$\mathbb{Z}_2$-equivariant Heegaard Floer theory and transverse knots



Sungkyung Kang
St Catherine's College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy
Michaelmas 2018

To my parents and my sister
for their unending support and encouragement

Acknowledgements

I am mostly indebted to **András Juhász**, who has been my supervisor during my years in Oxford. I have benefited a lot from his generosity and assistance, not only mathematically, but also socially and financially. Without his continued patience and guidance during my Ph.D., this thesis would not exist.

I am really grateful to various other people who spent some of their time to have mathematical discussions with me. In particular, I would like to thank those who have contributed to this thesis, via conversations or comments: **Robert Lipshitz**, **Marco Golla**, **Olga Plamenevskaya**, **Alexander Ritter**, and **Marco Marengon**.

I am also very grateful to my sister **Monica Jinwoo Kang**. Her wonderful physical knowledge as a string theorist helped me to widen my views on mathematics, especially on low-dimensional topology.

The next person whom I wish to give thanks is **Hyunwoo Kwon**, who has been my good friend in Korea for several years. I am indebted for his help on drawing figures and solving various technical problems involving LaTeX and LyX.

I would also give thanks to my other friends in Korea, who gave me emotional support every time I spend a vacation in Korea, namely: **Jaesun Shin**, **Seungkyun Park**, **Jungseop Kim**, **Jiwon Kim**, **Dongmin Jung**, **Hyunjin Lim**, **Hongtaek Jung**, **Seungwook Jang**, **Jinyeob Lee**, **Dongyeop Kang**, **Donghwi Seo**, and **Jineon Baek**.

Lastly, I wish to thank those people who supported me the most in the course of my Ph.D., namely my family, to whom this thesis is dedicated. A very warm acknowledgement goes to my mom, **Young Soon Lee**, my dad, **Ki Heung Kang**, and again, my sister, **Monica Jinwoo Kang**.

Statement of Originality

I declare that the work in this thesis is, to the best of my knowledge, original and my own work, except those which are indicated, cited, or commonly known. I also declare that this thesis has not been submitted for a degree at another university.

Abstract

This thesis consists roughly of two parts. In the first part, we study various properties of $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology. Then, in the second part, we apply the results of the first part to study transverse knot invariants in the standard contact S^3.

When (L, p) is a based link in S^3, the branched double cover $\Sigma(L)$ is a rational homology sphere which carries a natural $\mathbb{Z}_2$-action. Hendricks, Lipshitz, and Sarkar studied this action on the Heegaard Floer setting to define $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), p)$, the $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology of $\Sigma(L)$, using bridge diagrams of L drawn on a sphere. They also proved that its isomorphism class as an $\mathbb{F}_2[\theta]$-module is an isotopy invariant of (L, p), which also implies that we can drop p from its notation when L is a knot.

In this thesis, we prove that, when $L = K$ is a knot, then $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ satisfies naturality and admits cobordism maps for based knot cobordisms. Then we move on to show that $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ can be calculated in terms of bridge diagrams of L, which are now drawn on any weakly admissible Heegaard diagrams representing S^3. Based on this observation, we point out that, for knots K, we can consider $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ as a weak Heegaard invariant, and prove that it is actually a strong Heegaard invariant. We also prove that the definition of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ in terms of bridge diagrams and its alternative definition as a strong Heegaard diagram are equivalent, by constructing an invertible natural transformation between the two TQFT functors induced by those two definitions. Using that fact, we construct $\mathbb{Z}_2$-equivariant knot Floer cohomology, $\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K)$, and show that it is a refinement of both $\widehat{HFK}(-S^3, K)$ and $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$.

We then construct the $\mathbb{Z}_2$-equivariant contact class $c_{\mathbb{Z}_2}(\xi_K)$ of a transverse knot K inside the standard contact 3-sphere (S^3, ξ_{std}), as a well-defined element of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, and prove that it is a transverse knot invariant which refines the contact class $c(\xi_K)$ of the contact branched double cover $(\Sigma(K), \xi_K)$ of (S^3, ξ_{std}) along K. We also construct the $\mathbb{Z}_2$-equivariant LOSS invariant $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K)$ of a transverse knot K in (S^3, ξ_{std}), as an element of $\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K)$ and prove that it is also a transverse knot invariant which refines both $c_{\mathbb{Z}_2}(\xi_K)$ and the LOSS invariant $\hat{\mathcal{T}}(K)$.

Lastly, we give an interesting topological application of $c_{\mathbb{Z}_2}(\xi_K)$, by showing that it can be used to prove the nonvanishing of $c(\xi_K)$ when K achieves equality in a Bennequin-type inequality.

Contents

1 Introduction **1**

1.1 Overview of results 5

1.1.1 Genus-zero construction of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ and the $\mathbb{Z}_2$-equivariant contact class 5

1.1.2 Extension to higher-genus diagrams 8

1.2 Questions 10

1.2.1 Naturality of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L))$ when L is a link 10

1.2.2 Effectivity of $c_{\mathbb{Z}_2}$ 10

1.2.3 Higher-genus naturality of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ 12

2 Contact 3-manifolds, Heegaard Floer theory, and transverse knots **13**

2.1 Contact structures on 3-manifolds 13

2.1.1 Contact forms and contact structures 13

2.1.2 Convex surfaces 16

2.1.3 Tight and overtwisted 17

2.1.4 Open books and closed contact 3-manifolds 19

2.2 Heegaard Floer theory 21

2.2.1 Heegaard diagrams of 3-manifolds 21

2.2.2 Moduli space of J-holomorphic disks 23

2.2.3 Heegaard Floer homology 25

2.2.4 Computing Heegaard Floer homology 27

2.3 Contact invariant in Heegaard Floer cohomology 29

2.4 Transverse knots in standard contact S^3 31

2.5 Contact branched double covers along transverse knots 35

3 $\mathbb{Z}_2$-Equivariant Heegaard Floer cohomology **39**

3.1 Construction of $\mathbb{Z}_2$-equivariant Floer cohomology 39

3.1.1 Homotopy coherent diagrams of almost complex structures . . 40

3.1.2 The freed Floer complex 42

3.1.3 Maps between equivariant Floer complexes 46

3.1.3.1 Diffeomorphism maps 46

3.1.3.2 Continuation maps 46

3.1.3.3 Triangle maps 48

3.2 Equivariant transversality and localization 51

3.3 $\mathbb{Z}_2$-Equivariant Heegaard Floer cohomology of based links 52

3.3.1 Isotopy . . . 54
3.3.2 Handleslide . . . 54
3.3.3 Stabilization . . . 54
3.3.4 Diffeomorphism . . . 56
3.3.5 Equivariant transversality and localization . . . 56
3.4 Weak admissibility . . . 56
3.5 Some Floer-theoretic lemmas . . . 64
3.6 Naturality . . . 67
3.6.1 Commutative diagrams of basic moves . . . 67
3.6.2 Translating diagrams of basic moves to diagrams of $\widehat{HF}_{\mathbb{Z}_2}$ . . . 71
3.6.2.1 Diagrams of type (c) (and similarly for (a),(f),(h)) . . . 72
3.6.2.2 Diagrams of type (e) . . . 73
3.6.2.3 Well-definedness of general stabilization map . . . 73
3.6.2.4 Diagrams of type (b) (and similarly for (d),(g)) . . . 73
3.6.2.5 Diagrams of type (i) . . . 74
3.6.3 Finishing the proof . . . 77
3.7 Functoriality . . . 82
3.7.1 Cobordism maps for basic pieces . . . 82
3.7.2 Isotopy invariance of cobordism maps . . . 86
3.7.3 Statement of functoriality . . . 89

4 Calculating $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ from extended bridge diagrams 91
4.1 Extended bridge diagrams . . . 92
4.2 Extended Heegaard moves suffice . . . 97
4.3 Weak admissibility . . . 99
4.4 Special stabilizations induce isomorphisms . . . 104
4.5 Dropping the 2-bridge restriction . . . 107
4.6 A note on stabilizations of type II . . . 112

5 $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ is a strong Heegaard invariant 118
5.1 Weak and strong Heegaard invariants . . . 118
5.1.1 Isotopy diagrams and graphs . . . 119
5.2 A Floer-theoretic lemma . . . 122
5.3 Construction of a weak Heegaard invariant . . . 123
5.4 Functoriality axiom and well-definedness of general stabilization maps 125
5.5 Continuity axiom . . . 133
5.6 Commutativity axiom . . . 134
5.7 Handleswap invariance . . . 136
5.8 Naturality of the equivalence . . . 138

6 $\mathbb{Z}_2$-Equivariant contact class 145
6.1 Construction of $c_{\mathbb{Z}_2}(\xi_K)$ . . . 145
6.2 Invariance . . . 149
6.2.1 Basic moves of half-arc bases . . . 149
6.3 Functoriality . . . 154

6.3.1 Weak symplectic isotopy and constructible symplectic cobordisms 154
6.3.2 Functoriality of $c_{\mathbb{Z}_2}$ under constructible symplectic cobordisms 163
6.4 Properties 168

7 $\mathbb{Z}_2$-Equivariant knot Floer cohomology and the LOSS invariant 174
7.1 $\mathbb{Z}_2$-Equivariant Heegaard Floer cohomology 174
7.2 Properties of $\widehat{HFK}_{\mathbb{Z}_2}$ (and $\widehat{HFL}_{\mathbb{Z}_2}$) 176
7.3 Refining the LOSS invariant 180
7.4 $\hat{\mathcal{T}}_{\mathbb{Z}_2}$ is a refinement of both $c_{\mathbb{Z}_2}$ and $\hat{\mathcal{T}}$ 186

8 Topological application 192
8.1 Analyzing the spectral sequence from $\widehat{HF}$ to $\widehat{HF}_{\mathbb{Z}_2}$ 192
8.1.1 Comparison with the Khovanov side 196

List of Figures

2.1 Impossible local figures of front projections of positively transverse curves 33
2.2 The positive stabilization S_+ and the negative stabilization S_- performed to a Legendrian knot 34
2.3 A transverse stabilization performed to a transverse knot 34

3.1 Possible local pictures of nice Heegaard diagrams. A choice of a perturbation changes the figures on the left to the figures of nice diagrams, shown on the right. 58
3.2 The periodic domain $\tilde{D}$ near an invariant intersection point. 59
3.3 The periodic domain $\tilde{D}$ near a non-invariant intersection point. . . . 59
3.4 The periodic domain D near a non-invariant intersection point. . . . 61
3.5 The periodic domain $\tau_* D$ near a non-invariant intersection point. . . 61
3.6 The periodic domain $\tilde{D}$ near a non-invariant intersection point. . . . 61
3.7 The periodic domain D near a non-invariant intersection point. . . . 63
3.8 The periodic domain D near a non-invariant intersection point. . . . 63
3.9 The periodic domain D near a non-invariant intersection point. . . . 64
3.10 Three A-arcs lying close to each other 69
3.11 Three A-arcs, after three handleslides 69
3.12 The diagram after applying a Dehn twist along (the boundary of a neighborhood of) A_i 77
3.13 The triple-diagram 77
3.14 The nonconstant triangles in (the branched double cover of) the given triple-diagram 78
3.15 The triple-diagram representing a small isotopy of B-arcs 78
3.16 Taking saddles along two arcs a_Σ and a'_Σ, which differ by a handleslide 85
3.17 Saddle along a small arc near L 88
3.18 The bridge diagram of $L = L_0 \coprod U_0$ on a sphere with a long neck . . 88

4.1 Attaching a small A-arc and a B-arc at the point w. 106
4.2 The diagram on the left is the triple-diagram for the case when a stabilization map of type II is composed first and then a handleslide map of type IV is composed, and the diagram on the right is for the case when the maps are composed in the opposite order. 116

5.1 A simple handleswap diagram. 122
5.2 Attaching a small A-arc and a B-arc at the point w. 124

5.3 A sequence of handleslides, involving A-arcs and α-curves, used to define a stabilization map of type II. The circles with a letter H are those where a handle is attached. 127
5.4 The unique $\mathbb{Z}_2$-invariant cocycle $\mathbf{x}_n$. 127
5.5 A Heegaard triple-diagram (with triple intersections). Note that $\mathbf{x}_{n+1} = \mathbf{x}_n \cup \mathbf{x}$. 127
5.6 The Heegaard diagram H_s; the boundary is collapsed to form a genus 2 surface. 138
5.7 The triple-diagram near the connected sum region. The circle in the middle is where the diagram H_s is attached via connected sum neck. . 138
5.8 The diagrams $R(\mathcal{E})$(above) and $HR(\mathcal{E})$(below) near the A-arcs and B-arcs. Here, we assume for simplicity that the rightmost A-arc and the leftmost B-arc are the ones connected to the basepoint z. 142

6.1 A disk with four marked points and a half-arc basis 146
6.2 The branched double cover. 146
6.3 A picture describing a half-arc slide. 149
6.4 The induced Heegaard triple-diagram. 153
6.5 The family $\{U_t\}_{t\in(0,1]}$ of transverse unknots which converge to p as $t \to 0$157
6.6 The family of curves L_t at $t = 0$, $t = \frac{1}{2}$, and $t = 1$. 161
6.7 The projection of $\tilde{L}_{1/2}$ along the z-axis. 161
6.8 The associated Heegaard triple-diagram. 165
6.9 The intermediate slice. 166
6.10 The associated equivariant Heegaard diagram, near the last strand. . 171
6.11 Relevant triangles in the associated equivariant Heegaard triple-diagram, each of which has zero Maslov index. 172

7.1 The diagram H_n. Here we have set $n = 3$ for simplicity. 179
7.2 The page Σ of the open book h, near the Legendrian knot L. 182
7.3 The arc a_1 is obtained from a by a sequence of arc-slides, and a' is obtained from a_1 again by a sequence of arc-slides. During the process, the arc-slides we perform do not pass through the basepoint p. 185

Chapter 1

Introduction

Topology of manifolds can be divided into two parts: low-dimensional topology and high-dimensional topology. While low-dimensional topology involves manifolds of dimension at most 4, high-dimensional topology involves manifolds of dimension at least 5. This division is due to the celebrated h-cobordism theorem[50] of Smale. In the proof of the h-cobordism theorem, Smale uses the Whitney trick as a key idea, which is a trick which takes an immersed disk in the ambient manifold and replaces the given disk with a smoothly embedded one. The most important aspect of that trick is that it works only if the ambient manifold has dimension at least 5. If one works in a 4-manifolds, then one can only replace the given immersed disk into a topologically embedded one, as shown by Freedman[17]. If one works in manifolds of dimension less than 4, then one would not need to consider applying Whitney trick. This difference distinguishes high dimensions from low dimensions.

Low-dimensional topology can also be divided into 3-dimensional topology and 4-dimensional topology. Again, the topology of 3-manifolds and 4-manifolds are a bit different. While one does not need to distinguish the topological category and the smooth category in dimension three, due to the theorem of Moise[34], the difference between the two categories in four dimensions exists and is very subtle; analyzing this difference is one of the most central issues in 4-dimensional topology. Also,

the classification problem of 3-manifolds is very interesting, given the geometrization conjecture, proven by Perelman[41], while the world of 4-manifolds is so vast and diverse, as any finitely presented group can arise as the fundamental group of a closed 4-manifold.

However, the topology of 3-manifolds and 4-manifolds are closely related. This is because 3-manifolds are boundaries of 4-manifolds, so that we can consider 4-manifolds (with boundary) as cobordisms between 3-manifolds. This relation becomes easily visible when one tries to use gauge-theoretic invariants to study low-dimensional topology. Many gauge-theoretic invariants form 3+1-dimensional TQFTs (topological quantum field theories). In other words, they associates algebraic objects to each 3-manifold, and morphisms between those objects to each cobordism. This "functoriality" in low-dimensional topology allows one to study 3-dimensional topology through studying properties of 4-manifolds. More roughly, one can even say that the theory of 3-manifolds and 4-manifolds are inseparable. This is why we call that theory as low-dimensional topology.

One possible way of studying low-dimensional topology is through knot theory. As its name suggests, knot theory is the theory of knots and links, which are embeddings of one or more copies of S^1 into a 3-manifold. These objects are important in low-dimensional topology because they play a central role in analyzing 3-manifolds and 4-manifolds. For example, given a knot in a 3-manifold, one can perform Dehn surgery along it by choosing a framing and a surgery coefficient. This construction is very important not only in 3-dimensional topology, but also in 4-dimensional topology, as it can be seen as a sequence of 2-handle attachments on the boundary of a 4-manifold; this is one of the basic aspects of Kirby calculus. Also, given a knot in a 3-manifold, one can also take the cyclic branched cover of the given 3-manifold, branched along the given knot; this is also a very important construction in 3-dimensional topology.

Of course, the ways of studying low-dimensional topology is far from being unique. A possible alternate approach to low-dimensional topology is using contact geome-

try and symplectic geometry. Again, as their name suggests, contact geometry is geometry of contact structures, and symplectic geometry is geometry of symplectic structures. Symplectic geometry is interesting, partially because it is directly related to many fields of mathematics, including topology, algebraic geometry, and complex geometry, and also to string theory in physics. Contact geometry can be seen as a boundary theory of symplectic geometry; while symplectic geometry can be used to study 4-manifolds, contact geometry can be used to study 3-manifolds. Indeed, there are several problems in low-dimensional topology which can be solved by applying contact geometric techinques. Cerf's theorem[9] is a nice example of such problems; Eliashberg[12] produced an extremely short proof of it by applying the classification of contact structures in S^3 and his "filling by holomorphic disks" technique. Also, by studying weak symplectic fillings of contact 3-manifolds and a certain variant of Floer homology for 3-manifolds, Kronheimer and Mrowka[28] were able to prove Property P conjecture for knots.

As mentioned above, there are several gauge-theoretic invariants of 3-manifolds and 4-manifolds; they form a very powerful tool in low-dimensional topology. Many of them also have their counterpart in knot theory; for example, Heegaard Floer theory of Ozsvath-Szabo[37], which is essentially a 3+1-dimensional TQFT, has a knot theoretic counterpart, which is knot Floer theory, developed independently by Ozsvath-Szabo[36] and Rasmussen[46]. A nice aspect of those invariants is that they are usually compatible with 3-dimensional contact topology; for each Floer-theoretic invariant of 3-manifolds, i.e. instanton/monopole/Heegaard Floer homology[16, 29, 37], there is an invariant, called the contact class[2, 1, 38], which is a well-defined invariant inside those Floer homology groups. For instanton and monopole Floer homology, the contact class is functorial under symplectic cobordisms, which allows us to study symplectic topology using Floer theory. On the other hand, the symplectic functoriality of contact class in Heegaard Floer homology is not yet proven, partially due to its combinatorial nature, but that nature makes the Heegaard Floer contact

class the only computable one.

While knots play a central role in low-dimensional topology, Legendrian knots and transverse knots also play a central role in 3-dimensional contact topology. As one can perform Dehn surgery along a knot in a 3-manifold, one can also perform contact surgery along a Legendrian knot in a contact 3-manifold. It is conjectured[14] that classification of Legendrian and transverse knots can be used to distinguish certain contact 3-manifolds, which makes the classification problem particularly interesting in low-dimensional topology.

Some gauge-theoretic knot homology can be used to classify Legendrian knots and transverse knots. For example, the LOSS invariant[32], which is an element in knot Floer homology, can be used to distinguish certain Legendrian knots which are not detected by classical methods. Also, the knot contact homology[10] also admits a Legendrian knot invariant, which have been extremely successful in studying Legendrian knots. On the other hand, some gauge-theoretic Legendrian/transverse knot invariants, including the contact class of branched double cover[44] and the ψ-invariant[43], are conjectured to be not stronger than the classical invariants, but none of those conjectures are proven up to now.

This leads us to study $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology, which is a relatively new Floer-theoretic knot (co)homology, defined by Lipshitz, Hendricks and Sarkar[20]. For each knot K in S^3, the $\mathbb{Z}_2$-equivariant Heegaard Floer theory associates to it an isomorphism class $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ of $\mathbb{F}_2[\theta]$-modules, where θ is the formal variable induced by the $\mathbb{Z}_2$-equivariance. Given this new Floer-theoretic knot invariant, there is a standard list of questions to ask about it.

Naturality/Functoriality Is it a natural invariant, i.e. has a well-defined notion of elements? Does it admit cobordism maps?

Contact invariants Does it admit invariants of Legendrian knots or transverse knots, as other Floer-theoretic knot homologies do?

Relations How is it related to the other Floer-theoretic knot homologies?

In this thesis, we will give full, or partial, answers to the above three questions. Also, we will also extend the definition of $\mathbb{Z}_2$-equivariant Heegaard Floer theory using the notion of extended bridge diagrams, and answer some questions arising naturally from the extension. Furthermore, we will observe that the transverse knot invariant in $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology has an interesting topological application on tightness of contact structures on a branched double cover of S^3 along a knot.

1.1 Overview of results

1.1.1 Genus-zero construction of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ and the $\mathbb{Z}_2$-equivariant contact class

Suppose that a based link (L, p) in S^3 is given. Hendricks, Lipshitz, and Sarkar were able to define $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ from a bridge diagram of L as follows.

- Choose a bridge diagram of L on a sphere, so that p is one of the endpoints of a bridge.
- Erase the pair of bridges connected to p.
- Take the branched double cover of the whole diagram, branched along the set of endpoints of bridges, excluding p.
- Now the diagram we get is a pointed Heegaard diagram

In this thesis, we first prove the functoriality of $\widehat{HF}_{\mathbb{Z}_2}$ in chapter III. Using the techniques introduced by Juhasz and Thurston[23], we prove that the $\mathbb{F}_2[\theta]$-module $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), p)$ is natural in the sense that it admits an action of $MCG(S^3, K, p)$, when (K, p) is a based knot in S^3. To be precise, we have the following naturality statement.

Theorem 1.1.1 (Theorem 6.9 of [25]). *Let $Knot_*$ be the category whose objects are based oriented knots in S^3 and morphisms are orientation-preserving self-diffeomorphisms of S^3 which preserve the knot (as a set) and the basepoint. Then we have a functor*

$$\widehat{HF}_{\mathbb{Z}_2} : Knot_* \to Vect_{\mathbb{F}_2},$$

agreeing up to isomorphism with the invariants $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$.

Then, using the naturality of $\widehat{HF}_{\mathbb{Z}_2}$ for based knots, we prove that a based cobordism $S = (S_0, s)$ between two based knots (K_1, p_1) and (K_2, p_2) in S^3 functorially induces a map

$$\hat{f}_S : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K_2), p_2) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K_1), p_1),$$

which is an invariant of the isotopy class of S rel ∂S. Here, a based cobordism is a cobordism togther with a curve from p_1 to p_2.

After estabilishing the functoriality of $\widehat{HF}_{\mathbb{Z}_2}$, we construct an element $c_{\mathbb{Z}_2}(\xi_K) \in \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), p)$ in chapter VI, where K is a tranverse knot in the standard contact 3-sphere (S^3, ξ_{std}), by considering the contact branched double cover $(\Sigma(K), \xi_K)$ of (S^3, ξ_{std}), branched along K, as defined by Plamenevskaya[44]. We prove that this element is indeed a well-defined element inside the equivariant Floer cohomology, and becomes an invariant of the transverse (based) isotopy class of (L, p). We discuss the functoriality of the element $c_{\mathbb{Z}_2}(\xi_K)$ inside $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), p)$, when (K, p) is a based transverse knot in (S^3, p). Recall that any smooth cobordism between knots(or links) is a composition of isotopies, births, saddles, and deaths. We define their analogues (except for deaths) in the symplectic setting, which turn out to be well-defined up to a weaker version of symplectic isotopy, and restrict our attention to symplectically constructible symplectic cobordisms, which can roughly be defined to be based symplectic cobordisms which can be weakly symplectically isotoped to a composition of symplectic versions of isotopies, births, and saddles. Then we prove that $c_{\mathbb{Z}_2}(\xi_K)$ is preserved under the maps associated to symplectically constructible based

cobordisms.

To summarize, we prove the following theorem.

Theorem 1.1.2. *For a based link (K,p) in S^3, the $\mathbb{F}_2[\theta]$-module $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K),p)$ is natural. Given a based knot cobordism (S,s) in $S^3\times I$ with $\partial S = K_1\times\{0\}\cup K_2\times\{1\}$ and $\partial s = \{(p_1,0),(p_2,1)\}$, we have an $\mathbb{F}_2[\theta]$-module homomorphism*

$$\hat{f}_{(S,s)} : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K_2),p_2)\to\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K_1),p_1),$$

which is an invariant of the isotopy class of (S,s) and is functorial under composition. When (K,p) is a based transverse knot inside (S^3,ξ_{std}), we have an element

$$c_{\mathbb{Z}_2}(\xi_K)\in\widehat{HF}(\Sigma(K),p)$$

which is an invariant of the transverse isotopy class of (K,p). Furthermore, if (S,s) is a symplectically constructible symplectic cobordism in the $(S^3\times I, d(e^t\alpha_{std}))$ between based transverse knots, say, $\partial S = K_1\times\{0\}\cup K_2\times\{1\}$ and $\partial s = \{(p_1,0),(p_2,1)\}$, then we have

$$\hat{f}_{(S,s)}(c_{\mathbb{Z}_2}(\xi_{K_2})) = c_{\mathbb{Z}_2}(\xi_{K_1}).$$

Note that moving the basepoint by an isotopy preserves the element $c_{\mathbb{Z}_2}(\xi_K)$, although it lies in $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, which does depend on the choice of a basepoint on K. So we can drop the choice of a basepoint on K and say that $c_{\mathbb{Z}_2}(\xi_K)$ is an invariant of the transverse isotopy class of a given transverse knot K in (S^3,ξ_{std}).

Surprisingly, as we will see in chapter VIII, basic properties of $c_{\mathbb{Z}_2}(\xi_K)$ allows us to give a condition on the self-linking number of K which ensures the vanishing or the nonvanishing of the (non-equivariant) contact class $c(\xi_K)$. This can be seen as a Heegaard Floer analogue of a similar condition, for the Plamenveskaya ψ-invariant, proven in [45]. The two results are shown to be the same for transverse representatives of quasi-alternating knots.

Theorem 1.1.3. *Let K be a knot in S^3 and T be a transverse representative of K. Then $c(\xi_K) \neq 0$ if $d_3(\xi_K) = \frac{q_\tau(K)-1}{2}$ and $c(\xi_K) = 0$ if $d_3(\xi_K) > \frac{q_\tau(K)-1}{2} + v_\tau(K)$.*

Remark 1.1.4. The quantity $d_3(\xi_K)$ denotes the 3-dimensional invariant of the contact structure ξ_K. In general, given a closed contact 3-manifold (M, ξ) such that $c_1(\xi)$ is torsion, its 3-dimensional invariant is defined as

$$d_3(\xi) = \frac{1}{4}\left(c_1^2(X,J) - 3\sigma(X) - 2\chi(X)\right),$$

where (X, J) is a almost complex compact 4-manifold such that $M = \partial X$ and $\xi = TM \cap J(TM)$. Such a manifold (X, J) always exists, and the resulting quantity $d_3(\xi)$ depends only on the homotopy class of ξ, due to Lemma 4.16 of [19].

1.1.2 Extension to higher-genus diagrams

After studying the naturality and functoriality of $\widehat{HF}_{\mathbb{Z}_2}$, using bridge diagrams of based knots on a sphere, it is natural to ask whether $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ can be computed from bridge diagrams of a given knot K, which is now drawn on a closed surface of arbitrary genera, instead of a sphere. In chapter IV, we will see that this is always possible, by proving the following theorem. The definition of extended bridge diagrams can be found in chapter IV.

Theorem 1.1.5. *Let $\mathcal{E}$ be a weakly admissible extended bridge diagram representing a knot K in S^3. Then $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}) \simeq \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$.*

Now, given such a fact, we can use it to compute $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ from a weakly admissible knot Heegaard diagram of K, since a knot Heegaard diagram is always an extended bridge diagram. This construction implies that $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ is a weak Heegaard invariant of K, as defined in [23]. In chapter V, we will see that the $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology, as a weak Heegaard invariant, satisfies certain commutativity axioms, thereby proving that it is actually a strong Heegaard

invariant. Morover, we will also see that $\widehat{HF}_{\mathbb{Z}_2}$ as a natural invariant calculated from bridge diagrams on a sphere is naturally isomorphic to $\widehat{HF}_{\mathbb{Z}_2}$ as a strong Heegaard invariant; to be precise, we will prove the following theorem.

Theorem 1.1.6. *Consider the category $Knot_*$ whose objects are based knots in S^3 and morphisms are self-diffeomorphisms of S^3, and let*

$$\widehat{HF}_{\mathbb{Z}_2}^{bridge} \ : \ Knot_* \to Mod_{\mathbb{F}_2[\theta]}$$

be the functor defined in Theorem 1.1.1. Also, let

$$\widehat{HF}_{\mathbb{Z}_2}^{knot} \ : \ Knot_* \to Mod_{\mathbb{F}_2[\theta]}$$

be the functor defined by considering $\widehat{HF}_{\mathbb{Z}_2}$ as a strong Heegaard invariant. Then there exists an invertible natural transformation between $\widehat{HF}_{\mathbb{Z}_2}^{bridge}$ and $\widehat{HF}_{\mathbb{Z}_2}^{knot}$.

Finally, In chapter VII, we will construct a new invariant $\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K)$ associated to a knot K in S^3, whose isomorphism class is also an invariant of the isotopy class of K, and prove that there exists a following isomorphism for any knot K in S^3:

$$\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}] \simeq \widehat{HFK}(-S^3, K) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

Using this, we will also construct an element $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K) \in \widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K)$ associated to a transverse knot K in the standard contact sphere (S^3, ξ_{std}), which depends only on the transverse isotopy class of K, and see that it is a refinement of both the LOSS invariant, defined in [32], and the $\mathbb{Z}_2$-equivariant contact class, which will be defined in chapter VI.

1.2 Questions

1.2.1 Naturality of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L))$ when L is a link

As shown in chapter III, $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L))$ is natural when $L = K$ is a knot. So it is natural to ask whether the naturality holds in the general case, when L is a link with at least two components. The arguments used to prove the functoriality $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ for knots K actually tells us that, if one can show its naturality for general links, then its functoriality for link cobordisms would naturally follow.

Thus, to sum up, we can ask the following question.

Question. *Does the functor* $\widehat{HF}_{\mathbb{Z}_2} : Knots_* \to \mathbf{Mod}_{\mathbb{F}_2[\theta]}$ *in Theorem 1.1.1 allow a unique extension to the category* $Link_*$*, whose objects are based links in* S^3 *and morphisms are orientation-preserving self-diffeomorphisms of* S^3*?*

1.2.2 Effectivity of $c_{\mathbb{Z}_2}$

It is natural to ask whether the transverse knot invariant $c_{\mathbb{Z}_2}(\xi_K)$ is effective, in the sense that it can distinguish between two knots with the same self-linking number and topological knot type. There are several transverse knot invariants defined previously. For example, the Plamenevskaya invariant of transverse knots is an element of Khovanov homology, as shown in [43]; Wu[51] defined an $\mathfrak{sl}_n$ invariant of transverse knots, as an element of $\mathfrak{sl}_n$ homology. Lisca, Ozsvath, Stipsicz, and Szabo[32] defined the LOSS invariant of Legendrian and transverse knots, as an element of knot Floer homology, which was proved to be the same (up to automorphisms of $\widehat{HFK}$) as the HFK grid invariant by Baldwin and Vela-Vick[4]. Also, Ekholm, Etnyre, Ng, and Sullivan[10] defined transverse homology, which is a filtered version of knot contact homology. Baldwin and Sivek[1] defined a monopole version of the LOSS invariant, as an element of the sutured monopole homology, which is functorial under Lagrangian concordances and maps to the LOSS invariant via isomorphism between KHM and

HFK[3]. Among them, the LOSS invariant, its monopole version, and the transverse homology are proven to be effective. It is still not known whether the others are effective invariants of transverse knots.

Unfortunately, we do not know whether the invariant $c_{\mathbb{Z}_2}$ is effective. This makes us to ask the following question.

Question. *Is there a knot K in S^3 such that there exist two transverse knots K_1, K_2 in (S^3, ξ_{std}), topologically isotopic to K, with $\mathbf{sl}(K_1) = \mathbf{sl}(K_2)$, but*

$$c_{\mathbb{Z}_2}(K_1) \neq c_{\mathbb{Z}_2}(K_2)$$

as elements in $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$?

If the answer to the above question is yes, then we get a new effective transverse invariant. However, if the answer is no, we also have an interesting consequence. Given a knot K in S^3, let K_1, K_2 be two transverse knots, topologically isotopic to K. Then, we apply the following well-known theorem.

Theorem. *Any two topologically isotopic transverse knots are related by a sequence of (de)stabilizations.*

Using the above theorem, suppose that the nth and mth positive stabilizations of K_1 and K_2 are transversely isotopic, and assume that $n \geq m$. Then the $n-m$th positive stabilization K_1^{n-m} of K_1 satisfies $\mathbf{sl}(K_1^{n-m}) = \mathbf{sl}(K_2)$. Hence, by the assumed non-effectiveness of $c_{\mathbb{Z}_2}$ and its behavior under positive stabilizations, we get

$$c_{\mathbb{Z}_2}(\xi_{K_2}) = c_{\mathbb{Z}_2}(\xi_{K_1^{n-m}}) = c_{\mathbb{Z}_2}(\xi_{K_1}) \cdot \theta^{n-m}.$$

Hence, if T_K is the transverse representative of K with the minimal self-linking number, then for any other transverse representative T of K, we must have

$$c_{\mathbb{Z}_2}(\xi_T) = c_{\mathbb{Z}_2}(\xi_{T_K}) \cdot \theta^{n_T}$$

for some $n_T \geq 0$. Therefore we deduce that the subset

$$\{c_{\mathbb{Z}_2}(\xi_T) \,|\, T \text{ is a transverse representative of } K\} \subset \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$$

is a single θ-tower; its minimal-order element $c_{\mathbb{Z}_2}(\xi_{T_K}) \in \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ becomes a knot invariant.

1.2.3 Higher-genus naturality of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$

In this thesis, we prove that $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, where K is a (based) knot in S^3, is a natural invariant, when one consider diagrams of the following types.

- Genus-zero bridge diagrams of K, as shown in chapter III;
- Extended bridge diagrams of K with bridge number one, as shown in chapter V.

These two cases can be seen as the two extreme cases of extended bridge diagrams of K. If one takes the diagrams with the minimal genus, one gets genus-zero bridge diagrams; on the other hand, if one takes the diagrams with the minimal bridge number, then one gets extended digrams of bridge number one. This leads us to ask the following question.

Question. *Consider the graph of extended bridge diagrams $\mathcal{EB}_K$ representing a given based knot (K, p) in S^3, and the induced monodromy representation*

$$\rho_{EB} \,:\, \pi_1(\mathcal{EB}_K) \to Aut_{\mathbb{F}_2[\theta]}\left(\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))\right).$$

Does this representation factorize through the mapping class group $Mod(S^3, K, p)$?

Chapter 2

Contact 3-manifolds, Heegaard Floer theory, and transverse knots

2.1 Contact structures on 3-manifolds

2.1.1 Contact forms and contact structures

Let M be a 3-manifold. Given a nonvanishing 1-form α on M, we have a codimension 1 sub-bundle of TM, given by $\xi = \ker(\alpha)$. The Frobenius integrability condition $[\xi, \xi] \subset \xi$ is equivalent to the condition

$$\alpha \wedge d\alpha = 0.$$

We say that ξ is a contact structure if it can be locally represented as $\xi = \ker(\alpha)$ and is completely nonintegrable, i.e. $\alpha \wedge d\alpha$ is nowhere vanishing. The pair (M, ξ) is called a contact manifold, and a diffeomorphism $f : (M, \xi_M) \to (N, \xi_N)$ between contact manifolds is said to be a contactomorphism if $f_* \xi_M = \xi_N$. Any contact manifolds are orientible, and every closed orientible 3-manifold admits a contact structure, by the theorem of Martinet; see [33].

Contact manifolds have two important types of curves: Legendrian and transverse.

Given a contact manifold (M, ξ), a curve $\gamma \subset M$ is Legendrian if its tangent vectors are contained in ξ, and transverse if its tangent vectors are transverse to ξ. If γ is a homologically trivial Legendrian knot in M, then we have the contact framing $\gamma^*\xi$ coming from the contact structure ξ and the surface framing $\gamma^*T\Sigma$ coming from a Seifert surface Σ of γ. In this setting, the classical (numerical) invariants of γ are defined as follows.

- The twisting number of the contact framing relative to the surface framing is called the Thurston-Bennequin invariant, and denoted $\mathbf{tb}(\gamma)$.
- Since $\xi|_\Sigma$ is trivial as Σ is a surface with boundary, choose a trivialization of ξ on Σ and restrict it to γ. Then the tangent vectors of γ induces a closed curve in $\mathbb{R}^2 - \{0\}$, thus has a winding number. Its winding number is called the rotation number, and denoted $\mathbf{rot}(\gamma)$.

On the other hand, if γ is a homologically trivial transverse knot in M, then we have only one classical (numerical) invariant, defined as follows.

- Choose a Seifert surface Σ of γ. Since $\xi|_\Sigma$ is trivial, we can choose its nonzero section v. Let γ' be a slight perturbation of γ along the direction of v. Then the linking number between γ and γ' is called the self-linking number, and denoted $\mathbf{sl}(\gamma)$.

Example. On $\mathbb{R}^3$ with cartesian coordinates (x, y, z), we have a 1-form

$$\alpha_{std} = dz + xdy.$$

Then $\alpha_{std} \wedge d\alpha_{std} = dx \wedge dy \wedge dz$ is a volume form, i.e. α_{std} is a contact form. The contact structure $\xi_{std} = \ker(\alpha_{std})$ is called the standard contact structure on $\mathbb{R}^3$.

Example. On $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z), we have a 1-form

$$\alpha_{OT} = \cos(r)dz + r\sin(r)d\theta.$$

Since $r^2 d\theta$ is smooth, α_{OT} is also smooth. Since we have

$$\alpha_{OT} \wedge d\alpha_{OT} = \left(1 + \frac{\sin(2r)}{2r}\right) \cdot (r dr \wedge d\theta \wedge dz),$$

the form α_{OT} is a contact form. The contact structure $\xi_{OT} = \ker(\alpha_{OT})$ is called the standard overtwisted contact structure on $\mathbb{R}^3$.

A contact structure ξ is said to be co-oriented if TM/ξ is oriented. This is equivalent to saying that ξ has a global expression

$$\xi = \ker(\alpha)$$

for a globally defined 1-form α, since a global section of $(TM/\xi)^*$ gives a global section of T^*M. Such a form is said to be a contact form, and it is defined up to conformal equivalence. We will always assume that contact structures are co-oriented.

Two contact structures ξ_0 and ξ_1 on M is said to be isotopic if there is an isotopy $(\psi_t)_{t\in[0,1]}$ of M such that $\psi_0 = \mathrm{id}_M$ and $(\psi_1)_*\xi_0 = \xi_1$. As in symplectic geometry, we have a Moser-type theorem for isotopy of contact structures, as follows: if ξ_t is a smooth family of contact structures and M is closed, then ξ_0 and ξ_1 are isotopic. Hence the notion of isotopy of contact structures and 1-parameter family of contact structures are equivalent. Here, ξ_t is said to be a smooth family of contact structures if there exist contact forms α_t representing ξ_t, such that $\{\alpha_t\}$ is a smooth path of 1-forms on M.

Contact structures on 3-manifolds have a natrual interpretation as boundaries of symplectic structures on 4-manifolds. Given a contact 3-manifold (M, ξ) with $\xi = \ker(\alpha)$ consider the 4-manifold $W = M \times I$. The 2-form $\omega = d(e^t\alpha)$ satisfies

$$\omega \wedge \omega = e^{2t}(dt \wedge \alpha + d\alpha) \wedge (dt \wedge \alpha + d\alpha) = 2e^{2t} dt \wedge \alpha \wedge d\alpha.$$

Since α is a contact form, ω is a symplectic form, and we have a Liouville vector field

$X = \frac{\partial}{\partial t}$ satisfying $\omega \lrcorner X = \alpha$. We say that $M = M \times \{0\} \subset W$ is a convex boundary of W.

2.1.2 Convex surfaces

Given a contact manifold (M, ξ), a vector field X on M is called a contact vector field if it satisfies $\mathcal{L}_X \xi = 0$, i.e. the flow of X preserves ξ. Given a contact form α representing ξ, we have a canonical choice of a contact vector fields, called the Reeb vector field R_α, which is the unique vector field satisfying the following conditions:

- $(d\alpha) \lrcorner R_\alpha = 0$,
- $\alpha \lrcorner R_\alpha = 1$.

Since we have the following identity by Cartan's magic formula:

$$\mathcal{L}_{R_\alpha} \alpha = d(\alpha \lrcorner R_\alpha) + (d\alpha) \lrcorner R_\alpha = 0,$$

the Reeb vector field R_α preseves $\xi = \ker(\alpha)$, thus is a contact vector field. Note that the contact vector field condition is equivalent to the existence of a smooth function $\mu : M \to \mathbb{R}$ satisfying $\mathcal{L}_X \alpha = \mu \alpha$.

Now let $S \subset M$ be an embedded surface. We say that S is a convex surface if there exists a contact vector field X on M, defined near S, which is transverse to TS over S. Given a convex surface S and a transverse vector field X defined near S, we can easily extend X to a contact vector field on M, supported near S. Then the flow of X defines an identification $N(S) \simeq S \times \mathbb{R}$, where the contact structure ξ is invariant in the $\mathbb{R}$-direction. The converse is also true: given an $\mathbb{R}$-invariant contact structure on $S \times \mathbb{R}$, the vector field $\frac{\partial}{\partial z}$, where z is a parameter for $\mathbb{R}$, obviously becomes a contact vector field, transverse to $S \times \{0\}$. Any $\mathbb{R}$-invariant (i.e. vertically invariant) contact structure on $S \times \mathbb{R}$ can be written in the form

$$\xi = \ker(\beta + u dz),$$

where β is a 1-form on S and u is a smooth function on S. The contact condition for such a structure can be written as

$$ud\beta + \beta \wedge du > 0.$$

Given a convex surface S and a transverse contact vector field X, we define the dividing set Γ_S of S as the set of point $p \in S$ where X is tangent to ξ. If we write $\xi = \ker(\beta + udz)$, the dividing set can be written as the zero locus of the function u. The contact condition says that the dividing set Γ_S is transverse to the characteristic foliation S_ξ, which is the 1-dimensional foliation represented by the vector field Y satisfying

$$\Omega \lrcorner Y = \beta|_S,$$

for some area form Ω on S. The contact condition also implies that Γ_S is nonempty.

The definition of Γ_S depends on the choice of a transverse contact vector field X. However, for any two choices X, Y of transverse contact vector fields, the resulting dividing sets differ by an isotopy. Hence the isotopy class of Γ_S depends only on the contact structure ξ. It can also be proven that Γ_S is always a union of pairwise disjoint simple closed curves. Since a vanishing locus of a smooth function need not be a submanifold, this is not an obvious result.

Convex surfaces are "everywhere": given any smoothly embedded surface S_0 in a contact manifold (M, ξ), there exists an a convex surface S which is arbitrarily C^∞-close to S_0. For this reason, convex surface play an important role in classifications of tight contact structures, which will be explained later.

2.1.3 Tight and overtwisted

Recall that the standard overtwisted contact structure on $\mathbb{R}^3$ is given by

$$\xi_{OT} = \ker(\cos(r)dz + r\sin(r)d\theta),$$

in the cylindrical coordinate (r, θ, z). Let Δ be the disk $\{z = 0, r \leq \pi\} \subset \mathbb{R}^3$. Its boundary $\partial\Delta$ is a Legendrian curve, and the characteristic foliation on it consists of all radial lines, with singular points at the origin and at all boundary points. After pushing the interior of Δ slightly upwards, the singular points at the boundary disappear. We call Δ, in the perturbed or unperturbed form, the standard overtwisted disk.

Using the above example, we can define an embedded disk Δ in a contact manifold (M, ξ) as an overtwisted disk if its boundary $\partial\Delta$ is Legendrian, $\mathbf{tb}(\partial\Delta) = 0$, and the characteristic foliation Δ_ξ contains a unique singular point in the interior of Δ. We say that ξ is overtwisted if (M, ξ) contains an overtwisted disk, and is tight if it is not overtwisted.

Given these definitions, we are led to ask whether we can classify the overtwisted and tight structures on a 3-manifold, up to isotopy. It turns out that, while the classification of tight contact structures is very hard and interesting, the classification of overtwisted contact structures is easy. Eliashberg[11] gave a complete classification of overtwisted contact structures on any closed 3-manfolds, as we will discuss below.

Recall that a contact structure is a co-oriented tangent 2-plain field, sastisfying the complete nonintegrability condition. If two contact structures are isotopic, then they are homotopic as co-oriented tangent 2-plane fields. The theorem of Eliashberg says that, for any embedded oriented 2-disk $\Delta \subset M$ with a distinguished center 0_Δ, the inclusion map

$$i_\Delta : \Xi^{OT}(M, \Delta) \to \mathrm{Dist}(M, \Delta)$$

is a weak homotopy equivalence, where $\Xi^{OT}(M, \Delta)$ is the space of contact structures having Δ as an overtwisted disk (whose characteristic foliation is singular at 0_Δ) and $\mathrm{Dist}(M, \Delta)$ is the space of co-oriented tangent 2-plane fields that are tangent to Δ at 0_Δ. Since any two embeddings of a 2-disk into M are isotopic, we deduce that every co-oriented tangent 2-plane field is homotopic to an overtwisted contact structure, and two overtwisted contact structures on M are isotopic if and only if

they are homotopic as co-oriented tangent 2-plane fields. This result means that overtwisted contact structures on M are completely classified by homotopy classes of maps $M \to S^2$, which makes overtwisted structures very uninteresting.

2.1.4 Open books and closed contact 3-manifolds

Although the definition of contact structures may seem to be purely geometric and by no means topological, it is possible to study it in a purely combinatorial way. Given a connected oriented surface Σ with boundary $\partial\Sigma$ and an orientation-preserving diffeomorphism $f : \Sigma \to \Sigma$ satisfying $f|_{\partial\Sigma} = \mathrm{id}_{\partial\Sigma}$, we can construct the mapping torus T_f of f as follows:

$$T_f = \Sigma \times I / \sim,$$

where $(x, 1) \sim (f(x), 0)$ for $x \in \Sigma$ and $(p, 0) \sim (p, t)$ for $t \in I$ and $p \in \partial\Sigma$. The pair (Σ, f) is an called an abstract open book, and f is called the monodromy. The boundary $\partial\Sigma$ becomes a knot inside T_f, and is called the binding of the open book. The pair $(K = \partial\Sigma, p)$, where $p : T_f - K \to S^1$ is the naturally defined fibration, is called an open book of the closed oriented 3-manifold T_f; the fibers of p are called the pages of the open book.

The mapping torus T_f is a closed oriented 3-manifold, and any closed oriented 3-manifold can be constructed in this way. Now, given a contact structure ξ on T_f, we say that ξ is adapted to an open book (K, p) if there is a contact form α representing ξ satisfying the following conditions:

- $d\alpha$ is an area form on each page,
- K is a transverse knot in (M, ξ).

Given a 3-manifold M and an open book (K, p) representing it, let F be a page and $f : F \to F$ be a monodromy. Choose a 1-form β on F so that $d\beta$ is an area form and

β is positively transverse to $K = \partial F$, and consider the family of 1-form:

$$\beta_t = (1-t)\beta + tf^*\beta,\ 0 \leq t \leq 1.$$

For any $\epsilon > 0$, the 1-form $\alpha = dt + \epsilon\beta_t$ on $F \times I$ defines a 1-form on M, and satisfies

$$\begin{aligned} \alpha \wedge d\alpha &= (dt + \epsilon(1-t)\beta + \epsilon t f^*\beta) \wedge (\epsilon((1-t)d\beta + tf^*d\beta) + \epsilon dt \wedge (-\beta + f^*\beta)) \\ &= \epsilon dt \wedge ((1-t)d\beta + tf^*d\beta) + \epsilon^2(\cdots). \end{aligned}$$

Hence, if ϵ is sufficiently small, α becomes a contact form on M. By the definition of α, the contact structure $\xi = \ker(\alpha)$ is adapted to the given open book (K, p). Thus we proved the theorem of Thurston-Winkelnkemper: every open book decomposition of M admits a contact structure adapted to it. One can also prove that any two contact structures adapted to the same open book are isotopic.

Now we are led to ask whether any contact structure on M is adapted to some open book decomposition of M. This question is answered by the theorem of Giroux, which states that every contact structure is adapted to some open book, and any two such open books are related by isotopies and positive stabilizations (see [18]). Thus we have the following bijective correspondence:

$$\begin{array}{c} \{\text{contact structures up to isotopy}\} \\ \updownarrow \\ \{\text{open books up to isotopy and positive stabilization}\} \end{array}$$

Here, a positive stabilization of an abstract open book (Σ, f) is an open book (Σ', f') where:

- Σ' is obtained by attaching a 1-handle to Σ,
- $f' = f \circ T_c$ where c is a curve in Σ' which intersects once with the co-core of the attached 1-handle, and T_c is a Dehn twist along c.

Hence, using the theorem of Giroux, we are able to describe isotopy class of contact structures in terms of open books.

2.2 Heegaard Floer theory

2.2.1 Heegaard diagrams of 3-manifolds

Suppose that a closed, connected and oriented 3-manifold Y is given. A Heegaard splitting of Y is a decomposition $Y = U_0 \cup U_1$, where U_0 and U_1 are handlebodies with the common boundary $\Sigma = U_0 \cap U_1$. It can be easily proven that a Heegaard splitting always exists; one can see that directly by choosing a self-indexing Morse function $f : Y \to [0, 3]$, which has a unique maximum and minimum, and considering $U_0 = f^{-1}[0, 3/2]$, $U_1 = f^{-1}[3/2, 3]$, and $\Sigma = f^{-1}(3/2)$. If Σ is a genus g surface, we say that (U_0, U_1, Σ) is a Heegaard splitting of Y of genus g.

Given a Heegaard splitting (U_0, U_1, Σ), let $\alpha_1, \cdots, \alpha_g$ be the maximal system of disjoint simple closed curves on Σ, which are linearly independent in $H_1(\Sigma; \mathbb{Z})$, such that α_i bounds a disk in U_0 for each i. The diffeomorphism $\partial U_0 \simeq \partial U_1$ can be described as a self-diffeomorphism $\phi : \Sigma \to \Sigma$. Then we get another g-tuple $\{\beta_1, \cdots, \beta_g\}$ of simple closed curves on Σ, where $\beta_i = \phi(\alpha_i)$. Since an orientation reversing diffeomorphism sending each α_i to β_i is unique up to isotopy, we deduce that the triple $(\Sigma, \boldsymbol{\alpha} = \{\alpha_i\}, \boldsymbol{\beta} = \{\beta_i\})$ completely determines the original manifold Y. The triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is called a Heegaard diagram of Y. Note that, in Morse-theoretic viewpoint, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the intersection of Σ with the unstable and stable manifolds of critical points of index one and two, respectively.

Given a closed closed oriented 3-manifold Y, the choice of a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is not unique. There are three possible naturally defined operations on Heegaard diagrams.

Isotopy

We can change the curve bases $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ by isotopies, while leaving the surface Σ fixed. We say that $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ and $(\Sigma, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ are isotopic if there are two smooth 1-parameter families $\boldsymbol{\alpha}_t$ and $\boldsymbol{\beta}_t$ of g-tuples of pairwise disjoint simple closed curves.

Handleslide

Given two disjoint curves γ_1 and γ_1' on Σ, the handleslide of γ_1 over γ_1' is a simple closed curve γ_2, disjoint from γ_1 and γ_1', such that γ_1, γ_1', and γ_2 bound a pair of pants on Σ, We say that $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ is obtained from $(\Sigma, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ by handleslides if $\boldsymbol{\alpha}_1$ is obtained by handlesliding curves in $\boldsymbol{\alpha}_0$ over other curves in $\boldsymbol{\alpha}_0$, and the same is true for $\boldsymbol{\beta}_0$ and $\boldsymbol{\beta}_1$.

Stabilization

Let E be a torus, and α, β be two curves on E, meeting transversally in a single point. The stabilization of a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ is given by $(\Sigma', \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$, where $\Sigma' = \Sigma \sharp E$, $\boldsymbol{\alpha_1} = \boldsymbol{\alpha}_0 \cup \{\alpha\}$, and $\boldsymbol{\beta}_1 = \boldsymbol{\beta} \cup \{\beta\}$.

Destabilization

We say that $(\Sigma', \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ is a destabilization of $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ if $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ is a stabilization of $(\Sigma', \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$.

It is easy to see that these operations leave the resulting 3-manifolds fixed. The converse is also true; given two Heegaard diagrams $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ and $(\Sigma', \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ which represent the same 3-manifold Y, the diagram $(\Sigma', \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ can be obtained by applying the above four operations to the diagram $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$.[49] This fact means that, if a quantity associated to Heegaard diagrams is invariant under isotopies, handleslides, and stabilization, that quantity becomes a topological 3-manifold invariant.

