

Mad subalgebras of rings of differential operators on curves

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Abstract

We study the maximal abelian ad-nilpotent (mad) subalgebras of the domains \mathcal{D} Morita equivalent to the first Weyl algebra. We give a complete description both of the individual mad subalgebras and of the space of all such. A surprising consequence is that this last space is independent of \mathcal{D} . Our results generalize some classic theorems of Dixmier about the Weyl algebra.

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1. Introduction and statement of results

We begin by recalling some results of Dixmier (see [9]) about the (first) Weyl algebra A . We shall think of A as the algebra $\mathcal{D}(\mathbb{A}^1)$ of differential operators on the (complex) affine line, that is, as the algebra $\mathbb{C}[z, \partial_z]$ of polynomial differential operators in one variable z . We call an element $b \in A$ *ad-nilpotent* (“strictly nilpotent” in [9]) if for each $a \in A$ we have $(\text{ad } b)^k(a) = 0$ for some k . We call a maximal abelian subalgebra B of A a *mad subalgebra* if every element of B is ad-nilpotent. For example, $\mathbb{C}[z]$ is clearly a mad subalgebra of A , and $\mathbb{C}[\partial_z]$ is another. One of Dixmier’s main aims in [9] was to obtain information about the group $\text{Aut } A$ of \mathbb{C} -automorphisms

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of A ; to this end he studied the action of $\text{Aut } A$ on the set of mad subalgebras of A . One of his key results was that this action is *transitive*. Clearly, that implies

Theorem 1.1. [9] *Every mad subalgebra $B \subset A$ has the form $B = \mathbb{C}[x]$ for some $x \in A$.*

If B is a mad subalgebra, we shall call a choice of generator x for B a *framing* of B , and the pair (B, x) a *framed mad subalgebra* of A . Dixmier showed in fact (see [9, Lemme 8.9]) that $\text{Aut } A$ acts transitively on the set $\text{Mad } A$ of all framed mad subalgebras of A . Let Γ be the subgroup of $\text{Aut } A$ consisting of all automorphisms γ_p of the form

$$\gamma_p(z) = z, \quad \gamma_p(\partial_z) = \partial_z - p'(z) \quad (1.1)$$

where $p \in \mathbb{C}[z]$ (we may think of γ_p as conjugation by $e^{p(z)}$). It is easy to check that Γ is exactly the isotropy group of $z \in A$, or, equivalently, of the natural base-point $(\mathbb{C}[z], z) \in \text{Mad } A$. Dixmier's result can therefore be formulated as follows.

Theorem 1.2. [9] *There is a natural bijection*

$$\text{Aut}(A)/\Gamma \rightarrow \text{Mad } A.$$

We now wish to generalize Theorems 1.1 and 1.2 to the case where the Weyl algebra is replaced by the ring $\mathcal{D}(X)$ of differential operators on any affine curve. Clearly, the ring $\mathcal{O}(X)$ of regular functions on X is a mad subalgebra of $\mathcal{D}(X)$. Theorems of Makar-Limanov and Perkins (see [17,18]) show that $\mathcal{O}(X)$ is the *only* mad subalgebra except in the case when X is a *framed curve*, by which we mean that there is a regular bijective map $\pi: \mathbb{A}^1 \rightarrow X$ (thus topologically a framed curve is simply the affine line, but it may have an arbitrary finite number of cusps). From now on we suppose that X is a framed curve, since Theorems 1.1 and 1.2 can have interesting generalizations only in that case. Our first result is as follows.

Theorem 1.3. *Let B be any mad subalgebra of $\mathcal{D}(X)$, where X is a framed curve. Then $\text{Spec } B$ is a framed curve.*

This theorem is a sharper version of a result of [15], where it is shown that the normalization of $\text{Spec } B$ is always isomorphic to \mathbb{A}^1 ; that means that $\text{Spec } B$ is obtained from \mathbb{A}^1 by introducing cusps, but also, perhaps, identifying certain points of \mathbb{A}^1 to form double points, or even higher order multiple points. The new part of Theorem 1.3 is thus the assertion that multiple points do not occur. The question of whether $\text{Spec } B$ is necessarily a framed curve (that is, free of multiple points) was raised by P. Perkins (see [18]) in a special case where the mad subalgebra B is dual (in the sense of Section 5 below) to $\mathcal{O}(X)$. He raised also a more subtle question: setting $Y := \text{Spec } B$, is it true that $\mathcal{D}(X)$ is isomorphic to $\mathcal{D}(Y)$? Our proof of Theorem 1.3 yields also the answer to this question, namely “not always”; more precisely:

Theorem 1.4. *Let B be any mad subalgebra of $\mathcal{D}(X)$, and let $Y := \text{Spec } B$. Then there is a rank 1 torsion-free coherent sheaf \mathcal{L} over Y and an isomorphism $\varphi: \mathcal{D}_{\mathcal{L}}(Y) \rightarrow \mathcal{D}(X)$ such that $\varphi(\mathcal{O}(Y)) = B$.*

Here $\mathcal{D}_{\mathcal{L}}(Y)$ denotes the ring of differential operators on global sections of \mathcal{L} . If \mathcal{L} is not locally free, then $\mathcal{D}_{\mathcal{L}}(Y)$ is not necessarily isomorphic to $\mathcal{D}(Y)$ (see [6, Example 8.4]).

Theorems 1.3 and 1.4 give a satisfactory description of the individual mad subalgebras B in $\mathcal{D}(X)$; we now describe the “space” of all such B , in the spirit of Theorem 1.2. As we saw above, if B is any mad subalgebra of $\mathcal{D}(X)$, then its integral closure \bar{B} is isomorphic to a polynomial algebra $\mathbb{C}[x]$; as before, we call a choice of generator for \bar{B} a *framing* of B , and we denote the set of all framed mad subalgebras of $\mathcal{D}(X)$ by $\text{Mad } \mathcal{D}(X)$. Generalizing Theorem 1.2, we shall prove

Theorem 1.5. *For any framed curve X , there is a natural bijection*

$$\text{Aut}(A)/\Gamma \rightarrow \text{Mad } \mathcal{D}(X).$$

We found this result surprising: it implies that the space of mad subalgebras of $\mathcal{D}(X)$ is independent of X . The algebras $\mathcal{D}(X)$ are (up to isomorphism) exactly the domains Morita equivalent to the Weyl algebra; thus $\text{Mad } \mathcal{D}$ is a Morita invariant for this special class of algebras. It would be interesting to understand whether this is an instance of some more general principle.

The last theorem that we want to formulate in this Introduction describes the quotient space of $\text{Mad } \mathcal{D}(X)$ by the natural action of the automorphism group of $\mathcal{D}(X)$. Recall (see [24]) that for each $n \geq 0$ the *Calogero–Moser space* \mathcal{C}_n is the space of isomorphism classes of triples $(\mathcal{V}; \mathbb{X}, \mathbb{Y})$ where \mathcal{V} is an n -dimensional complex vector space, and \mathbb{X} and \mathbb{Y} are endomorphisms of \mathcal{V} such that $[\mathbb{X}, \mathbb{Y}] + I$ has rank 1. We make the group Γ act on \mathcal{C}_n by the formula

$$\gamma_p(\mathbb{X}, \mathbb{Y}) = (\mathbb{X} + p'(\mathbb{Y}), \mathbb{Y}). \quad (1.2)$$

Recall further (see [4,6,13,14]) that the algebras $\mathcal{D}(X)$ are classified up to isomorphism by a non-negative integer n which we call the *differential genus* of X : it can be thought of as the number of cusps of X , but counted with appropriate multiplicities (see formula (8.3) in [6]).

Theorem 1.6. *Let X be a framed curve, and let n be its differential genus. Then there is a natural bijection*

$$\mathcal{C}_n/\Gamma \rightarrow \text{Mad } \mathcal{D}(X)/\text{Aut } \mathcal{D}(X).$$

This theorem was announced (without proof) in [6]. The space \mathcal{C}_0 is a point, so in the case where $X = \mathbb{A}^1$ Theorem 1.6 reduces to Dixmier’s result that $\text{Aut } A$ acts transitively on $\text{Mad } A$. In general, \mathcal{C}_n is a smooth affine variety of dimension $2n$, and generic orbits of Γ are n -dimensional, so the theorem suggests that $\text{Mad } \mathcal{D}(X)/\text{Aut } \mathcal{D}(X)$ is an n -dimensional space. Unfortunately, we do not know any intrinsic way of assigning a dimension to this space; and even \mathcal{C}_n/Γ is not a good quotient in the sense of algebraic geometry (for example, because Γ has some orbits of dimension less than n , at least for $n > 2$).

Despite its modest appearance, Theorem 1.3 is the key result of this paper, the others being comparatively formal consequences. Its proof involves a curious mixture of familiar algebraic arguments and others connected with the theory of integrable systems; in particular, the Burch–Chaundy theory of commuting ordinary differential operators plays a crucial role. We offer two versions of the proof, in one of which (playing devil’s advocate) we have sought to reduce the role of the Burch–Chaundy theory to a minimum. We do not know how to eliminate it entirely: we leave the possibility of that as a worry for the reader. A detailed overview of the contents of the paper can be found in the introductory remarks to the individual sections that follow. Here

we just mention that Sections 3 and 4 make no claim to originality, but are an exposition of some of the results of [15]. We give the exposition in some detail, because we rely heavily on these results, and the account of them given in [15] is not quite satisfying (specifically, part of Section 3 of that paper is missing, and the reference to [18] in the proof of its Corollary 4.6 seems difficult to justify directly). The version presented here is based on notes graciously placed at our disposition by G. Letzter.

2. Mad subalgebras

In this section we give some definitions, including those of mad subalgebras and filtrations of an algebra A : these abstract the basic properties of the rings $\mathcal{D}(X)$ of differential operators on algebraic varieties (see Example 2.4 below). We also note some special features of the “1-dimensional” case where A satisfies the condition (2.1) below.

Let A be a noncommutative algebra over \mathbb{C} . As usual, for each $b \in A$ we write $\text{ad } b$ for the inner derivation of A defined by $(\text{ad } b)(a) := [b, a]$; we set

$$N_k(b) := \text{Ker}(\text{ad } b)^{k+1}, \quad N(b) := \bigcup_{k \geq 0} N_k(b).$$

It is easy to check that $N(b)$ is a filtered subalgebra of A . If $N(b) = A$, we say that b is a (*locally*) *ad-nilpotent* element of A , and we call the above filtration on $A = N(b)$ the *filtration induced on A by b* . In later sections we shall sometimes write $N_A(b)$ instead of $N(b)$ (if the algebra A is not clear from the context). For $b \in A$, we denote the centralizer of b by $C(b)$ (or, if necessary, by $C_A(b)$); thus $C(b) \equiv N_0(b)$ as defined above.

Proposition 2.1. *Suppose $C(b_1) = C(b_2)$. Then $N(b_1) = N(b_2)$ as filtered algebras.*

The proof depends on the following (purely set-theoretical) lemma.

Lemma 2.2. *Let $f_1, f_2: A \rightarrow A$ be two maps such that (i) f_1 and f_2 commute; (ii) $\text{Ker } f_1 = \text{Ker } f_2$. Then $\text{Ker } f_1^n = \text{Ker } f_2^n$ for all $n \geq 1$.*

Proof. An easy induction on n (f_1 and f_2 do not even have to be linear). \square

Proof of Proposition 2.1. The elements b_1 and b_2 commute, hence the derivations $\text{ad } b_1$ and $\text{ad } b_2$ commute. So Lemma 2.2 applies to give $N_k(b_1) = N_k(b_2)$ for all $k \geq 0$, which is what the proposition asserts. \square

More generally, if B is any subset of A , we can define the filtered subalgebra $N(B) = \bigcup_{k \geq 0} N_k(B)$, where

$$N_k(B) := \{a \in A: (\text{ad } b_0)(\text{ad } b_1) \dots (\text{ad } b_k)(a) = 0 \text{ for all } b_0, b_1, \dots, b_k \in B\}.$$

We are interested in the case when B is an abelian subalgebra of A : we say B is *ad-nilpotent* if $N(B) = A$. Choosing $b_0 = b_1 = \dots = b_k$ in the definition of $N_k(B)$, we see that if B is ad-nilpotent, then every element of B is ad-nilpotent. If B is ad-nilpotent, we call the natural filtration on $N(B) \equiv A$ the *filtration induced by B* . Clearly, in this filtration $\{A_k\}$ the ring A_0 is the *commutant* $C(B)$ of B .

