Non-Reductive Geometric Invariant Theory and Compactifications of Enveloped Quotients

Thomas James Keith Hawes
Balliol College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Trinity Term 2015

To all who have nurtured me over the years and uphold me in my life now.

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In this thesis we develop a framework for constructing quotients of varieties by actions of linear algebraic groups which is similar in spirit to that of Mumford's geometric invariant theory. This is done by extending the work of Doran and Kirwan in the unipotent setting to deal with more general non-reductive groups.

Given a linear algebraic group acting on an irreducible variety with a linearisation, an open subset of *stable* points is identified that admits a geometric quotient in the category of varieties. This lies within the *enveloped quotient*, which is a dense constructible subset of a scheme that is locally of finite type, called the *enveloping quotient*. Ways to compactify the enveloped quotient—and the quotient of the stable locus therein—are considered. In particular, the theory of reductive envelopes from Doran and Kirwan's work is extended to the more general non-reductive setting to give ways of constructing compactifications of the enveloped quotient by using the techniques of Mumford's geometric invariant theory for reductive groups.

We then look at two ways in which this non-reductive geometric invariant theory can be used in practice. Firstly, we consider a procedure for constructing quotients inductively, using the extra data of a choice of appropriate subnormal series of a group. Related to this is a method for constructing an approximation of the stable set. Secondly, we study the actions of certain extensions of unipotent groups by multiplicative groups on projective varieties with very ample linearisation. Here we identify an open subset of points that admits a geometric quotient by the action of the extended group and which is explicitly computable via Hilbert-Mumford-like criteria.

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Statement of Originality

This thesis contains no material that has already been accepted, or is concurrently being submitted, for any degree or diploma or certificate or other qualification in this University or elsewhere. To the best of my knowledge and belief this thesis contains no material previously published or written by another person, except where due reference is made in the text or commonly known.

Thomas J. K. Hawes

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0.1 Introduction

Group actions are a frequent occurrence within algebraic geometry. Many spaces that one might want to understand arise naturally as the quotient of a variety by a group action, with moduli spaces giving some of the most prominent examples [MFK94, New78, Gie83, KPT01]. Given a variety X over an algebraically closed field $\Bbbk$ of characteristic zero and a linear algebraic group H acting on X, a basic question therefore is to ask how one can construct the quotient X/H, and study it. By 'quotient' here we mean more precisely a *geometric quotient*, in the sense of [MFK94]: this is a variety X/H with an H-invariant morphism $X \to X/H$ whose fibres are the orbits of the action and that is universal with respect to H-invariant morphisms from X, amongst other properties. As is well known, there are lots of cases of interest where a geometric quotient for an action cannot possibly exist in the category of varieties. One way to address this is to enlarge one's category and work with more general geometric objects, such as algebraic spaces [Art71, Knu71] or even stacks [DM69, LMB00]. Another approach is to instead look for nonempty open subsets of X that admit a geometric quotient variety; such open subsets are guaranteed to exist by a theorem of Rosenlicht [Ros63]. It is this second approach we will be concerned with in this thesis.

In the case where $H = G$ is a reductive linear algebraic group this second approach was studied by Mumford in the first edition of [MFK94], resulting in his *geometric invariant theory* (GIT). (In this context see also the work of Seshadri [Ses63, Ses72, Ses77].) Mumford's GIT works particularly well in the case where X is projective. Given the additional choice of an ample linearisation $L \to X$ (that is, an ample line bundle $L \to X$ with a lift of the action of G to L) Mumford defines a G-invariant open subset of stable points X^{s} which has a geometric quotient variety X^{s}/G. This is contained in the G-invariant open subset of semistable points X^{ss} in X and there is a natural surjective G-invariant map from X^{ss} onto a projective variety $X/\!\!/G$ (canonical to the choice of L) which can be described as $\mathrm{Proj}(S^G)$, where $S = \bigoplus_{r \geq 0} H^0(X, L^{\otimes r})$ and S^G is the ring of invariant sections of non-negative tensor powers of $L \to X$. One therefore has the following diagram.

$$\begin{tikzcd}
X^{\mathrm{s}} \arrow[r, phantom, "\subseteq"] \arrow[d, "\mathrm{geo}"] & X^{\mathrm{ss}} \arrow[r, phantom, "\subseteq"] \arrow[d, two heads] & X \arrow[d, dashed] \\
X^{\mathrm{s}}/G \arrow[r, phantom, "\subseteq"] & X/\!\!/G \arrow[r, phantom, "="] & \mathrm{Proj}(S^G)
\end{tikzcd}$$

The variety $X/\!\!/G$, which is often referred to as the *GIT quotient*, provides a natural compactification of the quotient of the stable set. Tools abound for studying the spaces in this diagram. In [MFK94] Mumford gave numerical criteria for computing the sets of stable and semistable points in terms of the actions of one-parameter subgroups of G. When X is smooth the local geometry of the orbits in X^{ss} can be studied with the slice theorem of Luna [Lun73] and, in the case where the ground field is the complex numbers $\mathbb{C}$, links with symplectic geometry yield ways to compute the (rational intersection) cohomology of $X/\!\!/G$ [Kir84, Kir85, Kir86, Kir87, JK95, JKKW03]. Moreover, as studied in the work of 'variation of GIT' (VGIT) by Thaddeus [Tha96], Dolgachev and Hu [DH98] and Ressayre [Res00], the GIT quotients undergo birational transformations when the linearisation varies, which can often be described explicitly as certain kinds of flips.

Various authors have considered a similar approach of finding open subsets of 'stable' points that admit geometric quotients in the case where H is not reductive. This problem is very much more challenging, in essence because the representation theory of a non-reductive group is not as complete and well-behaved as in the reductive case. Any linear algebraic group H has a canonical normal unipotent subgroup H_u, called the unipotent radical of H, such that H/H_u is reductive, thus constructing quotients for H can, in principle, be reduced to studying the actions of unipotent groups. Fauntleroy [Fau83, Fau85] and Dixmier and Raynaud [DR81] give geometric descriptions of open subsets that admit geometric quotients, but these are typically difficult to find in practice (requiring knowledge of which points are separated by invariant functions) and often some extra condition on X, such as normality or quasi-factoriality, needs to be imposed. More algorithmic approaches have been taken in [GP93, GP98], [vdE93] and [SdS00], though here the geometric picture is somewhat more obscure in favour of computation. Another development, this time more on the algebraic side of the subject, is the search for *separating sets of* *invariants* to construct quotient morphisms of affine varieties X, made popular by Derksen and Kemper in [DeKe02] and pursued recently by the work of Dufresne and others [Duf13, DES14, DJ15, DuKr15] (although the use of separating sets of (rational) invariants seems to in fact go back to Rosenlicht [Ros56] and is used in the proof of his aforementioned theorem; see [VP94].) The idea here lies in the observation that one can find a *finite* set of invariants $\mathcal{S} \subseteq \mathcal{O}(X)^H$ such that two points in X get separated by the natural map $X \to \operatorname{Spec}(\mathcal{O}(X)^H)$ if, and only if, there is an element of $\mathcal{S}$ that separates the points. Therefore one does not need to find a generating set for $\mathcal{O}(X)^H$ to describe quotient maps.

Amongst the literature that studies the actions of unipotent groups is the work of Doran and Kirwan [DK07]. They consider the case where $H = U$ is a unipotent group acting on an irreducible projective variety X with ample linearisation. The overarching idea of [DK07] is to consider various notions of 'stability', 'semistability' and 'quotient' that are intrinsic to the linearisation $L \to X$ of U, and relate these to the geometric invariant theory of [MFK94] of certain reductive linearisations associated to $L \to X$. The main appeal of this approach is that it gives ways to use the tools available in the reductive setting to study quotients of unipotent group actions on (open subsets of) projective varieties. The material in [DK07] is foundational for this thesis and a detailed survey of the main results and definitions will feature in Chapter 1, so here we only provide a brief sketch of the main ideas.

The inclusion $S^U \hookrightarrow S$ defines a natural rational map

$$q : X \dashrightarrow \operatorname{Proj}(S^U)$$

to a scheme $\operatorname{Proj}(S^U)$ which is not necessarily noetherian, and its maximal domain of definition contains the open subsets X_f given by the non-vanishing of invariant sections f of positive tensor power of $L \to X$. Imposing various conditions on the sections f yields a subset of *stable* points X^{s}, a subset $X^{\mathrm{ss,fg}}$ of *finitely generated semistable* points that contains the stable set, and an open subscheme $X /\!\!/ U$ of $\operatorname{Proj}(S^U)$, locally of finite type over $\Bbbk$, called the *enveloping quotient*. The rational map q defines a dominant U-invariant morphism $q : X^{\mathrm{ss,fg}} \to X /\!\!/ U$ which restricts to give a geometric quotient $q : X^{\mathrm{s}} \to q(X^{\mathrm{s}})$ of the stable locus, and $q(X^{\mathrm{s}})$ is an open subset of $X /\!\!/ U$. While this looks similar to the picture resulting from Mumford's reductive GIT, the fact that U is unipotent presents two key differences: firstly, $X/\!\!/U$ is not a projective variety in general, although when S^U is finitely generated then $X/\!\!/U = \mathrm{Proj}(S^U)$ is projective; and secondly, $q : X^{\mathrm{ss,fg}} \to X/\!\!/U$ is not surjective in general, with the image of $X^{\mathrm{ss,fg}}$ under q potentially only a dense constructible subset of $X/\!\!/U$. This image $q(X^{\mathrm{ss,fg}})$ is called the *enveloped quotient*. (Despite the name, one should be aware that in general the enveloped quotient is not even a categorical quotient within the category of varieties).

The enveloping quotient $X/\!\!/U$ may not be quasi-compact (contrary to what is claimed in [DK07]), but when this is the case then it is in fact a quasi-projective variety; let us assume from now on that $L \to X$ is a U-linearisation such that the enveloping quotient is quasi-compact. Then it is natural to ask how to construct projective completions of the enveloping quotient, or in the case where S^U is a finitely generated algebra to ask how to study $\mathrm{Proj}(S^U)$. This is where Mumford's GIT for reductive groups enters the picture. Given a reductive group G containing U, one can consider the fibre bundle $G \times^U X$ associated to the principal U-bundle $G \to G/U$, which is a G-variety together with a canonical ample G-linearisation $G \times^U L \to G \times^U X$. In [DK07] the stable locus X^{s} for the unipotent linearisation $L \to X$ is shown to be equal to the intersection of X with the stable locus for the G-linearisation on $G \times^U X$, providing a first look towards the potential of studying the unipotent linearisation $L \to X$ in terms of a reductive linearisation. The variety $G \times^U X$ is however only quasi-projective, so in order to exploit the full effect of Mumford's GIT Doran and Kirwan consider the question of constructing suitable equivariant compactifications $\overline{G \times^U X}$ of $G \times^U X$, together with extensions $L' \to \overline{G \times^U X}$ of the G-linearisation, that can be used to study the original U-linearisation $L \to X$. When sufficiently many U-invariant sections over X extend to G-invariant sections over $\overline{G \times^U X}$ the pair $(\overline{G \times^U X}, L')$ defines a *reductive envelope*, and if L' is ample (or, more generally, if $(\overline{G \times^U X}, L')$ is a *fine* reductive envelope) then it is shown that

$$X \cap (\overline{G \times^U X})^{\mathrm{s}(L')} \subseteq X^{\mathrm{s}} \subseteq X^{\mathrm{ss,fg}} \subseteq X \cap (\overline{G \times^U X})^{\mathrm{ss}(L')}$$

and the reductive GIT quotient $\overline{G \times^U X}/\!\!/_{L'} G$ defines a compactification of the enveloping quotient $X/\!\!/U$. In particularly good cases one can even find reductive envelopes satisfying stronger conditions that ensure the outer-most containments in the above chain of inclusions are equalities, thus giving a way to compute the stable locus and finitely generated semistable locus for the original unipotent linearisation $L \to X$ using the methods of reductive GIT.

In this thesis we generalise the definitions and constructions in [DK07] to the case where H is any linear algebraic group, not necessarily unipotent or reductive, and study some ways this theory can be used in practice. Thus we develop a theoretical framework for studying non-reductive group actions that is in the same spirit as Mumford's GIT, with the basic guiding goal of obtaining results that are as close as possible to the diagram relating the stable and semistable sets and GIT quotient in the reductive setting. Indeed, our constructions reduce to Mumford's theory when H is a reductive group and $L \to X$ is an ample linearisation over a projective variety. The way we extend Doran and Kirwan's work is to make use of natural residual actions of the reductive group H/H_u and take quotients in stages—first by H_u, then by H/H_u.

Let us now give an outline of the contents of this thesis. We begin in Chapter 1 by recalling background material on linear algebraic groups and their quotients. We discuss various notions of quotient in the category of algebraic varieties and recall the concept of a linearisation, as introduced by Mumford in the first edition of [MFK94]. Some of the main differences between actions of reductive groups and unipotent groups are also highlighted. We then recall the main theorems of GIT for reductive groups in [MFK94], giving particular attention to the case of ample linearisations over projective varieties. We close the chapter with a detailed summary of the work of Doran and Kirwan in [DK07] on GIT for unipotent groups, discussing more fully their consideration of various notions of 'stability' and 'semistability' and also the construction of reductive envelopes.

In Chapter 2 we begin the work of extending the theory of [DK07] to more general linear algebraic groups H, focussing on constructing objects from the data of a linearisation

$L \to X$. Unlike in [DK07], we do not assume X is projective or irreducible, or that the linearisation L is ample. We directly generalise the naively semistable locus, the finitely generated semistable locus $X^{\mathrm{ss,fg}}$, the enveloping quotient (which we denote $X \wr\wr H$) and the enveloped quotient $q(X^{\mathrm{ss,fg}})$ (that is, the image of the finitely generated semistable locus under the natural map $q : X^{\mathrm{ss,fg}} \to X \wr\wr H$ to the enveloping quotient) to the general case. This gives a geometric set-up to work around the fact that the ring of invariant sections for the linearisation may not be finitely generated. To address the fact that the enveloping quotient is only a scheme *locally* of finite type in general, we introduce *inner enveloping quotients* in Definition 2.1.11, which are quasi-compact open subschemes of the enveloping quotient that contain the enveloped quotient $q(X^{\mathrm{ss,fg}})$. Inner enveloping quotients are not canonical to the linearisation, but have the advantage of being quasi-projective varieties. This is shown by introducing the concept of an *enveloping system* in Definition 2.1.16, which is closely related to the idea of a finite separating set of invariants, considered by other authors (e.g. [DeKe02]). In the case where the ring of invariant sections S^H is finitely generated, an inner enveloping quotient is given by $\mathrm{Proj}(S^H)$. A way in which the collection of inner enveloping quotients can be thought of as 'universal' with respect to H-invariant morphisms from the finitely generated semistable locus is also considered.

This all corresponds to the 'semistable' part of Mumford's theory for reductive groups. Building on the notion of stability in [DK07] and the existence of a residual action of H/H_u on an H_u-quotient, we define the *stable locus* X^{s} for a general linearisation of a linear algebraic group over an irreducible variety in Definition 2.3.2. When the group H is reductive or unipotent our definition reduces to that of Mumford or Doran and Kirwan, respectively. For any choice of inner enveloping quotient $\mathcal{U} \subseteq X \wr\wr H$, we show that the natural map $q : X^{\mathrm{ss,fg}} \to \mathcal{U}$ restricts to define a geometric quotient on the stable locus. In Theorem 2.4.2 we obtain a diagram

$$\begin{tikzcd}
X^{\mathrm{s}} \arrow[r, phantom, "\subseteq"] \arrow[d, "\mathrm{geo}"] & X^{\mathrm{ss,fg}} \arrow[r, phantom, "\subseteq"] \arrow[d] & X \arrow[d, dashed] \\
X^{\mathrm{s}}/H \arrow[r, phantom, "\subseteq"] & \mathcal{U} \arrow[r, phantom, "\subseteq"] & \mathrm{Proj}(S^H)
\end{tikzcd}$$

that is analogous to Mumford's in the reductive case, but in contrast the map $q : X^{\mathrm{ss,fg}} \to \mathcal{U}$ is not necessarily surjective, while there are many choices of inner enveloping quotient $\mathcal{U}$ containing X^{s}/H and $\mathcal{U}$ is not necessarily a projective variety.

The possible lack of projectivity of an inner enveloping quotient $\mathcal{U}$ naturally motivates the construction of their projective completions $\overline{\mathcal{U}}$. Any such completion contains the enveloped quotient $q(X^{\mathrm{ss,fg}})$ as a dense constructible subset, so we refer to $\overline{\mathcal{U}}$ as a *compactification of the enveloped quotient* (Definition 3.0.1). Chapter 3 extends the theory of reductive envelopes from [DK07], with the aim of constructing compactifications of the enveloped quotient. We consider the formation of fibre spaces $G \times^{H_u} X$, with G a reductive group, formed from homomorphisms $H \to G$ which restrict to give embeddings of the unipotent radical $H_u \hookrightarrow G$. Such homomorphisms give a diagonal action of the reductive group $H_r = H/H_u$ on $G \times^{H_u} X$ that commutes with the G-action, so $G \times^{H_u} X$ is a $G \times H_r$-variety that comes equipped with a canonical $G \times H_r$-linearisation. We thus extend the work of Doran and Kirwan by taking account of this extra H_r-action. Various kinds of *reductive envelope* $(\overline{G \times^{H_u} X}, \beta, L')$ are defined, where $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$ is an equivariant completion and $L' \to \overline{G \times^{H_u} X}$ is an extension of the $G \times H_r$-linearisation over $G \times^{H_u} X$, by requiring that invariant sections of certain choices of enveloping system over X extend to linear systems over the reductive envelope, satisfying assumptions of varying strength. In Theorem 3.1.14 we show that when the line bundle $L' \to \overline{G \times^{H_u} X}$ in the reductive envelope is ample, then the reductive GIT quotient $\overline{G \times^{H_u} X} /\!\!/_{L'} (G \times H_r)$ gives a compactification of the enveloped quotient, and there is a chain of inclusions

$$X \cap (\overline{G \times^{U} X})^{\mathrm{s}(L')} \subseteq X^{\mathrm{s}} \subseteq X^{\mathrm{ss,fg}} \subseteq X \cap (\overline{G \times^{U} X})^{\mathrm{ss}(L')}$$

in analogy with the unipotent case considered in [DK07].

We also generalise the concept of a *strong reductive envelope* from [DK07] to the more general setting. In Proposition 3.2.2 we show that from any strong reductive envelope $(\overline{G \times^{H_u} X}, \beta, L')$ we obtain equalities

$$X^{\mathrm{s}} = X \cap (\overline{G \times^{U} X})^{\mathrm{s}(L')}, \quad X^{\mathrm{ss,fg}} = X \cap (\overline{G \times^{U} X})^{\mathrm{ss}(L')},$$

which thus provides a way to compute the intrinsically defined stable locus and finitely generated semistable locus using methods from reductive GIT. The existence of strong reductive envelopes with ample L' is therefore especially good for the purposes of computation in our non-reductive geometric invariant theory. When the H-linearisation over X extends to one of the reductive group $G \times H_r$ in an appropriate way, we can use similar arguments to those found in [DK07] to reduce the construction of strong reductive envelopes with ample L' to a study of the homogeneous space G/H_u, first considered by [BBHM63] and studied by Grosshans in [Gro73, Gro97]. This works out particularly well when H_u is a *Grosshans subgroup* of G, for then S^H is a finitely generated algebra and an explicit description of X^{s}, $X^{\mathrm{ss,fg}}$ and $X /\!\!/ H = \mathrm{Proj}(S^H)$ can be obtained in terms of the reductive GIT of $L' \to \overline{G \times^{H_u} X}$ (Corollary 3.2.11). We end Chapter 3 with a study of the space of n unordered points on $\mathbb{P}^1$ under the action of a Borel subgroup in $\mathrm{SL}(2, \Bbbk)$. This serves to both illustrate of our general theory and give an informal look at the potential for studying the variation of non-reductive quotients in certain good cases (a topic we do not pursue in this thesis).

Chapter 4 marks a departure from the program of extending the work of [DK07] to explore an alternative way to construct a compactification of the enveloped quotient. This is based on choosing a subnormal series

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_m = H_u \trianglelefteq H_{m+1} = H, \tag{$\mathcal{C}$}$$

of the unipotent radical and taking quotients in stages, by considering naturally induced actions of H_{j+1}/H_j on H_j-quotients. For any choice of chain ($\mathcal{C}$) the stable locus is contained in the *generalised semistable locus* $X^{\mathrm{ss},\mathcal{C}}$ associated to ($\mathcal{C}$) (Definition 4.1.1), which is a subset of the finitely generated semistable locus that is more compatible with the approach of taking successive quotients. When X is quasi-projective and $L \to X$ is an ample linearisation, by taking a very positive embedding of X into a projective space via a linear system V we can turn the problem of taking a quotient of the unipotent radical H_u into one of studying successive projection rational maps between projective spaces,

$$\mathbb{P}(V^*) \dashrightarrow \mathbb{P}((V^{H_1})^*)) \dashrightarrow \cdots \dashrightarrow \cdots \dashrightarrow \mathbb{P}((V^{H_m})^*) = \mathbb{P}((V^{H_u})^*).$$

In this way we construct a compactification of the enveloped quotient $\overline{\mathcal{U}_m}$ for the linearisation of the unipotent radical H_u which has a canonical ample $H_r = H/H_u$-linearisation, and such that the associated GIT quotient $\overline{\mathcal{U}_m}/\!\!/H_r$ gives a compactification of the enveloped quotient for the original linearisation of H (Proposition 4.1.5). In a bid to make this more effective for computation, we restrict our attention to the case where the chain ($\mathcal{C}$) consists of characteristic subgroups and the quotient groups H_{j+1}/H_j are isomorphic to the additive group $\mathbb{G}_a$, for $j \leq m-1$. Using Weitzenböck's theorem [Wei32], for each $0 \leq j \leq m$ we construct strong reductive envelopes for a natural H_{j+1}/H_j-linearisation on each $\mathbb{P}((V^{H_j})^*)$ and show that the associated stable sets $\mathbb{P}((V^{H_j})^*)^{\mathrm{s}(H_{j+1}/H_j)}$ and finitely generated semistable sets $\mathbb{P}((V^{H_j})^*)^{\mathrm{ss,fg}(H_{j+1}/H_j)}$ can be used to approximate the stable locus X^{s} and generalised semistable locus $X^{\mathrm{ss},\mathcal{C}}$, respectively, for the original linearisation $L \to X$.

Finally we study linearisations of *positively graded unipotent groups* in Chapter 5. These are unipotent groups U that admit a $\mathbb{G}_m$-extension $\hat{U} = U \rtimes \mathbb{G}_m$ such that all the weights for the induced action on the Lie algebra of U are positive. Actions of such groups occur in a number of situations. This includes the study of invariant jet differentials [GG80, Dem97, DMR10, Mer10, BK12] in singularity theory (which relate to Kobayashi's hyperbolicity conjecture, cf. [Kob98]); the construction of moduli spaces of hypersurfaces in weighted projective spaces [Kir09], or more generally of hypersurfaces in toric varieties [BDHK16], and also the question of constructing quotients of the unstable strata arising in a reductive GIT scenario [Kem78, Hes78, Kir84, Nes84]. Given a very ample U-linearisation L over an irreducible projective variety X that extends to a linearisation of $\hat{U}$, we may consider the closed subvariety $Z_{\hat{U},L}$ of $\mathbb{G}_m$-fixed points $z \in X$ such that $\mathbb{G}_m$ acts on $L^*|_z$ with minimal possible weight, and the open subset $X^0_{\hat{U},L}$ of points in X that have a limit point in $Z_{\hat{U},L}$ under the action of $t \in \mathbb{G}_m$, as $t \to 0$. In Theorem 5.1.4 we show that, assuming a certain condition ($\mathfrak{C}$) is satisfied, $X^0_{\hat{U},L}$ admits a locally trivial U-quotient, and if $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ is nonempty then for a suitable rational character χ of $\hat{U}$, the stable locus and finitely generated semistable locus for the twisted linearisation $L^{(\chi)} \to X$ are both equal to $X^0_{\hat{U},L} \setminus$

$(U \cdot Z_{\hat{U},L})$, and the enveloping quotient $X \wr\!\wr_{L(\chi)} \hat{U}$ is a projective variety that is a geometric quotient for the $\hat{U}$-action on $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$. Under the assumption of ($\mathfrak{C}$), we apply this result to the natural projective completion

$$\overline{\hat{U} \times^U X} = \overline{\mathbb{G}_m \times X} = \mathbb{P}^1 \times X$$

of $\hat{U} \times^U X$ to show that the enveloping quotient $(\mathbb{P}^1 \times X) \wr\!\wr_{\mathcal{L}} \hat{U}$ of a certain $\hat{U}$-linearisation $\mathcal{L} \to \mathbb{P}^1 \times X$ defines a compactification of $X^0_{\hat{U},L}/U$. Moreover, the associated enveloping quotient map $(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})} \to (\mathbb{P}^1 \times X) \wr\!\wr_{\mathcal{L}} \hat{U}$ is a geometric quotient for the $\hat{U}$-action on $(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})}$ and $(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})}$ admits a similarly explicit description in terms of subvarieties $Z_{\hat{U},L}$ and $X^0_{\hat{U},L}$ (see Corollary 5.3.8). We should stress that our results here do not require any assumption of normality on X.

At the end we have included an appendix, where we give some basic facts that we felt would unnecessarily slow down the exposition in the main text.

0.2 Notation and Conventions

We work over a ground field $\Bbbk$ that is algebraically closed and of characteristic zero. By 'variety' we mean a reduced, separated scheme of finite type over $\Bbbk$; note that we do not assume varieties are irreducible unless otherwise stated, but we do insist they are separated. If a topological space satisfies the condition that every cover of it by open sets admits a finite subcover then we say it is 'quasi-compact'. By a 'point' in a scheme we will always mean a closed $\Bbbk$-valued point. A *compactification* $X \hookrightarrow \overline{X}$ of a variety X is a dominant open immersion into a projective variety $\overline{X}$.

When we talk about actions of groups on varieties or vector spaces, we always mean a *left action*, unless stated otherwise. In Chapter 5 we will make use of the additive notation for characters, though this is not done in earlier chapters.

When talking about line bundles we will usually be talking about the total space of an invertible sheaf of modules; the sheaf of sections of a line bundle L is denoted by $\underline{L}$, so that $L = \mathbf{Spec}(\underline{L}^*)$. An exception to this is when talking about twisting sheaves $\mathcal{O}(n)$ on varieties—here we don't make any notational distinction between the sheaf and its total space. Given a linearisation $L \to X$ of a group H and a character χ of H, the *twist* $L^{(\chi)} \to X$ of $L \to X$ by χ is the linearisation obtained by multiplying the fibres of the linearisation $L \to X$ by the character χ^{-1}; this will also be emphasised in Section 1.1.2 of the main text.

If $\phi : X \to Y$ is a morphism of schemes, then the natural pullback morphism of structure sheaves is denoted $\phi^{\#} : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$. On the other hand, if $L \to Y$ is a line bundle then we use the notation ϕ^* to denote pull-back $\underline{L} \to \phi_*(\underline{\phi^* L})$. Given an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$, then $\mathcal{F} \boxtimes \mathcal{G} = (\mathrm{pr}_X^* \mathcal{F}) \otimes_{\mathcal{O}_{X \times Y}} (\mathrm{pr}_Y^* \mathcal{G})$, where $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ are the projections.

Unless indicated otherwise, graded rings R are non-negatively $\mathbb{Z}$-graded, i.e. if R is a graded ring then the degree $d \in \mathbb{Z}$ piece S_d is trivial whenever $d < 0$. If $f \in R$ is a non-zero homogeneous element then $R_{(f)}$ is the subring of the localisation R_f consisting of degree 0 elements. Similarly, if M is a graded R-module and $f \in R$ a non-zero homogeneous element, then $M_{(f)}$ is the $R_{(f)}$-submodule of the localisation M_f consisting of degree 0 elements. If $r \in \mathbb{Z}$ is positive then $R^{(r)}$ denotes the Veronese subring of R whose degree m piece is $(R^{(r)})_m = R_{mr}$.

Associated to a vector space V we understand the projective space $\mathbb{P}(V)$ to be the space whose points correspond to one-dimensional subspaces of V. Another way to say this is that $\mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}^\bullet(V^*))$, where $\mathrm{Sym}^\bullet(V^*)$ is the symmetric algebra $\bigoplus_{m \geq 0} \mathrm{Sym}^m(V^*)$. With these conventions, if $L \to X$ is a very ample line bundle on a scheme X with a basepoint-free linear system $V \subseteq H^0(X, L)$, then there is a canonical morphism $X \to \mathbb{P}(V^*)$.

Finally, our main references for facts in algebraic geometry are [Har77] and [Sta15]. The latter is particularly useful for results regarding not-necessarily noetherian schemes.

Chapter 1

Background: Quotients of Varieties and Geometric Invariant Theory

In this chapter we collect some background material that will be used throughout the rest of the thesis. We begin in Section 1.1 by recalling basic definitions concerning linear algebraic groups, then discuss various kinds of quotient in the category of varieties and review the concept of a linearisation of an action. We also recall the definitions of reductive groups and unipotent groups and compare them from the point of view of the geometry of their actions and their invariant theory. In Section 1.2 we give a summary of the main results from Mumford's geometric invariant theory (GIT) for reductive groups, paying particular attention to the case of ample linearisations over projective varieties. Finally, in Section 1.3 we give a detailed survey of the paper [DK07], which will form the basis for our development of a geometric invariant theoretic approach to studying actions of more general linear algebraic groups in subsequent chapters.

Our main references for the material on linear algebraic groups are [Bor91, DG70], while for the material on quotients we have used [Ser58, Ses72, MFK94]. For reductive GIT we have used [MFK94, New78, Dol03].

1.1 Basics of Group Actions and Quotients

1.1.1 Linear Algebraic Groups and Quotients

We begin by recalling some of the basic theory of algebraic groups. Following [Bor91, Chapter 1] we define an *affine algebraic group* to be an affine variety H equipped with a group structure such that the multiplication map $H \times H \to H$ and inversion map $H \to H$ are morphisms of varieties. A *homomorphism* of algebraic groups $H_1 \to H_2$ is a morphism of varieties that is also a homomorphism of the group structures. If $H_1 \hookrightarrow H_2$ is a homomorphism that is also a closed immersion then we say that H_1 is a *closed subgroup* of H_2. A first example of an affine algebraic group is the general linear group $\mathrm{GL}(n, \Bbbk)$, for any integer $n \geq 0$. A *linear algebraic group* is an affine algebraic group that is a closed subgroup of $\mathrm{GL}(n, \Bbbk)$, for some $n \geq 0$. A basic result in the theory of affine algebraic groups says that every affine algebraic group is a linear algebraic group [Bor91, Proposition 1.10]. Throughout this thesis we concern ourselves with linear algebraic groups.

Example 1.1.1. Any finite group is a linear algebraic group. The group $\mathbb{G}_m := (\Bbbk \setminus \{0\}, \times)$ of non-zero elements of $\Bbbk$ under multiplication is a linear algebraic group (indeed, it is just $\mathrm{GL}(1, \Bbbk)$). The group $\mathbb{G}_a := (\Bbbk, +)$ of elements of $\Bbbk$ under addition is also a linear algebraic group: it is isomorphic to the group $\mathbb{U}_2 \subseteq \mathrm{GL}(2, \Bbbk)$ of upper-triangular matrices via $a \mapsto \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$.

Example 1.1.2. (Operations on linear algebraic groups [Bor91, Chapters 1 and 6].) A *normal subgroup* N of an affine algebraic group H is a closed subgroup that is normal as an abstract group. If H is a linear algebraic group and N a normal subgroup, then H/N is a linear algebraic group. Products of linear algebraic groups are linear algebraic groups. More generally, suppose H and G are two linear algebraic groups and $\lambda : G \to \mathrm{Aut}(H)$ is a homomorphism of abstract groups. The *semi-direct product* $H \rtimes_\lambda G$ is the linear algebraic group whose underlying structure as a variety is equal to $H \times G$, but with group multiplication defined by[1]

$$(h_1; g_1) \cdot (h_2; g_2) = (\lambda(g_2^{-1})(h_1)h_2; g_1g_2).$$

Now let X be a variety over $\Bbbk$ and H a linear algebraic group. An *action* of H on X is a (left) action $H \times X \to X$ that is also a morphism of varieties. In this case we will often refer to X as an "*H-variety*" and sometimes we will write $H \curvearrowright X$ to indicate the given action.

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[1] We write points in $H \rtimes_\lambda G$ as $(h; g)$ to emphasize the different group structure.

We usually write the morphism of an action as

$$H \times X \to X, \quad (h, x) \mapsto h \cdot x \quad (\text{or } (h, x) \mapsto hx)$$

if no confusion is likely to occur. We write

$$H \cdot x := \{hx \mid h \in H\}$$

for the *orbit* of a point $x \in X$ under the action of H and

$$\mathrm{Stab}_H(x) := \{h \in H \mid hx = x\}$$

for the *stabiliser* of x. Given a subset $Z \subseteq X$, we say Z is *H-stable*, or *H-invariant*, if $H \cdot z \in Z$ for all $z \in Z$.

Remark 1.1.3. For simplicity, throughout this thesis we will assume all actions are such that stabilisers of generic points are finite.

Given a homomorphism of linear algebraic groups $\rho : H_1 \to H_2$, an H_1-variety X and an H_2-variety Y, we say a morphism of $\phi : X \to Y$ is *equivariant (with respect to $\rho : H_1 \to H_2$)* if $\phi(hx) = \rho(h)\phi(x)$ for all $h \in H_1$ and all $x \in X$. If $H = H_1 = H_2$ and ρ is the identity homomorphism, then we simply say ϕ is *H-equivariant*, and if furthermore H acts trivially on Y (so that $\phi(hx) = \phi(x)$ for all $h \in H$ and all $x \in X$) then we say ϕ is *H-invariant*.

If a linear algebraic group H acts on a variety X then a fundamental question, if vaguely stated, is to ask: does there exist a variety Y that is a 'quotient' of X by the action of H? There are various definitions to make the term 'quotient' more precise, with varying agreement with one's geometric intuition. In the following definition we recall the kinds of 'quotient' we will use throughout this thesis. Before doing so, note that given an H-variety X and an H-stable open subset $U \subseteq X$, there is a canonically induced action of H on the ring of regular functions:

$$(h \cdot f)(x) := f(h^{-1}x) \quad \text{for all } \ x \in U, \ f \in \mathcal{O}(U), \ h \in H, \tag{1.1.1}$$

and one can consider the subring of invariant functions:

$$\mathcal{O}(U)^H = \{f \in \mathcal{O}(U) \mid h \cdot f = f \text{ for all } h \in H\}.$$

Definition 1.1.4. Let H be a linear algebraic group acting on a variety X.

1. A *categorical quotient* is a variety Y together with an H-invariant morphism $\phi : X \to Y$ satisfying the following universal property: any other H-invariant morphism $X \to Z$ admits a unique factorisation through ϕ.

2. A *good quotient* is an H-invariant morphism $\phi : X \to Y$ satisfying the following properties:

   (a) the morphism ϕ is surjective and affine;

   (b) the pull-back map $\phi^{\#} : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ induces an isomorphism of sheaves $\mathcal{O}_Y \cong (\phi_* \mathcal{O}_X)^H$, where $(\phi_* \mathcal{O}_X)^H(U) = \mathcal{O}_X(\phi^{-1}(U))^H$ for each open subset $U \subseteq Y$; and

   (c) if $W_1, W_2 \subseteq X$ are disjoint H-invariant closed subsets, then $\phi(W_1)$ and $\phi(W_2)$ are disjoint closed subsets of Y. (Note this implies $\phi : X \to Y$ is a submersion [MFK94, Chapter 0, §2, Remark 6].)

3. A *geometric quotient* is a good quotient $\phi : X \to Y$ that is also an *orbit space*; i.e. $\phi^{-1}(y)$ is a single H-orbit for each $y \in Y$. In this case we write $Y = X/H$.

4. A *principal H-bundle* (or a *locally trivial quotient*) is an H-invariant morphism $\phi : X \to Y$ such that, for every point $y \in Y$, there is a (Zariski) open neighbourhood $U_y \subseteq Y$ of y and a surjective unramified morphism $\widetilde{U_y} \to U_y$ such that there exists an H-equivariant isomorphism $H \times \widetilde{U_y} \cong \widetilde{U_y} \times_Y X$, where the fibred product $\widetilde{U_y} \times_Y X$ has the canonical H-action and $H \times \widetilde{U_y}$ has the *trivial H-bundle* action, induced by left multiplication by H on itself:

$$H \times (H \times \widetilde{U_y}) \to H \times \widetilde{U_y}, \quad (h, h_0, u) \mapsto (hh_0, u).$$

Definition 1.1.4, 1 is taken from [MFK94, Definition 0.5], while 2–3 are from [Ses72, Definitions 1.4 and 1.5] and 4 is [Ser58, Definition 2.2].

Remark 1.1.5. Because we work exclusively with linear algebraic groups, we may equivalently work with quotients that are locally trivial in the étale topology, or even in the fppf-topology, in Definition 1.1.4, 4.

We have the following chain of implications: principal bundle $\implies$ geometric quotient $\implies$ good quotient $\implies$ categorical quotient. The main non-trivial implication is the last one, whose proof may be found in [MFK94, Chapter 0, §2, Remark 6]. (Note that by virtue of the last implication one often refers to a good quotient as a "good categorical quotient".) In general, none of the reverse implications hold.

Example 1.1.6. (Good quotient $\not\Rightarrow$ geometric quotient.) Let $\mathbb{G}_m$ act on $X = \mathbb{k}^n$ by the usual scaling action, $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$. Then the unique map $\mathbb{k}^n \to \mathrm{pt} := \operatorname{Spec} \mathbb{k}$ to a single point is a good quotient for this action, but is clearly not a geometric quotient: the preimage of pt consists of many orbits.

Example 1.1.7. (Geometric quotient $\not\Rightarrow$ principal bundle.) Let $\mathbb{G}_m$ act on $\mathbb{k}^n \setminus \{0\}$ ($n > 1$) by the action $t \cdot (x_1, \ldots, x_n) = (t^r x_1, \ldots, t^r x_n)$, where $r \geq 2$ is an integer. Then the usual projection $\mathbb{k}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is a geometric quotient which is not a principal $\mathbb{G}_m$-bundle, because the action is not set-theoretically free. (Thus, it is possible for geometric quotients to exist for actions where some stabilisers are non-trivial.)

Example 1.1.8. (Categorical quotient $\not\Rightarrow$ good quotient.) Such examples are more difficult to come by, but do exist. The interested reader can refer to [ANH99, ANH01].

A very useful property of good and geometric quotients that distinguishes them from categorical quotients is that they are determined locally on the base variety. More precisely, [New78, Proposition 3.10]:

- an H-invariant morphism $\phi : X \to Y$ is a good (respectively, geometric) quotient if, and only if, there is is an open cover $\{U_i\}$ of Y such that each restriction $\phi : \phi^{-1}(U_i) \to U_i$ is a good (respectively, geometric) quotient of $H \curvearrowright \phi^{-1}(U_i)$; and

- if $\phi : X \to Y$ is a good (respectively, geometric) quotient, then for each open subset $U \subseteq Y$ the restriction $\phi : \phi^{-1}(U) \to U$ is a good (respectively, geometric) quotient of

$H \curvearrowright \psi^{-1}(U)$.

Given an action of H on X, there are certain topological restrictions on the action that must be fulfilled if a geometric quotient or a principal bundle structure is to exist. If a geometric quotient for $H \curvearrowright X$ exists then the action must be *closed*: that is, for each point $x \in X$ the orbit $H \cdot x$ is a closed subset of X. Furthermore, if a geometric quotient $X \to X/H$ exists, then by [MFK94, Proposition 0.9] and [Ses72, Theorem 6.1] $X \to X/H$ has the structure of a principal H-bundle if, and only if, the action of H on X is *free*: that is, the graph

$$H \times X \to X \times X, \quad (h, x) \mapsto (hx, x)$$

of the action morphism is a closed immersion. Checking freeness of an action can be made simpler by the following lemma from [EG98, §6.3, Lemma 8].

Lemma 1.1.9. *An action of a linear algebraic group H on X is free (in the above sense) if, and only if, it is set-theoretically free and* proper *(that is, the graph $H \times X \to X \times X$ of the action is a proper morphism).*

Example 1.1.6 shows that not every action of a linear algebraic groups on a variety need admit a geometric quotient. From here, there are a couple of possible ways to proceed if one wants to construct a quotient for the action. One way is to enlarge the category in which one works so that it contains a quotient object for the action. This is where the technology of stacks [DM69, LMB00] or, in the case of proper actions with finite stabilisers, algebraic spaces [Art71, Knu71, Kol97, KM97], comes into the picture. Another way, which we shall pursue in this thesis, is to look for nonempty invariant open subsets that admit a geometric quotient. This approach is validated by the following result of Rosenlicht.

Theorem 1.1.10. *[Ros63][2] Let H be a linear algebraic group acting on an irreducible variety X. Then there is a nonempty H-invariant open subset $U \subseteq X$ admitting a quasi-projective geometric quotient for the H-action.*

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[2] For slightly more modern treatments, see also [VP94, Theorem 4.4] and [Dol03, Theorem 6.2].

Rosenlicht's proof of Theorem 1.1.10 is non-constructive, so the question remains of how to explicitly find nonempty open subsets—ideally as large as poss-ible—that admit geometric quotients. This is the basic task of *geometric invariant theory*. We will discuss ways in which this has been done for certain kinds of linear algebraic group in the upcoming Sections 1.2 and 1.3. A natural way to try and construct open subsets of X that admit geometric quotients is to try and glue together quotients of smaller open subsets. However it is possible for this to result in non-separated quotient schemes. A way to navigate this is to instead consider open subsets X_f defined by the non-vanishing of some invariant rational function f and glue the maps $X_f \to \mathrm{Spec}(\mathcal{O}(X_f)^H)$, for then orbits in X_f are separated by orbits in the complement by an invariant function on the base. This is essentially what a *linearisation* achieves for us, to be discussed next.

1.1.2 Linearisations of Actions

The following definition, due to Mumford [MFK94, Definition 1.6], is fundamental for what follows.

Definition 1.1.11. Let H be a linear algebraic group acting on a variety X. A *linearisation* of the action is a line bundle $p : L \to X$ together with a choice of H-action on L such that

1. the bundle projection $p : L \to X$ is H-equivariant; and
2. for each $h \in H$ and $x \in X$, the induced map between the fibres

$$L|_x \to L|_{hx}, \quad l \mapsto hl$$

is linear.

Remark 1.1.12. If $L \to X$ is a linearisation for the action of H on X, we will often represent this using the notation $H \curvearrowright L \to X$, or say that $L \to X$ is an "*H-linearisation*" for short. In general we will not distinguish between the line bundle and the linearisation in our notation, unless this is likely to lead to confusion.

For practical purposes (e.g. the study of moduli problems) the following two classes of examples frequently arise.

Example 1.1.13. In the case where $X = \operatorname{Spec} A$ affine and $L = \mathcal{O}_X = X \times \Bbbk$ is the trivial line bundle, a linearisation of H on $\mathcal{O}_X$ corresponds to a choice of character $\chi : H \to \mathbb{G}_m$ [Dol03, Theorem 7.1 and Corollary 7.1] via

$$H \times (X \times \Bbbk) \to X \times \Bbbk, \quad (h, x, t) \mapsto (hx, \chi(h)t).$$

Example 1.1.14. Let V be a finite dimensional vector space over $\Bbbk$ and $\rho : H \to \mathrm{GL}(V)$ a homomorphism. Then H acts on $\mathbb{P}(V)$ in the obvious manner, and ρ defines a canonical choice of linearisation on the tautological line bundle $\mathcal{O}(-1) \to \mathbb{P}(V)$. This dually defines a linearisation on $\mathcal{O}(1) \to \mathbb{P}(V)$ (see below).

A linearisation $H \curvearrowright L \to X$ gives us a natural action on the sections of $L \to X$ over invariant open subsets $U \subseteq X$, by the formula

$$(h \cdot f)(x) = hf(h^{-1}x) \quad \text{for all } x \in U,\ f \in H^0(U, L),\ h \in H. \tag{1.1.2}$$

Given any invariant open subset $U \subseteq X$ we write

$$H^0(U, L)^H := \{f \in H^0(U, L) \mid h \cdot f = f \text{ for all } h \in H\}$$

for the sections invariant under the action (1.1.2). Elements of $H^0(U, L)^H$ are called *invariants* for the linearisation $L|_U \to U$.

There are also various natural operations on linearisations over an H-variety arising from the standard operations on line bundles. Given an H-linearisation $L \to X$, the dual line bundle $L^* \to X$ has a canonical linearisation, defined fibre-wise by pulling back linear maps along the action of H i.e. for any $x \in X$, an element $h \in H$ acts via

$$(L|_x)^* \to (L|_{hx})^*, \quad (h, \alpha) \mapsto \alpha \circ h^{-1} : L|_{hx} \to \Bbbk.$$

(Note the use of h^{-1} is to ensure the resulting H-action is a left action.) Also, given two H-linearisations $L_1 \to X$ and $L_2 \to X$, there is a canonical linearisation on the tensor product $L_1 \otimes L_2 \to X$ line bundle, induced by the map on fibres

$$((L_1)|_x \otimes (L_2)|_x) \to (L_1)|_{hx} \otimes (L_2)|_{hx}, \quad (h, l_1 \otimes l_2) \mapsto (hl_1) \otimes (hl_2),$$

for any $x \in X$ and $h \in H$.

A *character* of H is simply a group homomorphism $H \to \mathbb{G}_m$. Given a linearisation $L \to X$ and a character $\chi \in \operatorname{Hom}(H, \mathbb{G}_m)$ of H, we can define a linearisation $L^{(\chi)} \to X$, which is said to be the result of *twisting L by the character χ* and is defined to be the linearisation $L \otimes \mathcal{O}_X^{(\chi)}$, where $\mathcal{O}_X^{(\chi)}$ is the linearisation of the trivial bundle $\mathcal{O}_X = X \times \Bbbk$ defined by χ^{-1} (see Example 1.1.13). In other words, $L^{(\chi)} \to X$ is obtained by multiplying the fibres of the linearisation $L \to X$ by the character χ^{-1}.

Finally, if $\phi : X \to Y$ is an equivariant morphism between two H-varieties and $H \curvearrowright L \to Y$ is a linearisation, then there is a unique linearisation on the pullback line bundle $\phi^* L \to X$ making the natural bundle map $\phi^* L \to L$ equivariant. This linearisation makes the pullback map $\phi^* : H^0(Y, L) \to H^0(X, \phi^* L)$ an H-equivariant linear map with respect to the actions defined in (1.1.2).

Given a line bundle $L \to X$, define the *section ring* of $L \to X$ to be the commutative graded ring

$$S := \Bbbk[X, L] := \bigoplus_{r \geq 0} H^0(X, L^{\otimes r}),$$

where the multiplication is induced by the natural maps

$$H^0(X, L^{\otimes r_1}) \otimes H^0(X, L^{\otimes r_2}) \to H^0(X, L^{\otimes (r_1 + r_2)}).$$

The action in (1.1.2) defines a linear action of H on $\Bbbk[X, L]$ that respects the grading and distributes over the multiplicative structure. Given $r > 0$ and an invariant global section $f \in H^0(X, L^{\otimes r})^H$, the open set

$$X_f := \{x \in X \mid f(x) \neq 0\}$$

is H-invariant, and the H-action on $\mathcal{O}(X_f)$ corresponding to the naturally induced one on $S_{(f)}$ under the the canonical isomorphism $S_{(f)} \cong \mathcal{O}(X_f)$ is the one defined by the formula in (1.1.1). A linearisation thus gives a way of studying the invariant functions on certain open subsets of X, which is an important consideration for the construction of geometric quotients (cf. Definition 1.1.4).

Example 1.1.15. In the case of Example 1.1.13, where $X = \operatorname{Spec} A$ is affine, $L = \mathcal{O}_X$ and the action of H on $\mathcal{O}_X$ is defined by a character $\chi : H \to \mathbb{G}_m$, then S is the graded ring $\bigoplus_{r \geq 0} A$ (with the grading corresponding to r) and the ring of invariants S^H is the graded subring of *semi-invariants*,

$$\bigoplus_{r \geq 0} A^H_{\chi^r}, \quad A^H_{\chi^r} := \{f \in A \mid f(hx) = \chi(h)^r f(x) \text{ for all } x \in X, h \in H\};$$

see [Muk03, Chapter 6] or [VP94, §3].

Example 1.1.16. Suppose now X is a projective H-variety and $L \to X$ a very ample linearisation (that is, a linearisation which is very ample as a line bundle). Letting $V = H^0(X, L)^*$, the natural graded ring map $\operatorname{Sym}^\bullet H^0(X, L) \to \Bbbk[X, L]$ defines an embedding $\phi : X \hookrightarrow \mathbb{P}(V)$ (see the Appendix, Section A.1). Dualising the action of H on $H^0(X, L)$ of (1.1.2) defines a canonical linearisation $H \curvearrowright \mathcal{O}_{\mathbb{P}(V)}(1) \to \mathbb{P}(V)$ as in Example 1.1.14, with respect to which ϕ is H-equivariant and $L = \phi^* \mathcal{O}_{\mathbb{P}(V)}(1)$ as linearisations. If $L \to X$ is sufficiently positive then the restriction map $\phi^* : \Bbbk[\mathbb{P}(V), \mathcal{O}(1)] \to \Bbbk[X, L]$ is surjective by Serre vanishing [Har77, Chapter 3, Proposition 5.3], so that $\Bbbk[X, L]^H \cong (\Bbbk[\mathbb{P}(V), \mathcal{O}(1)] / \ker(\phi^*))^H$. Note that in general the induced restriction map on invariants $\phi^* : \Bbbk[\mathbb{P}(V), \mathcal{O}(1)]^H \to \Bbbk[X, L]^H$ is *not* surjective; that is, not every invariant section over X extends to one over $\mathbb{P}(V)$.

Remark 1.1.17. When X is a *normal* quasi-projective variety equipped with an action of a linear algebraic group H, then one can always find an equivariant embedding of X into some projective space $\mathbb{P}^m$, with the H-action on $\mathbb{P}^m$ defined by some representation $H \to \operatorname{GL}(m+1, \Bbbk)$ as in Example 1.1.14 [MFK94, Corollary 1.6]. Hence any normal variety equipped with a linear algebraic group action admits a very ample linearisation.

We saw earlier that, given a linear algebraic group H acting on a variety X, the operations of tensor product and dualising may be applied to H-linearisations. These give an abelian group structure to the set $\operatorname{Pic}^H(X)$ of isomorphism classes of H-linearised line bundles, such that the natural forgetful map $\operatorname{Pic}^H(X) \to \operatorname{Pic}(X)$ to the usual Picard group of X is a homomorphism. We shall see that many constructions in geometric invariant theory are independent of taking positive tensor powers of a linearisation, therefore it is useful to consider the following notion.

Definition 1.1.18. Given a linear algebraic group H acting on a variety X, we define a *rational linearisation* to be an element of $\mathrm{Pic}^H(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 1.1.19. Given an element $\mathcal{L} \in \mathrm{Pic}^H(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, we may write $\mathcal{L} = \frac{1}{n}L$ for some integer $n > 0$ and H-linearisation $L \in \mathrm{Pic}^H(X)$, and if $\tilde{n} \in \mathbb{Z}_{>0}$ and $\tilde{L} \in \mathrm{Pic}^H(X)$ are another such integer and linearisation then we have $L^{\otimes \tilde{n}} = \tilde{L}^{\otimes n}$ within $\mathrm{Pic}^H(X)$. This observation will be used define various geometric invariant theoretic notions for rational linearisations.

We conclude this section by noting a very useful observation regarding linearisations $L \to X$: the induced actions on $H^0(X, L)$ are *locally finite* (also called *rational* in [New78] and elsewhere). In other words, we have the following result (see [MFK94, Chapter 1, §1, Lemma], or [Bor91, Proposition 1.9] in the case $L = \mathcal{O}_X$).

Lemma 1.1.20. *Let H be a linear algebraic group, X an H-variety and $L \to X$ a linearisation. Given a finite-dimensional linear subspace $W \subseteq H^0(X, L)$, there is a finite-dimensional H-invariant subspace $V \subseteq H^0(X, L)$ containing W.*

1.1.3 Unipotent Groups and Reductive Groups

It turns out that the problem of constructing quotients and finding invariants for a given linearisation depends very much on the sort of linear algebraic group one is considering. Two particular sub-classes of group are of most interest: *unipotent* groups and *reductive* groups. These are defined as follows. (In the following definition, and throughout this thesis, we write $e \in H$ for the unit of the group H.)

Definition 1.1.21. Let H be a linear algebraic group.

1. [Bor91, Chapter 4] We say H is *unipotent* if there is a closed embedding $\rho : H \hookrightarrow \mathrm{GL}(n, \Bbbk)$, for some $n \geq 0$, such that $\rho(h) - \rho(e)$ is nilpotent in $\mathrm{GL}(n, \Bbbk)$ for each $h \in H$, i.e. $(\rho(h) - \rho(e))^m = 0$ for some $m \geq 0$ (depending on h).[3]

2. [Bor91, §11.21] The *unipotent radical* H_u of H is the maximal connected normal unipotent subgroup of H.

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[3] If this is the case, then in fact for *any* closed embedding $\rho : H \hookrightarrow G$ into any general linear group G one has $\rho(h) - \rho(e)$ nilpotent for each $h \in H$; cf. [Bor91, Theorem 4.4].

3. [Bor91, §11.21] We say H is *reductive* if $H_u = \{e\}$.

The following are well-known examples of reductive and unipotent groups; cf. [Spr94].

Example 1.1.22. (Reductive groups.) Finite groups are reductive. All semisimple groups are reductive ($\mathrm{GL}(n,\Bbbk)$, $\mathrm{SL}(n,\Bbbk)$, $\mathrm{Sp}(2n,\Bbbk)$, $\mathrm{SO}(n,\Bbbk)$...) Products of reductive groups are again reductive. In particular, groups isomorphic to products of $\mathbb{G}_m$ (which are called *tori*) are reductive.

Example 1.1.23. (Unipotent groups.) The group $\mathbb{G}_a$ is unipotent. The group

$$\mathbb{U}_n := \{(a_{ij}) \in \mathrm{GL}(n,\Bbbk) \mid a_{ii} = 1 \text{ for each } i = 1,\dots,n \text{ and } a_{ij} = 0 \text{ whenever } j < i\}$$

of strictly upper triangular inside $\mathrm{GL}(n,\Bbbk)$ is unipotent, for each $n \geq 1$. In fact, a group H is unipotent if, and only if, it is isomorphic to a closed subgroup of $\mathbb{U}_n$ for some $n \geq 1$ [Bor91, Theorem 4.8]. Products of unipotent groups are unipotent, and all subgroups of unipotent groups are unipotent.

Given a linear algebraic group H, the quotient by the unipotent radical,

$$H_r := H/H_u$$

is a reductive group. Moreover, given a variety X and a normal subgroup N of H, it is easy to show that if X has a geometric N-quotient X/N and X/N has a geometric H/N-quotient $(X/N)/(H/N)$, then the geometric quotient for the H-action on X exists, with $X/H = (X/N)/(H/N)$. This suggests that a natural way to construct geometric quotients by general linear algebraic groups is to try and understand the construction of quotients for unipotent groups and reductive groups.

Reductive and unipotent groups behave rather differently from the point of view of invariant theory. On the one hand, reductive groups have a well behaved representation theory: any representation of a reductive group can be written as a direct sum of irreducible representations [Spr94, §4.6.6]. This has a number of important consequences. Firstly, given a finitely generated $\Bbbk$-algebra A and a reductive group G acting on A in a locally finite fashion, by a theorem of Nagata [Nag64] the ring of invariants A^G is also finitely generated over $\Bbbk$. Thus $\mathrm{Spec}(A^G)$ is an affine *variety*. Related to this fact is the following (see [Nag64, Lemma 5.1.A]): if $I \subseteq A$ is a G-invariant ideal, then any invariant element of A/I lifts to an invariant in A; geometrically stated, any invariant regular function on a G-invariant closed subset $Z = V(I)$ of $\mathrm{Spec}\, A$ extends to a G-invariant regular function on the whole of $\mathrm{Spec}\, A$. Thirdly, given two distinct ideals of A invariant under the G-action, one can always find an element of A^G contained in one but not the other (this follows from a result of Haboush [Hab75], see also [New78, Lemma 3.3]). Geometrically this says that any two disjoint closed invariant subsets of $\mathrm{Spec}\, A$ can be separated by an invariant function—this implies property 2c of Definition 1.1.4, 2 of a good quotient. The upshot is that reductive group actions on affine varieties are amenable to constructing good quotients; see Theorem 1.2.1 in the next section.

All three of the above properties fail for non-reductive group actions. The issue of whether the ring of invariants is finitely generated, which is closely related to the fourteenth problem of Hilbert[4], has arguably received the most attention historically. The following celebrated example of Nagata [Nag59] demonstrates that the ring of invariants for a non-reductive group need not be finitely generated. (We follow the exposition of [Dol03, §4.3].)

Example 1.1.24. Given $n > 0$, let $X = \Bbbk^2 \oplus \cdots \oplus \Bbbk^2$ (n times) and consider the action of $H_1' \times \cdots \times H_n'$ on X, where $H_i' = \{\left(\begin{smallmatrix} c_i & a_i \\ 0 & c_i \end{smallmatrix}\right) \mid a_i, c_i \in \Bbbk,\ c_i \neq 0\}$ acts on the i-th factor of $\Bbbk^2$ in X via standard matrix multiplication. Let $H \subseteq H_1' \times \cdots \times H_n'$ be the subgroup obtained by demanding that $c_1 \cdots c_n = 1$ and $(a_1, \ldots, a_n)$ satisfy three suitable linear equations $\sum_j x_{i,j} a_j = 0$, ($i = 1, 2, 3$). Then for $n = 16$, the ring of invariants $\mathcal{O}(X)^H$ is not finitely generated over $\Bbbk$. It follows that $\mathcal{O}(X)^{H_u}$ is not finitely generated over $\Bbbk$, where H_u is the unipotent radical of H defined by $c_i = 1$ for each $i = 1, \ldots, n$; cf. [Nag60].

Unipotent group actions nevertheless have their own distinctive invariant theoretic flavour; indeed, topologically they can be better behaved than reductive groups. For example, every action of a unipotent group on an affine variety is closed. Somewhat more

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[4] A good survey of counterexamples to Hilbert's fourteenth problem from an invariant theoretic perspective is [Fre01].

strikingly, every unipotent group is *special* (in the sense of [Ser58, §4.1]): any principal bundle of a unipotent group is *Zariski*-locally trivial, not just locally trivial in the isotrivial topology [Ser58, Proposition 14]. Moreover, the following result is of particular note:

Proposition 1.1.25. *[AD07, Theorem 3.12] Suppose X is an affine variety acted upon by a unipotent group U and a locally trivial quotient $X \to X/U$ exists. Then X/U is affine if, and only if, $X \to X/U$ is a trivial U-bundle.*

Representations of unipotent groups can be studied via representations of their *Lie algebra*. Recall that the Lie algebra $\operatorname{Lie} H$ associated to a linear algebraic group H is the tangent space T_eH of H at the identity element e, together with its Lie bracket obtained by identifying T_eH with the left-invariant derivations $\mathcal{O}(H) \to \mathcal{O}(H)$ [Bor91, Chapter 3]. If $H = U$ is unipotent then $\operatorname{Lie} U$ is a nilpotent Lie algebra and there is an isomorphism of varieties

$$\exp : \operatorname{Lie} U \xrightarrow{\cong} U,$$

called the *exponential map* [DG70, Chapter II, §6.3]. Given any $\xi \in \operatorname{Lie} U$, the assignment $t \mapsto \exp(t\xi)$ defines a closed embedding *of algebraic groups* $\mathbb{G}_a \hookrightarrow U$, with the property that $\frac{d}{dt}|_{t=0} \exp(t\xi) = \xi$. If V is any U-representation, one obtains naturally a representation of the Lie algebra $\operatorname{Lie} U$ on V via differentiation; conversely, if V is a representation of $\operatorname{Lie} U$, then one obtains a U-action on V via

$$\exp(\xi) \cdot v = \sum_{n \geq 0} \tfrac{1}{n!} \xi^n(v), \quad \xi \in \operatorname{Lie} U, \ v \in V;$$

note the sum is finite because $\operatorname{Lie} U$ is nilpotent. Thus U-representations are equivalent to representations of the Lie algebra $\operatorname{Lie} U$. Given any U-representation V, the subspace of fixed points V^U is equal to the intersection $\bigcap_{\xi \in \operatorname{Lie} U} \ker(\xi : V \to V)$.

1.2 Geometric Invariant Theory for Actions of Reductive Groups

In the first edition of [MFK94] Mumford introduced his ‘geometric invariant theory’ (GIT) to give constructive ways of finding invariant open sets of points inside a G-variety X that admit geometric quotients, when G is a reductive linear algebraic group. In the case where

X is affine, the good invariant theoretic properties of the action of G on $\mathcal{O}(X)$ yield the following geometric statement.

Theorem 1.2.1. *[MFK94, Chapter 1, §2]*[5] *Let* $X = \operatorname{Spec} A$ *be an affine variety upon which a reductive group* G *acts. Then*

1. *the natural map* $\phi : X \to \operatorname{Spec}(A^G)$ *induced by the inclusion* $A^G \hookrightarrow A$ *is a good categorical quotient; and*

2. *the set* $U := \{x \in X \mid G \cdot x \text{ is closed and } \operatorname{Stab}_G(x) \text{ is finite}\}$ *is an open subset of* X*, and the restriction of* ϕ *to* U *gives a geometric quotient* $U \to \phi(U)$ *for the* G*-action on* U*, with* $\phi(U)$ *open in* $\operatorname{Spec}(A^G)$*.*

To deal with the more general case where G acts on any variety X, Mumford considered patching together the quotients of affine open subsets by using the fact that good and geometric quotients are local on the base. The affine opens which arise are defined in terms of the non-vanishing loci of invariant sections for some choice of linearisation $G \curvearrowright L \to X$, satisfying extra conditions.

Definition 1.2.2. [MFK94, Chapter 1, §4] Let X be a G-variety and $L \to X$ a linearisation. A point $x \in X$ is called

1. *semistable* if there is an invariant $f \in H^0(X, L^{\otimes r})^G$, with $r > 0$, such that $f(x) \neq 0$ and X_f is affine; and

2. *stable* if there is an invariant $f \in H^0(X, L^{\otimes r})^G$, with $r > 0$, such that X_f is affine, the G-action on X_f is closed and $\operatorname{Stab}_G(y)$ is finite for all $y \in X_f$.

We denote the subset of semistable (respectively, stable) points by $X^{\mathrm{ss}(L)}$ (respectively $X^{\mathrm{s}(L)}$), dropping the mention of L if there is no risk of confusion. Note that we have followed [New78, Chapter 3, §5] in requiring finite stabilisers for Definition 1.2.2, 2 of 'stable'; this corresponds to Mumford's definition of 'properly stable' in [MFK94, Definition 1.8].

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[5] While Theorem 1.2.1, 2 is not stated explicitly in [MFK94, Chapter 1, §2], it follows easily from the material there, together with the Closed Orbit Lemma [Bor91, Proposition 1.8] and the lower semi-continuity of the function $x \mapsto \dim(H \cdot x)$. See [New78, Proposition 3.8] for a proof.

Remark 1.2.3. The sets X^{ss} and X^{s} are G-invariant open subsets of X that may be defined for rational linearisations $\mathcal{L}$, in the following way: if $n > 0$ is an integer such that $L = n\mathcal{L}$ is in $\mathrm{Pic}^G(X)$, then define $X^{\mathrm{ss}(\mathcal{L})} = X^{\mathrm{ss}(L)}$ and $X^{\mathrm{s}(\mathcal{L})} = X^{\mathrm{s}(L)}$. Since $X_f = X_{f^m}$ for any global section f of a line bundle and any integer $m > 0$, this is well-defined by Remark 1.1.19.

The central result of Mumford's geometric invariant theory can now be stated.

Theorem 1.2.4. *[MFK94, Theorem 1.10] Let G be a reductive group acting on a variety X and $L \to X$ a linearisation for the action. Then*

1. *the semistable locus X^{ss} has a good categorical quotient $\phi : X^{\mathrm{ss}} \to X/\!\!/G$ onto a quasi-projective variety $X/\!\!/G$, and there is an ample line bundle $M \to X/\!\!/G$ pulling back to a positive tensor power of $L|_{X^{\mathrm{ss}}}$ under ϕ; and*
2. *the image of X^{s} under ϕ is an open subset of $X/\!\!/G$, and the restriction of ϕ to X^{s} gives a geometric quotient $\phi : X^{\mathrm{s}} \to \phi(X^{\mathrm{s}})$ for the action of G on X^{s}.*

The variety $X/\!\!/G$ is called the *GIT quotient* for the linearisation $G \curvearrowright L \to X$. By a result of Seshadri [Ses77, Proposition 9] the GIT quotient $X/\!\!/G$ can be regarded topologically as the quotient $X^{\mathrm{ss}}/\!\sim$ of X^{ss} under the 'S-equivalence' relation $\sim$, where $x_1 \sim x_2$ if, and only if, the closures of $G \cdot x_1$ and $G \cdot x_2$ in X^{ss} intersect nontrivially.

The affine case of Theorem 1.2.1 can be recovered from Theorem 1.2.4 by considering the linearisation of $L = \mathcal{O}_X \to X$ defined by the trivial character from Examples 1.1.13 and 1.1.15: in this case the constant function $1 \in H^0(X, L)$ is an invariant, so $X^{\mathrm{ss}(\mathcal{O}_X)} = X$, and it follows immediately that U from Theorem 1.2.1 is equal to the stable locus $X^{\mathrm{s}(\mathcal{O}_X)}$.

Another important special case of Theorem 1.2.4 is when X is *projective* and $L \to X$ is an *ample* linearised line bundle. In this case the ring of invariant sections S^G is a finitely generated $\Bbbk$-algebra by Nagata's theorem [Nag64], and $X/\!\!/G = \mathrm{Proj}(S^G)$ is a projective variety [MFK94, Page 40]. Furthermore the good quotient $\phi : X^{\mathrm{ss}} \to X/\!\!/G$ is a representative of the rational map $X \dashrightarrow \mathrm{Proj}(S^G)$ induced by the inclusion $S^G \hookrightarrow S$. Thus we have the following commutative diagram, with all inclusions open.

$$\begin{tikzcd}
X^{\mathrm{s}} \arrow[r, phantom, "\subseteq"] \arrow[d, "\phi", "\mathrm{geo}"'] & X^{\mathrm{ss}} \arrow[r, phantom, "\subseteq"] \arrow[d, two heads, "\phi", "\mathrm{good}"'] & X \arrow[d, dashed] \\
X^{\mathrm{s}}/G \arrow[r, phantom, "\subseteq"] & X/\!\!/G \arrow[r, phantom, "="] & \mathrm{Proj}(S^G)
\end{tikzcd} \tag{1.2.1}$$

In the case where X is projective and $L \to X$ is ample the GIT quotient $X/\!\!/G$ may therefore be regarded as a canonical compactification of the geometric quotient X^{s}/G of the stable locus.

Another appealing feature of the case where X is projective and $L \to X$ is ample is that we have an effective way to compute the semistable and stable loci, via the Hilbert-Mumford criterion. We present here two versions of this result. For the first, recall that if G is any reductive group, then a *one-parameter subgroup* of G (or *1-PS* for short) is simply a non-trivial homomorphism $\lambda : \mathbb{G}_m \to G$. Given a point $x \in X$ and a one-parameter subgroup λ, the limit[6] $x_0 := \lim_{t\to 0} \lambda(t) \cdot x$ is a well-defined point in X, because X is proper. The subgroup λ fixes x_0, hence $\mathbb{G}_m$ acts on the fibre $L|_{x_0}$ over x_0 via λ; define

$$\mu(x, \lambda) := \text{weight for the } \mathbb{G}_m\text{-action on } L|_{x_0}.$$

Then the Hilbert-Mumford criterion can be stated as follows.

Theorem 1.2.5. *[MFK94, Theorem 2.1] Let G be a reductive group, X a projective G-variety and $L \to X$ an ample linearisation. Then for any point $x \in X$,*

$$x \in X^{\mathrm{ss}} \iff \mu(x,\lambda) \geq 0 \textit{ for all 1-PS } \lambda : \mathbb{G}_m \to G;$$

$$x \in X^{\mathrm{s}} \iff \mu(x,\lambda) > 0 \textit{ for all 1-PS } \lambda : \mathbb{G}_m \to G.$$

The second form of the Hilbert-Mumford criterion we shall need makes use of an embedding in a projective space. Suppose still that X is a projective G-variety, with G reductive, but now assume $L \to X$ is a very ample linearisation. As in Example 1.1.16, X embeds into the projective space $\mathbb{P}(V)$ equivariantly, where $V = H^0(X, L)^*$. Fix a maximal torus $T \subseteq G$ and let $\mathrm{Hom}(T, \mathbb{G}_m)$ be the abelian group of characters of T. The action of

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[6] By '$\lim_{t\to 0} \lambda(t) \cdot x$' we mean the value at $0 \in \Bbbk$ of $\phi : \Bbbk \to X$, where ϕ is the unique extension of the morphism of varieties $t \mapsto \lambda(t) \cdot x$ to a morphism on $\Bbbk$.

T on V is diagonalisable [Bor91, Proposition 8.4], so we may decompose V into T-weight spaces:

$$V = \bigoplus_{\chi \in \mathrm{Hom}(T, \mathbb{G}_m)} V_\chi, \quad V_\chi := \{v \in V \mid t \cdot v = \chi(t)v \text{ for all } t \in T\}.$$

Given $x \in X$, write $x = [v] \in \mathbb{P}(V)$ with $v \in V \setminus \{0\}$ and let $v = \sum_\chi v_\chi$ with $v_\chi \in V_\chi$. Define the *weight polytope* of x to be

$$\Delta_x := \overline{\text{convex hull of } \{\chi \mid v_\chi \neq 0\}} \subseteq \mathrm{Hom}(T, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where the closure is taken with respect to the usual Euclidean topology on the vector space $\mathrm{Hom}(T, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$. Denote the interior of Δ_x inside $\mathrm{Hom}(T, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$ by Δ_x°. Then the Hilbert-Mumford criterion can be stated in the following way (see [Dol03, Theorem 9.2] and [Dol03, Theorem 9.3]).

Theorem 1.2.6. *Retain the preceding notation.*

1. *A point $x \in X$ is semistable (respectively, stable) for $G \curvearrowright L \to X$ if, and only if, for each $g \in G$ the point gx is semistable (respectively, stable) for the restricted linearisation $T \curvearrowright L \to X$.*

2. *For any point $x \in X$ we have*

$$x \text{ is semistable for } T \curvearrowright L \to X \iff 0 \in \Delta_x;$$
$$x \text{ is stable for } T \curvearrowright L \to X \iff 0 \in \Delta_x^\circ.$$

Thus we see that in the case where X is projective and $L \to X$ is very ample, semistability and stability can be computed in terms of weights for torus actions.

1.3 Doran and Kirwan's Geometric Invariant Theory for Unipotent Group Actions

Given the effectiveness of Mumford's GIT for studying quotients of actions of reductive groups, there has been interest in developing a similar 'geometric invariant theoretic' approach for unipotent group actions. Such a programme is taken up by Doran and Kirwan in their paper 'Towards non-reductive geometric invariant theory' [DK07], building on previous work such as [Fau83, Fau85, GP93, GP98, Win03]. Given an irreducible projective variety X with an action of a unipotent group U and an ample linearisation $L \to X$ for the action, they consider various notions of 'stability' (intrinsic to the data of the linearisation $L \to X$) that admit geometric quotients, formulate an analogue of the GIT quotient in this context, and relate these notions to reductive geometric invariant theory via the construction of certain associated bundles. Much of their approach will form the backbone of our development of a geometric invariant theory for more general linear algebraic groups in subsequent chapters. In this section we will therefore summarise their main definitions and results. We also point out some errors in their exposition, but leave details of how to correct them to further chapters (where much of their theory will be subsumed by ours).

We assume for the rest of this section that U is a unipotent group acting on an irreducible projective variety X with ample linearisation $L \to X$.

1.3.1 Intrinsic Notions of Semistability and Stability

As in Section 1.1.2, let $S = \Bbbk[X, L]$ be the section ring. The inclusion $S^U \hookrightarrow S$ defines a rational map

$$q : X \dashrightarrow \operatorname{Proj}(S^U)$$

which is U-invariant on its maximal domain of definition.

Definition 1.3.1. [DK07, Definition 4.1.1] Let U be a unipotent linear algebraic group acting on an irreducible projective variety X and $L \to X$ an ample linearisation. The *naively semistable locus* is the open subset

$$X^{\mathrm{nss}} := \bigcup_{f \in I^{\mathrm{nss}}} X_f$$

of X, where $I^{\mathrm{nss}} := \bigcup_{r>0} H^0(X, L^{\otimes r})^U$ is the set of invariant sections of positive tensor powers of L.

The rational map q restricts to define a U-invariant morphism $q : X^{\mathrm{nss}} \to \operatorname{Proj}(S^U)$. As Nagata showed [Nag59], the ring of invariants S^U need not be finitely generated over

$\Bbbk$, so $\mathrm{Proj}(S^U)$ is in general a non-noetherian scheme. It can also happen that this map is not surjective, with the image only a dense constructible subset of $\mathrm{Proj}(S^U)$ in general, even if $\mathrm{Proj}(S^U)$ is of finite type (an example of this phenomenon—which features later in Example 1.3.18—is given in [DK07, §6]). To address the first of these issues, Doran and Kirwan consider the following subset of X^{nss}.

