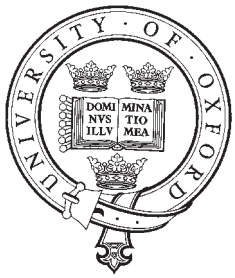
A Comparison Between Goal Reaching and Yaari's Models



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A thesis submitted in partial fulfillment of the MSc in

Mathematical and Computational Finance

August 1, 2009

I would like to dedicate this work to my parents and brother for their uninhibited support through all times.

Acknowledgements

I would like to thank Professor Xun Yu Zhou for his guidance in this project and Xue Dong He for helping me understand the models. I would also like to thank Dr. Geetha Venkatraman, Dr. Sanjeev Agrawal and Dr. Radhamohan for their support and inspiration.

Abstract

In this project we build on the work done by He and Zhou in their paper Portfolio Choice via Quantiles[3]. We first explain in brief their portfolio choice model for the complete markets and the solution to the Goal Reaching problem and Yaari's model. We study the effect of Sharpe ratio on the two and are able to analytically prove that for the Goal Reaching Model an increase in Sharpe ratio increases the probability of achieving the target. We study in depth a concrete example of a particular distortion function - the power function in the Yaari's model. We calibrate the Goal Reaching problem and the Yaari's model with power function as probability distortion to the Dow Jones Industrial Average and show the probabilities of achieving targets under the two models. We also compare the results from the two models when given the same targets, in the same time horizon. Finally we show that the Goal Reaching problem can be embedded within the Yaari's model for a particular class of probability distortion functions and under this framework the Goal Reaching p roblem is risk averse

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Chapter 1

Introduction

Most of the work in continuous-time portfolio choice theory has been done in expected utility maximisation since the papers of Samuelson[23] and Merton[20]. The two main approaches used to solve the utility model are:

- The stochastic control or dynamic programming approach, initially proposed by Merton [20][21], which transforms the problem into solving a partial differential equation called the Hamilton-Jacobi-Bellman (HJB) equation.

- The martingale approach. This approach, developed by Harrison and Kreps [5], Harrison and Pliska [5][6], and Pliska (1986)[22], employs a martingale characterisation to turn the dynamic wealth equation into a static budget constraint and then identifies the optimal terminal wealth via solving a static optimisation problem. Given a complete market an optimal strategy can be derived by replicating the optimal terminal wealth. This approach is elaborated in Karatzas and Shreve[13]. The problem for the Incomplete market can also be solved by this approach, albeit with certain constraints, see, e.g., Cvitanic and Karatzas (1992)[10] , Kramkov and Schachermayer (1999)[15], and Goll and Ruschendorf (2001)[8].

However, the use of this model as a risk preference measure is very limited as its basic priniciples aer often violated in the real world scenarios. Hence, many alternative preference measures have been put forth, notably Yaari's "dual theory of choice"[25] which attempts to resolve a number of puzzles and paradoxes associated with the expected utility theory, although, as Yaari himself says, "the dual theory would lead

to other paradoxes, which are solved by the classical Goal reaching problem[1]".

In his theory, instead of applying a utility which is essentially a "distortion" in payment, Yaari distorts the probability decumulative function of the payment. Yaari shows that this probability distortion function represents the risk preference in an opposite way, risk aversion is characterised by a convex - rather than concave - distortion.

Other theories developed along this line of involving subjective probability distortions include Lopes' SP/A model[17] [18] and, Kahneman and Tversky's prospect theory [12] [24], both in the modern behavioural decision-making paradigm.

Although it seems natural to include probability distortions in a portfolio choice model, such a distortion renders a nonlinear expectation that destroys the time-consistency necessary for the dynamic programming approach as well as the convexity necessary for the convex duality approach.

Some of the other portfolio choice problems proposed involve explicitly probability and VaR/CVaR/quantile instead of expectation in their objectives and/or constraints. For instance, the goal-reaching problem, initiated by Kulldorff [16] and Heath [9] and investigated extensively by Browne (1999, 2000)[1] [2], is to maximise the probability of the terminal cash flow in excess of a given level or a given benchmark. Other models could include VaR/CVaR/quantile as risk measures.

However as He and Zhou [3] point out, "these problems cannot be solved directly by conventional approaches. For example, we could write a probability as an expectation of an indicator function but the latter is inherently nonconvex. Dynamic programming, on the other hand, becomes inapplicable for problems explicitly involving VaR/quantile".

Kataoka's paper [14] in 1963 was probably the first to include a quantile criterion in a single-period portfolio choice model in which he studies a discrete - time model with uniform distribution of returns. Grigor'ev and Kan [27] and Dhaene et al. [26] both consider dynamic portfolio selection problems with explicitly present quantile-related criteria, but Dhaene restricts the portfolios to belong to the class of "constant mix" portfolios.

He and Zhou (2009)[3] showed that it is possible to establish a unified, general framework that covers all the aforementioned models (and many others). They exploit two main similarities in all the models:

- All the preference/performance measures involved are law-invariant, i.e, agents only care about the probability distribution of the cash flow, rather than the cash flow itself.

- All the preferences can be written as a distorted mean where both the payment and its distribution function are altered.

They use the fact that if we change the decision variable from the terminal cash flow X to $G(Z)$, where G is the quantile function of X and Z is any uniform random variable on [0, 1], then the preference reduces to a linear expectation, under a possibly different probability measure. This change of variable does not change the preference value because X and $G(Z)$ are always identical in law. However, the budget constraint is inherently $law - variant$, but, a dual argument originally due to Dybvig (1988)[4], arguing that maximising a performance measure is equivalent to minimising the associated cost, reveals that X can also be replaced by G(Z) in the budget constraint where Z is a particular uniform variable generated by the pricing kernel.

Based on this analysis, He and Zhou formulate a portfolio choice model where the optimal quantile function of the terminal payment is to be chosen. The corresponding optimal terminal cash flow can be recovered by a simple formula. In our study we restrict ourselves to complete markets and thus the optimal portfolio is the one replicating the obtained terminal payoff.

Using this formulation we explain the explicit solutions to the Goal Reaching problem and the Yaari's model as given by He and Zhou. We see that the solutions obtained for the two are very similar cosmetically - both reduce to a digital option which has an "all or nothing" structure, however we shall study that they are very different inherently.

We delve deeper into these two solutions and see how they actually translate to real markets. We look at a portfolio comprising of the Dow Jones Industrial Average and calibrate the two models to this portfolio and find the probability of achieving a target in certain years. We move further to try to find explicit relation of these models with the Sharpe Ratio of the markets. As we shall see this is not trivial for the Yaari's model and we have to resort to numerical methods to find explicit solution for individual cases.

We then study a particular example of probability distortion function - the Power Function for the Yaari's Model in much detail. Again we use Zhou and He's solution structure to this problem but extend it further and give numerical results of probabilities of achieving targets under this model. We study the relation between the parameters of the model and the probability of achieving targets. We also look at the reverse relation - the restriction that a probability target sets on the parameter γ of the Power Distortion function. We see a pattern developing numerically and then show analytically that γ has to always be greater than one. This result has a major consequence in terms of risks and shows that the Goal reaching model can be embedded within the Yaari's model.

The project is divided into five more chapters - in the second chapter we lay down the principles of the continuous time market and explain the models and their formulation via quantiles. The third gives the solution to the generalized problem using quantile approach and then the explicit solutions for the two models as explained by He and Zhou. The fourth chapter is devoted to finding the effect of Sharpe ratio on the two models and finding relations between the two models, while in the fifth we study the power probability function in detail. The final chapter summarizes our results.

Chapter 2

Portfolio Choice Formulation via Quantiles

2.1 Continuous-time Market and Terminology

Let $T > 0$ be given and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$ be a filtered probability space on which is defined a standard $\mathcal{F}_t$ -adapted n-dimensional Brownian motion $W(t) \equiv (W^1(t), \ldots, W^n(t))^\top$ with $W(0) = 0$. It is assumed that $\mathcal{F}_t = \sigma\{W(s) : 0 \le s \le t\}$, by all the P-null sets. From here onwards $A^\top$ denotes the transpose of a matrix A, and $a_+ := max(a, 0)$, $a_- := min(-a, 0)$ for $a \in \mathbb{R}$.

We define a continuous-time financial market following Karatzas and Shreve[13]. In the market there are $m + 1$ assets being traded continuously. One of the assets is a bank account whose price process $S_0(t)$ follows:

$$dS_0(t) = r(t)S_0(t)dt, \quad t \in [0, T]; \quad S_0(t) = s_0 > 0, \tag{2.1}$$

where the interest rate $r(\cdot)$ is an $\mathcal{F}_t$-progressively measurable, scalar-valued stochastic process with $\int_0^T |r(s)|ds < +\infty$ almost surely (a.s.). The other m assets are stocks whose price processes $S_i(t), i = 1, \ldots, m$, satisfy the following stochastic differential equation (SDE):

$$dS_i(t) = S_i(t) \left[b_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dw^j(t) \right], \quad t \in [0, T]; \quad S_i(0) = s_i > 0, \tag{2.2}$$

where $b_i(\cdot)$ and $\sigma_{ij}(\cdot)$, the appreciation and volatility rates respectively, are scalar-valued, $\mathcal{F}_t$-progressively measurable stochastic processes with

$$\int_0^T \left[\sum_{i=1}^m |b_i(t)| + \sum_{i=1}^m \sum_{j=1}^n |\sigma_{ij}(t)|^2 \right] dt.$$

Set the excess rate of return process

$$B(t) := (b_1(t) - r(t), \ldots, b_m(t) - r(t))^\top,$$

and define the volatility process $\sigma(t) := (\sigma_{ij}(t))_{m\times n}$. We also impose these basic assumptions on the market parameters:

Assumption 1: There exists an $\mathcal{F}_t$-progressively measurable, $\mathbb{R}^n$-valued process $\theta_0(\cdot)$ with $\mathbb{E}e^{\frac{1}{2}\int_0^T |\theta_0(t)|^2 dt} < +\infty$ such that

$$\sigma(t)\theta_0(t) = B(t), \text{ a.s., a.e., } t \in [0, T].$$

Assumption 2: There exist $s_2 \geq s_1 > 0$ such that $s_1 \leq S_0(t) \leq s_2$.