Definition 2.3. We say that B is a *mad* subalgebra of A if

- (i) B is ad-nilpotent;
- (ii) $C(B) = B$.

If B is a mad subalgebra of A , then the filtration $\{A_k\}$ induced by B has the properties

- (1) $A_{-1} = 0$ (that is, the filtration is *positive*);
- (2) if $b \in A_0$, $a \in A_k$, then $[b, a] \in A_{k-1}$;
- (3) if a has filtration degree k , then there is a $b \in A_0$ such that $[b, a]$ has filtration degree $k - 1$.

We call a filtration of A with these properties a *mad filtration*. It is easy to check that if $\{A_k\}$ is a mad filtration and we define $B := A_0$, then B is a mad subalgebra of A , and the given filtration coincides with the one induced by B . In this way the mad subalgebras of A correspond 1–1 to the mad filtrations.

Example 2.4. Our definitions of mad subalgebras and filtrations are modelled on the following situation. Let X be an irreducible complex affine variety, \mathcal{O} the ring of regular functions on X , and let $E := \text{End}_{\mathbb{C}} \mathcal{O}$. The filtered algebra $N_E(\mathcal{O})$ is (by definition) the ring $\mathcal{D}(X)$ of *differential operators* on X , and \mathcal{O} is a mad subalgebra of $\mathcal{D}(X)$. More generally, let \mathcal{L} be a rank 1 torsion-free coherent sheaf over X , and M the corresponding \mathcal{O} -module (of global sections); if $E := \text{End}_{\mathbb{C}} M$, then the filtered algebra $N_E(\mathcal{O})$ is (by definition) the ring $\mathcal{D}_{\mathcal{L}}(X)$ of differential operators on \mathcal{L} . Here it may happen that the centralizer $C_E(\mathcal{O})$ is slightly larger than \mathcal{O} : we call \mathcal{L} *maximal* if $C_E(\mathcal{O}) = \mathcal{O}$. In that case \mathcal{O} is a mad subalgebra of $\mathcal{D}_{\mathcal{L}}(X)$.

Remark. Every line bundle (locally free rank 1 coherent sheaf) over X is maximal, but the converse is not true (see, for example, [20, p. 46]). For this reason our notion of a “ring with mad filtration” is slightly more general than the “algebras of twisted differential operators” introduced in [3] (which model the case where \mathcal{L} is a line bundle).

We suppose from now on that our algebra A satisfies the condition

$$C(a) \text{ is commutative for each } a \in A \setminus \mathbb{C}. \quad (2.1)$$

This condition is very restrictive; for example, if A is the ring of differential operators on an affine variety X , then (2.1) is satisfied only if X is 1-dimensional. However, that is exactly the situation that concerns us in this paper. Many things become simpler if (2.1) holds: for example, we have $B = C(B)$ if (and only if) B is a maximal abelian subalgebra of A . Further, the maximal abelian subalgebras of A are exactly the centralizers of the elements $b \in A \setminus \mathbb{C}$, and $C(b)$ is the unique maximal abelian subalgebra containing b . It follows that the intersection of any two distinct maximal abelian subalgebras is \mathbb{C} . The following facts about *mad* subalgebras are all easy consequences of (2.1) and Proposition 2.1.

Proposition 2.5. Suppose that A satisfies (2.1), and let $b \in A \setminus \mathbb{C}$ be ad-nilpotent. Then $C(b)$ is a mad subalgebra of A .

Proposition 2.6. Suppose that A satisfies (2.1), and let B be a mad subalgebra of A . Then the filtration induced on A by B coincides with the filtration induced by any element of $B \setminus \mathbb{C}$.

In particular, if a has degree $k > 0$ in this filtration, then $[b, a]$ has degree (exactly) $k - 1$ for every $b \in B \setminus \mathbb{C}$.

Proposition 2.7. *Suppose that A satisfies (2.1), and let B be maximal among the abelian subalgebras of A all of whose elements are ad-nilpotent. Then B is a mad subalgebra of A .*

These propositions indicate that if (2.1) is satisfied, then various possible definitions of “mad” all coincide; in particular, the definition given in the present section agrees with the one we used in the Introduction.

3. Rings of differential operators

The next two sections provide a self-contained exposition of some of the results of [15]. The present section gathers together some preliminary facts, culled from [12,15,17,19]. The main points to note are Proposition 3.1, which ensures that the algebras studied later on all satisfy the condition (2.1); and the more technical-looking Proposition 3.5, which lies at the heart of the proofs in the following Section 4.

Let \mathbb{K} be a commutative field containing¹ \mathbb{C} , and let ∂ be a derivation of \mathbb{K} with kernel \mathbb{C} . Then we can form the ring $\mathbb{K}[\partial]$, consisting of expressions of the form

$$D = \sum_0^n f_i \partial^i, \quad f_i \in \mathbb{K},$$

with multiplication defined by the commutation relation

$$[\partial, f] = \partial(f) \quad \text{for all } f \in \mathbb{K}.$$

Clearly, the ring $\mathbb{K}[\partial]$ does not change if we replace ∂ by $f\partial$ for some nonzero $f \in \mathbb{K}$. We have in mind principally the case when \mathbb{K} is the function field of a curve, so that \mathbb{K} is an extension of \mathbb{C} of transcendence degree 1. In that case the \mathbb{C} -derivations of \mathbb{K} form a 1-dimensional \mathbb{K} -vector space, so the algebra $\mathbb{K}[\partial]$ has an intrinsic interpretation (independent of the choice of ∂) as the ring $\mathcal{D}(\mathbb{K})$ of differential operators on \mathbb{K} .

It is easy to show that $\mathbb{K}[\partial]$ is a Noetherian domain, hence it has a quotient field Q . It is sometimes helpful to think of Q as sitting inside the still larger field $\bar{Q} = \mathbb{K}((\partial^{-1}))$ of formal Laurent series

$$D = \sum_{-\infty}^n f_i \partial^i, \quad f_i \in \mathbb{K}. \quad (3.1)$$

If D has the form (3.1) with $f_n \neq 0$, we call $f_n \partial^n$ the *leading term* of D and f_n its *leading coefficient*. The following fact goes back to Schur (see [19]).

Proposition 3.1. \bar{Q} satisfies the condition (2.1).

¹ For the purposes of this section we could consider that \mathbb{C} denotes an arbitrary field of characteristic zero.

Sketch of proof. There are three cases.

(i) Suppose that $L \in \overline{Q}$ has leading term $a\partial^n$, where $n \neq 0$. If necessary we adjoin to \mathbb{K} an n th root α of a . Then one shows that L has an n th root $L^{1/n} = \alpha\partial + \dots$ in $\mathbb{K}(\alpha)((\partial^{-1}))$, and that the centralizer of L in this field consists of the Laurent series in $L^{1/n}$. Clearly, this is commutative. For more details, see [19].

(ii) Suppose $L = a + a_1\partial^{-1} + \dots$ has order 0 with $a \in \mathbb{C}$ (and $L \neq a$). Then $C(L) = C(L - a)$, which is commutative by case (i).

(iii) Suppose $L = a + a_1\partial^{-1} + \dots$ has order 0 with $a \notin \mathbb{C}$. If P has leading term $p\partial^m$ with $m \neq 0$, then $[P, L]$ has leading coefficient $mp\partial(a) \neq 0$; hence $C(L)$ consists of operators of order zero. If now $P_1, P_2 \in C(L)$, then $[P_1, P_2] \in C(L)$ is either zero or an operator of order < 0 . We just saw that the latter is impossible, hence $C(L)$ is commutative. Alternatively, to get a more precise result, we can argue as follows: equating coefficients of powers of ∂ in the expansion of $PL = LP$ shows that for each $p \in \mathbb{K}$ there is a unique operator of the form $P = p + p_1\partial^{-1} + \dots$ that commutes with L . It follows that $C(L)$ is isomorphic to \mathbb{K} . \square

Of course, it follows from Proposition 3.1 that any subalgebra of \overline{Q} , in particular Q , satisfies (2.1).

Our aim in the rest of this section is to prove the basic Proposition 3.5 below. We start with the following very simple lemma.

Lemma 3.2. *Let F be a field of characteristic 0 (not necessarily commutative), and let ∂ be a derivation of F . Suppose that $\text{Ker } \partial^2 \neq \text{Ker } \partial$. Then there is a $q \in F$ such that $\partial(q) = 1$; and for each $n \geq 1$, $\text{Ker } \partial^n$ is an n -dimensional (left or right) vector space over $\text{Ker } \partial$ with basis $\{1, q, \dots, q^{n-1}\}$.*

Proof. Choose $r \in \text{Ker } \partial^2 \setminus \text{Ker } \partial$, and let $s = \partial(r)$, so that $s \neq 0$ but $\partial(s) = 0$. Set $q = s^{-1}r$; then $\partial(q) = 1$. The rest is an easy induction on n (using the fact that $\partial(q^n) = nq^{n-1}$). \square

In particular, we can apply Lemma 3.2 in the case where $\partial = \text{ad } u$ for some $u \in F$; in this case it is tempting to denote the element q in the lemma by $-\partial_u$. We record the result for future reference.

Corollary 3.3. *Let $u \in F$ (where F is a noncommutative field of characteristic 0), and suppose that $N_F(u) \neq C_F(u)$. Then there is an element $\partial_u \in F$ such that $[\partial_u, u] = 1$ and $N_F(u) = C_F(u)[\partial_u]$.*

We return now to our ring $\mathbb{K}[\partial]$. The next lemma is a special case of Proposition 3.5.

Lemma 3.4. *Suppose that $D \in \overline{Q}$ has leading term ∂^n , where $n \neq 0$, and suppose D acts ad-nilpotently on some operator Θ with leading term $f\partial^m$, where $f \notin \mathbb{C}$. Then the equation $\partial(q) = 1$ has a solution in \mathbb{K} , and if L is any operator on which D acts ad-nilpotently, then the leading coefficient of L belongs to $\mathbb{C}[q]$.*

Proof. We have $[\partial^n, f\partial^m] = n\partial(f)\partial^{n+m-1} + (\text{lower order terms})$, hence for any $i \geq 1$ the coefficient of $\partial^{m+i(n-1)}$ in $(\text{ad } D)^i(\Theta)$ is $n^i\partial^i(f)$. So if D acts ad-nilpotently on Θ , we have $\partial^i(f) = 0$ for some $i \geq 1$, so that $\text{Ker } \partial^i \neq \mathbb{C}$. Hence $\text{Ker } \partial^2 \neq \text{Ker } \partial (= \mathbb{C})$, so Lemma 3.2 tells us that q exists as stated, and that $f \in \mathbb{C}[q]$. The last assertion in the lemma is trivial if the

leading coefficient of L is a scalar, and otherwise follows by the argument above (applied to L instead of Θ). \square

Finally, we want to remove the hypothesis in Lemma 3.4 that D has scalar leading coefficient. This assumption is not essential, because we can always reduce to that case by a “change of variable.” Recall that if $\widehat{\mathbb{K}}$ is a finite extension field of \mathbb{K} , then ∂ extends uniquely to a derivation of $\widehat{\mathbb{K}}$, still with kernel \mathbb{C} : we denote this extension by the same symbol ∂ . If $D \in \overline{Q}(\mathbb{K})$ has leading term $a\partial^n$, where $n \neq 0$, we can form the extension field $\widehat{\mathbb{K}} = \mathbb{K}(\alpha)$, where $\alpha^n = a$. Then $d := \alpha\partial$ is a derivation of $\widehat{\mathbb{K}}$, and we may write the elements of $\overline{Q}(\widehat{\mathbb{K}})$ as Laurent series in d (rather than ∂). The operator D then has leading term d^n , so we may apply Lemma 3.4 to $(\widehat{\mathbb{K}}, d)$ to get the following.