Definition 1.3.2. [DK07, Definition 4.2.6] Let U be a unipotent linear algebraic group acting on an irreducible projective variety X and $L \to X$ an ample linearisation. The *finitely generated semistable locus* is the open subset

$$X^{\mathrm{ss,fg}} := \bigcup_{f \in I^{\mathrm{ss,fg}}} X_f$$

of X^{nss}, where

$$I^{\mathrm{ss,fg}} := \left\{ f \in \bigcup_{r>0} H^0(X, L^{\otimes r})^U \mid \mathcal{O}(X_f)^U \text{ is a finitely generated } \Bbbk\text{-algebra} \right\}.$$

The image of $X^{\mathrm{ss,fg}}$ under the map $q : X^{\mathrm{nss}} \to \mathrm{Proj}(S^U)$ is contained in the open subscheme of $\mathrm{Proj}(S^U)$ obtained by patching together the affine open subsets $\mathrm{Spec}(\mathcal{O}(X_f)^U)$ for $f \in I^{\mathrm{ss,fg}}$:

Definition 1.3.3. [DK07, Definition 4.2.7] Let U be a unipotent linear algebraic group acting on an irreducible projective variety X and $L \to X$ an ample linearisation. The *enveloping quotient* is the open subscheme

$$X /\!\!/ U := \bigcup_{f \in I^{\mathrm{ss,fg}}} \mathrm{Spec}(\mathcal{O}(X_f)^U) \subseteq \mathrm{Proj}(S^U)$$

of $\mathrm{Proj}(S^U)$, together with the canonical map $q : X^{\mathrm{ss,fg}} \to X /\!\!/ U$. The image $q(X^{\mathrm{ss,fg}})$ of this map is called the *enveloped quotient*.

The enveloping quotient $X /\!\!/ U$ is canonically defined and, as we will shortly see, in some sense plays the role of Mumford's reductive GIT quotient [MFK94]. But there are two significant differences from the reductive case to be aware of (compare with the discussion after Theorem 1.2.4). Firstly, $X /\!\!/ U$ is not a projective variety in general; however if S^U is a finitely generated $\Bbbk$-algebra then $X^{\mathrm{ss,fg}} = X^{\mathrm{nss}}$ and $X/\!\!/U = \mathrm{Proj}(S^U)$ is a projective variety. Secondly, the map $q : X^{\mathrm{ss,fg}} \to X/\!\!/U$ is not necessarily surjective and the image $q(X^{\mathrm{ss,fg}})$ is not necessarily a variety. In particular, neither $X/\!\!/U$ nor $q(X^{\mathrm{ss,fg}})$ are a categorical quotient of $X^{\mathrm{ss,fg}}$ in general, cf. [DK07, Remark 4.2.8].

Remark 1.3.4. The enveloping quotient $X/\!\!/U$ is a scheme *locally* of finite type. In [DK07, Proposition 4.2.9] it is erroneously claimed that $X/\!\!/U$ is a quasi-projective variety. The basic problem is that it is not necessarily quasi-compact: the ideal in S^U generated by $I^{\mathrm{ss,fg}}$ may not satisfy the ascending chain condition and we cannot guarantee that finitely many of the affine open subsets $\mathrm{Spec}(\mathcal{O}(X_f)^U)$, for $f \in I^{\mathrm{ss,fg}}$, cover $X/\!\!/U$. (In the proof of [DK07, Proposition 4.2.9] the authors implicitly assume there is such a finite cover of $X/\!\!/U$ in order to construct an embedding of $X/\!\!/U$ into a projective space.) Geometrically speaking, the problem is that even though finitely many of the open sets X_f, with $f \in I^{\mathrm{ss,fg}}$, cover $X^{\mathrm{ss,fg}}$, the enveloping quotient map $q : X^{\mathrm{ss,fg}} \to X/\!\!/U$ is not surjective in general. However, if either S^U is a finitely generated $\Bbbk$-algebra or the enveloping quotient map $q : X^{\mathrm{ss,fg}} \to X/\!\!/U$ is surjective then the proof of [DK07, Proposition 4.2.9] goes through to show that $X/\!\!/U$ is a quasi-projective variety.

The finitely generated semistable locus $X^{\mathrm{ss,fg}}$ is analogous to Mumford's notion of semistability in reductive GIT (cf. Definition 1.2.2, 1), and indeed later in their paper the authors of [DK07] dub $X^{\mathrm{ss,fg}}$ as their notion of the 'semistable' locus for the linearisation $U \curvearrowright L \to X$ [DK07, Definition 5.3.7]. They also define various kinds of 'stable' set, each of which are subsets of $X^{\mathrm{ss,fg}}$ whose images under the enveloping quotient map $q : X^{\mathrm{ss,fg}} \to X/\!\!/U$ define geometric quotients. In the end, they consider the following 'locally trivial' version to be the most suitable notion of stability in the unipotent setting.

Definition 1.3.5. [DK07, Definition 4.2.6] Let U be a unipotent linear algebraic group acting on an irreducible projective variety X and $L \to X$ an ample linearisation. The set of *locally trivial stable* points (later called the set of *stable* points in [DK07, Definition 5.3.7]) is the set

$$X^{\mathrm{s}} = X^{\mathrm{lts}} = \bigcup_{f \in I^{\mathrm{lts}}} X_f,$$

where

$$I^{\mathrm{lts}} := \left\{ \; f \in \bigcup_{r>0} H^0(X, L^{\otimes m})^U \;\middle|\; \begin{array}{c} \mathcal{O}(X_f)^U \text{ is a finitely generated } \Bbbk\text{-algebra} \\ \text{and } q : X_f \to \operatorname{Spec}(\mathcal{O}(X_f)^U) \text{ is a} \\ \text{locally trivial geometric quotient} \end{array} \right\}.$$

Proposition 1.3.6. *[DK07, §4] Let U be a unipotent linear algebraic group acting on an irreducible projective variety X and $L \to X$ an ample linearisation. The image $q(X^{\mathrm{s}})$ of X^{s} under the enveloping quotient map $q : X^{\mathrm{ss,fg}} \to X/\!\!/U$ is an open subscheme of $X/\!\!/U$ that is a quasi-projective variety, and $q : X^{\mathrm{s}} \to q(X^{\mathrm{s}})$ is a geometric quotient.*

$$\begin{tikzcd}
X^{\mathrm{s}} \arrow[r, phantom, "\subseteq"] \arrow[d, "\mathrm{geo}"'] & X^{\mathrm{ss,fg}} \arrow[r, phantom, "\subseteq"] \arrow[d] & X^{\mathrm{nss}} \arrow[d, "q"] \\
q(X^{\mathrm{s}}) \arrow[r, phantom, "\subseteq"] & X/\!\!/U \arrow[r, phantom, "\subseteq"] & \operatorname{Proj}(S^U)
\end{tikzcd}$$

It is helpful to compare this to the case where G is reductive and $L \to X$ is an ample linearisation over a projective G-variety. The diagram in Proposition 1.3.6 is similar to (1.2.1), but the unipotence of U leads to two main differences already mentioned: the enveloping quotient $X/\!\!/U$ need not be projective and $q : X^{\mathrm{ss,fg}} \to X/\!\!/U$ is not in general a good categorical quotient.

Remark 1.3.7. It is clear from Remark 1.1.19 and the definitions that X^{nss}, $X^{\mathrm{ss,fg}}$, X^{s} and $X/\!\!/U$ may be defined for rational linearisations.

1.3.2 Extending to Reductive Linearisations

Having considered intrinsic notions of semistability and stability, Doran and Kirwan next relate this to a reductive linearisation, with an eye to finding ways to compute X^{lts}, $X^{\mathrm{ss,fg}}$ and $X/\!\!/U$. They do this by making use of fibre spaces associated to homogeneous spaces G/U, where G is a reductive group. We take a moment to recall the general construction of such fibre spaces.

Let H_1 and H_2 be linear algebraic groups and suppose $H_1 \hookrightarrow H_2$ is a closed embedding. For the moment suppose also that X is any H_1-variety. Then we may consider the diagonal action of H_1 on the product $H_2 \times X$:

$$H_1 \curvearrowright H_2 \times X, \quad h_1 \cdot (h_2, x) := (h_2 h_1^{-1}, h_1 x),$$

where $h_1 \in H_1$, $h_2 \in H_2$ and $x \in X$. If H_1 is a special group (in the sense of [Ser58, §4.1]), or if X satisfies some mild assumptions—for example, if X is quasi-projective, or more generally if any finite set of points in X is contained in an affine open subset—the geometric quotient $H_2 \times^{H_1} X$ for this action exists as a variety [7] (see [EG98, Proposition 23] in the case where H_1 is special, or [VP94, Theorem 4.19] otherwise). This quotient is the associated fibre space of the principal H_1-bundle $H_2 \to H_2/H_1$ with fibre X; see [Ser58, §3.2]. We shall write points in $H_2 \times^{H_1} X$ as equivalence classes $[h_2, x]$ of points $(h_2, x) \in H_2 \times X$. The action of H_2 on $H_2 \times^{H_1} X$ induced by left multiplication of H_2 on itself makes $H_2 \times^{H_1} X$ into an H_2-variety. Note that there is a natural closed immersion

$$\alpha : X \hookrightarrow H_2 \times^{H_1} X, \quad x \mapsto [e, x],$$

which is H_1-equivariant with respect to H_1 acting on $H_2 \times^{H_1} X$ through the action of H_2.

Suppose $L \to X$ is a linearisation for the H_1-action on X. Then this extends to a natural H_2-linearisation $H_2 \times^{H_1} L \to H_2 \times^{H_1} X$ which pulls back to $L \to X$ under α, using the same constructions as above. For brevity, we will usually abuse notation and write $L \to H_2 \times^{H_1} X$ for this linearisation instead of $H_2 \times^{H_1} L$, unless confusion is likely to arise. Observe that, because the projection $H_2 \times X \to H_2 \times^{H_1} X$ is a categorical quotient, pullback along α induces an isomorphism of graded rings

$$\alpha^* : \Bbbk[H_2 \times^{H_1} X, L]^{H_2} \xrightarrow{\cong} \Bbbk[X, L]^{H_1}.$$

Let us now return to the setting where U is a unipotent group acting on an irreducible projective variety X with ample linearisation $L \to X$. Given a closed embedding of U into some reductive group G (e.g. $G = \mathrm{GL}(n, \Bbbk)$ for a suitable n) Doran and Kirwan consider the G-linearisation $G \curvearrowright L = G \times^U L \to G \times^U X$ [DK07, §5]. This is an ample linearisation over the quasi-projective variety $G \times^U X$, so it makes sense to ask for semistability and stability, in the sense of Mumford's reductive GIT (Definition 1.2.2).

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[7] Indeed, $H_2 \times X \to H_2 \times^{H_1} X$ is in fact a principal H_1-bundle [Ser58, Proposition 4].

Definition 1.3.8. [DK07, Definition 5.1.6] Let U be a unipotent group contained in a reductive group G as a closed subgroup and let $L \to X$ be an ample U-linearisation over a projective U-variety X. Define the set of *Mumford stable* points to be

$$X^{\mathrm{ms}} := \alpha^{-1}((G \times^U X)^{\mathrm{s}})$$

and the set of *Mumford semistable* points to be

$$X^{\mathrm{mss}} := \alpha^{-1}((G \times^U X)^{\mathrm{ss}})$$

where (semi)stability of $G \times^U X$ is defined as in Definition 1.2.2 with respect to the G-linearisation $G \times^U L \to G \times^U X$, and $\alpha : X \hookrightarrow G \times^U X$ is the natural closed immersion.

These sets would appear to depend on the choice of G and embedding $U \hookrightarrow G$, but in fact this is not the case by virtue of the following result.

Proposition 1.3.9. *[DK07, Lemma 5.1.7 and Proposition 5.1.10] Given a unipotent group U, a reductive group G containing U as a closed subgroup and an ample U-linearisation $L \to X$ of a projective U-variety X, then*

$$X^{\mathrm{mss}} = X^{\mathrm{ms}} = X^{\mathrm{lts}}.$$

There are two main facts used in the proof that $X^{\mathrm{mss}} = X^{\mathrm{ms}}$: firstly, any stabiliser of a point with a closed G-orbit in $(G \times^U X)^{\mathrm{ss}}$ must have a *reductive* stabiliser (by Matsushima's criterion [VP94, Theorem 4.17]) that is also a subgroup of U, hence is trivial; and secondly, any U-orbit within a U-stable affine subvariety is necessarily closed. Both of these rely on the unipotency of U in an essential way. The equality $X^{\mathrm{ms}} = X^{\mathrm{lts}}$ is established by using descent to relate the notions of U-local triviality of suitable affine open subsets of X with G-local triviality of the corresponding subsets in $G \times^U X$.

Before continuing, let us state that *we assume from now on that the linearisation $U \curvearrowright L \to X$ is such that $X/\!\!/U$ is quasi-projective.* (See Remark 1.3.4.)

As we observed, $G \times^U X$ is only a *quasi-projective* variety, so computing the (semi)stable locus for the linearisation $G \times^U L \to G \times^U X$ is difficult in general; on the other hand, reductive GIT is very effective at dealing with ample linearisations over *projective* varieties. So the authors of [DK07] next turn to study certain G-equivariant compactifications $\overline{G \times^U X}$ of $G \times^U X$, together with extensions $L' \to \overline{G \times^U X}$ of the linearisation $L \to G \times^U X$, in a bid to compute the stable locus $X^{\mathrm{s}} = X^{\mathrm{lts}}$ for $U \curvearrowright L \to X$ and study compactifications of the enveloping quotient $X /\!\!/ U$. More precisely, they look for $G \curvearrowright L' \to \overline{G \times^U X}$ such that

1. the pre-image of the stable locus of $L' \to \overline{G \times^U X}$ under $X \hookrightarrow \overline{G \times^U X}$ is contained in $X^{\mathrm{s}} = X^{\mathrm{lts}}$; and
2. there is a naturally induced open embedding of $X /\!\!/ U$ into the reductive GIT quotient $\overline{G \times^U X} /\!\!/_{L'} G$.

The following definition, which can be regarded as an enhanced version of a collection of 'separating invariants' in [DeKe02, Definition 2.3.8], facilitates this.[8]

Definition 1.3.10. Let U be a unipotent group acting on a projective variety X, with ample linearisation $L \to X$, and let G be a reductive group containing U as a closed subgroup. A finite collection $A \subseteq \bigcup_{r>0} H^0(X, L^{\otimes r})^U$ is called a *finite fully separating set of invariants* if

1. $X^{\mathrm{nss}} = \bigcup_{f \in A} X_f$ and the set A is *separating*: whenever $x, y \in X^{\mathrm{nss}}$ are distinct points and there exist U-invariant sections $g_0, g_1 \in H^0(X, L^{\otimes r})^U$ (for some $r > 0$) such that $g_0(x) \neq 0$, $g_1(y) \neq 0$ and $[g_0(x) : g_1(x)] \neq [g_0(y) : g_1(y)]$ (as points in $\mathbb{P}^1$), then there are sections $f_0, f_1 \in A$ of some common tensor power of L such that $f_0(x) \neq 0$, $f_1(y) \neq 0$ and $[f_0(x) : f_1(x)] \neq [f_0(y) : f_1(y)]$;
2. for every $x \in X^{\mathrm{s}}$ there is $f \in A$ such that $f(x) \neq 0$ and $(G \times_U X)_F$ is affine, where F is the G-invariant section of a tensor power of $L \to G \times^U X$ such that $\alpha^* F = f$; and
3. we have $X /\!\!/ U \subseteq \bigcup_{f \in A} \operatorname{Spec}(\mathcal{O}(X_f)^U) \subseteq \operatorname{Proj}(S^U)$, and for every $x \in X^{\mathrm{ss,fg}}$ there is $f \in A$ such that $x \in X_f$ and $\mathcal{O}(X_f)^U \cong \Bbbk[A]_{(f)}$ (where $\Bbbk[A]$ is the graded subalgebra of $S^U = \Bbbk[X, L]^U$ generated by A).

The existence of a finite fully separating set of invariants follows by a suitable application of Hilbert's Basis Theorem inside $\Bbbk[X, L]$ and the quasi-compactness of X^{nss}, $X^{\mathrm{ss,fg}}$

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[8] The definition we give corrects a couple of small errors in [DK07, Definition 5.2.2].

and $X/\!\!/U$ (recall our assumption!) The salient conditions in Definition 1.3.10 relevant to points 1 and 2 above are the conditions 2 and 3, respectively. The idea now is to consider $L' \to \overline{G \times^U X}$ such that some finite fully separating set of invariants $A \subseteq \Bbbk[X, L]^U$ extends to a collection of G-invariant sections over $\overline{G \times^U X}$, with various further restrictions to increase their effectiveness for studying $U \curvearrowright L \to X$.

Definition 1.3.11. [DK07, Definitions 5.2.4–5.2.7] Let $U \curvearrowright L \to X$ be an ample linearisation of a unipotent group over an irreducible projective variety, G a reductive group containing U as a closed subgroup and A a finite fully separating set of invariants. Suppose $\beta : G \times^U X \hookrightarrow \overline{G \times^U X}$ is a dominant G-equivariant open immersion into a projective G-variety $\overline{G \times^U X}$ and $L' \to \overline{G \times^U X}$ a G-linearisation that restricts to $U \curvearrowright L \to X$ under $\beta \circ \alpha$. If every $f \in A$ extends to a G-invariant section of some positive tensor power of L' over $\overline{G \times^U X}$, the pair $(\overline{G \times^U X}, L')$ is called a *reductive envelope* for $U \curvearrowright L \to X$ (with respect to A). Furthermore,

1. if each $f \in A$ extends to a G-invariant F over $\overline{G \times^U X}$ such that $\overline{G \times^U X}_F$ is affine then $(\overline{G \times^U X}, L')$ is called a *fine reductive envelope*;
2. if L' is an ample line bundle then $(\overline{G \times^U X}, L')$ is called an *ample reductive envelope*; and
3. if each $f \in A$ extends to a G-invariant section F over $\overline{G \times^U X}$ which vanishes on the codimension 1 part of the boundary $\overline{G \times^U X} \setminus (G \times^U X)$ then $(\overline{G \times^U X}, L')$ is called a *strong reductive envelope*.

Clearly any ample reductive envelope is a fine reductive envelope. In [DK07, Proposition 5.2.8] it is shown that for any ample linearisation $U \curvearrowright L \to X$, there is some positive tensor power $L^{\otimes r}$ of L which possesses an ample reductive envelope. Associated to any reductive envelope $L' \to \overline{G \times^U X}$ are the *completely semistable locus*

$$X^{\overline{\mathrm{ss}}} = (\beta \circ \alpha)^{-1}(\overline{G \times^U X}^{\mathrm{ss}(L')})$$

and the *completely stable locus*

$$X^{\overline{\mathrm{s}}} = (\beta \circ \alpha)^{-1}(\overline{G \times^U X}^{\mathrm{s}(L')})$$

as in [DK07, Definition 5.2.11]. The main theorem concerning reductive envelopes, stated below, says that in the case where $L' \to \overline{G \times^U X}$ is fine, $X^{\overline{\mathrm{ss}}}$ and $X^{\overline{\mathrm{s}}}$ 'bookend' the sets $X^{\mathrm{lts}} \subseteq X^{\mathrm{ss,fg}}$ associated to the U-linearisation $L \to X$, and the GIT quotient $\overline{G \times^U X}/\!\!/_{L'}G$ contains the enveloping quotient $X/\!\!/U$.

Theorem 1.3.12. *[DK07, Theorem 5.3.1]*[9] *Let X be an irreducible projective variety with an ample linearisation $L \to X$ of a unipotent group U, and let $(\overline{G \times^{H_u} X}, L')$ be a fine reductive envelope, with open embedding $\beta : G \times^U X \hookrightarrow \overline{G \times^U X}$. Let $\pi : \overline{G \times^{H_u} X}^{\mathrm{ss},L'} \to \overline{G \times^{H_u} X}/\!\!/G$ be the GIT quotient map and suppose $X/\!\!/U$ is a quasi-projective variety. Then there is a commutative diagram:*

$$\begin{tikzcd}
X^{\overline{\mathrm{s}}} \arrow[r, phantom, "\subseteq"] \arrow[d, "q"] & X^{\mathrm{lts}} = X^{\mathrm{ms}} = X^{\mathrm{mss}} \arrow[r, phantom, "\subseteq"] \arrow[d, "q"] & X^{\mathrm{ss,fg}} \arrow[r, phantom, "\subseteq"] \arrow[d, "q"] & X^{\overline{\mathrm{ss}}} = X^{\mathrm{nss}} \arrow[d, "\pi\circ\beta\circ\alpha"] \\
q(X^{\overline{\mathrm{s}}}) \arrow[r, phantom, "\subseteq"] & q(X^{\mathrm{lts}}) \arrow[r, phantom, "\subseteq"] & X/\!\!/U \arrow[r, phantom, "\subseteq"] & \overline{G \times^{H_u} X}/\!\!/G
\end{tikzcd}$$

with all inclusions open.

Remark 1.3.13. Note that Theorem 1.3.12 holds for ample reductive envelopes in particular, and in this case $\overline{G \times^U X}/\!\!/_{L'}G$ is a projective variety which is a compactification of $X/\!\!/U$. If furthermore $\Bbbk[X, L]^U$ is a finitely generated $\Bbbk$-algebra, then $X^{\mathrm{ss,fg}} = X^{\mathrm{nss}} = X^{\overline{\mathrm{ss}}}$ and $X/\!\!/U \cong \overline{G \times^U X}/\!\!/_{L'}G$.

In the case where the reductive envelope is fine and strong with a completion $\overline{G \times^U X}$ that is normal, the sets $X^{\mathrm{s}} = X^{\mathrm{lts}}$ and $X^{\mathrm{ss,fg}}$ can be computed via the stable and semistable loci of the reductive envelope:

Theorem 1.3.14. *[DK07, Theorem 5.3.5] Retain the notation of Theorem 1.3.12. If furthermore $\overline{G \times^U X}$ is normal, and $(\overline{G \times^U X}, L')$ defines a fine strong reductive envelope, then $X^{\overline{\mathrm{s}}} = X^{\mathrm{lts}}$ and $X^{\overline{\mathrm{ss}}} = X^{\mathrm{ss,fg}}$.*

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[9] [DK07, Theorem 5.3.1] actually says more than presented here and needs a normality assumption on X to include this extra material. An examination of the proof shows that the version we give here does not need X to be normal.

Given their use for computing the stable locus and finitely generated semistable locus for a linearisation of a unipotent group, the question of how to construct ample strong reductive envelopes which are normal naturally arises. (Ampleness is desirable, because it means the associated sets $X^{\overline{\mathrm{ss}}}$ and $X^{\overline{\mathrm{s}}}$ can be computed via the Hilbert-Mumford criterion applied to $L' \to \overline{G \times^U X}$.) This question is considered in [DK07, §§5.3.1–5.3.2]. For certain kinds of completion $\overline{G \times^U X}$ Doran and Kirwan give a method for turning any G-linearisation $L' \to \overline{G \times^U X}$ into a strong reductive envelope. The kinds of completion considered are so-called *gentle* completions, defined as follows.

Definition 1.3.15. [DK07, Definition 5.3.8] Let X be a quasi-projective variety and $\beta : X \hookrightarrow \overline{X}$ a projective completion of X. The completion is said to be *gentle* if $\overline{X}$ is normal and every codimension 1 component of the boundary of X in $\overline{X}$ is a $\mathbb{Q}$-Cartier divisor.

Suppose $\overline{G \times^U X}$ is a gentle completion of $G \times^U X$ and $L' \to \overline{G \times^U X}$ is *any* G-linearisation extending $L \to G \times^U X$. Let $D_1, \ldots, D_m \subseteq \overline{G \times^U X}$ be the codimension 1 irreducible components of the complement of $G \times^U X$ in $\overline{G \times^U X}$ and define a $\mathbb{Q}$-Cartier divisor

$$D := \sum_{i=1}^{m} D_i.$$

Then for any sufficiently divisible integer $N > 0$ the divisor ND is Cartier and defines a line bundle $\mathcal{O}(ND)$ on $\overline{G \times^U X}$ which restricts to the trivial bundle on $G \times^U X$. Define

$$L'_N := L' \otimes \mathcal{O}(ND) \to \overline{G \times^U X}.$$

If G is connected, then the G-linearisation on $L \to G \times^U X$ extends uniquely to a G-linearisation on L'_N. The next proposition provides a useful way for turning $L' \to \overline{G \times^U X}$ into a strong reductive envelope.

Proposition 1.3.16. *[DK07, Proposition 5.3.10] Suppose G is a connected reductive group and, as above, suppose $\overline{G \times^U X}$ is a gentle completion of $G \times^U X$ and $L' \to \overline{G \times^U X}$ is an extension of the G-linearisation $L \to G \times^U X$. Given a finite fully separating set of invariants $\mathcal{A}$ on X, then $(\overline{G \times^U X}, L'_N)$ is a strong reductive envelope with respect to $\mathcal{A}$, for sufficiently divisible integers $N > 0$. If in fact $(\overline{G \times^U X}, L')$ defines a fine reductive envelope with respect to $\mathcal{A}$, then $(\overline{G \times^U X}, L'_N)$ defines a fine strong reductive envelope.*

The final piece of material from [DK07] we wish to recall here concerns a particular scenario in which the above construction is especially simple to describe explicitly.

Suppose X is normal and a reductive group G can be found that contains U as a closed subgroup in such a way that

- the homogeneous space G/U can be embedded in a normal affine variety $\overline{G/U}^{\mathrm{aff}}$ with codimension 2 complement; and
- the U-linearisation $U \curvearrowright L \to X$ extends to a G-linearisation $G \curvearrowright L \to X$.

(Note that the first of these conditions is equivalent to U being a *Grosshans subgroup* of G, about which more will be discussed in Section 3.2.1 of Chapter 3.) The extension of the linearisation leads to an isomorphism of G-linearisations

$$\begin{tikzcd}
G \times^U L \arrow[r, "\cong"] \arrow[d] & (G/U) \times L \arrow[d] & {[g,l] \mapsto (gU, gl)} \\
G \times^U X \arrow[r, "\cong"] & (G/U) \times X & {[g,x] \mapsto (gU, gx)}
\end{tikzcd}$$

with the corresponding G-linearisation on the right hand side being the product of the linearisation $G \curvearrowright L \to X$ and left multiplication on G/U. Note that because $\overline{G/U}^{\mathrm{aff}} \times X$ is normal and G reductive, the ring of invariants

$$\Bbbk[X, L]^U \cong (\mathcal{O}(G/U) \otimes \Bbbk[X, L])^G = (\mathcal{O}(\overline{G/U}^{\mathrm{aff}}) \otimes \Bbbk[X, L])^G$$

is a finitely generated $\Bbbk$-algebra. In particular, this implies $X/\!\!/U = \mathrm{Proj}(S^U)$ is a projective variety. One can choose a normal projective G-equivariant completion $\overline{G/U}$ of $\overline{G/U}^{\mathrm{aff}}$ whose boundary is a single codimension 1 component D_∞, and there is a very ample G-linearisation on the associated line bundle $\mathcal{O}(D_\infty) \to \overline{G/U}$ extending the canonical linearisation on $\mathcal{O}_{G/U} \to G/U$. As above, for any $N > 0$, let

$$L'_N = \mathcal{O}(ND_\infty) \boxtimes L \to \overline{G/U} \times X,$$

equipped with its natural G-linearisation. Then using Proposition 1.3.16 and Theorem 1.3.12, Doran and Kirwan show

Proposition 1.3.17. *[DK07, Lemma 5.3.14] In the above situation, the pair* $(\overline{G/U}\times X, L'_N)$ *defines an ample strong reductive envelope for sufficiently large* $N > 0$ *and* $X/\!\!/U = (\overline{G/U}\times X)/\!\!/_{L'_N}G$.

As a demonstration of this construction, the following example is studied at the end of [DK07].

Example 1.3.18. [DK07, §6] Let $U = (\mathbb{C}, +)$, embedded in $\mathrm{GL}(2, \mathbb{C})$ as the subgroup of upper triangular matrices, act on $V = \mathrm{Sym}^n\,\mathbb{C}^2$ via the standard representation of $\mathrm{GL}(2, \mathbb{C})$ on V, and consider the canonical U-linearisation on $L := \mathcal{O}(1) \to X := \mathbb{P}(V)$. (Note that X may be regarded as the space of degree n divisors on $\mathbb{P}^1$, and the action of U on X corresponds to moving points on $\mathbb{P}^1$ by the usual translation Möbius transformation.) This linearisation extends to one of $G = \mathrm{SL}(2, \mathbb{C})$ in the obvious way. The homogeneous space $G/U \cong \mathbb{C}^2 \setminus \{0\}$ via the usual transitive action of G on $\mathbb{C}^2 \setminus \{0\}$, and it has a normal G-equivariant affine completion $\mathbb{C}^2$. Embedding $\mathbb{C}^2$ into $\mathbb{P}^2$ by adding a hyperplane at infinity, we arrive in the setting of Proposition 1.3.17, so $\mathcal{O}_{\mathbb{P}^2}(N) \boxtimes L \to \mathbb{P}^2 \times X$, for $N \gg 0$, defines a strong ample reductive envelope for $U \curvearrowright L \to X$. Using the Hilbert-Mumford criterion on $\mathbb{P}^2 \times X$ and Theorem 1.3.14, one sees that

$$\begin{aligned} X^{\mathrm{s}} &= \{\text{divisors } \textstyle\sum_{i=1}^n p_i \text{ where} < n/2 \text{ of the } p_i \text{ coincide}\}, \\ X^{\mathrm{ss,fg}} &= \{\text{divisors } \textstyle\sum_{i=1}^n p_i \text{ where} \leq n/2 \text{ of the } p_i \text{ coincide}\}. \end{aligned}$$

In the case where n is odd then $X^{\mathrm{s}} = X^{\mathrm{ss,fg}}$ and X^{s}/U is an open subset of $X/\!\!/U = (\mathbb{P}^2 \times X)/\!\!/G$ with complement given by the reductive GIT quotient $(\{0\} \times X)/\!\!/G = X/\!\!/G$ for the classical action of $G = \mathrm{SL}(2, \mathbb{C})$ on X (linearised with respect to $\mathcal{O}(1) \to X$). In particular, the enveloping quotient map $X^{\mathrm{ss,fg}} \to X/\!\!/U$ is not surjective.

On the other hand, when n is even then X^{s} is a proper subset of $X^{\mathrm{ss,fg}}$ and the image of $X^{\mathrm{ss,fg}} \to X/\!\!/U$ is not a variety: it is equal to the union of X^{s}/U together with the point $\mathrm{pt} = (X/\!\!/G) \setminus (X^{\mathrm{s}(G)}/G)$ given by the quotient of the strictly semistable set for the G-linearisation on X.

Chapter 2

Geometric Invariant Theory for Non-Reductive Groups

Let X be a variety with an action of a linear algebraic group H. In this chapter we develop a theoretical framework for finding H-invariant open subsets of X which admit a geometric quotient. The approach we take is a geometric invariant theoretic one: given a linearisation $L \to X$ for the action, we consider H-invariant open subsets of X obtained by patching together subsets of the form X_f, where f is an H-invariant section of a positive tensor power of $L \to X$. A basic guiding goal is to develop a theory which, in the case where L is an ample linearisation over an irreducible projective variety X, reduces to Mumford's GIT [MFK94] when the group H is reductive and to Doran and Kirwan's theory [DK07] when H is unipotent. Due to the technical issues that arise when working with non-reductive groups (for example, non-finite generation of invariant rings) we follow the general approach taken [DK07, §4 and §5.1]. A number of our definitions and results are simple generalisations of those found in [DK07] to the context of not-necessarily unipotent groups. Having said this, we also address some errors that occur in [DK07, §4] and thus our work can be seen as giving some new perspectives on the unipotent picture. We also do not restrict ourselves to only working with projective varieties with ample linearisations, as is done in [DK07].

We begin in Section 2.1 by extending the finitely generated semistable locus $X^{\mathrm{ss,fg}}$ and the notions of enveloped quotients and enveloping quotients in Doran and Kirwan's theory to the more general non-reductive case (Definitions 2.1.1 and 2.1.6). As a way to address the observation that the enveloping quotient $X \wr\!\wr H$ need not be a variety (see Remark 1.3.4) we define *inner enveloping quotients* (Definition 2.1.11). These are subvarieties of the enveloping quotient that, in some sense, play the role of the GIT quotient from Mumford's theory for reductive groups; indeed, in the case where $H = G$ is reductive, X is projective and $L \to X$ is ample, there is only one inner enveloping quotient—namely, the GIT quotient $X/\!\!/G$). In general an inner enveloping quotient is not intrinsic to the data of the linearisation $L \to X$, but instead corresponds to an additional choice of a certain kind of linear system, called an *enveloping system*, introduced in Definition 2.1.16. We explore ways in which inner enveloping quotients give a certain 'universality' with respect to H-invariant morphisms from $X^{\mathrm{ss,fg}}$. In Section 2.2 we examine how the enveloping quotient behaves under naturally induced group actions. In particular, we note some of the difficulties that can arise when trying to take enveloping quotients 'in stages': first by a normal subgroup N of H and then by the quotient group H/N. In Section 2.3 we introduce the stable locus X^{s} for a general non-reductive linearisation over an irreducible variety X (Definition 2.3.2). This is intrinsic to the linearisation $L \to X$ and admits a geometric quotient by the H-action. Our notion of stability also reduces to Definition 1.2.2 in the reductive case and to Doran and Kirwan's Definition 1.3.5 in the unipotent case. Following the ideas of [DK07, §5], we relate our definition of stability X^{s} to stability for a certain reductive linearisation obtained by extension to a reductive structure group, which is important for the work in later chapters. Finally, in Section 2.4 we draw together all our definitions and key results into Theorem 2.4.2, which provides a summary of our geometric invariant theoretic picture for non-reductive groups.

2.1 Finitely Generated Semistability and Enveloping Quotients

Let H be a linear algebraic group acting on a variety X equipped with a linearisation $H \curvearrowright L \to X$. As in Chapter 1, we let

$$S = \Bbbk[X, L] = \bigoplus_{r \geq 0} H^0(X, L^{\otimes r})$$

be the graded $\Bbbk$-algebra of global sections of positive tensor powers of L and S^H be the subring of invariant sections under the action (1.1.2) of Section 1.1.2. The inclusion $S^H \hookrightarrow S$ defines an H-invariant rational map of schemes

$$q : X \dashrightarrow \operatorname{Proj}(S^H), \tag{2.1.1}$$

whose maximal domain of definition contains the open subset of points where some invariant section of a positive tensor power of L does not vanish (see Section A.1 of Appendix A). As we have seen with the work of Mumford [MFK94] and Doran and Kirwan [DK07], the basic technique of geometric invariant theory is, roughly speaking, to use the non-vanishing loci X_f of invariant sections f to construct H-invariant open subsets of X which admit geometric quotients in the category of varieties. Since any such geometric quotient must be a scheme of finite type, it makes sense to restrict which opens X_f to include.

Definition 2.1.1. Let H be a linear algebraic group acting on a variety X and $L \to X$ a linearisation of the action. The *naively semistable locus* is the open subset

$$X^{\mathrm{nss}} := \bigcup_{f \in I^{\mathrm{nss}}} X_f$$

of X, where $I^{\mathrm{nss}} := \bigcup_{r>0} H^0(X, L^{\otimes r})^H$ is the set of invariant sections of positive tensor powers of L. The *finitely generated semistable locus* is the open subset

$$X^{\mathrm{ss,fg}} := \bigcup_{f \in I^{\mathrm{ss,fg}}} X_f$$

of X^{nss}, where

$$I^{\mathrm{ss,fg}} := \left\{ f \in \textstyle\bigcup_{r>0} H^0(X, L^{\otimes r})^H \mid (S^H)_{(f)} \text{ is a finitely generated } \Bbbk\text{-algebra} \right\}.$$

These definitions generalise definitions 1.3.1 and 1.3.2 of the naively semistable and finitely generated semistable loci, respectively, from [DK07]. These definitions depend on the choice of the linearisation L and, when necessary, we shall indicate this by writing $X^{\mathrm{nss}(L)}$ and $X^{\mathrm{ss,fg}(L)}$.

Remark 2.1.2. It is clear from the definition that for any $r > 0$ the subset X^{nss} is unaffected by replacing the linearisation $L \to X$ with $L^{\otimes r} \to X$, and there is a canonical isomorphism $\mathrm{Proj}(\Bbbk[X, L]^H) \cong \mathrm{Proj}(\Bbbk[X, L^{\otimes r}]^H)$. The subset $X^{\mathrm{ss,fg}}$ is also unaffected by this replacement. Indeed, it is easy to see that $X^{\mathrm{ss,fg}(L^{\otimes r})} \subseteq X^{\mathrm{ss,fg}(L)}$. For the reverse containment, note that for any $f \in H^0(X, L^{\otimes m})^H$ $(m > 0)$ with $(\Bbbk[X, L]^H)_{(f)}$ a finitely generated $\Bbbk$-algebra, we have $f^r \in \Bbbk[X, L^{\otimes r}]^H$ with

$$(\Bbbk[X, L^{\otimes r}]^H)_{(f^r)} = (\Bbbk[X, L]^H)_{(f^r)} = (\Bbbk[X, L]^H)_{(f)}$$

a finitely generated $\Bbbk$-algebra, so that $X_f = X_{f^r} \subseteq X^{\mathrm{ss,fg}(L^{\otimes r})}$. It thus makes sense to define X^{nss}, $X^{\mathrm{ss,fg}}$ and the scheme $\mathrm{Proj}(\Bbbk[X, L]^H)$ for rational linearisations using Remark 1.1.19 (cf. Remark 1.2.3).

As we noted in Section 1.1.2 of Chapter 1, the most common linearisations one comes across are when X is affine and $L = \mathcal{O}_X$ is the trivial bundle, or else when X is a projective variety and L is an ample line bundle. We take a moment to consider the rational map (2.1.1) and Definition 2.1.1 in each of these cases.

Example 2.1.3. In the case where $X = \operatorname{Spec} A$ affine and $L = \mathcal{O}_X$, recall from Example 1.1.15 that the linearisation is defined by a character $\chi : H \to \mathbb{G}_m$ and that S^H is the graded subring of semi-invariants,

$$\bigoplus_{r \geq 0} A^H_{\chi^r}, \quad A^H_{\chi^r} := \{f \in A \mid f(hx) = \chi(h)^r f(x) \text{ for all } x \in X, h \in H\}.$$

The rational map $q : X = \operatorname{Spec} A \dashrightarrow \mathrm{Proj}(\bigoplus_{r \geq 0} A^H_{\chi^r})$ corresponds to the natural map $\bigoplus_{r \geq 0} A^H_{\chi^r} \to A$ induced by the inclusions $A^H_{\chi^r} \hookrightarrow A$, and X^{nss} is in this case the maximal domain of definition of q, consisting of points $x \in X$ where $f(x) \neq 0$ for some $f \in A^H_{\chi^r}$ with $r > 0$.

In the special case where $\chi = 1$ is the trivial character, then the ring of semi-invariants is just $\bigoplus_{r \geq 0} A^H$, so that $\mathrm{Proj}(\Bbbk[X, L]^H) = \mathrm{Spec}(A^H)$. Furthermore, we have $X^{\mathrm{nss}} = X$ because the constant function $1 \in H^0(X, L)^H$, and $X^{\mathrm{ss,fg}}$ is the union of X_f with $f \in A^H$ such that $(A^H)_f$ is a finitely generated $\Bbbk$-algebra.

Example 2.1.4. If now X is projective and L is ample, then each of the open subsets X_f arising in Definition 2.1.1 is affine, so the restriction of the rational map $q : X \dashrightarrow \operatorname{Proj}(S^H)$ to X^{nss} and $X^{\text{ss,fg}}$ defines an *affine* morphism. Moreover, by taking a sufficiently positive tensor power $L^{\otimes r}$ of L we may embed X equivariantly into the projective space $\mathbb{P}(V^*)$ using the complete linear system $V = H^0(X, L^{\otimes r})$, and the linearisation $L^{\otimes r}$ extends to $\mathcal{O}_{\mathbb{P}(V^*)}(1) \to \mathbb{P}(V^*)$; cf. Example 1.1.16. If L is very ample (so that we may take $r = 1$), I_X is the kernel of the restriction map $\Bbbk[\mathbb{P}(V^*), \mathcal{O}(1)] \to \Bbbk[X, L]$ and $R_X = \Bbbk[\mathbb{P}(V^*), \mathcal{O}(1)]/I_X$, then by Serre vanishing [Har77, Chapter 3, Proposition 5.3] for some $m > 0$ the m-th Veronese subring $(R_X)^{(m)} \subseteq R_X$ is isomorphic to $\Bbbk[X, L^{\otimes m}]$. Then $\operatorname{Proj}(\Bbbk[X, L]^H) \cong \operatorname{Proj}((R_X)^H)$, and in light of Remark 2.1.2 computing X^{nss} and $X^{\text{ss,fg}}$ for $H \curvearrowright L \to X$ is essentially equivalent to studying the action of H on R_X. Thus when L is ample and X is projective we can always reduce to the case where H acts on a projective space $\mathbb{P}^n$ via a representation $H \to \mathrm{GL}(n+1, \Bbbk)$ and $X \subseteq \mathbb{P}^n$ is a closed subvariety invariant under the action.

Remark 2.1.5. In general the finitely generated semistable locus $X^{\text{ss,fg}}$ is strictly contained in X^{nss}, due to the fact that the subring of invariant sections can be non-noetherian (even if S is a finitely generated $\Bbbk$-algebra). Indeed, when $X = \operatorname{Spec} A$ is an affine variety and $L = \mathcal{O}_X$ is equipped with the canonical H-linearisation (i.e. defined by the trivial character $1 : H \to \mathbb{G}_m$), then as seen in Example 2.1.3 $X^{\text{nss}} = X$ and $X^{\text{ss,fg}}$ is the union of all X_f where $(A^H)_f$ is finitely generated over $\Bbbk$. In [DeKe08, Proposition 2.10] Derksen and Kemper show that the set

$$I^{\text{ss,fg}} \cup \{0\} = \{f \in A^H \mid (A^H)_f \text{ is finitely generated}\} \cup \{0\}$$

is in fact a radical ideal of A^H. In [Gro76] Grosshans shows that if A is an integral domain then there is a nonzero $f \in A^H$ such that $(A^H)_f$ is finitely generated, so if A^H is not finitely generated then $I^{\text{ss,fg}}$ is a proper nonzero ideal. It follows that any irreducible affine example in which the ring of invariant global functions is not finitely generated will result in $\emptyset \neq X^{\text{ss,fg}} \neq X^{\text{nss}}$; for example, the Nagata counterexample in Example 1.1.24.

The rational map of (2.1.1) restricts to define a morphism on $X^{\mathrm{ss,fg}}$ whose image is contained in the following open subscheme of $\mathrm{Proj}(S^H)$.

Definition 2.1.6. Let H be a linear algebraic group and $H \curvearrowright L \to X$ a linearisation of an H-variety X. The *enveloping quotient* is the scheme

$$X \wr\wr H := \bigcup_{f \in I^{\mathrm{ss,fg}}} \mathrm{Spec}\left((S^H)_{(f)}\right) \subseteq \mathrm{Proj}(S^H)$$

together with the canonical map $q : X^{\mathrm{ss,fg}} \to X \wr\wr H$. We call the image $q(X^{\mathrm{ss,fg}})$ of this map the *enveloped quotient*.

When it is necessary to do so, we will include the data of the linearisation in an enveloping quotient by writing $X \wr\wr_L H$. Definition 2.1.6 is simply an extension of Doran and Kirwan's definition of enveloping quotient and enveloped quotient in [DK07] to the case of linearisations for any linear algebraic group. Note that we do not use the "$/\!\!/$" notation, since one can define the enveloping quotient for a linearisation of a reductive group and in general this is *not* equal to Mumford's reductive GIT quotient from [MFK94]. (More will be said about this in Section 2.1.2.) Observe that the enveloping quotient is a canonically defined reduced, separated scheme locally of finite type over $\Bbbk$.

Remark 2.1.7. As we noted earlier in Remark 1.3.4, the enveloping quotient is only a scheme *locally* of finite type in general. However, when S^H is a finitely generated $\Bbbk$-algebra, or when the enveloping quotient map $q : X^{\mathrm{ss,fg}} \to X \wr\wr H$ is surjective, then $X \wr\wr H$ is noetherian and hence a variety.

In the next lemma we make some initial observations about the rational map $q : X \dashrightarrow \mathrm{Proj}(S^H)$ of (2.1.1) associated to a linearisation $H \curvearrowright L \to X$. As well as using standard results about the Proj construction from Section A.1 of the Appendix, we need the following commutative algebra result from [DeKe08, Proposition 2.9]: if A is an integral domain over $\Bbbk$ and $a, b \in A \setminus \{0\}$ are such that A_a and A_b are both finitely generated $\Bbbk$-algebras and the ideal generated by a, b is equal to A, then A is also a finitely generated $\Bbbk$-algebra (and in fact $A = A_a \cap A_b$).

Lemma 2.1.8. *Suppose $S^H \neq \Bbbk$, let $\mathcal{Y} = \operatorname{Proj}(S^H)$ and let $\underline{L}$ denote the sheaf of sections of L on X. Then for each $r \geq 0$, pulling back along q defines inclusions of sheaves $q^* : \mathcal{O}_{\mathcal{Y}}(r) \hookrightarrow (q_*(\underline{L}^{\otimes r}|_{X^{\mathrm{nss}}}))^H \subseteq q_*(\underline{L}^{\otimes r}|_{X^{\mathrm{nss}}})$. If S is furthermore assumed to be an integral domain, then*

1. *for each $r \geq 0$ the twisting sheaf $\mathcal{O}_{\mathcal{Y}}(r)$ is identified with $(q_*(\underline{L}^{\otimes r}|_{X^{\mathrm{nss}}}))^H$ via q^*; and*
2. *if S is finitely generated over $\Bbbk$ then the ideal $\mathfrak{a} \subseteq S^H$ generated by $I^{\mathrm{ss,fg}}$ is a non-zero graded radical ideal of S^H satisfying $\mathfrak{a} \cap S^H_r = (I^{\mathrm{ss,fg}} \cap S^H_r) \cup \{0\}$ for each $r \geq 0$. In particular, $X^{\mathrm{ss,fg}} \neq \emptyset$.*

Proof. Fix $r \geq 0$ and let $f \in S^H_m = H^0(X, L^{\otimes m})^H$ for $m > 0$. Let $S(r)$ be the graded S-module with degree d piece equal to S_{d+r} for each $d \in \mathbb{Z}$ and let

$$M = \bigoplus_{n \geq 0} S(r)_{mn} = \bigoplus_{n \geq 0} H^0(X, L^{\otimes r} \otimes L^{\otimes mn})$$

with its $S^{(m)} = \Bbbk[X, L^{\otimes m}]$-module structure. Since X is quasi-compact and (quasi-) separated, by [Har77, Lemma 5.14] there is a canonical identification $H^0(X_f, L^{\otimes r}) = M_{(f)}$ of $\mathcal{O}(X_f)$-modules, where the module structure on the right hand side comes from the identification $\mathcal{O}(X_f) = (S^{(m)})_{(f)}$, while $H^0(\operatorname{Spec}((S^H)_{(f)}), \mathcal{O}_{\mathcal{Y}}(r)) = (M^H)_{(f)}$ with its $((S^H)^{(m)})_{(f)}$-module structure by definition. The pullback map

$$q^* : H^0(\operatorname{Spec}((S^H)_{(f)}), \mathcal{O}_{\mathcal{Y}}(r)) \to H^0(X_f, L^{\otimes r})$$

corresponds to the inclusion $(M^H)_{(f)} \hookrightarrow (M_{(f)})^H \subseteq M_{(f)}$ under these identifications. Hence

$$q^* : \mathcal{O}_{\mathcal{Y}}(r) \hookrightarrow (q_*(\underline{L}^{\otimes r}|_{X^{\mathrm{nss}}}))^H \subseteq q_*(\underline{L}^{\otimes r}|_{X^{\mathrm{nss}}})$$

is an inclusion of sheaves. If furthermore S is an integral domain, then in fact $(M^H)_{(f)} = (M_{(f)})^H$ (with notation as above): for if $g \in S(r)_{mn}$ is such that $g/f^n \in (M_{(f)})^H$ and $h \in H$, then $g/f^n = h \cdot (g/f^n) = (h \cdot g)/f^n$, whence $h \cdot g = g$. Statement 1 follows.

Now we prove 2, assuming S is an integral domain and finitely generated over $\Bbbk$. We first show that $I^{\mathrm{ss,fg}} \neq \emptyset$. Since $S^H \neq \Bbbk$ we can find a nonzero homogeneous $f \in S^H$ of positive degree. Then $A := S_{(f)}$ is a finitely generated integral domain over $\Bbbk$, so applying

Grosshans' localisation result [Gro76] (see Remark 2.1.5) to $A = S_{(f)}$ we conclude that there exists $a \in A^H \setminus \{0\}$ such that $(A^H)_a$ is finitely generated over $\Bbbk$. Because S is an integral domain we have $A^H = (S_{(f)})^H = (S^H)_{(f)}$, so $a = g/f^m$ for some integer $m \geq 0$ and $g \in S^H$ homogeneous of degree equal to $m \deg f$ and $(A^H)_a = (S^H)_{(fg)}$. Hence $fg \in I^{\mathrm{ss,fg}}$. Note that this implies $X^{\mathrm{ss,fg}} \neq \emptyset$.

It is immediate that $\mathfrak{a}$ is a graded ideal of S^H. The fact that it is radical follows from the equality $(S^H)_{(f)} = (S^H)_{(f^m)}$ for each $f \in S^H$ homogeneous and $m > 0$. It remains to show $\mathfrak{a} \cap S^H_r = I^{\mathrm{ss,fg}} \cap S^H_r$ for all $r \geq 0$. The inclusions $\mathfrak{a} \cap S^H_r \supseteq I^{\mathrm{ss,fg}} \cap S^H_r$ are obvious. For the reverse inclusions, it suffices to show that for any $g_1, g_2 \in I^{\mathrm{ss,fg}}$ and any $f \in H^0(X, L^{\otimes r'})^H$ ($r' > 0$), we have $fg_i \in I^{\mathrm{ss,fg}}$ and $\tilde{g} := g_1 + g_2 \in I^{\mathrm{ss,fg}}$ whenever $g_1 + g_2 \neq 0$. To this end, note that $fg_i \in I^{\mathrm{ss,fg}}$ because

$$(S^H)_{(fg_i)} = ((S^H)_{(g_i)})_{\frac{f^r}{g_i^{r'}}}$$

is the localisation of a finitely generated algebra. On the other hand, setting $a_i := g_i/\tilde{g} \in (S^H)_{(\tilde{g})}$, we see that each $((S^H)_{(\tilde{g})})_{a_i} = ((S^H)_{(\tilde{g}g_i)})$ is a finitely generated integral domain over $\Bbbk$. Since (a_1, a_2) is the unit ideal in $(S^H)_{(\tilde{g})}$, the ring $(S^H)_{(\tilde{g})}$ is therefore finitely generated by the result [DeKe08, Proposition 2.9] quoted before the statement of the lemma, so $\tilde{g} = g_1 + g_2 \in I^{\mathrm{ss,fg}}$. Thus, $\mathfrak{a} \cap H^0(X, L^{\otimes r})^H \subseteq I^{\mathrm{ss,fg}} \cap H^0(X, L^{\otimes r})^H$ for each $r \geq 0$. □

From the lemma, we see that for general X the natural map of sheaves $q^\# : \mathcal{O}_{\mathcal{Y}} \to q_*\mathcal{O}_{X^{\mathrm{nss}}}$ is injective with image contained in $(q_*\mathcal{O}_{X^{\mathrm{nss}}})^H$ and $q : X^{\mathrm{nss}} \to \mathcal{Y}$ is a dominant morphism. Also, if $\mathcal{U} \subseteq \mathcal{Y}$ is any nonempty open subscheme then the sheaves

$$\mathcal{O}_{\mathcal{Y}}(r)|_{\mathcal{U}}, \quad r \geq 0 \tag{2.1.2}$$

are quasi-coherent sheaves whose sections are included in the H-invariant sections of $L^{\otimes r}|_{q^{-1}(\mathcal{U})}$ under pullback by q.

Remark 2.1.9. When X is irreducible the ring of sections S is an integral domain, so for any section $f \in I^{\mathrm{nss}} = \bigcup_{r>0} H^0(X, L^{\otimes r})^H$ we have a natural identification $\mathcal{O}(X_f)^H = (S_{(f)})^H = (S^H)_{(f)}$, by Lemma 2.1.8, 1. Under this identification, the morphism $q : X_f \to$

$\mathrm{Spec}((S^H)_{(f)})$ defined by (2.1.1) corresponds to the natural map $X_f \to \mathrm{Spec}(\mathcal{O}(X_f)^H)$ induced by $\mathcal{O}(X_f)^H \hookrightarrow \mathcal{O}(X_f)$.

Notice also that, as a corollary of 2 of Lemma 2.1.8, any situation where a linear algebraic group H acts on a finitely generated graded integral $\Bbbk$-algebra S, with $S_0 = \Bbbk$, such that the ring of invariants S^H is not finitely generated over $\Bbbk$ will result in an example of a projective variety $X = \mathrm{Proj}\, S$ with ample linearisation $L \to X$ such that $X^{\mathrm{nss}} \neq X^{\mathrm{ss,fg}}$ and $X \wr\wr H \neq \mathrm{Proj}(S^H)$. This holds, in particular, for the projectivised version of the Nagata example (cf. Example 1.1.24).

2.1.1 'Universality' of the Enveloping Quotient

Given a linear algebraic group H acting on a variety X with linearisation $L \to X$, it is not necessarily the case that $q : X^{\mathrm{ss,fg}} \to X \wr\wr H$ is surjective; in particular, $X \wr\wr H$ is not in general a categorical quotient of $X^{\mathrm{ss,fg}}$. As a standard counterexample in the affine case, consider

Example 2.1.10. Let $X = \mathrm{SL}(2, \Bbbk)$ and let $H \subseteq X$ be the subgroup of strictly upper triangular matrices, acting on X via matrix multiplication. There is a unique linearisation of the trivial bundle $\mathcal{O}_X \to X$. By [Bor91, Theorem 6.8] the geometric quotient X/H exists; in fact, H is precisely the stabiliser of the standard action of $\mathrm{SL}(2, \Bbbk)$ on $\Bbbk^2 \setminus \{0\}$, therefore $X/H \cong \Bbbk^2 \setminus \{0\}$ and $\Bbbk[X, \mathcal{O}_X]^H \cong \Bbbk[z_0, z_1]$ is finitely generated. So in this case $X \wr\wr H = \mathrm{Spec}(\Bbbk[X, \mathcal{O}_X]^H) \cong \Bbbk^2$, and the image of $X^{\mathrm{ss,fg}} = X$ under the enveloping quotient map is identified with $\Bbbk^2 \setminus \{0\}$.

Another example of the failure of surjectivity of $q : X^{\mathrm{ss,fg}} \to X \wr\wr H$ was given in Example 1.3.18. There we also saw examples where the enveloped quotient $q(X^{\mathrm{ss,fg}})$ is not a variety, so in general a categorical quotient of $X^{\mathrm{ss,fg}}$ need not exist at all. This raises the question of whether there is any sort of way in which to view the enveloping quotient as 'universal' for H-invariant morphisms. Here we give one possible way to answer this. (We should say that our use of the word 'universal' here is informal—while we do prove a sort of uniqueness and existence result regarding morphisms induced by certain H-invariant morphisms from $X^{\mathrm{ss,fg}}$, we don't formulate this in terms of a universal property within some category, though it is surely possible to do so. This is simply because we won't have need for such a formal usage in what follows.)

The key observation is that, even though $X \wr\wr H$ may not be quasi-compact, the enveloped quotient—being the image $q(X^{\mathrm{ss,fg}})$ of $X^{\mathrm{ss,fg}}$—is quasi-compact as a subset of $X \wr\wr H$. So it is natural to look at *quasi-compact* open subschemes $\mathcal{U}$ of $X \wr\wr H$ that contain $q(X^{\mathrm{ss,fg}})$. Note that any such $\mathcal{U}$ is of finite type and is separated, being an open subscheme of $\mathrm{Proj}(S^H)$. Furthermore, by the well-known theorem of Chevalley [Har77, Chapter 2, Exercise 3.19] the image $q(X^{\mathrm{ss,fg}})$ is a constructible subset of $X \wr\wr H$; in fact, it is easy to see that we have an equality of sets

$$q(X^{\mathrm{ss,fg}}) = \bigcap \{\mathcal{U} \mid \mathcal{U} \subseteq X \wr\wr H \text{ is open, quasi-compact and contains } q(X^{\mathrm{ss,fg}})\}.$$

This suggests it is natural to study diagrams of the form

$$\begin{tikzcd}
 & & X^{\mathrm{ss,fg}} \arrow[dll] \arrow[d] & \subseteq & X^{\mathrm{nss}} \arrow[d, "q"] & \subseteq & X \\
\mathcal{U} & \subseteq & X \wr\wr H & \subseteq & \mathrm{Proj}(S^H) & &
\end{tikzcd}$$

where the inclusions are open and the $\mathcal{U}$ are quasi-compact.

More generally, *any* nonempty open set $\mathcal{U} \subseteq X \wr\wr H$ intersects $q(X^{\mathrm{ss,fg}})$ and is covered by basic affine open subsets of the form $\mathrm{Spec}((S^H)_{(f)})$ with f such that $(S^H)_{(f)}$ is finitely generated. Thus the pre-image of $\mathcal{U}$ under the enveloping quotient map q is a nonempty union of the associated open subsets X_f. So given any open subset $U \subseteq X$ that is a union of X_f with $f \in I^{\mathrm{ss,fg}}$ we can also consider its image $q(U)$ as an intersection of those quasi-compact open $\mathcal{U} \subseteq X \wr\wr H$ containing it. This motivates the following definition.

Definition 2.1.11. Let H be a linear algebraic group acting on a variety X with linearisation $L \to X$ and let $U \subseteq X^{\mathrm{ss,fg}}$ be a nonempty H-invariant open subset. An *inner enveloping quotient of* U is a quasi-compact open subscheme of $X \wr\wr H$ that contains the image $q(U)$ of U under the enveloping quotient map $q : X^{\mathrm{ss,fg}} \to X \wr\wr H$. An inner enveloping quotient of $U = X^{\mathrm{ss,fg}}$ is simply called an *inner enveloping quotient*.

Example 2.1.12. In the case where X is an irreducible projective H-variety and $L \to X$ an ample linearisation we can intrinsically define a collection of inner enveloping quotients, as follows. The section ring $S = \Bbbk[X, L]$ is an integral domain finitely generated over $\Bbbk$, so by Lemma 2.1.8, 2 the set $(I^{\mathrm{ss,fg}} \cap H^0(X, L^{\otimes r})^H) \cup \{0\}$ is a vector space over $\Bbbk$, for each $r > 0$. Thus, for any $r > 0$ such that $X^{\mathrm{ss,fg}} = \bigcup\{X_f \mid f \in I^{\mathrm{ss,fg}} \cap H^0(X, L^{\otimes r})^H\}$, the associated open subscheme

$$\mathcal{U}^{(r)} := \bigcup\{\mathrm{Spec}((S^H)_{(f)}) \mid f \in I^{\mathrm{ss,fg}} \cap H^0(X, L^{\otimes r})^H\} \subseteq X \wr\wr H$$

is an inner enveloping quotient: fixing any basis $\{f_i\}$ of $(I^{\mathrm{ss,fg}} \cap H^0(X, L^{\otimes r})^H) \cup \{0\}$ yields a finite open cover $\{\mathrm{Spec}((S^H)_{(f_i)})\}$ of $\mathcal{U}^{(r)}$ of quasi-compact open subsets.

From the discussion above we see it is natural to regard the image of an H-invariant open subset U of $X^{\mathrm{ss,fg}}$ under the enveloping quotient map $q : X^{\mathrm{ss,fg}} \to X \wr\wr H$ as sitting inside a 'germ' of inner enveloping quotients of U. The following Proposition makes this notion more precise.

Proposition 2.1.13. *Let H be a linear algebraic group acting on an irreducible variety X with linearisation $L \to X$, and let $U = \bigcup_{f \in \mathcal{S}} X_f$, where $\mathcal{S}$ a nonempty subset of $I^{\mathrm{ss,fg}}$. Suppose we are given the data of a quasi-projective variety Z together with a very ample line bundle $M \to Z$ and an H-invariant morphism $\phi : U \to Z$ with $\phi^* M \cong L^{\otimes r}|_U$ for some $r > 0$. Then*

1. *there is an inner enveloping quotient $\mathcal{U}$ of U and a morphism $\overline{\phi} : \mathcal{U} \to Z$ such that $\phi = \overline{\phi} \circ q|_U$ and $\overline{\phi}^* M \cong \mathcal{O}_{\mathcal{U}}(r)$; and*

2. *if $\mathcal{U}, \mathcal{U}' \subseteq X \wr\wr H$ are two inner enveloping quotients of U and $\psi : \mathcal{U} \to Z$ and $\psi' : \mathcal{U}' \to Z$ two morphisms such that $\psi \circ q|_U = \psi' \circ q|_U$, then ψ and ψ' agree on $\mathcal{U}' \cap \mathcal{U}$.*

Proof. (Proof of 1.) Let $\iota : Z \hookrightarrow \mathbb{P}^n$ be a locally closed immersion defined by sections $\sigma_0, \dots, \sigma_n \in H^0(Z, M)$, so that the composition $\iota \circ \phi : U \to \mathbb{P}^n$ is defined by the H-invariant sections $f_0 = \phi^*\sigma_0, \dots, f_n = \phi^*\sigma_n \in H^0(U, L^{\otimes r})^H$. Let $\mathcal{U}_0 = \bigcup_{f \in \mathcal{S}} \mathrm{Spec}((S^H)_{(f)}) \subseteq X \wr\wr H$. Then $U = q^{-1}(\mathcal{U}_0)$, so appealing to Lemma 2.1.8, 1 there are $g_0, \dots, g_n \in H^0(\mathcal{U}_0, \mathcal{O}(r))$ such that $q^* g_i = f_i$ for each i. The sections $g_0, \dots, g_n$ define a morphism $\Phi : \mathcal{U} \to \mathbb{P}^n$ on some nonempty quasi-compact open subscheme $\mathcal{U} \subseteq \mathcal{U}_0$ that contains $q(U)$, since the collection of $q^*g_i = f_i$ is basepoint free on U. By construction we have $\Phi \circ q = \iota \circ \phi$. Because q^* is injective, any section of a power of $\mathcal{O}_{\mathbb{P}^n}(1)$ that vanishes on Z will pull back under $\overline{\phi}$ to a zero section over $\mathcal{U}$, so the image of $\mathcal{U}$ under Φ is contained in the closure $\overline{\iota(Z)}$ of $\iota(Z)$ in $\mathbb{P}^n$; by shrinking $\mathcal{U}$ if necessary we may assume that $\Phi(\mathcal{U}) \subseteq \iota(Z) \cong Z$. Then $\overline{\phi} := \Phi|_{\mathcal{U}} : \mathcal{U} \to Z$ is a morphism such that $\overline{\phi} \circ q|_U = \phi$ and $\overline{\phi}^* M \cong \mathcal{O}_{\mathcal{U}}(r)$.

(Proof of 2.) Suppose we have $\psi : \mathcal{U} \to Z$ and $\psi' : \mathcal{U}' \to Z$ with $q(U) \subseteq \mathcal{U} \cap \mathcal{U}'$ and $\psi \circ q|_U = \psi' \circ q|_U$. Then $q(U)$ is a dense constructible subset of the noetherian scheme $\mathcal{U} \cap \mathcal{U}'$, so the interior $q(U)^\circ$ is a nonempty dense open subscheme of $\mathcal{U} \cap \mathcal{U}'$ on which ψ and ψ' agree. Since $X \wr\wr H$ is separated, so too is $\mathcal{U} \cap \mathcal{U}'$ and thus we have $\psi = \psi'$ on $\mathcal{U} \cap \mathcal{U}'$. $\square$

Remark 2.1.14. If the sections $\sigma_0, \ldots, \sigma_n \in H^0(Z, M)$ defining an embedding $Z \hookrightarrow \mathbb{P}^n$ in the statement of Proposition 2.1.13 are such that each $\phi^*\sigma_i$ extends to a global section of $L^{\otimes r} \to X$, then in fact one can prove 1 and 2 for reducible X and any H-invariant open subset $U \subseteq X$.

As a corollary of Proposition 2.1.13, we obtain a sort of universal property for the enveloping quotient $q : X^{\mathrm{ss,fg}} \to X \wr\wr H$ when X is irreducible (for reducible X an appropriate statement can be formulated from Remark 2.1.14). Given a quasi-projective variety Z embedded in some projective space and an H-invariant morphism $\phi : X^{\mathrm{ss,fg}} \to Z$ defined by sections of some positive power of $L|_{X^{\mathrm{ss,fg}}}$, there is an inner enveloping quotient $\mathcal{U} \subseteq X \wr\wr H$ of $X^{\mathrm{ss,fg}}$ and a morphism $\overline{\phi} : \mathcal{U} \to Z$ such that the diagram

$$\begin{tikzcd}
 & & X^{\mathrm{ss,fg}} \arrow[d, "q"] \arrow[dr, "\phi"] & \\
X \wr\wr H & \supseteq & \mathcal{U} \arrow[r, "\overline{\phi}"] & Z
\end{tikzcd}$$

commutes, and any other inner enveloping quotient $\mathcal{U}'$ and morphism $\overline{\phi}'$ with $\overline{\phi}' \circ q = \phi$ defines the same rational map $X \wr\wr H \dashrightarrow Z$ as $(\mathcal{U}, \overline{\phi})$.

Remark 2.1.15. The inner enveloping quotient $\mathcal{U}$ and the map $\overline{\phi} : \mathcal{U} \to Z$ constructed above depend on the choice of embedding of Z into a projective space $\mathbb{P}^n$, and the whole construction furthermore relies on the requirement that the morphism $\phi : X^{\mathrm{ss,fg}} \to Z \subseteq \mathbb{P}^n$

is defined by sections of some positive tensor power of $L \to X^{\mathrm{ss,fg}}$ (or for reducible X, of $L \to X$). Contrast this to Mumford's GIT quotient arising from a reductive group G acting on a variety X with linearisation L: then the GIT quotient $X^{\mathrm{ss}} \to X /\!\!/ G$ is a categorical quotient of the semistable locus X^{ss} *in the category of varieties*, so that a G-invariant morphism $X^{\mathrm{ss}} \to Z$ factors uniquely through $X^{\mathrm{ss}} \to X /\!\!/ G$ without any further assumptions on $X^{\mathrm{ss}} \to Z$. So we see that the universal property for the enveloping quotient $q : X^{\mathrm{ss,fg}} \to X \wr\wr H$ for a general linear algebraic group H is considerably weaker than in reductive GIT. The reason for this can be traced in large part to the fact that the enveloping quotient $q : X^{\mathrm{ss,fg}} \to X \wr\wr H$ is not surjective. In Chapter 5 we will find examples of enveloping quotients where $q : X^{\mathrm{ss,fg}} \to X \wr\wr H$ is surjective and indeed $X \wr\wr H$ is a geometric—and hence categorical—quotient of $X^{\mathrm{ss,fg}}$.

Any inner enveloping quotient $\mathcal{U} \subseteq X \wr\wr H$ is quasi-compact, so for sufficiently large integers $r > 0$ the twisting sheaf $\mathcal{O}_{\mathcal{U}}(r)$ defines a line bundle on $\mathcal{U}$; cf. Section A.1 of the Appendix. We shall soon see that, for r large enough, $\mathcal{O}_{\mathcal{U}}(r)$ is in fact very ample. In order to prove this—as well as a similar statement for inner enveloping quotients of more general open subsets of $X^{\mathrm{ss,fg}}$—it is convenient to make the following definition.

Definition 2.1.16. Let H be a linear algebraic group, X an H-variety and $L \to X$ a linearisation. For $r > 0$ and $\mathcal{S} \subseteq H^0(X, L^{\otimes r})^H$ a finite subset of invariant sections, we say a linear subspace $V \subseteq H^0(X, L^{\otimes r})$ is an *enveloping system adapted to* $\mathcal{S}$ if

1. it is finite dimensional, contains $\mathcal{S}$ and is stable under the H-action; and
2. for each $f \in \mathcal{S}$ the $\Bbbk$-algebra $(S^H)_{(f)}$ is finitely generated with generating set $\{\tilde{f}/f \mid \tilde{f} \in V^H\}$.

We call V simply an *enveloping system* if it is an enveloping system adapted to a subset $\mathcal{S}$ such that $X^{\mathrm{ss,fg}} = \bigcup_{f \in \mathcal{S}} X_f$.

The following basic result asserts that finding enveloping systems adapted to finite subsets is essentially equivalent to finding quasi-compact open subschemes of the enveloping quotient $X \wr\wr H$ and giving ways to embed them into projective spaces.

Proposition 2.1.17. *Suppose H is a linear algebraic group and $L \to X$ a linearisation of an H-variety X.*

1. *For any quasi-compact open subscheme $\mathcal{U} \subseteq X \wr\!\wr H$, there is an enveloping system $V \subseteq H^0(X, L^{\otimes r})^H$ adapted to a finite subset $\mathcal{S} \subseteq H^0(X, L^{\otimes r})^H$ with $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \operatorname{Spec}((S^H)_{(f)})$, for some $r > 0$ such that $\mathcal{O}_{\mathcal{U}}(r)$ is a very ample line bundle. Moreover, the natural map $V \to H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r))$ defines a locally closed embedding $\mathcal{U} \hookrightarrow \mathbb{P}(V^*)$.*

2. *Conversely, suppose $H^0(X, L^{\otimes r})$ contains an enveloping system V adapted to a finite subset $\mathcal{S} \subseteq H^0(X, L^{\otimes r})^H$, let $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \operatorname{Spec}((S^H)_{(f)}) \subseteq X \wr\!\wr H$ and let $\phi : U := \bigcup_{f \in \mathcal{S}} X_f \to \mathbb{P}((V^H)^*)$ be the H-invariant map defined by the inclusion $V^H \subseteq H^0(X, L^{\otimes r})$. Then there is a locally closed embedding $\overline{\phi} : \mathcal{U} \hookrightarrow \mathbb{P}((V^H)^*)$ such that $\phi = \overline{\phi} \circ q$ on U and $\overline{\phi}^* \mathcal{O}_{\mathbb{P}((V^H)^*)}(1) = \mathcal{O}_{\mathcal{U}}(r)$.*

3. *If $V \subseteq H^0(X, L^{\otimes r})$ is any enveloping system adapted to $\mathcal{S}$, then the image of the natural multiplication map $V^{\otimes n} \to H^0(X, L^{\otimes rn})$ defines an enveloping system adapted to the set $\{f^n \mid f \in \mathcal{S}\}$, for each $n > 0$.*

Proof. (Proof of 1.) The argument we use can essentially be found in [DK07, Proposition 4.2.2] and is based on a slight modification of the argument used to prove quasi-projectivity of the GIT quotient in reductive GIT (cf. [MFK94, Theorem 1.10]). For completeness, it runs as follows. Let $\mathcal{Y} = \operatorname{Proj}(S^H)$. Since $\mathcal{U}$ is quasi-compact, we may find finitely many invariants $f_1, \ldots, f_m \in I^{\mathrm{ss,fg}} \subseteq S$ such that the basic open subsets $\operatorname{Spec}((S^H)_{(f_i)})$ cover $\mathcal{U}$. Using the reducedness of S^H we can take powers of the f_i and assume, without loss of generality, that there is $r_0 > 0$ such that $f_i \in S^H_{r_0}$ for each i, so that $\mathcal{O}_{\mathcal{U}}(r_0)$ is trivial of rank 1 over $\operatorname{Spec}((S^H)_{(f_i)})$. The $\Bbbk$-algebras $(S^H)_{(f_i)}$ have finite generating sets, which we can write as $\{g_{i1}/(f_i^{r_1}), \ldots, g_{in_i}/(f_i^{r_1})\}$ for $g_{ij} \in S^H_{r_0 r_1}$ and some $n_i > 0$, with one common $r_1 > 0$ working for each $i = 1, \ldots, m$. Resetting $f_i = f_i^{r_1}$ for each i and letting $\mathcal{S} := \{f_1, \ldots, f_m\}$, we can assume that we have found $r > 0$ and a set

$$A := \mathcal{S} \cup \{g_{i,j} \mid i = 1, \ldots, m,\ j = 1, \ldots, n_i\}$$

of invariant sections such that $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \operatorname{Spec}((S^H)_{(f)})$, the sheaf $\mathcal{O}_{\mathcal{U}}(r)$ is locally free and $(S^H)_{(f_i)} = \Bbbk[g_{i,1}/f_i, \ldots, g_{i,n_i}/f_i]$ for each i. Taking $V \subseteq S^H_r = H^0(X, L^{\otimes r})^H$ to be the $\Bbbk$-span of the elements of A, we see that V is an enveloping system adapted to $\mathcal{S}$. The image of the natural map $V \to H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r))$ induced by the structure map $S^H_r \to H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(r))$ is basepoint-free on $\mathcal{U}$, so $V \to H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r))$ defines a morphism

$$\psi : \mathcal{U} \to \mathbb{P}(V^*)$$

such that $\psi^* \mathcal{O}_{\mathbb{P}(V^*)}(1) = \mathcal{O}_{\mathcal{U}}(r)$. Now $H^0(\mathcal{U}_{f_i}, \mathcal{O}_{\mathcal{U}}) \cong (S^H)_{(f_i)}$ and the restriction of ψ to $\mathcal{U}_{f_i}$ maps into the affine open subset $\mathbb{P}(V^*)_{f_i}$ of points of $\mathbb{P}(V^*)$ where $f_i \in H^0(\mathbb{P}(V^*), \mathcal{O}(1))$ doesn't vanish. So $\psi : \mathcal{U}_{f_i} \to \mathbb{P}(V^*)_{f_i}$ corresponds to the natural ring homomorphism

$$(\operatorname{Sym}^\bullet V)_{(f_i)} \to (S^H)_{(f_i)}$$

given by multiplying sections, which is surjective since the generators $g_{i,1}/f_i, \ldots, g_{n_i,1}/f_i$ of $(S^H)_{(f_i)}$ are contained in the image. Thus $\psi : \mathcal{U}_{f_i} \to \mathbb{P}(V^*)_{f_i}$ is a closed immersion. Since $\mathcal{U}$ is covered by the $\mathcal{U}_{f_i}$, the map $\psi : \mathcal{U} \to \mathbb{P}(V^*)$ is a locally closed immersion. In particular, $\mathcal{O}_{\mathcal{U}}(r)$ is very ample.