2.2.2 Moduli space of J-holomorphic disks

As above, let Y be a closed connected orientable 3-manifold, and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram of some genus g, representing Y. By moving $\boldsymbol{\beta}$ by an isotopy if necessary, we may assume that the intersections of curves in $\boldsymbol{\beta}$ and curves in $\boldsymbol{\alpha}$ are transverse. The symmetric product

$$\mathrm{Sym}^g(\Sigma) = \Sigma^g / S_g,$$

where S_g is a symmetric group, is a $2g$-dimensional manifold; one can see this by observing that

$$\mathbb{C}[[x_1, \cdots, x_g]]^{S_g} = \mathbb{C}[[\sum x_i, \sum x_i x_j, \cdots, \prod x_i]]$$

and applying geometric invariant theory. The tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, defined as the images of $\prod \alpha_i$ and $\prod \beta_i$ under the projection map $\Sigma^g \to \mathrm{Sym}^g\Sigma$, form g-dimensional submanifolds inside the symmetric product. Note that the given projection map is a covering outside the diagonal D, which is defined as

$$D = \{\{x_1, \cdots, x_g\} \in \mathrm{Sym}^g(\Sigma) \,|\, x_i \neq x_j \text{ for some } i \neq j\}.$$

At this stage, one might be tempted to use Lagrangian Floer theory directly to the triple $(\mathrm{Sym}^g(\Sigma), \mathbb{T}_\alpha, \mathbb{T}_\beta)$ to obtain a Floer-theoretic invariant of Y. This is actually possible; indeed, Perutz proved in [42] that there exists a symplectic form ω on $\mathrm{Sym}^g(\Sigma)$ such that the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are Lagrangian with respect to ω. However, this result is not easy, and its proof need some hard analysis. Instead, we will use a different approach, using certain kinds of almost complex structures on $\mathrm{Sym}^g(\Sigma)$.

Choose a Kahler structure $(\mathfrak{j}, \eta)$ on Σ. Then we have the almost complex structure $\mathrm{Sym}^g(\mathfrak{j})$ on $\mathrm{Sym}^g\Sigma$ and the Kahler form $\omega_0 = \eta^g$ on $\mathrm{Sym}^g(\Sigma) - D$. Using these data, we define a concept of nearly symmetric almost complex structures on $\mathrm{Sym}^g(\Sigma)$ as

follows.

Definition. Fix a finite collection a points $z_1, \cdots, z_m \in \Sigma - \cup\alpha_i - \cup\beta_i$ and an open set V with

$$(\{z_1, \cdots, z_m\} \times \mathrm{Sym}^{g-1}(\Sigma) \cup D) \subset V \subset \mathrm{Sym}^g(\Sigma)$$

and $\bar{V} \cap (\mathbb{T}_\alpha \cup \mathbb{T}_\beta) = \emptyset$. An almost complex structure J on $\mathrm{Sym}^g(\Sigma)$ is called $(\mathfrak{j}, \eta, V)$-symmetric if J is tamed by ω_0 outside $\bar{V}$ and agrees with $\mathrm{Sym}^g(\mathfrak{j})$ over V. The set of all such J is denoted $\mathcal{J}(\mathfrak{j}, \eta, V)$.

With respect to a nearly symmetric almost complex structure J, the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are totally real submanifolds. Given a 1-parameter family of almost complex structures in $\mathcal{J}(\mathfrak{j}, \eta, V)$ and two intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we consider the moduli space $\mathcal{M}_{J_s}(\mathbf{x}, \mathbf{y})$ of disks $u : \mathbb{D} = [0,1] \times i\mathbb{R} \to \mathrm{Sym}^g(\Sigma)$ in $\mathrm{Sym}^g(\Sigma)$ satisfying the following conditions.

- $u(\{1\} \times \mathbb{R}) \subset \mathbb{T}_\alpha$
- $u(\{0\} \times \mathbb{R}) \subset \mathbb{T}_\beta$
- $\lim_{t\to\infty} u(s+it) = \mathbf{x}$
- $\lim_{t\to\infty} u(s+it) = \mathbf{y}$
- $\frac{du}{ds} + J(s)\frac{du}{dt} = 0$.

These conditions mean that u is a J_s-holomorphic disk from $\mathbf{x}$ to $\mathbf{y}$.

Let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the set of homotopy classes of disks satisfying the above conditions except the last one; such a disk is called a topological Whitney disk. Then, for each homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we have a space $\mathcal{M}_{J_s}(\phi)$ of J_s-holomorphic disks, whose homotopy class is ϕ. We then have a natural $\mathbb{R}$-action on $\mathcal{M}_{J_s}(\phi)$ given by translation on the t-direction, whose quotient is called the space of unparametrized J_s-holomorphic curves:

$$\widehat{\mathcal{M}}_{J_s}(\phi) = \mathcal{M}_{J_s}(\phi)/\mathbb{R}.$$

One can use Atiyah-Singer theorem to show that the expected dimension of $\mathcal{M}_{J_s}(\phi)$ is given by the Maslov index $\mu(\phi)$. Using the usual transversality argument in Floer theory, we can prove that for a dense set of paths J_s in $\mathcal{J}(\mathfrak{j}, \eta, V)$, the moduli spaces $\mathcal{M}_{J_s}(\mathbf{x}, \mathbf{y})$ are smoothly cut out by defining equations. In particular, it is one-dimensional if $\mu(\phi) = 2$ and zero-dimensional if $\mu(\phi) = 1$.

These moduli spaces also carry a coherent system of orientations, which allows us to define the algebraic count of points $\sharp\mathcal{M}_{J_s}(\phi)$. However, we have a problem that a choice of coherent orientations is not unique, and depends on some additional choices of auxiliary data. This issue will be discussed in the next subsection.

2.2.3 Heegaard Floer homology

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram for Y and fix the following auxiliary data:

- A coherent orientation system $\mathfrak{o}$.
- A generic complex structure $\mathfrak{j}$ on Σ.
- A basepoint $z \in \Sigma - \cup\alpha_i - \cup\beta_i$.
- A generic path of nearly symmetric almost complex structures J_s on $\mathrm{Sym}^g(\Sigma)$.

Suppose first that $b_1(Y) = 0$. This condition is useful because it ensures that, for every pair of intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, there is at most one homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ such that $\mu(\phi) = 1$ and $\mathcal{M}(\phi)$ is nonempty. We now start by defining the abelian group $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ as follows:

$$\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathbb{Z}[\mathbb{T}_\alpha \cap \mathbb{T}_\beta].$$

Then we define a map $\partial : \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \to \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ by

$$\partial\mathbf{x} = \sum_{\mathbf{y}} \sharp\widehat{\mathcal{M}}_0(\mathbf{x}, \mathbf{y}) \cdot \mathbf{y},$$

where $\widehat{\mathcal{M}}_0(\mathbf{x},\mathbf{y}) = \widehat{\mathcal{M}}(\phi)$ for the unique homotopy class $\phi \in \pi_2(\mathbf{x},\mathbf{y})$ such that $n_z(\phi) = 0$ and $\mu(\phi) = 1$. Here, n_z denotes the intersection number with the submanifold $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$.

The above summation is finite in this case, which means that the map ∂ is a well-defined group homomorphism. However, even though the submanifolds $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ may not be Lagrangian, it can be shown that the contribution of bubbles to the algebraic count of moduli spaces are zero. Hence we can use the usual gluing argument in Lagrangian intersection Floer theory to conclude that $\partial^2 = 0$. Hence $(\widehat{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}),\partial)$ is a chain complex.

Now suppose that $b_1(Y) \neq 0$. In this case, the summation in the definition of ∂ may be infinite, so we need further assumptions on the given diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$. Given a Spin^c structure $\mathfrak{s}$ on Y, we say that the diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is weakly admissible with respect to $\mathfrak{s}$ if any nontrivial periodic domain $\mathcal{D}$ with $\langle c_1(\mathfrak{s}), H(\mathcal{D})\rangle = 0$ has both positive and negative coefficients when expressed as a formal linear combination of connected components(i.e. *regions*) of $\Sigma \setminus \bigcup(\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. Here, a periodic domain is a two-chain in Σ whose boundary is a linear combination of α-curves and β-curves, and given such a domain, we can cap off its boundary with disks in $Y - \Sigma$ to get a homology class $H(\mathcal{D}) \in H_2(Y;\mathbb{Z})$. This assumption is possible because any Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ can be modified by an isotopy to satisfy weak admissibility with respect to all possible Spin^c-structures.

Fix a Spin^c-structure $\mathfrak{s}$ on Y and suppose that $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is weakly admissible with respect to $\mathfrak{s}$. Then we can define the chain complex

$$\widehat{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathfrak{s}) = \mathbb{Z}[\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid s_z(\mathbf{x}) = \mathfrak{s}]$$

and the differential

$$\partial\mathbf{x} = \sum_{\mathbf{y}} \sharp\widehat{\mathcal{M}}_0(\mathbf{x},\mathbf{y}) \cdot \mathbf{y}.$$

It turns out that weak admissibility ensures the finiteness of the above sum. $\partial^2 = 0$

remains to holds in this case. Hence the chain complex $(\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}), \partial)$ and the total complex

$$\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \bigoplus_{\mathfrak{s}} (\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}), \partial)$$

are chain complexes.

Construction of this chain complex depends largely on the choice of auxiliary data. However, it turns out that any two choice of such data give chain homotopic chain complexes, so we deduce that the homotopy type of $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ depends only on the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. But it also turns out that if a Heegaard diagram is given by applying an isotopy, a handleslide, or a stabilization to another diagram, then the associated chain complexes are also chain homotopic. Therefore, given any Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ of a 3-manifold Y, the homology groups

$$\widehat{HF}(Y) = H_*(\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})), \ \widehat{HF}(Y, \mathfrak{s}) = H_*(\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}))$$

depend only on Y, up to isomorphism. These groups are called the Heegaard Floer homology groups of Y.

2.2.4 Computing Heegaard Floer homology

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram for a 3-manifold Y, $\mathfrak{j}$ an almost complex structure on Σ, and let $u : D^2 \to \mathrm{Sym}^g(\Sigma)$ be a J-holomorphic disk from $\mathbf{x}$ to $\mathbf{y}$. Then we can take a pullback of u by the g-fold branched cover $\Sigma \times \mathrm{Sym}^{g-1}(\Sigma) \to \mathrm{Sym}^g(\Sigma)$ to obtain a g-fold branched covering $p : S \to D^2$. Composing it with the projection map $\Sigma \times \mathrm{Sym}^{g-1}(\Sigma) \to \Sigma$ induces a holomorphic map $f : S \to \Sigma$. The maps p and f completely determine u.

Hence we deduce that, to find a J-holomorphic representative of a homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, it suffices to find a g-fold branched cover $p : S \to D^2$ and a holomorphic

map $f : S \to \Sigma$ such that $f_*[S] = \mathcal{D}(\phi)$, where $\mathcal{D}(\phi)$ is the 2-chain defined as

$$\mathcal{D}(\phi) = \sum_{\text{region } D} n_D(\phi) \cdot D.$$

Here, $n_D(\phi)$ is defined as $n_z(\phi)$ where z is any point in D. This observation was developed by Lipschitz to construct a cylindrical reformulation of HF, which counts J-holomorphic curves inside $\Sigma \times \mathbb{R} \times [0,1]$ with suitable boundary conditions, and does not refer to Lagrangian intersection Floer theory.

To explicitly calculate Heegaard Floer homology groups, in principle, we need to choose a generic path of nearly symmetric almost complex structures on $\mathrm{Sym}^g(\Sigma)$. Since finding such a path is almost impossible in explicit computations, we need to choose an almost complex structure with a special form, which is very explicit and easy to describe, and then prove that making such a choice is enough to carry out transversality arguments. In Proposition 3.9 of [37], Ozsvath and Szabo prove that if a homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is α-injective, then the moduli space $\mathcal{M}_{\mathrm{Sym}^g(\mathbf{j})}(\phi)$ is smoothly cut out by its defining equations. Here, ϕ is said to be α-injective if it satisfies the following conditions.

- The coefficients of the domain $\mathcal{D}(\phi)$ are either 0 or 1;
- For any i, there exists an interval in α_i, which has a nonzero coefficient in the 1-chain $\partial\mathcal{D}(\phi)$.

Note that, by a region, we mean a connected component of $\Sigma \backslash (\alpha, \beta - \text{curves})$.

Also, if every region having coefficient 1 inside $\mathcal{D}(\phi)$ is either a bigon or a rectangle, and $\mu(\phi) = 1$, then $\sharp\widehat{\mathcal{M}}_{\mathrm{Sym}^g(\mathbf{j})}(\phi) = \pm 1$. This enables us to compute the differentials explicitly, in the following way. We first always modify $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ by a sequence of isotopies and handleslides so that every region not containing the basepoint $z \in \Sigma$ is either a bigon or a square. After the modification, Sarkar and Wang[47] prove that the domain of every holomorphic disk not intersecting z is either an embedded bigon

or an embedded rectangle, which makes the computation possible. They also give an explicit algorithm for finding such a modification sequence, which is commonly called as the Sarkar-Wang algorithm.

2.3 Contact invariant in Heegaard Floer cohomology

Heegaard Floer theory is a very effective tool in contact topology. One of its reason would be the existence of a contact class $c(\xi) \in \widehat{HF}(-M)$, associated to a contact structure on ξ on M, which depends only on the isotopy class of ξ. In this section, we will review its construction, both in the original form by Ozsvath-Szabo[38] and also in the combinatorial form by Honda-Kazaz-Matic[21]. Note that, from now on, we will always take $\mathbb{F}_2$ as the coefficient field for Heegaard Floer (co)homology.

Suppose that a closed contact 3-manifold (M, ξ) is given. By Giroux correspondence theorem, we can always represent it as an open book decomposition. Let K be its binding, and F be its page. Then F is a Seifert surface of K, so K is nulhomologous; denote its genus by g. Then Ozsvath and Szabo proves that

$$\widehat{HFK}(-M, K, F, -g) \simeq \mathbb{F}_2,$$

so we can denote its generator by $c(K)$. Here, HFK denotes knot Floer homology, constructed by Ozsvath-Szabo[36] and Rasmussen[46], which is defined in the same way as in HF of 3-manifolds, but instead of using single-pointed Heegaard diagrams, we use doubly-pointed diagrams which represent the given nulhomologous knot. Now the image of $c(K)$ under the natural map

$$\widehat{HFK}(-M, K, F, -g) \to \widehat{HF}(-M),$$

given by the natural inclusion of chain complexes, is denoted $c(\xi)$. Ozsvath and Szabo showed that $c(\xi)$ is invariant under isotopy and positive stabilization, thus showing

that $c(\xi)$ is an invariant of the isotopy class of ξ. The cohomology class $c(\xi)$ is called the **contact class** of ξ.

Although the above original definition of $c(\xi)$ by Ozsvath and Szabo is quite abstract, there is another way to define it in a combinatorial way, due to Honda-Kazez-Matic. Given a closed contact 3-manifold (M, ξ), we again start from its open book decomposition (Σ, f), where Σ is a page and f is the monodromy. Then the monodromy f can be described combinatorially, as follows. Choose an arc basis $\mathbf{a} = \{a_1, \cdots, a_r\}$ of S; here, an arc basis is a collection of disjoint properly embedded arcs on Σ so that $\Sigma - \cup a_i$ is a disk. Then $\mathbf{b} = \{b_1, \cdots, b_r\}$, $b_i = f(a_i)$, is another arc basis of Σ. The bases $\mathbf{a}$ and $\mathbf{b}$ completely determine the open book up to isotopy.

Now we define a slight perturbation $\mathbf{a}' = \{a_1', \cdots, a_r'\}$ of $\mathbf{a}$ as an arc basis satisfying the following conditions:

- a_i' is isotopic to a_i by a small isotopy,
- The endpoints of a_i are isotoped along $\partial\Sigma$ in the direction given by the boundary orientation of Σ,
- a_i and a_i' intersect transversely in one point in the interior of Σ,
- With respect to the orientations induced from the isotopy, the arcs a_i and a_i' intersect positively.

Since Y is given by the mapping torus $\Sigma \times I/\sim$ of f, the surface $S = (\Sigma \times \{0, 1/2\})/(\partial\Sigma \times \{0, 1/2\}) \subset Y$ is a Heegaard surface. The set of simple closed curves $\boldsymbol{\alpha} = \{\alpha_1, \cdots, \alpha_r\}$, $\boldsymbol{\beta} = \{\beta_1, \cdots, \beta_r\}$ given by:

$$\begin{aligned} \alpha_i &= a_i \times \{0\} \cup a_i \times \{1/2\}, \\ \beta_i &= a_i' \times \{0\} \cup b_i \times \{1/2\}, \end{aligned}$$

are curve bases on S, and forms a Heegaard diagram $(S, \boldsymbol{\alpha}, \boldsymbol{\beta})$.

Denote the intersection point $(a_i \cap a_i') \times \{0\}$ by p_i. Then $\mathbf{p} = \{p_1, \cdots, p_r\} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is an element of the Heegaard Floer chain complex $\widehat{CF}(S, \boldsymbol{\alpha}, \boldsymbol{\beta})$. Since there are no J-holomorphic disks going out of $\mathbf{p}$, because of the convenient placement of the basepoint, we have $\partial \mathbf{p} = 0$, i.e. $\mathbf{p}$ is a cycle. Hence we get a Floer homology class

$$[\mathbf{p}] \in \widehat{HF}(-Y).$$

Honda, Kazez, and Matic proved in [21] that this element does not depend on auxiliary data needed for its definition, and is invariant under positive stabilization. This implies that, given a contact structure ξ on Y, we have a well-defined Floer homology class $c(\xi) \in \widehat{HF}(-Y)$ depending only on ξ. Also, they have proved that the contact class $c(\xi)$ which they have constructed coincides with the original definition of $c(\xi)$.

The most important property of contact invariants is that they detect tightness of contact structures. Let Y be a closed 3-manifold. If a contact structure ξ on Y is overtwisted, then its contact class $c(\xi) \in \widehat{HF}(-Y)$ vanishes. In other words, if a contact structure has nonvanishing contact invariant, then it is tight. However, one should notice that there are many examples of tight contact manifolds (M, ξ) with $c(\xi) = 0$.

2.4 Transverse knots in standard contact S^3

Recall that, as defined in previous sections, a Legendrian knot in a 3-manifold is a knot which is tangent to the contact planes, and a transverse knot is a knot which is transverse to the contact planes. The two concepts are closely related, since one can always obtain a pair of transverse knots from any Legendrian knot, by a process called positive/negative **transverse push-off**. The process goes as follows:

- Given a Legendrian knot γ in a contact 3-manifold, it always has a standard

neighborhood, which is of the form

$$(S^1 \times D^2, \ker(\cos(z)dx - \sin(z)dy)).$$

- In the above coordinates, the knot

$$\gamma_{\pm}(z) = (z, \pm\delta\sin(\theta), \pm\delta\cos(\theta))$$

  is a transverse knot for a sufficiently small $\delta > 0$.

- The transverse isotopy class of $\gamma_{\pm}$ does not depend on the choice of δ, so we define it as the positive/negative transverse pushoff of γ.

This process tells us that Legendrian knots and transverse knots are closely related. In fact, if K is a Legendrian knot in $S^3_{std} = (S^3, \xi_{std})$ and $K^{\pm}$ is its positive/negative transverse pushoff, then it is known that

$$\mathbf{sl}(K^{\pm}) = \mathbf{tb}(K) \mp \mathbf{rot}(K).$$

This fact relates the two (numerical) classical invariants of Legendrian knots and the only numerical classical invariant of transverse knots.

To visualize Legendrian knots and transverse knots (in S^3_{std}) as a 2-dimensional picture, it is nice to use their front projections. A front projection is the projection map $(x, y, z) \mapsto (y, z)$, defined on $\mathbb{R}^3 = S^3_{std}\backslash\{\infty\}$. Under this projection map, the Legendrian knot equation $\frac{dz}{dt} + x\frac{dy}{dt} = 0$ becomes $x = \frac{dz}{dy}$, so that we can lift any regular curve on the yz-plane with semi-cubical cusps and without vertical tangencies to a unique Legendrian curve in $\mathbb{R}^3$.

Similarly, the positively transverse knot equation $\frac{dz}{dt} + x\frac{dy}{dt} > 0$ tells us that a front

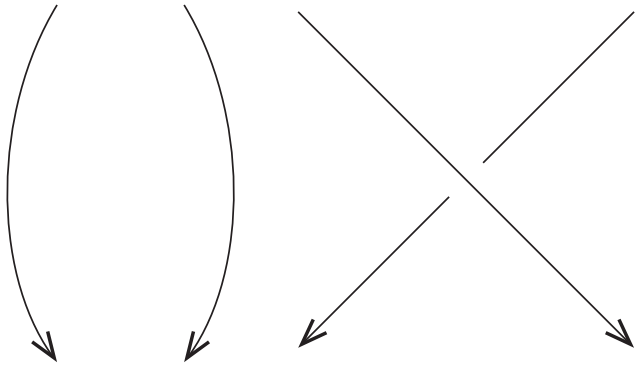


Figure 2.1: Impossible local figures of front projections of positively transverse curves

projection of a positively transverse knot in $\mathbb{R}^3$ must satisfy the following conditions.

$$\begin{cases} \text{if } y' = 0 \quad \text{then } z' > 0, \\ \text{if } y' > 0 \quad \text{then } x > -z'/y', \\ \text{if } y' < 0 \quad \text{then } x < -z'/y'. \end{cases}$$

The first statement says that there are no vertical tangencies oriented downwards. Also, the second and third statements say that the crossing in Figure 2.1 cannot appear. So, as in the case of Legendrian knots, we can lift any regular curve without local figures as in Figure 2.1 to a (positively) transverse curve in $\mathbb{R}^3$. However, unlike the Legendrian case, one can recover a positively transverse knot from its front projection only up to transverse isotopy.

Now it is natural to ask for a classification of Legendrian and transverse knots, which are isotopic to a given knot in S^3. These two classification problems are very closely related, because of the following reason. Given a Legendrian knot K in S^3_{std}, we can perform a **Legendrian stabilization** to it, defined as shown in Figure 2.2, which gives another Legendrian knot.

It is known that, given any Legendrian knot K and its negative stabilization K^-, the positive transverse push-offs of K and K^- are transversely isotopic. On the other

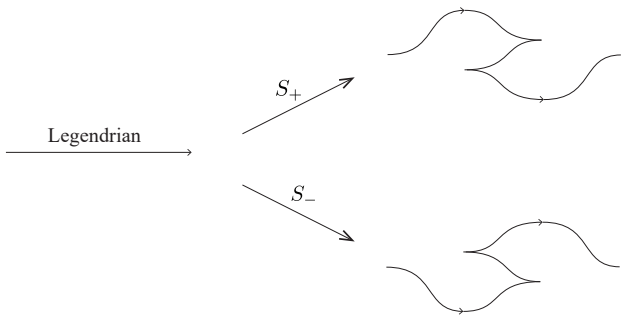


Figure 2.2: The positive stabilization S_+ and the negative stabilization S_- performed to a Legendrian knot

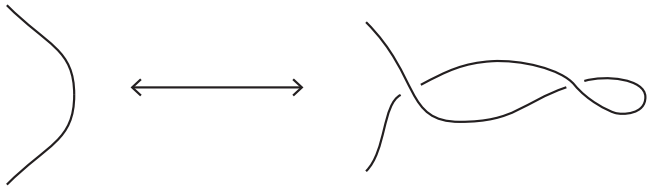


Figure 2.3: A transverse stabilization performed to a transverse knot

hand, if we denote the positive stabilization of K by K^+, then its positive transverse push-off $T(K^+)$ depends only on the transverse isotopy class of the positive transverse push-off $T(K)$ of K, and we have

$$\mathbf{sl}(T(K^+)) = \mathbf{sl}(T(K)) - 2;$$

so we call $T(K^+)$ as the **transverse stabilization** of $T(K)$. This operation can be described in terms of front projections as in Figure 2.3. This tells us that classifying Legendrian knots in S^3_{std} up to Legendrian isotopy and negative stabilization is equivalent to classifying transverse knots in S^3_{std} up to transverse isotopy. So instead of trying to classify Legendrian knots, we can instead work on a slightly easier question of classifying transverse knots.

One might expect that topological knot type and self-linking number, which form the complete set of classical transverse invariants, are sufficient to classify transverse knots in S^3_{std} up to transverse isotopy. Unfortunately this is not always the case. There

are many examples of pairs of transverse knots which are isotopic and have the same self-linking number but are not transversely isotopic. For example, the (2,3)-cable of the (2,3)-torus knot is shown to have a pair of such transverse representatives, due to Etnyre and Honda[15].

When a knot $K \subset S^3$ admits an example of such a pair of transverse representatives, we say that K is **transversely nonsimple**. Also, when a transverse knot invariant can be used to detect a transversely nonsimple knot type, we call that invariant **effective**.

2.5 Contact branched double covers along transverse knots

In this final section, we will recall the definition of contact branched double covers of S^3_{std} along a transverse link, as its construction will be used very frequently in the rest of the thesis. Given a link L in S^3, we can remove its neighborhood, take the double cover with respect to the meridian of its boundary, and then reglue a solid torus along the boundary to get the branched double cover $\Sigma(L)$ of S^3 along L. The covering transformation, i.e. natural $\mathbb{Z}_2$-action on $\Sigma(L)$ is an orientation-preserving homeomorphism. Now suppose that we are working with the standard contact sphere (S^3, ξ_{std}), and a transverse link L inside it. It has a standard contact neighborhood:

$$N(L) \simeq (S^1 \times D^2, \ker(\alpha = d\phi + r^2 d\theta)),$$

where ϕ parametrizes S^1 and (r, θ) are the polar coordinates on D^2. The pullback of α along the branched covering $p : (z, r, \theta) \mapsto (z, r^2, 2\theta)$ is given by

$$p^*\alpha = dz + 2r^4 d\theta,$$

which satisfies the contact condition away from the fixed locus $L = \{r = 0\}$. However, the branched double cover construction of Plamenevskaya[44] tells us that we may consider the interpolated 1-forms

$$\alpha_f = dz + f(r^2)d\theta,$$

for smooth increasing functions f which satisfy $f(r^2) = r^2$ near $r = 0$ and $f(r^2) = 2r^4$ away from $r = 0$. For such a function f, the form α_f is always contact. Also, for any two such functions f and g, the forms α_f and α_g are obviously isotopic by a radial isotopy. Hence, by gluing the solid torus to the double cover of $S^3 - N(L)$, we see that the contact branched double cover

$$(\Sigma(L), \xi_L) = (S^3 - \widetilde{N(L)}, \xi_{std}) \cup (S^1 \times D^2, \alpha_f)$$

is well-defined up to isotopy supported near L.

We also have a combinatorial interpretation of the above construction in terms of open book decompositions. Consider the case when L is braided along the z-axis in S^3; this is always possible by the following theorems.

Definition 2.5.1. Consider the genus 0 open book of S^3, as follows:

$$\pi : S^3 - \{z\text{-axis}\} \to S^1.$$

Then a link L in S^3 is braided if it does not intersect the z-axis and the map $\pi|_L : L \to S^1$ is a regular covering map.

Theorem 2.5.2. *(Bennequin, [5]) For any transverse link L in (S^3, ξ_{std}), there exists a transverse braid B around the z-axis, such that L is transversely isotopic to B.*

Theorem 2.5.3. *(Orevkov-Shevchishin, [35]) Two (closed) transverse braids around the z-axis are transversely isotopic as transverse links if and only if they are related by braid isotopies, conjugations in the braid group, and positive braid stabilizations.*

We adopt the usual notation for braids. If we consider braids with n strings, the corresponding braid group is generated by the standard generators $\sigma_1, \cdots, \sigma_{n-1}$, where σ_i creates a positive crossing between the ith and the $(i+1)$th strands. This notation can also be applied to transverse braids around the z-axis.

Now we will discuss how one can construct the contact branched double cover from a braid representative of a given transverse link, as pointed out by Plamenevskaya[44]. Suppose that the transverse link L is represented by a transverse braid B with a braid word $w = \sigma_{i_1}^{\pm 1} \cdots \sigma_{i_k}^{\pm 1}$. Consider the disk D with n points $p_1, \cdots, p_n$ in its interior and pairwise (interior-)disjoint simple arcs c_i connecting p_i and p_{i+1}. Then we define the following self-diffeomorphism of D:

$$h(w) = T_{i_1}^{\pm 1} \cdots T_{i_k}^{\pm 1} \in \mathrm{Diff}^+(D, \partial D, \{p_1, \cdots, p_n\}).$$

Here, T_i denotes the Dehn half-twist along the arc c_i. Clearly this definition depends on the choice of points p_i and arcs c_i, but since any two choices of such data are related by a self-diffeomorphism of D(due to the fact that any two such data are isotopic to each other), we see that $h(w)$ is well-defined up to conjugation. Since the self-diffeomorphism $h(w)$ is orientation-preserving and fixes the boundary ∂D pointwise, the pair (D, h) becomes an abstract open book of S^3.

We then consider the branched double cover S of D, branched along the points $p_1, \cdots, p_n$. The arcs c_i smoothly lift to smooth simple closed curves C_i and the self-diffeomorphism h of D lifts smoothly to a self-diffeomorphism $\tilde{h}$ of S, which is now given by products of positive/negative Dehn twists along the curves C_i. By the argument used above, the conjugacy class of $\tilde{h}$ is uniquely determined. Hence the abstract open book $\mathcal{P}_B = (S, \tilde{h})$ is defined up to diffeomorphism. Since this abstract open book has a natural $\mathbb{Z}_2$-action(i.e. covering transformation) and the monodromy $\tilde{h}$ is equivariant with respect to that action, $\mathcal{P}_B$ represents a uniquely determined closed contact 3-manifold (M_B, ξ_{M_B}) with a contact $\mathbb{Z}_2$-action, which turns out to be

the contact branched double cover of (S^3, ξ_{std}) along L. In other words, we get an open book description of $(\Sigma(L), \xi_L)$.

Note that the construction of (M_B, ξ_B) depends on the braid isotopy class of B, not only on its link isotopy class. However, by the definition of contact branched double covers, we know that if L, L' are transversely isotopic, then the double covers $(\Sigma(L), \xi_L)$ and $(\Sigma(L'), \xi_{L'})$ are ($\mathbb{Z}_2$-equivariantly) contactomorphic. Hence we see that, if two transverse braids B, B' are related by braid isotopies and positive stabilizations, then $(M_B, \xi_B) \simeq (M_{B'}, \xi_{B'})$. This means that the combinatorial construction of $(\Sigma(L), \xi_L)$ is well-defined, by Theorem 2.5.3.

Chapter 3

$\mathbb{Z}_2$-Equivariant Heegaard Floer cohomology

3.1 Construction of $\mathbb{Z}_2$-equivariant Floer cohomology

Hendricks, Lipshitz, and Sarkar[20] constructed the $\mathbb{F}_2[\theta]$-module $\widehat{HF}_{\mathbb{Z}_2}(L_0, L_1)$ when $\mathbb{Z}_2$ acts symplectically on a symplectic manifold (M, ω) and $L_0, L_1 \subset M$ are transversely intersecting Lagrangians which are fixed by the $\mathbb{Z}_2$-action and satisfy certain hypotheses, and use it to define $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), z)$ for the branched double cover $\Sigma(L)$, where (L, z) is a based link in S^3. In this section, we will briefly review their construction.

Before proceeding to the construction, we will first give a list of hypotheses needed to define $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology. Here, η is a fixed collection of homotopy classes of paths from L_0 to L_1; in Heegaard Floer terminology, this corresponds to fixing a collection of Spin^c-sturctures.

Assumption 3.1.1 (Modified version of Hypothesis 3.2, [20])**. (J-1)** *For any loop of*

paths

$$v : ([0,1] \times S^1, \{0\} \times S^1, \{1\} \times S^1) \to (M, L_0, L_1)$$

such that the homotopy class of the path $[v|_{[0,1]\times\{pt\}}]$ *is in* η*, the symplectic area of* v *and the Maslov index of* v *are both zero.*

(J-2) *The manifold* M *is either compact or else is convex at infinity (see section 1.7.1 of [13] for its definition). Further, the Lagrangians* L_0 *and* L_1 *are both compact.*

(J-3) *There is a* $\mathbb{Z}_2$*-action on* M *which acts by symplectomorphisms and preserves* L_0*,* L_1*, and* η *setwise.*

3.1.1 Homotopy coherent diagrams of almost complex structures

By a cylindrical complex structure on a symplectic manifold (M, ω), we mean a smooth 1-parameter family $J = J(t)$, $0 \leq t \leq 1$, of almost complex structures on M, compatible with ω. By an eventually cylindrical almost complex structure on (M, ω), we mean a smooth 1-parameter family $\tilde{J}(s)$, $s \in \mathbb{R}$, of cylindrical complex structures, which is constant outside a compact subset of $\mathbb{R}$, modulo translation by $\mathbb{R}$. Denote the set of eventually cylindrical almost complex structures by $\mathcal{J}$. Then it carries a natural topology by declaring that a sequence $\{\tilde{J}_i\}$ converges if and only if we can replace $\tilde{J}_i$ by their representatives so that every $\tilde{J}_i$ is constant outside a fixed compact set C and $\{\tilde{J}_i|_C\}$ is convergent in the C^∞ topology. Given an element $J \in \mathcal{J}$, its limit at $-\infty$ and ∞ are well-defined cylindrical complex structures, which we will denote as $J_{-\infty}$ and $J_{+\infty}$. Let $\mathcal{J}(J_{-\infty}, J_{+\infty})$ be the subspace of $\mathcal{J}$ consisting of $J' \in \mathcal{J}$ with $J'_{-\infty} = J_{-\infty}$ and $J'_{+\infty} = J_{+\infty}$.

We now define a category $\overline{\mathcal{J}}$ as follows. Its objects are cylindrical complex structures. For any cylindrical complex structures J and J', the morphism set $\overline{\mathcal{J}}(J, J')$

consists of finite nonempty sequences

$$(\tilde{J}^1, \cdots, \tilde{J}^n) \in \mathcal{J}(J, J_1) \times \cdots \times \mathcal{J}(J_{n-1}, J'),$$

where $J_1, \cdots, J_{n-1}$ are cylindrical complex structures, modulo the equivalence relation

$$(\tilde{J}^1, \cdots, \tilde{J}^{i-1}, \tilde{J}^i, \tilde{J}^{i+1}, \cdots, \tilde{J}^n) \sim (\tilde{J}^1, \cdots, \tilde{J}^{i-1}, \tilde{J}^{i+1}, \tilde{J}^n),$$

whenever $\tilde{J}^i$ is a constant path. Then we have a natural composition map

$$\circ : \overline{\mathcal{J}}(J, J') \times \overline{\mathcal{J}}(J', J'') \to \overline{\mathcal{J}}(J, J''),$$

defined by concatenation which makes $\overline{\mathcal{J}}$ into a category. This category can be given a natural topology, which turns it into a topological category.

Note that, since the space $\overline{\mathcal{J}}(J, J')$ are weakly contractible, we can also consider continuous multi-parameter families of elements in $\overline{\mathcal{J}}(J, J')$ and consider them as higher morphisms in the category $\overline{\mathcal{J}}$. Thus we can define the notion of homotopy coherent diagrams in $\overline{\mathcal{J}}$ as follows.

Definition 3.1.2. Given a small category $\mathcal{C}$, a homotopy coherent $\mathcal{C}$-diagram F in $\overline{\mathcal{J}}$ consists of the following data.

(i) For each $x \in \mathrm{ob}(\mathcal{C})$, an object $F(x)$ of $\overline{\mathcal{J}}$,

(ii) For each integer $n \geq 1$ and each composable sequence $x_0 \xrightarrow{f_1} x_1 \to \cdots \xrightarrow{f_n} x_n$ of morphisms in $\mathcal{C}$, a continuous family

$$F(f_n, \cdots, f_1) : [0,1]^{n-1} \to \overline{\mathcal{J}}(F(x_n), F(x_1)),$$

so that the conditions in Definition 3.3 of [20] are satisfied.

The source category $\mathcal{C}$ which we will use frequently is the groupoid $\mathcal{E}\mathbb{Z}_2$, which has two objects a and b, and four morphisms, as follows.

- $\mathrm{Hom}_{\mathcal{C}}(x,x) = \{\mathrm{id}_x\}$ for $x = a, b$,
- $\mathrm{Hom}_{\mathcal{C}}(a,b) = \{\alpha\}$ and $\mathrm{Hom}_{\mathcal{C}}(b,a) = \{\beta\}$.

3.1.2 The freed Floer complex

Definition 3.1.3. Given an morphism $\tilde{J} = (\tilde{J}^1, \cdots, \tilde{J}^n) \in \overline{\mathcal{J}}$, where each $\tilde{J}^i \in \mathcal{J}(J_{i-1}, J_i)$ is nonconstant, and points $x, y \in L_0 \cap L_1$, a $\tilde{J}$-holomorphic disk from x to y is a sequence

$$(v^{0,1}, \cdots, v^{0,m_0}, u^1, v^{1,1} \cdots, v^{1,m_1}, u^2 \cdots, u^n, v^{n,1}, \cdots, v^{n,m_n}),$$

where $m_0, \cdots, m_n$ are nonnegative integers and the following conditions are satisfied.

- Each $v^{i,j}$ is a J_i-holomorphic Whitney disk with boundary on L_0 and L_1, connecting some points $x^{i,j-1}$ and $x^{i,j}$ in $L_0 \cap L_1$.
- Each u^i is a $\tilde{J}^i$-holomorphic Whitney disk with boundary on L_0 and L_1, connecting some points x^{i-1} and x^i in $L_0 \cap L_1$.
- $x^{i,0} = x^i$ for $i \geq 1$, $x^{i,m_i} = x^{i+1}$ for all i, $x^{0,0} = x$, and $x^{n,m_n} = y$.

Given a map $\tilde{J} : [0,1]^k \to \overline{\mathcal{J}}$, points $x, y \in L_0 \cap L_1$, and a homotopy class $\phi \in \pi_2(x,y)$, let $\mathcal{M}(x,y;\tilde{J})$ denote the moduli space of pairs (p,u) where $p \in [0,1]^k$ and u is a $\tilde{J}(p)$-holomorphic disk from x to y, and $\mathcal{M}(\phi;\tilde{J})$ denotes the subspace of disks representing the class ϕ. Then we have a splitting, as follows.

$$\mathcal{M}(x,y;\tilde{J}) = \coprod_{\phi \in \pi_2(x,y)} \mathcal{M}(\phi;\tilde{J})$$

Since $\tilde{J}$ is a k-parameter family, the expected dimension of $\mathcal{M}(\phi;\tilde{J})$ is given by $\mu(\phi) + k$. When a homotopy coherent diagram $F : \mathcal{C} \to \overline{\mathcal{J}}$ is sufficiently generic, i.e. for any points $x, y \in L_0 \cap L_1$, a homotopy class $\phi \in \pi_2(x,y)$, and a composable

sequence of morphisms $f_1, \cdots, f_n$ in $\mathcal{C}$, the space $\mathcal{M}(\phi; F(f_n, \cdots, f_1))$ is transversely cut out, and its dimension is given by $\mu(\phi) + n - 1$.

Now, given a sufficiently generic homotopy coherent diagram $F : \mathcal{C} \to \overline{\mathcal{J}}$, for each $a \in \mathrm{ob}(\mathcal{C})$, let $G(a)$ be the Floer chain complex $(CF(L_0, L_1), \partial_{F(a)})$ with respect to the cylindrical complex structure $F(a)$. For a composable sequence of morphisms $f_1, \cdots, f_n$ in $\mathcal{C}$, and a k-dimensional face σ of $[0,1]^{n-1}$, we have a k-dimensional subfamily:

$$F(f_n, \cdots, f_1)|_\sigma : [0,1]^k \to \overline{\mathcal{J}}.$$

So we define:

$$G(f_n, \cdots, f_1)(\sigma \otimes x) = \sum_{y \in L_0 \cap L_1} \sum_{\phi \in \pi_2(x,y),\, \mu(\phi) = 1-k} |\mathcal{M}(\phi; F(f_n, \cdots, f_1)|_\sigma)| \cdot y.$$

Then $G : \mathcal{C} \to \mathrm{Kom}_{\mathbb{F}_2}$ turns out to be a homotopy coherent $\mathcal{C}$-diagram in the (∞-)category of complexes of $\mathbb{F}_2$-vector spaces, in the following sense.

Definition 3.1.4. Given a small category $\mathcal{C}$, a homotopy coherent $\mathcal{C}$-diagram F in $\mathrm{Kom}_{\mathbb{F}_2}$ consists of:

- For each $x \in \mathrm{ob}(\mathcal{C})$, a chain complex $F(x) \in \mathrm{ob}(\mathrm{Kom}_{\mathbb{F}_2})$.
- For each $n \geq 1$ and each composable sequence $f_1, \cdots, f_n$ of morphisms in $\mathcal{C}$, a chain map

  $$G(f_n, \cdots, f_1) : I_*^{\otimes n-1} \otimes G(x_0) \to G(x_n),$$

  where $I_* = C_*^{\mathrm{simplicial}}([0,1])$, such that $G(f_n, \cdots, f_1)(t_1 \otimes \cdots \otimes t_{n-1})$, where $t_i \in I_*$, is equal to:

  - $G(f_n, \cdots, f_2)(\pi(t_1) \otimes t_2 \otimes \cdots \otimes t_{n-1})$ if $f_1 = [0,1]$, where $\pi : I_* \to \mathbb{F}_2$ is the map induced the projection $[0,1] \to \{\mathrm{pt}\}$;
  - $G(f_n, \cdots, f_{i+1}, f_{i-1}, \cdots, f_1)(t_1 \otimes \cdots \otimes m(t_{i-1} \otimes t_i) \otimes \cdots \otimes t_{n-1})$ if $f_i = [0,1]$ and $1 < i < n$, where $m : I_* \otimes I_* \to I_*$ is the map induced by the

multiplication $[0,1] \times [0,1] \to [0,1]$,

- $G(f_{n-1}, \cdots, f_1)(t_1 \otimes \cdots \otimes t_{n-2} \otimes \pi(t_{n-1}))$ if $f_n = [0,1]$,
- $G(f_n, \cdots, f_{i+1} \circ f_i, \cdots, f_1)(t_1 \otimes \cdots \otimes t_{i-1} \otimes t_{i+1} \otimes \cdots \otimes t_{n-1})$ if $t_i = \{1\}$,
- $G(f_n, \cdots, f_{i+1})(t_{i+1} \otimes \cdots \otimes t_{n-1}) \circ G(f_i, \cdots, f_1)(t_1 \otimes \cdots \otimes f_{i-1})$ if $t_i = \{0\}$.

Since G is homotopy coherent, we can consider its homotopy colimit hocolim G, which is a single chain complex of $\mathbb{F}_2$-vector spaces.

Definition 3.1.5. When $\mathcal{C} = \mathcal{E}\mathbb{Z}_2$ and the homotopy coherent diagram $F : \mathcal{E}\mathbb{Z}_2 \to \overline{\mathcal{J}}$ is $\mathbb{Z}_2$-equivariant and sufficiently generic, the chain complex hocolim G is defined as the freed Floer complex, and denoted as $\widetilde{CF}_{\mathbb{Z}_2}(L_0, L_1)$. Since we have a natural $\mathbb{Z}_2$-action on the freed Floer complex, the complex

$$CF_{\mathbb{Z}_2}(L_0, L_1) = \mathrm{Hom}_{\mathbb{F}_2[\mathbb{Z}_2]}(\widetilde{CF}_{\mathbb{Z}_2}(L_0, L_1), \mathbb{F}_2)$$

is defined as the equivariant Floer cochain complex, and its cohomology

$$HF_{\mathbb{Z}_2}(L_0, L_1) = H^*(CF_{\mathbb{Z}_2}(L_0, L_1))$$

is defined as the equivariant Floer cohomology.

In the category $\mathcal{E}\mathbb{Z}_2$, any composable sequence of morphisms is either of the form $\alpha_n = (\alpha, \beta, \alpha, \cdots)$ or of the form $\beta_n = (\beta, \alpha, \beta, \cdots)$. Thus the elements of the freed Floer chain complex $\widetilde{CF}_{\mathbb{Z}_2}(L_0, L_1)$ are of the form $\alpha_n \otimes x$ or $\beta_n \otimes x$ for $x \in L_0 \cap L_1$. The differential is of the following form.

$$\partial(\alpha_n \otimes x) = \alpha_n \otimes (\partial x) + \beta_{n-1} \otimes x + \sum_{i=1}^{n} \alpha_{n-i} \otimes \begin{cases} G_{\beta,\alpha,\cdots}(x) & \text{if } n-i \text{ is odd} \\ G_{\alpha,\beta,\cdots}(x) & \text{if } n-i \text{ is even} \end{cases}$$

$$\partial(\beta_n \otimes x) = \beta_n \otimes (\partial x) + \alpha_{n-1} \otimes x + \sum_{i=1}^{n} \beta_{n-i} \otimes \begin{cases} G_{\alpha,\beta,\cdots}(x) & \text{if } n-i \text{ is odd} \\ G_{\beta,\alpha,\cdots}(x) & \text{if } n-i \text{ is even} \end{cases}$$

Here, $G_{\alpha,\beta,\cdots}(x)$ and $G_{\beta,\alpha,\cdots}(x)$ can be evaluated by counting holomorphic disks of Maslov index $2+i-n$ from x. Note that the holomorphic disks which we are counting here are the ones defined in 3.1.3.

The $\mathbb{Z}_2$-action on the freed Floer chain complex is given by

$$\tau(\alpha_n \otimes x) = \beta_n \otimes \tau x,$$

where $\mathbb{Z}_2 = \langle \tau \rangle$. Hence the elements of the equivariant Floer cochain complex $CF_{\mathbb{Z}_2}(L_0, L_1)$ are of the form $\theta^n \otimes x^*$, where θ is a formal variable and x^* is the dual of a given Floer generator x, and its differential is given by

$$d(\theta^n \otimes x^*) = \theta^n \otimes dx^* + \theta^{n+1} \otimes \tau^* x^* + \sum_{i=1}^{\infty} \theta^{n+i} \otimes (x^* \circ G_{\alpha,\beta,\cdots}).$$

We have an action of $\mathbb{F}_2[\theta]$ on $CF_{\mathbb{Z}_2}(L_0, L_1)$, given as follows.

$$\theta \cdot (\theta^n \otimes x^*) = \theta^{n+1} \otimes \tau^* x^*$$

Since the differential of $CF_{\mathbb{Z}_2}(L_0, L_1)$ is θ-equivariant, we get a natural $\mathbb{F}_2[\theta]$-module structure on $HF_{\mathbb{Z}_2}(L_0, L_1)$.

The quasi-isomorphism type of $CF_{\mathbb{Z}_2}(L_0, L_1)$, and thus the $\mathbb{F}_2[\theta]$-module isomorphism type of $HF_{\mathbb{Z}_2}(L_0, L_1)$, turns out to be invariant under the choice of sufficiently generic $\mathbb{Z}_2$-equivariant diagrams in $\overline{\mathcal{J}}$, $\mathbb{Z}_2$-equivariant Hamiltonian isotopies, and non-$\mathbb{Z}_2$-equivariant Hamiltonian isotopies which satisfy certain conditions. This will be explained in detail in the next subsection.

3.1.3 Maps between equivariant Floer complexes

3.1.3.1 Diffeomorphism maps

As in the ordinary(non-equivariant) Floer theory, equivariant Floer theory also admits certain types of natural maps between complexes. The map which can be most obviously defined is the diffeomorphism map. It can be defined in the natural way, as follows.

Given a $\mathbb{Z}_2$-equivariant symplectomorphism $\phi : M \to M'$ which sends $\mathbb{Z}_2$-invariant Lagrangians $L_0, L_1 \subset M'$ to $L_0' = \phi(L_0)$ and $L_1' = \phi(L_1)$, we have a naturally defined chain isomorphism

$$\phi_* : \widetilde{CF}_{\mathbb{Z}_2}(L_0, L_1; J) \to \widetilde{CF}_{\mathbb{Z}_2}(L_0', L_1'; \phi(J)),$$

for homotopy coherent diagrams J, when (M, L_0, L_1) satisfies Hypothesis 3.1.1.

Hence we get a natural map between equivariant Floer cohomology.

$$\phi^* : HF_{\mathbb{Z}_2}(L_0', L_1', \phi(J)) \xrightarrow{\sim} HF_{\mathbb{Z}_2}(L_0, L_1, J)$$

3.1.3.2 Continuation maps

Another Floer-theoretically defined map that can also be defined in the equivariant setting is the continuation map, which is defined when we change our choice of homotopy coherent diagrams J of eventually cylindrical almost complex structures. As in non-equivariant Floer theory, continuation maps are also isomorphisms in the equivariant setting, as stated in the proposition below.

Proposition 3.1.6 (Proposition 3.23, [20])**.** *Up to quasi-isomorphism of complexes over $\mathbb{F}_2[\mathbb{Z}_2]$, the freed Floer complex $\widetilde{CF}(L_0, L_1, J)$ is independent of the choice of $\mathbb{Z}_2$-equivariant sufficiently generic homotopy coherent diagram J, in $\overline{\mathcal{J}}$, used to define it.*

Since the proof of the above proposition contains the construction of continuation maps and thus is very important, we present a sketch of its proof here. Let $\mathscr{J}$ be the category with two objects, 0 and 1, and a single morphism $f : 0 \to 1$. Given homotopy coherent $\mathcal{E}\mathbb{Z}_2$-diagrams G_0 and G_1 in $\overline{\mathcal{J}}$ (respectively $\mathrm{Kom}_{\mathbb{F}_2}$), a *morphism* from G_0 to G_1 is a homotopy coherent $\mathscr{J} \times \mathcal{E}\mathbb{Z}_2$-diagram H in $\overline{\mathcal{J}}$ (respectively $\mathrm{Kom}_{\mathbb{F}_2}$) such that $H|_{\{i\}\times\mathcal{E}\mathbb{Z}_2} = G_i$ for $i = 0, 1$. For a morphism H of diagrams in $\mathrm{Kom}_{\mathbb{F}_2}$, we call it a *quasi-isomorphism* if

$$H(f \times \mathrm{id}_x) : G_0(x) \to G_1(x)$$

is a quasi-isomorphism. It is not hard to see that morphism of diagrams induce morphisms of their homotopy colimits, and that if a morphism H is a quasi-isomorphism then the induced map

$$\mathrm{hocolim}\ H : \mathrm{hocolim}\ G_0 \to \mathrm{hocolim}\ G_1$$

is also a quasi-isomorphism.

Now fix a sufficiently generic $\mathbb{Z}_2$-equivariant $\mathcal{E}\mathbb{Z}_2$-diagrams F_0 and F_1 in $\overline{\mathcal{J}}$, and let G_0, $G_1 : \mathcal{E}\mathbb{Z}_2 \to \mathrm{Kom}_{\mathbb{F}_2}$ be the maps induced by F_0 and F_1, respectively, by applying Floer theory as in section 1.2. Again, as in section 1.2, we can extend F_0 and F_1 to a diagram $H : \mathscr{J} \times \mathcal{E}\mathbb{Z}_2 \to \overline{\mathcal{J}}$ which is $\mathbb{Z}_2$-equivariant, where the $\mathbb{Z}_2$-action on $\mathscr{J}$ is trivial. This gives a diagram

$$K : \mathscr{J} \times \mathcal{E}\mathbb{Z}_2 \to \mathrm{Kom}_{\mathbb{F}_2}.$$

But since G_0 and G_1 maps edges of $\mathcal{E}\mathbb{Z}_2$ to Floer continuation maps, which we know are quasi-isomorphisms, the diagram K is a quasi-isomorphism from G_0 to G_1. So hocolim K is also a quasi-isomorphism, which is obviously $\mathbb{Z}_2$-equivariant. Hence it is a $\mathbb{F}_2[\mathbb{Z}_2]$-module quasi-isomorphism.

The key point of the above proof sketch is that the construction of the quasi-isomorphism hocolim K involves counting the number of points in the moduli spaces of disks of Maslov index $-n$, for $n \geq 0$, which is holomorphic with respect to paths of n-parameter families of eventually cylindrical almost complex structures. This will be used later when we prove the invariance of $\mathbb{Z}_2$-equivariant contact class.

3.1.3.3 Triangle maps

The correct version of triangle map that can be defined in the equivariant setting is the equivariant triangle map. A rather strange feature of the map is that it becomes an isomorphism not only for equivariantly Hamiltonian isotopic Lagrangians, but also for non-equivariantly Hamiltonian isotopic $\mathbb{Z}_2$-invariant Lagrangians which satisfy some conditions. It is worth noticing that this feature will play a crucial role when proving the invariance of $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology of based links.

Suppose that L_0, L_0', and L_1 are $\mathbb{Z}_2$-invariant Lagrangians, which are pairwise transverse, and there is a τ-invariant ω-compatible almost complex structure J on M which achieves transversality for all moduli spaces of holomorphic disks with boundary on (L_0, L_0') of Maslov index at most 1. Fix a cocycle $\Theta \in CF(L_0, L_0')$, which is $\mathbb{Z}_2$-invariant. Assuming that $L_0 \cap L_0' \cap L_1 = \emptyset$, define a category $\mathscr{D}$ as follows. (The case when $L_0 \cap L_0' \cap L_1$ is nonempty can be done by extending $\mathscr{D}$ to include all possible Hamiltonian perturbations of L_1)

- $\mathrm{ob}(\mathscr{D}) = \{0,1\} \times \mathrm{ob}(\overline{\mathcal{J}})$.
- For (i, J), $(i, J') \in \{i\} \times \mathrm{ob}(\overline{\mathcal{J}})$, $\mathrm{Hom}_{\mathscr{D}}((i, J), (i, J') = \mathrm{Hom}_{\overline{\mathcal{J}}}(J, J') = \mathcal{J}(J, J')$.
- $\mathrm{Hom}_{\mathscr{D}}((0, J), (1, J'))$ is the space of sequences $(\tilde{J}_{-i}, \cdots, \tilde{J}_{-1}, \tilde{J}_0, \tilde{J}_1, \cdots, \tilde{J}_j)$ where:
  - For $k \neq 0$, $\tilde{J}_k \in \mathcal{J}(J_k, J_{k+1})$ for some sequence $J_{-i}, \cdots, J_{j+1}$ of cylindrical complex structures.
  - $J_{-i} = J$ and $J_{j+1} = J'$.