Assumption 1 is slightly stronger than the standard no-arbitrage assumption due to the additional Novikov condition; see Karatzas and Shreve[13] for details. Assumption 2 holds when the risk-free rate is bounded.

Consider an agent, with an initial endowment $x_0 > 0$ and an investment horizon [0, T], whose total wealth at time $t \geq 0$ is denoted by $x(t)$. Assume that the trading of shares takes place continuously in a self-financing fashion and there are no transaction costs. Then $x(\cdot)$ satisfies (see, e.g., Karatzas and Shreve [13]),

$$dx(t) = \left[r(t)x(t) + B^\top(t)\pi(t)\right] + \pi(t)^\top \sigma(t) dW(t), \quad t \in [0, T], \tag{2.3}$$

where $\pi_i(t), i = 1, \ldots, m$, denotes the total market value of the agents wealth in the i-th asset at time t. The process $\pi \equiv (\pi_1(\cdot), \ldots, \pi_m(\cdot))^\top$ is called *portfolio* if it is $\mathcal{F}_t$-progressively measurable with

$$\int_0^T \left|\sigma(t)^\top \pi(t)\right|^2 dt < +\infty \text{ and } \int_0^T \left|B(t)^\top \pi(t)\right| dt < +\infty, \text{ a.s.},$$

and the corresponding discounted wealth process, $S_0(t)^{-1}x(t)$, is almost surely bounded from below although the bound may depend on $\pi(\cdot)$. This is called tameness and is standard in order to, among other things, exclude the doubling strategy[1].

[1]A doubling strategy is a process in which a person keeps doubling his bet till he wins. This ensures a win but mathematically assumes unbounded (infinite) wealth.

We could impose other constraints on the portfolios specific to a given problem, such as prohibition of shorting[2] or bankruptcy. A portfolio is called admissible if it satisfies all the given constraints. Let Π be the set of all admissible portfolios. It is important to note that Π does not depend on the initial position x_0. The agent evaluates each admissible portfolio $\pi(\cdot)$ via a certain performance (or preference) measure, denoted by $J(x_0, \pi(\cdot))$. The precise forms of $J(x_0, \pi(\cdot))$ are dictated by individual problems, and will be discussed individually. The objective of a portfolio selection problem is, for a given initial endowment x_0, to choose an optimal portfolio whose performance value achieves the supremum of $J(x_0, \pi(\cdot))$ over Π. Denote by $v(x_0)$ this supremum value.

Assumption 3: For an initial position x_0 and an admissible portfolio $\pi(\cdot)$, if $\hat{x}_0 > x_0$ then there is an admissible portfolio $\hat{\pi}(\cdot)$ such that $J(\hat{x}_0, \hat{\pi}(\cdot)) > J(x_0, \pi(\cdot))$.

This makes perfect economic sense: with more initial budget the agent will be able to do strictly better.

2.2 The Models

Here we shall discuss in brief the Expected Utility Maximization, the Goal - Reaching and the Yaari's model and explain the general principles behind them. Though these models appear quite different in terms of their economical interpretations and mathematical formulations they have certain underlying commonalities which were used by Zhou and He to propose a universal framework and approach covering all of them. Throughout our work we shall assume that the underlying continuous time market is complete or or equivalently the process $\theta_0(\cdot)$ in Assumption 1 is unique. We define $\rho(\cdot)$ the pricing kernel as

$$\rho(t) := \exp\left\{-\int_0^t \left[r(s) + \frac{1}{2}\left|\theta_0(s)\right|^2\right] ds - \int_0^t \theta_0^\top(s) dW(s)\right\}. \tag{2.4}$$

We denote $\rho := \rho(T)$. Under assumptions 1 and 2, $0 < \rho < +\infty$ a.s. and $0 < \mathbb{E}\rho < +\infty$. Let

$$\begin{aligned} \bar{\rho} &\equiv esssup\rho := sup\{a \in \mathbb{R} : P\{\rho > a\} > 0\} \\ \underline{\rho} &\equiv essinf\rho := inf\{a \in \mathbb{R} : P\{\rho < a\} > 0\}. \end{aligned} \tag{2.5}$$

[2]We allow the agent to go short,i.e, sell first, in this work

Thus for a martingale approach a portfolio choice problem in this economic setting reduces to finding the optimal terminal wealth.

Let $\mathbb{F}$ be the set of cumulative distribution functions (CDFs) of all the lower bounded random variables taking values on $\mathbb{R}$, i.e.,

$$\mathbb{F} = F(\cdot : \mathbb{R} \to [0,1], \text{ nondecreasing, } c\grave{a}dl\grave{a}g \ F(a-) = 0 \text{ for some } a \in \mathbb{R} \text{ and } F(+\infty) = 1.$$

The lower boundedness above corresponds to the required "tameness" of portfolios. For any $F(\cdot) \in \mathbb{F}$, denote by $F^{-1}(\cdot)$ its left-inverse, i.e.,

$$F^{-1}(t) = \inf\{x \in \mathbb{R} : F(x) > t\} = \sup\{x \in \mathbb{R} : F(x) < t\}, \quad t \in [0,1].$$

Let $\mathbb{G} := \{F^{-1}(\cdot) : F(\cdot) \in \mathbb{F}\}$ be the corresponding set of quantile functions, or $\mathbb{G} = \{G(\cdot) : [0,1] \to \mathbb{R}^+\}$, nondecreasing, left continuous, $G(0) = -\infty$, $G(0+) > -\infty$, where $G(1) := G(1-)$.

2.2.1 Expected Utility Maximisation Model

The model is given by,

$$\begin{aligned} & \max_X \mathbb{E}u(X), \\ \text{subject to} \quad & \mathbb{E}[\rho X] = x_0, \\ X \geq 0, X \text{ is } & \mathcal{F}_T \text{ measurable}, \end{aligned} \tag{2.6}$$

where $u(\cdot)$ is a utility function, $\mathbb{E}[\rho X] = x_0$ is the budget constraint, and $X \geq 0$ is the no-bankruptcy constraint (which may be absent in some models). This classical utility model was first given by Samuelson (1969)[23] and Merton (1969)[20] and has been studied in much depth since then. Assuming that the utility function is concave (representing the agent's risk-aversion), we can use Lagrange's technique to solve the above optimisation problem. X^* the solution to this static optimisation is the optimal terminal cash flow that ought to be achieved and the optimal portfolio will then be the one replicating X^*.

If we delve a little further into the preference measure of this model, $\mathbb{E}u(X)$, then recalling that for a random payoff $X \geq 0$ its mean is

$$\mathbb{E}[X] = \int_0^{+\infty} x dF_X(x), \tag{2.7}$$

where $F_X(\cdot)$ is the CDF of X, while

$$\mathbb{E}u(X) = \int_0^{+\infty} u(x)dF_X(x). \tag{2.8}$$

Thus comparing the two the expected utility (2.8) uses the utility function to distort the payment outcomes when evaluating the random payments. The way the distortion takes place reflects the agent's attitude towards risk, which is captured mathematically by the convexity and/or concavity of the utility function.

2.2.2 Goal Reaching Model

The model is given by,

$$\begin{aligned} &\max_X P(X \geq b), \\ \text{subject to} \quad &\mathbb{E}[\rho X] = x_0, \\ &X \geq 0, X \text{ is } \mathcal{F}_T \text{ measurable}, \end{aligned} \tag{2.9}$$

where $b > 0$ is the target to be reached by time T. This is called the goal-reaching problem, which was proposed by Kulldorff (1993)[16], Heath (1993)[9], and studied extensively (including various extensions) by Browne [1][2].

Economically, the goal-reaching model is different from the expected utility model and is discussed in detail in Browne (1999)[1]. It cannot be covered by the standard utility model as,

$$P(X \geq b) = \int_0^{\infty} \mathbf{1}_{(x \geq b)} dF_X(x), \tag{2.10}$$

and the indicator function is not concave $\mathbf{1}_{(x \geq b)}$. In his paper Browne uses dynamic programming and HJB equation to solve the problem.

2.2.3 Yaari's Model

The model is given by,

$$\begin{aligned} &\max_X \int_0^{\infty} w(P(X \geq b))dx, \\ \text{subject to} \quad &\mathbb{E}[\rho X] = x_0, \\ &X \geq 0, X \text{ is } \mathcal{F}_T \text{ measurable}, \end{aligned} \tag{2.11}$$

where $w : [0, 1] \to [0, 1]$ is a probability distortion function. It is generally a nonlinear, non-decreasing (preserving the order of the probabilities) function with $w(0) = 0$ and

$w(1) = 1$ (so there is no distortion on sure events).