(Proof of 2.) Suppose $V \subseteq H^0(X, L^{\otimes r})$ is an enveloping system adapted to $\mathcal{S} \subseteq H^0(X, L^{\otimes r})^H$ and let $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \operatorname{Spec}((S^H)_{(f)}) \subseteq \mathcal{Y} = \operatorname{Proj}(S^H)$. As above, the structure map $S^H_r \to H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(r))$ defines a linear map

$$\alpha : H^0(\mathbb{P}((V^H)^*), \mathcal{O}(1)) = V^H \to H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r))$$

such that the composition $q^* \circ \alpha$ is equal to $\phi^* : H^0(\mathbb{P}((V^H)^*), \mathcal{O}(1)) \to H^0(U, L^{\otimes r})$. Now $\mathcal{S} \subseteq V^H$, so $\mathcal{O}_{\mathcal{U}}(r)$ is globally generated by the sections in the image of α and thus α defines a morphism

$$\overline{\phi} : \mathcal{U} \to \mathbb{P}((V^H)^*)$$

such that $\overline{\phi}^* \mathcal{O}_{\mathbb{P}((V^H)^*)}(1) = \mathcal{O}_{\mathcal{U}}(r)$. By 2 of Definition 2.1.16, for each $f \in \mathcal{S}$ the algebra $(S^H)_{(f)}$ is generated by $\tilde{f}/f$, where $\tilde{f} \in V^H$, and now the same argument as in the proof of 1 above shows that $\overline{\phi}$ is a locally closed immersion, and we have $\phi = \overline{\phi} \circ q$.

(Proof of 3.) Given an enveloping system $V \subseteq H^0(X, L^{\otimes r})$ adapted to $\mathcal{S}$ and $n > 0$, the image V' of the natural multiplication map $V^{\otimes n} \to H^0(X, L^{\otimes nr})$ is an H-invariant subspace of $H^0(X, L^{\otimes nr})$ that contains the set of n-fold products of invariant sections $A' := \{f_1 \cdots f_n \mid f_k \in V^H\}$. For any $f \in \mathcal{S}$ the algebra $(S^H)_{(f^n)} = (S^H)_{(f)}$ is generated by A', since we have $\tilde{f}/f = (\tilde{f} f^{n-1})/f^n$ in $(S^H)_{(f)}$ for all $\tilde{f} \in V^H$. Hence V' is an enveloping system adapted to $\{f^n \mid f \in \mathcal{S}\}$. □

Remark 2.1.18. Given an enveloping system V consisting of invariant sections, it follows from Proposition 2.1.17, 1 that any basis of V will give a set of invariants of some positive tensor power of $L \to X$ that separates points *in* $X^{\mathrm{ss,fg}}$ (compare with Definition 1.3.10, 1).

We have already seen that when the ring of invariants S^H is finitely generated and $S_0 = \Bbbk$, then $X \wr\wr H = \operatorname{Proj}(S^H)$ is a projective variety. As a first application of enveloping systems, we can prove a sort of converse for irreducible X:

Corollary 2.1.19. *Suppose H is a linear algebraic group, X an irreducible H-variety and $L \to X$ a linearisation. If the enveloping quotient $X \wr\wr H$ is quasi-compact and complete, then $X \wr\wr H = \operatorname{Proj}(S^H)$. Furthermore, for suitably divisible integers $r > 0$ the sheaf $\mathcal{O}_{X\wr\wr H}(r)$ is an ample line bundle on $X \wr\wr H$ and the natural structure map*

$$\Bbbk[X, L^{\otimes r}]^H = (S^H)^{(r)} \to \Bbbk[X \wr\wr H, \mathcal{O}_{X\wr\wr H}(r)]$$

is an isomorphism. (In particular, $\Bbbk[X, L^{\otimes r}]^H$ is a finitely generated $\Bbbk$-algebra for such r and we have $X^{\mathrm{nss}} = X^{\mathrm{ss,fg}}$.)

Proof. Recall that $q : X^{\mathrm{nss}} \to \operatorname{Proj}(S^H)$ is a dominant morphism, as a result of Lemma 2.1.8. Because X is irreducible, by 2 of the same lemma $X^{\mathrm{ss,fg}}$ is a dense open subset of X^{nss}, so the enveloped quotient $q(X^{\mathrm{ss,fg}})$ is a dense constructible subset of $\operatorname{Proj}(S^H)$. Thus the enveloping quotient $X \wr\wr H$ is a dense open subscheme of $\operatorname{Proj}(S^H)$. Because $X \wr\wr H$ is quasi-compact and complete it is universally closed over $\operatorname{Spec} \Bbbk$, and since $\operatorname{Proj}(S^H)$ is separated over $\operatorname{Spec} \Bbbk$ the open immersion $X \wr\wr H \hookrightarrow \operatorname{Proj}(S^H)$ is a closed morphism [Sta15, Tag 01W0]. Thus $X \wr\wr H = \operatorname{Proj}(S^H)$. Using Proposition 2.1.17, 1 find $r' > 0$ and an enveloping system $V \subseteq H^0(X, L^{\otimes r'})^H$ so that the natural map $V \to H^0(X \wr\!\wr H, \mathcal{O}(r')) = H^0(\mathrm{Proj}(S^H), \mathcal{O}(r'))$ defines a closed immersion $\mathrm{Proj}(S^H) \hookrightarrow \mathbb{P}(V^*)$ (the fact the immersion is closed is implied from the completeness of $\mathrm{Proj}(S^H) = X \wr\!\wr H$). The line bundle $\mathcal{O}(r')$ on $\mathrm{Proj}(S^H)$ is (very) ample, so by Serre vanishing [Har77, Chapter 3, Proposition 5.3] there is $m_0 > 0$ such that for all $m \geq m_0$ the restriction map

$$H^0(\mathbb{P}(V^*), \mathcal{O}(m)) = \mathrm{Sym}^m V \to H^0(\mathrm{Proj}(S^H), \mathcal{O}(mr'))$$

is surjective. Letting r be any positive multiple of $m_0 r'$ and $m = r/r'$, we see that the restriction map $\Bbbk[\mathbb{P}(V^*), \mathcal{O}(m)] \to \Bbbk[\mathrm{Proj}(S^H), \mathcal{O}(r)]$ is surjective and therefore the $\Bbbk$-algebra $\Bbbk[\mathrm{Proj}(S^H), \mathcal{O}(r)]$ is finitely generated. The map $\Bbbk[\mathbb{P}(V^*), \mathcal{O}(m)] \to \Bbbk[\mathrm{Proj}(S^H), \mathcal{O}(r)]$ factors through the canonical structure map

$$\Bbbk[X, L^{\otimes r}]^H = (S^H)^{(r)} \to \Bbbk[\mathrm{Proj}(S^H), \mathcal{O}(r)],$$

thus this too is a surjective map onto a finitely generated $\Bbbk$-algebra. On the other hand, the composition of this map with pull-back along the natural map q from (2.1.1) agrees with restriction of sections $\Bbbk[X, L^{\otimes r}]^H \to \Bbbk[X^{\mathrm{nss}}, L^{\otimes r}]^H$, which is injective because X is irreducible. It follows that

$$\Bbbk[X, L^{\otimes r}]^H = (S^H)^{(r)} \cong \Bbbk[\mathrm{Proj}(S^H), \mathcal{O}(r)].$$

In particular, $\Bbbk[X, L^{\otimes r}]^H$ is a finitely generated $\Bbbk$-algebra. The equality $X^{\mathrm{nss}} = X^{\mathrm{ss,fg}}$ now follows from the definitions 2.1.1 of these sets and Remark 2.1.2. ☐

2.1.2 Comparison with Mumford's Reductive GIT

The definitions of the naively semistable locus, finitely generated semistable locus and enveloping quotient are direct generalisations of the corresponding notions in [DK07] for unipotent groups to the context of general varieties with actions of any linear algebraic group. As such they apply to the situation where $H = G$ is a reductive group, so we take a moment to compare these notions to those arising in Mumford's GIT [MFK94] for reductive groups.

Firstly, *if X is projective over an affine variety and $G \curvearrowright L \to X$ is a linearisation that is relatively ample as a line bundle, then $X^{\mathrm{nss}} = X^{\mathrm{ss,fg}}$ is equal to Mumford's semistable locus X^{ss} for $G \curvearrowright L \to X$ [MFK94, Definition 1.7], and the enveloping quotient is precisely the GIT quotient $X/\!\!/G = \mathrm{Proj}(S^G)$ of [MFK94, Theorem 1.10].* (Notice this includes the cases where X is affine with a linearisation of the trivial bundle $L = \mathcal{O}_X \to X$, or X is projective with ample linearisation $L \to X$.) Indeed, we have $I^{\mathrm{ss,fg}} = \bigcup_{r>0} H^0(X, L^{\otimes r})^G$ in this case: for any invariant section f of a positive tensor power of L, the ring $(S^G)_f$ is the localisation of a finitely generated ring by Nagata's theorem [Nag64], and so $(S^G)_{(f)}$ is also a finitely generated algebra, being the subring of invariants for the $\mathbb{G}_m$-action defining the grading on S^G. Thus $X \wr\wr G = \mathrm{Proj}(S^G)$. Because L is ample X_f is affine for each $f \in I^{\mathrm{ss,fg}}$, from which it follows that $X^{\mathrm{ss}} = X^{\mathrm{nss}} = X^{\mathrm{ss,fg}}$.

However, the similarities with Mumford's GIT when G is reductive do not extend beyond these cases. For a general variety X with possibly non-ample linearisation $L \to X$ of G, there may be invariant sections f whose loci of non-vanishing X_f are not affine. In Mumford's theory only those X_f that are affine are included in defining the semistable locus X^{ss}; cf. [MFK94, Definition 1.7]. So *for a general linearisation $G \curvearrowright L \to X$ with G reductive, Mumford's semistable locus X^{ss} is contained in $X^{\mathrm{ss,fg}}$ as a (possibly empty) open subset.* Given any inner enveloping quotient $q : X^{\mathrm{ss,fg}} \to \mathcal{U} \subseteq X \wr\wr G$, the restriction to Mumford's semistable locus X^{ss} coincides with the GIT quotient, thus $q(X^{\mathrm{ss}})$ is an open subvariety of $\mathcal{U}$. Hence *the GIT quotient $X/\!\!/G$ is a (possibly empty) quasi-compact open subscheme of any inner enveloping quotient inside $X \wr\wr G$.* Finally, as discussed in Remark 2.1.15, *the enveloping quotient $q : X^{\mathrm{ss,fg}} \to X \wr\wr G$ is not in general a categorical quotient in the category of varieties for the G-action on $X^{\mathrm{ss,fg}}$, whereas Mumford's GIT quotient $X^{\mathrm{ss}} \to X/\!\!/G$ is a categorical quotient for the G-action on X^{ss} [MFK94, Theorem 1.10].*

2.2 Natural Properties of Enveloping Quotients with Respect to Induced Group Actions

In this section we will study the enveloping quotient and inner enveloping quotients (adapted to some finite subset) with respect to various natural operations on groups, which will be needed for the sequel.

2.2.1 Restriction and Extension of the Structure Group

We first look at the case of restricting a linearisation under a surjective homomorphism $\rho : H_1 \to H_2$ of linear algebraic groups. More precisely, suppose X is an H_2-variety and $L \to X$ a line bundle with an H_2-linearisation. For precision, let us denote this linearisation as $\mathcal{L}_2 \to X$. Then the homomorphism ρ induces an H_1-linearisation on the line bundle $L \to X$, which we denote $\mathcal{L}_1 \to X$. It is immediately clear that

$$\Bbbk[X, \mathcal{L}_1]^{H_1} = \Bbbk[X, \mathcal{L}_2]^{H_2}$$

and also that for any given section $f \in H^0(X, \mathcal{L}_1^{\otimes r})^{H_1} = H^0(X, \mathcal{L}_2^{\otimes r})^{H_2}$, (with $r > 0$) we have that $(\Bbbk[X, \mathcal{L}_1]^{H_1})_{(f)}$ is a finitely generated $\Bbbk$-algebra if, and only if, $(\Bbbk[X, \mathcal{L}_2]^{H_2})_{(f)}$ is. It follows that there are canonical identifications

$$X^{\mathrm{nss}(\mathcal{L}_1)} = X^{\mathrm{nss}(\mathcal{L}_2)}, \quad X^{\mathrm{ss,fg}(\mathcal{L}_1)} = X^{\mathrm{ss,fg}(\mathcal{L}_2)}, \quad X \wr\wr_{\mathcal{L}_1} H_1 = X \wr\wr_{\mathcal{L}_2} H_2$$

and the natural maps

$$q_1 : X^{\mathrm{nss}(\mathcal{L}_1)} \to \mathrm{Proj}(\Bbbk[X, \mathcal{L}_1]^{H_1}), \quad q_2 : X^{\mathrm{nss}(\mathcal{L}_2)} \to \mathrm{Proj}(\Bbbk[X, \mathcal{L}_2]^{H_2})$$

of (2.1.1) coincide under these identifications.

Next we look at the case of extension of the structure group. Suppose we have an inclusion $H_1 \hookrightarrow H_2$ and $L \to X$ is an H_1-linearisation. Recall from Chapter 1, Section 1.3.2 that we may consider the fibre space $H_2 \times^{H_1} X$ associated to the principal H_1-bundle $H_2 \to H_2/H_1$, and if H_1 is special (in the sense of [Ser58, §4.1]) or X satisfies mild assumptions (for example, if X is quasi-projective, or more generally if any finite set of points in X is contained in an affine open subset) this space exists as a variety. Then $H_2 \times^{H_1} L \to H_2 \times^{H_1} X$

is a line bundle and there is a natural H_2-linearisation on $H_2 \times^{H_1} L \to H_2 \times^{H_1} X$ induced by left multiplication. This extends the H_1-linearisation $L \to X$ under the closed immersion

$$\alpha : X \hookrightarrow H_2 \times^{H_1} X, \quad x \mapsto [e, x].$$

As before, we will usually abuse notation and write $L \to H_2 \times^{H_1} X$ for this linearisation instead of $H_2 \times^{H_1} L$, unless confusion is likely to arise. Recall also that pullback along α induces an isomorphism of graded rings

$$\alpha^* : \Bbbk[H_2 \times^{H_1} X, L]^{H_2} \xrightarrow{\cong} \Bbbk[X, L]^{H_1},$$

thus there is an isomorphism of schemes

$$\overline{\alpha} : \mathrm{Proj}(\Bbbk[X, L]^{H_1}) \xrightarrow{\cong} \mathrm{Proj}(\Bbbk[H_2 \times^{H_1} X, L]^{H_2})$$

such that $\overline{\alpha}^*$ identifies the corresponding twisting sheaves. Let $q_{H_1} : X^{\mathrm{ss,fg}} \to X \wr\wr H_1$ be the enveloping quotient map for the linearisation $H_1 \curvearrowright L \to X$ and $q_{H_2} : (H_2 \times^{H_1} X)^{\mathrm{ss,fg}} \to (H_2 \times^{H_1} X) \wr\wr H_2$ be the enveloping quotient map for the linearisation $H_2 \curvearrowright H_2 \times^{H_1} L \to H_2 \times^{H_1} X$. Clearly pulling back along α establishes a bijection $I^{\mathrm{ss,fg}(H_2 \times^{H_1} L)} \longleftrightarrow I^{\mathrm{ss,fg}(L)}$. This implies that α restricts to give a closed immersion of $X^{\mathrm{ss,fg}(H_1)}$ into $(H_2 \times^{H_1} X)^{\mathrm{ss,fg}(H_2)}$ and $\overline{\alpha}$ restricts to an isomorphism of the enveloping quotients. Furthermore, we have the following commutative diagram.

$$\begin{tikzcd}
X^{\mathrm{ss,fg}(H_1)} \arrow[r, hook, "\alpha"] \arrow[d, "q_{H_1}"] & (H_2 \times^{H_1} X)^{\mathrm{ss,fg}(H_2)} \arrow[d, "q_{H_2}"] \\
X \wr\wr H_1 \arrow[r, "\cong", "\overline{\alpha}"'] & (H_2 \times^{H_1} X) \wr\wr H_2
\end{tikzcd}$$

2.2.2 Induced Actions of Quotient Groups on Enveloping Quotients

Let us return to the situation where a linear algebraic group H acts on a variety X and is equipped with a linearisation $L \to X$, but now also suppose H has a normal subgroup N. Then we may consider the restricted linearisation $N \curvearrowright L \to X$ and form its naively semistable locus $X^{\mathrm{nss}(N)}$, semistable finitely generated locus $X^{\mathrm{ss,fg}(N)}$ and enveloping quotient $q_N : X^{\mathrm{ss,fg}(N)} \to X \wr\wr N$. Because N is normal in H, the action of H on $S = \Bbbk[X, L]$

induces a natural H/N-action on the ring S^N of N-invariant sections. For any $h \in H$ and $f \in I^{\mathrm{nss}(N)}$, the action on X induces an isomorphism

$$X_f \xrightarrow{\cong} X_{h \cdot f}, \quad x \mapsto hx$$

with $h \cdot f \in I^{\mathrm{nss}(N)}$ and inverse given by acting by h^{-1}, so the action of H on X restricts to an action on $X^{\mathrm{nss}(N)}$. Moreover, for any $f \in H^0(X, L^{\otimes r})^N$ (with $r > 0$) and $\overline{h} = hN \in H/N$ the application of $\overline{h}$ induces an isomorphism

$$\overline{h} \cdot (-) : (S^N)_{(f)} \xrightarrow{\cong} (S^N)_{(h \cdot f)},$$

from which it follows that the action of H/N on S^N preserves $I^{\mathrm{ss,fg}(N)}$. Thus $X^{\mathrm{ss,fg}(N)}$ is also stable under the H-action.

Proposition 2.2.1. *Retain the notation of the preceding discussion. Then the action of H/N on S^N defines a canonical action of H/N on $\mathcal{Y} := \mathrm{Proj}(S^N)$ such that the map $q_N : X^{\mathrm{nss}(N)} \to \mathcal{Y}$ of* (2.1.1) *is equivariant with respect to the quotient $H \to H/N$. In addition, if $\mathcal{S} \subseteq I^{\mathrm{nss}(N)}$ is any subset that is stable under the canonical H/N-action on S^N, then the open subscheme $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \mathrm{Spec}((S^N)_{(f)})$ of $\mathcal{Y}$ is preserved under this action. (In particular, $X /\!\!/ N$ is preserved under the action.)*

Before proving Proposition 2.2.1 we need to introduce some notation and prove a lemma. Let

$$\Sigma : H \times X \to X$$

be the action morphism. Recall from [MFK94, Chapter 1, §3] that a linearisation of H on L is equivalent to a choice of line bundle isomorphism

$$\Theta : \Sigma^* L \xrightarrow{\cong} \mathcal{O}_H \boxtimes L = H \times L$$

over $H \times X$, satisfying an appropriate cocycle condition. This naturally extends to isomorphisms of tensor powers of the bundles, so we get isomorphisms

$$\theta : H^0(H \times X, \Sigma^*(L^{\otimes r})) \xrightarrow{\cong} H^0(H \times X, H \times L^{\otimes r}), \quad r \geq 0.$$

(Note we abuse notation and suppress mention of r in the map θ.) Composition of θ with Σ thus gives us maps

$$\Sigma_\theta^* : H^0(X, L^{\otimes r}) \xrightarrow{\theta \circ \Sigma^*} H^0(H \times X, H \times L^{\otimes r}) = \mathcal{O}(H) \otimes H^0(X, L^{\otimes r}), \quad r \geq 0, \tag{2.2.1}$$

where the last equality follows from the Künneth formula [Sta15, Tag 02KE]. For any $h \in H$, the composition

$$H^0(X, L^{\otimes r}) \xrightarrow{\Sigma_\theta^*} \mathcal{O}(H) \otimes H^0(X, L^{\otimes r}) \xrightarrow{\mathrm{ev}_h \otimes \mathrm{id}_X^*} H^0(X, L^{\otimes r})$$

satisfies

$$(\mathrm{ev}_h \otimes \mathrm{id}_X^*)(\Sigma_\theta^*(f)) = h^{-1} \cdot f \tag{2.2.2}$$

for all $f \in H^0(X, L^{\otimes r})$, for each $r \geq 0$.

Lemma 2.2.2. *Let $r \geq 0$ and suppose $V \subseteq H^0(X, L^{\otimes r})^N$ is an H-invariant subspace of sections. Then the image of V under Σ_θ^* lies in $\mathcal{O}(H)^N \otimes V$, where $\mathcal{O}(H)^N$ is formed with respect to the action of N on H by right multiplication. (In particular, this holds for $V = H^0(X, L^{\otimes r})^N$.)*

Proof. Suppose $f \in V$ is non-zero. Then we may write $\Sigma_\theta^* f = \sum_{j=1}^m a_j \otimes f_j$, with $m > 0$, $a_j \in \mathcal{O}(H)$ and $f_j \in H^0(X, L^{\otimes r})$ such that the a_j and the f_j are linearly independent over $\Bbbk$. For any $h \in H$ we have

$$h^{-1} \cdot f = (\mathrm{ev}_h \otimes \mathrm{id}_X^*)(\Sigma_\theta^*(f)) = \sum_j a_j(h) f_j.$$

We can find $h_1, \ldots, h_m \in H$ such that the matrix

$$(a_j(h_i))_{i,j}$$

is invertible:[1] indeed, the morphism $H \to \Bbbk^m$ defined by the a_j has image not contained in any proper linear subspace of $\Bbbk^m$, so there are $h_1, \ldots, h_m$ such that the $(a_1(h_i), \ldots, a_m(h_i))$ span $\Bbbk^m$. For such h_i, the system of linear equations

$$h_i^{-1} \cdot f = \sum_{j=1}^m a_j(h_i) f_j, \quad i = 1, \ldots, m$$

---

[1] A result like this is used in the proof of [New78, Lemma 3.1].

tells us that each f_j is in the span of $\{h_1^{-1} \cdot f, \dots, h_m^{-1} \cdot f\} \subseteq V$. So $f_j \in V$ for each j. Because $V \subseteq H^0(X, L^{\otimes r})^N$, by the associativity property of an action and the fact that N is normal in H we have

$$\sum_j a_j(hn) f_j = (n^{-1}h^{-1}) \cdot f = h^{-1} \cdot f = \sum_j a_j(h) f_j.$$

for any $n \in N$ and $h \in H$. Since the f_j are linearly independent $a_j(hn) = a_j(h)$ for all $n \in N$, $h \in H$, so $a_j \in \mathcal{O}(H)^N$ for each j. Hence $\Sigma_\theta^* f \in \mathcal{O}(H)^N \otimes V$. $\square$

Proof of Proposition 2.2.1. The proof is divided into two steps. We begin by constructing the morphism $\overline{\Sigma} : (H/N) \times \mathcal{Y} \to \mathcal{Y}$ which defines the desired action and show that it maps $(H/N) \times \mathcal{U}$ to $\mathcal{U}$, for $\mathcal{U}$ as in the statement of the proposition. We then show $\overline{\Sigma}$ satisfies the axioms for a group action and prove the equivariance of q_N.

(**Step 1:** Definition of $\overline{\Sigma}$ and restriction to $\mathcal{U}$.) Recall that

$$(H/N) \times \mathcal{Y} = \operatorname{Proj}(\mathcal{O}(H/N) \otimes S^N),$$

where the grading in $\mathcal{O}(H/N) \otimes S^N$ is induced by S^N, with $\mathcal{O}(H/N)$ having degree 0. The corresponding twisting sheaves $\mathcal{O}(r)$ are given by the exterior tensor product $\mathcal{O}_{H/N} \boxtimes \mathcal{O}_{\mathcal{Y}}(r)$ for each $r \geq 0$ [Sta15, Tag 01MX]. The quotient map $H \to H/N$ identifies $\mathcal{O}(H/N)$ with $\mathcal{O}(H)^N$ and by virtue of Lemma 2.2.2 the diagram

$$\begin{tikzcd}
S^N \arrow[r, "\Sigma_\theta^*|_{S^N}"] \arrow[d, hook, "q_N^*"] & \mathcal{O}(H)^N \otimes S^N \arrow[d, hook] \\
S \arrow[r, "\Sigma_\theta^*"] & \mathcal{O}(H) \otimes S
\end{tikzcd} \tag{2.2.3}$$

of graded rings is well defined and commutes, where Σ_θ^* is as in (2.2.1). Applying the Proj functor to the top horizontal map defines a rational map, which we claim is in fact a morphism

$$\overline{\Sigma} := \operatorname{Proj}(\Sigma_\theta^*|_{S^n}) : (H/N) \times \mathcal{Y} \to \mathcal{Y}.$$

To see this, we need to verify that if $f \in S^N$ is a homogenous element of positive degree, then there is a homogeneous prime ideal of $\mathcal{O}(H)^N \otimes S^N$ different to the irrelevant ideal and not containing $\Sigma_\theta^*(f)$. But since S^N is reduced there is a homogeneous prime $\mathfrak{p} \in \mathcal{Y} = \operatorname{Proj}(S^N)$ not containing f, and it follows that

$$(\mathrm{ev}_e \otimes \mathrm{id}_S)^{-1}(\mathfrak{p}) \in \operatorname{Proj}(\mathcal{O}(H)^N \otimes S^N)$$

is a homogeneous prime which does not contain $\Sigma_\theta^*(f)$ and is different to the irrelevant ideal.

Now let $\mathcal{S} \subseteq I^{\mathrm{nss}(N)}$ be a subset that is stable under the H-action on S. Notice that this includes the case $\mathcal{S} = I^{\mathrm{ss,fg}(N)}$ by virtue of the discussion before the statement of Proposition 2.2.1. Let $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \operatorname{Spec}((S^N)_{(f)}) \subseteq \mathcal{Y}$. We next show that $\overline{\Sigma}$ restricts to a morphism $(H/N) \times \mathcal{U} \to \mathcal{U}$. As noted in Section A.1 of the Appendix, the construction of $\overline{\Sigma}$ comes with a canonical map of $\mathcal{O}_{(H/N)\times\mathcal{Y}}$-algebras

$$\overline{\Sigma}^* \bigoplus_{r\geq 0} \mathcal{O}_{\mathcal{Y}}(r) \to \bigoplus_{r \geq 0} \mathcal{O}_{H/N} \boxtimes \mathcal{O}_{\mathcal{Y}}(r),$$

which gives rise to a homomorphism of graded rings

$$\overline{\Sigma}^* : \Bbbk[\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(1)] \to \mathcal{O}(H)^N \otimes \Bbbk[\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(1)]$$

that is compatible with $\Sigma_\theta^* : S^N \to \mathcal{O}(H)^N \otimes S^N$ under the natural structure maps $S^N \to \Bbbk[\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(1)]$ and $\mathcal{O}(H)^N \otimes S^N \to \mathcal{O}(H)^N \otimes \Bbbk[\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(1)]$. Let $h \in H$ and $\iota_{\overline{h}} : \mathcal{Y} \hookrightarrow (H/N) \times \mathcal{Y}$ be the inclusion $y \mapsto (\overline{h}, y)$. Then the following diagram commutes.

$$\begin{tikzcd}
S^N \arrow[r, "\Sigma_\theta^*|_{S^N}"] \arrow[d] & \mathcal{O}(H)^N \otimes S^N \arrow[r, "\mathrm{ev}_h \otimes \mathrm{id}_{S^N}"] \arrow[d] & S^N \arrow[d] \\
\Bbbk[\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(1)] \arrow[r, "\overline{\Sigma}^*"] & \mathcal{O}(H)^N \otimes \Bbbk[\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(1)] \arrow[r, "\iota_{\overline{h}}^*"] & \Bbbk[\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(1)]
\end{tikzcd} \tag{2.2.4}$$

Suppose $y \in \mathcal{U} \subseteq \mathcal{Y}$ and $f \in \mathcal{S}$ is such that $f(y) \neq 0$. Then $h \cdot f$ is contained in $\mathcal{S}$ and maps to f under the composition of the top row of morphisms in (2.2.4), and by commutativity we see that

$$(h \cdot f)(\overline{\Sigma}(\overline{h}, y)) \neq 0 \iff f(y) \neq 0,$$

where we think of f as a section of some power of $\mathcal{O}_{\mathcal{Y}}(1)$. Thus $\iota_{\overline{h}} \circ \overline{\Sigma}$ maps $\operatorname{Spec}((S^N)_{(f)})$ to $\operatorname{Spec}((S^N)_{(h\cdot f)}) \subseteq \mathcal{U}$. This show that $\overline{\Sigma}$ restricts to a map $(H/N) \times \mathcal{U} \to \mathcal{U}$. In the case where $\mathcal{S} = I^{\mathrm{ss,fg}(N)}$, we conclude that $\overline{\Sigma}$ restricts to a morphism $(H/N) \times X \wr\wr N \to X \wr\wr N$.

(**Step 2:** $\overline{\Sigma}$ is an action of (H/N) on $\mathcal{Y}$ and q_N is equivariant.) Let μ (respectively, $\overline{\mu}$) be the morphism defining group multiplication on H (respectively, on (H/N)). By using the Proj functor, the commutative diagrams that $\overline{\Sigma} : (H/N) \times \mathcal{Y} \to \mathcal{Y}$ needs to satisfy in order to be an action follow immediately from verifying that the diagrams (Identity) and (Associativity) commute.

$$\begin{tikzcd}
S^N \arrow[d, "\Sigma_\theta^*|_{S^N}"'] \arrow[dr, "\mathrm{id}_{S^N}"] & \\
\mathcal{O}(H)^N \otimes S^N \arrow[r, "\mathrm{ev}_e \otimes \mathrm{id}_{S^N}"'] & S^N
\end{tikzcd} \tag{Identity}$$

$$\begin{tikzcd}
S^N \arrow[r, "\Sigma_\theta^*|_{S^N}"] \arrow[d, "\Sigma_\theta^*|_{S^N}"] & \mathcal{O}(H)^N \otimes S^N \arrow[d, "\overline{\mu}^* \otimes \mathrm{id}_{S^N}"] \\
\mathcal{O}(H)^N \otimes S^N \arrow[r, "\mathrm{id}_{H/N}^* \otimes (\Sigma_\theta^*|_{S^N})"] & \mathcal{O}(H)^N \otimes \mathcal{O}(H)^N \otimes S^N
\end{tikzcd} \tag{Associativity}$$

Note that (Associativity) is well defined by Lemma 2.2.2. The diagram (Identity) is simply Lemma 2.2.2 applied to $\overline{h} = \overline{e} \in H/N$. To verify commutativity of diagram (Associativity), note that

$$\overline{\mu}^* : \mathcal{O}(H)^N \to \mathcal{O}(H)^N \otimes \mathcal{O}(H)^N$$

is just the restriction of $\mu^* : \mathcal{O}(H) \to \mathcal{O}(H) \otimes \mathcal{O}(H)$ to the subring $\mathcal{O}(H)^N$, so (Associativity) is obtained by restricting the diagram

$$\begin{tikzcd}
S \arrow[r, "\Sigma_\theta^*"] \arrow[d, "\Sigma_\theta^*"] & \mathcal{O}(H) \otimes S \arrow[d] \\
\mathcal{O}(H) \otimes S \arrow[r, "\mathrm{id}_H^* \otimes \Sigma_\theta^*"] & \mathcal{O}(H) \otimes \mathcal{O}(H) \otimes S
\end{tikzcd}$$

to subrings of N-invariant. But this diagram commutes because $\Sigma : H \times X \to X$ defines an action.

Finally, let $\pi : H \to H/N$ be the canonical quotient map. Applying Proj to the commuting diagram (2.2.3), we see that $\overline{\Sigma}$ makes the diagram

$$\begin{tikzcd}
H \times X^{\mathrm{nss}(N)} \arrow[r, "\Sigma"] \arrow[d, "\pi \times q_N"] & X^{\mathrm{nss}(N)} \arrow[d, "q_N"] \\
(H/N) \times \mathcal{Y} \arrow[r, "\overline{\Sigma}"] & \mathcal{Y}
\end{tikzcd}$$

commute. This is to say that q_N is equivariant with respect to the projection $\pi : H \to H/N$. $\square$

Proposition 2.2.3. *Retain the notation preceding Proposition 2.2.1. Let $\mathcal{S} \subseteq I^{\mathrm{ss,fg}(N)}$ be a finite subset such that $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \operatorname{Spec}((S^N)_{(f)})$ is stable under the H/N-action on $X /\!\!/ N$ of Proposition 2.2.1.*

1. *If $r > 0$ and $V \subseteq H^0(X, L^{\otimes r})^N$ is an H-stable enveloping system adapted to $\mathcal{S}$ for the restricted linearisation $N \curvearrowright L \to X$, then the immersion $\overline{\phi} : \mathcal{U} \hookrightarrow \mathbb{P}(V^*)$ of Proposition 2.1.17, 2 is equivariant, and pullback of the canonical linearisation $H/N \curvearrowright \mathcal{O}_{\mathbb{P}(V^*)}(1) \to \mathbb{P}(V^*)$ along $\overline{\phi}$ defines a linearisation $(H/N) \curvearrowright \mathcal{O}_{\mathcal{U}}(r) \to \mathcal{U}$ such that the natural morphism $L^{\otimes r}|_{q_N^{-1}(\mathcal{U})} \to \mathcal{O}_{\mathcal{U}}(r)$ is equivariant with respect to the projection $H \to H/N$.*

2. *Given $r > 0$ such that $\mathcal{O}_{\mathcal{U}}(r) \to \mathcal{U}$ is very ample, there is at most one H/N-linearisation on $\mathcal{O}_{\mathcal{U}}(r) \to \mathcal{U}$ making the natural map $L^{\otimes r}|_{q_N^{-1}(\mathcal{U})} \to \mathcal{O}_{\mathcal{U}}(r)$ equivariant with respect to the projection $H \to H/N$.*

Proof. (Proof of 1.) The action of H on V descends to an action of H/N on V, which defines a linearisation $(H/N) \curvearrowright \mathcal{O}(1) \to \mathbb{P}(V^*)$. We show that $\overline{\phi}$ is equivariant with respect to this action on $\mathbb{P}(V^*)$. Let $\Sigma : H \times X \to X$ denote the action morphism and $\overline{\Sigma} : (H/N) \times \mathcal{Y} \to \mathcal{Y}$ the action morphism on $\mathcal{Y} = \operatorname{Proj}(S^N)$ constructed in Proposition 2.2.1. Note that $\overline{\Sigma}$ restricts to a morphism $(H/N) \times \mathcal{U} \to \mathcal{U}$ by assumption. By Lemma 2.2.2 the linear map Σ_θ^* of (2.2.1) restricts to define a map $\Sigma_\theta^*|_V : V \to \mathcal{O}(H)^N \otimes V$. Applying the $\operatorname{Sym}^\bullet$ functor, we get a homomorphism of graded rings

$$\operatorname{Sym}^\bullet(\Sigma_\theta^*|_V) : \operatorname{Sym}^\bullet V \to \mathcal{O}(H)^N \otimes \operatorname{Sym}^\bullet V,$$

where $\mathcal{O}(H)^N$ is in degree zero in the latter ring. Note that applying Proj to this homomorphism recovers the linearisation of H/N on $\mathcal{O}(1) \to \mathbb{P}(V^*)$ just described. Furthermore, the following diagram of graded rings commutes (recall $(S^N)^{(r)}$ is the r-th Veronese subring of S^N).

$$\begin{tikzcd}
\operatorname{Sym}^\bullet V \arrow[r, "\operatorname{Sym}^\bullet(\Sigma_\theta^*|_V)"] \arrow[d, "\mathrm{mult}"] & \mathcal{O}(H)^N \otimes \operatorname{Sym}^\bullet V \arrow[d, "\mathrm{id}_H^* \otimes \mathrm{mult}"] \\
(S^N)^{(r)} \arrow[r, "\Sigma_\theta^*|_{S^N}"] \arrow[d] & \mathcal{O}(H)^N \otimes (S^N)^{(r)} \arrow[d] \\
\Bbbk[\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r)] \arrow[r, "\overline{\Sigma}^*"] & \mathcal{O}(H)^N \otimes \Bbbk[\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r)]
\end{tikzcd}$$

Under the identification $\operatorname{Sym}^\bullet V = \Bbbk[\mathbb{P}(V^*), \mathcal{O}(1)]$, the composition of the left-hand vertical arrows corresponds to pull-back along $\overline{\phi}$ and, by the Künneth isomorphism, the composition of the right-hand vertical arrows corresponds to pulling back along $\mathrm{id}_{H/N} \times \overline{\phi}$. Applying Proj to this diagram, it follows that $\overline{\phi} : \mathcal{U} \hookrightarrow \mathbb{P}(V^*)$ is H/N-equivariant.

Define $(H/N) \curvearrowright \mathcal{O}_{\mathcal{U}}(r) \to \mathcal{U}$ to be the linearisation obtained by pulling back $H/N \curvearrowright \mathcal{O}(1) \to \mathbb{P}(V^*)$ under $\overline{\phi}$. It is immediate that $q_N^*\mathcal{O}_{\mathcal{U}}(r) = L^{\otimes r}|_{q_N^{-1}(\mathcal{U})}$ as line bundles. Let $\psi : L^{\otimes r}|_{q_N^{-1}(\mathcal{U})} \to \mathcal{O}_{\mathcal{U}}(r)$ be the naturally induced map. To show ψ is equivariant with respect to $H \to H/N$, argue as follows. The image of $\overline{\phi}^* : H^0(\mathbb{P}(V^*), \mathcal{O}(1)) \to H^0(\mathcal{U}, \mathcal{O}(r))$ is an H/N-stable subspace of $H^0(\mathcal{U}, \mathcal{O}(r))$ that pulls back under q_N to the linear system $V \subseteq H^0(X, L^{\otimes r})^N$, which is basepoint free on $q_N^{-1}(\mathcal{U})$. Let $x \in q_N^{-1}(\mathcal{U})$, let $f \in V$ such that $f(x) \neq 0$ and let $F \in H^0(\mathcal{U}, \mathcal{O}(r))$ with $q_N^* F = f$. Then because q_N^* is equivariant with respect to the natural H/N-actions on V and $H^0(\mathcal{U}, \mathcal{O}(r))$, for any $h \in H$ we have

$$hf(x) = (h \cdot f)(hx) = (h \cdot (q_N^* F))(hx) = (q_N^*(\overline{h} \cdot F))(hx),$$

(where $\overline{h} = hN \in H/N$), whence

$$\psi(hf(x)) = (\overline{h} \cdot F)(q_N(hx)) = \overline{h}F(q_N(x)) = \overline{h}\psi(f(x)).$$

It follows by linearity that $\psi(hl) = \overline{h}\psi(l)$ for any $l \in L^{\otimes r}|_x$. Hence ψ is equivariant with respect to $H \to H/N$.

(Proof of 2.) Suppose now $r > 0$ is such that $\mathcal{O}_{\mathcal{U}}(r) \to \mathcal{U}$ is equipped with two H/N-linearisations $\mathcal{L}_1, \mathcal{L}_2$ such that the natural maps $L^{\otimes r}|_{q_N^{-1}(\mathcal{U})} \to \mathcal{L}_1$ and $L^{\otimes r}|_{q_N^{-1}(\mathcal{U})} \to \mathcal{L}_2$ are both equivariant with respect to the projection $H \to H/N$. Then the inclusions

$$q_N^* : H^0(\mathcal{U}, \mathcal{L}_1) \hookrightarrow H^0(q_N^{-1}(\mathcal{U}), L^{\otimes r})^N,$$
$$q_N^* : H^0(\mathcal{U}, \mathcal{L}_2) \hookrightarrow H^0(q_N^{-1}(\mathcal{U}), L^{\otimes r})^N$$

are both H/N-equivariant linear maps, therefore $H^0(\mathcal{U}, \mathcal{L}_1)$ and $H^0(\mathcal{U}, \mathcal{L}_2)$ agree as H/N-modules. Because $\mathcal{O}_{\mathcal{U}}(r) \to \mathcal{U}$ is very ample, by [MFK94, Chapter 1, §1, Lemma] we can find a finite dimensional H/N-stable complete linear system $W \subseteq H^0(\mathcal{U}, \mathcal{L}_1) = H^0(\mathcal{U}, \mathcal{L}_2)$

with which to equivariantly embed $\mathcal{U}$ into $\mathbb{P}(W^*)$. Then the restriction of the H/N-linearisation $\mathcal{O}_{\mathbb{P}(W^*)}(1) \to \mathbb{P}(W^*)$ to $\mathcal{U}$ is equal to both the linearisations $\mathcal{L}_1$ and $\mathcal{L}_2$, so that $\mathcal{L}_1 = \mathcal{L}_2$. $\square$

The next proposition can be regarded as an equivariant version of Proposition 2.1.13.

Proposition 2.2.4. *Let H be a linear algebraic group with a normal subgroup N, let X be an irreducible H-variety with H-linearisation $L \to X$ and suppose $U = \bigcup_{f \in \mathcal{S}} X_f$, where $\mathcal{S}$ a nonempty subset of $I^{\mathrm{ss,fg}}$ invariant under the H-action. Let Z be a quasi-projective H/N-variety Z with very ample linearisation $H/N \curvearrowright M \to Z$ and suppose there is a morphism $\phi : U \to Z$ that is equivariant with respect to $H \to H/N$ and such that $\phi^* M = L^{\otimes r}|_U$ as line bundles, for some $r > 0$, and the natural map $L^{\otimes r}|_U \to M$ is equivariant with respect to $H \to H/N$. Then there is an inner enveloping quotient $\mathcal{U}$ of U that is stable under the natural H/N-action on $X \wr\!\wr N$ and an H/N-equivariant morphism $\overline{\phi} : \mathcal{U} \to Z$ such that $\phi = \overline{\phi} \circ q|_U$, and $\overline{\phi}^* M$ and $\mathcal{O}_{\mathcal{U}}(r)$ define the same linearisation of H/N.*

Proof. The proof is similar to that of Proposition 2.1.13, 1. Using [MFK94, Chapter 1, §1, Lemma] find sections $\sigma_0, \dots, \sigma_n \in H^0(Z, M)$ which span an H/N-invariant complete linear system $W \subseteq H^0(Z, M)$. Let $f_i = \phi^* \sigma_i \in H^0(U, L^{\otimes r})^N$ for each i and define $\mathcal{U}_0 = \bigcup_{f \in \mathcal{S}} \operatorname{Spec}((S^N)_{(f_i)}) \subseteq X \wr\!\wr N$. By Lemma 2.1.8, 1 there is a linear subspace W' contained in $H^0(\mathcal{U}_0, \mathcal{O}(r))$ which maps isomorphically onto $\phi^*(W)$ under the inclusion $q_N^* : H^0(\mathcal{U}_0, \mathcal{O}(r)) \hookrightarrow H^0(U, L^{\otimes r})^N$. The subspace W' is H/N-stable and so the locus of points $\mathcal{U} \subseteq \mathcal{U}_0$ on which W' is basepoint free is an open subset of $X \wr\!\wr N$ containing $q_N(U)$ that is stable under the natural H/N-action on $X \wr\!\wr N$. Because q_N^* is injective there is a unique H/N-equivariant map $W \to W'$ which composes with q_N^* to give $\phi^* : W \to \phi^*(W)$, and the composition $W \to W' \hookrightarrow H^0(\mathcal{U}_0, \mathcal{O}(r))$ defines an H/N-equivariant morphism $\Phi : \mathcal{U} \to \mathbb{P}(W^*)$ such that $\Phi \circ q_N|_U = \phi$. Now the linear system W defines an immersion $\iota : Z \hookrightarrow \mathbb{P}(W^*)$, and as in the proof of Proposition 2.1.13, 1, the injectivity of q_N^* means the image of $\Phi : \mathcal{U} \to \mathbb{P}(W^*)$ is contained in the closure $\overline{\iota(Z)}$ of $\iota(Z)$ inside $\mathbb{P}(W^*)$. By taking the preimage of Z under Φ, we may shrink $\mathcal{U}$ if necessary to obtain an H/N-stable quasi-compact open subset of $X \wr\!\wr N$ containing $q_N(U)$ with $\Phi(\mathcal{U}) \subseteq \iota(Z)$. Thus there is an H/N-equivariant map $\overline{\phi} = \Phi : \mathcal{U} \to Z$ such that $\overline{\phi} \circ q_N|_U = \phi$ and $\overline{\phi}^* M = \mathcal{O}_{\mathcal{U}}(r)$ as line bundles. Since $q_N^* \overline{\phi}^* M = \phi^* M = L^{\otimes r}|_U$ as line bundles and $L^{\otimes r}|_U \to M$ is equivariant with respect to $H \to H/N$, by chasing around Cartesian diagrams one see that the natural map $L^{\otimes r}|_U \to \overline{\phi}^* M$ is equivariant with respect to the projection $H \to H/N$. By Proposition 2.2.3, 2 the ample line bundle $\overline{\phi}^* M$ defines the same H/N-linearisation as $\mathcal{O}_{\mathcal{U}}(r)$. □

Having taken the enveloping quotient $X \wr\!\wr N$, it makes sense to consider taking the enveloping quotient for any quasi-compact, H/N-stable open subset $\mathcal{U} \subseteq X \wr\!\wr N$ and H/N-linearisation $\mathcal{O}_{\mathcal{U}}(r)$, for $r \gg 0$, defined in Proposition 2.2.3. One would hope that we might be able to find $\mathcal{U}$ that yields a natural identification $\mathcal{U} \wr\!\wr (H/N) = X \wr\!\wr H$. This is unfortunately too much to hope for. The basic problem is that there may be H-invariant sections f over X where $(S^H)_{(f)}$ is finitely generated but $(S^N)_{(f)}$ is not.

Example 2.2.5. Consider any example where N is a linear algebraic group acting linearly on a finitely generated graded $\Bbbk$-algebra $A = \bigoplus_{d \geq 0} A_d$, with $A_0 = \Bbbk$, such that A^N is not finitely generated over $\Bbbk$ (for example, Nagata's Example 1.1.24). Let $X = \operatorname{Spec} A$ and $L = \mathcal{O}_X$ with the canonical N-linearisation. Because N respects the grading on A, there is a linearisation of $H = N \times \mathbb{G}_m$ on $L \to X$, where $\mathbb{G}_m \curvearrowright L \to X$ is the canonical linearisation defined by the grading on A. Now consider $f = 1 \in H^0(X, L)^H = A^H$. Then $(S^H)_{(1)} = (\Bbbk[X, L]^H)_{(1)} = A^H = \Bbbk$ is finitely generated over $\Bbbk$, because the only $\mathbb{G}_m$-invariants in A are the constant functions. But $(S^N)_{(1)} = A^N$ is not finitely generated over $\Bbbk$.

As a remedy to this issue, one can consider instead a definition of 'semistability' that ensures a step-by-step quotienting approach works. We will take up this theme in the next section when talking about stability and also later when we consider taking quotients in stages with respect to chains of subnormal groups. The basic idea runs as follows. The inclusion $S^H \hookrightarrow S^N$ induces a rational map

$$q_{H/N} : \operatorname{Proj}(S^N) \dashrightarrow \operatorname{Proj}(S^H) \tag{2.2.5}$$

that is invariant with respect to the canonical H/N-action on $\mathrm{Proj}(S^N)$ of Proposition 2.2.1. For the purpose of the discussion, let $X^{\mathrm{ss},N\text{-fg}}$ denote the union of all X_f such that $f \in \bigcup_{r>0} H^0(X, L^{\otimes r})^H$ is an H-invariant section with both of the $\Bbbk$-algebras $(S^N)_{(f)}$ and $(S^H)_{(f)}$ finitely generated. For each such f we have $\mathrm{Spec}((S^N)_{(f)}) \subseteq X /\!\!/ N$ and the restriction of $q_{H/N}$ to $\mathrm{Spec}((S^N)_{(f)})$ maps into $X /\!\!/ H$. Letting $\mathcal{U}$ be the union of the $\mathrm{Spec}((S^N)_{(f)})$ defined by such f, we see that the rational map $q_{H/N}$ restricts to give a well-defined morphism $q_{H/N} : \mathcal{U} \to X /\!\!/ H$. Furthermore, if $q_H : X^{\mathrm{ss,fg}(H)} \to X /\!\!/ H$ and $q_N : X^{\mathrm{ss,fg}(N)} \to X /\!\!/ N$ are the enveloping quotients for the H and N-linearisations on $L \to X$, respectively, then the diagram

$$\begin{tikzcd}
X^{\mathrm{ss,fg}(H)} \arrow[ddrr, "q_H"'] & \supseteq & X^{\mathrm{ss},N\text{-fg}} \arrow[d, "q_N"] & \subseteq & X^{\mathrm{ss,fg}(N)} \arrow[d, "q_N"] \\
 & & \mathcal{U} \arrow[d, "q_{H/N}"] & \subseteq & X /\!\!/ N \\
 & & X /\!\!/ H & &
\end{tikzcd}$$

commutes, with all inclusions open.

2.3 Stability for Non-Reductive Linearisations

We now turn to the question of defining an open subset of 'stable' points of X for a given linearisation $L \to X$ of a linear algebraic group H, which admits a geometric quotient under the H-action on X. This should extend the definitions of stability in the cases where H is reductive (Definition 1.2.2, 2) or unipotent (Definition 1.3.5). We will do this in the case where X is *irreducible*. This is because we want to make use of Remark 2.1.9 after Lemma 2.1.8, 1.

2.3.1 An Intrinsic Definition of Stability

Recall that for any linear algebraic group H there is a canonical normal unipotent subgroup H_u of H, called the unipotent radical of H, with the property that the quotient $H_r = H/H_u$ is a reductive group; see Definition 1.1.21, 2. According to Proposition 2.2.1, the enveloping quotient $X /\!\!/ H_u$ for the restricted linearisation $H_u \curvearrowright L \to X$ has a canonical H_r-action which makes the enveloping quotient map $q_{H_u} : X^{\mathrm{ss,fg}(H_u)} \to X /\!\!/ H_u$ equivariant with respect to the quotient $H \to H_r$. Recall that in fact the action comes from an action of H_r on $\mathrm{Proj}(S^{H_u})$ which has the property that the rational map

$$q_{H_r} : \mathrm{Proj}(S^{H_u}) \dashrightarrow \mathrm{Proj}(S^H) \tag{2.3.1}$$

defined in (2.2.5) is H_r-invariant.

Given an H-invariant section f of some positive tensor power of $L \to X$ such that $(S^{H_u})_{(f)}$ is a finitely generated $\Bbbk$-algebra, the basic open set $\mathrm{Spec}((S^{H_u})_{(f)})$ is stable under the H_r-action on $X /\!\!/ H_u$. The composition of $q_{H_u} : X_f \to \mathrm{Spec}((S^{H_u})_{(f)})$ with q_{H_r} coincides with the restriction of the natural map $q_H : X_f \to \mathrm{Spec}((S^H)_{(f)})$. Because X is irreducible, we have

$$(S^H)_{(f)} = \left((S^{H_u})^{H_r}\right)_{(f)} = \left((S^{H_u})_{(f)}\right)^{H_r}$$

(see Remark 2.1.9) and since H_r is reductive we thus have $(S^H)_{(f)}$ finitely generated over $\Bbbk$ and $\mathrm{Spec}((S^H)_{(f)}) \subseteq X /\!\!/ H$. To define a notion of stability for the linearisation $H \curvearrowright L \to X$ we would like the restriction of the enveloping quotient map $q_H : X^{\mathrm{ss,fg}(H)} \to X /\!\!/ H$ for $H \curvearrowright L \to X$ to give a geometric quotient $X_f \to \mathrm{Spec}((S^H)_{(f)})$ for the H-action on X_f. There are a number of ways one could go about doing this. For example, it is easy to see that if each of q_{H_u} and q_{H_r} define geometric quotients for the H_u and H_r-actions on X_f and $\mathrm{Spec}((S^{H_u})_{(f)})$, respectively, then the composition q_H is a geometric quotient for $H \curvearrowright X_f$. But we also want to build on the theory in [DK07], where the authors conclude that a good notion of stability for a unipotent group is to take the X_f which are affine and admit a locally trivial geometric quotient. So for us, it makes sense to further require that X_f is affine and $q_{H_u} : X_f \to \mathrm{Spec}((S^{H_u})_{(f)})$ is a principal bundle for the action of H_u on X_f. By [MFK94, Amplification 1.3], the induced action of the reductive group H_r on $\mathrm{Spec}((S^{H_u})_{(f)})$ has a geometric quotient if, and only if, all the orbits are closed in $\mathrm{Spec}((S^{H_u})_{(f)})$, and following the ideas of stability in reductive GIT it also natural to demand that the stabilisers for this action are finite. Because the action of H_u on X_f is free and H_u is normal in H, these last conditions can be lifted to the action of H on X_f using the following lemma.

Lemma 2.3.1. *Suppose H is a linear algebraic group, N is a normal subgroup of H and X is an H-variety (not necessarily assumed irreducible). Suppose all the stabilisers for the restricted action $N \curvearrowright X$ are finite and this action has a geometric quotient $\pi : X \to X/N$. Note that H/N acts canonically on X/N. Then*

1. *for all the H/N-orbits in X/N to be closed, it is necessary and sufficient that all the H-orbits in X are closed;*
2. *given $y \in X/N$, the stabiliser $\operatorname{Stab}_{H/N}(y)$ is finite if, and only if, $\operatorname{Stab}_H(x)$ is finite for some (and hence all) $x \in \pi^{-1}(y)$; and*
3. *if H/N is reductive and X/N is affine, then X/N has a geometric H/N-quotient if, and only if, all H-orbits in X are closed.*

Proof. (Proof of 1.) Let $x \in X$ and $y = \pi(x)$. We first show that $H \cdot x = \pi^{-1}((H/N) \cdot y)$. Clearly $H \cdot x \subseteq \pi^{-1}((H/N) \cdot y)$, because π is equivariant with respect to the projection $H \to H/N$. On the other hand, if $x' \in \pi^{-1}((H/N) \cdot y)$, then there is $h \in H$ such that $y = \overline{h}\pi(x') = \pi(hx')$. Since $\pi^{-1}(y) = N \cdot x$, there is therefore $n \in N$ such that $x' = h^{-1}nx \in H \cdot x$. Hence $H \cdot x = \pi^{-1}((H/N) \cdot y)$. Because π is a submersion, $H \cdot x$ is closed if, and only if, $(H/N) \cdot y$ is closed. Since π is surjective, this suffices to prove 1.

(Proof of 2.) Suppose $y \in X/N$ has finite stabiliser in H/N and again let $x \in \pi^{-1}(y)$. Then

$$\operatorname{Stab}_{H/N}(y) = \{g_1 N, \dots, g_m N\}$$

for some finite collection of representatives $g_1, \dots, g_m \in H$, which we fix once and for all. We also may assume the cosets $g_i N$ are pairwise disjoint. If $h \in \operatorname{Stab}_H(x)$ then $\overline{h} = hN \in \operatorname{Stab}_{H/N}(y)$, so h is contained in a unique coset $g_{i(h)}N$, where $i(h) \in \{1, \dots, m\}$. In this way we define a function

$$\operatorname{Stab}_H(x) \to \{g_1, \dots, g_m\}, \quad h \mapsto g_{i(h)}.$$

We claim the fibres of this function are finite. Indeed, let $h \in \operatorname{Stab}_H(x)$ and suppose $\tilde{h} \in \operatorname{Stab}_H(x)$ is such that $g_{i_0} := g_{i(h)} = g_{i(\tilde{h})}$, with $i_0 \in \{1, \dots, m\}$. Then we may find $n, \tilde{n} \in N$

such that $h = ng_{i_0}$ and $\tilde{h} = \tilde{n}g_{i_0}$, so $ng_{i_0}x = \tilde{n}g_{i_0}x$. It follows that, for some $p \in \mathrm{Stab}_N(g_{i_0}x)$, we have $\tilde{n} = pn$ and $\tilde{h} = ph$. Since all stabilisers for the N-action on X are finite, there are finitely many choices for $\tilde{h}$ and hence the fibre containing h is finite, as claimed. We conclude that $\mathrm{Stab}_H(x)$ is finite for any $x \in \pi^{-1}(y)$.

Conversely, suppose $x \in X$ has finite stabiliser in H and let $y = \pi(x) \in X/N$. Let $h \in H$ such that $\overline{h} = hN \in \mathrm{Stab}_{H/N}(y)$. Then $\pi(x) = \overline{h}\pi(x) = \pi(hx)$ and, because π is a geometric N-quotient and N is normal in H, there is $n \in N$ such that $hnx = x$. Hence $hn \in \mathrm{Stab}_H(x)$ and $\overline{h}$ is in the image of $\mathrm{Stab}_H(x)$ under the quotient map $H \to H/N$. Thus $\mathrm{Stab}_{H/N}(y)$ is finite.

(Proof of 3.) If a geometric quotient $X/N \to (X/N)/(H/N)$ exists then the composition $X \xrightarrow{\pi} X/N \longrightarrow (X/N)/(H/N)$ is a geometric quotient for the H-action on X, which implies that all the H-orbits in X are closed. Now suppose all the H-orbits in X are closed. Because X/N is affine and H/N is reductive the categorical quotient of X/N by H/N exists [MFK94, Theorem 1.1]. Every H/N-orbit in X/N is closed by 1, so the categorical quotient of X/N by H/N is a geometric quotient [MFK94, Amplification 1.3]. □

In light of the above lemma and the preceding discussion, we make the following definition.

Definition 2.3.2. Let H be a linear algebraic group acting on an irreducible variety X and $L \to X$ a linearisation for the action. The *stable locus* is the open subset

$$X^{\mathrm{s}} := \bigcup_{f \in I^{\mathrm{s}}} X_f$$

of X^{nss}, where $I^{\mathrm{s}} \subseteq \bigcup_{r>0} H^0(X, L^{\otimes r})^H$ is the subset of H-invariant sections satisfying the following conditions:

1. the open set X_f is affine;
2. the action of H on X_f is closed with all stabilisers finite groups; and
3. the restriction of the H_u-enveloping quotient map

$$q_{H_u} : X_f \to \mathrm{Spec}((S^{H_u})_{(f)})$$

is a principal H_u-bundle for the action of H_u on X_f.

Remark 2.3.3. It is clear that this definition of stability extends the definition of stability in [DK07] for unipotent groups (see Definition 1.3.5). In the case where H is reductive, then H_u is trivial and our definition reduces to Mumford's notion of properly stable points [MFK94] (see Definition 1.2.2, 2).

Remark 2.3.4. Observe that by Proposition 1.1.25 requiring 1 and 3 in Definition 2.3.2 is equivalent to demanding that X_f be an affine open subset of X that is a *trivial* H_u-bundle.

The significance of assuming that X is irreducible in Definition 2.3.2 is that it ensures

$$\operatorname{Spec}((S^H)_{(f)}) = \operatorname{Spec}(((S^{H_u})_{(f)})^{H_r}),$$

so that by reductive GIT for affine varieties $q_{H_r} : \operatorname{Spec}((S^{H_u})_{(f)}) \to \operatorname{Spec}((S^H)_{(f)})$ is at least a good categorical H_r-quotient when f is an H-invariant (cf. Theorem 1.2.1). If $f \in I^{\mathrm{s}}$, then conditions 1–3 in Definition 2.3.2, combined with Lemma 2.3.1, 3, tell us that

$$q_H : X_f \xrightarrow{q_{H_u}} \operatorname{Spec}((S^{H_u})_{(f)}) \xrightarrow{q_{H_r}} \operatorname{Spec}((S^H)_{(f)})$$

is a composition of geometric quotients, hence a geometric quotient for $H \curvearrowright X_f$. Because the property of being a geometric quotient is local on the base, it follows that the enveloping quotient $q_H : X^{\mathrm{ss,fg}(H)} \to X \wr\!\wr H$ restricts to define a geometric quotient

$$q_H : X^{\mathrm{s}} \to X^{\mathrm{s}}/H = q_H(X^{\mathrm{s}}).$$

This factors through the restriction of the enveloping quotient for H_u in a natural way, and we have the following commutative diagram, with all inclusions open.

$$\begin{tikzcd}
X^{\mathrm{ss,fg}(H)} \arrow[dd, "q_H"] & \supseteq & X^{\mathrm{s}} \arrow[d, "\mathrm{geo}"', "q_{H_u}"] & \subseteq & X^{\mathrm{ss,fg}(H_u)} \arrow[d, "q_{H_u}"] \\
 & & X^{\mathrm{s}}/H_u \arrow[d, "\mathrm{geo}"', "q_{H_r}"] & \subseteq & X \wr\!\wr H_u \\
X \wr\!\wr H & \supseteq & X^{\mathrm{s}}/H & &
\end{tikzcd}$$

Remark 2.3.5. If X is irreducible and $q_H : X^{\mathrm{ss,fg}(H)} \to \mathcal{U}$ is any inner enveloping quotient for the linearisation $H \curvearrowright L \to X$, then the geometric quotient X^{s}/H of X^{s} is naturally an open subvariety of $\mathcal{U}$.

One of the features of the stable locus defined in Definition 2.3.2 is that it behaves well under affine locally closed immersions, as the next lemma shows.

Lemma 2.3.6. *Let H be a linear algebraic group acting on an irreducible variety X with linearisation $L \to X$. Suppose Y is another irreducible variety and $\phi : Y \hookrightarrow X$ is an H-equivariant locally closed immersion that is an affine morphism. Then $\phi^{-1}(X^{\mathrm{s}})$ is an open subset of $Y^{\mathrm{s}(\phi^* L)}$, the image of $\phi^{-1}(X^{\mathrm{s}})$ under the enveloping quotient $q' : Y^{\mathrm{ss,fg}(\phi^* L)} \to Y \wr\!\wr_{\phi^* L} H$ is a geometric quotient for the H-action on $\phi^{-1}(X^{\mathrm{s}})$, and there is a locally closed immersion $\bar{\phi} : \phi^{-1}(X^{\mathrm{s}})/H \hookrightarrow X^{\mathrm{s}}/H$ such that the following diagram commutes (with unmarked inclusions open)*

$$\begin{tikzcd}
Y^{\mathrm{s}(\phi^* L)} \arrow[d, "q'"] & \supseteq & \phi^{-1}(X^{\mathrm{s}}) \arrow[r, hook, "\phi"] \arrow[d, "q'", "\mathrm{geo}"'] & X^{\mathrm{s}} \arrow[d, "q"] \\
Y^{\mathrm{s}(\phi^* L)}/H & \supseteq & \phi^{-1}(X^{\mathrm{s}})/H \arrow[r, hook, "\bar{\phi}"] & X^{\mathrm{s}}/H
\end{tikzcd}$$

Proof. Let $R = \Bbbk[Y, \phi^* L]$ and $S = \Bbbk[X, L]$. The set $\phi^{-1}(X^{\mathrm{s}})$ is covered by open subsets of the form $Y_{\phi^* f}$, where f is a section in $I^{\mathrm{s}} \subseteq \bigcup_{r>0} H^0(X, L^{\otimes r})^H$ of Definition 2.3.2, and by Remark 2.3.5 $q : X_f \to \operatorname{Spec}((S^H)_{(f)})$ is a trivial H_u-bundle. For each such f the open subset $Y_{\phi^* f} = \phi^{-1}(X_f)$ is affine because ϕ is affine and X_f is affine. It is clear that the action of H on $Y_{\phi^* f}$ is closed with all stabilisers finite. By restriction $Y_{\phi^* f}$ also has the structure of a trivial H_u-bundle, thus $Y_{\phi^* f}/H_u$ is affine and isomorphic to $\operatorname{Spec}((R^{H_u})_{(\phi^* f)})$ (because Y is irreducible). The restriction of the enveloping quotient for $H_u \curvearrowright \phi^* L \to Y$,

$$(q')_{H_u} : Y_{\phi^* f} \to \operatorname{Spec}((R^{H_u})_{(\phi^* f)}) = \operatorname{Spec}(\mathcal{O}(Y_{\phi^* f})^{H_u}) = Y_{\phi^* f}/H_u$$

therefore has the structure of a trivial H_u-bundle. Thus $Y_{\phi^* f} \subseteq Y^{\mathrm{s}(\phi^* L)}$ and the enveloping quotient map,

$$q' : Y_{\phi^* f} \to \operatorname{Spec}((R^H)_{(\phi^* f)})$$

is a geometric quotient for the H-action on $Y_{\phi^* f}$. On the other hand, by the submersion property of a geometric quotient the image of $q \circ \phi : Y_{\phi^* f} \to X_f/H$ is a locally closed subset of X_f/H that is also a geometric quotient for the H-action on $Y_{\phi^* f}$, hence there is a unique locally closed immersion

$$\mathrm{Spec}((R^H)_{(\phi^* f)}) \hookrightarrow X_f/H$$

factoring $(q \circ \phi)|_{Y_{\phi f}}$ through $q'|_{Y_{\phi f}}$.

By varying over suitable f, we see that $\phi^{-1}(X^{\mathrm{s}}) \subseteq Y^{\mathrm{s}(\phi^* L)}$ and the $\mathrm{Spec}((R^H)_{(\phi^* f)}) \hookrightarrow X_f/H$ glue to give the locally closed immersion

$$\overline{\phi} : q'(\phi^{-1}(X^{\mathrm{s}})) = \phi^{-1}(X^{\mathrm{s}})/H \hookrightarrow X^{\mathrm{s}}/H$$

making the required diagram commute. □

2.3.2 Relation to Stability of Reductive Extensions

We next consider how the notion of stability proposed in Definition 2.3.2 relates to stability for a reductive group acting on the fibre space $G \times^{H_u} X$ associated to certain embeddings $H_u \hookrightarrow G$, with G reductive. This is an extension of the work in [DK07, §5], summarised in Section 1.3.2 of Chapter 1.

It will be convenient to make the following definition.

Definition 2.3.7. Let H be a linear algebraic group and G a reductive group. A homomorphism $H \to G$ is called *H_u-faithful* if its restriction to the unipotent radical H_u of H defines a closed embedding $H_u \hookrightarrow G$.

Fix a reductive group G and an H_u-faithful homomorphism $\rho : H \to G$. Then we can consider the fibre space $G \times^{H_u} X$ associated to X and the homomorphism $\rho|_{H_u} : H_u \hookrightarrow G$, together with its natural closed immersion $\alpha : X \hookrightarrow G \times^{H_u} X$. Because H_u is special (in the sense of [Ser58, §4.1]), the space $G \times^{H_u} X$ is a variety [EG98, Proposition 23] and the natural projection

$$G \times X \to G \times^{H_u} X$$

is therefore a geometric quotient for the diagonal action of H_u in the category of varieties. Because H_u is normal in H the diagonal action of H on $G \times X$, induced by the action of H

on X and right multiplication on G through ρ, descends through this projection to give an action of $H_r = H/H_u$ on $G \times^{H_u} X$. Explicitly, this is given by

$$\overline{h} \cdot [g, x] = [g\rho(h)^{-1}, hx] \quad \text{for all } g \in G,\ x \in X,\ \overline{h} = hH_u \in H/H_u.$$

This action of H_r commutes with the G-action on $G \times^{H_u} X$, so we can view $G \times^{H_u} X$ as a $G \times H_r$-variety in a natural way. Notice that the inclusion $\alpha : X \hookrightarrow G \times^{H_u} X$ is equivariant with respect to the diagonal embedding $H \hookrightarrow G \times H_r$ induced by ρ and the quotient $H \to H_r$.

As noted in Section 1.3.2 of Chapter 1, there is a natural G-linearisation over $G \times^{H_u} X$ which extends the H_u-linearisation on L under the inclusion α. By abuse of notation, we denote this linearisation $L = G \times^{H_u} L \to G \times^{H_u} X$. The diagonal H_r-action on $G \times^{H_u} X$ canonically lifts to the line bundle L to define an H_r-linearisation on $L \to G \times^{H_u} X$ which commutes with the G-linearisation. Thus there is a natural linearisation

$$G \times H_r \curvearrowright L \to G \times^{H_u} X.$$

This provides an extension of the H-linearisation $H \curvearrowright L \to X$, when we let H act on $L = G \times^{H_u} L$ via the diagonal homomorphism $H \to G \times H_r$, in a similar fashion to $G \times^{H_u} X$. As such, pulling back sections along α induces an isomorphism

$$\alpha^* : \Bbbk[G \times^{H_u} X, L]^{G \times H_r} \xrightarrow{\cong} \Bbbk[X, L]^H.$$

We may now state

Proposition 2.3.8. *Let X be an irreducible H-variety with linearisation $H \curvearrowright L \to X$, let G be a reductive group with an H_u-faithful homomorphism $\rho : H \to G$ and consider the associated fibre bundle $G \times^{H_u} X$. Let α be the natural closed immersion of X into $G \times^{H_u} X$ and let $(G \times^{H_u} X)^{\mathrm{s}(L)}$ be the stable locus for the $G \times H_r$-linearisation $L \to G \times^{H_u} X$. Then*

$$X^{\mathrm{s}} = \alpha^{-1}((G \times^{H_u} X)^{\mathrm{s}(L)}).$$

Proof. Some parts of the proof use arguments from the proof of [DK07, Proposition 5.1.10], which we include for the sake of completeness; we will indicate this appropriately.

(Proof of $\alpha^{-1}((G \times^{H_u} X)^{\mathrm{s}(L)}) \subseteq X^{\mathrm{s}}$.) Suppose $x \in X$ and $\alpha(x) \in (G \times^{H_u} X)_F$, where F is a $G \times H_r$-invariant over $G \times^{H_u} X$ such that $(G \times^{H_u} X)_F$ is affine and the $G \times H_r$-action on $(G \times^{H_u} X)_F$ is closed with finite stabilisers. Let $f = \alpha^* F$ be the corresponding H-invariant, so that $x \in X_f$. Then α restricts to an H-equivariant closed immersion $X_f \hookrightarrow (G \times^{H_u} X)_F$, thus X_f is affine. For any $y \in X_f$ the orbit $H \cdot y = \alpha^{-1}((G \times H_r) \cdot \alpha(y))$ is closed in X_f and $\mathrm{Stab}_H(y) \subseteq \mathrm{Stab}_{G \times H_r}(\alpha(y))$ because $H \to G \times H_r$ is injective, so all H-orbits in X_f are closed and all stabilisers for the H-action on X_f are finite. It remains to show that the restriction of the H_u-enveloping quotient

$$q_{H_u} : X_f \to \mathrm{Spec}((S^{H_u})_{(f)})$$

gives X_f the structure of a principal H_u-bundle. But G clearly acts on $(G \times^{H_u} X)_F$ with finite stabilisers, and all G-orbits in $(G \times^{H_u} X)_F$ are closed because the action of $G \times H_r$ on $(G \times^{H_u} X)_F$ is proper [MFK94, Corollary 2.5]. Hence $(G \times^{H_u} X)_F$ is in the stable locus for the restricted linearisation $G \curvearrowright L \to G \times^{H_u} X$. We can now follow the argument given in [DK07, Proposition 5.1.10] to show that $q_{H_u} : X_f \to \mathrm{Spec}((S^{H_u})_{(f)})$ is a principal H_u-bundle and complete the proof that $\alpha^{-1}((G \times^{H_u} X)^{\mathrm{s}(L)}) \subseteq X^{\mathrm{s}}$. The action of G on $(G \times^{H_u} X)_F$ is set-theoretically free, because all its stabilisers are conjugate to subgroups of the unipotent group H_u and, since they are finite, are thus trivial. Furthermore the action of G on $(G \times^{H_u} X)_F$ is proper ([MFK94, Corollary 2.5] again) and so the action of G on $(G \times^{H_u} X)_F$ is free by Lemma 1.1.9. The subset $(G \times^{H_u} X)_F$ has an affine geometric quotient $(G \times^{H_u} X)_F / G \cong X_f / H_u$ by Theorem 1.2.1, 2, which by [MFK94, Proposition 0.9] is actually a locally trivial quotient. By descent [Ser58, Proposition 10] this means X_f has an affine locally trivial geometric quotient, isomorphic to $\mathrm{Spec}(\mathcal{O}(X_f)^{H_u}) = \mathrm{Spec}((S^{H_u})_{(f)})$. So $q_{H_u} : X_f \to \mathrm{Spec}((S^{H_u})_{(f)})$ is a locally trivial H_u-quotient.

(Proof of $X^{\mathrm{s}} \subseteq \alpha^{-1}((G \times^{H_u} X)^{\mathrm{s}(L)})$.) Let $x \in X_f$, where f is an H-invariant over X such that X_f is affine, has closed H-orbits with all stabilisers finite and $q_{H_u} : X_f \to \mathrm{Spec}((S^{H_u})_{(f)})$ is a principal H_u-bundle. Let F be the $G \times H_r$-invariant over $G \times^{H_u} X$ pulling back to f under α. As shown in the proof of [DK07, Proposition 5.1.10], $G \times^{H_u} (X_f) = (G \times^{H_u} X)_F$ is an affine open subset of $G \times^{H_u} X$. The argument runs as follows.

By [Ser58, Proposition 5], the natural morphism $(G \times^{H_u} X)_F = G \times^{H_u} (X_f) \to X_f/H_u$ is a principal G-bundle with affine base. By [MFK94, Proposition 0.7] this means $(G \times^{H_u} X)_F \to X_f/H_u$ is an affine morphism, hence $(G \times^{H_u} X)_F$ is affine, as claimed. Now, any $G \times H_r$-orbit in $(G \times^{H_u} X)_F$ is the image $G \times^{H_u} O$ of a subset of the form $G \times O \subseteq G \times X_f$ under the geometric quotient $G \times X_f \to G \times^{H_u} (X_f)$, where $O \subseteq X_f$ is an H-orbit. Since O is closed in X_f, so too is the $G \times H_r$-orbit $G \times^{H_u} O$ inside $(G \times^{H_u} X)_F$. Hence all $G \times H_r$-orbits in $(G \times^{H_u} X)_F$ are closed. Moreover, because any point in $(G \times^{H_u} X)_F$ is in the G-sweep of a point in X_f via α, any stabiliser for the $G \times H_r$-action on $(G \times^{H_u} X)_F$ is conjugate to an H-stabiliser for a point in X_f under the inclusion $H \hookrightarrow G \times H_r$ induced by ρ and $H \to H_r$. Hence all stabilisers for the $G \times H_r$-action on $(G \times^{H_u} X)_F$ are finite. It follows that $(G \times^{H_u} X)_F \subseteq (G \times^{H_u} X)^{\mathrm{s}(L)}$ and $x \in \alpha^{-1}((G \times^{H_u} X)^{\mathrm{s}(L)})$. □

Remark 2.3.9. For future reference we note the following fact, which was shown during the proof of Proposition 2.3.8: if $H \to G$ is an H_u-faithful homomorphism and f an H-invariant section over X with associated $G \times H_r$-invariant F over $G \times^{H_u} X$, then $(G \times^{H_u} X)_F = G \times^{H_u} (X_f)$ is affine if, and only if, X_f is affine and $X_f \to \operatorname{Spec}((S^{H_u})_{(f)})$ is a principal H_u-bundle.

