i=1}^p X_i\right)\ \gamma_-[1]\left(X_0 - 2\sinh(h)\sum_{i=p+1}^q X_i\right). \tag{3.8c}$$

Proof. We begin by showing equality of (3.8b) and (3.8c), and then equality to $Z_{N,q}$. Subsequently showing equality to (3.8a) completes the proof. By definition, we can write

[2] Thinking of $\{X_i\}_{i=1}^q$ as coordinates on configuration space, these simply correspond to linear canonical transformations on the corresponding phase space.

$$\begin{aligned}\gamma_\pm\left[\left(\gamma'_\pm[\mathrm{e}^{-N\mathrm{tr}U}]\right)^n\right](X_0) = &\int_{\mathbb{R}} \mathrm{d}P_\pm\ \mathrm{e}^{-N\mathrm{tr}P^2/2}\ \mathrm{e}^{N\mathrm{tr}X_0P_\pm/\sqrt{\mathrm{e}^{\pm 2h}-1}} \\ &\times \left(\prod_{i=1}^n \int_\Gamma \mathrm{d}X_i\ \mathrm{e}^{-N\mathrm{tr}U(X_i)}\ \mathrm{e}^{N\mathrm{tr}P_\pm X_i\sqrt{1-\mathrm{e}^{\mp 2h}}}\right) .\end{aligned} \tag{3.9}$$

In general, there are $\deg U' = k+1$ independent cycles Γ that render this iterated integral absolutely convergent for finite N. Hence we can apply the Fubini-Tonelli-theorem, that is, exchange the order of integration:

$$\gamma_\pm\left[\left(\gamma'_\pm[\mathrm{e}^{-N\mathrm{tr}U}]\right)^n\right](X_0) = \left(\prod_{i=1}^n \int_\Gamma \mathrm{d}X_i\ \mathrm{e}^{-N\mathrm{tr}U(X_i)}\right)\gamma_\pm[1]\left(X_0 \pm 2\sinh(h)\sum_{i=1}^n X_i\right) . \tag{3.10}$$

Inserting this result into (3.8b) proves equality to (3.8c). To obtain equality to $Z_{N,q}$, note that up to an overall multiplicative constant,

$$\int_{\mathbb{R}} \mathrm{d}X_0\ \gamma_+[1]\left(X_0 + 2\sinh(h)\sum_{i=1}^p X_i\right)\gamma_-[1]\left(X_0 - 2\sinh(h)\sum_{i=p+1}^q X_i\right) = \mathrm{e}^{N\mathrm{tr}\left(\sum_{i=1}^q X_i\right)^2/2} . \tag{3.11}$$

Inserting this result into (3.8c) and interchanging the order of integration between X_0 and X_i by the same argument proves equality to $Z_{N,q}$. It remains to show equivalence to (3.8a). Starting from (3.8c), we may use (3.9) to write the action of γ_+ on $(\gamma'_+[\mathrm{e}^{-N\mathrm{tr}U}])^p$ and of γ_- on $(\gamma'_-[\mathrm{e}^{-N\mathrm{tr}U}])^{q-p}$ as Gaussian integrals over two matrices P_+, P_-, respectively. Performing the integration over P_- and subsequently X_0, we are left with (3.8a). As a cross-check, it is straightforward to confirm that (3.8a) equals our initial definition in (2.4) of $Z_{N,q}$ by writing out the q^{th} power of $\gamma'_+[\mathrm{e}^{-N\mathrm{tr}U}]$ as a product of integrals over X_i, $i = 1\dots q$ and reversing the order of integration with P_+. □

3.3 Planar limit

This section is concerned with the explicit evaluation of $W_{(p)}(z)$ in the planar limit and is organised as follows: Subsection 3.3.1 expresses $W_{(p)}(z)$ via the p-independent spectrum of the matrix $Y \equiv \sqrt{1-\mathrm{e}^{-2h}}P_+$ in (3.8a). The latter is then provided explicitly in Subsection 3.3.2 – a problem first solved in [20, 82] and rederived here for arbitrary $q \neq 4$. Finally, in Subsection 3.3.3, we discuss how Voiculescu's formula (3.3) arises as a special case of our results in the limit of vanishing interaction strength of the Potts model. To streamline the presentation, for any pair of $N \times N$ matrices (X,P), we define the averages

$$\begin{aligned} G_P^X(z) &= \frac{1}{N}\frac{\partial}{\partial z}\ln\left\langle \det_{1\le k,l\le N} \mathrm{e}^{Nx_kp_l}\right\rangle_{p_N=z} , \quad z \notin \operatorname{supp}\rho_P , \\ G_X^P(z) &= \frac{1}{N}\frac{\partial}{\partial z}\ln\left\langle \det_{1\le k,l\le N} \mathrm{e}^{Nx_kp_l}\right\rangle_{x_N=z} , \quad z \notin \operatorname{supp}\rho_X . \end{aligned} \tag{3.12}$$

The key property of the above functions is that $G_P^X(G_X^P(z)) = z(1+\mathscr{O}(1/N))$ for large N [86, 82].

3.3.1 Saddle point equations

We begin by stating the main result of this section. This rests on Lemmas 3.3.4 and 3.3.5, which we derive from the $q+1$-matrix integral (3.8c) and the pair of 1-matrix integrals (3.8a), (3.8b) at large N, respectively. After presenting the latter, we conclude this section with the proof of the main statement.

Proposition 3.3.1. *Let the matrix P_+ be defined as in Lemma 3.2.1, and set $Y = \sqrt{1-\mathrm{e}^{-2h}}P_+$. Then for $N \to \infty$, the spectral density of the sum of p matrices distributed according to* (2.4) *is given by*

$$\rho_{(p)}(z) = \frac{1}{2\pi\mathrm{i}}\left[G_{(p)}^Y(z)_+ - G_{(p)}^Y(z)_-\right] , \tag{3.13}$$

where $G^Y_{(p)}(z)$ *is the functional inverse of*

$$G_Y^{(p)}(z) = \frac{p}{q}(z - W_Y(z)_-) + \frac{q-p}{q} W_Y(z)_+ \ . \tag{3.14}$$

Remark 3.3.2. Generally, $G^Y_{(p)}(z)$ is a multi-valued function so that we need to specify the sheet on which (3.13) is evaluated. This ambiguity is fixed by the condition that $\lim_{z\to\infty} z\, W_{(p)}(z) = 1$.

Corollary 3.3.3. *When* $G^Y_{(p)}(z)$ *satisfies an algebraic equation of the form* $F_{(p)}(z, G^Y_{(p)}(z)) = 0$*, then* $G^Y_{(q-p)}(z)$ *follows from*

$$F_{(p)}\left(G^Y_{(q-p)}(z) - z, G^Y_{(q-p)}(z)\right) = 0 \ . \tag{3.15}$$

As announced, we proceed to formulate the main lemmas involved in the proof of the above results:

Lemma 3.3.4. *In the limit* $N \to \infty$*, the matrix* $M = \mathrm{e}^{-h}\sum_{i=1}^{p} X_i + \mathrm{e}^{h}\sum_{i=p+1}^{q} X_i$ *satisfies*

$$W_{X_0}(z) = W_M(z - W_{X_0}(z)) \ . \tag{3.16}$$

Proof. This result follows from the translation invariance of the measure (2.4): Setting $\sigma = \mathrm{id}$ in (3.8c) without loss of generality, consider the shift by a small Hermitian matrix[3]

$$X_0 \longrightarrow X_0' = X_0 + \varepsilon\left(\frac{1}{z - X_0}\frac{1}{z' - M} + \mathrm{h.c.}\right) \ , \quad \varepsilon \ll 1 \ , \tag{3.17}$$

as a formal power series in z, z'. When $M = \mathrm{e}^{-h}\sum_{i=1}^{p} X_i + \mathrm{e}^{h}\sum_{i=p+1}^{q} X_i$, the variation of the product of the two Gaussian integrals

[3] See also [77] for an earlier application of this method of “loop equations” to the Potts matrix model.

$$I\left(\{X_i\}_{i=0}^{q}\right) \equiv \gamma_+[1]\left(X_0 + 2\sinh(h)\sum_{i=1}^{p} X_i\right)\,\gamma_-[1]\left(X_0 - 2\sinh(h)\sum_{i=p+1}^{q} X_i\right) \tag{3.18}$$

and the measure $\mathrm{d}X_0$ is respectively given to leading order by

$$I\left(\{X_i\}_{i=0}^{q}\right) \longrightarrow I(\{X_i\}_{i=0}^{q}) + \varepsilon \mathrm{tr}\frac{1}{z-X_0}\frac{1}{z'-M}(M-X_0) + \mathscr{O}(\varepsilon^2)\,, \tag{3.19a}$$

$$\mathrm{d}X_0 \longrightarrow \mathrm{d}X_0\left(1 + \varepsilon \mathrm{tr}\frac{1}{z-X_0}\mathrm{tr}\frac{1}{z-X_0}\frac{1}{z'-M} + \mathscr{O}(\varepsilon^2)\right)\,, \tag{3.19b}$$

Demanding invariance of $Z_{N,q}$ to first order in ε and approximating $\langle \mathrm{tr}A\mathrm{tr}B\rangle = \langle \mathrm{tr}A\rangle\langle \mathrm{tr}B\rangle + \mathscr{O}(1/N)$ yields

$$W_M(z') - W_{X_0}(z) = \frac{1}{N}\left\langle \mathrm{tr}\frac{1}{z-X_0}\frac{1}{z'-M}\right\rangle\left(W_{X_0}(z) - z + z'\right) + \mathscr{O}(1/N^2)\,. \tag{3.20}$$

Evaluating the above at $z' = z - W_{X_0}(z)$ proves the lemma. □

Lemma 3.3.5. *Let the matrices P_+, X_0 be defined as in Lemma 3.2.1, and set $P_+ = Y/\sqrt{1-\mathrm{e}^{-2h}}$ and $X_0 = 2\sinh(h)\bar{X}_0$. Then for $N\to\infty$,*

$$\mathrm{Re}\; G_Y^{\bar{X}_0}(z) = \left(\frac{2p}{q} - 1\right)\mathrm{Re}\; W_Y(z) + \left(\frac{1}{1-\mathrm{e}^{-2h}} - \frac{p}{q}\right)z\,,\quad z\in \mathrm{supp}\,\rho_Y\,. \tag{3.21}$$

Proof. This result follows from the saddle point approximation to the integrals (3.8a) and (3.8b) in Lemma 3.2.1: Setting

$$Y = U\mathrm{diag}(\{y_n\}_{n=1}^{N})U^\dagger\,,\quad \frac{X_0}{2\sinh(h)} = V\mathrm{diag}(\{x_n\}_{n=1}^{N})V^\dagger\,, \tag{3.22}$$

with $U,V\in U(N)$, we can perform the integration over $U^\dagger V$ using the well-known result

[87, 88]

$$\int_{U(N)} \mathrm{d}U\ \mathrm{e}^{\lambda N \mathrm{tr}[YUXU^\dagger]} = \text{const.} \times \frac{\det_{1\le k,l\le N} \mathrm{e}^{\lambda N y_k x_l}}{\Delta(x)\Delta(y)} \qquad \forall \lambda \in \mathbb{C}\ , \tag{3.23}$$

where $\mathrm{d}U$ is the normalised Haar measure. It follows that for the exponent of the integrand in (3.8b) to have an extremum, the eigenvalues in (3.22) must satisfy

$$0 = \frac{1}{N}\left[\frac{\partial}{\partial y_n} \ln \det_{k,l} \mathrm{e}^{N y_k x_l} + \sum_{k\neq n} \frac{1}{y_n - y_k} + p \frac{\partial}{\partial y_n} \ln \gamma_+[\mathrm{e}^{-N\mathrm{tr}U}]\left(\frac{Y}{\sqrt{1-\mathrm{e}^{-2h}}}\right)\right] - \frac{y_n}{1-\mathrm{e}^{-2h}}\ . \tag{3.24a}$$

On the other hand, from (3.8a) we find, that when (3.24a) holds, then also

$$0 = \frac{2}{N}\sum_{k\neq n} \frac{1}{y_n - y_k} + \frac{q}{N}\frac{\partial}{\partial y_n} \ln \gamma_+[\mathrm{e}^{-N\mathrm{tr}U}]\left(\frac{Y}{\sqrt{1-\mathrm{e}^{-2h}}}\right) - y_n\ , \tag{3.25}$$

which allows us to eliminate $\gamma_+[\mathrm{e}^{-N\mathrm{tr}U}](Y/\sqrt{1-\mathrm{e}^{-2h}})$ between the above and (3.24a)[4]. Taking $N \to \infty$ and using the definition (3.12) yields (3.21) as advertised. □

Proof of Proposition 3.3.1 and Corollary 3.3.3. We begin by deriving the form of (3.14), then (3.13). Firstly, the form of (3.14) follows from Lemma 3.3.5 after analytic continuation: Following an argument in [75], we note that the derivative w.r.t. x_N of the logarithm of (3.23) is an entire function of x_N, which implies that as $N \to \infty$, $G_X^Y(z)$ and $W_X(z)$ have the same discontinuity across the real axis. Applying this to our situation, we conclude that when $z \in \operatorname{supp} \rho_Y$,

$$G_Y^{\bar{X}_0}(z)_\pm = \mathrm{Re}\ G_Y^{\bar{X}_0}(z) \pm \mathrm{i}\pi\rho_Y(z)\ , \tag{3.26a}$$

$$G_{\bar{X}_0}^{Y}(z)_\pm = \mathrm{Re}\ G_{\bar{X}_0}^{Y}(z) \pm \mathrm{i}\pi\rho_{\bar{X}_0}(z)\ . \tag{3.26b}$$

[4]It is the analysis of this quantity that leads us to the exact solution for $W_Y(z)$ in the next subsection.

For $h > 0$, it follows from (3.8a) that Y is Hermitian, so $W_Y(z)$ has no singularities in the complex plane away from the real axis. Hence we can analytically continue (3.21) to $z \in \mathbb{C} \setminus \operatorname{supp} \rho_Y$ using

$$G_Y^{\bar{X}_0}(z)_+ - G_Y^{\bar{X}_0}(z)_- = W_Y(z)_+ - W_Y(z)_- \;, \tag{3.27}$$

which results in

$$G_Y^{\bar{X}_0}(z) = \frac{p}{q}\left(W_Y(z)_+ - z\right) - \frac{q-p}{q} W_Y(z)_- + \frac{z}{1-\mathrm{e}^{-2h}} \;. \tag{3.28}$$

Secondly, to obtain (3.13), note first that from Lemma 3.3.4, we find

$$W_{\bar{X}_0}(z) = W_{M/(2\sinh(h))}\left(z - \frac{1}{4\sinh(h)^2} W_{\bar{X}_0}(z)\right) \;, \tag{3.29}$$

where we used the property $W_X(z) = \lambda W_{\lambda X}(\lambda x)$ for real λ. In the limit $h \to \infty$, $M/(2\sinh(h)) \to \sum_{i=p+1}^{q} X_i$ and consequently, from the above,

$$W_{\bar{X}_0}(z) \to W_{(q-p)}\left(z + \mathscr{O}(\mathrm{e}^{-2h})\right) \quad \text{as } h \to \infty \;. \tag{3.30}$$

We infer that in this limit, $\rho_{\bar{X}_0}(z) \to \rho_{(q-p)}(z)$, which in conjunction with (3.26b) yields

$$\rho_{(q-p)}(z) = \lim_{h\to\infty} \frac{1}{2\pi\mathrm{i}}\left[G^Y_{\bar{X}_0}(z)_+ - G^Y_{\bar{X}_0}(z)_-\right] \;. \tag{3.31}$$

We thus obtain the desired expressions (3.13) and (3.14) from the above and (3.28) by identifying $G^Y_{\bar{X}_0}(z) = G^Y_{(q-p)}(z)$, and noting that according to (3.28), the analytic continuation of $G_Y^{(q-p)}(z)$ through $\operatorname{supp} \rho_Y$ is given by $z - G_Y^{(p)}(z)$,

$$G_Y^{(q-p)}(z)_\pm = z - G_Y^{(p)}(z)_\mp \;. \tag{3.32}$$

The functional inversion relation then follows from $G^Y_{\bar{X}_0} \circ G_Y^{\bar{X}_0} = \mathrm{id}$. Finally, to show Corollary 3.3.3, observe that according to (3.32), for an algebraic function F in two variables,

$$F_{(p)}\left(z, G^Y_{(p)}(z)\right) = 0 \quad \text{implies} \quad F_{(p)}\left(z' - G^{(q-p)}_Y(z'), z'\right) = 0\,, \tag{3.33}$$

since the analytic continuation merely takes us from one solution to the above equation to another. Evaluating at $z' = G^Y_{(q-p)}(z)$ proves the corollary. □

3.3.2 General solution

The main result in Proposition 3.3.1 is expressed via the functional inverse of the quantity (3.14). Generally, this functional inversion is most easily achieved by means of an explicit parametric form of $W_Y(z)$; Proposition 3.3.6 below provides just that for general $q \neq 4$ when $U(z)$ is cubic, i.e. $k = 1$.

Proposition 3.3.6. *Let* $k = 1$, $\nu = \arccos((q-2)/2)/\pi$ *and assume* $\operatorname{supp}\rho_Y = [z_-, z_+] \subset \mathbb{R}$ *as* $N \to \infty$. *Then* $W_Y(z) = W_Y^{\text{reg.}}(z) + W_Y^{\text{sing.}}(z)$ *with*[5]

$$\begin{aligned}
z(\sigma) &= \delta_U + \sqrt{(z_+ - \delta_U)(z_- - \delta_U)}\left(\frac{\vartheta_2(\pi\sigma|\tau)}{\vartheta_3(\pi\sigma|\tau)}\right)^2\,,\\
W_Y^{\text{reg.}}(z(\sigma)) &= \frac{1}{4-q}\left(\frac{qt_2}{t_3} + 2z(\sigma)\right)\,,\\
W_Y^{\text{sing.}}(z(\sigma)) &= \sum_{n\geq 0}\frac{f_n}{n!}\frac{\partial^n}{\partial\sigma^n}\left(\mathrm{e}^{\mathrm{i}\pi\nu\sigma}\frac{\vartheta_3(\pi\sigma + \pi\tau\nu/2|\tau)}{\vartheta_3(\pi\sigma|\tau)} + \mathrm{e}^{-\mathrm{i}\pi\nu\sigma}\frac{\vartheta_3(\pi\sigma - \pi\tau\nu/2|\tau)}{\vartheta_3(\pi\sigma|\tau)}\right)\,,
\end{aligned} \tag{3.34}$$

where τ *and* δ_U *are implicit functions of* t_2, t_3, *and the coefficients* f_n *are determined by the requirement* $\lim_{z\to\infty} zW_Y(z)$.

Proof. We can determine the spectrum of Y from the saddle point approximation (3.25) to the integral (3.8c), which is precisely the problem first considered in [20, 75]. To our knowledge, the first large N analysis of $\gamma'_+[\mathrm{e}^{-N\mathrm{tr}U}]$ appearing in (3.25) for cubic U was

[5] Our conventions for elliptic functions are those of Gradshtein and Ryzhik [73] and are spelled out in Appendix A.

done, if in a slightly different context, by Gross and Newman in [89]. Using [89, eqns. (2.10), (2.11)], equation (3.25) can be expressed as

$$\begin{aligned} z = 2\mathrm{Re}\, W_Y(z) + \frac{q}{2}\int_{z_-}^{z^+} \frac{\mathrm{d}z'}{\sqrt{z'-\delta_U}} \frac{\rho_Y(z')}{\sqrt{z-\delta_U}+\sqrt{z'-\delta_U}} \\ + q\frac{\sqrt{z-\delta_U}}{\sqrt{t_3}} - q\frac{t_2}{2t_3}\,, \qquad z\in[z_-,z_+]\,, \end{aligned} \tag{3.35}$$

where δ_U solves the implicit equation

$$\frac{t_2}{4t_3} + \delta_U = \frac{\sqrt{t_3}}{t_2^{3/2}}\int_{z_-}^{z^+} \mathrm{d}z\, \frac{\rho_Y(z)}{\sqrt{z-\delta_U}}\,. \tag{3.36}$$

Let us resolve the branch point at δ_U by the change of variables $w(z) = \sqrt{z-\delta_U}$, and denote $w(z_\pm) = w_\pm$. Introducing the auxiliary function

$$f(w) = \int_{w_-}^{w_+} \mathrm{d}w'\, \frac{\rho_Y(\delta_U + w'^2)}{w - w'}\,, \tag{3.37}$$

we derive the two identities

$$\begin{aligned} \mathrm{Re}\, W_Y(z) &= \mathrm{Re}\, f(w(z)) + f(-w(z))\,, \qquad z\in[z_-,z_+]\,, \\ f(-w(z)) &= -\frac{1}{2}\int_{z_-}^{z_+} \frac{\mathrm{d}z'}{\sqrt{z'-\delta_U}} \frac{\rho_Y(z')}{\sqrt{z-\delta_U}+\sqrt{z'-\delta_U}}\,. \end{aligned} \tag{3.38}$$

We can then rewrite (3.35) in the equivalent form

$$2\mathrm{Re}\, f(w) + (2-q)f(-w) = \delta_U + w^2 - q\frac{w}{\sqrt{t_3}} + q\frac{t_2}{2t_3}\,, \quad w\in[w_-,w_+]\,. \tag{3.39}$$

A particular polynomial solution to the above equation when $q \neq 4$ is

$$f_{\mathrm{reg.}}(w) = \frac{qt_2}{2t_3(4-q)} - \frac{w}{\sqrt{t_3}} + \frac{\delta_U + w^2}{4-q}\,. \tag{3.40}$$

The general solution will therefore differ from the above by a function $f_{\mathrm{sing.}}(w) = f_{\mathrm{reg.}}(w) -$

$f(w)$ holomorphic on $\mathbb{C} \setminus [w_-, w_+]$ satsifying the homogenous equation

$$2\mathrm{Re}\, f_{\text{sing.}}(w) + (2-q) f_{\text{sing.}}(-w) = 0 \ , \quad w \in [w_-, w_+] \ . \tag{3.41}$$

We recover $\rho_Y(z)$ by inverting the relationship (3.37), which, using the fact that $f_{\text{reg.}}(w)$ is analytic, becomes

$$\rho_Y(z) = \frac{1}{2\pi \mathrm{i}} \left[f_{\text{sing.}}\left(\sqrt{z-\delta_U}\right)_- - f_{\text{sing.}}\left(\sqrt{z-\delta_U}\right)_+ \right] \ , \quad z \in [z_-, z_+] \ . \tag{3.42}$$

From the above expressions it then follows that for $z \notin [z_-, z_+]$, $W_Y(z)$ is given by

$$\begin{aligned} W_Y(z) &= 2\int_{w_-}^{w^+} \zeta' \mathrm{d}w' \frac{\rho_Y(\delta_U + w'^2)}{z - \delta_U - w'^2} \\ &= \frac{1}{4-q}\left(\frac{q t_2}{t_3} + 2z\right) - f_{\text{sing.}}\left(\sqrt{z-\delta_U}\right) - f_{\text{sing.}}\left(-\sqrt{z-\delta_U}\right) \ ; \end{aligned} \tag{3.43}$$

The general solution to the homogeneous equation (3.41) was first derived in [90] in the context of the $O(n)$ model and is presented in more detail in Appendix A. There we recall how $f_{\text{sing.}}(w)$ can be parametrised in terms of elliptic functions as[6]

$$\begin{aligned} f_{\text{sing.}}(w(\sigma)) &= \sum_{n\geq 0} \frac{f_n}{n!}\frac{\partial^n}{\partial \sigma^n}\left(\mathrm{e}^{\mathrm{i}\pi\nu(\sigma-1)}\frac{\vartheta_3(\pi\sigma + \pi\tau\nu/2|\tau)}{\vartheta_3(\pi\sigma|\tau)} + \mathrm{e}^{-\mathrm{i}\pi\nu(\sigma-1)}\frac{\vartheta_3(\pi\sigma - \pi\tau\nu/2|\tau)}{\vartheta_3(\pi\sigma|\tau)}\right) \ , \\ w(\sigma) &= \sqrt{w_- w_+}\frac{\vartheta_2(\pi\sigma|\tau)}{\vartheta_3(\pi\sigma|\tau)} \ , \end{aligned} \tag{3.44}$$

where $\nu = \arccos((2-q)/2)/\pi$ and $\tau = \mathrm{i}K'/K$, with K and K' respectively given by the complete elliptic integral of the first and second kind (cf. eqn. (A.4)); the coefficients $\{f_n\}$ are entirely determined by the condition that $\lim_{z\to\infty} z\, W_Y(z) = 1$. Inserting the above

[6]By abuse of notation, we distinguish the functions $w(\sigma)$ and $w(z) = \sqrt{z-\delta_U}$ solely by their arguments.

parametrisation into (3.43) completes the proof. □

3.3.3 Derivation of Vouiculescu's formula for free convolution

Here we show how our results imply a non-trivial generalisation of Voiculescu's formula for free convolution of probability distributions to a non-free situation. This is essentially an adaption of the derivation in [81] to the case where the "external" matrix follows a Gaussian distribution[7]; gradually turning off the $O(q)$-symmetry breaking interactions of the Potts model, our formulae should reduce to Voiculescu's for free random variables. To confirm this is the case, it is convenient to consider the slight generalisation of (2.4),

$$\mathrm{d}\mu(X_1, X_2, \ldots X_q) = \frac{1}{Z^{\lambda}_{N,q}} \prod_{\langle ij \rangle} \mathrm{e}^{\lambda N \mathrm{tr} X_i X_j} \times \prod_{i=1}^{q} \mathrm{e}^{-N \mathrm{tr} V_i(x)} \mathrm{d}X_i \; , \quad \lambda \geq 0 \; , \tag{3.45}$$

which reduces to (2.4) for $\lambda \to 1$ and should yield Voiculescu's formula for $\lambda \to 0$[8]. Then the following holds for averages with respect to (3.45):

Proposition 3.3.7. *Take $N \to \infty$. Then as $\lambda \to 0$,*

$$G^M_{\lambda Y}(z) - W_Y(z) \to R^M(z) \; , \tag{3.46a}$$

$$W^{-1}_{(q)}(z) \to \sum_{i=1}^{q} W^{-1}_{X_i}(z) - \frac{q-1}{z} \; , \tag{3.46b}$$

where $R^M(z)$ denotes the R-transform (3.1) *of $W_M(z)$.*

Proof. According to Lemma 3.2.1, we can write the partition function in (3.45) via the

[7] Equivalent results were previously obtained by Zee in [91].