The preference measure in (2.11) was first put forward by Yaari (1987)[25] as a "dual theory of choice under risk" to the expected utility theory. Using Fubinis theorem it can be written as

$$\int_0^\infty w(P(X \geq x)) = \int_0^\infty x d[-w(1 - F_X(x))], \tag{2.12}$$

then unlike the expected utility (2.8) that distorts the payment, Yaari's measure (2.12) distorts the CDF of the payment instead. Yaari [25], Theorem 2, shows that the risk preference can also be captured by this distortion; specifically the agent is risk-averse if and only if w is convex.

Yaari's dual measure is one of the so-called non-expected utilities which, as Yaari (1987) argues, can explain a number of paradoxes associated with the expected utility theory, although it leads to new dual paradoxes at the same time. In our work we shall look at He and Zhou's study of Yaari's measure on the continuous time model. As they point out, "the technical difficulties in solving (2.11) include the non-concavity of (2.12) in X due to the distortion w, and the time-inconsistency of the measure because of (2.12) being essentially a nonlinear expectation".

2.3 Quantile Formulation

In this section we outline the main principles of the transformation of the above models into the Quantile form given by He and Zhou.

Among the three models discussed above the main problems arise with the Yaari's model as it involves non-expected utilities due to the probability distortions and hence the standard approaches such as convex duality and dynamic programming fail to apply. If we look at the commonalities in all the above models we see that the preference measures in each of them can be written in the following general form:

$$C(X) := \int_{-\infty}^\infty u(x) d[w(F_X(x))], \tag{2.13}$$

where $u(\cdot)$ and $w : [0, 1] \to [0, 1]$ are both nonlinear. One can visualize (2.13) as a modified mean of the cash flow X where both the cash flow and its probability distribution are distorted. While $C(X)$ appears to be a non-expected measure mainly

due to the presence of w, by letting $z = F_X(x)$ in (2.13) we have (assuming that w is differentiable)

$$C(X) = \int_0^1 u(F_X^{-1}(z))d(w(z)) = \int_0^1 u(F_X^{-1}(z))w'(z)dz = \mathbb{E}[u(G(Z))w'(Z)], \quad (2.14)$$

where Z is any uniform random variable on $(0,1)$ (we write $Z \sim U(0,1)$) and $G = F_X^{-1}$, the quantile function of X. Hence, by regarding G (a quantile) as the decision variable, instead of X (a random variable), linear expectation and concavity is recovered (if $u(\cdot)$ is concave, that is).

Assumption 4 ρ admits no atom

This assumption is satisfied, in particular, when $r(\cdot)$ and $\theta(\cdot) := \theta_0(\cdot)$ are deterministic with $\int_0^T |\theta(t)|^2 dt \neq 0$ (in which case ρ is a nondegenerate lognormal random variable). It allows us to express the budget constraint in terms of the quantile of X.

Denote by $F_\rho(\cdot)$ the CDF of ρ and $Z_\rho := 1 - F_\rho(\rho)$. Because ρ is atom-less, $Z_\rho \sim U(0,1)$ and we can express ρ in terms of Z_ρ as $\rho = F_\rho^{-1}(1 - Z_\rho)$ a.s.. We use the following lemma as derived in Jin and Zhou (2008)[11] Theorem $B.1$.

Lemma 1 *Suppose Assumption* 4 *holds. Then* $\mathbb{E}[\rho G(Z_\rho)] \leq \mathbb{E}[\rho X]$ *for any lower bounded random variable* X *whose quantile is* G*. Furthermore, if* $\mathbb{E}[\rho G(Z_\rho)] < \infty$*, then the inequality becomes equality if and only if* $X = G(Z_\rho)$*, a.s..*

$\mathbb{E}[\rho X]$ is the $t = 0$ price of a future ($t = T$) random cash flow X. This lemma basically says that one can always replace a random payment X by $Y := G(Z_\rho)$, which has the same probability law as X, yet with no greater (and possibly smaller) cost. Notice this replacement would not change the preference measure (2.14), and thus of the three models under discussion, due to the law-invariance. Hence a dual argument yields that if X is an optimal solution then $[\rho G^*(Z_\rho)] = \mathbb{E}[\rho X^*]$ where G^* is the quantile of X (if $y_0 := \mathbb{E}[\rho G^*(Z_\rho)] < \mathbb{E}[\rho X] =: x_0$, then $G^*(Z_\rho)$ would achieve the same performance value with a strictly smaller budget. By Assumption 3, the agent could strictly increase the performance value with the original budget $x_0 > y_0$, thus contradicting the optimality of X. This, in turn, leads to $X^* = G^*(Z_\rho)$, a.s. using Lemma 1.

The above argument shows that an optimal solution X. of all the models in this section, or indeed any continuous-time model satisfying Assumptions 3 and 4, must

be in the form $G^*(Z_\rho)$ where G^* is a quantile and Z_ρ is a particular uniform random variable such that $Z_\rho = 1 - F_\rho(\rho)$. In other words, to find an optimal solution we need only to search among the random variables of the form $G(Z_\rho)$ where $G \in \mathbb{G}$. Since $\rho = F_\rho^{-1}(1 - Z_\rho)$ a.s., we can replace the budget constraint $E[\rho X] = x_0$ by

$$\mathbb{E}[F_\rho^{-1}(1 - Z_\rho)G(Z_\rho)] = x_0. \tag{2.15}$$

Thus finally the general portfolio choice model via quantiles is:

$$\begin{aligned} \max_{G(\cdot)} U(G(\cdot)) &= \mathbb{E}[u(G(Z_\rho))w'(Z_\rho)], \\ \text{subject to} \quad \mathbb{E}[F_\rho^{-1}(1 - Z_\rho)G(Z_\rho)] &= x_0, \\ G(\cdot) &\in \mathbb{G} \cap \mathbb{M}, \end{aligned} \tag{2.16}$$

where $Z_\rho = 1 - F_\rho(\rho)$, $\mathbb{G}$ is the set of quantile functions of lower bounded random variables and $\mathbb{M}$ specifies some other constraints. For instance, the no-bankruptcy constraint $X \geq 0$ can be translated into $\mathbb{M} = \{G(\cdot) : G(0+) \geq 0\}$.

We can alternatively write this in the integral form :

$$\begin{aligned} \max_{G(\cdot)} U(G(\cdot)) &= \int_0^1 u(G(z)w'(z)dz, \\ \text{subject to} \quad \int_0^1 F_\rho^{-1}(1 - z)G(z)dz &= x_0, \\ G(\cdot) &\in \mathbb{G} \cap \mathbb{M}. \end{aligned} \tag{2.17}$$

This covers the three models under discussion and in fact many other models. However, it depends on market completeness as it involves explicitly a pricing kernel via Z_ρ.

Chapter 3

Solutions to the models by the Quantile Approach

In this chapter we first outline in brief the solution to the general case (2.17) and then use it result to solve the Goal - Reaching problem and Yaari's dual model explicitly, as explained by He and Zhou[3].

Using the Lagrange multiplier $\lambda \in \mathbb{R}$ we first remove the budget constraint

$$\max_{G(\cdot)} U_\lambda(G(\cdot)) = \int_0^1 u(G(z)w'(z)dz - \lambda \left(\int_0^1 F_\rho^{-1}(1-z)G(z)dz = x_0 \right) \qquad (3.1)$$
$$\text{subject to } G(\cdot) \in \mathbb{G} \cap \mathbb{M}.$$

The standard way to solve such a problem is to initially ignore the constraint, $G(\cdot) \in \mathbb{G} \cap M$, since in many cases the optimal solution of the resulting unconstrained problem could be modified (without affecting the objective value) to satisfy this constraint under some reasonable assumptions.

Once (3.1) is solved with an optimal solution $G^*_\lambda(\cdot)$, we then find $\lambda^* \in \mathbb{R}$ that binds the original budget constraint,

$$\int_0^1 F_\rho^{-1}(1-z)G^*_{\lambda^*}(z)dz = x_0.$$

The existence of such λ^*. can usually be obtained by examining the monotonicity and continuity of $f(\lambda) := \int_0^1 F_\rho^{-1}(1-z)G^*_{\lambda^*}(z)dz = x_0$ in λ. Moreover, strict monotonicity would imply a unique λ.

Finally, $G^*(\cdot) := G^*_{\lambda^*}(\cdot)$ can be proved to be the optimal solution to (2.16) or (2.17). This is shown in the following way. Let $v(x_0)$ and $v_\lambda(x_0)$ be respectively the

optimal value of (2.17) and (3.1). By their very definitions we have the following *weak duality*

$$v(x_0) \leq \inf_{\lambda \in \mathbb{R}} v_\lambda(x_0) \quad \forall x_0 \in \mathbb{R}.$$

Using that,

$$v(x_0) \leq \inf_{\lambda \in \mathbb{R}} v_\lambda(x_0) \leq v_{\lambda^*}(x_0) = U_{\lambda^*}(G^*(\cdot)) = U(G^*(\cdot)) \leq v(x_0),$$

we see that $G^*(\cdot)$ is optimal to (2.17) (and the *strong duality* $v(x_0) = inf_{\lambda \in \mathbb{R}} v_\lambda(x_0)$ holds).