It immediately follows from Proposition 2.3.8 that we have an equality

$$(G \times^{H_u} X)^{\mathrm{s}(L)} = G \times^{H_u} (X^{\mathrm{s}}).$$

Because G is a closed reductive subgroup of $G \times H_r$ it follows that $(G \times^{H_u} X)^{\mathrm{s}(L)}$ is contained in the stable locus for the restricted linearisation $G \curvearrowright L \to G \times^{H_u} X$ and hence has a geometric quotient for the G-action. The inclusion $\alpha_{H_u} : X^{\mathrm{s}} \hookrightarrow G \times^{H_u} (X^{\mathrm{s}})$ thus induces an H_r-equivariant isomorphism

$$X^{\mathrm{s}}/H_u \cong (G \times^{H_u} X)^{\mathrm{s}(L)}/G,$$

and since $(G \times^{H_u} X)^{\mathrm{s}(L)}/(G \times H_r) = ((G \times^{H_u} X)^{\mathrm{s}(L)}/G)/H_r$, we conclude that α_{H_u} descends further to an isomorphism

$$X^{\mathrm{s}}/H \cong (G \times^{H_u} X)^{\mathrm{s}(L)}/(G \times H_r).$$

So the significance of Proposition 2.3.8 is that it allows us to describe stability for the linearisation $H \curvearrowright L \to X$ and its geometric quotient in terms of stability and the associated quotient for the *reductive* linearisation $G \times H_r \curvearrowright L \to G \times^{H_u} X$.

Remark 2.3.10. Even if $H \curvearrowright L \to X$ is a linearisation of an ample line bundle over a projective variety, the induced fibre space $G \times^{H_u} X$ will only be quasi-projective with an ample linearisation, so care needs to be taken when computing stability. Similarly, if X is affine then $G \times^{H_u} X$ is not necessarily affine.

Remark 2.3.11. One can also consider the fibre space $G \times^{H} X$ associated to the H_u-faithful homomorphism $\rho : H \to G$, together with its natural G-linearisation $L_2 := G \times^H L \to G \times^H X$ and inclusion $\alpha_H : H \hookrightarrow G \times^H X$ (assuming these spaces exist as varieties). If $\ker \rho$ is finite, then is can be shown that $X^{\mathrm{s}} = \alpha_H^{-1}((G \times^H X)^{\mathrm{s}(L_2)})$ and the induced embedding $X^{\mathrm{s}} \hookrightarrow (G \times^H X)^{\mathrm{s}(L_2)}$ descends to an isomorphism $X^{\mathrm{s}}/H \cong ((G \times^H X)^{\mathrm{s}(L_2)})/G$. Since we will not use this in the sequel, we omit the details of the proofs.

2.4 Summary of the Intrinsic Picture

We shall shortly draw together the results of this chapter to give a picture which we believe is a good theoretical framework for doing geometric invariant theory for any linear algebraic group H acting on an irreducible variety X with linearisation $L \to X$. Before doing so, we make one final observation about the relationship between the notion of stability in Definition 2.3.2 and the various notions of semistability considered before. As already observed, we have

$$X^{\mathrm{s}} \subseteq X^{\mathrm{ss,fg}} \subseteq X^{\mathrm{nss}}.$$

This can be further refined using the ideas at the end of Section 2.2.2. The stable locus is patched together with affine open subsets X_f, for certain H-invariant sections f of a positive tensor power of $L \to X$ which, among other things, have the property that $(S^{H_u})_{(f)}$ is a finitely generated $\Bbbk$-algebra (cf. Definition 2.3.2, 3). Because H_r is reductive and X is irreducible then the full invariant algebra $(S^H)_{(f)}$ is finitely generated over $\Bbbk$. This idea suggests it is useful to consider another notion of 'semistability' that sits inside the finitely generated semistable locus.

Definition 2.4.1. Let H be a linear algebraic group and $H \curvearrowright L \to X$ a linearisation of an irreducible H-variety X. We define the *H_u-finitely generated semistable locus* to be the open subset

$$X^{\mathrm{ss},H_u-\mathrm{fg}} := \bigcup_{f \in I^{\mathrm{ss},H_u-\mathrm{fg}}} X_f,$$

of X^{nss}, where

$$I^{\mathrm{ss},H_u-\mathrm{fg}} = \{f \in \textstyle\bigcup_{r>0} H^0(X, L^{\otimes r})^H \mid (S^{H_u})_{(f)} \text{ is a finitely generated } \Bbbk\text{-algebra}\}.$$

It follows from the discussion above that $X^{\mathrm{s}} \subseteq X^{\mathrm{ss},H_u-\mathrm{fg}} \subseteq X^{\mathrm{ss,fg}}$. The image of $X^{\mathrm{ss},H_u-\mathrm{fg}}$ under the enveloping quotient map $q_{H_u} : X^{\mathrm{ss,fg}(H_u)} \to X \wr\wr H_u$ for the restricted linearisation $H_u \curvearrowright L \to X$ is contained in the H_r-invariant open subscheme

$$\bigcup_{f \in I^{\mathrm{ss},H_u-\mathrm{fg}}} \mathrm{Spec}((S^{H_u})_{(f)}) \subseteq X \wr\wr H_u.$$

This subscheme is not necessarily quasi-compact, but we can always find a finite subset $\mathcal{S} \subseteq I^{\mathrm{ss},H_u-\mathrm{fg}}$ of invariant sections such that the image $q_{H_u}(X^{\mathrm{ss},H_u-\mathrm{fg}})$ is contained in the quasi-compact open subscheme

$$\mathcal{U} := \bigcup_{f \in \mathcal{S}} \mathrm{Spec}((S^{H_u})_{(f)}) \subseteq X \wr\wr H_u.$$

Note that $\mathcal{U}$ is an inner enveloping quotient for $X^{\mathrm{ss},H_u-\mathrm{fg}}$ under the restricted linearisation $H_u \curvearrowright L \to X$. The rational map $q_{H_r} : \mathrm{Proj}(S^{H_u}) \dashrightarrow \mathrm{Proj}(S^H)$ of (2.3.1) is defined on $\mathcal{U}$ and gives a morphism

$$q_{H_r} : \mathcal{U} \to X \wr\wr H.$$

In fact, the image under q_{H_r} is precisely the reductive GIT quotient $\mathcal{U} /\!\!/ H_r$ for the natural H_r-linearisation $\mathcal{O}_{\mathcal{U}}(r) \to \mathcal{U}$ (for $r > 0$ sufficiently large) defined in Proposition 2.2.3, noting that the semistable set for this linearisation is the whole of $\mathcal{U}$. Indeed, we have

$$\mathcal{U} /\!\!/ H_r = \bigcup_{f \in \mathcal{S}} \mathrm{Spec}((S^H)_{(f)}) \subseteq X \wr\wr H.$$

Thus $q_{H_r} : \mathcal{U} \to \mathcal{U} /\!\!/ H_r$ is a good categorical quotient by H_r, and $q_H : X^{\mathrm{ss},H_u-\mathrm{fg}} \to \mathcal{U} /\!\!/ H_r \subseteq X \wr\wr H$ is an inner enveloping quotient for $X^{\mathrm{ss},H_u-\mathrm{fg}}$. It is also clear that the geometric quotient X^{s}/H of the stable locus X^{s} by H is naturally an open subvariety of $\mathcal{U} /\!\!/ H_r$.

We thus arrive at the main theorem for this chapter, which summarises the key points of our work so far.

Theorem 2.4.2. *Let H be a linear algebraic group acting on an irreducible variety X and $L \to X$ a linearisation for the action. Let $S = \Bbbk[X, L]$ and let*

$$q_H : \operatorname{Proj} S \dashrightarrow \operatorname{Proj}(S^H)$$

$$q_{H_u} : \operatorname{Proj} S \dashrightarrow \operatorname{Proj}(S^{H_u})$$

$$q_{H_r} : \operatorname{Proj}(S^{H_u}) \dashrightarrow \operatorname{Proj}(S^H)$$

be the rational maps defined by the obvious inclusions. Also let $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \operatorname{Spec}((S^{H_u})_{(f)})$, where $\mathcal{S}$ is a finite subset of $I^{\mathrm{ss},H_u-\mathrm{fg}}$ such that $X^{\mathrm{ss},H_u-\mathrm{fg}} = \bigcup_{f \in \mathcal{S}} X_f$.

1. *There is a commutative diagram*

$$\begin{tikzcd}
X^{\mathrm{s}} \arrow[d, "\mathrm{geo}"', "q_{H_u}"] & \subseteq & X^{\mathrm{ss},H_u-\mathrm{fg}} \arrow[d, "q_{H_u}"] & \subseteq & X^{\mathrm{ss,fg}} \arrow[dd, "q_H"] & \subseteq & X^{\mathrm{nss}} \arrow[dd, "q_H"] & \subseteq & X \arrow[d, dashed, "q_{H_u}"] \\
X^{\mathrm{s}}/H_u \arrow[d, "\mathrm{geo}"', "q_{H_r}"] & \subseteq & \mathcal{U} \arrow[d, "\mathrm{good}"', "q_{H_r}"] & & & & & & \operatorname{Proj}(S^{H_u}) \arrow[d, dashed, "q_{H_r}"] \\
X^{\mathrm{s}}/H & \subseteq & \mathcal{U} /\!\!/ H_r & \subseteq & X \wr\wr H & \subseteq & \operatorname{Proj}(S^H) & = & \operatorname{Proj}(S^H)
\end{tikzcd}$$

   with good or geometric quotients as indicated and all inclusions open. The induced morphism $q_H : X^{\mathrm{ss},H_u-\mathrm{fg}} \to \mathcal{U} /\!\!/ H_r$ is an inner enveloping quotient of $X^{\mathrm{ss},H_u-\mathrm{fg}}$.

2. *Given any reductive group G and H_u-faithful homomorphism $H \to G$, the induced fibre bundle $G \times^{H_u} X$ satisfies*

$$X^{\mathrm{s}} = \alpha^{-1}((G \times^{H_u} X)^{\mathrm{s}(L)}),$$

   which induces a natural isomorphism

$$X^{\mathrm{s}}/H \cong (G \times^{H_u} X)^{\mathrm{s}(L)}/(G \times H_r)$$

   where $\alpha : X \hookrightarrow G \times^{H_u} X$ and $L = G \times^{H_u} L$ are the natural inclusion and linearisation defined in Section 2.3.2.

Remark 2.4.3. One of the features of the intrinsic picture described in this chapter is that all the spaces involved in the statement of Theorem 2.4.2 are unchanged when we replace the linearisation $L \to X$ by any positive tensor power $L^{\otimes r} \to X$. (In the case of X^{nss} and $X^{\mathrm{ss,fg}}$ this was observed in Remark 2.1.2.) It thus makes sense to talk about the notions of stability, finitely generated semistability, enveloping quotients etc. for rational linearisations (see Remark 1.1.19).

Theorem 2.4.2 is a culmination of all the intrinsic notions we have discussed in this chapter and we believe it provides a good basis for doing geometric invariant theory for non-reductive groups. One reason for this is that, in the case where $L \to X$ is an ample linearisation over a projective variety X, it extends the main geometric invariant theoretic theorems in both the reductive and unipotent settings. However, given a general non-reductive linearisation the question remains of how one can compute the main objects discussed so far, particularly the stable locus X^{s}, finitely generated semistable locus $X^{\mathrm{ss,fg}}$, inner enveloping quotients of $X^{\mathrm{ss,fg}}$ and the geometric quotient X^{s}/H. The next two chapters are devoted to developing methods for doing this.

Chapter 3

Compactifications of Enveloped Quotients and Reductive Envelopes

In Chapter 2 we developed a theoretical framework for identifying open subsets of varieties that admit geometric quotients under a given group action. We approached this in the spirit of Mumford's geometric invariant theory [MFK94] for reductive groups, specifically by generalising the results and general approach of [DK07]. Due to the fact that a general linear algebraic group does not have such a well behaved invariant theory (for example, the possibility of non-finite generation of rings of invariants) there are marked differences with Mumford's theory for reductive groups.

For example, when $H = G$ is a reductive group acting on a projective variety X with an ample linearisation $L \to X$, the quotient of the stable locus (in the sense of Definition 1.2.2, 2) admits a canonical compactification $X /\!\!/ G$, which is a good categorical quotient of the semistable locus. We have $X /\!\!/ G = \operatorname{Proj}(\Bbbk[X, L]^G)$, where $\Bbbk[X, L]^G$ is a finitely generated $\Bbbk$-algebra by Nagata's theorem [Nag64], and set-theoretically $X /\!\!/ G$ can be described as the quotient of X^{ss} under the S-equivalence relation; see Section 1.2 of Chapter 1. This picture breaks down when H is not reductive, due to the fact that the ring of invariants $\Bbbk[X, L]^H$ is not necessarily finitely generated and the image of the enveloping quotient map $q : X^{\mathrm{ss,fg}} \to X \wr\!\wr H$ (within which the quotient X^{s}/H of the stable locus X^{s} is contained) is not necessarily a variety. To address the first issue, we introduced the notion of an inner enveloping quotient (Definition 2.1.11). Recall this is a choice of quasi-compact open subscheme $\mathcal{U} \subseteq X \wr\!\wr H$ which contains the image of the enveloping quotient map q :

$X^{\mathrm{ss,fg}} \to X \wr\!\wr H$ as a dense subset. Every inner enveloping quotient is *quasi-projective*, so it makes sense to talk about their projective completions.

So a reasonable way to try and recover a picture similar to Mumford's for reductive groups is to consider how to construct projective completions of inner enveloping quotients, which contain the enveloped quotient as a dense subset. We make

Definition 3.0.1. Let H be a linear algebraic group acting on a variety X with linearisation $L \to X$. We call a projective variety Z a *compactification of the enveloped quotient* if there is an inner enveloping quotient $\mathcal{U} \subseteq X \wr\!\wr H$ and a dominant open immersion $\mathcal{U} \hookrightarrow Z$.

The purpose of this chapter is to describe a method of constructing compactifications of the enveloped quotient. We will do this by generalising the work of Doran and Kirwan in [DK07, §5], where they construct reductive envelopes to compactify the enveloping quotient of unipotent groups, in the cases where the enveloping quotient is a quasi-projective variety (see Section 1.3.2 in Chapter 1 for a summary of this work). Our rough goal is as follows: for a reductive group G and an H_u-faithful homomorphism $H \to G$, we consider the construction of equivariant projective completions $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$, together with extensions of the linearisation $L \to X$ to a $G \times H_r$-linearisation $L' \to \overline{G \times^{H_u} X}$, such that we have inclusions

$$\begin{gathered} X \cap (\overline{G \times^{H_u} X}^{\mathrm{s}(L')}) \subseteq X^{\mathrm{s}} \subseteq X^{\mathrm{ss,fg}} \subseteq X \cap (\overline{G \times^{H_u} X}^{\mathrm{ss}(L')}) \\ \text{and} \quad q(X^{\mathrm{ss,fg}}) \subseteq \mathcal{U} \subseteq \overline{G \times^{H_u} X} /\!\!/_{L'} (G \times H_r), \end{gathered} \tag{3.0.1}$$

where $\mathcal{U}$ is some inner enveloping quotient of $X^{\mathrm{ss,fg}}$. Thus when L' is ample, the GIT quotient $\overline{G \times^{H_u} X} /\!\!/_{L'} (G \times H_r)$ is projective and provides a compactification of the enveloped quotient $q(X^{\mathrm{ss,fg}})$, in the sense of Definition 3.0.1. The stable and semistable loci for $L' \to \overline{G \times^{H_u} X}$ also give approximations of the intrinsically defined stable locus X^{s} and finitely generated semistable locus $X^{\mathrm{ss,fg}}$ for the non-reductive linearisation $H \curvearrowright L \to X$.

The layout of the chapter is as follows. In Section 3.1 we extend Doran and Kirwan's notion of a reductive envelope to the more general case where H is not necessarily unipotent, nor the linearisation $L \to X$ ample over a projective variety (Definition 3.1.4). These are defined by putting conditions on the triples $(\overline{G \times^{H_u} X}, \beta, L')$ which yield the kind of diagram above; the section culminates with a more precise statement of this in Theorem 3.1.14. In Section 3.2 we consider certain kinds of reductive envelope, called *strong reductive envelopes*, which yield *equalities* $X \cap \overline{G \times^{H_u} X}^{\mathrm{s}(L')} = X^{\mathrm{s}}$ and $X^{\mathrm{ss,fg}} = X \cap \overline{G \times^{H_u} X}^{\mathrm{ss}(L')}$. These are therefore interesting from the point of view of computing the stable and finitely generated semistable loci for $H \curvearrowright L \to X$. We also give an explicit way to construct strong reductive envelopes when some extra conditions on the linearisation $H \curvearrowright L \to X$ and group G are satisfied. Section 3.2 is again mostly a generalisation of the approach used in [DK07, §5.3.2] to the setting of not-necessarily unipotent groups. Finally, in Section 3.3 we undertake a detailed study of quotients of the space of unordered points on $\mathbb{P}^1$ by a non-reductive subgroup of the Möbius transformations, as an example of the theory developed in the previous two sections. To the best of our knowledge, this example has not appeared in the literature before.

3.1 Reductive Envelopes: General Theory and Main Theorem

Let H be a linear algebraic group acting on an irreducible variety X with linearisation $L \to X$ and suppose we have an H_u-faithful homomorphism $H \to G$ into a reductive group G. Consider the fibre space $G \times^{H_u} X$ associated to this homomorphism, as in Section 2.3.2 of Chapter 2. As there, we abuse notation and write L for the natural $G \times H_r$ linearisation $G \times^{H_u} L \to G \times^{H_u} X$, unless confusion is likely to occur, and also let $\alpha : X \hookrightarrow G \times^{H_u} X$ be the natural closed immersion. In this section we generalise Doran and Kirwan's notion of a reductive envelope to the more general setting where H is not necessarily unipotent. Let us sketch out what our ultimate aim is. We seek compactifications $\overline{G \times^{H_u} X}$ of $G \times^{H_u} X$, together with extensions $L' \to \overline{G \times^{H_u} X}$ of the linearisation $L \to G \times^{H_u} X$, such that the reductive GIT quotient $\overline{G \times^{H_u} X} /\!\!/_{L'} (G \times H_r)$ is a compactification of the enveloped quotient and the associated semistable and stable sets, $\overline{G \times^{H_u} X}^{\mathrm{ss}(L')}$ and $\overline{G \times^{H_u} X}^{\mathrm{s}(L')}$, allow us to approximate the finitely generated semistable locus $X^{\mathrm{ss,fg}}$ and stable locus X^{s}, as in (3.0.1) above. Furthermore, we seek to preserve as much of the intrinsic non-reductive

GIT picture as possible: we would like the reductive quotients of $L' \to \overline{G \times^{H_u} X}$ by G and $G \times H_r$ to reflect the diagram obtained in Theorem 2.4.2, 1. In order to do this, one needs to identify collections of invariant sections in $\Bbbk[X, L]$ that are large enough to detect the subsets X^{nss}, $X^{\mathrm{ss,fg}}$, $X^{\mathrm{ss},H_u-\mathrm{fg}}$ and the stable locus X^{s} and ensure these extend to sections over $\overline{G \times^{H_u} X}$.

To begin, we identify precisely the sorts of collections of invariants over X we will wish to extend.

Definition 3.1.1. Let H be a linear algebraic group, X an irreducible H-variety and $L \to X$ a linearisation. Also fix a reductive group G and an H_u-faithful homomorphism. We say an enveloping system V is *fully separating* if it is adapted to a finite subset $\mathcal{S} \subseteq V^H$ such that the following properties are satisfied:

1. $X^{\mathrm{nss}} = \bigcup_{f \in V^H} X_f$;
2. there is a subset $\mathcal{S}^{\mathrm{ss},H_u-\mathrm{fg}} \subseteq \mathcal{S}$ such that $X^{\mathrm{ss},H_u-\mathrm{fg}} = \bigcup_{f \in \mathcal{S}^{\mathrm{ss},H_u-\mathrm{fg}}} X_f$ and V defines an enveloping system adapted to $\mathcal{S}^{\mathrm{ss},H_u-\mathrm{fg}}$ for the restricted linearisation $H_u \curvearrowright L \to X$; and
3. for every $x \in X^{\mathrm{s}}$ there is $f \in \mathcal{S}$ with corresponding $G \times H_r$-invariant F over $G \times^{H_u} X$ such that $(G \times^{H_u} X)_F$ is affine. (Equivalently, for every $x \in X^{\mathrm{s}}$ there if $f \in \mathcal{S}$ such that X_f is affine and $X_f \to \operatorname{Spec}((S^{H_u})_{(f)})$ is a principal H_u-bundle; cf. Remark 2.3.9.)

It is not difficult to modify the proof of Proposition 2.1.17, 1 to prove the existence of fully separating enveloping systems, for any given linearisation $L \to X$ with X irreducible, and that such an enveloping system is stable under taking products of sections. This is done in the next lemma.

Lemma 3.1.2. *Given any irreducible H-variety X with linearisation $L \to X$ and any H_u-faithful homomorphism $H \to G$ with G reductive, for some $r > 0$ there exists a fully separating enveloping system $V \subseteq H^0(X, L^{\otimes r})^{H_u}$. Furthermore, for any fully separating enveloping system $V \subseteq H^0(X, L^{\otimes m})$ and any $n > 0$ the image of the natural multiplication map*

$$V^{\otimes n} \to H^0(X, L^{\otimes mn}),$$

is again a fully separating enveloping system.

Proof. By Proposition 2.1.17, 1 there is an enveloping system $V' \subseteq H^0(X, L^{\otimes r})^H$, for some $r > 0$, adapted to a finite subset $\mathcal{S}$ with $X^{\mathrm{ss,fg}} = \bigcup_{f \in \mathcal{S}} X_f$. We will augment this enveloping system by taking a suitably large multiple of r and replacing V' and $\mathcal{S}$ by their images under the natural multiplication map of sections, cf. Proposition 2.1.17, 3. We repeatedly use the fact that $X_f = X_{f^n}$ for any section f over X and integer $n > 0$, as well as the equalities $(S^H)_{(f)} = (S^H)_{(f^n)}$ and $(S^{H_u})_{(f)} = (S^{H_u})_{(f^n)}$ for invariant sections f.

Because X is quasi-compact, by taking a large multiple of r and replacing V' and $\mathcal{S}$ appropriately we may assume there are subsets $\mathcal{S}^{\mathrm{s}}$ and $\mathcal{S}^{\mathrm{ss},H_u-\mathrm{fg}}$ of $\mathcal{S}$ such that X^{s} and $X^{\mathrm{ss},H_u-\mathrm{fg}}$ are covered by open subsets of the form X_f with $f \in \mathcal{S}^{\mathrm{s}}$ and $f \in \mathcal{S}^{\mathrm{ss},H_u-\mathrm{fg}}$, respectively. We may also assume that r is chosen so that $H^0(X, L^{\otimes r})^H$ contains sections $f_1, \ldots, f_n$ with $X^{\mathrm{nss}} = \bigcup_{i=1}^n X_{f_i}$, and furthermore we can choose $\mathcal{S}^{\mathrm{s}}$ so that each $f \in \mathcal{S}^{\mathrm{s}}$ extends to F over $G \times^{H_u} X$ such that $(G \times^{H_u} X)_F$ is affine. Following an argument similar to the construction of the subset A in the proof of Proposition 2.1.17, 1, by taking another suitably large multiple of r and replacing V' and the sets $\mathcal{S}$ and $\{f_1, \ldots, f_n\}$ by their images under the multiplication map on sections we may assume there is a subset $A^{\mathrm{ss},H_u-\mathrm{fg}} \subseteq H^0(X, L^{\otimes r})^{H_u}$, containing $\mathcal{S}^{\mathrm{ss},H_u-\mathrm{fg}}$, such that $(S^{H_u})_{(f)}$ is generated by $\{\tilde{f}/f \mid \tilde{f} \in A^{\mathrm{ss},H_u-\mathrm{fg}}\}$ for each $f \in \mathcal{S}^{\mathrm{ss},H_u-\mathrm{fg}}$. By Lemma 1.1.20 there is a finite dimensional subspace $V \subseteq H^0(X, L^{\otimes r})^{H_u}$ that is stable under the H-action and contains $V' \cup A^{\mathrm{ss},H_u-\mathrm{fg}} \cup \{f_1, \ldots, f_n\}$. Then V is an enveloping system adapted to $\mathcal{S}$ such that properties 1–3 of Definition 3.1.1 are satisfied.

The statement about images of fully separating enveloping systems under multiplication maps follows immediately from Proposition 2.1.17, 3 and the equalities $X_{f^n} = X_f$, $(S^H)_{(f)} = (S^H)_{(f^n)}$ and $(S^{H_u})_{(f)} = (S^{H_u})_{(f^n)}$ for any H-invariant f and $n > 0$. $\square$

Example 3.1.3. If X is an irreducible projective H-variety and $L \to X$ an ample linearisation, then each space of sections $H^0(X, L^{\otimes r})$, where $r > 0$, is a finite dimensional vector space. Then given an H_u-faithful homomorphism $H \to G$, an easy consequence of Lemma

3.1.2 is that the space $V = H^0(X, L^{\otimes r})$ defines a fully separating enveloping system for sufficiently divisible $r > 0$.

We now turn to the definition of a reductive envelope. Given an H_u-faithful homomorphism $H \to G$ with G reductive, the idea is to extend the linearisation $G \times H_r$ on $G \times^{H_u} L \to G \times^{H_u} X$ over a suitable equivariant projective completion $\overline{G \times^{H_u} X}$ of $G \times^{H_u} X$. A key condition for obtaining the diagram (3.0.1) is to ensure enough invariants over X (or equivalently over $G \times^{H_u} X$) extend to invariants over $\overline{G \times^{H_u} X}$.

Definition 3.1.4. Let $H \curvearrowright L \to X$ be a linearisation of a linear algebraic group H and suppose $H \to G$ is an H_u-faithful homomorphism into a reductive group G. Let $\overline{G \times^{H_u} X}$ be a projective $G \times H_r$-variety with $G \times H_r$-equivariant dominant open immersion $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$ and $L' \to \overline{G \times^{H_u} X}$ a $G \times H_r$-linearisation that restricts to some positive tensor power of $L \to G \times^{H_u} X$ under β. We call $(\overline{G \times^{H_u} X}, \beta, L')$ a *reductive envelope* for the linearisation $H \curvearrowright L \to X$ if there is fully separating enveloping system V for $H \curvearrowright L \to X$ such that

1. each section in V^{H_u} extends under $\beta \circ \alpha$ to a G-invariant section of some tensor power of L' over $\overline{G \times^{H_u} X}$;
2. each section in V^H extends under $\beta \circ \alpha$ to a $G \times H_r$-invariant section of some tensor power of L' over $\overline{G \times^{H_u} X}$; and
3. for $f \in V^{H_u}$ with extension to F over $\overline{G \times^{H_u} X}$, the open subset $(\overline{G \times^{H_u} X})_F$ is affine.

If the line bundle L' is ample, then we call $(\overline{G \times^{H_u} X}, \beta, L')$ an *ample reductive envelope*.

Remark 3.1.5. In the case where H is unipotent, our notion of reductive envelope corresponds to that of a *fine* reductive envelope in [DK07]; cf. Definition 1.3.11.

Remark 3.1.6. The case where L' is ample is of most interest to us, because it ensures the GIT quotient $\overline{G \times^{H_u} X} /\!/_{L'} (G \times H_r)$ is a projective variety. It also means that condition 3 of Definition 3.1.4 is automatically satisfied, so verifying that the data $(\overline{G \times^{H_u} X}, \beta, L')$ defines a reductive envelope reduces to checking that invariant sections from a fully separating enveloping system extend to sections over $\overline{G \times^{H_u} X}$.

The next proposition asserts the existence of an ample reductive envelope for any given *ample* linearisation $L \to X$.

Proposition 3.1.7. *Let H be a linear algebraic group acting on an irreducible quasi-projective variety X and $L \to X$ an ample linearisation for the action. Then $H \curvearrowright L \to X$ possesses an ample reductive envelope for some reductive group G containing H as a closed subgroup.*

Proof. Begin by using Lemma 3.1.2 to find $r > 0$ with a fully separating enveloping system $V \subseteq H^0(X, L^{\otimes r})$. (Note that by Remark 2.3.9 it makes sense to talk about fully separating enveloping systems without reference to any reductive group G and H_u-faithful homomorphism $H \to G$.) The line bundle L is ample, so by taking a sufficiently large multiple of r, replacing V by its image under the natural multiplication map of sections and enlarging the resulting V if necessary using Lemma 1.1.20, we may assume that V is H-stable and defines a locally closed immersion $X \hookrightarrow \mathbb{P}(V^*)$. The action of H on V defines a closed embedding of H into the reductive group $G := \mathrm{GL}(V)$ and there is a canonical G-linearisation on $\mathcal{O}_{\mathbb{P}(V^*)}(1) \to \mathbb{P}(V^*)$ extending the H-linearisation $L^{\otimes r} \to X$. Note that the embedding $H \hookrightarrow G$ is H_u-faithful. Consider the fibre bundle $G \times^{H_u} \mathbb{P}(V^*)$, together with its $G \times H_r$-linearisation $G \times^{H_u} \mathcal{O}_{\mathbb{P}(V^*)}(1)$. Then there is an isomorphism of $G \times H_r$-varieties

$$G \times^{H_u} \mathbb{P}(V^*) \cong (G/H_u) \times \mathbb{P}(V^*),$$
$$[g, y] \mapsto (gH_u, gy),$$

and the corresponding $G \times H_r$-linearisation over $(G/H_u) \times \mathbb{P}(V^*)$ has underlying line bundle $\mathcal{O}_{G/H_u} \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$.

Because H_u is unipotent the homogeneous space G/H_u is quasi-affine [Gro97, Corollary 2.8 and Theorem 2.1], so there is a finite dimensional vector subspace $W \subseteq \mathcal{O}(G/H_u)$ defining a locally closed immersion of G/H_u into the affine space $\mathbb{A} = \mathrm{Spec}(\mathrm{Sym}^\bullet W)$. Right multiplication by H on G descends to define an H_r-action on G/H_u, and G acts by left multiplication on G/H_u. By [Bor91, Proposition 1.9] we may assume that W is invariant under the corresponding actions of H_r and G on $\mathcal{O}(G/H_u)$. This induces an action of $G \times H_r$ on $\mathbb{A}$ together with a linearisation on the trivial line bundle $\mathcal{O}_{\mathbb{A}} \to \mathbb{A}$ which restricts to the canonical $G \times H_r$-linearisation $\mathcal{O}_{G/H_u} \to G/H_u$ under $G/H_u \hookrightarrow \mathbb{A}$. Let $\Bbbk$ be a copy of the ground field equipped with the trivial $G \times H_r$-representation, set $\mathbb{P} := \mathbb{P}(W^* \oplus \Bbbk)$, and let $\beta_1 : G/H_u \hookrightarrow \overline{G/H_u}$ be the projective completion of G/H_u resulting from the embedding $G/H_u \hookrightarrow \mathbb{A}$ and the standard open immersion $\mathbb{A} \hookrightarrow \mathbb{P}$. Then the restriction $\mathcal{O}_{\overline{G/H_u}}(1) = \mathcal{O}_{\mathbb{P}}(1)|_{\overline{G/H_u}}$ of the canonical $G \times H_r$-linearisation $\mathcal{O}_{\mathbb{P}}(1) \to \mathbb{P}$ to $\overline{G/H_u}$ pulls back to the $G \times H_r$-linearisation $\mathcal{O}_{G/H_u} \to G/H_u$ under β_1. Consider the linearisation

$$G \times H_r \curvearrowright \mathcal{O}_{\overline{G/H_u}}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1) \to \overline{G/H_u} \times \mathbb{P}(V^*)$$

given by taking the product of $\mathcal{O}_{\overline{G/H_u}}(1)$ with the $G \times H_r$-linearisation on $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ defined by G and the trivial H_r-action on $\mathbb{P}(V^*)$. Let $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X} \subseteq \overline{G/H_u} \times \mathbb{P}(V^*)$ be the projective completion of $G \times^{H_u} X$ obtained by the composition of the embedding $G \times^{H_u} X \hookrightarrow G \times^{H_u} \mathbb{P}(V^*) \cong (G/H_u) \times \mathbb{P}(V^*)$ with the open immersion $\beta_1 \times \mathrm{id}_{\mathbb{P}(V^*)}$, and let L' be the restriction of $\mathcal{O}_{\overline{G/H_u}}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$ to $\overline{G \times^{H_u} X}$. Then $\beta^* L' = L^{\otimes r} \to G \times^{H_u} X$ as $G \times H_r$-linearisations.

To conclude we observe that the required extension properties 1–3 of Definition 3.1.4 hold for $(\overline{G \times^{H_u} X}, \beta, L')$, as follows. Any invariant section $f \in V \subseteq H^0(X, L^{\otimes r})$ extends to an invariant (which we also call f) of $\mathcal{O}_{\mathbb{P}(V^*)}(1) \to \mathbb{P}(V^*)$ by construction, and if f is H_u-invariant (respectively, H-invariant) over $\mathbb{P}(V^*)$ then it extends to the G-invariant (respectively, $G \times H_r$-invariant) section

$$1 \otimes f \in \mathcal{O}(G/H_u) \otimes H^0(\mathbb{P}(V^*), \mathcal{O}(1)) = H^0((G/H_u) \times \mathbb{P}(V^*), \mathcal{O}_{G/H_u} \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)).$$

(Here we have used the Künneth formula [Sta15, Tag 02KE].) But now if $\epsilon \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ is the homogeneous coordinate of $\mathbb{P} = \mathbb{P}(W^* \oplus \Bbbk)$ corresponding to the trivial $G \times H_r$-summand $\Bbbk$ then $1 \otimes f$ extends to $\epsilon \otimes f$ under β, which is G- or $G \times H_r$-invariant if $1 \otimes f$ is G- or $G \times H_r$-invariant, respectively. Thus $f \in V^{H_u}$ (respectively, $f \in V^H$) extends to the G-invariant (respectfully, $G \times H_r$-invariant) $(\epsilon \otimes f)|_{\overline{G \times^{H_u} X}}$ of $L' \to \overline{G \times^{H_u} X}$. This shows properties 1 and 2. Finally, property 3 holds because L' is (very) ample. ☐

Remark 3.1.8. In practise, the group $G = \mathrm{GL}(V)$ containing H constructed in the proof of Proposition 3.1.7 is too large to be computationally useful. In Section 3.2.1 we will look at a class of reductive envelopes where the reductive group G contains H_u as a *Grosshans subgroup* and the geometry of the homogeneous space G/H_u lends itself to more explicit calculations.

Associated to any reductive envelope $(\overline{G \times^{H_u} X}, \beta, L')$ we have open subsets of X obtained by pulling back the stable and semistable loci for the $G \times H_r$-linearisation $L' \to \overline{G \times^{H_u} X}$. As we will see shortly, one of the key properties of these sets is that they 'bookend' the intrinsically defined notions of stability and semistability considered in chapter 2. In analogy to [DK07, Definition 5.2.11] (see the statement of Theorem 1.3.12), we make the following definition.

Definition 3.1.9. Let $H \curvearrowright L \to X$ be a linearisation of a linear algebraic group H, let $H \to G$ be an H_u-faithful homomorphism into a reductive group G and suppose $(\overline{G \times^{H_u} X}, \beta, L')$ is a reductive envelope. The *completely semistable locus* is the set

$$X^{\overline{\mathrm{ss}}} := (\beta \circ \alpha)^{-1}((\overline{G \times^{H_u} X})^{\mathrm{ss}(L')})$$

and the *completely stable locus* is the set

$$X^{\overline{\mathrm{s}}} := (\beta \circ \alpha)^{-1}((\overline{G \times^{H_u} X})^{\mathrm{s}(L')}),$$

where $\overline{G \times^{H_u} X}^{\mathrm{ss}(L')}$ and $\overline{G \times^{H_u} X}^{\mathrm{s}(L')}$ are the semistable and stable loci, respectively, for the reductive linearisation $G \times H_r \curvearrowright L' \to \overline{G \times^{H_u} X}$.

The key properties of the completely semistable and completely stable loci for a given reductive enveloped are demonstrated in the next proposition.

Proposition 3.1.10. *Let H be a linear algebraic group with an H_u-faithful morphism $H \to G$, with G reductive, and let X be an irreducible quasi-projective H-variety with an ample linearisation $L \to X$. If $(\overline{G \times^{H_u} X}, \beta, L')$ is a reductive envelope for the linearisation, then $X^{\overline{\mathrm{ss}}} = X^{\mathrm{nss}}$ and $X^{\overline{\mathrm{s}}} \subseteq X^{\mathrm{s}}$.*

Proof. Let V be a fully separating enveloping system adapted to a finite subset $\mathcal{S} \subseteq V$ satisfying properties 1–3 of Definition 3.1.1, and suppose V satisfies the extension properties

1–3 of Definition 3.1.4 for the reductive envelope $(\overline{G \times^{H_u} X}, \beta, L')$. Then there is a basis $f_1, \dots, f_n$ of V^H such that $X^{\mathrm{nss}} = \bigcup_{i=1}^n X_{f_i}$ and each f_i extends to a $G \times H_r$-invariant F_i of some positive tensor power of $L' \to \overline{G \times^{H_u} X}$ such that $(\overline{G \times^{H_u} X})_{F_i}$ is affine. Hence $X^{\mathrm{nss}} \subseteq X^{\overline{\mathrm{ss}}}$. On the other hand, any $G \times H_r$-invariant of a tensor power of $L' \to \overline{G \times^{H_u} X}$ restricts to a $G \times H_r$-invariant over $G \times^{H_u} X$ under β, which in turn corresponds to an H-invariant over X via α. Hence $X^{\overline{\mathrm{ss}}} \subseteq X^{\mathrm{nss}}$ also.

Now suppose $x \in X^{\overline{\mathrm{s}}}$. Then there is a $G \times H_r$-invariant F of some positive tensor power of $L' \to \overline{G \times^{H_u} X}$ such that $(\overline{G \times^{H_u} X})_F$ is an affine open subset containing $(\beta \circ \alpha)(x)$, and the $G \times H_r$-action on $(\overline{G \times^{H_u} X})_F$ is closed with all stabilisers finite. By abuse of notation, write F for the section $\beta^* F$ over $G \times^{H_u} X$. Invoking Proposition 2.3.8, to prove $x \in X^{\mathrm{s}}$ it suffices to show that $(G \times^{H_u} X)_F \subseteq (G \times^{H_u} X)^{\mathrm{s}(L)}$, where stability is with respect to the canonical $G \times H_r$-linearisation $L \to G \times^{H_u} X$. Note that $(G \times^{H_u} X)_F$ is a $G \times H_r$-invariant open subset of $(\overline{G \times^{H_u} X})_F$, and $(\overline{G \times^{H_u} X})_F$ has a geometric $G \times H_r$-quotient $\pi : (\overline{G \times^{H_u} X})_F \to (\overline{G \times^{H_u} X})_F/(G \times H_r)$ with affine base (Theorem 1.2.1). The image $\pi((G \times^{H_u} X)_F)$ of $(G \times^{H_u} X)_F$ is an open subset of $(\overline{G \times^{H_u} X})_F/(G \times H_r)$, thus we may cover $\pi((G \times^{H_u} X)_F)$ with basic affine open subsets of $\pi((\overline{G \times^{H_u} X})_F)$. Each of these takes the form $\pi((\overline{G \times^{H_u} X})_{F\tilde{F}}) = (\overline{G \times^{H_u} X})_{F\tilde{F}}/(G \times H_r)$, for a $G \times H_r$-invariant section $\tilde{F}$ over $\overline{G \times^{H_u} X}$, by virtue of the canonical isomorphism

$$\mathcal{O}(\pi((G \times^{H_u} X)_F)) \xrightarrow{\pi^\#} (\mathcal{O}((G \times^{H_u} X)_F))^{G \times H_r} = \Bbbk[\overline{G \times^{H_u} X}, L']^{G \times H_r}_{(F)}.$$

(In the final equality we have used the fact that $\overline{G \times^{H_u} X}$ is irreducible, which is necessarily the case because $G \times^{H_u} X$ is irreducible and β is dominant.) Thus, for suitable $G \times H_r$-invariant sections F_i over $\overline{G \times^{H_u} X}$, we have

$$(G \times^{H_u} X)_F = \bigcup_i \pi^{-1}(\pi((\overline{G \times^{H_u} X})_{FF_i})) = \bigcup_i (G \times^{H_u} X)_{FF_i}$$

and by affineness of π each $\pi^{-1}(\pi((\overline{G \times^{H_u} X})_{FF_i})) = (G \times^{H_u} X)_{FF_i}$ is an affine open subset of $G \times^{H_u} X$. By restriction the $G \times H_r$-action on each $(G \times^{H_u} X)_{FF_i}$ is closed with finite stabilisers, hence $(G \times^{H_u} X)_{FF_i} \subseteq (G \times^{H_u} X)^{\mathrm{s}(L)}$ for each i. Therefore $(G \times^{H_u} X)_F \subseteq (G \times^{H_u} X)^{\mathrm{s}(L)}$, as desired. □

Corollary 3.1.11. *For a linearisation $H \curvearrowright L \to X$ with X projective and L ample, the restriction of the enveloping quotient map to the completely stable locus $X^{\overline{s}}$ for a reductive envelope $(\overline{G \times^{H_u} X}, \beta, L')$ defines a geometric quotient $q_H : X^{\overline{s}} \to X^{\overline{s}}/H$, and the composition*

$$X^{\overline{s}} \xhookrightarrow{\beta\circ\alpha} \overline{G \times^{H_u} X}^{s(L')} \longrightarrow \overline{G \times^{H_u} X}^{s(L')}/(G \times H_r)$$

induces a natural open immersion $X^{\overline{s}}/H \hookrightarrow \overline{G \times^{H_u} X}^{s(L')}/(G \times H_r)$.

Proof. Since $X^{\overline{s}}$ is an H-invariant open subset of X^s the map $q_H : X^s \to X^s/H$ restricts to define a geometric quotient $q_H : X^{\overline{s}} \to q_H(X^{\overline{s}}) \subseteq X^s/H$. By definition $G \times^{H_u} (X^{\overline{s}})$ is an open subset of $\overline{G \times^{H_u} X}^{s(L')}$ via β and hence

$$X^{\overline{s}}/H = G \times^{H_u} (X^{\overline{s}})/(G \times H_r) \subseteq \overline{G \times^{H_u} X}^{s(L')}/(G \times H_r)$$

with the inclusion open. □

Suppose we have a reductive envelope $(\overline{G \times^{H_u} X}, \beta, L')$ for the linearisation $H \curvearrowright L \to X$. Then we may consider the reductive GIT quotients

$$\pi_G : \overline{G \times^{H_u} X}^{ss(G)} \to \overline{G \times^{H_u} X}/\!\!/G,$$
$$\pi_{G\times H_r} : \overline{G \times^{H_u} X}^{ss(G\times H_r)} \to \overline{G \times^{H_u} X}/\!\!/(G \times H_r)$$

for the G- and $G \times H_r$-linearisations on L', respectively. According to Proposition A.2.1 of the Appendix, there is an induced ample H_r-linearisation $M' \to \overline{G \times^{H_u} X}/\!\!/G$ such that π_G maps $\overline{G \times^{H_u} X}^{ss(G\times H_r)}$ into $(\overline{G \times^{H_u} X}/\!\!/G)^{ss(M')}$, and if

$$\overline{\pi}_{H_r} : (\overline{G \times^{H_u} X}/\!\!/G)^{ss(M')} \to (\overline{G \times^{H_u} X}/\!\!/G)/\!\!/_{M'} H_r$$

is the reductive GIT quotient for the linearisation $H_r \curvearrowright M' \to \overline{G \times^{H_u} X}/\!\!/G$, then there is a canonical open immersion

$$\psi : \overline{G \times^{H_u} X}/\!\!/(G \times H_r) \hookrightarrow (\overline{G \times^{H_u} X}/\!\!/G)/\!\!/_{M'} H_r$$

such that the diagram

$$\begin{tikzcd}
 & \overline{G \times^{H_u} X}^{\mathrm{ss}(G\times H_r)} \arrow[d, "\pi_G"] \arrow[ddl, "\pi_{G\times H_r}"'] \\
 & (\overline{G \times^{H_u} X}/\!\!/G)^{\mathrm{ss}(M')} \arrow[d, "\overline{\pi}_{H_r}"] \\
\overline{G \times^{H_u} X}/\!\!/(G\times H_r) \arrow[r, hook, "\psi"] & (\overline{G \times^{H_u} X}/\!\!/G)/\!\!/_{M'} H_r
\end{tikzcd} \tag{3.1.1}$$

commutes.

Proposition 3.1.12. *Let H be a linear algebraic group acting on an irreducible variety X with linearisation $L \to X$, let $H \to G$ be an H_u-faithful homomorphism with G reductive and suppose $(\overline{G \times^{H_u} X}, \beta, L')$ is a reductive envelope for $H \curvearrowright L \to X$. Retain the notation above.*

1. *There is an inner enveloping quotient $\mathcal{V} \subseteq X \wr\!\wr H$, together with an open immersion*

   $$\theta_H : \mathcal{V} \hookrightarrow \overline{G \times^{H_u} X}/\!\!/(G\times H_r)$$

   such that $\theta_H^* \psi^* N' = \mathcal{O}_{\mathcal{V}}(n)$ for some $n > 0$, where $N' \to (\overline{G \times^{H_u} X}/\!\!/G)/\!\!/_{M'} H_r$ is a very ample line bundle pulling back to a positive tensor power of the line bundle $M' \to (\overline{G \times^{H_u} X}/\!\!/G)^{\mathrm{ss}(M')}$ under $\overline{\pi}_{H_r}$.

2. *There is an inner enveloping quotient $\mathcal{U} \subseteq X \wr\!\wr H_u$ of $X^{\mathrm{ss},H_u-\mathrm{fg}}$ that is stable under the canonical H_r-action on $X \wr\!\wr H_u$ of Proposition 2.2.1 and an H_r-equivariant open immersion*

   $$\theta_{H_u} : \mathcal{U} \hookrightarrow (\overline{G \times^{H_u} X}/\!\!/G)^{\mathrm{ss}(M')}$$

   such that $\theta_{H_u}^* M'$ defines the same linearised polarisation over $\mathcal{U}$ as the natural one on $\mathcal{O}_{\mathcal{U}}(n)$, for n as in 1. Furthermore, $\mathcal{U}$ is such that the natural rational map $q_{H_r} : \mathrm{Proj}(S^{H_u}) \dashrightarrow \mathrm{Proj}(S^H)$ of (2.3.1) *restricts to define a good categorical quotient $q_{H_r} : \mathcal{U} \to \mathcal{U}/\!\!/H_r$ for the H_r-action on $\mathcal{U}$, with $\mathcal{U}/\!\!/H_r$ contained in $\mathcal{V}$ as an open subscheme.*

3. *The following diagram commutes (where all unmarked inclusions are natural open immer-*

sions):

$$\begin{tikzcd}
 & X^{\mathrm{ss,fg}} \arrow[rr, hook] \arrow[dl, hook] \arrow[ddd, "q_H"] & & X^{\overline{\mathrm{ss}}} \arrow[ddd, "\pi_{G\times H_r}\circ\beta\circ\alpha"] \\
X^{\mathrm{ss},H_u-\mathrm{fg}} \arrow[rr, hook] \arrow[d, "q_{H_u}"] & & X^{\overline{\mathrm{ss}}} \arrow[ur, equal] \arrow[d, "\pi_G\circ\beta\circ\alpha"] & \\
\mathcal{U} \arrow[rr, hook, "\theta_{H_u}"] \arrow[dd, "q_{H_r}"] & & (\overline{G \times^{H_u} X}/\!\!/G)^{\mathrm{ss}(M')} \arrow[dd, "\overline{\pi}_{H_r}"] & \\
 & \mathcal{V} \arrow[rr, hook, "\theta_H"] & & \overline{G \times^{H_u} X}/\!\!/(G \times H_r) \arrow[dl, hook, "\psi"] \\
\mathcal{U}/\!\!/H_r \arrow[ur, hook] \arrow[rr, hook] & & (\overline{G \times^{H_u} X}/\!\!/G)/\!\!/_{M'}H_r &
\end{tikzcd}$$

Proof. We begin by fixing some notation. Suppose $L' \to \overline{G \times^{H_u} X}$ pulls back to $L^{\otimes r} \to X$ under $\beta \circ \alpha$ with $r > 0$. Let V be a fully separating enveloping system associated to $(\overline{G \times^{H_u} X}, \beta, L')$ with $V \subseteq H^0(X, L^{\otimes r_1})$ for some positive integral multiple r_1 of r. Let $\mathcal{S} \subseteq V$ be a finite subset to which V is adapted and such that properties 1–3 of Definition 3.1.1 are satisfied. From the construction of the GIT quotient [MFK94, Theorem 1.10] there is an ample line bundle

$$N' \to (\overline{G \times^{H_u} X}/\!\!/G)/\!\!/_{M'}H_r$$

that pulls back to $(M')^{\otimes m} \to (\overline{G \times^{H_u} X}/\!\!/G)^{\mathrm{ss}(M')}$ under $\overline{\pi}_{H_r}$ for some $m > 0$. Similarly, M' pulls back to $(L')^{\otimes l} \to \overline{G \times^{H_u} X}^{\mathrm{ss}(G)}$ for some $l > 0$. By replacing N' by a sufficiently positive tensor power of itself, we may assume the following: $N' \to (\overline{G \times^{H_u} X}/\!\!/G)/\!\!/_{M'}H_r$ and $(M')^{\otimes m} \to \overline{G \times^{H_u} X}/\!\!/G$ are very ample and there is $r_2 > 0$ such that $r_1 r_2 = n := lmr$. Using this second assumption and Lemma 3.1.2, we may use the multiplication map $V^{\otimes r_2} \to H^0(X, L^{\otimes n})$ to further assume that $\mathcal{S} \subseteq V \subseteq H^0(X, L^{\otimes n})$.

(Proof of 1.) We now construct the inner enveloping quotient $\mathcal{V}$ and open immersion θ_H. Let

$$\mathcal{V} = \bigcup_{f \in \mathcal{S}} \mathrm{Spec}((S^H)_{(f)}) \subseteq X \wr\!\wr H.$$

Recall from Definition 2.1.16 of an enveloping system that $\mathcal{S}$ satisfies $X^{\mathrm{ss,fg}} = \bigcup_{f \in \mathcal{S}} X_f$, and $(S^H)_{(f)}$ has generating set $\{\tilde{f}/f \mid \tilde{f} \in V^H\}$ for each $f \in \mathcal{S}$. Given $f \in \mathcal{S}$, by Definition

3.1.4, 3 of a reductive envelope, there is an extension $F \in H^0(\overline{G \times^{H_u} X}, (L')^{\otimes lm})^{G\times H_r}$ of f under $\beta \circ \alpha$ such that $(\overline{G \times^{H_u} X})_F$ is affine. Pulling back along $\pi_{G\times H_r}$ identifies the ring of regular functions on the affine open subset $\pi_{G\times H_r}((\overline{G \times^{H_u} X})_F) \subseteq \overline{G \times^{H_u} X}/\!\!/(G \times H_r)$ with $\mathcal{O}((\overline{G \times^{H_u} X})_F)^{G\times H_r}$, and because X is irreducible $q_H^\# : (S^H)_{(f)} \hookrightarrow \mathcal{O}(X_f)^H$ is an isomorphism by Lemma 2.1.8, 1. Therefore there is a unique ring homomorphism $\Theta_f : \mathcal{O}(\pi_{G\times H_r}((\overline{G \times^{H_u} X})_F)) \to (S^H)_{(f)}$ making the diagram

$$\begin{tikzcd}
\mathcal{O}(\pi_{G\times H_r}((\overline{G \times^{H_u} X})_F)) \arrow[r, "(\pi_{G\times H_r})^\#", "\cong"'] \arrow[d, "\Theta_f"] & \mathcal{O}((\overline{G \times^{H_u} X})_F)^{G\times H_r} \arrow[d, "(\beta\circ\alpha)^\#"] \\
(S^H)_{(f)} \arrow[r, "q_H^\#", "\cong"'] & \mathcal{O}(X_f)^H
\end{tikzcd}$$

commute. In fact, Θ_f is an isomorphism. Indeed, by Definition 2.1.16, 2 of an enveloping system $\mathcal{O}(X_f)^H = (S^H)_{(f)}$ is generated by the regular functions $q_H^\#(\tilde{f}/f)$, where $\tilde{f} \in V^H$. Each such $\tilde{f}$ extends to some $\tilde{F} \in H^0(\overline{G \times^{H_u} X}, (L')^{\otimes lm})^{G\times H_r}$ under $\beta \circ \alpha$ by Definition 3.1.4, 2 of a reductive envelope, and the regular function in $\mathcal{O}((\overline{G \times^{H_u} X})_F)^{G\times H_r}$ defined by $\tilde{F}/F$ pulls back to $q_H^\#(\tilde{f}/f)$ under $\beta \circ \alpha$. It follows that Θ_f is surjective. On the other hand, because $\beta : (G \times^{H_u} X)_F \hookrightarrow (\overline{G \times^{H_u} X})_F$ is a dominant morphism and $\alpha^\#$ identifies $\mathcal{O}((G \times^{H_u} X)_F)^{G\times H_r}$ with $\mathcal{O}(X_f)^H$ the map $(\beta \circ \alpha)^\#$ is injective, hence Θ_f is injective also.

It follows that Θ_f defines an isomorphism of affine varieties

$$(\theta_H)_f : \mathrm{Spec}((S^H)_{(f)}) \xrightarrow{\cong} \pi_{G\times H_r}((\overline{G \times^{H_u} X})_F)$$

with $(\theta_H)_f \circ q_H|_{X_f} = \pi_{G\times H_r} \circ \beta \circ \alpha|_{X_f}$. Because Θ_f is defined in terms of compositions of (inverses of) sheaf homomorphisms and taking invariants is natural with respect to equivariant inclusions, it can easily be shown that the maps $(\theta_H)_f$ glue over the $\mathrm{Spec}((S^H)_{(f)})$ with $f \in \mathcal{S}$ to define an open immersion

$$\theta_H : \mathcal{V} \hookrightarrow \overline{G \times^{H_u} X}/\!\!/(G \times H_r)$$

such that $\theta_H \circ q_H = \pi_{G\times H_r} \circ \beta \circ \alpha$ on $X^{\mathrm{ss,fg}}$. We can see that $\theta_H^* \psi^* N' = \mathcal{O}_{\mathcal{V}}(n)$, as follows. Because of the extension property 2 of Definition 3.1.4 of a reductive envelope, the space V^H extends isomorphically under $\beta \circ \alpha$ to a subspace W of $H^0(\overline{G \times^{H_u} X}, (L')^{\otimes lm})^{G\times H_r}$.

Descent of sections through the GIT quotient map $\pi_{G\times H_r}$ gives an injective linear map $W \to H^0(\overline{G \times^{H_u} X} /\!\!/ (G \times H_r), \psi^* N')$, yielding a rational map

$$\gamma_H : \overline{G \times^{H_u} X} /\!\!/ (G \times H_r) \dashrightarrow \mathbb{P}(W^*)$$

that defines a morphism on the image of $\mathcal{V}$ under θ_H. There is a natural isomorphism $\mathbb{P}((V^H)^*) \cong \mathbb{P}(W^*)$ induced by $(\beta \circ \alpha)^*$ and, by inspection, one sees that the composition $\gamma_H \circ \theta_H : \mathcal{V} \to \mathbb{P}(W^*)$ corresponds to the natural locally closed immersion $\mathcal{V} \hookrightarrow \mathbb{P}((V^H)^*)$ defined by V^H (cf. Proposition 2.1.17). Hence $\theta_H^* \psi^* N' = \mathcal{O}_{\mathcal{V}}(n)$.

(Proof of 2.) The map θ_{H_u} is constructed in a similar way to θ_H. By Definition 3.1.1, 2 there is $\mathcal{S}^{\mathrm{ss}, H_u-\mathrm{fg}} \subseteq \mathcal{S}$ such that V defines an enveloping system adapted to $\mathcal{S}^{\mathrm{ss}, H_u-\mathrm{fg}}$ for the restricted linearisation $H_u \curvearrowright L \to X$. Letting

$$\mathcal{U} = \bigcup_{f \in \mathcal{S}^{\mathrm{ss}, H_u-\mathrm{fg}}} \mathrm{Spec}((S^{H_u})_{(f)}) \subseteq X \wr\wr H_u,$$

it follows from 1 of Definition 3.1.4 that there are natural isomorphisms $\mathrm{Spec}((S^{H_u})_{(f)}) \cong \mathcal{O}(\pi_G((\overline{G \times^{H_u} X})_F))$ (for $f \in \mathcal{S}^{\mathrm{ss}, H_u-\mathrm{fg}}$ with extension F over $\overline{G \times^{H_u} X}$) which patch to define an open immersion

$$\theta_{H_u} : \mathcal{U} \hookrightarrow (\overline{G \times^{H_u} X /\!\!/ G})^{\mathrm{ss}(M')}$$

such that $\theta_{H_u} \circ q_{H_u} = \pi_G \circ \beta \circ \alpha$ on $X^{\mathrm{ss}, H_u-\mathrm{fg}}$, and $\theta_{H_u}^*(M')^{\otimes m} = \mathcal{O}_{\mathcal{U}}(n)$ as line bundles. The arguments are analogous to those for $\theta_H : \mathcal{V} \hookrightarrow \overline{G \times^{H_u} X} /\!\!/ (G \times H_r)$. Notice that each section in $\mathcal{S}^{\mathrm{ss}, H_u-\mathrm{fg}}$ is fixed by the H-action on $H^0(X, L^{\otimes n})^H$, so by Proposition 2.2.1 $\mathcal{U}$ is stable under the H_r-action on $X \wr\wr H_u$. Furthermore, the equality $\theta_{H_u} \circ q_{H_u} = \pi_G \circ \beta \circ \alpha$ implies that θ_{H_u} is H_r-equivariant on the image $q_{H_u}(X^{\mathrm{ss}, H_u-\mathrm{fg}})$. The interior $q_{H_u}(X^{\mathrm{ss}, H_u-\mathrm{fg}})^\circ$ inside $\mathcal{U}$ is therefore a dense open subset of $\mathcal{U}$ on which θ_{H_u} is equivariant, so it follows from the separatedness of $(\overline{G \times^{H_u} X /\!\!/ G})^{\mathrm{ss}(M')}$ that θ_{H_u} is equivariant on the whole of $\mathcal{U}$. One sees from $\theta_{H_u} \circ q_{H_u} = \pi_G \circ \beta \circ \alpha$ that the naturally induced map $L^{\otimes n}|_{X^{\mathrm{ss}, H_u-\mathrm{fg}}} \to \theta_{H_u}^*(M')^{\otimes m}$ is equivariant with respect to $H \to H_r$, and since $\theta_{H_u}^*(M')^{\otimes m} \cong \mathcal{O}_{\mathcal{U}}(n)$ as line bundles it follows from Proposition 2.2.3, 2 the H_r-linearisation on $\theta_{H_u}^*(M')^{\otimes m}$ defines the same rational linearisation as the natural one on $\mathcal{O}_{\mathcal{U}}(n) \to \mathcal{U}$.

Because $\mathcal{S}^{\mathrm{ss},H_u-\mathrm{fg}} \subseteq \mathcal{S}$, the rational map $q_{H_r} : \mathrm{Proj}(S^{H_u}) \dashrightarrow \mathrm{Proj}(S^H)$ defines an H_r-invariant morphism $q_{H_r} : \mathcal{U} \to \mathcal{V}$, whose restriction to $\mathrm{Spec}((S^{H_u})_{(f)})$ for $f \in \mathcal{S}^{\mathrm{ss},H_u-\mathrm{fg}}$ is the map

$$\mathrm{Spec}((S^{H_u})_{(f)}) \to \mathrm{Spec}((S^H)_{(f)}) = \mathrm{Spec}(((S^{H_u})_{(f)})^{H_r})$$

induced by the inclusion $((S^{H_u})_{(f)})^{H_r} \hookrightarrow (S^{H_u})_{(f)}$. By reductive GIT for affine varieties (Theorem 1.2.1), each of these restrictions is a good categorical quotient for the H_r-action on $\mathrm{Spec}((S^{H_u})_{(f)})$, and since good categorical quotients are local on the base it follows that $q_{H_r} : \mathcal{U} \to \mathcal{U}/\!\!/H_r = q_{H_r}(\mathcal{U})$ is a good categorical quotient for the H_r-action on $\mathcal{U}$.

(Proof of 3.) It remains to prove the commutativity of the diagram in 3. Most of this follows from the construction of θ_H and θ_{H_u}—all that is left is to show is the equality

$$\overline{\pi}_{H_r} \circ \theta_{H_u} = \psi \circ \theta_H \circ q_{H_r} : \mathcal{U} \to (\overline{G \times^{H_u} X}/\!\!/G)/\!\!/_{M'}H_r.$$

Note first that both of these morphisms are indeed well defined on $\mathcal{U}$. By construction of θ_H and diagram (3.1.1) we have

$$\psi \circ \theta_H \circ q_H = \psi \circ \pi_{G\times H_r} \circ \beta \circ \alpha = \overline{\pi}_{H_r} \circ \pi_G \circ \beta \circ \alpha$$

on $X^{\mathrm{ss,fg}}$. Since $\pi_G \circ \beta \circ \alpha = \theta_{H_u} \circ q_{H_u}$ and $q_H = q_{H_r} \circ q_{H_u}$ on $X^{\mathrm{ss},H_u-\mathrm{fg}}$, it follows that

$$\psi \circ \theta_H \circ q_{H_r} \circ q_{H_u} = \overline{\pi}_{H_r} \circ \theta_{H_u} \circ q_{H_u} : X^{\mathrm{ss},H_u-\mathrm{fg}} \to (\overline{G \times^{H_u} X}/\!\!/G)/\!\!/_{M'}H_r.$$

Applying Proposition 2.1.13, 2 to this morphism, we conclude the desired equality $\overline{\pi}_{H_r} \circ \theta_{H_u} = \psi \circ \theta_H \circ q_{H_r} : \mathcal{U} \to (\overline{G \times^{H_u} X}/\!\!/G)/\!\!/_{M'}H_r$. □

By appealing to Proposition A.2.1, 3 in the Appendix, we obtain a corollary which is particularly relevant for the aims of finding compactifications of the enveloped quotient.

Corollary 3.1.13. *If $H \curvearrowright L \to X$ is an ample linearisation and $(\overline{G \times^{H_u} X}, \beta, L')$ is an ample reductive envelope, then $\overline{G \times^{H_u} X}/\!\!/(G \times H_r) = (\overline{G \times^{H_u} X}/\!\!/G)/\!\!/_{M'}H_r$ is projective and $\theta_H : \mathcal{V} \hookrightarrow \overline{G \times^{H_u} X}/\!\!/(G \times H_r)$ defines a compactification of the enveloped quotient $q(X^{\mathrm{ss,fg}})$, as in Definition 3.0.1. Moreover, $\theta_{H_u} : \mathcal{U} \hookrightarrow \overline{G \times^{H_u} X}/\!\!/G$ defines an H_r-equivariant compactification of the inner enveloping quotient $q_{H_u} : X^{\mathrm{ss},H_u-\mathrm{fg}} \to \mathcal{U}$ of $X^{\mathrm{ss},H_u-\mathrm{fg}}$.*

We now come to the *raison d'être* of reductive envelopes within non-reductive GIT.

Theorem 3.1.14. *Let H be a linear algebraic group, let $L \to X$ be an ample linearisation of a quasi-projective H-variety X and suppose $(\overline{G \times^{H_u} X}, \beta, L')$ is a reductive envelope for the linearisation, formed with respect to an H_u-faithful homomorphism $H \to G$ with G reductive. Let*

$$\pi : \overline{G \times^{H_u} X}^{\mathrm{ss}(L')} \to \overline{G \times^{H_u} X} /\!\!/ (G \times H_r)$$
$$\pi : \overline{G \times^{H_u} X}^{\mathrm{s}(L')} \to \overline{G \times^{H_u} X}^{\mathrm{s}(L')} / (G \times H_r)$$

be the GIT quotient and geometric quotient of the semistable and stable locus, respectively, for the $G \times H_r$-linearisation $L' \to \overline{G \times^{H_u} X}$.

1. *There is an inner enveloping quotient $q_{H_u} : X^{\mathrm{ss}, H_u-\mathrm{fg}} \to \mathcal{U}$ of $X^{\mathrm{ss}, H_u-\mathrm{fg}}$, with $\mathcal{U}$ an H_r-invariant open subset of $X \wr\!\wr H_u$ with a good categorical H_r-quotient $q_{H_r} : \mathcal{U} \to \mathcal{U} /\!\!/ H_r$, and an inner enveloping quotient $q_H : X^{\mathrm{ss,fg}} \to \mathcal{V}$ making the diagram*

$$\begin{tikzcd}
X^{\overline{\mathrm{s}}} \arrow[r, phantom, "\subseteq"] \arrow[d, "\mathrm{geo}"', "q_H"] & X^{\mathrm{s}} \arrow[r, phantom, "\subseteq"] \arrow[d, "\mathrm{geo}"', "q_H"] & X^{\mathrm{ss}, H_u-\mathrm{fg}} \arrow[r, phantom, "\subseteq"] \arrow[d, "q_H"] & X^{\mathrm{ss,fg}} \arrow[r, phantom, "\subseteq"] \arrow[d, "q_H"] & X^{\overline{\mathrm{ss}}} = X^{\mathrm{nss}} \arrow[d, "\pi \circ \beta \circ \alpha"] \\
X^{\overline{\mathrm{s}}}/H \arrow[r, phantom, "\subseteq"] & X^{\mathrm{s}}/H \arrow[r, phantom, "\subseteq"] & \mathcal{U} /\!\!/ H_r \arrow[r, phantom, "\subseteq"] & \mathcal{V} \arrow[r, phantom, "\subseteq"] & \overline{G \times^{H_u} X} /\!\!/ (G \times H_r)
\end{tikzcd}$$

   commute, where all the inclusions are natural open immersions and the two left-most vertical arrows are geometric quotients.

2. *The inclusion $\beta \circ \alpha : X^{\overline{\mathrm{s}}} \hookrightarrow \overline{G \times^{H_u} X}^{\mathrm{s}(L')}$ induces a natural open immersion*

$$X^{\overline{\mathrm{s}}}/H \hookrightarrow \overline{G \times^{H_u} X}^{\mathrm{s}(L')} / (G \times H_r).$$

3. *If moreover the reductive envelope $(\overline{G \times^{H_u} X}, \beta, L')$ is ample, then $\overline{G \times^{H_u} X} /\!\!/ (G \times H_r)$ is a compactification of the enveloped quotient, as in Definition 3.0.1.*

Proof. This immediately follows from combining Theorem 2.4.2 with Proposition 3.1.10, Corollary 3.1.11, Proposition 3.1.12 and Corollary 3.1.13. □

We make a couple of remarks regarding this result.

Remark 3.1.15. Notice from Proposition 3.1.12 that the inner enveloping quotient $\mathcal{U}$ of $X^{\mathrm{ss},H_u-\mathrm{fg}}$ may be chosen so as to embed equivariantly into $(\overline{G \times^{H_u} X} /\!\!/ G)^{\mathrm{ss}(M')}$, where $M' \to \overline{G \times^{H_u} X} /\!\!/ G$ is the naturally induced ample linearisation on the GIT quotient of the linearisation $G \curvearrowright L' \to \overline{G \times^{H_u} X}$. When $(\overline{G \times^{H_u} X}, \beta, L')$ is an ample reductive envelope then $\overline{G \times^{H_u} X} /\!\!/ G$ provides an H_r-equivariant compactification of $\mathcal{U}$, and moreover the composition

$$X^{\mathrm{ss},H_u-\mathrm{fg}} \xrightarrow{q_{H_u}} \mathcal{U} \xrightarrow{q_{H_r}} \mathcal{U} /\!\!/ H_r$$

can be studied by doing reductive GIT on $\overline{G \times^{H_u} X}$ in stages—first by G, then by H_r.

Remark 3.1.16. If one happens to know that semistability and stability for a reductive envelope $(\overline{G \times^{H_u} X}, \beta, L')$ coincide, then $X^{\overline{\mathrm{s}}} = X^{\overline{\mathrm{ss}}}$ and we then have a string of equalities

$$X^{\overline{\mathrm{s}}} = X^{\mathrm{s}} = X^{\mathrm{ss},H_u-\mathrm{fg}} = X^{\mathrm{ss}} = X^{\mathrm{nss}} = X^{\overline{\mathrm{ss}}}.$$

From the point of view of constructing compactifications of enveloping quotients, the most important application of Theorem 3.1.14 is in the case where the reductive envelope $(\overline{G \times^{H_u} X}, \beta, L')$ is ample; see statement 3. In this case, the associated completely semistable and stable loci can be computed using the Hilbert-Mumford criterion and the GIT quotient $\overline{G \times^{H_u} X} /\!\!/ (G \times H_r)$ can be described set-theoretically as the quotient space of $\overline{G \times^{H_u} X}^{\mathrm{ss}(L')}$ modulo the S-equivalence relation (see Section 1.2 of Chapter 1).

3.2 Strong Reductive Envelopes

Given a linear algebraic group H acting on an irreducible variety X with ample linearisation $L \to X$, we saw in Proposition 3.1.10 that the completely stable and completely semistable loci associated to a reductive envelope $(\overline{G \times^{H_u} X}, \beta, L')$ give approximations of the intrinsically defined stable locus X^{s} and the finitely generated semistable locus $X^{\mathrm{ss,fg}}$: one has $X^{\overline{\mathrm{s}}} \subseteq X^{\mathrm{s}}$ and $X^{\mathrm{ss,fg}} \subseteq X^{\mathrm{nss}} = X^{\overline{\mathrm{ss}}}$. In this section we discuss particular kinds of reductive envelope $(\overline{G \times^{H_u} X}, \beta, L')$ which give equalities $X^{\overline{\mathrm{s}}} = X^{\mathrm{s}}$ and $X^{\overline{\mathrm{ss}}} = X^{\mathrm{ss,fg}}$, thus providing ways to compute the finitely generated stable set and stable set for the original linearisation $H \curvearrowright L \to X$ using methods from reductive GIT. In light of Theorem 3.1.14 and Proposition 2.3.8, to obtain these equalities we need to make sure that

- all points inside $(G \times^{H_u} X)^{\mathrm{s}(L)}$ are stable for the linearisation $L' \to \overline{G \times^{H_u} X}$; and
- any point in $X \subseteq \overline{G \times^{H_u} X}$ that is semistable for L' must lie in X_f for some invariant f over X with $(S^H)_{(f)}$ finitely generated over $\Bbbk$.

Following the ideas of [DK07, §5.2], we adopt the strategy of effectively forcing out any complications arising from the codimension 1 boundary components of $\overline{G \times^{H_u} X} \setminus (G \times^{H_u} X)$, by demanding that extensions of appropriate invariants over $G \times^{H_u} X$ vanish on these components. This, together with a normality assumption on $\overline{G \times^{H_u} X}$, turns out to be enough to get the desired equalities $X^{\overline{\mathrm{s}}} = X^{\mathrm{s}}$ and $X^{\mathrm{ss,fg}} = X^{\overline{\mathrm{ss}}}$.

Definition 3.2.1. Let H be a linear algebraic group acting on an irreducible variety X with linearisation $L \to X$. Let $H \to G$ be an H_u-faithful homomorphism with G a reductive group and let $(\overline{G \times^{H_u} X}, \beta, L')$ be a reductive envelope. We call $(\overline{G \times^{H_u} X}, \beta, L')$ a *strong reductive envelope* if there is a fully separating enveloping system V for $H \curvearrowright L \to X$ satisfying the extension properties 1–3 of Definition 3.1.4 and the further property that every $f \in V^H$ extends to a $G \times H_r$-invariant over $\overline{G \times^{H_u} X}$ that vanishes on each codimension 1 component of the boundary of $G \times^{H_u} X$ inside $\overline{G \times^{H_u} X}$.

Proposition 3.2.2. *Let H be a linear algebraic group acting on an irreducible variety X with ample linearisation $L \to X$ and $H \to G$ an H_u-faithful homomorphism into a reductive group G. Suppose $(\overline{G \times^{H_u} X}, \beta, L')$ is a strong reductive envelope with $\overline{G \times^{H_u} X}$ a normal variety. Then $X^{\mathrm{s}} = X^{\overline{\mathrm{s}}}$ and $X^{\mathrm{ss},H_u-\mathrm{fg}} = X^{\overline{\mathrm{ss}}}$.*

Proof. By Proposition 3.1.10 it suffices to show $X^{\mathrm{s}} \subseteq X^{\overline{\mathrm{s}}}$ and $X^{\overline{\mathrm{ss}}} \subseteq X^{\mathrm{ss},H_u-\mathrm{fg}}$. Let V be a fully separating enveloping system associated to $(\overline{G \times^{H_u} X}, \beta, L')$ satisfying the conditions in Definition 3.2.1 and let $D_1, \ldots, D_m$ be the codimension 1 components of $\overline{G \times^{H_u} X} \setminus \beta(G \times^{H_u} X)$.

Let $x \in X^{\mathrm{s}}$. By Definition 3.1.1, 3 of a fully separating enveloping system there is $f \in V^H$ with extension to a $G \times H_r$-invariant section F over $G \times^{H_u} X$ such that $f(x) \neq 0$ and $G \times^{H_u} (X_f) = (G \times^{H_u} X)_F$ is affine. By 3 of Definition 3.1.4 and the definition of a strong reductive envelope, there is a section of some positive tensor power of $L' \to$

$\overline{G \times^{H_u} X}$, which we also call F, such that $(\beta \circ \alpha)^* F = f$, the open set $(\overline{G \times^{H_u} X})_F$ is affine and F vanishes on $\bigcup_i D_i \subseteq \overline{G \times^{H_u} X}$. Thus the complement of $G \times^{H_u} (X_f) = \beta^{-1}((\overline{G \times^{H_u} X})_F)$ inside $(\overline{G \times^{H_u} X})_F$ has codimension at least 2. Because $(\overline{G \times^{H_u} X})_F$ is normal, pullback along β yields an isomorphism $\mathcal{O}((\overline{G \times^{H_u} X})_F) \cong \mathcal{O}(G \times^{H_u} (X_f))$ and, since both $(\overline{G \times^{H_u} X})_F$ and $G \times^{H_u} (X_f)$ are affine, the open inclusion $\beta : G \times^{H_u} (X_f) \hookrightarrow (\overline{G \times^{H_u} X})_F$ is therefore an isomorphism. But by Proposition 2.3.8, $G \times^{H_u} (X_f) = (\overline{G \times^{H_u} X})_F$ is contained in the stable locus for the $G \times H_r$-linearisation $L \to G \times^{H_u} X$, so that the $G \times H_r$-action on $(\overline{G \times^{H_u} X})_F$ is closed with all stabilisers finite. It follows that $(\overline{G \times^{H_u} X})_F \subseteq \overline{G \times^{H_u} X}^{\mathrm{s}(L')}$ and thus $x \in X^{\overline{\mathrm{s}}}$.