– $\tilde{J}_0 \in \mathcal{J}_\triangle$ agrees with J_0 on some cylindrical neighborhood $[n,\infty)\times[0,1]$ of p_2, J_1 on some cylindrical neighborhood of $[n,\infty)\times[0,1]$ of p_3, and J on some cylindrical neighborhood $[n,\infty)\times[0,1]$ of p_1.

Here, $\mathcal{J}_\triangle$ is the space of almost complex structures parametrized by $\triangle$, where $\triangle$ is a disk with three boundary punctures p_1, p_2, p_3, together with identifications of a small closed neighborhood of p_i with $[n,\infty)\times[0,1]$. Then, like $\overline{\mathcal{J}}$, the category $\mathscr{D}$ also becomes a topological category.

For families $\tilde{J} : [0,1]^\ell \to \mathcal{J}_\triangle$ and a homotopy class ϕ of triangles in (L_0, L_0', L_1), we can consider the moduli space

$$\mathcal{M}(\phi;\tilde{J}) = \cup_{t\in[0,1]^\ell}\mathcal{M}(\phi;\tilde{J}(t)).$$

For generic $\tilde{J}$, the moduli space $\mathcal{M}(\phi;\tilde{J})$ is transversely cut out, and so we can define a map

$$G(\tilde{J}) : CF(L_0,L_1;J_0) \to CF(L_0',L_1;J_1)$$

by the following equation. Here, $p\in\Theta$ means that p runs over all Floer generators appearing in the given cocycle Θ.

$$G(x) = \sum_{y\in L_0'\cap L_1}\sum_{p\in\Theta}\sum_{\phi\in\pi_2(x,y,p),\,\mu(\phi)=-\ell}\left|\mathcal{M}(\phi;\tilde{J})\right|\cdot y.$$

From any homotopy coherent diagram $F_0 : \mathcal{C}\to\mathscr{D}$, we can use the map G to construct a homotopy coherent diagram of $\mathcal{C}$ in $\mathrm{Kom}_{\mathbb{F}_2}$. Now, for sufficiently generic $\mathbb{Z}_2$-equivariant homotopy coherent diagrams $F, F' : \mathcal{E}\mathbb{Z}_2\to\overline{\mathcal{J}}$, we can extend $(\{0\}\times F)\cup(\{1\}\times F')$ to a sufficiently generic $\mathbb{Z}_2$-equivariant homotopy coherent diagram

$$G : \mathscr{I}\times\mathcal{E}\mathbb{Z}_2\to\mathscr{D}$$

where $\mathscr{I}$ has objects 0 and 1, and the only non-identity morphism in $\mathscr{I}$ is given by

$\mathrm{Hom}_{\mathscr{I}}(0,1) = \{f_{0,1}\}$. Applying Floer theory gives a homotopy coherent diagram

$$G'' : \mathscr{I} \times \mathcal{E}\mathbb{Z}_2 \to \mathrm{Kom}_{\mathbb{F}_2}.$$

This gives a map hocolim $G'' : \widetilde{CF}_{\mathbb{Z}_2}(L_0, L_1; F) \to \widetilde{CF}_{\mathbb{Z}_2}(L_0', L_1, F')$. Since it is $\mathbb{Z}_2$-equivariant, we get the equivariant triangle map between equivariant Floer cochain complexes:

$$F_\Theta : CF_{\mathbb{Z}_2}(L_0, L_1; F) \to CF_{\mathbb{Z}_2}(L_0', L_1, F').$$

In the case when L_0 is Hamiltonian isotopic, or A_∞ homotopy equivalent inside the Fukaya category, to L_0', and Θ is the top class in $HF(L_0, L_0')$, we can expect F_Θ to become a quasi-isomorphism. To be precise, the following theorem holds.

Theorem 3.1.7 (Proposition 3.25, [20])**.** *Suppose that L_0, L_0' and L_1 are $\mathbb{Z}_2$-invariant Lagrangians, which are pairwise transverse, and that L_0 is Hamiltonian isotopic to L_0' or, more generally, so that L_0 is A_∞ homotopy equivalent to L_1 in the Fukaya category. Suppose further that:*

(1) *The collection of homotopy classes η' induced by η and the Hamiltonian isotopies is preserved by the $\mathbb{Z}_2$-action. Of course, η is also required to be preserved by the $\mathbb{Z}_2$-action.*

(2) *There is a $\mathbb{Z}_2$-invariant ω-compatible almost complex structure J on M which achieves transversality for all moduli spaces of holomorphic bigons with boundary on (L_0, L_0') of Maslov index ≤ 1.*

(3) *The top class Θ in $HF(L_0, L_0')$ is represented by a $\mathbb{Z}_2$-invariant cycle.*

Then F_Θ is a quasi-isomorphism.

For more self-containedness, we will also present the version of the above theorem in the case of $\mathbb{Z}_2$-equivariant Hamiltonian isotopies.

Theorem 3.1.8 (Proposition 3.24, [20]). *If L_0' and L_1' are isotopic to L_0 and L_1 via compactly supported Hamiltonian isotopies through $\mathbb{Z}_2$-invariant Lagrangians, then there is a quasi-isomorphism*

$$\widetilde{CF}(L_0, L_1) \simeq \widetilde{CF}(L_0', L_1')$$

over $\mathbb{F}_2[\mathbb{Z}_2]$.

3.2 Equivariant transversality and localization

When a generic $\mathbb{Z}_2$-equivariant 1-parameter family of complex structures achieves transversality for all Whitney disks of Maslov index at most 1, then we have the following (quasi-)isomorphism.

$$CF_{\mathbb{Z}_2}(L_0, L_1) \simeq CF(L_0, L_1) \otimes^L_{\mathbb{F}_2[\mathbb{Z}_2]} \mathbb{F}_2.$$

More explicitly, for a $\mathbb{Z}_2$-equivariant sufficiently generic 1-parameter family of almost complex structures J, we have a quasi-isomorphism

$$CF_{\mathbb{Z}_2}(L_0, L_1) \simeq (0 \to CF^*(L_0, L_1, J) \xrightarrow{1+\tau^*} CF^*(L_0, L_1, J) \xrightarrow{1+\tau^*} \cdots),$$

where τ is the generator of the $\mathbb{Z}_2$-action and τ^* is the induced action on $CF^*(L_0, L_1)$. This is because the higher-order terms in the $\mathbb{Z}_2$-equivariant Floer differential vanish.

Of course, the assumption on the existence of such a generic $\mathbb{Z}_2$-equivariant 1-parameter family J is not automatic, but rather restrictive. However, Hendricks, Lipshitz, and Sarkar pointed out in [20] that the assumption is satisfied in the following two cases.

(EH-1) Let NM^{fix}, NL_0^{fix} and NL_1^{fix} denote the normal bundles to the $\mathbb{Z}_2$-invariant locus M^{fix}, L_0^{fix}, and L_1^{fix} of M, L_0, and L_1, respectively. Suppose that there

is an equivariant trivialization $NM^{fix} \simeq M^{fix} \times \mathbb{C}^k$, where τ acts on $\mathbb{C}^k$ by -1, which sends NL_0^{fix} and NL_1^{fix} to $L_0^{fix} \times \mathbb{R}^k$ and $L_1^{fix} \times i\mathbb{R}^k$, respectively. Then Seidel-Smith observe in [48] that there is a one-parameter family of $\mathbb{Z}_2$-invariant almost complex structures J on M achieving transversality for all moduli spaces of J-holomorphic disks.

(EH-2) Let $i : (M^{fix}, L_0^{fix}, L_1^{fix}) \to (M, L_0, L_1)$ denote the inclusion. Suppose that for any homotopy class $\phi : (D^2, \partial D^2) \to (M^{fix}, L_0 \cup L_1)$ of Whitney disks satisfies $\mu(i_*(\phi)) = 2\mu(\phi)$. Then for any one-parameter family of almost complex structures J_{fix} on M^{fix} which achieves transversality for holomorphic disks in M^{fix}, its generic $\mathbb{Z}_2$-invariant extension J on M achieves transversality for all holomorphic disks not contained entirely in M^{fix}, as pointed out in [27]. But the condition $\mu(i_*(\phi)) = 2\mu(\phi)$ implies that there are no nontrivial holomorphic curves in M^{fix} with Maslov index ≤ 1.

Furthermore, when the condition (EH-1) is satisfied, then Seidel and Smith also proved in [48] that fixing a homotopy class of a choice of stable normal trivializations gives a localization map

$$HF_{\mathbb{Z}_2}(L_0, L_1) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}] \xrightarrow{\sim} HF(L_0, L_1) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}],$$

which is an isomorphism determined up to multiplication by powers of θ. This fact will play a crucial role later, when we prove that $\mathbb{Z}_2$-equivariant contact class is not torsion.

3.3 $\mathbb{Z}_2$-Equivariant Heegaard Floer cohomology of based links

Using the equivariant Floer theory package that we have introduced earlier, we can now give a construction of $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology, which is a

cohomology theory of based links in S^3. A based link is a pair (L, p) where L is a link in S^3 and $p \in L$ is a choice of a basepoint. Given a genus 0 Heegaard surface $\Sigma \subset S^3$, a based link (L, p) is in a bridge position with respect to Σ if, for a Heegaard splitting $S^3 = H_a \cup_\Sigma H_b$, the connected arcs $\{a_i\}$ and $\{b_i\}$, $1 \leq i \leq n$, given by

$$\cup a_i = L \cap H_a, \cup b_i = L \cap H_b,$$

satisfy the following conditions.

- There exist disks D_{a_i} and D_{b_j} such that we have $a_i \subset \partial D_{a_i} \subset a_i \cup \Sigma$ and $b_j \subset \partial D_{b_j} \subset b_j \cup \Sigma$.
- The disks D_{a_i} and D_{b_j} can be chosen to have pairwise disjoint interiors.
- $p \in L \cap \Sigma$.

If $\partial D_{a_i} = a_i \cup A_i$ and $\partial D_{B_j} = b_j \cup B_j$ where A_i and B_j are simple arcs on Σ, we say that $(\{A_i\}, \{B_j\})$ is the bridge diagram for the based link (L, p). Given a bridge diagram, by taking the branched double cover of the whole diagram, with the branching locus given by $L \cap \Sigma$, and removing the curves which contain the basepoint p, gives a Heegaard diagram $(\tilde{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, p)$ together with the covering $\mathbb{Z}_2$-action, where the alpha(beta)-curves are given by the inverse images of the arcs $A_i(B_i)$.

The $\mathbb{Z}_2$-equivariant Floer cohomology theory can be applied to the symplectic $\mathbb{Z}_2$-action on the symmectric power $(\mathrm{Sym}^g(\tilde{\Sigma} - \{p\}), \mathbb{T}_\alpha, \mathbb{T}_\beta)$. It turns out that the $\mathbb{F}_2[\theta]$-isomorphism class of the equivariant Floer cohomology

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), p) = HF_{\mathbb{Z}_2}(\mathbb{T}_\alpha, \mathbb{T}_\beta)$$

is an invariant of the isotopy class of (L, p).

The proof of the invariance uses the fact that any two bridge diagram of a based link are related by three types of moves: isotopy, handleslide, and stabilization. An

isotopy of bridge diagram $(\{A_i\}, \{B_j\})$ is an isotopy of each arc A_i and B_j, while fixing their endpoints. A handleslide is a move which replaces A_i(or B_j) by A_i'(or B_j') with the same endpoints, when there exists another arc A_k such that A_i and A_i' bound a disk D which contains A_k, and D does not intersect any other A-arcs(or B-arcs). Finally, a stabilization is a move which adds an A-arc and a B-arc to an arc A_i(or B_j) near one of its endpoint.

3.3.1 Isotopy

An isotopy of A-arcs or B-arcs induces a $\mathbb{Z}_2$-equivariant Hamiltonian isotopy of the α-torus or the β-torus, respectively. Hence, by Theorem 3.1.8, an isotopy of bridge diagrams induces an isomorphism of $\widehat{HF}_{\mathbb{Z}_2}$.

3.3.2 Handleslide

A handleslide of A-arcs or B-arcs induces a Hamiltonian isotopy of the α-torus or the β-torus. Unfortunately, the Hamiltonian isotopy we get need not be an isotopy through $\mathbb{Z}_2$-invariant Lagrangians. However, the upshot is that it satisfies the conditions of Theorem 3.1.7, so we still have an isomorphism of $\widehat{HF}_{\mathbb{Z}_2}$.

3.3.3 Stabilization

This is the most tricky move; to define a map associated to a stabilization move, we have to consider it as a composition of a saddle cobordism followed by a death cobordism. For now, this can only be defined when the stabilization is performed near the basepoint. However, later we will see that a stabilization map can also be well-defined even when the stabilization is performed in a region which does not contain the basepoint.

Let $\mathcal{H}$ be the branched double cover of the given bridge diagram and let $\mathcal{H}_1$ be the branched double cover of the result of stabilizing the given diagram. Also, let $\mathcal{H}_0$ be

the connected sum of $\mathcal{H}$ and a 1-bridge diagram for the unknot, and $\mathcal{H}'$ be the result of applying a Reidemeister move I move to the pair of arcs in $\mathcal{H}_1$ which are created by the stabilization move. Then $\mathcal{H}_0$ and $\mathcal{H}_1$ are results of applying a zero-resolution and a one-resolution to $\mathcal{H}'$. So we have a skein exact triangle

$$\cdots \to \widehat{HF}(\mathcal{H}') \xrightarrow{f} \widehat{HF}(\mathcal{H}_0) \xrightarrow{g} \widehat{HF}(\mathcal{H}_1) \to \widehat{HF}(\mathcal{H}') \to \cdots .$$

From this exact sequence, we can make the following observations:

- The diagram $\mathcal{H}_0$ represents $\Sigma(L)\sharp(S^3 \times S^1)$; the diagrams $\mathcal{H}, \mathcal{H}'$, and $\mathcal{H}_1$ all represent $\Sigma(L)$.
- There are $\mathbb{Z}_2$-equivariant isomorphisms

  $$\widehat{CF}(\mathcal{H}_0) \to \widehat{CF}(\mathcal{H}) \otimes \widehat{CF}(S^2 \times S^1) \simeq \widehat{CF}(\mathcal{H}) \otimes \mathbb{F}_2 \langle \Theta_{top}, \Theta_{bot} \rangle$$

  where Θ_{top} is the top generator of $\widehat{HF}(S^2 \times S^1)$.
- The map g sends the kernel of the $H_1(S^2 \times S^1)$-action to 0, and $\widehat{HF}(\mathcal{H}) \otimes \mathbb{F}_2 \langle \Theta_{top} \rangle$ isomorphically to $\widehat{HF}(\mathcal{H}_1)$.

We already know that the quasi-isomorphism type of $\widehat{CF}(\Sigma(L))$ over $\mathbb{F}_2$ is an invariant of $\Sigma(L)$. The saddle map $\widehat{CF}(\mathcal{H}_0) \xrightarrow{g} \widehat{CF}(\mathcal{H}_1)$ is given by counting holomorphic triangles, and hence respects the $\mathbb{F}_2[\mathbb{Z}_2]$-module structure. Hence the composition

$$\widehat{CF}(\mathcal{H}) \to \widehat{CF}(\mathcal{H}_0) \xrightarrow{g} \widehat{CF}(\mathcal{H}_1),$$

where the first map sends $\widehat{CF}(\mathcal{H})$ to $\widehat{CF}(\mathcal{H}) \otimes \mathbb{F}_2 \langle \Theta_{top} \rangle$, is a $\mathbb{Z}_2$-equivariant quasi-isomorphism. Therefore we have a stabilization isomorphism $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{H}_1) \xrightarrow{\sim} \widehat{HF}_{\mathbb{Z}_2}(\mathcal{H})$.

3.3.4 Diffeomorphism

Although diffeomorphism moves are not needed to relate any two bridge diagrams of a given link, it is needed when we give a proof that $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology is natural, so we introduce it here for simplicity.

A diffeomorphism of bridge diagrams induces a diffeomorphism of its branched double cover, only up to $\mathbb{Z}_2$-action. However, in the cochain level, the $\mathbb{Z}_2$-action is quotiented out in the construction of $\mathbb{Z}_2$-equivariant Floer cohomology, when we took the Hom complex from the freed Floer complex to $\mathbb{F}_2$, over the ring $\mathbb{F}_2[\mathbb{Z}_2]$. Therefore it induces a well-defined diffeomorphism map between $\widehat{HF}_{\mathbb{Z}_2}$, which is obviously an isomorphism.

3.3.5 Equivariant transversality and localization

It is worth noting here that the $\mathbb{Z}_2$-action on the triple $(\mathrm{Sym}^g(\hat{\Sigma}\backslash\{p\}), \mathbb{T}_\alpha, \mathbb{T}_\beta)$ satisfies both assumptions (EH-1) and (EH-2). Thus we have a quasi-isomorphism

$$\widehat{CF}_{\mathbb{Z}_2}(\Sigma(L), p) \simeq \widehat{CF}(\Sigma(L), p) \otimes^L_{\mathbb{F}_2[\mathbb{Z}_2]} \mathbb{F}_2,$$

as well as a localization isomorphism

$$\widehat{CF}_{\mathbb{Z}_2}(\Sigma(L), p) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}] \simeq (\mathbb{F}_2^2)^{\otimes |L|-1} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

3.4 Weak admissibility

Recall that, to define the hat-version of Heegaard Floer homology and its cobordism maps, we need to assure that all diagrams involved satisfy weak admissibility. In particular, to define the Floer chain complex, we need weak admissibility for Heegaard diagrams. We also need admissibilities for higher polygons; the triangle maps need weakly admissible triple-diagrams and the proof of its associativity needs weakly

admissible quadruple-diagrams. Counting of higher polygons is not needed.

In this section, we will prove that all Heegaard (double, triple, quadruple)-diagrms that we will use in this paper will be weakly admissible, thus allowing us to freely use all hat-flavored aspects of Heegaard Floer theory. The Heegaard diagrams that we will use are involutive ones, which we will define as follows.

Definition 3.4.1. A Heegaard (double-)diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is involutive (with respect to an orientation-preserving involution τ) if τ fixes z and the alpha- and beta-curves setwise and $\tau|_{\alpha_i}$, $\tau|_{\beta_j}$ is orientation-reversing with two distinct fixed points.

A Heegaard triple-diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ is involutive with respect to τ if it is a small perturbation of a diagram $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, \boldsymbol{\gamma}_0, z)$ such that the tuples $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, z)$, $(\Sigma, \boldsymbol{\beta}_0, \boldsymbol{\gamma}_0, z)$, $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\gamma}_0, z)$ are involutive with respect to τ.

A Heegaard quadruple-diagram is involutive if it is a small perturbation of a diagram such that the triple-diagrams given by choosing any three of the curve systems among the given four are involutive with respect to τ.

For simplicity, we usually do not specify the action of an involution τ in figures, unless necessary. An example of possible local pictures of Heegaard triple-diagrams and quadruple-diagrams around the intersection points $\boldsymbol{\alpha}_0 \cap \boldsymbol{\beta}_0 \cap \boldsymbol{\gamma}_0 (\cap \boldsymbol{\delta}_0)$ are drawn in Figure 3.1. Note that any diagram given by a small perturbation can be obtained by permuting the labels of $\alpha, \beta, \cdots$ in the figures.

Lemma 3.4.2 (Lemma 4.2 of [25])**.** *Every involutive Heegaard diagram is weakly admissible.*

Proof. Suppose that a nontrivial periodic domain D with nonnegative coefficients is given. If we denote the involution by τ, the domain $\tilde{D} = D + \tau_* D$ is nontrivial, periodic, has nonnegative coefficients and is τ-invariant. Thus, in a neighborhood of any τ-invariant point $p \in \alpha_i \cap \beta_j$, the domain $\tilde{D}$ should have coefficients as described in Figure 3.2.

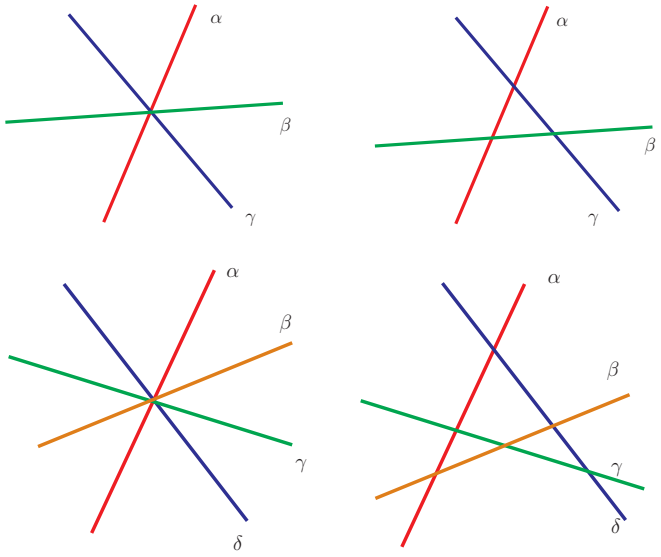


Figure 3.1: Possible local pictures of nice Heegaard diagrams. A choice of a perturbation changes the figures on the left to the figures of nice diagrams, shown on the right.

By periodicity, $2a = 2b$, i.e. $a = b$. But then, near an adjacent intersection point $q \in \alpha_i \cap \beta_k$, the domain $\tilde{D}$ is given as in Figure 3.3. By periodicity, $a + c = a + d$, i.e. $c = d$. Continuing in this manner, we see that if two components of $\Sigma - \cup\alpha_i - \cup\beta_j$ share a segment of a beta-curve, then the coefficients of $\tilde{D}$ for those components are the same. But by the same argument, we can prove the same for components sharing a segment of a alpha-curve. This implies that all coefficients of $\tilde{D}$ should be the same. Since $n_z(\tilde{D}) = 0$, we deduce that $\tilde{D} = 0$. Contradiction. □

Lemma 3.4.3 (Lemma 4.3 of [25])**.** *Every involutive Heegaard triple-diagram* $\mathcal{D} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ *is weakly admissible.*

Proof. Suppose that a nontrivial triply-periodic domain D with nonnegative coefficients in $\mathcal{D}$ is given. If we denote the involution by τ, $\tau_* D$ would be well-defined away from the triple intersections. A typical local picture near a triple intersection point is drawn in Figure 3.4.

We denote the coefficients by a, b, c, d, e, f, g as in Figure 3.4, and claim that

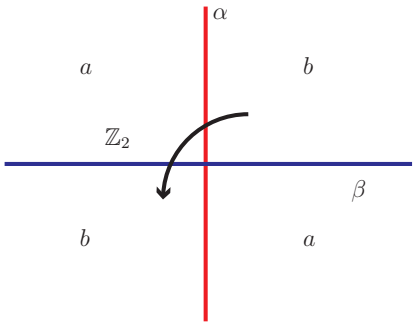


Figure 3.2: The periodic domain $\tilde{D}$ near an invariant intersection point.

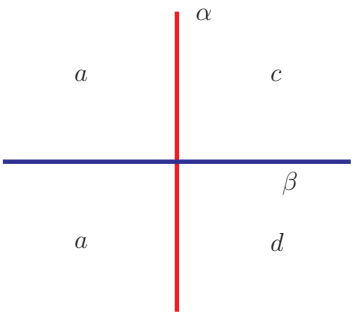


Figure 3.3: The periodic domain $\tilde{D}$ near a non-invariant intersection point.

there exists an integer g' which makes the domain described in Figure 3.5 achieve periodicity. Define g' as $g' = a + c - b$. Then we have

$$c + e - d = b + g' - a + e - d = g' + g - f + f - g = g',$$

and similarly $a + e - f = g'$, so our choice of g' makes the domain periodic; denote the resulting periodic domain by $\tau_* D$. Then $\tau_* D$ may not have nonnegative coefficients, since we do not know whether $g' \geq 0$ holds. However, we have

$$\begin{aligned} g + g' &= g + a + c - b \\ &= g + f + b - g + d + b - g - b \\ &= f + d + b - g \\ &= e + b \geq 0. \end{aligned}$$

Hence $\tilde{D} = D + \tau_* D$ is a periodic domain with nonnegative coefficients. Also, since $a, \cdots, f = 0$ implies $g = 0$, we see that $\tilde{D} \neq 0$. Now, the local picture of $\tilde{D}$ near triple intersections is given as in Figure 3.6. By periodicity, we get

$$\begin{aligned} \tilde{a} + \tilde{b} &= \tilde{c} + \tilde{d}, \\ \tilde{a} + \tilde{c} &= \tilde{b} + \tilde{d}, \\ \tilde{a} + \tilde{d} &= \tilde{b} + \tilde{c}. \end{aligned}$$

Thus $\tilde{a} = \tilde{b} = \tilde{c} = \tilde{d}$. Now, by the argument used to prove the previous lemma, we see that all coefficients of $\tilde{D}$ are the same. Since $n_z(\tilde{D}) = 0$, we get $\tilde{D} = 0$, which is a contradiction. □

Lemma 3.4.4 (Lemma 4.4 of [25]). *Every involutive Heegaard quadruple-diagram is weakly admissible.*

Proof. We continue to use the above approach and start from Figure 3.7 for a non-

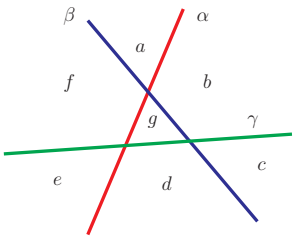


Figure 3.4: The periodic domain D near a non-invariant intersection point.

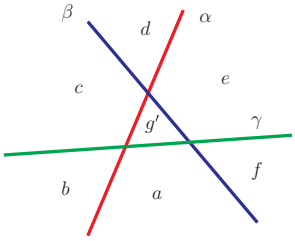


Figure 3.5: The periodic domain $\tau_* D$ near a non-invariant intersection point.

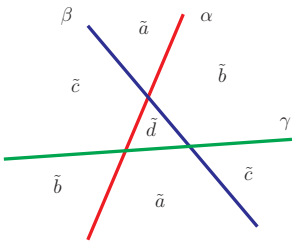


Figure 3.6: The periodic domain $\tilde{D}$ near a non-invariant intersection point.

trivial quadruply-periodic domain D with nonnegative coefficients. Suppose that we are given a, b, c, d, e, f, g, h and try to find x, y, z so that the resulting domain becomes periodic. If such x, y, z exists, then we must have

$$y = b + h - f,$$
$$z = b + g - d,$$
$$x = c + e - a.$$

The remaining three equations are

$$c + y - h - x = 0,$$
$$x + g - e - z = 0,$$
$$b + x - y - z = 0.$$

Using the first three set of equations, we get the followng.

$$\begin{aligned} 0 = c + y - h - x &= c + b + h - f - h - c - e + a \\ &= b - f - e + a, \\ 0 = x + g - e - z &= c + e - a + g - e - b - g + d \\ &= c + d - a - b, \\ 0 = b + x - y - z &= b + c + e - a - b - h + f - b - g + d \\ &= c + d + e + f - a - b - g - h. \end{aligned}$$

Hence we get

$$a + b = c + d = e + f = g + h,$$

which we will call as the filling condition. Now, if a given a, b, c, d, e, f, g, h satisifes the filling condition, we can reverse the above argument to deduce that there exists

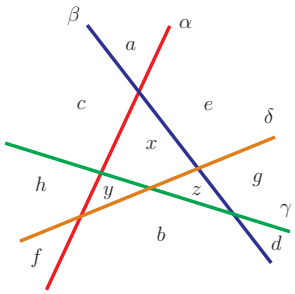


Figure 3.7: The periodic domain D near a non-invariant intersection point.

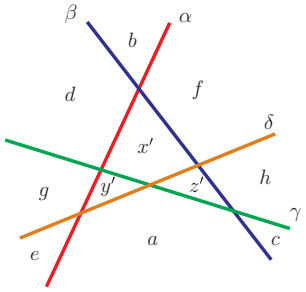


Figure 3.8: The periodic domain D near a non-invariant intersection point.

a unique choice of x, y, z which makes the resulting domain periodic.

Here we notice that the filling condition is invariant respect to the change

$$a \leftrightarrow b, c \leftrightarrow d, e \leftrightarrow f, g \leftrightarrow h.$$

This implies that there exists a unique choice of integers x', y', z' making the domain described in Figure 3.8 periodic, where coefficients for all other domains are transformed by the involution τ, whch makes the given quadruple-diagram nice. Denote the resulting domain by $\tau_* D$. Then $\tilde{D} = D + \tau_* D$ has the local picture as in Figure 3.9, near any τ-invariant intersection point, by the filling condition. Note that we obviously have $\tilde{D} \neq 0$. Now, by the argument used for Heegaard (double) diagrams, we deduce that all coefficients of $\tilde{D}$ are the same. Since $n_z(\tilde{D}) = 0$, we must have $\tilde{D} = 0$, which is a contradiction. □

By the above results, we see that we can now freely talk about counting holo-

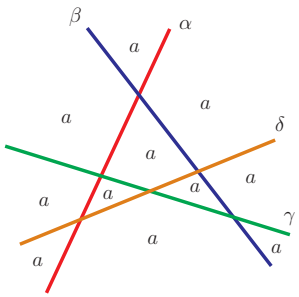


Figure 3.9: The periodic domain D near a non-invariant intersection point.

morphic disks, triangles, and squares in involutive diagrams. Note that, as shown in the previous sections, counting triangles of Maslov indices at most zero gives a map between equivariant Floer (co)homologies.

3.5 Some Floer-theoretic lemmas

In this section, we give three Floer-theoretic lemmas which will be used to prove the naturality and functoriality of $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology. The first two of them are about compatibility of equivariant isotopy maps with equivariant isotopy maps or triangle maps.

Lemma 3.5.1 (Lemma 6.2 of [25]). *Let $(S, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $(S, \boldsymbol{\beta}, \boldsymbol{\gamma})$ be $\mathbb{Z}_2$-Heegaard diagrams given by taking branched double covers of bridge diagrams of links drawn on a sphere Σ, and suppose that $\{\boldsymbol{\alpha}_t\}_{t\in[0,1]}$ is a $\mathbb{Z}_2$-invariant isotopy so that $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}$. Suppose further that $\boldsymbol{\alpha}_t, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are pairwise transverse for $t = 0$ and $t = 1$. Then, for each $\mathbb{Z}_2$-invariant cycle $\theta_{\beta,\gamma} \in \widehat{CF}(S, \boldsymbol{\beta}, \boldsymbol{\gamma})$ and each element $x_{\alpha_0,\beta}$ in $H_*(RHom_{\mathbb{F}_2[\mathbb{Z}_2]}(\widetilde{CF}_{\mathbb{Z}_2}(S, \boldsymbol{\alpha}_0, \boldsymbol{\beta}), \mathbb{F}_2))$, we have*

$$\Gamma_{\{\boldsymbol{\alpha}_t\},\gamma}(\hat{f}_{\alpha_0,\beta,\gamma}(x_{\alpha_0,\beta} \otimes \theta_{\beta,\gamma})) = \hat{f}_{\alpha,\beta,\gamma}(\Gamma_{\{\boldsymbol{\alpha}_t\},\beta}(x_{\alpha_0,\beta}) \otimes \theta_{\beta,\gamma}),$$

where $\hat{f}$ denote the equivariant triangle maps and Γ denote the equivariant isotopy map.

Proof. Recall that we used a topological category $\mathscr{D}$ when constructing equivariant triangle map. Denote the three edges of the triangle $\triangle$ by $e_\alpha, e_\beta, e_\gamma$, and parametrize the edge e_α by a function $E_\alpha : \mathbb{R} \to \triangle$. Given a parametrized family

$$\tilde{J} : [0,1]^\ell \to \mathcal{J}_\triangle,$$

consider the moduli spaces

$$\mathcal{M}_\tau(\tilde{J}) = \bigcup_{t\in[0,1]^\ell} \left\{ u : \triangle \to \mathrm{Sym}^g(\Sigma) \;\middle|\; \begin{array}{c} u \circ e_\alpha(t) \in \mathbb{T}_{\alpha_{t+\tau}},\, u(e_\beta) \subset \mathbb{T}_\beta,\, u(e_\gamma) \subset \mathbb{T}_\gamma \\ u \text{ is } J(t) - \text{holomorphic} \end{array} \right\},$$
$$\mathcal{M}(\tilde{J}) = \bigcup_{\tau\in\mathbb{R}} \mathcal{M}_\tau(\tilde{J}),$$

and split them into homotopy classes $\phi \in \pi_2^{\{\mathbb{T}_{\alpha_t}\},\mathbb{T}_\beta,\mathbb{T}_\gamma}(\mathbf{x},\mathbf{y},\mathbf{z})$ for $\mathbf{x} \in \mathbb{T}_{\alpha_0} \cap \mathbb{T}_\beta$, $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, and $\mathbf{z} \in \mathbb{T}_\gamma \cap \mathbb{T}_\alpha$, as follows.

$$\mathcal{M}_\tau(\tilde{J}) = \bigcup_\phi \mathcal{M}_\tau(\phi;\tilde{J}),\ \mathcal{M}(\tilde{J}) = \bigcup_\phi \mathcal{M}(\phi;\tilde{J})$$

Then we define the map $G(\tilde{J}) : \widehat{CF}(S,\boldsymbol{\alpha}_0,\boldsymbol{\beta}) \to \widehat{CF}(S,\boldsymbol{\alpha},\boldsymbol{\gamma})$ as follows:

$$G(\tilde{J})(\mathbf{x}) = \sum_{\mathbf{z}\in\mathbb{T}_\gamma\cap\mathbb{T}_\alpha} \sum_{\mathbf{y}\in\theta_{\beta,\gamma}} \sum_{\phi\in\pi_2^{\{\mathbb{T}_{\alpha_t}\},\mathbb{T}_\beta,\mathbb{T}_\gamma}(\mathbf{x},\mathbf{y},\mathbf{z}),\, \mu(\phi)=-1-\ell} \left|\mathcal{M}(\phi;\tilde{J})\right| \cdot \mathbf{z}.$$

As in the proof of Proposition 3.25 of [20], the function G induces a map

$$F_G : \widetilde{CF}_{\mathbb{Z}_2}(S,\boldsymbol{\alpha}_0,\boldsymbol{\beta}) \to \widetilde{CF}_{\mathbb{Z}_2}(S,\boldsymbol{\alpha},\boldsymbol{\gamma}).$$

There are three types of ends in the moduli space $\mathcal{M}(\phi;\tilde{J})$ when $\mu(\phi) = -\ell$. The first type is the degeneration of the almost complex structure to the boundary $\tilde{J}|_{\partial[0,1]^\ell}$, which does not contribute to the total count of ends; since we are using $\mathbb{Z}_2$-equivariant diagrams of almost complex structures, the count of such ends must be

even and hence zero in $\mathbb{F}_2$. The second and third types are the ones in the proof of Proposition 8.14 in [37], which contribute to

$$\Gamma_{\{\boldsymbol{\alpha}_t\},\gamma}(\hat{f}_{\alpha_0,\beta,\gamma}(\mathbf{x}\otimes\theta_{\beta,\gamma}))+\hat{f}_{\alpha,\beta,\gamma}(\Gamma_{\{\boldsymbol{\alpha}_t\},\beta}(\mathbf{x})\otimes\theta_{\beta,\gamma})$$

and $\partial F_G(\mathbf{x})+F_G(\partial\mathbf{x})$, respectively. Therefore we deduce that

$$\Gamma_{\{\boldsymbol{\alpha}_t\},\gamma}(\hat{f}_{\alpha_0,\beta,\gamma}(x_{\alpha_0,\beta}\otimes\theta_{\beta,\gamma}))+\hat{f}_{\alpha,\beta,\gamma}(\Gamma_{\{\boldsymbol{\alpha}_t\},\beta}(x_{\alpha_0,\beta})\otimes\theta_{\beta,\gamma})=\partial F_G(x_{\alpha_0,\beta})+F_G(\partial x_{\alpha_0,\beta}).$$

Since $\theta_{\beta,\gamma}$ is $\mathbb{Z}_2$-invariant, the map F_G is also $\mathbb{Z}_2$-invariant. Therefore we get the desired result. □

Lemma 3.5.2 (Lemma 6.3 of [25])**.** *Let $(S,\boldsymbol{\alpha},\boldsymbol{\beta})$ be a $\mathbb{Z}_2$-Heegaard diagram given by taking branched double cover of a bridge diagram of a link drawn on a sphere Σ, and suppose that $\{\boldsymbol{\alpha}_t\}_{t\in[0,1]}$ is a $\mathbb{Z}_2$-invariant isotopy so that $\boldsymbol{\alpha}_1=\boldsymbol{\alpha}$. Suppose further that $\boldsymbol{\alpha}_t,\boldsymbol{\beta},\boldsymbol{\gamma}$ are pairwise transverse for $t=0,\frac{1}{2},1$. Then, for each element $x_{\alpha_0,\beta}\in H_*(\widetilde{CF}_{\mathbb{Z}_2}(S,\boldsymbol{\alpha}_0,\boldsymbol{\beta})\otimes_{\mathbb{F}_2[\mathbb{Z}_2]}\mathbb{F}_2)$, we have*

$$\Gamma_{\{\boldsymbol{\alpha}_t\}_{t\in\left[\frac{1}{2},1\right]},\beta}(\Gamma_{\{\boldsymbol{\alpha}_t\}_{t\in\left[0,\frac{1}{2}\right]},\beta}(x_{\alpha_0,\beta}))=\Gamma_{\{\boldsymbol{\alpha}_t\}_{t\in[0,1]},\beta}(x_{\alpha_0,\beta}),$$

where Γ denotes the equivariant isotopy map.

Proof. As in the proof of 3.5.1, we can mimic the proof of Theorem 7.3 of [37] to make it work in the equivariant setting. □

The third lemma gives a refinement of equivariant transversality, which will be used a lot throughout the thesis. Here, the setting is that we are given a bridge diagram of a based link, drawn on S^2. So, Σ is the Heegaard surface of the Heegaard diagram H we get by taking branched double cover, and we are considering equivariant transversality for the $\mathbb{Z}_2$-action on H.

Lemma 3.5.3 (Theorem 5.1 of [25]). *For a generic 1-parameter family of almost complex structures* $\mathfrak{j}$ *on* S^2*, the* $\mathbb{Z}_2$*-equivariant cylindrical complex structure* $Sym^g(\tilde{\mathfrak{j}})$ *on* $Sym^g(\Sigma)$*, where* g *is the genus of* Σ*, achieves equivariant transversality.*

Proof. Since the $\mathbb{Z}_2$-invariant locus $(\mathbb{T}_\alpha \cap \mathbb{T}_\beta)^{inv}$ consists of 0-dimensional components of $(\mathrm{Sym}^g(\Sigma))^{inv}$, for any choice of an almost complex structure J on $\mathrm{Sym}^g(\Sigma)$, any J-holomorphic disk connecting two points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is not completely contained in $(\mathrm{Sym}^g(\Sigma))^{inv}$. Thus, as in the proof of Proposition 5.13 in [27], the argument used in the proof of Corollary 1.12 in [20] actually gives transversality for all homotopy classes of Whitney disks in this case. □

3.6 Naturality

In this section, we will prove that, when K is a knot and p is a basepoint in K, the equivariant Floer cohomology $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), p)$ is well-defined up to natural isomorphism, so that it admits a natural action of the mapping class group $\mathrm{MCG}(S^3, K) = \pi_0 \mathrm{Diff}^+(S^3, K)$. Recall that any two bridge diagrams of a given based link are related by isotopies, handleslides, and (de)stabilizations, applied to arcs which do not contain the basepoint; for simplicity, we make the following definition.

Definition 3.6.1. Let $\{A_i\}, \{B_i\}$ denote the A- and B-arcs of a bridge diagram of a based link (L, p). A basic move on L is an isotopy, a handleslide, or a (de)stabilization applied to either a single A-arc or a single B-arc, which does not contain p. An A(B)-equivalence is an isotopy or a handleslide applied to a single A(B)-arc which does not contain p.

3.6.1 Commutative diagrams of basic moves

We can easily point out some naturally arising commutative triangles, squares, and hexagons, consisting of basic moves. Those diagrams are described below. Please

note that, by an A(B)-arc, we mean an A(B)-arc of a given bridge diagram of a given based link, which does not contain the basepoint.

(a) A-equivalences and B-equivalences commute with each other

Given a bridge diagram $D = (\{A_i\}, \{B_i\})$ of a based link (L, p), we can consider applying an A-equivalence on A_i and a B-equivalence on B_j. Suppose that applying an A-equivalence on A_i of D transforms it into D_a and applying a B-equivalence on B_j of D transforms it into D_b. Denote the result of applying both equivalences on D gives D_{ab}. Then we have a following dintinguished square.

$$\begin{array}{ccc} D & \xrightarrow[B]{} & D_b \\ {\scriptstyle A}\downarrow & & {\scriptstyle A}\downarrow \\ D_a & \xrightarrow[B]{} & D_{ab} \end{array}$$

(b) Commutative triangles of A(B)-equivalences

Suppose that applying an A-equivalence on a bridge diagram D gives D_1 , applying another A-equivalence on D_1 gives D_2, and there exists an A-equivalence which transforms D into D_2. Suppose further that, if two of the three A-equivalences are handleslides, then they are handleslides along the same A-arc. Then we get a distinguished triangle. The same thing also holds for B-equivalences.

$$\begin{array}{ccc} D & \xrightarrow[A]{} & D_1 \\ & {\scriptstyle A}\searrow & {\scriptstyle A}\downarrow \\ & & D_2 \end{array}$$

(c) Handleslides on different arcs commute

Suppose that applying a handleslide on an A-arc A_i of a bridge diagram D gives D_i, applying a handleslide on an A-arc A_j gives D_j, and applying both on D gives D_{ij}. If $i \neq j$, then we have a distinguished square. The same thing also holds for

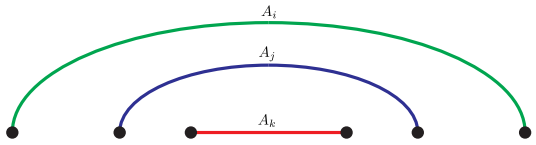


Figure 3.10: Three A-arcs lying close to each other

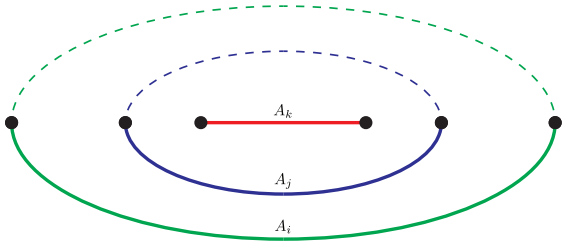


Figure 3.11: Three A-arcs, after three handleslides

B-equivalences.

$$\begin{array}{ccc} D & \xrightarrow[A]{} & D_i \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A} \\ D_j & \xrightarrow[A]{} & D_{ij} \end{array}$$

(d) Commutative hexagon of handleslides

Suppose that there are three A-arcs A_i, A_j, A_k lying close to each other in a bridge diagram D, as in Figure 3.10. Then we have two choices when handlesliding A_i and A_j over A_k to reach Figure 3.11; we can either move A_j over A_k first and then move A_i over A_k and A_j, or move A_i over A_j and A_k first and then move A_j over A_k.

This gives a distinguished hexagon, and the same thing holds for B-arcs. Here, A_i/A_j denotes the handleslide of A_i over A_j.

$$\begin{array}{ccccc} & & D_1 & \xrightarrow[A_i/A_k]{} & D_2 & & \\ & \nearrow^{A_j/A_k} & & & & \searrow^{A_i/A_j} & \\ D & & & & & & D_3 \\ & \searrow_{A_i/A_j} & & & & \nearrow_{A_j/A_k} & \\ & & D_1' & \xrightarrow[A_i/A_k]{} & D_2' & & \end{array}$$

(e) A(B)-equivalences commute with stabilizations

Suppose that applying an A(B)-equivalence on an A(B)-arc $A_i(B_j)$ in the bridge diagram D gives D_1, and applying a stabilization on an arc of D, which does not contain the basepoint and is different from A_i, gives D^s, and applying a stabilization on the corresponding arc of D_1 gives D_1^s. Then D_1^s can be obtained from D^s by an A(B)-equivalence, and so we get a distinguished square.

$$\begin{array}{ccc} D & \xrightarrow[A]{} & D_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ D^s & \xrightarrow[A]{} & D_1^s \end{array}$$

(f) Stabilizations applied on different arcs commute

Consider applying stabilization on two different arcs of a bridge diagram D. There are two possible orders, which give us a distinguished square.

$$\begin{array}{ccc} D & \xrightarrow[s]{} & D_2 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ D_1 & \xrightarrow[s]{} & D_3 \end{array}$$

(g) Commutative triangle of two stabilizations and an isotopy

Given a point $x \in \partial A_i \cap \partial B_j$ of a bridge diagram $D = (\{A_i\}, \{B_j\})$, suppose that applying a stabilization on A_i near x gives D_1 and applying a stabilization on B_j near x gives D_2. Then D_1 and D_2 differ by an isotopy. So we get a distinguished triangle.

$$\begin{array}{ccc} D & \xrightarrow[s]{} & D_2 \\ {\scriptstyle s}\downarrow & \nearrow_{\text{isotopy}} & \\ D_1 & & \end{array}$$

Given a bridge diagram D of a based link (L, p), whose set of endpoints of arcs is given by $S \subset \Sigma$, we can also define maps on the equivariant Floer cohomology,

associated to diffeomorphisms $\phi \in \mathrm{Diff}^+(\Sigma)$ which fix p and S pointwise in a natural way. Then we get few more types of distinguished diagrams.

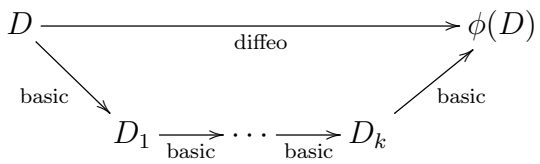
(h) Diffeomorphism and basic moves commute

$$\begin{array}{ccc} D & \xrightarrow[\text{diffeo}]{} & \phi(D) \\ {\scriptstyle\text{basic}}\downarrow & & \downarrow{\scriptstyle\text{basic}} \\ D_1 & \xrightarrow[\text{diffeo}]{} & \phi(D') \end{array}$$

Here, we use diffeomorphisms $\phi \in \mathrm{Diff}^+(\Sigma, S, p)$.

(i) Diffeomorphism can be undone by basic moves

Suppose that a diffeomorphism $\phi \in \mathrm{Diff}^+(\Sigma, S, p)$ maps a bridge diagram D of (L, p) to $\phi(D)$. Since they represent the same based link, we can obtain $\phi(D)$ from D by a sequence of basic moves. So we get a distinguished diagram.



3.6.2 Translating diagrams of basic moves to diagrams of $\widehat{HF}_{\mathbb{Z}_2}$

Given the list of commutative diagrams consisting of basic moves, the next thing to do is to prove that applying $\widehat{HF}_{\mathbb{Z}_2}$ to its vertices and edges preserves commutativity. Among the basic moves, the only one which does not yet have a well-defined associated isomorphism between $\widehat{HF}_{\mathbb{Z}_2}$ is the stabilization move performed in a region not containing the basepoint. For such a move, we can still define the associated stabilization map by by taking a family of almost complex structures of type $\mathrm{Sym}^g(\mathfrak{j})$ as in Lemma 3.5.3, where the 1-parameter family $\mathfrak{j}$ is split and has long neck near the point at which a stabilization is performed. The question is whether such a map is indeed an isomorphism; this question will be answered right after proving that

applying $\widehat{HF}_{\mathbb{Z}_2}$ to the commutative diagrams presented in the previous subsection preserves commutativity.

3.6.2.1 Diagrams of type (c) (and similarly for (a),(f),(h))

Note that, by the three lemmas presented in the previous section. we only have to prove that the corresponding commutative diagrams of $\widehat{CF}$ groups are satisfied up to $\mathbb{Z}_2$-equivariant chain homotopies; this principle also applied to the other types of diagrams.

For the distinguished diagrams of type (c), we already know that it is satisfied on the $\widehat{CF}$ level, up to a chain homotopy. The chain homotopy is given by counting holomorphic squares, after a perturbation as in Figure 4 of [20]. Here, we can always perturb a given bridge diagram by an isotopy using Lemma 3.5.1. After making such a perturbation, we have no constant triangle of negative Maslov index, and thus the equivariant triangle map is induced by the ordinary triangle map with respect to generic 1-parameter families of almost complex structures. Since holomorphic squares contained in the $\mathbb{Z}_2$-fixed locus are constant squares, and such squares have Maslov index 0, they are not counted in the square map. This implies, by the arguments of the proof of Proposition 5.13 in [27], that a generic $\mathbb{Z}_2$-invariant 1-parameter family of almost complex structures achieves transversality for squares of Maslov index -1, which tells us that the holomorphic square map is well-defined for generic $\mathbb{Z}_2$-invariant families. Hence the given chain homotopy is $\mathbb{Z}_2$-equivariant, i.e. the given square diagram commutes up to $\mathbb{Z}_2$-equivariant chain homotopy on the $\widehat{CF}$ level. Therefore we get a commuting square diagram of corresponding $\widehat{HF}_{\mathbb{Z}_2}$ groups. The same argument can be used to prove the commutativity for distinguished squares of type (a), (f), and (h).

3.6.2.2 Diagrams of type (e)

By Theorem 2.14 of [39], we can follow the proof of Lemma 2.15 in [39] to show that distinguished squares of type (e) also commute.

3.6.2.3 Well-definedness of general stabilization map

Suppose that a bridge diagram $\mathcal{H}_s$ is obtained by performing a stabilization to another diagram $\mathcal{H}$, where we do not require the stabilization to be performed near the basepoint. Then, using the commutativity of diagrams of type (e), we can find a bridge diagram $\mathcal{H}'$, related to $\mathcal{H}$ via A-equivalences and B-equivalences and another bridge diagram $\mathcal{H}'_s$, related to $\mathcal{H}_s$ via same types of A-equivalences and B-equivalences, so that $\mathcal{H}'_s$ is the result of performing a stabilization on $\mathcal{H}'$ near its basepoint, and that we have a commutative diagram of the following form. Notice that, for simplicity, we are mixing between bridge diagrams and their branched double covers.

$$\begin{array}{ccc} \widehat{HF}_{\mathbb{Z}_2}(\mathcal{H}_s) & \longrightarrow & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{H}'_s) \\ \downarrow & & \downarrow \\ \widehat{HF}_{\mathbb{Z}_2}(\mathcal{H}) & \longrightarrow & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{H}') \end{array}$$

Here, we know that all arrows except the left one are isomorphisms, so the left arrow must also be an isomorphism. Therefore the general stabilization map

$$\widehat{HF}_{\mathbb{Z}_2}(\mathcal{H}_s) \to \widehat{HF}_{\mathbb{Z}_2}(\mathcal{H})$$

is an isomorphism. This finishes the work of associating isomorphisms to each basic move.

3.6.2.4 Diagrams of type (b) (and similarly for (d),(g))

For the distinguished diagrams of type (b), the A-equivalence from D to D_1 is given by evaluating the triangle map using a $\mathbb{Z}_2$-invariant cocycle Θ_{D,D_1} which represent

the top class, and similarly consider $\mathbb{Z}_2$-invariant cocycles Θ_{D,D_2} and Θ_{D_1,D_2}. By the technique used to prove the commuativity of distinguished diagrams of type (c), it suffices to prove that the image of $\Theta_{D,D_1} \otimes \Theta_{D_1,D_2}$ under the triangle map is the same as the cocycle Θ_{D,D_2}. Since the image of $\Theta_{D,D_1} \otimes \Theta_{D_1,D_2}$ must also be a $\mathbb{Z}_2$-invariant cocycle which represents the top class, the proof will be finished if Θ_{D,D_2} is the only $\mathbb{Z}_2$-invariant cocycle which represents the top class. Since the Heegaard diagram for an isotopy or a saddle obviously admits a unique representative of its top class, we are done. The same argument can be used to prove the commutativity of distinguished squares of type (d) and (g).

3.6.2.5 Diagrams of type (i)

This is the trickiest case; we have to prove that diffeomorphisms isotopic to the identity induces the identity map of $\widehat{HF}_{\mathbb{Z}_2}$ in order to give prove commutativity for diagrams of type (i).

Lemma 3.6.2 (Lemma 6.5 of [25]). *Given a bridge diagram $(\{A_i\}, \{B_i\})$ of a based link (L, p) in S^3, drawn on a sphere $\Sigma = S^2$, let $\{p_1, \cdots, p_n\}$ be the set of endpoints of arcs A_i and B_i, which are not equal to the basepoint p. Given a 1-parameter family of self-diffeomorphisms $\{\phi_t\}_{t\in[0,1]}$ of Σ, such that $\phi_0 = id_\Sigma$, each ϕ_t fixes p pointwise and $\{p_1, \cdots, p_n\}$ setwise, and the images of A- and B-curves under ϕ_1 intersect transversely with the original A- and B-curves, consider the following two maps. First, the isotopy map induced by the isotopy $\{\phi_t(A_i)\}$ and $\{\phi_t(B_i)\}$ of A- and B-arcs:*

$$\Gamma : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), p) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), p).$$

Next, the diffeomorphism map

$$\phi^* : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), p) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), p).$$

Then we have $\Gamma = \phi^*$.