[8] Of course, the parameter λ is redundant in that we may equivalently obtain (2.4) by a suitable resacling of X_i and $\{t_m\}_{m=2}^{k+2}$; we are thus not departing from the initial parameter space of the model.

fiducial matrix $Y=\sqrt{1-\mathrm{e}^{-2h}}P_+$ as

$$\begin{aligned} Z^{\lambda}_{N,q} &= \int_{\mathbb{R}} \mathrm{d}Y\ \mathrm{e}^{-N\mathrm{tr}Y^2/2} \prod_{i=1}^{q} \int_{\Gamma} \mathrm{d}X_i\ \mathrm{e}^{-N\mathrm{tr}[V_i(X_i)+X_i^2/2-\lambda X_i Y]} && (3.47a) \\ &\equiv \int_{\mathbb{R}} \mathrm{d}Y\ \mathrm{e}^{-N\mathrm{tr}Y^2/2} \left(\prod_{i=1}^{q} \int_{\Gamma} \mathrm{d}X_i\ \mathrm{e}^{-N\mathrm{tr}[V_i(X_i)+X_i^2/2]} \right) \mathrm{e}^{\lambda N \mathrm{tr} Y(X_1+X_2+\ldots X_q)}\ . && (3.47b) \end{aligned}$$

Diagonalising the matrices and integrating over the unitary group, we can write, taking the limit $N\to\infty$,

$$\frac{1}{N}\frac{\partial}{\partial z}\ln \int_{\Gamma} \mathrm{d}X_i\ \mathrm{e}^{-N\mathrm{tr}[V_i(X_i)+X_i^2/2-\lambda X_i Y]}\Bigg|_{y_N=z} = G^{X_i}_{\lambda Y}(z) - W_{\lambda Y}(z)\ ,\quad i=1\ldots q\ , \tag{3.48}$$

where we used the definition (3.12); comparing (3.47a) and (3.47b), this implies

$$G^{X_1+X_2+\ldots X_q}_{\lambda Y}(z) - W_{\lambda Y}(z) = \sum_{i=1}^{q}\left(G^{X_i}_{\lambda Y}(z)-W_{\lambda Y}(z)\right)\ . \tag{3.49}$$

Now consider the limit $\lambda\to 0$. On the one hand,

$$\lim_{\lambda\to 0}\frac{\mathrm{e}^{\lambda N x_i y_j}}{\Delta(x)\Delta(y)} = 1\ , \tag{3.50}$$

from which it follows that

$$\begin{aligned} \lim_{\lambda\to 0}\left(G^{\lambda Y}_{X_i}(z) - W_{X_i}(z)\right) &= 0\ , && (3.51a)\\ \lim_{\lambda\to 0}\left(G^{\lambda Y}_{X_1+X_2+\ldots X_q}(z) - W_{(q)}(z)\right) &= 0\ . && (3.51b) \end{aligned}$$

On the other hand, as can be seen from (3.25) in this limit, the matrix Y will follow a Gaussian distribution, so its spectral density approaches the semi-circle (2.14). As a result, the spectral density of the rescaled matrix λY approaches a delta function so that $W_{\lambda Y}(z) \to 1/z$. Together with the relation $G_X^Y \circ G_Y^X = \text{id}$, this means that indeed

$$G_{\lambda Y}^M(z) - W_{\lambda Y}(z) \to R^M(z) \quad \text{as } \lambda \to 0\ , \tag{3.52}$$

from comparison with the definition (3.1). Lastly, inserting the above into (3.49) yields (3.46b). □

For $q = 2$, (3.46b) indeed gives Voiculescu's formula (3.3). It is in this sense that the function $G_{\lambda Y}^M(z) - W_{\lambda Y}(z)$ lifts the notion of the R-transform, so that (3.49) represents a nontrivial extension of Voiculescu's formula to the addition of *correlated* random matrices, distributed according to (3.45). It is instructive to compare (3.49) for $\lambda = 1$ to the expressions in Proposition 3.3.1 of the previous section more explicitly. Since from (3.47a) and (3.47b)

$$G_Y^{X_i}(z) = \frac{q-1}{q} W_Y(z)_+ + \frac{1}{q}\left(z - W_Y(z)_-\right)\ , \quad G_Y^{X_1+\ldots X_q}(z) = z - W_Y(z)_-\ , \tag{3.53}$$

we observe upon comparison to (3.14) that indeed

$$G_Y^{(1)}(z) = G_Y^{X_i}(z)\ , \quad G_Y^{(q)}(z) = G_Y^{X_1+\ldots X_q}(z)\ . \tag{3.54}$$

Hence, for the S_q-invariant case[9] $V_i(z) \equiv U(z) - z^2/2\ \forall i$, our main result in Proposition 3.3.1 *further* generalises this result to the sum of $p \leq q$ matrices: the function $G_Y^{(p)}(z) - W_Y(z)$ generalises the R-transform of $W_{(p)}(z)$, and (3.14) generalises Voiculescu's formula.

[9]The above expressions indicate that the generalisation of Proposition 3.3.1 to $\lambda \neq 1$ and $U_i \neq U_j$ for $i \neq j$ is straightforward.

3.4 Case studies

Here we consider the cases $(q,k) = (1,2)$, $(2,1)$, and $(3,1)$, which describe hard dimers, the A_3 and the D_4 model on planar triangulations, respectively. For the first two models, the functions $W_{(p)}(z)$ have been known for a while [92, 23, 24] – the fact that our general formula in Proposition (3.3.1) reproduces these results lends crecedence to our extension to the D_4-model. Unlike models with irrational values of $\arccos((q-2)/2)/\pi$, all of these share the simplification that they can be described by polynomial equations: We derive explicit expressions for the polynomials $F_{(p)}(x,y)$ satisfying

$$F_{(p)}\left(z, G^{Y}_{(p)}(z)\right) = 0 , \quad 1 \leq p \leq q , \tag{3.55}$$

which define a family of algebraic curves $\mathscr{C}_{(p)} = \{(x,y) \in \mathbb{C}^2 | F_{(p)}(x,y) = 0\}$. In Appendix B, we describe the resulting analytic structure of $G^{(p)}_{Y}(z)$ and $G^{Y}_{(p)}(z)$. The coefficients in (3.55) may be fixed as follows: As stated in the introduction, herein we restrict ourselves to solutions for which the spectral densities have connected support, translating into a single cut in the complex z-plane for the Stieltjes transform $W_{(p)}(z)$. To ensure this property, the condition that the curve $\mathscr{C}_{(p)}$ be of genus zero is sufficient, though not in general necessary, as is clear from the geometry of Riemann surfaces[10]. Nonetheless, we remark that this slightly stronger condition on the solution set guarantees the existence of a (non-unique) rational parametrisation of the curve. Requiring consistency of the latter with the deduced asymptotic behaviour for large z in turn determines the constants $c^{(p)}_{i,j}$ that appear in the expressions for $F_{(p)}(x,y)$ entirely as functions of $\{t_m\}_{m=2}^{k+2}$.

[10]See also [93] for a discussion of the relationship between the connectedness of the spectral density and a vanishing genus of the spectral curve.

3.4.1 $(q,k)=(1,2)$ – Hard dimers

This model describes hard dimers on planar triangulations and was first solved on the sphere by Staudacher [92]. According to (2.4) and (3.8a), the partition function can be written as both a one- and two-matrix integral,

$$Z_{N,1} = \int \mathrm{d}X\ \mathrm{e}^{-N\mathrm{tr}[U(X)-X^2/2]} \tag{3.56a}$$

$$= \int \mathrm{d}Y\ \mathrm{e}^{-N\mathrm{tr}Y^2/2} \int \mathrm{d}X\ \mathrm{e}^{-N\mathrm{tr}[U(X)-XY]}\ . \tag{3.56b}$$

Using the defintion (3.12) in the planar limit, the above expressions imply the following relations:

$$z = W_Y(z)_- + G_Y^X(z)_+\ , \tag{3.57a}$$

$$U'(z) = W_{(1)}(z)_- + G_X^Y(z)_+\ , \tag{3.57b}$$

$$U'(z) = W_{(1)}(z)_- + W_{(1)}(z)_+ + z\ , \tag{3.57c}$$

The first line (3.57a) is indeed consistent with (3.14) in Proposition 3.3.1 and the relations (3.54) in Subsection 3.3.3. Via the relation $G_Y^X \circ G_X^Y = \mathrm{id}$, equations (3.57a) and (3.57b) dictate the analytic structure and asymptotic behaviour of $G_Y^X(z)$, the result of which is spelled out in Appendix B, Example B.0.1. This allows us to compute the spectral curve using (3.14),

$$\begin{aligned} F_{(1)}(x,y) = x^4 - x^3 y + \frac{t_3}{t_4}x^3 + \frac{y^2}{t_4} - \frac{t_3}{t_4}x^2 y + \frac{t_2+t_4}{t_4}x^2 \\ - \frac{t_2+1}{t_4}xy - c_{0,0}^{(1)}x + c_{1,1}^{(1)}y + c_{1,0}^{(1)}\ . \end{aligned} \tag{3.58}$$

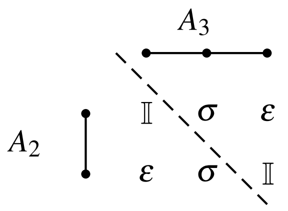
According to Corollary 3.3.3, the functions $G_{(p)}^Y(z)$ then satisfy

$$F_{(1)}\left(G^{Y}_{(0)}(z)-z,G^{Y}_{(0)}(z)\right)=0\ ,\quad F_{(1)}\left(z,G^{Y}_{(1)}(z)\right)=0\ , \tag{3.59}$$

which in turn determines their analytic structure and asymptotic behaviour on all sheets – see Appendix B. Finally, comparing to (3.57c), we conclude that

$$G^{Y}_{(1)}(z)_{+}=z+W_{(1)}(z)\ ,\quad G^{Y}_{(1)}(z)_{-}=t_4z^3+t_3z^2+t_2z-W_{(1)}(z)\ . \tag{3.60}$$

3.4.2 $(q,k)=(2,1)$ – Ising model



	$A_2\times A_3$	$X_{(p\vert\sigma)}$
$\mathbb{I}$	$(1,1)$	X_1
ε	$(2,1)$	X_2
σ	$(2,2)$	X_1+X_2

Figure 3.1: Integrable boundary conditions for the Ising model ($q=2$) on a fixed lattice are labelled by the nodes of the graph $A_2\times A_3$; the dashed line separates two equivalent choices of a fundamental domain.

This corresponds to the Ising model on planar triangulations, which is the A_3 model in the classification of [84] and was first solved on the sphere by Kazakov and Boulatov [12, 13] using the much-studied $\mathbb{Z}_2$-symmetric Hermitian two-matrix model. The 3 integrable boundary conditions of this model are captured by the linear combinations of X_i shown in Figure 3.1 [57]: $W_{(1)}(z)$ captures the $S_2\simeq\mathbb{Z}_2$-doublet $\{\mathbb{I},\varepsilon\}$, $W_{(2)}(z)$ the $\mathbb{Z}_2$-singlet $\{\sigma\}$. From (3.8a), we see that in this case the partition function can be written as

$$Z_{N,2}=\int \mathrm{d}X_1\mathrm{d}X_2\ \mathrm{e}^{-N\mathrm{tr}[U(X_1)+U(X_2)]}\mathrm{e}^{N\mathrm{tr}(X_1+X_2)^2/2} \tag{3.61a}$$

$$=\int \mathrm{d}Y\ \mathrm{e}^{-N\mathrm{tr}Y^2/2}\left(\int \mathrm{d}X\ \mathrm{e}^{-N\mathrm{tr}[U(X)-XY]}\right)^2\ . \tag{3.61b}$$

On the other hand, changing variables to $X_{\pm}=X_1\pm X_2+t_2/t_3$ and integrating out X_-, we obtain the equivalent one-matrix representation going back to [94],

$$Z_{N,2} = \text{const.} \times \int \frac{\mathrm{d}X_+ \ \mathrm{e}^{-N\mathrm{tr}U_+(X_+)}}{\sqrt{\mathrm{Det}(X_+ \otimes \mathbb{I} + \mathbb{I} \otimes X_+)}} \ , \tag{3.62}$$

where $U'_+(z) = t_3 z^2/4 - z - t_2(4-t_2)/(4t_3)$ and capital Det denotes the determinant on $N^2 \times N^2$ matrices. Using the defintion (3.12) in the planar limit, the above expressions respectively imply the following set of equations:

$$z = W_Y(z)_- - W_Y(z)_+ + 2G_Y^X(z)_+ \ , \tag{3.63a}$$

$$U'(z) = W_{(1)}(z)_- + G_X^Y(z)_+ \ , \tag{3.63b}$$

$$U'(z) = W_{(1)}(z)_- + G_{X_1}^{X_2}(z)_+ + z \ , \tag{3.63c}$$

$$U'_+(z + t_2/t_3) = W_{(2)}(z)_- + W_{(2)}(z)_+ + W_{(2)}(-z) \ . \tag{3.63d}$$

Again, the first line is consistent with (3.14) in Proposition 3.3.1 and the relations (3.54) in Subsection 3.3.3. Equations (3.63a) and (3.63b) dictate the analytic structure and asymptotic behaviour of $G_Y^X(z)$, cf. Appendix B, Example B.0.2. As before, this allows us to compute the spectral curve using (3.14),

$$\begin{aligned} F_{(1)}(x,y) = x^4 - 2x^3y - \frac{1}{t_3}y^3 + \frac{1-t_2}{t_3^2}y^2 + x^2y^2 - \frac{t_2+2}{t_3}x^2y + \frac{t_3^2 - t_2^2}{t_3^2}x^2 \\ + \frac{t_2+2}{t_3}xy^2 + \frac{t_2^2 - t_3^2}{t_3^2}xy + c_{1,1}^{(1)}x + c_{1,0}^{(1)} \ , \end{aligned} \tag{3.64a}$$

$$\begin{aligned} F_{(2)}(x,y) = x^4 + \frac{4t_2}{t_3}x^3 + \frac{4}{t_3}y^3 - x^2y^2 - \frac{4+2t_2}{t_3}x^2y - \frac{2t_2}{t_3}xy^2 + \frac{4t_2^2 + 2t_3^2}{t_3^2}x^2 + \frac{8t_2}{t_3^2}y^2 \\ - \frac{4t_2(2+t_2)}{t_3^2}xy - c_{0,0}^{(2)}x + c_{1,1}^{(2)}y + c_{1,0}^{(2)} \ . \end{aligned} \tag{3.64b}$$

According to Corollary 3.3.3, the functions $G_{(p)}^Y(z)$ then satisfy

$$\begin{aligned} F_{(1)}\left(z, G^Y_{(1)}(z)\right) &= 0\,, \quad F_{(1)}\left(G^Y_{(1)}(z)-z, G^Y_{(1)}(z)\right) = 0\,, \\ F_{(2)}\left(z, G^Y_{(2)}(z)\right) &= 0\,, \quad F_{(2)}\left(G^Y_{(0)}(z)-z, G^Y_{(0)}(z)\right) = 0\,. \end{aligned} \tag{3.65}$$

Again we may use the above to compute the analytic structure and asymptotic behaviour of $G^Y_{(p)}(z)$ on all sheets – see Appendix B. Comparing to (3.63c) and (3.63d), we conclude that

$$G^Y_{(1)}(z)_+ = z + G^{X_2}_{X_1}(z)\,, \quad G^Y_{(1)}(z)_- = t_3 z^2 + t_2 z - W_{(1)}(z)\,, \tag{3.66a}$$

$$G^Y_{(2)}(z)_+ = z + W_{(2)}(z)\,, \quad G^Y_{(2)}(z)_- = t_3 z^2/4 + t_2 z/2 - W_{(2)}(z) - W_{(2)}(-z)\,. \tag{3.66b}$$

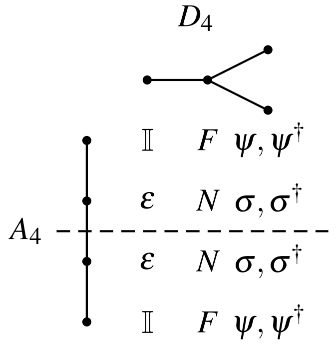
Our results reproduce the analytic structure found in [23, 24, 31] as well as the relation between the $p=1$ and $p=2$ boundary conditions reported in [32]: at the level of the polynomial equation, the correspondence with the quantities defined therein is

$$W_Y(z) \leftrightarrow W_A(a)\,, \quad G^{(1)}_Y(z) \leftrightarrow x(a)\,, \quad G^{(2)}_Y(z) \leftrightarrow m(a)\,. \tag{3.67}$$

The polynomial $E(x,y) = -t_3 F_{(1)}(x, x+y)$ is of order 3 in both x and y,

$$\begin{aligned} E(x,y) = x^3 + y^3 - t_3 x^2 y^2 - \frac{1-t_2}{t_3}(x^2 y + y^2 x) - \frac{1-t_2}{t_3}(x^2+y^2) \\ - \frac{2-2t_2+t_2^2-t_3^2}{t_3} xy - t_3 c^{(1)}_{1,1}(x+y) - t_3 c^{(1)}_{1,0}\,, \end{aligned} \tag{3.68}$$

and satisfies $E(x,y) = E(y,x)$ and $E(z, G^{X_2}_{X_1}(z)) = 0$, as follows from comparison of (3.63a) and (3.63b). This is the usual spectral curve of the two-matrix model introduced by Eynard [93].



	$A_4\times D_4$	$X_{(p\|\sigma)}$
$\mathbb{I}$	$(1,1)$	X_1
ψ	$(1,3)$	X_2
$\psi^\dagger$	$(1,4)$	X_3
F	$(1,2)$	$X_1+X_2+X_3$
ε	$(2,1)$	X_2+X_3
σ	$(2,3)$	X_1+X_3
$\sigma^\dagger$	$(2,4)$	X_1+X_2
N	$(2,2)$	–

Figure 3.2: Integrable boundary conditions for the 3-states-Potts model ($q=3$) on a fixed lattice are labelled by the nodes of the graph $A_4\times D_4$; the dashed line separates two equivalent choices of a fundamental domain.

3.4.3 $(q,k)=(3,1)$ – 3-states Potts model

This model is equivalent to the D_4 lattice model on planar triangulations, for which $W_{(1)}(z)$ was first calculated by Daul in [20]. The full list of boundary conditions of the D_4 lattice model is given in Figure 3.2 [74, p.60]: $W_{(1)}(z)$ captures the S_3-triplet $\{\mathbb{I},\psi,\psi^\dagger\}$, $W_{(2)}(z)$ the S_3-triplet $\{\varepsilon,\sigma,\sigma^\dagger\}$, and $W_{(3)}(z)$ the singlet $\{F\}$; thanks to Corollary 3.3.3, the spectral curve for the latter also defines another singlet $W_{(0)}(z)$, which may be conjectured to describe the one remaing independent boundary condition $\{N\}$, though herein we will not attempt to prove its equvialence to the microscopic definition given in [74]. From (3.8a), we see that the partition function can be written as

$$Z_{N,3}=\int \mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3\ \mathrm{e}^{-N\mathrm{tr}[U(X_1)+U(X_2)+U(X_3)]}\mathrm{e}^{N\mathrm{tr}(X_1+X_2+X_2)^2/2} \tag{3.69a}$$

$$=\int \mathrm{d}Y\ \mathrm{e}^{-N\mathrm{tr}Y^2/2}\left(\int \mathrm{d}X\ \mathrm{e}^{-N\mathrm{tr}[U(X)-XY]}\right)^3 . \tag{3.69b}$$

Again we may set $X_\pm = X_1\pm X_2+t_2/t_3$ and integrate out X_-, which gives

$$Z_{N,3}=\text{const.}\times\int\frac{\mathrm{d}X_+\mathrm{d}X_3\ \mathrm{e}^{-N\mathrm{tr}[U_+(X_+)+U(X_3)-X_+X_3]}}{\sqrt{\mathrm{Det}(X_+\otimes\mathbb{I}+\mathbb{I}\otimes X_+)}}\ , \tag{3.70}$$

where $U_+(z)$ is as in the previous section. Using the defintion (3.12) in the planar limit, the above expressions respectively imply the following set of equations:

$$z = W_Y(z)_- - 2W_Y(z)_+ + 3G_Y^X(z)_+ \ , \tag{3.71a}$$

$$U'(z) = W_{(1)}(z)_- + G_X^Y(z)_+ \ , \tag{3.71b}$$

$$U'(z) = W_{(1)}(z)_- + G_{X_3}^{X_1+X_2}(z)_+ + z \ , \tag{3.71c}$$

$$U'_+(z + t_2/t_3) = W_{(2)}(z)_- + G_{X_1+X_2}^{X_3}(z)_+ + W_{(2)}(-z) \ . \tag{3.71d}$$

Once again, the first line is consistent with (3.14) in Proposition 3.3.1 and the relations (3.54) in Subsection 3.3.3. The analytic structure and asymptotic behaviour of all relevant functions can be determined as before - cf. Appendix B, Example B.0.3. The resulting spectral curves are

$$\begin{aligned}
F_{(1)}(x,y) = {} & x^6 + x^5\left(-\frac{6t_2}{t_3} - 6y\right) - \frac{4y^5}{t_3} + x^4\left(13y^2 + \frac{24t_2 - 6}{t_3}y + \frac{9t_2^2 + 2t_3^2}{t_3^2}\right) \\
& + \frac{17 - 18t_2}{t_3^2}y^4 + x^3\left(\frac{24 - 28t_2}{t_3}y^2 + \frac{-12t_2^2 + 24t_2 - 8t_3^2}{t_3^2}y - 12y^3 - c_{0,0}^{(1)}\right) \\
& + x^2\left(yc_{1,1}^{(1)} + c_{1,0}^{(1)} + \frac{(6t_2 - 30)\,y^3}{t_3} + \frac{\left(-15t_2^2 - 54t_2 + 10t_3^2 + 9\right)y^2}{t_3^2} + 4y^4\right) \\
& + x\left(-y^2c_{2,2}^{(1)} - yc_{2,1}^{(1)} - c_{2,0}^{(1)} + \frac{(4t_2 + 12)\,y^4}{t_3} + \frac{\left(18t_2^2 + 24t_2 - 4t_3^2 - 18\right)y^3}{t_3^2}\right) \\
& + y^3c_{3,3}^{(1)} + y^2c_{3,2}^{(1)} + yc_{3,1}^{(1)} + c_{3,0}^{(1)} \ ,
\end{aligned} \tag{3.72}$$

$$
\begin{aligned}
F_{(3)}(x,y) = x^6 &+ x^5\left(\frac{18t_2}{t_3}+6y\right)+\frac{108y^5}{t_3}+x^4\left(9y^2+\frac{72t_2-18}{t_3}y+\frac{117t_2^2+6t_3^2}{t_3^2}\right)\\
&+\frac{702t_2-243}{t_3^2}y^4+x^3\left(-4y^3+\frac{36t_2-72}{t_3}y^2+\frac{24t_3^2+252t_2^2-216t_2}{t_3^2}y-c^{(3)}_{0,0}\right)\\
&+x^2\left(-12y^4-\frac{90t_2+54}{t_3}y^3+\frac{81-486t_2-135t_2^2+18t_3^2}{t_3^2}y^2+c^{(3)}_{1,1}y+c^{(3)}_{1,0}\right)\\
&+x\left(\frac{36(1-t_2)}{t_3}y^4+\frac{162-234t_2^2-12t_3^2}{t_3^2}y^3-c^{(3)}_{2,2}y^2-c^{(3)}_{2,1}y-c^{(3)}_{2,0}\right)\\
&+c^{(3)}_{3,3}y^3+c^{(3)}_{3,2}y^2+c^{(3)}_{3,1}y+c^{(3)}_{3,0}\,.
\end{aligned}
\tag{3.73}
$$

