QUASI-STANDARD C*-ALGEBRAS AND NORMS OF INNER DERIVATIONS

BODLEIAN LIBRARY OXFORD
DEPOSITED THESIS

Douglas W. B. Somerset

Queen's College
Oxford

A thesis submitted for the degree of Doctor of Philosophy
of the University of Oxford

Trinity Term 1989

QUASI-STANDARD C*-ALGEBRAS AND NORMS OF INNER DERIVATIONS

Douglas W. B. Somerset

Queen's College
Oxford

A thesis submitted for the degree of Doctor of Philosophy
of the University of Oxford

Trinity Term 1989

ABSTRACT

In the first half of the thesis a necessary and sufficient condition is given for a separable C*-algebra to be *-isomorphic to a maximal full algebra of cross-sections over a base-space such that the fibre algebras are primitive throughout a dense subset. The condition is that the relation of inseparability for pairs of points in the primitive ideal space should be an open equivalence relation.

In the second half of the thesis a characterisation is given of those C*-algebras A for which each self-adjoint inner derivation $D(a, \mathrm{A})$ satisfies

$$\|D(a,\mathrm{A})\| = 2 \inf \{\|a - z\| : z \in \mathrm{Z}(\mathrm{A}), \text{ the centre of A}\}.$$

This time the characterisation is that A should be quasicentral and the relation of inseparability for pairs of points in the primitive ideal space should be an equivalence relation. Those C*-algebras for which every inner derivation satisfies the equation are characterised in a similar way.

ACKNOWLEDGEMENTS

Much of the work in Chapters 2 and 3 of this thesis is in a paper entitled 'Quasi-standard C*-algebras' to appear in the Mathematical Proceedings of the Cambridge Philosophical Society, and written jointly with Dr R.J. Archbold of the University of Aberdeen, and a lot of the credit for this work is due to him. In particular, he provided the final versions of the proofs, most of which are simpler or more elegant or more accurate than my earlier versions. I wish to thank him for his help and encouragement.

It is a pleasure to thank my supervisor Dr Charles Batty for the enormous help that he has given me in the preparation of this thesis and over the last three years. His reluctance to believe that anything is true until it has been proved has been a great stimulus.

I also wish to thank Michael Vynnycky, who helped me with the word-processing on the condition that his name appeared on this page. Finally, I am grateful to SERC for their financial support.

CONTENTS

INTRODUCTION

The contents of this thesis fall naturally into two main sections, as intimated in the title. Since these are not directly connected I will give a separate introduction to each section.

The indirect connection between the two sections is that in both we consider the topological decomposition of a C*-algebra over its centre. This is a standard technique for dealing with C*-algebras with non-trivial centre, see for example [Kaplansky 1951], [Misonou 1952], [Glimm 1960], and it has recently become more useful with the introduction of primal ideals into C*-algebra theory [Archbold and Batty 1986].

In addition to the two main sections, Chapter 4 contains a miscellany of results on AW*-algebras, also involving topological decompositions over the centre. Brief introductions to these results are given in Chapter 4.

1. QUASI-STANDARD C*-ALGEBRAS

This section, Chapters 2 and 3, is devoted to the study of the topological decomposition itself. Some elementary properties are covered in Chapter 2 while the main new results are in Chapter 3.

Broadly speaking, the idea is that the theory of C*-algebras can be divided into the study of primitive C*-algebras (and, in particular, simple C*-algebras) and the study of how a general C*-algebra is related to its primitive quotients. The second of these problems may be regarded as a search for non-commutative analogues of the Gelfand-Naimark theorem for commutative C*-algebras.

One standard approach, going back to [Godement 1949] and [Kaplansky 1951] is to try to represent (or decompose) a C*-algebra as an algebra of cross-sections (or operator fields in Fell's terminology) using varying fibres over a locally compact, Hausdorff space, called *the base-space* (see [Fell 1961: pp 235-242] for precise details).

There are four desirable properties which such representation may have:

(i) the norm of each cross-section should be continuous

(ii) the algebra of cross-sections should be maximal (this corresponds to the surjectivity of the Gelfand transform in the commutative case)

(iii) the fibre algebras should be primitive throughout a dense subset of the base-space

(iv) the base-space should be homeomorphic to the maximal ideal space of the centre of the algebra (at least when the algebra is unital).

If A is a C*-algebra and *Prim A*, the set of primitive ideals of A, is Hausdorff in the Jacobson topology, then there is a natural representation of A with base-space Prim A having all four properties. In general, however, the four properties are conflicting and it is not always possible to find a representation with all of them. Property (iv) determines a unique base-space,not always locally compact, which we call *Glimm A*, and which we study in Chapter 2. Roughly speaking, to obtain property (iii) one might need a larger base-space than Glimm A, while to obtain properties (i) and (ii) simultaneously one might need a smaller base-space.

In the best circumstances, however, the natural representation on Glimm A has all four properties and we study this situation in Chapter 3. We introduce a class of C*-algebras called quasi-standard C*-algebras, the defining condition being that the relation of inseparability for pairs of points in Prim A should be an open equivalence relation. In Theorem 3.4 we give some characterisations of quasi-standardness in terms of primal ideals, showing, in particular, that this is exactly the class of C*-algebras which arose in [Archbold 1987; Theorem 5.3], see below. This leads to the main result of the section, Corollary 3.6, which states that a separable C*-algebra is quasi-standard if and only if the natural topological representation over Glimm A has properties (i) to (iv). We then give some interesting examples of quasi-standard C*-algebras.

We now mention some connections between quasi-standard C*-algebras and the earlier work on the subject:

a) In the terminology of Chapter 1, [Fell 1961] took as a base-space the τ_s-closure of Prim A. The representation thus obtained has properties (i) and (iii), but has (ii) (or (iv)) if and only if Prim A is Hausdorff [Fell 1961; Theorem 2.3]. Let *Min-Primal A* denote the topologised set of minimal primal ideals of A, see Chapter 1. Then Min-Primal A is contained in the τ_s-closure of Prim A [Archbold 1987; Corollary 4.3], and furthermore if A is separable then Min-Primal A has a dense subset consisting of primitive ideals [Archbold 1987; Corollary 4.6]. Archbold therefore took the τ_s-closure of Min-Primal A as a base-space. The representation thus obtained has property (i), and property (iii) when A is separable, and property (ii) (or (iv)) if and only if A is quasi-standard [Archbold 1987; Theorem 5.3] and Theorem 3.4.

b) [Tomiyama 1962] and [Lee 1976] investigated representations satisfying (i) and (ii). We collect some of their results in Theorem 2.1. Theorem 3.5 shows that if such a representation also satisfies (iii) then the algebra is quasi-standard.

c) [Dauns and Hofmann 1968] studied the representation with property (iv). In general the cross-sections for this representation are only upper semicontinuous, and (iii) need not hold either. A modified version of (ii) holds if the algebra is unital [Dauns and Hofmann 1968; Corollary 8.13] or, more generally, quasicentral [Dauns 1974], but it has not been proved for all C*-algebras.

d) [Ching 1976] introduced the classes of *standard* and *pre-standard* C*-algebras (from which quasi-standard C*-algebras take their name), and showed that algebras in these classes have good topological representations. It is shown in [Archbold and

Somerset] that every pre-standard C*-algebra is quasi-standard, but not conversely, and some of the details of this are given in Chapter 3.

In summary, the class of quasi-standard C*-algebras forms a well-behaved extension of the class of C*-algebras with Hausdorff primitive ideal space.

2. NORMS OF INNER DERIVATIONS OF C*-ALGEBRAS

In this section, Chapters 5, 6 and 7, the topological decomposition over the centre is used, together with primal ideals, to obtain some new results on an old subject.

Let A be a C*-algebra with centre Z(A). If $a \in$ A, the bounded linear mapping $x \mapsto ax - xa$, $x \in$ A, is called the inner derivation of A induced by a, and we denote it by $D(a,\mathrm{A})$. It follows easily from the triangle inequality that

$$\|D(a, \mathrm{A})\| \leq 2d(a, \mathrm{Z}(\mathrm{A})) \qquad (1)$$

where $d(a,\mathrm{Z}(\mathrm{A}))$ denotes the distance from a to Z(A) in the normed space A.

Considerable attention has been paid to the question of when equality holds in (1). It was shown in [Stampfli 1970] that equality holds when A is a primitive C*-algebra with identity, and this fact was used in [Zsido 1973] to show that equality holds if A is a von Neumann algebra. Special cases of this latter result had been proved in [Kadison, Lance and Ringrose 1967], [Gajendragadkar 1972], and [Hall 1972]. More generally, Halpern showed that equality holds in (1) if A is an AW*-algebra, and a proof of this was published in [Elliott 1978], while [Apostol and Zsido 1973; Remark after Lemma 4.2] showed that the same is true if A is a quotient of a von Neumann algebra. On the other hand, there is an example in [Kadison, Lance and Ringrose 1967] showing that the inequality in (1) can be strict.

In a different direction, [Johnson 1971] and [Kyle 1977] have examined the same inequality in the context of the algebra of bounded operators on arbitrary Banach space.

To examine the possible behaviour in more detail, [Archbold 1978] introduced constants K(A) and $K_s(A)$, defined to be the smallest numbers in $[0,\infty]$ such that

$$d(a, Z(A)) \leq K(A)\|D(a, A)\| \text{ for all } a \in A$$

and

$$d(a, Z(A)) \leq K_s(A)\|D(a, A)\| \text{ for all } a \in A_{sa}.$$

It is easy to check that $\frac{1}{2}K(A) \leq K_s(A) \leq K(A)$. If A is commutative then K(A)=0; otherwise it follows from (1) that $K(A) \geq \frac{1}{2}$. Work in [Kadison, Lance and Ringrose 1967] implies that $K(A) < \infty$ if and only if the set of inner derivations is closed, with respect to the operator norm, in the set of all derivations of A.

Archbold first studied the stability of K(A). If J is an ideal in A he showed that $K(J) \leq 2K(A)$, and he gave an example where $K(A)=\frac{1}{2}$ and $K(A/J)=\infty$. He then studied $K(A\otimes B)$ in terms of K(A) and K(B), see also [Batty 1978]. Next he showed that if A is a weakly central C*-algebra then $K_s(A) \leq 1$, and he gave an example of a unital, weakly central, AF-algebra with $K(A)=K_s(A)=1$ He then showed that $K(A)=\frac{1}{2}$ if A is n-homogeneous ($n \geq 2$). Finally he studied $K_s(A)$ when A is an AF-algebra.

In this section of the thesis we continue the study of K(A) and $K_s(A)$. We will now briefly describe the main results.

In Chapter 5 we use Stampfli's theorem to characterise those C*-algebras which have $K_s(A)=\frac{1}{2}$ and those which have $K(A)=\frac{1}{2}$. The characterisations are in terms of the primal ideal structure. Specifically, we show in Theorem 5.8 that $K_s(A)=\frac{1}{2}$ if and only if A is a non-commutative, quasicentral C*-algebra such that whenever *two* primitive ideals of A contain the same Glimm ideal their intersection is primal. This condition is equivalent, by work in Chapter 3, to requiring that the relation of inseparability should be an equivalence relation on Prim A. In Theorems 5.10 and 5.12 we show that $K(A)=\frac{1}{2}$ if and only if A is a non-commutative, quasicentral C*-algebra such that whenever *three* primitive ideals of A contain the same Glimm ideal their intersection is primal. We also show that if A is non-commutative and $K(A) \neq \frac{1}{2}$ then either $K_s(A) = \frac{1}{2}$ and $K(A) = \frac{1}{\sqrt{3}}$ or $K(A) \geq K_s(A) \geq 1$. One corollary of these results is that a non-commutative, unital C*-algebra A is prime if and only if the centre of A is trivial and $K(A) = \frac{1}{2}$. Another corollary is that $K(A) = \frac{1}{2}$ if A is a quotient of an AW*-algebra.

In Chapter 6 we study unital C*-algebras with trivial centre. We show that if such an algebra A has only finitely many minimal primal ideals then $K_s(A)$ is completely determined by the ideal structure of A, and is always a multiple of a half. We also obtain some partial results on K(A) under the same hypotheses. Finally, we apply these results to C*-algebras with non-trivial centre, showing that if A is an arbitrary weakly central C*-algebra then $K(A) \leq 1$.

In Chapter 7 we consider the class of *open* C*-algebras, which are those such that the topological decomposition over the base-space Glimm A has the properties (i) and (ii), mentioned earlier in the introduction. We show that if A is a separable, open C*-algebra with $K(A) < \infty$ then K(A) and $K_s(A)$ are completely determined by the values of $K(A/G)$ and $K_s(A/G)$ on a dense subset of the base-space Glimm A.

1. NOTATION AND PRELIMINARIES

In this chapter we introduce notation and various definitions and results which are needed later. Let A be a C*-algebra.

1.1 We will use the following notation:

A^+	the set of positive elements of A
A_{sa}	the set of self-adjoint elements of A
Z(A)	the centre of A
A^*	the dual space of A
S(A)	the set of states of A with the w*-topology
P(A)	the set of pure states of A with the w*-topology
Prim A	the set of primitive ideals of A with the Jacobson topology

1.2 If I is an ideal (assumed to be closed and two-sided unless otherwise stated) in A then Φ_I :A→A/I denotes the canonical *-homomorphism. If $a \in$ A then we will usually write a_I or $a + I$ for $\Phi_I(a)$, whichever is most convenient.

If X is a collection of ideals of A then ker X will denote the ideal $\bigcap\{I : I \in X\}$.

1.3 The sets $\{P \in$ Prim A: $\|a + P\| > \epsilon\}$, $a \in$ A, $\epsilon \geq 0$, are open and form a base for the Jacobson topology on Prim A. For $\epsilon > 0$ the sets $\{P \in$ Prim A: $\|a + P\| \geq \epsilon\}$ are compact (but not necessarily closed) [Pedersen 1979; 4.4.4].

1.4 An ideal J is said to be *primal* [Archbold and Batty 1986] if whenever $n \geq 2$ and $I_1, I_2, ...I_n$ are ideals of A with zero product then $J \supseteq I_i$ for at least one value of i. Clearly every prime ideal is primal, as is every ideal which contains a primal ideal. The set of primal ideals of A is denoted by *Primal A*.

The following characterisation of primality [Archbold and Batty 1986; Proposition

3.2] is very useful: J is primal if and only if there is a net (P_α) in Prim A which converges to every point in Prim A/J (regarded as a closed subset of Prim A [Pedersen 1979; 4.1.10]).

A Zorn's Lemma argument shows that each primal ideal contains a minimal primal ideal. The set of minimal primal ideals of A is denoted by *Min-Primal A*.

1.5 The set *Id A* of all ideals of A can be equipped with the strong and weak topologies, τ_s and τ_w. The precise definitions and origins of these topologies are given in [Archbold 1987; Section 2]. The main features are that a net (I_α) is τ_s-convergent to I in Id A if and only if $\|a + I_\alpha\| \to \|a + I\|$ for all $a \in$ A, whilst a base for τ_w is given by the family of sets of the form

$$U(F) = \{I \in \text{Id A} : \text{ for all } J \in F \; I \not\supseteq J\}$$

where F is a finite (possibly empty) set of ideals of A. When restricted to Prim A, τ_w coincides with the Jacobson topology. The restrictions of τ_s and τ_w to Primal A are usually distinct (τ_s is Hausdorff while τ_w need not be), but the restrictions to Min-Primal A coincide [Archbold 1987; Corollary 4.3]. We denote by τ this common topology on Min-Primal A.

1.6 Let X be a topological space. Let C^b(X) (respectively C_0(X)) denote the C*-algebra of continuous, bounded, complex functions (respectively, continuous complex functions vanishing at infinity) on X, with the supremum norm.

If $\sim$ is any equivalence relation on X let X/$\sim$ denote the set of equivalence classes. If q denotes the quotient map from X to X/$\sim$ then the *quotient topology* on X/$\sim$ is the strongest topology on X/$\sim$ for which q is continuous.

If X is a metric space, $x, y \in$ X, and Y$\subseteq$X then $d(x,y)$ and $d(x$,Y) denote the distances of x from y and Y.

1.7 The Dauns-Hofmann theorem [Pedersen 1979; 4.4.8] states that if R is the centre of the multiplier algebra of A then there is a *-isomorphism γ: R$\rightarrow$ C^b(Prim A) given by $\gamma(z)(P) = \|z+P\|$ for $z \in$ R and $P \in$ Prim A (regarded as an open subset of Prim A+R [Pedersen 1979; 4.1.10]).

1.8 A C*-algebra A is said to *quasicentral* [Delaroche 1968] if no primitive ideal of A contains the centre of A. Clearly every unital C*-algebra is quasicentral. The following result is an amalgam of [Delaroche 1967; Proposition 1], [Archbold 1975; Proposition 1] and unpublished work of Laursen.

THEOREM 1.1 Let A be a C*-algebra. The following are equivalent:

(i) A is quasicentral

(ii) the map γ of 1.7 gives a *-isomorphism of Z(A) onto C_0(Prim A)

(iii) A has an approximate unit contained in Z(A)

(iv) A = AZ(A)

Proof: (i)$\Rightarrow$(ii) This is a special case of [Delaroche 1967; Proposition 1], see below.

(ii)$\Rightarrow$(iii) This is a trivial variation on the proof of the first half of [Archbold 1975; Proposition 1].