Now suppose $x \in X^{\overline{\mathrm{ss}}}$. By Definition 3.1.1, 1 of a fully separating enveloping system we have $X^{\overline{\mathrm{ss}}} = X^{\mathrm{nss}} = \bigcup_{f \in V^H} X_f$, so by Definition 3.2.1 there is $f \in V^H$ with extension to a $G \times H_r$-invariant section F of some positive tensor power of $L' \to \overline{G \times^{H_u} X}$, with $(\overline{G \times^{H_u} X})_F$ affine, such that $x \in X_f$ and F vanishes on $\bigcup_i D_i$. As above, the complement of $G \times^{H_u} (X_f)$ is therefore codimension at least 2 in $(\overline{G \times^{H_u} X})_F$, so from the normality of $(\overline{G \times^{H_u} X})_F$ it follows that the pullback map $\beta^\# : \mathcal{O}((\overline{G \times^{H_u} X})_F) \to \mathcal{O}(G \times^{H_u} (X_f))$ is a $G \times H_r$-equivariant isomorphism. Thus $(\beta \circ \alpha)^\#$ yields isomorphisms

$$\mathcal{O}((\overline{G \times^{H_u} X})_F)^G \xrightarrow{\cong} \mathcal{O}(X_f)^{H_u} = (S^{H_u})_{(f)},$$
$$\mathcal{O}((\overline{G \times^{H_u} X})_F)^{G \times H_r} \xrightarrow{\cong} \mathcal{O}(X_f)^H = (S^H)_{(f)}.$$

Since $(\overline{G \times^{H_u} X})_F$ is affine and G and $G \times H_r$ are reductive, the $\Bbbk$-algebras $(S^{H_u})_{(f)}$ and $(S^H)_{(f)}$ are therefore finitely generated and thus $x \in X_f \subseteq X^{\mathrm{ss}, H_u-\mathrm{fg}}$. □

Remark 3.2.3. Observe that as a corollary to Proposition 3.2.2, for any given linearisation $H \curvearrowright L \to X$ a necessary condition for the existence of a strong reductive envelope with normal $\overline{G \times^{H_u} X}$ is that $X^{\mathrm{ss}, H_u-\mathrm{fg}} = X^{\mathrm{ss,fg}} = X^{\mathrm{nss}}$.

For the remainder of this section we will consider how we can construct strong reductive envelopes. Recall that this means (1) choosing an equivariant compactification $\overline{G \times^{H_u} X}$ of $G \times^{H_u} X$, together with (2) an extension $L' \to \overline{G \times^{H_u} X}$ of some positive tensor power of the linearisation $L \to G \times^{H_u} X$ such that (3) (roughly stated) enough invariant sections over $G \times^{H_u} X$ extend to sections over $\overline{G \times^{H_u} X}$ vanishing on the boundary divisors. In general (2) depends heavily on the singularities of the completion in (1)—an issue we don't wish to explore in depth here. It suffices for us to note that if $\overline{G \times^{H_u} X}$ is $\mathbb{Q}$-factorial then some positive tensor power of $L \to G \times^{H_u} X$ extends over the boundary, and if $\overline{G \times^{H_u} X}$ is even factorial (for example, smooth) then L itself extends. Moreover, if X is irreducible and H and G are *connected* linear algebraic groups such that the action of $G \times H_r$ on $G \times^{H_u} X$ extends to one on $\overline{G \times^{H_u} X}$, then any line bundle $L' \to \overline{G \times^{H_u} X}$ extending $L^{\otimes r} \to G \times^{H_u} X$ (with $r > 0$) has a unique linearisation extending $L^{\otimes r} \to G \times^{H_u} X$; see the proof of [MFK94, Converse 1.13].

On the other hand, given any extension $L' \to \overline{G \times^{H_u} X}$ of (a power of) the $G \times H_r$-linearisation $L \to G \times^{H_u} X$, then as in [DK07] the question of (3) can be approached by making a fairly mild assumption on the nature of the completion $\overline{G \times^{H_u} X}$: namely, that it is a *gentle* completion of $G \times^{H_u} X$, in the sense of [DK07, Definition 5.3.8]. Recall this means that $\overline{G \times^{H_u} X}$ is a normal projective variety such that every codimension 1 component of the boundary of $G \times^{H_u} X$ in $\overline{G \times^{H_u} X}$ is a $\mathbb{Q}$-Cartier divisor (see Definition 1.3.15). What follows is a generalisation of the constructions in [DK07, §§5.3.1–5.3.2] (described in Section 1.3.2) that applies to our current setting.

Remark 3.2.4. In [DK07] it is erroneously claimed that if $X \subseteq \overline{X}$ is a gentle completion then some positive tensor power of a given a line bundle $L \to X$ extends to a line bundle on $\overline{X}$. (The problem is that, while the Cartier divisor on X defined by L extends to a *Weil* divisor E on $\overline{X}$, assuming only that the boundary divisors are $\mathbb{Q}$-Cartier is not in general sufficient to deduce that E is $\mathbb{Q}$-Cartier.)

Let us now assume that H and G are *connected* linear algebraic groups, with G reductive, and $\rho : H \to G$ is a fixed H_u-faithful homomorphism. Let $\overline{G \times^{H_u} X}$ be a gentle completion of $G \times^{H_u} X$ and $L' \to \overline{G \times^{H_u} X}$ a $G \times H_r$-linearisation extending $L^{\otimes r} \to G \times^{H_u} X$, for some $r > 0$. Also let $D_1, \dots, D_m \subseteq \overline{G \times^{H_u} X}$ be the codimension 1 irreducible components of the complement of $G \times^{H_u} X$ in $\overline{G \times^{H_u} X}$ and define a $\mathbb{Q}$-Cartier divisor

$$D := \sum_{i=1}^{m} D_i.$$

Then for any sufficiently divisible integer $N > 0$ the divisor ND is Cartier and defines a line bundle $\mathcal{O}(ND)$ on $\overline{G \times^{H_u} X}$ which restricts to the trivial bundle on $G \times^{H_u} X$. As in Section 1.3.2, given any line bundle $M \to \overline{G \times^{H_u} X}$, define

$$M_N := M \otimes \mathcal{O}(ND) \to \overline{G \times^{H_u} X}. \tag{3.2.1}$$

In the case $M = L'$ then, because $G \times H_r$ is connected, the $G \times H_r$-linearisation on $L^{\otimes r} \to G \times^{H_u} X$ extends uniquely to a $G \times H_r$-linearisation on L'_N.

The next lemma consists of an expanded version of the argument found in [DK07, Proposition 5.3.10].

Lemma 3.2.5. *Let $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$ be a gentle completion and $L' \to \overline{G \times^{H_u} X}$ a line bundle extending L. Retain the preceding notation.*

1. *Let f be a section of $L^{\otimes rr'} \to G \times^{H_u} X$ for some $r' > 0$. Then for $N > 0$ a sufficiently divisible integer, f extends to a section of $(L'_N)^{\otimes r'} \to \overline{G \times^{H_u} X}$ under β.*
2. *For any $N > 0$ such that $N \sum_i D_i$ is Cartier and for any integers $r', m > 0$, there is a natural inclusion $H^0(\overline{G \times^{H_u} X}, (L'_N)^{\otimes r'}) \hookrightarrow H^0(\overline{G \times^{H_u} X}, (L'_{mN})^{\otimes r'})$ whose image consists of sections of $(L'_N)^{\otimes r'}$ that vanish on each of the D_i.*

Proof. Throughout the proof we denote the field of rational functions on an irreducible variety Y by $\Bbbk(Y)$.

(Proof of 1.) Because the completion $\overline{G \times^{H_u} X}$ of $G \times^{H_u} X$ is normal, the section f extends canonically over components of codimension at least 2, so we need only show that f extends over codimension 1 boundary components. The variety $\overline{G \times^{H_u} X}$ is also irreducible, so L' corresponds to a Cartier divisor [Har77, Proposition 6.15]; that is, there is a finite collection of pairs (U_j, t_j), with $U_j \subseteq \overline{G \times^{H_u} X}$ open and $t_j \in \Bbbk(U_j) = \Bbbk(\overline{G \times^{H_u} X})$, such that L' is represented by a Weil divisor whose restriction to U_j is the principal divisor defined by t_j. Note that $(L')^{\otimes r'}$ is then represented by the collection $(U_j, t_j^{r'})$. There is a positive integer N such that the order of the vanishing of t_j along each D_i is less than N for each j. Thinking of sections of $L^{\otimes rr'}$ as sections of the constant sheaf of rational functions on $G \times^{H_u} X$ in the standard way [Har77, Chapter 2, §6], we can write $f|_{U_j \cap (G \times^{H_u} X)} = b_j / t_j^{r'}$ with b_j a regular function on $U_j \cap (G \times^{H_u} X)$. We can also assume without loss of generality that ND is a Cartier divisor of $\overline{G \times^{H_u} X}$, since the completion $\overline{G \times^{H_u} X}$ is $\mathbb{Q}$-factorial. Let (V_k, s_k) represent ND, with $V_k \subseteq \overline{G \times^{H_u} X}$ open and $s_k \in \Bbbk(V_k) = \Bbbk(\overline{G \times^{H_u} X})$, again with the index set for k being finite, and set $a_{jk} = b_j s_k^{r'} \in \Bbbk(\overline{G \times^{H_u} X})$. Now b_j may have poles along $U_j \cap V_k \cap (\bigcup_i D_i)$; on the other hand each $s_k^{r'}$ vanishes over $U_j \cap V_k \cap (\bigcup_i D_i)$ with order $r'N$, so by further increasing N if necessary we may assume each a_{jk} defines a regular function on $U_j \cap V_k$. Thus each $a_{jk}/(t_j s_k)^{r'}$ is a section of $(L'_N)^{\otimes r'}$ over $U_j \cap V_k$, whose restriction to $(G \times^{H_u} X) \cap U_j \cap V_k$ is defined by $b_j / t_j^{r'}$. One can check that all the $a_{jk}/(t_j s_k)^{r'}$ (for all j and k) agree on overlaps, so they patch together to give a global section of $(L'_N)^{\otimes r'}$ which extends f.

(Proof of 2.) Assume N is large enough so that ND is Cartier, and let $r' > 0$. Continuing with the notation used above to prove 1, the sheaf $\mathcal{O}(ND)$ is represented by the collection (V_k, s_k), with $s_k \in \mathcal{O}(V_k)$ such that $\mathcal{O}(ND)|_{V_k} = \mathcal{O}_{V_k}\langle 1/s_k \rangle$ as sheaves of $\mathcal{O}_{V_k}$-modules. Then for each $m > 0$ there is a well-defined inclusion $\mathcal{O}(r'ND) \hookrightarrow \mathcal{O}(mr'ND)$, whose restriction to V_k corresponds to the multiplication-by-$s_k^{r'(m-1)}$ map $\mathcal{O}_{V_k} \to \mathcal{O}_{V_k}$. Note that sections in the image of this map vanish of each of the D_i. Because $(L')^{\otimes r'}$ is locally free the natural map of sheaves

$$(\underline{L'})^{\otimes r'} \otimes \mathcal{O}(r'ND) \to (\underline{L'})^{\otimes r'} \otimes \mathcal{O}(mr'ND)$$

is again injective [Har77, Chapter 3, Proposition 9.2], and since taking global sections is left-exact we see that this yields an injection

$$H^0(\overline{G \times^{H_u} X}, (L'_N)^{\otimes r'}) \hookrightarrow H^0(\overline{G \times^{H_u} X}, (L'_{mN})^{\otimes r'}).$$

It is immediate that any section in the image of this map vanishes on each of the D_i. $\square$

Lemma 3.2.5 says that given any collection of sections of some power of $L^{\otimes r} \to G \times^{H_u} X$ we can always modify the extension L' so that these sections extend to sections of this modification over $\overline{G \times^{H_u} X}$ which vanish on the boundary. With additional assumptions on L', we can use it to produce a strong reductive envelope.

Proposition 3.2.6. *Let H be a connected linear algebraic group acting on an irreducible variety X with linearisation $L \to X$ and let $H \to G$ be an H_u-faithful homomorphism into a connected reductive group G. Suppose $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$ is a gentle $G \times H_r$-equivariant projective completion and let $L' \to \overline{G \times^{H_u} X}$ be any $G \times H_r$-linearisation extending some positive tensor power of the H-linearisation $L \to X$. If either*

- *$(\overline{G \times^{H_u} X}, \beta, L')$ defines a reductive envelope for $H \curvearrowright L \to X$; or*
- *the line bundle L'_n of* (3.2.1) *is ample, for sufficiently divisible integers $n > 0$;*

then for sufficiently divisible integers $N > 0$ the triple $(\overline{G \times^{H_u} X}, \beta, L'_N)$ defines a strong reductive envelope.

Proof. Suppose $L' \to \overline{G \times^{H_u} X}$ pulls back to $L^{\otimes r} \to X$ under $\beta \circ \alpha$ and again let $D_1, \ldots, D_m$ be the irreducible codimension 1 components of the boundary of the gentle completion $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$. We first show the following: given $r' > 0$ and an enveloping system $V \subseteq H^0(X, L^{\otimes rr'})$, for sufficiently divisible integers $N > 0$ (depending on V) each section $f \in V^{H_u}$ (respectively, $f \in V^H$) extends to a G-invariant (respectively, $G \times H_r$-invariant) section F of $(L'_N)^{\otimes r'} \to \overline{G \times^{H_u} X}$ under $\beta \circ \alpha$ which vanishes on each D_i. To this end, let $f \in V^{H_u}$ (respectively, $f \in V^H$). By Lemma 3.2.5 there is an integer $N_f > 0$ such that f extends to a section F of $(L'_{N_f})^{\otimes r'}$ over $\overline{G \times^{H_u} X}$ which vanishes on the codimension 1 boundary components of $G \times^{H_u} X$ inside $\overline{G \times^{H_u} X}$. Note that F must be G-invariant (respectively, $G \times H_r$-invariant): by the normality of $\overline{G \times^{H_u} X}$ the section f extends canonically to an invariant over the boundary components of codimension at least 2 in $\overline{G \times^{H_u} X}$, and since F vanishes on the remaining boundary components it too must be invariant. Now take bases $\mathcal{B}_{H_u}$ of V^{H_u} and $\mathcal{B}_H$ of V^H and let $N > 0$ be any positive integer which is properly divisible by all the N_f for $f \in \mathcal{B}_{H_u} \cup \mathcal{B}_H$. Then by Lemma 3.2.5, 2 any $f \in V^{H_u} \cup V^H$ extends to a section F of $(L'_N)^{\otimes r'}$, invariant in the appropriate sense, which vanishes on the codimension 1 boundary components of $G \times^{H_u} X$ in $\overline{G \times^{H_u} X}$.

Now suppose $(\overline{G \times^{H_u} X}, \beta, L')$ defines a reductive envelope for $H \curvearrowright L \to X$ and let V be an associated fully separating enveloping system satisfying 1–3 of Definition 3.1.4 of a reductive envelope. As just shown, for sufficiently divisible $N > 0$ each $f \in V^{H_u} \cup V^H$ extends to an invariant F (in the appropriate sense) of some positive tensor power of $L'_N \to \overline{G \times^{H_u} X}$ which vanishes on the codimension 1 complement $\bigcup_i D_i$ of $G \times^{H_u} X$ in $\overline{G \times^{H_u} X}$. To show that $(\overline{G \times^{H_u} X}, \beta, L'_N)$ is a strong reductive envelope, we are left to show that each such $(\overline{G \times^{H_u} X})_F$ is affine (see Definition 3.1.4, 3). But by Definition 3.1.4 of a reductive envelope applied to L', the section f does extend to an invariant F' of some positive tensor power of $L' \to \overline{G \times^{H_u} X}$ with $(\overline{G \times^{H_u} X})_{F'}$ is affine. Observe that the restrictions of L'_N and L' to $\overline{G \times^{H_u} X} \setminus (\bigcup_i D_i)$ are equal, and $G \times^{H_u} X$ and $\overline{G \times^{H_u} X} \setminus (\bigcup_i D_i)$ differ only in codimension at least 2, so by normality of $\overline{G \times^{H_u} X}$ the sections F and F' are equal over $\overline{G \times^{H_u} X} \setminus (\bigcup_i D_i)$. It follows that

$$(\overline{G \times^{H_u} X})_F = (\overline{G \times^{H_u} X})_{F'} \setminus (\textstyle\bigcup_i D_i).$$

Because $(\overline{G \times^{H_u} X})_{F'}$ is affine and the complement of the support of a Cartier divisor on an affine variety is again affine [Sta15, Tag 01WQ], $(\overline{G \times^{H_u} X})_F$ is therefore also affine. Hence, $(\overline{G \times^{H_u} X}, \beta, L'_N)$ is a strong reductive envelope, for sufficiently divisible $N > 0$.

Finally, consider the case where L'_n is ample for sufficiently divisible $n > 0$. Appealing to Lemma 3.1.2 there is an integer $r' > 0$ such that $H^0(X, L^{\otimes rr'})$ contains a fully separating enveloping system V, and for sufficiently divisible $N > 0$ each $f \in V^{H_u} \cup V^H$ extends to an invariant F (in the appropriate sense) of some positive tensor power of $L'_N \to \overline{G \times^{H_u} X}$ which vanishes on each D_i. We can choose $N > 0$ sufficiently divisible so that that L'_N is ample and thus each $(\overline{G \times^{H_u} X})_F$ affine. Then $(\overline{G \times^{H_u} X}, \beta, L'_N)$ is a strong ample reductive envelope. □

Corollary 3.2.7. *In the setting of Proposition 3.2.6, suppose we are in the situation that X is projective, $L \to X$ is ample and L'_n is ample for sufficiently divisible integers $n > 0$. If $S^H = \Bbbk[X, L]^H$*

is a finitely generated $\Bbbk$*-algebra, then for sufficiently divisible integers* $N > 0$ *the inclusion* $\beta \circ \alpha :$ $X^{\mathrm{ss,fg}} \hookrightarrow (\overline{G \times^{H_u} X})^{\mathrm{ss}(L'_N)}$ *induces a natural isomorphism*

$$X \wr\!\wr H = \mathrm{Proj}(S^H) \cong \overline{G \times^{H_u} X} /\!\!/_{L'_N} (G \times H_r).$$

Proof. Note that $X \wr\!\wr H$ is canonically equal to the projective variety $\mathrm{Proj}(S^H)$ by definition and is an inner enveloping quotient of $X^{\mathrm{ss,fg}} = X^{\mathrm{nss}}$. As in the proof of Proposition 3.2.6, by appealing to Lemma 3.1.2 we may find $r' > 0$ such that $H^0(X, L^{\otimes rr'})$ contains a fully separating enveloping system V. By taking the image of V under a suitable multiplication map (cf. Lemma 3.1.2) and enlarging using Lemma 1.1.20 if necessary, we can assume that V is a fully separating system that also contains a (finite) collection of generators $f_1, \dots, f_m$ of the ring $\Bbbk[X, L^{\otimes rr'}]^H$. Then

$$X \wr\!\wr H \cong \mathrm{Proj}(\Bbbk[X, L^{\otimes rr'}]^H) = \bigcup_{j=1}^{m} \mathrm{Spec}((S^H)_{(f_j)})$$

and each $(S^H)_{(f_j)}$ is generated by $\{\tilde{f}/f_j \mid \tilde{f} \in V^H\}$. Without loss of generality we may therefore view V as a fully separating enveloping system which is adapted to to a subset $\mathcal{S}$ containing $\{f_1, \dots, f_m\}$. For sufficiently divisible $N > 0$ we have L'_N ample and, as in the proof of Proposition 3.2.6, each $f \in V^H$ extends to a $G \times H_r$-invariant F of some positive tensor power of $L'_N \to \overline{G \times^{H_u} X}$. The inclusion $\beta \circ \alpha : X^{\mathrm{ss,fg}} \hookrightarrow (\overline{G \times^{H_u} X})^{\mathrm{ss}(L'_N)}$ is therefore well-defined, and defines a dominant open immersion

$$\theta_H : X \wr\!\wr H \hookrightarrow \overline{G \times^{H_u} X} /\!\!/_{L'_N} (G \times H_r);$$

cf. the proof of Proposition 3.1.12, 1. Since $X \wr\!\wr H$ is proper and $\overline{G \times^{H_u} X} /\!\!/_{L'_N} (G \times H_r)$ is separated, the image of θ_H is also closed in $\overline{G \times^{H_u} X} /\!\!/_{L'_N} (G \times H_r)$ [Sta15, Tag 01W0] and thus is the whole of $\overline{G \times^{H_u} X} /\!\!/_{L'_N} (G \times H_r)$. Therefore θ_H defines an isomorphism $X \wr\!\wr H \cong \overline{G \times^{H_u} X} /\!\!/_{L'_N} (G \times H_r)$. □

The situation from Proposition 3.2.6 where the linearisation $L'_N \to \overline{G \times^{H_u} X}$ is ample for sufficiently divisible N is potentially the most useful in applications, because it does not require any verification that $(\overline{G \times^{H_u} X}, \beta, L')$ forms a reductive envelope (in particular, verifying that enough invariants extend to sections F with $(\overline{G \times^{H_u} X})_F$ affine). For the proposition to apply though one needs to know that $\overline{G \times^{H_u} X}$ provides a gentle completion of $G \times^{H_u} X$ and this is not always easy to verify. In the next section we shall consider a special case which allows us to construct strong ample reductive envelopes $(\overline{G \times^{H_u} X}, \beta, L')$ with gentle completions of $G \times^{H_u} X$.

3.2.1 Special Case: Extension to a G-Linearisation and Grosshans Subgroups

In [DK07, §5.3.2] Doran and Kirwan considered the case where a linearisation of a unipotent group extended to that of a reductive group, with the aim of constructing strong ample reductive envelopes. Here we generalise this work to the setting of not-necessarily unipotent groups. For this section we suppose that X is a *projective* irreducible H-variety. If $\rho : H \to G$ is an H_u-faithful homomorphism into a reductive group G and the linearisation $H \curvearrowright L \to X$ can be partially extended to a G-linearisation (in a way we will make precisely momentarily), one can reduce the question of constructing ample strong reductive envelopes to understanding the geometry of the homogeneous space G/H_u. (An example of the line of argument we shall use has in fact already been used in the proof that ample reductive envelopes exist, Proposition 3.1.7.) To make this more precise, suppose there is a linearisation $G \curvearrowright L \to X$ satisfying the following condition:

(C1) The linearisation $H_u \curvearrowright L \to X$ arising from restricting the H-linearisation extends to the G-linearisation through $\rho|_{H_u}$.

The extension condition (C1) yields an isomorphism of $G \times H_r$-linearisations:

$$\begin{tikzcd}
G \times^{H_u} L \arrow[r, "\cong"] \arrow[d] & (G/H_u) \times L \arrow[d] & {[g,l] \mapsto (gH_u, gl)} \\
G \times^{H_u} X \arrow[r, "\cong"] & (G/H_u) \times X & {[g,x] \mapsto (gH_u, gx)}
\end{tikzcd} \tag{3.2.2}$$

The corresponding G-linearisation on the right hand side of this diagram is the one given by taking the product of the linearisation on $L \to X$ and left multiplication on G/H_u. The H_r-linearisation is more complicated and in general cannot be expressed as the product of linearisations over G/H_u and X. We make an additional assumption to demand this:

(C2) There is a linearisation $G \times H_r \curvearrowright \tilde{L} \to X$, with $\tilde{L} = L$ as line bundles, such that the $G \times H_r$-linearisation $(G/H_u) \times L \to (G/H_u) \times X$ arising from (3.2.2) coincides with the product of $\tilde{L} \to X$ and the $G \times H_r$-action on G/H_u given by left multiplication by G and right multiplication by H_r[1].

Example 3.2.8. Suppose the H_u-faithful homomorphism ρ is just a closed embedding $H \hookrightarrow G$. Then the H_r-linearisation on $(G/H_u) \times L \to (G/H_u) \times X$ under the isomorphism (3.2.2) is simply the product of the right multiplication action on G/H_u together with $\tilde{L} \to X$, where $\tilde{L} \to X$ is the line bundle $L \to X$ equipped with the trivial H_r-linearisation.

Assuming X is projective and conditions (C1) and (C2) are satisfied, then a natural way to compactify $G \times^{H_u} X \cong (G/H_u) \times X$ is to study $G \times H_r$-equivariant projective completions $\overline{G/H_u}$ of G/H_u. For example, if L is ample and it is known that $\overline{G/H_u}$ can be chosen to be normal with $\mathbb{Q}$-Cartier prime boundary divisors $E_1, \ldots, E_m$ such that $\mathcal{O}(N \sum_i E_i) \to \overline{G/H_u}$ is ample for sufficiently divisible $N > 0$, then the codimension 1 components of the boundary $\overline{G/H_u} \times X$ are precisely the $E_i \times X$. These are $\mathbb{Q}$-Cartier divisors, so if X is further assumed to be normal then $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$ is a gentle completion and Proposition 3.2.6 applies to $L' = \mathcal{O}(N \sum_i E_i) \boxtimes \tilde{L}$ (with $\tilde{L} \to X$ the $G \times H_r$-linearisation of (C2)).

This works out particularly well in the case where H_u is a *Grosshans subgroup* of the reductive group G. Recall from [Gro97, §4] that this means the pair $H_u \subseteq G$ satisfies the following equivalent conditions:

- $\mathcal{O}(G/H_u)$ is a finitely generated $\Bbbk$-algebra;
- there is a finite dimensional G-module W and a vector $w \in W$ such that $H_u = \mathrm{Stab}_G(w)$ and G/H_u embeds in the closure $\overline{G \cdot w} \subseteq W$ with complement having codimension at least 2, via the natural map $G/H_u \to G \cdot w$.

(The equivalence is shown in [Gro97, Theorem 4.3].) In this case, since $\mathcal{O}(G/H_u) = \mathcal{O}(G)^{H_u}$ is a normal ring $\mathrm{Spec}(\mathcal{O}(G/H_u)) = \mathrm{Spec}(\mathcal{O}(G)^{H_u})$ is a normal affine variety upon which

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[1] That is, $(g, \overline{h}) \cdot g_0 H_u = g g_0 \rho(h)^{-1} H_u$ for all $\overline{h} = hH_u \in H_r = H/H_u$, $g \in G$ and all $g_0 H_u \in G/H_u$.

$G \times H_r$ naturally acts, and there is a canonical open immersion

$$G/H_u \hookrightarrow \mathrm{Spec}(\mathcal{O}(G/H_u))$$

that is $G \times H_r$-equivariant. Moreover, by [Gro97, Theorem 4.3] $\mathrm{Spec}(\mathcal{O}(G/H_u))$ contains a point p such that $G \cdot p$ is open in $\mathrm{Spec}(\mathcal{O}(G/H_u))$, the stabiliser of p in G is equal to H_u and the compliment of $G \cdot p$ in $\mathrm{Spec}(\mathcal{O}(G/H_u))$ has codimension at least 2 in $\mathrm{Spec}(\mathcal{O}(G/H_u))$. Thus if H_u is a Grosshans subgroup of G then there is a *normal*, affine $G \times H_r$-equivariant completion $\overline{G/H_u}^{\mathrm{aff}}$ of G/H_u whose boundary has no codimension 1 components.

Remark 3.2.9. In the situation where (C1) holds we have

$$\Bbbk[X, L]^{H_u} = \Bbbk[G \times^{H_u} X, L]^G = \Bbbk[G/H_u \times X, L]^G \cong (\mathcal{O}(G/H_u) \otimes \Bbbk[X, L])^G,$$

where the last isomorphism follows from the Künneth formula [Sta15, Tag 02KE]. If H_u is a Grosshans subgroup of G and $H \curvearrowright L \to X$ is an ample linearisation of a projective variety, then by Nagata's theorem [Nag64] it follows that $\Bbbk[X, L]^{H_u}$ is finitely generated. Then $\Bbbk[X, L]^H = (\Bbbk[X, L]^{H_u})^{H_r}$ is also finitely generated, hence $X^{\mathrm{ss}, H_u-\mathrm{fg}} = X^{\mathrm{ss,fg}} = X^{\mathrm{nss}}$ by definition and $X \wr\!\wr H = \mathrm{Proj}(\Bbbk[X, L]^H)$ is a projective variety.

Any normal affine completion $\overline{G/H_u}^{\mathrm{aff}}$ of G/H_u admits an equivariant closed immersion into some $G \times H_r$-module W, say. As in the proof of Proposition 3.1.7, let $\Bbbk$ be a copy of the ground field equipped with the trivial $G \times H_r$-action and consider the closure $\overline{G/H_u}$ of $\overline{G/H_u}^{\mathrm{aff}}$ under the natural open immersion $W \hookrightarrow \mathbb{P} := \mathbb{P}(W \oplus \Bbbk)$. The complement $D_\infty := \overline{G/H_u} \setminus \overline{G/H_u}^{\mathrm{aff}}$ is an effective Cartier divisor corresponding to the hyperplane line bundle $\mathcal{O}_{\overline{G/H_u}}(1) = \mathcal{O}_{\mathbb{P}}(1)|_{\overline{G/H_u}}$. Let $\nu : \widetilde{G/H_u} \to \overline{G/H_u}$ be the normalisation of $\overline{G/H_u}$. Then $\widetilde{G/H_u}$ naturally contains $\overline{G/H_u}^{\mathrm{aff}}$ as an open subset with $\widetilde{G/H_u} \setminus \overline{G/H_u}^{\mathrm{aff}} = \nu^{-1}(D_\infty)$, and because ν is a finite map $\nu^{-1}(D_\infty)$ is a divisor corresponding to the ample line bundle $\mathcal{O}_{\widetilde{G/H_u}}(1) = \nu^* \mathcal{O}_{\overline{G/H_u}}(1)$. The naturally induced $G \times H_r$-linearisation on $\mathcal{O}_{\overline{G/H_u}}(1) \to \overline{G/H_u}$ canonically defines a linearisation on the normalisation $\mathcal{O}_{\widetilde{G/H_u}}(1) \to \widetilde{G/H_u}$ [Ses63, Chapter 1], which pulls back to the canonical $G \times H_r$-linearisation on $\mathcal{O}_{\overline{G/H_u}^{\mathrm{aff}}} \to \overline{G/H_u}^{\mathrm{aff}}$.

Example 3.2.10. As a simple example, consider the situation where H is a linear algebraic group with $H_u \cong \mathbb{G}_a$ and $\rho : H \to G = \mathrm{SL}(2, \Bbbk)$ is an H_u-faithful homomorphism. Without loss of generality we may assume ρ maps H_u onto the subgroup of strictly upper-triangular matrices of G and, as is well known, H_u is a Grosshans subgroup of G. Indeed, consider the defining representation of G on $\Bbbk^2$. The orbit of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is $\Bbbk^2 \setminus \{0\}$ and has stabiliser equal to H_u, so that $G/H_u \cong \Bbbk^2 \setminus \{0\}$. This has a $G \times H_r$-equivariant normal affine completion $\overline{G/H_u}^{\mathrm{aff}} := \Bbbk^2$, containing G/H_u with codimension 2 complement. By adding a hyperplane D_∞ at infinity, we obtain an equivariant normal projective completion $\overline{G/H_u} = \mathbb{P}^2$ and then $\mathcal{O}(D_\infty) = \mathcal{O}_{\mathbb{P}^2}(1)$ has a natural $G \times H_r$-linearisation extending the one on $\mathcal{O} \to \Bbbk^2 \setminus \{0\}$.

To summarise: in the case where H_u is a Grosshans subgroup of G we may find a gentle, $G \times H_r$-equivariant completion $\overline{G/H_u}$ of G/H_u whose only codimension 1 boundary component is an effective Cartier divisor D_∞ corresponding to an ample, $G \times H_r$-linearised line bundle $\mathcal{O}_{\overline{G/H_u}}(1) \to \overline{G/H_u}$ which restricts to the canonical $G \times H_r$-linearisation on $\mathcal{O}_{G/H_u} \to G/H_u$. Given this, and assuming condition (C2), let

$$\beta : G \times^{H_u} X \cong (G/H_u) \times X \hookrightarrow \overline{G/H_u} \times X$$

be the resulting open immersion and let $L' = \mathcal{O}_{\overline{G/H_u}} \boxtimes \tilde{L} \to \overline{G/H_u} \times X$ be the $G \times H_r$-linearisation required by (C2). The only codimension 1 boundary component of $G \times^{H_u} X$ inside $\overline{G/H_u} \times X$ is the Cartier divisor $D_\infty \times X$, so for any $N > 0$ (3.2.1) yields the linearisation $L'_N = \mathcal{O}(ND_\infty) \boxtimes \tilde{L}$ and we obtain a triple

$$(\overline{G/H_u} \times X, \beta, \mathcal{O}(ND_\infty) \boxtimes \tilde{L}) \tag{3.2.3}$$

such that $\overline{G/H_u} \times X$ is a gentle completion of $G \times^{H_u} X$ and the $G \times H_r$-linearisation $L'_N \to \overline{G/H_u} \times X$ extends the H-linearisation $L \to X$ under $\beta \circ \alpha$. In the case where $L \to X$ is ample, we obtain the following corollary to Proposition 3.2.6 and Corollary 3.2.7.

Corollary 3.2.11. *Let H be a connected linear algebraic group acting on a normal, irreducible projective variety X with ample linearisation $L \to X$. Suppose there is a connected reductive group*

G with an H_u-faithful homomorphism $H \to G$ such that H_u embeds into a Grosshans subgroup of G and conditions (C1) *and* (C2) *hold. Then there is a gentle $G \times H_r$-equivariant projective completion $\overline{G/H_u}$ of G/H_u whose only codimension 1 boundary component is an ample, effective Cartier divisor D_∞. Furthermore, given any such completion $\overline{G/H_u}$ of G/H_u, for sufficiently large $N > 0$ the triple $(\overline{G/H_u} \times X, \beta, L'_N = \mathcal{O}(ND_\infty) \boxtimes \tilde{L})$ of* (3.2.3) *is an ample strong reductive envelope for $H \curvearrowright L \to X$, and*

$$X \wr\wr H \cong (\overline{G/H_u} \times X) /\!\!/_{L'_N} (G \times H_r).$$

Remark 3.2.12. Because the various intrinsic notions of conventional reductive GIT and non-reductive GIT may be defined for rational linearisations, one can work with rational linearisations in the setting of reductive envelopes. For example, in Corollary 3.2.11 if one assumes $H \curvearrowright L \to X$ and $\tilde{L} \to X$ are rational linearisations satisfying the natural rational versions of (C1) and (C2), then for $N \gg 0$ some positive integral multiple of the rational linearisation $L'_N = \mathcal{O}(ND_\infty) \boxtimes \tilde{L} \to \overline{G/H_u} \times X$ will define a strong reductive envelope for the corresponding multiple of $L \to X$. The stable locus, finitely generated semistable locus and enveloping quotient for $H \curvearrowright L \to X$ can thus still be computed using the rational linearisation L'_N, which is often more convenient to work with in computations.

Corollary 3.2.11 is a useful result that will be used a number of times later in this thesis, beginning with the extended example in the upcoming Section 3.3.

3.3 An Example: n Unordered Points on $\mathbb{P}^1$

In this section we undertake a detailed study of an example to demonstrate the use of strong reductive envelopes for computing the semistable and stable loci in a non-reductive GIT set-up, and to construct compactifications of the enveloped quotient. We will work over the field $\Bbbk = \mathbb{C}$ for concreteness, though in fact the computations hold over any algebraically closed field of characteristic zero. We will study the space of n unordered points on $\mathbb{P}^1$ up to compositions of translations and dilations. This extends Example 1.3.18 from [DK07, §6], which only looked at the translation actions. Including the dilations gives a somewhat richer picture, due to the possibility of variation of linearisations and their associated birational transformations on the quotients (as we shall explore in Section 3.3.3).

We first fix some notation. Let $G := \mathrm{SL}(2,\mathbb{C})$ and consider its action on $\mathbb{P}^1$ via Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{P}^1 \to \mathbb{P}^1, \quad [z_0 : z_1] \mapsto [az_0 + bz_1 : cz_0 + dz_1], \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

Fix an integer $n > 0$. Then G acts dually on the complete linear system associated to $\mathcal{O}_{\mathbb{P}^1}(n)$, which we identify as $X := \mathbb{P}(V)$ with $V := \mathbb{C}[x,y]_n$ the vector space of homogeneous polynomials of degree n in two indeterminates x and y. We write points of X as $[\sigma(x,y)]$ with $\sigma(x,y) \in V$; note that $[\sigma(x,y)]$ defines an effective divisor of zeros on $\mathbb{P}^1$, which can be thought of as a collection of n unordered points on $\mathbb{P}^1$ with multiplicities.

We consider the action of the subgroup

$$H := \left\{ \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \;\middle|\; t \in \mathbb{C}^\times,\ a \in \mathbb{C} \right\}$$

of G on X. Geometrically H corresponds to the Möbius transformations on $\mathbb{P}^1$ that are compositions of scalings and translations, all of which fix $\infty = [1:0] \in \mathbb{P}^1$. The unipotent radical of H is

$$H_u = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{C} \right\} \cong \mathbb{C}^+$$

while the quotient $H_r = H/H_u$ is isomorphic to the torus

$$T := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \;\middle|\; t \in \mathbb{C}^\times \right\} \cong \mathbb{C}^\times$$

via the composition $T \hookrightarrow H \to H_r$. We will typically identify H_r with T in this way throughout the example.

We next consider the possible linearisations L of H over X. As line bundles, any linearisation is isomorphic to $\mathcal{O}_X(m)$ for $m \in \mathbb{Z}$ [Har77, Corollary 6.17]. The action of H on V defines, for each $m \in \mathbb{Z}$, a canonical linearisation $\mathcal{O}_X(m)^{\mathrm{can}} \to X$ on the line bundle $\mathcal{O}_X(m)$. Because X is proper over $\mathbb{C}$ and irreducible, any other linearisation on $\mathcal{O}_X(m)$ is obtained by twisting the canonical linearisation by a character of H [Dol03, Corollary 7.1].

We take a moment to recall our conventions here: if $\chi : H \to \mathbb{C}^\times$ is a character of H, then let $\mathcal{O}_X^{(\chi)}$ denote the linearisation on the trivial bundle $\mathcal{O}_X = X \times \mathbb{C}$ defined by the character χ^{-1}:

$$H \times X \times \mathbb{C} \to X \times \mathbb{C}, \quad (h, x, z) \mapsto (hx, \chi^{-1}(h)z).$$

Then any other linearisation on $\mathcal{O}_X(m)$ is of the form $\mathcal{O}_X(m)^{(\chi)} := \mathcal{O}_X(m)^{\mathrm{can}} \otimes \mathcal{O}_X^{(\chi)}$. Because unipotent groups have no nontrivial characters, the inclusion $T \hookrightarrow H$ induces an identification between the groups of characters of H and T, so that χ is of the form

$$\chi : \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \mapsto t^r, \quad \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \in H$$

for some weight $r \in \mathbb{Z}$. For each $m, r \in \mathbb{Z}$ we therefore define linearisations

$$L_{m,r} := \mathcal{O}_X(m)^{(\chi)}, \quad \text{where } \chi \text{ has weight } r \in \mathbb{Z}.$$

of H over X. It is these linearisations that we will study in this example.

When $m \leq 0$ the sets $X^{\mathrm{s}(L_{m,r})}$, $X^{\mathrm{nss}(L_{m,r})}$, $X^{\mathrm{ss,fg}(L_{m,r})}$, $X^{\mathrm{ss},H_u-\mathrm{fg}(L_{m,r})}$ and the enveloping quotient $X \wr\!\wr_{L_{m,r}} H$ for the linearisations $L_{m,r}$ can be easily described by inspection. Indeed, if $m < 0$ then $\mathcal{O}_X(m)$ has no nonzero global sections as a line bundle, so that the stable and naively semistable locus, and enveloping quotient, are all empty. On the other hand, if $m = 0$, then the ring of invariants $\mathbb{C}[X, L_{0,r}]^H$ is isomorphic to $\mathrm{Sym}^\bullet \mathbb{C}$ if $r = 0$, and $\mathbb{C}$ (in degree 0) otherwise. Thus when $r = 0$, we have $X^{\mathrm{s}(L_{0,0})} = \emptyset$ and $X^{\mathrm{ss},H_u-\mathrm{fg}(L_{0,0})} = X^{\mathrm{ss,fg}(L_{0,0})} = X^{\mathrm{nss}(L_{0,0})} = X$, while $X \wr\!\wr_{L_{0,0}} H = \mathrm{pt}$; on the other hand, if $r \neq 0$ then all of $X^{\mathrm{s}(L_{0,r})}$, $X^{\mathrm{ss},H_u-\mathrm{fg}(L_{0,r})}$, $X^{\mathrm{ss,fg}(L_{0,r})}$, $X^{\mathrm{nss}(L_{0,r})}$ and $X \wr\!\wr_{L_{0,r}} H$ are empty.

In what follows we therefore consider the linearisations $L_{m,r}$ with $m > 0$. In the next section we shall use the methods of Corollary 3.2.11 of Section 3.2.1 to construct strong ample reductive envelopes for these linearisations, which will allow us to compute the stable locus $X^{\mathrm{s}(L_{m,r})}$, the various semistable loci $X^{\mathrm{ss},H_u-\mathrm{fg}(L_{m,r})}$, $X^{\mathrm{ss,fg}(L_{m,r})}$, $X^{\mathrm{nss}(L_{m,r})}$ (which will all be equal) and a compactification of the enveloped quotient; cf. Proposition 3.2.2 and Theorem 3.1.14.

3.3.1 The Strong Reductive Envelopes

Fix $m, r \in \mathbb{Z}$, with $m > 0$, and let $\chi : H \to \mathbb{C}^\times$ be the character of H of weight r. Consider the inclusion $H \hookrightarrow G$, which is clearly an H_u-faithful homomorphism. By construction, the restricted linearisation of the unipotent radical $H_u \curvearrowright L_{m,r} \to X$ extends to a linearisation of $G = \mathrm{SL}(2, \mathbb{C})$, and we are in the setting of Example 3.2.8. Furthermore, as we saw in Example 3.2.10, H_u is a Grosshans subgroup of G, and $G/H_u \cong \mathbb{C}^2 \setminus \{0\}$ via the defining representation of G on $\mathbb{C}^2$. This has a $G \times H_r$-equivariant normal affine completion $\overline{G/H_u}^{\mathrm{aff}} := \mathbb{C}^2$, containing G/H_u with codimension 2 complement.

Remark 3.3.1. Because the restricted linearisation $H_u \curvearrowright L_{m,r} \to X$ extends to one of $G = \mathrm{SL}(2, \mathbb{C})$ and H_u is a Grosshans subgroup of G, by Remark 3.2.9 the ring of invariants $\mathbb{C}[X, L_{m,r}]^H$ is finitely generated, so that $X \wr\!\wr_{L_{m,r}} H = \mathrm{Proj}(\Bbbk[X, L_{m,r}]^H)$ is a projective variety. We also have $X^{\mathrm{ss}, H_u-\mathrm{fg}(L_{m,r})} = X^{\mathrm{ss,fg}(L_{m,r})} = X^{\mathrm{nss}(L_{m,r})}$.

In what follows we regard elements of $\mathbb{C}^3$ as column vectors. As in Example 3.2.10, by adding a hyperplane at infinity we obtain a normal (in fact, smooth) $G \times H_r$-equivariant projective completion $\mathbb{P}^2$ of $\overline{G/H_u}^{\mathrm{aff}}$. Here we write $\mathbb{P}^2 = \{[v_0 : v_1 : v_2] \mid 0 \neq (v_0, v_1, v_2)^{\mathrm{t}} \in \mathbb{C}^3\}$ with the hyperplane at infinity defined by $v_0 = 0$. The action of $G \times H_r = G \times T$ on $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^3)$ is the one induced by the representation given in block form

$$(g, \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)) \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix} \end{array}\right) \in \mathrm{GL}(3, \mathbb{C}), \quad g \in G,\ \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \in T$$

where $\mathrm{GL}(3, \mathbb{C})$ acts on $\mathbb{C}^3$ by left multiplication. This representation canonically defines a $G \times H_r$-linearisation $\mathcal{O}_{\mathbb{P}^2}(N) \to \mathbb{P}^2$ which restricts to the canonical linearisation on $\mathcal{O}_{G/H_u} \to G/H_u$, for each $N > 0$.

Let $\beta : G \times^{H_u} X \cong (\mathbb{C}^2 \setminus \{0\}) \times X \hookrightarrow \mathbb{P}^2 \times X$ be the naturally induced open immersion and for integers $N > 0$ let

$$L'_{m,r,N} := \mathcal{O}_{\mathbb{P}^2}(N) \boxtimes L_{m,r} \to \mathbb{P}^2 \times X,$$

equipped with its natural $G \times H_r$-linearisation. By Corollary 3.2.11 the triple

$$(\mathbb{P}^2 \times X, \beta, L'_{m,r,N})$$

defines a strong ample reductive envelope for $H \curvearrowright L_{m,r} \to X$, for $N > 0$ sufficiently large depending on m and r. The stable locus $X^{\mathrm{s}(L_{m,r})}$ and finitely generated semistable locus $X^{\mathrm{ss,fg}(L_{m,r})}$ for the linearisation $L_{m,r}$ may therefore be computed as the completely stable and completely semistable loci, respectively, associated to the $G \times H_r$-linearisation $L'_{m,r,N}$ by Proposition 3.2.2. We also have

$$X \wr\!\wr_{L_{m,r}} H = \operatorname{Proj}(\Bbbk[X, L_{m,r}]^H) = (\mathbb{P}^2 \times X) /\!\!/_{L'_{m,r,N}} (G \times H_r),$$

by Corollary 3.2.11. We therefore next compute the semistable and stable loci for $L'_{m,r,N} \to \mathbb{P}^2 \times X$.

3.3.2 Semistability and Stability for $L'_{m,r,N} \to \mathbb{P}^2 \times X$

In order to compute semistability and stability for the $G \times H_r$-linearisations $L'_{m,r,N}$ over $\mathbb{P}^2 \times X$ we will use the Hilbert-Mumford criterion as stated in Theorem 1.2.6. To do this we use the maximal torus $T_1 \times T_2, \subseteq G \times H_r$, where T_1 is the maximal torus T of $G = \mathrm{SL}(2, \mathbb{C})$ and $T_2 := T \cong H_r$. The group of characters of $T_1 \times T_2$ is then identified with $\mathbb{Z} \times \mathbb{Z}$ in the natural way.

The set of fixed points for the action $T_1 \times T_2$-action on $\mathbb{P}^2 \times X$ is

$$\{([1:0:0], [x^{n-i}y^i]),\ ([0:1:0], [x^{n-i}y^i]),\ ([0:0:1], [x^{n-i}y^i]) \mid i = 0, \dots, n\}.$$

Table 3.1 gives the weights for each fixed point with respect to the linearisation $L'_{m,r,N}$ and a general plot of these weights is given in Figure 3.1.

Fixed point ($i = 0, \dots, n$)	Weight in $\mathrm{Hom}(T_1 \times T_2, \mathbb{C}^\times) = \mathbb{Z} \times \mathbb{Z}$
$([1:0:0], [x^{n-i}y^i])$	$(m(2i-n), r)$
$([0:1:0], [x^{n-i}y^i])$	$(N + m(2i-n), -N + r)$
$([0:0:1], [x^{n-i}y^i])$	$(-N + m(2i-n), -N + r)$

Table 3.1: Weights of the fixed points of $T_1 \times T_2 \curvearrowright \mathbb{P}^2 \times X$ with respect to the linearisation $L'_{m,r,N}$.

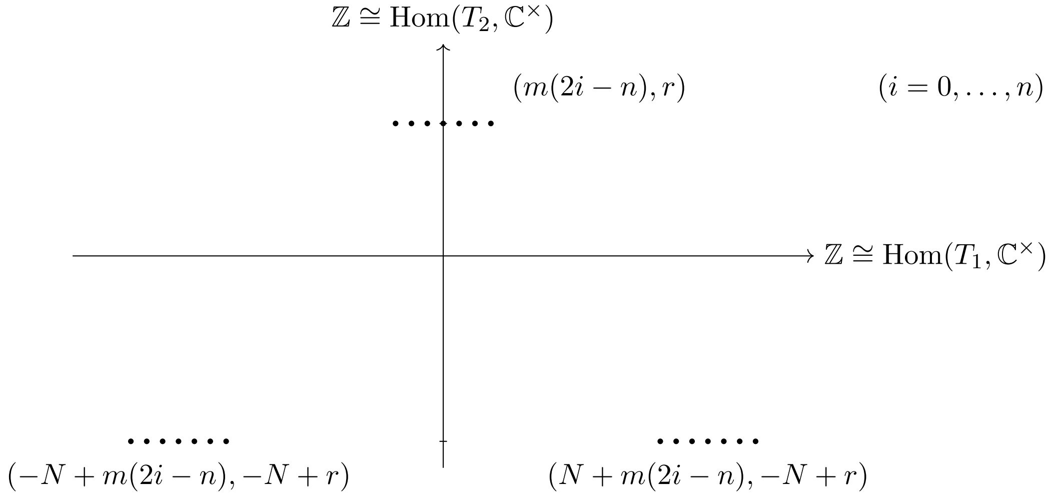

Figure 3.1: The weight diagram for $T_1 \times T_2 \curvearrowright L'_{m,r,N} \to \mathbb{P}^2 \times X$.

By the Hilbert-Mumford criterion Theorem 1.2.6, a point

$$p = ([v_0 : v_1 : v_2], [\sigma(x,y)]) \in \mathbb{P}^2 \times X$$

is semistable (respectively, stable) for the restricted linearisation $T_1 \times T_2 \curvearrowright L'_{m,r,N} \to \mathbb{P}^2 \times X$ if, and only if, the origin of $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}) = \mathbb{R} \times \mathbb{R}$ is contained in the weight polytope $\Delta_p \subseteq \mathbb{R} \times \mathbb{R}$ (respectively, the interior Δ_p°) associated to p. Since we are only interested in taking N very large with respect to m and r, we find that p can only be $T_1 \times T_2$-(semi)stable by satisfying the following criteria, split into four cases:

Case $v_0 v_1 v_2 \neq 0$:

$$0 \in \Delta_p \iff \begin{array}{l} [1:0] \text{ and } [0:1] \text{ are both zeros of } \sigma(x,y) \\ \text{with multiplicity } \leq (n + \frac{r}{m})/2 \end{array}$$

$$0 \in \Delta_p^\circ \iff \begin{array}{l} [1:0] \text{ and } [0:1] \text{ are both zeros of } \sigma(x,y) \\ \text{with multiplicity } < (n + \frac{r}{m})/2 \end{array}$$

Case $v_0 v_1 \neq 0$, $v_2 = 0$:

$$0 \in \Delta_p \iff \begin{array}{l} [1:0] \text{ is a zero of } \sigma(x,y) \text{ with multiplicity } \leq (n - \frac{r}{m})/2 \text{ and} \\ [0:1] \text{ is a zero of } \sigma(x,y) \text{ with multiplicity } \leq (n + \frac{r}{m})/2 \end{array}$$

$$0 \in \Delta_p^\circ \iff \begin{array}{l} [1:0] \text{ is a zero of } \sigma(x,y) \text{ with multiplicity } < (n - \frac{r}{m})/2 \text{ and} \\ [0:1] \text{ is a zero of } \sigma(x,y) \text{ with multiplicity } < (n + \frac{r}{m})/2 \end{array}$$

Case $v_0 v_2 \neq 0$, $v_1 = 0$:

$$0 \in \Delta_p \iff \begin{array}{l} [1:0] \text{ is a zero of } \sigma(x,y) \text{ with multiplicity } \leq (n+\frac{r}{m})/2 \text{ and} \\ [0:1] \text{ is a zero of } \sigma(x,y) \text{ with multiplicity } \leq (n-\frac{r}{m})/2 \end{array}$$

$$0 \in \Delta_p^\circ \iff \begin{array}{l} [1:0] \text{ is a zero of } \sigma(x,y) \text{ with multiplicity } < (n+\frac{r}{m})/2 \text{ and} \\ [0:1] \text{ is a zero of } \sigma(x,y) \text{ with multiplicity } < (n-\frac{r}{m})/2 \end{array}$$

Case $v_0 \neq 0$, $v_1 = v_2 = 0$ and $r = 0$:

$$0 \in \Delta_p \iff \begin{array}{l} [1:0] \text{ and } [0:1] \text{ are both zeros of } \sigma(x,y) \\ \text{with multiplicity } \leq n/2 \end{array}$$

$$0 \in \Delta_p^\circ \iff \begin{array}{l} [1:0] \text{ and } [0:1] \text{ are both zeros of } \sigma(x,y) \\ \text{with multiplicity } < n/2 \end{array}$$

By the Hilbert-Mumford criterion, the point p is (semi)stable for the whole $G \times H_r$-linearisation if, and only if, $(g, \overline{h}) \cdot p$ is $T_1 \times T_2$-(semi)stable for each $(g, \overline{h}) \in G \times H_r$. Using this, one deduces the following:

Proposition 3.3.2. *Let $m > 0$ and r be integers. Then for sufficiently large $N > 0$ (depending on m and r) the semistable and stable loci for the $G \times H_r$-linearisations $L'_{m,r,N} \to \mathbb{P}^2 \times X$ are as follows:*

(Case $r < 0$ or $\frac{r}{m} > n$): *Then $(\mathbb{P}^2 \times X)^{\mathrm{s}(L'_{m,r,N})} = (\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,r,N})} = \emptyset$.*

(Case $r = 0$): *Then $(\mathbb{P}^2 \times X)^{\mathrm{s}(L'_{m,r,N})} = \emptyset$ and*

$$(\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,r,N})} = \left\{ ([1:v_1:v_2], [\sigma(x,y)]) \;\middle|\; \begin{array}{c} \sigma(x,y) \text{ has no zeros} \\ \text{of multiplicity} > \frac{n}{2} \end{array} \right\}.$$

(Case $0 < \frac{r}{m} < n$): *Then*

$$(\mathbb{P}^2 \times X)^{\mathrm{s}(L'_{m,r,N})} = \left\{ ([1:v_1:v_2], [\sigma(x,y)]) \;\middle|\; \begin{array}{c} (v_1,v_2) \neq (0,0), [v_1:v_2] \text{ is a zero of} \\ \sigma(x,y) \text{ of multiplicity} < (n-\frac{r}{m})/2 \\ \text{and all zeros of } \sigma(x,y) \text{ have} \\ \text{multiplicity} < (n+\frac{r}{m})/2 \end{array} \right\},$$

$$(\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,r,N})} = \left\{ ([1:v_1:v_2], [\sigma(x,y)]) \;\middle|\; \begin{array}{c} (v_1,v_2) \neq (0,0), [v_1:v_2] \text{ is a zero of} \\ \sigma(x,y) \text{ of multiplicity} \leq (n-\frac{r}{m})/2 \\ \text{and all zeros of } \sigma(x,y) \text{ have} \\ \text{multiplicity} \leq (n+\frac{r}{m})/2 \end{array} \right\}.$$

(Case $\frac{r}{m} = n$): *Then $(\mathbb{P}^2 \times X)^{\mathrm{s}(L'_{m,r,N})} = \emptyset$ and*

$$(\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,r,N})} = \left\{ ([1:v_1:v_2], [\sigma(x,y)]) \;\middle|\; \begin{array}{c} [v_1:v_2] \text{ is not a} \\ \text{zero of } \sigma(x,y) \end{array} \right\}.$$

For fixed m, r, N the completely semistable and completely stable loci are by Definition 3.1.9 the intersections of the semistable and stable loci for $G \times H_r \curvearrowright L'_{m,r,N} \to \mathbb{P}^2 \times X$ under the inclusion

$$\beta \circ \alpha : X \hookrightarrow \mathbb{P}^2 \times X, \quad [\sigma(x,y)] \mapsto ([1:1:0], [\sigma(x,y)]),$$

Using Corollary 3.2.11, we therefore deduce

Corollary 3.3.3. *For integers $m > 0$ and r, the semistable and stable loci for the linearisations $H \curvearrowright L_{m,r} \to X$ are as follows:*

(Case $r < 0$ or $\frac{r}{m} > n$): *Then $X^{\mathrm{s}(L_{m,r})} = X^{\mathrm{ss,fg}(L_{m,r})} = \emptyset$.*

(Case $r = 0$): *Then $X^{\mathrm{s}(L_{m,r})} = \emptyset$ and*

$$X^{\mathrm{ss,fg}(L_{m,r})} = \{[\sigma(x,y)] \in X \mid \sigma(x,y) \textit{ has no zeros of multiplicity} > n/2\}$$

(Case $0 < \frac{r}{m} < n$): *Then*

$$X^{\mathrm{s}(L_{m,r})} = \left\{ [\sigma(x,y)] \in X \;\middle|\; \begin{array}{c} [1:0] \textit{ is a zero of } \sigma(x,y) \textit{ with multiplicity} \\ < (n - \frac{r}{m})/2 \textit{ and all other zeros} \\ \textit{of } \sigma(x,y) \textit{ have multiplicity} < (n + \frac{r}{m})/2 \end{array} \right\},$$

$$X^{\mathrm{ss,fg}(L_{m,r})} = \left\{ [\sigma(x,y)] \in X \;\middle|\; \begin{array}{c} [1:0] \textit{ is a zero of } \sigma(x,y) \textit{ with multiplicity} \\ \leq (n - \frac{r}{m})/2 \textit{ and all other zeros} \\ \textit{of } \sigma(x,y) \textit{ have multiplicity} \leq (n + \frac{r}{m})/2 \end{array} \right\}.$$

(Case $\frac{r}{m} = n$): *Then $X^{\mathrm{s}(L_{m,r})} = \emptyset$ and*

$$X^{\mathrm{ss,fg}(L_{m,r})} = \{[\sigma(x,y)]) \in X \mid [1:0] \textit{ is not a zero of } \sigma(x,y)\}.$$

3.3.3 Variation of the Enveloping Quotients

We conclude this example by studying the variation of the enveloping quotients $X /\!\!/_{L_{m,r}} H$ as we range over the different possible ample linearisations $L_{m,r} \to X$. This is closely related to studying the variation of the reductive GIT quotients (in the sense of the VGIT of [Tha96, DH98]) of the linearisations $L'_{m,r,N} \to \mathbb{P}^2 \times X$. In order to keep the exposition brief we focus only on the non-reductive linearisations $L_{m,r} \to X$ and reference the results required from VGIT without further justification.

We shall now always assume N is sufficiently large with respect to m and r to satisfy the conclusions of Corollary 3.2.11 and Proposition 3.3.2. Note that two linearsations $L_{m,r}$ and $L_{m',r'}$ will have the same stable locus, finitely generated semistable locus and enveloping quotient if $\frac{r}{m} = \frac{r'}{m'}$ (see Remarks 1.1.19 and 2.4.3). By inspecting Corollary 3.3.3 we see that the changes in stability and finitely generated semistability occur when $\frac{r}{m} = 0$, or $n - \frac{r}{m} \in 2\mathbb{Z}$, or $\frac{r}{m} = n$. (Clearly we only need consider the cases where $\frac{r}{m} \in \mathbb{Q} \cap [0, n]$.) It makes sense therefore to consider four cases: (1) when $r = 0$; (2) when $0 < \frac{r}{m} < n$ and $n - \frac{r}{m} \in \mathbb{Q} \setminus 2\mathbb{Z}$; (3) when $0 < \frac{r}{m} < n$ and $n - \frac{r}{m} \in 2\mathbb{Z}$; and (4) when $\frac{r}{m} = n$.

3.3.3.1 Case $r = 0$

Observe from Corollary 3.3.3 that $X^{\mathrm{ss,fg}(L_{m,0})}$ is equal to the semistable locus for the canonical $G = \mathrm{SL}(2, \mathbb{C})$-linearisation on $\mathcal{O}_X(1) \to X = \mathbb{P}(V)$ [Dol03, §10.2] (which is the classical reductive GIT problem of configurations of n unordered points on $\mathbb{P}^1$ up to *all* Möbius transformations). Notice also that there is a G-equivariant retraction

$$(\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,0,N})} = \mathbb{C}^2 \times X^{\mathrm{ss,fg}(L_{m,0})} \to \{0\} \times X^{\mathrm{ss,fg}(L_{m,0})} \cong X^{\mathrm{ss,fg}(L_{m,0})}$$

(where $\mathbb{C}^2 \subseteq \mathbb{P}^2$ corresponds to the gentle affine completion $\overline{G/H_u}^{\mathrm{aff}}$ of G/H_u), defined by taking a limit along the flow of $t \in T_2 \cong \mathbb{C}^\times$ as $t \to \infty$. Thus two points $[\sigma(x, y)], [\tilde{\sigma}(x, y)]$ from $X^{\mathrm{ss,fg}(L_{m,0})}$ get identified in $X \wr\wr_{L_{m,0}} H = (\mathbb{P}^2 \times X) /\!\!/_{L'_{m,0,N}} (G \times H_r)$ if, and only if, $[\sigma(x, y)]$ and $[\tilde{\sigma}(x, y)]$ are S-equivalent for the standard action of G on X (see Section 1.2). Writing $X /\!\!/ G$ for the GIT quotient of the canonical linearisation $G \curvearrowright \mathcal{O}_X(1) \to X$, it follows that the inclusion $X^{\mathrm{ss,fg}(L_{m,0})} \hookrightarrow (\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,0,N})}$ induces an isomorphism

$$X \wr\wr_{L_{m,0}} H \cong X /\!\!/ G.$$

In particular, we see that the dimension of $X \wr\wr_{L_{m,0}} H$ is one less than the anticipated dimension.

3.3.3.2 Case $0 < \frac{r}{m} < n$ and $n - \frac{r}{m} \in \mathbb{Q} \setminus 2\mathbb{Z}$

In this case we see from Proposition 3.3.2 and Corollary 3.3.3 that $(\mathbb{P}^2 \times X)^{\mathrm{s}(L'_{m,r,N})} = (\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,r,N})}$ and $X^{\mathrm{ss,fg}(L_{m,r})} = X^{\mathrm{s}(L_{m,r})}$. Moreover, by inspection we see that

$$(\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,r,N})} \subseteq (\mathbb{C}^2 \setminus \{0\}) \times X \cong G \times^{H_u} X,$$

thus $(\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,r,N})} \cong G \times^{H_u} (X^{\mathrm{ss,fg}(L_{m,r})})$ and $(\mathbb{P}^2 \times X)^{\mathrm{s}(L'_{m,r,N})} \cong G \times^{H_u} (X^{\mathrm{s}(L_{m,r})})$. From Corollary 3.2.11 we thus have

$$X \wr\!\wr_{L_{m,r}} H = (\mathbb{P}^2 \times X) /\!\!/_{L'_{m,r,N}} (G \times H_r) = (\mathbb{P}^2 \times X)^{\mathrm{s}(L'_{m,r,N})}/(G \times H_r) = X^{\mathrm{s}(L_{m,r})}/H.$$

In particular, the enveloping quotient map $X^{\mathrm{ss,fg}(L_{m,r})} \to X \wr\!\wr_{L_{m,r}} H$ is a geometric quotient of $X^{\mathrm{ss,fg}(L_{m,r})}$ and the enveloped quotient is equal to the enveloping quotient, which itself is the canonical choice of inner enveloping quotient (Definition 2.1.11). Indeed, the quotient of $X^{\mathrm{ss,fg}(L_{m,r})}$ by H is even projective. So in the case $0 < \frac{r}{m} < n$ and $n - \frac{r}{m} \in \mathbb{Q} \setminus 2\mathbb{Z}$ we obtain the best possible geometric picture we could hope for.

3.3.3.3 Case $0 < \frac{r}{m} < n$ and $n - \frac{r}{m} \in 2\mathbb{Z}$

Now $\frac{r}{m}$ lies on a 'wall' (in the sense of Thaddeus [Tha96, Theorem 2.3]) and $X^{\mathrm{s}(L_{m,r})}$ is a proper subset of $X^{\mathrm{ss,fg}(L_{m,r})}$. As in the above case $n - \frac{r}{m} \notin 2\mathbb{Z}$ we still have

$$(\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,r,N})} \cong G \times^{H_u} (X^{\mathrm{ss,fg}(L_{m,r})}), \quad (\mathbb{P}^2 \times X)^{\mathrm{s}(L'_{m,r,N})} \cong G \times^{H_u} (X^{\mathrm{s}(L_{m,r})})$$

and $X \wr\!\wr_{L_{m,r}} H = (\mathbb{P}^2 \times X) /\!\!/_{L'_{m,r,N}} (G \times H_r)$. In particular, $X^{\mathrm{s}(L_{m,r})}/H \cong (\mathbb{P}^2 \times X)^{\mathrm{s}(L'_{m,r,N})}/(G \times H_r)$ and the enveloping quotient map $q : X^{\mathrm{ss,fg}(L_{m,r})} \to X \wr\!\wr_{L_{m,r}} H$ is surjective by Theorem 3.1.14 (which means that again the notions of enveloped quotient, inner enveloping quotient and enveloping quotient all coincide in this case). We next compute the complement of $X^{\mathrm{s}(L_{m,r})}/H$ inside $X \wr\!\wr_{L_{m,r}} H$. Let $[\sigma(x,y)] \in X^{\mathrm{ss,fg}(L_{m,r})} \setminus X^{\mathrm{s}(L_{m,r})}$. We claim that

$$[x^{(n+\frac{r}{m})/2} y^{(n-\frac{r}{m})/2}] \in \overline{H \cdot [\sigma(x,y)]} \cap X^{\mathrm{ss,fg}(L_{m,r})} \quad \text{(closure taken in } X\text{)}.$$

Indeed, by inspection of Corollary 3.3.3 either (1) $\sigma(x,y)$ has $[1:0]$ as a root of multiplicity $(n - \frac{r}{m})/2$; or (2) $\sigma(x,y)$ has a root $[u_1 : u_2] \neq [1:0]$ of multiplicity $(n + \frac{r}{m})/2$. In the first case, the limit of any zero of $\sigma(x,y)$ different to $[1:0]$ under the flow of $\left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{smallmatrix}\right) \in T_1 \subseteq G$ as $t_1 \to 0$ is equal to $[0:1]$, so $[x^{(n+\frac{r}{m})/2}y^{(n-\frac{r}{m})/2}] \in \overline{H \cdot [\sigma(x,y)]} \cap X^{\mathrm{ss,fg}(L_{m,r})}$. In the second case, there is $h \in H_u \cong \mathbb{C}^+$ taking $[u_1:u_2]$ to $[0:1]$, and any other zero of $\sigma(x,y)$ is taken to a point of the form $[v_1:v_2]$ with $v_1 \neq 0$. Any such $[v_1:v_2]$ flows to $[1:0]$ under $\left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{smallmatrix}\right)$ as $t_1 \to \infty$, so that $\overline{H \cdot [\sigma(x,y)]} \cap X^{\mathrm{ss,fg}(L_{m,r})}$ also contains $[x^{(n+\frac{r}{m})/2}y^{(n-\frac{r}{m})/2}]$. This proves our claim.

It follows that any two points of $X^{\mathrm{ss,fg}(L_{m,r})} \setminus X^{\mathrm{s}(L_{m,r})}$ are S-equivalent inside $(\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,r,N})}$ and so

$$X \wr\!\wr_{L_{m,r}} H = (X^{\mathrm{s}(L_{m,r})}/H) \amalg \mathrm{pt},$$

where pt is the image of $X^{\mathrm{ss,fg}(L_{m,r})} \setminus X^{\mathrm{s}(L_{m,r})}$ under the enveloping quotient map $q : X^{\mathrm{ss,fg}(L_{m,r})} \to X \wr\!\wr_{L_{m,r}} H$. Note that multiple orbits get collapsed to pt, so the enveloping quotient fails to be a geometric quotient.

3.3.3.4 Case $\frac{r}{m} = n$

In this case $X^{\mathrm{s}(L_{m,r})} = \emptyset$, while for $X^{\mathrm{ss,fg}(L_{m,r})}$ a similar analysis to the above case $0 < \frac{r}{m} < n$ and $n - \frac{r}{m} \in 2\mathbb{Z}$ holds: again we have $(\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,r,N})} \cong G \times^{H_u} (X^{\mathrm{ss,fg}(L_{m,r})})$, and any $[\sigma(x,y)] \in X^{\mathrm{ss}(L_{m,r})}$ has limit point equal to $[x^n]$ inside $X^{\mathrm{ss}(L_{m,r})}$ under the action of T_1. So we see that any two points in $X^{\mathrm{ss,fg}(L_{m,r})}$ are S-equivalent inside $(\mathbb{P}^2 \times X)^{\mathrm{ss}(L'_{m,r,N})}$. Since $X \wr\!\wr_{L_{m,r}} H = (\mathbb{P}^2 \times X)/\!\!/_{L'_{m,r,N}}(G \times H_r)$, we deduce that

$$X \wr\!\wr_{L_{m,r}} H = \mathrm{pt}.$$

3.3.3.5 Birational Transformations of $X^{\mathrm{s}(L_{m,r})}/H$

Finally, we examine the birational transformations induced on the geometric quotients $X^{\mathrm{s}(L_{m,r})}/H$ as $\frac{r}{m}$ crosses the 'walls' of integers congruent to n modulo 2 between 0 and n, or else equal to 0 or n.

We first consider the case where $\frac{r}{m} \in \mathbb{Z} \cap (0,n)$ and $\frac{r}{m} \equiv n \mod 2$, with $n \geq 3$. Let $0 < \epsilon = \frac{r'}{m'} < 1$ be a small rational number, with $r', m' > 0$, and let

$$L^+_{m,r} \to X, \quad L^-_{m,r} \to X$$

be the H-linearisations obtained by perturbing $L_{m,r}$ by $\pm\epsilon$, respectively (i.e. $L^{\pm}_{m,r}$ corresponds to the rational number $\frac{r}{m} \pm \epsilon$). By inspecting Corollary 3.3.3 we see there are inclusions

$$X^{\mathrm{s}(L^{-}_{m,r})} \subseteq X^{\mathrm{ss,fg}(L_{m,r})} \supseteq X^{\mathrm{s}(L^{+}_{m,r})}, \quad X^{\mathrm{s}(L^{-}_{m,r})} \supseteq X^{\mathrm{s}(L_{m,r})} \subseteq X^{\mathrm{s}(L^{+}_{m,r})},$$

from which we obtain surjective proper birational morphisms

$$\psi_{-} : X \wr\!\wr_{L^{-}_{m,r}} H \to X \wr\!\wr_{L_{m,r}} H,$$

$$\psi_{+} : X \wr\!\wr_{L^{+}_{m,r}} H \to X \wr\!\wr_{L_{m,r}} H$$

fitting into the following commutative diagram (with all unmarked inclusions natural open immersions):

$$\begin{tikzcd}
X^{\mathrm{s}(L^{-}_{m,r})} \arrow[r, phantom, "\subseteq"] \arrow[d] & X^{\mathrm{ss,fg}(L_{m,r})} \arrow[r, phantom, "\supseteq"] \arrow[d, "q"] & X^{\mathrm{s}(L^{+}_{m,r})} \arrow[d] \\
X \wr\!\wr_{L^{-}_{m,r}} H \arrow[r, two heads, "\psi_{-}"] & X \wr\!\wr_{L_{m,r}} H & X \wr\!\wr_{L^{+}_{m,r}} H \arrow[l, two heads, "\psi_{+}"'] \\
& X^{\mathrm{s}(L_{m,r})}/H \arrow[ul, hook] \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}$$

(Cf. [Tha96, Theorem 3.3].) If $n = 3$ then in fact

$$X \wr\!\wr_{L^{-}_{m,r}} H \overset{\psi_{-}}{\cong} X \wr\!\wr_{L_{m,r}} H \overset{\psi_{+}}{\cong} X \wr\!\wr_{L^{+}_{m,r}} H$$

are all isomorphic to $\mathbb{P}^1$.[2] Otherwise ψ_{-} and ψ_{+} are both small contractions, and the induced birational morphism

$$X \wr\!\wr_{L^{-}_{m,r}} H \dashrightarrow X \wr\!\wr_{L^{+}_{m,r}} H$$

is a blow-down of $E_{-} := \psi_{-}^{-1}(\mathrm{pt})$ followed by a blow-up of $E_{+} := \psi_{+}^{-1}(\mathrm{pt})$, where $\mathrm{pt} = X \wr\!\wr_{L_{m,r}} H \setminus (X^{\mathrm{s}(L_{m,r})}/H)$ [Tha96, Theorem 3.5].

In the case where $n \geq 4$ we claim that E_{+} and E_{-} are isomorphic to the following weighted projective spaces[3]: $E_{+} \cong \mathbb{P}(1, 2, \ldots, s)$ and $E_{-} \cong \mathbb{P}(1, 2, \ldots, n-s)$, where $s = (n - \frac{r}{m})/2$. Indeed, recall that

$$E_{-} = (X^{\mathrm{s}(L^{-}_{m,r})} \setminus X^{\mathrm{s}(L_{m,r})})/H, \quad E_{+} = (X^{\mathrm{s}(L^{+}_{m,r})} \setminus X^{\mathrm{s}(L_{m,r})})/H.$$

---

[2] To see this, note that $X \wr\!\wr_{L_{m,r}} H = (\mathbb{P}^2 \times X) /\!\!/_{L'_{m,r,N}} (G \times H_r)$ is one dimensional, hence isomorphic to $\mathbb{P}^1$ [Kem80]; similarly for the linearisations $L^{\pm}_{m,r}$.

[3] The fact that E_{+} and E_{-} are weighted projective spaces also follows from [Tha96, Theorem 5.6].