Proposition 3.5. *Suppose that $D \in \overline{Q}$ has leading term $a\partial^n$, where $n \neq 0$, and suppose D acts ad-nilpotently on some operator $\Theta \in \overline{Q}$ with leading term $f\partial^m$, where $f^n/a^m \notin \mathbb{C}$. Let $\widehat{\mathbb{K}} = \mathbb{K}(\alpha)$, where $\alpha^n = a$. Then the equation $\alpha\partial(q) = 1$ has a solution $q \in \widehat{\mathbb{K}}$, and if L is an operator with leading term $\beta\partial^r$ on which D acts ad-nilpotently, then $\beta \in \alpha^r \mathbb{C}[q]$.*

4. Rings with several mad subalgebras

In this section we conclude our reworking of some parts of [15] which were not treated convincingly in that paper. The main results are Theorems 4.1 and 4.5.

For the rest of the paper Q will denote the quotient field of the Weyl algebra, and \mathcal{D} will be a subalgebra of Q with the properties

$$\text{the quotient field of } \mathcal{D} \text{ is } Q; \quad (4.1)$$

$$\mathcal{D} \text{ contains more than one mad subalgebra.} \quad (4.2)$$

We fix a mad subalgebra $B \subset \mathcal{D}$. We may regard its field of fractions $\text{Frac } B$ as a subfield of Q ; in particular, the integral closure \overline{B} of B (in $\text{Frac } B$) is a subalgebra of Q .

Theorem 4.1. *There is an $x \in Q$ such that $\overline{B} = \mathbb{C}[x]$.*

The proof uses the following lemma, which is well known (see, for example, [10], [11, p. 256], [18, p. 281]). However, we shall give a self-contained proof.

Lemma 4.2. *Let $B \neq \mathbb{C}$ be a subalgebra of a polynomial algebra $\mathbb{C}[q]$. Then (i) B is finitely generated; (ii) the integral closure \overline{B} of B has the form $\mathbb{C}[x]$ for some $x \in \mathbb{C}[q]$. In other words, B is the coordinate ring of a curve with normalization isomorphic to \mathbb{A}^1 .*

Proof. (i) follows from the fact that every sub-semigroup of \mathbb{N} is finitely generated (the degrees of the polynomials in B form such a semigroup), while we can see (ii) from general principles as follows. By Lüroth’s Theorem, $\text{Spec } B$ is a rational curve, hence $\text{Spec } \overline{B}$ is isomorphic to \mathbb{A}^1 with (perhaps) a finite number of points removed. Because $\mathbb{C}[q]$ is integrally closed, we have $\overline{B} \subseteq \mathbb{C}[q]$: this inclusion corresponds to a map $f: \mathbb{A}^1 \rightarrow \text{Spec } \overline{B}$ with dense image $\mathbb{A}^1 \setminus S$ for some finite set S . To see that S is empty, we regard f as a rational map $\mathbb{P}^1 \rightarrow \mathbb{P}^1$. Because \mathbb{P}^1 is a smooth curve, this map is regular everywhere; so its image is closed, hence equal to \mathbb{P}^1 ; and

the point at infinity maps to only one point. Thus f maps \mathbb{A}^1 onto \mathbb{A}^1 ; that is, S is empty and $f(\mathbb{A}^1) = \text{Spec } \bar{B} \simeq \mathbb{A}^1$. \square

Proof of Theorem 4.1. Let \mathbb{K} be the centralizer of B in Q ; by Proposition 3.1, \mathbb{K} is a commutative field. Choose any $u \in \mathbb{K} \setminus \mathbb{C}$; then $C_Q(u) = \mathbb{K}$ is commutative and $N_Q(u) = N_Q(B)$ is not (because it contains \mathcal{D} , which is not commutative by (4.1)). So by Corollary 3.3 we can choose $\partial_u \in Q$ such that $[\partial_u, u] = 1$ and $N_Q(B) = \mathbb{K}[\partial_u]$. The derivation $\text{ad } \partial_u$ of \mathbb{K} has kernel \mathbb{C} , for this kernel is the intersection of $C_Q(u)$ and $C_Q(\partial_u)$; since u and ∂_u do not commute, their centralizers are distinct, and hence have intersection \mathbb{C} . We may therefore think of Q as embedded in the field $\mathbb{K}((\partial_u^{-1}))$ and apply the results of Section 3.

By assumption (4.2), we may choose an ad-nilpotent element $D \in \mathcal{D} \setminus B$. Using the inclusion $\mathcal{D} \subseteq \mathbb{K}[\partial_u]$, we think of D as differential operator in ∂_u . It cannot be an operator of order zero, because B is a *maximal* commutative subalgebra of \mathcal{D} . Thus D has positive order in ∂_u , so we may apply Proposition 3.5 (with $m = r = 0$) to obtain a q in some extension field of \mathbb{K} such that $B \subseteq \mathbb{C}[q]$. The theorem now follows from Lemma 4.2. \square

As in the Introduction, we call a choice of x as in Theorem 4.1 a *framing* of B . Clearly, if x is a framing of B , then we have $\text{Frac } B = \mathbb{C}(x) \subseteq \mathbb{K}$, where (as in the proof above) we have set $\mathbb{K} := C_Q(B)$. In fact it is now easy to see

Theorem 4.3. *If x is a framing of B , then $\mathbb{C}(x) = \mathbb{K}$.*

For the proof of this, we choose the framing x to be the element denoted by u in the proof of Theorem 4.1, and choose ∂_x as in the proof of that theorem, that is, such that $[\partial_x, x] = 1$ and $N_Q(B) = \mathbb{K}[\partial_x]$. We shall think of the elements of \mathcal{D} as “operators” (with coefficients in \mathbb{K}).

Lemma 4.4. *Let $L \in \mathcal{D}$ (considered as an element of $\mathbb{K}[\partial_x]$) have leading coefficient $\beta \in \mathbb{K}$. Then $\beta \in \mathbb{C}(x)$.*

Proof. By induction on the order of L . If L has order zero, that is, $L \in \mathbb{K}$, then L commutes with the elements of B . Since B is a *maximal* commutative subalgebra of \mathcal{D} , that shows that $L \in B$, so in this case the lemma just claims that $B \subset \mathbb{C}(x)$, which is certainly true. Now suppose inductively that the assertion is true for operators of order $n - 1$, and let $L \in \mathcal{D}$ have leading term $\beta \partial_x^n$. Fix any $b \in B \setminus \mathbb{C}$; then $[L, b]$ belongs to \mathcal{D} and has leading term $n\beta(\partial b / \partial x) \partial_x^{n-1}$. Hence $n\beta(\partial b / \partial x) \in \mathbb{C}(x)$, so $\beta \in \mathbb{C}(x)$. \square

Proof of Theorem 4.3. Let $f \in \mathbb{K}$. By the assumption (4.1), we have $fL_1 = L_2$ for some $L_i \in \mathcal{D}$. So $f\beta_1 = \beta_2$, where β_i is the leading coefficient of L_i , and the result follows from Lemma 4.4. \square

If x is framing of B and $[\partial_x, x] = 1$, we shall call the pair (x, ∂_x) a *fat framing* of B . Thus we have shown that a fat framing always exists, and we have inclusions

$$B \subset \mathcal{D} \subset N_Q(B) = \mathbb{C}(x)[\partial_x] \subset Q. \quad (4.3)$$

The following theorem is a much stronger version of Lemma 4.4.

Theorem 4.5. *Let (x, ∂_x) be a fat framing of B , and let $L \in \mathcal{D}$ be written as an element of $\mathbb{C}(x)[\partial_x]$, using the corresponding embedding (4.3). Then the leading coefficient of L belongs to $\mathbb{C}[x]$.*

In what follows, for each $\lambda \in \mathbb{C}$ we denote by \mathfrak{v}_λ the corresponding valuation of $\mathbb{C}(x)$; that is, if the Laurent expansion at λ of a rational function f has the form

$$f(x) = \alpha(x - \lambda)^k + (\text{higher degree terms})$$

(with $\alpha \neq 0$), then $\mathfrak{v}_\lambda(f) = k$. Note that $\mathfrak{v}_\lambda(f') = k - 1$, provided $k \neq 0$.

Lemma 4.6. *Let $D \in \mathbb{C}(x)[\partial_x]$ have leading term $a(x)\partial_x^n$, where $n > 0$, and suppose that D acts ad-nilpotently on the rational function $p(x)$. Fix any $\lambda \in \mathbb{C}$, and set*

$$r := \mathfrak{v}_\lambda(a), \quad s := \mathfrak{v}_\lambda(p).$$

Suppose that $s \neq 0$. Then $ns = i(n - r)$ for some $i \in \mathbb{N}$.

Proof. Let $(\text{ad } D)^i(p) = p_i(x)\partial_x^{i(n-1)} + (\text{lower order terms})$, so that $p_0 = p$, $p_1 = np'a$, and

$$p_{i+1} = nap'_i - i(n-1)a'p_i \quad \text{for } i \geq 1.$$

If $\mathfrak{v}_\lambda(p_i) := q$, so that

$$p_i = \alpha(x - \lambda)^q + \cdots, \quad a = \beta(x - \lambda)^r + \cdots,$$

where α and β are nonzero and the \cdots denote terms of higher degree in $x - \lambda$, we find

$$p_{i+1} = \alpha\beta\{nq - ir(n-1)\}(x - \lambda)^{q+r-1} + \cdots.$$

So for each i , either $\mathfrak{v}_\lambda(p_{i+1}) = \mathfrak{v}_\lambda(p_i) + r - 1$, or $nq - ir(n-1) = 0$. Since D is ad-nilpotent on p , the latter must occur for some i : let i now denote the *first* number for which it occurs. The assumption $s \neq 0$ implies that $\mathfrak{v}_\lambda(p_1) = r + s - 1$ and

$$q = \mathfrak{v}_\lambda(p_i) = s + i(r - 1)$$

so $n[s + i(r - 1)] = ir(n - 1)$, which simplifies to give the lemma. \square

Corollary 4.7. *Let $D \in \mathbb{C}(x)[\partial_x]$ have leading term $a(x)\partial_x^n$, where $n > 0$. Suppose D acts ad-nilpotently on some algebra $B \subseteq \mathbb{C}[x]$ with $\overline{B} = \mathbb{C}[x]$. Then $a \in \mathbb{C}[x]$.*

Proof. Fix $\lambda \in \mathbb{C}$. For any $s \gg 0$, the algebra B contains a polynomial p with $\mathfrak{v}_\lambda(p) = s$. Applying Lemma 4.6 to any such p , we find that $r < n$. Then applying the lemma with two consecutive values of s and subtracting, we find that $n/(n - r) \in \mathbb{N}$, in particular $r \geq 0$. This shows that a is regular at every point $\lambda \in \mathbb{C}$, that is, a is a polynomial. \square

Now we can give the

Proof of Theorem 4.5. As in the proof of Theorem 4.1, we fix an ad-nilpotent element $D \in \mathcal{D} \setminus B$; let its leading term be $a(x)\partial_x^n$, where $n > 0$. Let $L \in \mathcal{D}$ have leading coefficient $\beta(x)$. Then if q, α are as in Proposition 3.5, we have $\beta \in \mathbb{C}[q, \alpha]$. We have $\alpha^n = a$, and by Corollary 4.7, $a \in \mathbb{C}[x]$; hence α is integral over $\mathbb{C}[x]$. Also, q is integral over $\mathbb{C}[x]$ (for if x has degree t as a polynomial in q , then $\{1, x, \dots, x^{t-1}\}$ generate $\mathbb{C}[q]$ as $\mathbb{C}[x]$ -module). Hence every element of $\mathbb{C}[q, \alpha]$, in particular β , is integral over $\mathbb{C}[x]$. But $\beta \in \mathbb{C}(x)$, hence $\beta \in \mathbb{C}[x]$. \square

Remark 4.8. We have not used the assumption that \mathcal{Q} is the Weyl quotient field, so the results of this section would still be valid for any of the fields \mathcal{Q} studied in Section 3. However, this extra generality would be illusory, because these fields do not contain any subalgebras \mathcal{D} satisfying (4.1) and (4.2) (see [12]).