Proof. The argument used in the proof of Lemma 9.5 of [23] and Proposition 9.8 of [37] directly generalizes to the equivariant setting. Hence we see that the lemma holds when the given isotopy $\{\phi_t\}$ is sufficiently small. Hence, by Lemma 3.5.2, we deduce that the lemma holds for any isotopy. □

Lemma 3.6.3 (Lemma 6.6 of [25])**.** *Given a bridge diagram $(\{A_i\},\{B_i\})$ of a based link (L,p) in S^3, drawn on a sphere $\Sigma = S^2$, let $\{p_1,\cdots,p_n\}$ be the set of endpoints of arcs A_i and B_i, which are not equal to the basepoint p. Given a diffeomorphism ϕ of Σ which fix p pointwise and $\{p_1,\cdots,p_n\}$ setwise, choose a sequence of basic moves from $(\{A_i\},\{B_i\})$ to $(\{\phi(A_i)\},\{\phi(B_i)\})$, and denote the induced map between $\widehat{HF}_{\mathbb{Z}_2}$ as follows:*

$$T_\phi : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L),p) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L),p).$$

Similarly construct a map $T_{\phi^{-1}}$ by choosing a sequence of basic moves from the bridge diagram $(\{\phi(A_i)\},\{\phi(B_i)\})$ to $(\{A_i\},\{B_i\})$. Then we have $T_\phi \circ T_{\phi^{-1}} = id$.

Proof. By Lemma 3.6.2, we can assume, without losing generality, that the given sequences of basic moves do not contain isotopies. Since the maps induced by A-equivalences and B-equivalences commute, it suffices to prove that the composition of all A-equivalence maps arising in T_ϕ and $T_{\phi^{-1}}$ is the identity, since it will also imply the same thing for B-equivalence maps.

The Heegaard diagram given by taking the branched double cover of $(\Sigma,\{A_i\},\{A_i'\})$ has unique $\mathbb{Z}_2$-invariant cocycle whch represents the top class, where A_i' is a slight perturbation of A_i so that $A_i \cap A_i' = \partial A_i$. Therefore we can use the arguments in the proof of commutativity for other types of diagrams to conclude that the composition of all A-equivalence maps in $T_\phi \circ T_{\phi^{-1}}$ is the identity, and so $T_\phi \circ T_{\phi^{-1}} = \mathrm{id}$. □

With the above lemmas, we can now prove that diagrams of type (i) remain commutative after applying $\widehat{HF}_{\mathbb{Z}_2}$.

Proposition 3.6.4. *The equivariant Floer cohomology of based knots in S^3 satisfies the commutative diagrams of type (i)*

Proof. For any positive integer n, the pure mapping class group of a disk with n punctures is given by the pure braid group on n strands, and taking its quotient by the Dehn twists along the boundary gives the mapping class group of a sphere with $n+1$ punctures. Thus, if we consider the standard generating set $\{T_i\}$ of the pure braid group B_n, where T_i denotes a positive twist of the ith and the $i+1$th strand, the set $\{T_i^2\}$ normally generates the pure braid group PB_n, and thus generates the group $\mathrm{PMod}(S_{0,n+1})$. Hence, by Lemma 3.6.3, given a bridge diagram of a based knot (K,p) on a sphere Σ, we only have to prove that the commutativity diagrams of type (i) holds for $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K),p)$ only for the full Dehn twists along an A-arc or a B-arc, which does not contain p, and any choice of a sequence of basic moves.

Choose such an A-arc A_i. The bridge diagram given by applying a Dehn twist along A_i is drawn in Figure 3.12. Its branched double cover admits a unique $\mathbb{Z}_2$-invariant cycle which represents the top generator in homology. Thus we can compute the composition of the maps associated to our choice of basic moves by computing the equivariant triangle map for the Heegaard triple-diagram, which is given by taking the branched double cover of the diagram drawn in Figure 3.13. Note that A_i is assumed to not intersect any B-arcs other than the two B-arcs adjacent to it, and the basepoint is placed near the leftmost point in Figure 3.13; this is possible because $\widehat{HF}_{\mathbb{Z}_2}$ satisfies commutative diagrams of type (h).

We claim that the only nonconstant triangles, each of which consists of a green arc, a blue arc, and a constant red arc, and involves at least one of the two endpoints of A_i, are those shown in Figure 3.14. Let T be such a triangle. Then, without loss of generality, we may assume that it uses the blue arc and the green arc connected to the leftmost point in Figure 3.13. If T uses the leftmost red arc instead, then by the assumption on the placement of the basepoint, T must intersect the basepoint, so this case is impossible. Hence T must use the constant red arc at that point. Then, the triangle we get is the one shown in Figure 3.14.

Now, the other triangles are exactly the ones which arise when calculating the

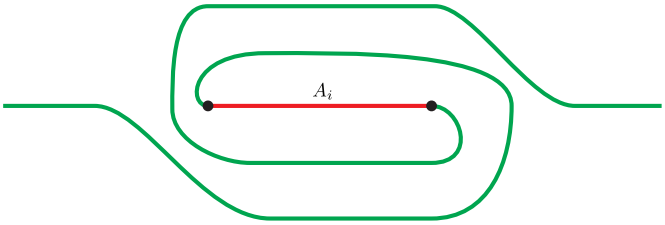


Figure 3.12: The diagram after applying a Dehn twist along (the boundary of a neighborhood of) A_i

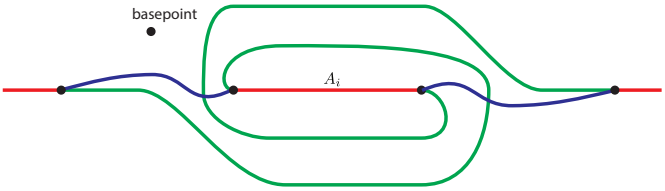


Figure 3.13: The triple-diagram

triangle map for the triple-diagram in Figure 3.15, except for the two shaded regions. Thus, the triangle map for the triple-diagram in Figure 3.13 is the composition of the triangle map for the triple-diagram in Figure 3.15, followed by the diffeomorphism map induced by the Dehn twist along A_i.

However, using the argument of Proposition 9.8 in [37], we see that the triangle map for Figure 3.15 agrees with the diffeomorphism map, induced by a diffeomorphism which is isotopic to the identity. Therefore, we see that the triangle map must give the same result as the diffeomorphism map. □

3.6.3 Finishing the proof

We can now present the proof of naturality of $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology. For a based link (L, p) in S^3 and a genus 0 Heegaard surface $\Sigma \subset S^3$, let $\mathcal{B}_{L,\Sigma,p}$ be the 2-dimensional cell complex defined as follows.

- The 0-cells are bridge diagrams of (L, p) on Σ,
- The 1-cells are basic moves and diffeomorphism maps,

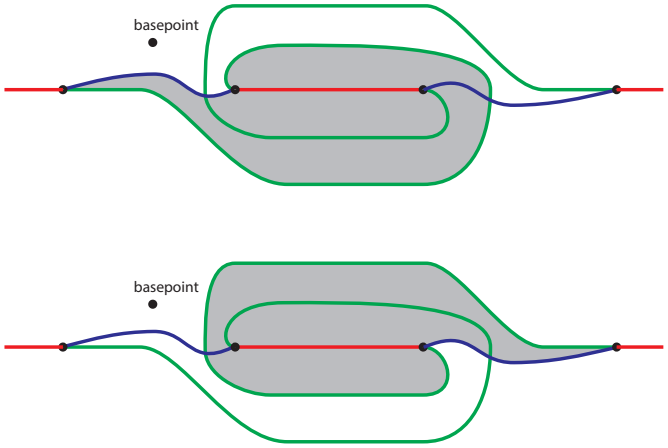


Figure 3.14: The nonconstant triangles in (the branched double cover of) the given triple-diagram



Figure 3.15: The triple-diagram representing a small isotopy of B-arcs

- The 2-cells are commutative diagrams of type (a)-(i).

If we denote the space of parametrized based links in S^3 which are isotopic to (L, p) as $\mathrm{Emb}_L\left(\coprod S^1, S^3\right)$, we have a canonical map

$$\mathcal{R} : \mathcal{B}_{L,\Sigma,p} \to \mathrm{Emb}_L\left(\coprod S^1, S^3\right).$$

Lemma 3.6.5 (Lemma 6.8 of [25])**.** *The map $\mathcal{R}$ is 1-connected.*

Proof. Let x, y be the north and south pole of S^3, and choose a projection function $p_\Sigma : S^3 - \{x, y\} \to \Sigma$, together with a height function $h_\Sigma : S^3 - \{x, y\} \to \mathbb{R}$, so that $p_\Sigma \times h_\Sigma$ is a diffeomorphism, $p_\Sigma|_\Sigma = \mathrm{id}_\Sigma$, and $h_\Sigma|_\Sigma = 0$. A generic point in $\mathrm{Emb}_L\left(\coprod S^1, S^3\right)$ is a based links in S^3, isotopic to (L, p), which do not pass through x nor y, intersects Σ transversely, no two points (x_1, x_2) on the link have the same image under p_Σ if $x_1 \in \Sigma$ or $x_2 \in \Sigma$, and no three points on the link have the same image under p_Σ. Such a link can be canonically isotoped to a bridge position by moving it across Σ so that, for each pair of points (x_1, x_2) on the link which satisfies

$p_\Sigma(x_1) = p_\Sigma(x_2)$ and $h_\Sigma(x_1) < h_\Sigma(x_2)$, we have

$$h_\Sigma(x_1) < 0 < h_\Sigma(x_2),$$

while fixing the points in $L \cap \Sigma$. The codimension 1 points are the links which satisfy one of the following cases.

(1a) The link projects to the cusp $y^2 = x^3$

(1b) Exactly two generic points x_1, x_2 on the link have the same image under p_Σ, and the projected imaged of the segments of the link near x_1 and x_2 are tangent to each other, where the order of tangency is 1

(1c) Exactly three generic points x_1, x_2, x_3 on the link have the same image under p_Σ

(1d) The link is tangent to Σ at a generic point x_Σ, where the order of tangency is 1

The case (1a) corresponds to isotopies, (1b) and (1c) corresponds to handleslides, and (1d) corresponds to stabilizations. Note that a path of generic points, which does not pass through codimension 1 singularities, corresponds to isotopy maps or diffeomorphism maps. The choice of a diffeomorphism map is unique up to distinguished squares of type (i).

Now the codimension 2 points are given as follows.

(2a) Exactly four generic points x_1, x_2, x_3, x_4 on the link have the same image under p_Σ

(2b) Exactly three points x_1, x_2, x_3 on the link have the same image under p_Σ, where the segments of the link near x_1, x_2 are tangent to each other, where the order of tangency is 1

codim 2 pts	(2a)	(2b)	(2c)	(2d)	(2e)	(2f)	(2g)	(2h)	(2i)	(2j)	none
Monodromy	(d)	(b)	(c)	(c)	(a)	(i)	(g)	(e)	(f)	(g)	(b),(h)

Table 3.6.1: Monodromies of codimension 2 points in terms of distinguished diagrams

(2c) Exactly two points x_1, x_2 on the link have the same image under p_Σ, where the segments of the link near x_1, x_2 are tangent to each other, and the order of tangency is 2

(2d) Two codimension 1 states of type (1c) occur at two different points of Σ

(2e) Exactly two points x_1, x_2 on the link have the same image under p_Σ, where the segments of the link near x_1 projects to the cusp $y^2 = x^3$

(2f) The link projects to Σ as the degenerate cusp $y^2 = x^5$

(2g) A codimension 1 state of type (1d) and a state of type (1a) occur at two different points of Σ

(2h) A codimension 1 state of type (1d) and a state of type (1b) or (1c) occur at two different points of Σ

(2i) Two codimension 1 states of type (1d) occur at two different points of Σ

(2j) The link is tangent to Σ, where the order of tangency is 2

The monodromies of codimension 2 points are given in Table 3.6.1. Note that (none) means a monodromy along a boundary of a 2-cell which does not contain codimension 2 points. Therefore, using the triangulation technique of [23], we deduce that the map $\mathcal{R}$ induces isomorphisms of π_0 and π_1. □

Now we stick to the case when $L = K$ is a knot. By the above lemma, the map

$$\mathcal{R} : \mathcal{B}_{K,\Sigma,p} \to \mathrm{Emb}_K(S^1, S^3)$$

is 1-connected. Consider the space $E_{K,p}$ of unparametrized based knots isotopic to (K,p). Since we have a fibration

$$\mathrm{Diff}^+(S^1,p) \to \mathrm{Emb}_K(S^1,S^3) \to E_{K,p}$$

and the group $\mathrm{Diff}^+(S^1,p)$ is contractible, the map $\mathrm{Emb}_K(S^1,S^3) \to E_{K,p}$ is a homotopy equivalence. Hence we have an isomorphism $\pi_1(\mathcal{B}_{K,\Sigma,p}) \simeq \pi_1(E_{K,p})$.

The natural action of the diffeomorphism group on $E_{K,p}$ gives a fibration

$$\mathrm{Diff}^+(S^3,K,p) \to \mathrm{Diff}^+(S^3) \to E_{K,p}.$$

Since $\mathrm{Diff}^+(S^3)$ is path-connected, $\pi_1\mathrm{Diff}^+(S^3) \simeq \mathbb{Z}_2$ with the generator given by rotation, and $\pi_0\mathrm{Diff}^+(S^3,K,p)$ is the mapping class group $MCG(S^3,K,p)$, we get an exact sequence

$$\begin{array}{ccccccc} \mathbb{Z}_2 & \longrightarrow & \pi_1(E_{K,p}) & \longrightarrow & MCG(S^3,K,p) & \longrightarrow & 1. \\ & & \sim\downarrow & & & & \\ & & \pi_1(\mathcal{B}_{K,\Sigma,p}) & & & & \end{array}$$

However, since we can place the genus 0 Heegaard surface of S^3 so that the generator of $\mathbb{Z}_2$ acts on it by a full rotation, and such a rotation induces the identity map of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K),p)$ by Lemma 3.6.2, the monodromy representation

$$\pi_1(\mathcal{B}_{K,\Sigma,p}) \to \mathrm{Aut}_{\mathbb{F}_2[\theta]}\left(\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K),p)\right)$$

factors through $MCG(S^3,K,p)$. Therefore we have a natural action of the mapping class group on the equivariant Floer cohomology $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K),p)$.

Theorem 3.6.6 (Theorem 6.9 of [25])**.** *Let $Knot_*$ be the category whose objects are based oriented knots in S^3 and morphisms are orientation-preserving self-diffeomorphisms*

of S^3 which preserve the knot (as a set) and the basepoint. Then we have a functor

$$\widehat{HF}_{\mathbb{Z}_2} : \mathit{Knot}_* \to \mathit{Vect}_{\mathbb{F}_2},$$

agreeing up to isomorphism with the invariants $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$.

A similar argument shows that the same statement holds for links, when each component has a basepoint. However, since we were working with based links, where only one component has a basepoint, we cannot say that $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), p)$ is natural with respect to the based link (L, p). This causes a slight problem when proving functoriality, so we will only consider whose both ends are knots. In this case, we can "push off" the excessive monodromies coming from absence of basepoints toward an end and then cancel them out. We will deal it in detail in the next section.

3.7 Functoriality

In this section, we will show that $\widehat{HF}_{\mathbb{Z}_2}$ admits cobordism maps for link cobordisms whose both ends are knots, thereby proving its functoriality for such cobordisms. We are dealing with based knots, rather than knots without any extra information, so we need a notion of based knot cobordism. The precise definition is given as follows.

Definition 3.7.1. A based cobordism (S, s) between based links (L_1, p_1) and (L_2, p_2) is an oriented cobordism $S \subset S^3 \times I$ between L_1 and L_2, together with a smooth curve $s : I \to S$ such that $s(0) = p_1$ and $s(1) = p_2$.

3.7.1 Cobordism maps for basic pieces

To construct cobordism maps, it is rather natural to first consider the case when the based cobordism (S, s) is of very simple type, and then compose them to deal with the general case. There are three possible cases of "simple" cobordisms, which we will call as basic pieces. The basic pieces can be defined as the follows.

1. cylindrical pieces

2. birth/death of a component without a basepoint

3. saddle along an arc joining two points on the link

To define a map associated to a based cobordism, the most natural strategy is to chop it into simple pieces. Given a based cobordism (S, s), consider the projection map $p : S^3 \times I \to I$. Then we can isotope the pair (S, s) so that it satisfies the following conditions.

- $p|_S$ is Morse on S,
- $p \circ s$ is regular.
- No two critical points of $p|_S$ have the same value under p.

Once such conditions are satisfied, we can cut S horizontally to reduce it into basic pieces, and thus we can represent (S, s) as a composition of basic pieces. Note that the third condition can be satisfied because, if there are two critical points of p with the same value, then we can perturb p slightly to make them have different values, and if the perturbation is sufficiently C^1-small, then the first and second conditions remain hold.

In [20], it is proved that when S is a basic piece from (L_1, p_1) to (L_2, p_2), there exists a map

$$\hat{f}_S : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_2), p_2) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_1), p_1),$$

which is compatible with the cobordism map between $\widehat{HF}^*(\Sigma(L_i))$, via the naturally defined spectral sequence

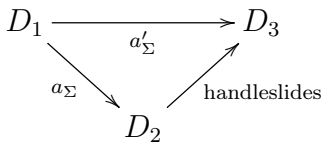
$$E_1 = \widehat{HF}^*(\Sigma(L)) \otimes \mathbb{F}_2[\theta] \Rightarrow \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L)).$$

However, the construction of this map depends on choices of auxiliary data, and we should prove that our maps are invariant under such choices.

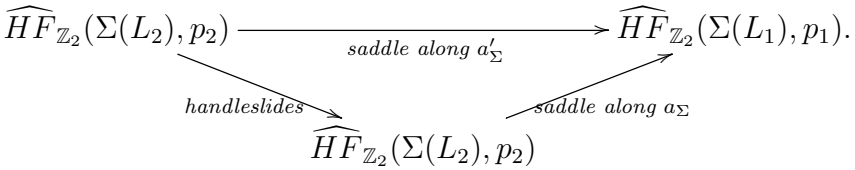
Suppose that S is a saddle along an arc a from $x_1 \in L_1$ to $x_2 \in L_2$, which is indeed a based link cobordism from (L_1, p_1) to (L_2, p_2), provided $x_i \neq p_i$. Choose a genus zero Heegaard surface $\Sigma \subset S^3$. To define a map $\hat{f}_S$ associated to S, we need to draw bridge diagrams of (L_i, p_i) on the surface Σ and then project the arc a on Σ.

Definition 3.7.2. A saddle diagram of S consists of bridge diagrams $(\{A_j^i\}, \{B_k^i\})$ of (L_i, p_i) on Σ for $i = 0, 1$ and an arc $a_\Sigma \subset \Sigma$ from an X-arc $A_j^0(B_j^0)$ to an X-arc $A_j^1(B_j^1)$, where X is either A or B, so that the 1-subcomplex of S^3 given by pushing the A-arcs inside Σ and the B-arcs outside Σ is isotopic to $L_1 \cup a \cup L_2$, and $\text{int}(a_\Sigma) \cap A_j^i(B_k^i) = \emptyset$ for all i and $j(k)$.

There are several choice of saddle diagrams which represent S. However, given two bridge diagrams of (L_i, p_i) for $i = 0, 1$, which coincide outside the saddle region, we only have to choose the placement of the arc a_Σ. Any two choice of a_Σ are related by isotopies and handleslides(over A(B)-arcs). If a_Σ and a'_Σ are related by a single handleslide, then the bridge diagrams representing the saddle moves along a_Σ and a'_Σ are related by a composition of two handleslides, as shown in Figure 3.16. This can be represented as a distinguished diagram of bridge diagrams as follows. Note that we are using saddle diagrams in a perturbed form, as in [20].



Lemma 3.7.3 (Lemma 6.12 of [25])**.** *The above diagram induces a commutative diagram of corresponding $\widehat{HF}_{\mathbb{Z}_2}$ groups:*



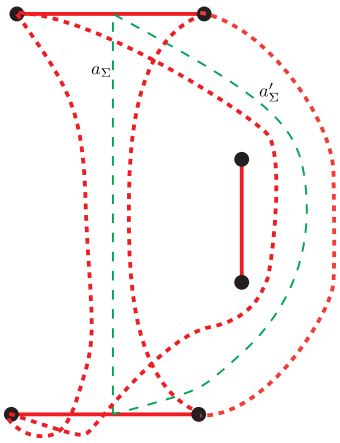


Figure 3.16: Taking saddles along two arcs a_Σ and a'_Σ, which differ by a handleslide

Proof. The Heegaard diagram for a saddle along a'_Σ admits a unique $\mathbb{Z}_2$-invariant top generator. □

The above lemma means that the saddle map is invariant under the choice of a_Σ. However, it remains to show the invariant under the choice of bridge diagrams of L_1 and L_2. For that, we need to show that the following commutative diagrams induce commutative diagrams of $\widehat{HF}_{\mathbb{Z}_2}$ groups.

- Basic moves and saddles commute.
- Diffeomorphisms and saddles commute.

$$\begin{array}{ccc} D & \xrightarrow[\text{saddle}]{} & D_2 \\ {\scriptstyle \text{basic moves/diffeomorphisms}}\downarrow & & \downarrow{\scriptstyle \text{basic moves/diffeomorphisms}} \\ D_1 & \xrightarrow[\text{saddle}]{} & D_3 \end{array} \tag{3.7.1}$$

Lemma 3.7.4 (Lemma 6.13 of [25])**.** *The diagram (3.7.1) induces commutative diagrams of $\widehat{HF}_{\mathbb{Z}_2}$ groups:*

$$\begin{array}{ccc} \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_2),p_2) & \xrightarrow[saddle]{} & \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_1),p_1) \\ {\scriptstyle basic\ moves/diffeomorphisms}\downarrow & & \downarrow{\scriptstyle basic\ moves/diffeomorphisms} \\ \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_2),p_2) & \xrightarrow[saddle]{} & \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_1),p_1) \end{array}$$

Proof. The lemma is obvious for diffeomorphisms, and Lemma 3.5.1 implies that the lemma holds fo isotopies. The proof for handleslides and stabilizations are the same as the proof of commutativity of diagrams of type (c) and (e), respectively. □

Using the above lemma, we see that the saddles give well-defined saddle maps.

Theorem 3.7.5 (Theorem 6.14 of [25])**.** *The basic pieces give well-defined maps between* $\widehat{HF}_{\mathbb{Z}_2}$ *groups.*

Proof. The saddle maps are well-defined by the above lemma. The birth map and the death map correspond to taking/untaking a connecting sum with an invariant Heegaard diagram of $S^1 \times S^2$ on the branched double cover. Since they commute with basic moves (and, obviously, diffeomorphisms) when the connected sum neck is very long, the theorem follows. □

3.7.2 Isotopy invariance of cobordism maps

Now consider the case when we are given a based cobordism S from a based knot (K_1, p_1) to a based knot (K_2, p_2). Then we can isotope S, slice it into basic pieces, convert them into maps between $\widehat{HF}_{\mathbb{Z}_2}$ groups, and then compose them to get a cobordism map

$$\hat{f}_S \ : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K_2), p_2) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K_1), p_1).$$

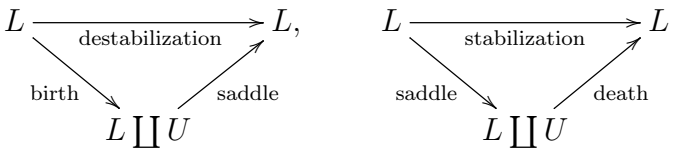
We claim that $\hat{f}_S$ does not depend on the process of slicing S into basic pieces. The idea is to use the 4-dimensional Reidemeister theorem due to Carter-Saito.

Lemma 3.7.6 (Lemma 6.15 of [25])**.** $\hat{f}_S$ *depends only on the isotopy class of based cobordisms* $S = (S_0, s)$ *rel* $s \cup \partial S_0$.

Proof. Isotoping S and then slicing it into basic pieces is equivalent to representing the cobordism S as a movie from (K_1, p_1) to (K_2, p_2). It is known that any two movies of S are related through a sequence of 15 possible types of movie moves, which are defined by Carter and Saito in the paper [8]. Note that, although the result of Carter

and Saito is about non-based cobordisms, it can also be applied directly to based cobordisms, by taking saddles moves to occur away from the basepoint.

Among the 15 types of movie moves, the only type that needs a proof for functoriality of equivariant Floer cohomology is the one drawn in Figure 30 of [8], which corresponds to the following commutative triangles.



But the composition of a birth map followed by a saddle map is precisely the definition of the stabilization map, which proves that the triangle on the left induces a commutative triangle of $\widehat{HF}_{\mathbb{Z}_2}$ groups; see the proof of Theorem 1.24 of [20] for the definition.

The triangle on the right induces a commuative triangle of $\widehat{HF}_{\mathbb{Z}_2}$ groups if the composite cobordism

$$S : \; L \xrightarrow[\text{saddle}]{} L \coprod U \xrightarrow[\text{death}]{} L \xrightarrow[\text{birth}]{} L \coprod U \xrightarrow[\text{saddle}]{} L$$

induces identity on equivariant Heegaard Floer cohomology. First, if L is a union of two unlinked unknots, then this is obvious. Next, if $L = L_0 \coprod U_0$, where U_0 is an unlinked unknot component, the basepoint lies in L_0, and the saddle moves are taken on the component U_0, then the question reduces to the previous case by making a long neck between the bridge diagrams of L_0 and U_0, as drawn in 3.18.

For general link L, consider the saddle move on L, as in Figure 3.17 (the green dashed line is the saddle arc) so that the resulting link L' is isotopic to $L \coprod$ unknot where the basepoint does not lie on the unknot component. Since the saddle map

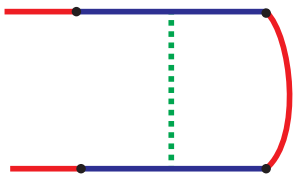


Figure 3.17: Saddle along a small arc near L

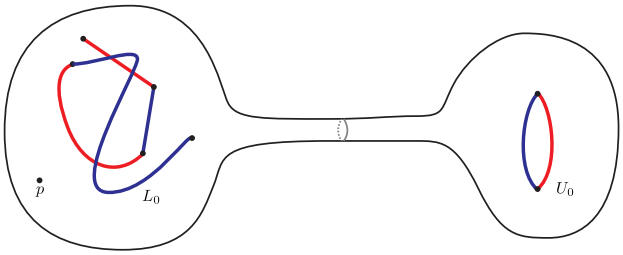


Figure 3.18: The bridge diagram of $L = L_0 \coprod U_0$ on a sphere with a long neck

$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), p) \xrightarrow{\hat{f}_{S_a}} \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L'), p)$ postcomposed with the birth map

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L'), p) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), p)$$

is the stabilization map, which is an isomorphism, we see that $\hat{f}_{S_a}$ is injective. Now, by the proof of commutativity of diagrams of type (c), the following diagram commutes.

$$\begin{array}{ccc} \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L)) & \xrightarrow[\hat{f}_S]{} & \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L)) \\ \downarrow{\scriptstyle \hat{f}_{S_a}} & & \downarrow{\scriptstyle \hat{f}_{S_a}} \\ \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L')) & \xrightarrow[\hat{f}_{S \coprod (L \times I)}]{} & \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L')) \end{array}$$

But we already know that $\hat{f}_{S \coprod (L \times I)}$ is the identity. Therefore, since $\hat{f}_{S_a}$ is injective, the map $\hat{f}_S$, too, is the identity. □

Theorem 3.7.7 (Theorem 6.16 of [25]). *$\hat{f}_S$ depends only on the isotopy class of based cobordisms $S = (S_0, s)$ rel ∂S_0.*

Proof. Denote the projection map $S^3 \times I \to I$ by p. We only have to prove that, given a knot cobordism S_0 and two curves s and s' such that $\partial s = \partial s'$, s is isotopic to s' rel ∂s, and p is regular when restricted to s and s', we have

$$\hat{f}_{(S_0,s)} = \hat{f}_{(S_0,s')}.$$

Isotope S_0 rel $s \cup s'$, so that $p|_{S_0}$ is Morse and the critical points of $p|_{S_0}$ have distinct images under p. We can isotope s slightly and horizontally so that it intersects s' transversely. We know that s and s' cobound a disk $D \subset S$, which is innermost in the sense that $D \cap (s \cup s') = \partial D$. Using Lemma 3.7.6, we can isotope S so that $p|_D$ has no critical points. Then we get an isotopy $\{s_t\}_{t\in[0,1]}$ from s to s' rel ∂s, such that h is regular on s_t for each $t \in [0,1]$. Since isotopy maps (on Σ) commute with all other types of maps, we deduce that the based cobordisms (S, s_t) induce the same map for all t. Therefore, by induction on $|s \cap s'|$, we get $\hat{f}_{(S_0,s)} = \hat{f}_{(S_0,s')}$. □

3.7.3 Statement of functoriality

To rephrase what we have proven, we now define a category **bCob** as follows.

- The objects of **bCob** are based oriented knots in S^3.
- The morphisms of **bCob** between based oriented knots K_1, K_2 are isotopy classes of based oriented cobordisms from K_1 to K_2.

Then what we proved up to now can be rephrased as follows.

Theorem 3.7.8 (Theorem 6.17 of [25])**.** $\widehat{HF}_{\mathbb{Z}_2} : \mathbf{bCob} \to \mathbf{Mod}_{\mathbb{F}_2[\theta]}$ *is a functor.*

It is worth noting that, by using the same argument, we can prove that a based cobordism of links also induces a cobordism map, but it is well-defined only up to monodromy. In other words, if S is a based cobordism from (L_1, p_1) to (L_2, p_2) and

we denote the monodromy of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_i), p_i)$ by

$$\rho_i \,:\, \pi_1(\mathcal{B}_{L_i,\Sigma,p_i}) \to \mathrm{Aut}_{\mathbb{F}_2[\theta]}(\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_i), p_i)),$$

then the cobordism map

$$\hat{f}_S \,:\, \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_2), p_2) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_1), p_1)$$

is well-defined up to composition with elements in the images of ρ_1 and ρ_2.

Chapter 4

Calculating $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ from extended bridge diagrams

Suppose that a based knot K in S^3 is given. Then, as we have seen in the previous chapter, we can represent K as a bridge diagram on a sphere, and taking its branched double cover along the points where K and the sphere intersect gives a Heegaard diagram of the branched double cover $\Sigma(K)$ of S^3 along K. Furthermore, we now also know that this diagram admits a natural $\mathbb{Z}_2$-action which fixes the basepoint and the α,β-curves, and its equivariant Floer cohomology $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, which is an $\mathbb{F}_2[\theta]$-module, is a natural and functorial invariant of knots.

Given these facts, it is natural to ask what happens when we start with a bridge diagram of K, not drawn on a sphere, but on a closed oriented surface of arbitrary genus. Can we similarly construct its $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology? If we can, then is it isomorphic to $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$? In this chapter, we will give positive answers to these questions.

4.1 Extended bridge diagrams

If we are given a bridge diagram of a given knot $K \subset S^3$ drawn on a closed oriented surface Σ, as in the genus-zero case, we can still try to take its branched double cover, and expect it to become a Heegaard diagram representing $\Sigma(K)$. However, when Σ has positive genus, the branched double cover can never be a Heegaard diagram, as it does not have sufficiently many α-curves and β-curves. We can resolve this problem by starting with a bridge diagram drawn on a Heegaard diagram, where the given Heegaard diagram represents S^3. For simplicity, we start with the following definitions.

Definition 4.1.1. Suppose that a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is given. A based bridge diagram on $\mathcal{H}$ is a 4-tuple (F, A, B, z), where $F \subset \Sigma$ is a finite subset of points in Σ, A, B are sets of simple arcs on Σ, and $z \in F$, such that the following properties are satisfied.

- For any $\gamma \in \boldsymbol{\alpha} \cup \boldsymbol{\beta}$, we have $F \cap \gamma = \emptyset$.
- For any two distinct elements $\boldsymbol{a}, \boldsymbol{a}' \in A \cup \boldsymbol{\alpha}$, we have $\boldsymbol{a} \cap \boldsymbol{a}' = \emptyset$, and the same statement holds for elements in $B \cup \boldsymbol{\beta}$.
- For any $\boldsymbol{a} \in A \cup \boldsymbol{\alpha}$ and $\boldsymbol{b} \in B \cup \boldsymbol{\beta}$, the intersection $\boldsymbol{a} \cap \boldsymbol{b}$ is transverse.
- For any $\boldsymbol{c} \in A \cup B$, the two endpoints of $\boldsymbol{c}$ are distinct and $\boldsymbol{c} \cap F = \partial \boldsymbol{c}$.
- For any $p \in F$, there exists a unique element $\boldsymbol{a}$ of A which satisfies $p \in \partial \boldsymbol{a}$, and the same statement holds for B.

Given a pointed bridge diagram (F, A, B, z) on a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, we call the elements of A as **A-arcs**, the elements of B as **B-arcs**, and z as the **basepoint**. Note that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is a pointed Heegaard diagram.

Suppose that a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ describes a 3-manifold M. Then, given a based bridge diagram $\mathcal{P} = (F, A, B, z)$ on $\mathcal{H}$, we can construct a based link

(L, p) lying inside M as follows. Let $M = H_1 \cup H_2$ be a Heegaard splitting of M given by the Heegaard surface Σ. Suppose that we call H_1 as the "outside" of Σ and H_2 as the "inside" of Σ. Then, we can isotope the A-arcs of $\mathcal{P}$ slightly ourwards and the B-arcs of $\mathcal{P}$ slightly inwards, while leaving the set F fixed. Concatenating the isotoped arcs gives us a link $L \subset M$. Since the basepoint p now lies on L, we recover a based link (L, p) in M, which is uniquely determined up to (based) isotopy.

Definition 4.1.2. Suppose that a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ describes a 3-manifold M. We say that a based link (L, p) in M is represented by a based bridge diagram $\mathcal{P} = (F, A, B, z)$ if the process described above gives a based link which is (based) isotopic to (L, p).

Note that the linking number between any component of the link L and any α-curve(or β-curve) is zero.

Definition 4.1.3. An extended bridge diagram is a pair $\mathcal{E} = (\mathcal{H}, \mathcal{P})$, where $\mathcal{H}$ is a Heegaard diagram and $\mathcal{P}$ is a based bridge diagram on $\mathcal{H}$. We write $\mathcal{H}$ as $\mathcal{H}(\mathcal{E})$ and $\mathcal{P}$ as $\mathcal{P}(\mathcal{E})$. If $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathcal{P} = (A, B, F, p)$, the pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, p)$ will be denoted as $\mathcal{H}_{st}(\mathcal{E})$. Also, the 3-manifold represented by $\mathcal{H}$ is denoted as $M(\mathcal{E})$, and the based link in $M(\mathcal{E})$ represented by $\mathcal{P}$ is denoted as $L(\mathcal{E})$.

As in the genus-zero case, one can expect that, given any Heegaard diagram representing S^3, any based link in S^3 can represented as a based bridge diagram on it. This is indeed possible, as one can see from the following proposition. Note that, for more generality, we will work on **arbitrary 3-manifolds** in this section.

Proposition 4.1.4 (Proposition 2.3 of [24])**.** *Suppose that a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ describes a 3-manifold M. Then every based link in M can be represented by a based bridge diagram on $\mathcal{H}$.*

Proof. Let $M = H_1 \cup H_2$ be the Heegaard splitting of M, induced by $\mathcal{H}$. Then for each $i = 1, 2$, there exists a 1-subcomplex $C_i \subset \mathrm{int}(H_i)$ so that $H_1 - C_1 \simeq \Sigma \times [0, \infty)$

and $H_2 - C_2 \simeq \Sigma \times (-\infty, 0]$. Since C_1 and C_2 are both 1-dimensional and M is a 3-manifold, we can isotope (L, p) so that $p \in \Sigma$, L does not intersect $C_1 \cup C_2$ and L intersect transversely with Σ. After further isotoping L, we may assume that every component c^i of $L \cap H_i$ admits a disk $D_{c^i} \subset H_i$ so that $c^i \subset \partial D_{c^i} \subset c^i \cup \Sigma$, and for any two distinct components c^i_1, c^i_2 of $L \cap H_i$, we have $D_{c^i_1} \cap D_{c^i_2} = \emptyset$. We can also assume that the disks D_{c^i} does not intersect the family of compressing disks in H_i , determined by the alpha- and beta-curves of $\mathcal{H}$, possibly after applying another isotopy to L, while leaving the basepoint p fixed. For each component $c^i \in \pi_0(L \cap H_i)$, write $\partial D_{c^i} = c^i \cup p(c^i)$ where $u^i \subset \Sigma$, i.e. $p(c^i)$ is a simple arc on Σ which is a projection of c^i. By assumption, the arcs $p(c^i)$ does not intersect the alpha- or beta-curves, and any two distinct arcs $p(c^i_1)$ and $p(c^i_2)$ do not intersect.

Now, after isotoping the arcs $p(c^i)$, we can assume that any two curves $p(c^i)$ and $p(c^j)$ intersect transversely if $i \neq j$, i.e. $\{i, j\} = \{1, 2\}$. Consider the following sets:

$$A = \{p(c) \,|\, c \in \pi_0(L \cap H_1)\},$$
$$B = \{p(c) \,|\, c \in \pi_0(L \cap H_2)\},$$
$$F = \bigcup_{u \in A \cup B} \partial u.$$

Then $p \in F$, and the based bridge diagram (A, B, F, p) represents the given based link (L, p) in M. □

Given an extended bridge diagram $\mathcal{E} = (\mathcal{H}, \mathcal{P})$, we have an associated triple (M, L, p), where M is the 3-manifold represented by $\mathcal{H}$, and (L, p) is a based link in M, represented by $\mathcal{P}$. Obviously, given a 3-maniold together with a based link inside it, we have lots of extended bridge diagrams which represents it. In particular, we have a set of operations on extended bridge diagrams which leave the associated 3-manifold and based link fixed, which we will call as **extended Heegaard moves**. Also, we will call isotopies and handleslides involving a change of $\alpha(\beta)$-curves as **$\alpha(\beta)$-equivalences**, and those involving a change of A(B)-arcs as **A(B)-equivalences**.

Finally, we will call α,β-equivalences and A,B-equivalences as **basic moves**.

Isotopies

Given an extended bridge diagram, we can isotope its A-arcs, B-arcs, alpha-curves and beta-curves.

Handleslides of type I

Given an extended bridge diagram, we can replace an alpha(beta)-curve α with another simple closed curve α' through an ordinary handleslide of Heegaard diagrams. Here, the handleslide region must not intersect any of the A/B-arcs.

Handleslides of type II

Given an extended bridge diagram, we can replace an alpha(beta)-curve α with another simple closed curve α' if the following conditions are satisfied.

- The curve α' does not intersect with any of the A(B)-arcs and the alpha(beta)-curves.
- There exists an A-arc A so that α, α' cobound a cylinder whose interior contains A and does not intersect with any of the A-arcs, B-arcs, alpha-curves, and the beta-curves, except A.

Handleslides of type III

Given an extended bridge diagram, we can replace an A(B)-arc a with another simple arc a' if the following conditions are satisfied.

- $\partial a = \partial a'$.
- The interior of a' does not intersect with any of the A(B)-arcs and the alpha(beta)-curves.

- There exists an alpha(beta)-curve α so that a, a', α bound a cylinder $C \subset \Sigma$, whose interior does not intersect with any of the A-arcs, B-arcs, alpha-curves, and the beta-curves.

Handleslides of type IV

Given an extended bridge diagram, we can replace an A(B)-arc a with another simple arc a' if the following conditions are satisfied.

- The interior of a' does not intersect with any of the A(B)-arcs and the alpha(beta)-curves.
- There exists an A(B)-arc a_0 such that a, a' bound a disk $D \subset \Sigma$, whose interior contains a_0 and does not intersect with any of the A-arcs, B-arcs, alpha-curves, and the beta-curves, except for a_0.

(De)stabilizations of type I

Given an extended bridge diagram $((\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (F, A, B, p))$, we can stabilize its Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ at a point $q \in \Sigma$ such that $q \notin c$ for any $c \in A \cup B$. The based bridge diagram (F, A, B, p) remains the same.

(De)stabilizations of type II

Given an extended bridge diagram $((\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (F, A, B, p))$, where $|F| > 2$, choose an A-arc a such that $p \notin \partial a$, and pick one of its endpoints, $z \in \partial a$. Choose two distinct points x, y lying in the interior of z, so that the following conditions hold.

- $a - \{x, y\}$ has three components a_1, b_1, a', which are simple arcs on Σ.
- $\partial a_1 = \{z, x\}$, $\partial b_1 = \{x, y\}$, and $\partial a' = \{y\} \cup (\partial a - \{z\})$.
- a_1 and b_1 do not intersect with any of the B-arcs and beta-curves.

Then $((\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (F \cup \{x, y\}, (A - \{a\}) \cup \{a_1, a'\}), B \cup \{b_1\}, p))$ is again an extended bridge diagram.

Diffeomorphism

Given an extended bridge diagram $((\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (F, A, B, p))$ and a diffeomorphism $\phi \in \mathrm{Diff}^+(\Sigma)$, we can apply ϕ on everything to get another extended bridge diagram.

4.2 Extended Heegaard moves suffice

As in the genus-zero case, we can expect that the any two extended bridge diagrams representing the same 3-manifold and the same based link are related by a sequence of extended Heegaard moves. This can be proven using a correspondence between extended bridge diagrams and multi-pointed knot(link) Heegaard diagrams, as we will see in this section.

Suppose that an extended bridge diagram $\mathcal{E} = (\mathcal{H}, \mathcal{P})$, $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $\mathcal{P} = (F, A, B, z)$ is given. For each arc $a \in A \cup B$, the boundary of a small neighborhood $\nu(a)$ of a is a simple closed curve on Σ. Also, we can uniquely define a partition $F = \mathbf{x} \sqcup \mathbf{y}$, so that the following conditions hold.

- $p \in \mathbf{x}$;
- For each $a \in A \cup B$, we have $|\mathbf{x} \cap \partial a| = |\mathbf{y} \cap \partial a| = 1$.

Consider the sets $\boldsymbol{\alpha}' = \{\partial \nu(a) \,|\, a \in A\}$ and $\boldsymbol{\beta}' = \{\partial \nu(b) \,|\, b \in B\}$. Then we can form a tuple $\mathcal{KH}(\mathcal{E}) = (\Sigma, \boldsymbol{\alpha} \cup \boldsymbol{\alpha}', \boldsymbol{\beta} \cup \boldsymbol{\beta}', \mathbf{x}, \mathbf{y}, z)$, which is a multi-pointed link Heegaard diagram with a distinguished basepoint z. Note that the based link represented by $\mathcal{KH}(\mathcal{E})$ is the same as the one represented by $\mathcal{E}$.

Now suppose that a multi-pointed link Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{x}, \mathbf{y}, z)$ together with a distinguished basepoint $z$is given, where $z \in \mathbf{x}$. Then each connected component R of $\Sigma \setminus \bigcup \boldsymbol{\alpha}$ satisfies $|R \cap \mathbf{x}| = |R \cap \mathbf{y}| = 1$, so that we can choose a simple

arc γ, contained in the interior of R, so that γ connects the two basepoints inside R, and similarly for $\boldsymbol{\beta}$ instead of $\boldsymbol{\alpha}$. Also, $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an overcomplete Heegaard diagram, thus as shown in Section 6.2 of [23], there exist subsets $\boldsymbol{\alpha}_0 \subset \boldsymbol{\alpha}$ and $\boldsymbol{\beta}_0 \subset \boldsymbol{\beta}$ such that $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ represents the same underlying 3-manifold as $\mathcal{E}$, and any two such choices are related by a sequence of isotopies and handleslides. These data gives an extended bridge diagram $\mathcal{BD}(\mathcal{H})$, defined uniquely up to isotopies and handleslides (of types I, II, and III).

Theorem 4.2.1. *If two extended bridge diagrams which represent the same 3-manifold and isotopic based links, they are related by extended Heegaard moves.*

Proof. Suppose that the given two extended bridge diagrams $\mathcal{E}_1$ and $\mathcal{E}_2$ representing the same 3-manifold and isotopic based links have the same number of bridges. Then the multi-pointed link Heegaard diagrams $\mathcal{KH}(\mathcal{E}_1)$ and $\mathcal{KH}(\mathcal{E}_2)$ again represent the same link, so they should be related by a sequence of isotopies, handleslides, and (de)stabilizations. Now recall the following facts:

- Multi-pointed link Heegaard diagram uniquely determines an extended bridge diagram up to isotopies and handleslides.
- A Heegaard move of overcomplete Heegaard diagrams can always be translated to a Heegaard move of ordinary Heegaard diagrams. (see Section 6.4 of [23])

Those facts imply that we can translate a sequence of Heegaard moves from $\mathcal{KH}(\mathcal{E}_1)$ to $\mathcal{KH}(\mathcal{E}_2)$ to a sequence of extended Heegaard moves from $\mathcal{E}_1$ to $\mathcal{E}_2$. Finally, if $\mathcal{E}_1$ and $\mathcal{E}_2$ have different number of bridges, we can apply stabilizations of type II to make the number of bridges equal. □

The above theorem has an interesting corollary, which will be used frequently in this section and the next section.

Definition 4.2.2. An extended bridge diagram $\mathcal{E} = (\mathcal{H}, \mathcal{P})$, $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $\mathcal{P} = (F, A, B, z)$ is **simple** if $\bigcup(A \cup B) \subset R$, where R denotes the connected component of $\Sigma \backslash \bigcup(\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ such that $z \in R$.

Corollary 4.2.3. *Let M be a (closed, connected, oriented) 3-manifold and $L \subset M$ be a link which is contained in a ball. Then any extended bridge diagram representing L can be transformed to a simple diagram by a sequence of extended Heegaard moves.*

Proof. Since L is contained in a ball, there exists a simple extended bridge diagram $\mathcal{E}$ representing L. Then, by Theorem 4.2.1, any extended bridge diagram representing L can be transformed to $\mathcal{E}$ by a sequence of extended Heegaard moves. □

4.3 Weak admissibility

As in the genus-zero case, in order to talk freely about counting holomorphic disks, triangles, and squares, we have to settle the issue of weak admissibility of Heegaard diagrams, given by taking branched double covers of extended bridge diagrams. This issue will be dealt with in this section. As a result, from the next section, we will implicitly assume all extended bridge diagrams to be **weakly admissible**, by which we will mean that its base is weakly admissible.

Suppose that an extended bridge diagram $\mathcal{E} = (\mathcal{H}, \mathcal{P})$, $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $\mathcal{P} = (F, A, B, z)$ is given, where the Heegaard diagram $\mathcal{H}$ represents a 3-manifold M and the based bridge diagram $\mathcal{P}$ represents the isotopy class of a based link (L, z) in M. Then we construct a 4-tuple $\mathcal{H}_d(\mathcal{E}) = (\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, z)$, which is defined as follows.

- $\tilde{\Sigma}$ is the branched double cover of Σ along F.
- $\tilde{\boldsymbol{\alpha}} = \left(\bigcup_{\alpha \in \boldsymbol{\alpha}} \{\text{connected components of } \alpha\}\right) \cup \{p^{-1}(a) \,|\, a \in A,\, z \notin \partial a\}$, where $p : \tilde{\Sigma} \to \Sigma$ is the branched covering map, and $\tilde{\boldsymbol{\beta}}$ is defined similarly.
- z is the basepoint of the given based link (L, z).

The 4-tuple $\mathcal{H}_d(\mathcal{E})$, by construction, is a Heegaard diagram for the based 3-manifold $(\Sigma_L(M), z)$, which is the branched double cover of the based 3-manifold (M, z) along the link L. The covering transformation of the branched cover $\Sigma_L(M) \to M$ induces

an orientation-preserving $\mathbb{Z}_2$-action on $\mathcal{H}_d(\mathcal{E})$. We will say that $\mathcal{H}_d(\mathcal{E})$ is the branched double cover of $\mathcal{H}$ along $\mathcal{P}$.

Proposition 4.3.1 (Proposition 2.8 of [24]). *For any extended bridge diagram $\mathcal{E}$, the pointed Heegaard diagram $\mathcal{H}_d(\mathcal{E})$ is weakly admissible if $\mathcal{H}_{pt}(\mathcal{E})$ is weakly admissible.*

Proof. We will continue using the notations that we have been using in the above discussion. Consider the branched double covering map $p_\Sigma : \tilde{\Sigma} \to \Sigma$. Then, for any connected component R of $\Sigma - \left(\bigcup_{c\in\boldsymbol{\alpha}\cup\boldsymbol{\beta}} c\right)$, such that $z \notin R$, we define its pullback $p_\Sigma^*(R)$ as follows.

- If $p_\Sigma^{-1}(R)$ is connected, it is a connected component of $\tilde{\Sigma} - \left(\bigcup_{\tilde{c}\in\tilde{\boldsymbol{\alpha}}\cup\tilde{\boldsymbol{\beta}}} \tilde{c}\right)$, so we define $p_\Sigma^*(R)$ as $p_\Sigma^{-1}(R)$.
- If $p_\Sigma^{-1}(R)$ is disconnected, it consists of a $\mathbb{Z}_2$-orbit of some connected component of $\tilde{\Sigma} - \left(\bigcup_{\tilde{c}\in\tilde{\boldsymbol{\alpha}}\cup\tilde{\boldsymbol{\beta}}} \tilde{c}\right)$, where $\mathbb{Z}_2$ acts as covering transformations. Denote that orbit as $\{T, \sigma T\}$, where $\mathbb{Z}_2 = \langle\sigma\rangle$. Then we define $p_\Sigma^*(R)$ as $T + \sigma T$.

This definition extends linearly to a group homomorphism

$$p_\Sigma^* : \mathcal{D}(\mathcal{H}_{pt}(\mathcal{E})) \to \mathcal{D}(\mathcal{H}_d(\mathcal{E})),$$

where $\mathcal{D}(\mathcal{H})$, for a pointed Heegaard diagram $\mathcal{H}$, is defined to be the free abelian group of domains in $\mathcal{H}$ which do not intersect the basepoint. The map p_Σ^* clearly preserves periodicity.

Suppose that $\mathcal{H}_{pt}(\mathcal{E})$ is weakly admissible and there exists a positive periodic domain $D \in \mathcal{D}(\mathcal{H}_d(\mathcal{E}))$. Then $D + \sigma D$ is also a positive periodic domain in $\mathcal{H}_d(\mathcal{E})$. Using the proof of weak admissibility in the genus-zero case in the previous chapter, we see that there exists a positive periodic domain $D_0 \in \mathcal{D}(\mathcal{H}_{pt}(\mathcal{E}))$ such that $p_\Sigma^* D_0 = D + \sigma D$. Since $\mathcal{H}_{pt}(\mathcal{E})$ is assumed to be weakly admissible, we must have $D_0 = 0$ and thus $D + \sigma D = 0$. Since both D and σD are positive, this implies $D = 0$, a contradiction. Therefore $\mathcal{H}_d(\mathcal{E})$ must be weakly admissible. □

We will now proceed to weak admissibility of Heegaard triple diagrams and quadruple diagrams, which are perturbations of branched double covers of extended bridge diagrams. More precisely, the diagrams we will deal with are defined as follows.

Definition 4.3.2. A 5-tuple $(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, z)$ is called an involutive Heegaard 5-tuple if the conditions below are satisfied.

- $\tilde{\Sigma}$ is a branched double cover of a surface Σ along a branching locus F, such that $z \in F$.
- The 4-tuples $\tilde{\mathcal{H}}_{\alpha\beta} = (\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, z)$, $\tilde{\mathcal{H}}_{\beta\gamma} = (\tilde{\Sigma}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, z)$, $\tilde{\mathcal{H}}_{\alpha\gamma} = (\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\gamma}}, z)$ are pointed Heegaard diagrams.
- There exist families of simple closed curves $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ and families of simple arcs A, B, C on Σ such that $\mathcal{T}_0 = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ is a Heegaard triple diagram, $\mathcal{P}_{AB} = (F, A, B, z)$, $\mathcal{P}_{BC} = (F, B, C, z)$, and $\mathcal{P}_{AC} = (F, A, C, z)$ are based bridge diagrams on the Heegaard diagrams $\mathcal{H}_{\alpha\beta} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $\mathcal{H}_{\beta\gamma} = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma})$, and $\mathcal{H}_{\alpha\gamma} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma})$, respectively, and the branched double covers of $\mathcal{H}_{\alpha\beta}, \mathcal{H}_{\beta\gamma}, \mathcal{H}_{\alpha\gamma}$ along $\mathcal{P}_{AB}, \mathcal{P}_{BC}, \mathcal{P}_{CA}$ are $\tilde{\mathcal{H}}_{\alpha\beta}, \tilde{\mathcal{H}}_{\beta\gamma}, \tilde{\mathcal{H}}_{\alpha\gamma}$, respectively.

A pointed Heegaard triple diagram $\mathcal{T}$ is nearly involutive if it is given by a small perturbation of alpha-, beta-, and gamma-curves of some involutive Heegaard 5-tuple $(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, z)$. We say that the pointed Heegaard triple diagram $\mathcal{T}_0$ is the **base** of the nearly involutive triple diagram $\mathcal{T}$.

Proposition 4.3.3 (Proposition 2.10 of [24])**.** *A nearly involutive Heegaard triple diagram is weakly admissible if its base is weakly admissible.*

Proof. The proof is the same as in Proposition 4.3.1, except that we are using triple diagrams instead of ordinary diagrams. Using the proof of Lemma 3.4.3, we see that the argument for ordinary diagrams can also be used for triple diagrams. □

Definition 4.3.4. A pointed 6-tuple $(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\delta}}, z)$ is an involutive Heegaard 6-tuple if any of the four 5-tuples given by excluding one out of four curve bases

$\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\delta}}$ are involutive. A pointed Heegaard quadruple diagram $\mathcal{Q}$ is nearly admissible if it is given by a small perturbation of alpha-, beta-, gamma-, and delta-curves of some involutive Heegaard 6-tuple $(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\delta}}, z)$. If the bases of the triple diagrams given by excluding one out of four curve bases $\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\delta}}$ are given by a surface Σ and curve bases $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}$ on Σ, then we say that the pointed Heegaard quadruple diagram $\mathcal{Q}_0 = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, z)$ is the base of the nearly involutive quadruple diagram $\mathcal{Q}$.