In the case of E_+, any $H_u \cong \mathbb{C}^+$-orbit in $X^{\mathrm{s}(L^+_{m,r})} \setminus X^{\mathrm{s}(L_{m,r})}$ contains a unique point $[\sigma(x,y)]$ such that $[0:1]$ is a zero of $\sigma(x,y)$ of multiplicity $n-s$ and $[1:0]$ is a zero of multiplicity $0 \leq l < s$. Thus the locally closed subset

$$Z_+ = \{[a_0x^n + a_1x^{n-1}y + \cdots + a_sx^{n-s}y^s] \in X \mid a_s \neq 0 \text{ and } a_i \neq 0 \text{ for some } 0 \leq i < s\}$$

of X provides a slice to the H_u-action on $X^{\mathrm{s}(L^+_{m,r})} \setminus X^{\mathrm{s}(L_{m,r})}$ which is stable under the T_1-action. Now $Z_+ \cong \mathbb{C}^s \setminus \{0\}$ in the obvious way and one can check that the corresponding T_1-action on $\mathbb{C}^s \setminus \{0\}$ has weights $-2s, -2s-2, \ldots, -2$. It follows that $E_+ = Z_+/T_1$ is isomorphic to $\mathbb{P}(1,2,\ldots,s)$. The proof that $E_- \cong \mathbb{P}(1,2,\ldots,n-s)$ is similar.

In the case where $r=0$, we of course have $X^{\mathrm{ss,fg}(L^-_{m,0})} = X^{\mathrm{s}(L^-_{m,0})} = \emptyset$ and $X^{\mathrm{s}(L^+_{m,0})} \subseteq X^{\mathrm{ss,fg}(L_{m,0})} = X^{\mathrm{ss}(G)}$, where recall $X^{\mathrm{ss}(G)}$ is the semistable locus for the canonical linearisation $G \curvearrowright \mathcal{O}_X(1) \to X$. In the case where n is even, we see that $X^{\mathrm{s}(G)} \subseteq X^{\mathrm{s}(L^+_{m,0})} \subseteq X^{\mathrm{ss}(G)} = X^{\mathrm{ss,fg}(L_{m,0})}$, so there is a commutative diagram

$$\begin{tikzcd}
X^{\mathrm{s}(L^+_{m,0})} \arrow[r, phantom, "\subseteq"] \arrow[d] & X^{\mathrm{ss}(G)} \arrow[r, phantom, "="] \arrow[d, "\text{S-equivalence}"] & X^{\mathrm{ss,fg}(L_{m,0})} \arrow[d] \\
X^{\mathrm{s}(L^+_{m,0})}/H \arrow[r, "\psi"] & X/\!\!/G \arrow[r, phantom, "="] & X \wr\!\wr_{L_{m,0}} H
\end{tikzcd}$$

with $\psi : X^{\mathrm{s}(L^+_{m,0})}/H \to X/\!\!/G$ surjective. The restriction $\psi : \psi^{-1}(X^{\mathrm{s}(G)}/G) \to X^{\mathrm{s}(G)}/G$ is a geometric quotient for the action of the unipotent subgroup $(H_u)^{\mathrm{opp}}$ of strictly lower triangular matrices in G opposite to H_u. On the other hand, the same computation as that of E_+ in the previous paragraph shows that the preimage of the unique point $X/\!\!/G \setminus (X^{\mathrm{s}(G)}/G)$ under ψ is equal to the weighted projective space $\mathbb{P}(1,2,\ldots,\frac{n}{2})$. When n is odd a similar diagram holds, except now $X^{\mathrm{s}(G)} = X^{\mathrm{ss}(G)}$ and $\psi : X^{\mathrm{s}(L^+_{m,0})}/H \to X/\!\!/G$ is a geometric quotient for the natural $(H_u)^{\mathrm{opp}}$-action on $X^{\mathrm{s}(L^+_{m,0})}/H$.

Lastly, the case where $\frac{r}{m} = n$ is trivial: we now have $X^{\mathrm{ss,fg}(L^+_{m,r})} = X^{\mathrm{s}(L^+_{m,r})} = \emptyset$ and $X^{\mathrm{s}(L^-_{m,r})} \subseteq X^{\mathrm{ss,fg}(L_{m,r})}$ induces the unique map $X \wr\!\wr_{L^-_{m,r}} H \to X \wr\!\wr_{L_{m,r}} H = \mathrm{pt}$.

Chapter 4

Quotienting in Stages

In Chapter 3 we considered ways of constructing compactifications of the enveloped quotient for a linearisation $L \to X$ of a linear algebraic group H by means of reductive envelopes, extending the work of Doran and Kirwan [DK07]. We showed in Proposition 3.1.7 that, in principle, we can always find an ample reductive envelope, whose associated reductive GIT quotient gives a compactification of the enveloped quotient and such that the associated completely semistable set $X^{\overline{\mathrm{ss}}}$ and completely stable set $X^{\overline{\mathrm{s}}}$ give good approximations to the intrinsically defined finitely generated semistable locus and stable locus:

$$X^{\overline{\mathrm{s}}} \subseteq X^{\mathrm{s}} \subseteq X^{\mathrm{ss,fg}} \subseteq X^{\overline{\mathrm{ss}}};$$

see Theorem 3.1.14. In general, however, it can be very difficult to find an ample reductive envelope that is easy to work with in practice; the reductive group constructed in Proposition 3.1.7 to create the reductive envelope there is hopelessly large. The purpose of this chapter is to consider an alternative way in which we can construct compactifications of the enveloped quotient and obtain approximations to the sets X^{s} and $X^{\mathrm{ss,fg}}$, which does not rely on constructing an ample reductive envelope for the linearisation $L \to X$.

The basic idea is to additionally choose a subnormal series of the unipotent radical H_u of H,

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_m = H_u \trianglelefteq H_{m+1} = H \qquad (\mathcal{C})$$

and take enveloping quotients of open subsets of X 'in stages': first by the action of H_1, then successively by naturally induced actions of H_{j+1}/H_j, for $j = 1, \dots, m$. The details to make this more precise form the content of Section 4.1. A new kind of semistability is introduced in Definition 4.1.1, called *generalised semistability*, which is intrinsic to the data of the linearisation and choice of chain ($\mathcal{C}$). The generalised semistable locus $X^{\mathrm{ss},\mathcal{C}}$ is an H-invariant open subset of X that contains the stable locus X^{s} and is better equipped to handle the stage-by-stage quotienting process. Combined with this, we consider very positive embeddings of X into projective spaces $\mathbb{P}(V^*)$ defined by *generalised enveloping systems* V (Definition 4.1.2) and the induced chain of projection maps

$$\mathbb{P}(V^*) \dashrightarrow \mathbb{P}((V^{H_1})^*)) \dashrightarrow \cdots \dashrightarrow \cdots \dashrightarrow \mathbb{P}((V^{H_m})^*) = \mathbb{P}((V^{H_u})^*).$$

In this way we construct compactifications of enveloping quotients for each of the restricted linearisations $H_j \curvearrowright L \to X$ which receive induced ample H_{j+1}/H_j-linearisations. In particular, the compactification $\overline{\mathcal{U}_m}$ of the enveloped quotient $\mathcal{U}_m$ for the linearisation of the unipotent radical H_u receives a canonical ample $H_r = H/H_u$-linearisation, and we will see that the associated GIT quotient $\overline{\mathcal{U}_m}/\!\!/H_r$ gives a compactification of the enveloped quotient for the original linearisation of H (Proposition 4.1.5).

In Section 4.2 we apply these constructions to the case where X is projective and the chain ($\mathcal{C}$) consists of characteristic subgroups of H with H_{j+1}/H_j isomorphic to the additive group for $j \leq m-1$. In this case a theorem of Weitzenböck [Wei32] allows us to construct ample strong reductive envelopes for the natural H_{j+1}/H_j-linearisation on the projective space $\mathbb{P}((V^{H_j})^*)$ in the case where $j \leq m-1$. We use this to define H-stable open subsets U_1 and U_2 of X such that

$$U_1 \subseteq X^{\mathrm{s}} \subseteq X^{\mathrm{ss},\mathcal{C}} \subseteq U_2.$$

The subsets U_1 and U_2 are constructed from $\mathbb{P}((V^{H_j})^*)^{\mathrm{s}}$ and $\mathbb{P}((V^{H_j})^*)^{\mathrm{ss,fg}}$ for the H_{j+1}/H_j-linearisation, respectively, so one can try to approximate X^{s} and $X^{\mathrm{ss},\mathcal{C}}$ by using the theory of strong reductive envelopes on $\mathbb{P}((V^{H_j})^*)$.

The results in this chapter form a somewhat less complete story than those of previous chapters. Indeed, they are perhaps best thought of as sketching out a possible programme for making non-reductive geometric invariant theory more computationally effective in the case where the theory of reductive envelopes is not so readily available.

We continue with the basic notation of Chapters 2 and 3: that H is a linear algebraic group acting on a variety X and $L \to X$ is a linearisation for this action, with $S = \Bbbk[X, L]$ is the section ring. We will typically assume in addition that X is *irreducible* and *quasi-projective* and that $L \to X$ is an *ample* linearisation, though these assumptions will still appear in statements of results when they are needed.

4.1 Quotienting in Stages

In addition to the data of a linearisation $H \curvearrowright L \to X$ we assume we have fixed a chain of subnormal groups of H:

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_m = H_u \trianglelefteq H_{m+1} = H, \tag{$\mathcal{C}$}$$

where $m \geq 0$. For each $j = 0, \ldots, m$, let

$$Q_{j+1} := H_{j+1}/H_j$$

and for each $j = 0, \ldots, m+1$ let

$$q_{H_j} : X^{\mathrm{ss,fg}(H_j)} \to X \wr\wr H_j$$

be the enveloping quotient map for the restricted linearisation $H_j \curvearrowright L \to X$. Note that $1 = H_0 \trianglelefteq \cdots \trianglelefteq H_m$ defines a subnormal series for the unipotent radical H_u of H, and $Q_{m+1} = H_r$ is the reductive quotient H/H_r. We introduce a new notion of semistability which is more compatible with the idea of taking a quotient by H in stages using the chain ($\mathcal{C}$).

Definition 4.1.1. Let H be a linear algebraic group acting on a variety X with linearisation $L \to X$. The *generalised semistable locus* (with respect to the chain ($\mathcal{C}$)) is the subset

$$X^{\mathrm{ss},\mathcal{C}} := \bigcup_{f \in I^{\mathrm{ss},\mathcal{C}}} X_f$$

of X^{nss}, where

$$I^{\mathrm{ss},\mathcal{C}} := \left\{ f \in \bigcup_{r>0} H^0(X, L^{\otimes r})^H \;\middle|\; \begin{array}{c} (S^{H_j})_{(f)} \text{ is a finitely generated} \\ \Bbbk\text{-algebra for each } j = 1, \dots, m+1 \end{array} \right\}.$$

It follows immediately from the definition that $X^{\mathrm{ss},\mathcal{C}} \subseteq X^{\mathrm{ss},\mathrm{fg}(H_j)}$ for each $j = 1, \dots, m+1$. In particular,

$$X^{\mathrm{ss},\mathcal{C}} \subseteq X^{\mathrm{ss},H_u-\mathrm{fg}} \subseteq X^{\mathrm{ss},\mathrm{fg}},$$

and indeed when the subnormal series ($\mathcal{C}$) is equal to the chain

$$1 = H_0 \trianglelefteq H_1 = H_u \trianglelefteq H_2 = H$$

then we have $X^{\mathrm{ss},\mathcal{C}} = X^{\mathrm{ss},H_u-\mathrm{fg}}$. In the case where X is irreducible we also have

$$X^{\mathrm{s}} \subseteq X^{\mathrm{ss},\mathcal{C}},$$

which can be seen as follows. Given $x \in X^{\mathrm{s}}$, by Remark 2.3.5 there is a section $f \in H^0(X, L^{\otimes r})^H$ (for some $r > 0$) such that X_f is an affine open neighbourhood of x and the natural map $X_f \to \operatorname{Spec}(\mathcal{O}(X_f)^{H_u})$ has the structure of a trivial principal H_u-bundle; in particular, $\mathcal{O}(X_f)^{H_u}$ is a finitely generated $\Bbbk$-algebra. Hence there is an equivariant isomorphism $X_f \cong H_u \times \operatorname{Spec}(\mathcal{O}(X_f)^{H_u})$, so for each $j = 1, \dots, m$ the naturally induced map $X_f \to (H_j \backslash H_u) \times \operatorname{Spec}(\mathcal{O}(X_f)^{H_u})$ (where $(H_j \backslash H_u)$ is the coset space for the left multiplication action of H_j on H_u) is a geometric quotient for the H_j-action on X_f. But $H_j \backslash H_u$ is affine [Bor91, Corollary 6.9], so $\mathcal{O}(X_f)^{H_j} \cong \mathcal{O}(H_j \backslash H_u) \otimes \mathcal{O}(X_f)^{H_u}$ is a finitely generated $\Bbbk$-algebra, for each $j = 1, \dots, m$. Finally, $\mathcal{O}(X_f)^H = (\mathcal{O}(X_f)^{H_u})^{H_r}$ is finitely generated by Nagata's theorem [Nag64] and hence $x \in X^{\mathrm{ss},\mathcal{C}}$.

In a similar vein, we also extend our definition of an enveloping system to one that is compatible with the notion of taking successive quotients by actions of H_j/H_{j-1}.

Definition 4.1.2. Let H be a linear algebraic group acting on a variety X with linearisation $L \to X$. Given $r > 0$, a finite dimensional linear subspace $V \subseteq H^0(X, L^{\otimes r})$ is called a *generalised enveloping system* (for the chain ($\mathcal{C}$)) if the following properties are satisfied:

1. there is a finite subset $\mathcal{S}^{\mathcal{C}} \subseteq V^H$ such that $X^{\mathrm{ss},\mathcal{C}} = \bigcup_{f \in \mathcal{S}^{\mathcal{C}}} X_f$ and, for each $j = 1, \dots, m$, the linear system V^{H_j} defines an enveloping system adapted to $\mathcal{S}^{\mathcal{C}}$ for the restricted linearisation $H_j \curvearrowright L \to X$ (cf. Definition 2.1.16); and

2. for each $j = 1, \dots, m+1$ there is a finite subset $\mathcal{S}_j \subseteq V^{H_j}$ containing $\mathcal{S}^{\mathcal{C}}$ such that
    - the open subset $\mathcal{U}_j = \bigcup_{f \in \mathcal{S}_j} \operatorname{Spec}((S^{H_j})_{(f)}) \subseteq X /\!\!/ H_j$ is stable under the canonical H_{j+1}/H_j-action on $X /\!\!/ H_j$, for each $j = 1, \dots, m$;
    - $X^{\mathrm{ss},\mathrm{fg}(H_j)} = \bigcup_{f \in \mathcal{S}_j} X_f$; and
    - the space V defines an enveloping system adapted to $\mathcal{S}_j$.

Lemma 4.1.3. *Given a linear algebraic group H with subnormal series ($\mathcal{C}$) and linearisation $H \curvearrowright L \to X$, there is $r > 0$ such that $H^0(X, L^{\otimes r})$ contains a generalised enveloping system. Furthermore, given a generalised enveloping system $V \subseteq H^0(X, L^{\otimes r})$, for any $r > 0$, then for any $n > 0$ the image of the natural multiplication map*

$$V^{\otimes n} \to H^0(X, L^{\otimes rn})$$

is again a generalised enveloping system.

Proof. Using the quasi-compactness of $X^{\mathrm{ss},\mathcal{C}}$ and Proposition 2.1.17, begin by finding a $r' > 0$, a finite subset $\mathcal{S}^{\mathcal{C}} \subseteq H^0(X, L^{\otimes r'})^H$ and a finite dimensional subspace $V' \subseteq H^0(X, L^{\otimes r'})^H$ containing $\mathcal{S}^{\mathcal{C}}$, such that $X^{\mathrm{ss},\mathcal{C}} = \bigcup_{f \in \mathcal{S}} X_f$ and V' defines an enveloping system adapted to $\mathcal{S}^{\mathcal{C}}$. Similarly, for each $j = 1, \dots, m+1$ find an integer $r_j > 0$ and finite subset $\mathcal{S}_j \subseteq H^0(X, L^{\otimes r_j})^{H_j}$ such that $X^{\mathrm{ss},\mathrm{fg}(H_j)} = \bigcup_{f \in \mathcal{S}_j} X_f$ with $(S^{H_j})_{(f)}$ a finitely generated $\Bbbk$-algebra for each $f \in \mathcal{S}_j$. Without loss of generality, for $j = 1, \dots, m$ we may assume that the associated open subscheme

$$\mathcal{U}_j = \bigcup_{f \in \mathcal{S}_j} \operatorname{Spec}((S^{H_j})_{(f)}) \subseteq X /\!\!/ H_j$$

is stable under the canonical H_{j+1}/H_j-action on $X /\!\!/ H_j$. Indeed, recall from Lemma 2.1.8, 2 that $I^{\mathrm{ss},\mathrm{fg}(H_j)} \cap H^0(X, L^{\otimes r_j})^{H_j}$ is an H_{j+1}-stable subspace of $H^0(X, L^{\otimes r_j})^{H_j}$ containing $\mathcal{S}_j$ (where $I^{\mathrm{ss},\mathrm{fg}(H_j)}$ is the set of H_j-invariant sections of positive tensor powers of $L \to X$

such that $(S^{H_j})_{(f)}$ is finitely generated over $\Bbbk$). By Lemma 1.1.20 the subset $\mathcal{S}_j$ is therefore contained in a finite dimensional H_{j+1}-stable subspace $W_j \subseteq I^{\mathrm{ss,fg}(H_j)} \cap H^0(X, L^{\otimes r_j})^{H_j}$. By replacing $\mathcal{S}_j$ by a basis of W_j and defining $\mathcal{U}_j$ as above, it follows that $\mathcal{U}_j$ is stable under the H_{j+1}/H_j-action on $X \wr\!\wr H_j$.

By replacing $\mathcal{S}_j$ by $\{f^n \mid f \in \mathcal{S}_j\}$ for suitably large $n > 0$, and replacing r_j by nr_j, we may also find a subspace $V_j \subseteq H^0(X, L^{\otimes r_j})^{H_j}$ such that V_j is an enveloping system adapted to $\mathcal{S}_j$, for each $j = 1, \ldots, m+1$. Applying Proposition 2.1.17, 3 we may assume V' and each V_j ($j = 1, \ldots, m+1$) is contained in $H^0(X, L^{\otimes r})$, for one big $r > 0$. Now any finite dimensional H-invariant subspace $V \subseteq H^0(X, L^{\otimes r})$ that contains V' and each V_j is a generalised enveloping system with respect to the chain ($\mathcal{C}$). Note also that we may safely assume $\mathcal{S}^{\mathcal{C}} \subseteq \mathcal{S}_j$ for each $j = 1, \ldots, m+1$.

It follows immediately from Proposition 2.1.17, 3 that, if $V \subseteq H^0(X, L^{\otimes r})$ is any generalised enveloping system, then the image of the natural map $V^{\otimes n} \to H^0(X, L^{\otimes rn})$ is too, for any $n > 0$. $\square$

Remark 4.1.4. If X is projective and $L \to X$ is an ample H-linearisation then it follows immediately from Lemma 4.1.3 that for sufficiently divisible integers $r > 0$ the space of sections $H^0(X, L^{\otimes r})$ defines a generalised enveloping system. If we wish we may also arrange for $L^{\otimes r} \to X$ to be very ample.

Suppose now $L \to X$ is an ample H-linearisation and, for some $r > 0$, we are given a generalised enveloping system $V \subseteq H^0(X, L^{\otimes r})$ which is also a complete linear system that defines an equivariant closed immersion $X \hookrightarrow \mathbb{P}(V^*)$. (Such a generalised enveloping system always exists for large enough $r > 0$, by the definition of ampleness and Lemma 4.1.3.) Roughly speaking, V contains enough invariant sections to 'see' associated inner enveloping quotients of $X^{\mathrm{ss,fg}(H_j)}$ for each $j = 1, \ldots, m+1$, as well as successive inner enveloping quotients of the generalised semistable locus $X^{\mathrm{ss},\mathcal{C}}$. We shall see shortly that, by using the embedding $X \hookrightarrow \mathbb{P}(V^*)$, we can study the inner enveloping quotients of each restricted linearisation $H_j \curvearrowright L \to X$, by looking at the chain of projection rational maps

$$\mathbb{P}(V^*) \dashrightarrow \mathbb{P}((V^{H_1})^*) \dashrightarrow \cdots \dashrightarrow \mathbb{P}((V^{H_m})^*) = \mathbb{P}((V^{H_u})^*)$$

induced by the successive inclusions $V^{H_u} = V^{H_m} \hookrightarrow V^{H_{m-1}} \hookrightarrow \ldots \hookrightarrow V^{H_1} \hookrightarrow V$. Moreover, each stage contains inner enveloping quotients of the generalised semistable locus $X^{\mathrm{ss},\mathcal{C}}$ which, intuitively, can be thought of as providing a 'core' that persists throughout this stage-by-stage quotienting process. The result of this will be a compactification of the enveloped quotient for the linearisation of the unipotent radical H_u, together with a very ample linearisation of the reductive group H_r with which we can take a reductive GIT quotient. The definition of a generalised enveloping system is set up so that this GIT quotient contains an inner enveloping quotient for the full linearisation $H \curvearrowright L \to X$, thus giving us a compactification of the enveloped quotient, in the sense of Definition 3.0.1.

Let us now make all this more precise. Let $\mathcal{S}^{\mathcal{C}}$ and $\mathcal{S}_j$ ($j = 1, \ldots, m+1$) be as in Definition 4.1.2. For each $j = 1, \ldots, m+1$, let

$$\mathcal{U}_j := \bigcup_{f \in \mathcal{S}_j} \operatorname{Spec}((S^{H_j})_{(f)}) \subseteq X \wr\wr H_j, \quad q_{H_j} : X^{\mathrm{ss},\mathrm{fg}(H_j)} \to \mathcal{U}_j$$

be the inner enveloping quotient defined by $\mathcal{S}_j$ for the restricted linearisation $H_j \curvearrowright L \to X$, and let

$$\phi_j : X^{\mathrm{ss},\mathrm{fg}(H_j)} \to \mathbb{P}((V^{H_j})^*)$$

be the morphism defined by the line bundle $L^{\otimes r}$ and natural inclusion $V^{H_j} \hookrightarrow H^0(X, L^{\otimes r})$. This descends through q_{H_j} to define a locally closed immersion

$$\overline{\phi}_j : \mathcal{U}_j \hookrightarrow \mathbb{P}((V^{H_j})^*), \tag{4.1.1}$$

and the natural $Q_{j+1} = H_{j+1}/H_j$-linearisation on $\mathcal{O}(1) \to \mathbb{P}((V^{H_j})^*)$ restricts to give the canonical Q_{j+1}-linearisation on $\mathcal{O}_{\mathcal{U}_j}(r) \to \mathcal{U}_j$ making the natural map $L^{\otimes r}|_{X^{\mathrm{ss},\mathrm{fg}(H_j)}} \to \mathcal{O}_{\mathcal{U}_j}(r)$ equivariant with respect to $H_{j+1} \to H_{j+1}/H_j$ (see Proposition 2.2.3, 1).

Now $X^{\mathrm{ss},\mathcal{C}} \subseteq X^{\mathrm{ss},\mathrm{fg}(H_j)}$ by definition, with the elements of $\mathcal{S}^{\mathcal{C}}$ being fixed by the action of H_j. The open subscheme

$$\mathcal{U}_j^{\mathcal{S}^{\mathcal{C}}} := \bigcup_{f \in \mathcal{S}^{\mathcal{C}}} \operatorname{Spec}((S^{H_j})_{(f)})$$

of $\mathcal{U}_j$ is therefore an inner enveloping quotient of $X^{\mathrm{ss},\mathcal{C}}$ for $H_j \curvearrowright L \to X$ (cf. Definition 2.1.11) that is invariant under the H_{j+1}/H_j-action on $\mathcal{U}_j$. Let

$$\Pi_j : \mathbb{P}(V^*) \dashrightarrow \mathbb{P}((V^{H_j})^*)$$

be the projection defined by the inclusion $V^{H_j} \hookrightarrow V$, with maximal domain of definition $\mathrm{dom}(\Pi_j)$. Each element of $\mathcal{S}_j \subseteq V^{H_j} \subseteq V$ may be considered a section in $H^0(\mathbb{P}(V^*), \mathcal{O}(1))$, and when done so we see that $X^{\mathrm{ss},\mathrm{fg}(H_j)}$ maps into $\mathrm{dom}(\Pi_j)$ under the inclusion $X \hookrightarrow \mathbb{P}(V^*)$, and in fact we have the following commutative diagram.

$$\begin{tikzcd}
X^{\mathrm{ss},\mathrm{fg}(H_j)} \arrow[r, hook] \arrow[d, "q_{H_j}"] & \mathrm{dom}(\Pi_j) \arrow[d, "\Pi_j"] \arrow[r, phantom, "\subseteq"] & \mathbb{P}(V^*) \\
\mathcal{U}_j \arrow[r, hook, "\overline{\phi}_j"] & \mathbb{P}((V^{H_j})^*) &
\end{tikzcd}$$

Now consider the projection

$$\pi_{j+1} : \mathbb{P}((V^{H_j})^*) \dashrightarrow \mathbb{P}((V^{H_{j+1}})^*)$$

defined by the inclusion $V^{H_{j+1}} \hookrightarrow V^{H_j}$ and let $\mathrm{dom}(\pi_{j+1})$ be its maximal domain of definition. Of course, we have

$$\Pi_{j+1} = \pi_{j+1} \circ \cdots \circ \pi_1 : \mathrm{dom}(\Pi_{j+1}) \to \mathbb{P}((V^{H_j})^*)$$

for each $j = 0, \ldots, m$. Because $\mathcal{S}^{\mathcal{C}} \subseteq V^{H_{j+1}} \subseteq V^{H_j}$, we see that $\mathcal{U}_j^{\mathcal{S}^{\mathcal{C}}}$ maps into the open subset $\{p \in \mathbb{P}((V^{H_j})^*) \mid f(p) \neq 0 \text{ for some } f \in \mathcal{S}^{\mathcal{C}}\} \subseteq \mathrm{dom}(\pi_{j+1}) \subseteq \mathbb{P}((V^{H_j})^*)$ under $\overline{\phi}_j$. Furthermore, for $j = 1, \ldots, m-1$ the natural rational map

$$q_{Q_{j+1}} : \mathrm{Proj}(S^{H_j}) \dashrightarrow \mathrm{Proj}(S^{H_{j+1}})$$

of (2.2.5) in Section 2.2.2 restricts to define a dominant Q_{j+1}-invariant morphism

$$q_{Q_{j+1}} : \mathcal{U}_j^{\mathcal{S}^{\mathcal{C}}} \to \mathcal{U}_{j+1}^{\mathcal{S}^{\mathcal{C}}}$$

such that the diagram

$$\begin{tikzcd}
X^{\mathrm{ss},\mathcal{C}} \arrow[r, "q_j"] \arrow[dr, "q_{H_{j+1}}"'] & \mathcal{U}_j^{\mathcal{S}^{\mathcal{C}}} \arrow[d, "q_{Q_{j+1}}"] \\
& \mathcal{U}_{j+1}^{\mathcal{S}^{\mathcal{C}}}
\end{tikzcd}$$

commutes. By chasing the appropriate diagrams, one sees that

$$(\overline{\phi}_{j+1} \circ q_{Q_{j+1}})^* : V^{H_{j+1}} = H^0(\mathbb{P}((V^{H_{j+1}})^*), \mathcal{O}(1)) \longrightarrow H^0(\mathcal{U}_j^{\mathcal{S}^{\mathcal{C}}}, \mathcal{O}(r))^{Q_{j+1}}$$

is equal to the restriction of $\overline{\phi}_j^* : V^{H_j} = H^0(\mathbb{P}((V^{H_j})^*), \mathcal{O}(1)) \longrightarrow H^0(\mathcal{U}_j^{\mathcal{S}^{\mathcal{C}}}, \mathcal{O}(r))$ to the subspace $V^{H_{j+1}}$ of V^{H_j}. It follows that the diagram

$$\begin{tikzcd}
\mathcal{U}_j^{\mathcal{S}^{\mathcal{C}}} \arrow[r, hook, "\overline{\phi}_j"] \arrow[d, "q_{Q_{j+1}}"'] & \mathrm{dom}(\pi_{j+1}) \arrow[d, "\pi_{j+1}"] & \subseteq & \mathbb{P}((V^{H_j})^*) \\
\mathcal{U}_{j+1}^{\mathcal{S}^{\mathcal{C}}} \arrow[r, hook, "\overline{\phi}_{j+1}"] & \mathbb{P}((V^{H_{j+1}})^*) & &
\end{tikzcd}$$

commutes.

The main point of this construction is that the generalised enveloping system V yields the following commutative diagram (of morphisms and rational maps) with which to study successive enveloping quotients of the generalised semistable locus $X^{\mathrm{ss},\mathcal{C}}$ (and the stable locus X^{s} within it).

$$\begin{tikzcd}
 & \mathcal{U}_1^{\mathcal{S}^{\mathcal{C}}} \arrow[d, "q_{Q_2}"] & \subseteq & \mathcal{U}_1 \arrow[r, hook, "\overline{\phi_1}"] & \mathbb{P}((V^{H_1})^*) \arrow[d, dashed, "\pi_2"] & \curvearrowleft & H_2/H_1 \\
 & \vdots \arrow[d] & & \vdots & \vdots \arrow[d, dashed] & & \\
X^{\mathrm{ss},\mathcal{C}} \arrow[uur, "q_{H_1}"] \arrow[r, "q_{H_j}"] \arrow[dr, "q_{H_{j+1}}"'] \arrow[dddr, "q_{H_u}"'] & \mathcal{U}_j^{\mathcal{S}^{\mathcal{C}}} \arrow[d, "q_{Q_{j+1}}"] & \subseteq & \mathcal{U}_j \arrow[r, hook, "\overline{\phi}_j"] & \mathbb{P}((V^{H_j})^*) \arrow[d, dashed, "\pi_{j+1}"] & \curvearrowleft & H_{j+1}/H_j \\
 & \mathcal{U}_{j+1}^{\mathcal{S}^{\mathcal{C}}} \arrow[d] & \subseteq & \mathcal{U}_{j+1} \arrow[r, hook, "\overline{\phi}_{j+1}"'] & \mathbb{P}((V^{H_{j+1}})^*) \arrow[d, dashed] & \curvearrowleft & H_{j+2}/H_{j+1} \\
 & \vdots \arrow[d, "q_{Q_m}"] & & \vdots & \vdots \arrow[d, dashed, "\pi_m"] & & \\
 & \mathcal{U}_m^{\mathcal{S}^{\mathcal{C}}} & \subseteq & \mathcal{U}_m \arrow[r, hook, "\overline{\phi}_m"] & \mathbb{P}((V^{H_u})^*) & \curvearrowleft & H_r
\end{tikzcd} \tag{4.1.2}$$

Each $\mathcal{U}_j$ is an H_{j+1}/H_j-stable inner enveloping quotient of $X^{\mathrm{ss},\mathrm{fg}(H_j)}$, and at each stage $\overline{\phi}_j$ maps the open subscheme $\mathcal{U}_j^{\mathcal{S}^{\mathcal{C}}}$ into $\mathrm{dom}(\pi_{j+1})$ equivariantly. We may use the $\overline{\phi}_j$ to compactify the inner enveloping quotients $\mathcal{U}_j$ inside $\mathbb{P}((V^{H_j})^*)$.

At the m-th stage, we end up with a commutative diagram

$$\begin{tikzcd}
X^{\mathrm{ss},\mathcal{C}} \arrow[r, phantom, "\subseteq"] \arrow[d] & X^{\mathrm{ss},\mathrm{fg}(H_u)} \arrow[d, "q_{H_u}"] \arrow[dr, "\phi_m"] & & \\
\mathcal{U}_m^{\mathcal{S}^{\mathcal{C}}} \arrow[r, phantom, "\subseteq"] & \mathcal{U}_m \arrow[r, hook, "\overline{\phi}_m"] & \overline{\mathcal{U}_m} \arrow[r, phantom, "\subseteq"] & \mathbb{P}((V^{H_u})^*)
\end{tikzcd}$$

and the resulting projective variety $\overline{\mathcal{U}_m} \subseteq \mathbb{P}((V^{H_u})^*)$ is stable under the $H_r = H_{m+1}/H_m$-action on $\mathbb{P}((V^{H_u})^*)$. There is a very ample H_r-linearisation over $\overline{\mathcal{U}_m}$ coming from restricting the natural H_r-linearisation on $\mathcal{O}(1) \to \mathbb{P}((V^{H_u})^*)$. The next proposition shows that the associated GIT quotient $\overline{\mathcal{U}_m}/\!\!/H_r$ yields a compactification of the inner enveloping quotient $\mathcal{U}_{m+1}$ of $X^{\mathrm{ss},\mathrm{fg}(H)}$.

Proposition 4.1.5. *Let H be a linear algebraic group with subnormal series ($\mathcal{C}$), let X be an irreducible quasi-projective H-variety and let $L \to X$ be an ample linearisation; also retain the preceding notation. Let $L_m = \mathcal{O}_{\mathbb{P}((V^{H_u})^*)}(1)|_{\overline{\mathcal{U}_m}} \to \overline{\mathcal{U}_m}$ and let $\pi_{H_r} : \overline{\mathcal{U}_m}^{\mathrm{ss}(L_m)} \to \overline{\mathcal{U}_m}/\!\!/_{L_m} H_r$ be the GIT quotient for the induced linearisation $H_r \curvearrowright L_m \to \overline{\mathcal{U}_m}$. Then there is an open immersion $\theta_{m+1} : \mathcal{U}_{m+1} \hookrightarrow \overline{\mathcal{U}_m}/\!\!/_{L_m} H_r$ making the following diagram commute:*

$$\begin{tikzcd}
X^{\mathrm{ss},\mathcal{C}} \arrow[r, phantom, "\subseteq"] \arrow[d] & X^{\mathrm{ss},\mathrm{fg}(H)} \cap X^{\mathrm{ss},\mathrm{fg}(H_u)} \arrow[r, phantom, "\subseteq"] \arrow[d, "q_H"] & \phi_m^{-1}(\overline{\mathcal{U}_m}^{\mathrm{ss}(L_m)}) \arrow[d, "\pi_{H_r} \circ \phi_m"] \\
\mathcal{U}_{m+1}^{\mathcal{S}^{\mathcal{C}}} \arrow[r, phantom, "\subseteq"] & \mathcal{U}_{m+1} \arrow[r, hook, "\theta_{m+1}"] & \overline{\mathcal{U}_m}/\!\!/_{L_m} H_r
\end{tikzcd}$$

with all unmarked inclusions open.

Proof. First note that the the image of the pullback morphism

$$\phi_m^* : H^0(\overline{\mathcal{U}_m}, L_m) \to H^0(X^{\mathrm{ss},\mathrm{fg}(H_u)}, L^{\otimes r})$$

contains $V^{H_u} \subseteq H^0(X, L^{\otimes r}) \subseteq H^0(X^{\mathrm{ss},\mathrm{fg}(H_u)}, L^{\otimes r})$, because

$$\phi_m^* : H^0(\mathbb{P}((V^{H_u})), \mathcal{O}(1)) = V^{H_u} \hookrightarrow H^0(X^{\mathrm{ss},\mathrm{fg}(H_u)}, L^{\otimes r})$$

factors through $H^0(\mathbb{P}((V^{H_u})), \mathcal{O}(1)) \to H^0(\overline{\mathcal{U}_j}, L_m)$ by construction. Taking H_r-invariants, we see that the image of $H^0(\overline{\mathcal{U}_j}, L_m)^{H_r}$ under ϕ_m^* contains V^H, and in particular $\mathcal{S}_{m+1}$, hence $X^{\mathrm{ss},\mathrm{fg}(H)} \cap X^{\mathrm{ss},\mathrm{fg}(H_u)} \subseteq \phi_m^{-1}(\overline{\mathcal{U}_m}^{\mathrm{ss}(L_m)})$. Now, we may view $\overline{\mathcal{U}_m}/\!\!/_{L_m} H_r$ as naturally equal to $\mathrm{Proj}(\Bbbk[\overline{\mathcal{U}_m}, L_m]^{H_r})$, with the GIT quotient map π_{H_r} being induced by the inclusion of graded rings $\pi_{H_r}^* : \Bbbk[\overline{\mathcal{U}_m}, L_m]^{H_r} \hookrightarrow \Bbbk[\overline{\mathcal{U}_m}, L_m]$; cf. [MFK94, Page 40] and Section A.1 of the Appendix. Similarly, the morphism $\phi_m : X^{\mathrm{ss,fg}(H_u)} \to \overline{\mathcal{U}_m} = \operatorname{Proj}(\Bbbk[\overline{\mathcal{U}_m}, L_m])$ may be regarded as the morphism induced by the inclusion

$$\phi_m^* : \Bbbk[\overline{\mathcal{U}_m}, L_m] \hookrightarrow \Bbbk[X, L^{\otimes r}]^{H_u} \hookrightarrow \Bbbk[X^{\mathrm{ss,fg}(H_u)}, L^{\otimes r}]$$

(note that this ring map is injective because $\phi_m = \overline{\phi}_m \circ q_m$ is dominant) and the composition $\pi_{H_r} \circ \phi_m : \phi_m^{-1}(\overline{\mathcal{U}_m}^{\mathrm{ss}(L_m)}) \to \overline{\mathcal{U}_m} /\!/_{L_m} H_r$ is induced by the ring map $\phi_m^* \circ \pi_{H_r}^*$. Clearly $\phi_m^* \circ \pi_{H_r}^*$ factors through an inclusion

$$\Theta : \Bbbk[\overline{\mathcal{U}_m}, L_m]^{H_r} \hookrightarrow \Bbbk[X, L^{\otimes r}]^H$$

which contains in its image each $f \in \mathcal{S}_{m+1}$. Since

$$\mathcal{U}_{m+1} = \bigcup_{f \in \mathcal{S}_{m+1}} \operatorname{Spec}((\Bbbk[X, L^{\otimes r}]^H)_{(f)}) \subseteq \operatorname{Proj}(\Bbbk[X, L^{\otimes r}]^H),$$

the composition of Θ with the natural structure map $\Bbbk[X, L^{\otimes r}]^H \to \Bbbk[\mathcal{U}_j, \mathcal{O}(r)]$ defines a morphism

$$\theta_{m+1} : \mathcal{U}_{m+1} \to \overline{\mathcal{U}_m} /\!/_{L_m} H_r$$

such that $\theta_{m+1} \circ q_{m+1} = \pi_{H_r} \circ \phi_m$ on $X^{\mathrm{ss,fg}(H)} \cap X^{\mathrm{ss,fg}(H_u)}$ (cf. Section A.1 of the Appendix). For each $f \in \mathcal{S}_{m+1}$, the homomorphism

$$(\Bbbk[\overline{\mathcal{U}_m}, L_m]^{H_r})_{(f)} \hookrightarrow (\Bbbk[X, L^{\otimes r}]^H)_{(f)}$$

induced by Θ is surjective as well as injective, because the generating set $\{\tilde{f}/f \mid \tilde{f} \in V^H\}$ of $(\Bbbk[X, L^{\otimes r}]^H)_{(f)}$ is contained in its image. Hence θ_{m+1} restricts to an isomorphism

$$\operatorname{Spec}((\Bbbk[X, L^{\otimes r}]^H)_{(f)}) \cong \operatorname{Spec}((\Bbbk[\overline{\mathcal{U}_m}, L_m]^{H_r})_{(f)}) \subseteq \overline{\mathcal{U}_m} /\!/_{L_m} H_r$$

for each $f \in \mathcal{S}_{m+1}$, from which it follows that $\theta_{m+1} : \mathcal{U}_{m+1} \to \overline{\mathcal{U}_m} /\!/_{L_m} H_r$ is an open immersion. □

Let us summarise the main point of the above work. Given an ample linearisation $L \to X$ of a linear algebraic group H and a subnormal series of the form $(\mathcal{C})$, by choosing a large enough tensor power $L^{\otimes r}$ of the linearisation and a generalised enveloping system $V \subseteq H^0(X, L^{\otimes r})$ (or even taking $V = H^0(X, L^{\otimes r})$ for projective X) we can construct compactifications of the enveloped quotient for $H \curvearrowright L \to X$ in two steps. Firstly, a compactification $\overline{\mathcal{U}_m}$ of the enveloped quotient for the linearisation of the unipotent radical $H_u = H_m$ can be found by considering the chain of projections

$$\mathbb{P}(V^*) \dashrightarrow \mathbb{P}((V^{H_1})^*) \dashrightarrow \cdots \dashrightarrow \mathbb{P}((V^{H_m})^*) = \mathbb{P}((V^{H_u})^*),$$

and secondly, taking the reductive GIT quotient $\overline{\mathcal{U}_m} /\!\!/ H_r$ of $\overline{\mathcal{U}_m}$ arising from the H_r-linearisation on the projective space $\mathbb{P}((V^{H_u})^*)$ results in a compactification of the enveloped quotient for the full linearisation. Moreover, there are natural maps from the generalised semistable locus $X^{\mathrm{ss},\mathcal{C}}$ to each of the projective spaces $\mathbb{P}((V^{H_j})^*)$ and to $\overline{\mathcal{U}_m} /\!\!/ H_r$.

Thus we have reduced the problem of constructing compactifications of enveloped quotients to understanding the nested sequence of inclusions

$$V^{H_u} = V^{H_m} \hookrightarrow V^{H_{m-1}} \hookrightarrow \ldots \hookrightarrow V^{H_1} \hookrightarrow V$$

and the use of Mumford's reductive GIT. Of course, it is still a difficult problem in general to understand the invariant subspaces V^{H_j} algebraically, so it is desirable to know if there are more geometric methods available to study this picture. We discuss a possible approach to this in the next section.

4.2 Successive Quotients by $\mathbb{G}_a$

We continue to assume that X is an irreducible variety with ample H-linearisation $L \to X$, but will now further assume that X is *projective*. The aim of this section is to define H-stable open subsets U_1 and U_2 of X such that

$$U_1 \subseteq X^{\mathrm{s}} \subseteq X^{\mathrm{ss},\mathcal{C}} \subseteq U_2.$$

The motivation here is to achieve something similar to the completely stable locus and completely semistable locus for a reductive envelope—see Theorem 3.1.14 of Chapter 3. Observe that in such a situation U_1 admits a geometric quotient for the H-action, and if it is known by other methods that $U_1 = U_2$ then the equality $X^{\mathrm{s}} = X^{\mathrm{ss},\mathcal{C}}$ holds. While we will show how such U_1 and U_2 can be constructed, we will see that these are not as computationally effective as one might want. We will discuss a possible solution to this.

Our method is to couple the ideas of Section 4.1 with the theory of reductive envelopes developed in [DK07]. More precisely, we shall apply the approach of Section 4.1 for chains $(\mathcal{C})$ of the form

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_m = H_u \trianglelefteq H_{m+1} = H, \tag{$\mathcal{C}_a$}$$

which satisfy the following properties:

(Ch1) the subgroup H_j is normal in H for each $j = 1, \dots, m$; and

(Ch2) for each $j = 0, \dots, m-1$ the quotient $Q_{j+1} = H_{j+1}/H_j \cong \mathbb{G}_a$.

Such chains have two features that make them attractive from a computational point of view. Firstly, we have the following theorem of Weitzenböck at our disposal.

Theorem 4.2.1. *[Wei32] If W is any finite dimensional $\mathbb{G}_a$-representation then there is an action of $\mathrm{SL}(2,\Bbbk)$ on W extending the $\mathbb{G}_a$-action under the standard inclusion $\mathbb{G}_a \hookrightarrow \mathrm{SL}(2,\Bbbk)$ as the subgroup of upper triangular matrices.*

It follows that any $\mathbb{G}_a$-linearisation of $\mathcal{O}(1)$ over a projective space can be extended to an $\mathrm{SL}(2,\Bbbk)$-linearisation. Because $\mathbb{G}_a$ is a Grosshans subgroup of $\mathrm{SL}(2,\Bbbk)$, one can study the non-reductive GIT of this $\mathbb{G}_a$-linearisation using the theory of strong reductive envelopes, as discussed in Section 3.2.1 of Chapter 3. In the case where V is a generalised enveloping system for the chain $(\mathcal{C}_a)$ and $W = (V^{H_j})^*$ for some $j = 1, \dots, m$, we will see that pulling back the finitely generated semistable locus for $\mathbb{G}_a \cong Q_{j+1} \curvearrowright \mathcal{O}(1) \to \mathbb{P}((V^{H_j})^*)$ under the natural rational map $X \dashrightarrow \mathbb{P}((V^{H_j})^*)$ yields an open subset containing the generalised semistable locus $X^{\mathrm{ss},\mathcal{C}_a}$ associated to $(\mathcal{C}_a)$.

The second advantage of chains of the form $(\mathcal{C}_a)$ is that each projection

$$H_{j+1} \to Q_{j+1} \cong \mathbb{G}_a$$

is split by a suitable closed embedding $\mathbb{G}_a \hookrightarrow H_{j+1}$, via the exponential map on $\mathrm{Lie}(H_u)$ (see Section 1.1.3 of Chapter 1). The upcoming Lemma 4.2.2 thus implies that one can try to build open subsets $U_1 \subseteq X^{\mathrm{s}}$ by iteratively requiring that each induced action of Q_{j+1} on the H_j-quotient U_1/H_j be locally trivial, and then shrinking U_1 appropriately. We will conjecture that one way to do this is by pulling back the stable loci for the linearisations $Q_{j+1} \curvearrowright \mathcal{O}(1) \to \mathbb{P}((V^{H_j})^*)$ just discussed to X.

Note that we do not claim to be the first to prove the following result, but include a proof for lack of a suitable reference.

Lemma 4.2.2. *Let H be a unipotent linear algebraic group with normal subgroup N such that the projection $H \to H/N$ splits and let X be an affine H-variety. Suppose X has the structure of a principal N-bundle, and the quotient X/N is a principal H/N-bundle, for the canonical action of H/N on X. Then X is a principal H-bundle.*

Proof. Let $\pi_N : X \to X/N$ be the quotient map for the N-action on X and $\pi_{H/N} : X/N \to (X/N)/(H/N)$ the quotient map for the H/N-action on X/N. Also let $H_1 \subseteq H$ be a subgroup that splits the projection $H \to H/N$, so that

$$N \rtimes H_1 \xrightarrow{\cong} H, \quad (n; h) \mapsto nh,$$

where the multiplication in the semi-direct product is defined by $(n_1; h_1) \cdot (n_2; h_2) = (n_1 h_1 n_2 h_1^{-1}; h_1 h_2)$. Note that the composition $\pi_{H/N} \circ \pi_N : X \to (X/N)/(H/N) = X/H$ is a geometric quotient for the H-action on X. Because H and N are unipotent the quotients $\pi_{H/N}$ and π_H are locally trivial in the Zariski topology [Ser58, Proposition 14], so by choosing sufficiently fine open covers it suffices to treat the case where X and X/N are trivial bundles for H and H_1, respectively, where we identify H_1 with H/N in the natural way. So let $X = N \times (X/N)$ and $(X/N) = H_1 \times (X/H)$, with the N-action on X (respectively, H_1-action on X/N) induced by left multiplication on N (respectively, H_1) and the quotient maps π_N and $\pi_{H/N}$ given by projecting to the second factor in both cases. Also let

$$s_N : N \times (X/N) \to N, \quad s_{H_1} : H_1 \times (X/H) \to H_1$$

be the projections to the first factors, and let

$$\sigma : X/H \to N \times H_1 \times (X/H), \quad z \mapsto (e, e, z)$$

be the obvious section to π (note π is the projection to the factor X/H). Given $x \in X$, there are unique $n \in N$ and $h \in H_1$ such that $x = nh\sigma(\pi(x))$. The assignments

$$\phi_N : X \to N, \quad x = nh\sigma(\pi(x)) \mapsto n,$$
$$\phi_{H_1} : X \to H_1, \quad x = nh\sigma(\pi(x)) \mapsto h$$

are morphisms of varieties: for each $x \in X$ we have

$$\phi_{H_1}(x) = s_{H_1}(\pi_N(x)), \quad \phi_N(x) = s_N(x)(s_N(\phi_{H_1}(x)\sigma(\pi(x))))^{-1}.$$

It is clear that ϕ_N and ϕ_{H_1} are N-equivariant and also that ϕ_{H_1} is H_1-equivariant, while $\phi_N(hx) = h\phi_N(x)h^{-1}$ for all $x \in X$ and $h \in H_1$. Therefore

$$X \xrightarrow{\cong} H \times (X/H), \quad x \mapsto (\phi_N(x)\phi_{H_1}(x), \pi(x))$$

defines an H-equivariant isomorphism, where $H \times (X/H)$ is the trivial H-bundle with base X/H. □

Let us now proceed with the construction. As in Section 4.1, use Lemma 4.1.3 and the fact that $L \to X$ is ample to find a generalised enveloping system $V \subseteq H^0(X, L^{\otimes r})$, for some $r > 0$, such that V defines a closed embedding $X \hookrightarrow \mathbb{P}(V^*)$. We shall retain some of the notation used in Section 4.1 to construct the diagram (4.1.2). So for each $j = 1, \dots, m+1$, there are finite subsets $\mathcal{S}^{\mathcal{C}_a} \subseteq V^H$ and $\mathcal{S}_j \subseteq V^{H_j}$ with the properties described in Definition 4.1.2, with

$$\begin{aligned} \mathcal{U}_j^{\mathcal{S}^{\mathcal{C}_a}} &:= \bigcup_{f \in \mathcal{S}^{\mathcal{C}_a}} \operatorname{Spec}((S^{H_j})_{(f)}) \\ &\subseteq \mathcal{U}_j := \bigcup_{f \in \mathcal{S}_j} \operatorname{Spec}((S^{H_j})_{(f)}), \quad j = 1, \dots, m+1. \end{aligned}$$

For each $j = 1, \dots, m$ the natural rational map $\operatorname{Proj}(S^{H_j}) \dashrightarrow \operatorname{Proj}(S^{H_{j+1}})$ defines a Q_{j+1}-invariant morphism $q_{Q_{j+1}} : \mathcal{U}_j^{\mathcal{S}^{\mathcal{C}_a}} \to \mathcal{U}_{j+1}^{\mathcal{S}^{\mathcal{C}_a}}$.

For brevity, we shall define

$$\mathbb{P}_j := \mathbb{P}((V^{H_j})^*)$$

for what follows; note that $\mathbb{P}_0 = \mathbb{P}(V^*)$. Recall the morphism

$$\phi_j : X^{\mathrm{ss,fg}(H_j)} \to \mathbb{P}_j$$

defined by the inclusion $V^{H_j} \hookrightarrow H^0(X, L^{\otimes r})^{H_j}$ and the induced locally closed immersion

$$\overline{\phi}_j : \mathcal{U}_j \hookrightarrow \mathbb{P}_j$$

such that $\overline{\phi}_j \circ q_j = \phi_j$, where $q_j : X^{\mathrm{ss,fg}(H_j)} \to \mathcal{U}_j$ is the inner enveloping quotient map. There is also the projection map

$$\Pi_j : \mathbb{P}_0 \dashrightarrow \mathbb{P}_j$$

induced by the inclusion $V^{H_j} \hookrightarrow V$, whose maximal domain of definition $\mathrm{dom}(\Pi_j)$ contains $X^{\mathrm{ss,fg}(H_j)}$, and Π_j coincides with ϕ_j on $X^{\mathrm{ss,fg}(H_j)}$. (Note that $\Pi_0 = \mathrm{id}_{\mathbb{P}_0}$ is just the identity map on $\mathbb{P}_0$.)

For each $j = 0, \dots, m$ the natural action of $Q_{j+1} = H_{j+1}/H_j$ on V^{H_j} canonically induces a linearisation

$$Q_{j+1} \curvearrowright \mathcal{O}_{\mathbb{P}_j}(1) \to \mathbb{P}_j$$

which restricts to the canonical Q_{j+1}-linearisation on $\mathcal{O}_{\mathcal{U}_j}(r) \to \mathcal{U}_j$. (Recall that $\mathcal{U}_j$ is stable under the canonical action of Q_{j+1} on $X \wr\wr H_j$ by 2 of Definition 4.1.2 of an enveloping system.) In the case where $j = m$ then this is a linearisation of the reductive group $H_r = Q_{m+1}$, so we have

$$\mathbb{P}_m^{\mathrm{ss,fg}(Q_{m+1}, \mathcal{O}_{\mathbb{P}_j}(1))} = \mathbb{P}_m^{\mathrm{ss}(H_r, \mathcal{O}_{\mathbb{P}_j}(1))}, \quad \mathbb{P}_m^{\mathrm{s}(Q_{m+1}, \mathcal{O}_{\mathbb{P}_j}(1))} = \mathbb{P}_m^{\mathrm{s}(H_r, \mathcal{O}_{\mathbb{P}_j}(1))},$$

where $\mathbb{P}_m^{\mathrm{ss}(H_r, \mathcal{O}_{\mathbb{P}_j}(1))}$ and $\mathbb{P}_m^{\mathrm{s}(H_r, \mathcal{O}_{\mathbb{P}_j}(1))}$ are the semistable and stable locus, respectively, in the sense of reductive GIT (cf. Definition 1.2.2).

On the other hand, if $j \leq m - 1$ then according to (Ch2) we have $Q_{j+1} \cong \mathbb{G}_a$, so by Weitzenböck's theorem 4.2.1 the action on V^{H_j} extends to one of $G := \mathrm{SL}(2, \Bbbk)$ under a suitable closed embedding $Q_{j+1} \hookrightarrow G$, thus the above linearisation extends to a linearisation $G \curvearrowright \mathcal{O}_{\mathbb{P}_j}(1) \to \mathbb{P}_j$. This extension can be used to construct a strong ample reductive envelope for the Q_{j+1}-linearisation $\mathcal{O}_{\mathbb{P}_j}(1) \to \mathbb{P}_j$, either using the results in [DK07, §5.3.2] or Section 3.2.1 in Chapter 3. Let us briefly recall how this is done. The extension gives a G-equivariant isomorphism of linearisations

$$\begin{tikzcd}
G \times^{Q_{j+1}} \mathcal{O}_{\mathbb{P}_j}(1) \arrow[r, "\cong"] \arrow[d] & (G/Q_{j+1}) \times \mathcal{O}_{\mathbb{P}_j}(1) \arrow[d] & {[g,l] \mapsto (gQ_{j+1}, gl)} \\
G \times^{Q_{j+1}} \mathbb{P}_j \arrow[r, "\cong"] & (G/Q_{j+1}) \times \mathbb{P}_j & {[g,x] \mapsto (gQ_{j+1}, gx),}
\end{tikzcd}$$

with $G/Q_{j+1} \cong \mathbb{k}^2 \setminus \{0\}$. By equivariantly completing $\mathbb{k}^2 \setminus \{0\}$ to $\mathbb{P}^2$ we obtain an open immersion

$$\beta_j : G \times^{Q_{j+1}} \mathbb{P}_j \cong G/Q_{j+1} \times \mathbb{P}_j = (\mathbb{k}^2 \setminus \{0\}) \times \mathbb{P}_j \hookrightarrow \mathbb{P}^2 \times \mathbb{P}_j$$

together with an extension of the G-linearisation,

$$G \curvearrowright \mathcal{O}_{\mathbb{P}_j}(1)_{N_j} := \mathcal{O}_{\mathbb{P}^2}(N_j) \boxtimes \mathcal{O}_{\mathbb{P}_j}(1) \to \mathbb{P}^2 \times \mathbb{P}_j$$

for integers $N_j > 0$ (see Example 3.2.10 and Section 3.3.1 for more explicit details). By Corollary 3.2.11, for sufficiently large $N_j > 0$ the triple $(\mathbb{P}^2 \times \mathbb{P}_j, \beta_j, \mathcal{O}_{\mathbb{P}_j}(1)_{N_j})$ defines a strong ample reductive envelope for the Q_{j+1}-linearisation $\mathcal{O}_{\mathbb{P}_j}(1) \to \mathbb{P}_j$, and so by Proposition 3.2.2 under the naturally induced embedding $\mathbb{P}_j \hookrightarrow \mathbb{P}^2 \times \mathbb{P}_j$ we have

$$\begin{aligned}
\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1}, \mathcal{O}_{\mathbb{P}_j}(1))} &= \mathbb{P}_j \cap (\mathbb{P}^2 \times \mathbb{P}_j)^{\mathrm{ss}(\mathcal{O}_{\mathbb{P}_j}(1)_{N_j})}, \\
\mathbb{P}_j^{\mathrm{s}(Q_{j+1}, \mathcal{O}_{\mathbb{P}_j}(1))} &= \mathbb{P}_j \cap (\mathbb{P}^2 \times \mathbb{P}_j)^{\mathrm{s}(\mathcal{O}_{\mathbb{P}_j}(1)_{N_j})},
\end{aligned}$$

where $\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1}, \mathcal{O}_{\mathbb{P}_j}(1))}$ and $\mathbb{P}_j^{\mathrm{s}(Q_{j+1}, \mathcal{O}_{\mathbb{P}_j}(1))}$ denote the finitely generated semistable locus and stable locus, respectively, for $Q_{j+1} \curvearrowright \mathcal{O}_{\mathbb{P}_j}(1) \to \mathbb{P}_j$.

Remark 4.2.3. By Remark 3.2.9 of Chapter 3 the ring of invariants $\mathbb{k}[\mathbb{P}_j, \mathcal{O}_{\mathbb{P}_j}(1)]^{Q_{j+1}}$ is necessarily a finitely generated $\mathbb{k}$-algebra, so we have $\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1}, \mathcal{O}_{\mathbb{P}_j}(1))} = \mathbb{P}_j^{\mathrm{nss}(Q_{j+1}, \mathcal{O}_{\mathbb{P}_j}(1))}$, where $\mathbb{P}_j^{\mathrm{nss}(Q_{j+1}, \mathcal{O}_{\mathbb{P}_j}(1))}$ is the naively semistable locus, defined in Definition 2.1.1 of Chapter 2.

The aim now is to relate the stable locus X^{s} and generalised semistable locus $X^{\mathrm{ss},\mathcal{C}_a}$ to the finitely generated semistable loci and stable loci for the linearisations $Q_{j+1} \curvearrowright \mathcal{O}_{\mathbb{P}_j}(1) \to \mathbb{P}_j$, for each $j = 1, \dots, m+1$. In the following proposition we will talk about preimages of subsets of $\mathbb{P}_j$ under the projection $\Pi_j : \mathbb{P}_0 \dashrightarrow \mathbb{P}_j$; by definition,

$$\Pi_j^{-1}(U) = \{p \in \mathrm{dom}(\Pi_j) \mid \Pi_j(p) \in U\}$$

for any $U \subseteq \mathbb{P}_j$.

Proposition 4.2.4. *Assume H is a linear algebraic group with a subnormal series $(\mathcal{C}_a)$ satisfying (Ch1) and (Ch2), let X be an irreducible projective H-variety and let $L \to X$ be an ample linearisation. Also retain the preceding notation. Then we have inclusions*

$$X^{\mathrm{ss},\mathcal{C}_a} \cap \bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{s}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}) \subseteq X^{\mathrm{s}} \subseteq X^{\mathrm{ss},\mathcal{C}_a} \subseteq X \cap \bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}),$$

where $\Pi_j : \mathbb{P}_0 \dashrightarrow \mathbb{P}_j$ are the projection maps. Furthermore, $\bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{s}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))})$ and $\bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))})$ are both H-stable open subsets of X.

Proof. First observe that because the chain $(\mathcal{C}_a)$ consists of characteristic subgroups of H by (Ch1), for each j there is a natural action of H on $\mathbb{P}_j$ with respect to which the projection Π_j is equivariant and the subsets $\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}(1))}$ and $\mathbb{P}_j^{\mathrm{s}(Q_{j+1},\mathcal{O}(1))}$ are H-stable. It follows that $\bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{s}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))})$ and $\bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))})$ are subsets that are stable under the H-action on X.

For each $j = 0, \dots, m-1$ we have $\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))} = \mathbb{P}_j^{\mathrm{nss}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}$ by Remark 4.2.3, and in the case $j = m$ we have $\mathbb{P}_m^{\mathrm{ss,fg}(Q_{m+1},\mathcal{O}_{\mathbb{P}_m}(1))} = \mathbb{P}_m^{\mathrm{ss}(H_r,\mathcal{O}_{\mathbb{P}_m}(1))}$ (where the linearisation is one of the reductive group H_r). This means that, for each $j = 0, \dots, m$, the set $\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}$ contains all points $p \in \mathbb{P}_j$ where there is some $f \in H^0(\mathbb{P}_j, \mathcal{O}(1))^{Q_{j+1}} = V^{H_{j+1}}$ not vanishing at p, thus

$$\Pi_j^{-1}(\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}) \supseteq \mathrm{dom}(\Pi_{j+1}) \subseteq \mathbb{P}_0.$$

So in fact

$$\bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}) \supseteq \mathrm{dom}(\Pi_{m+1}).$$

But $X^{\mathrm{ss},\mathcal{C}_a} \subseteq \mathrm{dom}(\Pi_{m+1})$, because $V^H = V^{H_{m+1}}$ contains a subset $\mathcal{S}^{\mathcal{C}_a}$ such that $X^{\mathrm{ss},\mathcal{C}_a}$ is covered by X_f with $f \in \mathcal{S}^{\mathcal{C}_a}$. Thus

$$X^{\mathrm{ss},\mathcal{C}_a} \subseteq X \cap \bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}).$$

We prove the containment $X^{\mathrm{ss},\mathcal{C}_a} \cap \bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{s}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}) \subseteq X^{\mathrm{s}}$ by showing that

$$U_k := X^{\mathrm{ss},\mathcal{C}_a} \cap \bigcap_{j=0}^{k} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{s}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}) \subseteq X^{\mathrm{s}(H_{k+1})},$$

(where $X^{\mathrm{s}(H_{k+1})}$ is the stable locus for the restricted linearisation $H_{k+1} \curvearrowright L \to X$) via an induction on k, for $k = 0, \dots, m$. In the case $k = 0$, the map $\Pi_0 : \mathbb{P}_0 \to \mathbb{P}_0$ is the identity, so $U_0 = X^{\mathrm{ss},\mathcal{C}_a} \cap \mathbb{P}_0^{\mathrm{s}(H_1,\mathcal{O}_{\mathbb{P}_0}(1))}$. Because $X \hookrightarrow \mathbb{P}_0$ is a closed immersion, we have $U_0 \subseteq X \cap \mathbb{P}_0^{\mathrm{s}(H_1,\mathcal{O}_{\mathbb{P}_0}(1))} \subseteq X^{\mathrm{s}(H_1)}$ by virtue of Lemma 2.3.6 of Chapter 2.

Now suppose $0 \leq k \leq m-1$ and $U_k \subseteq X^{\mathrm{s}(H_{k+1})}$. Then the enveloping quotient map $q_{H_{k+1}} : X^{\mathrm{ss,fg}(H_{k+1})} \to X \wr\!\wr H_{k+1}$ restricts to give a diagram

$$\begin{tikzcd}
U_k \arrow[r, phantom, "\subseteq"] \arrow[d, "\mathrm{geo}"] & X^{\mathrm{ss},\mathcal{C}_a} \arrow[r, phantom, "\subseteq"] \arrow[d] & X^{\mathrm{ss,fg}(H_{k+1})} \arrow[d, "q_{H_{k+1}}"] & \\
q_{H_{k+1}}(U_k) \arrow[r, phantom, "\subseteq"] & \mathcal{U}_{k+1}^{\mathcal{S}^{\mathcal{C}_a}} \arrow[r, phantom, "\subseteq"] & \mathcal{U}_{k+1} \arrow[r, hook, "\overline{\phi}_{k+1}"] & \mathbb{P}_{k+1}
\end{tikzcd}$$

where $q_{H_{k+1}} : U_k \to q_{H_{k+1}}(U_k)$ is a geometric quotient and all unmarked inclusions are open. By abuse of notation, let $\mathcal{U}_{k+1}^{\mathcal{S}^{\mathcal{C}_a}} \cap \mathbb{P}_{k+1}^{\mathrm{s}(Q_{k+2},\mathcal{O}(1))}$ be the intersection of $\mathcal{U}_{k+1}^{\mathcal{S}^{\mathcal{C}_a}}$ with the open subset $(\overline{\phi}_{k+1})^{-1}(\mathbb{P}_{k+1}^{\mathrm{s}(Q_{k+2},\mathcal{O}(1))})$ of $\mathcal{U}_{k+1}$. We claim that

$$\mathcal{U}_{k+1}^{\mathcal{S}^{\mathcal{C}_a}} \cap \mathbb{P}_{k+1}^{\mathrm{s}(Q_{k+2},\mathcal{O}(1))} = \bigcup_i \mathrm{Spec}((S^{H_{k+1}})_{(f_i)})$$

for some collection f_i of global H_{k+2}-invariant sections of positive tensor powers of $L \to X$ such that the natural Q_{k+2}-invariant maps

$$\mathrm{Spec}((S^{H_{k+1}})_{(f_i)}) \to \mathrm{Spec}((S^{H_{k+2}})_{(f_i)})$$

define principal Q_{k+2}-bundles in the case $k \leq m-2$, or geometric $Q_{m+1} = H_r$-quotients with the stabilisers for the action of H_r on $\mathrm{Spec}((S^{H_m})_{(f_i)})$ all finite in the case $k = m-1$.

Indeed, it follows from Definition 4.1.2, 1 of a generalised enveloping system that, for each $f \in \mathcal{S}^{\mathcal{C}_a}$ (which is an H-invariant), the map $\overline{\phi}_{k+1}$ restricts to define a Q_{k+2}-equivariant closed immersion $\mathrm{Spec}((S^{H_{k+1}})_{(f)}) \hookrightarrow (\mathbb{P}_{k+1})_F$ (where $(\mathbb{P}_{k+1})_F$ is the non-vanishing locus in $\mathbb{P}_{k+1}$ of $F = f$ thought of as a section in $H^0(\mathbb{P}_{k+1}, \mathcal{O}(1))^{H_{k+2}}$). Let $\tilde{F}$ be a Q_{k+2}-invariant section of some positive tensor power of $\mathcal{O}_{\mathbb{P}_{k+1}}(1) \to \mathbb{P}_{k+1}$ such that $(\mathbb{P}_{k+1})_{\tilde{F}}$ has an affine locally trivial Q_{k+2}-quotient (if $k \leq m-2$) or has an affine geometric $Q_{m+1} = H_r$-quotient with the stabilisers of the action of H_r on $(\mathbb{P}_{k+1})_{\tilde{F}}$ all finite (if $k = m-1$). Letting $\tilde{f} = (\phi_{k+1})^*\tilde{F}$ be the corresponding H_{k+2}-invariant section over X, we see that $\mathrm{Spec}((S^{H_{k+1}})_{(f\tilde{f})})$ is a Q_{k+2}-stable affine closed subset of $(\mathbb{P}_{k+1})_{F\tilde{F}}$ via $\overline{\phi}_{k+1}$. Hence, $\mathrm{Spec}((S^{H_{k+1}})_{(f\tilde{f})})$ also has an affine Q_{k+2}-quotient which is locally trivial if $k \leq m-2$, or just geometric with all H_r-stabilisers in $\mathrm{Spec}((S^{H_m})_{(f\tilde{f})})$ finite if $k = m-1$. Because X is irreducible, this quotient map is therefore given by the natural morphism

$$\mathrm{Spec}((S^{H_{k+1}})_{(f\tilde{f})}) \to \mathrm{Spec}((S^{H_{k+2}})_{(f\tilde{f})}).$$

The claim is now established by letting the f_i be a collection of sections over X of the form $f\tilde{f}$ just described, such that the $\mathrm{Spec}((S^{H_{k+1}})_{(f_i)})$ cover $\mathcal{U}^{\mathcal{S}^{\mathcal{C}_a}}_{k+1} \cap \mathbb{P}^{\mathrm{s}(Q_{k+2},\mathcal{O}(1))}_{k+1}$.

From the claim we see that the natural map

$$q_{Q_{k+2}} : \mathcal{U}^{\mathcal{S}^{\mathcal{C}_a}}_{k+1} \to \mathcal{U}^{\mathcal{S}^{\mathcal{C}_a}}_{k+2}$$

restricts to define a geometric quotient

$$q_{Q_{k+2}} : \mathcal{U}^{\mathcal{S}^{\mathcal{C}_a}}_{k+1} \cap \mathbb{P}^{\mathrm{s}(Q_{k+2},\mathcal{O}(1))}_{k+1} \to \bigcup_i \mathrm{Spec}((S^{H_{k+2}})_{(f_i)}),$$

which is locally trivial if $k \leq m-2$. Consider

$$U_{k+1} = U_k \cap \Pi^{-1}_{k+1}(\mathbb{P}^{\mathrm{s}(Q_{k+2},\mathcal{O}(1))}_{k+1})$$

and its image $q_{H_{k+1}}(U_{k+1}) \subseteq \mathcal{U}^{\mathcal{S}^{\mathcal{C}_a}}_{k+1} \cap \mathbb{P}^{\mathrm{s}(Q_{k+2},\mathcal{O}(1))}_{k+1}$ under the H_{k+1}-quotient $q_{H_{k+1}} : U_k \to q_{H_{k+1}}(U_k)$; this is an H_{k+2}-stable open subset of $\mathcal{U}^{\mathcal{S}^{\mathcal{C}_a}}_{k+1} \cap \mathbb{P}^{\mathrm{s}(Q_{k+2},\mathcal{O}(1))}_{k+1}$. Then $q_{H_{k+2}}(U_{k+1}) = q_{Q_{k+2}}(q_{H_{k+1}}(U_{k+1}))$ is an open subset of $\bigcup_i \mathrm{Spec}((S^{H_{k+2}})_{(f_i)})$, so is covered by affine open subsets $\mathrm{Spec}((S^{H_{k+2}})_{(f_{i,l})})$, where $f_{i,l} = f_i f_l$ with f_l some H_{k+2}-invariant over X. Thus

$$U_{k+1} = q^{-1}_{H_{k+2}}(q_{H_{k+2}}(U_{k+1})) = \bigcup_{i,l} X_{f_{i,l}}$$

and each $X_{f_{i,l}}$ is affine because $L \to X$ is ample over a projective variety. Furthermore, the restriction

$$q_{H_{k+2}} : X_{f_{i,l}} \overset{q_{H_{k+1}}}{\longrightarrow} \operatorname{Spec}((S^{H_{k+1}})_{(f_{i,l})}) \overset{q_{Q_{k+2}}}{\longrightarrow} \operatorname{Spec}((S^{H_{k+2}})_{(f_{i,l})})$$

is a geometric H_{k+2}-quotient, by composition of geometric quotients. If $k \leq m-2$ then it is a locally trivial quotient, by virtue of Lemma 4.2.2. In the case $k = m-1$, the $H_{k+1} = H_u$-quotient

$$q_{H_u} : X_{f_{i,l}} \to q_{H_r}^{-1}(\operatorname{Spec}((S^H)_{(f_{i,l})})) = \operatorname{Spec}((S^{H_u})_{(f_{i,l})})$$

is a locally trivial H_u-quotient with the action of H_r on the affine $X_{f_{i,l}}/H_u$ being closed and having all stabilisers finite, so by Lemma 2.3.1 the action of H on $X_{f_{i,l}}$ is closed with all stabilisers finite. Thus each $X_{f_{i,l}} \subseteq X^{\mathrm{s}(H_{k+2})}$ and so $U_{k+1} \subseteq X^{\mathrm{s}(H_{k+2})}$, as desired. □

Corollary 4.2.5. *In the setting of Proposition 4.2.4, if* $\mathbb{P}_j^{\mathrm{s}(Q_{j+1},\mathcal{O}(1))} = \mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}(1))}$ *for each* $j = 0, \dots, m$*, then*

$$X^{\mathrm{ss},\mathcal{C}_a} \cap \bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{s}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}) = X^{\mathrm{s}} = X^{\mathrm{ss},\mathcal{C}_a}.$$

From Proposition 4.2.4 we see that setting

$$U_1 = X^{\mathrm{ss},\mathcal{C}_a} \cap \bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{s}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}), \quad U_2 = X \cap \bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))})$$

gives us H-stable open subsets of X such that $U_1 \subseteq X^{\mathrm{s}} \subseteq X^{\mathrm{ss},\mathcal{C}_a} \subseteq U_2$. Unfortunately these sets are not easy to compute in general. The first issue is the fact that they rely on taking a generalised enveloping system $V \subseteq H^0(X, L^{\otimes r})$ for some big $r > 0$ from the outset, and the sets $\mathbb{P}_j^{\mathrm{ss,fg}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}$ and $\mathbb{P}_j^{\mathrm{s}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))}$ therefore require an understanding of the action of $Q_{j+1} \cong \mathbb{G}_a$ on V^{H_j} for each j. Besides this, observe that U_1 is defined by taking an intersection with the generalised semistable locus $X^{\mathrm{ss},\mathcal{C}_a}$, which again is hard to compute. We would ideally like to instead define U_1 as an intersection of the variety X with $\bigcap_{j=0}^{m} \Pi_j^{-1}(\mathbb{P}_j^{\mathrm{s}(Q_{j+1},\mathcal{O}_{\mathbb{P}_j}(1))})$, but the obstruction to doing this seems to be the need to find appropriate H-invariant sections to guarantee that U_1 is contained in the stable locus X^{s}—this is what taking the intersection with the generalised semistable locus did for us in the proof of Proposition 4.2.4. A possible way around this might be to assume some nice condition on X, for example that the punctured cone $\hat{X} \subseteq \Bbbk^{n+1} \setminus \{0\}$ over $X \subseteq \mathbb{P}^n$ has a ring of regular functions $\mathcal{O}(\hat{X})$ that is a unique factorisation domain; for then any H-invariant codimension 1 closed subset of X is defined by the vanishing of an invariant section[1]. One can then try to use geometric reasoning to build invariants.

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[1] This observation was pointed out to us by Brent Doran.