(iii)$\Rightarrow$(iv) The smallest ideal of A containing Z(A) must contain AZ(A). However AZ(A) is a closed subspace by the Cohen factorisation theorem, see for example [Doran and Belfi 1986; B7.1], and it is clearly closed under multiplication by elements of A. Hence AZ(A) is the smallest ideal containing Z(A). But (iii) implies that A is the smallest closed ideal containing Z(A). Hence A = AZ(A).

(iv)$\Rightarrow$(i) If (iv) holds then clearly no proper ideal of A can contain Z(A).

Each C*-algebra A has a unique largest quasicentral ideal B, which is equal to AZ(A) by the proof of Theorem 1.1. The map γ of 1.7 maps Z(A) *-isomorphically onto C_0(Prim B) (where Prim B is regarded as an open subset of Prim A) [Delaroche 1967;

Proposition 1].

1.9 A C^*-algebra A, possibly without an identity, is said to be *weakly central* [Misonou 1952], [Rickart 1960; Definition 2.7.6], if no maximal, modular ideal contains the centre of A and if for any two maximal, modular ideals M and N, $M = N \iff M \cap Z(A) = N \cap Z(A)$. (An ideal I is said to be modular if A/I has an identity). If A does not have an identity then it is easy to check that A is weakly central if and only if $A + C1$ is weakly central.

It was shown in [Vesterstrøm 1971] that a unital C^*-algebra is weakly central if and only if whenever π is a *-homomorphism of A $Z(\pi(A)) = \pi(Z(A))$. The proof adapts easily to show that the same characterisation holds for non-unital C^*-algebras.

Let *Max A* denote the space of maximal modular ideals of A, equipped with the Jacobson topology. Then it is easy to show that if A is weakly central the map $M \mapsto M \cap Z(A)$ gives a homeomorphism from Max A onto Prim Z(A). Hence Z(A) is *-isomorphic to $C_0(\text{Max A})$.

Let I be an ideal in a weakly central C^*-algebra A. Then the map $\delta : M \mapsto M \cap Z(A)$ gives a homeomorphism from the open set $\{M \in \text{Max A} : M \not\supseteq I\}$ onto Max I [Rickart 1960; 2.6.6]. Let $P, Q \in \text{Max}\, I$. Let $f \in C_0(\text{Max A})$ with $f(\delta^{-1}(P)) = 1$, $f(\delta^{-1}(Q)) = 0$, and with f vanishing on the set $\{M \in \text{Max A} : M \supseteq I\}$. By the paragraph above there is a $z \in Z(A)$ such that $f(M) = \|z + M\|$ for each $M \in$ Max A. If $R \in$ Prim A and $R \supseteq I$ then either R is non-modular, in which case $R \supseteq Z(A)$, or R is contained in a maximal modular ideal S, with $R \cap Z(A) = S \cap Z(A)$. In this case $f(S) = 0$, so $z \in S \cap Z(A)$. In either case $z \in R$. Hence $z \in I$. It follows that I is weakly central.

1.10 If H is a Hilbert space then L(H) will denote the C^*-algebra of all bounded operators on H, and LC(H) will denote the C^*-algebra of all compact operators on H.

1.11 The reader is assumed to be acquainted with the definition of C^*-algebras of cross-sections (also called C^*-algebras of operator fields, continuous fields of C^*-algebras and C^*-bundles). The main references for this area are [Fell 1961], [Tomiyama 1962], [Dauns and Hofmann 1968], [Dixmier 1977; Chapter 10], [Dupré and Gillette 1983], [Fell and Doran 1988] and [Rieffel 1989], see also [Baker 1983].

2. THE COMPLETE REGULARISATION OF PRIM A

In this chapter we establish various elementary properties of the complete regularisation of Prim A. This has previously been considered, in various guises, by [Misonou 1952], [Glimm 1960], [Dauns and Hofmann 1968], [Dauns 1969], [Dauns and Hofmann 1969], [Vesterstrøm 1973], [Dauns 1974] and [Becker 1984], see also [Tomiyama 1962] and [Lee 1976].

2.1 For $P, Q \in$Prim A let $P \approx Q$ if and only if $f(P) = f(Q)$ for all $f \in C^b$(Prim A). Then $\approx$ is an equivalence relation on Prim A and the equivalence classes are closed subsets of Prim A. It follows that there is a one-to-one correspondence between Prim A$/\approx$ and a set of ideals of A given by

$$[P] \iff \cap[P] \qquad\qquad (P \in \text{Prim A})$$

where $[P]$ denotes the equivalence class of P. The set of ideals obtained in this way is denoted by Glimm A, and we identify Glimm A with Prim A$/\approx$ via the above correspondence. The quotient map ϕ_A:Prim A$\to$ Glimm A is known as the *complete regularisation map*. Note that if $P, Q \in$ Prim A, $G \in$ Glimm A and $P \supseteq G = \cap[Q]$ then, since $[Q]$ is closed, $P \in [Q]$ and so $\phi_A(P) = \phi_A(Q) = G$. It follows that if $P \in$ Prim A and $P \supseteq G \in$ Glimm A then $\phi_A(P) = G$.

2.2 There are two natural topologies on Glimm A (in addition to the τ_s and τ_w topologies, which we will consider later on). These are the quotient topology τ_q, which is Hausdorff, and the completely regular topology τ_{cr} which is the weakest topology for which the functions on Glimm A induced by C^b(Prim A) are all continuous. For a general topological space X it can happen that τ_{cr} differs from the finer τ_q topology on X$/\approx$ [Gillman and Jerison 1960; 3J.3], but it is not clear whether this can occur for

X=Prim A. A number of conditions are known which ensure that $\tau_{cr} = \tau_q$ on Glimm A. For example if A is unital then Prim A is compact, so Glimm A is τ_q -compact. Since the weaker topology τ_{cr} is Hausdorff, it must equal τ_q. More generally, if A is quasicentral then for each $P \in$ Prim A there is a positive $f \in C_0$(Prim A) such that $f(P) = 1$, see 1.8. The set X=$\{Q \in$ Prim A: $f(Q) \geq \frac{1}{2}\}$ is compact and contains the $\approx$-saturated, open neighbourhood $\{Q \in$ Prim A: $f(Q) > \frac{1}{2}\}$ of P, so $\phi_A(X)$ is a compact, τ_q-neighbourhood of $\phi_A(P)$ in Glimm A. Thus (Glimm A, τ_q) is a locally compact, Hausdorff space, so τ_q is equal to the weak topology defined on Glimm A by C_0(Glimm A, τ_q). It follows that $\tau_{cr} \geq \tau_q$, so $\tau_{cr} = \tau_q$. The topologies τ_{cr} and τ_q also coincide if ϕ_A is either τ_{cr}-open or τ_q-open (and so we may speak unambiguously of ϕ_A being open). This is immediate if ϕ_A is τ_{cr}-open, while if ϕ_A is τ_q-open then (Glimm A, τ_q) is locally compact, so the argument just given when A is quasicentral shows that $\tau_{cr} = \tau_q$.

2.3 Now suppose that A is quasicentral with centre Z(A). Suppose $P, Q \in$ Prim A and there exists $g \in C^b$(Prim A) such that $g(P) \neq g(Q)$. Let $f \in C_0$(Prim A) with $f(P) \neq 0$, 1.8. Then either $f(P) \neq f(Q)$ or $f(P) = f(Q)$ in which case $f(P)g(P) \neq f(Q)g(Q)$. Since $f.g \in C_0$(Prim A) it follows that, for $P, Q \in$ Prim A, $P \approx Q$ if and only if $f(P) = f(Q)$ for all $f \in C_0$(Prim A) if and only if $P \cap$Z(A)=$Q \cap$Z(A), by the Dauns-Hofmann theorem 1.7 and 1.8. Hence each Glimm ideal of A is quasicentral, by Theorem 1.1, and there is a bijection Ψ: Glimm A$\rightarrow$Prim Z(A) such that $\Psi(G) = G \cap$Z(A), $G \in$ Glimm A, and $\Psi^{-1}(J)$ =AJ, $J \in$ Prim Z(A). Thus our use of the terminology Glimm A arises from [Glimm 1960; Section 4]. Since the mapping $\rho : P \mapsto P \cap$Z(A) of Prim A onto Prim Z(A) is continuous for the Jacobson topologies, it follows that Ψ is continuous for the topology $\tau_q = \tau_{cr}$. If U is a basic open subset of Glimm A of the form U=$\{G \in$ Glimm A: $\eta > f(G) > \epsilon\}, f \in C_0$(Glimm A), $\eta, \epsilon \in \mathbf{R}$, then Ψ(U)=$\rho(\phi_A^{-1}$(U)) which is open in Prim Z(A). It follows that if A is quasicentral then Glimm A is homeomorphic to

Prim Z(A). This is a special case of [Dauns 1969; Proposition 4.19], and it immediately yields a short proof of a strengthened version of the main result of [Dauns 1974]: if A is quasicentral and K is a compact subset of Glimm A then there exists a $z \in$ Z(A) and a $\lambda > 0$ such that

$$K \subseteq \{G \in \text{Glimm A} : \|z + G\| \geq \lambda\}.$$

2.4 Let I be an ideal in a C*-algebra A. Then Prim I is homeomorphic to $\{P \in \text{Prim A}: P \not\supseteq I\}$ [Pedersen 1979; 4.1.10]. Since a function in $C^b(\text{Prim } I)$ need not have an extension to a function in $C^b(\text{Prim A})$ the equivalence relation $\approx$ on Prim I might be finer than the restriction of the equivalence relation $\approx$ on Prim A to $\{P \in \text{Prim A}: P \not\supseteq I\}$. However if I is quasicentral then the relation $\approx$ on Prim I is determined by Z(I), by 2.3, so it is equal to the restriction of $\approx$ on Prim A to $\{P \in \text{Prim A}: P \not\supseteq I\}$.

If I is the largest quasicentral ideal in a C*-algebra A and $P \in$ Prim A then $P \supseteq I \iff P \supseteq \text{Z(A)} \iff \phi_A(P) \supseteq \text{Z(A)}$. Hence $\phi_A(\{P \in \text{Prim A} : P \not\supseteq I\}) = \{G \in \text{Glimm A} : G \not\supseteq \text{Z(A)}\}$, and since $\{P \in \text{Prim A} : P \not\supseteq I\}$ is $\approx$-saturated and open $\{G \in \text{Glimm A} : G \not\supseteq \text{Z(A)}\}$ is τ_q-open in Glimm A.

2.5 We now return to the case of a general C*-algebra A to consider the topologies τ_w and τ_s on Glimm A. If I is an ideal in A then an elementary argument, see for example [Archbold 1987: Lemma 5.1], shows that $\phi_A(P \in \text{Prim A}: P \not\supseteq I\} = \{G \in \text{Glimm A}: G \not\supseteq I\}$. It follows that $\tau_w \geq \tau_q$ on Glimm A. Another elementary argument, see for example [Dauns 1974; Proposition 1.6] shows that if $a \in$A, $\lambda \geq 0$ then

$$\phi_A(\{P \in \text{Prim A} :\| a + P \| > \lambda\}) = \{G \in \text{Glimm A} :\| a + G \| > \lambda\}$$

and

$$\phi_A(\{P \in \text{Prim A} :\| a + P \| \geq \lambda\}) = \{G \in \text{Glimm A} :\| a + G \| \geq \lambda\}.$$

Therefore, by 1.3 the function $G \mapsto \| a + G \|, G \in$ Glimm A, is upper semicontinuous on Glimm A for τ_{cr} and τ_q, and lower semicontinuous, and therefore continuous, for τ_w, since $\tau_w \geq \tau_q$. This shows that $\tau_w = \tau_s$ on Glimm A. In general the function $G \mapsto \| a + G \|$ is not continuous for τ_{cr} and τ_q, equivalently τ_{cr} and τ_q are strictly weaker than $\tau_w = \tau_s$. The following result is essentially an amalgam of [Tomiyama 1962; Theorem 3.1] and [Lee 1976; Theorem 4], see also [Dauns 1974; Corollary 1.9] and [Vesterstrøm 1973; Lemma 4.1].

THEOREM 2.1 Let A be a C*-algebra. The following conditions are equivalent:

(i) ϕ_A is an open map

(ii) for each $a \in$ A the function $G \mapsto \|a + G\|$, $G \in$ Glimm A, is τ_{cr}-continuous on Glimm A, equivalently $\tau_{cr} = \tau_s$ on Glimm A

(iii) $\tau_q = \tau_s$ on Glimm A

(iv) the topology τ_q is given by the hull-kernel process on Glimm A

(v) (Glimm A, τ_q) is locally compact, and the mapping $a \mapsto \hat{a}$, where $\hat{a}(G) = a + G$, $G \in$ Glimm A, is a *-isomorphism of A onto a maximal, full algebra of cross-sections over (Glimm A, τ_q).

Proof:(i)⇒(iv) This follows from the second part of the proof of [Lee 1976; Theorem 4].

(iv)⇒(v) This follows from [Tomiyama 1962; Theorem 3.1].

(v)⇒(iii) This is immediate.

(iii)⇒(i) This follows from the first part of the proof of [Lee 1976; Theorem 4]. We have shown that (i),(iii),(iv) and (v) are equivalent. Clearly (ii)⇒(iii). Finally if (iii) holds then $\tau_{cr} = \tau_q$, since (iii)⇒(i), so (ii) holds. Q.E.D.

Some other conditions equivalent to the openness of ϕ_A, when A is unital, are given in [Vesterstrøm 1973; Theorem 4.3].

If A is a C*-algebra satisfying the five equivalent conditions of Theorem 2.1 then A is said to be *open* . It seems unlikely that ideals in open C*-algebras need be open, but I do not have a counterexample. However, if I is a quasicentral ideal in an open C*-algebra A then I is open by 2.4. All AW*-algebras are open [Elliott 1978; Lemma 4.7], but there are quotients of AW*-algebras which are not open, Theorem 4.4. This is contrary to the claim of [Misonou 1952; Theorem 1] that all weakly central C*-algebras are open; another counterexample occurs in [Archbold 1987; Example 4.8]. However if A is a unital, weakly central, open C*-algebra and I is a quasicentral ideal in A then A/I is open, Corollary 4.3.

COROLLARY 2.2 A quasicentral C*-algebra A is open if and only if the intersection of an arbitrary collection of Glimm ideals of A is quasicentral.

Proof: For any subset X⊆Glimm A set J_X=ker X and let Q_X be the largest quasicentral ideal in J_X. Using the homeomorphism between Glimm A and Prim Z(A), 2.3, we see that $\bar{X}$ (the closure of X in Glimm A) is equal to $\{G \in \text{Glimm A}: G \cap Z(A) \supseteq J_X \cap Z(A)\}$. But since each Q_X is quasicentral it follows from Theorem 1.1 (iv) that

$$G \cap Z(A) \supseteq J_X \cap Z(A) \iff G \supseteq J_X \cap Z(A) \iff G \supseteq Q_X.$$

Thus $\bar{X} = \{G \in \text{Glimm A}: G \supseteq Q_X\}$.

If each J_X is quasicentral then $J_X = Q_X$ so the topology on Glimm A is given by the hull-kernel process. By (iv)⇒(i) of Theorem 2.1, A is open. Conversely, if for some X⊆Glimm A $Q_X \neq J_X$ let $P \in$ Prim A with $P \supseteq Q_X$ but $P \not\supseteq J_X$. Then $\phi_A(P) \supseteq Q_X$, so $\phi_A(P) \in \bar{X}$ but $\phi_A(P) \not\supseteq J_X$. Therefore the topology on Glimm A is not given by the hull-kernel process, so A is not open, by (i)⇒(iv) of Theorem 2.1. Q.E.D.

As an illustration, let A be the algebra of Example 5.14. Here each P_{3n} is a Glimm ideal but $\bigcap\{P_{3n}\}$ is not quasicentral, so A is not open.

2.6 Finally, we consider some connections between Glimm A and Min-Primal A.

LEMMA 2.3 Let J be a proper primal ideal in a C*-algebra A. Then there exists a unique Glimm ideal G such that $G \subseteq J$.

Proof: Let $P \in$ Prim A/J, regarded as a subset of Prim A. By [Archbold and Batty 1986; Proposition 3.2], see 1.4, $P \approx Q$ for all $Q \in$ Prim A/J. Let $G = \phi_A(P)$. Then $G = \bigcap[P] \subseteq \bigcap\{Q : Q \in \text{Prim } A/J\} = J$. Finally if $G' \in$ Glimm A and $G' \subseteq J$ then $G' \subseteq P$ so $G' = \phi_A(P)$, as noted after the definition of Glimm A, 2.1. Hence $G' = G$. Q.E.D.

It follows easily that if each Glimm ideal of A is primal then Glimm A and Min-Primal A coincide as sets. In this case the topology τ on Min-Primal A is the same as the topology τ_s on Glimm A, so that τ_q on Glimm A is coarser than τ on Min-Primal A, with equality if and only if A is open, Theorem 2.1. We shall investigate this in more detail in the next chapter.

In [Dauns and Hofmann 1968; Example 9.2] Glimm A and Min-Primal A coincide as sets, but τ on Min-Primal A is locally compact, while $\tau_q = \tau_{cr}$ on Glimm A is not. On the other hand, by generalising the construction in [Archbold 1987; Example 4.8] it is possible to produce a unital C*-algebra A (so that $\tau_q = \tau_{cr}$ is a compact topology on Glimm A) with Glimm A equal to Min-Primal A as sets, but τ not locally compact on Min-Primal A.