5. The dual mad subalgebra

The main aim of this section is to show that if a certain finiteness condition ((5.3) below) is satisfied, then the mad subalgebra B of \mathcal{D} possesses a *dual* mad subalgebra \check{B} . If \mathcal{D} is the Weyl algebra $\mathbb{C}[x, \partial_x]$ and $B = \mathbb{C}[x]$, then $\check{B} = \mathbb{C}[\partial_x]$; in general, the relationship between B and \check{B} is similar to this, but \check{B} is not necessarily isomorphic to B .

We retain the assumptions of the preceding section; thus \mathcal{D} is an algebra satisfying (4.1) and (4.2), B is a mad subalgebra of \mathcal{D} , and (x, ∂_x) is a fat framing of B , so that \mathcal{D} becomes a subalgebra of $\mathbb{C}(x)[\partial_x]$, as in (4.3). The filtration $\{\mathcal{D}_\bullet\}$ induced on \mathcal{D} by the usual filtration (by order in ∂_x) on $\mathbb{C}(x)[\partial_x]$ coincides with that induced by B ; in particular, it is independent of the choice of fat framing. We regard the associated graded algebra

$$\mathrm{gr}_\partial \mathcal{D} := \bigoplus_{k \geq 0} \mathcal{D}_k / \mathcal{D}_{k-1}$$

as embedded in $\mathbb{C}(x)[\xi]$ via the symbol map (if $L \in \mathcal{D}$ has leading term $a(x)\partial_x^k$, its symbol is $a(x)\xi^k$). According to Theorem 4.5, we have

$$\mathrm{gr}_\partial \mathcal{D} \subseteq \mathbb{C}[x, \xi]. \quad (5.1)$$

Following [18], we now consider the *x-filtration* on $\mathbb{C}(x)[\partial_x]$ (and the filtration it induces on \mathcal{D}). By definition, an operator $\sum a_i(x)\partial_x^i$ has *x-filtration* $\leq k$ if $\deg_x a_i \leq k$ for all i (if f and g are polynomials in x , we define $\deg_x(f/g) := \deg_x f - \deg_x g$). We identify the associated graded algebra $\mathrm{gr}_x \mathbb{C}(x)[\partial_x]$ with $\mathbb{C}[x, x^{-1}, \xi]$, and we regard $\mathrm{gr}_x \mathcal{D}$ as embedded in $\mathbb{C}[x, x^{-1}, \xi]$ via the “*x*-symbol map” (defined in the obvious way). Theorem 4.5 shows that in fact $\mathrm{gr}_x \mathcal{D} \subseteq \mathbb{C}[x, \xi]$, in particular, that the induced *x-filtration* on \mathcal{D} is positive. We define

$$\check{B} := \{D \in \mathcal{D} : \deg_x D = 0\}. \quad (5.2)$$

Proposition 5.1. *Either $\check{B} = \mathbb{C}$ or \check{B} is a mad subalgebra of \mathcal{D} .*

Proof. If $\check{B} \neq \mathbb{C}$, it is easy to check that the *x-filtration* is a mad filtration on \mathcal{D} (see [18, p. 286]). \square

The following example shows that the undesirable case $\check{B} = \mathbb{C}$ can indeed occur.

Example 5.2. Let $\mathcal{D} \subset \mathbb{C}[x, \partial_x]$ be the subalgebra of the Weyl algebra consisting of all operators that can be written as polynomial differential operators in the variable $w := x^{1/2}$. Then \mathcal{D} contains $x (= w^2)$ and $x\partial_x (= \frac{1}{2}w\partial_w)$, so clearly \mathcal{D} satisfies (4.1). Also, \mathcal{D} contains the mad subalgebra $B := \mathbb{C}[x]$, and the ad-nilpotent element $\partial_w^2 = 4x\partial_x^2 + 2\partial_x$, hence \mathcal{D} satisfies (4.2). But \mathcal{D} contains no operator (of positive order) with constant coefficients, hence $\check{B} = \mathbb{C}$.

To exclude the possibility that $\check{B} = \mathbb{C}$, we make one more (very strong) assumption about the pair (\mathcal{D}, B) , namely

$$\mathrm{gr}_{\partial} \mathcal{D} \text{ has finite codimension in } \mathbb{C}[x, \xi]. \quad (5.3)$$

Proposition 5.3. Suppose that $\mathcal{D} \subseteq \mathbb{C}(x)[\partial_x]$ satisfies (4.1), (5.1) and (5.3). Then the x -symbol map defines an isomorphism from \check{B} onto a subalgebra of finite codimension in $\mathbb{C}[\xi]$. In particular, $\check{B} \neq \mathbb{C}$.

Proof. This follows from [15, Proposition 2.4], which shows that $\mathrm{gr}_x \mathcal{D}$ and $\mathrm{gr}_{\partial} \mathcal{D}$ have the same finite codimension in $\mathbb{C}[x, \xi]$. Under the x -symbol embedding $\mathrm{gr}_x \mathcal{D} \hookrightarrow \mathbb{C}[x, \xi]$ the elements of $\mathbb{C}[\xi]$ come exactly from \check{B} . Thus if \check{B} had infinite codimension in $\mathbb{C}[\xi]$, then $\mathrm{gr}_x \mathcal{D}$ would have infinite codimension in $\mathbb{C}[x, \xi]$, contradicting [15]. \square

We call \check{B} the *dual mad subalgebra* to (B, x, ∂_x) . It does not depend on the choice of framing x . Indeed, any other framing has the form $ax + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$, so (despite the terminology) the x -filtration on $N_Q(B)$, and hence on \mathcal{D} , does not depend on this choice. On the other hand, \check{B} does depend on the choice of ∂_x : a different choice has the form $\partial_x + q$ with $q \in \mathbb{C}(x)$, and if q has positive degree the corresponding x -filtration, and hence also \check{B} , may change drastically. However, we do have the following.

Lemma 5.4. If we change ∂_x to $\partial_x + q$, where $q \in \mathbb{C}(x)$ has negative degree in x , then the x -filtration on $N_Q(B)$, and hence also the dual mad subalgebra \check{B} , remain unchanged.

Proposition 5.3 implies that \check{B} contains an operator of every sufficiently high order, that is, that \check{B} is an *algebra of rank 1* in the sense of Burchnell–Chaundy theory (cf. [7]). By (5.1), the leading coefficient of every operator in \check{B} is constant; however, in the Burchnell–Chaundy theory it is convenient to consider algebras of differential operators that are normalized to have their first *two* coefficients constant. We call a fat framing (x, ∂_x) of B *good* if the corresponding \check{B} has this property.

Proposition 5.5. Let (x, ∂) be any fat framing of B . Then B has a good fat framing (x, ∂_x) with the same dual subalgebra \check{B} .

Proof. Choose any $L \in \check{B}$ of positive order and with leading coefficient 1: it has the form

$$L = \partial^n + (c + nq)\partial^{n-1} + (\text{lower order terms})$$

where $c \in \mathbb{C}$ and $\deg_x q < 0$. Let $\partial_x := \partial + q$: then by Lemma 5.4 the fat framings (x, ∂_x) and (x, ∂) determine the same \check{B} , and we have

$$L = \partial_x^n + c\partial_x^{n-1} + (\text{lower order terms});$$

that is, the first two coefficients of L are now constant. Any operator that commutes with L also has this property, hence all the elements of \check{B} now have their first two coefficients constant; that is, (x, ∂_x) is good. \square

Remark 5.6. The notion of a “good” fat framing introduced above may seem a little artificial. To appreciate it better, let us reconsider the case where \mathcal{D} is the Weyl algebra A . By Theorem 1.1, in this case we have $x \in A$ for any framing x of a mad subalgebra; a fat framing (x, ∂_x) is good exactly when $\partial_x \in A$ too. It follows easily from Theorem 1.2 that the group $\text{Aut } A$ acts *freely and transitively* on the set of triples (B, x, ∂_x) , where B is a mad subalgebra of A and (x, ∂_x) is a good fat framing of B . We shall see in Section 10 that the same is true for any of our algebras \mathcal{D} , except that $\text{Aut } \mathcal{D}$ has to be replaced by the larger group $\text{Pic } \mathcal{D}$ (in the case of the Weyl algebra these two groups coincide).

6. The adelic Grassmannian

In this section we summarize various facts about the adelic Grassmannian Gr^{ad} which we need to prove our main results. We make no attempt to indicate proofs, except for Theorem 6.1, which we have not been able to find stated explicitly in the literature.

6.1. The Grassmannian

We recall the definition of Gr^{ad} . For each $\lambda \in \mathbb{C}$, we choose a λ -primary subspace of $\mathbb{C}[z]$, that is, a linear subspace V_λ such that

$$(z - \lambda)^N \mathbb{C}[z] \subseteq V_\lambda \quad \text{for some } N.$$

We suppose that $V_\lambda = \mathbb{C}[z]$ for all but finitely many λ . Let $V = \bigcap_\lambda V_\lambda$ (such a space V is called *primary decomposable*) and, finally, let

$$W = \prod_\lambda (z - \lambda)^{-k_\lambda} V \subset \mathbb{C}(z),$$

where k_λ is the codimension of V_λ in $\mathbb{C}[z]$. By definition, Gr^{ad} consists of all $W \subset \mathbb{C}(z)$ obtained in this way. For each $W \in \text{Gr}^{\text{ad}}$ we set

$$A_W := \{f \in \mathbb{C}[z]: fW \subseteq W\}; \quad (6.1)$$

then the inclusion $A_W \subseteq \mathbb{C}[z]$ corresponds to a framed curve $\pi: \mathbb{A}^1 \rightarrow X$ and the A_W -module W corresponds to a rank 1 torsion-free coherent sheaf \mathcal{L} over X . Indeed, in this way the points of Gr^{ad} correspond bijectively to isomorphism classes of such triples (π, X, \mathcal{L}) . For more details see [23].

6.2. The Baker function and the Burchnell–Chaundy theory

Associated to each $W \in \text{Gr}^{\text{ad}}$ is its *Baker function* ψ_W (see [20] or [23]). It has the form

$$\psi_W(x, z) = e^{xz} \left\{ 1 + \sum_i f_i(x) g_i(z) \right\}, \quad (6.2)$$

where the f_i, g_i are rational functions that vanish at infinity. For each $f \in A_W$ there is a unique differential operator $L_f \in \mathbb{C}(x)[\partial_x]$ such that

$$L_f \psi_W(x, z) = f(z) \psi_W(x, z); \quad (6.3)$$

the map $f \mapsto L_f$ defines an isomorphism from A_W to a maximal commutative rank 1 subalgebra \mathcal{A}_W of $\mathbb{C}(x)[\partial_x]$. Clearly, the operators L_f are normalized to have their first two coefficients constant.

We shall need (briefly) the larger Grassmannian Gr^{rat} of [23]: it is similar to Gr^{ad} , except that the normalization map $\pi: \mathbb{A}^1 \rightarrow X$ is not required to be bijective. The Baker function now does not necessarily have the form (6.2), and the operators L_f may not have rational coefficients; however, we can expand ψ_W in a series

$$\psi_W(x, z) = e^{xz} \left\{ 1 + \sum_i a_i(x) z^{-i} \right\} \quad (6.4)$$

in which the coefficients a_i are rational functions of x and some exponentials $e^{\lambda x}$ (the numbers λ occurring are the inverse images under π of the multiple points of X). Every normalized rank 1 algebra of differential operators \mathcal{A} with $\text{Spec } \mathcal{A}$ rational can be obtained from a point of Gr^{rat} in the way explained above for Gr^{ad} . The following theorem is almost proved in [23].