We can derive the uniqueness of the optimal solution from that of (3.1). Having done this and for $\lambda = \lambda^*$, and λ^* such that $G^*_{\lambda^*}(\cdot)$ binds the budget constraint. Then $G^*_{\lambda^*}(\cdot)$ is the unique optimal solution to (2.16). To verify, assume there exists another optimal solution $\tilde{G}^*_{\lambda^*}(\cdot)$ to (2.16). Then

$$U_{\lambda^*}(\tilde{G}^*(\cdot)) = U_{\lambda^*}(G^*_{\lambda^*}(\cdot)) = v(x_0) = U(\tilde{G}^*(\cdot)) = U_{\lambda^*}(\tilde{G}^*(\cdot)).$$

Hence, by the uniqueness of the optimal solution to (3.1), it can be concluded that $\tilde{G}^*(\cdot) = G^*_{\lambda^*}(\cdot)$.

Once (2.16) has been solved with the optimal solution $G^*(\cdot)$, the corresponding optimal terminal cash flow can be recovered by

$$X^* = G^*(Z_\rho) \equiv G^*(1 - F_\rho(\rho)). \tag{3.2}$$

The above expression shows that the optimal terminal wealth of the model (2.16) is anti-comonotonic with the pricing kernel ρ in a complete market. "This underlines one of the most important, common properties of the quantile model (which covers a wide range of portfolio selection problems from neoclassical to behavioural). It is also important in treating incomplete markets", He and Zhou [3].

We now apply this general approach to the Goal Reaching Problem and the Yaari's model in complete markets. In order to do so we have to make one more assumption on the market defined in Section 2.1

Assumption 5 $m = n$ and $\sigma(t)$ is invertible a.s., a.e. $t \in [0, T]$.

3.1 Solution to the Goal Reaching Model

Let us consider the goal-reaching problem defined in section 2.2.2. Browne (1999)[1] solved this problem, assuming that the investment opportunity set is deterministic, using rather a method based on the HJB equation and the associated verification theorem. Zhou and He assuming a deterministic investment opportunity set, gave a much simpler approach to the problem using their quantile formulation.

Firstly, if $x_0 \geq b\mathbb{E}[\rho]$ then a trivial optimal solution is $X^* = b$ and the optimal value is 1. Therefore let us only study the case $0 < x_0 < b\mathbb{E}[\rho]$, i.e, the target is at least greater than the risk free return. Note that

$$P(X \geq b) = \int_{\mathbb{R}} \mathbf{1}_{\{x \geq b\}} dF_X(x) = \int_0^1 \mathbf{1}_{\{G(z) \geq b\}} dz,$$

and $X \geq 0$ is equivalent to $G(0+) \geq 0$. Hence problem (2.9) using (2.17) can be formulated in the following equivalent quantile version:

$$\begin{aligned} \max_{G(\cdot)} U(G(\cdot)) &= \int_0^1 \mathbf{1}_{\{G(z) \geq b\}} dz, \\ \text{subject to} \quad \int_0^1 F_\rho^{-1}(1-z) G(z) dz &= x_0, \\ G(\cdot) \in \mathbb{G}, G(0+) &\geq 0. \end{aligned} \tag{3.3}$$

Introducing the Lagrange multiplier $\lambda > 0$ (as we only need to consider positive multipliers in this case), we get,

$$\begin{aligned} \max_{G(\cdot)} U_\lambda(G(\cdot)) = \int_0^1 \left[\mathbf{1}_{\{G(z) \geq b\}} - \lambda F_\rho^{-1}(1-z) G(z)\right] dz + \lambda x_0, \\ \text{subject to} \quad G(\cdot) \in \mathbb{G}, G(0+) \geq 0. \end{aligned} \tag{3.4}$$

Ignoring the constraints, the optimal value of the pointwise maximisation of the above integrand with argument $x = G(z)$ is $\max\{1 - \lambda F_\rho^{-1}(1-z)b, 0\}$ attained at $x^* = b\mathbf{1}_{\{1-\lambda F_\rho^{-1}(1-z)b \geq 0\}}$; and the solution is unique whenever $1 - \lambda F_\rho^{-1}(1-z)b > 0$. Define,

$$G^*_{\lambda^*}(z) := b\mathbf{1}_{\{1-\lambda F_\rho^{-1}(1-z)b \geq 0\}}, \qquad 0 < z < 1,$$

which is nondecreasing in z. It may not be left continuous; however, the value of $U_\lambda(G(\cdot))$ is unchanged if $G(\cdot)$ is altered only at countable points on $[0, 1]$, thus we can

take the left-continuous modification of $G^*_{\lambda^*}(z)$ to be the optimal solution of (3.4), and the optimal solution is unique up to a null Lebesgue measure. On the other hand, the modification above would generate the same random payment (3.2) since ρ has no atom. So the above $G^*_{\lambda^*}(\cdot)$ can be regarded as the optimal solution to (3.4).

In order to find $\lambda^* \geq 0$ binding the budget constraint so as to conclude that this is the optimal solution to (3.3) put back $G^*_{\lambda^*}(z)$ in the constraint of (3.3) and consider the function,

$$\begin{aligned} f(\lambda) &:= \int_0^1 F_\rho^{-1}(1-z) G^*_{\lambda^*}(z) dz \\ &= b \int_0^1 F_\rho^{-1}(1-z) \mathbf{1}_{\{F_\rho^{-1}(1-z) \leq 1/(\lambda b)\}} dz \\ &= b \int_0^\infty x \mathbf{1}_{\{x \leq 1/(\lambda b)\}} dF_\rho(x) \\ &= b\mathbb{E}\left[\rho \mathbf{1}_{\{x \leq 1/(\lambda b)\}}\right], \quad \lambda > 0. \end{aligned}$$

Clearly $f(\cdot)$ is nonincreasing, continuous on $(0, \infty)$ with $\lim_{\lambda \downarrow 0} f(\lambda) = b\mathbb{E}[\rho]$ and $\lim_{\lambda \uparrow \infty} f(\lambda) = 0$. Thus for any $0 < x_0 < b\mathbb{E}[\rho]$, there exists $\lambda^* > 0$ such that $f(\lambda^*) = x_0$ or the budget constraint holds. Then by the genereal scheme the corresponding $G^*_{\lambda^*}(\cdot)$ solves (3.3) and the terminal payment $G^*_{\lambda^*}(1 - F_\rho(\rho)) = b\mathbf{1}_{\{\rho \leq c^*\}}$ where $c^* \equiv (\lambda^*)^{-1}$ is such that the initial budget constraint binds, solves the original problem (2.9). Finally, the optimal solution is unique and the optimal value is

$$P(X^* \geq b) = P(\rho \leq c^*) = F_\rho(c^*).$$

For future reference in the paper we summarise the above solution into a theorem as given by Zhou and He,

Theorem 1 Assume that $0 < x_0 < bE[\rho]$. Then the unique solution to the goal-reaching problem (2.9) is $X^* = b\mathbf{1}_{\{\rho \leq c^*\}}$ where $c^* > 0$ is the one such that $E[\rho X^*] = x_0$. The optimal value is $F_\rho(c^*)$.

The optimal terminal wealth profile for the goal-reaching problem is a digital option. Optimally the agent either obtains a fixed payment upon a winning event or else loses all the money on a losing event at the end of the investment horizon dictated by the pricing kernel not exceeding a critical level c^*. Moreover, since $X^* = b\mathbf{1}_{\{\rho \leq c^*\}}$, the payoff b in case of a winning is fixed while the winning probability, $P(\rho \leq c^*)$, monotonically decreases with b/x_0, a quantity that measures the aspiration of the agent.

$$\mathbb{E}\left[\rho \mathbf{1}_{\{x \leq c^*\}}\right] = (b/x_0)^{-1}.$$

Hence there is a tradeoff between the winning chance and the winning amount

$$b\mathbb{E}\left[\rho \mathbf{1}_{\{x \leq c^*\}}\right] = x_0. \tag{3.5}$$

This in turn suggests that, although the notion of risk preference is not explicitly presented in the goal-reaching model, it is *implied* in the following sense: the more risk-averse the agent is, the more weight should be put on the *winning chance* and the less on the *winning amount*.

3.2 Solution to Yaari's Dual Model

In this section we discuss the solution to the Yaari's dual model given by He and Zhou[3] using their Quantile formulation technique. We assume $x_0 > 0$ to exclude a trivial case. By (2.12), (2.13), and (2.14) the problem has the following quantile formulation:

$$\begin{aligned} \max_{G(\cdot)} U(G(\cdot)) &= \int_0^1 \mathbf{1}_{\{G(z)w'(1-z)\}} dz \\ \text{subject to} \quad & \int_0^1 F_\rho^{-1}(1-z)G(z)dz = x_0, \\ & G(\cdot) \in \mathbb{G}, \quad G(0+) \geq 0. \end{aligned} \tag{3.6}$$

In order to move ahead Zhou and He impose a following assumption on the distortion function $w(\cdot)$

Assumption 6 $w(\cdot) : [0,1] \to [0,1]$ is continuous and strictly increasing with $w(0) = 0$, $w(1) = 1$. Furthermore, $w(\cdot)$ is continuously differentiable on $(0,1)$.

Ley $v(x_0)$ the optimal value of (3.6). The model is called ill-posed if $v(x_0) = \infty$, otherwise it is well-posed. An ill-posed model is one where the incentives implied by the model are wrong, and in the context of portfolio choice an ill-posed model usually leads to trading strategies that take the greatest possible leverages (hence the agent is most aggressive); see Jin and Zhou (2008)[11] for a detailed discussion and treatment of the ill-posedness. A well posed Yaari's model requires some consistency between the probability distortion and the market. This is made precise in the following theorem.