3. QUASI-STANDARD C*-ALGEBRAS

In this chapter we introduce a second relation $\sim$ on Prim A and study its connection with the problem of representing a C*-algebra as a continuous field of C*-algebras.

For $P, Q \in$ Prim A let $P \sim Q$ if P and Q cannot be separated by disjoint open subsets of Prim A. By [Archbold and Batty 1986; Proposition 3.2], see 1.4, $P \sim Q$ if and only if $P \cap Q$ is primal. If $P \sim Q$ then clearly $P \approx Q$, but the converse can fail, and $\sim$ need not even be an equivalence on Prim A. For example, in the C*-algebra of sequences $x = (x_n)_{n \geq 1}$ of 2×2 complex matrices such that, as $n \to \infty$

$$x_{2n} \to \begin{pmatrix} \lambda(x) & 0 \\ 0 & \mu(x) \end{pmatrix} \text{ and } x_{2n+1} \to \begin{pmatrix} \mu(x) & 0 \\ 0 & \sigma(x) \end{pmatrix}$$

we have ker $\lambda \sim$ker μ and ker $\mu \sim$ker σ but ker $\lambda \not\sim$ker σ.

For each $P \in$ Prim A the set $\{Q \in \text{Prim A}: Q \sim P\}$ is closed. Let $P^{\#} = \bigcap\{Q \in \text{Prim A}: Q \sim P\}$.

LEMMA 3.1 Let A be a C*-algebra and let $a \in$ A. The function $P \mapsto \|a + P^{\#}\|$, $P \in$ Prim A, is upper semicontinuous on Prim A.

Proof: Let (P_α) be a net in Prim A converging to a point P, and suppose that $\|a + P_\alpha^{\#}\| \geq k$ for some $k \in \mathbf{R}^+$. Then for each α there is a $Q_\alpha \in$ Prim A with $Q_\alpha \sim P_\alpha$ such that $\|a + Q_\alpha\| \geq k$, see [Dixmier 1977; 3.3.6]. The set X=$\{R \in \text{Prim A}: \|a+P\| \geq k\}$ is compact, 1.3, so (Q_α) has a subnet (Q_β) converging to some $Q \in$ X. But $Q \sim P$, so $\|a + P^{\#}\| \geq k$. Q.E.D.

The function in Lemma 3.1 need not be continuous, see Theorem 3.4 (v).

PROPOSITION 3.2 Let A be a quasicentral C*-algebra. If $\sim$ is an equivalence relation on Prim A then $\sim$ is equal to $\approx$ on Prim A.

Proof: Let q:Prim A$\to$Prim A/$\sim$ denote the quotient map, so that for $P \in$ Prim A, q(P) = $P^{\#}$. Let $P, Q \in$ Prim A with $P \not\sim Q$. Since A is quasicentral there is a $z \in$ Z(A) such that U={$R \in$ Prim A:$\|z + R\| > \frac{1}{2}$} is an open neighbourhood of both P and Q, 1.8. Since $z \in$ Z(A) U is $\sim$-saturated. Since $\sim$ is an equivalence relation q(P)+q(Q)=A, so there exist $a \in$ q(P) and $b \in$ q(Q) such that $a + b = z$. Let X={$R \in$ Prim A:$\|a + \text{q}(R)\| < \frac{1}{4}$} and Y={$R \in$ Prim A:$\|b + \text{q}(R)\| < \frac{1}{4}$}. Then X and Y are open by Lemma 3.1, so U$\cap$X and U$\cap$Y are disjoint, open, $\sim$-saturated neighbourhoods of P and Q respectively. By definition of the quotient topology, q(U$\cap$X) and q(U$\cap$Y) are disjoint, open neighbourhoods of q(P) and q(Q) respectively. Hence Prim A/$\sim$ is a Hausdorff space. It is also locally compact since, with z as above, the set W={$R \in$ Prim A;$\|z + R\| \geq \frac{1}{2}$} is compact,1.3, so that q(W) is a compact neighbourhood of q(P) and q(Q). It follows that the continuous bounded functions $f \circ$q, $f \in C_0$(Prim A/$\sim$), separate the $\sim$-equivalence classes of Prim A, so $\sim$ is equal to $\approx$. Q.E.D

If $\sim$ is an equivalence relation but A is not quasicentral it is no longer clear whether $\sim$ need equal $\approx$, or even whether Prim A/$\sim$ need be Hausdorff, although it must be a T_1 space since the equivalence classes are closed. However, if $\sim$ is an open equivalence relation then Prim A/$\sim$ is a locally compact, Hausdorff space, see [Ching 1976; proof of Lemma 7], so the argument just given when A was quasicentral shows that $\sim$ equals $\approx$ in this case as well.

One situation in which $\sim$ equals $\approx$ is when every Glimm ideal of A is primal. For then if $P, Q \in$ Prim A and $P \approx Q$ then $\phi_A(P) = \phi_A(Q)$ is primal, so there is a net (P_α) converging to both P and Q [Archbold and Batty 1986; Proposition 3.2]. Hence $P \sim Q$. It is not necessary, however, for every Glimm ideal to be primal in order for $\sim$ to equal $\approx$. In Example 5.14 $\sim$ is equal to $\approx$ but the Glimm ideal $H = Q_1 \cap Q_2 \cap Q_3$ is not primal. Nevertheless, if $\sim$ is an open equivalence relation then every Glimm ideal is primal, as we now show. The proof is due to Charles Batty.

PROPOSITION 3.3 Let A be a C*-algebra. If $\sim$ is an open equivalence relation on Prim A then $\sim$ is equal to $\approx$ and every Glimm ideal of A is primal.

Proof: We have already observed that $\sim$ is equal to $\approx$ when $\sim$ is an open equivalence relation. A is therefore an open C*-algebra and the topologies τ_{cr}, τ_q and τ_s coincide on Glimm A, by Theorem 2.1.

Let $G \in$ Glimm A. Let $n \geq 2$ and suppose that $J_1, J_2, ..., J_n$ are ideals of A with zero product. For each i let $V_i = \phi_A^{-1}(\phi_A(\text{Prim })J_i))$, an open subset of Prim A, since ϕ_A is open. We prove by induction on r that

$$V_1 \cap \ldots \cap V_r \cap \text{Prim } J_{r+1} \cap \ldots \cap \text{Prim } J_n = \emptyset \qquad (*)$$

For $r = 0$ this follows immediately since $J_1 J_2 \ldots J_n = \{0\}$. Let

$$U_0 = \text{Prim } J_2 \cap \ldots \cap \text{Prim } J_n$$

$$U_k = V_1 \cap \ldots \cap V_k \cap \text{Prim } J_{k+2} \cap \ldots \cap \text{Prim } J_n \qquad (1 \leq k \leq n-2)$$

$$U_{n-1} = V_1 \cap \ldots \cap V_{n-1}.$$

Suppose that $(*)$ is true when $r = k$ for some k with $0 \leq k \leq n$. Then U_k and Prim J_{k+1} are disjoint, open subsets of Prim A. Thus if $P \in U_k$ and $Q \in \text{Prim } J_{k+1}$ then $P \not\sim Q$ and so $\phi_A(P) \neq \phi_A(Q)$. Hence $P \notin V_{k+1}$ and so $U_k \cap V_{k+1} = \emptyset$, that is $(*)$ holds for $r = k+1$. By induction $(*)$ holds for $r = n$ so $\bigcap V_i = \emptyset$.

Now suppose that $J_i \not\subseteq G$ for i=1,2,...,n. Fix $P \in \text{Prim } A/G$. For each i there exists $P_i \in \text{Prim } A/G$ such that $P_i \not\supseteq J_i$. Then $P_i \in \text{Prim } J_i$ and $\phi_A(P_i) = G = \phi_A(P)$, so $P \in V_i$ for each i, contradicting the fact that $\bigcap V_i = \emptyset$. Thus $J_i \subseteq G$ for at least one value of i, and so G is primal. Q.E.D.

We now define a C*-algebra A to be *quasi-standard* if $\sim$ is an open equivalence relation on Prim A. The reason for this name will be given at the end of the chapter.

The next theorem gives some characterisations of quasi-standard C*-algebras. Condition (iv) is the hypothesis used in [Archbold 1987; Lemma 5.1].

THEOREM 3.4 Let A be a C*-algebra. The following conditions are equivalent:

(i) A is quasi-standard

(ii) A is open and every Glimm ideal of A is primal

(iii) every Glimm ideal of A is primal and the topology τ_q on Glimm A coincides with the topology τ on Min-Primal A

(iv) Min-Primal A is τ_s-closed in the set of proper primal ideals of A, and each primitive ideal of A contains a unique minimal primal ideal

(v) for all $a \in$ A the function $P \mapsto \|a + P^{\#}\|$ $(P \in$ Prim A$)$ is continuous on Prim A.

Proof: (i)⇒(ii) This follows from Proposition 3.3.

(ii)⇒(iii) This follows from the remark after Lemma 2.3.

(iii)⇒(iv) Let (J_α) be a net of minimal primal ideals τ_s-converging to a proper primal ideal I. Let $a \in$ A$\backslash I$. The set X=$\{G \in$ Glimm A:$\|a + G\| \geq \frac{1}{2}\|a + I\|\}$ is τ_q-compact, being the image under ϕ_A of the compact set $\{P \in$ Prim A:$\|a{+}P\| \geq \frac{1}{2}\|a{+}I\|\}$, and X contains (J_α) eventually, so there is a subnet of (J_α) which is τ_q-convergent to some $G \in$ X. Since $\tau_q = \tau_s$ and τ_s is a Hausdorff topology $G = I$, so Min-Primal A is τ_s-closed in the set of proper primal ideals of A. Finally, each primitive ideal contains a unique Glimm ideal, 2.1, and this is the unique minimal primal ideal which it contains.

(iv)⇒(i) It was shown in [Archbold 1987; Lemma 5.1] that the map Φ, taking each primitive ideal to the unique minimal primal ideal that it contains, is continuous and open. Let $P, Q \in$ Prim A. If $P \sim Q$ then $P \cap Q$ is primal. Let J be a minimal primal ideal contained in $P \cap Q$. By the uniqueness, $\Phi(P) = J = \Phi(Q)$. Conversely, if $\Phi(P) = \Phi(Q)$ then there is a net in Prim A converging to both P and Q [Archbold and Batty 1986; Proposition 3.2], so $P \sim Q$. It follows that $\sim$ is an open equivalence relation.

(ii)⇒(v) Since each Glimm ideal of A is primal $\sim$ is equal to $\approx$ on Prim A, so each $P^{\#}$ is a Glimm ideal. The function $P \mapsto \|a + P^{\#}\|$ is therefore the composition of the two continuous functions $P \mapsto \phi_A(P)$ and $\phi_A(P) \mapsto \|a + \phi_A(P)\|$, Theorem 2.1, so it too is continuous.

(v)⇒(i) Let $P, Q \in$ Prim A with $P^{\#} \neq Q^{\#}$. Suppose, without loss of generality, that $P^{\#} \not\supseteq Q^{\#}$. Let $a \in Q^{\#} \setminus P^{\#}$ with $\|a + P^{\#}\| = 1$. The sets X=$\{R \in$ Prim A:$\|a + R^{\#}\| > \frac{1}{2}\}$ and Y=$\{R \in$ Prim A:$\|a + R^{\#}\| < \frac{1}{2}\}$ are open, by assumption, with $P \in$ X and $Q \in$ Y, so $P \not\sim Q$. Hence $\sim$ is an equivalence relation. Now it is easy to check that that if X is the open set $\{R \in$ Prim A:$\|a + R\| > \epsilon\}$ ($a \in$ A, $\epsilon \geq 0$) the $\sim$-saturation of X is $\{R \in$ Prim A:$\|a + R^{\#}\| > \epsilon\}$, which is open by assumption. Since sets of the form $\{R \in$ Prim A:$\|a + R\| > \epsilon\}$ ($a \in$ A, $\epsilon \geq 0$) give a basis for the Jacobson topology on Prim A, 1.3, it follows that $\sim$ is an open equivalence relation. Q.E.D.

If I is a non-zero ideal in a C*-algebra A then Prim I is (homeomorphic to) an open subset of Prim A, so it follows directly from the definition that if A is quasi-standard then so is I. On the other hand, although every AW*-algebra A is quasi-standard (by Theorem 3.4 (ii), since it is open [Elliott 1978; Lemma 4.7] and each Glimm ideal of A is prime, see Chapter 4, fact (i)), we will show in Theorem 4.4 that there are quotients of AW*-algebras which are not quasi-standard, see also Proposition 3.7. We will give some more examples of quasi-standard C*-algebras in a moment.

Recall that if a C*-algebra A is a maximal, full algebra of cross-sections over a base-space X with fibre A_x at x $\in$ X then there is a one-to-one correspondence between X and a set of ideals of A given by x $\Longleftrightarrow$ $I_x = \{a \in$ A: $a_x = 0\}$ (x $\in$ X).For each x $\in$ X the map $a + I_x \mapsto a_x$ ($a \in$ A) is a *-isomorphism of A/I_x onto A_x.

THEOREM 3.5 Let A be a C*-algebra. The following conditions are equivalent:

(i) A is quasi-standard

(ii) Min-Primal A is locally compact and A is *-isomorphic to a maximal, full algebra of cross-sections over Min-Primal A

(iii) A is *-isomorphic to a maximal, full algebra of cross-sections in which the base-space contains a dense set of proper primal ideals.

Proof: (i)⇒(ii) This follows from Theorem 3.4 and either Theorem 2.1 or [Archbold 1987; Theorem 5.3].

(ii)⇒(iii) This is immediate.

(iii)⇒(i) We may suppose that A is a maximal, full algebra of cross-sections over a locally compact, Hausdorff space X and that a dense subset of X consists of proper primal ideals. The topology on X is finer than τ_s and therefore finer than τ_w. Since Primal A is τ_w-closed in Id A [Archbold 1987; Proposition 3.1], each ideal in X is primal. By [Lee 1976; Theorem 4] there is a continuous, open map Φ of Prim A onto X such that $\Phi(P) \subseteq P$ ($P \in$ Prim A) and the maximality implies that for $P \in$ Prim A $\Phi(P)$ is the unique element of X contained in P, see for example [Archbold 1987; Theorem 5.3].

Let $P, Q \in$ Prim A. If $\Phi(P) = \Phi(Q)$ then $P, Q \supseteq \Phi(P)$ which is primal, so $P \sim Q$ by [Archbold and Batty 1986, Proposition 3.2]. If $\Phi(P) \neq \Phi(Q)$ then, since Φ is continuous and X is a Hausdorff space, $P \not\sim Q$. Hence $\sim$ is an open equivalence relation and A is quasi-standard. Q.E.D.

COROLLARY 3.6 Let A be a separable C*-algebra. The following conditions are equivalent:

(i) A is quasi-standard

(ii) A is *-isomorphic to a maximal, full algebra of cross-sections in which the base-space contains a dense set of primitive ideals.

Proof: (i)$\Rightarrow$(ii) This follows from Theorem 3.5 (i)$\Rightarrow$(ii) and the fact that if A is separable then Min-Primal contains a dense subset of primitive ideals [Archbold 1987; Corollary 4.6].

(ii)$\Rightarrow$(i) This follows from Theorem 3.5 (iii)$\Rightarrow$(i) since primitive ideals are primal. Q.E.D.

There is a construction due to [Dixmier 1961; Proposition 4] of an inseparable (non-unital) C^*-algebra such that no minimal primal ideal is primitive [Archbold 1987; p.533-4]. It can be checked directly that this algebra is quasi-standard. Thus the implication (i)$\Rightarrow$(ii) in Corollary 3.6 need not hold without the assumption of separability.

We now illustrate Corollary 3.6 with some examples. Let A be the group C^*-algebra of the discrete Heisenberg group [Anderson and Paschke]. Then A is *-isomorphic to a maximal, full algebra of cross-sections over the unit circle T. If $\lambda \in T$ is not a root of unity then the fibre at λ is the corresponding irrational rotation algebra, so the ideal of A corresponding to λ is primitive. By Corollary 3.6, A is quasi-standard (see [Archbold 1987; Example 5.4] for another proof).

Next, let A be the group C^*-algebra of the continuous Heisenberg group. Then A is *-isomorphic to a maximal, full algebra of cross-sections over $\mathbf{R}$ [Lee 1977, Section III], and for $\lambda \in \mathbf{R}\backslash\{0\}$ the fibre at λ is *-isomorphic to the algebra of compact operators on a separable Hilbert space. By Corollary 3.6, A is quasi-standard.

For a third example, let A be the Lee algebra of [Lee 1977; Section IV]. Then A is *-isomorphic to a maximal, full algebra of cross-sections over the one-point compactification of the positive integers. The fibre at the integer n is *-isomorphic to the algebra of $n \times n$ complex matrices, so A is quasi-standard by Corollary 3.6.

The next result is a development of [Archbold and Batty 1986; p.59] and [Archbold 1987; Proposition 4.7]

PROPOSITION 3.7 Let B be an arbitrary C*-algebra. Then there is a quasi-standard C*-algebra A with a minimal primal ideal J such that B is isomorphic to A/J. If B is separable then A may be constructed so as to be separable.