According to Corollary 3.3.3, the functions $G^Y_{(p)}(z)$ then satisfy

$$
\begin{aligned}
F_{(1)}\left(z,G^Y_{(1)}(z)\right) &= 0\,, \quad F_{(1)}\left(G^Y_{(2)}(z)-z,G^Y_{(2)}(z)\right)=0\,,\\
F_{(3)}\left(z,G^Y_{(3)}(z)\right) &= 0\,, \quad F_{(3)}\left(G^Y_{(0)}(z)-z,G^Y_{(0)}(z)\right)=0\,.
\end{aligned}
\tag{3.74}
$$

As before, the above fixes the analytic structure and asymptotic behaviour of $G^Y_{(p)}(z)$ on all sheets – see Appendix B. Comparing to (3.71c) and (3.71d), we conclude that

$$
G^Y_{(1)}(z)_+ = z+G^{X_1+X_2}_{X_3}(z)\,,\quad G^Y_{(1)}(z)_- = t_3z^2+t_2z-W_{(1)}(z)\,, \tag{3.75a}
$$

$$
G^Y_{(2)}(z)_+ = z+G^{X_3}_{X_1+X_2}(z)\,,\quad G^Y_{(2)}(z)_- = t_3z^2/4+t_2z/2-W_{(2)}(z)-W_{(2)}(-z)\,. \tag{3.75b}
$$

Similarly, one can show

$$
G^Y_{(3)}(z)_+ = z+W_{(3)}(z)\,,\quad G^Y_{(3)}(z)_- = t_3z^2/9+t_2z/3+\mathscr{O}(z^{-1})\,. \tag{3.76}
$$

$F_{(1)}(x,y)$ corresponds to the spectral curve first described in [20, 75]; the remaining expressions are new results. The polynomials $-t_3F_{(p)}(x,x+y)=4E_{(p)}(x,y)$ are of degree one less in x. For example,

$$
\begin{aligned}
E_{(1)}(x,y) = x^5 + y^5 + x^4 &\left(-\frac{t_3}{4}y^2 - \frac{2t_2-20}{4}y - \frac{8-24t_2}{4t_3} \right) - \frac{17-18t_2}{4t_3}y^4 \\
&+ x^3 \left(\frac{t_3\widetilde{c}_1}{4} - t_3y^3 - \frac{14t_2-34}{4t_3}y^2 - \frac{12t_2^2-84t_2+32}{4t_3}y \right) \\
&+ x^2 \left(\frac{t_3\widetilde{c}_2}{4}y - \frac{\widetilde{c}_3}{4} - t_3y^4 - \frac{11\,(t_2-1)}{2}y^3 - \frac{39t_2^2-90t_2-2t_3^2+57}{4t_3}y^2 \right) \\
&+ x \left(\frac{t_3\left(3c^{(1)}_{2,2}-c^{(1)}_{3,3}\right)}{4}y^2 - \frac{t_3\left(2c^{(1)}_{3,2}-c^{(1)}_{2,1}\right)}{4}y - \frac{4t_2t_3-8}{4}y^4 - \frac{9t_2^2-24t_2-2t_3^2+25}{2t_3}y^3 \right) \\
&- \frac{1}{4}t_3y^3c^{(1)}_{3,3} - \frac{1}{4}t_3y^2c^{(1)}_{3,2} - \frac{t_3(c^{(1)}_{3,1}-c^{(1)}_{2,0})}{4}x - \frac{1}{4}t_3yc^{(1)}_{3,1} - \frac{1}{4}t_3c^{(1)}_{3,0}\,,
\end{aligned}
\tag{3.77}
$$

where

$$\widetilde{c}_1 = c^{(1)}_{2,2} - c^{(1)}_{1,1} - c^{(1)}_{3,3} + c^{(1)}_{0,0}\,, \tag{3.78a}$$

$$\widetilde{c}_2 = c^{(1)}_{2,2} - 2c^{(1)}_{1,1} - 3c^{(1)}_{3,3}\,, \tag{3.78b}$$

$$\widetilde{c}_3 = c^{(1)}_{1,0} - c^{(1)}_{2,1} + c^{(1)}_{3,2}\,. \tag{3.78c}$$

This expression satisfies $E_{(2)}(x,y) = E_{(1)}(y,x)$ and is equivalent to the polynomial $Q(x_3,x_+)$ reported previously by the author in [34]. Upon inspection of (3.71b), (3.71c) and (3.71d) we conclude that

$$0 = E_{(2)}\left(z, G^{X_3}_{X_1+X_2}(z)\right) = E_{(2)}\left(G^{X_1+X_2}_{X_3}(z), z\right)\,. \tag{3.79}$$

3.5 Critical behaviour

This section discusses the critical behaviour of $W_{(p)}(z)$ for $0 < q < 4$. The existence of a second-order phase transition for the Potts model in this regime has been demonstrated

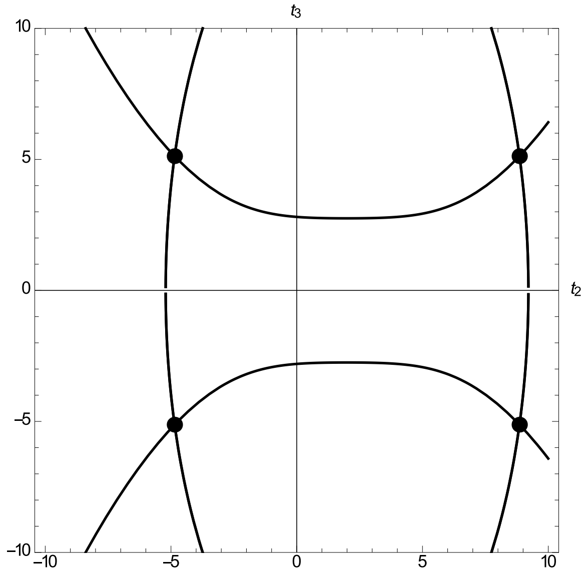


Figure 3.3: A portion of the phase diagram of the 3-states Potts model on planar triangulations. Along the critical lines, $\partial_y^4 E_{(1)}$, $\partial_x \partial_y^3 E_{(1)}$ and $\partial_x^3 \partial_y E_{(1)}$ vanish, with the polynomial $E_{(1)}$ as in (3.77); at the critical points, $\partial_x^4 E_{(1)}$ vanishes in addition.

on a fixed lattice by Baxter [95, 96]; here we describe their random-lattice counterparts[11]. According to Proposition 3.3.1, it suffices to determine the critical behaviour of $W_Y(z)$ for $1 > \nu > 0$. Its possible critical exponents are determined by the multiplicity of the singularity at the left edge z_- of the spectral density $\rho_Y(z)$, which controls the large-order behaviour of the generating function $W_Y(z)$.

Let us begin with the case of triangulations covered in Proposition 3.3.6. Then $z_- = \delta_U$ when both t_2 and t_3 are at their critical values $t_{2,c}$, $t_{3,c}$, with $t_{m>3} = 0$. When ν is rational, exact expressions for the critical lines and points can be obtained easily by requiring sufficiently many derivatives of the polynomial (3.55) to vanish; the result is depicted for the example $(q,k) = (3,1)$ in Figure 3.5. For example, from expressions (3.58), (3.64a) and (3.72), we find

[11] When $q > 4$, these critical points do not exist, though presumably another critical point emerges as for the $O(n)$ model on planar triangulations, for which $\gamma_s = 1/2$ when $n > 2$ [90, 97].

$$(t_{2,c}, t_{3,c}) = \begin{cases} \left(1 \pm 2\sqrt{3}, \pm\sqrt{2}\right) , & q = 1 , \\ \left(2 \pm 2\sqrt{7}, \pm\sqrt{10}\right) , & q = 2 , \\ \left(3 \pm \sqrt{47}, \pm\sqrt{105}/2\right) , & q = 3 . \end{cases} \tag{3.80}$$

Let us parametrise the vicinity of this point by eliminating δ_U in favour of the scaling parameter $\varepsilon = z_- - \delta_U$ and investigate the limit $\varepsilon \to 0$. We would like to expand $W_Y(z)$ in powers of ε, keeping $(z - z_-)/\varepsilon$ finite. Setting again $w(z) = \sqrt{z - \delta_U}$, this requires the expansion of $f(w)$ in (3.43) in powers of $\sqrt{\varepsilon} \equiv w_-$, keeping $w/\sqrt{\varepsilon}$ finite. As we show in Appendix A, equation (A.21), the terms of $\mathscr{O}(\varepsilon^{n\pm\nu/2})$ in the expansion of $f_{\text{sing.}}(w)$ can be written as

$$\varepsilon^{n\pm\nu/2}\left(t_n^{(\pm)} T_{2n\pm\nu}(-w/\sqrt{\varepsilon}) + u_n^{(\pm)} U_{2n\pm\nu}(-w/\sqrt{\varepsilon})\right) , \tag{3.81}$$

where $T_\nu(w)$ (resp. $U_\nu(w)$) is the Chebyshev function of the first (resp. second) kind as defined in equation (A.19). Using (A.20), the term of same order in the expansion of the discontinuity $f(w)_+ - f(w)_-$ across $w/\sqrt{\varepsilon} \in [1, \infty)$ becomes

$$-2\mathrm{i}\sin(\pi\nu)\varepsilon^{n\pm\nu/2}\left(u_n^{(\pm)} \frac{T_{2n+1\pm\nu}(w/\sqrt{\varepsilon})}{\sqrt{1 - w^2/\varepsilon}} + t_n^{(\pm)}\sqrt{1 - w^2/\varepsilon}\, U_{2n-1\pm\nu}(w/\sqrt{\varepsilon})\right) . \tag{3.82}$$

Comparing to (3.42) and requiring that $\rho_Y(z) \to 0$ as $z \to z_-$ reveals that this expression must vanish as $w \to \pm\sqrt{\varepsilon}$, which implies that $u_n^{(\pm)} = 0$ for all n. Using $(z - z_-)/\varepsilon \equiv w^2/\varepsilon - 1$ and the relation (3.43) together with

$$T_{2-\nu}\left(\sqrt{1-\eta}\right) + T_{2-\nu}\left(-\sqrt{1-\eta}\right) = 2\cos\left(\frac{\pi\nu}{2}\right) T_{2-\nu}\left(\sqrt{\eta}\right) , \tag{3.83}$$

gives the following expansion of $W_Y(z)$:

$$W_Y(z_- - \varepsilon\eta) = W_Y(z_-) + C\varepsilon^{1-\nu/2} T_{2-\nu}(\sqrt{\eta}) - \varepsilon\frac{2\eta}{4-q} + \mathscr{O}(\varepsilon^{1+\nu/2}) \ , \tag{3.84}$$

where $W_Y(z_-) = (2z_- + qt_{2,c}/t_{3,c})/(4-q)$ and C is a normalisation constant. The expansion of $G_Y^{(p)}(z)$ now follows immediately from Proposition 3.3.1; the leading non-analytic term reads

$$C\varepsilon^{1-\nu/2}\left[T_{2-\nu}(\sqrt{\eta}) - \frac{2p}{q}\cos\left(\frac{\pi\nu}{2}\right)T_{2-\nu}\left(\sqrt{1-\eta}\right)\right] \ . \tag{3.85}$$

The string exponent γ_s predicted by $W_{(p)}(z-z_c) \sim (z-z_c)^{1-\gamma_s}$ is in agreement with previous findings [20, 75, 77], namely

$$\gamma_s = \frac{\nu}{\nu-2} \ . \tag{3.86}$$

In particular, $\gamma_s = -1/2$, $-1/3$, $-1/5$ and 0 for $q = 1$, 2, 3 and 4 respectively, which is consistent with Liouville theory interacting with conformal matter of central charges $c_M = 0$, $1/2$, $4/5$ and 1 according to (2.34). Whilst in the first two cases the conformal field theory is unique, there exist two distinct modular invariants at $c_M = 4/5$, corresponding to the (A_4, A_5) Virasoro minimal model and the (A_4, D_4) minimal model, which admits a conserved spin-3 current is diagonal under the extended $\mathscr{W}_3$-algebra [48, 50]. In light of the S_3-symmetry of the partition function (3.69a) and the resulting spectrum of boundary conditions [25] – cf. Figure 3.2 – we expect our equations to describe the latter coupled to Liouville theory, not the former.

3.6 Discussion

Let us summarise our results. Starting from the matrix integral representation of the Potts model on a random lattice in Lemma 3.2.1, we employed the saddle point apprixomation to express $W_{(p)}(z)$ via the p-independent average $W_Y(z)$ in Proposition 3.3.1. For the case of planar triangulations, Proposition 3.3.6 provides an explicit elliptic parametrisation of the latter for arbitrary $q \neq 4$. Just as equation (3.8a) defines an analytic continuation of the partition function to the complex q-plane, equation (3.14) may thus be used to define the analytic continuation of $W_{(p)}(z)$ in the complex p- and q-plane. What is more, Corollary 3.3.3 showed that $W_{(p)}(z)$ and $W_{(q-p)}(z)$ can be related algebraically – in the case studies in Section 3.4, this resulted in the p- and $(q-p)$-boundary conditions being described by a single spectral curve defined by the zero locus of (3.55). Remarkably, equations (3.66a), (3.75a) and (3.75b) indicate that $G^Y_{(p)}(z) - z$ and $G^Y_{(q-p)}(z) - z$ are functional inverses, generalising the well-known duality[12] interchanging the two matrices of the symmetric Hermitian two-matrix model [68, 69].

Our results naturally pave the way for a number of further developments: Firstly, going beyond the planar limit, as was done in [99] for the $O(n)$ model on random lattices, it would be interesting to explore if and when the curves defined by (3.55) can be used as a valid part of the initial data of the topological recursion algorithm [66], which allows to compute averages to all orders in $1/N$. Secondly, for general values of h in Lemma 3.2.1, the remarkably simple result in Lemma 3.3.4 should enable us to investigate the boundary renormalisation group flow relating boundary conditions with different p. This flow is expected to induce a partial order on the spectrum of boundary states in accordance with the boundary analogue of the c-theorem [100], as conjectured in [101] and finally proven by Friedan and Konechny [102]; it would be interesting to derive this fact directly from the matrix model, thus extending the work of [23, 24, 32].

Finally, it would be instructive to check if the universal results of Section 3.5 can be reproduced by other means, e.g. by explicitly constructing the corresponding conformal field theory. Remarkably, as exemplified by the case of the D_4 model, this appears to require a

[12] Note that this involution is in general distinct from the Kramers-Wannier duality [98] on the dynamical lattice: e.g. for $q = 3$, the latter interchanges $p = 1$ with $p = 3$, and $p = 2$ with $p = 0$, mixing singlets and triplets [25].

non-diagonal partition function in the Liouville sector in general. From this perspective, various other corners of the (q,k)-parameter space also warrant more detailed investigations. Of particular interest would be the computation of the scaling behaviour for strongly coupled models with $q > 4$: one might wonder if there exist analogues of the critical points of the $O(n)$ model on a random lattice with $n > 2$ reported in [90, 97]. For the models with $q = 5^2 - 6n$ for $n \in \{1,2,3\}$ in the infinite k limit, $c_M = 18$, 12 and 6, respectively, and Liouville theory allows a truncation to a tachyon-free spectrum [103, 104]; the matrix model might help in reconciling the conflicting CFT predictions [105] and [106]. Finally, it might be of interest to enquire about the existence of the 't Hooft limit $q \to \infty$, $k \to \infty$, $q/(q+k)$ fixed, which has been studied for the pure $\mathscr{W}_q$ minimal models in [107].

Chapter 4

The Critical Potts Model Coupled to Liouville Theory

4.1 Overview

In this chapter, we investigate the spectrum of the $\mathscr{W}_q$ conformal minimal models coupled to gravity in two dimensions. We shall consider the theory on the sphere and the disk. Requiring the overall conformal anomaly to vanish allows for the interpretation of these theories as a family of bosonic string backgrounds. A new feature with respect to the so-called minimal string (i.e., a Virasoro minimal model coupled to Liouville theory) is the presence of conversed currents with integer spins up to q on the worldsheet which – with the exception of the stress-energy tensor – remain *ungauged*. The matter sector consequently enjoys an extended non-linear symmetry admitting additional globally conserved charges given by the generators of the so-called $\mathscr{W}_q$ algebra. A major motivation for invoking the latter is that it can be understood as a continuous extension of a discrete symmetry arising from the continuum limit of a critical statistical model on a random lattice. In particular, when $q \leq 4$, such theories are expected to provide a description of the scaling limit of the Potts model on a random planar lattice with discrete symmetry group S_q. Unlike two-dimensional $\mathscr{W}$-gravity, which is obtained from these systems by coupling the remaining currents to higher-spin gauge fields, and for which the critical target space dimension increases with q [46], the former enter a strong-coupling regime with tachyonic instabilities when the central charge of the minimal model exceeds one.

The unitary $\mathscr{W}_q$ CFTs also provide a dual description of three-dimensional spin-q gravity coupled to scalar matter, with Newton coupling $G_N = \frac{3}{2}\ell c_M$, where c_M denotes the central charge (2.42) of the minimal model and $-1/\ell^2$ is the cosmological constant: for example, the torus partition function of the CFT equals the semiclassical partition function of the gravity theory in thermal Anti-de Sitter space with radius ℓ [107]. The perturbative excitations of the latter are described by Chern-Simons theory with an $\mathfrak{sl}(q,\mathbb{R}) \oplus \mathfrak{sl}(q,\mathbb{R})$-valued connection; coupling the boundary CFT to Liouville theory such that the overall conformal anomaly vanishes corresponds to switching from Dirichlet to Neumann boundary conditions for the metric on the boundary of AdS [108]. In this way, our computations also solve a problem in three-dimensional spin-q gravity with negative cosmological constant, with the boundary metric allowed to fluctuate but the asymptotic behaviour of the higher spin gauge fields held fixed. The presence of tachyonic excitations for $c_M \geq 1$ indicates the perturbative instability of the boundary condition for large enough G_N/ℓ.

This chapter is organised at follows: After determining the spectrum of theory on the sphere in Section 4.2 from the cohomology of the BRST operator associated with the diffeomorphism symmetry, we proceed to introduce a family of conformally invariant boundary conditions from the tensor product of the Liouville and minimal model boundary states Section 4.3. We present evidence that upon analytic continuation of the boundary cosmological constant, this construction in fact overcounts the number of distinct boundary conditions, as has previously been observed for the minimal string in [29]. Lastly, a summary and discussion of our results is provided in Section 4.4.

4.2 Bulk states

Here we discuss the spectrum of physical states on $\mathbb{CP}^1$. For $q = 2$, our results reduce to those for the Virasoro minimal models coupled to gravity, for which the spectrum was first determined in [17, 109]. These results were rederived in [36] in a more elementary free field formalism and our approach will be close in spirit.

The reparametrisation invariance requires the sum $c_M + c_L + c_{gh}$ to vanish, which according to (2.42) and (2.55) gives a condition on the background charges, fixing the Liouville coupling b for given q, p and p':

$$Q_L^2 - Q_M^2 = \frac{26-q}{12} \,, \tag{4.1}$$

where here and in what follows we abbreviated $Q_M^2 = Q_0^2 \rho \cdot \rho$. When $c_M \leq 1$, the susceptibility exponent $\gamma_s = 1 - b^{\pm 2}$ then follows from (2.34). For later comparison with the matrix model with cubic potential, we print this relationship for the simplest model with $k = 1$ in (2.47), i.e. $p' - p = p - q = 1$:

$$\gamma_s = \frac{1}{12}\left(1 - \frac{6}{(q+2)} \pm \frac{\sqrt{(4-q)(52+23q)}}{(q+2)}\right) \tag{4.2}$$

The above formula produces the values $\gamma_s = -1/2$, $-1/3$,$-1/5$ and 0 for $q = 1$, 2, 3 and 4, respectively if we pick the negative branch of the square root such that the weak coupling regime $2/3 \leq b^2 \leq 1$ corresponds to $1 \leq q \leq 4$[1]; the strong-weak duality $b \to 1/b$ permutes these two branches. When (4.1) holds, the BRST charge d defined in (2.35) becomes nilpotent and denoting the irreducible $\mathscr{W}_q$-module with conformal dimension (2.45) by $\mathscr{M}(\lambda)$, we define the holomorphic part of the physical Hilbert space of the $\mathscr{W}_q$ minimal model coupled to gravity as

[1] This is also the only choice for which the cosmological constant operator obeys the Seiberg bound.

$$\bigoplus_{\lambda\in\mathscr{B}^{(q)}_{p,p'}}\bigoplus_{n\in\mathbb{Z}} H^n(\mathscr{M}(\lambda)\otimes\mathscr{F}_L(P)\otimes\mathscr{F}_{gh},\mathrm{d}) \;, \tag{4.3}$$

where $\mathscr{F}_L(P)$ and $\mathscr{F}_{gh}$ are defined in (2.58) and (2.64), respectively and H^n denotes the subspace of Ker d$/$Im d with ghost number n. In what follows, we shall argue that for a suitable choice of the fundamental domain $\mathscr{B}^{(q)}_{p,p'}$, the following result holds:

Proposition 4.2.1. *Let the complex $\mathscr{C}^\perp(\lambda)$ be defined as*

$$\begin{aligned}\mathscr{C}^\perp(\lambda) &= \bigoplus_{N^i\in\mathbb{Z}}\bigoplus_{w\in S_q}\mathscr{F}^\perp\left(P_{\lambda^w-pp'N^ie_i},\lambda^w-pp'N^ie_i\right) \;,\\ P^2_{\lambda^w-pp'N^ie_i} &= \frac{q-2}{12}-\frac{1}{pp'}\left(\lambda^w-pp'N^ie_i\right)^2 \;.\end{aligned} \tag{4.4}$$

Then for any $\lambda\in\mathscr{B}^{(q)}_{p,p'}$,

$$H^n\left(\mathscr{M}(\lambda)\otimes\mathscr{F}_L(P)\otimes\mathscr{F}_{gh},\mathrm{d}\right)\simeq H^n\left(\mathscr{C}^\perp(\lambda),\mathrm{d}'\right) \;. \tag{4.5}$$

The Fock space of transverse oscillations $\mathscr{F}^\perp(P,\lambda)$ in the above is defined in (4.11) and λ^w is as in (2.49). In particular, the highest-weight states created by the 'tachyon' operators

$$\mathscr{T}_\lambda(z) = c(z)V^L_{Q_L+\mathrm{i}P_\lambda}(z)V^M_{Q_0\rho-\frac{1}{\sqrt{pp'}}\lambda}(z) \;. \tag{4.6}$$

are always contained in the cohomology of d'. When $P^2_{\lambda^w-pp'N^ie_i}<0$, these states are non-normalisable and the sign of the square root can be fixed by the Seiberg bound $\mathrm{i}P>0$. To see explicitly when this prescription breaks down, note $P^2_{\lambda^w-pp'N^ie_i}\le P^2_\rho=(c_M-1)/12$, where $\lambda=\rho$ labels the dressed identity field, for which

$$\frac{1}{12}\frac{q-4}{q+2}\le P^2_\rho\le\frac{q-2}{12} \;. \tag{4.7}$$

Hence, "macroscopic" states with finite real P are absent from the spectrum only for $c_M\le 1$, signalling the well-known Kosterlitz-Thouless transition of Liouville theory at $c_L=25$.

When $c_M > 1$, we expect the complex values of γ_s yielded by (4.2) – and, more generally, the KPZ relations – to be unreliable.