Theorem 2 Under *Assumption 6*, model (3.6) is ill-posed if $\liminf_{z\downarrow 0} \frac{w'(z)}{F_\rho^{-1}(z)} = +\infty$ and well posed if $\liminf_{z\uparrow 0} \frac{w'(z)}{F_\rho^{-1}(z)} < +\infty$. For the proof of the above theorem we refer the reader to He and Zhou[3].

If w is concave and differentiable, and $\underline{\rho} = 0$, then

$$\liminf_{z\downarrow 0} \frac{w'(z)}{F_\rho^{-1}(z)} \geq \liminf_{z\downarrow 0} \frac{w'(1/2)}{F_\rho^{-1}(z)} = +\infty.$$

So a concave distortion leads to an ill-posed problem or the agent is most aggressive in taking the risk. However, this is perfectly consistent with Yaaris theory that a concave distortion is equivalent to the risk-seeking preference.

To solve the problem in (3.6) using Lagrange Method, Zhou and He introduce another assumption in terms of a new function $M(\cdot)$ which they define as $M(z) := \frac{w'(z)}{F_\rho^{-1}(z)}$, $0 < z < 1$.

Assumption 7 $M(\cdot) = \frac{w'(z)}{F_\rho^{-1}(z)}$, $0 < z < 1$ is continuous on $(0,1)$, and there exists $z_0 \in (0,1)$ such that $M(\cdot)$ is strictly increasing on $(0, z_0)$ and strictly decreasing on $(z_0, 1)$.

Assumption 7 can be weakened to the one where $M(\cdot)$ may have a finite number of monotonic pieces. Thus under this assumption and with Theorem 2, the problem (3.6) is well-posed.

Consider the following family of problems with the parameter $\lambda > 0$ being the Lagrange multiplier:

$$\max_{G(\cdot)} U_\lambda(G(\cdot)) = \int_0^1 G(z) \left[w'(1-z) - \lambda F_\rho^{-1}(1-z)\right] dz + \lambda x_0.$$
$$\text{subject to } G(\cdot) \in \mathbb{G}, \quad G(0+) \geq 0. \tag{3.7}$$

Denote by $v_\lambda(x_0)$ the optimal value of (3.7). Zhou and He give the following proposition to solve this completely.

Proposition 1 Let Assumptions $6-7$ hold. Then there is the unique root $\lambda^* > 0$ of the following function on $(0, +\infty)$:

$$h(\lambda) := \int_0^1 \left[w'(1-z) - \lambda F_\rho^{-1}(1-z)\right]_+ dz - \int_0^1 \left[w'(1-z) - \lambda F_\rho^{-1}(1-z)\right]_- dz.$$

Moreover,

1. If $0 < \lambda < \lambda^*$ then, $v_\lambda(x_0) = +\infty$
2. If $\lambda > \lambda^*$, then $v_\lambda = \lambda x_0$ and the unique optimal solution to (3.7) is $G^*_\lambda(\cdot) \equiv 0$
3. If$\lambda = \lambda^*$, then $v_\lambda(x_0) = \lambda x_0$, and the set of optimal solutions to (3.7) is $\{G(\cdot) \in \mathbb{G} : G(z) = b\mathbf{1}_{\{z(\lambda^*)<z\leq 1\}}, b > 0\}$ where $0 < z(\lambda^*) \leq z_0$ is the one satisfying $M(z(\lambda^*)) = \lambda^*$.

For the proof of the above we refer the reader to He and Zhou[3]. Using the above they give the solution to the problem, (2.11) or its Quantile formulation (3.6) which they encapsulate in the theorem below,

Theorem 3 Suppose Assumptions 6−7 hold, and let λ be the one in Proposition 1. Then

1. The strong duality holds, i.e., $v(x_0) = \inf_{\lambda>0} v_\lambda(x_0) = \lambda^* x_0$.
2. $X^* = b^*\mathbf{1}_{\{\rho\leq c\}}$ is the unique optimal solution to (2.11), where c is the unique root of the following function

$$\varphi(x) := xw(F_\rho(x)) - w'(F_\rho(x)) \int_0^x sdF_\rho(s), \tag{3.8}$$

on $(F_\rho^{-1}(1 - z_0), \bar{\rho})$ and $b^* > 0$ is the one binding the initial budget constraint, i.e., $\mathbb{E}[\rho X^*] = x_0$.
Again for the proof of the above we refer the reader to He and Zhou's paper[3].

Thus we see that the solution to the Yaari's model is also a digital option, which cosmetically looks very similar to the solution of the Goal Reaching problem (Theorem 1). However they are very different. Although the terminal wealth X^* in both cases reduces to similar digital options, the components have different meanings. In the Goal Reaching problem, the target, b, was given as input to the model and represented the aspiration of the agent. However the $b \equiv b^*$ in the Yaari's case comes from the model itself once the agent chooses the probability distortion function and then he has no control over it.

In the next sections we shall delve into this more deeply and motivated by the similar structure of the solution under the two models try to see if the Goal Reaching

problem can be linked in some sense to the more generalistic Yaari's Model. But first we shall see what effect does the Sharpe Ratio have on the probability of achieving the targets within the two models.

Chapter 4

Effect of Sharpe Ratio

The *Sharpe Ratio* is a measure of the excess return per unit of risk in an investment asset or a trading strategy and thus forms an important benchmark in rating a portfolio. Recalling from our discussion of continuous time market, we defined *B(t)* to be our excess rate of return process, and from Assumption 1, we have that there exists a $\theta(\cdot)$ such that $\sigma(t)\theta(t) = B(t)$. This θ is the Sharpe ratio of our portfolio[1]. Intuitively it seems that if the Sharpe ratio increases the probability of achieving the target should also increase.

4.1 Goal Reaching Problem

From equation (3.5) we have that

$$\mathbb{E}\left[\rho \mathbf{1}_{\{x \leq c^*\}}\right] = x_0/b, \tag{4.1}$$

using our definition of the pricing kernel $\rho(t)$ from the expression (2.4) let us first compute $\rho(t) \leq c^*$

$$\begin{aligned} & \exp\left\{-\int_0^t \left[r(s) + \frac{1}{2}\left|\theta(s)\right|^2\right] ds - \int_0^t \theta(s) dW(s)\right\} \leq c^* \\ \text{iff} \quad & \left\{\int_0^t \left[r(s) + \frac{1}{2}\left|\theta(s)\right|^2\right] ds + \int_0^t \theta(s) dW(s)\right\} \geq -\ln c^* \\ \text{iff} \quad & \int_0^t \theta(s) dW(s) \geq \left[-\ln c^* - \int_0^t \left[r(s) + \frac{1}{2}\left|\theta(s)\right|^2\right] ds\right]. \end{aligned} \tag{4.2}$$

[1]In our study we shall consider the parameter to be those of the portfolio as a whole, like an index, thus reducing it to a single dimension and thus $\theta^\top(t) = \theta(t) = B(t)/\sigma(t)$

But $\int_0^t \theta(s)dW(s)$ is normally distributed such that $\int_0^t \theta(s)dW(s) \sim \Phi\left(0, \sqrt{\int_0^t |\theta(s)|^2 ds}\right)$, where $\Phi(\cdot)$ is the CDF of the Standard Normal distribution. Then (4.2) follows if and only if

$$Z \geq \left[\frac{-\ln c^* - \int_0^t \left[r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds}{\sqrt{\int_0^t |\theta(s)|^2 ds}} \right], \tag{4.3}$$

where Z follows the standard normal distribution.

Substituting this along with the definition of $\rho(\cdot)$, (2.4), in (4.1)

$$\exp\left\{ -\int_0^t \left[r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds \right\} \mathbb{E}\left[e^{-Z\int_0^t |\theta(s)|^2 ds} \mathbf{1}_{\left[Z \geq \frac{-\ln c^* - \int_0^t \left[r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds}{\sqrt{\int_0^t |\theta(s)|^2 ds}} \right]} \right] = x_0/b. \tag{4.4}$$

Solving this expectation seperately first,

$$\begin{aligned} &\exp\left\{ -\int_0^t \left[r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds \right\} \mathbb{E}\left[e^{-Z\int_0^t |\theta(s)|^2 ds} \mathbf{1}_{\left[Z \geq \frac{-\ln c^* - \int_0^t \left[r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds}{\sqrt{\int_0^t |\theta(s)|^2 ds}} \right]} \right] \\ &= \exp\left\{ -\int_0^t \left[r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds \right\} \frac{1}{\sqrt{2\pi}} \int_{\frac{-\ln c^* - \int_0^t \left[r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds}{\sqrt{\int_0^t |\theta(s)|^2 ds}}}^{\infty} e^{-\sqrt{\int_0^t |\theta(s)|^2 ds}} e^{-z^2/2} dz \end{aligned} \tag{4.5}$$

on completing the squares within the integral we reduce the above to,

$$\exp\left\{ -\int_0^t \left[r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds \right\} \frac{1}{\sqrt{2\pi}} \int_{\frac{-\ln c^* - \int_0^t \left[r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds}{\sqrt{\int_0^t |\theta(s)|^2 ds}}}^{\infty} e^{-\frac{[z + \sqrt{\int_0^t |\theta(s)|^2 ds}]}{2} + \frac{\int_0^t \theta(s)^2 ds}{2}} dz. \tag{4.6}$$