Proof: We may assume that B is a C*-algebra of operators on a Hilbert space H. Let A be the C*-algebra of all sequences $x = (x_n)_{n\geq 1}$ of elements in L(H) which are norm-convergent to an element $b(x)$ in B. Then A is a full algebra of cross-sections over the one-point compactification of the positive integers, and A is maximal by [Fell 1961; p.242, Corollary]. The ideal corresponding to a positive integer is primitive, so A is quasi-standard by (iii)⇒(i) of Theorem 3.5. If J is the ideal corresponding to the point at infinity (the ideal of null sequences) then J is a Glimm ideal, hence minimal primal, and clearly A/J is isomorphic to B.

If B is separable then we can construct A to be separable as in [Archbold 1987; Proposition 4.7] and apply Theorem 3.5 as in the previous paragraph. Q.E.D.

The next result is sometimes useful for seeing whether an ideal is essential.

PROPOSITION 3.8 Let A be a quasi-standard C*-algebra. An ideal I in A is essential if and only if $\phi_A(\text{Prim } I)$ is dense in Glimm A.

Proof: If I is an essential ideal then Prim I is dense in Prim A so $\phi_A(\text{Prim } I)$ is dense in Glimm A.

Conversely, if I is not essential then there is an open subset U of Prim A disjoint from Prim I. If $P \in$ U and $Q \in$ Prim I then $P \not\sim Q$, so $\phi_A(P) \neq \phi_A(Q)$. Hence $\phi_A(\text{U})$ and $\phi_A(\text{Prim } I)$ are disjoint, open subsets of Glimm A, and $\phi_A(\text{Prim } I)$ is not dense. Q.E.D.

Finally, in this chapter, we explain the origin of the name 'quasi-standard'. We start by introducing some terminology from [Ching 1976]. A C*-algebra A is said to be *bounded* if every primitive ideal contains a minimal primitive ideal. If A is bounded the *basic structure space* A^b of A is defined to be the set of minimal primitive ideals, with the Jacobson topology. A is said to be *-*bounded* if it is bounded and if whenever S is a non-empty subset of A^b such that $P \supseteq \ker S$ for some $P \in$ Prim A there exists $P_0 \in A^b$ such that $P \supseteq P_0 \supseteq \ker S$. A is said to be *normal* if it is *-bounded and each primitive ideal contains a unique minimal primitive ideal.

We can now give the main definitions from [Ching 1976]. A C*-algebra A is *standard* if it is normal and A^b is a Hausdorff space. A is *pre-standard* if it is normal and the restriction of $\sim$ to A^b is an open equivalence relation. (Note that if $P, Q \in A^b$ then $P \not\sim Q$ if and only if P and Q can be separated by disjoint, open subsets of A^b).

It is shown in [Ching 1976; Theorem 1] that if A is standard then A is *-isomorphic to a maximal, full algebra of cross-sections with each fibre a primitive C*-algebra. It is not difficult to show that, conversely, each such maximal, full algebra of cross-sections is standard. These facts should be compared with Theorem 3.5 and Corollary 3.6.

Let B be any C*-algebra with a primitive ideal containing two distinct minimal primitive ideals, for example $((LC(H) \oplus LC(H)) + \mathbf{C}1$. Then the C*-algebra A, as constructed in Proposition 3.7, is quasi-standard, but not normal, and therefore not pre-standard. On the other hand it is shown in [Archbold and Somerset; Theorem 4.1] that every pre-standard C*-algebra is quasi-standard. Thus the theory of quasi-standard C*-algebras is a natural extension of the theory of pre-standard C*-algebras, requiring no assumptions on the existence of minimal primitve ideals.

4. AW*-ALGEBRAS

In this chapter we prove three theorems on AW*-algebras. There is no particular connection between the results. The first is on the existence of ideals in Type I_∞ AW*-algebras, Theorem 4.1. The second is on the existence of non-open quotients of properly infinite AW*-algebras, Theorem 4.4. The third is on the automatic continuity of homomorphisms from an AW*-algebra to a Banach algebra, Theorem 4.5.

We start by recalling some facts about AW*-algebras.

(i) If A is an AW*-algebra then the ideals containing each Glimm ideal of A are linearly ordered. Hence A is weakly central, and each ideal containing a Glimm ideal is prime. [Glimm 1960; Lemma 11], [Stratila and Zsido 1979; 4.23], [Berberian 1972; §24 Exercise 4].

(ii) If A is a Type I AW*-algebra and G is a Glimm ideal of A then G is primitive and there is a faithful, irreducible representation π of A/G on a Hilbert space H such that $\pi(A/G)$ contains LC(H). [Glimm 1960; Theorem 4], [Tomiyama 1962; Section 4], [Halpern 1966].

(iii) If A is a Type I AW*-algebra then Glimm A is a dense, open subset of Prim A, and Glimm A is disjoint from Max A if and only if A is properly infinite. [Halpern 1966; Theorem 3]

(iv) If A is a properly infinite AW*-algebra then Max A is a closed subset of Prim A. The ideal $J = \bigcap\{M : M \in \text{Max A}\}$ is called the *strong radical* of A, and it contains each finite projection. For each $M \in$ Max A $M = J + \phi_A(M)$. If A is semi-finite then the interior of Max A is empty. [Halpern 1969; Proposition 2.3], [Ringrose 1978; Corollary 2.2].

(v) If A is a finite, Type I AW*-algebra then A has the form $A \cong \Pi_{n=1}^{\infty} C(X_n) \otimes M_n$ where each X_n is a Stonean space, and M_n is the C*-algebra of $n \times n$ complex matrices. If X is equal to the disjoint union of the X_n's then Prim Z(A) $\cong \beta X$, the Stone-Cech compactification of X. If $M \in$ Max A and $M \cap$ Z(A) $\in X_n$ for some n then clearly $A/M \cong M_n$, while if $M \cap$ Z(A) $\in \beta X \setminus X$ then A/M is a Type II_1 AW*-factor. [Wright 1954; Theorem 5.1]

(vi) The centre of an AW*-algebra is an AW*-algebra [Berberian 1972; §4 Corollary 2].

Looking at (iii) and (iv), it seems plausible that Prim A is equal to the union of the disjoint subsets Max A and Glimm A when A is a Type I_∞ AW*-algebra. The first theorem shows that this is not usually the case.

THEOREM 4.1 Let A be a Type I_∞ AW*-algebra with countably decomposable centre. If G is a non-isolated point of Glimm A then there is a primitive ideal, properly containing G, which is not a maximal ideal.

Proof: Since A is weakly central there is a unique maximal ideal M of A containing G. Let π be a faithful representation of A/G on a Hilbert space H such that $\pi(A/G)$ contains LC(H), see fact (ii). Let $P = \Phi_G^{-1}(\pi^{-1}(\text{LC(H)}))$, so that P is a prime ideal in A containing G. We will show that $P \neq M$ from which it follows that there is a primitive ideal properly between G and M. If e is a finite projection then $e \in M$ by fact (iv). Therefore to show that $P \neq M$ it is sufficient to find a finite projection not in P.

Let $\{X_i\}_{i \in I}$ be a maximal collection of non-empty, disjoint, clopen subsets of Glimm A not containing G. Since the centre of A is countably decomposable, and G is not an isolated point of Glimm A, the collection must be countably infinite, so we may assume that the index set $I = \mathbf{N}$. Furthermore if $X = \bigcup_{i=1}^{\infty} X_i$ then since G

is non-isolated and Glimm A is totally disconnected, (vi), G is in the closure of X in Glimm A. By [Gillman and Jerison 1960; 1H.6] this closure is isomorphic to βX.

For each $i \in \mathbf{N}$ let e_i be a projection in A such that $e_i A e_i \cong C(X_i) \otimes M_i$ and such that $\|e_i + G\| = 0$ for $G \notin X_i$. Set $e = \sum e_i$. Then e is a finite projection [Stratila and Zsido 1979; Lemma 4.14] so $e \in M$.

Set B $= e\mathrm{A}e$. Then B $\cong \Pi_{i=1}^{\infty} C(X_i) \otimes M_i$ and Z(B) $= e$Z(A) [Stratila and Zsido 1979; 3.15]. If $J = G \cap \mathrm{Z(A)}$ then $G = \mathrm{A}J$, by 2.3, so $eGe = e\mathrm{A}Je = e\mathrm{A}eJ = \mathrm{B}eJ$. Since A is weakly central eJ is a maximal ideal of Z(B), so eGe is a Glimm ideal of B. Let N be the unique maximal ideal of B containing eGe. Because $eJ \in \beta X \setminus X$ B$/N$ is a Type II_1 AW*-factor, by (v), so in particular B$/eGe$ is infinite dimensional.

Now if $e \in P$ then $\pi(e)$ would be a finite projection in $\pi(\mathrm{A}/G)$, by definition of P, so $e_G(\mathrm{A}/G)e_G$ would be a finite dimensional algebra. But $e_G(\mathrm{A}/G)e_G \cong \mathrm{B}/eGe$ which is infinite dimensional. Hence $e \notin P$, so $P \neq M$. Q.E.D.

Some other results on ideals in AW*-algebras can be found in [Feldman 1958], [Wils 1970] amd [Halpern 1973].

The second theorem is the one promised in Chapters 2 and 3 showing that an AW*-algebra can have quotients which are not open. In fact Theorems 4.1 and 4.4 imply that Type I_∞ AW*-algebras usually have many such quotients.

LEMMA 4.2 Let A be a unital, weakly central C*-algebra and I an ideal in A with $\pi : \mathrm{A} \mapsto \mathrm{A}/I$ the canonical homomorphism. Then π maps the set $\{G \in$ Glimm A: $G \supseteq I \cap \mathrm{Z(A)}\}$ homeomorphically onto Glimm A$/I$.

Proof: Set $X = \{G \in$ Glimm A: $G \supseteq I \cap \mathrm{Z(A)}\}$. By 2.3, 1.9, and 1.8, the maps $G \mapsto G \cap \mathrm{Z(A)} \mapsto \pi(G \cap \mathrm{Z(A)}) \mapsto \pi(G \cap \mathrm{Z(A)}).\pi(\mathrm{A})$ from $X \to \{P \in \mathrm{Prim\ Z(A)} : P \supseteq I \cap \mathrm{Z(A)}\} \to \mathrm{Prim\ Z}(\pi(\mathrm{A})) \to \mathrm{Glimm}\ \pi(\mathrm{A})$ are all homeomorphisms. But by Theorem

1.1 (iv) $G = (G \cap Z(A)).A$ so $\pi(G) = \pi(G \cap Z(A)).\pi(A)$. Hence π is a homeomorphism from X onto Glimm A/I. Q.E.D.

COROLLARY 4.3 If A is a unital, weakly central, open C*-algebra and I is a quasi-central ideal in A then A/I is open.

Proof: If $G \in$ Glimm A then since I is quasicentral $G \supseteq I \cap Z(A) \iff G \supseteq I$. Hence, with the notation of Lemma 4.2, if $a \in A$ and $G \supseteq I \cap Z(A)$ then $\|\pi(a)+\pi(G)\| = \|a + G\|$. Therefore since A is open the function $G' \mapsto \|a_I + G'\|$ ($G' \in$ Glimm A/I) is continuous on Glimm A/I, using Lemma 4.2. By Theorem 2.1 A/I is open. Q.E.D.

THEOREM 4.4 Let A be a properly infinite AW*-algebra and let J be the strong radical of A. If Q is a non-maximal ideal of A containing a non-isolated Glimm ideal then $A/(Q \cap J)$ is not open.

Proof: Set $B = A/(Q \cap J)$ and let π: A→B be the canonical homomorphism. Then Prim B is homeomorphic to the closed subset hull $Q \cup$ hull J of Prim A. Recall that hull J equals Max A, fact (iv), and our assumption implies that J is non-zero, while if $P \in$ hull Q then $\phi_A(P) = \phi_A(Q) = H$, say, by fact (i). Since $(Q \cap J) \cap Z(A) \subseteq J \cap Z(A) = 0$ it follows from Lemma 4.2 that π maps Glimm A homeomorphically onto Glimm B. Hence $\pi(H)$ is a non-isolated point of Glimm B. Let X be the open subset of Prim B given by $X = \{P \in \text{Prim B} : P \not\supseteq \pi(J)\}$. Then $\phi_B(X) = \{G' \in \text{Glimm B} : G' \not\supseteq \pi(J)\}$. If $G' \in$ Glimm B then since $J \supseteq Q \cap J$ $G' \supseteq \pi(J) \iff \pi^{-1}(G') \supseteq J \iff G + (Q \cap J) \in$ Max A, where G is the unique Glimm ideal in A such that $\pi(G) = G'$. But hull $G + (Q \cap J) =$ hull $G \cap$ (hull $Q \cup$ Max A), so $G + (Q \cap J) \in$ Max A $\iff G \neq H$, using fact (i). Hence $\phi_B(X) = \pi(H)$ which is not an open subset of B since $\pi(H)$ is not an isolated point. Q.E.D.

If A is Type I_∞ or Type II_∞ then Max A has empty interior, fact (iv). Since hull Q also has empty interior, Proposition 3.8 implies that $Q \cap J$ is an essential ideal in A, so A is the multiplier algebra of $Q \cap J$ [Pedersen 1984] and B is the corona algebra. The centre of B is an AW*-algebra, since it is isomorphic to the centre of A. The ideal $Q \cap J$ is quasi-standard, by the remark after Theorem 3.4.

The third theorem is on automatic continuity for homomorphisms of AW*-algebras. It turns out, rather suprisingly, that the continuity or otherwise of a homomorphism from an AW*-algebra into a Banach algebra is determined by the centre of the AW*-algebra.

THEOREM 4.5 Let A be an AW*-algebra and θ a homomorphism from A into a Banach algebra B. Then θ is continuous if and only if the restriction of θ to Z(A) is continuous.

This result is to be found in [Laursen 1983; Theorem 11], but there is an assumption in the proof there which is not generally correct. We first show how the theorem can be proved using another result of Laursen's, and we then find out exactly when the assumption is not true.

The next proposition is an unpublished result due to Laursen.

PROPOSITION 4.6 Let A be a unital C*-algebra and suppose that every ideal in A of finite codimension is quasicentral. Let θ be a homomorphism from A into a Banach algebra B. If the restriction of θ to Z(A) is continuous then θ is continuous.

Proof: Let J be the continuity ideal of θ, that is, the largest (not necessarily closed) ideal in A on which θ is continuous [Laursen 1983; Remark 1]. By [Sinclair

1976; Theorem 9.3] $\bar{J}$ (the closure of J) has finite codimension.

Since the restriction of θ to Z(A) is continuous, A and B become Banach Z(A)-modules [Sinclair 1976; Definition 6.1] with Z(A) acting on A by multiplication in the usual way, and Z(A) acting on B by $z.b = \theta(z).b$ for $z \in$ Z(A) and $b \in$ B. It now follows from [Sinclair 1976; Corollary 9.4] that θ is continuous on $(\bar{J} \cap \mathrm{Z(A)})\mathrm{A}$. But $\bar{J}$ is quasicentral, by assumption, so $\bar{J} = (\bar{J} \cap \mathrm{Z(A)})\mathrm{A}$, by 1.8. Hence θ is continuous on a closed subspace of finite codimension, so θ is continuous on all of A [Sinclair 1976; Lemma 1.4]. Q.E.D.

In fact it is sufficient for Proposition 4.6 to assume that every maximal ideal of finite codimension is quasicentral. For then any ideal of finite codimension is equal to the intersection of a finite number of maximal ideals, which are quasicentral by assumption, and it follows easily from Theorem 1.1 (iv) that the intersection of a finite number of quasicentral ideals is quasicentral.

Proof of Theorem 4.5: If θ is continuous then, trivially, the restriction of θ to Z(A) is continuous.

Suppose, conversely, that the restriction of θ to Z(A) is continuous. It is known from [Albrecht and Dales 1983; Theorem 4.2] that if A has no finite summand of Type I then every homomorphism into a Banach algebra is continuous. We may assume therefore that A is finite and of Type I. But now it is clear from fact (v) that every maximal ideal of finite codimension is quasicentral. It follows from Proposition 4.6 that θ is continuous on all of A. Q.E.D.

We now look at the incorrect assumption. This is that if A is a unital, weakly central C*-algebra and if $a \in$ A then the function $M \mapsto \|a + M\|$ ($M \in$ Max A) is continuous on Max A. This is not true, in general.

THEOREM 4.7 Let A be a unital, weakly central C*-algebra. The following are equivalent:

(i) Max A is closed in Prim A

(ii) for each $a \in$ A the function $M \mapsto \|a + M\|$ ($M \in$ Max A) is continuous on Max A.

Proof: (i)$\Rightarrow$(ii) For each $a \in$ A and $\epsilon \geq 0$ the set $X = \{P \in \text{Prim A} : \|a + P\| \geq \epsilon\}$ is compact, 1.3, so if Max A is closed then $X \bigcap$ Max A is a compact subset of Max A. Since Max A is Hausdorff, $X \bigcap$ Max A is closed in Max A. But the set $\{P \in \text{Max A} : \|a + P\| \leq \epsilon\}$ is always closed in Max A, by 1.3, so the function $M \mapsto \|a + M\|$ is continuous.