Theorem 6.1. *Let $\mathcal{B} \subseteq \mathbb{C}(x)[\partial_x]$ be any rank 1 commutative algebra of differential operators with first two coefficients constant, and² such that the curve $\text{Spec } \mathcal{B}$ is rational. Then there is a unique $W \in \text{Gr}^{\text{ad}}$ such that $\mathcal{B} \subseteq \mathcal{A}_W$.*

Proof. Let \mathcal{A} be the maximal commutative algebra of differential operators containing \mathcal{B} . Then $\text{Spec } \mathcal{A}$ is still a rational curve (with normalization \mathbb{A}^1), so it is known that $\mathcal{A} = \mathcal{A}_W$ for some $W \in \text{Gr}^{\text{rat}}$. The assertion that W can be chosen to be in Gr^{ad} is equivalent to saying that $\text{Spec } \mathcal{A}$ is a framed curve. According to [23], that in turn is equivalent to the fact that if the Baker function of W is expanded in the form (6.4), then all the a_i are rational functions of x . But if \mathcal{A}_W contains an operator of positive order with rational coefficients, then this must be the case. To see that, let

$$L = \partial_x^n + c \partial_x^{n-1} + \sum_{i=0}^{n-2} u_i(x) \partial_x^i \quad (n > 0, c \in \mathbb{C})$$

² This last assumption is almost certainly superfluous, but we do not know a reference.

be such an operator; then we have an equation

$$L\psi_W(x, z) = (z^n + cz^{n-1} + \cdots)\psi_W(x, z). \quad (6.5)$$

Substituting in the expansion (6.4) of ψ_W and equating coefficients of powers of z , we get a recursion relation of the form

$$a'_r = \{\text{some polynomial in derivatives of the } u_i \text{ and the } a_j \text{ with } j < r\}.$$

Now suppose inductively that the a_j are rational for $j < r$. The recursion relation then shows that a_r is the sum of a rational function and (possibly) some logarithmic terms $\lambda \log(x - \mu)$. But a_r is a meromorphic function, hence the logarithmic terms must be absent. Thus all a_i are rational, as claimed. \square

6.3. The algebras $\mathcal{D}(W)$

For each $W \in \text{Gr}^{\text{ad}}$, we define

$$\mathcal{D}(W) := \{D \in \mathbb{C}(z)[\partial_z] : D.W \subseteq W\}. \quad (6.6)$$

If W corresponds to the triple (π, X, \mathcal{L}) , then we can interpret $\mathcal{D}(W)$ as the ring $\mathcal{D}_{\mathcal{L}}(X)$ of differential operators on sections of the sheaf \mathcal{L} (embedded in $\mathbb{C}(z)[\partial_z]$ via the “framing” π). It is fairly well-known (cf. [21], or see Appendix A) that the algebras $\mathcal{D}(W)$ satisfy all the assumptions we have made about \mathcal{D} in the previous sections: we shall prove later (see Corollary 7.2) that in fact the $\mathcal{D}(W)$ are (up to isomorphism) the *only* algebras that satisfy these assumptions.

The paper [8] provides further information about the algebras $\mathcal{D}(W)$: let us list the results that we need from that paper. If V and W are linear subspaces of $\mathbb{C}(z)$, we set

$$\mathcal{D}(V, W) := \{D \in \mathbb{C}(z)[\partial_z] : D.V \subseteq W\}. \quad (6.7)$$

If $V, W \in \text{Gr}^{\text{ad}}$, then clearly $\mathcal{D}(V, W)$ is a $\mathcal{D}(W)$ – $\mathcal{D}(V)$ -bimodule (the actions being given by composition).

Theorem 6.2. [8] *Each isomorphism class of right ideals in the Weyl algebra $A \equiv \mathbb{C}[z, \partial_z]$ has a unique representative of the form $\mathcal{D}(\mathbb{C}[z], W)$ with $W \in \text{Gr}^{\text{ad}}$.*

Generalizing slightly the definition in Section 6.1, let us call a linear subspace $V \subset \mathbb{C}(z)$ primary decomposable if $V = fW$ for some $f \in \mathbb{C}(z)$, $W \in \text{Gr}^{\text{ad}}$.

Theorem 6.3. [8] *A subspace $V \subset \mathbb{C}(z)$ is primary decomposable if and only if $\mathcal{D}(\mathbb{C}[z], V) \cdot \mathbb{C}[z] = V$.*

Theorem 6.4. [8] *For each $W \in \text{Gr}^{\text{ad}}$, the algebra $\mathcal{D}(W)$ can be identified with the endomorphism ring of the corresponding A -module $\mathcal{D}(\mathbb{C}[z], W)$.*

Since the Weyl algebra A is hereditary and simple, every ideal in it is a progenerator; so Theorem 6.4 implies that all the algebras $\mathcal{D}(W)$ are Morita equivalent to A ; in particular, all the

$\mathcal{D}(W)$ are *simple*. Furthermore, all the bimodules $\mathcal{D}(V, W)$ are invertible, and for any $U, V, W \in \text{Gr}^{\text{ad}}$, we have

$$\mathcal{D}(V, U)\mathcal{D}(W, V) = \mathcal{D}(W, U). \quad (6.8)$$

6.4. The action of Γ

In the theory of integrable systems, a key role is played by the action on Gr^{ad} of the group Γ from the Introduction. Recall that for each polynomial $p(z)$ we have the element $\gamma_p \in \Gamma$ defined by (1.1): it acts on the Weyl algebra, or more generally on the algebra $\mathbb{C}(z)[\partial_z]$ as formal conjugation by $e^{p(z)}$. Roughly speaking, the action of Γ on Gr^{ad} is given by scalar multiplications; that is, we define $\gamma_p W := e^{p(z)} W$. Of course, since $e^{p(z)}$ is not a rational function, this does not immediately make sense: to interpret it correctly we have temporarily to replace W by a suitable completion (see, for example, [5, Section 2]). This difficulty need not concern us here, because we are interested mainly in the induced action of Γ on the spaces $\mathcal{D}(V, W)$, which makes sense without any completions. Namely, we have

$$\mathcal{D}(\gamma_p^{-1} V, \gamma_p^{-1} W) = e^{-p(z)} \mathcal{D}(V, W) e^{p(z)},$$

so that γ_p induces a bijective map

$$\gamma_p : \mathcal{D}(\gamma_p^{-1} V, \gamma_p^{-1} W) \rightarrow \mathcal{D}(V, W)$$

defined by $\gamma_p(D) := e^{p(z)} D e^{-p(z)}$. In particular, taking $V = W$, we have isomorphisms of algebras

$$\gamma : \mathcal{D}(\gamma^{-1} W) \rightarrow \mathcal{D}(W)$$

for each $\gamma \in \Gamma$, $W \in \text{Gr}^{\text{ad}}$. We refer to [5] for a more thorough discussion of these points.

6.5. The bispectral involution

The *bispectral involution* $W \mapsto b(W)$ on Gr^{ad} can be characterized by the formula

$$\psi_{b(W)}(x, z) = \psi_W(z, x). \quad (6.9)$$

Generalizing (6.3), one can show (see [5]) that for each $D \in \mathcal{D}(W)$ there is a unique differential operator Θ in the variable x such that

$$D(z) \cdot \psi_W(x, z) = \Theta(x) \cdot \psi_W(x, z). \quad (6.10)$$

The map $D \mapsto \Theta$ defines³ an anti-isomorphism from $\mathcal{D}(W)$ to $\mathcal{D}(b(W))$. To write it more explicitly, we introduce the *formal integral operator* (in x) K_W with the property that (formally) $\psi_W = K_W \cdot e^{xz}$. If ψ_W is given by (6.2), then we have

$$K_W = 1 + \sum_i f_i(x) g_i(\partial_x); \quad (6.11)$$

note that K_W belongs to the Weyl quotient field Q . If we denote by b also the anti-automorphism of Q defined by $b(x) = \partial_x$, $b(\partial_x) = x$, then the formula (6.9) takes the form

$$K_{b(W)} = b(K_W),$$

and (6.10) says that the anti-isomorphism $\beta: \mathcal{D}(W) \rightarrow \mathcal{D}(b(W))$ defined above is given by the formula

$$\beta(D) = K_W b(D) K_W^{-1}. \quad (6.12)$$

The connection of the bispectral involution with the construction in Section 5 is as follows.

Proposition 6.5. *The algebra $\beta^{-1}(\mathcal{A}_{b(W)}) \equiv \mathcal{A}_{b(W)}$ is the mad subalgebra of $\mathcal{D}(W)$ dual to A_W .*

Proof. This follows at once from (6.12) and the fact that $K_W - 1$ has negative x -filtration. \square

7. Proof of Theorems 1.3 and 1.4

We now come back to the situation of Section 5: thus we have the mad subalgebra $B \subset \mathcal{D}$ together with a good fat framing (x, ∂_x) of B . The dual subalgebra \check{B} then satisfies the conditions of Theorem 6.1, so it determines a point of the adelic Grassmannian. We denote this point by $b(W)$ (where b is the bispectral involution on Gr^{ad}) so that $\check{B} \subseteq \mathcal{A}_{b(W)}$. As in Section 6.5, we allow ourselves the imprecision of using x to denote the variable in the definition of Gr^{ad} , so that W is a subspace of $\mathbb{C}(x)$, and both \mathcal{D} and $\mathcal{D}(W)$ are subalgebras of $\mathbb{C}(x)[\partial_x]$. With that understanding, the main result of this section is as follows.

Theorem 7.1. *With W defined as above, we have $\mathcal{D} = \mathcal{D}(W)$.*

Proof. For each $L \in \check{B}$ we have an equation of the form $L\psi_{b(W)} = f(z)\psi_{b(W)}$, or equivalently, $LK_{b(W)} = K_{b(W)}f(\partial_x)$ (f is a polynomial). Thus $K_{b(W)}^{-1}\check{B}K_{b(W)}$ is a subalgebra of $\mathbb{C}[\partial_x]$. Since \check{B} acts ad-nilpotently on \mathcal{D} , the algebra $K_{b(W)}^{-1}\check{B}K_{b(W)}$ acts ad-nilpotently on $K_{b(W)}^{-1}\mathcal{D}K_{b(W)}$, hence

$$K_{b(W)}^{-1}\mathcal{D}K_{b(W)} \subseteq N_Q(\partial_x) = \mathbb{C}(\partial_x)[x].$$

Applying the anti-involution b , we deduce that

$$K_W b(\mathcal{D}) K_W^{-1} \subseteq \mathbb{C}(x)[\partial_x];$$

³ After restoring the notation z for x ; this kind of confusion will recur several times in what follows.

by [5, Proposition 8.2], that is equivalent to

$$\mathcal{D} \subseteq \mathcal{D}(W).$$

To see that we have equality here, we use the following lemma of Levasseur and Stafford (see [16]): let $R \subseteq S$ be Noetherian domains such that (i) R and S have the same quotient field; (ii) one of R and S is simple; (iii) S is finitely generated as an R -module (both left and right). Then $R = S$. Let us check that the hypotheses of the lemma are satisfied for $\mathcal{D} \subseteq \mathcal{D}(W)$. Certainly, these are both domains with quotient field Q , and $\mathcal{D}(W)$ is simple. Because the finiteness condition (5.3) is satisfied, $\mathbb{C}[x, \xi]$ is a finitely generated module over $\mathrm{gr}_{\partial} \mathcal{D}$ (or $\mathrm{gr}_{\partial} \mathcal{D}(W)$), so by [1, Proposition 7.8], these are finitely generated \mathbb{C} -algebras, hence Noetherian rings. It follows that \mathcal{D} and $\mathcal{D}(W)$ are also (both left and right) Noetherian. Finally, to see the property (iii), note that we have

$$\mathrm{gr}_{\partial} \mathcal{D} \subseteq \mathrm{gr}_{\partial} \mathcal{D}(W) \subseteq \mathbb{C}[x, \xi],$$

and $\mathrm{gr}_{\partial} \mathcal{D}$ has finite codimension in $\mathbb{C}[x, \xi]$, hence (*a fortiori*) in $\mathrm{gr}_{\partial} \mathcal{D}(W)$. Thus $\mathrm{gr}_{\partial} \mathcal{D}(W)$ is a finitely generated module over $\mathrm{gr}_{\partial} \mathcal{D}$, so property (iii) follows. \square

If we now combine Theorem 7.1 with the main result of [15], we obtain the following.