The result (4.5) follows from an application of the following useful result on doubly graded complexes:

Lemma 4.2.2 ([110, 111])**.** *Let* d *and* d' *be commuting, nilpotent differentials on a complex* $\mathscr{C}$ *and suppose* $H^n(\mathscr{C},\mathrm{d}) = 0$ *for* $n \neq 0$ *and* $H^n(\mathscr{C},\mathrm{d}') = 0$ *for* $n \neq 0$*. Then*

$$H^n\left(H^0(\mathscr{C},\mathrm{d}'),\mathrm{d}\right) \simeq H^n\left(H^0(\mathscr{C},\mathrm{d}),\mathrm{d}'\right) \ . \tag{4.8}$$

The remainder of this section is devoted to demonstrating that in the case at hand, the conditions in the above lemma are indeed satisfied. To this end, we introduce the complex

$$\mathscr{C}(P,\lambda) = \bigoplus_{N^i \in \mathbb{Z}} \bigoplus_{w \in S_q} \mathscr{F}(P,\lambda^w - pp'N^i e_i) \ , \quad \lambda \in \mathscr{B}^{(q)}_{p,p'} \ , \tag{4.9}$$

where we defined an extended Fock space from the tensor product of (2.48), (2.58) and (2.64),

$$\mathscr{F}(P,\lambda) = \mathscr{F}_M(\lambda) \otimes \mathscr{F}_L(P) \otimes \mathscr{F}_{gh} \ . \tag{4.10}$$

Firstly, on this complex, we must have $\mathrm{d}^2 = (\mathrm{d}')^2 = 0$ and also $[\mathrm{d},\mathrm{d}'] = 0$, since d' acts nontrivially only on $\mathscr{F}_M(\lambda)$ and $[\mathrm{d}', L_0^M] = 0$ by construction. Secondly, recall from the introduction in Subsection 2.2.2 that $H^n(\mathscr{C}(P,\lambda),\mathrm{d}') = 0$ for $n \neq 0$ is already implied in the free-field resolution of $\mathscr{M}(\lambda)$ along the lines of [53, 111, 52]. To show applicability of Lemma 4.2.2, it thus remains to characterise $H^n(\mathscr{F}(P,\lambda),\mathrm{d})$. The result is

Lemma 4.2.3. *Pick a basis in root space such that* $\rho = (|\rho|, 0, \ldots, 0)$*, and let*

$$\mathscr{F}^{\perp}(P,\lambda) = \mathrm{span}\left\{ |P\rangle_L \otimes |0\rangle_{gh} \otimes \prod_{i=2}^{q-1} \prod_{n_j^{(i)}=1}^{k_i} a^i_{-n_j^{(i)}} |\lambda\rangle_M \,|\, k_i \geq 0,\ 0 < n_1^{(i)} \leq \cdots \leq n_{k_i}^{(i)} \right\} \ . \tag{4.11}$$

Then $H^n(\mathscr{F}(P,\lambda),\mathrm{d}) = \delta_{n,0}\mathscr{F}^{\perp}(P_\lambda,\lambda)$, *where* P_λ *is the solution to*

$$P_\lambda^2 = \frac{q-2}{12} - \frac{\lambda^2}{pp'} \; . \tag{4.12}$$

Proof. Let us begin by considering those states that satisfy $b_0|\psi\rangle = 0$. On this subspace, the total energy $L_0 = L_0^M + L_0^L + L_0^{gh} = \{b_0, \mathrm{d}\}$ also annihilates physical states[2] and moreover commutes with d so that we may restrict our attention to the subspace

$$\mathscr{F}_{\mathrm{rel}}(P,\lambda) = \mathscr{F}(P,\lambda) \cap \mathrm{Ker}\, L_0 \cap \mathrm{Ker}\, b_0 \tag{4.13}$$

and determine the so-called relative cohomology of the restriction $\mathrm{d}_{\mathrm{rel}}$ of d to this subspace, which is [36]

$$\mathrm{d}_{\mathrm{rel}} = \mathrm{d} + b_0 \sum_{n\neq 0} n \; : c_n c_{-n} : - c_0 L_0 \; . \tag{4.14}$$

Since b_0 (anti-)commutes with all modes besides c_0, the full – or "absolute" – cohomology is given by

$$H^n(\mathscr{F}(P,\lambda),\mathrm{d}) = H^n(\mathscr{F}_{\mathrm{rel}}(P,\lambda),\mathrm{d}_{\mathrm{rel}}) \oplus c_0 H^{n-1}(\mathscr{F}_{\mathrm{rel}}(P,\lambda),\mathrm{d}_{\mathrm{rel}}) \; . \tag{4.15}$$

To expose the physical modes, we pick a basis in root space such that $\rho = (\sqrt{\rho\cdot\rho}, 0, \ldots, 0)$ and transform the fields into 'lightcone' variables,

$$\begin{aligned} q^{\pm} &= \frac{1}{\sqrt{2}}(\phi_0^1 \pm \mathrm{i}\varphi_0) \; , \\ p_n^{\pm} &= \frac{1}{\sqrt{2}}\left(a_0^1 \pm \mathrm{i}\alpha_0 - (n+1)(Q_M \mp Q_L)\right) \; , \\ \alpha_n^{\pm} &= \frac{1}{\sqrt{2}}(a_n^1 \pm \mathrm{i}\alpha_n) \; , \quad n \neq 0 \; . \end{aligned} \tag{4.16}$$

We note the resulting commutation relations

[2]This is sometimes called the Hamiltonian constraint in quantum gravity.

$$[q^{\pm}, p_0^{\mp}] = \mathrm{i}\,, \qquad [\alpha_n^{\pm}, \alpha_m^{\mp}] = n\delta_{m+n,0}\,, \tag{4.17}$$

and refer to the remaining a_n^i, $i = 2 \ldots q-1$ as the transverse modes. In these variables, the restriction to the kernel of L_0 reads

$$L_0|\psi\rangle = (L_0^{\perp} + L_0^{\parallel})|\psi\rangle = 0\,, \tag{4.18}$$

where, using the expressions (2.41), (2.54) and (2.63),

$$\begin{aligned} L_0^{\parallel} =&\ p_0^+ p_0^- + \sum_{n\neq 0} (\,:\alpha_n^+ \alpha_{-n}^-: + n :c_{-n} b_n:) + \frac{2-q}{24}\,, \\ L_0^{\perp} =&\ \frac{1}{2}\sum_{n\in\mathbb{Z}} \sum_{i=2}^{q-1} :a_n^i a_{-n}^i:\,, \end{aligned} \tag{4.19}$$

and (4.14) can be decomposed as $\mathrm{d}_{\mathrm{rel}} = \mathrm{d}_+^{\parallel} + \mathrm{d}_-^{\parallel} + \mathrm{d}^{\parallel} + \mathrm{d}^{\perp}$, where

$$\mathrm{d}_{\pm}^{\parallel} = \sum_{n\neq 0} p_n^{\pm}\ :c_{-n}\alpha_n^{\mp}:\,, \tag{4.20a}$$

$$\mathrm{d}^{\parallel} = \sum_{\substack{n,m\neq 0\\ m+n\neq 0}} :c_{-n}\left(\alpha_{-m}^+ \alpha_{m+n}^- + \frac{1}{2}(m-n)c_{-m}b_{m+n}\right):\,, \tag{4.20b}$$

$$\mathrm{d}^{\perp} = \frac{1}{2}\sum_{i=2}^{q-1}\sum_{n\neq 0} :c_{-n} a_m^i a_{n-m}^i:\,. \tag{4.20c}$$

We observe that $\mathrm{d}_+^{\parallel}$ is in fact nilpotent on the extended Fock space and (anti-)commutes with both b_0 and the total energy. We can therefore determine the cohomology $\mathrm{d}_+^{\parallel}$ on $\mathscr{F}$ first and thereafter restrict to $\mathscr{F}_{\mathrm{rel}}$. The procedure to determine the cohomology of $\mathrm{d}_+^{\parallel}$ on $\mathscr{F}$ is analogous to that for the critical bosonic string: We distinguish the cases

1. Either $p_n^+|\lambda\rangle_M \otimes |P\rangle_L \neq 0$ or $p_n^-|\lambda\rangle_M \otimes |P\rangle_L \neq 0\ \forall\, n \in \mathbb{Z}\setminus\{0\}$,
2. Otherwise, i.e. $p_{n_\pm}^{\pm}|\lambda\rangle_M \otimes |P\rangle_L = 0$ for a pair of non-zero integers (n_+, n_-).

In the first case, we may assume w.l.o.g. that the operator

$$\mathscr{O} = \sum_{n\neq 0} (p_n^+)^{-1} : \alpha_{-n}^+ b_n : \tag{4.21}$$

exists. We observe that only states annihilated by

$$\{\mathscr{O}, \mathrm{d}_+^{\|}\} = \sum_{n\neq 0} (: \alpha_n^+ \alpha_{-n}^- : \; + n \; : c_{-n} b_n :) \tag{4.22}$$

may be physical since any other state closed under $\mathrm{d}_+^{\|}$ is also $\mathrm{d}_+^{\|}$-exact. But the above expression is the level operator for the modes $\alpha_n^\pm$, b_n and c_n. This implies

$$H^n\left(\mathscr{F}(P,\lambda), \mathrm{d}_+^{\|}\right) = \delta_{n,0}\mathscr{F}^\perp(P,\lambda) \;, \tag{4.23}$$

where $\mathscr{F}^\perp(P,\lambda)$ is given by (4.11). Finally, we need to restrict to the subspace $\mathscr{F}_{\mathrm{rel}}$ by imposing (4.18), i.e.

$$\left(p_0^+ p_0^- + L_0^\perp + \frac{2-q}{24}\right) |\lambda\rangle_M \otimes |P\rangle_L \otimes |0\rangle_{gh} = 0 \;. \tag{4.24}$$

From the expressions of the Liouville and matter conformal weights and using (4.1), we see that the above equation holds iff $P = \pm P_\lambda$, where P_λ solves the equation (4.12).

The second case occurs iff the following equations hold simultaneously:

$$\frac{n_+ - n_-}{2} Q_M - \frac{n_+ + n_-}{2} Q_L = \mathrm{i}P \;, \tag{4.25a}$$

$$\frac{n_+ + n_-}{2} Q_M - \frac{n_+ - n_-}{2} Q_L = -\frac{1}{\sqrt{pp'}} \frac{\lambda\cdot\rho}{\sqrt{\rho\cdot\rho}} \;. \tag{4.25b}$$

Such a state would survive the projection onto Ker L_0 when $\frac{1}{24}(n_+ n_-(26-q) + q - 2)$ is a positive integer. However, for each such $\lambda \in \mathscr{B}_{p,p'}^{(q)}$ there exists another choice of fundamental domain $\widetilde{\mathscr{B}}^{(q)}_{p,p'}$ and $\widetilde{\lambda} \in \widetilde{\mathscr{B}}^{(q)}_{p,p'}$ such that

$$L_0^M|\lambda\rangle_M = L_0^M|\widetilde{\lambda}\rangle_M \ , \tag{4.26a}$$

$$\lambda^w - pp'N^i e_i \neq \widetilde{\lambda}^w - pp'N^i e_i \quad \forall w \in S_q \ , \quad N^i \in \mathbb{Z} \ . \tag{4.26b}$$

For example, $\mathscr{B}^{(q)}_{p,p'}$ may be chosen such that $\lambda \cdot e_i \equiv p'r^i - ps^i \geq 1$ [51]; another choice can be obtained from it by imposing $\widetilde{\lambda} \cdot e_i \leq -1$ for some i instead[3]. We may therefore always choose a resolution of $\mathscr{M}(\lambda) = H^0(\mathscr{C}(\widetilde{\lambda}), d')$ for which (4.25) is never satisfied. On the latter, the operator (4.21) is well defined, thus yielding the same result as in the previous case. □

Remark 4.2.4. It is instructive to see how the known results for the Virasoro minimal model coupled to gravity are recovered from the above results when $q = 2$: then the Liouville coupling satisfies $b^2 = p/p'$ and we obtain the particularly simple relationship $\gamma_s = 1 - p'/p$ since $c_M \leq 1$. The Fock module of transverse oscillations $\mathscr{F}^\perp(\lambda)$ reduces to $\mathbb{C}$, with $|\lambda\rangle_M \otimes |P_\lambda\rangle \otimes |0\rangle_{gh}$ the only state. Introducing the Kac labels $1 \leq r < p$, $1 \leq s < p'$ by setting $e_1 \cdot \lambda^w = p'wr - ps$, we find that the transverse complex (4.4) becomes

$$\begin{aligned} \mathscr{C}^\perp(r,s) &= \bigoplus_{w\in\{-1,1\}} \bigoplus_{N\in\mathbb{Z}} \mathrm{span}\left\{ \left| \lambda^w - Npp'e_1 ; P_{\lambda^w - Npp'e_1} \right\rangle \right\} \ , \\ P_{\lambda^w - Npp'e_1} &= \pm \frac{\mathrm{i}}{\sqrt{2pp'}} \left(wp'r - ps + 2pp'N \right) \ . \end{aligned} \tag{4.27}$$

The physical states are then given by the cohomology of d' on $\mathscr{C}^\perp(r,s)$, which is exactly the procedure first used in [36] to determine the spectrum; hence we recover the familiar result of Lian and Zuckerman [17] for the physical states of the Virasoro minimal model coupled to gravity.

[3] For $q = 2$, this simply corresponds to the reflection symmetry of the Kac table.

4.3 Boundary states

Using the operator-state correspondence and modular invariance, we associate to each conformal boundary condition a "boundary state" in the of $L_n - \bar{L}_{-n}$ of the physical Hilbert space; identities in this section are therefore implied to hold modulo BRST exact terms. Because the Virasoro algebra acts diagonally on the tensor factors in (4.10), conformal invariance has to be preserved in each sector independently. Our ansatz for a conformally invariant boundary state, given $\lambda \in \mathscr{B}^{(q)}_{p,p'}$, is thus the tensor product of (2.67), (2.75) and (2.77),

$$|\sigma\rangle_\lambda = |\lambda\rangle_C \otimes |\sigma\rangle_{\text{FZZT}} \otimes |B\rangle_{gh} \,. \tag{4.28}$$

We now present some evidence that the above definition actually overcounts the number of independent boundary conditions if we analytically continue the boundary cosmological constant (2.73), as was observed in the case $q = 2$ in [29]. Consider the one-point function of a tachyon operator (4.6) on the upper half plane with matter boundary condition $|\sigma\rangle_{\lambda'}$ on the real line,

$$\langle \mathscr{T}_\lambda \rangle_{\lambda'}(\sigma) = \lim_{z,\bar{z}\to\infty} \langle 0| \mathscr{T}_\lambda(z) \bar{\mathscr{T}}_{\bar{\lambda}}(\bar{z}) |\sigma\rangle_{\lambda'} \,. \tag{4.29}$$

In particular, the one-point function of the dressed identity computes the first derivative of the partition function on the disk with boundary condition λ,

$$\langle \mathscr{T}_\rho \rangle_\lambda(\sigma) = \left.\frac{\partial D(\mu,\mu_B;\lambda)}{\partial \mu}\right|_{\mu_B} \,. \tag{4.30}$$

The factorisation of the one-point function into a product of matter and Liouville contributions implies

$$\langle \mathscr{T}_\lambda \rangle_{\lambda'}(\sigma) = \frac{S_{\lambda\lambda'}}{S_{\lambda\rho}} \langle \mathscr{T}_\lambda \rangle_\rho(\sigma) \,, \tag{4.31}$$

where the modular S-matrix is as in (2.68). The ratio of S-matrix elements can be written in terms of $SU(q)$ characters,

$$\frac{S_{\lambda\lambda'}}{S_{\rho\lambda'}} = \chi_{r^i\omega_i-\rho}\left(\frac{2\pi\lambda'}{p}\right)\chi_{s^i\omega_i-\rho}\left(-\frac{2\pi\lambda'}{p'}\right) . \tag{4.32}$$

Applying the Weyl character formula to the above results in

$$\frac{S_{\lambda\lambda'}}{S_{\rho\lambda'}} = \sum_{\mu\in\Omega_{r^i\omega_i-\rho}} \mathrm{mult}_{r^i\omega_i-\rho}(\mu)\mathrm{e}^{2\pi\mathrm{i}\lambda'\cdot\mu/p} \sum_{\nu\in\Omega_{s^i\omega_i-\rho}} \mathrm{mult}_{s^i\omega_i-\rho}(\nu)\mathrm{e}^{-2\pi\mathrm{i}\lambda'\cdot\nu/p'} , \tag{4.33}$$

where $\Omega_{r^i\omega_i-\rho}$ (resp. $\Omega_{s^i\omega_i-\rho}$) denotes the set of weights of the $\widehat{\mathfrak{su}}(q)_k$ (resp. $\widehat{\mathfrak{su}}(q)_{k+1}$) representation of highest weight $r^i\omega_i-\rho$ (resp. $s^i\omega_i-\rho$) and $\mathrm{mult}_{r^i\omega_i-\rho}(\mu)$ denotes the S_q-invariant multiplicity of the corresponding state. We summarise the above with the abbreviated notation

$$\frac{S_{\lambda\lambda'}}{S_{\rho\lambda'}} = \sum_{\mu,\nu} \mathrm{mult}_{\lambda-(p'-p)\rho}(\mu,\nu)\mathrm{e}^{2\pi\mathrm{i}\lambda'\cdot(\mu/p-\nu/p')} . \tag{4.34}$$

To obtain relations between different boundary states, we introduce operators that change the boundary conditions λ and σ when acting on $|\sigma\rangle_\lambda$:

$$D_L(\sigma')|\sigma\rangle_\lambda = |\sigma'+\sigma\rangle_\lambda , \qquad D_M(\lambda)|\sigma\rangle_\rho = |\sigma\rangle_\lambda . \tag{4.35}$$

The σ-translation operator can be represented explicitly as

$$D_L(\sigma) = \mathrm{e}^{\pi\sigma(\bar{\alpha}_0-\alpha_0)} . \tag{4.36}$$

We claim the operator changing the matter boundary condition can be written as

$$D_M(\lambda) = \sum_{\mu,\nu} \mathrm{mult}_{\lambda-(p'-p)\rho}(\mu,\nu)\mathrm{e}^{\frac{2\pi\mathrm{i}}{\sqrt{pp'}}(Q_0\rho-a_0)\cdot(p'\mu-p\nu)} . \tag{4.37}$$

To see this, note first that the coherent states are eigenstates of $D_M(\lambda)$,

$$D_M(\lambda')|B(\lambda^w)\rangle_\Lambda = |B(\lambda^w)\rangle_\Lambda \sum_{\mu,\nu} \mathrm{mult}_{\lambda'-(p'-p)\rho}(\mu,\nu) \mathrm{e}^{2\pi \mathrm{i}(\lambda+(1-w)r^i\omega_i)\cdot(\mu/p-\nu/p')} . \tag{4.38}$$

To determine the action of $D_M(\lambda)$ on the Ishibashi states, we need to sum this expression over the Felder complex according to (2.66). To this end, we write w-dependent phase contribution as

$$r^i(w\omega_i)\cdot(p'\mu-\nu) = r^i\left(p'\mu^j - p\nu^j\right)\omega_i\cdot(w^{-1}\omega_j) . \tag{4.39}$$

Noting that the infinite sum in (2.66) over the N^i produces the irrelevant phase $2\pi \mathrm{i}\sum_i N^i(p'\nu^i - p\mu^i)$, we thus find

$$\begin{aligned} D_M(\lambda')|\lambda;\Lambda\rangle\rangle_M &= \sum_{w\in S_q}\sum_{N^j\in\mathbb{Z}} \kappa_N^w D_M(\lambda')\left|B(\lambda^w - pp'N^je_j)\right\rangle_\Lambda \\ &= \frac{S_{\lambda'\lambda}}{S_{\rho\lambda}}\sum_{w\in S_q}\sum_{N^j\in\mathbb{Z}} \kappa_N^w\left|B(\lambda^w - pp'N^je_j)\right\rangle_\Lambda \\ &= \frac{S_{\lambda'\lambda}}{S_{\rho\lambda}}|\lambda;\Lambda\rangle\rangle_M , \end{aligned} \tag{4.40}$$

where we used the identity (4.34). In conjunction with (2.67) it follows that $D_M(\lambda)$ takes the identity Cardy state to the state $|\lambda\rangle_C$ as advertised. Now, by construction, the bulk tachyons (4.6) create eigenstates $\langle\mathscr{T}_\lambda| = \lim_{z,\bar{z}\to\infty}\langle 0|\mathscr{T}_\lambda(z)\bar{\mathscr{T}}_{\bar{\lambda}}(\bar{z})$ of both $D_L(\sigma)$ and $D_M(\lambda)$,

$$\langle\mathscr{T}_\lambda|D_L(\sigma') = \mathrm{e}^{\mathrm{i}\pi\sigma'(P_\lambda - P_{\bar{\lambda}})}\langle\mathscr{T}_\lambda| , \tag{4.41a}$$

$$\langle\mathscr{T}_\lambda|D_M(\lambda) = \sum_{\mu,\nu}\mathrm{mult}_{\lambda'-(p'-p)\rho}(\mu,\nu)\mathrm{e}^{2\pi\mathrm{i}\lambda\cdot(\mu/p-\nu/p')}\langle\mathscr{T}_\lambda| . \tag{4.41b}$$

where P_λ is given by the on-shell Liouville weight (4.12). Using the above observations, we may express a given tachyon one-point function as a sum over insertions of $D_L(\sigma)$,

$$\begin{aligned}\langle \mathscr{T}_\lambda \rangle_{\lambda'}(\sigma) &= \langle \mathscr{T}_\lambda | D_M(\lambda') | \sigma \rangle_\rho \\ &= \sum_{\mu,\nu} \mathrm{mult}_{\lambda' - (p'-p)\rho}(\mu,\nu) \mathrm{e}^{2\pi\mathrm{i}\lambda\cdot(\mu/p - \nu/p')} \langle \mathscr{T}_\lambda | \sigma \rangle_\rho \\ &= \sum_{\mu,\nu} \mathrm{mult}_{\lambda' - (p'-p)\rho}(\mu,\nu) \langle \mathscr{T}_\lambda | D_L\left(\Delta^{(\lambda)}_{\mu,\nu}\right) | \sigma \rangle_\rho \ ,\end{aligned} \tag{4.42}$$

where we abbreviated

$$\Delta^{(\lambda)}_{\mu,\nu} = \pm \frac{2}{(P_\lambda - P_{\bar{\lambda}})} \lambda \cdot (\mu/p - \nu/p') \ . \tag{4.43}$$

Indeed, for any such solution we can replace the corresponding boundary state as a sum over states with trivial matter configurations when inserting an arbitrary tachyon in the bulk:

$$\langle \mathscr{T}_\lambda \rangle_{\lambda'}(\sigma) = \sum_{\mu,\nu} \mathrm{mult}_{\lambda' - (p'-p)\rho}(\mu,\nu) \langle \mathscr{T}_\lambda \rangle_\rho (\sigma + \Delta^{(\lambda)}_{\mu,\nu}) \ , \quad \lambda,\ \lambda' \in \mathscr{B}^{(q)}_{p,p'} \ . \tag{4.44}$$

Note that at this stage, the sum rule for this decomposition can depend on the bulk insertion. Together with the factorisation property (4.31), the above relations place constraints on the σ-dependence of disc one-point functions: Setting $\lambda = \rho$ in the sum rule above and comparing with (4.31), we find a set of functional equations labelled by λ for the one-point function of the area operator $\mathscr{T}_\rho$:

$$\begin{aligned}\frac{S_{\rho\lambda}}{S_{\rho\rho}} \langle \mathscr{T}_\rho \rangle_\rho(\sigma) = &\sum_{\mu \in \Omega_{r^i\omega_i - \rho}} \mathrm{mult}_{r^i\omega_i - \rho}(\mu) \exp\left(-\frac{\rho\cdot\mu}{pP_\rho}\frac{\partial}{\partial\sigma}\right) \\ &\times \sum_{\nu \in \Omega_{s^i\omega_i - \rho}} \mathrm{mult}_{s^i\omega_i - \rho}(\nu) \exp\left(\frac{\rho\cdot\nu}{p'P_\rho}\frac{\partial}{\partial\sigma}\right) \langle \mathscr{T}_\rho \rangle_\rho(\sigma) \ .\end{aligned} \tag{4.45}$$

where $P_\rho^2 = (c_M - 1)/12$ and $\lambda = (p'r^i - ps^i)\omega_i \in \mathscr{B}^{(q)}_{p,p'}$ as before. More compactly, in terms of $SU(q)$ characters,

$$0=\left[\chi_{r^i\omega_i-\rho}\left(-\frac{2\pi\rho}{pP_\rho}\frac{\partial}{\partial\sigma}\right)\chi_{s^i\omega_i-\rho}\left(\frac{2\pi\rho}{p'P_\rho}\frac{\partial}{\partial\sigma}\right)-d_\lambda\right]\langle\mathscr{T}_\rho\rangle_\rho(\sigma)\ ,\quad \lambda\in\mathscr{B}^{(q)}_{p,p'}\ , \tag{4.46}$$

where $d_\lambda = S_{\rho\lambda}/S_{\rho\rho}$ is sometimes called the ground state degeneracy or quantum dimension of the state λ. We can then use (4.31) to obtain the disc one-point of $\mathscr{T}_\rho$ for all other matter configurations on the boundary. Note that in general, not all the above equations are independent.