(ii)$\Rightarrow$(i) Let J be the strong radical of A and set B = A/J. Let $P \in$ Prim B and let N be the unique maximal ideal of B containing P. Let $b \in N^+$. It follows straight from our assumption that the function $M \mapsto \|b + M\|$ is continuous on Max B, so, by 1.9, there is a $z \in$ Z(B) such that $\|b + M\| = z + M$ for all $M \in$ Max B. Since Max B is dense in Prim B it follows that $b \leq z$. But $0 = \|b + N\| = z + N$ so $z \in N$. Since $N \cap \text{Z(A)} = P \cap \text{Z(A)}$ $z \in P$. But $b \leq z$, so $b \in P$. Hence $P = N$ and Max B = Prim B, that is Max A is a closed subset of Prim A. Q.E.D.

If A is a finite AW*-algebra then Max A is dense in Prim A [Wright 1954; Theorem 2.7], [Handelman, Higgs and Lawrence 1980; Theorem 3.1]. Hence Max A is closed in Prim A if and only if Prim A is Hausdorff. There is an example in [Dixmier 1981; III Chapter 5 Exercise 6] of a finite, Type I von Neumann algebra B such that Prim B is not Hausdorff.

5. THE CASES $K_s(A)=\frac{1}{2}$ AND $K(A)=\frac{1}{2}$

In this chapter we obtain necessary and sufficient conditions for $K_s(A)=\frac{1}{2}$ and $K(A)=\frac{1}{2}$ in terms of the primal ideal structure of A.

We start with an elementary lemma whose proof is left to the reader.

LEMMA 5.1 Let A be a C*-algebra. Let X be a collection of ideals of A such that ker X=$\{0\}$. Then for all $a \in A$

$$\|D(a, A)\| = \sup \{\| D(a_I, A/I) \| : I \in X\}.$$

The next theorem, which we state without proof, is a combination of results from [Stampfli 1970]. It is crucial for the rest of the thesis.

THEOREM 5.2 Let A be a unital C*-algebra.

(i) For each $a \in A$ there is a unique scalar $\lambda(a)$ such that for all $\mu \in \mathbf{C}$

$$\|a - \lambda(a)\|^2 + | \lambda(a) - \mu |^2 \leq \|a - \mu\|^2.$$

(ii) If A is also a primitive C*-algebra then for all $a \in A$

$$\| D(a, A) \| = 2 \| a - \lambda(a) \| .$$

If a is self-adjoint then $\lambda(a) = \frac{1}{2}(\alpha + \beta)$ where α is the largest point in the spectrum of a and β is the smallest.

We now give an expression for the distance of an element from the centre in an arbitrary C*-algebra. Recall that if G is an ideal in A then $\Phi_G : A \rightarrow A/G$ denotes the canonical *-homomorphism.

PROPOSITION 5.3 Let A be a C*-algebra. For all $a \in$ A

$$d(a, \mathrm{Z}(\mathrm{A})) = \sup \{d(a_G, \Phi_G(\mathrm{Z}(\mathrm{A}))) : G \in \text{Glimm A}\}.$$

Remark: If A is quasicentral the formula has the simpler form

$$d(a, \mathrm{Z}(\mathrm{A})) = \sup \{\| a_G - \lambda(a_G) \| : G \in \text{Glimm A}\}.$$

Proof of Proposition: Set $\alpha = \sup \{d(a_G, \Phi_G(\mathrm{Z}(\mathrm{A}))) : G \in \text{Glimm A}\}$. Clearly $d(a,\mathrm{Z}(\mathrm{A})) \geq \alpha$. Conversely, for any $\epsilon > 0$ let M be the τ_q-compact set $\{G \in \text{Glimm A}: \| a+G\| \geq \alpha+\epsilon\}$, see 2.5. Then M is contained in the τ_q-open set W= $\{G \in \text{Glimm A} : G \not\supseteq \mathrm{Z}(\mathrm{A})\}$, see 2.4. If M is empty then $\|a\| = d(a,0) < \alpha + \epsilon$. Otherwise, for each fixed $H \in$ M, $\Phi_H(\mathrm{Z}(\mathrm{A}))$ is one-dimensional, A$/H$ is unital and there is a $z \in \mathrm{Z}(\mathrm{A})$ such that $\|a_H - z_H\| = \|a_H - \lambda(a_H)\| \leq \alpha$. The function $G \mapsto \|a_G - z_G\|$ is τ_q-upper semicontinuous on Glimm A, 2.5, so there is an open neighbourhood N of H in W such that for all $G \in$ N $\|a_G - z_G\| < \alpha + \epsilon$.

The open neighbourhoods of this form give an open cover of M, so by the τ_q-compactness of M there are a finite number of points $H(1), H(2), ..., H(n)$ in M with corresponding elements $z(1), z(2), ..., z(n)$ in Z(A) such that the sets

$$\mathrm{N(i)} = \{G \in \mathrm{W} : \|a_G - z(i)_G\| < \alpha + \epsilon\} \ (\mathrm{i} = 1, 2, ..., \mathrm{n})$$

form an open cover of M. Let $f(1), f(2), ..., f(n)$ be a partition of unity equal to one on M and subordinate to the cover{N(1),N(2),...,N(n)}, [Rudin 1966, Theorem 2.13]. By 1.8 and 2.4, there are elements $g(1), g(2), ..., g(n) \in \mathrm{Z}(\mathrm{A})$ such that $g(i)_G = f(i)(G)$ for each i and for $G \in$ Glimm A. Set $z = \sum g(i)z(i)$. Then for each $G \in$ Glimm A

$$\|a_G - z_G\| = \|a_G - \sum g(i)_G z(i)_G\|$$

$$\leq \|a_G - \sum f(i)(G)a_G\| + \|\sum f(i)(G)(a_G - z(i)_G)\|$$

$$\leq (1 - \sum f(i)(G))\|a_G\| + \sum f(i)(G)\|a_G - z(i)_G\|$$

If $G \in$ M then $\sum f(i)(G) = 1$, so $\|a_G - z_G\| < \alpha + \epsilon$, while if $G \notin$ M then $\|a_G\| < \alpha + \epsilon$, so again $\|a_G - z_G\| < \alpha + \epsilon$. Hence $\|a - z\| < \alpha + \epsilon$. Since ϵ was arbitrary $d(a,\mathrm{Z(A)}) \leq \alpha$. Q.E.D.

The next proposition shows that in our search for C*-algebras with $\mathrm{K}_s(\mathrm{A}) = \frac{1}{2}$ we may restrict attention to quasicentral C*-algebras.

PROPOSITION 5.4 Let A be a C*-algebra. If A is not quasicentral then $\mathrm{K}_s(\mathrm{A}) \geq 1$.

Proof: Let J=AZ(A) be the largest quasicentral ideal in A, 1.8, and let $a \in \mathrm{A} \backslash J$ with $a \geq 0$ and $\|a\| = \|a_J\| = 1$. Then clearly $d(a,\mathrm{Z(A)}) = 1$.

Let $\tilde{\mathrm{A}}$ =A+C1. Then

$$\|D(a, \mathrm{A})\| = \|D(a, \tilde{\mathrm{A}})\| \leq 2d(a, \mathbf{C}1) = 1$$

Hence $\mathrm{K}_s(\mathrm{A}) \geq 1$. Q.E.D.

It was shown in [Archbold 1978, Proposition 3.1] that if J is an ideal in a C*-algebra A then $\mathrm{K}(J) \leq 2\mathrm{K}(\mathrm{A})$.

PROPOSITION 5.5 Let A be a C*-algebra and let J be a quasicentral ideal in A. Then $\mathrm{K}(J) \leq \mathrm{K}(\mathrm{A})$.

Proof: Let $a \in$ A and $z \in$ Z(A). By Theorem 1.1 (iii) J has an approximate unit (z_λ) contained in $\mathrm{Z}(J)$. Then the inequality

$$\|a - z_\lambda z\| \leq \|a - az_\lambda\| + \|z_\lambda\| \; \|a - z\|$$

where $z \in$ Z(A) shows that $d(a, \mathrm{Z}(J)) = d(a, \mathrm{Z(A)})$. Since $\|D(a, J)\| = \|D(a, \mathrm{A})\|$ [Archbold 1978, Proposition 3.1] it follows that $\mathrm{K}(J) \leq \mathrm{K}(\mathrm{A})$. Q.E.D.

If A is a C*-algebra without identity then it is clear that K(A)≥K(A+C1). If A is quasicentral then Proposition 5.5 shows that K(A)=K(A+C1).

The construction of [Fell and Doran 1988; Chapter VII, 7.2] exhibits any unital C*-algebra A as a quotient of a unital primitive C*-algebra B. By Theorem 5.2 K(B)=$\frac{1}{2}$. This shows that the constant K is very unstable under quotients. Alternatively, A may be obtained as a quotient of a unital, quasi-standard C*-algebra C, Proposition 3.7, and we will show in Corollary 5.11 (ii) that K(C)=$\frac{1}{2}$.

Now let A be any C*-algebra and let G(A) denote the set of pure functionals of A, that is, the set of extreme points of the unit ball of A*, equipped with the weak *-topology [Archbold and Shultz 1989]. Then it is shown in [Archbold and Shultz 1989; Proposition 1.1] that if $g \in$ G(A) then there is an irreducible representation π of A on a Hilbert space H such that, for $a \in$ A, $g(a) = (\pi(a)\xi, \eta)$ where ξ and η are unit vectors in H. If ϕ is the pure state defined on A by $\phi(a) = (\pi(a)\xi, \xi)$ ($a \in$ A) then $\phi = |g|$ where $|g|$ is the absolute value of g [Dixmier 1977; 12.2.7]. Conversely, if π is any irreducible representation of A on a Hilbert space H and ξ and η are unit vectors in H then the map $a \mapsto (\pi(a)\xi, \eta)$, $a \in$ A, defines a pure functional on A.

Let $a \in$ A. Since $a^*a \geq 0$ there is a pure state ϕ such that $\phi(a^*a) = \|a^*a\| = \|a\|^2$. Let π be the G.N.S. representation of ϕ on a Hilbert space H and let ξ be a unit vector in H such that, for $b \in$ A, $\phi(b) = (\pi(b)\xi, \xi)$. Then $\|a\|^2 = \phi(a^*a) = (\pi(a)\xi, \pi(a)\xi) = \|\pi(a)\xi\|^2$. Let g be the pure functional defined on A by $g(b) = (\pi(b)\xi, \pi(a)\xi/\|a\|), b \in$ A. Then $g(a) = \|a\|$.

For $g \in$ G(A) let Δ_A:G(A)→P(A) be the map $\Delta_A(g) = |g|$ and Γ_A:G(A)→Prim A be the map $\Gamma_A(g)$= ker π, where π is an irreducible representation such that $g(a) = (\pi(a)\xi, \eta)$. Then Δ_A is continuous by [Effros 1963; Lemma 3.5], and the restriction of

Γ_A to P(A) is continuous and open [Dixmier 1977; 3.4.11]. Since $\Gamma_A = \Gamma_A.\Delta_A$ it follows that Γ_A is continuous. I am grateful to Charles Batty for the elementary proof of the next lemma.

LEMMA 5.6 The maps Δ_A and Γ_A, defined above, are open.

Proof: It is sufficient to show that Δ_A is open, for then $\Gamma_A = \Gamma_A.\Delta_A$ is the composition of two open maps. Let $g \in$ G(A) and let U$=\{f \in$ G(A)$:|f(a_i) - g(a_i)| < \epsilon_i \ (i = 1, 2, ..., n)\}$ $(a_1, a_2, ..., a_n \in$ A, $\epsilon_1, \epsilon_2, ..., \epsilon_n > 0)$ be a basic neighbourhood of g in G(A). There is an irreducible representation π of A on a Hilbert space H, with unit vectors $\xi, \eta \in$ H, such that $g(a) = (\pi(a)\xi, \eta)$ for $a \in$ A. By the Kadison Transitivity theorem [Pedersen 1979; 3.13.2] there is a unitary u in A+C1 such that $\pi(u)\eta = \xi$. Let $\phi(a) = (\pi(a)\xi, \xi)$ so that $\phi = \Delta_A(g) \in$ P(A) and $g(a) = \phi(ua)$. Let V$=\{\psi \in$ P(A)$:|\psi(ua_i) - \phi(ua_i)| < \epsilon_i \ (i = 1, 2, ..., n)\}$. Then V is an open neighbourhood of ϕ in P(A). For $\psi \in$ V let $f(a) = \psi(ua)$. Then $f \in$ U and $\Delta_A(f) = \psi \in$ V. Hence Δ_A(U) is open. Q.E.D.

We will use the following well-known characterisation of open maps, see [Fell and Doran 1988; Chapter II, 13.2] for example. Let Y and Z be topological spaces and f: Y→Z a surjection. Then f is open if and only if whenever $y \in$ Y and (z_i) is a net of elements in Z converging to f(y) there is a subnet (z_j) of (z_i) and a net (y_j) of elements in Y such that (y_j) converges to y and f(y_j)=z_j for each j.

PROPOSITION 5.7 Let A be a quasicentral C*-algebra. Let S be any dense subset of Prim A. Then for any $a \in$ A

$$\frac{1}{2}\|D(a, \mathrm{A})\| = \sup\{\|a_P - \lambda(a_P)\| : P \in \mathrm{S}\} = \sup\{\|a_P - \lambda(a_P)\| : P \in \mathrm{Primal\ A}\}$$

Proof: The first equality follows from Lemma 5.1 and Theorem 5.2 (ii). We may therefore assume that S = Prim A. Since each proper primal ideal is primitive it is trivial that

$$\sup\{\|a_P - \lambda(a_P)\| : P \in \text{Primal A}\} \geq \sup\{\|a_P - \lambda(a_P)\| : P \in \text{Prim A}\}.$$

Conversely, let $a \in$ A and let I be a fixed proper primal ideal. Let (P_α) be a net of primitive ideals converging to every point in Prim A/I, [Archbold and Batty 1986; Proposition 3.2]. By Theorem 5.2(i) $|\lambda(a_{P_\alpha})| \leq \|a_{P_\alpha}\| \leq \|a\|$, so we may assume, by compactness,and by passing to a subnet of (P_α) if necessary, that there is a $\mu \in \mathbf{C}$ such that $\lambda(a_{P_\alpha}) \to \mu$.

Let f be a pure functional on A such that $\Gamma_A(f) \supseteq I$ and

$$|f(a_I - \mu)| = \|a_I - \mu\|$$

Using Lemma 5.6, and again passing to a subnet of (P_α) if necessary, we can find a net (f_α) of pure functionals of A such that (f_α) tends to f in G(A) and such that $\Gamma_A(f_\alpha) = P_\alpha$ for each α. Hence for any $\epsilon > 0$ there is an α such that $|\lambda(a_{P_\alpha}) - \mu| < \epsilon$ and $|f_\alpha(a_{P_\alpha} - \mu) - f(a_I - \mu)| < \epsilon$. For this α

$$\|a_{P_\alpha} - \lambda(a_{P_\alpha})\| \geq |f_\alpha(a_{P_\alpha} - \lambda(a_{P_\alpha}))|$$

$$\geq |f_\alpha(a_{P_\alpha} - \mu)| - \epsilon$$

$$\geq |f(a_I - \mu)| - 2\epsilon$$

$$= \|a_I - \mu\| - 2\epsilon$$

$$\geq \|a_I - \lambda(a_I)\| - 2\epsilon.$$

Since ϵ was arbitrary the proposition is proved. Q.E.D.

When a is normal a slightly simpler proof of Proposition 5.7 can be given, using pure states.

We can now characterise the C*-algebras with $K_s = \frac{1}{2}$.

THEOREM 5.8 Let A be a non-commutative C*-algebra. If A is quasicentral and $\sim$ is an equivalence relation on Prim A then $K_s(A) = \frac{1}{2}$. Otherwise $K_s(A) \geq 1$.

Proof: Suppose first that A is quasicentral and that $\sim$ is an equivalence relation on Prim A. Then by Proposition 3.2 $\sim$ is equal to $\approx$ on Prim A. Let $a \in A_{sa}$ and let $G \in$ Glimm A. Let P be a primitive ideal containing G such that a_P and a_G have the same largest point in their spectrum, and let Q be a primitive ideal containing G such that $a_Q$and a_G have the same smallest point in their spectrum. Then $P \approx Q$ so $P \sim Q$ so $R = P \cap Q$ is primal by [Archbold and Batty 1986; Proposition 3.2]. By Proposition 5.7 $\|a_R - \lambda(a_R)\| \leq \frac{1}{2}\|D(a,A)\|$. But $\|a_R - \lambda(a_R)\| = \|a_G - \lambda(a_G)\|$, so since G was arbitrary, Proposition 5.3 implies that $d(a,Z(A)) \leq \frac{1}{2}\|D(a,A)\|$. Hence $K_s(A) = \frac{1}{2}$ (since A is non-commutative).

Conversely, we have shown in Proposition 5.4 that if A is not quasicentral then $K_s(A) \geq 1$, so assume that A is quasicentral but that $\sim$ is not an equivalence relation on Prim A. Hence $\sim$ does not equal $\approx$ on Prim A, so there is a Glimm ideal G with disjoint open sets X and Y such that $X \cap \text{Prim } A/G \neq \emptyset$ and $Y \cap \text{Prim } A/G \neq \emptyset$. By passing to subsets of X and Y, if necessary, we may assume that there are orthogonal positive elements x and y in A with $\|x\| = \|x_G\| = 1$ and $\|y\| = \|y_G\| = 1$ such that $X = \{S \in \text{Prim } A : \|x_S\| > 0\}$ and $Y = \{S \in \text{Prim } A : \|y_S\| > 0\}$. Set $a = x - y$. Then $d(a,Z(A)) \geq \|a_G - \lambda(a_G)\| = 1$ while $\frac{1}{2}\|D(a,A)\| = \sup\{\|a_P - \lambda(a_P)\| : P \in \text{Prim } A\} \leq 1$. Hence $K_s(A) \geq 1$. Q.E.D.