Proposition 4.3.5 (Proposition 2.12 of [24])**.** *A nearly involutive Heegaard quadruple diagram is weakly admissible if its base is weakly admissible.*

Proof. The proof is the same as in Proposition 4.3.3, except that we are now using the proof of Lemma 3.4.4, instead of the proof of Lemma 3.4.3. □

The propositions 4.3.1, 4.3.3, and 4.3.5 tell us that, when dealing with nearly involutive diagrams, we do not have to care about their weak admissibility, as long as their bases are weakly admissible. Hence, for the sake of simplicity, we will call a extended bridge diagram $\mathcal{E}$ **weakly admissible** if the pointed Heegard diagram $\mathcal{H}_{pt}(\mathcal{E})$ is weakly admissible. Note that this implies weak admissibility of $\mathcal{H}_d(\mathcal{E})$.

Now, given an extended bridge diagram

$$\mathcal{E} = ((\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (F, A, B, z)),$$

such that $\mathcal{H}_{pt}(\mathcal{E})$ is weakly admissible, we can apply the construction of [20] to the induced $\mathbb{Z}_2$-action on the triple $(\mathrm{Sym}^{\tilde{g}}(\tilde{\Sigma} - \{z\}), \mathbb{T}_{\tilde{\boldsymbol{\alpha}}}, \mathbb{T}_{\tilde{\boldsymbol{\beta}}})$, where $\mathcal{H}_d(\mathcal{E}) = (\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, z)$ and $\tilde{g}$ is the genus of $\tilde{\Sigma}$. What we get is the equivariant Floer cohomology

$$\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}) = \widehat{HF}_{\mathbb{Z}_2}(\mathrm{Sym}^{\tilde{g}}(\tilde{\Sigma} - \{z\}), \mathbb{T}_{\tilde{\boldsymbol{\alpha}}}, \mathbb{T}_{\tilde{\boldsymbol{\beta}}}),$$

which is a $\mathbb{F}_2[\theta]$-module in a natural way.

Lemma 4.3.6 (Lemma 2.13 of [24]). *Given any two weakly admissible extended bridge diagrams $\mathcal{E}, \mathcal{E}'$ representing the same bridge link (L, p) inside the same 3-manifold M, there exists a sequence of extended Heegaard moves which relates $\mathcal{E}$ and $\mathcal{E}'$, such that every extended bridge diagram $\mathcal{E}_0$ appearing in an intermediate step is weakly admissible.*

Proof. From Proposition 4.3.1 and the proof of Theorem 4.2.1, we see that we do not have to consider the positions of bridges in extended bridge diagrams. So we only have to care about extended Heegaard moves of pointed Heegaard diagrams. Since any two weakly admissible diagrams representing the same 3-manifold are related by isotopies, handleslides, stabilizations, and diffeomorphisms while preserving weak admissibility by Proposition 2.2 of [37], we are done. □

Before we move on to the next section, we make the following observation. Given an extended Heegaard move of extended bridge diagrams, whose source and target are both weakly admissible, we can associate to it an $\mathbb{F}_2[\theta]$-module homomorphism between the corresponding $\mathbb{Z}_2$-equivariant Floer cohomology, as defined in [20]. From Lemma 4.3.6, we know that any two weakly admissible extended bridge diagrams $\mathcal{E}, \mathcal{E}'$ which represent the same based link inside S^3, we have a map

$$\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}') \to \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}),$$

which is defined as a composition of maps associated to extended Heegaard moves. The arguments used in the section 6 of [20] can be extended directly to show that the maps associated to extended Heegaard moves, except for stabilizations, are isomorphisms.

4.4 Special stabilizations induce isomorphisms

It is clear, from Floer-theoretic lemmas given in the previous chapter, that every basic move of extended Heegaard diagrams induces an isomorphism between $\widehat{HF}_{\mathbb{Z}_2}$. Also, stabilizations of type I induce isomorphisms of $\widehat{HF}_{\mathbb{Z}_2}$, under the condition that they are performed near the basepoint. The problem is that, although stabilizations of type II also induce maps between $\widehat{HF}_{\mathbb{Z}_2}$ under the same condition, it is not easy to see why they should also be isomorphisms. In this section, we will circumvent this problem by introducing a new type of move which can be easily shown to induce an isomorphism of $\widehat{HF}_{\mathbb{Z}_2}$ and then observing that it can be used instead of stabilizations of type II without any loss of generality.

Definition 4.4.1. An extended bridge diagram $\mathcal{E} = ((\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (F, A, B, z))$ is said to be in a **nice position** if there exists an arc $c \in A \cup B$ such that the following conditions hold.

- $z \in \partial c$.
- The interior of c does not intersect with any of the B-arcs and beta-curves.

Given an extended bridge diagram $\mathcal{E} = ((\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (F, A, B, z))$ which is in a nice position, choose an arc c as in Definition 4.4.1, and assume without loss of generality that c is an A-arc. Let p be an endpoint of c such that $\partial c = \{p, z\}$. Choose two distinct interior points x, y of c such that the following condition is satisfied.

- $c - \{x, y\}$ has three connected components c_{px}, c_{xy}, c_{yz}, each of which is a simple arc, satisfying $\partial c_{px} = \{p, x\}$, $\partial c_{xy} = \{x, y\}$, and $\partial c_{yz} = \{y, z\}$.

Then, the pair $\mathcal{E}' = ((\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (F, (A - \{c\}) \cup \{c_{xy}, c_{yz}\}, B \cup \{c_{px}\}, z))$ is an extended bridge diagram, which represents the same based link in a same 3-manifold as $\mathcal{E}$.

Definition 4.4.2. We define the above operation as **special stabilization**, i.e. applying a special stabilization to $\mathcal{E}$ gives $\mathcal{E}'$.

Lemma 4.4.3 (Lemma 2.17 of [24]). *Let (K,p) be a based knot in S^3. Then any two extended bridge diagrams representing (K,p) are related by isotopies, handleslides of type I, II, III, IV, stabilizations of type I near the basepoint, diffeomorphisms, and special stabilizations.*

Proof. We only have to prove that we can use special stabilizations instead of stabilizations of type II. Given any extended bridge diagram $\mathcal{E}$ representing (K,p) in S^3, we can apply handleslide of type II, III, and IV to place it in a nice position. Since K is a knot and thus has only one component, applying a stabilization of type II at any point has the same effect as applying a special stabilization and then moving the newly created pair of arcs to that point via isotopies and handleslide of type III. Therefore a stabilization of type II has the same effect as a composition of handleslides of type III, IV, and special stabilizations. □

Lemma 4.4.4 (Lemma 2.18 of [24]). *Let (K,p) be a based knot in S^3. Suppose that an extended bridge diagram $\mathcal{E}$, which is in a nice position, represents (K,p) in M, and applying a special stabilization to $\mathcal{E}$ gives another diagram $\mathcal{E}'$. Then there exists an associated isomorphism:*

$$\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}') \xrightarrow{\sim} \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}).$$

Proof. The extended bridge diagrams $\mathcal{E}$ and $\mathcal{E}'$, near the basepoint p, are drawn in Figure 4.1. Since γ_{zx} is the only B-arc/β-curve which intersects the A-arc γ_{xy} of $\mathcal{H}_d(\mathcal{E}')$, any Floer generator $\mathbf{x}'$ of $\mathcal{H}_d(\mathcal{E}')$ can be written as

$$\mathbf{x}' = \mathbf{x} \cup \{x\},$$

where $\mathbf{x}$ is a uniquely determined Floer generator of $\mathcal{H}_d(\mathcal{E})$.

Now assume that we are using almost complex structures of the form $\mathrm{Sym}^{\tilde{g}}(\mathfrak{j})$, where $\mathfrak{j}$ is an almost complex structure on the Heegaard surface $\tilde{\Sigma}$ of $\mathcal{H}_d(\mathcal{E})$ (thus also

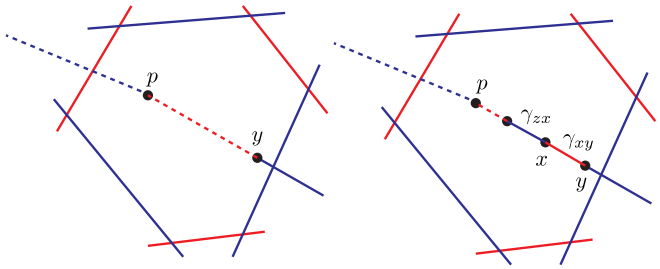


Figure 4.1: Attaching a small A-arc and a B-arc at the point w.

of $\mathcal{H}_d(\mathcal{E}')$) and $\tilde{g}$ is the genus of $\tilde{\Sigma}$. This is possible since such structures are enough to acheive transversality for all moduli spaces of holomorphic disks. Since the regions of $\mathcal{H}_d(\mathcal{E}')$ which contains x on its boundary also contains the basepoint z in its interior, any holomorphic disks from $\mathbf{x} \cup \{x\}$ to $\mathbf{y} \cup \{x\}$, where $\mathbf{x}, \mathbf{y}$ are Floer generators of $\mathcal{H}_d(\mathcal{E})$, are actually holomorphic disks from $\mathbf{x}$ to $\mathbf{y}$, as the point x cancels out. Hence we have a diffeomorphism

$$\mathcal{M}(\mathbf{x}, \mathbf{y}) \simeq \mathcal{M}(\mathbf{x} \cup \{x\}, \mathbf{y} \cup \{x\}).$$

This implies that the map between the equivariant Floer complexes,

$$\widehat{CF}_{\mathbb{Z}_2}(\mathcal{E}) \to \widehat{CF}_{\mathbb{Z}_2}(\mathcal{E}'),$$
$$\mathbf{x} \mapsto \mathbf{x} \cup \{x\}$$

is an isomorphism. Therefore the induced map $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}') \to \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E})$ is also an isomorphism. □

Using the above lemmas, we can now prove that, when K is a knot, then we can calculate $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ using almost all (weakly admissible) extended bridge diagrams $\mathcal{E}$. The caveat is that, since special destabilizations cannot be performed on diagrams with only two bridges, we get the restriction that $\mathcal{E}$ should have at least two A-arcs. This restriction will be dropped in the next section.

Corollary 4.4.5 (Corollary 2.22 of [24]). *Let $\mathcal{E}$ be an extended bridge diagrams representing a knot in S^3, which has at least two A-arcs. Then $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}) \simeq \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$.*

Proof. This follows directly from Lemma 4.4.3 and Lemma 4.4.4. □

4.5 Dropping the 2-bridge restriction

In this section, we will drop the 2-bridge condition on Corollary 4.4.5. The idea is that, for certain types of diagrams, although they do not achieve equivariant transversality, they behave as if they actually achieve it. Then our result will directly follow.

Recall that, given a knot K in S^3, we can combinatorially compute its knot Floer homology $\widehat{HFK}(S^3, K)$ in the following way. Choose any knot Heegaard diagram H_0 representing K. Theorem 1.2 of [47] tells us that we can convert H_0 into a nice diagram H using only isotopies and handleslides. Here, we say that H is nice if the follwing condition holds.

- Every region of H bounded by α- and β-curves, which does not contain basepoints of H, is either a bigon or a square. Then, by Theorem 3.3 and Theorem 3.4 of [47], for any Floer generators $\mathbf{x}$,$\mathbf{y}$ and a Whitney disk $\phi \in \pi_2^0(\mathbf{x}, \mathbf{y})$ with $\mu(\phi) = 1$, there exists a holomorphic representative of ϕ if and only if the domain $\mathcal{D}(\phi)$ of ϕ is either a bigon or a square which does not contain basepoints of H, and the moduli space of holomorphic representatives of ϕ is a point. This allows us to describe $\widehat{CFK}(S^3, K)$ and thus compute $\widehat{HFK}(S^3, K)$ in a combinatorial way.

We will now prove that we can also compute $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$ in a combinatorial way, without using desirable bridge diagrams of K as in Figure 1 of [20]. The key idea is to use diagrams satisfying the conditions similar to nice diagrams. For the technical terms arising in Lemma 4.5.2 involving details of the construction of $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology, one can find their definitions in section 3 of [20].

Definition 4.5.1. A knot Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ representing a based knot (K, z) in S^3 is **very nice** if the following conditions are satisfied.

- H is a nice diagram.
- The region of H containing w is a bigon.

Lemma 4.5.2 (Lemma 4.2 of [24]). *Let $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ be a very nice knot Heegaard diagram of a based knot (K, z) in S^3, and let $\hat{H} = (\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, z)$ be the diagram obtained by taking branched double cover of H along $\{z, w\}$ and forgetting w. Then there exists a $\mathbb{Z}_2$-equivariant homotopy coherent diagram $F : \mathscr{E}\mathbb{Z}_2 \to \overline{\mathcal{J}}$ of eventually cylindrical almost complex structures on $Sym^g(\tilde{\Sigma})$, where g is the genus of $\tilde{\Sigma}$, satisfying the following conditions.*

- *For every object $a \in Ob(\mathscr{E}\mathbb{Z}_2)$, every pair of generators $x, y \in \mathbb{T}_{\tilde{\boldsymbol{\alpha}}} \cap \mathbb{T}_{\tilde{\boldsymbol{\beta}}}$, and every $\phi \in \pi_2(x, y)$, the moduli space $\mathcal{M}(\phi; F(a))$ is transversely cut out.*
- *For every composable sequence $f_n, \cdots, f_1$ of morphisms of $\mathscr{E}\mathbb{Z}_2$ with $n \geq 2$, every pair of generators $x, y \in \mathbb{T}_{\tilde{\boldsymbol{\alpha}}} \cap \mathbb{T}_{\tilde{\boldsymbol{\beta}}}$, and every $\phi \in \pi_2(x, y)$ with $\mu(\phi) = 1-n$, whose domain does not intersect z, we have*

$$\mathcal{M}(\phi; F(f_n, \cdots, f_1)) = \emptyset.$$

Also, the same statement is true for the $\mathbb{Z}_2$-invariant Heegaard diagram $\mathcal{H}_d(\widetilde{H})$.

Proof. Choose any $\mathbb{Z}_2$-invariant complex structure $\mathfrak{j}_0$ on $\tilde{\Sigma}$. Then, by Theorem 3.4 of [47], a generic perturbation of α- and β-curves achieves transversality of moduli spaces $\mathcal{M}(\phi; \mathrm{Sym}^g(\mathfrak{j}_0))$ for any bigons or squares ϕ which does not intersect z. If the perturbation is sufficiently small, then it can be seen as an action of a self-diffeomorphism of $\tilde{\Sigma}$ which is sufficiently close to the identity. Thus there exists a 1-parameter family of complex structure $\mathfrak{j}$, isotopic to $\mathfrak{j}_0$, such that $\mathcal{M}(\phi; \mathrm{Sym}^g(\mathfrak{j}))$ achieves transversality for any bigons or squares ϕ which does not intersect z.

Let σ denote the deck transformation of the branched double covering $\tilde{\Sigma} \to \Sigma$. Since $\mathfrak{j}_0$ is $\mathbb{Z}_2$-invariant and $\mathfrak{j}$ is sufficiently close to $\mathfrak{j}_\circ$, $\sigma\mathfrak{j}$ is also close to $\mathfrak{j}_0$. So we can choose a generic $\mathbb{Z}_2$-coherent homotopy coherent diagram F of eventually cylindrical almost complex structures on $\tilde{\Sigma}$, consisting only of families of complex structures of the form $\mathrm{Sym}^g(\mathfrak{j}_s)$ for complex structures $\mathfrak{j}_s$ on $\tilde{\Sigma}$ which are C^∞-close to $\mathfrak{j}_0$, so that the first condition is satisfied. Also, since the moduli space of complex structures on $\tilde{\Sigma}$ has dimension $6g-6$ and thus finite-dimensional, we may assume that every element of those families is isotopic to $\mathfrak{j}_0$. Then such complex structures are pullbacks of $\mathfrak{j}_0$ under diffeomorphisms isotopic to the identity, so elements of $\mathcal{M}(\phi; F(f_n, \cdots, f_1))$ can be seen as a holomorphic disks in $\mathrm{Sym}^g(\tilde{\Sigma})$ with dynamic boundary conditions. Such disks must intersect positively with diagonals $\{p\} \times \mathrm{Sym}^{g-1}(\tilde{\Sigma})$ by holomorphicity when p is not contained in a neighborhood of α- and β-curves. Hence, if $\mathcal{M}(\phi; F(f_n, \cdots, f_1)) \neq \emptyset$, the domain $\mathcal{D}(\phi)$ of ϕ must be nonzero and positive.

Now, since H is assumed to be very nice, $\hat{H}$ is a nice diagram. Thus, by the proof of Theorem 3.3 of [47], positivity of $\mathcal{D}(\phi)$ and the condition $\mu(\phi) = 1 - n < 0$ implies that ϕ is either a bigon or a square, whose domain does not intersect z. However, such domains have Maslov index zero, so we get a contradiction. Furthermore, since the domains of $\mathcal{H}_d(\widetilde{H})$ are in 1-1 correspondence, preserving Maslov indices, with domains of $\hat{H}$, the same conclusion holds for $\mathcal{H}_d(\widetilde{H})$. □

Remark 4.5.3. The above definition also appears in [20], where the authors use the terminology "doubly nice", instead of our terminology "very nice". They also gave a proof of Lemma 4.5.2, which is more combinatorial than our proof.

Lemma 4.5.4 (Lemma 4.3 of [24])**.** *Let H be a very nice knot Heegaard diagram which represents a based knot (K, z) in S^3. Then we have*

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z) \simeq H^*(RHom_{\mathbb{F}_2[\mathbb{Z}_2]}(\widehat{CF}(\hat{H}), \mathbb{F}_2)) \simeq H^*(RHom_{\mathbb{F}_2[\mathbb{Z}_2]}(\widehat{CF}(\mathcal{H}_d(\widetilde{H})), \mathbb{F}_2)).$$

Proof. Let F be a $\mathbb{Z}_2$-equivariant homotopy coherent diagram arising in Lemma 4.5.2.

Then all higher terms of the differential of freed Floer complex $\widetilde{CF}_{\mathbb{Z}_2}(\mathcal{H}_d(\widetilde{H}); F)$ vanish by construction, and thus we get

$$\widehat{CF}_{\mathbb{Z}_2}(\Sigma(K), z) \simeq \widehat{CF}_{\mathbb{Z}_2}(\mathcal{H}_d(\widetilde{H})) \simeq \mathrm{RHom}_{\mathbb{F}_2[\mathbb{Z}_2]}(\widehat{CF}(\mathcal{H}_d(\widetilde{H})), \mathbb{F}_2).$$

Also, since $\mathcal{H}_d(\widetilde{H})$ is given by performing a $\mathbb{Z}_2$-invariant stabilization to the Heegaard diagram $\hat{H}$, the stabilization map

$$\widehat{CF}(\hat{H}) \to \widehat{CF}(\mathcal{H}_d(\widetilde{H}))$$

is a $\mathbb{Z}_2$-equivariant quasi-isomorphism for a suitable choice of almost complex structures. Therefore, by taking cohomology, we get the desired result. □

Now we will briefly prove that any based knot can be represented by a very nice knot Heegaard diagram by using Sarkar-Wang algorithm in an equivariant way.

Lemma 4.5.5 (Lemma 4.4 of [24])**.** *Every knot Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ representing a based knot (K, z) in S^3 is related to a very nice diagram by a sequence of isotopies and handleslides.*

Proof. This is already shown in the proof of Proposition 6.25 of [20]. Note that, in [20], the authors are using the terminology "doubly nice" instead of "very nice". □

To sum up, we have proved the following theorem.

Theorem 4.5.6 (Theorem 4.5 of [24])**.** *There is a combinatorial way to compute $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$ of based knots (K, z) from knot Heegaard diagrams of K.*

Proof. Choose any knot Heegaard diagram H_0 representing (K, z). Then, by Lemma 4.5.5, we can replace it with a very nice diagram H. By Lemma 4.5.4, one can compute $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$ using $\widehat{CF}(\hat{H})$. But the chain complex $\widehat{CF}(\hat{H})$ can be computed in a combinatorial way, by Theorem 3.3 and Theorem 3.4 of [47]. This gives a combitorial description of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$. □

Finally, recall that any (weakly admissible, doubly-pointed) knot Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ representing a based knot (K, z) in S^3, we have defined $\hat{H}$ as the $\mathbb{Z}_2$-equivariant Heegaard diagram defined by taking branched double cover of H along $\{z, w\}$ and forgetting w, and defined $\widetilde{H}$ as the extended bridge diagram obtained from H by forgetting w and adding an A-arc and a B-arc which start at w and do not intersect each other in their interior. In the previous section, we have proved that

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z) \simeq \widehat{HF}_{\mathbb{Z}_2}(\mathcal{H}_d(\widetilde{H})).$$

Now, using Theorem 4.5.6, we can remove the pair of an A-arc and a B-arc from $\widetilde{H}$.

Theorem 4.5.7 (Theorem 4.6 of [24])**.** *Let H be a weakly admissible knot Heegaard diagram representing a based knot (K, z) in S^3. Then we have*

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z) \simeq \widehat{HF}_{\mathbb{Z}_2}(\hat{H}).$$

Proof. By 4.5.5, H is related by a sequence of isotopies and handleslides to a very nice diagram H_0. Then $\widehat{H_0}$ is related by a sequence of equivariant isotopies and handleslides to $\hat{H}$. Hence, by Lemma 3.24 and Lemma 3.25 of [20], we have

$$\widehat{HF}_{\mathbb{Z}_2}(\hat{H}) \simeq \widehat{HF}_{\mathbb{Z}_2}(\widehat{H_0}).$$

But since H_0 is very nice, $\widehat{HF}_{\mathbb{Z}_2}(\widehat{H_0}) \simeq \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$ by 4.5.4. □

As a result, we can drop the 2-bridgeness condition from Corollary 4.4.5, as stated below.

Theorem 4.5.8 (Theorem 1.3 of [24])**.** *Let $\mathcal{E}$ be an extended bridge diagrams representing a knot in S^3, which has at least two A-arcs. Then $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}) \simeq \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$.*

4.6 A note on stabilizations of type II

Before proceeding to the next chapter, it is worth discussing here about the reason why stabilization maps of type II, which are defined a priori in the case where the stabilization is performed near the basepoint, is always an isomorphism. This discussion is necessary to give a proof of the well-definedness of general stabilization maps of type II, i.e. stabilization maps of type II in the case when the stabilization occurs away from the basepoint.

One might wonder why don't we define the general stabilization map using the special stabilization map. While it is possible to "define" the general stabiliization map of type II in such a way, proving its invariance under auxiliary choices seems to be a difficult question. So we will take an alternative approach, by defining the stabilization maps of type II as a composition of a birth map with a saddle map, as we did in the genus zero bridge diagram approach.

There are two technical obstacles lying in this approach. The first one is that the birth map is canonically defined only when it occurs near the basepoint; in the general case, one has to perform a birth near the basepoint and then move it via handleslides. The other obstacle is that we do not know whether the stabilization map of type II defined via this approach is an isomorphism.

In this section, we will prove that our stabilization map of type II is indeed an isomorphism when the stabilization occurs near the basepoint; the general case will be dealt with in the next chapter. Our strategy is to reduce the situation to the case of simple diagrams. Indeed, when we are given an extended bridge diagram which is very similar to a genus-zero bridge diagram, then we can easily see that applying stabilizations of type II to such a diagram always induce isomorphisms. To be precise, we have the following lemma. Note that simple extended bridge diagrams always satisfy its assumptions.

Lemma 4.6.1 (Lemma 2.14 of [24]). *Let* $\mathcal{E} = ((\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (F, A, B, z))$ *be an extended*

bridge diagram, and let $F = F_1 \cup F_2$ *be the unique partition of* F *such that every* $c \in A \cup B$ *satisfies* $|\partial c \cap F_1| = |\partial c \cap F_2| = 1$. *Suppose that every connected component* R *of* $\Sigma \setminus \bigcup(\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ *such that* $z \notin R$ *satisfies* $|R \cap F_1| = |R \cap F_2|$. *Then applying a stabilization of type II to* $\mathcal{E}$ *induces an isomorphism of equivariant Floer cohomology.*

Proof. Recall that, in the paper [20], the proof that stabilizations of bridge diagrams induce isomorphisms use equivariant transversality. That proof can be directly extended to our case, so if the $\mathbb{Z}_2$-action on $\mathcal{H}_d(\mathcal{E})$ achieves equivariant transversality, then a stabilization of type II induces an isomorphism.

For any domain D of $\mathcal{H}_d(\mathcal{E})$ from a Floer generator $\mathbf{x}$ to a generator $\mathbf{y}$, the Maslov index formula in the paper [30] reads:

$$\mu(D) = n_{\mathbf{x}}(D) + n_{\mathbf{y}}(D) + e(D),$$

where $n_{\mathbf{x}}, n_{\mathbf{y}}$ are the point measures at $\mathbf{x}$ and $\mathbf{y}$, respectively, and e is the Euler measure. Let $p : \tilde{\Sigma} \to \Sigma$ be the branched double covering map with the branching locus F. Suppose that D is $\mathbb{Z}_2$-invariant. Then $\mathbf{x}$ and $\mathbf{y}$ are also $\mathbb{Z}_2$-invariant, and thus we may assume without loss of generality that $\mathbf{x} = \mathbf{x}' \cup F_1$ and $\mathbf{y} = \mathbf{y}' \cup F_1$, where $\mathbf{x}', \mathbf{y}'$ are Floer generators in $\mathcal{H}_{pt}(\mathcal{E})$. By the assumption that every $c \in A \cup B$ satisfies $|\partial c \cap F_1| = |\partial c \cap F_2|$, the Maslov index of the domain p^*D, as defined in the proof of Proposition 4.3.1, is given as follows.

$$\begin{aligned}
\mu(p^*D) &= n_{\mathbf{x}}(p^*D) + n_{\mathbf{y}}(p^*D) + e(p^*D) \\
&= (2n_{\mathbf{x}}(D) + \sum_{y \in F_1} n_y(D)) + (2n_{\mathbf{y}}(D) + \sum_{y \in F_1} n_y(D)) + (2e(D) - \sum_{y \in F} n_y(D)) \\
&= 2\mu(D) + \left(\sum_{y \in F_1} n_y(D) - \sum_{y \in F_2} n_y(D) \right) \\
&= 2\mu(D).
\end{aligned}$$

Therefore, hypothesis (EH-2) is satisfied, and thus $\mathcal{E}$ achieves equivariant transver-

sality. □

We also need a way to "reduce" the situation to the diagrams considered in Lemma 4.6.1. For that, we need the following Floer-theoretic lemma.

Lemma 4.6.2. *Let $(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\delta}}, z)$ be an involutive Heegaard 6-tuple with base $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, z)$, and $\theta_{\beta,\gamma}, \theta_{\gamma,\delta}$ be $\mathbb{Z}_2$-invariant cycles in $\widehat{CF}(\tilde{\Sigma}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, z), \widehat{CF}(\tilde{\Sigma}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\delta}}, z)$, respectively. Suppose that, for any Floer generator $\mathbf{x}$ of $(\tilde{\Sigma}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\delta}}, z)$, every element of $\pi_2(\theta_{\beta,\gamma}, \theta_{\gamma,\delta}, \mathbf{x})$ has Maslov index at least 0 and achieves transversality for generic $\mathbb{Z}_2$-equivariant families of almost complex structures. Then, for any $x_{\alpha,\beta} \in H_*(\widetilde{CF}_{\mathbb{Z}_2}(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \otimes_{\mathbb{F}_2[\mathbb{Z}_2]} \mathbb{F}_2)$, we have*

$$\hat{f}_{\alpha,\gamma,\delta}(\hat{f}_{\alpha,\beta,\gamma}(x_{\alpha,\beta} \otimes \theta_{\beta,\gamma}) \otimes \theta_{\gamma,\delta}) = \hat{f}_{\alpha,\beta,\delta}(x_{\alpha,\beta} \otimes f_{\beta,\gamma,\delta}(\theta_{\beta,\gamma} \otimes \theta_{\gamma,\delta})),$$

where $\hat{f}$ denotes the equivariant triangle maps, defined in [20], and f denotes the ordinary triangle map, with repsect to their sub-indices.

Proof. By assumption, the cycle $f_{\beta,\gamma,\delta}(\theta_{\beta,\gamma} \otimes \theta_{\gamma,\delta}) \in \widehat{CF}_{\mathbb{Z}_2}(\tilde{\Sigma}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\delta}}, z)$ is $\mathbb{Z}_2$-invariant, so that the term

$$\hat{f}_{\alpha,\beta,\delta}(x_{\alpha,\beta} \otimes f_{\beta,\gamma,\delta}(\theta_{\beta,\gamma} \otimes \theta_{\gamma,\delta}))$$

is well-defined. Using square maps from non-equivariant Floer theory, we can construct the equivariant square map

$$\hat{s}_{\alpha,\beta,\gamma,\delta} : \widetilde{CF}_{\mathbb{Z}_2}(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, z) \to \widetilde{CF}_{\mathbb{Z}_2}(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\delta}}, z),$$

by mimicing the construction of equivariant triangle maps, as in the proof of Lemma 3.1.7. If a sequence of holomorphic squares in $(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\delta}}, z)$ having $\theta_{\beta,\gamma}, \theta_{\gamma,\delta}$ as two of their vertices degenerates to a concatenation of a holomorphic triangle in $(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\delta}}, z)$ and another holomorphic triangle in $(\tilde{\Sigma}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\delta}}, z)$, the singular vertex should be the points in the cycle $f_{\beta,\gamma,\delta}(\theta_{\beta,\gamma} \otimes \theta_{\gamma,\delta})$ by assumption. Hence, if we choose

a cycle representative $\mathbf{x}_{\alpha,\beta}$ of $x_{\alpha,\beta}$, the quantity

$$\hat{f}_{\alpha,\gamma,\delta}(\hat{f}_{\alpha,\beta,\gamma}(\mathbf{x}_{\alpha,\beta}\otimes\theta_{\beta,\gamma})\otimes\theta_{\gamma,\delta})+\hat{f}_{\alpha,\beta,\delta}(\mathbf{x}_{\alpha,\beta}\otimes f_{\beta,\gamma,\delta}(\theta_{\beta,\gamma}\otimes\theta_{\gamma,\delta}))$$

should be the same as the quantity

$$d\hat{s}_{\alpha,\beta,\gamma,\delta}(\mathbf{x}_{\alpha,\beta}\otimes\theta_{\beta,\gamma}\otimes\theta_{\gamma,\delta})+\hat{s}_{\alpha,\beta,\gamma,\delta}(d\mathbf{x}_{\alpha,\beta}\otimes\theta_{\beta,\gamma}\otimes\theta_{\gamma,\delta}).$$

Therefore, by passing to the homology, we get the desired result. □

Now, given an weakly extended bridge diagram $\mathcal{E}$ which represents a knot in S^3, we have a following commutative diagram of weakly admissible extended bridge diagrams, where the vertical arrows are stabilizations of type II and horizontal arrows are either isotopies, handleslides of type I, II, III, IV, stabilizations of type I, or special stabilizations, and $\mathcal{E}_n$, $\mathcal{E}_n^{st}$ are both simple.

$$\begin{array}{ccccccc}
\mathcal{E} & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{E}_{n-1} \longrightarrow \mathcal{E}_n \\
\downarrow{\scriptstyle \text{stab}} & & \downarrow{\scriptstyle \text{stab}} & & & & \downarrow{\scriptstyle \text{stab}} \quad \downarrow{\scriptstyle \text{stab}} \\
\mathcal{E}^{st} & \longrightarrow & \mathcal{E}_1^{st} & \longrightarrow & \cdots & \longrightarrow & \mathcal{E}_{n-1}^{st} \longrightarrow \mathcal{E}_n^{st}
\end{array}$$

Translating this diagram into equivariant Floer cohomology and maps between them gives the following diagram.

$$\begin{array}{ccccccc}
\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_n^{st}) & \xrightarrow{\sim} & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_{n-1}^{st}) & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_1^{st}) \xrightarrow{\sim} \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}^{st}) \\
\downarrow{\scriptstyle \sim} & & \downarrow & & & & \downarrow \quad \downarrow \\
\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_n) & \xrightarrow{\sim} & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_{n-1}) & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_1) \xrightarrow{\sim} \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E})
\end{array}$$

All horizontal arrows are isomorphisms. We also know that, since $\mathcal{E}$ is simple, the leftmost vertical arrow is also an isomorphism by Lemma 4.6.1. So, if this diagram is commutative, we can deduce that the map $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}^{st}) \to \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E})$, which is the map induced by a stabilization of type II applied to $\mathcal{E}$, is an isomorphism.

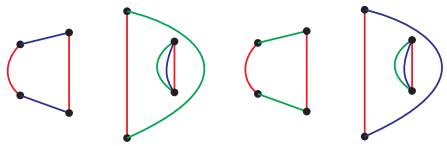


Figure 4.2: The diagram on the left is the triple-diagram for the case when a stabilization map of type II is composed first and then a handleslide map of type IV is composed, and the diagram on the right is for the case when the maps are composed in the opposite order.

Lemma 4.6.3. *The maps induced by extended Heegaard moves, and special stabilizations commute with the maps induced by stabilizations of type II.*

Proof. The statement is obvious for diffeomorphisms and special stabilizations. Also, the statement for isotopy maps is a direct consequence of Lemma 6.2 of [25]. The remaining cases involve associativity of equivariant triangle maps, and thus we would like to use Lemma 4.6.2.

Here, we will only give a proof for the commutation of stabilization maps of type II with handleslides of type IV, since other cases can be proven using the same argument. To give a proof for this case, we consider the two triple-diagram in Figure 4.2. In (the branched double covers of) the given triple-diagrams, there are clearly no nonconstant triangles, and the constant triangles have Maslov index 0. Also, the induced (non-equivariant) triangle maps sends the top classes to the unique $\mathbb{Z}_2$-invariant top class. Therefore, by Lemma 4.6.2, the maps induced by extended Heegaard moves and special stabilizations commute with the maps induced by stabilizations of type II. □

Theorem 4.6.4. *Given an extended bridge diagram $\mathcal{E}$ which represents a knot in S^3, let $\mathcal{E}'$ be the extended bridge diagram we get by applying an extended Heegaard move to $\mathcal{E}$. Suppose that both $\mathcal{E}$ and $\mathcal{E}'$ have at least two A-arcs. Then the associated map $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}') \to \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E})$ is an isomorphism.*

Proof. We only have to consider the case when $\mathcal{E}'$ is obtained from $\mathcal{E}$ by a stabilization

of type II. In this case, recall that we had the following diagram.

$$\begin{array}{ccccccccc}
\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_n^{st}) & \xrightarrow{\sim} & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_{n-1}^{st}) & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_1^{st}) & \xrightarrow{\sim} & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}^{st}) \\
\downarrow \sim & & \downarrow & & & & \downarrow & & \downarrow \\
\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_n) & \xrightarrow{\sim} & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_{n-1}) & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_1) & \xrightarrow{\sim} & \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E})
\end{array}$$

We know from Lemma 4.6.3 that this diagram is commutative. Therefore, all vertical arrows should be isomorphisms. In particular, the map $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}') \to \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E})$, induced by a stabilization of type II, is an isomorphism. □

It is worth noting that we can drop the 2-bridge condition from Theorem 4.6.4. However, doing so would require us to use tools that will be introduced in the next chapter. So we will use the theorem without dropping that condition. This is possible since, as we will see in the next chapter, we can do everything under the restriction of having the 2-bridgeness condition.

Chapter 5

$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ is a strong Heegaard invariant

The results that we have proven in the previous chapter imply that, given any weakly admissible doubly-pointed Heegaard diagram of a knot $K \subset S^3$, taking $\widehat{HF}_{\mathbb{Z}_2}$ of its branched double cover (with the second basepoint forgotten) allows us to compute $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$. From this fact, a few related questions naturally arise. If we try to define $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ purely in terms of doubly-pointed Heegaard diagrams of K, without referring to genus-zero bridge diagrams of K, do we still get a natural invariant of K? If we do, then is it *naturally* isomorphic to the original definition of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$? We will give positive answers to those questions in this chapter.

5.1 Weak and strong Heegaard invariants

In this section, we review the concept of weak and strong Heegaard invariants, defined by Juhasz and Thurston in [23]. Since we are working directly with knot Heegaard diagrams, instead of more general sutured Heegaard diagrams, the definitions in this section are modified slightly to fit in this context.

5.1.1 Isotopy diagrams and graphs

Definition 5.1.1. An isotopy diagram $(\Sigma, [\boldsymbol{\alpha}], [\boldsymbol{\beta}], z, w)$ is an isotopy class of a knot Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$. In other words, an isotopy diagram specifies isotopy classes of α-curves and β-curves instead of specifying precise locations of α-curves and β-curves.

Definition 5.1.2. We say that an isotopy diagram (Σ, A, B, z, w) represents an isotopy class $[K, p]$ of a based knot (K, p) in S^3 if there exists a knot Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ representing (K, p), where $z = p$, $A = [\boldsymbol{\alpha}]$, and $B = [\boldsymbol{\beta}]$.

Notice that we can also apply handleslides, (de)stabilizations, and diffeomorphisms to isotopy diagrams, instead of knot Heegaard diagrams. We do not have to apply isotopy to isotopy diagrams as such diagrams are already given up to isotopy.

Now we briefly recall the notion of a graph, from a categorical viewpoint.

Definition 5.1.3. A graph G consists of

- a class $|G|$ whose elements are called the objects (or vertices) of the graph.
- for each pair (A, B) of objects of G, a set $G(A, B)$ whose elements are called the morphisms (or arrows) from A to B.

Definition 5.1.4. A morphism of graphs $F : G \to K$ between two graphs G and K consists of

- a map $F : |G| \to |K|$,
- for each pair (A, B) of objects in G, a map

$$F : G(A, B) \to K(F(A), F(B)).$$

Notice that every category is a graph, and every functor between categories is also a morphism of graphs.

Definition 5.1.5. Let $\mathcal{G}$ be the graph whose class of vertices $|\mathcal{G}|$ consists of isotopy diagrams and, for $H_1, H_2 \in |\mathcal{G}|$, the edges $\mathcal{G}(H_1, H_2)$ can be written as a union of four sets

$$\mathcal{G}(H_1, H_2) = \mathcal{G}_\alpha(H_1, H_2) \cup \mathcal{G}_\beta(H_1, H_2) \cup \mathcal{G}_{stab}(H_1, H_2) \cup \mathcal{G}_{diff}(H_1, H_2).$$

The set $\mathcal{G}_\alpha(H_1, H_2)$ consists of a single arrow if H_1 and H_2 are α-equivalent and is empty otherwise. The set $\mathcal{G}_\beta(H_1, H_2)$ is defined similarly using β-equivalence. The set $\mathcal{G}_{stab}(H_1, H_2)$ consists of a single arrow if H_2 is obtained by a (de)stabilization of H_1 and is empty otherwise. Finally, $\mathcal{G}_{diff}(H_1, H_2)$ consists of diffeomorphisms from H_1 to H_2.

Definition 5.1.6. Let $\mathcal{K}$ be the set of isotopy types of based knots in S^3, and let $\mathcal{C}$ be any category. We denote by $\mathcal{G}(\mathcal{K})$ the full subgraph of $\mathcal{G}$ spanned by isotopy diagrams H representing an object in $\mathcal{K}$. A ($\mathcal{C}$-valued) weak Heegaard invariant (of $\mathcal{K}$) is a morphism of graphs $F : \mathcal{G}(\mathcal{K}) \to \mathcal{C}$ such that for every morphism e of $\mathcal{G}(\mathcal{K})$, its image $F(e)$ is an isomorphism.

In other words, a ($\mathcal{C}$-valued) weak Heegaard invariant (of $\mathcal{K}$) is a rule which assigns an object of $\mathcal{C}$ to each isotopy diagram and an isomorphism of $\mathcal{C}$ to each Heegaard move between isotopy diagrams. An important point here is that the assignment need not be functorial in any sense. Indeed, even when we are given a rule in which the isomorphisms induced by Heegaard moves are not uniquely defined, we can make arbitrary choices among those isomorphisms to form a weak Heegaard invariant.

Now we will define the notion of a strong Heegaard invariant. In short, it is a weak Heegaard invariant which satisfies few properties which ensures naturality of the invariant.

Definition 5.1.7. Let $H_i = (\Sigma_i, [\boldsymbol{\alpha}_i], [\boldsymbol{\beta}_i], z, w)$ be isotopy diagrams for $i = 1, \cdots, 4$.

A distinguished rectangle in $\mathcal{G}$ is a subgraph

$$\begin{array}{ccc} H_1 & \xrightarrow{e} & H_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ H_3 & \xrightarrow{h} & H_4 \end{array}$$

of $\mathcal{G}$ that satisfies one of the following properties.

1. Both e and h are α-equivalences, while both f and g are β-equivalences.

2. Both e and h are α- or β-equivalences, while f and g or both stabilizations.

3. Both e and h are α- or β-equivalences, while f and g are both diffeomorphisms. In this case, we necessarily have $\Sigma_1 = \Sigma_2$ and $\Sigma_3 = \Sigma_4$, and we require in addition that the diffeomorphisms $\Sigma_1 \xrightarrow{f} \Sigma_3$ and $\Sigma_2 \xrightarrow{g} \Sigma_4$ are the same.

4. The maps e, f, g, h are all stabilizations, such that there are disjoint disks $D_1, D_2 \subset \Sigma_1$ and disjoint punctured tori $T_1, T_2 \subset \Sigma_4$ satisfying $\Sigma_1 \backslash (D_1 \cup D_2) = \Sigma_4 \backslash (T_1 \cup T_2)$, $\Sigma_2 = (\Sigma_1 \backslash D_1) \cup T_1$, and $\Sigma_3 = (\Sigma_1 \backslash D_2) \cup T_2$.

5. The maps e, h are stabilizations, while f, g are diffeomorphisms. Furthermore, there are disks $D \subset \Sigma_1$ and $D' \subset \Sigma_4$ and punctured tori $T \subset \Sigma_2$ and $T' \subset \Sigma_4$ such that $\Sigma_1 \backslash D = \Sigma_2 \backslash T$, $\Sigma_3 \backslash D' = \Sigma_4 \backslash T'$, and the diffeomorphisms f, g satisfy $f(D) = D'$, $g(T) = T'$, and $f|_{\Sigma_1 \backslash D} = g|_{\Sigma_2 \backslash T}$.

Definition 5.1.8. (Definition 2.33 of [23]) A weak Heegaard invariant F is a strong Heegaard invariant if the following axioms hold.

- (Functoriality) The restriction of F to $\mathcal{G}_\alpha, \mathcal{G}_\beta, \mathcal{G}_{\text{diff}}$ are functors, and for any stabilization e and its reverse destabilization e', we have $F(e') = F(e)^{-1}$.

- (Commutativity) For any distinguished rectangle D, the diagram $F(D)$ is commutative.

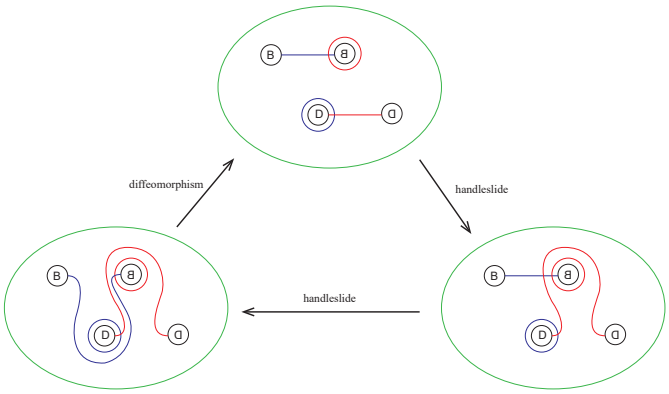
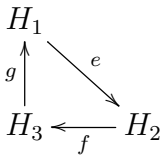


Figure 5.1: A simple handleswap diagram.

- (Continuity) If e is a loop-edge of $\mathcal{G}$ which is a diffeomorphism isotopic to the identity, then $F(e)$ is the identity.
- (Handleswap invariance) For every simple handleswap



in $\mathcal{G}$, we have $F(g) \circ F(f) \circ F(e) = \mathrm{id}$.

Here, a **simple handleswap** is a diagram in $\mathcal{G}$ which can be locally represented as in Figure 5.1.

5.2 A Floer-theoretic lemma

In this section, we state a Floer-theoretic lemma which were not used in the previous chapters but will be used in this chapter. As its proof is basically the same as in Lemma 4.6.2, except that we do not need additional conditions on triangles, we do not write down its proof here.

Lemma 5.2.1 (Lemma 2.19 of [24]). *Let $(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\delta}}, z)$ be an involutive Heegaard 6-tuple with base $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, z)$, and $\theta_{\alpha,\beta}, \theta_{\gamma,\delta}$ be $\mathbb{Z}_2$-invariant cycles in $\widehat{CF}(\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, z)$ and $\widehat{CF}(\tilde{\Sigma}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\delta}}, z)$, respectively. Then, for any $x_{\beta,\gamma} \in H_*(\widetilde{CF}_{\mathbb{Z}_2}(\tilde{\Sigma}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}) \otimes_{\mathbb{F}_2[\mathbb{Z}_2]} \mathbb{F}_2)$, we have*

$$\hat{f}_{\alpha,\gamma,\delta}(\hat{f}_{\alpha,\beta,\gamma}(\theta_{\alpha,\beta} \otimes x_{\beta,\gamma}) \otimes \theta_{\gamma,\delta}) = \hat{f}_{\alpha,\beta,\delta}(\theta_{\alpha,\beta} \otimes f_{\beta,\gamma,\delta}(x_{\beta,\gamma} \otimes \theta_{\gamma,\delta})),$$

where $\hat{f}$ denotes the equivariant triangle maps, defined in [20], and f denotes the ordinary triangle map, with respect to their subindices.

5.3 Construction of a weak Heegaard invariant

Given a based knot (K, z) in S^3, choose a knot Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$, so that one of the two basepoints of H (which is z here) is the basepoint of the given based knot. Recall that, by distinguishing the two basepoints z and w, we may say that $\mathcal{H}$ represents the based knot (K, z).

Definition 5.3.1. We say that a knot Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ represents a based knot (K, p) in S^3 if H repesents K and $z = p$. Here, we call w as the **second basepoint**.

Given such a diagram H, we can naturally define an element $w \in H^2(\Sigma \backslash \{z, w\}; \mathbb{F}_2)$ using the following formula.

$$w([\alpha_i]) = w([\beta_j]) = 0 \text{ for each } \alpha_i \in \boldsymbol{\alpha} \text{ and } \beta \in \boldsymbol{\beta}$$
$$w([c]) = 1 \text{ where } c \text{ is a small circle around } z$$

Since $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ represents S^3, the homology classes $[\alpha_i]$ and $[\beta_j]$ linearly span $H_2(\Sigma \backslash \{z, w\}; \mathbb{F}_2)$, so the above formula defines w uniquely. The element w defines a map $\pi_1(\Sigma \backslash \{z - w\}) \to \mathbb{Z}_2$, thus induces a branched double covering $\tilde{\Sigma} \xrightarrow{f} \Sigma$, branched

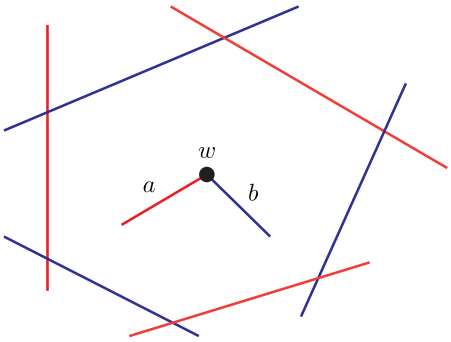


Figure 5.2: Attaching a small A-arc and a B-arc at the point w.

along $\{z, w\}$. Denote the sets of inverse images of alpha-curves and beta-curves by $\tilde{\boldsymbol{\alpha}}$ and $\tilde{\boldsymbol{\beta}}$. Also, add a small A-arc a and a B-arc b at w, as drawn in Figure 5.2. Then the Heegaard diagram $\tilde{H} = ((\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}), (\{z, w\}, \{a\}, \{b\}, z))$ is a Heegaard diagram of $\Sigma(K)$ with a $\mathbb{Z}_2$-action.

Since H can also be regarded as a 1-bridge knot diagram of K with a hidden pair of bridges a' and b' connected to z, the pair $\mathcal{E}_H = ((\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}), (\{z, w\}, \{a, a'\}, \{b, b'\}, z))$ is an extended bridge diagram representing K. By the definition of $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_H)$, we have

$$\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_H) \simeq \widehat{HF}_{\mathbb{Z}_2}(\mathcal{H}_d(\tilde{H})).$$

Together with Corollary 4.4.5, this gives a way to compute the $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology of a knot from its knot Heegaard Floer cohomology.

Lemma 5.3.2 (Lemma 3.2 of [24])**.** *Let H be a knot Heegaard diagram of a based knot (K, z) in S^3. Then $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z) \simeq \widehat{HF}_{\mathbb{Z}_2}(\tilde{H})$.*

Proof. By Corollary 4.4.5, we have $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z) \simeq \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_H) \simeq \widehat{HF}_{\mathbb{Z}_2}(\tilde{H})$. □

Recall that any two knot Heegaard diagrams of a given based knot are related by a sequence of Heegaard moves, namely, isotopies, handleslides, (de)stabilizations, and diffeomorphisms. We claim that these basic moves induce isomorphisms of $\widehat{HF}_{\mathbb{Z}_2}$, thereby proving that the equivariant Heegaard Floer cohomology is a weak Heegaard invariant.

Theorem 5.3.3 (Theorem 3.3 of [24]). *Let H, H' be two knot Heegaard diagrams of a based knot (K, z), related by a Heegaard move. Then there exists an induced isomorphism*

$$\hat{f}^{basic}_{\mathcal{H}\to\mathcal{H}'} : \widehat{HF}_{\mathbb{Z}_2}(\tilde{H}) \to \widehat{HF}_{\mathbb{Z}_2}(\tilde{H}').$$

Proof. The isotopy case is obvious, and the handleslide case follows directly from Proposition 3.24 of [20], which proves the invariance of equivariant Floer cohomology under equivariant Hamiltonian isotopies of Lagrangians. The stabilization case is just an application of the standard neck-stretching argument, after assuming that it happens near the basepoint. The diffeomorphism case was already covered in subsection 1.3.1 of chapter III. □

Corollary 5.3.4 (Corollary 3.4 of [24]). *The $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology of based knots in S^3 is a weak Heegaard invariant.*

Proof. The statement of Theorem 5.3.3 is precisely the definition of a weak Heegaard invariant. □

5.4 Functoriality axiom and well-definedness of general stabilization maps

Now we will start showing that the $\mathbb{Z}_2$-equivariant Heegaard Floer cohomology is not only a weak invariant, but also a strong Heegaard invariant, by checking its axioms one by one.

Lemma 5.4.1 (Lemma 3.7 of [24]). *Consider the following commutative diagram $\mathcal{D}$ of extended bridge diagram of a based knot in S^3.*

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{e} & \mathcal{E}_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathcal{E}_3 & \xrightarrow[e']{} & \mathcal{E}_4 \end{array}$$

Suppose that e, e' are α-equivalences or A-equivalences, and f, f' are β-equivalences or B-equivalences. Then $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{D})$ is a commutative diagram, i.e.

$$\widehat{HF}_{\mathbb{Z}_2}(e) \circ \widehat{HF}_{\mathbb{Z}_2}(f') = \widehat{HF}_{\mathbb{Z}_2}(f) \circ \widehat{HF}_{\mathbb{Z}_2}(e').$$

Proof. This is a direct application of Lemma 5.2.1. □

Using the above lemma, we first prove that stabilization maps of type II are well-defined in the general case.

Proposition 5.4.2 (Proposition 3.8 of [24])**.** *Let $\mathcal{E}_1, \mathcal{E}_2$ be two extended bridge diagrams of a based knot (K, z) in S^3, related by a single stabilization of type II, and let $\mathcal{E}_1 \xrightarrow{s} \mathcal{E}_2$ be such a stabilization. Then the induced map*

$$\widehat{HF}_{\mathbb{Z}_2}(s) \,:\, \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_2) \to \widehat{HF}(\mathcal{E}_1)$$

is defined uniquely.

Proof. Recall that the general type II stabilization map is defined as the composition of a stabilization of type II near the basepoint, together with a sequence of handleslides of type II and IV. To prove the proposition, we have to show that, if we perform a sequence of handleslides as in Figure 5.3, along a simple closed curve, then the composition of all handleslide maps involved is the identity. By Lemma 5.4.1, we only have to prove that the composition of all handleslide maps involving A-arcs and α-curves is the identity.

Choose a sequence $\mathcal{E}_1 \to \mathcal{E}_2 \to \cdots \to \mathcal{E}_n$ of handleslides, involved in defining a stabilization map of type II. Without loss of generality, we can assume that all handleslides are those involving A-arcs and α-curves. We claim that the composed map $F_n \,:\, \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_1) \to \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_n)$ is the same as the equivariant triangle map given by the unique ($\mathbb{Z}_2$-invariant) top cocycle $\mathbf{x}_n$, defined as in Figure 5.4. Note that this directly implies the proposition by the continuity axiom of $\widehat{HF}_{\mathbb{Z}_2}$, which will be

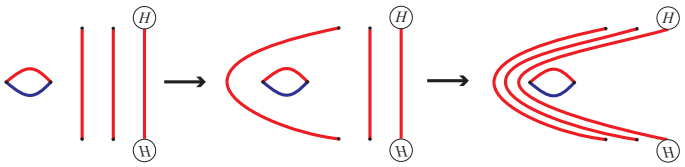


Figure 5.3: A sequence of handleslides, involving A-arcs and α-curves, used to define a stabilization map of type II. The circles with a letter H are those where a handle is attached.