Taking $e^{\frac{\int_0^t |\theta(s)|^2 ds}{2}}$ outside the integral and using the substitution $y = z + \theta\sqrt{t}$ we get

$$e^{-\int_0^t r(s) ds} \frac{1}{\sqrt{2\pi}} \int_{\frac{-\ln c^* - \int_0^t \left[r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds + \int_0^t \frac{1}{2} |\theta(s)|^2 ds}{\sqrt{\int_0^t |\theta(s)|^2 ds}}}^{\infty} e^{-\frac{y^2}{2}} dy. \tag{4.7}$$

$$e^{-\int_0^t r(s) ds} \frac{1}{\sqrt{2\pi}} \int_{\frac{-\ln c^* - \int_0^t \left[r(s) - \frac{1}{2} |\theta(s)|^2 \right] ds}{\sqrt{\int_0^t |\theta(s)|^2 ds}}}^{\infty} e^{-\frac{y^2}{2}} dy. \tag{4.8}$$

$$= e^{-\int_0^t r(s)ds}\mathbb{P}\left\{ Z \geq \frac{-\ln c^* - \int_0^t \left[r(s) - \frac{1}{2}\left|\theta(s)\right|^2\right] ds}{\sqrt{\int_0^t |\theta(s)|^2 ds}} \right\} \tag{4.9}$$

$$= e^{-\int_0^t r(s)ds}\mathbb{P}\left\{ Z \leq \frac{\ln c^* \int_0^t \left[r(s) - \frac{1}{2}\left|\theta(s)\right|^2\right] ds}{\sqrt{\int_0^t |\theta(s)|^2 ds}} \right\} \tag{4.10}$$

$$= e^{-\int_0^t r(s)ds}\Phi\left[\ln c^* + \frac{\int_0^t \left[r(s) - \frac{1}{2}\left|\theta(s)\right|^2\right] ds}{\sqrt{\int_0^t |\theta(s)|^2 ds}} \right]. \tag{4.11}$$

Thus on substituting this in the budget constraint (4.4) we immediately get that

$$\ln c^* = \Phi^{-1}\left[\frac{x_0}{be^{-\int_0^t r(s)ds}}\right]\sqrt{\int_0^t |\theta(s)|^2 ds} - \int_0^t \left[r(s) - \frac{1}{2}\left|\theta(s)\right|^2\right] ds. \tag{4.12}$$

Hence we have found an explicit relation between c^* and the Sharpe ratio, $\theta(\cdot)$. In order to move forward and find how Sharpe ratio affects c^* let us assume the simple case when the parameters are independent of time $\theta = \theta(\cdot)$ and $r = r(\cdot)$. Then let us define

$$j(\theta) := \ln c^* = \Phi^{-1}\left[\frac{x_0}{be^{-rt}}\right]\theta\sqrt{t} - \left[rt - \frac{1}{2}\theta^2 t\right]. \tag{4.13}$$

Since c^* and $\ln c^*$ have the same monotonicity we can study $j(\theta)$ instead. Now

$$\frac{dj(\theta)}{d(\theta)} = \Phi^{-1}\left[\frac{x_0}{be^{-rt}}\right]\sqrt{t} + \theta\sqrt{t}. \tag{4.14}$$

But then,

$$\begin{aligned} &\frac{dj(\theta)}{d\theta} > 0 \\ \text{if and only if} \quad & b < \frac{x_0 e^{rt}}{\Phi(-\theta\sqrt{t})}. \end{aligned} \tag{4.15}$$

This means that if the target is too high then an increase in the Sharpe ratio of the market decreases the value of c^*. In order to study the effect of Sharpe ratio on Yaari's model we need to study the effect of Sharpe ratio on $\mathbb{P}(\rho < c^*)$.

Now $\mathbb{P}(\rho < c^*)$ from equation (4.3), and the symmetry of the standard normal distribution about 0 is equal to

$$\mathbb{P}(\rho < c^*) = \mathbb{P}\left(Z \leq \left[\frac{\ln c^* + \int_0^t \left[r(s) + \frac{1}{2}\left|\theta(s)\right|^2\right] ds}{\sqrt{\int_0^t |\theta(s)|^2 ds}} \right] \right) \tag{4.16}$$

Table 4.1: Results of using Goal Reaching Model on a portfolio comprising of Dow Jones Industial Average with an initial investment of 100 and target 140

Results of Goal Reaching Model				
Year	μ	Sharpe Ratio	c^*	Probability
1998	0.1652	1.0062	3.1724	0.95729
1999	0.2184	1.9687	26.1928	0.99629
2000	0.0218	0.515	1.5438	0.88849
2001	-0.0442	1.545	9.2270	0.98841
2002	-0.0989	2.2629	60.0693	0.99847
2003	-0.0031	0.3951	1.343225859	0.86432

Let $\mu = -\int_0^t \left[r(s) + \frac{1}{2}|\theta(s)|^2\right] ds$ and $\delta = \sqrt{\int_0^t |\theta(s)|^2 ds}$, then

$$\mathbb{P}(\rho < c^*) = \Phi\left(\frac{\ln c^* - \mu}{\delta}\right) \tag{4.17}$$

Let,

$$f(\delta, c^*(\delta)) = \mathbb{P}(\rho < c^*) = \Phi\left(\frac{\ln c^* - \mu}{\delta}\right). \tag{4.18}$$

Using equation (4.13) we substitute $\ln c^*$ in the above to get

$$\Phi\left(\frac{\ln c^* - \mu}{\delta}\right) = \Phi\left(\Phi^{-1}\left[\frac{x_0}{be^{-\int_0^t r(s)ds}}\right] + \delta\right) \tag{4.19}$$

and this clearly increases as δ increases. Thus summarizing our work above in the following theorem:

Theorem 4 In the Goal Reaching model, given a particular target b or aspiration of the agent the probability of achieving the target in a particular market increases with increase in the Sharpe ratio of the market.

We calibrate this model to the Dow Jones Industrial Average as our portfolio. We take yearly targets and use historical volatility, calculated by the sliding window technique, which we assume to remain constant over our time horizon with a constant interest rate, $r = 6.18\%$. We take weekly closing prices. Note that we allow short positions, i.e., selling first and thus take the absolute values of the Sharpe ratio. Table 4.1 shows results of the portfolio.

4.2 Yaari's Model

Unlike the Goal Reaching problem, in this case we do not have an explicit relation of c^* with the Sharpe ratio. Thus we must rely on equation (3.8) and try to use the implicit function rule to find a relation

$$\varphi(x) := xw(F_\rho(x)) - w'(F_\rho(x)) \int_0^x s dF_\rho(s).$$

From the definition of ρ and our work in the previous section we see that it follows the lognormal distribution

$$\rho(t) \sim \text{Lognormal}\left(-\int_0^t \left[r(s) + \frac{1}{2} |\theta_0(s)|^2 \right] ds, \quad \sqrt{\int_0^t |\theta(s)|^2 ds} \right) \sim \text{Lognormal}(\mu, \delta), \tag{4.20}$$

and thus,

$$\int_0^x s dF_\rho(s) = e^{\mu + \frac{\delta^2}{2}} \Phi\left(\frac{\ln x - \mu}{\delta} - \delta \right). \tag{4.21}$$

But by our definition of μ and δ, $\mu + \frac{\delta^2}{2} = -\int_0^t r(s) ds$. Then (3.8) can be written as,

$$\varphi(x) := xw\left[\Phi\left(\frac{\ln x - \mu}{\delta} \right) \right] - w'\left[\Phi\left(\frac{\ln x - \mu}{\delta} \right) \right] e^{-\int_0^t r(s) ds} \Phi\left(\frac{\ln x - \mu}{\delta} - \delta \right). \tag{4.22}$$

We know that $c := c^*$ is the root of this equation. Then using the implicit rule, we can write (for want of a better way to express)

$$0 = \frac{\varphi(c(\delta), \delta)}{\delta} = \frac{\partial \varphi(c(\delta), \delta)}{\partial \delta} + \frac{\partial \varphi(c(\delta), \delta)}{\partial x} \frac{\partial c(\delta)}{\partial \delta}$$
$$\text{i.e.,} \qquad \frac{\partial c(\delta)}{\partial \delta} = \frac{\partial \varphi(c(\delta), \delta)}{\partial c} \Big/ \frac{\partial \varphi(c(\delta))}{\partial \delta}. \tag{4.23}$$

It can be easily checked that,

$$\frac{\partial \varphi(c)}{\partial c} = w[\Phi(z)] + \frac{1}{\delta} w'[\Phi(z)]\phi(z)$$
$$- \frac{e^{-\int_0^t r(s) ds}}{c\delta} \left[w''[\Phi(z)]\phi(z)\Phi(z - \delta) + w'[\Phi(z)]\phi(z - \delta) \right], \tag{4.24}$$

where ϕ is the probability distribution function of the standard normal distribution and

$$\frac{d\varphi(c)}{d\delta} = cw'[\Phi(z)]\phi(z)\left(1-\frac{z}{\delta}\right)$$
$$-e^{-\int_0^t r(s)ds}\left\{w''[\Phi(z)]\phi(z)\left(1-\frac{z}{\delta}\right)\Phi(z-\delta)+w'[\Phi(z)]\phi(z-\delta)\frac{z}{\delta}\right\}, \qquad (4.25)$$

where $z = \dfrac{\ln c - \mu}{\delta}$.