It follows from Proposition 3.2 and [Archbold and Batty 1986; Proposition 3.2] that the condition that $\sim$ should be an equivalence relation on Prim A is the same (at least when A is quasicentral) as requiring that whenever two primitive ideals contain the same Glimm ideal then their intersection should be primal, see Theorem 5.10.

In the proof of Theorem 5.8 we made use of the fact that when a is self-adjoint $\lambda(a)$ is the midpoint between the largest and smallest points in the spectrum of a. When a is a general element there seems to be no such simple characterisation of $\lambda(a)$. To cope with this, we introduce a sort of generalised numerical range, as follows.

For a unital C*-algebra A let N(A) be the closed, convex hull of the set of pure functionals g of A with $g(1) \geq 0$. For $a \in$ A let U(a,A)=$\{f(a); f \in \mathrm{N(A)}\}$. Note that $\|a\| = \sup \{|f(a)| : f \in \mathrm{N(A)}\}$.

If X is a compact subset of the plane there is a unique circle S of smallest radius containing X called the *circumcircle* of X. The centre of S is called the *circumcentre* of X. If a is a normal element in a unital C*-algebra then $\lambda(a)$ is the circumcentre of the spectrum of a. We will make use of two properties of S:

(i) Every closed semi-circle of S intersects X

(ii) There are three points x,y,z in X$\cap$S such that S is the circumcircle of the set {x,y,z} (in some cases less than three points might suffice).

I am grateful to Charles Batty for help with the next lemma.

LEMMA 5.9 Let A be a unital C*-algebra and let $a \in$ A. Let $\mu(a)$ denote the circumcentre of U(a,A).

(i) $|\lambda(a)|^2 \leq 4|\mu(a)|\ \|a\|$

(ii) if $|\mu(a)| \leq \frac{1}{2}\|a\|$ then $|\lambda(a)|^2 \leq 4|\mu(a)|(\|a\| - |\mu(a)|)$

(iii) $|\lambda(a) - \mu(a)|^2 \leq 2|\lambda(a)|\ \|a\|$.

Proof: (i) Let r denote the radius of the circumcircle of U(a,A). Since there is a point in U(a,A) with modulus equal to $\|a\|$ $r \geq \|a\| - |\mu(a)|$. By property (i) of circumcentres there is an $f \in$ N(A) such that $|f(a) - \mu(a)| = r$ and the angle between $f(a) - \mu(a)$ and $\lambda(a) - \mu(a)$ is at least $\pi/2$. Then since $0 \leq f(1) \leq 1$

$$|f(a) - (\lambda(a)f(1) + \mu(a)(1 - f(1))| \geq |f(a) - \mu(a)|.$$

But $\|a-\lambda(a)\| \geq |f(a-\lambda(a))| = |f(a)-\lambda(a)f(1)| \geq |f(a)-(\lambda(a)f(1)+\mu(a)(1-f(1))| - |\mu(a)(1-f(1))| \geq |f(a)-\mu(a)| - |\mu(a)| = r - |\mu(a)| \geq \|a\| - 2|\mu(a)|$. Therefore $\|a\| - \|a-\lambda(a)\| \leq 2|\mu(a)|$.

Hence it follows from Theorem 5.2 (i) that

$$|\lambda(a)|^2 \leq \|a\|^2 - \|a-\lambda(a)\|^2$$

$$= (\|a\| - \|a-\lambda(a)\|)(\|a\| + \|a-\lambda(a)\|) \leq 4|\mu(a)|\ \|a\|.$$

(ii) If $|\mu(a)| \leq \frac{1}{2}\|a\|$ then

$$|\lambda(a)|^2 \leq \|a\|^2 - \|a-\lambda(a)\|^2 \leq \|a\|^2 - (\|a\| - 2|\mu(a)|)^2$$

$$= 4|\mu(a)|(\|a\| - |\mu(a)|).$$

(iii) As before let r denote the radius of the circumcircle of U(a,A). Let $f \in$ N(A) with the angle between $f(a)$ and $\mu(a)$ at least $\pi/2$. Then $\|a\|^2 \geq |f(a)|^2 \geq r^2 + |\mu(a)|^2$. In particular $\|a\| \geq r$. Suppose that $g \in$ N(A) and $|g(a-\mu(a))| \geq r$. Then $|g(a-\mu(a))| = |g(a)-\mu(a)g(1)| \geq r \geq |g(a)-\mu(a)|$. If $g(1) = 1$ then $|g(a-\mu(a))| = |g(a)-\mu(a)| \leq r \leq \|a\|$. Otherwise the fact that $|g(a)-\mu(a)g(1)| \geq |g(a)-\mu(a)|$ implies that $\|a\| \geq |g(a)| \geq |g(a-\mu(a))|$. Hence $\|a\| \geq \|a-\mu(a)\|$.

Now, by Theorem 5.2 (i),

$$|\lambda(a)-\mu(a)|^2 \leq \|a-\mu(a)\|^2 - \|a-\lambda(a)\|^2$$

$$\leq \|a\|^2 - \|a-\lambda(a)\|^2$$

$$= (\|a\| - \|a-\lambda(a)\|)(\|a\| + \|a-\lambda(a)\|)$$

$$\leq 2|\lambda(a)|\ \|a\|.$$

Q.E.D.

If A is non-commutative then the inequality in (ii) is attained when $a = 1$.

THEOREM 5.10 Let A be a non-commutative C*-algebra. Suppose that A is quasi-central and that whenever three primitive ideals of A contain the same Glimm ideal then their intersection is a primal ideal. Then K(A)=$\frac{1}{2}$.

Proof: Let $a \in$ A. By Propositions 5.3 and 5.7 it is sufficient to show that for any Glimm ideal G and any $\epsilon > 0$ there is a primal ideal S such that $\|a_S - \lambda(a_S)\| \geq \|a_G - \lambda(a_G)\| - \epsilon$.

So let $G \in$ Glimm A. For any $z \in$ Z(A) $D((a-z),\text{A}) = D(a,\text{A})$ and $d((a-z),\text{Z(A)}) = d(a,\text{Z(A)})$, so we can assume, replacing a by $a-z$ for some $z \in$ Z(A) with $z_G = \lambda(a_G)$, that $\lambda(a_G)$=0. It follows from Lemma 5.9 (iii) that 0 is the circumcentre of U(a_G,A/G). If λ is an extreme point of U(a_G,A/G) then the set $\{f \in \text{N}(\text{A}/G) : f(a_G) = \lambda\}$ is a face in N(A/G), so there exists an extreme point g of N(A/G) such that $g(a_G) = \lambda$. Combining this with property (ii) of circumcentres we can obtain three extreme points f, g, h of N(A/G) such that $|f(a_G)| = |g(a_G)| = |h(a_G)| = \|a_G\|$ and such that the circumcircle of the set $\{f(a_G), g(a_G), h(a_G)\}$ is the same as the circumcircle of U(a_G,A/G). By Milman's theorem, see for example [Dixmier 1977; B.14], there are nets (f_α), (g_α) and (h_α) in G(A/G), based on the same directed set, such that (f_α) converges to f, (g_α) converges to g and (h_α) converges to h in N(A/G). For each α let $P_\alpha = \Gamma_A(f_\alpha.\Phi_G)$, $Q_\alpha = \Gamma_A(g_\alpha.\Phi_G)$ and $R_\alpha = \Gamma_A(h_\alpha.\Phi_G)$ (so that P_α, Q_α and R_α are primitive ideals of A containing G). By assumption $S_\alpha = P_\alpha \cap Q_\alpha \cap R_\alpha$ is primal for each α.

If $\mu(a_{S_\alpha})$ denotes the circumcentre of U(a_{S_α},A/S_α) then it is clear that $\mu(a_{S_\alpha})$ converges to 0. It follows from Lemma 5.9 (i) that $\lambda(a_{S_\alpha})$ converges to 0. Choose α such that $\lambda(a_{S_\alpha}) < \epsilon/2$ and $|f_\alpha(a_{S_\alpha}) - f(a_G)| < \epsilon/2$. Then $\|a_{S_\alpha} - \lambda(a_{S_\alpha})\| \geq |f_\alpha(a_{S_\alpha} - \lambda(a_{S_\alpha}))| \geq |f(a_G)| - \epsilon = \|a_G\| - \epsilon$. Q.E.D.

If A is a quasicentral C*-algebra and every Glimm ideal of A is primal then clearly A satisfies the above hypotheses.

COROLLARY 5.11 Let A be a non-commutative C*-algebra. If either

(i) A is prime and has an identity, or

(ii) A is quasi-standard and quasicentral, or

(iii) A is a quasicentral ideal in a quotient of an AW*-algebra

then $K(A)=\frac{1}{2}$.

Proof: As we have just remarked, it is sufficent to show that each Glimm ideal of A is primal. In case (i) this is immediate. In case (ii) this follows from Theorem 3.4. For case (iii), it follows from Lemma 4.2 and Fact (i) of Chapter 4 that if A is a quotient of an AW*-algebra then every Glimm ideal of A is prime, and the same is therefore true if A is a quasicentral ideal in a quotient of an AW*-algebra, by 2.4. Q.E.D.

It has been shown by [Elliott and Zsido 1982; Proposition 3.1], see also [Batty 1984; Proposition 6], that each element of a prime C*-algebra is contained in a separable, prime subalgebra. This subalgebra is primitive, since it is separable, so Theorem 5.2 can be applied to give a different proof of Corollary 5.11 (i). Alternatively, one can use Theorem 5.8 and Corollary 5.11 (i) to give a proof of Elliott and Zsido's result, see Proposition 6.9.

Two cases of Theorem 5.10 are particularly interesting. If A is a unital, non-commutative C*-algebra with trivial centre then it follows from Theorems 5.10 and 5.8 and Corollary 5.11 that either A is prime and $K(A)=K_s(A)=\frac{1}{2}$, or $K(A)\geq K_s(A)\geq 1$. Similarly, if A is a non-commutative, quasicentral, open C*-algebra then either A is quasi-standard and $K(A)=K_s(A)=\frac{1}{2}$, or $K(A)\geq K_s(A)\geq 1$.

We now prove a converse to Theorem 5.10.

THEOREM 5.12 Let A be a C*-algebra. If A has three primitive ideals, containing the same Glimm ideal of A, whose intersection is not primal then K(A)$\geq \frac{1}{\sqrt{3}}$.

Proof: By Proposition 5.4 and Theorem 5.8 we may assume that A is quasicentral and that $\sim$ is an equivalence relation on Prim A. Let P, Q and R be three primitive ideals of A containing the same Glimm ideal G such that $P \cap Q \cap R$ is not primal. Then there exist open sets U,V,W$\subseteq$Prim A such that U$\cap$V$\cap$W $= \emptyset$ and $P \in$U$\setminus$V$\cup$W, $Q \in$V$\setminus$W$\cup$U and $R \in$W$\setminus$U$\cup$V. Because A is quasicentral and $\sim$ is a equivalence relation, $\sim$ is equal to $\approx$, Proposition 3.2, so U$\cap$V, V$\cap$W and W$\cap$U must be non-empty. Rephrasing this in terms of ideals, there exist ideals I, J and K with $IJK = \{0\}$, $IJ \neq \{0\}$, $JK \neq \{0\}$ and $KI \neq \{0\}$ such that $P \supseteq J + K$ but $P \not\supseteq I$, $Q \supseteq K + I$ but $Q \not\supseteq J$ and $R \supseteq I + J$ but $R \not\supseteq K$.

Let $a \in I^+ \setminus P$, $b \in J^+ \setminus Q$ and $c \in K^+ \setminus R$ with $\|a\| = \|a_P\| = \|b\| = \|b_Q\| = \|c\| = \|c_R\| = 1$. Then $ab \in IJ$, $bc \in JK$ and $ca \in KI$. By [Akemann and Pedersen 1977; Corollary 2.4] there are elements $a_1, b_1 \in IJ_{sa}$, $b_2, c_2 \in JK_{sa}$ and $c_3, a_3 \in KI_{sa}$ such that $0 = (a - a_1)(b - b_1) = (b - b_2)(c - c_2) = (c - c_3)(a - a_3)$. Set $d = a - a_1 - a_3$, $f = b - b_1 - b_2$ and $g = c - c_2 - c_3$. Then $d \in I_{sa} \setminus P$, $f \in J_{sa} \setminus Q$ and $g \in K_{sa} \setminus R$ and $df = fg = gd = 0$. Furthermore $\|d_P\| = \|f_Q\| = \|g_R\| = 1$ and d_P, f_Q and g_R are positive. Replacing d, f and g by d^2, f^2 and g^2 and using functional calculus we may assume, in addition, that d, f and g are positive and that $\|d\| = \|f\| = \|g\| = 1$.

Set $h = d + e^{\frac{2\pi i}{3}} f + e^{\frac{4\pi i}{3}} g$. Then $d(h, \mathrm{Z(A)}) = \|h\| = \|h_G\| = 1$. If $S \in$ Prim A then S contains at least one of I, J and K so $\|h_S - \lambda(h_S)\| \leq \frac{\sqrt{3}}{2}$. Hence $\|D(h, \mathrm{A})\| \leq \sqrt{3}$ (with equality in fact) so K(A)$\geq \frac{1}{\sqrt{3}}$. Q.E.D.

THEOREM 5.13 Let A be a C*-algebra. If K_s(A)$=\frac{1}{2}$ then K(A)$\leq \frac{1}{\sqrt{3}}$.

Proof: As in Theorem 5.10 it is sufficient to show that if $a \in A$ then for each Glimm ideal G and each $\epsilon > 0$ there is a primal ideal S such that $\|a_S - \lambda(a_S)\| \geq \frac{\sqrt{3}}{2}\|a_G - \lambda(a_G)\| - \epsilon$. Furthermore we may assume as before that $\lambda(a_G) = 0$. It follows immediately from property (i) of circumcentres, together with Lemma 5.9 (iii), that there exist extreme points $f, g \in N(A/G)$ with $|f(a_G) - g(a_G)| \geq \sqrt{3}\|a_G\|$ and $|f(a_G)| = |g(a_G)| = \|a_G\|$. As in Theorem 5.10 there are nets (f_α) and (g_α) in $G(A/G)$, based on the same directed set, with (f_α) converging to f and (g_α) converging to g. For each α let $P_\alpha = \Gamma_A(f_\alpha.\Phi_G)$ and $Q_\alpha = \Gamma_A(g_\alpha.\Phi_G)$ (so that P_α and Q_α are primitive ideals of A containing G). By the remark after Theorem 5.8 $R_\alpha = P_\alpha \cap Q_\alpha$ is primal for each α. Choose α such that $|f_\alpha(a_G) - g_\alpha(a_G)| \geq \sqrt{3}\|a_G\| - \epsilon$. It follows that for every $\mu \in \mathbf{C}$ either $|f_\alpha(a_{R_\alpha} - \mu)| = |f_\alpha(a_{R_\alpha}) - \mu f_\alpha(1)| \geq \frac{\sqrt{3}}{2}\|a_G\| - \epsilon$ or $|g_\alpha(a_{R_\alpha} - \mu)| \geq \frac{\sqrt{3}}{2}\|a_G\| - \epsilon$. Hence $\|a_{R_\alpha} - \lambda(a_{R_\alpha})\| \geq \max\{|f_\alpha(a_{R_\alpha} - \lambda(a_{R_\alpha}))|, |g_\alpha(a_{R_\alpha} - \lambda(a_{R_\alpha}))\} \geq \frac{\sqrt{3}}{2}\|a_G\| - \epsilon = \frac{\sqrt{3}}{2}\|a_G - \lambda(a_G)\| - \epsilon$. Q.E.D.

We now show that it is possible to have K(A)=$\frac{1}{2}$ and K_s(A)=$\frac{1}{\sqrt{3}}$.

EXAMPLE 5.14 [Archbold 1987; Example 4.12] Let A be the set of all sequences $x = (x_n)_{n \geq 1}$ of 2×2 complex matrices such that, as $n \to \infty$,

$$x_{3n} \to \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{pmatrix}, x_{3n+1} \to \begin{pmatrix} \lambda_2(x) & 0 \\ 0 & \lambda_3(x) \end{pmatrix}, x_{3n+2} \to \begin{pmatrix} \lambda_3(x) & 0 \\ 0 & \lambda_1(x) \end{pmatrix}.$$

With pointwise operations and the supremum norm, A is a C*-algebra. The space Prim A consists of the ideals P_n $(n \geq 1)$ and Q_i $(i = 1, 2, 3)$, where

$$P_n = \{x \in A : x_n = 0\} \text{ and } Q_i = \{x \in A : \lambda_i(x) = 0\}.$$

The space Min-Primal A consists of the ideals P_n $(n \geq 1)$ together with $Q_1 \cap Q_2$, $Q_2 \cap Q_3$ and $Q_3 \cap Q_1$. The space Glimm A consists of the ideals P_n $(n \geq 1)$ and $H = Q_1 \cap Q_2 \cap Q_3$.

Hence $\sim$ is an equivalence relation on Prim A, so $K_s(A)=\frac{1}{2}$ by Theorem 5.8, but H is not primal, so $K(A)=\frac{1}{\sqrt{3}}$ by Theorems 5.12 and 5.13.