Chapter 5

Linearisations of Positive $\mathbb{G}_m$-Extensions of Unipotent Groups

In this final chapter we apply the techniques of our non-reductive geometric invariant theory in Chapters 2 and 3 to study linearisations of positive extensions of unipotent groups, defined in Definition 5.1.1. These are $\mathbb{G}_m$-extensions $\hat{U} = U \rtimes \mathbb{G}_m$ of unipotent groups U where $\mathbb{G}_m$ acts on $\operatorname{Lie} U$ with all weights positive. There are a number of interesting moduli problems in algebraic geometry: for example, in the study of invariant jet differentials [GG80, Dem97, DMR10, Mer10, BK12], or the construction of moduli spaces of hypersurfaces in weighted projective spaces [Kir09], or more generally of hypersurfaces in toric varieties [BDHK16]. It also naturally arises when trying to take quotients of the unstable strata arising in a reductive GIT scenario [Kem78, Hes78, Kir84, Nes84]. Here, an unstable stratum S_β of the linearisation of a reductive group G is equal to $G \times^{P_\beta} Y_\beta$ for a suitable subvariety Y_β and parabolic subgroup $P_\beta \subseteq G$, so that taking a quotient of S_β by G corresponds, roughly speaking, to taking a quotient of Y_β by P_β. The group P_β contains a natural positive extension of its unipotent radical which is normal in P_β, so positive $\mathbb{G}_m$-extensions hold the key to understanding the quotient of S_β by G.

Our main result is Theorem 5.1.4, where we consider irreducible projective varieties X with very ample U-linearisations $L \to X$ that extend to a linearisation of a positive extension $\hat{U}$ of U, such that a certain condition ($\mathfrak{C}$) is satisfied (see Section 5.1). In this case we may consider the open subset $X^0_{\hat{U},L}$ of points in X that have a limit point in $Z_{\hat{U},L}$ under the action of $t \in \mathbb{G}_m$, as $t \to 0$, where $Z_{\hat{U},L}$ is the set of $\mathbb{G}_m$-fixed points $z \in X$ such that $\mathbb{G}_m$

acts on $L^*|_z$ with minimal possible weight (see Definition 5.1.2). Assuming ($\mathfrak{C}$), we show that $X^0_{\hat{U},L}$ admits a locally trivial U-quotient $X^0_{\hat{U},L}/U$ under the enveloping quotient map for the U-linearisation $L \to X$. Furthermore, assuming $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ is nonempty, we prove that, for a suitable rational character χ of $\hat{U}$, the stable locus and finitely generated semistable locus for the twisted linearisation $L^{(\chi)} \to X$ are both equal to $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$, and the enveloping quotient $X \wr\wr_{L^{(\chi)}} \hat{U}$ is a projective variety that is a geometric quotient for the $\hat{U}$-action on $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$. Notice that the open subset $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ can be computed purely in terms of the actions of U and the one parameter subgroup $\mathbb{G}_m \subseteq \hat{U}$ on X, so this theorem can be seen as a sort of Hilbert-Mumford-type result for linearisations of positive extensions of unipotent groups. As an application of Theorem 5.1.4, under the assumption of ($\mathfrak{C}$) we also construct a compactification of $X^0_{\hat{U},L}/U$ as the enveloping quotient of a suitable $\hat{U}$-linearisation $\mathcal{L}$ over the natural compactification

$$\overline{\hat{U} \times^U X} = \overline{\mathbb{G}_m \times X} = \mathbb{P}^1 \times X$$

of $\hat{U} \times^U X$. This satisfies the further property that the associated enveloping quotient map $(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})} \to (\mathbb{P}^1 \times X) \wr\wr_{\mathcal{L}} \hat{U}$ is a geometric quotient for the $\hat{U}$-action on $(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})}$ and $(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})}$ admits a similarly explicit description in terms of subvarieties $Z_{\hat{U},L}$ and $X^0_{\hat{U},L}$.

In Section 5.1 we introduce the definitions and notation needed to state Theorem 5.1.4. Section 5.3 is devoted to the proof of this theorem and its application to constructing a compactification of $X^0_{\hat{U},L}/U$ (see Corollary 5.3.8). Broadly speaking, the proof of Theorem 5.1.4 can be split into two parts. The first is concerned with showing that $X^0_{\hat{U},L}$ is contained in the stable locus for the U-linearisation $L \to X$, utilising condition ($\mathfrak{C}$) and an induction on the dimension of U. The second part uses the U-invariant sections of a sufficiently large tensor power of $L \to X$ to construct a projective completion $\overline{X^0_{\hat{U},L}/U}$ of $X^0_{\hat{U},L}/U$, together with a very ample $\mathbb{G}_m$-linearisation. By twisting the $\hat{U}$-linearisation on $L \to X$ appropriately and using the techniques of Mumford's reductive GIT [MFK94] on the resulting linearisation over $\overline{X^0_{\hat{U},L}/U}$, we deduce that $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ is equal to the stable and finitely generated semistable loci for the twisted $\hat{U}$-linearisation over X, and the enveloping quotient is a geometric quotient of $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ that is isomorphic to the (projective) GIT quotient $\overline{X^0_{\hat{U},L}/U} /\!\!/ \mathbb{G}_m$.

Section 5.2 provides something of a tangent to the main thrust of the chapter. Here we consider a proof of a weaker version of the main theorem, in the case where $U = \mathbb{G}_a$ is the additive group and a stronger condition ($\mathfrak{C}^*$) is satisfied (see Theorem 5.2.2). The main ingredient here is an application of the results on strong reductive envelopes from Section 3.2.1 in Chapter 3 to study the non-reductive geometric invariant theory of $\hat{U}$-linearisations of the hyperplane line bundle on projective spaces; the result for more general projective varieties X is then deduced by restricting from the projective space case. Thus Section 5.2 provides a further illustration of how the material from Chapter 3 can be used to study the actions of certain non-reductive groups. Note that the results from Section 5.2 are not required for proof of the main theorem 5.1.4 in Section 5.3.[1]

5.1 Preparation and Statement of Main Result

Throughout this chapter X will be an irreducible *projective* variety and $L \to X$ a *very ample* line bundle, unless otherwise stated. Let $\lambda : \mathbb{G}_m \to \mathrm{Aut}(U)$ be a one-parameter group of automorphisms (i.e. a homomorphism of abstract groups) and let

$$\hat{U} := U \rtimes_\lambda \mathbb{G}_m$$

be the semi-direct product, whose points are denoted $(u;t)$ (where $u \in U$ and $t \in \mathbb{G}_m$) and with multiplication given as follows:

$$(u_1;t_1) \cdot (u_2;t_2) = ((\lambda(t_2^{-1})(u_1))u_2; t_1 t_2), \quad u_i \in U, \ t_i \in \mathbb{G}_m.$$

Note that the pointwise derivative of λ defines a $\mathbb{G}_m$-action on $\mathrm{Lie}\, U$, which may be diagonalised with weights in $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$. We will consider the following kinds of extension of U:

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[1] The unusual layout of this chapter has its roots in earlier attempts to prove the main theorem 5.1.4. The original idea was prove Theorem 5.1.4 under the stronger assumption ($\mathfrak{C}^*$) in an inductive fashion, where the material in Section 5.2 would provide both the base case and the main ingredient of the inductive step of the proof. Late in the writing of this thesis an error was discovered with this approach, out of which the current Theorem 5.1.4 (requiring us only to assume condition ($\mathfrak{C}$)) and its proof in Section 5.3 were born. Given that the techniques used in Section 5.2 follow on naturally from the preceding chapters and are different to those of Section 5.3, we felt justified in keeping the material from Section 5.2 in this thesis.

Definition 5.1.1. Let U be a unipotent group and let $\lambda : \mathbb{G}_m \to \mathrm{Aut}(U)$ be a one-parameter group of automorphisms. We say U is *positively graded* if the induced action of $\mathbb{G}_m$ on $\mathrm{Lie}\, U$ has all weights positive. We refer to $\hat{U} = U \rtimes_\lambda \mathbb{G}_m$ as a *positive extension* of U.

Throughout this chapter, we will be concerned with studying the non-reductive geometric invariant theory of linearisations of positively graded unipotent groups U, in the case where the linearisation $U \curvearrowright L \to X$ extends to $\hat{U} \curvearrowright L \to X$, where $\hat{U}$ is a positive extension of U. We take a moment to set some notation and conventions that will be used throughout the chapter.

Firstly, we will use *additive notation* when working with characters of a group H. Thus, if $\chi_1, \chi_2 \in \mathrm{Hom}(H, \mathbb{G}_m)$ then $\chi_1 + \chi_2$ is the character sending $h \mapsto \chi_1(h)\chi_2(h) \in \mathbb{G}_m$, and $-\chi_1$ is the character sending $h \mapsto \chi_1(h)^{-1} \in \mathbb{G}_m$. By a *rational character* we mean an element of the $\mathbb{Q}$-vector space $\mathrm{Hom}(H, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$.

It will also be convenient to work with *rational linearisations*. Recall from Definition 1.1.18 that a rational linearisation on an H-variety X is an element of the $\mathbb{Q}$-vector space $\mathrm{Pic}^H(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. As observed in Remark 2.4.3, stability, finitely generated semistability, the enveloping quotient and all other spaces discussed in Theorem 2.4.2 are defined for rational linearisations: if $\mathcal{L} \in \mathrm{Pic}^H(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ then finitely generated semistability, stability, the enveloping quotient etc. can be defined for $\mathcal{L}$ by choosing an integer $n > 0$ such that $n\mathcal{L}$ lives in $\mathrm{Pic}^H(X)$ and considering the corresponding notions for the linearisation $n\mathcal{L} \to X$ (this is well-defined by virtue of Remark 1.1.19). Note that in this situation we will usually write $\mathcal{L}^{\otimes n}$ for $n\mathcal{L}$.

The following kind of rational linearisation will arise frequently. The group of characters $\mathrm{Hom}(H, \mathbb{G}_m)$ embeds into $\mathrm{Pic}^H(X)$ by sending a character χ to the linearisation $\mathcal{O}_X^{(\chi)} \to X$ defined by acting fibre-wise on the trivial bundle $\mathcal{O}_X \to X$ by multiplication by $-\chi$. This induces a map of vector spaces $\mathrm{Hom}(H, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Pic}^H(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. If now χ is a rational character and $\mathcal{O}_X^{(\chi)} \to X$ denotes the image of χ under this map, then given any rational linearisation $\mathcal{L} \to X$ we define $\mathcal{L}^{(\chi)} := \mathcal{L} \otimes \mathcal{O}_X^{(\chi)} \to X$ inside $\mathrm{Pic}^H(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Notice in this case that for sufficiently divisible integers $n > 0$ the rational linearisation $(\mathcal{L}^{(\chi)})^{\otimes n}$

may be written as $(\mathcal{L}^{\otimes n})^{(n\chi)}$, where $\mathcal{L}^{\otimes n}$ defines an element of $\mathrm{Pic}^H(X)$ and $n\chi$ defines a character of H.

We shall also talk about twisting representations by characters. If V is a representation of a linear algebraic group H and $\chi : H \to \mathbb{G}_m$ is a (non-rational) character, then we denote by $V^{(\chi)}$ the representation of H which is equal to V as a vector space and has action

$$H \times V^{(\chi)} \to V^{(\chi)}, \quad (h, v) \mapsto \chi(h) h \cdot v$$

where $h \cdot v$ denotes the given action of H on V.[2]

Let us finally note that the natural inclusion $\mathbb{G}_m \hookrightarrow \hat{U}$ induces a canonical identification $\hat{U}/U = \mathbb{G}_m$. We shall make implicit use of this, particularly when talking about characters of $\hat{U}$: the character $\hat{U} \to \mathbb{G}_m$ of weight $a \in \mathbb{Z}$ is the morphism $(u; t) \mapsto t^a$.

We next use the extension of the U-linearisation on $L \to X$ to one of $\hat{U}$ to define some subvarieties of X. The extension gives an induced $\mathbb{G}_m$-action on the space $V := H^0(X, L)^*$; let

$$\varpi := \text{minimal weight in } \mathbb{Z} \text{ for the } \mathbb{G}_m\text{-action on } V$$

and

$$V_{\min} := \{v \in V \mid t \cdot v = t^{\varpi} v \text{ for all } t \in \mathbb{G}_m\}$$

the associated weight space. Then $\mathbb{P}(V_{\min})$ may be naturally regarded as a linear subspace of $\mathbb{P}(V)$, and $X \cap \mathbb{P}(V_{\min})$ is a nonempty subvariety, consisting of all fixed points $x \in X$ such that $\mathbb{G}_m$ acts on the fibre of L^* over x with weight ϖ. Indeed, we may equivalently define

$$\varpi = \min \left\{ a \in \mathbb{Z} \;\middle|\; \begin{array}{c} \text{there is } x \in X \text{ fixed by } \mathbb{G}_m \text{ such} \\ \text{that } \mathbb{G}_m \text{ acts on } L^*|_x \text{ with weight } a \end{array} \right\}. \tag{5.1.1}$$

Definition 5.1.2. Suppose $\hat{U}$ is a $\mathbb{G}_m$-extension of a unipotent group U and $\hat{U} \curvearrowright L \to X$ a very ample linearisation over a projective $\hat{U}$-variety X. As above, let ϖ be the minimal $\mathbb{G}_m$-weight on fibres of L^* over $\mathbb{G}_m$-fixed points in X. Define

$$Z_{\hat{U},L} := X \cap \mathbb{P}(V_{\min}) = \left\{ x \in X \;\middle|\; \begin{array}{c} x \text{ is a } \mathbb{G}_m\text{-fixed point and} \\ \mathbb{G}_m \text{ acts on } L^*|_x \text{ with weight } \varpi \end{array} \right\}$$

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[2] Note this is dual to the convention used when twisting a linearisation by a character. The reason for this is that, if V is a finite-dimensional H-representation and $H \curvearrowright \mathcal{O}(1) \to \mathbb{P}(V)$ the canonical linearisation as in Example 1.1.14, then for χ a character of H we have the twisted linearisation $\mathcal{O}(1)^{(\chi)} \to \mathbb{P}(V)$ equal to the canonical linearisation $\mathcal{O}(1) \to \mathbb{P}(V^{(\chi)}) = \mathbb{P}(V)$ induced by $V^{(\chi)}$.

and

$$X^0_{\hat{U},L} := \{x \in X \mid \lim_{t\to 0} t\cdot x \in Z_{\hat{U},L}\} \quad (t \in \mathbb{G}_m \subseteq \hat{U}).$$

Remark 5.1.3. It is clear that the subvarieties $Z_{\hat{U},L}$ and $X^0_{\hat{U},L}$ are unaffected when replacing $\hat{U} \curvearrowright L \to X$ by any element of the positive $\mathbb{Q}$-ray defined by L in $\operatorname{Pic}^{\hat{U}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Also, $X^0_{\hat{U},L}$ is a $\hat{U}$-stable open subset of X, while the U-sweep $U \cdot Z_{\hat{U},L}$ of $Z_{\hat{U},L}$ is also stable under the $\hat{U}$-action.

It will be necessary to require the following assumption:

$$\operatorname{Stab}_U(z) = \{e\} \text{ for every } z \in Z_{\hat{U},L}. \tag{$\mathfrak{C}$}$$

We can now state the main result of this chapter:

Theorem 5.1.4. *Let X be an irreducible projective variety acted upon by a unipotent group U and let $L \to X$ be a very ample linearisation. Suppose the linearisation extends to a linearisation of a positive extension $\hat{U} = U \rtimes_\lambda \mathbb{G}_m$ of U satisfying condition ($\mathfrak{C}$). Then*

1. *we have $X^0_{\hat{U},L} \subseteq X^{\mathrm{s}(U,L)}$, with the restriction of the enveloping quotient map for $U \curvearrowright L \to X$ defining a locally trivial U-quotient $X^0_{\hat{U},L} \to X^0_{\hat{U},L}/U$.*

Suppose furthermore that $X^0_{\hat{U},L} \neq U \cdot Z_{\hat{U},L}$, and let $L^{(\chi)}$ be the rational $\hat{U}$-linearisation resulting from twisting $\hat{U} \curvearrowright L \to X$ by the rational character χ of weight $-\varpi - \epsilon$, where ϖ is the minimal weight for the $\mathbb{G}_m$-action on $V := H^0(X,L)^*$ and ϵ is a positive rational number. Then for sufficiently small $\epsilon > 0$,

2. *there are equalities $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}) = X^{\mathrm{s}(L^{(\chi)})} = X^{\mathrm{ss,fg}(L^{(\chi)})}$;*

3. *the enveloping quotient $X \wr\wr_{L^{(\chi)}} \hat{U}$ is a projective variety and for suitably divisible integers $r > 0$ the ring of invariants $\Bbbk[X, (L^{(\chi)})^{\otimes r}]^{\hat{U}}$ for the linearisation $(L^{(\chi)})^{\otimes r} \to X$ is finitely generated, with*

$$X \wr\wr_{L^{(\chi)}} \hat{U} = \operatorname{Proj}(\Bbbk[X, (L^{(\chi)})^{\otimes r}]^{\hat{U}});$$

and

4. *the enveloping quotient map $X^{\mathrm{ss,fg}(L^{(\chi)})} \to X \wr\!\wr_{L^{(\chi)}} \hat{U}$ is a geometric quotient for the $\hat{U}$-action on $X^{\mathrm{ss,fg}(L^{(\chi)})}$.*

Let us also mention here that in Section 5.3.2 we shall use Theorem 5.1.4 to show that the geometric quotient $X^0_{\hat{U},L}/U$ can be compactified via a certain enveloping quotient, which itself is a geometric quotient of the associated finitely generated semistable locus (see Corollary 5.3.8).

We make a few comments regarding the significance of Theorem 5.1.4. Firstly, for any very ample linearisation of a projective U-variety that can be extended to a linearisation of a positive extension $\hat{U}$ of U, it identifies an open subset of X that admits a projective geometric quotient and can be computed by studying the induced $\mathbb{G}_m$-action. This can be seen as a sort of Hilbert-Mumford-like result. (Indeed, this viewpoint can be made mathematically concrete: by using the classical Hilbert-Mumford criterion (see Section 1.2) it is not difficult to see that the semistable and stable loci for the rational $\mathbb{G}_m$-linearisation $L^{(\chi)} \to X$ are both equal to $X^0_{\hat{U},L} \setminus Z_{\hat{U},L}$, so the open subset $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ is the complement of the U-sweep of the unstable locus for the $\mathbb{G}_m$-linearisation.) Secondly, Theorem 5.1.4 provides us with examples of settings within non-reductive GIT where the enveloping quotient is a good categorical quotient—compare with the situation in Mumford's reductive GIT [MFK94], specifically Theorem 1.2.4, 1 in Chapter 1. Thirdly, we can interpret Theorem 5.1.4 as saying something about the algebraic invariant theory of actions of positive extensions $\hat{U}$ of unipotent groups U: if $\hat{U}$ acts on a finitely generated graded $\mathbb{k}$-algebra R with $R_0 = \mathbb{k}$, then while it may be the case that the ring of invariants $R^{\hat{U}}$ is not a finitely generated $\mathbb{k}$-algebra, after modifying the action by a suitable character we can always achieve finite generation of the ring of invariants of a certain Veronese subalgebra; cf. Theorem 5.1.4, 3.

The proof of Theorem 5.1.4 in the generality stated will be proved in Section 5.3. In the upcoming Section 5.2 we will prove Theorem 5.1.4 for the case where $U \cong \mathbb{G}_a$ under a stronger assumption than that of condition ($\mathfrak{C}$), using the theory of strong reductive envelopes developed in Section 3.2 of Chapter 3. Note that the proof of the main theorem given in Section 5.3 does not rely on any of the material in Section 5.2.

5.2 Quotients by Positive Extensions of $\mathbb{G}_a$ via Reductive Envelopes

In this section we examine the case where $U = \mathbb{G}_a$ is the additive group and $\hat{U} \curvearrowright L \to X$ is a linearisation of a positive extension $\hat{U}$ of $\mathbb{G}_a$. Then the one-parameter family of automorphisms $\lambda : \mathbb{G}_m \to \mathrm{Aut}(\mathbb{G}_a)$ acts on $\mathrm{Lie}\,\mathbb{G}_a = \Bbbk$ with nonzero positive weight $\ell \in \mathbb{Z}_{>0}$; for this section only, when necessary we shall indicate this weight by writing one of

$$\hat{U} = \hat{U}^{[\ell]} = U \rtimes_\ell \mathbb{G}_m.$$

We wish to prove a weaker version of Theorem 5.1.4 using the theory of strong reductive envelopes (see Section 3.2 of Chapter 3). Apart from the restriction $U = \mathbb{G}_a$, we will need to require the following assumption on $\hat{U} \curvearrowright L \to X$:

$$\mathrm{Stab}_U(v) = \{e\} \text{ for all } v \in V_{\min} \setminus \{0\}. \tag{$\mathfrak{C}^*$}$$

Remark 5.2.1. Notice that, because $Z_{\hat{U},L} = X \cap \mathbb{P}(V_{\min})$, we have $(\mathfrak{C}^*) \implies (\mathfrak{C})$.

The aim of this section is to prove the following theorem.

Theorem 5.2.2. *Let X be an irreducible projective variety acted upon by $U = \mathbb{G}_a$ and let $L \to X$ be a very ample linearisation. Suppose the linearisation extends to one of $\hat{U} = \hat{U}^{[\ell]} = U \rtimes_\ell \mathbb{G}_m$ with $\ell > 0$ and let $L^{(\chi)}$ be the rational $\hat{U}$-linearisation resulting from twisting $\hat{U} \curvearrowright L \to X$ by the rational character χ of weight $-\varpi - \epsilon$, where ϖ is the minimal weight for the $\mathbb{G}_m$-action on $V := H^0(X, L)^*$ and ϵ is any rational number such that $0 < \epsilon < 1/2$. If condition $(\mathfrak{C}^*)$ is satisfied, then*

1. *there are equalities $X^{\mathrm{s}(L^{(\chi)})} = X^{\mathrm{ss,fg}(L^{(\chi)})} = X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$;*
2. *the enveloping quotient $X \wr\wr_{L^{(\chi)}} \hat{U}$ is a projective variety and for suitably divisible integers $r > 0$ the ring of invariants $\Bbbk[X, (L^{\otimes r})^{(r\chi)}]^{\hat{U}}$ is finitely generated, with*

$$X \wr\wr_{L^{(\chi)}} \hat{U} = \mathrm{Proj}(\Bbbk[X, (L^{\otimes r})^{(r\chi)}]^{\hat{U}});$$

and

3. *the enveloping quotient map* $X^{\mathrm{ss,fg}(L^{(\chi)})} \to X /\!\!/_{L^{(\chi)}} \hat{U}$ *is a geometric quotient for the* $\hat{U}$*-action on* $X^{\mathrm{ss,fg}(L^{(\chi)})}$.

The proof of Theorem 5.2.2 will take some time, so we first provide a sketch of the argument. We first look to prove the theorem in the case where $X = \mathbb{P}(V)$ and $L = \mathcal{O}(1) \to \mathbb{P}(V)$, for a finite dimensional $\hat{U}$-representation V. This is done by explicit calculation: we construct a strong ample reductive envelope for the twisted rational linearisation $\mathcal{O}(1)^{(\chi)} \to \mathbb{P}(V)$ using Corollary 3.2.11 of Section 3.2.1, and use the Hilbert-Mumford criterion to compute stability and semistability for this reductive envelope. Through the choice of twist, stability and semistability for the reductive envelope will turn out to be equivalent conditions. Proposition 3.2.2 is then invoked to show that the stable and finitely generated semistable loci are equal for $\hat{U} \curvearrowright \mathcal{O}(1)^{(\chi)} \to \mathbb{P}(V)$, and Corollary 3.2.11 is used to show that the associated enveloping quotient is projective. By understanding the stable locus explicitly, we prove that the enveloping quotient map $\mathbb{P}(V)^{\mathrm{ss,fg}(\mathcal{O}(1)^{(\chi)})} \to \mathbb{P}(V) /\!\!/_{\mathcal{O}(1)^{(\chi)}} \hat{U}$ is a geometric quotient for the $\hat{U}$-action on $\mathbb{P}(V)^{\mathrm{ss,fg}(\mathcal{O}(1)^{(\chi)})}$. (As an aside, the argument for the projective case $X = \mathbb{P}(V)$ will provide a generalisation of part of the calculation of the example of n unordered points on $\mathbb{P}^1$ under the action of a Borel subgroup of $\mathrm{SL}(2, \mathbb{C})$ in Section 3.3.) Theorem 5.2.2 is then proved in full generality by embedding X into a projective space and using the fact that stability behaves well under closed immersions (cf. Lemma 2.3.6).

5.2.1 The Case $(X, L) = (\mathbb{P}(V), \mathcal{O}(1))$

Let V be a finite-dimensional representation of $\hat{U} = U \rtimes_{\ell} \mathbb{G}_m$ and $X = \mathbb{P}(V)$, with $\hat{U} \curvearrowright L = \mathcal{O}(1) \to \mathbb{P}(V)$ the canonical linearisation. As usual we write points in $\mathbb{P}(V)$ as equivalence classes $[v]$ of nonzero vectors $v \in V$ under the scaling action of $\mathbb{G}_m$ on V. As before, let $V_{\min}$ be the $\mathbb{G}_m$-weight space of minimal weight ϖ, so that $\mathbb{P}(V)^0_{\hat{U},\mathcal{O}(1)}$ is the open subset of points flowing to $\mathbb{P}(V_{\min})$ under the action of $t \in \mathbb{G}_m$, as $t \to 0$; cf. Definition 5.1.2.

The version of Theorem 5.2.2 we wish to prove is

Proposition 5.2.3. *Retain the preceding notation. Let χ be the rational character of $\hat{U}$ of weight $-\varpi - \epsilon$, where ϵ is any rational number such that $0 < \epsilon < 1/2$, and let $\mathcal{O}(1)^{(\chi)} \to \mathbb{P}(V)$ be the rational linearisation obtained by twisting $\mathcal{O}(1) \to \mathbb{P}(V)$ by χ. If $V_{\min}$ does not contain any fixed points for the $\mathbb{G}_a$-action on V, then*

1. *there are equalities $\mathbb{P}(V)^{\mathrm{s}(\mathcal{O}(1)^{(\chi)})} = \mathbb{P}(V)^{\mathrm{ss,fg}(\mathcal{O}(1)^{(\chi)})} = \mathbb{P}(V)^{0}_{\hat{U},\mathcal{O}(1)} \setminus (U \cdot \mathbb{P}(V_{\min}))$;*
2. *the enveloping quotient $\mathbb{P}(V) \wr\wr_{\mathcal{O}(1)^{(\chi)}} \hat{U}$ is a projective variety and for suitably divisible integers $r > 0$ the ring of invariants $\Bbbk[\mathbb{P}(V), \mathcal{O}(r)^{(r\chi)}]^{\hat{U}}$ is finitely generated; and*
3. *the enveloping quotient map $\mathbb{P}(V)^{\mathrm{ss,fg}(\mathcal{O}(1)^{(\chi)})} \to \mathbb{P}(V) \wr\wr_{\mathcal{O}(1)^{(\chi)}} \hat{U}$ is a geometric quotient for the $\hat{U}$-action on $\mathbb{P}(V)^{\mathrm{ss,fg}(\mathcal{O}(1)^{(\chi)})}$.*

Let us give an indication for the reason for twisting the linearisation by the character χ is to ensure the minimal weight is just less than 0, so that the weight diagram for the resulting linearisation $\mathcal{O}(1)^{(\chi)} \to \mathbb{P}(V)$ looks something like Figure 5.1.

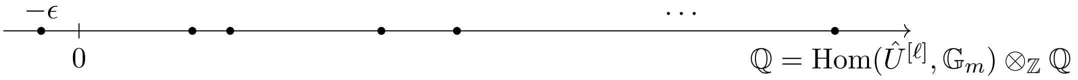

Figure 5.1: Example of distribution of rational weights for $\hat{U}^{[\ell]}$-linearisation $\mathcal{O}(1)^{(\chi)} \to \mathbb{P}(V)$.

This will ensure the corresponding arrangement of weights for the strong reductive envelope is such that stability and semistability for the reductive linearisation coincide, and thus stability and finitely generated semistability for $\mathcal{O}(1)^{(\chi)} \to \mathbb{P}(V)$ coincide. To relate this back to the example of n unordered points on $\mathbb{P}^1$ under the standard Borel subgroup of $\mathrm{SL}(2, \Bbbk)$ (which corresponds to the case $\ell = 2$ here), in the notation of Corollary 3.3.3 we have twisted by the rational character of weight $\frac{r}{m} = n - \epsilon$ (cf. (Case $0 < \frac{r}{m} < n$) of Corollary 3.3.3).

For the remainder of this section we will prove Proposition 5.2.3. In order to study the $\hat{U} = \hat{U}^{[\ell]}$-linearisation $\mathcal{O}(1) \to \mathbb{P}(V)$ we shall use the following trick. Consider the surjective homomorphism

$$\eta_\ell : \hat{U}^{[2\ell]} \to \hat{U}^{[\ell]}, \quad (u; t) \mapsto (u; t^2).$$

Restriction through η_ℓ defines a rational $\hat{U}^{[2\ell]}$-linearisation on $\mathcal{O}(1)^{(\chi)} \to \mathbb{P}(V)$ (cf. Section 2.2.1), which is equal to the rational linearisation

$$\hat{U}^{[2\ell]} \curvearrowright \mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$$

obtained by twisting the canonical linearisation $\hat{U}^{[2\ell]} \curvearrowright \mathcal{O}(1) \to \mathbb{P}(V)$ induced by the $\hat{U}^{[2\ell]}$-representation V by the rational character $\chi^{[2]} := \chi \circ \eta_\ell$ of $\hat{U}^{[2\ell]}$. Recall that if $r > 0$ is an integer such that $r\chi \in \operatorname{Hom}(\hat{U}^{[\ell]}, \mathbb{G}_m)$, then there is an equality

$$\Bbbk[\mathbb{P}(V), \mathcal{O}(r)^{(r\chi)}]^{\hat{U}^{[\ell]}} = \Bbbk[\mathbb{P}(V), \mathcal{O}(r)^{(r\chi^{[2]})}]^{\hat{U}^{[2\ell]}} \subseteq \Bbbk[\mathbb{P}(V), \mathcal{O}(r)],$$

from which it follows that the finitely generated semistable loci for the linearisations $\hat{U}^{[\ell]} \curvearrowright \mathcal{O}(1)^{(\chi)} \to \mathbb{P}(V)$ and $\hat{U}^{[2\ell]} \curvearrowright \mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$ coincide, and the same is true for the enveloping quotients. Furthermore, we have

Lemma 5.2.4. *We have equality of the stable loci,* $X^{\mathrm{s}(\hat{U}^{[2\ell]}, \mathcal{O}(1)^{(\chi^{[2]})})} = X^{\mathrm{s}(\hat{U}^{[\ell]}, \mathcal{O}(1)^{(\chi)})}$.

Proof. As observed, the rings of invariants for the $\hat{U}^{[2\ell]}$-linearisations $\mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$ and $\hat{U}^{[\ell]} \curvearrowright \mathcal{O}(1)^{(\chi)} \to \mathbb{P}(V)$ are equal. Let f be an invariant section with X_f affine. Because $\eta_\ell : \hat{U}^{[2\ell]} \to \hat{U}^{[\ell]}$ has finite kernel, the action of $\hat{U}^{[\ell]}$ on X_f is closed with all stabilisers finite if, and only if, the same is true for the action of $\hat{U}^{[2\ell]}$, and because η_ℓ restricts to identify the unipotent radicals of $\hat{U}^{[\ell]}$ and $\hat{U}^{[2\ell]}$, the natural morphism $X_f \to \operatorname{Spec}(\mathcal{O}(X_f)^{\hat{U}^{[\ell]}}) = \operatorname{Spec}(\mathcal{O}(X_f)^{\hat{U}^{[2\ell]}})$ is a principal $(\hat{U}^{[\ell]})_u$-bundle if, and only if, it is a principal $(\hat{U}^{[2\ell]})_u$-bundle. Thus, by Definition 2.3.2, the stable loci are equal. $\square$

In order to prove Proposition 5.2.3 we may therefore work with the linearisations $\hat{U}^{[2\ell]} \curvearrowright \mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$, without loss of generality. The $\hat{U}^{[2\ell]}$-representation V defined by η_ℓ admits a decomposition

$$V \overset{\hat{U}^{[2\ell]}}{\cong} \bigoplus_{i=1}^{q} \Bbbk^{(a_i)} \otimes \operatorname{Sym}^{l_i} \Bbbk^2, \tag{5.2.1}$$

of $\hat{U}^{[2\ell]}$-modules, where

- $\Bbbk^{(a_i)}$ is the one dimensional representation of $\hat{U}^{[2\ell]}$ defined by the character $\hat{U}^{[2\ell]} \to \mathbb{G}_m$ of weight $a_i \in \mathbb{Z}$;

- $\operatorname{Sym}^{l_i} \Bbbk^2$ is the standard irreducible representation of $G := \mathrm{SL}(2,\Bbbk)$ of highest weight $l_i \geq 0$, upon which $\hat{U}^{[2\ell]}$ acts via the surjective homomorphism

  $$\rho_\ell : \hat{U}^{[2\ell]} \to \hat{U}^{[2]}, \quad (u;t) \mapsto (u;t^\ell)$$

  and the identification of $\hat{U}^{[2]}$ with the Borel subgroup $B \subseteq G$ of upper triangular matrices given by

  $$\hat{U}^{[2]} = \mathbb{G}_a \rtimes \mathbb{G}_m \to B, \quad (u;t) \mapsto \left(\begin{smallmatrix} t & tu \\ 0 & t^{-1} \end{smallmatrix}\right);$$

  and

- because the action of $\hat{U}^{[2\ell]}$ factors through $\eta_\ell : \hat{U}^{[2\ell]} \to \hat{U}^{[\ell]}$, we have $a_i \equiv \ell\, l_i \mod 2$ for each $i = 1, \ldots, q$.

We wish to use the construction (3.2.3) of Section 3.2.1 to produce a strong reductive envelope for the linearisation $\hat{U}^{[2\ell]} \curvearrowright \mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$. The first task is to ensure conditions (C1) and (C2) of Section 3.2.1 are satisfied. Observe that $\rho_\ell : \hat{U}^{[2\ell]} \to \hat{U}^{[2]} \cong B \subseteq G$ restricts to give the standard inclusion of $U = \mathbb{G}_a = (\hat{U}^{[2\ell]})_u$ inside G as the subgroup of strictly upper triangular matrices, so ρ_ℓ is an $(\hat{U}^{[2\ell]})_u$-faithful homomorphism, in the sense of Definition 2.3.7. The action of $(\hat{U}^{[2\ell]})_u$ on V extends to one of G by demanding that G act on $\operatorname{Sym}^{l_i} \Bbbk^2$ in the usual manner and trivially on $\Bbbk^{(a_i)}$, for each i; we denote this action by "$\underset{G}{\cdot}$". Because $\chi^{[2\ell]}$ restricts to the trivial rational character of $(\hat{U}^{[2\ell]})_u$, the canonically induced linearisation of G on $\mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$ extends $(\hat{U}^{[2\ell]})_u \curvearrowright \mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$ through ρ_ℓ. Thus condition (C1) holds.

As in (3.2.2) of Section 3.2.1 there is therefore an isomorphism of rational $G \times (\hat{U}^{[2\ell]})_r$-linearisations

$$G \times^U (\mathcal{O}(1)^{(\chi^{[2]})}) \cong (G/U) \times (\mathcal{O}(1)^{(\chi^{[2]})}).$$

The next task is to verify condition (C2).

Lemma 5.2.5. *Let $\mathcal{P} = \mathcal{O}(1) \to \mathbb{P}(V)$ as line bundles and consider the $G \times (\hat{U}^{[2\ell]})_r$-linearisation on $\mathcal{P}$ canonically induced by the action $\underset{G}{\cdot}$ of G on V and the following action of $(\hat{U}^{[2\ell]})_r = \mathbb{G}_m$:*

$$\begin{aligned} t \cdot v = \textstyle\sum_i (t^{a_i} z_i) \otimes s_i, \quad & v \in V,\ t \in \mathbb{G}_m, \qquad (5.2.2) \\ & v = \textstyle\sum_i z_i \otimes s_i \in \bigoplus_{i=1}^{q} \mathbb{k}^{(a_i)} \otimes \operatorname{Sym}^{l_i} \mathbb{k}^2 \\ & \text{via (5.2.1).} \end{aligned}$$

Then the rational linearisation of $G \times (\hat{U}^{[2\ell]})_r$ on $(G/U) \times (\mathcal{O}(1)^{(\chi^{[2]})}) \to (G/U) \times \mathbb{P}(V)$ is equal to the product of the twisted rational linearisation $\mathcal{P}^{(\chi^{[2\ell]})} \to \mathbb{P}(V)$ with the $G \times (\hat{U}^{[2\ell]})_r$-action on G/U given by left multiplication by G and right multiplication by $(\hat{U}^{[2\ell]})_r = \mathbb{G}_m$. Thus condition (C2) of Section 3.2.1 is satisfied.

Proof of Lemma 5.2.5. We first describe the rational linearisation $\mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$ more explicitly, as follows. Fix an integer $n > 0$ such that $n\chi^{[2]}$ defines a character, i.e. $n\chi^{[2]} \in \operatorname{Hom}(\hat{U}^{[2\ell]}, \mathbb{G}_m) \subseteq \operatorname{Hom}(\hat{U}^{[2\ell]}, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$. The canonical induced action of $\hat{U}^{[2\ell]}$ on $\operatorname{Sym}^n V$ may be twisted by $n\chi^{[2]}$ to form a representation $(\operatorname{Sym}^n V)^{(n\chi^{[2]})}$, which itself defines a linearisation of $\hat{U}^{[2\ell]}$ on $\mathcal{O}_{\mathbb{P}(V)}(n) \to \mathbb{P}(V)$; note this linearisation is equal to $\hat{U}^{[2\ell]} \curvearrowright (\mathcal{O}(1)^{(\chi^{[2]})})^{\otimes n} \to \mathbb{P}(V)$. The isomorphism of rational $G \times (\hat{U}^{[2\ell]})_r$-linearisations

$$G \times^U (\mathcal{O}(1)^{(\chi^{[2]})}) \cong (G/U) \times (\mathcal{O}(1)^{(\chi^{[2]})})$$

thus arises from the isomorphism

$$G \times^U ((\operatorname{Sym}^n V)^{(\chi^{[2]})}) \cong (G/U) \times ((\operatorname{Sym}^n V)^{(\chi^{[2]})}),$$

where this isomorphism is obtained by using the extension of the U-action on V to the action $\underset{G}{\cdot}$. The corresponding action of G on the right hand side is the diagonal action induced by left multiplication on G/U and the natural action of G on $(\operatorname{Sym}^n V)^{(\chi^{[2]})}$ (note $(\operatorname{Sym}^n V)^{(\chi^{[2]})} = \operatorname{Sym}^n V$ as G-modules). The action of $(\hat{U}^{[2\ell]})_r = \mathbb{G}_m$ on $(gU, \nu) \in (G/U) \times ((\operatorname{Sym}^n V)^{(\chi^{[2]})})$ is formally given by

$$t(gU, \nu) = \left(g\rho_\ell(t)^{-1} U \, , \, (n\chi^{[2]})(t) \, ((g\rho_\ell(t)^{-1}) \underset{G}{\cdot} (t \underset{\hat{U}^{[2\ell]}}{\cdot} (g^{-1} \underset{G}{\cdot} \nu))) \right), \quad t \in \mathbb{G}_m, \qquad (\star)$$

where " $\underset{\hat{U}^{[2\ell]}}{\cdot}$ " denotes the (untwisted) action of $\hat{U}^{[2\ell]}$ on $\operatorname{Sym}^n V$. We can understand this as follows. First, fix $g \in G$, let $v \in V$, use (5.2.1) to write

$$v = \textstyle\sum_i z_i \otimes s_i, \quad z_i \in \Bbbk^{(a_i)},\ s_i \in \operatorname{Sym}^{l_i} \Bbbk^2,\ i = 1, \dots, q$$

and consider the corresponding action of $t \in \mathbb{G}_m$ on v:

$$(t, v) \mapsto (g\rho_\ell(t)^{-1}) \underset{G}{\cdot} (t \underset{\hat{U}^{[2\ell]}}{\cdot} (g^{-1} \underset{G}{\cdot} v)),$$

where again $\underset{\hat{U}^{[2\ell]}}{\cdot}$ denotes the (untwisted) action of $\hat{U}^{[2\ell]}$ on V. Using (5.2.1), we see that

$$\begin{aligned}
(g\rho_\ell(t)^{-1}) \underset{G}{\cdot} (t \underset{\hat{U}^{[2\ell]}}{\cdot} (g^{-1} \underset{G}{\cdot} v)) &= (g\rho_\ell(t)^{-1}) \underset{G}{\cdot} (t \underset{\hat{U}^{[2\ell]}}{\cdot} (\textstyle\sum_i z_i \otimes (g^{-1} \underset{G}{\cdot} s_i))) \\
&= (g\rho_\ell(t)^{-1}) \underset{G}{\cdot} (\textstyle\sum_i (t^{a_i} z_i) \otimes (t \underset{\hat{U}^{[2\ell]}}{\cdot} (g^{-1} \underset{G}{\cdot} s_i))) \\
&= (g\rho_\ell(t)^{-1}) \underset{G}{\cdot} (\textstyle\sum_i (t^{a_i} z_i) \otimes ((\rho_\ell(t) g^{-1}) \underset{G}{\cdot} s_i)) \\
&= \textstyle\sum_i (t^{a_i} z_i) \otimes s_i.
\end{aligned}$$

which is precisely the action (5.2.2). Now taking a symmetric power and multiplying by the character $n\chi^{[2]}$, it follows that the action ($\star$) defines a $G \times (\hat{U}^{[2\ell]})_r$-linearisation on $(G/U) \times \mathcal{O}(n) \to (G/U) \times \mathbb{P}(V)$ that is equal to $((G/U) \times \mathcal{P}^{(\chi^{[2]})})^{\otimes n}$. The lemma follows. $\square$

The homomorphism ρ_ℓ embeds $U = (\hat{U}^{[2\ell]})_u$ into G as a Grosshans subgroup. Indeed, recall from Example 3.2.10 that there is an isomorphism $G/U \cong \Bbbk^2 \setminus \{0\}$ given by considering the orbit of $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \Bbbk^2$ under the defining representation of G. The inclusion $\Bbbk^2 \setminus \{0\} \hookrightarrow \Bbbk^2$ defines a normal affine completion $\overline{G/H_u}^{\mathrm{aff}}$ which contains G/U with codimension 2 complement. We may therefore appeal to Corollary 3.2.11 in order to construct a strong ample reductive envelope of (a suitable tensor power of) the rational $\hat{U}^{[2\ell]}$-linearisation $\mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$; cf. Remark 3.2.12. To this end, we construct a $G \times \hat{U}^{[2\ell]}$-equivariant compactification of G/U in a fashion similar to that done in the n unordered points example in Section 3.3.1, as follows. Regarding elements of $\Bbbk^3$ as column vectors, add a hyperplane at infinity to obtain a smooth $G \times (\hat{U}^{[2\ell]})_r$-equivariant projective completion $\mathbb{P}^2$ of $\overline{G/U}^{\mathrm{aff}} = \Bbbk^2$: if $\mathbb{P}^2 = \{[v_0 : v_1 : v_2] \mid 0 \neq (v_0, v_1, v_2)^{\mathrm{t}} \in \Bbbk^3\}$ with the hyperplane at infinity defined by $v_0 = 0$ then the action of $G \times (\hat{U}^{[2\ell]})_r = G \times \mathbb{G}_m$ on $\mathbb{P}^2 = \mathbb{P}(\Bbbk^3)$

is the one defined by the representation given in block form

$$(g,t) \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g \begin{pmatrix} t^{-\ell} & 0 \\ 0 & t^{-\ell} \end{pmatrix} \end{array} \right) \in \mathrm{GL}(3,\Bbbk), \quad g \in G,\ t \in \mathbb{G}_m,$$

where $\mathrm{GL}(3,\Bbbk)$ acts on $\Bbbk^3$ by left multiplication. For any integer $N > 0$, this representation canonically defines a $G \times (\hat{U}^{[2\ell]})_r$-linearisation on $\mathcal{O}_{\mathbb{P}^2}(N) \to \mathbb{P}^2$ which restricts to the canonical linearisation on $\mathcal{O}_{G/U} \to G/U$.

Let $\beta : G \times^U \mathbb{P}(V) \cong (\Bbbk^2 \setminus \{0\}) \times \mathbb{P}(V) \hookrightarrow \mathbb{P}^2 \times \mathbb{P}(V)$ be the induced open immersion and for $N > 0$ let[3]

$$\mathcal{P}'_N := \mathcal{O}_{\mathbb{P}^2}(N) \boxtimes \mathcal{P}^{(\chi^{[2]})} \to \mathbb{P}^2 \times \mathbb{P}(V)$$

equipped with its natural rational $G \times (\hat{U}^{[2\ell]})_r$-linearisation, where recall the $G \times (\hat{U}^{[2\ell]})_r$-linearisation $\mathcal{P}^{(\chi^{[2]})} \to \mathbb{P}(V)$ is defined as in Lemma 5.2.5. By Corollary 3.2.11 the triple

$$(\mathbb{P}^2 \times \mathbb{P}(V), \beta, \mathcal{P}'_N)$$

defines a strong ample reductive envelope for $\hat{U}^{[2\ell]} \curvearrowright \mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$, when $N > 0$ is sufficiently large. Moreover, because U is a Grosshans subgroup of G, by Remark 3.2.9 the ring of invariants of any suitably positive tensor power of the rational linearisation $\mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$ is a finitely generated $\Bbbk$-algebra and by Corollary 3.2.11

$$\mathbb{P}(V) \wr\!\wr_{\mathcal{O}(1)^{(\chi^{[2]})}} \hat{U}^{[2\ell]} \cong (\mathbb{P}^2 \times \mathbb{P}(V)) /\!\!/_{\mathcal{P}'_N} (G \times (\hat{U}^{[2\ell]})_r)$$

is a projective variety.

By Proposition 3.2.2 the stable locus $\mathbb{P}(V)^{\mathrm{s}(\mathcal{O}(1)^{(\chi^{[2]})})}$ and finitely generated semi-stable locus $\mathbb{P}(V)^{\mathrm{ss,fg}(\mathcal{O}(1)^{(\chi^{[2]})})}$ for the linearisation $\mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$ may be computed as the completely stable and completely semistable loci, respectively, associated to the rational $G \times (\hat{U}^{[2\ell]})_r$-linearisation $\mathcal{P}'_N$. We next compute the semistable and stable loci for the reductive envelope $G \times (\hat{U}^{[2\ell]})_r \curvearrowright \mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$, using the Hilbert-Mumford criterion. From now on we shall continually use the isomorphism (5.2.1) to write elements of V as elements of $\bigoplus_{i=1}^q \Bbbk^{(a_i)} \otimes \mathrm{Sym}^{l_i} \Bbbk^2$ (but recall that the action of $G \times (\hat{U}^{[2\ell]})_r$ on V under

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[3] More precisely, we ought to denote this linearisation $(\mathcal{P}^{(\chi^{[2]})})'_N$ (cf. (3.2.1) of Section 3.2), but this is rather cumbersome.

consideration to study $\mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$ is defined in Lemma 5.2.5). Note that under this description of V, the minimal $\mathbb{G}_m$-weight for the $\hat{U}^{[\ell]}$-action (not $\hat{U}^{[2\ell]}$-action!) on V is

$$\varpi = \min\{(a_i - \ell\, l_i)/2 \mid i = 1, \dots, q\}.$$

Let us temporarily call an index $i \in \{0, \dots, q\}$ *exceptional* if $\varpi = (a_i - \ell\, l_i)/2$.

Lemma 5.2.6. *Stability and semistability are equivalent for the reductive rational linearisation* $G \times (\hat{U}^{[2\ell]})_r \curvearrowright \mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$*, and a point* $p = ([1 : w_1 : w_2], [v]) \in \mathbb{P}^2 \times \mathbb{P}(V)$ *is stable if, and only if, both* $p \in (\mathbb{k}^2 \setminus \{0\}) \times \mathbb{P}(V)$ *and, when one uses* (5.2.1) *to write* $v = \sum_i z_i \otimes s_i \in \bigoplus_i \mathbb{k}^{(a_i)} \otimes \operatorname{Sym}^{l_i} \mathbb{k}^2$ *with each* $s_i \neq 0$*, the following two conditions hold:*

- *there is an exceptional* i *such that* $z_i \neq 0$ *and* s_i *is not divisible by* $(w_1, w_2) \in \mathbb{k}^2 \setminus \{0\}$*; and*
- *either there is a non-exceptional* i *such that* $z_i \neq 0$*, or for each* $(\tilde{w}_1, \tilde{w}_2) \in \mathbb{k}^2 \setminus \{0\}$ *with* $[\tilde{w}_1 : \tilde{w}_2] \neq [w_1 : w_2]$ *as points in* $\mathbb{P}^1$ *there is an exceptional* i *such that* $z_i \neq 0$ *and* $s_i \neq (\tilde{w}_1, \tilde{w}_2)^{l_i} \in \operatorname{Sym}^{l_i} \mathbb{k}^2$.

Proof of Lemma 5.2.6. We shall deduce this by using the Hilbert-Mumford criterion as given in Theorem 1.2.6 using the maximal torus $T_1 \times T_2 \subseteq G \times (\hat{U}^{[2\ell]})_r$, where T_1 is the subgroup of diagonal matrices in G and $T_2 = \mathbb{G}_m = (\hat{U}^{[2\ell]})_r$. The group of characters of $T_1 \times T_2$ is identified with $\mathbb{Z} \times \mathbb{Z}$ in the natural way. Introduce the following notation: for $i = 1, \dots, q$ let $e_{i,1} = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), e_{i,2} = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ be the standard basis of $\mathbb{k}^2$, so that

$$e_{i,1}^{l_i}, \dots, e_{i,1}^{j} e_{i,2}^{l_i - j}, \dots, e_{i,2}^{l_i} \in \operatorname{Sym}^{l_i} \mathbb{k}^2$$

is a basis of T_1-weight vectors in $\operatorname{Sym}^{l_i} \mathbb{k}^2$, and consider the basis

$$1 \otimes e_{i,1}^{l_i - j} e_{i,2}^{j}, \quad j = 0, \dots, l_i, \quad i = 1, \dots, q$$

of weight vectors for the $T_1 \times T_2$-action on V. Without loss of generality, we may apply the Hilbert-Mumford criterion by using the projective space into which $\mathbb{P}^2 \times \mathbb{P}(V)$ is embedded via $\mathcal{P}'_N$ and computing rational weights. The fixed points in $\mathbb{P}^2 \times \mathbb{P}(V)$ for the $T_1 \times T_2$-action, along with the corresponding rational weights with respect to the embedding defined by $\mathcal{P}'_N$, are given in Table 5.1.

Fixed point $(j = 0, \dots, l_i, i = 1, \dots, q)$	Rational weight in $\mathrm{Hom}(T_1 \times T_2, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q} \times \mathbb{Q}$
$([1:0:0], [1 \otimes e_{i,1}^{j} e_{i,2}^{l_i - j}])$	$(2j - l_i, a_i - 2\varpi - 2\epsilon)$
$([0:1:0], [1 \otimes e_{i,1}^{j} e_{i,2}^{l_i - j}])$	$(2j - l_i, a_i - 2\varpi - 2\epsilon) + (N, -\ell N)$
$([0:0:1], [1 \otimes e_{i,1}^{j} e_{i,2}^{l_i - j}])$	$(2j - l_i, a_i - 2\varpi - 2\epsilon) + (-N, -\ell N)$

Table 5.1: Rational weights of the fixed points of $T_1 \times T_2 \curvearrowright \mathbb{P}^2 \times \mathbb{P}(V)$ with respect to the linearisation $\mathcal{P}'_N$.

Consider the rational weight $\vartheta := (2j - l_i, a_i - 2\varpi - 2\epsilon)$ for the fixed point $([1 : 0 : 0], [1 \otimes e_{i,1}^{j} e_{i,2}^{l_i - j}])$. Note that either ϑ is contained in the interior of the cone

$$C := \{(c_1, c_2) \in \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \mid \ell\, c_1 + c_2 \geq 0 \text{ and } -\ell\, c_1 + c_2 \geq 0\},$$

or ϑ lies outside C and i, j satisfy $\varpi = (a_i - \ell\, l_i)/2$ and $j \in \{0, l_i\}$: because $0 < \epsilon < 1/2$ we see that

$$\ell(2j - l_i) + (a_i - 2\varpi - 2\epsilon) \begin{cases} = -2\epsilon < 0 & \text{iff } j = 0 \text{ and } \varpi = (a_i - \ell\, l_i)/2 \\ > 0 & \text{otherwise} \end{cases}$$

while

$$-\ell(2j - l_i) + (a_i - 2\varpi - 2\epsilon) \begin{cases} = -2\epsilon < 0 & \text{iff } j = l_i \text{ and } \varpi = (a_i - \ell\, l_i)/2 \\ > 0 & \text{otherwise} \end{cases}.$$

We also claim that $a_i - 2\varpi - 2\epsilon > 0$ for all $i = 1, \dots, q$. Indeed, suppose $a_i - 2\varpi - 2\epsilon \leq 0$ for some $i = 1, \dots, q$. Because $0 < 2\epsilon < 1$ and $a_i - 2\varpi \in \mathbb{Z}$, this is equivalent to $a_i - 2\varpi \leq 0$. But $2\varpi \leq a_i - \ell\, l_i$, so $\ell\, l_i \leq a_i - 2\varpi \leq 0$. Because $\ell > 0$ we must have $l_i = 0$, and by examining the above possible cases for the value of $\ell(2j - l_i) + (a_i - 2\varpi - 2\epsilon)$ we see that $\varpi = a_i/2$ and i is exceptional. This implies there is a line $\Bbbk^{(\varpi)} = \Bbbk^{(\varpi)} \otimes \mathrm{Sym}^0 \Bbbk^2 \subseteq V_{\min}$ fixed by U, which contradicts the assumption that $V_{\min}$ does not contain a point fixed by the U-action. This verifies the claim.

We thus see that for sufficiently large $N > 0$ the weights for the rational $T_1 \times T_2$-linearisation $\mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$ are arranged in the fashion of Figure 5.2. (Notice that the only weights that lie outside the chambers are the extremal weights for rows corresponding to exceptional indices. This makes calculating semistability and stability for the torus $T_1 \times T_2$ easy.) In particular, the weight polytope $\Delta_p \subseteq \mathrm{Hom}(T_1 \times T_2, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$ for a point $p =$

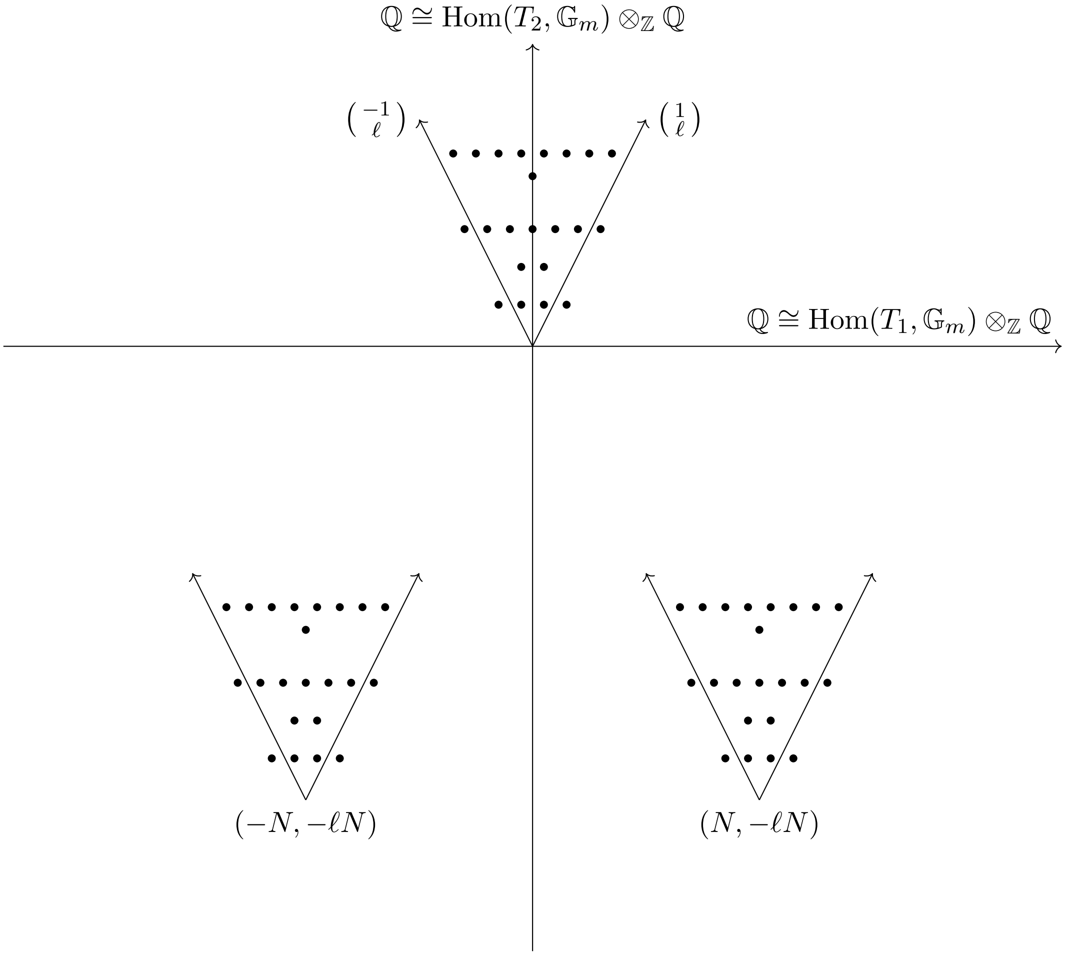

Figure 5.2: Example of distribution of rational weights for $T_1 \times T_2 \curvearrowright \mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$.

$([w_0 : w_1 : w_2], [v]) \in \mathbb{P}^2 \times \mathbb{P}(V)$ contains the origin precisely when the interior Δ_p° does and so semistability and stability for the rational linearisation $T_1 \times T_2 \curvearrowright \mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$ coincide. Using the isomorphism (5.2.1), write

$$v = \textstyle\sum_{i=1}^{q} z_i \otimes s_i, \quad z_i \in \Bbbk^{(a_i)}, \quad 0 \neq s_i = \textstyle\sum_{j=0}^{l_i} v_{i,j} e_{i,1}^{j} e_{2,i}^{l_i - j}, \ v_{i,j} \in \Bbbk.$$

Then one finds that p is $T_1 \times T_2$-*unstable* precisely when $p \notin (\Bbbk^2 \setminus \{0\}) \times \mathbb{P}(V)$ (i.e. $w_0 = 0$ or $w_1 = w_2 = 0$) or else by satisfying one of the following criteria, split into three cases:

Case $w_0 w_1 w_2 \neq 0$:

$$0 \notin \Delta_p \iff \begin{array}{l} \text{Either } v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = 0), \\ \text{or } v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = l_i). \end{array}$$

Case $w_0 w_1 \neq 0$**,** $w_2 = 0$**:**

$$0 \notin \Delta_p \iff \begin{array}{l} \text{Either } i \text{ exceptional } \implies v_{i,0} = 0, \\ \text{or } v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = 0). \end{array}$$

Case $w_0 w_2 \neq 0$**,** $w_1 = 0$**:**

$$0 \notin \Delta_p \iff \begin{array}{l} \text{Either } i \text{ exceptional } \implies v_{i,l_i} = 0, \\ \text{or } v_{i,j} \neq 0 \implies (i \text{ exceptional and } j = l_i). \end{array}$$

By the Hilbert-Mumford criterion, the point p is (semi)stable for the whole fractional $G \times (\hat{U}^{[2\ell]})_r$-linearisation if, and only if, $(g,t) \cdot p$ is $T_1 \times T_2$-(semi)stable for each $(g,t) \in G \times (\hat{U}^{[2\ell]})_r$. Thus stability and semistability are equivalent for $G \times (\hat{U}^{[2\ell]})_r \curvearrowright \mathcal{P}'_N \to \mathbb{P}^2 \times \mathbb{P}(V)$, and because G acts transitively on pairs of distinct points in $\mathbb{P}^1$ it follows that $p \in \mathbb{P}^2 \times \mathbb{P}(V)$ is stable precisely when $p \in (\Bbbk^2 \setminus \{0\}) \times \mathbb{P}(V)$ and the two conditions in the statement of Lemma 5.2.6 are fulfilled. □

We are now in a position to complete the proof of Proposition 5.2.3. Lemma 5.2.6 tells us that $(\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{s}(\mathcal{P}'_N)} = (\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{ss}(\mathcal{P}'_N)} \subseteq (\Bbbk^2 \setminus \{0\}) \times \mathbb{P}(V) \cong G \times^U \mathbb{P}(V)$, so we have

$$G \times^U (\mathbb{P}(V)^{\overline{\mathrm{s}}}) = G \times^U (\mathbb{P}(V)^{\overline{\mathrm{ss}}}) = (\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{ss}(\mathcal{P}'_N)},$$

where $\mathbb{P}(V)^{\overline{\mathrm{s}}}$ and $\mathbb{P}(V)^{\overline{\mathrm{ss}}}$ are the completely stable and completely semistable locus, respectively, for the reductive envelope $(\mathbb{P}^2 \times \mathbb{P}(V), \beta, \mathcal{P}'_N)$. The GIT quotient map $(\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{ss}(\mathcal{P}'_N)} \to (\mathbb{P}^2 \times \mathbb{P}(V)) /\!\!/_{\mathcal{P}'_N} (G \times (\hat{U}^{[2\ell]})_r)$ is a geometric quotient with the inclusion $\beta \circ \alpha : \mathbb{P}(V)^{\overline{\mathrm{s}}} \hookrightarrow (\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{ss}(\mathcal{P}'_N)}$ inducing an isomorphism

$$\mathbb{P}(V)^{\overline{\mathrm{s}}} / \hat{U}^{[2\ell]} \cong (G \times^U (\mathbb{P}(V)^{\overline{\mathrm{s}}})) / (G \times (\hat{U}^{[2\ell]})_r) = (\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{ss}(\mathcal{P}'_N)} / (G \times (\hat{U}^{[2\ell]})_r).$$

Because $(\mathbb{P}^2 \times \mathbb{P}(V), \beta, \mathcal{P}'_N)$ is strong, by Proposition 3.2.2 we have

$$\mathbb{P}(V)^{\mathrm{s}(\hat{U}^{[2\ell]}, \mathcal{O}(1)^{(\chi^{[2]})})} = \mathbb{P}(V)^{\overline{\mathrm{s}}} = \mathbb{P}(V)^{\overline{\mathrm{ss}}} = \mathbb{P}(V)^{\mathrm{ss,fg}(\hat{U}^{[2\ell]}, \mathcal{O}(1)^{(\chi^{[2]})})},$$

while by Corollary 3.2.11,

$$(\mathbb{P}^2 \times \mathbb{P}(V)) /\!\!/_{\mathcal{P}'_N} (G \times (\hat{U}^{[2\ell]})_r) \cong \mathbb{P}(V) \wr\wr_{\mathcal{O}(1)^{(\chi^{[2]})}} \hat{U}^{[2\ell]}.$$

From Theorem 3.1.14 we conclude that the enveloping quotient map

$$\mathbb{P}(V)^{\mathrm{ss,fg}(\hat{U}^{[2\ell]},\mathcal{O}(1)^{(\chi^{[2]})})} = \mathbb{P}(V)^{\mathrm{s}(\hat{U}^{[2\ell]},\mathcal{O}(1)^{(\chi^{[2]})})} \to \mathbb{P}(V) /\!\!/_{\mathcal{O}(1)^{(\chi^{[2]})}} \hat{U}^{[2\ell]}$$

is a geometric quotient for the $\hat{U}^{[2\ell]}$-action on $\mathbb{P}(V)^{\mathrm{ss,fg}(\hat{U}^{[2\ell]},\mathcal{O}(1)^{(\chi)})}$ onto a projective variety.

Because the non-reductive geometric invariant theories of $\hat{U}^{[\ell]} \curvearrowright \mathcal{O}(1)^{(\chi)} \to \mathbb{P}(V)$ and $\hat{U}^{[2\ell]} \curvearrowright \mathcal{O}(1)^{(\chi^{[2]})} \to \mathbb{P}(V)$ agree (see Lemma 5.2.6), we see that there is an equality $\mathbb{P}(V)^{\mathrm{ss,fg}(\hat{U}^{[\ell]},\mathcal{O}(1)^{(\chi)})} = \mathbb{P}(V)^{\mathrm{s}(\hat{U}^{[\ell]},\mathcal{O}(1)^{(\chi)})}$ and the enveloping quotient map for the original $\hat{U}^{[\ell]}$-linearisation

$$\mathbb{P}(V)^{\mathrm{ss,fg}(\hat{U}^{[\ell]},\mathcal{O}(1)^{(\chi)})} \to \mathbb{P}(V) /\!\!/_{\mathcal{O}(1)^{(\chi)}} \hat{U}^{[\ell]}$$

is also a geometric quotient for the $\hat{U}^{[\ell]}$-action on $\mathbb{P}(V)^{\mathrm{ss,fg}(\hat{U}^{[\ell]},\mathcal{O}(1)^{(\chi)})}$, with the enveloping quotient $\mathbb{P}(V) /\!\!/_{\mathcal{O}(1)^{(\chi)}} \hat{U}^{[\ell]}$ a projective variety. Moreover, for sufficiently divisible integers $r > 0$ the ring of invariants $\Bbbk[\mathbb{P}(V), \mathcal{O}(r)^{(r\chi)}]^{\hat{U}^{[\ell]}}$ is finitely generated over $\Bbbk$ by Corollary 2.1.19 and $\mathbb{P}(V) /\!\!/_{\mathcal{O}(1)^{(\chi)}} \hat{U}^{[\ell]} \cong \mathrm{Proj}(\Bbbk[\mathbb{P}(V), \mathcal{O}(r)^{(r\chi)}]^{\hat{U}^{[\ell]}}$. This establishes 2 and 3 of Proposition 5.2.3.

It remains to show the remainder of 1 of Proposition 5.2.3: that

$$\mathbb{P}(V)^{\mathrm{s}(\hat{U}^{[\ell]},\mathcal{O}(1)^{(\chi)})} = \mathbb{P}(V)^{0}_{\hat{U},\mathcal{O}(1)} \setminus (U \cdot \mathbb{P}(V_{\mathrm{min}})).$$

Recall that $\mathbb{P}(V)^{\mathrm{s}(\hat{U}^{[\ell]},\mathcal{O}(1)^{(\chi)})} = \mathbb{P}(V)^{\bar{\mathrm{s}}}$ is equal to the preimage of $(\mathbb{P}^2 \times \mathbb{P}(V))^{\mathrm{s}(\mathcal{P}'_N)}$ under the inclusion

$$\mathbb{P}(V) \hookrightarrow \mathbb{P}^2 \times \mathbb{P}(V), \quad [v] \mapsto ([1:1:0],[v]).$$

According to Lemma 5.2.6 we therefore have $[v] \in \mathbb{P}(V)^{\mathrm{s}(\hat{U}^{[\ell]},\mathcal{O}(1)^{(\chi)})}$ if, and only if, when one uses (5.2.1) to write $v = \sum_i z_i \otimes s_i \in \bigoplus_i \Bbbk^{(a_i)} \otimes \mathrm{Sym}^{l_i} \Bbbk^2$ with each $s_i \neq 0$, the following two conditions are satisfied:

- there is an exceptional i such that $z_i \neq 0$ and s_i is not divisible by $(1,0) \in \Bbbk^2$; and
- either there is a non-exceptional i such that $z_i \neq 0$, or for each $(w_1, w_2) \in \Bbbk^2 \setminus \{0\}$ with $[w_1 : w_2] \neq [1:0]$ as points in $\mathbb{P}^1$ there is an exceptional i such that $z_i \neq 0$ and $s_i \neq (w_1, w_2)^{l_i} \in \mathrm{Sym}^{l_i} \Bbbk^2$.

We can interpret each of these conditions geometrically, as follows. Under the isomorphism of vector spaces $V \cong \bigoplus_{i=1}^{q} \Bbbk^{(a_i)} \otimes \mathrm{Sym}^{l_i} \Bbbk^2$ the weight vectors for the induced $\mathbb{G}_m \subseteq \hat{U}^{[\ell]}$-action on V take the form $1 \otimes e_{1,i}^{j} e_{2,i}^{l_i - j}$, where $1 \leq i \leq q$ and $0 \leq j \leq l_i$, with the weight of $1 \otimes e_{1,i}^{j} e_{2,i}^{l_i - j}$ equal to $(a_i - \ell\, l_i + 2j)/2 \in \mathbb{Z}$. Moreover, the weight space $V_{\min}$ of minimal weight ϖ is spanned by all $1 \otimes e_{2,i}^{l_i}$ with i an exceptional index, and the U-sweep $U \cdot V_{\min}$ of $V_{\min}$ is contained in the $\hat{U}^{[\ell]}$-subspace

$$\bigoplus_{i \text{ exceptional}} \Bbbk^{(a_i)} \otimes \mathrm{Sym}^{l_i} \Bbbk^2 \subseteq V.$$

Now, if $v = \sum_i z_i \otimes s_i$ with each $s_i \neq 0$, then the existence of an exceptional i with $z_i \neq 0$ and s_i not divisible by $(1,0)$ is equivalent to $\lim_{t \to 0} t \cdot [v] \in \mathbb{P}(V_{\min})$ (where we take $t \in \mathbb{G}_m \subseteq \hat{U}^{[\ell]}$ in the limit). So the first of the above conditions is equivalent to requiring that $[v] \in \mathbb{P}(V)^0_{\hat{U},\mathcal{O}(1)}$. Now consider the second condition. The existence of a non-exceptional i such that $z_i \neq 0$ is equivalent to $v \notin \bigoplus_{i \text{ exceptional}} \Bbbk^{(a_i)} \otimes \mathrm{Sym}^{l_i} \Bbbk^2$, which itself implies $[v] \notin U \cdot \mathbb{P}(V_{\min})$. On the other hand, because of the transitivity of the U-action on $\Bbbk = \mathbb{P}^1 \setminus \{[1:0]\}$ and the fact that $V_{\min}$ is spanned by $1 \otimes e_{i,2}^{l_i}$ with i exceptional, we see that $[v] \in U \cdot \mathbb{P}(V_{\min})$ if, and only if, $v \in \bigoplus_{i \text{ exceptional}} \Bbbk^{(a_i)} \otimes \mathrm{Sym}^{l_i} \Bbbk^2$ and there is some $(w_1, w_2) \in \Bbbk^2 \setminus \{0\}$ with $[w_1 : w_2] \neq [1:0] \in \mathbb{P}^1$ such that $s_i = (w_1, w_2)^{l_i} \in \mathrm{Sym}^{l_i} \Bbbk^2$ for all exceptional i. Thus, the second condition is equivalent to demanding $[v] \notin U \cdot \mathbb{P}(V_{\min})$. It follows that

$$\mathbb{P}(V)^{\mathrm{s}(\hat{U}^{[\ell]}, \mathcal{O}(1)^{(\chi)})} = \mathbb{P}(V)^0_{\hat{U},\mathcal{O}(1)} \setminus (U \cdot \mathbb{P}(V_{\min})),$$

as required.

This completes the proof of Proposition 5.2.3.

5.2.2 Proof of Theorem 5.2.2 for General (X, L)

Having established Theorem 5.2.2 for the case $X = \mathbb{P}(V)$ and $L = \mathcal{O}_{\mathbb{P}(V)}(1)$, we now turn to its proof in the full generality stated. So suppose $L \to X$ is a very ample line bundle over an irreducible projective variety equipped with a $\hat{U}$-linearisation, where $\hat{U}$ is a positive extension of $U = \mathbb{G}_a$, let $V = H^0(X, L)^*$ and let $\gamma : X \hookrightarrow \mathbb{P}(V)$ be the canonical closed immersion. Let ϖ be the minimal weight for the induced $\mathbb{G}_m$-action on V and suppose condition $(\mathfrak{C}^*)$ is satisfied (so that the associated weight space $V_{\min}$ does not contain any fixed points for the U-action on V). Finally, let χ be the rational character of weight $-\varpi-\epsilon$, with $0<\epsilon<1/2$ a rational number.

Because $\hat{U} \curvearrowright L \to X$ satisfies $(\mathfrak{C}^*)$, by Proposition 5.2.3 the twisted linearisation $\hat{U} \curvearrowright \mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)} \to \mathbb{P}(V)$ has an enveloping quotient

$$q : \mathbb{P}(V)^{\mathrm{ss,fg}(\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)})} = \mathbb{P}(V)^{\mathrm{s}(\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)})} \to \mathbb{P}(V) \wr\!\wr_{\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)}} \hat{U}$$

which is a geometric quotient for the $\hat{U}$-action on $\mathbb{P}(V)^{\mathrm{ss,fg}(\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)})}$, and the quotient $\mathbb{P}(V) \wr\!\wr^{\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)}} \hat{U}$ is a projective variety. Furthermore,

$$\mathbb{P}(V)^{\mathrm{s}(\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)})} = \mathbb{P}(V)^0_{\hat{U},\mathcal{O}(1)} \setminus (U \cdot \mathbb{P}(V_{\min})),$$

from which it follows that

$$\gamma^{-1}(\mathbb{P}(V)^{\mathrm{s}(\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)})}) = X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}).$$

As $\hat{U}$-linearisations $\gamma^*(\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)}) = L^{(\chi)} \to X$, so it follows from Lemma 2.3.6 that $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ is an open subset of $X^{\mathrm{s}(L^{(\chi)})}$ whose image under the enveloping quotient

$$q : X^{\mathrm{ss,fg}(L^{(\chi)})} \to X \wr\!\wr_{L^{(\chi)}} \hat{U}$$

is a geometric quotient for the $\hat{U}$-action on $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ that embeds naturally as a closed subvariety of $\mathbb{P}(V)^{\mathrm{s}(\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)})}/\hat{U} = \mathbb{P}(V) \wr\!\wr_{\mathcal{O}_{\mathbb{P}(V)}(1)^{(\chi)}} \hat{U}$. Hence $q(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}))$ is itself a projective variety. In particular it is complete, and since $X \wr\!\wr_{L^{(\chi)}} \hat{U}$ is separated over $\operatorname{Spec} \Bbbk$ it follows that the inclusion $q(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})) \hookrightarrow X \wr\!\wr_{L^{(\chi)}} \hat{U}$ is a closed map [Sta15, Tag 01W0]. On the other hand, because X is irreducible $q(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}))$ is a dense open subset of $X \wr\!\wr_{L^{(\chi)}} \hat{U}$, hence

$$q(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})) = X \wr\!\wr_{L^{(\chi)}} \hat{U}.$$

In particular $X \wr\!\wr_{L^{(\chi)}} \hat{U}$ is a projective variety, so by Corollary 2.1.19 the ring of invariants $\Bbbk[X, (L^{\otimes r})^{(r\chi)}]^{\hat{U}}$ is finitely generated for suitably divisible integers $r > 0$. Finally, we can see

$$X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}) = X^{\mathrm{s}(L^{(\chi)})} = X^{\mathrm{ss,fg}(L^{(\chi)})}$$

as follows. First observe that $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}) = X^{\mathrm{s}(L^{(\chi)})}$: for $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ is open in $X^{\mathrm{s}(L^{(\chi)})}$, so the restriction $q : X^{\mathrm{s}(L^{(\chi)})} \to q(X^{\mathrm{s}(L^{(\chi)})}) = X \wr\!\wr_{L^{(\chi)}} \hat{U}$ is a geometric quotient and $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}) = X^{\mathrm{s}(L^{(\chi)})}$ because q restricted to $X^{\mathrm{s}(L^{(\chi)})}$ is an orbit map. Note that by construction of the morphism $q : X^{\mathrm{ss,fg}(L^{(\chi)})} \to X \wr\!\wr_{L^{(\chi)}} \hat{U}$ the subset $X^{\mathrm{s}(L^{(\chi)})}$ of $X^{\mathrm{ss,fg}(L^{(\chi)})}$ satisfies $X^{\mathrm{s}(L^{(\chi)})} = q^{-1}(q(X^{\mathrm{s}(L^{(\chi)})}))$ (see the Section A.1 of the Appendix). Since $q(X^{\mathrm{s}(L^{(\chi)})}) = X \wr\!\wr_{L^{(\chi)}} \hat{U}$, it follows that $X^{\mathrm{s}(L^{(\chi)})} = X^{\mathrm{ss,fg}(L^{(\chi)})}$. We also conclude that $q : X^{\mathrm{ss,fg}(L^{(\chi)})} \to X \wr\!\wr_{L^{(\chi)}} \hat{U}$ is a geometric quotient for the $\hat{U}$-action on $X^{\mathrm{ss,fg}(L^{(\chi)})}$.