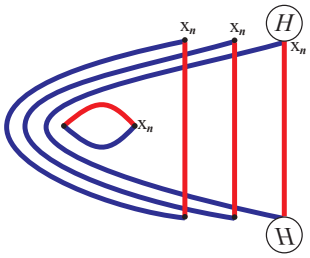


Figure 5.4: The unique $\mathbb{Z}_2$-invariant cocycle $\mathbf{x}_n$.

proven in Section 5.5. This is not a circular reasoning because stabilization maps of type II are not used in the proof of the continuity axiom.

We proceed by induction. When $n = 2$, the hypothesis is obviously true. Suppose that the hypothesis is true for some $n \geq 2$. Then, since there are no nonconstant triangles with two prescribed vertices $\mathbf{x}_n$ and $\mathbf{x}$ in the (projected) Heegaard triple-diagram in Figure 5.5, we see that the map F_{n+1} is precisely the equivariant triangle map given by the $\mathbb{Z}_2$-invariant cocycle $\mathbf{x}_{n+1} = \mathbf{x}_n \cup \mathbf{x}$, by Lemma 4.6.2. This finishes the inductive step. □

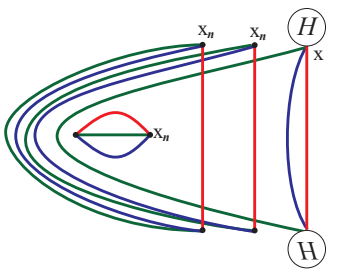


Figure 5.5: A Heegaard triple-diagram (with triple intersections). Note that $\mathbf{x}_{n+1} = \mathbf{x}_n \cup \mathbf{x}$.

Having settled the well-definedness of stabilization maps of type II, we can now prove that $\widehat{HF}_{\mathbb{Z}_2}$ satisfies the functoriality axiom.

Lemma 5.4.3 (Lemma 3.9 of [24]). *The restriction of the weak Heegaard invariant $\widehat{HF}_{\mathbb{Z}_2}$ to $\mathcal{G}_\alpha$ is a functor.*

Proof. In this proof, we will use knot bridge diagrams of nontrivial genera and their $\widehat{HF}_{\mathbb{Z}_2}$. The idea is to use the naturality of $\widehat{HF}_{\mathbb{Z}_2}$ for bridge diagrams on a sphere, which is already established by the author in [25].

Let $\{f_i : H_i \to H_{i+1} \,|\, i = 1, \cdots, n\}$ be a composable sequence of α-equivalences of knot Heegaard diagrams of a based knot (K, z) in S^3. Each f_i is either an isotopy of α-curves or an α-handleslide, which can be translated to a 1-parameter family of Morse-Smale pairs on S^3, as in Proposition 6.35 of [23]. Composing them gives a 1-parameter family $(f_t, v_t)_{t\in[0,1]}$ of simple pairs, where (f_t, v_t) is Morse-Smale except for finitely many t. Also, there exists a Heegaard surface $\Sigma \subset S^3$ which intersects transversely with K, contains the basepoint z, and is shared by every (f_t, v_t), since no stabilizations or destabilizations occur.

Given such a family and a Heegaard surface Σ, we now choose an isotopy $\{K_t\}_{t\in[0,1]}$ of the given based knot (K, z) while fixing the basepoint z, such that $K_0 = K$, K_1 is an isotopic copy of K which is contained in a very small neighborhood of z and intersects transversely with Σ. We can further assume that the given isotopy $\{K_t\}$ is generic, so that K_t is transverse to Σ for all but finitely many t. Since codimension-1 singularities of $\{K_t\}$ correspond to simple tangencies, we know that, when passing through those "finitely many" t, a pair of intersection points in $K_t \cap \Sigma$ is either created or annihilated.

We now take the product of the two 1-parameter families mentioned above to get a smooth 2-parameter family $\mathcal{F}$:

$$\mathcal{F} = \{(f_t, v_t, K_s)\}_{(t,s)\in[0,1]\times[0,1]}.$$

We can then perturb $\mathcal{F}$ in the space

$$\mathcal{S} = \{\text{simple pairs in } S^3\} \times \{\text{knots in } S^3,\ \text{isotopic to } K \text{ and containing } z\}$$

to another generic 2-parameter family $\mathcal{F}'$ close to $\mathcal{F}$ in $\mathcal{S}$, which would have finitely many codimension-2 singularities and no higher singularity. But since both (f_t, v_t) and $\{K_s\}$ were generic, by taking the perturbation to be sufficiently small, we may assume that every codimension-2 singularity arising in $\mathcal{F}'$ is a combination of a codimension-1 singularity in the space of simple pairs in S^3 and a codimension-1 singularity in the space of based knots in S^3. Furthermore, by closeness, no stabilization/destabilization would occur as a codimension-1 singularity. Therefore we only have to consider the following two possible types of codimension-2 singularities in $\mathcal{S}$.

1. An α-handleslide occurs and K_s is (simple) tangent to Σ; K_s is transverse to all stable/unstable manifolds of v_t.

2. An α-handleslide occurs and K_s intersects transversely with the unstable manifold of a order 1 singularity of v_t or the stable manifold of an order 2 singularity of v_t.

3. An α-handleslide occurs and K_s is tangent to the stable manifold of a order 1 singularity of v_t or the unstable manifold of an order 2 singularity of v_t.

4. An α-handleslide occurs, K_s is transverse to Σ and all stable/unstable manifolds of v_t, and the projection of K_s to Σ via the flow of v_t has a simple tangency.

The monodromies of these singularities can be translated as shown below. Note that, in the case 1, we have replaced stabilizations of type II by special stabilizations, as a stabilization of type II can be seen as the composition of a special stabilization followed by a sequence of handleslides of type II, IV, and (A/α)-isotopies.

1. Commutation of α-handleslides with stabilizations of type II

2. Commutation of α-handleslides with handleslides of type II

3. Commutation of α-handleslides with compositions of stabilizations of type II and handleslides of type III

4. Commutation of α-handleslides with compositions of stabilizations of type II and handleslides of type IV

For a type 2 monodromy, Lemma 4.6.2 can be used to prove its commutativity, as in the proof of Lemma 4.6.3. The type 1 monodromy is clearly commutative from the construction of maps associated to special stabilizations, which was given in the proof of Lemma 4.4.4. A similar argument can be applied to monodromies of type 3 or 4.

It remains to show that α-handleslide maps and isotopy maps commute with isotopy maps. Note that this commutation corresponds to the regions of $\mathcal{F}'$ which does not contain codimension-2 singularity, which can actually be proven by a direct application of the proof of naturality of $\widehat{HF}_{\mathbb{Z}_2}$ for genus-zero bridge diagrams.. Therefore, as in its proof, we have shown that the diagram in Table 5.4.1 commutes after applying $\widehat{HF}_{\mathbb{Z}_2}$, where $\mathcal{E}_i^j$ are extended bridge diagrams and arrows are extended Heegaard moves. Note that leftmost column and the rightmost column must be identical, i.e. $\mathcal{E}_1^i = \mathcal{E}_n^i$ for all $i = 1, \cdots, k$, and the extended bridge diagrams $\mathcal{E}_1^k, \cdots, \mathcal{E}_n^k$ in the bottom row satisfy the property that all A-arcs and B-arcs are contained in the region, bounded by α-curves and β-curves, which contains the basepoint. Also, the handleslides among the arrows in the bottom row can only be handleslides of type I.

Since the extended bridge diagrams appearing in the top row are of the form $\mathcal{B}\sharp\mathcal{H}$ for a genus 0 bridge diagram $\mathcal{B}$ and a (weakly admissible) Heegaard diagram $\mathcal{H}$, where the connected sum is taken near the basepoint z of $\mathcal{H}$, they clearly achieve hypothesis (EH-2) in [20], they satisfy equivariant transversality, so that we have a

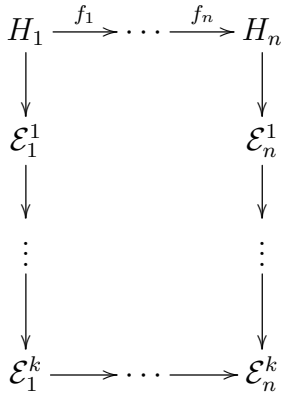


Table 5.4.1: A diagram of extended bridge diagrams and extended Heegaard moves, which becomes commutative after taking $\widehat{HF}_{\mathbb{Z}_2}$. What we have actually shown is that the interior of this diagram can be filled with smaller diagrams which represent monodromies around the singularities of $\mathcal{F}'$, and that such diagrams remain commutative after taking $\widehat{HF}_{\mathbb{Z}_2}$.

natural isomorphism

$$\widehat{HF}_{\mathbb{Z}_2}(\mathcal{B}\sharp\mathcal{H}) \simeq \widehat{HF}_{\mathbb{Z}_2}(\mathcal{B}) \otimes_{\mathbb{F}_2} \widehat{HF}(\mathcal{H}) \simeq \widehat{HF}_{\mathbb{Z}_2}(\mathcal{B}),$$

since $\mathcal{H}$ represents S^3 and so $\widehat{HF}(\mathcal{H}) \simeq \mathbb{F}_2$. Thus, by the naturality of $\widehat{HF}_{\mathbb{Z}_2}$ for genus-0 bridge diagrams of based knots in S^3, we deduce that the composition

$$\widehat{HF}_{\mathbb{Z}_2}(f_1) \circ \cdots \circ \widehat{HF}_{\mathbb{Z}_2}(f_n) \,:\, \widehat{HF}_{\mathbb{Z}_2}(H_n) \to \widehat{HF}_{\mathbb{Z}_2}(H_1)$$

does not depend on the choice of a sequence $f_1, \cdots, f_n$. Therefore the restriction of $\widehat{HF}_{\mathbb{Z}_2}$ to $\mathcal{G}_\alpha$ is a functor. □

Remark 5.4.4. The main idea of the proof of Lemma 5.4.3 is that we only have to avoid dealing with monodromies of Heegaard diagrams, as they tend to be very complex and hard to deal with. Thus, its proof can be directly extended to extended bridge diagrams representing based knots in S^3 to show that any loop consisting of A-equivalences and α-equivalences induce the identity map in $\widehat{HF}_{\mathbb{Z}_2}$. This is because, given an A-equivalence e_A and an α-equivalence e_α, such that $e_A \circ e_\alpha$ is well-defined,

we can always find another pair of an A-equivalence e'_A and an α-equivalence e'_α such that $e_A \circ e_\alpha = e'_\alpha \circ e'_A$. As a result, we only have to deal with additional codimension-2 singularities of bridge diagrams, which induce higher-genus versions of Reidemeister moves. Furthermore, applying 5.4.1 gives us that any loop of basic moves induce the identity map in $\widehat{HF}_{\mathbb{Z}_2}$.

The above lemma covers one part of the functoriality axiom. This partial result can be used to prove the well-definedness of stabilization maps of type I in the general case.

Proposition 5.4.5 (Proposition 3.11 of [24]). *Let $\mathcal{E}_1, \mathcal{E}_2$ be two extended bridge diagrams of a based knot (K, z) in S^3, related by a single stabilization of type I, and let $\mathcal{E}_1 \xrightarrow{s} \mathcal{E}_2$ be such a stabilization. Then the induced map*

$$\widehat{HF}_{\mathbb{Z}_2}(s) \,:\, \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_2) \to \widehat{HF}(\mathcal{E}_1)$$

is defined uniquely.

Proof. Let Σ be the Heegaard surface of $\mathcal{H}_{\mathrm{pt}}(\mathcal{E}_1)$ and choose a point w which is very close to the basepoint z. Let $\mathcal{E}$ be the extended bridge diagram given by performing a stabilization of type I on $\mathcal{E}_1$ at w. Write the induced map as

$$\widehat{HF}_{\mathbb{Z}_2}(s_w) \,:\, \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}) \to \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}_1).$$

Recall that, to construct an induced map for a stabilization of type I performed at another point on Σ which is not close to z, we have to compose $\widehat{HF}_{\mathbb{Z}_2}(s_w)$ with maps induced by α- and β-handleslides. To prove that $\widehat{HF}_{\mathbb{Z}_2}(s)$ is unique, we have to prove that, for any sequence of α-handleslides which starts from and ends at $\mathcal{E}$, the induced automorphism of $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E})$ is the identity. Note that this is because α-handleslide maps and β-handleslide maps commute, due to Lemma 5.4.1. But this is a direct consequence of Lemma 5.4.3. Therefore $\widehat{HF}_{\mathbb{Z}_2}(s)$ is uniquely defined. □

Now we can use the above proposition to show that $\widehat{HF}_{\mathbb{Z}_2}$ satisfies the functoriality axiom.

Lemma 5.4.6 (Lemma 3.12 of [24]). *The weak Heegaard invariant $\widehat{HF}_{\mathbb{Z}_2}$ satisfies functoriality condition of Definition 5.1.8.*

Proof. By Lemma 5.4.3, we know that the restriction of $\widehat{HF}_{\mathbb{Z}_2}$ to $\mathcal{G}_\alpha$ is a functor. So, by the same argument applied to β-curves instead of α-curves, we also see that the restriction of $\widehat{HF}_{\mathbb{Z}_2}$ to $\mathcal{G}_\beta$ is also a functor. If e is a stabilization and e' is its reverse, we can apply Proposition 5.4.5 to reduce to the case when the (de)stabilization is performed near the basepoint. But then $\widehat{HF}_{\mathbb{Z}_2}(e) \circ \widehat{HF}_{\mathbb{Z}_2}(e') = \mathrm{id}$ holds by choosing a suitable family of almost complex structures. The diffeomorphism part is obvious. □

5.5 Continuity axiom

Lemma 5.5.1 (Lemma 3.13 of [24]). *The weak Heegaard invariant $\widehat{HF}_{\mathbb{Z}_2}$ satisfies the continuity condition of Definition 5.1.8.*

Proof. We will prove a slightly more general statement, that for any extended bridge diagram $\mathcal{E}$ of a based knot (K, z) in S^3 and a self-diffeomorphism ϕ of the Heegaard surface of $\mathcal{E}$ which is isotopic to the identity, the map

$$\widehat{HF}_{\mathbb{Z}_2}(\phi) \in \mathrm{Aut}_{\mathbb{F}_2[\theta]}(\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z))$$

induced by ϕ is the identity.

If $\mathcal{E}$ is a bridge diagram on a sphere, then by the genus-0 naturality of $\widehat{HF}_{\mathbb{Z}_2}$, we know that $\widehat{HF}_{\mathbb{Z}_2}(\phi) = \mathrm{id}$. Next, suppose that $\mathcal{E}$ is simple. Then it satisfies the following property.

- There exists a small disk D on the Heegaard surface Σ of $\mathcal{H}_{\mathrm{pt}}(\mathcal{E})$, near its basepoint z, so that all A- and B-arcs of $\mathcal{H}_{\mathrm{pt}}(\mathcal{E})$ are contained in D and all α- and β-arcs are contained in $\Sigma \backslash D$.

We may assume that ϕ fixes the curve ∂D. Then, by taking ∂D as the connected-sum neck, we can represent $\mathcal{E}$ as a connected sum:

$$\mathcal{E} = \mathcal{B} \sharp \mathcal{H},$$

where $\mathcal{B}$ is a bridge diagram of (K, z) on a sphere and $\mathcal{H}$ is a Heegaard diagram representing S^3. As in the proof of 5.4.3, we have a decomposition

$$\widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}) \simeq \widehat{HF}_{\mathbb{Z}_2}(\mathcal{B}) \otimes_{\mathbb{F}_2} \widehat{HF}(\mathcal{H}, \mathrm{pt}) \simeq \widehat{HF}_{\mathbb{Z}_2}(\mathcal{B})$$

by taking the limit (neck length) $\to \infty$. By assumption, the action of ϕ on $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{H})$ reduces to $\widehat{HF}_{\mathbb{Z}_2}(\mathcal{B})$. But since $\mathcal{B}$ is a genus-0 diagram, we already know that ϕ acts trivially on it. Hence $\widehat{HF}_{\mathbb{Z}_2}(\phi) = \mathrm{id}$ in this case.

Finally, we work out the general case. As in the proof of Lemma 5.4.3, we know that there exists a sequence of extended Heegaard moves from $\mathcal{E}$ to another extended bridge diagram $\mathcal{E}'$ which satisfies the above property. Now, by the definition of a diffeomorphism map (it just acts by diffeomorphism), we know that it commutes with maps induced by all extended Heegaard moves. Therefore, the problem reduces to the former case and we are done. □

Remark 5.5.2. In the general case argument in the proof of Lemma 5.5.1, we can choose a sequence of extended Heegaard moves from $\mathcal{E}$ to $\mathcal{E}'$ so that it consists only of basic moves and special (de)stabilizations. By doing so, we can avoid using Proposition 5.4.2, so that we don't end up with a circular reasoning.

5.6 Commutativity axiom

Lemma 5.6.1 (Lemma 3.15 of [24])**.** *The weak Heegaard invariant $\widehat{HF}_{\mathbb{Z}_2}$ satisfies commutativity condition of Definition 5.1.8.*

Proof. Commutativity for distinguished rectangles of type 1 is proven in Lemma 5.4.1. For the remaining cases, recall that a stabilization map are defined by composing the map induced by a stabilization near the basepoint followed by maps induced by α-equivalences and β-equivalences. Also, given a knot Heegaard diagram $\mathcal{H}$ representing a based knot (K, z) and another diagram $\mathcal{H}'$ given by stabilizing $\mathcal{H}$ near z, the stabilization map is given explicitly by

$$\begin{aligned}\widehat{HF}_{\mathbb{Z}_2}(\tilde{\mathcal{H}}) &\to \widehat{HF}_{\mathbb{Z}_2}(\tilde{\mathcal{H}'}),\\ \mathbf{x} &\mapsto \mathbf{x} \cup p^{-1}(\{c\}),\end{aligned}$$

where p is the branched double covering map and c is the new intersection point introduced by the stabilization, by taking family of almost complex structure of the form $\mathrm{Sym}^g(\mathrm{j})$. The reason is that any holomorphic disk involving the Floer generator in $p^{-1}(\{c\})$ must intersect the basepoint. But this argument also works for counting holomorphic triangles, and thus we deduce that the maps induced by α-equivalences, β-equivalences, and stabilizations must commute. This proves the commutativity of distinguished rectangles of type 2.

Now, we can apply the functoriality condition, which is already proven in Lemma 5.4.6, together with commutativity of distinguished rectangles of type 2, to reduce the distinguished rectangles of type 3,4,5 to the case when all stabilizations involved are performed near the basepoint. Then, by the above description of maps induced by stabilizations near the basepoint z, they must commute with stabilization maps and diffeomorphism maps. Therefore $\widehat{HF}_{\mathbb{Z}_2}$ satisfies commutativity for all distinguished rectangles. □

Remark 5.6.2. It is worth noting here that the proof of Lemma 5.6.1 can be directly extended to stabilization of type I on extended bridge diagrams. The result we get is that stabilization maps of type I and diffeomorphism maps commutes with any maps induced by extended Heegaard diagrams.

5.7 Handleswap invariance

In this section, we will check that $\widehat{HF}_{\mathbb{Z}_2}$ satisfies the handleswap invariance and conclude that it is a strong Heegaard invariant.

Lemma 5.7.1 (Lemma 3.17 of [24]). *The weak Heegaard invariant $\widehat{HF}_{\mathbb{Z}_2}$ satisfies handleswap invariance condition of Definition 5.1.8.*

Proof. Let $H = H_0 \sharp H_s$ be a knot Heegaard diagram representing a based knot (K, z) in S^3, where H_0 is doubly-pointed, H_s is the diagram drawn in Figure 5.6, and suppose that we perform a simple handleswap on H_s. If the connected sum is taken near z, then as in the proof of Lemma 9.30 in [23], we are done.

In the general case, let Σ_0 and Σ_s be the Heegaard surfaces of H_0 and H_s, respectively, and let p be the point in Σ_0 at which we take the connected sum. Define a subset of $\Sigma_0 \backslash (\{z\} \cup (\alpha, \beta\text{-curves of } H_0)$ as follows.

$$X = \{p \in \Sigma_0 \backslash (\{z\} \cup (\alpha, \beta\text{-curves of } H_0) \,|\, \text{handleswap invariance holds for } H\}$$

Then, by functoriality(Lemma 5.4.6) and continuity(Lemma 5.6.1), we know that X is a union of some connected components.. Also, we already know that the connected component containing z is an element of X, so that X is nonempty.

Now let $p \in \Sigma_0 \backslash (\alpha, \beta\text{-curves of } H_0)$ be a point satisfying the following property.

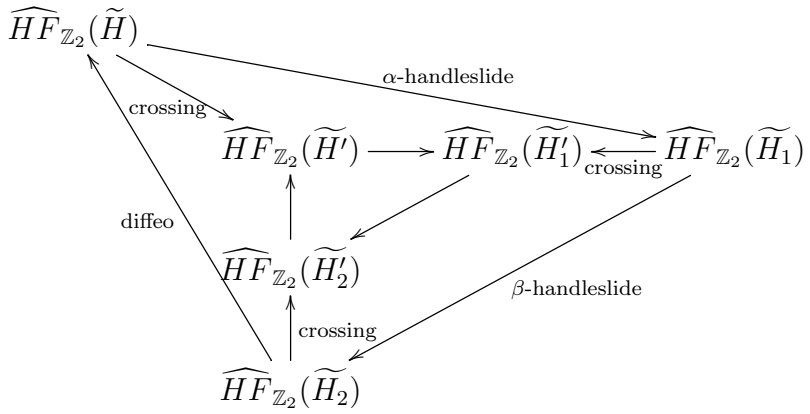
- Let R be the connected component of $\Sigma_0 \backslash (\{z\} \cup (\alpha, \beta\text{-curves of } H_0))$ containing p. Then $\overline{R} \cap X$ contains a segment of either an α-curve or a β-curve.

Then, without loss of generality, we may assume that there exists a point $p' \in X$ such that, when we denote the connected sums of H_0 and H_s performed at p and p' as H and H', respectively, the knot Heegaard diagrams H and H' are related by a sequence of four α-handleslides. Hence, by Lemma 3.25 of [20], the equivariant triangle map

$$F_{H,H'} : \widehat{HF}_{\mathbb{Z}_2}(\widetilde{H'}) \to \widehat{HF}_{\mathbb{Z}_2}(\widetilde{H}),$$

defined using the top class $\mathbf{x}$ drawn in Figure 5.7, is an isomorphism; we will call this map as the **α-crossing map**. We will not prove that the crossing map is the same as the composition of four α-handleslide maps, as we do not need it to prove this lemma. Note that, although we are drawing knot Heegaard diagrams, we are actually working with their branched double coverings.

Now consider the simple handleswap diagram involving H, and another simple handleswap diagram involving H'. Taking $\widehat{HF}_{\mathbb{Z}_2}$ gives the following diagram, where every edge is an isomorphism. Note that the innermost and the outermost triangle are simple handleswap diagrams and all crossing maps are α-crossing maps.



Since diffeomorphism maps clearly commutes with crossing maps, the leftmost face is commutative. Also, by Lemma 5.2.1, we see that the face in the right-bottom corner is also commutative, and the same argument works for the commutativity of the topmost face.

Now, the central triangle is commutive by assumption. Thus, every face of the diagram is commutative. Since all edges of the diagram are isomorphisms, the peripheral triangle must also be commutative. Hence $p \in X$ by the definition of X. Therefore, by induction on the number of α- and β-curves needed to cross to travel from p to z, we deduce that

$$X = \Sigma_0 \backslash (\{z\} \cup (\alpha, \beta\text{-curves of } H_0),$$

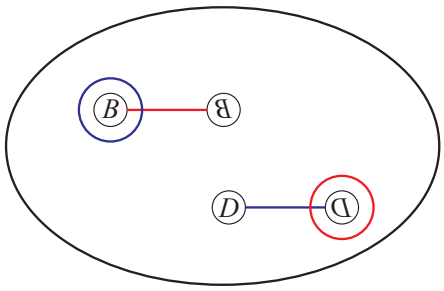


Figure 5.6: The Heegaard diagram H_s; the boundary is collapsed to form a genus 2 surface.

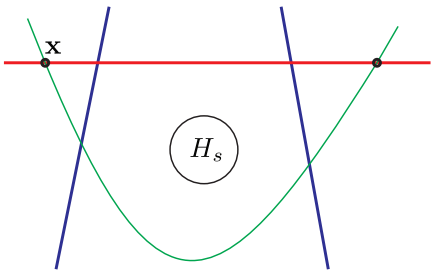


Figure 5.7: The triple-diagram near the connected sum region. The circle in the middle is where the diagram H_s is attached via connected sum neck.

i.e. $\widehat{HF}_{\mathbb{Z}_2}$ satisfies handleswap invariance condition of Definition 5.1.8. □

Theorem 5.7.2 (Theorem 3.18 of [24]). *There exists a strong Heegaard invariant of based knots in S^3, which is isomorphic to $\widehat{HF}_{\mathbb{Z}_2}$.*

Proof. By Lemma 5.4.6, Lemma 5.5.1, Lemma 5.6.1, and Lemma 5.7.1, $\widehat{HF}_{\mathbb{Z}_2}$ satisfies all conditions of Definition 5.1.8. Therefore, it is a strong Heegaard invariant. □

5.8 Naturality of the equivalence

The important point of Theorem 5.7.2 is that, instead of treating $\widehat{HF}$ directly as a Heegaard invariant, we have used the word "isomorphic". The reason is as follows. While it is true that we have constructed a strong Heegaard invariant $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$ of based knots (K, z) in S^3, defined using knot Heegaard diagrams, we also have the original definition of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$ which involves (genus zero) bridge diagrams of

(K,z) on S^2. Both versions of $\widehat{HF}_{\mathbb{Z}_2}$ are natural invariants, i.e. define functors

$$\widehat{HF}_{\mathbb{Z}_2}^{knot},\ \widehat{HF}_{\mathbb{Z}_2}^{bridge}\ :\ \mathrm{Knot}_* \to \mathrm{Mod}_{\mathbb{F}_2[\theta]},$$

where Knot_* is the category whose objects are based knots in S^3 and morphisms are diffeomorphisms. Also, Theorem 5.7.2 tells us that both of them have the same isomorphism type (of $\mathbb{F}_2[\theta]$-modules). However, we have not yet proven whether they are isomorphic through a natural isomorphism, i.e. there exists an invertible natural transformation η between functors $\widehat{HF}_{\mathbb{Z}_2}^{knot}$ and $\widehat{HF}_{\mathbb{Z}_2}^{bridge}$. We will construct such a natural transformation in this section. Note that, while dealing with this issue, we will strictly distinguish the two invariants $\widehat{HF}_{\mathbb{Z}_2}^{knot}$ and $\widehat{HF}_{\mathbb{Z}_2}^{bridge}$.

Let $\mathcal{E} = ((\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (P, A, B, z))$ be an extended bridge diagram representing a based knot (K,z) in S^3. Choose a point $p \in \mathrm{int}(A_i) \cap \mathrm{int}(B_j)$ for some $A_i \in A$, $B_j \in B$, where $z \notin A_i \cup B_j$; such a point will be called as a **proper crossing** of $\mathcal{E}$. Let $D \subset \Sigma$ be a small disk neighborhood of p, which does not intersect α-curves and β-curves, and intersects A- and B-arcs only with A_i and B_j, satisfying $D \cap (A_i \cup B_j) = \{p\}$. Then, on a puncture torus T whose boundary ∂T is identified with ∂D, draw two disjoint simple arcs, denoted A_i^p and B_j^p, so that $\partial A_i^p = \partial A_i$ and $\partial B_j^p = \partial B_j$. Also, draw two disjoint simple closed curves, denoted α_p and β_p, so that the following conditions hold.

- $A_i^p \cap \alpha_p = B_j^p \cap \beta_p = \emptyset$.
- α_p intersects B_j^p transversely at one point.
- β_p intersects A_i^p transversely at one point.

Now define the following objects.

- $\Sigma^p = (\Sigma \backslash D) \cup T$
- $\boldsymbol{\alpha}^p = \boldsymbol{\alpha} \cup \{\alpha_p\}$, $\boldsymbol{\beta}^p = \boldsymbol{\beta} \cup \{\beta_p\}$

- $A^p = (A \backslash \{A_i\}) \cup \{A_i^p\}$, $B^p = (B \backslash \{B_j\}) \cup \{B_j^p\}$

Then the pair

$$R_p(\mathcal{E}) = ((\Sigma^p, \boldsymbol{\alpha}^p, \boldsymbol{\beta}^p), (P, A^p, B^p, z))$$

also represents (K, z). Furthermore, $\mathcal{E}$ and $R_p(\mathcal{E})$ are related by a sequence consisting of a stabilization of type I and two handleslides of type III. Hence, by taking $\widehat{HF}_{\mathbb{Z}_2}$ and composing the induced maps, we get an isomorphism, which we will call as the resolution map at p:

$$\mathcal{R}_{p,\mathcal{E}} : \widehat{HF}_{\mathbb{Z}_2}(R_p(\mathcal{E})) \to \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}).$$

Lemma 5.8.1 (Lemma 3.19 of [24]). *Given an extended bridge diagram $\mathcal{E}$ representing a based knot (K, z) in S^3, and its proper crossing p, the resolution map $\mathcal{R}_{p,\mathcal{E}}$ depends only on the choice of p.*

Proof. The definition of $\mathcal{R}_{p,\mathcal{E}}$ involves choosing a point in $D \backslash (A_i \cup B_j)$, at which a stabilization of type I would occur, and also choosing which one of two handleslides of type I/II is to be applied first. Take any two possible choices, and denote the two maps given by taking compositions of the induced maps as $\mathcal{R}^1_{p,\mathcal{E}}$ and $\mathcal{R}^2_{p,\mathcal{E}}$. Then the composition $(\mathcal{R}^2_{p,\mathcal{E}})^{-1} \circ \mathcal{R}^1_{p,\mathcal{E}}$, which is an automorphism of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$, can be decomposed as

$$\begin{aligned}(\mathcal{R}^2_{p,\mathcal{E}})^{-1} \circ \mathcal{R}^1_{p,\mathcal{E}} &= (\text{basic moves}) \circ (\text{destabilization}) \circ (\text{stabilization}) \circ (\text{basic moves}) \\ &= (\text{basic moves}) \circ (\text{special destab}) \circ (\text{special stab}) \circ (\text{basic moves}) \\ &= \text{composition of basic moves}\end{aligned}$$

where Lemma 5.4.1 is applied in the first line. Now, by Remark 5.4.4, this map should be the identity. Therefore $\mathcal{R}^1_{p,\mathcal{E}} = \mathcal{R}^2_{p,\mathcal{E}}$. □

Lemma 5.8.2 (Lemma 3.20 of [24]). *Let p, q be two distinct proper crossings of $\mathcal{E}$. Then we have*

$$\mathcal{R}_{p,\mathcal{E}} \circ \mathcal{R}_{q,R_p(\mathcal{E})} = \mathcal{R}_{q,\mathcal{E}} \circ \mathcal{R}_{p,R_q(\mathcal{E})}.$$

In other words, resolution maps of proper crossings commute.

Proof. Using Lemma 5.6.1 and Remark 5.8.1, we can see that the map

$$(\mathcal{R}_{p,\mathcal{E}} \circ \mathcal{R}_{q,R_p(\mathcal{E})})^{-1} \circ \mathcal{R}_{q,\mathcal{E}} \circ \mathcal{R}_{p,R_q(\mathcal{E})}$$

is a composition of maps induced by a loop of basic moves, thus is the identity. □

Given an extended bridge diagram $\mathcal{E}$ representing a based knot (K, z) in S^3, let $\mathcal{I}$ be the set of all proper crossings of $\mathcal{E}$. Choose an enumeration $\mathcal{I} = \{x_1, \cdots, x_n\}$. Define an extended bridge diagram $R(\mathcal{E})$, which depends only on $\mathcal{E}$, as follows.

$$R(\mathcal{E}) = R_{x_{n-1}}(\cdots(\mathcal{R}_{x_1}(\mathcal{E}))\cdots)$$

Also, define an isomorphism $\mathcal{R}_{\mathcal{E}} : \widehat{HF}_{\mathbb{Z}_2}(R(\mathcal{E})) \to \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E})$ as follows.

$$\mathcal{R}_{\mathcal{E}} = \mathcal{R}_{x_1,\mathcal{E}} \circ \cdots \circ \mathcal{R}_{x_n,R_{x_{n-1}}(\cdots(\mathcal{R}_{x_1}(\mathcal{E}))\cdots)}$$

Then, by Lemma 5.8.1 and Lemma 5.8.2, $\mathcal{R}_{\mathcal{E}}$ does not depend on the choice of an enumeration of $\mathcal{I}$, thus depends only on $\mathcal{E}$.

Now observe that the A- and B-arcs of $R(\mathcal{E})$ which do not contain z also do not intersect themselves in their interior. So, after performing handleslides as in Figure 5.8 on $R(\mathcal{E})$, it can then be destabilized (of type II) to a knot Heegaard diagram $HR(\mathcal{E})$ representing (K, z). Denote the composition of destabilization maps as

$$\mathcal{D}_{\mathcal{E}} : \widehat{HF}_{\mathbb{Z}_2}(\widetilde{HR(\mathcal{E})}) \to \widehat{HF}_{\mathbb{Z}_2}(R(\mathcal{E})).$$

Then, by Lemma 4.6.3, $\mathcal{D}_{\mathcal{E}}$ also depends only on $\mathcal{E}$.

Definition 5.8.3. Given an extended bridge diagram $\mathcal{E}$ representing a based knot

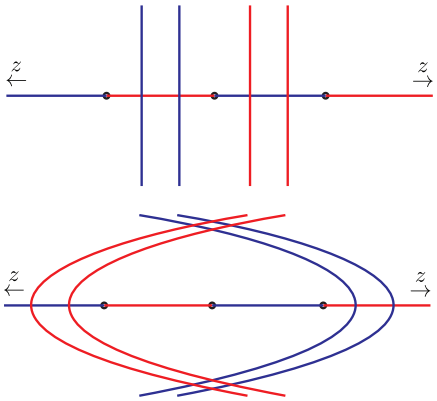


Figure 5.8: The diagrams $R(\mathcal{E})$(above) and $HR(\mathcal{E})$(below) near the A-arcs and B-arcs. Here, we assume for simplicity that the rightmost A-arc and the leftmost B-arc are the ones connected to the basepoint z.

(K, z) in S^3, we define its **translation isomorphism** as

$$\mathcal{T}_{\mathcal{E}} = (\mathcal{R}_{\mathcal{E}} \circ \mathcal{D}_{\mathcal{E}})^{-1} : \widehat{HF}_{\mathbb{Z}_2}(\mathcal{E}) \to \widehat{HF}_{\mathbb{Z}_2}(\widetilde{HR(\mathcal{E})}).$$

We will now prove that translation isomorphisms define a natural transformation between $\widehat{HF}_{\mathbb{Z}_2}^{knot}$ and $\widehat{HF}_{\mathbb{Z}_2}^{bridge}$.

Lemma 5.8.4 (Lemma 3.22 of [24]). *Let $\mathcal{E} \xrightarrow{s} \mathcal{E}'$ be an extended Heegaard move between extended bridge diagrams $\mathcal{E}, \mathcal{E}'$ representing a based knot (K, z) in S^3.*

1. *If s is a basic move, then the knot Heeegaard diagrams $HR(\mathcal{E})$ and $HR(\mathcal{E}')$ are related by a sequence of basic moves and stabilizations of type I.*

2. *If s is a stabilization of type I, then $HR(\mathcal{E})$ and $HR(\mathcal{E}')$ are also related by a stabilization of type I.*

3. *If s is a stabilization of type II, then $HR(\mathcal{E}) = HR(\mathcal{E}')$.*

4. *If s is a diffeomorphism, then $HR(\mathcal{E})$ and $HR(\mathcal{E}')$ are also related by a diffeomorphism.*

Proof. Cases 2, 3, and 4 are obvious. For case 1, it can be easily seen that $R(\mathcal{E})$ and $R(\mathcal{E}')$ are related by a sequence of basic moves and stabilizations of type I. Since HR

is obtained from R by performing handleslides of type III, we deduce that $HR(\mathcal{E})$ and $HR(\mathcal{E}')$ are also related by a sequence of basic moves and stabilizations of type I. □

Theorem 5.8.5 (Theorem 3.23 of [24]). *Translation isomorphisms define an invertible natural transformation between the functors*

$$\widehat{HF}_{\mathbb{Z}_2}^{knot}, \widehat{HF}_{\mathbb{Z}_2}^{bridge} : Knot_* \to Mod_{\mathbb{F}_2[\theta]}.$$

Proof. Let $\mathcal{B} \xrightarrow{s} \mathcal{B}'$ be an extended Heegaard diagram between bridge diagrams $\mathcal{B}, \mathcal{B}'$ representing a based knot (K, z) in S^3, and denote the isomorphism induced by s by

$$\widehat{HF}_{\mathbb{Z}_2}^{bridge}(s) : \widehat{HF}_{\mathbb{Z}_2}^{bridge}(\mathcal{B}') \to \widehat{HF}_{\mathbb{Z}_2}^{bridge}(\mathcal{B}).$$

If s is a basic move or a stabilization, then by Lemma 5.8.4, there exists either a (possibly empty) sequence of basic moves and stabilizations from $HR(\mathcal{B})$ and $HR(\mathcal{B}')$; denote the induced map by

$$\widehat{HF}_{\mathbb{Z}_2}^{knot}(HR(s)) : \widehat{HF}_{\mathbb{Z}_2}^{bridge}(\widetilde{HR(\mathcal{B}')}) \to \widehat{HF}_{\mathbb{Z}_2}^{bridge}(\widetilde{HR(\mathcal{B})}).$$

Then $(\widehat{HF}_{\mathbb{Z}_2}^{bridge}(s) \circ \mathcal{T}_{\mathcal{B}})^{-1} \circ \mathcal{T}_{\mathcal{B}'} \circ \widehat{HF}_{\mathbb{Z}_2}^{knot}(HR(s))$ is the map induced by a loop consisting of basic moves and stabilizations. Hence, by Lemma 4.6.3, Remark 5.6.2, and Remark 5.4.4, we see that it is equal to the identity map, i.e.

$$\widehat{HF}_{\mathbb{Z}_2}^{bridge}(s) \circ \mathcal{T}_{\mathcal{B}} = \mathcal{T}_{\mathcal{B}'} \circ \widehat{HF}_{\mathbb{Z}_2}^{knot}(HR(s)).$$

Thus the translation maps induce a well-defined map

$$\mathcal{T}_{(K,z)} : \widehat{HF}_{\mathbb{Z}_2}^{bridge}(\Sigma(K), z) \to \widehat{HF}_{\mathbb{Z}_2}^{knot}(\Sigma(K), z).$$

Now, morphisms of the category Knot_* are self-diffeomorphism of S^3, and they are

realized as a sequence of Heegaard moves in the level of Heegaard diagrams. Therefore the correspondence

$$\mathcal{T} : (K, z) \mapsto \mathcal{T}_{(K,z)}$$

is a natural transformation from $\widehat{HF}_{\mathbb{Z}_2}^{bridge}$ to $\widehat{HF}_{\mathbb{Z}_2}^{knot}$. Since $\mathcal{T}_{(K,z)}$ is an isomorphism for each based knot (K, z) in S^3, the natural transformation $\mathcal{T}$ is invertible. $\square$

Chapter 6

$\mathbb{Z}_2$-Equivariant contact class

Recall that, when a closed contact 3-manifold (M, ξ) is given, then we have the associated element $c(\xi) \in \widehat{HF}(-M)$, which depends only on the isotopy class of ξ. In this chapter, we will construct its $\mathbb{Z}_2$-equivariant refinement. In other words, for any given transverse knot K in S^3_{std}, we will construct an element $c_{\mathbb{Z}_2}(\xi_K) \in \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, which depends only on the transverse isotopy class of K, and show that it has some nice properties.

6.1 Construction of $c_{\mathbb{Z}_2}(\xi_K)$

Choose a system of pairwise disjoint simple arcs a_i^0 in D, where a_i^0 connects p_i to a boundary point. Pick a point p_1 among $p_1, \cdots, p_n$ and regard it as a basepoint; $z := p_1$ (such a system is called a **half-arc basis**). The arcs a_i^0 lift to non-separating smooth simple arcs a_i in S, which pass through p_i and connect two points in ∂S. We claim that $\{a_i\}_{i\neq 1}$ is an arc basis on the surface S, so that the data $(S, \{a_i\}_{i\neq 1}, \tilde{h}, z = p_1)$ defines an open book diagram of (M_B, ξ) in the sense of [21], which is invariant under the covering transformation. To prove this, recall that a pairwise disjoint system of simple arcs $\{a_i\} \subset S$ is called an arc basis if the two endpoints of each a_i lie on ∂S and $S - \cup a_i$ is a disk. Since each a_i is a lift of a_i^0 and $D - \cup_{i\neq 1} a_i^0$ is a disk with one

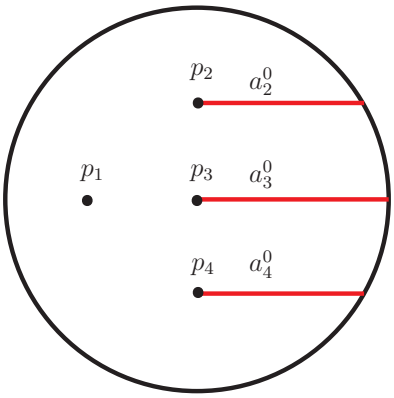


Figure 6.1: A disk with four marked points and a half-arc basis

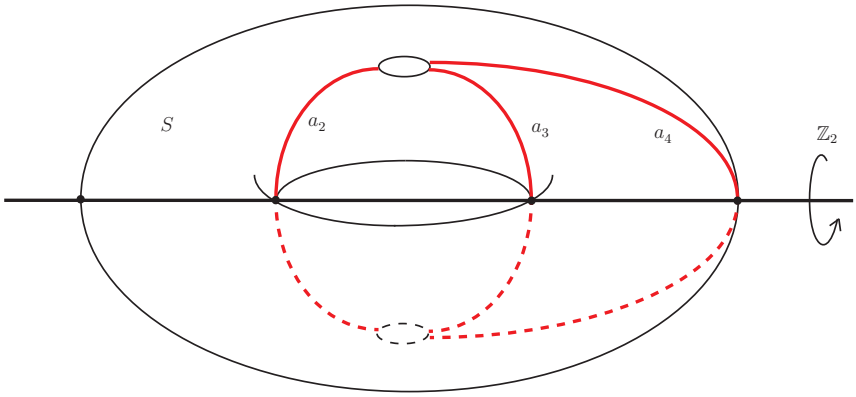


Figure 6.2: The branched double cover.

branching point p_1 in its interior, its inverse image $S - \cup_{i\neq 1} a_i$ is the double cover of a disk branched along an interior point, which is a disk. This proves our claim.

Example. Suppose that we are given a disk with four marked points $p_1, \cdots, p_4$ in its interior and a half-arc basis $\{a_2^0, a_3^0, a_4^0\}$ given as in Figure 6.1.

After taking the branched double cover, we get the twice-puncture torus with three arcs a_2, a_3, a_4. From Figure 6.2, We see that $S - (a_2 \cup a_3 \cup a_4)$ is a disk, i.e. $\{a_2, a_3, a_4\}$ gives an arc basis on S.

Thus, if we define the following α- and β-curves on the Heegaard surface

$$\Sigma = (S \times \{0,1\})/\partial S \times \{0,1\},$$

we get a $\mathbb{Z}_2$-invariant Heegaard diagram of M_B:

$$\begin{aligned} \alpha_i &= a_i \cup a_i, \\ \beta_i &= b_i \cup \tilde{h}(b_i), \end{aligned}$$

where b_i is a slight perturbation of a_i in positive direction along an orientation of ∂S and $|a_i \cap b_i| = 1$. For simplicity, we denote the sets of α_i and β_i as $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. If we denote the half-arc basis which we have started with by $\mathcal{A}$, then the contact element $EH(\xi_K, \mathcal{A})$ is defined as the element $\{p_1, \cdots, p_n\}$ in the Heegaard Floer cochain complex of M_B, as in section 3.1 of [21].

$$EH(\xi_K, \mathcal{A}) \in CF^*(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \simeq CF^*(M_B),$$

which is a $\mathbb{Z}_2$-invariant cocycle. This element induces the following element in the equivariant Floer cochain complex

$$EH_{\mathbb{Z}_2}(\xi_K, \mathcal{A}) = \theta^0 \otimes EH(\xi_K) \in \widehat{CF}_{\mathbb{Z}_2}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z).$$

When the choice of a half-arc basis $\mathcal{A}$ is not important, we will drop $\mathcal{A}$ and write $EH_{\mathbb{Z}_2}(\xi_K)$ for simplicity.

Remark. As we have seen above, given a half-arc basis $\{a_i^0\}$ and a monodromy h of D^2 which fixes the points p_i, we can take its branched double cover along the points p_i to get an arc basis $\{a_i\}$ in the open book diagram $(S, \tilde{h})$, and applying the Honda-Kazez-Matić construction to it gives a $\mathbb{Z}_2$-invariant Heegaard diagram. Now, if we apply the Honda-Kazez-Matić construction directly to the given half-arc basis, what

we get is a bridge diagram of a link in S^3, drawn on a genus 0 Heegaard surface, as follows.

$$\begin{aligned} S^2 \simeq \Sigma =& (D^2 \times \{0,1\})/(\partial D^2 \times \{0,1\}), \\ A_i =& a_i^0 \cup a_i^0, \\ B_i =& b_i^0 \cup \tilde{h}(b_i^0). \end{aligned}$$

Here, b_i^0 is a slight perturbation of a_i^0 along the positive direction of ∂D^2, so that a_i^0 and b_i^0 intersect only at the endpoint of a_i^0 which lies in the interior of D^2. Then, taking its branched double cover along the set $\{p_1, p_2, \cdots\} \times \{0,1\}$ also gives a $\mathbb{Z}_2$-invariant Heegaard diagram. The two Heegaard diagrams we get are identical. To summarize, we have a following commutative diagram of objects which we consider in this paper.

$$\begin{array}{ccc} \text{half-arc diagram} & \xrightarrow{\text{branched double cover}} & \text{arc diagram} \\ \Big\downarrow{\scriptstyle \text{HKM construction}} & & \Big\downarrow{\scriptstyle \text{HKM construction}} \\ \text{bridge diagram} & \xrightarrow{\text{branched double cover}} & \text{Heegaard diagram} \end{array}$$

Now, Theorem 3.5.3 tells us that we only have to work with $\mathbb{Z}_2$-invariant almost complex structures of the form $\mathrm{Sym}^g(\tilde{\mathfrak{j}})$. For such almost complex structures, the argument in Section 3.1 of [21] tells us that there are no holomorphic disks going towards $EH(\xi_K)$. Thus, if we denote the generator of $\mathbb{Z}_2$ as τ, the higher order terms in the $\mathbb{Z}_2$-equivariant differential vanishes for $EH(\xi_K)$, except for the term $\theta^1 \otimes EH(\xi_K)$ which comes from the constant disk of Maslov index 0. So the following equality holds.

$$d_{\mathbb{Z}_2}(EH_{\mathbb{Z}_2}(\xi_K)) = \theta^0 \otimes d(EH(\xi_K)) + \theta^1 \otimes (EH(\xi_K) + \tau^* EH(\xi_K)) = 0.$$

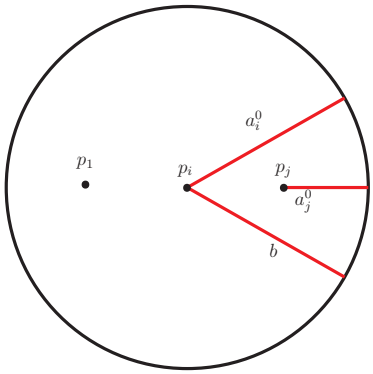


Figure 6.3: A picture describing a half-arc slide.

Hence $EH_{\mathbb{Z}_2}(\xi_K)$ is a cocycle, i.e. defines a cohomology class

$$c_{\mathbb{Z}_2}(\xi_K) := [EH_{\mathbb{Z}_2}(\xi_K)] \in \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)).$$

6.2 Invariance

6.2.1 Basic moves of half-arc bases

As in the case of arc bases on open book diagrams, half-arc bases also admit a certain set of moves. One of them is obviously an isotopy of half-arcs; the other one is a half-arc slide, which is a "half" version of an arc slide of arc bases. To be precise, half-arc slides are defined as follows.

Definition 6.2.1. Given a half-arc basis $\{a_i^0\}_{i\neq 1}$ on a disk D, suppose that an (half-)arc b starting at p_i and ending in ∂D satisfies the property that there exits a unique $j \neq i, 1$ such that p_j is contained in the region bounded by ∂D, a_i^0, and b. If a_j^0 is also contained in that region, we say that $\{a_k^0\}_{k\neq 1,i} \cup \{b\}$ is obtained by performing a half-arc slide of a_i^0 along a_j^0.

Proposition 6.2.2 (Proposition 5.3 of [25])**.** *Let $\{a_i\}_{1<i\leq n}, \{b_i\}_{1<i\leq n}$ on a disk D be two half-arc bases, where a_i and b_i connect an interior point $p_i \in int(D)$ with ∂D.*

Then they are related by a sequence of isotopies and half-arc slides.

Proof. We can isotope the arcs so that $\partial a_i = \partial b_i$ for all $1 < i \leq n$. Then, for each k, the closed curve γ given by the concatenation of a_k and b_k gives an element of $\pi_1(D - \{p_1\}, p_k) \simeq \mathbb{Z}$. If the homotopy class of $[\gamma]$ is s times the generator, then we can apply $n|s|$ half-arc slides on b_k so that a_k is homotopic to b_k in $D - \{p_1\}$. Thus we can apply this process for each $1 < k \leq n$, so that a_k is homotopic to b_k rel endpoints in $D - \{p_1\}$.

Now assume that, for any i, j with $1 < i, j \leq n$, a_i and b_i intersect transversely, and denote the total number of intersection points between a-half-arcs and b-half-arcs by N. We can find a disk D which is innermost, i.e. no half-arcs intersect its interior. Then, we can apply an isotopy along the disk to remove a pair of intersection points; the number of remaining intersection points in $N - 2$. Therefore, by induction on N, we see that, after a sequence of isotopies, we may assume that a_i and b_i cobound a disk D_i for each $1 < i \leq n$, and the disks D_i are pairwise disjoint. Then, we can isotope the half-arcs along the disks D_i to isotope a_i to b_i. □

Now we will prove invariance of $c_{\mathbb{Z}_2}(\xi_K)$ with respect to half-arc slides. Lifting the whole picture to S shows that, in the branched double cover, a half-arc slide corresponds to an arcslide in the sense of [21], which then corresponds to an α-handleslide followed by a β-handleslide. Hence, if $\{a_i\}$ is obtained by performing an arcslide to $\{\tilde{a}_i\}$, and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}$ are the associated α- and β-curves on the invariant Heegaard surface Σ, then we have the following quasi-isormorphism, which is induced by a composition of an equivariant triangle map for an α-handleslide followed by an equivariant triangle map for a β-handleslide:

$$\widetilde{CF}_{\mathbb{Z}_2}(\Sigma, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \xrightarrow{\sim} \widetilde{CF}_{\mathbb{Z}_2}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}).$$

Since this quasi-isomorphism is clearly $\mathbb{Z}_2$-equivariant, we get the following induced

quasi-isomorphism between equivariant Floer cochain complexes.

$$\widehat{CF}_{\mathbb{Z}_2}(\Sigma, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \xrightarrow{\sim} \widehat{CF}_{\mathbb{Z}_2}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{6.2.1}$$

Theorem 6.2.3 (Theorem 5.4 of [25]). *The map (6.2.1) sends $EH_{\mathbb{Z}_2}(\xi_K)$ to $EH_{\mathbb{Z}_2}(\xi_K)$.*

Proof. Note that performing a half-arc slide to a half-arc basis corresponds to performing an arcslide to the induced arc-basis in the branched double cover. Thus the Heegaard triple-diagrams we get are the same as the diagrams which arise in the proof of the invariance of contact classes under arcslides, as in [21]. Since all holomorphic triangles involved are small(see the proof of Lemma 3.5 in [21]), all holomorphic triangles we count here have Maslov index 0, and their moduli spaces consist of a single point by Riemann mapping theorem. Therefore we deduce that $EH_{\mathbb{Z}_2}(\xi_K)$ is mapped to $EH_{\mathbb{Z}_2}(\xi_K)$. □

The invariance under perturbations of almost complex structures is proved similarly.