Remark 4.3.1. Once again, it is instructive to consider how our results reduce to those for the Virasoro minimal model coupled to gravity upon setting $q=2$. The root space of $SU(2)$ is one-dimensional, and the Weyl character formula yields a sum over $\widehat{\mathfrak{su}}(2)_k\times\widehat{\mathfrak{su}}(2)_{k+1}$ representation weights. Explicitly, let the Kac indices (r,s) and (k,l) be defined by $e_1\cdot\lambda = p'r-ps$ and $e_1\cdot\lambda' = p'k-pl$. The ratio of S-matrix elements (4.34) can be written in terms of $SU(2)$ characters $\chi_j(\theta)=\mathrm{tr}_j\exp(2\mathrm{i}\theta J_3)$,

$$\begin{aligned}\frac{S_{(r,s),(k,l)}}{S_{(1,1),(k,l)}}&=\chi_{\frac{r-1}{2}}\left(\frac{\pi\lambda'}{p}\right)\chi_{\frac{s-1}{2}}\left(\frac{-\pi\lambda'}{p'}\right)\\&=(-1)^{k(s-1)+r(l-1)}\chi_{\frac{r-1}{2}}\left(\frac{\pi kp'}{p}\right)\chi_{\frac{s-1}{2}}\left(-\frac{\pi lp}{p'}\right)\ .\end{aligned}\tag{4.47}$$

The dependence of the σ-translations (4.43) on the bulk insertion cancels and we find from (4.44)

$$\langle\mathscr{T}_{r,s}\rangle_{k,l}(\sigma)=\sum_{m=1-k,2}^{k-1}\sum_{n=1-l,2}^{l-1}\langle\mathscr{T}_{r,s}\rangle_{1,1}\left(\sigma+\mathrm{i}m/b+\mathrm{i}nb\right)\ , \tag{4.48}$$

where we increment m and n in steps of 2. Hence we conclude that all boundary states can be replaced with superpositions of the identity Cardy state, in agreement with [29, 112].

Consider (4.46) for $(r,s)=(1,2)$ and $(2,1)$, respectively: these imply that as a function of the variables ζ and η defined by the relations (2.72) and (2.73), $\langle\mathscr{T}_{1,1}\rangle$ satisfies

$$
\begin{aligned}
\mathrm{Re}\,\langle \mathscr{T}_{1,1}\rangle(\zeta) &= \cos(\pi p'/p)\langle \mathscr{T}_{1,1}\rangle_{1,1}(-\zeta)\,, \qquad \zeta \in [1,\infty)\,, \\
\mathrm{Re}\,\langle \mathscr{T}_{1,1}\rangle(\eta) &= \cos(\pi p/p')\langle \mathscr{T}_{1,1}\rangle_{1,1}(-\eta)\,, \qquad \eta \in [1,\infty)\,,
\end{aligned}
\tag{4.49}
$$

which is the same equation as (2.101) arising at the critical point of the Hermitian two-matrix matrix model, the solutions to which are studied in the second part of Appendix A.

4.4 Discussion

Let us summarise the results of this chapter: in Section 4.2, we determined the the spectrum of physical states on the sphere using a generalisation of the free-field formalism used in [52, 105]. Our main result as summarised in Proposition 4.2.1 demonstrated the absence of states that would arise in the cohomology of the usual bosonic string – i.e. free bosons coupled to gravity – thanks to the symmetries of the model. Nevertheless, the cohomology includes operators that create boundaries in the worldsheet when the central charge of the minimal model exceeds one, signalling the expected Kosterlitz-Thouless transition of Liouville theory. Another notable feature that distinguishes the spectrum of the model from that of the minimal string is that though no state can carry overall spin, when $q > 2$, the full Hilbert space does include states with non-zero spins in both the matter and the Liouville sector, a phenomenon which resembles the "deconfinement of chirality" described in [103, 106].

We then introduced a family of conformally invariant boundary conditions (4.28) in Section 4.3, parametrised by the primary fields $\lambda \in \mathscr{B}^{(q)}_{p,p'}$ of the $\mathscr{W}_q$ minimal model and the cosmological constant on the boundary. The relation (4.44) following from the subsequent analysis of the one-point function of tachyon operators (4.6) revealed that on the disk, we can replace any boundary state with a sum over boundary states with $\lambda = \rho$ and complex values of the boundary cosmological constant, lending evidence to the fact that the tensor product (4.28) overcounts physically distinct boundary conditions. This generalises the observation made in [29] for the minimal string, to which our results reduce for $q = 2$. Moreover, in conjunction with the factorisation of matter and Liouville contributions, this provided an immediate derivation of a series of functional difference equations (4.46) obeyed by the tachyon one-point functions.

Our results are also of relevance to the holographic description of higher-spin gravity with negative cosmological constant alluded to in the introduction. In particular, the CFT on the disk D defines the holographic dual of a 3-manifold $\mathscr{M}_3$ with $\partial\mathscr{M}_3 = D \cup \mathscr{M}_2$, where $\partial D = \partial\mathscr{M}_2$ [113]. Notably, this construction has been invoked in [114] for a proposal of local observables on $\mathscr{M}_3$. Unlike [113], where the usual Dirichlet boundary condition is imposed on the metric on D, here we impose Neumann boundary conditions on both D and $\mathscr{M}_2$. An important consequence of this modification is the emergence of the relation (4.44), rendering boundary conditions corresponding to excited matter states semiclassically indistinguishable from a quantum superposition of boundary conditions corresponding to the matter ground state.

A central question raised by this analysis is whether the degeneracy implied by (4.44) persists in more complicated amplitudes and thus holds on the entire physical Hilbert space, as conjectured for $q = 2$ in [29] and subsequently challenged in [31, 32]: Here we have only considered one-point functions of tachyons on the disk; to see if the identification of boundary states holds generally on the Hilbert space requires more work. This leads us to the investigation in the following chapter: there we will consider the case $q = 2$ and show how this degeneracy is lifted upon inclusion of ‘infinite-genus’ worldsheets, or more precisely, effects contributing non-perturbatively in the string coupling constant.

Chapter 5

Wronskians, Duality and Cardy Branes

5.1 Overview

Here we consider the double scaling limit of the ensemble (2.4) with $q = 2$, where V_1 and V_2 are polynomials of degree p and p' respectively. As discussed in the introduction, Section 2.2, the universality classes of the critical points in the phase diagram spanned by the coefficients of V_1 and V_2 are labelled by pairs (p, p') of coprime integers and are described by Liouville theory coupled to a Virasoro minimal model [19, 115, 18]. In Subsection 2.2.2, we saw that the conformally invariant boundary states of Liouville theory fall into two classes: the discrete set of Zamolodchikov-Zamolodchikov (ZZ) [28] branes, and the Fateev-Zamolodchikov-Zamolodchikov-Teschner [26, 28, 56] (FZZT) brane $|\sigma\rangle_{\text{FZZT}}$ defined in (2.75). Their tensor product with the Cardy boundary states $|r,s\rangle_{\text{C}}$ of the minimal model yields the complete brane spectrum of the theory. As Seiberg and Shih pointed out [29], the resulting set of $(p-1)(p'-1)/2$ distinct FZZT branes – one per primary field of the minimal model – appears to be at odds with the merely two obvious boundary conditions that can be imposed in the matrix model description, corresponding to the resolvents of the matrices X_1 and X_2, which compute the partition function of a worldsheet with a single connected boundary. The solution to this paradox put forward in [29] is based on the conjecture that all boundary states can be written as superpositions of a single boundary state with analytically continued values of the boundary cosmological constant (2.73):

$$|\sigma\rangle_{\text{FZZT}} \otimes |r,s\rangle_{\text{C}} = \sum_{m=-(r-1)}^{r-1} \sum_{n=-(s-1)}^{s-1} |\sigma + \mathrm{i}m/b + \mathrm{i}nb\rangle_{\text{FZZT}} \otimes |1,1\rangle_{\text{C}} \,. \tag{5.1}$$

In the above, $b^2 = p/p'$ and we increment m and n in steps of 2. Indeed, the relation (4.48) for the one-point functions on the disk derived in the previous chapter is consistent with this proposal; see also [112, 116, 117, 118, 119] for an extensive amount of evidence. It was hence concluded that the resolvents of the matrices X_1 and X_2 suffice to capture all boundary conditions and there is no contradiction. Later, the $X_1 + X_2$-resolvent was computed directly from the matrix model in [31, 32] for the unitary $(p, p+1)$ series of critical points and found to describe the $(r,s) = (1, p-1)$ boundary condition, where the validity of (5.1) was challenged for worldsheets of non-planar topology. However, the lack of an independent construction of the complete brane spectrum in the matrix model has until now obstructed attempts at a satisfactory solution of these debates.

Here we point out that generally, the analytic continuation of an asymptotic expansion need not coincide with the asymptotic expansion of the analytically continued function, which is the well-known Stokes' phenomenon [120, 121], and we are led to wonder about the fate of this observation beyond perturbation theory in the string coupling g_s. Indeed, in the operator formalism, the non-perturbative differential equations (2.91) and (2.92) allow complete sets of p and p' *independent* solutions for the Baker-Akhiezer functions, respectively, only *one* of which describes the double-scaling limit of the expectation value of the resolvent of X_1 resp. X_2. It was later discovered that the remaining independent solutions in fact provide a consistent set of boundary conditions for normal matrices with eigenvalues supported on appropriate arcs away from the real axis [122, 123], suggesting their relevance for Stokes' phenomenon displayed by the resolvent operator. This motivates our study of the Wronskian for the non-perturbative linear differential equations (2.91) and (2.92). The main purpose of this investigation is to answer the questions

1. What differential equations does the Wronskian satisfy?

2. Which new observables are captured by the Wronskian?

The remainder of this chapter is organised as follows: In Section 5.2, we will detail the reasoning for considering the Wronskian associated with the system of differential equations (2.91). Employing a mild generalisation thereof[1] dating back to Schmidt [128], we subsequently answer the first question. We then use the results to turn to the second question, providing evidence for the conjecture that the set of independent Wronskians is organised in a Kac table whose entries are in one-to-one correspondence with the primary fields of the minimal model. In Section 5.3, we then show how this table reproduces the relation (5.1) in the semiclassical limit. Together, these observations strongly suggest that the Wronskian provides a non-perturbative description of the general FZZT brane with $(r,s) \neq (1,1)$, and that the degeneracy (5.1) is resolved by additional degrees of freedom whose independence is invisible in perturbation theory in g_s. We close with a discussion of results and possible further developments in Section 5.4.

5.2 Generalised Wronskian

This section proceeds as follows: In Subsection 5.2.1, we explain how the relation (5.1) hints at the Wronskian associated with the linear differential equation for the Baker-Akhiezer function. In Subsection 5.2.2, we derive the analogues of (2.91) and (2.92), allowing us to introduce an isomonodromy system akin to (2.99a), each of which defines a spectral curve. In Subsection 5.2.3, we use the properties of the duality transformation $(p,p') \to (p',p)$ to determine the complete set of observables defined by the Wronskian; the fact that a Kac table for the latter emerges directly from the matrix model without reference to the

[1] see also [124, 125, 126, 127] for recent work.

worldsheet conformal field theory provides the first piece of evidence that our construction provides a non-perturbative description of the FZZT branes with general Cardy labels $(r,s) \neq (1,1)$, which we will refer to as *Cardy branes* for short.

5.2.1 Wronskians and Cardy branes

Recall from the introduction that the single-trace operator $\mathrm{tr}(x-X)$ creates a connected boundary in the worldsheet corresponding to the state $|\sigma\rangle_{\mathrm{FZZT}} \otimes |1,1\rangle_{\mathrm{C}}$, and the determinant operator $\det(x-X)$ creates the associated brane at target space position x. From the expansion (2.82), it can be seen that a linear combination of such boundary states as in (5.1) indicates that the matrix model operator corresponding to the general *Cardy brane* $|\sigma\rangle_{\mathrm{FZZT}} \otimes |r,s\rangle_{\mathrm{C}}$ factorises into a product of more elementary operators. Indeed, such a relation is expected from Polchinski's general combinatorial picture applied to the present context [45, 129]. It was proven by Morozov in [65] that the average of a general product of characteristic polynomials

$$
\begin{aligned}
\alpha_n^{(M)}(x_1,x_2,\dots x_M) &= \left\langle \prod_{k=1}^{M} \det(x_k - X) \right\rangle_{n\times n} \\
\beta_n^{(M)}(y_1,y_2,\dots y_M) &= \left\langle \prod_{k=1}^{M} \det(y_k - Y) \right\rangle_{n\times n}
\end{aligned}
\tag{5.2}
$$

can be written in terms of the orthogonal polynomials $\{\alpha_n\}_{n=1}^{N}$ defined in (2.83) as

$$
\alpha_n^{(M)}(x_1,x_2,\dots x_M) = \frac{\det_{1\le k,l\le M} \alpha_{n+1-k}(x_l)}{\det_{1\le k,l\le M} x_k^{l-1}} \ , \tag{5.3}
$$

and similarly for $\beta_n^{(M)}$. As shown in [130], in the double-scaling limit, equation (5.3) can be written in terms of the alternating polynomial $a_\lambda(z) = \det_{1\le a,b\le n} z_b^{a-1+\lambda_a}$,

$$
\left\langle \prod_{k=1}^{M} \det(x_k - X) \right\rangle_{n\times n} = \frac{a_\varnothing(\partial)}{a_\varnothing(\zeta)} \prod_{k=1}^{M} \psi^{(1)}(t;\zeta_k) \quad \text{as } N\to\infty\ ,\ \varepsilon\to 0\ , \tag{5.4}
$$

with $\zeta_k = (x_k - x_c)/\varepsilon$ finite. In the above, $\partial_i \psi^{(j)} = \partial_t \psi^{(j)} \delta_{ij}$.

To proceed, we make the further observation that the translations $\sigma \to \sigma + \mathrm{i}m/b + \mathrm{i}nb$ in (5.1) move between sheets of the spectral curve defined by the zero locus of the semiclassical limit of the polynomials (2.100a) – a fact which has been widely discussed, including [112, 116, 117, 118, 119]. This semiclassical spectral curve provides the initial data for the topological recursion algorithm described in [66], which computes the asymptotic expansion of arbitrary correlation function to any finite order in g_s – it thus appears that to all orders in the perturbative expansion, all p branches $\{\psi^{(j)}(t;\zeta)\}_{j=1}^{p}$ of the solution can indeed be obtained from a single principal branch $\psi^{(1)}(t;\zeta)$ by mere analytic continuation $\zeta \to \mathrm{e}^{2\pi\mathrm{i}}\zeta$. This indicates that by performing the asymptotic expansion, we lose the information required to distinguish one solution from the other. We therefore generalise the expression (5.4) to account for the complete set of independent solutions $\{\psi^{(j)}(t;\zeta)\}_{j=1}^{p}$ resp. $\{\chi^{(j)}(t;\zeta)\}_{j=1}^{p'}$ to (2.91) resp. (2.92),

$$
\begin{aligned}
W_{\varnothing}^{(M)}[\psi](t;\zeta) &= a_{\varnothing}(\partial) \prod_{k=1}^{M} \psi^{(j_k)}(t;\zeta_k)\Big|_{\zeta_1=\zeta_2=\ldots\zeta_M=\zeta}, \\
W_{\varnothing}^{(M)}[\chi](t;\eta) &= a_{\varnothing}(\partial) \prod_{k=1}^{M} \chi^{(j_k)}(t;\eta_k)\Big|_{\eta_1=\eta_2=\ldots\eta_M=\eta}.
\end{aligned}
\tag{5.5}
$$

We may regard the above as the antisymmetrised ground state wave function of M coincident branes and their duals. Notably, due to the fermionic statistics, the existence of such "brane stacks" requires the presence of an additional quantum number – the label j distinguishing the independent solutions.

5.2.2 Differential equations and spectral curve

Here we derive the differential equations satisfied by the observables (5.5). We also introduce the corresponding spectral curves and define an extension of the charge conjugation, which will turn out useful when we consider the duality transformation (2.97) in the next subsection. To this end, we need to keep track of derivatives of $W_{\varnothing}^{(n)}(t;\zeta)$ with respect to the spectral parameters ζ_k in (5.5). This is conveniently achieved by the following

Definition 5.2.1. Denote the set of Young diagrams with n rows by Λ_n and the subset of $\binom{p}{n}$ diagrams with at most $p-n$ boxes in each row by $\Lambda_{p,n}$; denote the number of boxes in the a^{th} row of λ by λ_a, and call $|\lambda| = \sum_{a=1}^{n} \lambda_a$ the *size* of the diagram λ.

Definition 5.2.2. Let $\{\psi^{(j)}\}_{j=1}^{p}$ denote the p solutions to (2.91). Given $\lambda \in \Lambda_n$, we define the *generalised Wronskian*

$$W_{\lambda}^{(n)}(t;\zeta) = \det_{1\leq a,b\leq n} \left(\partial_t^{n-a+\lambda_{n-a+1}} \psi^{(j_b)}(t;\zeta_b) \right) \Big|_{\zeta_1=\zeta_2=\ldots\zeta_n=\zeta} . \tag{5.6}$$

For notational simplicity we keep the dependence on $\{j_b\}_{b=1}^{n}$ implicit. Note that from the properties of the determinant it follows immediately that $\partial_t W_{\lambda}^{(n)}(t;\zeta) = 0$ for $n \geq p$. We note another useful representation of $W_{\lambda}^{(n)}$ also reported in [126, 127]:

Lemma 5.2.3. *Let $S_{[\lambda_1,\lambda_2,\ldots\lambda_n]}(z) \equiv S_{\lambda}(z)$ denote the Schur polynomial in n variables $\{z_k\}_{k=1}^{n}$. Then the generalised Wronskian can be expressed as*

$$W_{\lambda}^{(n)}(t;\zeta) = S_{\lambda}(\partial) W_{\varnothing}^{(n)}(t;\zeta) , \quad \partial_k = \frac{1}{k}\sum_{i=1}^{n} \partial_{(i)}^{k} , \tag{5.7}$$

where $\partial_{(i)}\psi^{(j)} = \partial_t \psi^{(j)}\delta_{ij}$ and $\varnothing$ denotes the diagram with $\lambda_a = 0$ $\forall a$.

Proof. It suffices to note that $W_{\lambda}^{(n)}$ can be expressed in terms of the alternating polynomial $a_{\lambda}(\partial) = \det_{1\leq a,b\leq n} \partial_{(b)}^{a-1+\lambda_a}$ in the derivatives $\partial_{(i)}\psi^{(j)} = \delta_{ij}\partial_t\psi^{(j)}$:

$$\begin{aligned} W_{\lambda}^{(n)}(t;\zeta) &= (-1)^{n(n-1)/2} \det_{1\leq a,b\leq n} \left(\partial_{(j_b)}^{\lambda_a+a-1} \right) \prod_{k=1}^{n} \psi^{(j_k)}(t;\zeta) \\ &= a_{\lambda}(\partial) \prod_{k=1}^{n} \psi^{(j_k)}(t;\zeta) . \end{aligned} \tag{5.8}$$

Using the defining relation $S_{\lambda}(z) = a_{\lambda}(z)/a_{\varnothing}(z)$ then proves the statement. □

In light of the discussion in the preceeding subsection, we may think of $W_{\lambda\neq\varnothing}^{(n)}$ as the excited state created by the operator $S_{\lambda}(\partial)$ acting on the ground state (5.5). We can now state the differential equations satisfied by $W_\lambda^{(n)}$ by analogy with (2.91):

Proposition 5.2.4. *The functions $W_\lambda^{(n)}(t;\zeta)$ satisfy*

$$\zeta W_\lambda^{(n)}(t;\zeta) = \mathbb{P}_{\lambda,j}^{(n)}(t;\partial) W_\varnothing^{(n)}(t;\zeta)\ , \quad j=1,2,\dots n\ , \tag{5.9a}$$

$$\partial_\zeta W_\lambda^{(n)}(t;\zeta) = \beta_{p,p'} \sum_{j=1}^{n} \mathbb{Q}_{\lambda,j}^{(n)}(t;\partial) W_\varnothing^{(n)}(t;\zeta)\ , \tag{5.9b}$$

with the p^{th} and $(p')^{\text{th}}$ order differential operators

$$\begin{aligned} \mathbb{P}_{\lambda,j}^{(n)}(t;\partial) &= 2^{p-1} S_{[\lambda_1,\dots,\lambda_j+p,\dots,\lambda_n]}(\partial) + \sum_{m=2}^{p+\ell_j} U_m^{(\ell_j)}(t) S_{[\lambda_1,\dots,\lambda_j+p-m,\dots,\lambda_n]}(\partial)\ , \\ \mathbb{Q}_{\lambda,j}^{(n)}(t;\partial) &= 2^{p'-1} S_{[\lambda_1,\dots,\lambda_j+p',\dots,\lambda_n]}(\partial) + \sum_{m=2}^{p'+\ell_j} V_m^{(\ell_j)}(t) S_{[\lambda_1,\dots,\lambda_j+p'-m,\dots,\lambda_n]}(\partial)\ , \end{aligned} \tag{5.10}$$

where $\ell_j = \lambda_j + j - 1$ is the hook length of the first box in each row and

$$U_m^{(\ell)}(t) = \sum_{k=0}^{\min[m-2,\ell]} \binom{\ell}{k} \left(\partial_t^k u_{m-k}(t)\right)\ , \quad V_m^{(\ell)}(t) = \sum_{k=0}^{\min[m-2,\ell]} \binom{\ell}{k} \left(\partial_t^k v_{m-k}(t)\right)\ . \tag{5.11}$$

Proof. We first demonstrate (5.9a). Using (2.91a) and expanding $\partial_t^n f(t) = \sum_{k=0}^n \binom{n}{k} (\partial_t^k f(t)) \partial^{n-k}$, we can express

$$\begin{aligned} 2^{p-1} S_{[\lambda_1,\dots,\lambda_n+p]} &= -\sum_{m=2}^{p} \sum_{k=0}^{\ell_n} \binom{\ell_n}{k} \left[\left(\partial_t^k u_m(t)\right) - \delta_{mp}\delta_{k0}\zeta\right] S_{[\lambda_1,\dots,\lambda_n+p-m-k]}(\partial) \\ &= -\sum_{m=2}^{p+\ell_n} \sum_{k=0}^{\min[m-2,\ell]} \binom{\ell_n}{k} \left[\left(\partial_t^k u_{m-k}(t)\right) - \delta_{mp}\delta_{k0}\zeta\right] S_{[\lambda_1,\dots,\lambda_n+p-m]}(\partial)\ , \end{aligned} \tag{5.12}$$

with $u_{n<0}(t) = v_{n<0}(t) = 0$, which implies (5.9a) for $j=n$. For $j \neq n$, we first use

$$\begin{aligned}S_{[\lambda_1,\dots\lambda_j,\lambda_{j+1},\dots\lambda_n]}(\partial) &= -S_{[\lambda_1,\dots\lambda_{j+1}+1,\lambda_j-1,\dots\lambda_n]}(\partial) \\ &= (-1)^{n-j}S_{[\lambda_1,\dots,\lambda_{j+1}+1,\lambda_{j+2}+1\dots\lambda_n,\lambda_j-n+j]}(\partial)\end{aligned} \tag{5.13}$$

and then apply the previous result to $S_{[\lambda_1,\dots\lambda_j+p\dots\lambda_n]}(\partial)$. Equation (5.9b) can be obtained by explicit evaluation of the derivative of $W^{(n)}_\lambda$ w.r.t. ζ, which gives a sum of n terms, one for the action of ∂_ζ on each row of the matrix $\partial_t^{a-1+\lambda_a}\psi^{(j_b)}$. In each term, we may use (2.91b) and subsequently commute the derivatives to the right of $v_n(t)$ using the same procedure as for (5.9a), which immediately yields (5.9b). □

Corollary 5.2.5. $\mathscr{M}_{p,n} = (\mathrm{span}\{W^{(n)}_\lambda\}_{\lambda\in\Lambda_{p,n}},+)$ *is a module of the ring* $\mathscr{R}_n$ *of symmetric polynomials in n variables over* $\mathbb{R}$*.*