Corollary 7.2. *Let \mathcal{D} be a subalgebra of the Weyl quotient field Q satisfying (4.1) and (4.2). Then either all mad subalgebras B of \mathcal{D} satisfy the finiteness condition (5.3), or else none of them does. Furthermore, in the former case \mathcal{D} is isomorphic to $\mathcal{D}(X)$ for some framed curve X .*

Proof. Suppose that \mathcal{D} possesses one mad subalgebra satisfying (5.3). According to Theorem 7.1, this implies that \mathcal{D} is isomorphic to $\mathcal{D}(W)$ for some $W \in \mathrm{Gr}^{\mathrm{ad}}$, and hence to $\mathcal{D}(X)$ for some framed curve X (see [4] or [6]). It is well known (see [21]) that the pair $(\mathcal{D}(X), \mathcal{O}(X))$ satisfies (5.3); the main result of [15] states that for the algebras $\mathcal{D}(X)$ the codimension of $\mathrm{gr}_{\partial} \mathcal{D}$ in $\mathbb{C}[x, \xi]$ is independent of the choice of mad subalgebra B ; in particular, it is always finite, as claimed. That completes the proof. \square

It is now easy to give

Proof of Theorems 1.3 and 1.4. Let $\mathcal{D}(X) \subset \mathbb{C}(z)[\partial_z]$ be the ring of differential operators on a framed curve, and let B be a mad subalgebra of $\mathcal{D}(X)$. By the above, we may choose a good fat framing of B ; then Theorem 7.1 gives us a point W of $\mathrm{Gr}^{\mathrm{ad}}$ and an isomorphism $\varphi: \mathcal{D}(W) \rightarrow \mathcal{D}(X)$ taking A_W onto B . Thus $Y := \mathrm{Spec} B \simeq \mathrm{Spec} A_W$ is a framed curve, as claimed in Theorem 1.3. And Theorem 1.4 follows at once, because $\mathcal{D}(W) \simeq \mathcal{D}_{\mathcal{L}}(Y)$ for some sheaf \mathcal{L} over Y (see Section 6.3). \square

8. An alternative proof of Theorem 7.1

As we have just seen, our Theorems 1.3 and 1.4 are proved by combining the main result of [15] with Theorem 7.1. The proof of Theorem 7.1 given in the preceding section depends heavily on machinery inspired by the theory of integrable systems. In the present section we want to give a proof that makes the minimum possible use of this machinery; namely, we shall use from it only the following consequence of Theorem 6.1.

Proposition 8.1. *Let $\mathcal{B} \subseteq \mathbb{C}(x)[\partial_x]$ be any rank 1 commutative algebra of differential operators with first two coefficients constant, and such that the curve $\text{Spec } \mathcal{B}$ is rational. Then there is a rational function $\psi_0(x)$ whose annihilator in \mathcal{B} is a maximal ideal of \mathcal{B} .*

Proof. Let $\psi(x, z)$ be a joint eigenfunction for \mathcal{B} of the form (6.2), so that for each $L \in \mathcal{B}$ we have an equation $L\psi(x, z) = f_L(z)\psi(x, z)$. Suppose first that ψ is regular at $z = 0$, and set $\psi_0 := \psi(x, 0)$. Then $\psi_0 \in \mathbb{C}(x)$, and $L\psi_0 = f_L(0)\psi_0$ for all $L \in \mathcal{B}$, so the annihilator of ψ_0 is the kernel of the character $L \mapsto f_L(0)$ of \mathcal{B} . If ψ has a pole of order k at $z = 0$ we replace it by $z^k\psi$ and argue as above. \square

Returning to our algebra \mathcal{D} with its pair of mad subalgebras (B, \check{B}) , we may apply Proposition 8.1 to the rank 1 algebra \check{B} : let $V := \mathcal{D}.\psi_0$ be the cyclic sub- \mathcal{D} -module of $\mathbb{C}(x)$ generated by the corresponding function ψ_0 . We aim to show that V coincides with the space $W \in \text{Gr}^{\text{ad}}$ of the preceding section. In contrast to what we had there, it is clear from the definition of V that $\mathcal{D} \subseteq \mathcal{D}(V)$; however, it is not clear that $V \in \text{Gr}^{\text{ad}}$. The crucial step towards proving that is the following.

Lemma 8.2. *V is finite over B .*

Proof. Let $I \subset \mathcal{D}$ be the annihilator of ψ_0 in \mathcal{D} , and let $\mathfrak{m} = I \cap \check{B}$: according to Proposition 8.1, \mathfrak{m} is a maximal ideal in \check{B} . Clearly, I contains the extension $\mathcal{D}\mathfrak{m}$ of \mathfrak{m} to \mathcal{D} (in fact $I = \mathcal{D}\mathfrak{m}$, but we do not need to prove that here). Thus $V \simeq \mathcal{D}/I$ is a quotient module of $\mathcal{D}/\mathcal{D}\mathfrak{m}$, so it is enough to prove that $\mathcal{D}/\mathcal{D}\mathfrak{m}$ is finite over B . We regard $\mathcal{D}/\mathcal{D}\mathfrak{m}$ as a filtered \mathcal{D} -module (via the x -filtration): the associated graded module can then be identified⁴ with $\text{gr}_x \mathcal{D}/(\text{gr}_x \mathcal{D})\mathfrak{m}$. Thus it is enough if we prove that this is finite over $\text{gr}_x B$. Choose $p(\xi) \in \text{gr}_x \check{B}$ so that $p(\xi)\mathbb{C}[x, \xi] \subseteq \text{gr}_x \mathcal{D}$ (that is possible, because $\mathbb{C}[x, \xi]/\text{gr}_x \mathcal{D}$ is a finite-dimensional $\text{gr}_x \check{B}$ -module, so its annihilator is a nonzero ideal in $\text{gr}_x \check{B}$). Let $\mathfrak{n} := \mathbb{C}[\xi]p(\xi)\mathfrak{m}$ (thus \mathfrak{n} is a nonzero ideal in $\mathbb{C}[\xi]$). We have

$$\mathbb{C}[x, \xi]\mathfrak{n} \equiv \mathbb{C}[x, \xi]p(\xi)\mathfrak{m} \subseteq (\text{gr}_x \mathcal{D})\mathfrak{m} \subseteq \text{gr}_x \mathcal{D} \subseteq \mathbb{C}[x, \xi]. \quad (8.1)$$

Now, $M := \mathbb{C}[x, \xi]/\mathbb{C}[x, \xi]\mathfrak{n}$ is a finite $\mathbb{C}[x]$ -module (in fact it is free of rank equal to the codimension of \mathfrak{n} in $\mathbb{C}[\xi]$), and $\mathbb{C}[x]$ is a finite $\text{gr}_x B$ -module (because $\text{gr}_x B$ has finite codimension in $\mathbb{C}[x]$). Thus M is a finite $\text{gr}_x B$ -module, and hence Noetherian (because $\text{gr}_x B$ is Noetherian). Thus the subquotient (see (8.1)) $\text{gr}_x \mathcal{D}/(\text{gr}_x \mathcal{D})\mathfrak{m}$ of M is again a Noetherian $\text{gr}_x B$ -module. \square

For the rest of this section V could be any sub- \mathcal{D} -module of $\mathbb{C}(x)$ that is finite⁵ over B : however, we shall see at the end of this section that V is in fact uniquely determined by these properties. Being finite over B means that V is the space of sections of a rank 1 torsion-free sheaf over the curve $\text{Spec } B$. The next proposition is thus (part of) Proposition 7.1 in [20], but we shall give a self-contained proof.

Proposition 8.3. *Let $V \subset \mathbb{C}(x)$ be as above. Then there are nonzero polynomials $p, q \in \mathbb{C}[x]$ such that $pV \subseteq \mathbb{C}[x]$ and $q\mathbb{C}[x] \subseteq V$.*

⁴ To simplify the notation, we do not distinguish between $\mathfrak{m} \subset \check{B}$ and its isomorphic image in $\text{gr}_x \check{B}$.

⁵ Readers who wish to avoid using the Burchnell–Chaundy theory have only to prove the existence of such a V .

Proof. Let $\{v_1, \dots, v_k\}$ generate V as a B -module, and let $v_i = f_i/g_i$, where f_i and g_i are polynomials. Then if p is the product (or least common multiple) of the g_i , clearly $pv_i \in \mathbb{C}[x]$ for all i , hence $pV \subseteq \mathbb{C}[x]$. Now let $\{f_1, \dots, f_r\}$ generate $\mathbb{C}[x]$ as a B -module, and let $f_i = b_i/c_i$, where $b_i, c_i \in B$. If a is the product (or least common multiple) of the c_i , then $af_i \in B$ for all i , hence $a\mathbb{C}[x] \subseteq B$. On the other hand, let v be any nonzero element of V , and let $v = b/c$, where $b, c \in B$. Then $cv = b \in V \cap B$; since V is a B -module, it follows that $Bb = bB \subseteq V$. Thus if $q := ba$ then we have $q\mathbb{C}[x] = ba\mathbb{C}[x] \subseteq bB \subseteq V$. \square

Now set

$$P := \mathcal{D}(\mathbb{C}[x], V) \equiv \{D \in \mathbb{C}(x)[\partial_x] : D \cdot \mathbb{C}[x] \subseteq V\}. \quad (8.2)$$

Clearly, P is a right sub- A -module of $\mathbb{C}(x)[\partial_x]$.

Corollary 8.4. P is a (fractional) right ideal of A having nonzero intersection with $\mathbb{C}(x)$.

Proof. Let p be as in Proposition 8.3. Then $pP \cdot \mathbb{C}[x] \subseteq pV \subseteq \mathbb{C}[x]$, hence $pP \subseteq A$, so pP is an (integral) right ideal in A . And if q is as in Proposition 8.3, then by definition $q \in P$; thus P has nonzero intersection even with $\mathbb{C}[x]$. \square

Finally, we set

$$E := \{D \in \mathbb{C}(x)[\partial_x] : DP \subseteq P\}. \quad (8.3)$$

Lemma 8.5. We may identify E with the endomorphism ring $\text{End}_A P$.

Proof. Every A -module endomorphism of P extends uniquely to a Q -linear endomorphism of the 1-dimensional (right) vector space $P \otimes_A Q \simeq Q$: it follows that we may (as is usual) identify $\text{End}_A P$ with the algebra

$$E' := \{D \in Q : DP \subseteq P\}.$$

By Corollary 8.4, we may choose a nonzero element $q \in P \cap \mathbb{C}(x)$; then if $D \in E'$, we have $Dq \in P \subseteq \mathbb{C}(x)[\partial_x]$, hence $D \in \mathbb{C}(x)[\partial_x]$. Thus $E' = E$. \square

Lemma 8.6. $\text{gr}_{\partial} E \subseteq \mathbb{C}[x, \xi]$.

Proof. As in the proof of Corollary 8.4, we have $pP \subseteq A$ for suitable $p \in \mathbb{C}[x]$; hence $p \text{gr}_{\partial} P \subseteq \text{gr}_{\partial} A = \mathbb{C}[x, \xi]$. Thus $\text{gr}_{\partial} P$ is a fractional ideal of $\mathbb{C}[x, \xi]$, and hence is a finitely generated $\mathbb{C}[x, \xi]$ -module (since this ring is Noetherian). Let $\{p_1, \dots, p_m\}$ generate $\text{gr}_{\partial} P$ as a $\mathbb{C}[x, \xi]$ -module, and let $d \in \text{gr}_{\partial} E$. It follows from the definition (8.3) of E that $d \text{gr}_{\partial} P \subseteq \text{gr}_{\partial} P$, so we have equations of the form $dp_i = \sum_1^m f_{ij} p_j$ for some $f_{ij} \in \mathbb{C}[x, \xi]$. Multiplying on the left by the adjoint of the matrix $(d\delta_{ij} - f_{ij})$, we find that $\det(d\delta_{ij} - f_{ij})$ annihilates the p_i , hence it is zero. That shows that d is integral over $\mathbb{C}[x, \xi]$; but this ring is integrally closed, hence $d \in \mathbb{C}[x, \xi]$, as claimed. \square

The next result completes our alternative proof of Theorem 7.1.