Once it is known that $K(A)$ need not equal $K_s(A)$ it is natural to enquire how large $K(A)/K_s(A)$ can be. We remarked in the introduction that it must be less than or equal to 2, but Theorems 5.12 and 5.13, together with some results of the next chapter, suggest that perhaps $\frac{2}{\sqrt{3}}$, as above, is the largest possible value. I have not been able to establish this. There is, however, a partial result.

Define $K_n(A)$ to be the smallest number in $[0, \infty]$ such that

$$d(a, Z(A)) \leq K_n(A)\|D(a, A)\| \quad \text{for all normal } a \in A.$$

Clearly $K_s(A) \leq K_n(A) \leq K(A)$. No example is known with $K_n(A) \neq K(A)$.

PROPOSITION 5.15 If A is a quasicentral C*-algebra then $K_n(A) \leq \frac{2}{\sqrt{3}} K_s(A)$.

Proof: As in previous proofs it is sufficient to show that if a is a normal element of A and G is a Glimm ideal then for any $\epsilon > 0$ there is a primal ideal R such that $\|a_R - \lambda(a_R)\| \geq \frac{\sqrt{3}}{4k_s(A)}\|a_G - \lambda(a_G)\| - \epsilon$.

Since a is normal $\lambda(a_G)$ is the circumcentre of the spectrum of a, so it follows immediately from property (i) of circumcentres that there are points μ_1 and μ_2 in the spectrum of a such that $|\mu_1 - \mu_2| \geq \sqrt{3}\|a_G - \lambda(a_G)\|$. Multiplying a by a suitable scalar we may assume that $\mu_1 - \mu_2 \geq \sqrt{3}\|a_G - \lambda(a_G)\|$. Let b be the real part of a. Then $\|b_G - \lambda(b_G)\| \geq \frac{1}{2}(\mu_1 - \mu_2)$. By Theorem 5.2 and the definition of $K_s(A)$ there is a primitive ideal R such that $\|b_R - \lambda(b_R)\| \geq \frac{1}{2K_s(A)}\|b_G - \lambda(b_G)\| - \epsilon$. But $\|a_R - \lambda(a_R)\| \geq \|b_R - \lambda(b_R)\|$. Hence $\|a_R - \lambda(a_R)\| \geq \frac{\sqrt{3}}{4K_s(A)}\|a_G - \lambda(a_G)\| - \epsilon$. Q.E.D.

We remark that the proof can be modified to cope with the case when A is not quasi-central.

The final example shows that even if the set of primitive, Glimm ideals is dense in Glimm A this places no restriction on the size of K(A).

EXAMPLE 5.16 Let m be any fixed integer. Let A be the set of all sequences $x = (x_n)_{n\geq 1}$ of 2×2 complex matrices such that,

$$x_{mn} \to \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{pmatrix} x_{mn+1} \to \begin{pmatrix} \lambda_2(x) & 0 \\ 0 & \lambda_3(x) \end{pmatrix} \cdots$$

$$\text{and } x_{mn+m-1} \to \begin{pmatrix} \lambda_m(x) & 0 \\ 0 & \lambda_{m+1}(x) \end{pmatrix}$$

as $n \to \infty$. With pointwise operations and the supremum norm, A is a C*-algebra. The space Prim A consists of the ideals P_n $(n \geq 1)$ and Q_i $(i = 1, 2, ..., m+1)$, where

$$P_n = \{x \in \text{A} : x_n = 0\} \text{ and } Q_i = \{x \in \text{A} : \lambda_i = 0\}.$$

Glimm A consists of the ideals P_n $(n \geq 1)$ together with $H = Q_1 \cap Q_2 \cap ... \cap Q_{m+1}$. Hence the set of primitive, Glimm ideals is dense in Glimm A. However, if a is the self-adjoint element

$$a_{mn} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, a_{mn+1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, ..., a_{mn+m-1} = \begin{pmatrix} m-1 & 0 \\ 0 & m \end{pmatrix}$$

then it is easy to check, using Propositions 5.3 and 5.7, that $d(a,\text{Z(A)}) = m/2$ while $\|D(a,\text{A})\| = 1$. Hence $\text{K(A)} \geq \text{K}_s\text{(A)} \geq m/2$. In fact it can easily be shown that $\text{K(A)} = \text{K}_s\text{(A)} = m/2$.

6. C*-ALGEBRAS WITH TRIVIAL CENTRE

In this chapter we concentrate mainly on unital C*-algebras A with trivial centre. We obtain upper and lower bounds for $K_s(A)$ in terms of the ideal structure of A, and we show that these bounds are equal if A has only finitely many minimal primal ideals. It follows that in this case $K_s(A)$ is completely determined by the ideal structure of A, and is always a multiple of a half. We then obtain some partial results for $K(A)$. Finally we apply these results to C*-algebras with non-trivial centre, showing that if A is any weakly central C*-algebra then $K(A) \leq 1$.

We start with some definitions. Let A be a unital C*-algebra with trivial centre. Define two ideals P and Q of A to be *adjacent*, denoted $P * Q$, if $P + Q \neq A$. This makes Id A, and hence any subset X of Id A, into a graph. Given two points P and Q in X, a path in X of length n from P to Q is a sequence $(P_0, P_1, ..., P_n)$ in X such that $P_0 = P$, $P_n = Q$ and $P_i * P_{i-1}$ $(i = 1, 2, ..., n)$. The *distance* from P to Q in X, $d_X(P,Q)$, is defined to be the length of the shortest path in X from P to Q (if no such path exists then $d(P,Q) = \infty$), and the *diameter* of X, diam X, is the supremum of $d_X(P,Q)$ for $P, Q \in X$.

If Y and Z are finite subsets of Id A such that P and Q are not adjacent for each $P \in Y$, $Q \in Z$, then ker Y and ker Z are not adjacent. Since A has trivial centre, Prim A is connected (as a topological space). Therefore if X is finite and ker X $= \{0\}$, it follows that X is connected as a graph, and hence diam X $< \infty$.

A short argument shows that diam Min-Primal A $\leq$ diam Prim A. Let $D(A) =$ diam Min-Primal A. Note that $D(A) = 0 \iff$ A has a unique minimal primal ideal $\iff$ A is prime $\iff K_s(A) = \frac{1}{2}$ or $A = \mathbf{C}$ (by Theorem 5.8).

A *chain* in Id A is a path $(P_0, P_1, ..., P_n)$ such that $P_0 \cap P_1 \cap ... \cap P_n = \{0\}$, P_1

is not contained in P_0, P_{n-1} is not contained in P_n, and P_i is not adjacent to P_j if $|i-j| > 1$. Let C(A) be the length of the longest chain in Id A (with C(A) $= \infty$ if there are arbitrarily long chains).

The definition of primality, 1.4, implies that each primal ideal must contain at least one ideal in each chain. Conversely, the definition of a chain implies that each ideal in a chain is contained in a minimal primal ideal. If S and T are adjacent primal ideals with $S \supseteq P_i$ and $T \supseteq P_j$ then clearly P_i and P_j are adjacent. It follows that C(A) $\leq$ D(A).

If B is any unital C*-algebra and $b \in$ B we define the *numerical range V(b,B)* of b by V(b,B) $= \{\phi(b) : \phi \in$ S(B)$\}$. Then V(b,B) is a compact, convex set, $\lambda(b) \in$ V(b,B), and if b is normal V(b,B) is the convex hull of the spectrum of b [Doran and Belfi 1986; pp.181-182]. If $b \in B_{sa}$ let $|$V(b,B)$|$ denote the length of the interval V(b,B). Note that $|V(b,B)| = 2\|b - \lambda(b)\|$.

We now prove an algebraic lemma.

LEMMA 6.1 Let A be a C*-algebra and let $X = \{R_0, R_1, ..., R_n\}$ be a chain of length n in Id A. Let $a^i \in$ A $(0 \leq i \leq n)$ (i is a superscript here, not a power), and suppose that $a^i_{R_i+R_{i+1}} = a^{i+1}_{R_i+R_{i+1}}$ for $0 \leq i \leq n-1$. Then there exists a unique $a \in$ A such that $a_{R_i} = a^i_{R_i}$ for each i.

Proof: The uniqueness of a follows from the fact that ker X is zero. We prove the existence of a by induction, as follows. Suppose that there is a $b^j \in$ A such that $b^j_{R_i} = a^i_{R_i}$ for $i = 0, 1, ..., j$. Set $Q_j = R_0 \cap R_1 \cap ... \cap R_j$. Since R_{j+1} is not adjacent to R_i for $i = 0, 1, ..., j-1$, $R_0 \cap R_1 \cap ... \cap R_{j-1} + R_{j+1} =$ A so $Q_j + R_{j+1} = R_j + R_{j+1}$. Hence $b^j_{Q_j+R_{j+1}} = a^j_{Q_j+R_{j+1}} = a^{j+1}_{Q_j+R_{j+1}}$, that is, $b^j - a^{j+1} \in Q_j + R_{j+1}$. Hence there exist $c^j \in Q_j$ and $d^{j+1} \in R_{j+1}$ such that $b^j - a^{j+1} = c^j + d^{j+1}$, so $c^j_{R_{j+1}} = b^j_{R_{j+1}} - a^{j+1}_{R_{j+1}}$. Set $b^{j+1} = b^j - c^j$. Then for $0 \leq i \leq j$ $b^{j+1}_{R_i} = b^j_{R_i} = a^i_{R_i}$, while $b^{j+1}_{R_{j+1}} = b^j_{R_{j+1}} - c^j_{R_{j+1}} =$

$a^{j+1}_{R_{j+1}}$. Hence b^{j+1} satisfies the induction hypothesis. It follows by induction that $a = b^n$ has the required property. Q.E.D.

PROPOSITION 6.2 Let A be a unital C*-algebra with trivial centre. Then

$$\frac{1}{2}(\mathrm{C(A)}+1) \leq \mathrm{K}_s(\mathrm{A}) \leq \frac{1}{2}(\mathrm{D(A)}+1).$$

Proof: If $\mathrm{D(A)} = \infty$ the second inequality is trivial, so suppose that $\mathrm{D(A)} < \infty$. Let $a \in \mathrm{A}_{sa}$ and let α and β be the largest and smallest points in the spectrum of a. There is a primitive ideal R such that $\alpha \in \mathrm{V}(a_R,\mathrm{A}/R)$ so if S is a minimal primal ideal contained in R then $\alpha \in \mathrm{V}(a_S,\mathrm{A}/S)$. Similarly there is a minimal primal ideal T such that $\beta \in \mathrm{V}(a_T,\mathrm{A}/T)$. If $P, Q \in$ Min-Primal A with $P * Q$ then $\mathrm{V}(a_P,\mathrm{A}/P) \cap \mathrm{V}(a_Q,\mathrm{A}/Q) \supseteq \mathrm{V}(a_{P+Q},\mathrm{A}/(P+Q)) \neq \emptyset$. It follows that there must be at least one $P \in$ Min-Primal A with $|\mathrm{V}(a_P,\mathrm{A}/P)| \geq 2(\mathrm{D(A)}+1)^{-1}\|a-\lambda(a)\|$. Hence $\|D(a,\mathrm{A})\| \geq |\mathrm{V}(a_P,\mathrm{A}/P)| \geq 2(\mathrm{D(A)}+1)^{-1}\|a-\lambda(a)\|$, using Proposition 5.7, so $\mathrm{K}_s(\mathrm{A}) \leq \frac{1}{2}(\mathrm{D(A)}+1)$.

For the first inequality, let $\mathrm{X} = \{R_0, R_1, ..., R_n\}$ be a chain of length n in Id A. We will construct an element $a \in \mathrm{A}^+$ such that $\mathrm{V}(a,\mathrm{A}) = [0, n+1]$, $\mathrm{V}(a_{R_i},\mathrm{A}/R_i) = [i, i+1]$ $(i = 0, 1, ..., n)$ and $\mathrm{V}(a_{R_i+R_{i+1}},\mathrm{A}/R_i + R_{i+1}) = \{i+1\}$ $(i = 0, 1, ..., n-1)$. It follows that if P is a primal ideal then $|\mathrm{V}(a_P,\mathrm{A}/P)| \leq 1$. Hence $\|D(a,\mathrm{A})\| \leq 1$, by Proposition 5.7, while $d(a, \mathrm{Z(A)}) = \frac{n+1}{2}$. This implies that $\mathrm{K}_s(\mathrm{A}) \geq \frac{n+1}{2}$. Since the construction works for a chain of any length it follows that $\mathrm{K}_s(\mathrm{A}) \geq \frac{1}{2}(\mathrm{C(A)}+1)$.

We construct a as follows. Since R_1 is not contained in R_0 there exists $r^0 \in (R_0+R_1)^+$ with $\|r^0_{R_0}\| = 1$ (0 is a superscript here). Set $a^0 = 1 - r^0$. Then $a^0_{R_0+R_1} = 1$ and $\mathrm{V}(a^0_{R_0},\mathrm{A}/R_0) = [0,1]$. For $1 \leq i \leq n-1$ note that

$$\frac{\mathrm{A}}{R_{i-1}+R_i} = \frac{(R_{i-1}+R_i)+(R_i+R_{i+1})}{R_{i-1}+R_i}$$

$$\cong \frac{R_i+R_{i+1}}{(R_{i-1}+R_i)\cap(R_i+R_{i+1})}.$$

Hence there exists $r^i \in (R_i + R_{i+1})^+$ with $\|r^i\| = 1$ such that $r^i_{R_{i-1}+R_i} = 1$. Set $a^i = i+1-r^i$. Then $a^i_{R_{i-1}+R_i} = i$, $a^i_{R_i+R_{i+1}} = i+1$ and $\mathrm{V}(a^i_{R_i}, \mathrm{A}/R_i) = [i, i+1]$. Finally, since R_{n-1} is not contained in R_n there exists $r^n \in (R_{n-1}+R_n)^+$ with $\|r^n_{R_n}\| = 1$. Set $a^n = n + r^n$. Then $a^n_{R_{n-1}+R_n} = n$ and $\mathrm{V}(a^n_{R_n}, \mathrm{A}/R_n) = [n, n+1]$.

The elements a_i $(i = 0, 1, ..., n)$ satisfy the hypotheses of Lemma 6.1 so it follows from Lemma 6.1 that there exists $a \in \mathrm{A}$ such that $a_{R_i} = a^i_{R_i}$ for each i. It is easy to check that a has the required properties. Q.E.D.

THEOREM 6.3 Let A be a unital C*-algebra with trivial centre. If Min-Primal A is finite then $\mathrm{K}_s(\mathrm{A}) = \frac{1}{2}(\mathrm{D}(\mathrm{A})+1)$.

Proof: We remarked earlier that if X is a finite collection of ideals with zero intersection then X is a connected graph and diam $\mathrm{X} \leq \infty$. In particular, since Min-Primal A is finite D(A) must be finite. Let $P, Q \in$ Min-Primal A with $\mathrm{d}(P,Q) = \mathrm{D}(\mathrm{A})$. Set $R_0 = P$ and set $R_i = \bigcap\{R \in \text{Min-Primal A: } \mathrm{d}(P,R) = i\}$ $(i = 1, 2, ..., \mathrm{D}(\mathrm{A}))$. If $R_0 = P$ contained R_1 then ker Min-Primal A$\backslash\{P_0\}$ would be zero, so P_0 would contain another minimal primal ideal, contradicting the fact that it is minimal primal. Hence R_0 does not contain R_1. A similar argument shows that $R_{\mathrm{D}(\mathrm{A})}$ does not contain $R_{\mathrm{D}(\mathrm{A})-1}$. It is easy to check that $(R_0, R_1, ..., R_{\mathrm{D}(\mathrm{A})})$ satisfies the other conditions for a chain. The result now follows from Proposition 6.2. Q.E.D.

The theorem above, together with Theorem 5.8, suggests that perhaps $\mathrm{K}_s(\mathrm{A})$ is a multiple of a half for any C*-algebra A, but I have not been able to prove this.

We now illustrate Theorem 6.3 with some examples, which are modifications of [Kadison, Lance and Ringrose 1967; 6.2] and [Archbold 1972; p.676].

EXAMPLE 6.4 Let H_i $(i = 1, 2, ..., n)$ be separable Hilbert spaces and let E_i and F_i

$(i = 1, 2, ..., n)$ be projections on H_i with infinite dimensional ranges and sum 1. Let B be the abelian C*-algebra of all bounded operators on $H = \oplus_{i=1}^{n} H_i$ of the form

$$a_1 E_1 + a_2(F_1 + E_2) + a_3(F_2 + E_3) + ... + a_n(F_{n-1} + E_n) + a_{n+1} F_n.$$

Let $C = \oplus_{i=1}^{n} LC(H_i)$. Let A=B+C.

Then Z(A) = **C**1 and A has exactly n minimal primal ideals, namely the kernels of the restrictions of A to the Hilbert spaces H_i $(i = 1, 2, ..., n)$, which we will call P_i $(i = 1, 2, ..., n)$. For $1 \leq i \leq n-1$ $P_i + P_{i+1}$ is contained in the maximal ideal consisting of all elements having $a_{i+1} = 0$, while $P_i + P_j = A$ if $|i - j| \geq 2$. It follows that D(A) $= n - 1$, so $K_s(A) = \frac{n}{2}$, by Theorem 6.3.