This completes the proof of Theorem 5.2.2.

5.3 The General Case

We now turn to the proof of Theorem 5.1.4 in the case where U is any unipotent group and $\hat{U} = U \rtimes_\lambda \mathbb{G}_m$ is a positive $\mathbb{G}_m$-extension, formed by a homomorphism $\lambda : \mathbb{G}_m \to \mathrm{Aut}(U)$ with all weights for the induced $\mathbb{G}_m$-action on $\mathrm{Lie}\, U$ positive. As discussed at the beginning of this chapter, we will adopt a different line of argument to that of Section 5.2. This will enable us to work under a weaker assumption than the condition $(\mathfrak{C}^*)$ used in the proof of Theorem 5.2.2 in the case $U \cong \mathbb{G}_a$.

We continue to assume $L \to X$ is a very ample linearisation of U over an irreducible projective variety X and that there is an extension of this linearisation to one of $\hat{U}$. As usual, we let $S = \Bbbk[X, L]$. Note that, because X is irreducible, for any subgroup $H \subseteq \hat{U}$ and any H-invariant section f of a positive tensor power of $L \to X$ we have a canonical identification $(S^H)_{(f)} = \mathcal{O}(X_f)^H$ (see Lemma 2.1.8, 1), which we shall make implicit use of throughout this section. We will make use of the following notation: if H_1, H_2 are two linear algebraic groups and $L \to X$ is any line bundle over any variety that has (rational) linearisations of both H_1 and H_2, then we shall denote the stable locus of $H_i \curvearrowright L \to X$ by $X^{\mathrm{s}(H_i,L)}$ (as well as using similar notation for the finitely generated semistable loci etc.)

Recall from Section 5.1 that the linearisation $\hat{U} \curvearrowright L \to X$ satisfies condition $(\mathfrak{C})$ if $\mathrm{Stab}_U(z) = \{e\}$ for every $z \in Z_{\hat{U},L}$. The aim of this section is to prove Theorem 5.1.4, which we restate below for ease of reference.

Theorem 5.3.1 (Restatement of Theorem 5.1.4). *Let X be an irreducible projective variety acted upon by a unipotent group U and let $L \to X$ be a very ample linearisation. Suppose the linearisation extends to a linearisation of a positive extension $\hat{U} = U \rtimes_{\lambda} \mathbb{G}_m$ of U satisfying condition ($\mathfrak{C}$). Then*

1. *we have $X^0_{\hat{U},L} \subseteq X^{\mathrm{s}(U,L)}$, with the restriction of the enveloping quotient map for $U \curvearrowright L \to X$ defining a locally trivial U-quotient $X^0_{\hat{U},L} \to X^0_{\hat{U},L}/U$.*

Suppose furthermore that $X^0_{\hat{U},L} \neq U \cdot Z_{\hat{U},L}$, and let $L^{(\chi)}$ be the rational $\hat{U}$-linearisation resulting from twisting $\hat{U} \curvearrowright L \to X$ by the rational character χ of weight $-\varpi - \epsilon$, where ϖ is the minimal weight for the $\mathbb{G}_m$-action on $V := H^0(X, L)^*$ and ϵ is a positive rational number. Then for sufficiently small $\epsilon > 0$,

2. *there are equalities $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}) = X^{\mathrm{s}(L^{(\chi)})} = X^{\mathrm{ss,fg}(L^{(\chi)})}$;*
3. *the enveloping quotient $X \wr\!\wr_{L^{(\chi)}} \hat{U}$ is a projective variety and for suitably divisible integers $r > 0$ the ring of invariants $\Bbbk[X, (L^{(\chi)})^{\otimes r}]^{\hat{U}}$ for the linearisation $(L^{(\chi)})^{\otimes r} \to X$ is finitely generated, with*

   $$X \wr\!\wr_{L^{(\chi)}} \hat{U} = \mathrm{Proj}(\Bbbk[X, (L^{(\chi)})^{\otimes r}]^{\hat{U}});$$

   and
4. *the enveloping quotient map $X^{\mathrm{ss,fg}(L^{(\chi)})} \to X \wr\!\wr_{L^{(\chi)}} \hat{U}$ is a geometric quotient for the $\hat{U}$-action on $X^{\mathrm{ss,fg}(L^{(\chi)})}$.*

Due to its length, the proof of this theorem will be presented in the following section, where we shall argue by establishing intermediate results to aid readability. We first take a moment to sketch out the crux of the argument and establish some preliminaries.

Firstly, recall that if $\hat{U}$ is a positive extension of U and $\xi \in \mathrm{Lie}\, U$ is a $\mathbb{G}_m$-weight vector, then it has positive weight $\ell > 0$, say. If W is any representation of $\hat{U}$, then ξ defines a derivation $\xi : W \to W$ and any weight vector in W of weight $\omega \in \mathbb{Z}$ gets sent to a weight vector of weight $\omega + \ell$ under ξ. In particular, if $W_{\max}$ denotes the $\mathbb{G}_m$-weight space in W of maximal possible weight, then we have

$$W_{\max} \subseteq \bigcap_{\xi \in \operatorname{Lie} U} \ker(\xi : W \to W) = W^U;$$

see Section 1.1.3 of Chapter 1.

Now, given a linearisation $\hat{U} \curvearrowright L \to X$ of $\hat{U}$, let

$$H^0(X,L)_{\max} := \{\sigma \in H^0(X,L) \mid t \cdot \sigma = t^{-\varpi}\sigma \text{ for all } t \in \mathbb{G}_m \subseteq \hat{U}\};$$

note this is the $\mathbb{G}_m$-weight space in $H^0(X,L)$ of maximal possible weight. The open subset $X^0_{\hat{U},L}$ is covered by the affine open subsets X_σ, with $\sigma \in H^0(X,L)_{\max}$. Each X_σ is invariant under the $\hat{U}$-action, because $H^0(X,L)_{\max} \subseteq H^0(X,L)^U$ and σ is a $\mathbb{G}_m$-weight vector. The open subsets X_σ enjoy a prominent rôle in the proof of Theorem 5.3.1.

We first establish $X^0_{\hat{U},L} \subseteq X^{\mathrm{s}(U,L)}$ (point 1 of Theorem 5.3.1). This is done inductively, using the philosophy of taking successive quotients by smaller unipotent groups; cf. Chapter 4. More precisely, by diagonalising the action of $\mathbb{G}_m$-action on $\operatorname{Lie} U$ and using the exponential map (see Section 1.1.3 of Chapter 1) we may choose a subnormal series

$$1 = U_0 \trianglelefteq U_1 \trianglelefteq \cdots \trianglelefteq U_m = U$$

which is preserved by each automorphism in the family $\lambda : \mathbb{G}_m \to \operatorname{Aut}(U)$ and such that each successive quotient $U_{j+1}/U_j \cong \mathbb{G}_a$, with λ acting on $\operatorname{Lie}(U_{j+1}/U_j)$ with positive weight. We will inductively show that each X_σ (with $\sigma \in H^0(X,L)_{\max}$) has a (locally) trivial U_j-quotient that is affine, using a combination of ($\mathfrak{C}$) and Lemma 4.2.2.

This results in a locally trivial U-quotient $q_U : X^0_{\hat{U},L} \to X^0_{\hat{U},L}/U$. We then use a sufficiently divisible power of $L \to X$ to embed $X^0_{\hat{U},L}/U$ into a projective space $\mathbb{P}$, in a $\mathbb{G}_m$-equivariant manner. By twisting the linearisation on $L \to X$ by an appropriate rational character χ of $\hat{U}$, we obtain a $\mathbb{G}_m$-linearisation M over the closure $\overline{X^0_{\hat{U},L}/U}$ of $X^0_{\hat{U},L}/U$ in $\mathbb{P}$, that pulls back to a positive tensor power of the twisted rational linearisation $L^{(\chi)} \to X$ and has the properties that

$$\overline{X^0_{\hat{U},L}/U}^{\mathrm{s}(M)} = \overline{X^0_{\hat{U},L}/U}^{\mathrm{ss}(M)}$$

and

$$q_U^{-1}(\overline{X^0_{\hat{U},L}/U}^{\mathrm{s}(M)}) = X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}) \subseteq X^{\mathrm{s}(L^{(\chi)})}.$$

These equalities and inclusions are proved using reductive GIT, especially the Hilbert-Mumford criterion. From here it is then straightforward to show that $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ has a projective geometric $\hat{U}$-quotient under the enveloping quotient map $q : X^{\mathrm{ss,fg}(L^{(\chi)})} \to X \wr\wr_{L^{(\chi)}} \hat{U}$, isomorphic to $\overline{X^0_{\hat{U},L}/U} /\!\!/_M \mathbb{G}_m$. The rest of Theorem 5.3.1 then follows by applying results from Chapter 2 on non-reductive GIT.

5.3.1 Proof of Theorem 5.3.1

Suppose we are given a positive $\mathbb{G}_m$-extension $\hat{U} = U \rtimes_\lambda \mathbb{G}_m$ of U and a linearisation $\hat{U} \curvearrowright L \to X$ extending the U-linearisation satisfying ($\mathfrak{C}$). We first set about showing that $X^0_{\hat{U},L} \subseteq X^{\mathrm{s}(U,L)}$ (Theorem 5.3.1, 1). The proof will be based on an inductive argument that relies on using the following geometric interpretation of [Dix96, Lemma 4.7.5].

Lemma 5.3.2. *(Cf. [Dix96, Lemma 4.7.5].) Suppose X is an affine variety with an action of $\mathbb{G}_a$ and let $\xi \in \mathrm{Lie}(\mathbb{G}_a)$. If there is $f \in \mathcal{O}(X)$ such that $\xi(f) = 1 \in \mathcal{O}(X)$, then X is a trivial $\mathbb{G}_a$-bundle.*

Recall that $X^0_{\hat{U},L}$ is the union of basic affine opens X_σ with $\sigma \in H^0(X,L)_{\max}$. Given nonzero $\sigma \in H^0(X,L)_{\max}$, we can embed X into a projective space $\mathbb{P}^N \cong \mathbb{P}(H^0(X,L)^*)$ (for suitable $N > 0$) via L using a collection of linear sections which are weight vectors for the $\mathbb{G}_m$-action, and which includes σ. Then X_σ is contained in an affine coordinate patch $\mathbb{A}^N = (\mathbb{P}^N)_\sigma$ such that the action of $\mathbb{G}_m$ on $\mathbb{A}^N$ is diagonal, with all weights ≥ 0; or equivalently, $\mathbb{G}_m$ acts on $\mathcal{O}(X_\sigma)$ with all weights ≤ 0. Note also that each point $x \in X_\sigma$ has a limit point in $X_\sigma \cap Z_{\hat{U},L}$ under the action of $t \in \mathbb{G}_m$ as $t \to 0$.

By considering the action of $\hat{U}_1 = U_1 \rtimes_\lambda \mathbb{G}_m$ on X_σ, one is therefore naturally led to the following lemma.

Lemma 5.3.3. *Let X be an affine variety with action of $\hat{\mathbb{G}}_a = \mathbb{G}_a \rtimes \mathbb{G}_m$, where $\mathbb{G}_m$ acts on $\mathrm{Lie}(\mathbb{G}_a)$ with positive weight, and let ξ be a generator of $\mathrm{Lie}(\mathbb{G}_a)$. Suppose $\mathbb{G}_m$ acts on $\mathcal{O}(X)$ with* *all weights non-positive. Then every point in X has a limit in X under the action of $t \in \mathbb{G}_m$ as $t \to 0$; let Z be the set of such limit points in X. If $\mathrm{Stab}_{\mathbb{G}_a}(z) = \{e\}$ for each $z \in Z$, then there is $f \in \mathcal{O}(X)$ such that $\xi(f) = 1 \in \mathcal{O}(X)$.*

Proof. We first show that every point in X has a limit under the action of $t \in \mathbb{G}_m$, as $t \to 0$. Fix $x \in X$. To say that $\lim_{t\to 0} t \cdot x$ exists in X means that the morphism $\phi_x : \mathbb{G}_m \to X$, $\phi_x(t) = t \cdot x$, extends to a morphism $\phi_x : \Bbbk \to X$ under the usual open inclusion $\mathbb{G}_m \subseteq \Bbbk$ (and then $\lim_{t\to 0} t \cdot x := \phi_x(0)$). This is equivalent to saying that the pullback homomorphism $(\phi_x)^\# : \mathcal{O}(X) \to \mathcal{O}(\mathbb{G}_m) = \Bbbk[t, t^{-1}]$ factors through the localisation map $\Bbbk[t] \to \Bbbk[t, t^{-1}]$. But if $a \in \mathcal{O}(X)$ is a $\mathbb{G}_m$-weight vector of weight $m \leq 0$ then

$$((\phi_x)^\#(a))(t) = a(t \cdot x) = (t^{-1} \cdot a)(x) = t^{-m} a(x)$$

with $-m \geq 0$. Since $\mathcal{O}(X)$ is generated by such weight vectors, we see that $(\phi_x)^\# : \mathcal{O}(X) \to \Bbbk[t, t^{-1}]$ indeed factors through $\Bbbk[t] \to \Bbbk[t, t^{-1}]$. Let Z be the set of limit points in X.

Using the $\mathbb{G}_m$-action, write $\mathcal{O}(X) = \bigoplus_{m \leq 0} W_m$ as a negatively graded algebra, where $W_m \subseteq \mathcal{O}(X)$ is the subspace of weight vectors in $\mathcal{O}(X)$ of weight $m \leq 0$. Suppose $\mathbb{G}_m$ acts on ξ with weight $\ell > 0$. Then we have

$$\xi(W_m) \begin{cases} = 0 & \text{if } m > -\ell, \\ \subseteq W_0 & \text{if } m = -\ell, \\ \subseteq \bigoplus_{m<0} W_m & \text{if } m < -\ell. \end{cases}$$

Let $\tilde{W} = \xi(W_{-\ell})$ be the image of the weight space $W_{-\ell}$ under ξ, and consider the vector subspace

$$I := \tilde{W} \oplus \bigoplus_{m<0} W_m.$$

We claim that I is a $\hat{\mathbb{G}_a}$-stable ideal of $\mathcal{O}(X)$. Indeed, I is $\mathbb{G}_m$-stable and closed under the action of ξ by construction, so we immediately see that it is stable under the $\hat{\mathbb{G}_a}$-action. Let $f \in I$ and $a \in \mathcal{O}(X)$. We need to show that $af \in I$, for which we may assume that $a \in W_p$ for some $p \leq 0$, without loss of generality. Now if $p < 0$, then because multiplication respects the grading we have $af \in \bigoplus_{m<0} W_m \subseteq I$. So suppose $p = 0$. Write $f = \tilde{f} + g$ where $\tilde{f} \in \tilde{W}$ and $g \in \bigoplus_{m<0} W_m$, so $af = a\tilde{f} + ag$. On the one hand, $ag \in \bigoplus_{m<0} W_m \subseteq I$.

On the other hand, there is $h \in W_{-\ell}$ such that $\xi(h) = \tilde{f}$, and because $\xi(a) = 0$ we therefore have $\xi(ah) = a\tilde{f}$, with $ah \in W_{-\ell}$, thus $a\tilde{f} \in \tilde{W} \subseteq I$. Hence $af \in I$, and the claim is established.

To finish the proof, we will show that $I = \mathcal{O}(X)$. We may find a non-trivial $\mathbb{G}_m$-invariant complementary subspace W' of W_0 such that $\mathcal{O}(X) = W' \oplus I$ as vector spaces. It is easy to see that

$$Z = \{x \in X \mid f(x) = 0 \text{ for all } f \in \textstyle\bigoplus_{m<0} W_m\}$$

and so the subvariety $V(I) := \{x \in X \mid f(x) = 0 \text{ for all } f \in I\}$ defined by I is contained in Z. Suppose now, for a contradiction, that I is a proper ideal of $\mathcal{O}(X)$ and $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}(X)$ that contains I. (Note that $\mathfrak{m}$ defines a point in $V(I)$.) Given $a \in \mathfrak{m}$, write $a = a' + f$ with $a' \in W'$ and $f \in I$. Note that because $I \subseteq \mathfrak{m}$ we have $a' \in \mathfrak{m}$. Since $a' \in W' \subseteq \mathcal{O}(X)^{\hat{\mathbb{G}_a}}$ and I is stable under the $\hat{\mathbb{G}_a}$-action, we have $\hat{\mathbb{G}_a} \cdot a' \subseteq \mathfrak{m}$. So $\mathfrak{m}$ is stable under the $\hat{\mathbb{G}_a}$-action. But then $\mathfrak{m}$ defines a point of $V(I) \subseteq Z$ that is fixed by $\hat{\mathbb{G}_a}$, which is a contradiction. Hence $I = \mathcal{O}(X)$. In particular, the constant function $1 \in W_0 = \tilde{W}$, so there is $f \in \mathcal{O}(X)$ such that $\xi(f) = 1$. □

We are now in a position to prove

Proposition 5.3.4. *(Theorem 5.3.1, 1.) For each nonzero $\sigma \in H^0(X, L)_{\max}$, the natural map $X_\sigma \to \operatorname{Spec}(\mathcal{O}(X_\sigma)^U)$ is a trivial U-quotient for the U-action on X_σ. Thus, $X^0_{\hat{U},L} \subseteq X^{\mathrm{s}(U,L)}$ and the restriction of the enveloping quotient map for $U \curvearrowright L \to X$ restricts to define a locally trivial U-quotient of $X^0_{\hat{U},L}$.*

Proof. As discussed after the statement of Theorem 5.3.1, we may choose a subnormal series

$$1 = U_0 \trianglelefteq U_1 \trianglelefteq \cdots \trianglelefteq U_m = U$$

which is preserved by each automorphism in the family $\lambda : \mathbb{G}_m \to \operatorname{Aut}(U)$ and such that each successive quotient $U_{j+1}/U_j \cong \mathbb{G}_a$, with λ acting on $\operatorname{Lie}(U_{j+1}/U_j)$ with positive weight. We will prove that $X_\sigma \to \operatorname{Spec}(\mathcal{O}(X_\sigma)^{U_j})$ is a trivial U_j-quotient for each $1 \leq j \leq m$ by induction on j.

For the base case, let $\xi_1 \in \operatorname{Lie}(U_1)$ be non-zero. As observed before Lemma 5.3.3, the affine subset X_σ satisfies the conditions needed to apply Lemma 5.3.3 with respect to the semidirect product $\hat{U}_1 = U_1 \rtimes \mathbb{G}_m$, so there is $f \in \mathcal{O}(X)$ such that $\xi_1(f) = 1$. It follows from Lemma 5.3.2 that the natural map $X_\sigma \to \operatorname{Spec}(\mathcal{O}(X_\sigma)^{U_1})$ is a trivial U_1-quotient.

For the induction step, suppose the canonical map $q_j : X_\sigma \to \operatorname{Spec}(\mathcal{O}(X_\sigma)^{U_j})$ is a trivial U_j-quotient, for $1 \leq j \leq m-1$. The action of $U_{j+1} \rtimes_\lambda \mathbb{G}_m$ on X_σ descends to an action of $(U_{j+1}/U_j) \rtimes_\lambda \mathbb{G}_m$ on $\operatorname{Spec}(\mathcal{O}(X_\sigma)^{U_j})$, where the semidirect product $(U_{j+1}/U_j) \rtimes_\lambda \mathbb{G}_m$ is formed with respect to the natural map $\mathbb{G}_m \to \operatorname{Aut}(U_{j+1}/U_j)$ induced by λ. Fixing a $\mathbb{G}_m$-weight vector $\xi_{j+1} \in \operatorname{Lie}(U_{j+1}) \setminus \operatorname{Lie}(U_j)$, we obtain a generator of $\operatorname{Lie}(U_{j+1}/U_j) = \operatorname{Lie}(U_{j+1})/\operatorname{Lie}(U_j)$ which acts on $\mathcal{O}(X_\sigma)^{U_j}$ by restricting the action of ξ_{j+1} on $\mathcal{O}(X_\sigma)$ to the subring $\mathcal{O}(X_\sigma)^{U_j}$. It is immediate that all weights for the natural $\mathbb{G}_m$-action on $\mathcal{O}(X_\sigma)^{U_j}$ are non-positive so, by Lemma 5.3.3, given a point $y \in \operatorname{Spec}(\mathcal{O}(X_\sigma)^{U_j})$ the limit of y under the natural action of $t \in \mathbb{G}_m$ as $t \to 0$ exists. If $y = q_j(x)$ for $x \in X_\sigma$, then because q_j is $\mathbb{G}_m$-equivariant we have

$$\lim_{t\to 0} t \cdot y = \lim_{t \to 0} q_j(t \cdot x) = q_j\left(\lim_{t\to 0} t \cdot x\right),$$

thus we see that all points in $\operatorname{Spec}(\mathcal{O}(X_\sigma)^{U_j})$ have limit in $q_j(X_\sigma \cap Z_{\hat{U},L})$. Let $z \in X_\sigma \cap Z_{\hat{U},L}$ and suppose $u \in U_{j+1}$ is such that $(uU_j) \in \operatorname{Stab}_{U_{j+1}/U_j}(q_j(z))$. Then there is $\tilde{u} \in U_j$ such that $u^{-1}\tilde{u}z = z$. Since $\operatorname{Stab}_U(z)$ is trivial, we conclude that $u = \tilde{u} \in U_j$, so $uU_j = eU_j$. Hence we may apply Lemma 5.3.3 to the action of $(U_{j+1}/U_j) \rtimes_\lambda \mathbb{G}_m$ on $\operatorname{Spec}(\mathcal{O}(X_\sigma)^{U_j})$ to conclude that there is $f \in \mathcal{O}(X_\sigma)^{U_j}$ such that $\xi_{j+1}(f) = 1$. By Lemma 5.3.2, the natural map $\operatorname{Spec}(\mathcal{O}(X_\sigma)^{U_j}) \to \operatorname{Spec}(\mathcal{O}(X_\sigma)^{U_{j+1}})$ is a trivial U_{j+1}/U_j-bundle. Since the projection $U_{j+1} \to U_j$ splits, the composition

$$X_\sigma \to \operatorname{Spec}(\mathcal{O}(X_\sigma)^{U_j}) \to \operatorname{Spec}(\mathcal{O}(X_\sigma)^{U_{j+1}})$$

is a principal U_{j+1}-bundle by Lemma 4.2.2, which is in fact trivial by Proposition 1.1.25. This establishes the induction step.

Therefore $X_\sigma \to \operatorname{Spec}(\mathcal{O}(X_\sigma)^U)$ is a trivial U-quotient. The rest of the proposition follows immediately from Definition 2.3.2 of the stable locus for $U \curvearrowright L \to X$. $\square$

Having established 1 of Theorem 5.3.1, we now turn to proving statements 2–4 of the same theorem. So assume from now on that $X^0_{\hat{U},L} \neq U \cdot Z_{\hat{U},L}$. Also let $q_U : X^{\mathrm{ss,fg}(U,L)} \to X /\!\!/ U$ be the enveloping quotient map for the linearisation $U \curvearrowright L \to X$. As noted above, we have $X^0_{\hat{U},L} \subseteq X^{\mathrm{s}(U,L)}$, so the enveloping quotient map restricts to a geometric quotient

$$q_U : X^0_{\hat{U},L} \to X^0_{\hat{U},L}/U \subseteq X /\!\!/ U,$$

which can locally be described as $X_\sigma \to \mathrm{Spec}(\mathcal{O}(X_\sigma)^U)$, for $\sigma \in H^0(X,L)_{\max}$. Let us fix a basis $\sigma_1, \ldots, \sigma_n$ of $H^0(X,L)_{\max}$. Each of the algebras $\mathcal{O}(X_{\sigma_i})^U$ is finitely generated over $\Bbbk$, so we may find $s > 0$ such that

$$W := H^0(X, L^{\otimes s})^U$$

defines an enveloping system adapted to the subset $\mathcal{S} = \{\sigma_1^s, \ldots, \sigma_n^s\}$ (see the proof of Proposition 2.1.17, 1). Recall that this means each of the $\Bbbk$-algebras $(S^U)_{(\sigma_i^s)} = (S^U)_{(\sigma_i)}$ has generating set given by $\{f/(\sigma_i^s) \mid f \in H^0(X, L^{\otimes s})^U\}$. For each $i = 1, \ldots, n$, let Σ_i denote the section in $H^0(\mathbb{P}(W^*), \mathcal{O}(1))$ corresponding to σ_i^s under the identification $H^0(\mathbb{P}(W^*), \mathcal{O}(1)) = H^0(X, L^{\otimes s})^U$. The inclusion $W \hookrightarrow H^0(X, L^{\otimes s})$ defines a morphism

$$\phi : X^0_{\hat{U},L} \to \mathbb{P}(W^*)$$

which descends to a locally closed immersion

$$\overline{\phi} : X^0_{\hat{U},L}/U \hookrightarrow \mathbb{P}(W^*)$$

such that each of the restrictions $\overline{\phi} : \mathrm{Spec}(\mathcal{O}(X_{\sigma_i})^U) \hookrightarrow \mathbb{P}(W^*)_{\Sigma_i}$ is a closed immersion (this is because the pullback maps $\mathcal{O}(\mathbb{P}(W^*)_{\Sigma_i}) \to \mathcal{O}(X_{\sigma_i})^U$ are surjective, by definition of W). Furthermore, the canonical $\mathbb{G}_m$-linearisation on $\mathcal{O}_{\mathbb{P}(W^*)}(1) \to \mathbb{P}(W^*)$ is compatible with the restricted linearisation $\mathbb{G}_m \curvearrowright L^{\otimes s} \to X^0_{\hat{U},L}$ under ϕ, and the embedding $\overline{\phi} : X^0_{\hat{U},L}/U \hookrightarrow \mathbb{P}(W^*)$ is equivariant with respect to the canonically induced $\mathbb{G}_m$-action on $X^0_{\hat{U},L}/U$. Let $\overline{X^0_{\hat{U},L}/U}$ be the closure of $X^0_{\hat{U},L}/U$ inside $\mathbb{P}(W^*)$ via $\overline{\phi}$ and, by abuse of notation, let $\overline{\phi} : X^0_{\hat{U},L}/U \hookrightarrow \overline{X^0_{\hat{U},L}/U}$ be the induced open immersion.

Let $(W^*)_{\min}$ be the weight space of minimal possible weight for the natural $\mathbb{G}_m$-action on W^*, let $\mathbb{P}((W^*)_{\min})$ be the associated linear subspace of $\mathbb{P}(W^*)$ and let $\mathbb{P}(W^*)^0_{\mathbb{G}_m}$ be the open subset of points in $\mathbb{P}(W^*)$ that flow to $\mathbb{P}((W^*)_{\min})$ under the action of $t \in \mathbb{G}_m$, as $t \to 0$.

Lemma 5.3.5. *The locally closed immersion* $\overline{\phi} : X^0_{\hat{U},L}/U \hookrightarrow \mathbb{P}(W^*)$ *has image contained in* $\mathbb{P}(W^*)^0_{\mathbb{G}_m}$*, and the induced embedding* $X^0_{\hat{U},L}/U \hookrightarrow \mathbb{P}(W^*)^0_{\mathbb{G}_m}$ *is a closed immersion.*

Proof. We introduce some notation. Given a tuple $K = (k_1, \dots, k_n) \in \mathbb{N}^n$ of non-negative integers such that $k_1 + \cdots + k_n = s$, let $\sigma^K := \sigma_1^{k_1} \cdots \sigma_n^{k_n}$ and let Σ_K be the section in $H^0(\mathbb{P}(W^*), \mathcal{O}(1))$ that corresponds to σ^K under the identification $H^0(\mathbb{P}(W^*), \mathcal{O}(1)) = H^0(X, L^{\otimes s})^U$. Observe that the maximal weight space $H^0(\mathbb{P}(W^*), \mathcal{O}(1))_{\max}$ for the $\mathbb{G}_m$-action on $H^0(\mathbb{P}(W^*), \mathcal{O}(1))$ is spanned by the Σ_K with $K = (k_1, \dots, k_n)$ running over all tuples in $\mathbb{N}^n$ such that $k_1 + \cdots + k_n = s$, thus $\mathbb{P}(W^*)^0_{\mathbb{G}_m}$ is covered by the associated affine open subsets $\mathbb{P}(W^*)_{\Sigma_K}$. Because $q_U : X^0_{\hat{U},L} \to X^0_{\hat{U},L}/U$ is surjective, we also have

$$(\overline{\phi})^{-1}(\mathbb{P}(W^*)_{\Sigma_K}) = q_U(\phi^{-1}(\mathbb{P}(W^*)_{\Sigma_K})) = q_U(X_{\sigma^K}) = \operatorname{Spec}(\mathcal{O}(X_{\sigma^K})^U).$$

In particular, choosing K with i-th entry equal to s and zero in each other entry (so that $\Sigma_K = \Sigma_i$), we see that $(\overline{\phi})^{-1}(\mathbb{P}(W^*)_{\Sigma_K}) = \operatorname{Spec}(\mathcal{O}(X_{\sigma_i})^U)$, which cover $X^0_{\hat{U},L}/U$ as i runs from 1 to n. Hence, the image of $X^0_{\hat{U},L}/U$ under $\overline{\phi}$ is contained in $\mathbb{P}(W^*)^0_{\mathbb{G}_m}$.

For each tuple $K = (k_1, \dots, k_n)$ (with $k_1 + \cdots + k_n = s$), we claim that the restriction

$$\overline{\phi} : \operatorname{Spec}(\mathcal{O}(X_{\sigma^K})^U) \hookrightarrow \mathbb{P}(W^*)_{\Sigma_K}$$

is a closed immersion of affine varieties. Note that showing this is enough to prove the lemma, because closed immersions are local on the base and the $\mathbb{P}(W^*)_{\Sigma_K}$ cover $\mathbb{P}(W^*)^0_{\mathbb{G}_m}$. To prove the claim, it is equivalent to show that each pullback $(\overline{\phi})^{\#} : \mathcal{O}(\mathbb{P}(W^*)_{\Sigma_K}) \to \mathcal{O}(X_{\sigma^K})^U$ is surjective. Under the usual identifications $(\operatorname{Sym}^\bullet W)_{(\Sigma_K)} = \mathcal{O}(\mathbb{P}(W^*)_{\Sigma_K})$ and $(S^U)_{(\sigma^K)} = \mathcal{O}(X_{\sigma^K})^U$, this amounts to showing that the homomorphism

$$\psi_K : (\operatorname{Sym}^\bullet W)_{(\Sigma_K)} \to (S^U)_{(\sigma^K)},$$

induced by $\frac{f}{\Sigma_K} \mapsto \frac{f}{\sigma^K}$ for $f \in W = H^0(X, L^{\otimes s})^U$, is surjective.

To this end, after relabelling if necessary we may assume, without loss of generality, that $K = (k_1, \ldots, k_p, 0, \ldots, 0)$, with $1 \leq p \leq n$ and $k_i > 0$ for $i = 1, \ldots, p$. Then as subalgebras of the field of rational functions $\Bbbk(X) = S_{((0))}$, we have

$$(S^U)_{(\sigma^K)} = (S^U)_{(\sigma_1 \cdots \sigma_p)} = (S^U)_{(\sigma_1)} \left[\tfrac{\sigma_1}{\sigma_2}, \ldots, \tfrac{\sigma_1}{\sigma_p}\right]$$

where the last ring is the subalgebra generated by $(S^U)_{(\sigma_1)}$ and the rational functions $\frac{\sigma_1}{\sigma_2}, \ldots, \frac{\sigma_1}{\sigma_p}$ (which are regular on X_{σ^K}). Observe that

$$\frac{\sigma_1^{k_1-1}\sigma_2^{k_2}\cdots\sigma_p^{k_p}\sigma_i}{\sigma^K} = \frac{\sigma_i}{\sigma_1} \in (S^U)_{(\sigma_1)}, \quad i = 2, \ldots, p,$$

and (where $\widehat{\sigma_i^{k_i}}$ means 'omit $\sigma_i^{k_i}$' in what follows)

$$\frac{\sigma_1^{k_1}\cdots\widehat{\sigma_i^{k_i}}\cdots\sigma_p^{k_p}\sigma_i^{k_i-1}\sigma_1}{\sigma^K} = \frac{\sigma_1}{\sigma_i} \quad i = 2, \ldots, p.$$

Also, for each $f \in H^0(X, L^{\otimes s})^U$ we have

$$\frac{f}{\sigma^K} \cdot \left(\frac{\sigma_2}{\sigma_1}\right)^{k_2} \cdots \left(\frac{\sigma_p}{\sigma_1}\right)^{k_p} = \frac{f}{\sigma_1^s} \in (S^U)_{(\sigma_1)}.$$

and recall by choice of $s > 0$ that the algebra $(S^U)_{(\sigma_1)}$ is generated by the $\frac{f}{\sigma_1^s}$ for $f \in H^0(X, L^{\otimes s})^U$. We therefore see that the image of ψ_K contains $(S^U)_{(\sigma_1)}$, along with the extra generators $\frac{\sigma_1}{\sigma_2}, \ldots, \frac{\sigma_1}{\sigma_p}$, and so conclude that ψ_K is surjective, as claimed. This completes the proof. □

From Lemma 5.3.5 we see that the $\overline{\phi}$ induces a $\mathbb{G}_m$-equivariant isomorphism of quasi-projective varieties $\overline{\phi} : X^0_{\hat{U},L}/U \xrightarrow{\cong} \overline{X^0_{\hat{U},L}/U} \cap \mathbb{P}(W^*)^0_{\mathbb{G}_m}$. Let us now use this identification freely for the rest of the argument, writing

$$X^0_{\hat{U},L}/U = \overline{X^0_{\hat{U},L}/U} \cap \mathbb{P}(W^*)^0_{\mathbb{G}_m}.$$

Any point $X^0_{\hat{U},L}/U$ has limit under the action of $t \in \mathbb{G}_m$, as $t \to 0$, contained in the closed subset $\overline{X^0_{\hat{U},L}/U} \cap \mathbb{P}((W^*)_{\min}) \subseteq \overline{X^0_{\hat{U},L}/U} \cap \mathbb{P}(W^*)^0_{\mathbb{G}_m}$ under this isomorphism. In particular, given $x \in X^0_{\hat{U},L}$ the point $q_U(x) \in \overline{X^0_{\hat{U},L}/U} \cap \mathbb{P}((W^*)_{\min})$ if, and only if,

$$q_U(x) = \lim_{t\to 0} t \cdot q_U(x) = q_U\left(\lim_{t\to 0} t \cdot x\right).$$

Thus, for each $x \in X^0_{\hat{U},L}$, we have

$$q_U(x) \in \overline{X^0_{\hat{U},L}/U} \cap \mathbb{P}((W^*)_{\min}) \iff x \in U \cdot Z_{\hat{U},L}. \tag{5.3.1}$$

By assumption we have $X^0_{\hat{U},L} \neq U \cdot Z_{\hat{U},L}$, so as a consequence of (5.3.1) we may conclude that $\mathbb{P}((W^*)_{\min}) \neq \mathbb{P}(W^*)^0_{\mathbb{G}_m}$ and hence that the $\mathbb{G}_m$-action on $W = H^0(X, L^{\otimes s})^U$ as at least two distinct weights. Note that the maximum weight for the $\mathbb{G}_m$-action on $H^0(X, L^{\otimes s})^U$ is equal to $-s\varpi$. Let $\epsilon > 0$ be a rational number such that $-s(\varpi + \epsilon)$ lies strictly between $-s\varpi$ and the next largest weight for the $\mathbb{G}_m$-action on W (which must be $\leq -s\varpi - 1$). Let χ be the rational character of $\hat{U}$ of weight $-\varpi - \epsilon$ and consider the rational $\mathbb{G}_m$-linearisation $\mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)} \to \mathbb{P}(W^*)$. The rational weights of $H^0(\mathbb{P}(W^*), \mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)})^*$ are arranged such that the minimal weight is less than 0 and the next smallest weight is greater than 0, so it follows immediately from the Hilbert-Mumford criterion (Theorem 1.2.6) that the stable locus for $\mathbb{G}_m \curvearrowright \mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)} \to \mathbb{P}(W^*)$ is equal to the semistable locus, which is equal to $\mathbb{P}(W^*)^0_{\mathbb{G}_m} \setminus \mathbb{P}((W^*)_{\min})$. Let

$$M := \mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)}|_{\overline{X^0_{\hat{U},L}/U}} \to \overline{X^0_{\hat{U},L}/U}$$

be the restriction of the rational $\mathbb{G}_m$-linearisation $\mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)} \to \mathbb{P}(W^*)$ to $\overline{X^0_{\hat{U},L}/U}$. Then by restriction of (semi)stable loci we have

$$\begin{aligned}
\overline{X^0_{\hat{U},L}/U}^{\mathrm{s}(M)} &= \overline{X^0_{\hat{U},L}/U} \cap \mathbb{P}(W^*)^{\mathrm{s}(\mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)})} \\
&= \overline{X^0_{\hat{U},L}/U} \cap \mathbb{P}(W^*)^{\mathrm{ss}(\mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)})} \\
&= \overline{X^0_{\hat{U},L}/U}^{\mathrm{ss}(M)},
\end{aligned}$$

which are furthermore equal to

$$(\overline{X^0_{\hat{U},L}/U} \cap \mathbb{P}(W^*)^0_{\mathbb{G}_m}) \setminus (\overline{X^0_{\hat{U},L}/U} \cap \mathbb{P}((W^*)_{\min}) = (X^0_{\hat{U},L}/U) \setminus ((X^0_{\hat{U},L}/U) \cap \mathbb{P}((W^*)_{\min})).$$

By (5.3.1) we therefore have

$$q_U^{-1}\left(\overline{X^0_{\hat{U},L}/U}^{\mathrm{s}(M)}\right) = X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}).$$

Let $L^{(\chi)} \to X$ be the rational $\hat{U}$-linearisation obtained by twisting the linearisation $L \to X$ by the rational character χ. Note then that the rational $\mathbb{G}_m$-linearisations $L^{(s\chi)} \to X^0_{\hat{U},L}$ and $\mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)} \to \mathbb{P}(W^*)$ are compatible via $\phi : X^0_{\hat{U},L} \to \mathbb{P}(W^*)$.

Proposition 5.3.6. *We have* $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}) \subseteq X^{\mathrm{s}(\hat{U},L^{(\chi)})}$.

Proof. From the arguments proceeding the statement of the proposition we have the equality

$$X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}) = \phi^{-1}(\mathbb{P}(W^*)^{\mathrm{ss}(\mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)})}),$$

so we will show that $\phi^{-1}(\mathbb{P}(W^*)^{\mathrm{ss}(\mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)})}) \subseteq X^{\mathrm{s}(\hat{U},L^{(\chi)})}$. Suppose F is contained in $H^0(\mathbb{P}(W^*), \mathcal{O}(d))^{\mathbb{G}_m}$, with $d > 0$. Then we may regard F as a linear combination of degree d monomials in $\mathbb{G}_m$-weight vectors in W, and each such monomial must be $\mathbb{G}_m$-invariant. So, in covering $\mathbb{P}(W^*)^{\mathrm{ss}(\mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)})}$ by open subsets of the form $\mathbb{P}(W^*)_F$, we may assume that F is an invariant monomial of weight vectors, without loss of generality. Note also that such a monomial must be divisible by some $\Sigma_K \in H^0(\mathbb{P}(W^*), \mathcal{O}(1))_{\max} = H^0(X, L^{\otimes s})_{\max}$. It follows that $\phi^* F \in H^0(X, (L^{(\chi)})^{\otimes ds})^{\hat{U}}$ is equal to $g\sigma$, with $\sigma \in H^0(X, L)_{\max}$ and $g \in H^0(X, (L^{(\chi)})^{\otimes (ds-1)})^U$. In particular, the map $q_U : X_{\phi^* F} \to \operatorname{Spec}(\mathcal{O}(X_{\phi^* F})^U)$ is equal to the canonical morphism

$$X_{g\sigma} \to \operatorname{Spec}((\mathcal{O}(X_\sigma)^U)_a),$$

where $(\mathcal{O}(X_\sigma)^U)_a$ is the localisation of $\mathcal{O}(X_\sigma)^U$ at the function $a = \frac{g}{\sigma^{ds-1}}$. This must be a locally trivial U-quotient, being the restriction of the locally trivial quotient $q_U : X_\sigma \to \operatorname{Spec}(\mathcal{O}(X_\sigma)^U)$. Since $\operatorname{Spec}(\mathcal{O}(X_{\phi^* F})^U)$ maps into $\mathbb{P}(W^*)^{\mathrm{ss}(\mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)})}$ under the embedding $\overline{\phi}$, by restriction we see that the action of $\mathbb{G}_m$ on $\operatorname{Spec}(\mathcal{O}(X_{\phi^* F})^U)$ is closed with all stabilisers finite. The same is true for the $\mathbb{G}_m$-action on $X_{\phi^* F}$ by Lemma 2.3.1. Finally, the open set $X_{\phi^* F}$ is affine because $L^{(\chi)} \to X$ is ample as a line bundle, so we conclude that $X_{\phi^* F} \subseteq X^{\mathrm{s}(\hat{U},L^{(\chi)})}$. Thus $\phi^{-1}(\mathbb{P}(W^*)^{\mathrm{ss}(\mathcal{O}_{\mathbb{P}(W^*)}(1)^{(s\chi)})})$ is contained in $X^{\mathrm{s}(\hat{U},L^{(\chi)})}$, as desired. □

We are now in a position to complete the proof of Theorem 5.3.1. The GIT quotient $\pi_{\mathbb{G}_m} : \overline{X^0_{\hat{U},L}/U}^{\mathrm{ss}(M)} \to \overline{X^0_{\hat{U},L}/U} /\!\!/_M \mathbb{G}_m$ is a geometric quotient for the action of $\mathbb{G}_m$ on

$\overline{X^0_{\hat{U},L}/U}^{\mathrm{s}(M)} = \overline{X^0_{\hat{U},L}/U}^{\mathrm{ss}(M)}$, hence the composition

$$X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}) \xrightarrow{q_U} \overline{X^0_{\hat{U},L}/U}^{\mathrm{s}(M)} \xrightarrow{\pi_{\mathbb{G}_m}} \overline{X^0_{\hat{U},L}/U} /\!\!/_M \mathbb{G}_m$$

provides a geometric quotient for the $\hat{U}$-action on $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$, with projective quotient variety $(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}))/\hat{U} \cong \overline{X^0_{\hat{U},L}/U} /\!\!/_M \mathbb{G}_m$. On the other hand, by Proposition 5.3.6 we know that $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ is an open $\hat{U}$-stable subset of the stable locus $X^{\mathrm{s}(L^{(\chi)})}$ for the rational linearisation $\hat{U} \curvearrowright L^{(\chi)} \to X$, so by uniqueness of geometric quotients we may identify $(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}))/\hat{U}$ with the image of $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L})$ under the enveloping quotient map $q : X^{\mathrm{ss,fg}(L^{(\chi)})} \to X \wr\wr_{L^{(\chi)}} \hat{U}$. Note that, because X is irreducible and the enveloping quotient map is dominant, $(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}))/\hat{U}$ is a dense open subscheme of $X \wr\wr_{L^{(\chi)}} \hat{U}$. On the other hand, the quotient $(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}))/\hat{U}$ is projective, thus universally closed over $\operatorname{Spec} \Bbbk$, and so $(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}))/\hat{U}$ is also a closed subscheme of $X \wr\wr_{L^{(\chi)}} \hat{U}$, since the latter is separated over $\operatorname{Spec} \Bbbk$ [Sta15, Tag 01W0]. Hence $(X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}))/\hat{U} = X \wr\wr_{L^{(\chi)}} \hat{U}$. Because $X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}) \subseteq X^{\mathrm{s}(L^{(\chi)})}$ it follows that $X^{\mathrm{s}(L^{(\chi)})}/\hat{U} = X \wr\wr_{L^{(\chi)}} \hat{U}$. Also, a consequence of the definition of the stable locus $X^{\mathrm{s}(L^{(\chi)})}$ is that it satisfies $q^{-1}(q(X^{\mathrm{s}(L^{(\chi)})})) = X^{\mathrm{s}(L^{(\chi)})}$, so we in fact have

$$X^0_{\hat{U},L} \setminus (U \cdot Z_{\hat{U},L}) = X^{\mathrm{s}(L^{(\chi)})} = X^{\mathrm{ss,fg}(L^{(\chi)})},$$

which is 2 of Theorem 5.3.1. The enveloping quotient map $q : X^{\mathrm{ss,fg}(L^{(\chi)})} \to X \wr\wr_{L^{(\chi)}} \hat{U}$ is therefore a geometric quotient for the $\hat{U}$-action on $X^{\mathrm{ss,fg}(L^{(\chi)})}$, proving 4 of Theorem 5.3.1. Finally, by Corollary 2.1.19, for sufficiently divisible $r > 0$ the ring of invariants $\Bbbk[X, (L^{(\chi)})^{\otimes r}]^{\hat{U}}$ for the $\hat{U}$-linearisation $(L^{(\chi)})^{\otimes r} \to X$ is a finitely generated $\Bbbk$-algebra, which gives 3 of Theorem 5.3.1. This completes the proof of Theorem 5.3.1.

Remark 5.3.7. We expect that we can relax condition $(\mathfrak{C})$ to prove results similar to Theorem 5.1.4 for more general linearisations of positive extensions of unipotent groups, by removing the condition that all points in $Z_{\hat{U},L}$ have trivial U-stabiliser, though this will require some work. To give an idea of why this is so, consider the problem of studying $\hat{U}$-quotients of $(X, L) = (\mathbb{P}(V), \mathcal{O}(1))$ with $U = \mathbb{G}_a$ (cf. Proposition 5.2.3). The basic problem here is that there may now be points that lie in $\mathbb{P}(V)^{\mathrm{ss,fg}} \setminus \mathbb{P}(V)^{\mathrm{s}}$, so that the enveloping quotient $\mathbb{P}(V)^{\mathrm{ss,fg}} \to \mathbb{P}(V) \wr\!\wr_{\mathcal{O}(1)^{(\chi)}} \hat{U}$ may not be a geometric quotient. One could try to deal with this by using the partial desingularisation process of [Kir85] on the reductive envelope $\mathbb{P}^2 \times \mathbb{P}(V)$ to arrive at blown-up variety $\widetilde{\mathbb{P}(V)}$ with ample $\hat{U}$-linearisation such that finitely generated semistability and stability coincide. We expect that a similar idea could be applied to the setting where U is a more general unipotent group and $L \to X$ is a very ample linearisation over a projective variety X.

5.3.2 Application: A Compactification of $X^0_{\hat{U},L}/U$

We close this chapter by applying Theorem 5.1.4 to construct a compactification of the quotient $X^0_{\hat{U},L}/U$, in the case where $L \to X$ is a very ample linearisation of an irreducible projective U-variety X which admits an extension to a linearisation of a positive extension $\hat{U} = U \rtimes_\lambda \mathbb{G}_m$-linearisation satisfying condition ($\mathfrak{C}$).

The argument runs as follows. As in (3.2.2) of Section 3.2.1, the extension of the U-linearisation to one of $\hat{U}$ gives an isomorphism of $\hat{U}$-linearisations

$$\begin{tikzcd}
\hat{U} \times^U L \arrow[r, "\cong"] \arrow[d] & (\hat{U}/U) \times L \arrow[d] & {[h,l] \mapsto (hU, hl)} \\
\hat{U} \times^U X \arrow[r, "\cong"] & (\hat{U}/U) \times X & {[h,x] \mapsto (hU, hx).}
\end{tikzcd}$$

Note that $\hat{U}/U = \mathbb{G}_m$ sits inside $\mathbb{P}^1$ equivariantly via $z \mapsto [z:1]$, when we linearise $\mathcal{O}_{\mathbb{P}^1}(1) \to \mathbb{P}^1$ using the representation

$$\hat{U} \times \Bbbk^2 \to \Bbbk^2, \quad ((t;u),(z_0,z_1)) \mapsto (tz_0, z_1).$$

Consider the linearisation

$$\hat{U} \curvearrowright \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes L \to \mathbb{P}^1 \times X.$$

Observe that this restricts to $\hat{U} \times^U L$ as linearisations under the open immersion

$$\beta : \hat{U} \times^U X \cong (\hat{U}/U) \times X = \mathbb{G}_m \times X \hookrightarrow \mathbb{P}^1 \times X.$$

By the Künneth formula [Sta15, Tag 02KE] we have an identification

$$H^0(\mathbb{P}^1 \times X, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes L)^* = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^* \otimes H^0(X, L)^* = \Bbbk^2 \otimes V$$

of $\hat{U}$-representations. The minimal weight for the induced $\mathbb{G}_m$-action on $\Bbbk^2 \otimes V$ is equal to ϖ and the associated weight space is $\Bbbk\langle(0,1)\rangle \otimes V_{\min}$, where $V_{\min}$ is the weight space of minimal weight ϖ for the $\mathbb{G}_m$-action on V. The subvariety of $\mathbb{P}^1 \times X$ cut out by the projective subspace $\mathbb{P}(\Bbbk\langle(0,1)\rangle \otimes V_{\min})$ is equal to $Z_{\hat{U},L} \times \{0\}$, so by applying Theorem 5.1.4 to $\hat{U} \curvearrowright L \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \to X \times \mathbb{P}^1$ we see that, as long as $Z_{\hat{U},L}$ does not contain any points fixed by U, the rational linearisation

$$\mathcal{L} \to \mathbb{P}^1 \times X \tag{5.3.2}$$

obtained by twisting $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes L$ by the rational character of weight $-\varpi - \epsilon$, for a sufficiently small rational number $\epsilon > 0$, satisfies

$$(\mathbb{P}^1 \times X)^{\mathrm{s}(\mathcal{L})} = (\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})} = (\Bbbk \times X^0_{\hat{U},L}) \setminus (\{0\} \times (U \cdot Z_{\hat{U},L})),$$

where $\Bbbk \cong \mathbb{P}^1 \setminus \{[1:0]\}$ and $X^0_{\hat{U},L}$ is the subset of points of X that flow to $Z_{\hat{U},L}$ under the action of $t \in \mathbb{G}_m = \hat{U}/U$, as $t \to 0$. In addition, for suitably divisible $r > 0$ such that $\mathcal{L}^{\otimes r} = (\mathcal{O}_{\mathbb{P}^1}(r) \boxtimes L^{\otimes r})^{(r\chi)}$ is a linearisation, the ring of invariants $\Bbbk[\mathbb{P}^1 \times X, \mathcal{L}^{\otimes r}]^{\hat{U}}$ is a finitely generated algebra over $\Bbbk$, and the enveloping quotient map

$$(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})} \to (\mathbb{P}^1 \times X) /\!\!/_{\mathcal{L}} \hat{U} = \mathrm{Proj}(\Bbbk[\mathbb{P}^1 \times X, \mathcal{L}^{\otimes r}]^{\hat{U}})$$

is a geometric quotient for the $\hat{U}$-action, with $(\mathbb{P}^1 \times X) /\!\!/_{\mathcal{L}} \hat{U}$ a projective variety.

Let $\alpha : X \hookrightarrow \hat{U} \times^U X$ be the canonical closed immersion. By inspection we have

$$X^0_{\hat{U},L} = (\beta \circ \alpha)^{-1}((\mathbb{P}^1 \times X)^{\mathrm{s}(\mathcal{L})}),$$

so $\hat{U} \times^U (X^0_{\hat{U},L})$ is an $\hat{U}$-stable open subset of $(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})}$ and hence admits a geometric quotient. Thus $\beta \circ \alpha$ descends to give a compactification

$$X^0_{\hat{U},L}/U \cong (\hat{U} \times^U (X^0_{\hat{U},L}))/\hat{U} \subseteq (X \times \mathbb{P}^1)^{\mathrm{ss,fg}(\mathcal{L})}/\hat{U} = (X \times \mathbb{P}^1) /\!\!/_{\mathcal{L}} \hat{U}$$

of the quotient $X^0_{\hat{U},L}/U$.

In summary, we have proved the following corollary to Theorem 5.1.4.

Corollary 5.3.8. *Let X be an irreducible variety acted upon by a unipotent group U and let $L \to X$ be a very ample linearisation. Suppose the linearisation extends to a linearisation of a positive extension $\hat{U}$ of U satisfying condition ($\mathfrak{C}$). Then the rational linearisation $\hat{U} \curvearrowright \mathcal{L} \to \mathbb{P}^1 \times X$ of* (5.3.2) *satisfies the following properties:*

1. *for suitably divisible integers $r > 0$ the ring of invariants $\Bbbk[\mathbb{P}^1 \times X, \mathcal{L}^{\otimes r}]^{\hat{U}}$ for the linearisation $\mathcal{L}^{\otimes r} = (\mathcal{O}_{\mathbb{P}^1}(r) \boxtimes L^{\otimes r}))^{(r\chi)}$ is finitely generated over $\Bbbk$;*

2. *we have the equality $(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})} = (\mathbb{P}^1 \times X)^{\mathrm{s}(\mathcal{L})}$, and*

$$(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})} = (\Bbbk \times X^0_{\hat{U},L}) \setminus (\{0\} \times (U \cdot Z_{\hat{U},L}));$$

3. *the enveloping quotient map*

$$(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})} \to (\mathbb{P}^1 \times X) \wr\wr_{\mathcal{L}} \hat{U}$$

   is a geometric quotient for the $\hat{U}$-action on $(\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})} = (\mathbb{P}^1 \times X)^{\mathrm{s}(\mathcal{L})}$ and $(\mathbb{P}^1 \times X) \wr\wr_{\mathcal{L}} \hat{U} = \operatorname{Proj}(\Bbbk[\mathbb{P}^1 \times X, \mathcal{L}^{\otimes r}]^{\hat{U}})$ is a projective variety; and

4. *there is an inclusion $X^0_{\hat{U},L} \subseteq X^{\mathrm{s}(U,L)}$, and the natural locally closed immersion $X^0_{\hat{U},L} \hookrightarrow (\mathbb{P}^1 \times X)^{\mathrm{ss,fg}(\mathcal{L})}$ descends to an open immersion*

$$X^0_{\hat{U},L}/U \hookrightarrow (\mathbb{P}^1 \times X) \wr\wr_{\mathcal{L}} \hat{U}.$$

Appendix A

Additional Background Material

Here we collect some basic facts that will be needed in the main body of this thesis.

A.1 Proj of a Graded Ring

Throughout this thesis we often work with schemes of the form $\operatorname{Proj} S$ for a graded ring S that is not necessarily noetherian or generated in degree 1 over S_0. We therefore recall some basic constructions and facts about such schemes; our basic reference is [Sta15, Tag 01LE: Constructions of Schemes]. Let $X = \operatorname{Proj} S$. For $f \in S$ a homogeneous element of positive degree, we denote by $D_+(f)$ the basic open subset of X consisting of homogeneous prime ideals $\mathfrak{p} \in \operatorname{Proj} S$ not containing f. For each $d \in \mathbb{Z}$ the graded S-module $S(d)$ defines a quasi-coherent sheaf of $\mathcal{O}_X$-modules, $\mathcal{O}_X(d)$, in the usual way; see [Har77, Chapter2, §2]. These sheaves are in general not locally free, or even coherent, because S may not be generated in degree 1 as an S_0-algebra, or be locally noetherian. Moreover, the natural maps

$$\mathcal{O}_X(d) \otimes \mathcal{O}_X(e) \to \mathcal{O}_X(d+e), \quad d, e \in \mathbb{Z}$$

are not necessarily isomorphisms. However, for $f \in S_d$ a non-zero homogeneous element of degree $d > 0$, the sheaves $\mathcal{O}_X(nd)|_{D_+(f)}$ are isomorphic to $\mathcal{O}_{D_+(f)}$ over $D_+(f)$, for all integers n. It follows that if $U \subseteq X$ is a quasi-compact open subscheme, then there is an integer $d > 0$ such that the restriction $\mathcal{O}_U(d)$ of $\mathcal{O}_X(d)$ to U is locally free of rank 1, and for any $m, n \in \mathbb{Z}$ the maps $\mathcal{O}_U(md) \otimes \mathcal{O}_U(n) \to \mathcal{O}_U(md+n)$ are isomorphisms.

There are two common ways of constructing a morphism into $X = \operatorname{Proj} S$. The first runs as follows. Suppose $\psi : S \to R$ is a homomorphism of graded rings and let $Y = \operatorname{Proj} R$. Then there is a canonical morphism

$$\Psi : U_\psi \to Y$$

from the open subscheme

$$U_\psi = \bigcup \{D_+(\psi(f)) \mid f \in S \text{ homogeneous of positive degree}\}$$

of X, and a map of graded $\mathcal{O}_{U_\psi}$-algebras

$$\theta : \Psi^* \left(\bigoplus_{d \in \mathbb{Z}} \mathcal{O}_Y(d) \right) \to \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{U_\psi}(d).$$

The pair Ψ, θ is characterised by the following two properties. Firstly, for each $f \in S$ homogeneous of positive degree we have $\Psi^{-1}(D_+(f)) = D_+(\psi(f))$, and the restriction $\Psi : D_+(\psi(f)) \to D_+(f)$ is the morphism of affine schemes determined by the ring map $S_{(f)} \to R_{(\psi(f))}$ induced by ψ. Secondly, for each $d \in \mathbb{Z}$ the following diagram commutes

$$\begin{tikzcd}
S_d \arrow[r, "\psi"] \arrow[d] & R_d \arrow[r] & H^0(Y, \mathcal{O}_Y(d)) \arrow[d, "\text{res}"] \\
H^0(X, \mathcal{O}_X(d)) \arrow[rr, "\theta"] & & H^0(U_\psi, \mathcal{O}_{U_\psi}(d))
\end{tikzcd}$$

where the unmarked arrows are the natural ones arising from the construction of the twisting sheaves $\mathcal{O}(d)$.

The second way of constructing a map into X generalises the first. Suppose Y is any scheme equipped with an invertible sheaf $\mathcal{L}$ and

$$\psi : S \to \bigoplus_{d \geq 0} H^0(Y, \mathcal{L}^{\otimes d})$$

a homomorphism of graded rings. Then setting

$$U_{\mathcal{L},\psi} = \bigcup \{Y_{\psi(f)} \mid f \in S \text{ homogeneous of positive degree}\},$$

there is a canonical morphism

$$\Psi_{\mathcal{L}} : U_{\mathcal{L},\psi} \to X$$

and a map of graded $\mathcal{O}_{U_{\mathcal{L},\psi}}$-algebras

$$\theta : \Psi_{\mathcal{L}}^* \left(\bigoplus_{d \in \mathbb{Z}} \mathcal{O}_Y(d) \right) \to \bigoplus_{d \in \mathbb{Z}} \mathcal{L}^{\otimes d}|_{U_{\mathcal{L},\psi}},$$

characterised by the following requirements. First, for every $f \in S$ homogeneous of positive degree we have $\Psi_{\mathcal{L}}^{-1}(D_+(f)) = Y_{\psi(f)}$, and secondly, for each $d \geq 0$ the diagram

$$\begin{tikzcd}
S_d \arrow[r, "\psi"] \arrow[d] & H^0(Y, \mathcal{L}^{\otimes d}) \arrow[d, "\text{res}"] \\
H^0(X, \mathcal{O}_X(d)) \arrow[r, "\theta"] & H^0(U_{\mathcal{L},\psi}, \mathcal{L}^{\otimes d})
\end{tikzcd}$$

commutes. Furthermore, if Y is quasi-compact then there is $d_0 > 0$ such that

$$\theta : \Psi_{\mathcal{L}}^* \mathcal{O}_Y(d) \xrightarrow{\cong} \mathcal{L}^{\otimes d}|_{U_{\mathcal{L},\psi}}$$

is an isomorphism of sheaves, for all $d \geq d_0$. (This last statement follows from [Sta15, Tag 01N4].)

A.2 Linearisations of Products of Reductive Groups

We discuss GIT quotients of a direct product of reductive groups. For this section, suppose G_1 and G_2 are reductive groups and X is a $G_1 \times G_2$-variety equipped with a $G_1 \times G_2$-linearisation $L \to X$. Via the natural embeddings $G_i \hookrightarrow G_1 \times G_2$, $i = 1, 2$, this data is equivalent to saying that the variety X and the line bundle L are equipped with two commuting linearisations $G_i \curvearrowright L \to X$. In particular, it makes sense to consider the semistable loci $X^{\mathrm{ss}(G_1)}$ and $X^{\mathrm{ss}(G_1 \times G_2)}$ with respect to the linearisations $G_1 \curvearrowright L \to X$ and $G_1 \times G_2 \curvearrowright L \to X$ respectively, together with their reductive GIT quotients

$$\pi_{G_1} : X^{\mathrm{ss}(G_1)} \to X /\!\!/ G_1,$$
$$\pi_{G_1 \times G_2} : X^{\mathrm{ss}(G_1 \times G_2)} \to X /\!\!/ (G_1 \times G_2).$$

In the case where $L \to X$ is ample and X is projective over an affine variety the following result is well known (cf. [OST99] and [Sch08, Section 1.5.3] for the case $X = \mathbb{P}^n$ and also [Tha96]), though proofs in the general case are hard to come by. For the reader's convenience, we include here a proof for the more general case when X is any variety and $L \to X$ is any line bundle.

Proposition A.2.1. *Retain the notation above.*

1. *The set $X^{\mathrm{ss}(G_1)}$ is stable under the G_2-action on X and there is a canonical action of G_2 on $X/\!\!/G_1$ such that π_{G_1} is G_2-equivariant.*

2. *There is a natural ample G_2-linearisation $M \to X/\!\!/G_1$ such that, for some $n > 0$, $\pi_{G_1}^* M = L^{\otimes n}|_{X^{\mathrm{ss}(G_1)}}$ as G_2-linearisations, and $X^{\mathrm{ss}(G_1\times G_2)} \subseteq \pi_{G_1}^{-1}((X/\!\!/G_1)^{\mathrm{ss}(M)})$. Letting*

$$\overline{\pi}_{G_2} : (X/\!\!/G_1)^{\mathrm{ss}(M)} \to (X/\!\!/G_1)/\!\!/_M G_2$$

denote the reductive GIT quotient with respect to this linearisation, there is a canonical open immersion $\psi : X/\!\!/(G_1 \times G_2) \hookrightarrow (X/\!\!/G_1)/\!\!/_M G_2$ such that the following diagram commutes:

$$\begin{tikzcd}
 & X^{\mathrm{ss}(G_1\times G_2)} \arrow[d, "\pi_{G_1}"] \arrow[ddl, "\pi_{G_1\times G_2}"'] \\
 & (X/\!\!/G_1)^{\mathrm{ss}(M)} \arrow[d, "\overline{\pi}_{G_2}"] \\
X/\!\!/(G_1\times G_2) \arrow[r, hook, "\psi"] & (X/\!\!/G_1)/\!\!/_M G_2
\end{tikzcd}$$

3. *If X is further assumed to be projective, then $X^{\mathrm{ss}(G_1\times G_2)} = \pi^{-1}((X/\!\!/G_1)^{\mathrm{ss}(M)})$ and ψ is an isomorphism.*

Proof. (Proof of 1.) Suppose $f \in H^0(X, L^{\otimes r})^{G_1}$, for some $r > 0$, such that X_f is affine. For any $g_2 \in G_2$ the section $g_2 \cdot f$ is again G_1-invariant, and acting on X by g_2 induces an isomorphism (with inverse given by g_2^{-1}) $X_f \to X_{g_2\cdot f}$, so that $X_{g_2\cdot f}$ is also affine. Hence the G_2-action on X restricts to define an action $\sigma : G_2 \times X^{\mathrm{ss}(G_1)} \to X^{\mathrm{ss}(G_1)}$. Recall that the GIT quotient $\pi_{G_1} : X^{\mathrm{ss}(G_1)} \to X/\!\!/G_1$ is a categorical quotient for the action of G_1 on $X^{\mathrm{ss}(G_1)}$. Let G_1 act on $G_2 \times X^{\mathrm{ss}(G_1)}$ by demanding that G_1 acts trivially on G_2. Then the composition

$$G_2 \times X^{\mathrm{ss}(G_1)} \xrightarrow{\sigma} X^{\mathrm{ss}(G_1)} \xrightarrow{\pi_{G_1}} X/\!\!/G_1$$

is G_1-invariant by virtue of the fact that G_1 is normal in $G_1 \times G_2$, so there is a canonical map $\overline{\sigma} : G_2 \times X/\!\!/G_1 \to X/\!\!/G_1$ such that the diagram

$$\begin{tikzcd}
G_2 \times X^{\mathrm{ss}(G_1)} \arrow[r, "\sigma"] \arrow[d, "\mathrm{id}_{G_2}\times\pi_{G_1}"] & X^{\mathrm{ss}(G_1)} \arrow[d, "\pi_{G_1}"] \\
G_2 \times X/\!\!/G_1 \arrow[r, "\overline{\sigma}"] & X/\!\!/G_1
\end{tikzcd}$$

commutes. Using the universal property of categorical quotients it is easy to verify that $\overline{\sigma}$ defines an action of G_2 on $X/\!\!/G_1$—we omit the details.

(Proof of 2.) The construction of the GIT quotient $X/\!\!/G_1$ comes with an ample line bundle $M \to X/\!\!/G_1$ such that $\pi_{G_1}^* M = L^{\otimes n}|_{X^{\mathrm{ss}(G_1)}}$, for some $n > 0$ [MFK94, Theorem 1.10]. In fact, the natural map $L^{\otimes n}|_{X^{\mathrm{ss}(G_1)}} \to M$ thus arising is a good categorical quotient of the action of G_1 on $L^{\otimes n}|_{X^{\mathrm{ss}(G_1)}}$. (This can be shown by following through the proof of the following more general statement [New78, Proposition 3.12]: if G is a reductive group acting on varieties X and Y, if $X \to Y$ is an affine G-equivariant morphism and Y possesses a good categorical quotient by G, then so too does X.) Following an argument similar to that in the proof of 1, one sees that there is a canonical G_2-action on M such that $L^{\otimes n}|_{X^{\mathrm{ss}(G_1)}} \to M$ is G_2-equivariant and the line bundle projection $M \to X/\!\!/G_1$ is equivariant.

We next show that $X^{\mathrm{ss}(G_1 \times G_2)} \subseteq \pi_{G_1}^{-1}((X/\!\!/G_1)^{\mathrm{ss}(M)})$. Let $x \in X^{\mathrm{ss}(G_1 \times G_2)}$. Then without loss of generality there is an invariant section $f \in H^0(X, L^{\otimes mn})^{G_1 \times G_2}$ with $m > 0$ such that $x \in X_f$ and X_f is affine. Clearly π_{G_1} is defined at x. Because both $\pi_{G_1} : X^{\mathrm{ss}(G_1)} \to X/\!\!/G_1$ and $L^{\otimes mn}|_{X^{\mathrm{ss}(G_1)}} \to M^{\otimes m}$ are G_2-equivariant maps that are categorical quotients for the G_1-actions, pulling back along π_{G_1} defines a canonical G_2-equivariant isomorphism

$$\pi_{G_1}^* : H^0(X/\!\!/G_1, M^{\otimes m}) \xrightarrow{\cong} H^0(X^{\mathrm{ss}(G_1)}, L^{\otimes mn})^{G_1}.$$

Hence there is $F \in H^0(X/\!\!/G_1, M^{\otimes m})^{G_2}$ such that $\pi_{G_1}^{-1}((X/\!\!/G_1)_F) = X_f$. The map π_{G_1} restricts to a good categorical quotient $\pi_{G_1} : X_f \to (X/\!\!/G_1)_F$ for the G_1-action on X_f, and since X_f is affine so too is $(X/\!\!/G_1)_F$ by Theorem 1.2.1, 1. Thus $(X/\!\!/G_1)_F \subseteq (X/\!\!/G_1)^{\mathrm{ss}(M)}$ and $\pi_{G_1}(x) \in (X/\!\!/G_1)^{\mathrm{ss}(M)}$.

The composition $\overline{\pi}_{G_2} \circ \pi_{G_1} : X^{\mathrm{ss}(G_1 \times G_2)} \to (X/\!\!/G_1)/\!\!/_M G_2$ is $G_1 \times G_2$-invariant, so induces a unique morphism $\psi : X/\!\!/(G_1 \times G_2) \to (X/\!\!/G_1)/\!\!/_M G_2$ making the required diagram commute. Recall from the construction of the GIT quotient that $X/\!\!/(G_1 \times G_2)$ is covered by affine open subsets $\pi_{G_1 \times G_2}(X_f) = \mathrm{Spec}(\mathcal{O}(X_f)^{G_1 \times G_2})$, for $f \in H^0(X, L^{\otimes mn})^{G_1 \times G_2}$ with $m > 0$. The morphism ψ maps $\pi_{G_1 \times G_2}(X_f)$ to the affine open subset $\overline{\pi}_{G_2}((X/\!\!/G_1)_F)$ of

$(X/\!\!/G_1)/\!\!/_M G_2$, where as above F is a G_2-invariant section such that $\pi_{G_1}^* F = f|_{X^{\mathrm{ss}(G_1)}}$; this map and corresponds to the isomorphism of rings

$$\mathcal{O}(\overline{\pi}_{G_2}((X/\!\!/G_1)_F)) \xrightarrow{\overline{\pi}_{G_2}^*} \mathcal{O}((X/\!\!/G_1)_F)^{G_2} \xrightarrow{\pi_{G_1}^*} \mathcal{O}(X_f)^{G_1 \times G_2}.$$

Hence ψ restricts to an isomorphism $\pi_{G_1 \times G_2}(X_f) \to \overline{\pi}_{G_2}((X/\!\!/G_1)_F)$. Patching over all such $\pi_{G_1 \times G_2}(X_f)$ shows that ψ is an open immersion.

(Proof of 3.) Suppose now X is projective and L is ample. Then the GIT quotient $X/\!\!/G_1$ is canonically isomorphic to $\mathrm{Proj}(\Bbbk[X, L^{\otimes n}]^{G_1})$, with $\Bbbk[X, L^{\otimes n}]^{G_1}$ finitely generated and $M \to X/\!\!/G_1$ corresponding to the twisting sheaf $\mathcal{O}(1)$ on $\mathrm{Proj}(\Bbbk[X, L^{\otimes n}]^{G_1})$ [MFK94, Page 40]. The GIT quotient $\pi_{G_1} : X^{\mathrm{ss}(G_1)} \to X/\!\!/G_1$ is the morphism defined by the inclusion $\Bbbk[X, L^{\otimes n}]^{G_1} \hookrightarrow \Bbbk[X, L^{\otimes n}]$. Moreover, by Serre vanishing [Har77, Chapter 3, Proposition 5.3], for sufficiently large $m > 0$ the natural map $H^0(X, L^{\otimes mn})^{G_1} \to H^0(X/\!\!/G_1, M^{\otimes m})$ is surjective. Now suppose $x \in X^{\mathrm{ss}(G_1)}$ maps to $(X/\!\!/G_1)^{\mathrm{ss}(M)}$ under π_{G_1}. Then there is $F \in H^0(X/\!\!/G_1, M^{\otimes m})^{G_2}$ such that $F(\pi_{G_1}(x)) \neq 0$, with m sufficiently large so that $\pi_{G_1}^* F = f|_{X^{\mathrm{ss}(G_1)}}$ for some global invariant section $f \in H^0(X, L^{\otimes mn})^{G_1 \times G_2}$, so that $x \in X_f \subseteq X^{\mathrm{ss}(G_1 \times G_2)}$. Thus $X^{\mathrm{ss}(G_1 \times G_2)} = \pi_{G_1}^{-1}((X/\!\!/G_1))$. The induced map $\pi_{G_1} : X^{\mathrm{ss}(G_1 \times G_2)} \to (X/\!\!/G_1)^{\mathrm{ss}(M)}$ is therefore a categorical quotient for the G_1-action on $X^{\mathrm{ss}(G_1 \times G_2)}$, and so its composition with the categorical G_2-quotient $\overline{\pi}_{G_2} : (X/\!\!/G_1)^{\mathrm{ss}(M)} \to (X/\!\!/G_1)/\!\!/_M G_2$ is a categorical quotient for the full $G_1 \times G_2$-action on $X^{\mathrm{ss}(G_1 \times G_2)}$. It follows that the canonically induced map $\psi : X/\!\!/(G_1 \times G_2) \to (X/\!\!/G_1)/\!\!/_M G_2$ is an isomorphism. □

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