Proof. Note first that every $r \in \mathscr{R}_n$ can be expanded in Schur polynomials. Hence Lemma 5.2.3 provides a map $\mathscr{R}_n \times \mathscr{M}_{p,n} \longrightarrow \mathscr{M}_{p,n}$,

$$S_\lambda(\partial)W^{(n)}_\mu(t;\zeta) = \sum_{\nu\in\Lambda_{p,n}} f^{(n)\nu}_{\lambda\mu}(t;\zeta)W^{(n)}_\nu(t;\zeta)\ , \quad \lambda,\mu\in\Lambda_{p,n}\ , \tag{5.14}$$

where the $f^{(n)\nu}_{\lambda\mu}(t;\zeta)$ are determined by the Littlewood-Richardson rule

$$S_\lambda S_\mu = \sum_{\substack{|\nu|=|\lambda|+|\mu| \\ \nu\in\Lambda_n}} c^\nu_{\lambda\mu}S_\nu\ . \tag{5.15}$$

Whenever $\nu_a > p-n$ for some a on the right-hand side, we apply Proposition 5.2.4 repeatedly to obtain a linear combination of S_λ with $\lambda\in\Lambda_{p,n}$. Since $\mathscr{M}_{p,n}$ is an abelian group under addition, it is an $\mathscr{R}_n$-module. □

We now look for a suitable generalisation of the isomonodromy description (2.99a) and the spectral curve (2.100a). To this end, one first chooses an ordering on $\Lambda_{p,n}$, for example

$$\lambda < \lambda' \quad \Leftrightarrow \quad |\lambda| < |\lambda'| \quad \text{or} \quad \sum_{a=0}^{n-1} \lambda_{n-a}(p-n)^a < \sum_{a=0}^{n-1} \lambda'_{n-a}(p-n)^a \,. \tag{5.16}$$

We then have the following result:

Proposition 5.2.6. *Let* $\vec{W}^{(n)}(t;\zeta) = (W^{(n)}_{\varnothing}, W^{(n)}_{\square}, \ldots)^T$ *be the* $\binom{p}{n}$*-vector with entries ordered according to* (5.16)*. Then there exist* $\binom{p}{n} \times \binom{p}{n}$ *matrices* $\mathscr{B}^{(n)}(t;\zeta)$ *and* $\mathscr{Q}^{(n)}(t;\zeta)$ *such that*

$$\begin{aligned} \partial_t \vec{W}^{(n)}(t;\zeta) &= \mathscr{B}^{(n)}(t;\zeta)\vec{W}^{(n)}(t;\zeta) \,, && (5.17a) \\ \partial_\zeta \vec{W}^{(n)}(t;\zeta) &= \mathscr{Q}^{(n)}(t;\zeta)\vec{W}^{(n)}(t;\zeta) \,. && (5.17b) \end{aligned}$$

Proof. Since $\partial_t = S_{\square}(\partial)$, we can use Corollary 5.2.5 to find $\mathscr{B}^{(n)}$:

$$\partial_t W^{(n)}_{\lambda} = \sum_{\mu \in \Lambda_{p,n}} f^{(n)\mu}_{\square,\lambda} W^{(n)}_{\mu} \,, \quad \lambda, \mu \in \Lambda_{p,n} \quad \Rightarrow \quad (\mathscr{B}^{(n)})^{\mu}_{\lambda} = f^{(n)\mu}_{\square,\lambda} \,.$$

To show the existence of $\mathscr{Q}^{(n)}$, we use first use (5.9b) to expand the right hand side in Schur polynomials and thereafter apply Corollary 5.2.5. □

Definition 5.2.7. We introduce the characteristic polynomials

$$\begin{aligned} F^{(n)}(t;\zeta,z) &= \det\left(z\mathbb{I}_{\binom{p}{n}\times\binom{p}{n}} - \mathscr{B}^{(n)}(t;\zeta) \right) \,, \\ G^{(n)}(t;\zeta,Q) &= \det\left(Q\mathbb{I}_{\binom{p}{n}\times\binom{p}{n}} - \mathscr{Q}^{(n)}(t;\zeta) \right) \,, \end{aligned} \tag{5.18}$$

and define the *spectral curve* of the system (2.91),

$$\mathscr{C}^{(n)}_{p,p'}(t) = \{(P,Q) \in \mathbb{C}^2 | G^{(n)}(t;P,Q) = 0\} \,. \tag{5.19}$$

Observe that for given n, the spectral curves for $\mathscr{C}^{(n)}_{p,p'}(t)$ and $\mathscr{C}^{(p-n)}_{p,p'}(t)$ are of the same degree. To pave the way for a definition of the duality transformation, it is useful to relate these two systems by extending the definition of the charge conjugation (2.94) as follows: For given $\lambda \in \Lambda_{p,n}$, we define complement $\lambda^{\perp}$ and conjugate $\lambda^{\vee}$ via

$$\lambda_a^{\perp} = (p-n) - \lambda_{n-a+1} \ , \qquad a = 1, \dots n \tag{5.20a}$$

$$\lambda_a^{\vee} = \max_{1 \leq b \leq r} \{ b | \lambda_{r-b+1} \geq a \} \ , \qquad a = 1, \dots p-n \tag{5.20b}$$

and make the following

Definition 5.2.8. We define the charge conjugation

$$\begin{aligned} \mathscr{C} : \mathscr{M}_{p,n} &\longrightarrow \mathscr{M}_{p,p-n} \ , \\ W_\lambda^{(n)} &\longmapsto \mathscr{C}[W_\lambda^{(n)}] = W_{\mathscr{C}(\lambda)}^{(p-n)} \ , \end{aligned} \tag{5.21}$$

where $\mathscr{C}(\lambda) = (-1)^{|\lambda|}(\lambda^{\perp})^{\vee}$.

We close this subsection with a few examples illustrating the above construction; to facilitate the presentation, we relegate the explicit equations for the Lax operators to Appendix C. These immediately determine the full non-perturbative spectral curve via Definition 5.2.7. Below, we print the corresponding polynomials (5.18) in the semiclassical limit $g_s \to 0$. For later comparison with the conformal field theory prediction (5.1), we evaluate this limit in the conformal background, in which (2.101) holds.

Example 5.2.9. $(p, p') = (3, 2)$. *The only allowed cases* $n = 1, 2$ *are equivalent to the* 3×3 *Lax systems discussed in [131] and references therein. In the bases*

$$\vec{W}^{(1)}(t;\zeta) = \left(W_{\varnothing}^{(1)}, W_{\square}^{(1)}, W_{\square\square}^{(1)} \right)^T \ , \quad \vec{W}^{(2)}(t;\zeta) = \left(W_{\varnothing}^{(2)}, W_{\square}^{(2)}, W_{\substack{\square \\ \square}}^{(2)} \right)^T \ , \tag{5.22}$$

the Lax operators satisfy $\mathscr{B}^{(2)} = -\mathscr{C}^{-1}\left(\mathscr{B}^{(1)}\right)^T \mathscr{C}$ *and* $\mathscr{Q}^{(2)} = -\mathscr{C}^{-1}\left(\mathscr{Q}^{(1)}\right)^T \mathscr{C}$*, where the charge conjugation matrix* $\mathscr{C} : \mathscr{M}_{3,1} \longrightarrow \mathscr{M}_{3,2}$ *has components* $\mathscr{C}_{ab} = (-1)^{a+1}\delta_{4-a,b}$ *and* $\mathscr{B}^{(n)}$ *and* $\mathscr{Q}^{(n)}$ *are given in Appendix C, Example C.0.4. The eigenvalues of* $\mathscr{B}^{(n)}$, $\mathscr{Q}^{(n)}$ *are given by the zeroes of*

$$\begin{aligned} \pm F^{(n)}(t;\zeta,z) &= \pm z^3 - \frac{\zeta}{4} \pm \frac{3v_2}{4} z + \frac{3\dot{v}_2}{8} \, , \\ \pm G^{(n)}(t;\zeta,Q) &= \pm Q^3 - \frac{\zeta^2}{2} \mp Q\left(\frac{3v_2^2}{4} + \frac{\ddot{v}_2}{2}\right) - \frac{v_2^3}{4} - \frac{v_2\ddot{v}_2}{2} + \frac{\dot{v}_2^2}{8} \, , \end{aligned} \tag{5.23}$$

where the upper (resp. lower) sign holds for $n = 1$ *(resp.* $n = 2$*), in agreement with [131].*

Example 5.2.10. $(p, p') = (4, 3)$*. For* $n = 1$ *and* $n = 3$*, we again recover the familiar* 4×4 *Lax systems discussed in [131]. On the other hand, in the nontrivial case* $n = 2$ *the system* (5.17) *is 6-dimensional. In the basis*

$$\vec{W}^{(2)}(t;\zeta) = \left(W^{(2)}_{\varnothing}, W^{(2)}_{\ydiagram{1}}, W^{(2)}_{\ydiagram{2}}, W^{(2)}_{\ydiagram{1,1}}, W^{(2)}_{\ydiagram{2,1}}, W^{(2)}_{\ydiagram{2,2}}\right)^T \, , \tag{5.24}$$

the charge conjugation matrix $\mathscr{C} : \mathscr{M}_{4,2} \longrightarrow \mathscr{M}_{4,2}$ *is given by*

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, . \tag{5.25}$$

We find $\mathscr{B}^{(2)} = -\mathscr{C}^{-1}\left(\mathscr{B}^{(2)}\right)^T \mathscr{C} + \mathscr{O}(g_s)$*,* $\mathscr{Q}^{(2)} = -\mathscr{C}^{-1}\left(\mathscr{Q}^{(3)}\right)^T \mathscr{C}$*, with* $\mathscr{B}^{(2)}$ *and* $\mathscr{Q}^{(2)}$ *given in Appendix C, Example C.0.5. Now consider the limit* $g_s \to 0$*. In the conformal background,* $v_2(t) \to -1$ *and* $u_4(t) \to 1$*. The eigenvalues of* $\mathscr{B}^{(n)}$, $\mathscr{Q}^{(n)}$ *are then given by the zeroes of*

$$\begin{aligned} F^{(2)}(t;\zeta,z) &= z^6 - \frac{2}{3}z^4 + \frac{1}{2}z^2\zeta - \frac{7}{18}z^2 \, , \\ G^{(2)}(t;\zeta,Q) &= Q^6 - \frac{2}{27}Q^4 + 2Q^2\zeta^3 - \frac{16}{3}Q^2\zeta^2 + \frac{85}{18}Q^2\zeta - \frac{2023}{1458}Q^2. \end{aligned} \tag{5.26}$$

Example 5.2.11. $(p,p')=(5,2)$. *For the nontrivial cases $n=2,3$, we pick a basis*

$$\begin{aligned}\vec{W}^{(2)}(t;\zeta) &= \left(W^{(2)}_{\varnothing}, W^{(2)}_{\ydiagram{1}}, W^{(2)}_{\ydiagram{2}}, W^{(2)}_{\ydiagram{1,1}}, W^{(2)}_{\ydiagram{3}}, W^{(2)}_{\ydiagram{2,1}}, W^{(2)}_{\ydiagram{3,1}}, W^{(2)}_{\ydiagram{2,2}}, W^{(2)}_{\ydiagram{3,2}}, W^{(2)}_{\ydiagram{3,3}}\right)^T , \\ \vec{W}^{(3)}(t;\zeta) &= \left(W^{(3)}_{\varnothing}, W^{(3)}_{\ydiagram{1}}, W^{(3)}_{\ydiagram{1,1}}, W^{(3)}_{\ydiagram{2}}, W^{(3)}_{\ydiagram{1,1,1}}, W^{(3)}_{\ydiagram{2,1}}, W^{(3)}_{\ydiagram{2,1,1}}, W^{(3)}_{\ydiagram{2,2}}, W^{(3)}_{\ydiagram{2,2,1}}, W^{(3)}_{\ydiagram{2,2,2}}\right)^T ,\end{aligned} \tag{5.27}$$

in which the charge conjugation matrix $\mathscr{C}:\mathscr{M}_{5,3}\longrightarrow\mathscr{M}_{5,2}$ is given by

$$\mathscr{C}=\begin{pmatrix} 0 & \dots & & & \mathrm{i}\sigma_2 \\ \vdots & & & \mathbb{I}_{2\times 2} & \\ & & -\mathbb{I}_{2\times 2} & & \\ & \mathbb{I}_{2\times 2} & & & \vdots \\ -\mathrm{i}\sigma_2 & & & \dots & 0 \end{pmatrix} , \tag{5.28}$$

where σ_2 denotes the 2^{nd} Pauli matrix. We have $\mathscr{B}^{(2)}=-\mathscr{C}^{-1}\left(\mathscr{B}^{(3)}\right)^T\mathscr{C}+\mathscr{O}(g_s)$, $\mathscr{Q}^{(2)}=-\mathscr{C}^{-1}\left(\mathscr{Q}^{(3)}\right)^T\mathscr{C}$, with $\mathscr{B}^{(2)}$ and $\mathscr{Q}^{(2)}$ given in Appendix C, Example C.0.6. Now consider the limit $g_s\to 0$. In the conformal background, and $v_2(t)\to 1$, $u_4(t)\to -5/2$ and $u_3(t)$ and $u_5(t)$ both vanish. The eigenvalues of $\mathscr{B}^{(n)}$, $\mathscr{Q}^{(n)}$ are given by the zeroes of

$$\begin{aligned}F^{(n)}(t;\zeta,z) &= \frac{1}{256}(256z^{10}-960z^8+960z^6\pm 176z^5\zeta-300z^4+25z^2\pm 5z\zeta-\zeta^2), \\ G^{(n)}(t;\zeta,Q) &= Q^{10}-\frac{15}{4}Q^8+\frac{15}{4}Q^6\pm Q^5\left(\frac{11}{8}\zeta^2-\frac{11}{16}\right)-\frac{75}{64}Q^4 \\ &\quad \pm Q^3\left(\frac{5}{16}-\frac{5}{8}\zeta^2\right)+\frac{25}{256}Q^2\pm Q\left(\frac{5}{128}-\frac{5}{256}\right) ,\end{aligned} \tag{5.29}$$

where the upper (resp. lower) sign holds for $n=2$ (resp. $n=3$).

5.2.3 Kac table and duality

In this section, we provide evidence that the set of differential equations that characterise the generalised Wronskians for given (p,p') fall into a Kac table whose entries are in one-to-one correspondence with the $(p-1)(p'-1)/2$ Cardy states of the $(A_{p-1}, A_{p'-1})$ minimal model. To keep track of which Baker-Akhiezer function we take the Wronskian of, we refine our notation as follows:

$$W_{\lambda}^{(n)}[f](\cdot) = \det_{1\leq a,b\leq n}\left(\partial_t^{n-a+\lambda_{n+a-1}} f^{(j_b)}(t;\cdot)\right) . \tag{5.30}$$

Moreover, because we are only interested in relations between entire modules $\{\mathscr{M}_{p,n}\}_{n=1}^{p}$ whose elements satisfy the same differential equations, we shall at times omit the subscript λ in $W_{\lambda}^{(n)}$. Our derivation is based on the property of the Virasoro minimal model that the duality transformation (2.97) takes the boundary state $|r,s\rangle_{\mathrm{C}}$ of the (p,p') model to the equivalent state $|s,r\rangle_{\mathrm{C}}$ of the (p',p) model: Under the assumption that the composition of the Laplace transform (2.96) with the charge conjugation (5.21) extends this duality transformation non-perturbatively, we can fill the entries on the boundary of the Kac table shown to the left of Figure 5.1 via

$$\begin{aligned}
\Psi^{(r,1)}(\zeta) &= W^{(r)}[\psi](\zeta) , & \Psi^{(r,p'-1)}(\zeta) &= W^{(r)}[\mathscr{L}\chi](\zeta) , \\
\Psi^{(1,s)}(\zeta) &= \mathscr{L}W^{(p'-s)}[\chi](\zeta) , & \Psi^{(p-1,s)}(\zeta) &= \mathscr{L}W^{(p'-s)}[\mathscr{L}\psi](\zeta) .
\end{aligned} \tag{5.31}$$

Introducing the dual wave functions $\widetilde{\Psi}^{(s,r)}(\eta) = \mathscr{L}\mathscr{C}[\Psi^{(r,s)}](\eta)$, we can complete the entries on the boundary of the image of the Kac table under the duality transformation shown to the right of Figure 5.1 in the analogous manner:

$$\begin{aligned}
\widetilde{\Psi}^{(1,r)}(\eta) &= \mathscr{L}W^{(p-r)}[\psi](\eta) , & \widetilde{\Psi}^{(p'-1,r)}(\eta) &= \mathscr{L}W^{(p-r)}[\mathscr{L}\chi](\eta) , \\
\widetilde{\Psi}^{(s,1)}(\eta) &= W^{(s)}[\chi](\eta) , & \widetilde{\Psi}^{(s,p-1)}(\eta) &= W^{(s)}[\mathscr{L}\psi](\eta) .
\end{aligned} \tag{5.32}$$

Let us discuss some evidence in favour of this proposal. Firstly, note that our extension of the duality transformation $\mathscr{C}\circ\mathscr{L}$ evidently preserves the string equation (2.90) and hence

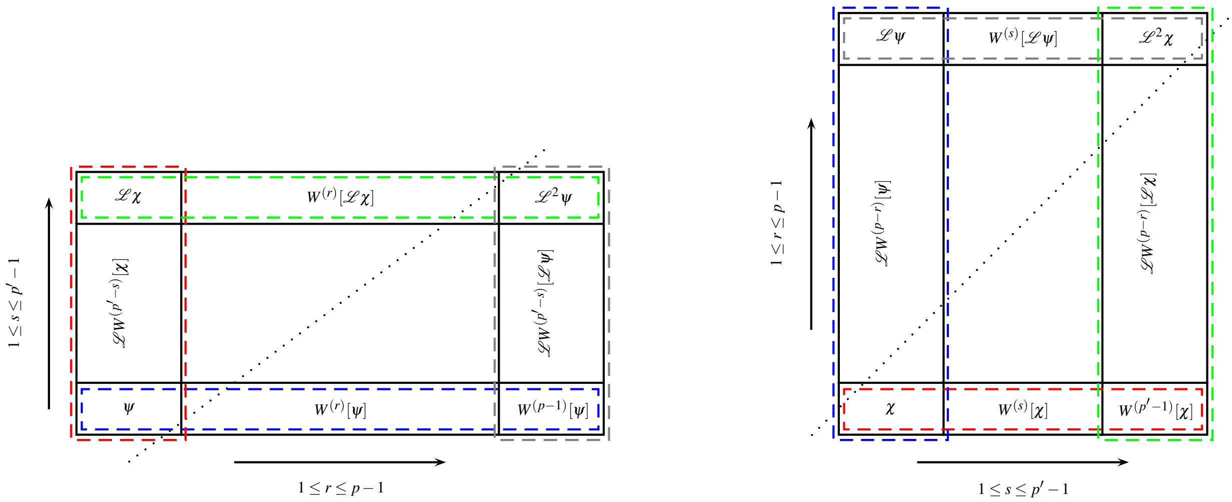


Figure 5.1: Definition of the boundary of the Kac table (left) and its image under the duality transformation (right). Regions of the same color are related by a Laplace transform.

the bulk physics. Secondly, since $W^{(r)}[\psi]$ and $W^{(s)}[\chi]$ are constant for $r \geq p$ resp. $s \geq p'$, the corresponding differential equations are trivial and the table in Figure 5.1 is bounded in the appropriate way. Thirdly, since the $(p-1)^{\text{th}}$-degree Wronskian for a linear differential equation of order p satisfies the transpose of the originial differential equation, we deduce from the definition of the charge conjugation (5.21) that

$$\begin{aligned} \mathscr{L}\mathscr{C}[\chi](\zeta) &= \mathscr{L}W^{(p'-1)}[\chi](\eta) = \psi(\zeta) \,, \\ \mathscr{L}\mathscr{C}[\psi](\eta) &= \mathscr{L}W^{(p-1)}[\psi](\eta) = \chi(\eta) \,, \end{aligned} \tag{5.33}$$

where equality means that the corresponding modules are characterised by the same differential equations; such a relation has also been pointed in [131]. In the same way, it follows that

$$\begin{aligned} \mathscr{C}[\mathscr{L}\chi](\zeta) &= W^{(p-1)}[\mathscr{L}\chi](\zeta) = \psi(-\zeta) \,, \\ \mathscr{C}[\mathscr{L}\psi](\eta) &= W^{(p'-1)}[\mathscr{L}\psi](\eta) = \chi(-\eta) \,. \end{aligned} \tag{5.34}$$

These relations prove the consistency of our proposal at the corner entries of the table. Lastly, a consistency check for other entries on the boundary of the table is provided by a comparison of the spectral curves: after performing a Laplace transform (2.96) and charge conjugation (5.21) on (5.17), we obtain a new isomondromy system of *different* size, with operators $\widetilde{\mathscr{B}}^{(n)}(t;\eta)$ and $\widetilde{\mathscr{Q}}^{(n)}(t;\eta)$, where now $0 < n < p'$. If this system is to provide a dual description of the same brane, we expect the zero locus of their characteristic polynomial to define the same spectral curve, i.e.

$$\widetilde{G}^{(n)}(t;P,Q) \propto G^{(p-n)}(t;Q,P) \ . \tag{5.35}$$

Below we provide a check of the above relation for a few simple examples.