Theorem 8.7. *We have $\mathcal{D} = \mathcal{D}(V) = E$. Moreover, $V \in \text{Gr}^{\text{ad}}$.*

Proof. Clearly, we have inclusions of algebras

$$\mathcal{D} \subseteq \mathcal{D}(V) \subseteq E. \quad (8.4)$$

To see that we have equality here we use the Levasseur–Stafford lemma, just as in the proof of Theorem 7.1. Let us check that the conditions of that lemma are satisfied by the pair of algebras $\mathcal{D} \subseteq E$. First, since \mathcal{D} satisfies (4.1) and $E \subset Q$, it is obvious that \mathcal{D} and E both have quotient field Q . Next, the algebra A is hereditary and simple, and P is an ideal of A (see Corollary 8.4): hence P is a progenerator, so by Lemma 8.5 E is Morita equivalent to A . It follows that E is simple and Noetherian. Finally, using (8.4) and Lemma 8.6, we get inclusions of commutative algebras

$$\text{gr}_{\partial} \mathcal{D} \subseteq \text{gr}_{\partial} \mathcal{D}(V) \subseteq \text{gr}_{\partial} E \subseteq \mathbb{C}[x, \xi]. \quad (8.5)$$

By (5.3), all the codimensions here are finite; exactly as in the proof of Theorem 7.1, it follows that \mathcal{D} is Noetherian and that E is finite over \mathcal{D} . Thus all the assumptions of the Levasseur–Stafford lemma hold.

It remains to see that $V \in \text{Gr}^{\text{ad}}$. We show first that V is primary decomposable: according to Theorem 6.3, it is equivalent to show that $P \cdot \mathbb{C}[x] = V$ (where P is as in (8.2)). Now, if $D \in \mathcal{D} = \mathcal{D}(V)$, it follows from the definition (8.2) of P that $DP \subseteq P$; hence $DP \cdot \mathbb{C}[x] \subseteq P \cdot \mathbb{C}[x]$, that is, $P \cdot \mathbb{C}[x]$ is a left sub- \mathcal{D} -module of V . Next, let p and q be as in Proposition 8.3; then $qpV \subseteq q\mathbb{C}[x] \subseteq V$, so $qp \in \mathcal{D}$. Also, since $q \in P$, we have

$$qpV \subseteq Pp \cdot V \subseteq P \cdot \mathbb{C}[x].$$

Thus qp is a nonzero element in the annihilator of the \mathcal{D} -module $V/P \cdot \mathbb{C}[x]$. Since \mathcal{D} is simple, this annihilator must be all of \mathcal{D} , in particular it must contain 1. This shows that $V/P \cdot \mathbb{C}[x]$ is the zero module, that is, $P \cdot \mathbb{C}[x] = V$, as claimed.

Finally, the fact that $V \in \text{Gr}^{\text{ad}}$ is a consequence of our assumption that the operators in \check{B} are normalized with first two coefficients constant. Indeed, since V is primary decomposable, we have $V = fW$ for some $W \in \text{Gr}^{\text{ad}}$, $f \in \mathbb{C}(x)$. Clearly $\mathcal{D}(V) = f\mathcal{D}(W)f^{-1}$; thus we have $\mathcal{D} = f\mathcal{D}(W)f^{-1}$. Conjugating by f does not change either the ∂ - or the x -filtration on $\mathbb{C}(x)[\partial_x]$, thus $\check{B} = f\mathcal{A}_{b(W)}f^{-1}$. So the algebras \check{B} and $f^{-1}\check{B}f$ both consist of operators with first two coefficients constant. But the second coefficients differ by nonzero multiples of $f'f^{-1}$, so that is possible only if f is constant, and hence $V = W$. \square

To end this section, we give the promised proof of the uniqueness of the space V that we have constructed. The proof depends on the fact that \mathcal{D} is simple. We note first

Lemma 8.8. *There exists at most one (nonzero) simple sub- \mathcal{D} -module of $\mathbb{C}(x)$.*

Proof. Suppose that V and V' are two such modules. Since $\text{Frac } B = \mathbb{C}(x)$, if we fix nonzero elements $v \in V$, $v' \in V'$, we can find nonzero $a, b \in B$ such that $av = bv'$. Thus $V \cap V' \neq 0$, hence (since V and V' are both simple) $V = V \cap V' = V'$. \square

The uniqueness of V follows from Lemma 8.8 and the next proposition.

Proposition 8.9. *Let V be any nonzero sub- \mathcal{D} -module of $\mathbb{C}(x)$ that is finite over B . Then V is simple.*

Proof. Let $U \subseteq V$ be a nonzero sub- \mathcal{D} -module: fix any nonzero element $u \in U$. Let $\{v_1, \dots, v_k\}$ generate V as a B -module. Since $\text{Frac } B = \mathbb{C}(x)$, we can find nonzero $a_i, b_i \in B$ such that $a_i v_i = b_i u$ (for $1 \leq i \leq k$). It follows that if a is the product of the a_i , then $av_i \in U$ for all i , hence $aV \subseteq U$. Thus the annihilator (in \mathcal{D}) of V/U is nonzero; because \mathcal{D} is simple, it must be the whole of \mathcal{D} , so $U = V$. Hence V is simple. \square

9. Proof of Theorem 1.6

For the rest of the paper \mathcal{D} will denote an algebra which is isomorphic to $\mathcal{D}(W)$ for some $W \in \text{Gr}^{\text{ad}}$. Let us define $\text{Grad } \mathcal{D}$ to be the set of all isomorphisms $\sigma_W : \mathcal{D}(W) \rightarrow \mathcal{D}$ for various $W \in \text{Gr}^{\text{ad}}$ (more precisely, $\text{Grad } \mathcal{D}$ is the set of all pairs (W, σ_W) where $W \in \text{Gr}^{\text{ad}}$ and σ_W is an isomorphism as above). On the other hand, let $\text{Fad } \mathcal{D}$ denote the set of all triples (B, x, ∂_x) where B is a mad subalgebra of \mathcal{D} and (x, ∂_x) is a good fat framing of B . The set $\text{Fad } \mathcal{D}(W)$ has the natural base-point (A_W, z, ∂_z) : thus there is an obvious map $\alpha : \text{Grad } \mathcal{D} \rightarrow \text{Fad } \mathcal{D}$ which assigns to $\sigma_W \in \text{Grad } \mathcal{D}$ the point

$$\alpha(\sigma_W) := (\sigma_W(A_W), \sigma_W(z), \sigma_W(\partial_z)) \in \text{Fad } \mathcal{D}$$

(the map σ_W extends to an isomorphism of quotient fields $Q \rightarrow \text{Frac } \mathcal{D}$, which we denote by the same symbol). We can reformulate Theorem 7.1 as follows.

Theorem 9.1. *The above map $\alpha : \text{Grad } \mathcal{D} \rightarrow \text{Fad } \mathcal{D}$ is bijective.*

Proof. Let $(B, x, \partial_x) \in \text{Fad } \mathcal{D}$, and let $\theta : \mathbb{C}(x)[\partial_x] \rightarrow \mathbb{C}(z)[\partial_z]$ be the isomorphism which sends x to z and ∂_x to ∂_z . Theorem 7.1 states that θ maps \mathcal{D} isomorphically onto one of the algebras $\mathcal{D}(W)$, so the restriction of θ^{-1} to $\mathcal{D}(W)$ gives us a point of $\text{Grad } \mathcal{D}$. It is clear that this construction defines the inverse map to α . \square

Observe now that the group $\text{Aut } \mathcal{D} \times \Gamma$ acts naturally on each of spaces $\text{Grad } \mathcal{D}$ and $\text{Fad } \mathcal{D}$ (recall from the Introduction that Γ is the group of maps γ_p defined by (1.1)). Given $\sigma_W \in \text{Grad } \mathcal{D}$, we can compose it with any $\sigma \in \text{Aut } \mathcal{D}$ and $\gamma \in \Gamma$, as follows:

$$\mathcal{D}(\gamma^{-1}W) \xrightarrow{\gamma} \mathcal{D}(W) \xrightarrow{\sigma_W} \mathcal{D} \xrightarrow{\sigma} \mathcal{D},$$

where the first map is explained in Section 6.4. This clearly defines an action of $\text{Aut } \mathcal{D} \times \Gamma$ on $\text{Grad } \mathcal{D}$. The action of $\text{Aut } \mathcal{D}$ on $\text{Fad } \mathcal{D}$ is induced from its natural action on \mathcal{D} ; we let $\gamma_p \in \Gamma$ act on $\text{Fad } \mathcal{D}$ as formal conjugation by $e^{p(x)}$, that is, we set

$$\gamma_p(B, x, \partial_x) = (B, x, \partial_x - p'(x)). \quad (9.1)$$

Directly from the definitions, we can check:

Proposition 9.2. *The bijection α in Theorem 9.1 is equivariant with respect to the above actions of $\text{Aut } \mathcal{D} \times \Gamma$.*

It follows that α induces bijections between the quotient spaces of $\text{Grad } \mathcal{D}$ and $\text{Fad } \mathcal{D}$ by any one of the groups $\text{Aut } \mathcal{D}$, Γ or $\text{Aut } \mathcal{D} \times \Gamma$. The latter two possibilities will yield Theorems 1.5 and 1.6, respectively, but we consider first the quotient by $\text{Aut } \mathcal{D}$. The obvious map $\text{Grad } \mathcal{D} \rightarrow \text{Gr}^{\text{ad}}$ (sending σ_W to W) clearly induces an injection from $\text{Grad } \mathcal{D} / \text{Aut } \mathcal{D}$ into Gr^{ad} . Its image consists of all $W \in \text{Gr}^{\text{ad}}$ such that $\mathcal{D}(W)$ is isomorphic to \mathcal{D} : as explained in [4,6], this consists of one of the *Calogero–Moser strata* $C_n \subset \text{Gr}^{\text{ad}}$. We therefore obtain

Corollary 9.3. *The bijection α of Theorem 9.1 induces a bijection*

$$C_n \rightarrow \text{Fad } \mathcal{D} / \text{Aut } \mathcal{D},$$

where n is the integer determining the isomorphism class of \mathcal{D} .

We now divide out further by the action of Γ . According to [5], the action of Γ on the space C_n is as defined by (1.2); while the formula (9.1) shows that the quotient map $\text{Fad } \mathcal{D} \rightarrow \text{Fad } \mathcal{D} / \Gamma$ can be identified with the forgetful map from $\text{Fad } \mathcal{D}$ to $\text{Mad } \mathcal{D}$, sending (B, x, ∂_x) to (B, x) . Hence Corollary 9.3 yields the following slightly sharpened version of Theorem 1.6.

Corollary 9.4. *The bijection of Corollary 9.3 induces a bijection*

$$C_n / \Gamma \rightarrow \text{Mad } \mathcal{D} / \text{Aut } \mathcal{D}.$$

We can also divide out just by the action of Γ , giving a bijection from $\text{Grad } \mathcal{D} / \Gamma$ to $\text{Mad } \mathcal{D}$. As mentioned above, this leads to Theorem 1.5, but more work is needed to identify $\text{Grad } \mathcal{D}$ with the space $\text{Aut } A$ in that theorem.

10. Proof of Theorem 1.5

To obtain Theorem 1.5 from the considerations in the preceding section, we need one more ingredient; namely, we need to see that the obvious action of $\text{Aut } \mathcal{D}$ on $\text{Grad } \mathcal{D}$ extends to an action of the larger group $\text{Pic } \mathcal{D}$. We first review some general facts about $\text{Pic } \mathcal{D}$ (which are valid for an arbitrary \mathbb{C} -algebra \mathcal{D}). For more details, see (for example) [2, Chapter 2].

Recall that $\text{Pic } \mathcal{D}$ is the group (under tensor product) of isomorphism classes of invertible \mathcal{D} – \mathcal{D} -bimodules (over \mathbb{C} , that is, we consider only bimodules on which the left and right \mathbb{C} -vector space structures coincide). There is a natural homomorphism from $\text{Aut } \mathcal{D}$ to $\text{Pic } \mathcal{D}$ which assigns to $\sigma \in \text{Aut } \mathcal{D}$ the bimodule ${}_{\bar{\sigma}}\mathcal{D}_1$ (that is, \mathcal{D} itself, but with the left action twisted by the inverse $\bar{\sigma}$ of σ). The kernel of this map is exactly the group of *inner* automorphisms of \mathcal{D} . At the cost of breaking the left/right symmetry, we can describe the elements of $\text{Pic } \mathcal{D}$ in the following way. Let M be an invertible \mathcal{D} – \mathcal{D} -bimodule: if we momentarily forget the left action of \mathcal{D} , then M becomes a (progenerative) right \mathcal{D} -module $M_{\mathcal{D}}$. The forgotten left action of \mathcal{D} is then defined by some isomorphism

$$\sigma : \text{End}_{\mathcal{D}}(M_{\mathcal{D}}) \rightarrow \mathcal{D},$$

which is again unique up to composition with an inner automorphism of \mathcal{D} .