Taking $\alpha = \dfrac{w(\cdot)}{w'(\cdot)}$ and $\beta = \dfrac{w''(\cdot)}{w'(\cdot)}$.

$$\frac{d\varphi(c)}{dc} = \alpha[\Phi(z)] + \frac{\phi(z)}{\delta} - \frac{e^{-\int r(s)ds}}{c\delta}\left\{\beta[\Phi(z)]\phi(z)\Phi(z-\delta)+\phi(z-\delta)\right\}, \qquad (4.26)$$

and

$$\frac{d\varphi(c)}{d\delta} = c\phi(z)\left(1-\frac{z}{\delta}\right)$$
$$-e^{-\int_0^t r(s)ds}\left\{\beta[\Phi(z)]\phi(z)\Phi(z-\delta)\left(1-\frac{z}{\delta}\right)+\phi(z-\delta)\frac{z}{\delta}\right\}. \qquad (4.27)$$

Clearly, this does not reduce to a nice explicit solution as in the case of the Goal Reaching problem, but one must realize that in this case we have generealized our concept of probability and are working under a general probability distortion function. However in the next chapter we shall see a concrete example for a particular distortion function in which we shall be able to numerically find the effect of the Sharpe ratio.

Chapter 5

Yaari's Model with Power Function as Probability Distortion

As we were not able to find a nice relation analytically between the Sharpe ratio and the probability of achieving the target under a general probability distortion in this chapter we study an example of a particular probability distortion function, $w(z) = z^\gamma$, for some $\gamma > 1$ and so $w(\cdot)$ is convex and reflects the risk aversion of the investor according to Yaari's theory. He and Zhou also study this example in their paper and are able to find an interval in which the root lies. We shall take it forward and calibrate the model to market data and use numerical methods to compute the actual value of c^* and thus the probability of achieving targets in that market.

Under probability disortion $w(z) = z^\gamma$ function we see that our (3.8) or (4.22), becomes

$$\varphi(x) := x\left[\Phi\left(\frac{\ln x - \mu}{\delta}\right)\right]^\gamma - \gamma\left[\Phi\left(\frac{\ln x - \mu}{\delta}\right)\right]^{\gamma-1} e^{-\int r(s)ds}\Phi\left(\frac{\ln x - \mu}{\delta} - \delta\right). \tag{5.1}$$

Then He and Zhou showed that in this case c^* the root of the above equation is such that

$$c^* \in \left(\exp\left(\mu - \frac{\delta^2 t}{(\gamma-1)}\right), \gamma\exp\left(\mu + \frac{\delta^2 t}{(\gamma-1)}\right)\right),$$

where μ and δ are as defined in (4.20)

As in the Goal Reaching problem we calibrate this model to the Dow Jones Industrial Average as our portfolio. As an example we use $\gamma = 2$ and $\gamma = 3$. We take yearly targets and use historical volatility, assumed constant over our period with a constant interest rate, $r = 6.18\%$. We take weekly closing prices. Note that we allow

Table 5.1: Results of using Yaari's Model on a portfolio comprising of Dow Jones Industial Average with an initial investment of \$100 and $\gamma = 2$

Results of Yaari's Model					
Year	μ	Sharpe Ratio	b*	c*	Probability
1998	0.1652	1.0062	552.302	0.6527	0.5596
1999	0.2184	1.9687	Inf	3.79E-04	0.0014
2000	0.0218	0.515	180.5268	1.4378	0.8631
2001	-0.0442	1.545	4.40E+15	0.0011	0.00015
2002	-0.0989	2.2629	Inf	1.94E-04	0.0042
2003	-0.0031	0.3951	161.4454	1.5692	0.9331

Table 5.2: Results of using Yaari's Model on a portfolio comprising of Dow Jones Industial Average with an initial investment of \$100 and $\gamma = 3$

Results of Yaari's Model					
Year	μ	Sharpe Ratio	b*	c*	Probability
1998	0.1652	1.0062	190.90	1.7961	0.8747
1999	0.2184	1.9687	6.60E+14	0.0018	0.0139
2000	0.0218	0.515	137.48	2.4319	0.9817
2001	-0.0442	1.545	2.74E+09	0.0097	0.0139
2002	-0.0989	2.2629	4.79E+17 0	.0009	0.0262
2003	-0.0031	0.3951	130.62	2.5481	0.9966

short positions, i.e., selling first and thus take the absolute values of the Sharpe ratio. The results are given in Table 5.1 and Table 5.2.

5.1 Relation Between γ and c^*

In this section we explore deeper the effect of c^* on γ. In an economic sense this question can be posed as - how does an increasing probability of achieving a target affect the risk of the agent, in Yaari's framework?

We have that if c^* is the root of equation (5.1) then

$$\varphi(c^*) := c^* \left[\Phi\left(\frac{\ln c^* - \mu}{\delta}\right)\right]^{\gamma} - \gamma \left[\Phi\left(\frac{\ln c^* - \mu}{\delta}\right)\right]^{\gamma-1} e^{-\int_0^t r(s)ds} \Phi\left(\frac{\ln c^* - \mu}{\delta} - \delta\right) \tag{5.2}$$

$$\text{i.e.,} \quad \left[c^*[\Phi(z)] - \gamma e^{-\int_0^t r(s)ds}[\Phi(z-\delta)]\right] \Phi(z)^{\gamma-1} = 0, \tag{5.3}$$

where $z = \frac{\ln c^* - \mu}{\delta}$.

Table 5.3: Comparison of Probabiility of achieving various targets under the Goal Reaching problem and Yaari's models

Comparison of Probabiility of achieving targets under the two models			
b	Goal Reaching	Yaari's Model	γ under Yaari's Model
125	0.99508	0.9922	12.6662
150	0.98182	0.9764	7.4589
175	0.96535	0.9575	5.7255
200	0.94836	0.9383	4.9047
225	0.9311	0.9207	4.4466
250	0.91353	0.9009	4.1659

Now $\Phi(z) = 0$ would imply that $z \to -\infty$ which would mean $c^* \to 0$. Neglecting this case we get that

$$c^*[\Phi(z)] - \gamma e^{-\int_0^t r(s)ds}[\Phi(z-\delta)] = 0, \tag{5.4}$$

which would immediately give

$$\gamma = c^* e^{\int_0^t r(s)ds} \frac{\Phi(z)}{\Phi(z-\delta)}. \tag{5.5}$$

Note that this means given a target b we can immediately find c^* using the equation (4.12), and thus γ. This actually stems from Theorem 3 which says that we have two equations - Theorem 3(1) and the initial budget constraint and three variables b, c and γ. Thus, now we can even compare the probabilities for a particular target under the Goal Reaching Problem and Yaari's models. Of course Yaari's model would give us an accompanying γ.

We plot in Figure 5.1 the variation of γ as c^* changes keeping theta constant as 1.545. We get similar plots for various theta and γ is always greater than 1. Thus let us try to find analytically if it is true that the value of this γ is always greater than 1 whatever the value of c^*.

From our definition of z we see that $c^* = e^{\delta z + \mu} = e^{\delta z - \int_0^t r(s)ds - \frac{\delta^2}{2}}$, then

$$\gamma(z) = e^{\delta z - \frac{\delta^2}{2}} \frac{\Phi(z)}{\Phi(z-\delta)}. \tag{5.6}$$

Since $\lim_{z \to -infty} \gamma$ is of the form $\frac{0}{0}$ using L'Hopital's rule we see that,

$$\lim_{z \to -\infty} \gamma(z) = \lim_{z \to -\infty} e^{\delta^2/2} \left[\frac{\delta e^{\delta z}\Phi(z) + e^{\delta z}\phi(z)}{\phi(z-\delta)} \right] \tag{5.7}$$

$$= \lim_{z\to-\infty}\left[\frac{\delta e^{\delta z-\delta^2/2}\Phi(z)}{e^{-[z-\delta]^2/2}/\sqrt{2\pi}} + \frac{e^{-\delta z-\delta^2/2-z^2/2}}{e^{-[z-\delta]^2/2}}\right] \tag{5.8}$$

$$= \lim_{z\to-\infty}\left[\delta\sqrt{2\pi}e^{z^2/2}\Phi(z)+1\right] > 1. \tag{5.9}$$

Thus $\lim_{z\to-\infty}$ exists and is greater than 1. Thus we can define

$$g(z) = e^{\delta z-\frac{\delta^2}{2}}\Phi(z) - \Phi(z-\delta), \tag{5.10}$$

and say that $\gamma(z) > 1$ if and only if $g(z) > 0$. Note that $\lim_{z\to-\infty} g(z) = 0$. Also,

$$g'(z) = \delta e^{\delta z-\frac{\delta^2}{2}}\Phi(z),$$

which is always positive and thus $g(z) > 0$ for all $z \in (-\infty, \infty)$ and therefore $\gamma > 1$ for any value of c^*. Hence,if we fix b then we can immediately get the corresponding c^* from equation (4.12) and then γ from equation (5.5) is always greater than 1. This means we can embed the Goal Reaching problem within the Yaari's model with the class of power distortions and thus under the frameowrk of Yaari's model the Goal Reaching problem is risk averse. The risk aversion comes from the fact that $\gamma > 1$. For a detailed study we refer the reader to Yaari[25].

To sum up we conclude the chapter with a theorem incorporating our results.

Theorem 5 A goal-reaching agent is risk-averse in Yaari's model with the class of power distortions.