Theorem 6.2.4 (Theorem 5.5 of [25]). *The map induced by changing the choice of almost complex structures sends $EH_{\mathbb{Z}_2}(\xi_K)$ to $EH_{\mathbb{Z}_2}(\xi_K)$.*

Proof. The induced map, which is defined in the proof of Proposition 3.23 of [20], counts holomorphic disks going towards EH with Maslov index at most 0. By Theorem 3.5.3, we can choose cylindrical complex structures of the form $\mathrm{Sym}^g(\tilde{\mathrm{j}})$ to compute equivariant Floer cohomology. However, by the argument of Section 3.1 in [21], such disks must intersect the basepoint. This completes the proof. □

Corollary 6.2.5 (Corollary 5.6 of [25]). *The map induced by an isotopy, from a half-arc basis $\mathcal{A}_0$ to another basis $\mathcal{A}$, sends $EH_{\mathbb{Z}_2}(\xi_K, \mathcal{A})$ to $EH_{\mathbb{Z}_2}(\xi_K, \mathcal{A}_0)$.*

Proof. An isotopy of half-arc basis can be replaced by a 1-parameter family of self-diffeomorphisms of D^2, starting from the identity. Such a family induces a 1-parameter

family of cylindrical complex structures which we use to compute equivariant Floer cohomology. By Theorem 6.2.4, we see that the the induced isomorphism maps $EH_{\mathbb{Z}_2}(\xi_K, \mathcal{A})$ to $EH_{\mathbb{Z}_2}(\xi_K, \mathcal{A}_0)$. □

Theorem 6.2.6 (Theorem 5.7 of [25]). *The map induced by an isotopy of the monodromy h sends $EH_{\mathbb{Z}_2}(\xi_K)$ to $EH_{\mathbb{Z}_2}(\xi_K)$.*

Proof. Let $\{h_t\}$ be an isotopy of monodromy functions. As in Theorem 7.3 of [37], we can reduce to the case where the isotopy $\{\tilde{h}_t\}$ of self-diffeomorphisms of Σ is a $\mathbb{Z}_2$-equivariant Hamiltonian isotopy. Then we can apply the proof of Lemma 3.3 in [21] to deduce that the equivariant isotopy map sends $EH_{\mathbb{Z}_2}(\xi_K)$ to $EH_{\mathbb{Z}_2}(\xi_K)$. □

Thus we proved the invariance under the choice of Floer-theoretic auxiliary data, so it remains to prove the invariance under a basepoint change and positive braid stabilization. Before proving the invariance under positive braid stabilization. we prove the invariance under the choice of a basepoint and stabilizations.

Theorem 6.2.7 (THeorem 5.8 of [25]). *The map induced by changing the (invariant) basepoint sends $EH_{\mathbb{Z}_2}(\xi_K)$ to $EH_{\mathbb{Z}_2}(\xi_K)$.*

Proof. According to the definition in the previous section, we choose the basepoint z to be one of the points $p_1, \cdots, p_n$, which form the fixed locus of the $\mathbb{Z}_2$-action. If we choose another such basepoint z', then since our transverse braid forms a knot, there exists a positive integer k satisfying $h^k(z) = z'$. Now applying the self-diffeomorphism $\tilde{h}^k$ to the open book diagram $(\Sigma, \{a_i\}, \tilde{h}, z)$ gives $(\Sigma, \{\tilde{h}^k(a_i)\}, \tilde{h}^k\tilde{h}\tilde{h}^{-k} = \tilde{h}, \tilde{h}^k(z) = z')$. But since we can always change the half-arc basis $\{h^k(a_i^0)\}$ back to $\{a_i^0\}$ via half-arc slides, we can change the arc basis $\{\tilde{h}^k(a_i)\}$ back to $\{a_i\}$ via arcslides, in the same manner. Since the maps induced by arcslides and diffeomorphisms preserve $EH^*_{\mathbb{Z}_2}$, the theorem follows. □

Theorem 6.2.8 (Theorem 5.9 of [25]). *The map induced by a positive braid stabilization sends $EH_{\mathbb{Z}_2}(\xi_K)$ to $EH_{\mathbb{Z}_2}(\xi_K)$.*

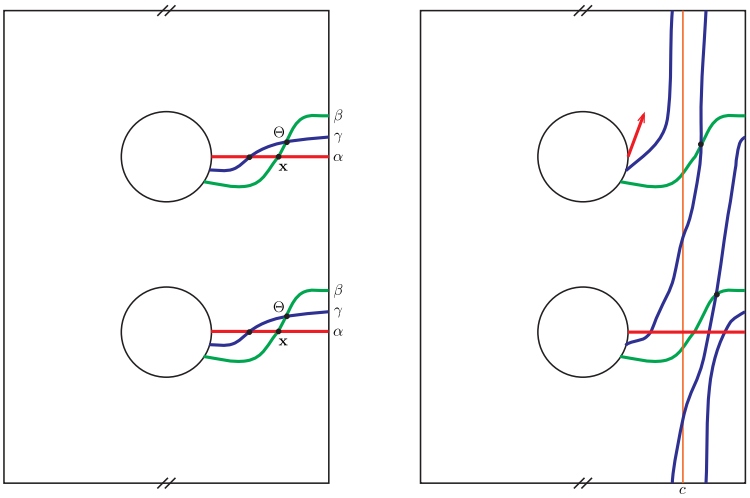


Figure 6.4: The induced Heegaard triple-diagram.

Proof. The induced map between $CF^*_{\mathbb{Z}_2}$ is induced by taking RHom at the following $\mathbb{Z}_2$-equivariant quasi-isomorphism of chain complexes(see the proof of Theorem 1.24 in [20] for details):

$$\widehat{CF}(\Sigma(B)) \to \widehat{CF}(\Sigma(B \coprod \text{unknot})) \to \widehat{CF}(\Sigma(B_+)),$$

where B is the original braid and B_+ is its positive stabilization. Dualizing this gives

$$\widehat{CF}^*(\Sigma(B_+)) \to \widehat{CF}^*(\Sigma(B \coprod \text{unknot})) \to \widehat{CF}^*(\Sigma(B)).$$

The second map preserves EH by its definition. The first map is the saddle map induced by a Legendrian (-1)-surgery along a lift c of a small Legendrian arc connecting B with a trivial braid. The Heegaard triple diagram for the saddle is drawn in Figure 6.4. Then, by the convenient placement of the basepoint, we see that all holomorphic triangles connecting $\mathbf{x}$ and Θ are small. Therefore the induced isomorphism between $CF^*_{\mathbb{Z}_2}$ preserves $EH_{\mathbb{Z}_2}$. □

Combining these invariance theorems, we get the complete invariance of $EH_{\mathbb{Z}_2}(\xi_K)$ and its cohomology class $c_{\mathbb{Z}_2}(\xi_K)$.

Theorem 6.2.9 (Theorem 5.10 of [25]). *The cohomology class $c_{\mathbb{Z}_2}(\xi_K) \in \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ depends only on the transverse isotopy class of the transverse knot K.*

Proof. By Theorem 6.2.4, the class $c_{\mathbb{Z}_2}(\xi_K)$ is independent of the choice of almost complex structures. From Theorem 6.2.3, Theorem 6.2.6 and Corollary 6.2.5, we see that $c_{\mathbb{Z}_2}(\xi_K)$ is invariant under isotopy and half-arc slide. Thus $c_{\mathbb{Z}_2}(\xi_K)$ does not depend on the choice of half-arc basis by Proposition 6.2.2, which means that it only depends on the choice of a transverse braid representative of the given transverse knot K. However, Theorem 6.2.8tells us that $c_{\mathbb{Z}_2}(\xi_K)$ is also invariant under positive braid stabilizations. Therefore, by Theorem 2.5.3 and the naturality of $\widehat{HF}_{\mathbb{Z}_2}$, we deduce that $c_{\mathbb{Z}_2}(\xi_K)$ is an invariant of the transverse isotopy class of K. □

Definition 6.2.10. The class $c_{\mathbb{Z}_2}(\xi_K)$, which is an invariant of the transverse isotopy class of K in (S^3, ξ_{std}), is called the equivariant contact class of $(\Sigma(K), \xi_K)$.

6.3 Functoriality

In this section, we will dicuss the functoriality of the invariant $c_{\mathbb{Z}_2}(\xi_K)$ that we have defined in the previous section. Since we are dealing with transverse knots in S^3_{std}, the right notion of cobordisms that we should use is symplectic cobordism, contained in the symplectization of S^3_{std}, whose both ends are transverse knots rather than links.

Due to the combinatorial nature of the definition of $c_{\mathbb{Z}_2}$, we will not able to prove that $c_{\mathbb{Z}_2}$ is functorial under all such symplectic cobordisms. However, we are still able to prove functoriality by restricting our attention to a smaller class of symplectic cobordisms, which we will call as constructible symplectic cobordisms.

6.3.1 Weak symplectic isotopy and constructible symplectic cobordisms

Recall that symplectic cobordism of links is defined as follows.

Definition 6.3.1. Let L_0, L_1 be transverse links in (S^3, ξ_{std}). A symplectic cobordism from L_0 to L_1 is an embedded symplectic surface $S \subset (S^3 \times [0,R], d(-e^t\xi_{std}))$ where R is a positive real, $S \cap (S^3 \times \{i\}) = L_i$ for $i = 0, 1$, and S is cylindrical near both ends.

Example 6.3.2. Suppose that we are given an isotopy $rels \cup s'\{L_t\}$ of transverse links, $0 \leq t \leq 1$. Given a symplectization

$$(S^3 \times [0,1], \omega_R = d(e^{Rt}\alpha_{std})) \simeq (S^3 \times [0,R], d(e^t\alpha_{std}))$$

for $R > 0$, where $\alpha_{std} = dz + xdy$(on $\mathbb{R}^3$), consider the cobordism

$$S_R = \bigcup_{0 \leq t \leq 1} L_t \times \{Rt\}.$$

Choose a point $(p,t) \in S^3 \times [0,R]$. Then the tangent plane $T_{(p,Rt)}S_R$ is spanned by the vector $\partial_t + w_{p,t}$ and the line T_pL_t, for some vector $w_{p,t} \in T_pS^3$. Let v_p be a tangent vector spanning T_pL_t, which is chosen to vary smoothly. Then we have

$$\begin{aligned}(\omega_R)_{(p,Rt)}(\partial_t + w_{p,t}, v_p) &= e^{Rt}(Rdt \wedge \xi_{std} + d\alpha_{std})(\partial_t + w_{p,t}, v_p) \\ &= Re^{Rt}(\alpha_{std}(v_p) + \frac{1}{R}d\alpha_{std}(\partial_t + w_{p,t}, v_p)).\end{aligned}$$

Since L_t are transverse to ξ_{std}, by compactness, the value of $\alpha_{std}(v_p)$ is bounded away from zero. Hence, for sufficiently large R, the cobordism S is symplectic. In other words, S_R is symplectic for sufficiently large $R > 0$; we will call such surfaces as isotopy cylinders.

Now suppose that we are given a cobordism $S \subset (S^3 \times [0,R], d(-e^t\alpha_{std}))$ which is symplectic and cylindrical near both ends. Let $L = S \cap (S^3 \times \{0\}) \subset S^3$ and choose a generator $v_p \in T_pL$ for $p \in S^3$. Since S is symplectic, by assumption, we must have

$$0 \neq d(e^t\alpha_{std})(\partial_t, v_p) = e^t(dt \wedge \alpha_{std} + d\alpha_{std})(\partial_t, v_p) = e^t\alpha_{std}(v_p).$$

This implies $\alpha_{std}(v_p) \neq 0$, i.e. L is transverse to ξ_{std}. Therefore, we see that symplectic cobordisms form a good notion of cobordisms between transverse links.

Before we explicitly construct "basic pieces" of based symplectic cobordisms which achieve the functoriality for equivariant contact classes, we need to define the notion of weak symplectic isotopies between symplectic cobordisms. They are defined as follows.

Definition 6.3.3. Two symplectic cobordisms S_1, S_2 are weakly symplectically isotopic if they are symplectically isotopic after concatenating with (trivial) cylindrical cobordisms on both ends.

A based symplectic cobordism is a based cobordism (S, s) where S is symplectic. Two based symplectic cobordisms (S, s) and (S', s') are weakly symplectically isotopic if S and S' are weakly symplectically isotopic and, after performing the isotopy, the homotopy class $[s] - [s']$ is contained in the image of the map $\pi_1(\partial S) \to \pi_1(S)$.

Thus we are led to define a category $\mathbf{sCob}_w$ as follows:

- objects of $\mathbf{sCob}_w$ are based transverse knots;
- morphisms of $\mathbf{sCob}_w$ are weak symplectic based isotopy classes of based symplectic cobordisms.

Then we have a well-defined natural functor

$$\mathbf{sCob}_w \to \mathbf{bCob},$$

which forgets all the contact and symplectic structures involved and reverses the directions of cobordisms. In other words, a weak symplectic based isotopy class of based symplectic cobordisms can be seen as a based isotopy class of based cobordisms.

Now we construct the "basic pieces" of symplectic cobordisms, which will only be defined up to weak symplectic based isotopy. The idea is to mimic the symplectic

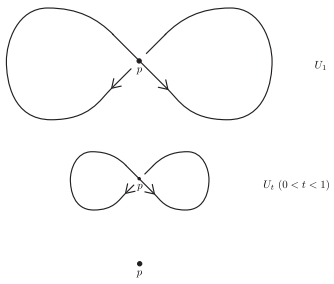


Figure 6.5: The family $\{U_t\}_{t\in(0,1]}$ of transverse unknots which converge to p as $t \to 0$

handle constructions on the branched double cover side, so we can construct the symplectic models of "birth" and "saddle" cobordisms. Note that a birth corresponds to a 1-handle and a saddle corresponds to a 2-handle, while a death corresponds to a 3-handle; 4-dimensional symplectic 3-handles do not exist.

The birth case is easy:

Definition 6.3.4. Given a transverse based link L in (S^3, ξ_{std}), choose a point $p \in S^3$ whch lies outside L. Consider a 1-parameter family $\{U_t\}_{t\in(0,1]}$ which converges to p, so that in the front projection $(x, y, z) \mapsto (y, z)$, the family is represented by Figure 6.5. Then the symplectic birth cobordism from L is defined as

$$B(L) = L \times [0, R] \cup \left(\bigcup_{t\in[0,1]} U_{1-t} \times \{Rt\} \right) \subset S^3 \times [0, R],$$

where $R > 0$ is chosen so that $B(L)$ becomes symplectic.

Lemma 6.3.5 (Lemma 7.5 of [25])**.** *There exists $R > 0$ such that $B(L)$ in Definition 6.3.4 becomes symplectic, and the weak symplectic based isotopy type of $B(L)$ depends only on L.*

Proof. The proof is very similar to the proof of Theorem 6.3.8, so we do not write it down here. □

We will now define symplectic saddle cobordisms. In the case of ordinary links and cobordisms, a saddle is defined by the following topological constructions.

- (C1) Given a link L, choose an arc a which intersects transversely with L at ∂a, such that $a \cap L = \partial a$.
- (C2) Remove a small neighborhood of ∂a in L and replace it by two arcs parallel to a. The saddle between those two links is the desired saddle cobordism of L along a.

In our case, the link L is transverse, and the arcs a_i should be Legendrian. The reason is explained in Example 6.3.6.

Example 6.3.6. Let L be a transverse link in (S^3, ξ_{std}). As we have seen previously, ξ_{std} lifts to a contact structure ξ_L on the branched double cover $\Sigma(L)$, which is determined uniquely up to isotopy(actually, $\mathbb{Z}_2$-eqivariant isotopy). As a smooth manifold, $\Sigma(L)$ is defined by removing a standard neighborhood

$$(N(L), \xi_{std}|_{N(L)}) \simeq (S^1 \times D^2, \ker(dz + r^2 d\theta)),$$

taking double cover with respect to the meridian ∂D^2, and then regluing a copy of $S^1 \times D^2$ where the covering transformation acts by $(r, \theta) \mapsto (r^2, 2\theta)$.

Suppose that a small smooth curve $\gamma \subset \Sigma(L)$ defined near a point $p \in L$ is invariant under $\mathbb{Z}_2$. Then, in the parametrization $N(L) \simeq S^1 \times D^2$, the tangent line $T_p\gamma$ must be spanned by the vector ∂_r. Since the contact structure ξ_L on $\Sigma(L)$ is defined near L by $\ker(\alpha_L)$ where $\alpha_L = dz + f(r)d\theta$ for a good increasing f, we deduce that, for any $v \in T_p\gamma$,

$$(\alpha_L)_p(v) = dz(v) + f(r)d\theta(v) = 0.$$

Therefore γ should be Legendrian at $\gamma \cap L$, whch implies that it is very natural to work with Legendrian arcs while dealing with saddle cobordisms of transverse links.

Now let L be a based transverse link with basepoint $z \in L$, and suppose that we are given a Legendrian arc a satisfying the conditions (C1) and (C2), which does not contain the basepoint z. Then the set $L \cup a$ is an embedded graph in S^3, such that it consists of transverse edge-cycles together with one Legendrian edge connecting points on the transverse cycles, and has a basepoint in some transverse edge. We shall impose a further condition on a to ensure that the lift γ of a becomes a smooth Legendrian knot in $(\Sigma(L), \xi_L)$:

- There exists a standard neighborhood $(N(L), \xi_{std}|_L) \simeq \coprod(S^1 \times D^2, \ker(dz + r^2 d\theta))$ such that the curve segment $a \cap N(L)$ is a straight radial line (i.e. point towards the r-axis).

When this condition is also satisfied, we shall call such a graph (L, a, z) a nice graph.

Suppose that a nice graph $G = (L, a, z)$ is given. By the definition of nice graphs, there exists a standard neighborhood $(N(L), \xi_{std}|_L) \simeq \coprod(S^1 \times D^2, \ker(dz + r^2 d\theta))$ such that the curve segment $a \cap N(L)$ is a straight radial line (i.e. towards the r-axis). So we have a contact 1-form representative α of ξ_{std} such that

$$\alpha|_{N(L)} = dz + r^2 d\theta,$$

in the given neighborhood parametrization. Now, by the standard neighborhood theorem for Legendrian arcs, we have a neighborhood parametrization, which is uniquely determined up to isotopy:

$$(N(a), \alpha|_{N(a)}) \simeq ([0, \pi/2] \times D^2, \ker(\cos(z)dx + \sin(z)dy)).$$

Here, by a (closed) neighborhood of the arc a, we mean an embedded closed disk-bundle over a. Sometimes we will choose a slight extension of the above parametriza-

tion, so that we have a contact embedding of the cylinder

$$\left(\left[-\epsilon, \frac{\pi}{2}+\epsilon\right] \times D^2, \ker(\cos(z)dx + \sin(z)dy)\right).$$

Example 6.3.7. Consider the contact manifold

$$U = ([0, \pi/2] \times D^2, \alpha = \cos(z)dx + \sin(z)dy),$$

which is the standard neighborhood of the Legendrian arc $[0, \pi] \times \{0\}$. If a curve $\gamma = (\gamma_x, \gamma_y, \gamma_z)$ satisfies

$$\frac{\gamma_y'}{\gamma_x'} = 2\tan(z),$$

Then $\alpha(\gamma') = (1 + \sin^2 z) \cdot v \neq 0$ where $v = \frac{\gamma_x'}{\cos(z)} = \frac{\gamma_y'}{2\sin(z)}$. So, given a regular curve

$$a(t) = (a_x(t), a_y(t)) \in D^2,$$

the induced curve

$$V_a = \left(a_x, a_y, \arctan\left(\frac{a_y'}{2a_x'}\right)\right) \in D \times [0, \pi/2],$$

if defined, is always transverse. Also note that we have

$$d\alpha_U = dz \wedge (-\sin(z)dx + \cos(z)dy).$$

Remark. In this section, when we draw a regular curve a on D^2, we will mean the induced transverse curve V_a in a slightly extended cylinder $\left[-\epsilon, \frac{\pi}{2}+\epsilon\right] \times D^2$ for a very small $\epsilon > 0$.

Since L is a Reeb orbit of α and the Reeb orbits of $\cos\theta dx + \sin\theta dy$ passing through a point in $I \times \{\mathbf{0}\}$ are radial lines, we deduce that the curve segments $L \cap N(a_i)$ are given by the y-axes on $D^2 \times \{\pm\pi/2\}$. Then we consider a family of curves

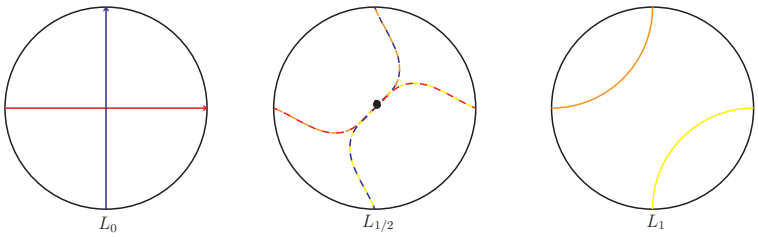


Figure 6.6: The family of curves L_t at $t = 0$, $t = \frac{1}{2}$, and $t = 1$.

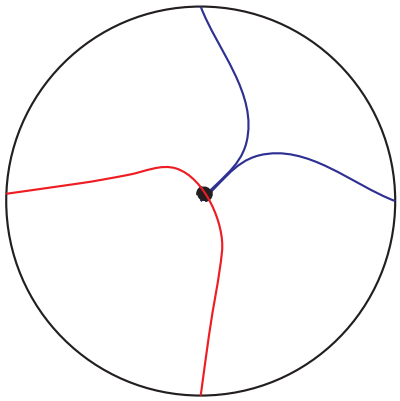


Figure 6.7: The projection of $\tilde{L}_{1/2}$ along the z-axis.

$\{L_t\}_{t\in[0,1]}$defined as in Figure 6.6, embedded in the cylinder $S^3 \times [0, R]$.

However, since the diagram in the middle gives two curves which are tangent at a point with tangent line $\mathbb{R}\partial_z$, we see that taking the union $\cup_{t\in[0,1]} L_t \times \{Rt\}$ of such curves does not give us a surface in $S^3 \times [0, R]$. To resolve this problem, we perform a slight pertubation to give a new family $\{\tilde{L}_t\}$, where the projection of $\tilde{L}_{1/2} \subset D^2 \times I$ to D^2 is given as in Figure 6.7.

If the perturbation is small enough, $\tilde{L}_t$ is transverse for $t \neq \frac{1}{2}$ and $\tilde{L}_t - \{p\}$ is transverse for $t = \frac{1}{2}$. Also, the two tangent vectors to $\tilde{L}_{1/2}$ at p define a positive orientation of D^2. We can now form a symplectic cobordism from such a family.

Theorem 6.3.8 (Theorem 7.8 of [25]). *When R is sufficiently large, the cobordism $S_R = \cup_{t\in[0,1]} \tilde{L}_t \times \{Rt\}$ is symplectic, and its weak symplectic isotopy class depends only on the given transverse link L and a Legendrian arc a.*

Proof. For simplicity, fix $R = 1$ and let the symplectic structures on $S^3 \times I$ vary:

$$\omega_R = d(e^{Rt}\alpha_{std}) = e^{Rt}(Rdt \wedge \alpha_{std} + d\alpha_{std}).$$

We are asking whether the given cobordism, which we will denote by S, is symplectic with respect to the symplectic form

$$\frac{1}{Re^{Rt}}\omega_R = dt \wedge \alpha_{std} + \frac{1}{R}d\alpha_{std}.$$

Choose a smooth nonvanishing tangent bivector field σ on S. By the transversality assumption, the term $dt \wedge \alpha_{std}(\sigma)$ is everywhere nonnegative, and it vanishes only at the saddle point, which we will call p. On the other hand, the term $d\alpha_{std}(\sigma)$ may not be everywhere nonnegative, but it takes a positive value at p. So, suppose that $d\alpha_{std}(\sigma) > 0$ in the r-ball centered at the saddle point, and let

$$\max_S |d\alpha_{std}(\sigma)| < M, \min_{S - B_r(p)} dt \wedge \alpha_{std}(\sigma) > m.$$

Then, if $R > \frac{M}{m}$, we get

$$\left| \left(dt \wedge \alpha_{std} + \frac{1}{R}d\alpha_{std} \right)(\sigma) \right| \geq m - \frac{M}{R} > 0$$

on $S - B_r(p)$. But this quantity is also positive inside $B_r(p)$ by our assumptions. Therefore, for all such R, the cobordism becomes symplectic. The statement about the weak symplectic isotopy class follows from the fact that, given a smooth 1-parameter family of auxiliary choices that we have made, we can choose a very large stretching factor R, which gives us an 1-parameter family of symplectic saddle cobordisms. □

Remark. The same argument can also be used for any surfaces constructed by a similar procedure. In particular, it works for symplectic births.

Definition 6.3.9. Denote such a cobordism, with the curve $\{z\} \times [0, R]$ on it, by $S = S(L, a, z)$. Then S has (L, z) as a concave end and a based transverse link $L' = (C(L, a), z)$ as a convex end. The (transverse isotopy class of) based transverse link $(C(L, a), z)$ is called the transverse surgery of (L, z) along a. The (weak symplectic isotopy class of) based symplectic cobordism $S(L, a, z)$ is called the symplectic saddle of (L, z) along a.

With those constructions in mind, we will say that the based symplectic cobordisms which can be constructed from the symplectic births and saddles are symplectically constructible. More precisely, we have the following definition.

Definition 6.3.10. The weak symplectic isotopy class of based symplectic cobordisms, obtained from gluing the classes of symplectic births and saddles, are called constructible classes. The cobordisms contained in constructible classes are called symplectically constructible based cobordisms.

6.3.2 Functoriality of $c_{\mathbb{Z}_2}$ under constructible symplectic cobordisms

Now we prove that the maps between $\widehat{HF}_{\mathbb{Z}_2}$, associated to symplectically constructible based cobordisms, maps the equivariant contact class of the concave end to the equivariant contact class of the convex end. We start from the simplest case, when the based link L is already braided along the z-axis and the Legendrian arc a is in a very nice position.

Definition 6.3.11. Let L be a transverse link in (S^3, ξ_{std}), which is braided along the z-axis. A simple Legendrian arc a is basic if there exists a genus-zero open book supporting (S^3, ξ_{std}), whose binding is the z-axis, such that a is contained in a single page.

Lemma 6.3.12 (Lemma 7.12 of [25])**.** *Let (L, a, z) be a nice graph, where L is braided*

along the z-axis, and suppose that a is basic. Let

$$\hat{f}_{S(L,a,z)} : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(C(L,a)),z) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L),z)$$

be the map associated to the based cobordism class $S(L,a,z)$. Then we have

$$\hat{f}_{S(L,a,z)}(c_{\mathbb{Z}_2}(\xi_{C(L,a)},z)) = c_{\mathbb{Z}_2}(\xi_L,z).$$

Proof. Since a is basic, we know that the contact branched double cover of (S^3,ξ_{std}) along $C(L,a)$ is the contact (-1)-surgery of $(\Sigma(L),\xi_L)$ along the lift γ of a. The map $\hat{f}_{S(L,\{a_i\},z)}$ is given by the equivariant triangle map. The Heegaard triple-diagram we get is described in Figure 6.8.

As in the proof of invariance under positive stabilizaton, by the convenient placement of a basepoint, all triangles connecting $\mathbf{x}$ and Θ in the diagram are small. So we deduce that the higher order terms in the equivariant triangle map vanish, since they count triangles of Maslov index at most 0. Thus we get the desired equality:

$$\begin{aligned}\hat{f}_{S(L,a,z)}(\xi_{C(L,a)},z) &= c_{\mathbb{Z}_2}(\xi_L,z) + \text{higher order terms}\\ &= c_{\mathbb{Z}_2}(\xi_L,z).\end{aligned}$$

□

In the general case when L is not in a braid position and a is arbitrary, we argue that we can alway isotope the whole situation to the above case, where L is braided and a is basic. For that we will have to deform our nice graph into a 4-valent transverse graph.

Definition 6.3.13. A based 4-valent transverse graph embedded in (S^3,ξ_{std}) is a 4-valent directed graph Γ with exactly one 4-valent vertex and several 2-valent vertices,

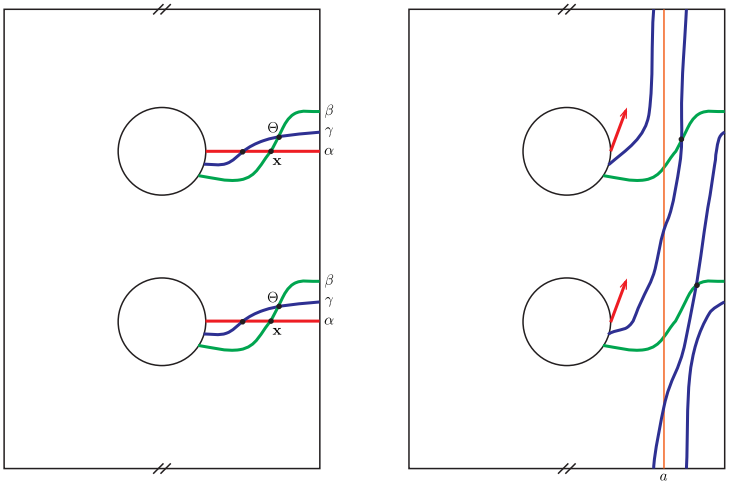


Figure 6.8: The associated Heegaard triple-diagram.

together with a basepoint z on an edge of Γ and an embedding

$$\Gamma \hookrightarrow S^3,$$

such that each edge is transverse, the two adjacent edges of 2-valent vertices glue smoothly, and the 4-valent vertex has an labelling $l_1^{\pm}, l_2^{\pm}$ of its adjacent edges so that, as directed smooth curves, $l_1^{\pm}$ glue smoothly with $l_2^{\pm}$.

Recall that, given a nice graph (L, a, z), we have defined the surgered link $C(L, a)$ and the symplectic saddle $S(L, a, z)$ by a sequence of diagrams drawn on D^2. By considering the intermediate slice (before applying a perturbation to make it a well-defined symplectic surface), which is defined as in Figure 6.9, we can see that L transforms to $C(L, a)$ through a based 4-valent transverse graph $G(L, a, z)$ in (S^3, ξ_{std}).

Now, given a nice graph (L, a, z), we want to isotope it, through a 1-parameter family of nice graphs, to another nice graph (L', a', z'), where L' is braided along the z-axis and a' is basic. To do that, we first isotope the based 4-valent transverse graph $G(L, a, z)$, so that its edges are in braid position. This is always possible due to the following extension of Bennequin's theorem.

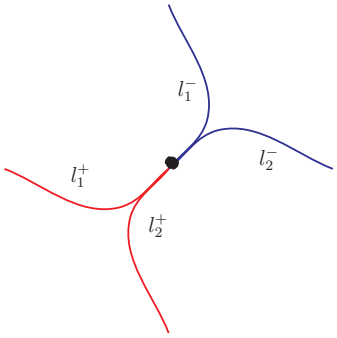


Figure 6.9: The intermediate slice.

Theorem 6.3.14 (Extension of Bennequin's theorem, [6])**.** *Any smooth transverse graph can be transversely isotoped into a braided position along the z-axis.*

Lemma 6.3.15 (Lemma 7.15 of [25])**.** *Every nice graph (L, a, z) can be isotoped to another nice graph (L', a', z) through nice graphs, so that L' is braided and a' is basic.*

Proof. By Theorem 6.3.14, we can transversely isotope the based 4-valent transverse graph $G(L, a, z)$, so that the isotopy is supported away from its unique 4-valent vertex, say v. Since (L, a, z) can be isotoped so that it agrees with $G(L, a, z)$ away from the vertex v by the definition of $G(L, a, z)$, we see that the transverse isotopy of $G(L, a, z)$ can also be applied to (L, a, z) so that a is very short and L is braided. Therefore, by isotoping the pages of the open book of S^3 in a small neighborhood of v, we see that a can be made basic, while remaining L braided. □

Now we can prove the functoriality in the most general setting.

Lemma 6.3.16 (Lemma 7.16 of [25])**.** *Given a nice graph (L, a, z), let*

$$\hat{f}_{S(L,\{a_i\},z)} : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(C(L,a),z) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L),z)$$

be the map associated to the based cobordism class $S(L, a, z)$. Then we have

$$\hat{f}_{S(L,a,z)}(c_{\mathbb{Z}_2}(\xi_{C(L,a)}, z)) = c_{\mathbb{Z}_2}(\xi_L, z).$$

Proof. Using Bennequin's theorem, we can isotope the decorated transverse 4-valent graph

$$(\Gamma, V, z) = G(L, a, z)$$

into a braided position along the z-axis, with respect to some genus-0 open book having the z-axis as its binding. Furthermore, we can isotope the vector field V so that it is tangent to the pages of the open book. Then its detachment is a nice graph (L', a', z'), whch is isotopic to the original nice graph (L, a, z), such that L' is braided along the z-axis and a is basic. Therefore, by the above lemma and the invariance of equivariant contact classes under isotopies, we must have

$$\hat{f}_{S(L,a,z)}(c_{\mathbb{Z}_2}(\xi_{C(L,a)}, z)) = c_{\mathbb{Z}_2}(\xi_L, z).$$

□

Theorem 6.3.17 (Theorem 7.17 of [25])**.** *Given any symplectically constructible (weak symplectic isotopy) class S of based symplectic cobordisms, with its concave and convex ends given by transverse isotopy classes (L_1, z_1) and (L_2, z_2) of based transverse links, we have*

$$\hat{f}_S(c_{\mathbb{Z}_2}(\xi_{L_2}, z_2)) = c_{\mathbb{Z}_2}(\xi_{L_1}, z_1),$$

where $\hat{f}_S : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_2), z_2) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_1), z_1)$ is the cobordism map, induced by S.

Proof. We already have the functoriality for both symplectic birth cobordisms and symplectic saddle cobordisms, and the functoriality for cylindrical cobordisms is obvious. Therefore, by composition, we get the functoriality for all symplectically con-

structible cobordisms. □

Corollary 6.3.18 (Corollary 7.18 of [25]). *We have a functor*

$$(\widehat{HF}_{\mathbb{Z}_2}, c_{\mathbb{Z}_2}) : \mathbf{sCob}^c_w \to (\mathbb{F}_2[\theta] \downarrow \mathbf{Mod}_{\mathbb{F}_2[\theta]}),$$

where $\mathbf{sCob}^c_w$ *is the wide subcategory of* $\mathbf{sCob}_w$ *spanned by symplectically constructible cobordism classes and* $\mathbb{F}_2[\theta] \downarrow \mathbf{Mod}_{\mathbb{F}_2[\theta]}$ *is the category of modules over* $\mathbb{F}_2[\theta]$ *with a* θ*-tower generated by a distinguished element.*

Proof. This is just a category-theoretic statement for the theorem above. □

6.4 Properties

The first property of $c_{\mathbb{Z}_2}(\xi_L, z)$ is that it can be used to recover the ordinary contact class $c(\xi_L)$.

Theorem 6.4.1 (Theorem 8.1 of [25]). *The natural map*

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), z) \to \widehat{HF}(-\Sigma(L))$$

sends $c_{\mathbb{Z}_2}(\xi_L, z)$ *to* $c(\xi_L)$.

Proof. The chain level map is given by truncating all terms with nontrivial θ-degree, so it sends $EH_{\mathbb{Z}_2}(\xi_L) = EH(\xi_L) \otimes \theta^0$ to $EH(\xi_L)$. Hence, on the cohomology level, $c_{\mathbb{Z}_2}(\xi_L)$ is sent to $c(\xi_L)$. □

Also, as in Section 6.1 of [20], we have a localization isomorphism

$$\theta^{-1}\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \simeq \widehat{HF}(S^3) \otimes \mathbb{F}_2[\theta, \theta^{-1}],$$

where K is a knot and θ^{-1} means that we are formally inverting θ, i.e. we define

$$\theta^{-1}\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) = \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}].$$

In turns out that the image of the equivariant contact class under the localization isomorphism takes a very simple form.

Theorem 6.4.2 (Theorem 8.2 of [25]). *Let K be a transverse knot in (S^3, ξ_{std}). Then the localization map*

$$\theta^{-1}\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \xrightarrow{\sim} \widehat{HF}(S^3) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \simeq \mathbb{F}_2[\theta, \theta^{-1}],$$

which is defined up to multiplication by powers of θ, sends $c_{\mathbb{Z}_2}(\xi_K)$ to a power of θ.

Proof. By the construction of the bare localization map in [48], the localization map can be written as follows:

$$c_{\mathbb{Z}_2}(\xi_K) \mapsto (c(\xi_{std}) + \text{higher order terms}) \otimes \theta^d,$$

for some d. But since there are no holomorphic disks going towards $EH^*(\xi_K)$, the higher order terms vanish. Therefore the localization map sends $c_{\mathbb{Z}_2}(\xi_K)$ to $c(\xi_{std}) \otimes \theta^d = \theta^d$. □

This theorem gives us a lower bound for the d_3-invariants of the branched double covers along transverse knots in (S^3, ξ_{std}). Recall that $q_\tau(K)$ is defined as:

$$q_\tau(K) = 2 \cdot \min\{\mathrm{gr}(x) \,|\, x \in \widehat{HF}_{\mathbb{Z}_2}(-\Sigma(K)),\, \theta^k x \neq 0 \text{ for all } k \geq 0\}.$$

Corollary 6.4.3 (Corollary 8.3 of [25]). *For a knot K in S^3, denote the set of all transverse representatives of K as $\mathcal{T}_K$. Then we have*

$$\frac{q_\tau(K) - 1}{2} \leq \min_{T \in \mathcal{T}_K} d_3(\xi_T).$$

Proof. For any transverse representative $T \in \mathcal{T}_K$, the absolute $\mathbb{Q}$-grading $\mathrm{gr}(EH(\xi_T))$ of the contact element of $(\Sigma(K), \xi_T)$, which is the same as $\mathrm{gr}(c_{\mathbb{Z}_2}(\xi_T))$, is given by $\frac{1}{2} + d_3(\xi_T)$; see Proposition 4.6 of [38]. By the above theorem, $c_{\mathbb{Z}_2}(\xi_T)$ cannot be annihilated by a power of θ, since the localization map is an isomorphism of $\mathbb{F}_2[\theta]$-modules. Therefore we have $\frac{q_\tau(K)-1}{2} \leq \min_{T\in\mathcal{T}_K} d_3(\xi_T)$ for all such T. □

The functoriality of equivariant contact classes for symplectically constructible cobordisms also gives us some results about symplectic representatives of link cobordisms.

Theorem 6.4.4 (Theorem 8.4 of [25])**.** *Let (L_1, z_1) and (L_2, z_2) be two based transverse links in (S^3, ξ_{std}). If the based isotopy class of a based cobordism S from (L_1, z_1) to (L_2, z_2) has a symplectically constructible representative, then we must have*

$$\hat{f}_S(c_{\mathbb{Z}_2}(L_2, z_2)) = c_{\mathbb{Z}_2}(L_1, z_1),$$

where $\hat{f}_S$ is the cobordism map induced by S.

Proof. This follows directly from the functoriality and the fact that $\hat{f}_S$ depends only on the based isotopy class of S. □

We can also explicitly calculate the equivariant contact class for some very simple transverse knots.

Example 6.4.5. Consider the trivial transverse braid U and its positive/negative stabilizations P, N, respectively. We will see from the proof of the next theorem that $c_{\mathbb{Z}_2}(\xi_U) = c_{\mathbb{Z}_2}(\xi_P) = 1$ but $c_{\mathbb{Z}_2}(\xi_N) = \theta$. This reflects the fact that, while the contact elements of ξ_U and ξ_P have Maslov degree zero, the contact element of ξ_N has Maslov degree one, in the Floer chain complex of S^3.

Note that, while performing a positive stabilization to a transverse link (on any of its components) does not change its transverse isotopy class, performing a negative

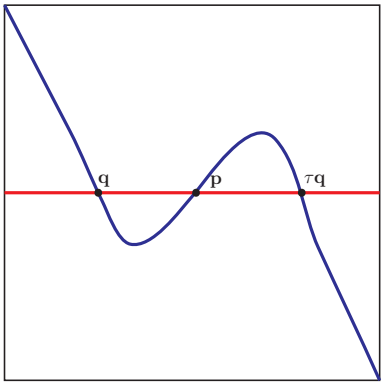


Figure 6.10: The associated equivariant Heegaard diagram, near the last strand.

stabilization does change its transverse isotopy class. However, the topological isotopy class does not change under negative stabilizations, so the equivariant contact class of a transverse link and its positive stabilization lie in the same group. It turns out that the behavior of the equivariant contact class under a negative stabilization is very simple.

Theorem 6.4.6 (Theorem 8.6 of [25]). *Let L be a transverse link in (S^3, ξ_{std}) and denote its negative stabilization(i.e. transverse stabilization), applied to any of its components, by L^-. Then we have*

$$c_{\mathbb{Z}_2}(\xi_{L^-}) = \theta \cdot c_{\mathbb{Z}_2}(\xi_L).$$

Proof. Put L in a braided position. Then the negative stabilization L^- is given by adding a negative twist to the last two strands in $L \coprod U$ where U is the trivial braid. So the equivariant Heegaard diagram for $\Sigma(L^-)$ near the last strand is given by Figure 6.10. Put $\mathbf{x} = EH(\xi_L) \otimes \mathbf{q}$. Then, in the dual of the freed Floer complex, we have

$$\begin{aligned} d_{\mathbb{Z}_2}\mathbf{x} &= EH(\xi_L) \otimes \mathbf{p} \otimes \theta^0 + EH(\xi_L) \otimes (\mathbf{q} + \tau\mathbf{q}) \otimes \theta^1 \\ &= EH_{\mathbb{Z}_2}(\xi_L) + EH(\xi_L) \otimes (\mathbf{q} + \tau\mathbf{q}) \otimes \theta^1. \end{aligned}$$

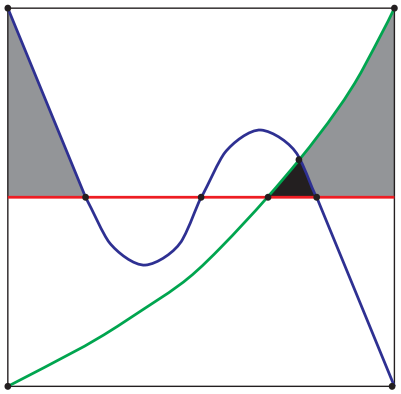


Figure 6.11: Relevant triangles in the associated equivariant Heegaard triple-diagram, each of which has zero Maslov index.

Now consider the equivariant triple Heegaard diagram in Figure 6.11, which describes the negative stabilization L^- and the positive stabilization L^+ of L: The two shaded triangles are the only holomorphic triangles in the above diagram, so by the associativity of equivariant triangle maps, working with equivariant contact classes in equivariant HFs should be the same as working with contact classes in ordinary HFs. In other words, we have

$$c_{\mathbb{Z}_2}(\xi_{L^+}) = [EH_{\mathbb{Z}_2}(\xi_{L^+})] = [EH(\xi_L) \otimes (\mathbf{q} + \tau\mathbf{q}) \otimes \theta^0] \in \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L)).$$

Therefore we get

$$c_{\mathbb{Z}_2}(\xi_{L^-}) = \theta \cdot c_{\mathbb{Z}_2}(\xi_{L^+}) = \theta \cdot c_{\mathbb{Z}_2}(\xi_L).$$

□

Finally, using the naturality and functoriality of equivariant Heegaard Floer cohomology, we can construct an isotopy invariant of slice disks of a given slice knot in S^3, as follows. Note that a similar invariant can be constructed using functoriality of knot Floer homology under decorated cobordisms, as the $t_{S,P}$ invariant in [22].

Theorem 6.4.7 (Theorem 8.7 of [25]). *Let $K \subset S^3$ be a (smoothly) slice knot, and $D \subset B^4$ be a slice disk which bounds K. Choose any point $p \in D$, its neighborhood*

$B(p) \subset B^4$, and draw a smooth simple arc s on the annulus $D^2 - B(p)$, so that $s \cap K$ is a point and $(D^2 - B(p), s)$ is a based cobordism between $(K, s \cap K)$ and the based unknot. Consider the induced cobordism map:

$$\hat{f}_{(D^2-B(p),s)} : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(unknot), pt) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), s \cap K).$$

Then the element $\hat{f}_{(D^2-B(p),s)}(\mathbf{x})$, where $\mathbf{x} \in \widehat{HF}_{\mathbb{Z}_2}(\Sigma(unknot), pt)$, is nonvanishing and depends only on the isotopy class of D rel K.

Proof. The fact that $\hat{f}_{(D^2-B(p),s)}(\mathbf{x})$ depends only on the isotopy class of D rel K following from Theorem 3.7.8, since $D^2 - B(p)$ is an annulus and thus the choice of s is unique up to homotopy. The fact that $\hat{f}_{(D^2-B(p),s)}(\mathbf{x})$ is nonvanishing follows from Theorem 6.4.2 and Lemma 6.11 of [20]. □

Chapter 7

$\mathbb{Z}_2$-Equivariant knot Floer cohomology and the LOSS invariant

Theorem 4.5.7 tells us that, given an admissible knot Heegaard diagram H of a based knot (K, z) in S^3, we can calculate $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$ by taking branched double cover of H, removing its second basepoint, and then taking $\widehat{HF}_{\mathbb{Z}_2}$ of the $\mathbb{Z}_2$-invariant diagram we obtain.

Now suppose that we do not remove the second basepoint. Then the diagram we obtain is a knot Heegaard diagram, which now represents the preimage of K in the branched double cover $\Sigma(K)$. In this chapter, we will observe that applying $\widehat{HF}_{\mathbb{Z}_2}$ to this diagram gives another knot invariant, and use it to construct a $\mathbb{Z}_2$-equivariant refinement of the LOSS invariant. Note that, to achieve more generality, we work with links in S^3, instead of knots.

7.1 $\mathbb{Z}_2$-Equivariant Heegaard Floer cohomology

Definition 7.1.1. Given a link Heegaard diagram H representing a link L in S^3, we denote by $B(H)$ the Heegaard diagram representing L in $\Sigma(L)$, obtained by taking branched double cover of H along its two basepoints.

Lemma 7.1.2 (Lemma 5.2 of [24]). *Let H be a link Heegaard diagram representing a link in S^3 and write $B(H) = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{x}, \mathbf{z})$. Then for a generic 1-parameter family J of $\mathbb{Z}_2$-equivariant almost complex structures on $Sym^g(\Sigma\backslash(\mathbf{x} \cup \mathbf{z}))$, every pair of generators $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and every homotopy class $\phi \in \pi_2(x, y)$ which does not intersect $\mathbf{x}$ and $\mathbf{z}$, the moduli space $\mathcal{M}(\phi; J)$ is transversely cut out.*

Proof. Since we are branching over $\mathbf{x} \cup \mathbf{z}$ and ϕ does not intersect with $\mathbf{x}$ and $\mathbf{z}$, condition (EH-2) in Section 3.2 is satisfied. □

Lemma 7.1.3 (Lemma 5.3 of [24]). *For any two weakly admissible link Heegaard diagrams H_1, H_2 representing a link in S^3, we have*

$$\widehat{HF}_{\mathbb{Z}_2}(B(H_1)) \simeq \widehat{HF}(B(H_2)).$$

Proof. We only have to consider the following three cases.

1. H_1 and H_2 are related by either an α-isotopy or a β-isotopy.
2. H_1 and H_2 are related by a handleslide.
3. H_2 is obtained by stabilizing H_1.

Cases 1 and 2 can be dealt in the same way as in the case of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$, so we assume that H_2 is obtained by stabilizing H_1. By Lemma 7.1.2, we have

$$\begin{aligned}\widehat{CF}_{\mathbb{Z}_2}(B(H_1)) &\simeq \mathrm{RHom}_{\mathbb{F}_2[\mathbb{Z}_2]}(\widehat{CF}(B(H_1)), \mathbb{F}_2),\\ \widehat{CF}_{\mathbb{Z}_2}(B(H_2)) &\simeq \mathrm{RHom}_{\mathbb{F}_2[\mathbb{Z}_2]}(\widehat{CF}(B(H_2)), \mathbb{F}_2).\end{aligned}$$

But $B(H_2)$ is the diagram obtained by performing a ($\mathbb{Z}_2$-invariant) pair of stabilizations on $B(H_1)$, so the stabilization map

$$\widehat{CF}(B(H_1)) \to \widehat{CF}(B(H_2))$$

is a $\mathbb{Z}_2$-equivariant quasi-isomorphism. Therefore $\widehat{CF}_{\mathbb{Z}_2}(B(H_1))$ is quasi-isomorphic to $\widehat{CF}_{\mathbb{Z}_2}(B(H_2))$. □

Lemma 7.1.3 suggests us to make the following definition.

Definition 7.1.4. Let L be a link in S^3 and H be a Heegaard diagram representing L. Then we define $\mathbb{Z}_2$-equivariant knot Floer cohomology $\widehat{HFL}_{\mathbb{Z}_2}(\Sigma(L), L)$ as the isomorphism class of the $\mathbb{F}_2[\theta]$-module $\widehat{HF}_{\mathbb{Z}_2}(B(H))$. When L is a knot, then we denote $\widehat{HFL}_{\mathbb{Z}_2}$ by $\widehat{HFK}_{\mathbb{Z}_2}$.

Remark 7.1.5. As in the case of $\widehat{HF}_{\mathbb{Z}_2}$ of based knots, $\widehat{HFK}_{\mathbb{Z}_2}$ of links can also be computed in a combinatorial way. The difference is that, for $\widehat{HFK}_{\mathbb{Z}_2}$, it suffices to use any nice diagrams, instead of very nice diagrams, and the proof is much simpler. Given a Heegaard diagram H_0 representing a link L in S^3, we first apply Sarkar-Wang algorithm on H_0 to turn it into a nice diagram H. Then $B(H)$ is also nice, since we are not throwing away any basepoints, so $\widehat{CF}(B(H))$ can be combinatorially computed. Then, by Lemma 7.1.3, we have

$$\widehat{HFL}_{\mathbb{Z}_2}(\Sigma(L), L) \simeq H^*(\mathrm{RHom}_{\mathbb{F}_2[\mathbb{Z}_2]}(\widehat{CF}(B(H)), \mathbb{F}_2)),$$

so we can compute $\widehat{HFL}_{\mathbb{Z}_2}$ in a purely combinatorial way.

7.2 Properties of $\widehat{HFK}_{\mathbb{Z}_2}$ (and $\widehat{HFL}_{\mathbb{Z}_2}$)

The equivariant link Floer cohomology $\widehat{HFL}_{\mathbb{Z}_2}$ is, as its name suggests, a refinement of both the equivariant Floer cohomology $\widehat{HF}_{\mathbb{Z}_2}$ of based knots in S^3 and the ordinary knot Floer cohomology $\widehat{HFL}$ of links in S^3.

Theorem 7.2.1 (Theorem 5.6 of [24])**.** *For any link L in S^3, there exists a spectral sequence*

$$\widehat{HFL}(\Sigma(L), L) \otimes \mathbb{F}_2[\theta] \Rightarrow \widehat{HFL}_{\mathbb{Z}_2}(\Sigma(L), L),$$

and the isomorphism classes of its pages are isotopy invariants of L.

Proof. This is a direct consequence of Lemma 7.1.2. □

Theorem 7.2.2 (Theorem 5.7 of [24]). *For any knot K in S^3, there exists a spectral sequence*

$$\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K) \Rightarrow \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$$

for any $z \in K$, and the isomorphism class of its pages are isotopy invariants of K.

Proof. Let $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ be any knot Heegaard diagram representing K. Then, as in the non-equivariant case, the second basepoint w induces a filtration on $\widehat{CF}_{\mathbb{Z}_2}(\hat{H})$, and its isomorphism class as a filtered chain complex is an invariant of K. The spectral sequence we get takes the form

$$\widehat{HF}_{\mathbb{Z}_2}(B(H)) \Rightarrow \widehat{HF}_{\mathbb{Z}_2}(\hat{H}).$$

Now, $\widehat{HF}_{\mathbb{Z}_2}(B(H)) \simeq \widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K)$ by the definition of $\widehat{HFK}_{\mathbb{Z}_2}$, and by Theorem 4.5.7, we have $\widehat{HF}_{\mathbb{Z}_2}(\hat{H}) \simeq \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z)$. □

Now we will prove a version of localization theorem for $\mathbb{Z}_2$-equivariant link Floer cohomology. Recall from Proposition 6.3 of [20] that, for any based link (L, z) in S^3, we have

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), z) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}] \simeq (\mathbb{F}_2^2)^{|L|-1} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}],$$

where $|L|$ is the number of connected components of L. We will prove that a similar statement is true for $\widehat{HFL}(\Sigma(L), L)$.

Theorem 7.2.3 (Theorem 5.8 of [24]). *For any link L in S^3, we have an isomorphism*

$$\widehat{HFL}_{\mathbb{Z}_2}(\Sigma(L), L) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}] \simeq \widehat{HFL}_{\mathbb{Z}_2}(-S^3, L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

Proof. Let $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ be a weakly admissible Heegaard diagram representing L, and for each positive integer n, denote by H_n the Heegaard diagram obtained by

adding n α-curves, n β-curves, and $2n$ basepoints in a small neighborhood of w, as drawn in Figure 7.1. The proof of Lemma 7.1.2 also applies to the diagram H_n, so if we denote the branched double cover of H_n along its basepoints by $B(H_n)$, then we get an isomorphism

$$\widehat{CF}_{\mathbb{Z}_2}(B(H_n)) \simeq \mathrm{RHom}_{\mathbb{F}_2[\mathbb{Z}_2]}(\widehat{CF}(B(H_n)), \mathbb{F}_2).$$

Since we have $\widehat{CF}(B(H_n)) \simeq \widehat{CF}(B(H)) \otimes C_n^*$ for a finite-dimensional chain complex C_n^*, where $\mathbb{Z}_2$ acts trivially on C_n^*, we have an isomorphism

$$\widehat{HF}_{\mathbb{Z}_2}(B(H_n)) \simeq \widehat{HFL}_{\mathbb{Z}_2}(\Sigma(L), L) \otimes V_n$$

for a $\mathbb{F}_2$-vector space V_n of dimension 2^n. Hence V_n satisfies

$$\widehat{HF}(H_n) \simeq \widehat{HFL}(-S^3, L) \otimes V_n.$$

When n is sufficiently large, there exists a multi-pointed Heegaard diagram H_0, whose Heegaard surface is S^2, so that H_n and H_0 are related by a sequence of isotopies, handleslides, and (de)stabilizations, which would imply $\widehat{HF}_{\mathbb{Z}_2}(B(H_0)) \simeq \widehat{HF}_{\mathbb{Z}_2}(B(H_n))$ by equivariant transversality, and $\widehat{HF}(H_0) \simeq \widehat{HF}(H_n)$. Since H_0 is a Heegaard diagram drawn on S^2, it satisfies condition (EH-1) in [20], so we have a localization isomorphism

$$\widehat{HF}_{\mathbb{Z}_2}(B(H_0)) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}] \simeq \widehat{HF}(H_0) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

We now consider the isomorphisms we have got. First, we have the following isomorphism:

$$\widehat{HF}(H_0) \simeq \widehat{HF}(H_n) \simeq \widehat{HFL}(-S^3, L) \otimes V_n.$$

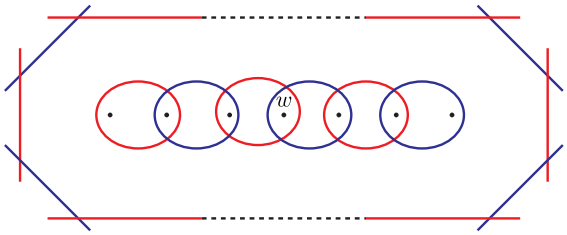


Figure 7.1: The diagram H_n. Here we have set $n = 3$ for simplicity.