Example 5.2.12. $(p,p') = (3,2)$. *After a Laplace transform, a complete basis is given by* $\vec{\widetilde{W}}^{(n)}(t;\eta) = \left(\mathscr{L}[W_{\varnothing}^{(n)}], \partial_t \mathscr{L}[W_{\varnothing}^{(n)}]\right)$, *with associated* 2×2 *Lax pair*

$$\begin{aligned} \widetilde{\mathscr{B}}^{(2)} = -\mathscr{C}(\widetilde{\mathscr{B}}^{(1)})^T\mathscr{C}^{-1} \ &= \frac{1}{2}\begin{pmatrix} 0 & 2 \\ -v_2-\eta & 0 \end{pmatrix} \ , \\ \widetilde{\mathscr{Q}}^{(2)} = -\mathscr{C}(\widetilde{\mathscr{Q}}^{(1)})^T\mathscr{C}^{-1} \ &= \frac{1}{2}\begin{pmatrix} -\dot{v}_2 & 2v_2+4\eta \\ -(v_2+\eta)(v_2-2\eta)+\ddot{v}_2 & \dot{v}_2 \end{pmatrix} \ . \end{aligned} \tag{5.36}$$

The spectral curve reads

$$\widetilde{G}^{(n)}(t;\eta,P) = \pm 2\eta^3 + P^2 \pm \eta\left(\frac{3v_2^2}{2} + \ddot{v}\right) + \frac{v_2^3}{2} + \ddot{v}_2 v_2 - \frac{\dot{v}_2^2}{4} \ , \tag{5.37}$$

where the upper (resp. lower) sign holds for $n=1$ *(resp.* $n=2$*). Comparison with* (5.23) *shows that* (5.35) *is indeed satisfied.*

Example 5.2.13. $(p,p') = (4,3)$. *After a Laplace transform, a complete basis is given by* $\vec{\widetilde{W}}^{(2)}(t;\eta) = \left(\mathscr{L}[W_{\varnothing}^{(2)}], \mathscr{L}[W_{\square}^{(2)}], \mathscr{L}[W_{\square^2}^{(2)}]\right)$ *as a consequence of* (2.90). *To leading order in* g_s, *the associated* 3×3 *Lax pair is given by*

$$
\begin{aligned}
\widetilde{\mathscr{B}}^{(2)} = -\mathscr{C}(\widetilde{\mathscr{B}}^{(1)})^T\mathscr{C}^{-1} \quad &= \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ -\eta & -v_2 & \end{pmatrix} , \\
\widetilde{\mathscr{Q}}^{(2)} = -\mathscr{C}(\widetilde{\mathscr{Q}}^{(1)})^T\mathscr{C}^{-1} \quad &= \frac{1}{2}\begin{pmatrix} \frac{v_2^2}{9} - \frac{1}{2\eta} & -\eta - \frac{1}{9\eta}v_3^2 & \frac{1}{3}v_2 - \frac{v_2^2}{9\eta} \\ -\frac{1}{6}v_2\eta + \frac{v_2^2}{18} & -\frac{v_2^2}{18} & \frac{2v_3}{9\eta} - \eta \\ \frac{1}{2}\eta^2 + \frac{1}{18}v_3^2 & \frac{v_2}{3}\eta & \frac{v_2^2}{18} \end{pmatrix} .
\end{aligned}
\tag{5.38}
$$

Taking the semiclassical limit and evaluating in the conformal background, the spectral curve simplifies to

$$
\lim_{g_s \to 0} \widetilde{G}^{(2)}(t;\eta,P) = \frac{1}{2}\left[T_3(-P) - T_4(\frac{\eta}{\sqrt{2}})\right] . \tag{5.39}
$$

Comparing the latter to (5.26)*, it follows that* $P^2 \times \widetilde{G}^{(2)}(t;Q,P) \propto G^{(2)}(t;P,Q)$ *and* (5.35) *is indeed satisfied.*

5.3 Semiclassical limit

In the previous section, we saw that the generalised Wronskians allow us to define a set of averages involving independent degrees of freedom that are in one-to-one correspondence with the entries on the boundary of the Kac table. Here we study how this table reproduces the relation (5.1) in the semiclassical limit $g_s \to 0$, providing another piece of evidence in favour of our definition in Section 5.2.3. Inspection of (2.101) reveals that in the conformal background, the p solutions to $\lim_{g_s\to 0} G^{(1)}(t;\zeta,Q) = 0$ can be parametrised as $\zeta(\tau) = \cosh(p\tau)$ and $Q = Q^{(j)}(\tau)$, where

$$
Q^{(j)}(\tau) = \cosh[p'(\tau - 2\pi\mathrm{i}(j-1)/p)] , \quad 1 \le j \le p . \tag{5.40}
$$

In general, the zeroes of $\lim_{g_s \to 0} G^{(n)}(t;\zeta,Q) = G^{(n)}_{\text{cl.}}(\zeta,Q)$ are then obtained from the set of linear combinations

$$Q^{(j_1,j_2,\dots j_n)}(\tau) = \sum_{k=1}^{n} Q^{(j_k)}(\tau) \tag{5.41}$$

on the fundamental domain $\{(j_1,j_2,\dots j_n) \mid 1 \le j_1 < j_2 < \dots < j_n \le p\}$. This is to be compared with the relation (5.1), which involves the following analytic continuations of ζ and η:

$$\zeta_{l,k} = \cosh\left[p(\tau + \mathrm{i}\pi l/p' + \mathrm{i}\pi k/p)\right] \ , \tag{5.42a}$$

$$\eta_{k,l} = \cosh\left[p'(\tau + \mathrm{i}\pi l/p' + \mathrm{i}\pi k/p)\right] \ . \tag{5.42b}$$

Comparing the explicit form of the modular S-matrix of the (p,p') minimal model

$$S_{(r,s)(m,n)} = 2\sqrt{\frac{2}{pp'}}(-1)^{sm+rn+1}\sin(\pi rmp'/p)\sin(\pi snp/p') \tag{5.43}$$

with the relations

$$\sum_{m=-(s-1)}^{s-1} \frac{\zeta_{0,m}}{\zeta} = \frac{\sin(\pi sp/p')}{\sin(\pi p/p')} \ , \qquad \sum_{n=-(r-1)}^{r-1} \frac{\eta_{0,n}}{\eta} = \frac{\sin(\pi rp'/p)}{\sin(\pi p'/p)} \ , \tag{5.44}$$

where m and n are incremented in steps of 2, we conclude that the product of the above two quantities can be written as

$$d_{r,s} = \frac{S_{(r,s)(1,1)}}{S_{(1,1)(1,1)}} \ . \tag{5.45}$$

The numbers $d_{r,s}$ are sometimes called the quantum dimension of the state (r,s); The definition of Cardy states (2.67) gave rise to the interpretation of these numbers as ground state degeneracies, and their logarithm as “boundary entropies” – see also the end of Section 4.3

of the previous chapter. From the latter, it is evident that the branch $j_k = k-(n+1)/2 \bmod p$ gives a description of the $(r,s)=(1,n)$ Cardy brane consistent with the conformal field theory prediction (5.1) when evaluated in the conformal background[2], since

$$Q^{(j_1,j_2,\dots j_n)}(\tau) = d_{1,n} Q^{(0)}(\tau)\ . \tag{5.46}$$

However, because the isomonodromy system (5.17) and (5.17) has size $\binom{p}{n}$, the spectral curve will in general contain other factors besides the above branch. For $(r,s)=(1,2)$, this can be seen in the Examples 5.2.10 and 5.2.11, where the spectral curve factorises as

$$(p,p') = (4,3)\ :\quad G^{(2)}_{\text{cl.}}(\zeta,Q) = \text{const.}\times Q^2\left(T_4\left(\frac{Q}{\sqrt{2}}\right) - T_3(-\zeta)\right)\ , \tag{5.47a}$$

$$(p,p') = (5,2)\ :\quad G^{(2)}_{\text{cl.}}(\zeta,Q) = \text{const.}\times \prod_{\pm}\left(T_5\left(\frac{2}{1\pm\sqrt{5}}Q\right) - T_2(\mp\zeta)\right)\ . \tag{5.47b}$$

We close this section by demonstrating that this observation generalises to arbitrary (p,p') according to the following

Proposition 5.3.1. *Let $T_p(\cosh\tau) = \cosh(p\tau)$ denote the p^{th} Chebyshev polynomial of the first kind. Then up to normalisation, the semiclassical curve for $n=2$ can be written as*

$$G^{(2)}_{\text{cl.}}(\zeta,Q) = \begin{cases} \prod_{a=1}^{(p-1)/2}\left(T_p\left(\frac{Q}{2\cos(\pi p' a/p)}\right) - T_{p'}((-1)^a\zeta)\right)\ , & p \text{ odd}\ , \\ Q^{p/2}\prod_{a=1}^{(p-2)/2}\left(T_p\left(\frac{Q}{2\cos(\pi p' a/p)}\right) - T_{p'}((-1)^a\zeta)\right)\ , & p \text{ even}\ . \end{cases} \tag{5.48}$$

Proof. The zeroes of $G^{(2)}_{\text{cl.}}(\zeta,Q)$ are parametrised by $p'(p'-1)/2$ functions $Q^{(j_1,j_2)}(\tau)$ on the fundamental domain $\{(j_1,j_2)\mid 1\le j_1<j_2\le p\}$. Eliminating j_2 in favour of $a=j_2-j_1$, we find

$$Q^{(j_1,j_2)}(\tau) = 2\cos(2\pi p' a/p) Q^{(j_1+a)}(\tau)\ , \tag{5.49}$$

[2] This result is also consistent with the relation (4.48) derived in the previous chapter.

which is the solution to

$$T_p\left(\frac{Q}{2\cos(\pi p'a/p)}\right)-T_{p'}(\zeta)=0\ . \tag{5.50}$$

We can now distinguish the following two cases:

1. When p is odd, we may choose $1\le j_1\le p$, $1\le a\le(p-1)/2$ as a fundamental domain, giving the first line in (5.48).

2. When p is even, we may choose $1\le j_1\le p$, $1\le a\le(p-2)/2$, together with $1\le j_1\le p/2$ for $a=p/2$ as a fundamental domain. Since p' is odd, $Q^{(j_1,j_2)}(\tau)=0$ for $a=p/2$, giving the factor $Q^{p/2}$ in the second line of (5.48). For $1\le a\le(p-2)/2$, $Q^{(j_1,j_2)}(\tau)$ again solves (5.50), giving the remainder of the second line in (5.48).

□

5.4 Discussion

Let us summarise our results. In Section 5.2, we first motivated our definition of the Wronskian (5.6) to describe the independent degrees of freedom arising in the non-perturbative description outlined in the introduction, Subsection 2.3. We then derived the non-perturbative differential equations satisfied by the latter in Proposition 5.2.4. The construction of the isomonodromy system using Proposition 5.2.6 consequently allowed for the construction of the spectral curves according to Definition 5.2.7. A Kac table of independent branes with entries in one-to-one correspondence with the primary fields of the minimal model then emerged naturally from the properties of the Wronskian in conjunction with the duality transformation (2.97). Finally, in Section 5.3, we then showed how the semiclassical limit $g_s\to 0$ of the Wronskian includes a branch consistent with the degeneracy (5.1) predicted by conformal field theory. Altogether, these results provide ample evidence that due to Stokes' phenomenon, the non-perturbative the general Cardy brane with $(r,s)\neq(1,1)$

can *not* be described by analytic continuation of a single principal solution $(r,s) = (1,1)$, but is instead a bound state of *independent* degrees of freedom, whose wave function is given by the Wronskian functions (5.6), and more generally (5.31).

There are many possible extensions of this work that we have not touched upon. Various computations and consistency checks have only been performed for particular examples; a more general proof of these statements would surely provide deeper insight. We have also omitted the entries in the bulk of the Kac table. To define the wave functions for the corresponding branes, additional successive Wronskian operations must be performed on products of the Wronskians considered herein; we leave an investigation of this more complicated case for future work [37]. It would also be interesting to extend our results to the non-diagonal theories with $q > 2$ studied in Chapters 3 and 4, which would allow for a check of the results reported in [132].

Finally, a potential application of our results pertains to the analogy of the Baker-Akhiezer function $\psi(t;\zeta)$ with the correlator of a gauge theory dual to a spacetime with a horizon [133]. Our results suggest that whilst perturbatively, we can obtain a description of the physics behind the horizon by analytic continuation through the branch cut in the complex ζ-plane, the non-perturbative correlator exhibits Stokes' phenomenon signalling the presence of independent degrees of freedom, as also alluded to in [29]. It would be interesting to explore the implications of our results for this topic, which may pave the way for an extension of these considerations to more complicated backgrounds such as Witten's black hole captured by the $SL(2,\mathbb{R})/U(1)$ coset model [134], which also has a matrix description [135].

Chapter 6

Summary

In this thesis, we have developed novel descriptions of boundary conditions for statistical models on random surfaces employing the measure (2.4) in the planar, scaling, and double scaling limit. We began by introducing each of these limits in Chapter 2, detailing their connection to statistical physics on planar lattices, conformal field theory and finally non-perturbative string theory in a low target space dimension. In each of these cases, we paid particular attention to the description of boundaries in the emsembles of random surfaces that arise in these limits. Following this compressed review came the three chapters containing the bulk of the author's original work, the main results of which we henceforth summarise:

In Chapter 3, we derived the large-N spectral density of sums of random matrices of the form $X_1 + X_2 + \ldots X_p$, $1 \leq p \leq q$, distributed according to (2.4) by generalising Voiculescu's formula (3.3) to a situation beyond free probability (Proposition 3.3.1) and explained the interpretation of these quantities as disk partition functions of the q-states Potts model with p allowed, equally weighted colors on a connected boundary. Besides finding a remarkable algebraic relation between the boundary conditions with p and $q-p$ colors (Corollary 3.3.3) and providing an elliptic parametrisation of the general solution for arbitrary $q \neq 4$ (Proposition 3.3.6), we derived the explicit polynomial equations satisfied by the latter for specific examples with $q = 1, 2, 3$. The scaling relations obtained for these cases were found to be consistent with a description in terms of Liouville theory coupled to a minimal model with central charge $c_M = 0, 1/2$ and $4/5$, setting the stage for Chapter 4.

Therein, we considered the non-diagonal $\mathscr{W}_q$ minimal model with conserved higher-spin currents coupled to Liouville theory as a description for the universality classes corresponding to the critical points in the phase diagram of the model (2.4). Using the free-field resolution of the $\mathscr{W}_q$-modules, we considered the cohomology of the nilpotent BRST operator associated with worldsheet diffeomorphisms (Proposition 4.2.1) and deduced the presence of the tachyon operators (4.6) in the spectrum of observables. We proceeded to consider the one-point function of the latter on the disk and showed that the naïve tensor product of the Liouville FZZT and matter Cardy states appears to overcount the number of physically distinct boundary conditions if one allows for complex values of the boundary cosmological constant. This degeneracy provided a simple way to derive a functional difference equation for the tachyon one-point functions that turned out to agree with the equations obtained from the scaling limit of the matrix model.

Finally, in Chapter 5, we investigated the double scaling limit of (2.4) for $q = 2$ to understand the fate of this degeneracy beyond perturbation theory in the string coupling. We argued that the resolvent operator may exhibit Stokes' phenomenon and proposed a generalised Wronskian as an observable that can resolve the degeneracy non-perturbatively. Without reference to conformal field theory, we determined the differential equations that govern the Wronskians (Propositions 5.2.4 and 5.2.6) and found a maximum of $(p-1)(p'-1)/2$ independent Wronskians, one per Cardy state of the minimal model. Moreover, we could explicitly show for various examples that each entry of the resulting Kac table consistently reproduces the relation (5.1) predicted by conformal field theory (Proposition 5.3.1). We argued that this is strong evidence that the above-mentioned degeneracy is an artefact of the asymptotic expansion and the usual determinant operator is insufficient to capture all information about the theory. Instead, the degeneracy is resolved non-perturbatively by the independent degrees of freedom comprising the Wronskian.

Altogether, these developments have led to the description of a multitude of nontrivial, well-defined boundary conditions whose properties, to the knowledge of the author, have not been previously described in the otherwise vast existing literature on the subject. This suggests that despite the excellent understanding we have of these models – largely thanks to their intimate connection to the theory of integrable systems – many of their properties remain to be worked out. Indeed, as seen from the discussions in Section 3.6, 4.4 and 5.4, this work has also prompted numerous follow-up questions that warrant further investigation, including the scaling behaviour of strongly coupled models with $c_M > 1$, the inclusion of magnetic fields on the boundary and the extension of the Wronskian to the interior of the Kac table. It has also hinted at diverse connections to other fields, such as free probability theory, higher-spin gravity in three dimensions and physics behind black hole horizons. Extending the insights of this thesis more comprehensively to some of the more complicated models may be more challenging; not all of them may share the simplicity of the Hermitian matrix model. Nevertheless, it is reasonable to expect that some features discovered herein may persist in more generality and it is the hope of the author that this thesis enticed the reader about the potential of these avenues to further our general understanding of the mathematical description of random geometry, boundaries and branes.

Appendix A

Auxiliary Saddle Point Problem

Given $|\nu| \leq 1$, consider a function $f(w)$ holomorphic on $\mathbb{C} \setminus [\alpha, \beta]$ for some connected $[\alpha, \beta] \subset \mathbb{R}$, satisfying

$$\mathrm{Re}\, f(w) = \cos(\pi\nu) f(-w)\ , \quad w \in [\alpha, \beta]\ . \tag{A.1}$$

The general solution to this equation was first described in [90] and we will derive it below; thereafter we investigate the limit $\alpha/\beta \to 0$.

General solution. We begin by showing that any function satisfying (A.1) is uniquely specified by the behaviour at its singularites. To this end, it is useful to introduce a new coordinate σ by

$$\begin{aligned} \mathbb{C} \setminus [\pm\alpha, \pm\beta] &\longrightarrow (0,1) \times [0, \tau) \subset \mathbb{C}\ , \\ w \longmapsto \sigma(w) &= \frac{1}{2K} \int_1^{w/\alpha} \mathrm{d}t\, \left((1-t^2)(1-(\alpha t/\beta)^2)\right)^{-1/2}\ . \end{aligned} \tag{A.2}$$

By definition of the Jacobi elliptic function[1] $\mathrm{sn}(u|k)$ of elliptic modulus k, the inverse map is

[1] Our conventions for elliptic functions are those of Gradshtein and Ryzhik [73].

$$
\begin{aligned}
(0,1)\times[0,\tau) &\longrightarrow \mathbb{C}\setminus[\pm\alpha,\pm\beta]\,, \\
\sigma &\longmapsto w(\sigma)=\alpha\,\mathrm{sn}\,(2K\sigma+K|\,\alpha/\beta)\,.
\end{aligned}
\tag{A.3}
$$

Here, K and K' are given by the complete elliptic integrals of the first and second kind, respectively:

$$
\begin{aligned}
K &= \int_0^1 \mathrm{d}t\,\left((1-t^2)(1-(\alpha t/\beta)^2)\right)^{-1/2}\,, \\
K' &= \int_0^\infty \mathrm{d}t\,\left((1+t^2)(1+(\alpha t/\beta)^2)\right)^{-1/2}\,.
\end{aligned}
\tag{A.4}
$$

This change of variables correpsonds to parametrising the two-cut complex w-plane on the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with modular parameter

$$
\tau=\mathrm{i}\frac{K'}{K}\,. \tag{A.5}
$$

The coordinate w is invariant under $\sigma\to-\sigma$ and (anti-) periodic along the respective cycles of the torus:

$$
w(\sigma+m+n\tau)=(-1)^m w(\sigma)\,, \qquad (m,n)\in\mathbb{Z}^2\,. \tag{A.6}
$$

We also require the Jacobi theta functions

$$
\begin{aligned}
\vartheta_1(u|\tau) &= \frac{1}{\mathrm{i}}\sum_{n\in\mathbb{Z}}(-1)^n \mathrm{e}^{\mathrm{i}\pi\tau(n+1/2)^2}\mathrm{e}^{\mathrm{i}u(2n+1)}\,, \\
\vartheta_2(u|\tau) &= \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi\tau(n+1/2)^2}\mathrm{e}^{\mathrm{i}u(2n+1)}\,, \\
\vartheta_3(u|\tau) &= \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi\tau n^2}\mathrm{e}^{2\mathrm{i}un}\,.
\end{aligned}
\tag{A.7}
$$

In particular, ϑ_1 is an entire function with a unique simple zero at $u=0$ mod $\mathbb{Z}\oplus\pi\tau\mathbb{Z}$, satisfying

$$\begin{aligned} \vartheta_1\left(u+\pi(m+n\tau)|\tau\right) &= (-1)^{mn}\mathrm{e}^{-\mathrm{i}n(\pi\tau+2u)}\vartheta_1(u|\tau)\ , \qquad (m,n)\in\mathbb{Z}^2\ , \\ \vartheta_1(-u|\tau) &= -\vartheta_1(u|\tau)\ , \\ \vartheta_1(u/\tau|-1/\tau) &= -\sqrt{\mathrm{i}\tau}\mathrm{e}^{\mathrm{i}\pi u^2/\tau}\vartheta_1(u|\tau)\ . \end{aligned} \tag{A.8}$$

We also note the equivalent representation of $w(\sigma)$ in terms of ϑ_i,

$$w(\sigma) = \sqrt{\alpha\beta}\frac{\vartheta_2(\pi\sigma|\tau)}{\vartheta_3(\pi\sigma|\tau)}\ . \tag{A.9}$$

Analytic continuation of $f(w(\sigma))$ requires boundary conditions on the rectangle $(0,1)\times[0,\tau)$:

1. Analyticity across $[0,\alpha]\cup[\beta,\infty]$ allows us to continue $f(w)$ to the infinite strip $(0,1)\times\mathrm{i}\mathbb{R}$ by

$$f(w(\tau+\sigma)) = f(w(\sigma))\ , \quad \sigma\in(0,1)\times[0,\tau)\ . \tag{A.10}$$

2. Analyticity across $[-\beta,-\alpha]$ allows us to extend this definition to $(0,2)\times\mathrm{i}\mathbb{R}$ using

$$f(w(1+\sigma)) = f(w(1-\sigma))\ , \quad \sigma\in(0,1)\times\mathrm{i}\mathbb{R}\ . \tag{A.11}$$

3. Finally, using all the above, the functional equation (A.1) implies

$$f(w(1+\sigma)) = \frac{f(w(\sigma))+f(w(2+\sigma))}{2\cos(\pi\nu)}\ , \quad \sigma\in(0,2)\times\mathrm{i}\mathbb{R}\ . \tag{A.12}$$

Solving the latter condition allows us to continue $f(w(\sigma))$ to a meromorphic function on

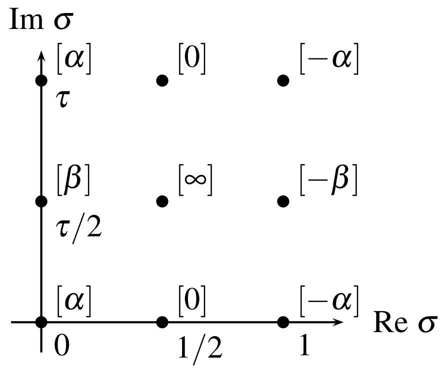


Figure A.1: A fundamental domain for $w \in \mathbb{C} \setminus [\pm\alpha, \pm\beta]$ is given by $\sigma \in (0,1) \times [0,\tau)$. The images of special points under the map $z(\sigma)$ are indicated in square brackets.

the entire complex σ-plane, on which $f(w)$ satisfies two (quasi-)periodicity conditions:

$$0 = (\mathrm{e}^{-\partial_\sigma} - \mathrm{e}^{\mathrm{i}\pi\nu})(\mathrm{e}^{-\partial_\sigma} - \mathrm{e}^{-\mathrm{i}\pi\nu}) f(w(\sigma)) \;, \tag{A.13a}$$

$$0 = (\mathrm{e}^{-\tau\partial_\sigma} - 1) f(w(\sigma)) \;. \tag{A.13b}$$

We find it convenient to follow [80] in introducing the unique function in $\mathrm{Ker}(\mathrm{e}^{-\partial_\sigma} - \mathrm{e}^{\mathrm{i}\pi\nu})$ with a simple pole of unit residue at $\sigma = 0$ and no other singularities mod $\mathbb{Z} \oplus \tau\mathbb{Z}$ as

$$g(\sigma;\nu) = \frac{\vartheta_1'(0|\tau)}{\vartheta_1(\pi\nu\tau/2|\tau)} \frac{\vartheta_1(\pi\sigma + \pi\nu\tau/2|\tau)}{\vartheta_1(\pi\sigma|\tau)} \mathrm{e}^{\mathrm{i}\pi\nu\sigma} \;, \tag{A.14}$$

which has a simple zero at $\sigma = -\nu\tau/2$; any solution to equations (A.13a) and (A.13b) may be expressed as a linear combination of derivatives $g(\sigma;\pm\nu)$ with shifted argument. The reflection relation (A.11) fixes the relative coefficient, so that the general solution to (A.1) can be expressed as

$$f(w) = \sum_{n\geq 0} \frac{a_n}{n!} \frac{\partial^n}{\partial\sigma_0^n} \left(\mathrm{e}^{-\mathrm{i}\pi\nu/2} g(\sigma(w) - \sigma_0; \nu) - \mathrm{e}^{\mathrm{i}\pi\nu/2} g(\sigma(w) - \sigma_0; -\nu) \right) \;, \tag{A.15}$$

where the requirement that $f(w)$ be free of singularities on $\mathbb{C} \setminus [\alpha, \beta]$ demands $\sigma_0 = (\tau +$

$1)/2$, and the coefficients a_n are to be determined by boundary conditions supplementing the problem (A.1), using

$$\lim_{z\to\infty}(\sigma-\sigma_0)^{n+1}\frac{1}{n!}\frac{\partial^n}{\partial\sigma_0^n}g(\sigma-\sigma_0;\nu)=1\ . \tag{A.16}$$

In particular, if $f(w)$ has a pole of order m at $w=\infty$, then $a_n=0$ for $n>m$.

The limit $\alpha/\beta\searrow 0$. In Section 3.5 we will be interested in the limit $\alpha/\beta\searrow 0$, in which $\tau\to\mathrm{i}\infty$, and thus

$$\lim_{\tau\to\mathrm{i}\infty}w(\sigma)/\alpha=\cos(\pi\sigma)\ ,\qquad \lim_{\tau\to\mathrm{i}\infty}g(\sigma;\nu)=\frac{\pi\,\mathrm{e}^{\mathrm{i}\pi(\nu-1)\sigma}}{\sin(\pi\nu)}\ . \tag{A.17}$$

In this limit, $f(w)$ is holomorphic on $w/\alpha\in\mathbb{C}\setminus[1,\infty)$, and the equation (A.1) becomes

$$(\mathrm{e}^{-\partial_\sigma}-\mathrm{e}^{\mathrm{i}\pi\nu})(\mathrm{e}^{-\partial_\sigma}-\mathrm{e}^{-\mathrm{i}\pi\nu})f(\alpha\cos(\pi\sigma))=0\ . \tag{A.18}$$

A convenient basis for the solution space is given by the Chebyshev functions. These are represented on the unit disk as

$$\begin{aligned} T_\nu(x)&=\cos(\pi\nu\phi)\ ,\\ U_\nu(x)&=\frac{\sin(\pi(\nu+1)\phi)}{\sin(\pi\phi)}\ ,\qquad x=\cos(\pi\phi)\ . \end{aligned} \tag{A.19}$$

From the above definition it is easy to verify that both $T_\nu(x)$ and $U_\nu(x)$ satisfy equation (A.1), and $T_{1/\nu}(x)$ is the functional inverse of $T_\nu(x)$. For non-integer ν, these functions have a branch cut on $x\in[-1,-\infty)$, with discontinuity

$$\begin{aligned} T_\nu(x)_+-T_\nu(x)_-&=-2\mathrm{i}\sin(\pi\nu)\sqrt{1-x^2}\,U_{\nu-1}(-x)\ ,\\ U_\nu(x)_+-U_\nu(x)_-&=-2\mathrm{i}\frac{\sin(\pi\nu)}{\sqrt{1-x^2}}T_{\nu+1}(-x)\ . \end{aligned} \tag{A.20}$$

When $\nu\in\mathbb{N}$, the right-hand side vanishes and we recover the definition of the Chebyshev polynomials of the first and second kind. As a result when $\nu = p/q$ is rational, $y = T_\nu(x)$ is the solution to the polynomial equation $T_q(y) - T_p(x) = 0$. Since (A.1) restricts the scaling exponents $f(w) \sim (-w)^\kappa$ to the form $\kappa = 2n \pm \nu$, $n \in \mathbb{Z}$ we can expand $f(w)$ as

$$f(w) = \sum_{n\geq 0} \sum_{\pm} \alpha^{2n\pm\nu} (t_n^{(\pm)} T_{2n\pm\nu}(-w/\alpha) + u_n^{(\pm)} U_{2n\pm\nu}(-w/\alpha)) \ , \quad |w/\alpha| \leq 1 \ , \quad \text{(A.21)}$$

with constants $t_n^{(\pm)}$, $u_n^{(\pm)}$ to be determined by boundary condtions.