Now we return to our case, where \mathcal{D} is isomorphic to one of the algebras $\mathcal{D}(W)$. In this case the remarks above can be simplified a little. We note first

Lemma 10.1. *The algebras $\mathcal{D}(W)$ (where $W \in \text{Gr}^{\text{ad}}$) have no nontrivial inner automorphisms.*

Proof. A differential operator $D \in \mathcal{D}(W)$ can be invertible only if it has order zero, that is, if it is a function. But by Proposition A.2, the only functions in $\mathcal{D}(W)$ are polynomials, hence the only invertible elements of $\mathcal{D}(W)$ are the scalars. \square

It follows that we may regard $\text{Aut } \mathcal{D}$ as a subgroup of $\text{Pic } \mathcal{D}$ via the natural homomorphism described above. Next, we have the Cannings–Holland description of the ideal classes of $\mathcal{D}(W)$.

Lemma 10.2. *For any $W \in \text{Gr}^{\text{ad}}$, each isomorphism class of right ideals of $\mathcal{D}(W)$ has a unique representative of the form*

$$\mathcal{D}(W, V) := \{D \in \mathbb{C}(z)[\partial_z] : D.W \subseteq V\}$$

with $V \in \text{Gr}^{\text{ad}}$.

Proof. In the case when $W = \mathbb{C}[z]$, so that $\mathcal{D}(W)$ is the Weyl algebra A , this is exactly Theorem 6.2: each ideal class in A has a unique representative of the form $\mathcal{D}(\mathbb{C}[z], V)$. But $\mathcal{D}(W)$ is Morita equivalent to A via the invertible bimodule $\mathcal{D}(W, \mathbb{C}[z])$; it follows that each ideal class in $\mathcal{D}(W)$ has a unique representative of the form

$$\mathcal{D}(\mathbb{C}[z], V)\mathcal{D}(W, \mathbb{C}[z]) = \mathcal{D}(W, V),$$

as claimed. \square

With these preliminaries, we can define the action of $\text{Pic } \mathcal{D}$ on $\text{Grad } \mathcal{D}$. Let $[M] \in \text{Pic } \mathcal{D}$, $\sigma_W \in \text{Grad } \mathcal{D}$. It follows from Lemmas 10.1 and 10.2 that $[M]$ has a unique representative of the form $\mathcal{D}(W, V)$ with the structure of right \mathcal{D} -module determined via σ_W and the structure of left \mathcal{D} -module determined via some isomorphism $\sigma_V : \mathcal{D}(V) \rightarrow \mathcal{D}$. For short, in what follows we shall say that $[M]$ is *represented* by this triple $(\mathcal{D}(W, V), \sigma_W, \sigma_V)$. We define

$$[M].\sigma_W = \sigma_V. \quad (10.1)$$

Theorem 10.3. *The formula (10.1) defines a free transitive action of $\text{Pic } \mathcal{D}$ on $\text{Grad } \mathcal{D}$.*

Proof. Straightforward. The main point is to check that we do indeed have a group action, that is, if $[M], [N] \in \text{Pic } \mathcal{D}$ and $\sigma_W \in \text{Grad } \mathcal{D}$, then

$$[N].([M].\sigma_W) = [N \otimes_{\mathcal{D}} M].\sigma_W.$$

That amounts to showing that if $[M]$ is represented by $(\mathcal{D}(W, V), \sigma_W, \sigma_V)$ and $[N]$ by $(\mathcal{D}(V, U), \sigma_V, \sigma_U)$ then $[N \otimes_{\mathcal{D}} M]$ is represented by $(\mathcal{D}(W, U), \sigma_W, \sigma_U)$. The map $D_1 \otimes D_2 \mapsto D_1 D_2$ provides the required isomorphism of bimodules from $\mathcal{D}(V, U) \otimes \mathcal{D}(W, V)$ to $\mathcal{D}(W, U)$. It is trivial to show that the action is free and transitive. \square

Now recall that we have an action of the group Γ on $\text{Grad } \mathcal{D}$, commuting with the action of $\text{Aut } \mathcal{D}$. A little more generally, we have

Proposition 10.4. *The above action of $\text{Pic } \mathcal{D}$ on $\text{Grad } \mathcal{D}$ commutes with the action of Γ .*

Proof. Let $[M] \in \text{Pic } \mathcal{D}$ and $\sigma_W \in \text{Grad } \mathcal{D}$. Let $[M].\sigma_W = \sigma_V$, so that $[M]$ is represented by $(\mathcal{D}(W, V), \sigma_W, \sigma_V)$. If $\gamma \in \Gamma$, we have to show that $[M].(\sigma_W \gamma) = \sigma_V \gamma$; equivalently, that $[M]$ is also represented by $(\mathcal{D}(\gamma^{-1}W, \gamma^{-1}V), \sigma_W \gamma, \sigma_V \gamma)$. It is easy to check that the map

$$\gamma: \mathcal{D}(\gamma^{-1}W, \gamma^{-1}V) = \gamma^{-1}\mathcal{D}(W, V) \rightarrow \mathcal{D}(W, V)$$

explained in Section 6.4 is a bimodule isomorphism; hence the result. \square

Now let us fix a base-point $\sigma_W \in \text{Grad } \mathcal{D}$; according to Theorem 10.3, the map

$$\text{Pic } \mathcal{D} \rightarrow \text{Grad } \mathcal{D} \quad (10.2)$$

which sends $[M]$ to $[M].\sigma_W$ is bijective. Fixing a base-point gives us also a distinguished invertible \mathcal{D} – A –bimodule $P := \mathcal{D}(\mathbb{C}[z], W)$, where it is understood that the structure of left \mathcal{D} –module on P is defined via the isomorphism σ_W . By (6.8), the inverse A – \mathcal{D} –bimodule is $P^* := \mathcal{D}(W, \mathbb{C}[z])$. According to [22], the natural map $\text{Aut } A \rightarrow \text{Pic } A$ is an isomorphism; on the other hand, P defines an isomorphism from $\text{Pic } A$ to $\text{Pic } \mathcal{D}$, sending (the class of) an A – A –bimodule M to $P \otimes_A M \otimes_A P^*$. Combining the composite isomorphism $\text{Aut } A \simeq \text{Pic } \mathcal{D}$ with the bijection (10.2), we obtain a bijective map

$$\beta: \text{Aut } A \rightarrow \text{Grad } \mathcal{D}. \quad (10.3)$$

Lemma 10.5. *Under the bijection β , the action of Γ on $\text{Grad } \mathcal{D}$ corresponds to its action by right multiplication on $\text{Aut } A$.*

Proof. Because of Proposition 10.4, it is enough to show that if $\gamma \in \Gamma$ then $\beta(\gamma) = \sigma_W \gamma$ (recall that σ_W is our chosen base-point in $\text{Grad } \mathcal{D}$). Since γ corresponds to the bimodule $M := P \otimes_A \bar{\gamma} A_1 \otimes_A P^*$ in $\text{Pic } \mathcal{D}$, we have to see that this bimodule is represented by $(\mathcal{D}(W, \gamma^{-1}W), \sigma_W, \sigma_W \gamma)$. It is easy to check that the map

$$D_1 \otimes a \otimes D_2 \mapsto \gamma^{-1}(D_1)aD_2$$

defines the desired bimodule isomorphism from M to $\mathcal{D}(W, \gamma^{-1}W)$. \square

Finally, we can now consider the composite bijection

$$\text{Aut } A \xrightarrow{\beta} \text{Grad } \mathcal{D} \xrightarrow{\alpha} \text{Fad } \mathcal{D}. \quad (10.4)$$

Using Lemma 10.5, we can divide both sides of this bijection by Γ to obtain the following more precise version of Theorem 1.5.

Theorem 10.6. *The bijection (10.4) induces a bijection*

$$\text{Aut}(A)/\Gamma \rightarrow \text{Mad } \mathcal{D}.$$

Remark 10.7. The bijection in Theorem 10.6 depends on the choice of base-point in $\text{Grad } \mathcal{D}$; however, in practice there is usually a natural choice. For example, if $\mathcal{D} = \mathcal{D}(W)$ for some $W \in \text{Gr}^{\text{ad}}$, then it is natural to take the identity map in $\mathcal{D}(W)$ as the base-point. Similarly, if $\mathcal{D} = \mathcal{D}(X)$, where X is a framed curve with ring of functions $\mathcal{O}(X) \subseteq \mathbb{C}[z]$, then there is a unique monic polynomial $p(z)$ such that $W := p^{-1}\mathcal{O}(X)$ belongs to Gr^{ad} , and it is natural to take as base-point the isomorphism $\sigma_W : \mathcal{D}(W) \rightarrow \mathcal{D}(X)$ defined by $\sigma_W(D) := pDp^{-1}$. This remark perhaps justifies our use of the word “natural” in Theorem 1.5.

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Appendix A. Some properties of $\mathcal{D}(W)$

Here we provide proofs that the algebras $\mathcal{D}(W)$ (for $W \in \text{Gr}^{\text{ad}}$) have the properties needed for us to apply the results of Sections 4 and 5. The proofs of the first two propositions use only the fact that we have

$$p\mathbb{C}[z] \subseteq W \subseteq q^{-1}\mathbb{C}[z] \quad (\text{A.1})$$

for suitable polynomials p, q ; thus these propositions hold also for spaces W in the larger Grassmannian Gr^{rat} .

Proposition A.1. *The field of fractions of $\mathcal{D}(W)$ is \mathcal{Q} .*

Proof. It follows from (A.1) that if $D \in \mathbb{C}[z, \partial_z]$, then

$$pDq.W \subseteq pD.\mathbb{C}[z] \subseteq p\mathbb{C}[z] \subseteq W,$$

that is, $p\mathbb{C}[z, \partial_z]q \subseteq \mathcal{D}(W)$. It follows that the quotient field of $\mathcal{D}(W)$ contains the Weyl algebra $\mathbb{C}[z, \partial_z]$; it is therefore the whole of \mathcal{Q} , as claimed. \square

Proposition A.2. $\text{gr}_{\partial} \mathcal{D}(W) \subseteq \mathbb{C}[z, \zeta]$.

Proof. Arguing as in the previous proof, it follows from (A.1) that $\mathcal{D}(W)$ is contained in $q^{-1}\mathbb{C}[z, \partial_z]p^{-1}$; hence the leading coefficient of every element $L \in \mathcal{D}(W)$ has denominator at worst pq . But this is true of L^n for every $n \geq 1$, hence that leading coefficient must be a polynomial, as claimed. \square

Our last proposition (which is *not* valid for all $W \in \text{Gr}^{\text{rat}}$) is less easy to prove. The proof given in [21] for $\mathcal{D}(X)$ (where X is a framed curve) generalizes easily to our case; here we give another proof, using the existence of the bispectral involution on Gr^{ad} .

Proposition A.3. *The pair $(\mathcal{D}(W), A_W)$ satisfies the condition (5.3).*

Proof. We observed in the proof of Proposition A.1 that $p\mathbb{C}[z, \partial_z]q \subseteq \mathcal{D}(W)$ for suitable polynomials p and q ; it follows that

$$p(z)q(z)\mathbb{C}[z, \zeta] \subseteq \text{gr}_{\partial} \mathcal{D}(W). \quad (\text{A.2})$$

Similarly, there are polynomials r and s such that $r(z)\mathbb{C}[z, \partial_z]s(z) \subseteq \mathcal{D}(b(W))$, hence (applying the anti-automorphism b of Q)

$$s(\partial_z)\mathbb{C}[z, \partial_z]r(\partial_z) \subseteq b\mathcal{D}(b(W)) = K_{b(W)}^{-1}\mathcal{D}(W)K_{b(W)}.$$

Since $K - 1$ has negative ∂ -filtration, it follows that

$$\zeta^N \mathbb{C}[z, \zeta] \subseteq \text{gr}_{\partial} \mathcal{D}(W), \quad (\text{A.3})$$

(where $N := \deg r + \deg s$). It follows at once from (A.2) and (A.3) that $\text{gr}_{\partial} \mathcal{D}(W)$ has finite codimension in $\mathbb{C}[z, \zeta]$. \square

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