Next, we have the following isomorphisms:

$$\begin{aligned}\widehat{HF}(H_0) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta,\theta]^{-1} \simeq & \widehat{HF}_{\mathbb{Z}_2}(B(H_0)) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta,\theta^{-1}] \\ \simeq & \widehat{HF}_{\mathbb{Z}_2}(B(H_n)) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta,\theta^{-1}] \\ \simeq & \widehat{HFL}_{\mathbb{Z}_2}(\Sigma(L),L) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta,\theta^{-1}] \otimes_{\mathbb{F}_2} V_n.\end{aligned}$$

Hence we have the following isomorphism of $\mathbb{F}_2[\theta,\theta^{-1}]$-modules:

$$\widehat{HFL}(-S^3,L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta,\theta^{-1}] \otimes V_n \simeq \widehat{HFL}_{\mathbb{Z}_2}(\Sigma(L),L) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta,\theta^{-1}] \otimes_{\mathbb{F}_2} V_n.$$

In other words, if we denote $\widehat{HFL}(S^3,L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta,\theta^{-1}]$ by M, $\widehat{HFL}_{\mathbb{Z}_2}(\Sigma(L),L) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta,\theta^{-1}]$ by N, then we have

$$M^{2^n} \simeq N^{2^n}.$$

However, M and N are finitely generated $\mathbb{F}_2[\theta,\theta^{-1}]$-modules, and since $\mathbb{F}_2[\theta]$ is a PID, its localization $\mathbb{F}_2[\theta,\theta^{-1}]$ is also a PID. Therefore, by the classfication of finitely generated modules over a PID, we must have $M \simeq N$, i.e.

$$\widehat{HFL}(-S^3,L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta,\theta^{-1}] \simeq \widehat{HFL}_{\mathbb{Z}_2}(\Sigma(L),L) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta,\theta^{-1}].$$

□

Theorem 7.2.3 gives a new proof of Theorem 1.14 of [20] as a direct corollary.

Corollary 7.2.4 (Corollary 5.9 of [24]). *Given a link L in S^3, there exists a spectral*

sequence

$$\widehat{HFL}(\Sigma(L),L)\otimes_{\mathbb{F}_2}\mathbb{F}_2[\theta,\theta^{-1}]\Rightarrow\widehat{HFL}_{\mathbb{Z}_2}(-S^3,L)\otimes_{\mathbb{F}_2[\theta]}\mathbb{F}_2[\theta,\theta^{-1}],$$

whose pages are isotopy invariants of L.

Proof. Tensoring the spectral sequence of Theorem 7.2.1 with $\mathbb{F}_2[\theta,\theta^{-1}]$ gives a spectral sequence

$$\widehat{HFL}(\Sigma(L),L)\otimes_{\mathbb{F}_2}\mathbb{F}_2[\theta,\theta^{-1}]\Rightarrow\widehat{HFL}_{\mathbb{Z}_2}(\Sigma(L),L)\otimes_{\mathbb{F}_2[\theta]}\mathbb{F}_2[\theta,\theta^{-1}],$$

and by Theorem 7.2.3, we have an isomorphism

$$\widehat{HFL}_{\mathbb{Z}_2}(\Sigma(L),L)\otimes_{\mathbb{F}_2[\theta]}\mathbb{F}_2[\theta,\theta^{-1}]\simeq\widehat{HFL}(-S^3,L)\otimes_{\mathbb{F}_2}\mathbb{F}_2[\theta,\theta^{-1}].$$

□

7.3 Refining the LOSS invariant

Let $K\subset S^3$ be a transverse knot with respect to the standard contact structure $\xi_{std}=\ker(dz-xdy)$. In the previous chapter, we have constructed an invariant $c_{\mathbb{Z}_2}(\xi_K)$ as an element of the module $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, which depends only on the transverse isotopy class of K. On the other hand, we have the LOSS invariant $\hat{c}(K)$, defined in [32], which is an element of $\widehat{HFK}(-S^3,K)$ and depends only on the transverse isotopy class of K. Therefore it is natural to ask how those two invariants are related.

We have shown in the previous section that the equivariant knot Floer cohomology $\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K),K)$ is a refinement of both $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ and $\widehat{HFK}(-S^3,K)$. As a topological application of that fact, we will show in this section that there exists a transverse invariant of K, as an element of $\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K),K)$, which is a simultaneous refinement of both the equivariant contact class $c_{\mathbb{Z}_2}(\xi_K)$ and the LOSS invariant $\hat{c}(K)$.

We first recall some facts regarding relations between braids and transverse knots. Let (M, ξ) be a contact 3-manifold and $h : M \backslash B \to S^1$ be an open book supporting ξ. Then a transverse knot(or link) $K \subset M$ is said to be in a **braid position** with respect to B if $h|_K$ is regular, and when K is in a braid position, we call it as a transverse braid. When (M, ξ) is the standard contact S^3, the theorem of Bennequin[6] tells us that every transverse knot is transversely isotopic to a transverse braid along the z-axis. This fact can be directly generalized to the most general case, when (M, ξ) is any contact 3-manifold and h is any open book; in fact, the theorem of Pavelescu[40] tells us that any transverse knot in M is transversely isotopic to a braid position with respect to h, and two transverse braids, braided with respect to B, are transversely isotopic if and only if they are related by a sequence of braid isotopies, positive Markov moves, and their inverses. This fact leads us to the following definition, made by Baldwin, Vela-Vick, and Vertesi in [4].

Definition 7.3.1. Given a contact 3-manifold (M, ξ), an open book $h : M \backslash B \to S^1$ supporting ξ, and a transverse knot K which is in a braid position with respect to B, let S be a page of h, $F = S \cap K$, and ϕ be the monodromy of h which fixes F pointwise. Then the isotopy class of the monodromy ϕ rel $F \cup \partial S$ uniquely determines the transverse isotopy class of K. So we say that the (transverse isotopy class of) transverse knot K is **encoded** by the **pointed open book** (S, F, ϕ).

Recall that, given a contact 3-manifold (M, ξ), any two open books of M which support ξ are related by a sequence of isotopies and positive (de)stabilizations. Then, given any pointed open book (S, F, ϕ) which encodes a transverse knot K in (M, ξ), we can perform a positive stabilization with respect to any simple arc $a \subset S \backslash F$ satisfying $\partial a \subset \partial S$. Then the pointed open book (S', F', ϕ') we obtain by the positive stabilization still encodes the same transverse knot K; note that this is a special case of Corollary 2.5 of [4].

Using the facts we have stated above, we will give some useful observations on transverse knots in the standard contact S^3.

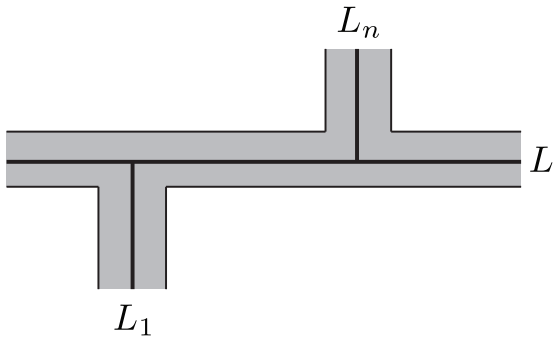


Figure 7.2: The page Σ of the open book h, near the Legendrian knot L.

Lemma 7.3.2 (Lemma 6.2 of [24]). *Let $K \subset (S^3, \xi_{std})$ be a transverse knot. Then there exists an open book decomposition of S^3 supporting ξ_{std}, such that its binding is disjoint from K and each of its page intersects K transversely at one point.*

Proof. Let L be a Legendrian knot in (S^3, ξ_{std}), whose positive transverse pushoff is K. Choose a contact cell decomposition of (S^3, ξ_{std}), so that L is contained in its 1-skeleton S. Denote the 1-cells of S which intersects L by $L_1, \cdots, L_n$, and the open book induced by the cell decomposition as $h : S^3 \backslash B \to S^1$, where B is the binding. Then a page Σ of h is drawn in Figure 7.2. Note that $N(L) \cap \partial(h^{-1}(p)) \subset K$, where $N(L)$ is a small neighborhood of L; this is because K is a positive pushoff of L.

Now perturb K to a knot K', so that $h|_{K'}$ is monotone but very close to zero in neighborhoods of $L_1, \cdots, L_n$. Then K' is clearly in a braid position with respect to B. Also, since we can choose our perturbation to be sufficiently C^1-small, K' is also a transverse knot. Therefore K' is a 1-strand transverse braid along B, so applying a self-contactomorphism of (S^3, ξ_{std}) which takes K to K' gives a desired open book. □

Lemma 7.3.3 (Lemma 6.3 of [24]). *Let $K \subset (S^3, \xi_{std})$ be a transverse knot and $(B_1, h_1), (B_2, h_2)$ be open book decompositions of S^3, supporting ξ_{std}, such that K is transversely isotopic to a 1-strand transverse braid along B_i, for each $i = 1, 2$. Denote the induced (single-)pointed open books by (S_1, p_1, ϕ_1) and (S_2, p_2, ϕ_2), respectively.*

Then (S_1, p_1, ϕ_1) and (S_2, p_2, ϕ_2) are related by a sequence of isotopies and positive (de)stabilizations.

Proof. By applying a self-contactomorphism of (S^3, ξ_{std}) to the given open books, we may assume that K intersects the pages of (B_1, h_1) and (B_2, h_2) transversely at one point. Then, by performing positive stabilizations to (B_1, h_1) and (B_2, h_2), we may assume that they are induced by a contact cell decomposition of (S^3, ξ_{std}), so that (B_1, h_1) is isotopic to (B_2, h_2). Also, by performing additional positive stabilizations, we may further assume that, for a Legendrian knot L which has K as its transverse push-off, the given contact cell decomposition contains L in its 1-skeleton. Then, by the argument used in the proof of Lemma 7.3.2, we see that (S_1, p_1, ϕ_1) and (S_2, p_2, ϕ_2) are isotopic. □

To sum up, we have proven the following proposition.

Proposition 7.3.4 (Proposition 6.4 of [24])**.** *Any transverse knot in (S^3, ξ_{std}) is encoded by a single-pointed open book, and any two single-pointed open books encoding the same transverse knot in (S^3, ξ_{std}) are related by a sequence of isotopies and positive (de)stabilizations.*

Proof. This is a direct consequence of Lemma 7.3.2 and Lemma 7.3.3. □

Now we will see how to translate a single-pointed open book, which encodes a transverse knot K in (S^3, ξ_{std}), to an element in $\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K)$. Given such a single-pointed open book (S, p, ϕ), choose an arc-basis $\mathbf{a} = \{a_1, \cdots, a_n\}$ of S, where the arcs in $\mathbf{a}$ do not contain p. Then consider the following objects.

- $\Sigma = (S \times \{0, 1\})/\sim$ where $(x, 0) \sim (x, 1)$ for any $x \in \partial S$
- For each $i = 1, \cdots, n$, b_i is the arc in S given by perturbing a_i slightly, so that a_i intersects b_i transversely at one point x_i, where the intersection is positive.
- $\boldsymbol{\alpha} = \{(a_i \times \{0\}) \cup (a_i \times \{1\}) | i = 1, \cdots, n\}$

- $\boldsymbol{\beta} = \{(b_i \times \{0\}) \cup (\phi(b_i) \times \{1\}) | i = 1, \cdots, n\}$
- $z = (p, 0)$, $w = (p, 1)$ in Σ

Then the 5-tuple $H_{(S,p,\phi,\mathbf{a})} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ is a knot Heegaard diagram representing K in S^3, and its branched double cover $\hat{H}$ along $\{z, w\}$ is a $\mathbb{Z}_2$-invariant knot Heegaard diagram representing K in $\Sigma(K)$, so that we have

$$\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K) \simeq \widehat{HF}_{\mathbb{Z}_2}(\hat{H}) \simeq H^*(\mathrm{RHom}_{\mathbb{F}_2[\mathbb{Z}_2]}(\widehat{CF}(\hat{H}), \mathbb{F}_2))$$

by Lemma 7.1.2. So, if we denote the inverse image of $\{x_1, \cdots, x_n\}$ under the branched covering map by $\mathbf{x}_{(S,p,\phi,\mathbf{a})}$, then since $\mathbf{x}_{(S,p,\phi,\mathbf{a})}$ is a $\mathbb{Z}_2$-invariant cocycle in $\widehat{CF}(\hat{H})$, the element $\mathbf{x}_{(S,p,\phi,\mathbf{a})} \otimes 1$ in $\mathrm{RHom}_{\mathbb{F}_2[\mathbb{Z}_2]}(\widehat{CF}(\hat{H}), \mathbb{F}_2)$ is a cocycle, which then corresponds to a cocycle in $\widehat{CFK}_{\mathbb{Z}_2}(\Sigma(K), K)$. We will denote its cohomology class as $\hat{\mathcal{T}}_{\mathbb{Z}_2}(S, p, \phi, \mathbf{a})$. Also, we will denote $\widehat{HF}_{\mathbb{Z}_2}(\hat{H})$ as $\widehat{HFK}_{\mathbb{Z}_2}(S, p, \phi, \mathbf{a})$, and call the tuple $(S, p, \phi, \mathbf{a})$ as a **single-pointed arc diagram**.

Lemma 7.3.5 (Lemma 6.5 of [24])**.** *Let S be a compact oriented surface with boundary, p be a point in the interior of S, and $\mathbf{a}_1, \mathbf{a}_2$ be arc bases of S whose arcs do not contain p. Then $\mathbf{a}_1$ and $\mathbf{a}_2$ are related by a sequence of isotopies and arc-slides, performed outside p.*

Proof. We only have to prove that we can replace an isotopy through p by a sequence of isotopies and handleslides performed in $S \backslash \{p\}$. To prove this, choose an arc $a \in \mathbf{a}_1$ and suppose that we want to isotope a to another simple arc a', where $p \notin a'$, $a \cap a' = \emptyset$, and a, a' cobound a strip $R \subset S$ where $p \in R$. Cutting S along the arcs in $\mathbf{a}_1 \backslash \{a\}$ gives an annulus A, where the two endpoints of a lie in different components of ∂A . Here, $A \backslash (a \cup a')$ is a disjoint union of a disk D with the strip R. Then, by a sequence of isotopies and arc-slides in D, we can replace a by a', as in Figure 7.3. Since $p \notin D$, we see that the isotopies we have used are performed outside p. □

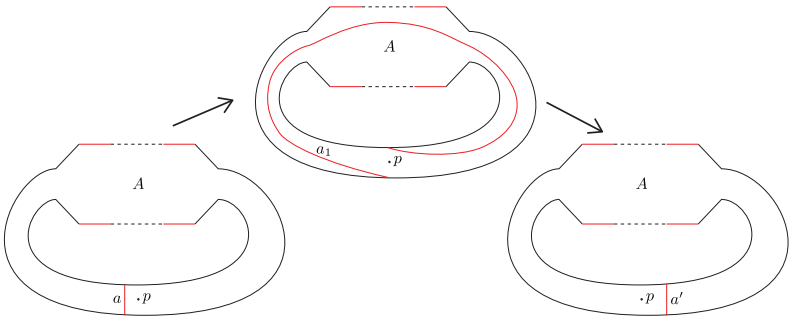


Figure 7.3: The arc a_1 is obtained from a by a sequence of arc-slides, and a' is obtained from a_1 again by a sequence of arc-slides. During the process, the arc-slides we perform do not pass through the basepoint p.

Definition 7.3.6. When two single-pointed arc diagrams $(S, p, \phi, \mathbf{a})$ and $(S', p', \phi', \mathbf{a}')$ are related by either an isotopy outside p, an arc-slide outside p, or a stabilization, we say that they are related by a **basic move**. Note that a basic move induces a map

$$\widehat{HFK}_{\mathbb{Z}_2}(S', p', \phi', \mathbf{a}') \to \widehat{HFK}_{\mathbb{Z}_2}(S, p, \phi, \mathbf{a}).$$

Lemma 7.3.7 (Lemma 6.7 of [24])**.** *Suppose that two single-pointed arc diagrams* $(S, p, \phi, \mathbf{a})$ *and* $(S', p', \phi', \mathbf{a}')$ *are related by a basic move. Then the induced map*

$$\widehat{HFK}_{\mathbb{Z}_2}(S', p', \phi', \mathbf{a}') \to \widehat{HFK}_{\mathbb{Z}_2}(S, p, \phi, \mathbf{a})$$

takes $\hat{\mathcal{T}}_{\mathbb{Z}_2}(S', p', \phi', \mathbf{a}')$ *to* $\hat{\mathcal{T}}_{\mathbb{Z}_2}(S, p, \phi, \mathbf{a})$.

Proof. The branched double covers of $(S, p, \phi, \mathbf{a})$ and $(S', p', \phi', \mathbf{a}')$, along p and p', respectively, are related by two basic moves, performed in a $\mathbb{Z}_2$-invariant way. Thus the conclusion follows from the argument used to prove the invariance of the contact classes of contact 3-manifolds; see Section 3 of [21] for details. □

From the things we have proven so far, we see that we have found a well-defined transverse invariant. Although we have not proven the naturality of $\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K)$, we can still define its well-defined element, which depends only on the transverse iso-

topy class of K in S^3.

Theorem 7.3.8 (Theorem 6.8 of [24]). *The element $\hat{\mathcal{T}}_{\mathbb{Z}_2}(S,p,\phi,\mathbf{a})$ depends only on the transverse isotopy class of transverse knots which $(S,p,\phi,\mathbf{a})$ encodes.*

Proof. This follows directly from Lemma 7.3.5 and Lemma 7.3.7. □

Note that, unlike the invariance of $c_{\mathbb{Z}_2}$, the invariance of $\hat{\mathcal{T}}_{\mathbb{Z}_2}$ is not the "full" invariance, since we did not prove the naturality of $\widehat{HFK}_{\mathbb{Z}_2}$. However, we can still think of it as a well-defined element, under the assumption that we only deal with knot Heegaard diagrams induced by the open book diagrams used in Lemma 7.3.2. Hence we can still say, in this restricted setting, that $\hat{\mathcal{T}}_{\mathbb{Z}_2}$ is indeed a transverse knot invariant, so we finish this section with the following definition.

Definition 7.3.9. Given a (transverse isotopy class of a) transverse knot K in (S^3,ξ_{std}), we define $\hat{\mathcal{T}}_{\mathbb{Z}_2}(S,p,\phi,\mathbf{a})$, for any single-pointed arc diagram $(S,p,\phi,\mathbf{a})$ encoding K, as $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K)$.

7.4 $\hat{\mathcal{T}}_{\mathbb{Z}_2}$ is a refinement of both $c_{\mathbb{Z}_2}$ and $\hat{\mathcal{T}}$

In this section, we will see that the transverse invariant $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K)$ refines both the hat-flavored LOSS invariant $\hat{\mathcal{T}}(K)\in\widehat{HFK}(S^3,K)$, and the $\mathbb{Z}_2$-equivariant contact class $c_{\mathbb{Z}_2}(\xi_K)\in\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$.

Theorem 7.4.1 (Theorem 6.10 of [24]). *For any transverse knot K in (S^3,ξ_{std}), there exists a localization isomorphism*

$$\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K),K)\otimes_{\mathbb{F}_2[\theta]}\mathbb{F}_2[\theta,\theta^{-1}]\xrightarrow{\sim}\widehat{HFK}(S^3,K)\otimes_{\mathbb{F}_2}\mathbb{F}_2[\theta,\theta^{-1}],$$

as defined in Theorem 7.2.3, which maps $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K)\otimes 1$ to $\hat{\mathcal{T}}(K)\otimes 1$.

Proof. Let $(S,p,\phi,\mathbf{a})$ be a single-pointed arc diagram encoding K, so that K is in a braid position with respect to the binding $B=\partial S$. For a positive integer n, consider

the transverse braid K_n given by applying postive Markov moves n times to K, and denote the $(n+1)$-pointed arc diagram encoding the braid K_n by $(S, \mathbf{p}_n, \phi, \mathbf{a}_n)$. Here, $\mathbf{p}_n = \{p, p_1, \cdots, p_n\}$ where $p_1, \cdots, p_n$ are the points added by Markov moves, and $\mathbf{a}_n$ is the arc basis given by adding to $\mathbf{a}$ small arcs connecting the points p_i to the component of ∂S where the ith Markov move was taken. Then, under the isomorphism

$$\widehat{HFK}_{\mathbb{Z}_2}(S, p, \phi, \mathbf{a}) \otimes_{\mathbb{F}_2} V_n \xrightarrow{\sim} \widehat{HFK}_{\mathbb{Z}_2}(S, \mathbf{p}_n, \phi, \mathbf{a}_n),$$

the element $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K) \otimes c$, where c denotes the top class in $V_n = (\mathbb{F}_2^2)^{\otimes n}$, is mapped to the cohomology class of the cocycle $\mathbf{x}_{H_{(S,\mathbf{p}_n,\phi,\mathbf{a}_n)}}$.

Now, when n is sufficiently big, $(S, \mathbf{p}_n, \phi, \mathbf{a}_n)$ is related by a sequence of isotopies, arc-slides, and positive (de)stabilizations to a pointed diagram of genus zero. So, by the argument used in the proof of Theorem 7.2.3, together with Lemma 7.3.7 applied to multi-pointed open books, we have a localization isomorphism

$$\widehat{HFK}_{\mathbb{Z}_2}(S, \mathbf{p}_n, \phi, \mathbf{a}_n) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}] \xrightarrow{\sim} \widehat{HFK}(S, \mathbf{p}_n, \phi, \mathbf{a}_n) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}],$$

which maps $\mathbf{x}_{H_{(S,\mathbf{p}_n,\phi,\mathbf{a}_n)}} \otimes 1$ to $\hat{t}(K_n) \otimes 1$, where $\hat{t}(K_n)$ is the image of the BRAID invariant of the transverse braid K_n, defined in [4], under the natural map from HFK^- to $\widehat{HFK}$. Also, by the LOSS=BRAID theorem (Theorem 5.1 of [4]) and the construction of localization map in [48], the isomorphism

$$\widehat{HFK}(S^3, K) \otimes V_n \xrightarrow{\sim} \widehat{HFK}(S, \mathbf{p}_n, \phi, \mathbf{a}_n)$$

maps $\hat{\mathcal{T}}(K) \otimes c$ to $\hat{t}(K_n)$. Therefore there exists an isomorphism

$$\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}] \xrightarrow{\sim} \widehat{HFK}(S^3, K) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$$

which maps $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K) \otimes 1$ to $\hat{\mathcal{T}}(K) \otimes 1$. $\square$

Proving that $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K)$ is a refinement of $c_{\mathbb{Z}_2}(K)$ is a bit more difficult, as it needs an extension of the definition of $c_{\mathbb{Z}_2}(\xi_K)$. Recall that, for a transverse knot K in (S^3, ξ_{std}), the $\mathbb{Z}_2$-equivariant contact class $c_{\mathbb{Z}_2}(\xi_K)$ is defined as follows.

- Choose a multi-pointed genus zero open book $(D, \mathbf{p} = \{p_1, \cdots, p_n\}, \phi)$ which encodes K.
- Choose a system of pairwise disjoint arcs $\mathbf{a} = \{a_1, \cdots, a_n\}$ such that, for each $i = 2, \cdots, n$, a_i starts from p_i and ends at a point in ∂D.
- Taking the branched double cover of $(D, \phi, \mathbf{a} \backslash \{a_1\})$, branched along $\mathbf{p}$, gives a $\mathbb{Z}_2$-invariant arc diagram $(\Sigma, \phi, \tilde{\mathbf{a}})$, where the point p_1 now works as a basepoint.
- Applying Honda-Kazez-Matic construction gives an element $EH_{\mathbb{Z}_2}(\xi_K) \in \widehat{CF}_{\mathbb{Z}_2}(\Sigma, \phi, \tilde{\mathbf{a}})$, which turns out to be a cocycle which depends only on the transverse isotopy class of K.
- The cohomology class of $EH_{\mathbb{Z}_2}(\xi_K)$ is denoted as $c_{\mathbb{Z}_2}(\xi_K)$.

We will now extend this definition to multi-pointed open books of arbitrary genera.

Definition 7.4.2. Let S be a compact oriented surface with boundary and $\mathbf{p} \subset \mathrm{int}(S)$ be a finite subset, say $\mathbf{p} = \{p_1, \cdots, p_n\}$. Then an **extended arc basis** of the multi-pointed surface $(S, \mathbf{p})$ is a set $\mathbf{a} = \{a_1^w, \cdots, a_m^w, a_1^h, \cdots, a_n^h\}$ of pairwise disjoint simple arcs, which satisfies the following conditions. Here, the arcs a_i^w are called **whole arcs**, and a_j^h are called **half arcs**.

- $\{a_1^w, \cdots, a_m^w\}$ is an arc basis of S.
- For each $i = 1, \cdots, n$, the arc a_i^h starts from p_i and ends at a point in ∂S.

Given a multi-pointed open book $(S, \mathbf{p}, \phi)$ encoding a transverse knot K in (S^3, ξ_{std}), choose an extended arc basis $\mathbf{a}$ of $(S, \mathbf{p})$ together with a distinguished element $p_1 \in \mathbf{p}$.

Then, as in the genus zero case, taking the branched double cover of $(S, \phi, \mathbf{a}\backslash\{a_1^h\})$ along $\mathbf{p}$, where a_1^h is the half arc in $\mathbf{a}$ which contains p_1, gives a $\mathbb{Z}_2$-invariant arc diagram $(\Sigma, \phi, \tilde{\mathbf{a}})$, where p_1 is now a basepoint in $\Sigma\backslash \cup \tilde{\mathbf{a}}$. Applying Honda-Kazez-Matic construction then gives a canonical element $EH_{\mathbb{Z}_2}(S, \mathbf{p}, \phi, \mathbf{a}, p_1)$.

Definition 7.4.3. The argument used to prove that $EH_{\mathbb{Z}_2}(\xi_K)$ is a cocycle can be directly applied to show that $EH_{\mathbb{Z}_2}(S, \mathbf{p}, \phi, \mathbf{a}, p_1)$ is also a cocycle in $\widehat{CF}_{\mathbb{Z}_2}(\Sigma, \phi, \tilde{\mathbf{a}}, p_1)$. So we denote its cohomology class as $c_{\mathbb{Z}_2}(S, \mathbf{p}, \phi, \mathbf{a}, p_1)$.

When $(S, \mathbf{p}, \phi)$ is a multi-pointed open book, $p_1 \in \mathbf{p}$ is a distinguished point, and $\mathbf{a}$ is an extended arc basis of $(S, \mathbf{p})$, then we call the tuple $(S, \mathbf{p}, \phi, \mathbf{a})$ as an **extended arc diagram**. The difference between using genus zero extended arc diagrams and using diagrams of arbitrary genera is that we now have four possible types of arc-slides. As usual, we call isotopies and arc-slides as basic moves. Note that the HKM construction applied to an extended arc diagram gives an extended bridge diagram, and arc-slides of type I/II/III/IV correspond to the same types of handleslides.

Isotopy

We can perform an isotopy to arcs in an extended arc basis $\mathbf{a}$ of a multi-pointed surface $(S, \mathbf{p})$. Here, the isotopy is not allowed to pass through points in $\mathbf{p}$.

Arc-slide of type I

We can perform an (ordinary) arc-slide of an whole arc along another whole arc in an extended arc basis $\mathbf{a}$, outside the basepoint p_1 and the half-arcs in $\mathbf{a}$.

Arc-slide of type II

Given a whole arc a^w and a half arc a^h in an extended arc basis $\mathbf{a}$, we can replace a^w by another whole arc a_1^w if a_1^w does not intersect the arcs in $\mathbf{a}\backslash\{a^w\}$, a_1^w intersects

a^w transversely at one point, and $S\backslash(a^w \cup a_1^w)$ contains a triangle component T such that $a^h \subset T$.

Arc-slide of type III

Given a half arc a^h and a whole arc a^w in an extended arc basis $\mathbf{a}$, we can replace a^h by another arc a_1^h if $a^h \cap a_1^h$ is a point in $\mathbf{p}$, the interior of a_1^h does not intersect the arcs in $\mathbf{a}$, and the arcs a^h, a^w, a_1^h bound a strip in S.

Arc-slide of type IV

Given two half arcs a_1^h, a_2^h in an extended arc basis $\mathbf{a}$, we can replace a_1^h by another arc $a_{1'}^h$ if $a_1^h \cap a_{1'}^h$ is a point in $\mathbf{p}$, the interior of $a_{1'}^h$ does not intersect the arcs in $\mathbf{a}$, and $S\backslash(a_1^h \cup a_{1'}^h)$ contains a triangle component T satisfying $a_2^h \subset T$.

Lemma 7.4.4 (Lemma 6.13 of [24]). *Any two arc bases of a multi-pointed surface $(S, \mathbf{p})$ are related by a sequence of isotopies and arc-slides.*

Proof. This can be seen easily by combining the argument used in the proof of Lemma 7.3.5 together with the proof of Proposition 6.2.2. □

Proposition 7.4.5 (Proposition 6.14 of [24]). *Let $(S, \mathbf{p}, \phi, \mathbf{a}, p_1)$ be an extended arc diagram, where $(S, \mathbf{p}, \phi)$ encodes a transverse knot K in (S^3, ξ_{std}). Then $c_{\mathbb{Z}_2}(S, \mathbf{p}, \phi, \mathbf{a}, p_1)$ depends only on the transverse isotopy class of K.*

Proof. The proof is essentially the same as in the proof of invariance of $c_{\mathbb{Z}_2}(\xi_K)$ for genus zero open books. □

Finally, we can prove that for transverse knots K in (S^3, ξ_{std}), $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K)$ is a refinement of $c_{\mathbb{Z}_2}(\xi_K)$.

Theorem 7.4.6 (Theorem 6.15 of [24]). *For any transverse knot K in (S^3, ξ_{std}), the natural map*

$$\widehat{HFK}_{\mathbb{Z}_2}(\Sigma(K), K) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$$

maps $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K)$ *to* $c_{\mathbb{Z}_2}(\xi_K)$.

Proof. Let $(S, p, \phi, \mathbf{a})$ be a single–pointed arc diagram, where (S, p, ϕ) encodes K. Choose a simple arc a^h on $S\backslash \cup \mathbf{a}$ which starts from p and ends at a point in ∂S. Then the image of $\mathbf{x}_{H_{(S,p,\phi,\mathbf{a})}}$ in the chain level is given by $EH_{\mathbb{Z}_2}(S, p, \phi, \mathbf{a} \cup \{a^h\}, p)$. Since $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K)$ is the cohomology class of $\mathbf{x}_{H_{(S,p,\phi,\mathbf{a})}}$, and the cohomology class of $EH_{\mathbb{Z}_2}(S, p, \phi, \mathbf{a} \cup \{a^h\}, p)$ is equal to $c_{\mathbb{Z}_2}(\xi_K)$ by 7.4.5, we see that $\hat{\mathcal{T}}_{\mathbb{Z}_2}(K)$ is mapped to $c_{\mathbb{Z}_2}(\xi_K)$. □

Chapter 8

Topological application

In this final chapter, we will see that the existence and properties of $c_{\mathbb{Z}_2}$ allow us to derive an interesting topological application, regarding the branched double covers along transverse knots in S^3_{std}.

8.1 Analyzing the spectral sequence from $\widehat{HF}$ to $\widehat{HF}_{\mathbb{Z}_2}$

Recall that, for any knot $K \subset S^3$, we have a spectral sequence

$$E_1 = \widehat{HF}(-\Sigma(K)) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta] \Rightarrow \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)),$$

constructed in Section 6.1 of [20], whose pages depend only on the isotopy class of K. This spectral sequence is induced by the θ-filtration on $\widehat{CF}_{\mathbb{Z}_2}(\Sigma(K))$, which can be written up to quasi-isomorphism as

$$\widehat{CF}_{\mathbb{Z}_2}(\Sigma(K)) = (\widehat{CF}(-\Sigma(K)) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta], d_{\mathbb{Z}_2}),$$
$$d_{\mathbb{Z}_2}(\mathbf{x} \otimes \theta^i) = d\mathbf{x} \otimes \theta^i + (\mathbf{x} + \tau\mathbf{x}) \otimes \theta^{i+1},$$

where d denotes the differential on the cochain complex $\widehat{CF}(-\Sigma(K))$ and τ denotes the generator of the $\mathbb{Z}_2$-action.

From the construction of the transverse knot invariant $c_{\mathbb{Z}_2}(\xi_K)$ we see that, for a given transverse braid representative of K along the z-axis, we have an element

$$EH_{\mathbb{Z}_2}(\xi_K) = EH(\xi_K) \otimes \theta^0 \in \widehat{CF}_{\mathbb{Z}_2}(\Sigma(K)),$$

which is a $d_{\mathbb{Z}_2}$-cocycle. The same element represents $c(\xi_K) \otimes \theta^0$ in the E_1 page, so we see that the element $c(\xi_K) \otimes \theta^0$ in the E_1 page of our spectral sequence induces an element on each page.

However, we have to be careful here: the limit of $c(\xi_K) \otimes \theta^0$ on the E^∞ page is not $c_{\mathbb{Z}_2}(\xi_K)$. This is because our spectral sequence actually does not converge directly to $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, but instead converges to the associated graded module

$$\mathbf{gr}_\theta \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) = \bigoplus_{i=0}^{\infty} \theta^i \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))/\theta^{i+1}\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)).$$

Thus, considering the bigrading on each page of the spectral sequence, we see that the limit of $c(\xi_K) \otimes \theta^0$ is the image $c_{\mathbb{Z}_2}(\xi_K)$ under the following map:

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \twoheadrightarrow \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))/\theta\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \hookrightarrow \mathbf{gr}_\theta \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)).$$

Similarly, for every nonnegative integer n, the limit of $c(\xi_K) \otimes \theta^n$ is the image of $\theta^n \cdot c_{\mathbb{Z}_2}(\xi_K)$ under the map

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \twoheadrightarrow \theta^n\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))/\theta^{n+1}\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \hookrightarrow \mathbf{gr}_\theta \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)).$$

For simplicity, we will denote the image by $c^n_{\mathbb{Z}_2}(\xi_K)$.

Now suppose that the transverse knot K achieves equality in the inequality of Corollary 6.4.3, i.e. we have

$$\mathbf{gr}(c_{\mathbb{Z}_2}(\xi_K)) = d_3(\xi_K) + \frac{1}{2} = \frac{q_\tau(K)}{2}.$$

Then $c_{\mathbb{Z}_2}(\xi_K)$ lies in the lowest possible grading among all elements of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ which are not annihilated by any powers of θ. Under this setting, we see that $c_{\mathbb{Z}_2}(\xi_K)$ cannot be written as a multiple of θ by the lowest-grading condition, so we have $c^0_{\mathbb{Z}_2}(\xi_K) \neq 0$ in $\mathbf{gr}_\theta \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, and similarly we have $c^n_{\mathbb{Z}_2}(\xi_K)$ for every nonnegative integer n. This simple fact can now be used to prove the following nonvanishing condition for $c(\xi_K)$.

Theorem 8.1.1 (Theorem 9.1 of [25])**.** *Let K be a knot in S^3, and suppose that a transverse representative T of K satisfies $d_3(\xi_K) = \frac{q_\tau(K)-1}{2}$. Then the following statements hold.*

1. *$c(\xi_T) \neq a + \tau^* a$ for every $a \in \widehat{HF}(\Sigma(K))$, where τ denotes the deck transformation of the branched covering map $\Sigma(K) \to S^3$.*

2. *The cardinality of the set*

$$\left\{ c(\xi_T) \,\middle|\, T \text{ is a transverse representative of } K \text{ satisfying } d_3(\xi_K) = \frac{q_\tau(K)-1}{2} \right\}$$

is at most half of the cardinality of the $\left(d_3(\xi_T) + \frac{1}{2}\right)$-graded component of $\widehat{HF}(-\Sigma(K), \mathfrak{s}_0^K)$, where $\mathfrak{s}_0^K$ is the Spin^c-structure on $\Sigma(K)$ induced by the unique spin structure on $\Sigma(K)$.

Proof. The statement $c(\xi_T) \neq a + \tau^* a$ for every $a \in \widehat{HF}(\Sigma(K))$ means that the element $c(\xi_K) \otimes \theta^1$ in the E_1 page also survives in the E_2 page. By the condition we imposed on K, we have $c^n_{\mathbb{Z}_2}(\xi_K) \neq 0$ for every nonnegative integer n, which means that the element $c(\xi_K) \otimes \theta^n$ in the E_1 page must survive on every page. This proves (1).

Also, statement (3) follows from the fact that the rank of the $(q_\tau(K)+N)$-graded component of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), \mathfrak{s}_0^K)$ converges to 1 under the limit $N \to \infty$ (and that $c(\xi_K) \otimes \theta^N$ must survive in every page). Note that this a direct corollary of the

existence of the localization isomorphism

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}] \xrightarrow{\sim} \mathbb{F}_2[\theta, \theta^{-1}].$$

□

On the other hand, by analyzing the structure of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, we can find a condition which implies the vanishing of $c(\xi_T)$ for a transverse representative T of K. We start by observing that taking a quotient of $\widehat{CF}_{\mathbb{Z}_2}(\Sigma(K))$ by the θ-action gives the ordinary Floer cochain complex $\widehat{CF}(-\Sigma(K))$:

$$\widehat{CF}_{\mathbb{Z}_2}(\Sigma(K)) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2 \simeq \widehat{CF}(-\Sigma(K)).$$

We already know from Theorem 6.4.1 that the induced map

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \to \widehat{HF}(-\Sigma(K))$$

maps $c_{\mathbb{Z}_2}(\xi_K)$ to $c(\xi_K)$. Now, since the above map is an $\mathbb{F}_2[\theta]$-module homomorphism and the θ-action on its codomain $\widehat{HF}(-\Sigma(K))$ is trivial, we get a map

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))/\theta\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \to \widehat{HF}(-\Sigma(K)),$$

which then maps $c^0_{\mathbb{Z}_2}(\xi_K)$ to $c(\xi_K)$. Hence, if $c_{\mathbb{Z}_2}(\xi_K)$ is divisible by θ, then $c(\xi_K) = 0$. This can be used to give a condition which implies the vanishing of $c(\xi_K)$.

Definition 8.1.2. Given a knot K in S^3, we define $v_\tau(K)$ to be the biggest nonnegative integer N such that there exists an element in the $(q_\tau(K) + N)$-graded piece of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), \mathfrak{s}^K_0)$ which is not annihilated by any powers of θ and not divisible by θ; such an integer always exists by the existence of the localization isomorphism. Here, $\mathfrak{s}^K_0$ is defined as in the statement (2) of Theorem 8.1.1.

Theorem 8.1.3 (Theorem 9.3 of [25]). *Let K be a knot in S^3 and T be a transverse representative of K. Then $c(\xi_K) \neq 0$ if $d_3(\xi_K) = \frac{q_\tau(K)-1}{2}$ and $c(\xi_K) = 0$ if $d_3(\xi_K) > \frac{q_\tau(K)-1}{2} + v_\tau(K)$.*

Proof. If $d_3(\xi_K) > v_\tau(K) - \frac{1}{2}$, then $c_{\mathbb{Z}_2}(\xi_K)$ is divisible by θ, so $c(\xi_K) = 0$. On the other hand, if $d_3(\xi_K) = \frac{q_\tau(K)-1}{2}$, then $c(\xi_K) \neq a + \tau^* a$ for any $a \in \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ by Theorem 8.1.1, so $c(\xi_K) \neq 0$ as $0 + \tau^* 0 = 0$. □

8.1.1 Comparison with the Khovanov side

Theorem 8.1.3 can be seen as a Heegaard Floer analogue of the following theorem of Plamenevskaya.

Theorem 8.1.4 (Theorem 1.2, [45]). *If K is a transverse knot such that $\mathbf{sl}(K) = s(K) - 1$, then $\psi(K) \neq 0$, where $s(K)$ stands for the Rasmussen invariant[46], and $\psi(K)$ stands for the Plamenevskaya invariant[43]. The converse holds if K is $Kh_{\mathbb{F}_2}$-thin.*

As its corollary, Plamenevskaya gives a vanishing/nonvanishing property of $c(\xi_K)$ when K is a transverse representative of a quasi-alternating knot.

Corollary 8.1.5 (Corollary 1.3, [45]). *Let K be a transverse representative of a quasi-alternating knot. Then $c(\xi_K) \neq 0$ if and only if $\mathbf{sl}(K) = \sigma(K) - 1$.*

We will now show that Corollary 8.1.5 is also a direct consequence of Theorem 8.1.3, by giving another proof of it.

Proof. When K is a quasi-alternating knot and T is a transverse representative of K which satisfies $\mathbf{sl}(T) = s(K) - 1$, then by Proposition 6 of [44], we have

$$d_3(\xi_T) = -\frac{3}{4}\sigma(X) - \frac{1}{2}\mathbf{sl}(T) = -\frac{3}{4}\sigma(X) - \frac{1}{2}\sigma(K) - \frac{1}{2},$$

where X is a 4-manifold consisting only of 2-handles, as defined in section 3.1 of [44]. Note that the original formula of Plamenevskaya is wrong by an additive factor of $\frac{1}{2}$;

it is easy to check it by testing it with the transverse unknot, which has self-linking number -1.

Now, by the construction of X, it is a branched double cover of the 4-ball B^4 along a smooth surface which bounds K, but with the opposite orientation, so by Theorem 3.1 of [26], we have $\sigma(X) = -\sigma(K)$. Hence we get

$$d_3(\xi_T) = \frac{3}{4}\sigma(K) - \frac{1}{2}\sigma(K) - \frac{1}{2} = \frac{1}{4}\sigma(K) - \frac{1}{2}.$$

On the other hand, since $\Sigma(K)$ is an L-space, we have

$$\frac{q_\tau(K)}{2} = d(K, \mathfrak{s}_0^K),$$

which is then equal to $\frac{\sigma(K)}{4}$ by Theorem 1 of [31]; note that we are using the sign convention which makes the right handed trefoil have signature 2. Also, $v_\tau(K) = 0$ by the same reason. Hence we finally get

$$d_3(\xi_T) + \frac{1}{2} = \frac{q_\tau(K)}{2} = \frac{q_\tau(K)}{2} + v_\tau(K).$$

Therefore, by Theorem 8.1.3, $c(\xi_T) \neq 0$. Similarly, we can prove that $c(\xi_T) = 0$ if $\mathbf{sl}(T) > s(K) - 1$ by replacing equalities with inequalities. □

Surprisingly, we can prove that a similar thing happens for a different class of knots.

Theorem 8.1.6 (Theorem 9.6 of [25])**.** *Let K be a quasi-positive knot and T be a transverse representative of K with minimal self-linking number. Then $c(\xi_K) \neq 0$.*

Proof. By the theorem of Boileau and Orevkov[7], we know that K bounds a symplectic cap S in B^4, which then gives K a structure of a transverse knot. Moreover, the proof of their theorem additionally tells us that S can be constructed as a rescaling of a concatenation of symplectic births and saddles. Hence, for any point p

in the interior of S and its small neighborhood $N(p)$, the surface $S\backslash N(p)$ is in fact a constructible symplectic cobordism between K and the unknot U.

Choose a basepoint $p_K \in K$, $p_U \in U$, and a simple arc $a \subset S\backslash N(p)$ which connects p_K and p_U. Then we have a cobordism map

$$\hat{f} = \hat{f}_{(S\backslash N(p),a)} : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), p_K) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(U), p_U) \simeq \mathbb{F}_2[\theta],$$

which sends $c_{\mathbb{Z}_2}(\xi_K)$ to $c_{\mathbb{Z}_2}(\xi_U) = 1 \in \mathbb{F}_2[\theta]$ by the constructibility of $S\backslash N(p)$. Hence the grading shift of $\hat{f}$ is precisely minus the absolute Maslov grading of $c_{\mathbb{Z}_2}(\xi_K)$.

Now, since we know that $\hat{f}$ becomes the identity when localized by inverting θ, we have the inequality

$$d_3(\xi_K) + \frac{1}{2} = \mathbf{gr}(c_{\mathbb{Z}_2}(\xi_K)) = -(\text{grading shift of } \hat{f}) \leq \frac{1}{2}(q_\tau(K) - q_\tau(U)) = \frac{1}{2}q_\tau(K).$$

On the other hand, we already know that the inequality $\frac{q_\tau(K)-1}{2} \leq d_3(\xi_K)$ holds. Hence the inequalities here are in fact equalities:

$$\frac{q_\tau(K) - 1}{2} = d_3(\xi_K).$$

Therefore, by Theorem 8.1.3, the result follows. □

References

[1] J. A. Baldwin and S. Sivek. A contact invariant in sutured monopole homology. In *Forum of Mathematics, Sigma*, **4**. Cambridge University Press, 2016.

[2] J. A. Baldwin and S. Sivek. Instanton floer homology and contact structures. *Selecta Mathematica*, **22**(2):939–978, 2016.

[3] J. A. Baldwin and S. Sivek. On the equivalence of contact invariants in sutured floer homology theories. *arXiv preprint arXiv:1601.04973*, 2016.

[4] J. A. Baldwin, D.S. Vela-Vick, and V. Vértesi. On the equivalence of legendrian and transverse invariants in knot floer homology. *Geometry & Topology*, **17**(2):925–974, 2013.

[5] D. Bennequin. *Entrelacements et équations de Pfaff*. PhD thesis, Université de Paris VII, 1982.

[6] J. S. Birman and W. W. Menasco. Stabilization in the braid groups ii: Transversal simplicity of knots. *Geometry & Topology*, **10**(3):1425–1452, 2006.

[7] M. Boileau and S. Orevkov. Quasi-positivité d'une courbe analytique dans une boule pseudo–convexe. *Comptes Rendus de l'Academie des Sciences Series I Mathematics*, **332**(9):825–830, 2001.

[8] J. S. Carter and M. Saito. Reidemeister moves for surface isotopies and their interpretation as moves to movies. *Journal of Knot Theory and its Ramifications*, **2**(03):251–284, 1993.

[9] J. Cerf. Sur les difféomorphismes de la sphère de dimension trois. *Lecture notes in Math*, **53**, 1968.

[10] T. Ekholm, J. Etnyre, L. Ng, and M. Sullivan. Filtrations on the knot contact homology of transverse knots. *Mathematische Annalen*, **355**(4):1561–1591, 2013.

[11] Y. Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Inventiones mathematicae*, **98**(3):623–637, 1989.

[12] Y. Eliashberg. Contact 3-manifolds twenty years since j. martinet,s work. *Ann. Inst. Fourier (Grenoble)*, **42**(1-2):165–192, 1992.

[13] Y. Eliashberg and M. Gromov. Convex symplectic manifolds, several complex variables and complex geometry, part 2 (santa cruz, ca, 1989), 135–162. In *Proc. Sympos. Pure Math*, **52**, 1991.

[14] J. Etnyre and L. Ng. Problems in low dimensional contact topology. In *Proceedings of the 2001 Georgia International Geometry and Topology Conference*, 2003.

[15] J. B. Etnyre and K. Honda. Cabling and transverse simplicity. *Annals of mathematics*, pages 1305–1333, 2005.

[16] A. Floer. An instanton-invariant for 3–manifolds. *Communications in mathematical physics*, **118**(2):215–240, 1988.

[17] M. H. Freedman. The topology of four-dimensional manifolds. *J. Differential Geom*, **17**(3):357–453, 1982.

[18] E. Giroux. Géométrie de contact: de la dimension trois vers les dimensions supérieures. *arXiv preprint math/0305129*, 2003.

[19] Robert E Gompf. Handlebody construction of stein surfaces. *Annals of mathematics*, pages 619–693, 1998.

[20] K. Hendricks, R. Lipshitz, and S. Sarkar. A flexible construction of equivariant floer homology and applications. *Journal of Topology*, **9**(4):1153–1236, 2016.

[21] K. Honda, W. H. Kazez, and G. Matic. On the contact class in heegaard floer homology. *arXiv preprint math/0609734*, 2006.

[22] A. Juhász and M. Marengon. Concordance maps in knot floer homology. *Geometry & Topology*, **20**(6):3623–3673, 2016.

[23] A. Juhász and D. P. Thurston. Naturality and mapping class groups in heegaard floer homology. *arXiv preprint arXiv:1210.4996*, 2012.

[24] S. Kang. $\mathbb{Z}_2$-equivariant heegaard floer cohomology of knots in S^3 as a strong heegaard invariant. *arXiv preprint arXiv:1810.01919*, 2018.

[25] S. Kang. A transverse knot invariant from $\mathbb{Z}_2$-equivariant heegaard floer cohomology. *arXiv preprint arXiv:1802.00351*, 2018.

[26] L. H. Kauffman and L. R. Taylor. Signature of links. *Transactions of the American Mathematical Society*, **216**:351–365, 1976.

[27] M. Khovanov and P Seidel. Quivers, floer cohomology, and braid group actions. *Journal of the American Mathematical Society*, **15**(1):203–271, 2002.

[28] P. B. Kronheimer and T. S. Mrowka. Witten,s conjecture and property p. *Geometry & Topology*, **8**(1):295–310, 2004.

[29] P. B. Kronheimer and T. S. Mrowka. *Monopoles and three-manifolds*, **10**. Cambridge University Press, 2007.

[30] R. Lipshitz. A cylindrical reformulation of heegaard floer homology. *Geometry & Topology*, **10**(2):955–1096, 2006.

[31] P. Lisca and B. Owens. Signatures, heegaard floer correction terms and quasi–alternating links. *Proceedings of the American Mathematical Society*, **143**(2):907–914, 2015.

[32] P. Lisca, P. Ozsváth, A. I. Stipsicz, and Z. Szabó. Heegaard floer invariants of legendrian knots in contact three–manifolds. *arXiv preprint arXiv:0802.0628*, 2008.

[33] J. Martinet. Formes de contact sur les variétés de dimension 3. In *Proceedings of Liverpool Singularities Symposium II*, pages 142–163. Springer, 1971.

[34] E. E. Moise. Affine structures in 3-manifolds: V. the triangulation theorem and hauptvermutung. *Annals of mathematics*, pages 96–114, 1952.

[35] S. Y. Orevkov and V. V. Shevchishin. Markov theorem for transversal links. *Journal of knot theory and its ramifications*, **12**(07):905–913, 2003.

[36] P. Ozsváth and Z. Szabó. Holomorphic disks and knot invariants. *Advances in Mathematics*, **186**(1):58–116, 2004.

[37] P. Ozsváth and Z. Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Annals of Mathematics*, pages 1027–1158, 2004.

[38] P. Ozsváth and Z. Szabó. Heegaard floer homology and contact structures. *Duke Mathematical Journal*, **129**(1):39–61, 2005.

[39] P. Ozsváth and Z. Szabó. Holomorphic triangles and invariants for smooth four-manifolds. *Advances in Mathematics*, **202**(2):326–400, 2006.

[40] E. Pavelescu. Braiding knots in contact 3-manifolds. *Pacific journal of mathematics*, **253**(2):475–487, 2012.

[41] G. Perelman. The entropy formula for the ricci flow and its geometric applications. *arXiv preprint math/0211159*, 2002.

[42] T. Perutz. Hamiltonian handleslides for heegaard floer homology. *arXiv preprint arXiv:0801.0564*, 2008.

[43] O. Plamenevskaya. Transverse knots and khovanov homology. *arXiv preprint math/0412184*, 2004.

[44] O. Plamenevskaya. Transverse knots, branched double covers and heegaard floer contact invariants. *Journal of Symplectic Geometry*, **4**(2):149–170, 2006.

[45] O. Plamenevskaya. Khovanov homology and tight contact structures. *arXiv preprint arXiv:0802.3835*, 2008.

[46] J. Rasmussen. Floer homology and knot complements. *arXiv preprint math/0306378*, 2003.

[47] S. Sarkar and J. Wang. An algorithm for computing some heegaard floer homologies. *Annals of mathematics*, pages 1213–1236, 2010.

[48] P. Seidel and I. Smith. Localization for involutions in floer cohomology. *Geometric and Functional Analysis*, **20**(6):1464–1501, 2010.

[49] J. Singer. Three-dimensional manifolds and their heegaard diagrams. *Transactions of the American Mathematical Society*, **35**(1):88–111, 1933.

[50] S. Smale. On the structure of manifolds. *American Journal of Mathematics*, **84**(3):387–399, 1962.

[51] H. Wu. Braids, transversal links and the khovanov-rozansky theory. *Transactions of the American Mathematical Society*, **360**(7):3365–3389, 2008.