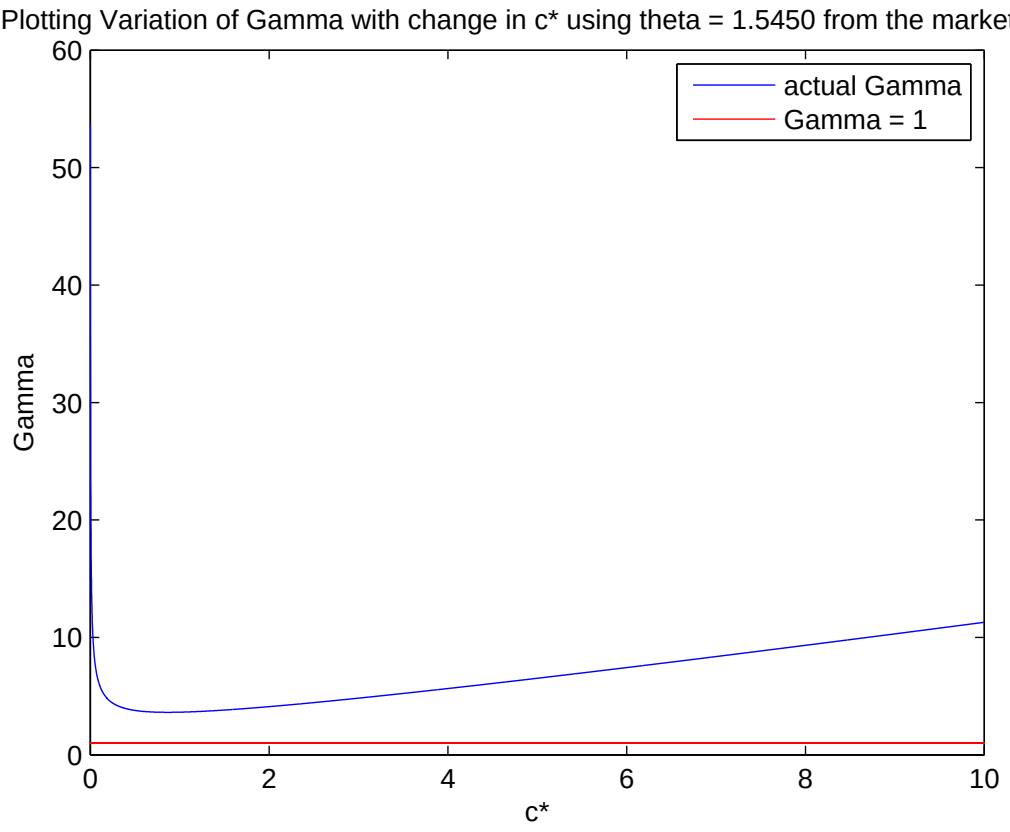


Figure 5.1: Effect of variation of c^* on γ

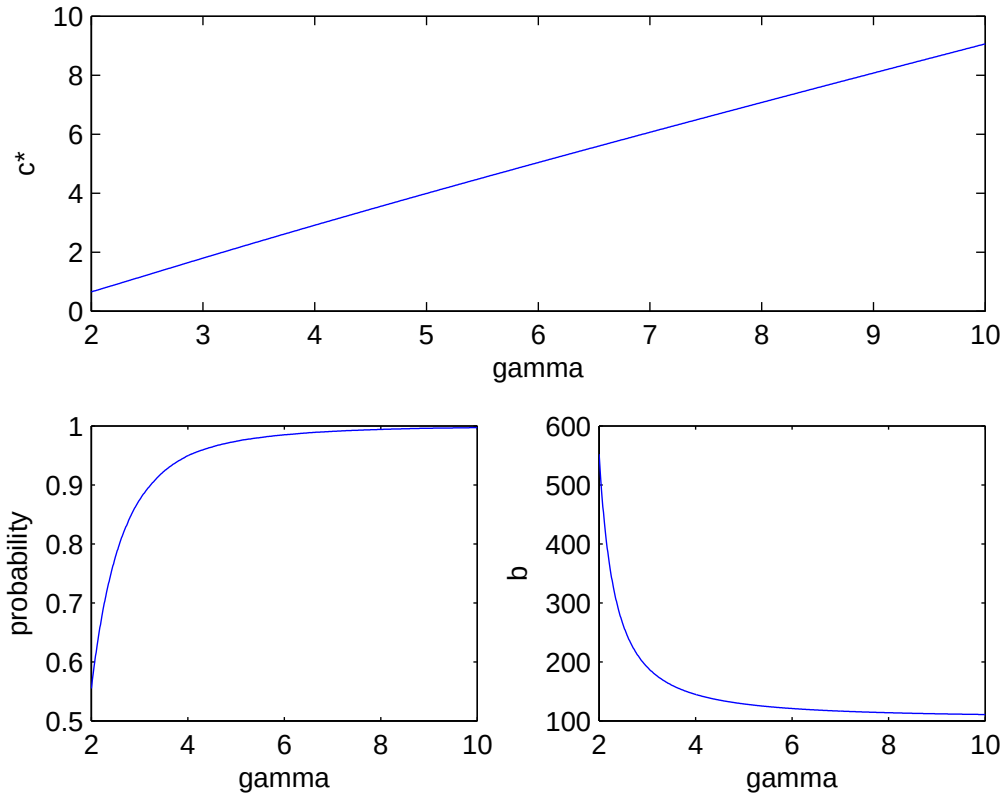


Figure 5.2: Effect of γ

5.2 Plots showing Effect of variation of γ and θ in Yaari's Model

Figure 5.2 and 5.3 show how b, c^* and the probability of achieving the target vary under the influence of γ and θ, respectively.

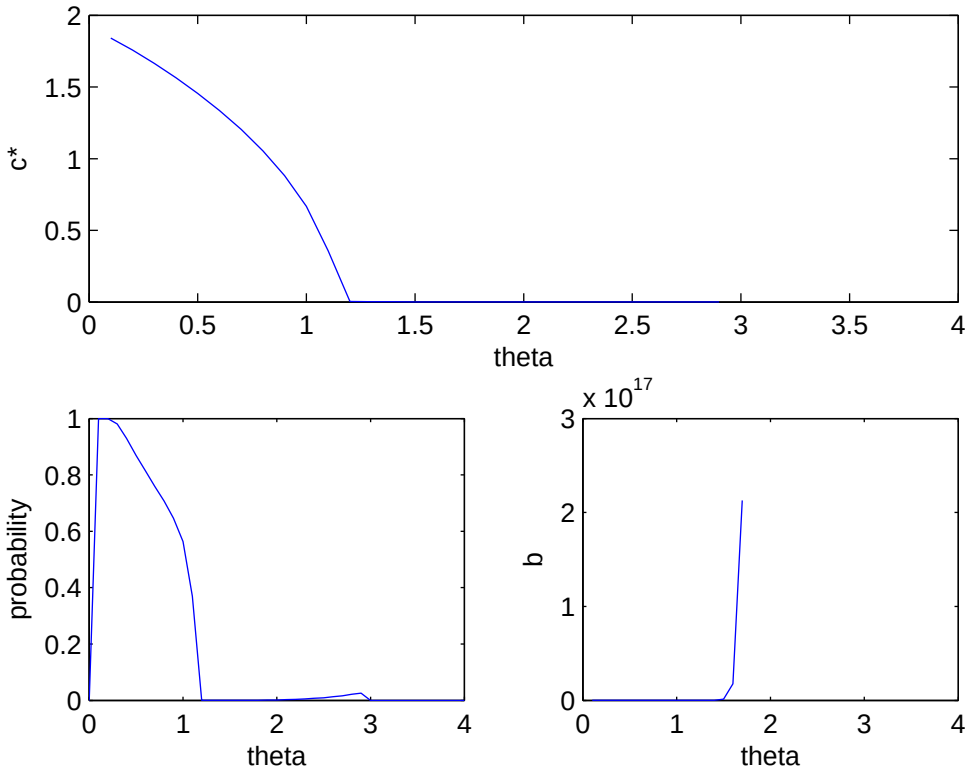


Figure 5.3: Effect of θ

Chapter 6

Conclusion

We started off with He and Zhou's paper [3] as our platform and discussed the generalized Quantile formulation of portfolio choice problems and their solutions as proposed by them. We then studied the explicit solution to the Goal Reaching problem and the Yaari's model and saw that they have very similar looking solutions, yet very different in meaning. However this motivated our study to look whether they had certain deeper relations.

In the second half of our work we started off by first calibrating the two models with the Dow Jones Industrial Average and saw the probability of achieving the targets. This was the first time when we actually saw the difference between the two models in terms of results. While in Goal Reaching problem, given a target, the model returned us the probability of achieving it, whereas in the case of Yaari, the model gave us both the target and the probability of achieving it once we chose our probability distortion function. This inherently comes from the fact that Yaari's model allows us to control the risk behaviour of the portfolio, giving us two equations Theorem 3(1) and three variables b, c^*, and γ.

We also looked at the effect that Sharpe ratio had on a portfolio under the two models, and proved analytically in the Goal Reaching problem that the probability of achieving a target increases with increasing Sharpe ratio. In the Yaari's model however we were not able to find any nice explicit formula or relation, between the two.

We then studied the example of the Power function as Probability distortion under the Yaari's model. As mentioned above we calibrated our model to the Dow Jones Industrial Average and compared the results for different gammas in the same year.

We then found the analytic relation between γ and c^*, and motivated by patterns in numerical results showed analytically that the γ has to always be greater than 1 and thus proving that the Goal Reaching problem can be embedded within the Yaari's Model under the class of power distortion functions and hence it is risk averse under Yaari's framework.

We finally ended by showing the effect of varying γ and θ on b, c^* and the probability of achieving a target.

An interesting topic for further research can be looking at more general class of probability distortion functions for which we can embed the Goal Reaching problem within the Yaari's model.

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