Appendix B

Analytic Structure and Asymptotics

We illustrate the analytic structure of $G_Y^{(p)}(z)$ and $G_{(p)}^Y(z)$ by graphs in which nodes depict sheets and lines between nodes depict branch cuts that connect the sheets. Of the latter, double lines represent finite cuts and single lines represent cuts that extend to infinity.

Example B.0.1. $(q,k)=(1,2)$. *From equations* (3.57a) *and* (3.57b)*, we compute the analytic structure and asymptotic behaviour of* $G_{(p)}^Y(z)$*,*

$$\begin{array}{ll}
G_Y^{(1)}(z)_- & z-z^{-1}+\mathcal{O}(z^{-2}) \\
G_Y^{(1)}(z)_+ & \omega^2 t_4^{-1/3} z^{1/3} - \frac{t_3}{3t_4} + \omega \frac{t_3^2-3t_2t_4}{9t_4^{5/3}} z^{-1/3} - \omega^2 \frac{2t_3^3-9t_2t_3t_4}{81t_4^{7/3}} z^{-2/3} + \frac{1}{3}z^{-1} + \mathcal{O}(z^{-4/3}) \\
 & \omega t_4^{-1/3} z^{1/3} - \frac{t_3}{3t_4} + \omega^2 \frac{t_3^2-3t_2t_4}{9t_4^{5/3}} z^{-1/3} - \omega \frac{2t_3^3-9t_2t_3t_4}{81t_4^{7/3}} z^{-2/3} + \frac{1}{3}z^{-1} + \mathcal{O}(z^{-4/3}) \\
 & t_4^{-1/3} z^{1/3} - \frac{t_3}{3t_4} + \frac{t_3^2-3t_2t_4}{9t_4^{5/3}} z^{-1/3} - \frac{2t_3^3-9t_2t_3t_4}{81t_4^{7/3}} z^{-2/3} + \frac{1}{3}z^{-1} + \mathcal{O}(z^{-4/3})
\end{array}$$

where $\omega=\mathrm{e}^{\mathrm{i}\pi/3}$*. From* (3.58)*, we may compute the asymptotic behaviour of* $G_{(p)}^Y(z)$ *on all sheets:*

$$\begin{array}{ll}
G^Y_{(0)}(z)_+ & z^{-1}+\mathscr{O}(z^{-2}) \\
G^Y_{(0)}(z)_- & z+\omega^2 t_4^{-1/3}z^{1/3}-\frac{t_3}{3t_4}+\omega\frac{t_3^2-3(t_2-1)t_4}{9t_4^{5/3}}z^{-1/3}-\omega^2\frac{2t_3^3-9(t_2-1)t_3t_4}{81t_4^{7/3}}z^{-2/3}+\frac{1}{3}z^{-1}+\mathscr{O}(z^{-4/3}) \\
 & z+\omega t_4^{-1/3}z^{1/3}-\frac{t_3}{3t_4}+\omega^2\frac{t_3^2-3(t_2-1)t_4}{9t_4^{5/3}}z^{-1/3}-\omega\frac{2t_3^3-9(t_2-1)t_3t_4}{81t_4^{7/3}}z^{-2/3}+\frac{1}{3}z^{-1}+\mathscr{O}(z^{-4/3}) \\
 & z+t_4^{-1/3}z^{1/3}-\frac{t_3}{3t_4}+\frac{t_3^2-3(t_2-1)t_4}{9t_4^{5/3}}z^{-1/3}-\frac{2t_3^3-9(t_2-1)t_3t_4}{81t_4^{7/3}}z^{-2/3}+\frac{1}{3}z^{-1}+\mathscr{O}(z^{-4/3})
\end{array}$$

$$\begin{array}{ll}
G^Y_{(1)}(z)_+ & t_4z^3+t_3z^2+t_2z-z^{-1}+\mathscr{O}(z^{-2}) \\
G^Y_{(1)}(z)_- & z+z^{-1}+\mathscr{O}(z^{-2})
\end{array}$$

Example B.0.2. $(q,k)=(2,1)$. *From equations* (3.63a) *and* (3.63b)*, we compute the analytic structure and asymptotic behaviour of* $G^Y_{(p)}(z)$*,*

$$\begin{array}{ll}
 & z+t_3^{-1/2}z^{1/2}+\frac{t_2}{2t_3}+\frac{t_2^2}{8t_3^{3/2}}z^{-1/2}-\frac{1}{2}z^{-1}+\mathscr{O}(z^{-3/2}) \\
G_Y^{(1)}(z)_- & z-t_3^{-1/2}z^{1/2}+\frac{t_2}{2t_3}-\frac{t_2^2}{8t_3^{3/2}}z^{-1/2}-\frac{1}{2}z^{-1}+\mathscr{O}(z^{-3/2}) \\
G_Y^{(1)}(z)_+ & t_3^{-1/2}z^{1/2}-\frac{t_2}{2t_3}+\frac{t_2^2}{8t_3^{3/2}}z^{-1/2}+\frac{1}{2}z^{-1}+\mathscr{O}(z^{-3/2}) \\
 & -t_3^{-1/2}z^{1/2}-\frac{t_2}{2t_3}-\frac{t_2^2}{8t_3^{3/2}}z^{-1/2}+\frac{1}{2}z^{-1}+\mathscr{O}(z^{-3/2})
\end{array}$$

$$\begin{array}{ll}
G_Y^{(2)}(z)_- & z-z^{-1}+\mathscr{O}(z^{-2}) \\
G_Y^{(2)}(z)_+ & 2t_3^{-1/2}z^{1/2}-\frac{t_2}{t_3}+\frac{t_2^2}{4t_3^{3/2}}z^{-1/2}+\mathscr{O}(z^{-3/2}) \\
 & -2t_3^{-1/2}z^{1/2}-\frac{t_2}{t_3}-\frac{t_2^2}{4t_3^{3/2}}z^{-1/2}+\mathscr{O}(z^{-3/2}) \\
 & -z-\frac{2t_2}{t_3}+z^{-1}+\mathscr{O}(z^{-2})
\end{array}$$

From the resulting polynomials $F_{(p)}(x,y)$*, we may also compute the asymptotic behaviour of* $G^Y_{(p)}(z)$ *on all sheets. For example, from* (3.64a)*,*

$$\begin{array}{ll}
G^Y_{(1)}(z)_- & t_3z^2+t_2z-z^{-1}+\mathscr{O}(z^{-2}) \\
G^Y_{(1)}(z)_+ & z+t_3^{-1/2}z^{1/2}+\frac{t_2-1}{t_3}+\frac{(t_2-1)^2}{8t_3^{3/2}}z^{-1/2}+\frac{1}{2}z^{-1}+\mathscr{O}(z^{-3/2}) \\
 & z-t_3^{-1/2}z^{1/2}+\frac{t_2-1}{t_3}-\frac{(t_2-1)^2}{8t_3^{3/2}}z^{-1/2}+\frac{1}{2}z^{-1}-\mathscr{O}(z^{-3/2})
\end{array}$$

$$\begin{array}{ll} G^{Y}_{(2)}(z)_{+} & z+z^{-1}+\mathscr{O}(z^{-2}) \\ G^{Y}_{(2)}(z)_{-} & \frac{t_3}{4}z^2+\frac{t_2}{2}z+\mathscr{O}(z^{-2}) \\ & -z-\frac{2t_2}{t_3}-z^{-1}+\mathscr{O}(z^{-2}) \end{array}$$

Example B.0.3. $(q,k)=(3,1)$. *From equations* (3.71a) *and* (3.71b)*, we compute the analytic structure and asymptotic behaviour of* $G^{Y}_{(p)}(z)$*,*

$$\begin{array}{ll} & 2z+t_3^{-1/2}z^{1/2}+\frac{5t_2}{2t_3}+\frac{t_2^2}{8t_3^{3/2}}z^{-1/2}-\frac{1}{2}z^{-1}+\mathscr{O}(z^{-3/2}) \\ & 2z-t_3^{-1/2}z^{1/2}+\frac{5t_2}{2t_3}-\frac{t_2^2}{8t_3^{3/2}}z^{-1/2}-\frac{1}{2}z^{-1}-\mathscr{O}(z^{-3/2}) \\ & z+2t_3^{-1/2}z^{1/2}+\frac{t_2}{t_3}+\frac{t_2^2}{4t_3^{3/2}}z^{-1/2}+\mathscr{O}(z^{-3/2}) \\ G_Y^{(1)}(z)_{-} & z-2t_3^{-1/2}z^{1/2}+\frac{t_2}{t_3}-\frac{t_2^2}{4t_3^{3/2}}z^{-1/2}-\mathscr{O}(z^{-3/2}) \\ G_Y^{(1)}(z)_{+} & t_3^{-1/2}z^{1/2}-\frac{t_2}{2t_3}+\frac{t_2^2}{8t_3^{3/2}}z^{-1/2}+\frac{1}{2}z^{-1}+\mathscr{O}(z^{-3/2}) \\ & -t_3^{-1/2}z^{1/2}-\frac{t_2}{2t_3}-\frac{t_2^2}{8t_3^{3/2}}z^{-1/2}+\frac{1}{2}z^{-1}-\mathscr{O}(z^{-3/2}) \end{array}$$

$$\begin{array}{ll} G_Y^{(3)}(z)_{-} & z-z^{-1}+\mathscr{O}(z^{-2}) \\ G_Y^{(3)}(z)_{+} & 3t_3^{-1/2}z^{1/2}-\frac{3t_2}{2t_3}+\frac{3t_2^2}{8t_3^{3/2}}z^{-1/2}-\frac{1}{2}z^{-1}+\mathscr{O}(z^{-3/2}) \\ & -3t_3^{-1/2}z^{1/2}-\frac{3t_2}{2t_3}-\frac{3t_2^2}{8t_3^{3/2}}z^{-1/2}-\frac{1}{2}z^{-1}-\mathscr{O}(z^{-3/2}) \\ & -2z+3t_3^{-1/2}z^{1/2}-\frac{9t_2}{2t_3}+\frac{3t_2^2}{8t_3^{3/2}}z^{-1/2}+\frac{1}{2}z^{-1}+\mathscr{O}(z^{-3/2}) \\ & -2z-3t_3^{-1/2}z^{1/2}-\frac{9t_2}{2t_3}-\frac{3t_2^2}{8t_3^{3/2}}z^{-1/2}+\frac{1}{2}z^{-1}+\mathscr{O}(z^{-3/2}) \\ & -3z-6\frac{t_2}{t_3}+z^{-1}+\mathscr{O}(z^{-2}) \end{array}$$

From the resulting polynomials $F_{(p)}(x,y)$*, we may also compute the asymptotic behaviour of* $G^{Y}_{(p)}(z)$ *on all sheets. For example, from* (3.72)*,*

$$
\begin{array}{ll}
 & z - t_3^{-1/2} z^{1/2} - \frac{t_2-1}{2t_3} - \frac{(t_2-1)^2}{8t_3^{3/2}} z^{1/2} + \frac{1}{2} z^{-1} - \mathscr{O}(z^{-3/2}) \\
G_{(2)}^{Y}(z)_+ & z + t_3^{-1/2} z^{1/2} - \frac{t_2-1}{2t_3} + \frac{(t_2-1)^2}{8t_3^{3/2}} z^{1/2} + \frac{1}{2} z^{-1} + \mathscr{O}(z^{-3/2}) \\
G_{(2)}^{Y}(z)_- & \frac{t_3}{4} z^2 + \frac{t_2}{2} z + \mathscr{O}(z^{-2}) \\
 & -z + \mathrm{i} t_3^{-1/2} z^{1/2} - \frac{t_2-1}{2t_3} + \mathrm{i} \frac{(t_2-1)^2}{8t_3^{3/2}} z^{1/2} - \frac{1}{2} z^{-1} + \mathscr{O}(z^{-3/2}) \\
 & -z - \mathrm{i} t_3^{-1/2} z^{1/2} - \frac{t_2-1}{2t_3} - \mathrm{i} \frac{(t_2-1)^2}{8t_3^{3/2}} z^{1/2} - \frac{1}{2} z^{-1} \mathscr{O}(z^{-3/2})
\end{array}
$$

Appendix C

Lax Operators

Here we provide the explicit form of the Lax matrices for $(p,p') = (3,2)$, $(4,3)$ and $(5,2)$, respectively, with the abbrevation $u_{p,\zeta} = u_p - \zeta$.

Example C.0.4. $(p,p') = (3,2)$. *The compatibility condition* (2.90) *implies* $u_2(t) = 3v_2(t)$ *and* $u_3(t) = 3\dot{v}_2(t)/2$*, where* $v_2(t)$ *solves the first Painlevé equation:* $\ddot{v}_2(t) = 6v_2(t) + t$*. For* $n = 1,2$*, we find*

$$\begin{aligned}
\mathscr{B}^{(2)} &= -\mathscr{C}^{-1}\left(\mathscr{B}^{(1)}\right)^T \mathscr{C} = \frac{1}{4}\begin{pmatrix} 0 & 4 & 0 \\ -u_2 & 0 & 4 \\ u_{3,\zeta} & 0 & 0 \end{pmatrix}, \\
\mathscr{Q}^{(2)} &= -\mathscr{C}^{-1}\left(\mathscr{Q}^{(1)}\right)^T \mathscr{C} = \frac{1}{4}\begin{pmatrix} 2v_2 & 0 & -8 \\ -\dot{v}_2 + 2\zeta & 2v_2 & 0 \\ -\ddot{v}_2 & \dot{v}_2 + 2\zeta & -4v_2 \end{pmatrix}.
\end{aligned} \tag{C.1}$$

Example C.0.5. $(p,p') = (4,3)$. *From* (2.90)*,* $u_2(t) = 8v_2(t)$*,* $u_3(t) = 8v_3(t)/3 + 8\dot{v}_2/6$*. We consider the case* $n = 2$*: the Lax operators read*

$$\mathscr{B}^{(2)} = -\mathscr{C}^{-1}\left(\mathscr{B}^{(2)}\right)^T \mathscr{C} = \frac{1}{8}\begin{pmatrix} 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 8 & 0 & 0 \\ -u_3 & -u_2 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ u_{4,\zeta} & 0 & 0 & -u_2 & 0 & 8 \\ 0 & u_{4,\zeta} & 0 & u_3 & 0 & 0 \end{pmatrix}, \tag{C.2}$$

and $\mathscr{Q}^{(2)} = \mathscr{Q}^{(2)}_{\text{cl.}} + \mathscr{Q}^{(2)[1]} + \mathscr{Q}^{(2)[2]}$, *where*

$$\mathscr{Q}^{(2)}_{\text{cl.}} = \frac{1}{8}\begin{pmatrix} -8u_3 & -u_2 & 0 & 0 & -32 & 0 \\ -4u_{4,\zeta} & -8u_3 & -u_2 & 3u_2 & 0 & -32 \\ u_2\left(u_3 - \frac{3}{8}u_2\right) & 8u_{4,\zeta} + \frac{5}{8}u_2^2 & -8u_3 & 0 & 3u_2 & 0 \\ 0 & 8u_{4,\zeta} & 0 & -8u_3 & -u_2 & 0 \\ u_2 u_{4,\zeta} & 0 & 0 & 4u_{4,\zeta} + \frac{u_2^2}{8} & -8u_3 & -u_2 \\ 0 & \frac{1}{2}\left(\frac{u_2}{8}u_{4,\zeta}\right) & 0 & -\frac{u_2 u_3}{8} & -\frac{u_3}{2} - u_{4,\zeta} & -8u_3 \end{pmatrix}, \tag{C.3}$$

$$\mathscr{Q}^{(2)[1]} = \frac{1}{8}\begin{pmatrix} \dot{u}_2 & 0 & \cdots & & \cdots & 0 \\ -12\dot{u}_3 & 2\dot{u}_2 & \ddots & & & \vdots \\ 16\dot{u}_4 - 2\dot{u}_3 & -18\dot{u}_3 & \dot{u}_2 & & & \\ 0 & -2\dot{u}_3 & 0 & \dot{u}_2 & \ddots & \vdots \\ 0 & 0 & 2 - 4\dot{u}_4 & 14\dot{u}_3 & 8\dot{u}_2 & 0 \\ 0 & 0 & -4\dot{u}_4 & 4\dot{u}_4 & 15\dot{u}_2 - 8\dot{u}_3 & -16\dot{u}_2 \end{pmatrix}, \tag{C.4}$$

$$
\mathscr{Q}^{(2)[2]} = \frac{1}{8}\begin{pmatrix}
0 & \cdots & & & \cdots & 0 \\
3\ddot{u}_2 & \ddots & & & & \vdots \\
9\ddot{u}_2 + 8\ddot{u}_3 & \ddot{u}_2 & & & & \\
2\ddot{u}_3 & 0 & & & & \\
-2\ddot{u}_3 & 0 & 2\ddot{u}_3 & 5\ddot{u}_2 & \ddots & \vdots \\
0 & 4\ddot{u}_4 & -2\ddot{u}_3 & 10\ddot{u}_3 - 3\dddot{u}_2 & 0 & 0
\end{pmatrix}. \tag{C.5}
$$

Example C.0.6. $(p,p') = (5,2)$. *In this case* (2.90) *requires* $u_2(t) = 20v_2(t)$*,* $u_3(t) = 30\dot{v}_2(t)$ *and* $u_5(t) = v_2(t)\dot{v}_2(t)/2$*. We consider the cases* $n = 2$ *and* 3*: the Lax operators read*

$$
\mathscr{B}^{(2)} = -\mathscr{C}\left(\mathscr{B}^{(3)}\right)^T \mathscr{C}^{-1} = \frac{1}{16}\begin{pmatrix}
0 & 16 & 0 & \cdots & & & & & \cdots & 0 \\
\vdots & \ddots & 16 & 16 & & & & & & \vdots \\
 & & & & 16 & 16 & & & & \\
 & & & & & 16 & & & & \\
-u_4 & -u_3 & -u_2 & & & & 16 & & & \\
 & & & & & & 16 & 16 & & \\
u_{5,\zeta} & & & -u_3 & & -u_2 & & & 16 & \vdots \\
 & & & & & & & & 16 & 0 \\
\vdots & u_{5,\zeta} & & u_4 & & & & -u_2 & \ddots & 16 \\
0 & \cdots & u_{5,\zeta} & & & u_4 & & u_3 & \cdots & 0
\end{pmatrix} \tag{C.6}
$$

and $\mathscr{Q}^{(n)} = \mathscr{Q}^{(n)}_{\text{cl.}} + \mathscr{Q}^{(n)[1]} + \mathscr{Q}^{(n)[2]}$*, with*

$$\mathscr{Q}^{(2)}_{\text{cl.}} = \frac{1}{8}\begin{pmatrix} 16v_2 & 0 & 16 & -16 & 0 & \dots & & & \dots & 0 \\ 0 & 16v_2 & \ddots & & 16 & & & & & \vdots \\ -u_4 & \ddots & -4v_2 & & & & & 16 & & \\ 0 & & & 16v_2 & & & 16 & -16 & & \\ -u_{5,\zeta} & -u_4 & & & -4v_2 & & & & 16 & \vdots \\ u_{5,\zeta} & & & & & -4v_2 & & & & 0 \\ & & & -u_4 & & & -4v_2 & & & 16 \\ & u_{5,\zeta} & & u_4 & & & & -4v_2 & \ddots & -16 \\ \vdots & & & u_{5,\zeta} & & & & \ddots & -4v_2 & 0 \\ 0 & \dots & & & -u_{5,\zeta} & u_{5,\zeta} & -u_4 & u_4 & 0 & -24v_2 \end{pmatrix}, \tag{C.7}$$

which satisfies $\mathscr{Q}^{(2)}_{\text{cl.}} = -\mathscr{C}\left(\mathscr{Q}^{(3)}_{\text{cl.}}\right)^T \mathscr{C}^{-1}$*, and the leading quantum corrections*

$$\mathscr{Q}^{(2)[1]} = \frac{1}{4}\begin{pmatrix} 0 & \dots & & & & & & & \dots & 0 \\ 4\dot{v}_2 & \ddots & & & & & & & & \vdots \\ & -3\dot{v}_2 & & & & & & & & \\ \vdots & 4\dot{v}_2 & & & & & & & & \\ & & \dot{v}_2 & & & & & & & \\ & & 4\dot{v}_2 & -3\dot{v}_2 & & & & & & \\ & & & & 4\dot{v}_2 & -11\dot{v}_2 & & & & \\ & & & & & 8\dot{v}_2 & & & & \\ \vdots & & & & & & 4\dot{v}_2 & \dot{v}_2 & \ddots & \vdots \\ 0 & \dots & & & & & \dots & & -3\dot{v}_2 & 0 \end{pmatrix} \tag{C.8}$$

$$\mathscr{Q}^{(3)[1]} = \frac{1}{4}\begin{pmatrix} 0 & \dots & & & & & & & \dots & 0 \\ -3\dot{v}_2 & \ddots & & & & & & & & \vdots \\ & 4\dot{v}_2 & & & & & & & & \\ & \dot{v}_2 & & & & & & & & \\ & & 4\dot{v}_2 & & & & & & & \\ & & \dot{v}_2 & 8\dot{v}_2 & & & & & & \\ & & & & \dot{v}_2 & 4\dot{v}_2 & & & & \\ & & & & & -3\dot{v}_2 & & & & \\ \vdots & & & & & & -3\dot{v}_2 & 4\dot{v}_2 & \ddots & \vdots \\ 0 & \dots & & & & & & & \dots & 0 \end{pmatrix} \tag{C.9}$$

and all higher-order corrections subsumed in

$$\mathscr{Q}^{(2)[2]} = \begin{pmatrix} 0 & \dots & & & & & & & \dots & 0 \\ 0 & \ddots & & & & & & & & \vdots \\ 3\ddot{v}_2 & \ddots & & & & & & & & \\ -\ddot{v}_2 & & & & & & & & & \\ 4\dddot{v}_2 & 6\ddot{v}_2 & & & & & & & & \\ -\dddot{v}_2 & & & & & & & & & \\ & & & 6\ddot{v}_2 & & & & & & \\ & & \ddot{v}_2 & -3\dddot{v}_2 & & & & & & \\ \vdots & & & -4\dddot{v}_2 & \ddot{v}_2 & & & \ddots & \ddots & \vdots \\ 0 & \dots & & & & -4\dddot{v}_2 & \ddot{v}_2 & -6\ddot{v}_2 & 0 & 0 \end{pmatrix} \tag{C.10}$$

$$
\mathscr{Q}^{(3)[2]} = \begin{pmatrix}
0 & \dots & & & & & & & \dots & 0 \\
0 & \ddots & & & & & & & & \vdots \\
3\ddot{v}_2 & \ddots & & & & & & & & \\
6\ddot{v}_2 & & & & & & & & & \\
\dddot{v}_2 & -\ddot{v}_2 & & & & & & & & \\
-4\dddot{v}_2 & & & & & & & & & \\
 & & & -\ddot{v}_2 & & & & & & \\
 & -4\dddot{v}_2 & -6\ddot{v}_2 & 3\ddot{v}_2 & & & & & & \\
\vdots & & & & -6\ddot{v}_2 & & & \ddots & \ddots & \vdots \\
0 & \dots & & & 4\dddot{v}_2 & -\dddot{v}_2 & -3\ddot{v}_2 & & 0 & 0
\end{pmatrix} \tag{C.11}
$$

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