The original C*-algebra A of [Kadison, Lance and Ringrose 1967; 6.2] was the same as Example 6.4 but with an infinite seqence of Hilbert spaces. For this C*-algebra K(A) $= \infty$.

EXAMPLE 6.5 As in Example 6.4, with $n = 3$ and $a_2 = a_4$ for all elements. This time D(A) = 1 so Theorem 6.3 implies that $K_s(A) = 1$. Note that $P_1 + P_2 + P_3$ is contained in the maximal ideal of all elements with $a_2 = 0$.

It is easy to modify Example 6.4 to obtain any connected, finite graph. For one more example:

EXAMPLE 6.6 As in Example 6.4, with $n = 4$ and $a_1 = a_4$ for all elements. Then D(A) = 2, so $K_s(A) = \frac{3}{2}$, by Theorem 6.3.

We now turn our attention to the study of K(A) when A is unital with trivial centre. This is more complicated and our results are only partial, even when Min-Primal A is finite.

For $a \in$ A and $P \in$ Min-Primal A let $X(a,P)$ denote the set $\{\mu \in \mathbf{C} : |\mu - \lambda(a_P)| \leq \|a_P - \lambda(a_P)\|\}$. If $P \in$ Min-Primal A and M is an ideal containing P then, by Theorem 5.2 (i),

$$\|a_M - \lambda(a_M)\|^2 + |\lambda(a_M) - \lambda(a_P)|^2 \leq \|a_M - \lambda(a_P)\|^2$$

$$\leq \|a_P - \lambda(a_P)\|^2.$$

Hence $\lambda(a_M) \in X(a,P)$. It follows that if $P, Q \in$ Min-Primal A and $P * Q$ then $X(a,P) \cap X(a,Q)$ is non-empty.

Let $X(a) = \bigcup\{X(a,P) : P \in \mathrm{Min-Primal\ A}\}$ and let $\chi(a)$ be the circumcentre of the set $\{\lambda(a_P) : P \in \mathrm{Min-Primal\ A}\}$. Since $\|D(a, \mathrm{A})\|$ is equal to the supremum of the diameters of the disks $X(a,P)$, Proposition 5.7, we have the following two elementary inequalities:

(i) If D(A)$< \infty$ then

$$\sup\ \{|\chi(a) - \lambda(a_P)| : P \in \mathrm{Min-Primal\ A}\} \leq \frac{1}{\sqrt{3}}\mathrm{D(A)}\|D(a,\mathrm{A})\|.$$

(ii) If Min-Primal A is finite, of cardinality n, then

$$\max\ \{|\chi(a) - \lambda(a_P)| : P \in \mathrm{Min-Primal\ A}\} \leq \frac{n-1}{2}\|D(a,\mathrm{A})\|.$$

THEOREM 6.7 Let A be a unital C*-algebra with trivial centre. Then K(A) $\leq \frac{1}{\sqrt{3}}$D(A) $+ \frac{1}{2}$. If Min-Primal A is finite, of cardinality n, then K(A) $\leq \frac{n}{2}$.

Proof: We prove the first inequality. The second is proved in the same way. Let $a \in$ A with $\chi(a)$ as defined above. Then

$$\|a - \chi(a)\| = \sup\ \{\|a_P - \chi(a)\| : P \in \mathrm{Min-Primal\ A}\}$$

$$\leq \sup\ \{\|a_P - \lambda(a_P)\| + |\lambda(a_P) - \chi(a)| : P \in \mathrm{Min-Primal\ A}\}$$

$$\leq \sup \{\|a_P - \lambda(a_P)\| + \frac{1}{\sqrt{3}}\mathrm{D}(\mathrm{A})\|D(a, \mathrm{A})\| : P \in \text{Min} - \text{Primal A}\}$$

(by inequality (1) above)

$$= (\frac{1}{\sqrt{3}}\mathrm{D}(\mathrm{A}) + \frac{1}{2})\|D(a, \mathrm{A}\|.$$

Hence $\mathrm{K}(\mathrm{A}) \leq \frac{1}{\sqrt{3}}\mathrm{D}(\mathrm{A}) + \frac{1}{2}$. Q.E.D.

As one application of this theorem, let A be the C*-algebra of Example 6.4. Then A has n minimal primal ideals, so $\frac{n}{2} = \mathrm{K}_s(\mathrm{A}) \leq \mathrm{K}(\mathrm{A}) \leq \frac{n}{2}$. Hence $\mathrm{K}(\mathrm{A}) = \mathrm{K}_s(\mathrm{A}) = \frac{n}{2}$.

Usually, however, the inequalities in the theorem are not attained, and this is not surprising. For instance, if A is the C*-algebra of Example 6.6 then $\mathrm{D}(\mathrm{A}) = 2$ and $n = 4$ (there are two different n's but they both equal four) so Theorem 6.7 implies that K(A) is less than or equal to the minimum of $\frac{2}{\sqrt{3}} + \frac{1}{2}$ and 2, which is $\frac{2}{\sqrt{3}} + \frac{1}{2}$. But it is easy to see, from considering the possible ways of arranging four disks subject to the intersection conditions, that in fact $\mathrm{K}(\mathrm{A}) \leq \frac{3}{2}$, so $\mathrm{K}(\mathrm{A}) = \frac{3}{2}$.

In Example 6.6 A didn't have enough minimal primal ideals to attain the inequality, but there is another thing which can restrict K(A), and that is when A has ideals which contain more than two minimal primal ideals. The next theorem illustrates this.

We will use the following geometrical fact: suppose that X is a collection of closed disks in C such that if $A, B, C \in \mathrm{X}$ then $A \cap B \cap C$ is non-empty. Then $\bigcap\{D : D \in \mathrm{X}\}$ is non-empty.

THEOREM 6.8 Let A be a unital C*-algebra and suppose that for any three minimal primal ideals P, Q, R $P + Q + R \neq \mathrm{A}$. Then $\mathrm{K}(\mathrm{A}) \leq 1$.

Proof: If P and Q are minimal primal ideals of A then $P \cap \mathrm{Z}(\mathrm{A})$ and $Q \cap \mathrm{Z}(\mathrm{A})$ are maximal ideals of Z(A), 2.3 and Lemma 2.3, so either $P \cap \mathrm{Z}(\mathrm{A}) = Q \cap \mathrm{Z}(\mathrm{A})$ or $1 \in P + Q$ so $P + Q = \mathrm{A}$. It therefore follows from the hypotheses that the centre of A is trivial.

Let $a \in$ A. The hypotheses imply that if P, Q and R are minimal primal ideals of A then $X(a,P) \cap X(a,Q) \cap X(a,R)$ is non-empty. It follows from the geometrical fact mentioned above that $\bigcap\{X(a,P) : P \in \text{Min} - \text{Primal A}\}$ is non-empty. Let μ be a point in this intersection. Then

$$\begin{aligned} \|a - \mu\| = & \sup\{\|a_P - \mu\| : P \in \text{Min} - \text{Primal A}\} \\ \leq & \sup\{\|a_P - \lambda(a_P)\| + |\lambda(a_P) - \mu| : P \in \text{Min} - \text{Primal A}\} \\ \leq & \sup\{\|a_P - \lambda(a_P)\| + \frac{1}{2}\|D(a, \text{A})\| : P \in \text{Min} - \text{Primal A}\} \\ & = \|D(a, \text{A})\|. \end{aligned}$$

Hence $\text{K(A)} \leq 1$. Q.E.D.

We know from Theorem 5.8 that if A above is not prime then in fact $\text{K(A)} = \text{K}_s(\text{A})$ $= 1$. In particular, if A is the C*-algebra of Example 6.5 then $\text{K(A)} = \text{K}_s(\text{A}) = 1$.

PROPOSITION 6.9 Let A be a unital C*-algebra with trivial centre and $0 < \text{K(A)} < \infty$. If B is a separable subalgebra of A then there is a separable subalgebra C of A with trivial centre, containing B, such that $\text{K(C)} \leq \text{K(A)}$.

Proof: If $1 \in$ B set $\text{B}_0 = \text{B}$, otherwise set $\text{B}_0 = \text{B} + \mathbb{C}1$. For $n \geq 1$ we define increasing, separable C*-algebras B_n inductively, as follows. Suppose that we have defined B_{n-1}. Let $\{b^i_{n-1}\}_{i\geq 1}$ be a countable, dense subset of B_{n-1} (i is a superscript here). For each i let $c^i_{n-1} \in$ A with $\|c^i_{n-1}\| = 1$ such that $\|b^i_{n-1}c^i_{n-1} - c^i_{n-1}b^i_{n-1}\| \geq \|D(b^i_{n-1}, \text{A})\| - \frac{1}{n}$. Let B_n be the C*-algebra generated by B_{n-1} and the set $\{c^i_{n-1}\}_{i\geq 1}$. Let C be the closure of $\bigcup_{n\geq 0} \text{B}_n$. Then B_n and C are separable.

Let $b \in$ C and let $\epsilon \geq 0$ be given. Choose n such that $\frac{1}{n} < \epsilon$ and such that there exists $d \in \text{B}_n$ with $\|b - d\| < \epsilon$. There exists an i such that $\|d - b^i_n\| < \epsilon$. Then

$$\text{K(A)}\|D(b, \text{C})\| \geq \text{K(A)}\|bc^i_n - c^i_n b\| \geq \text{K(A)}\|D(b^i_n, \text{A})\| - \frac{\text{K(A)}}{n+1} - 4\epsilon\text{K(A)}$$

$$\geq d(b_n^i, \mathrm{C}1) - \frac{\mathrm{K(A)}}{n+1} - 4\epsilon\mathrm{K(A)} \geq d(b, \mathrm{C}1) - 5\epsilon\mathrm{K(A)} - 2\epsilon.$$

Since ϵ is arbitrary it follows that Z(C) = C1 and K(C)$\leq$K(A). Q.E.D.

Clearly an identical result is true for $\mathrm{K}_s(\mathrm{A})$. Note that if A is prime then it follows from Theorem 5.8 that C constructed above is also prime, and hence primitive, see [Elliott and Zsido 1982; Proposition 3.1] and [Batty 1984; Proposition 6]. It is also easy to see that one can force K(C) to equal K(A) by adjoining a countable collection of elements.

We now return to C*-algebras with non-trivial centre. The following proposition allows us to use Theorem 6.8.

PROPOSITION 6.10 Let A be a C*-algebra and suppose that for each $G \in$ Glimm A $\mathrm{Z}(\mathrm{A}/G) = \Phi_G(\mathrm{Z}(\mathrm{A}))$. Then

$$\mathrm{K(A)} \leq \sup \{\mathrm{K}(\mathrm{A}/G) : G \in \text{Glimm A}\}.$$

Proof: Suppose that $\sup \{\mathrm{K}(\mathrm{A}/G) : G \in \text{Glimm A}\} = L < \infty$, since otherwise the result is trivial. Let $a \in$ A. Then, using Proposition 5.3,

$$d(a, \mathrm{Z(A)}) = \sup \{d(a_G, \Phi_G(\mathrm{Z(A)})) : G \in \text{Glimm A}\}$$

$$= \sup \{d(a_G, \mathrm{Z}(\mathrm{A}/G)) : G \in \text{Glimm A}\} \quad \text{(by assumption)}$$

$$\leq \sup \{L \, \|D(a_G, \mathrm{A}/G)\| : G \in \text{Glimm A}\}$$

$$\leq L\|D(a, \mathrm{A})\| \quad \text{(by Lemma 5.1)}.$$

Hence $\mathrm{K(A)} \leq L$. Q.E.D.

Recall the definition of a weakly central C*-algebra from 1.9. It was proved in [Archbold 1978; Theorem 4.1] that if A is a unital weakly central C*-algebra then $K_s(A) \leq 1$, and that the same is true for any ideal in A. Since ideals in weakly central C*-algebras are themselves weakly central, according to 1.9, Archbold's result says that $K_s(A) \leq 1$ for any weakly central C*-algebra.

THEOREM 6.11 If A is a weakly central C*-algebra then $K(A) \leq 1$.

Proof: It follows from Vesterstrøm's result mentioned in 1.9 that A satisfies the hypotheses of Proposition 6.10. We may assume therefore that A has trivial centre. If A is unital then $K(A) \leq 1$ by Theorem 6.8, since A has a unique maximal ideal.

Let B be a non-unital, primitive algebra with $b \in B$. Then

$$d(b, Z(B)) = \|b\| \leq \|b - \lambda(b)\| + |\lambda(b)| \qquad (\text{in } B + \mathbf{C}1)$$

$$\leq 2\ \|b - \lambda(b)\| \qquad (\text{since } 0 \text{ is in the spectrum of } B)$$

$$= \|D(b, B)\| \qquad (\text{by Theorem } 5.1).$$

Hence $K(B) \leq 1$. Now, if A is non-unital then every quotient of A is also non-unital, by Vesterstrøm's result, so for $a \in A$

$$d(a, Z(A)) = \|a\| = \sup\ \{\|a_P\| : P \in \text{Prim } A\}$$

$$\leq \sup\ \{\|D(a_P, A/P)\| : P \in \text{Prim } A\}$$

$$(\text{since every primitive quotient is non} - \text{unital})$$

$$= \|D(a, A)\|.$$

Again $K(A) \leq 1$. Q.E.D.

7. OPEN C*-ALGEBRAS

In this chapter we consider K(A) and K_s(A) when A is an open C*-algebra, that is, when the complete regularisation map ϕ_A is open, see Theorem 2.1. We show that if A is a separable, open C*-algebra then K(A) and K_s(A) are completely determined by the values of K(A/G) and K_s(A/G) on a dense subset of the Glimm ideals G. The proofs for K_s(A) are virtually identical to those for K(A) and we omit them.

LEMMA 7.1 If A is an open C*-algebra then for each $a \in$ A the function $G \mapsto \|D(a_G,A/G)\|$ ($G \in$ Glimm A) is lower semicontinuous on Glimm A. Hence if X is a dense subset of Glimm A $\|D(a,A)\| = \sup\{\|D(a_G,A/G)\| : G \in X\}$.

Proof: For each $a \in$ A the function $G \mapsto \|a_G\|$ is continuous on Glimm A, Theorem 2.1, so the function $G \mapsto \|D(a_G,A/G)\|$ is lower semicontinuous, since it is the supremum of a collection of continuous functions. Q.E.D.

PROPOSITION 7.2 Let A be a quasicentral, open C*-algebra. For each $a \in$ A there is a $z \in$ Z(A) such that for all $G \in$ Glimm A

$$z_G = \lambda(a_G).$$

Proof: It is enough, by Theorem 1.1, to show that for each $a \in$ A the function $G \mapsto \lambda(a_G)$ is continuous on Glimm A and vanishes at infinity. The second of these requirements follows from Theorem 5.2 (i) (taking $\mu = 0$) and 1.3.

Let $H \in$ Glimm A and suppose that (G_α) is a net in Glimm A converging to H. Let $a \in$ A. We may assume by compactness, and by passing to a subnet of (G_α) if necessary, that $\lambda(a_{G_\alpha})$ converges to some number, μ say, in **C**. From Theorem 5.2 (i) we know that

$$\|a_H - \lambda(a_H)\|^2 + |\lambda(a_H) - \mu|^2 \leq \|a_H - \mu\|^2.$$

Let $\epsilon > 0$ be given. By Theorem 2.1 (ii) there is a neighbourhood N of H such that for all $G \in$ N

$$\|a_G - \lambda(a_H)\|^2 + |\lambda(a_H) - \mu|^2 < \|a_G - \mu\|^2 + \epsilon.$$

But for each G_α

$$\|a_{G_\alpha} - \lambda(a_{G_\alpha})\|^2 + |\lambda(a_{G_\alpha}) - \lambda(a_H)|^2 \leq \|a_{G_\alpha} - \lambda(a_H)\|^2$$

so for $G_\alpha \in$ N

$$\|a_{G_\alpha} - \lambda(a_{G_\alpha})\|^2 + |\lambda(a_{G_\alpha}) - \lambda(a_H)|^2 + |\lambda(a_H) - \mu|^2 < \|a_{G_\alpha} - \mu\|^2 + \epsilon.$$

But $|\ \|a_{G_\alpha} - \lambda(a_{G_\alpha})\|^2 - \|a_{G_\alpha} - \mu\|^2|$ becomes arbitrarily small since $\lambda(a_{G_\alpha}) \to \mu$. Since ϵ is also arbitrary it follows that $\mu = \lambda(a_H)$, which proves the proposition. Q.E.D.

THEOREM 7.3 Let A be a quasicentral, open C*-algebra. If X is a dense subset of Glimm A such that the centre of A/G is trivial for all $G \in$ X then K(A) $\leq$ sup $\{$K(A/G) : $G \in$ X$\}$.

Proof: We may assume that sup $\{$K(A/G): $G \in$ X$\}$ $= L < \infty$. Let $a \in$ A. Then

$$d(a, \mathrm{Z}(\mathrm{A})) = \sup\{\|a_G - \lambda(a_G)\| : G \in \mathrm{X}\} \qquad \text{(by Proposition 7.2)}$$

$$\leq \sup\{L\|D(a_G, \mathrm{A}/G)\| : G \in \mathrm{X}\} \qquad \text{(since each A/}G\text{ has trivial centre)}$$

$$= L\|D(a, \mathrm{A})\| \qquad \text{(Lemma 7.1).}$$

Hence K(A) $\leq L$. Q.E.D.

Theorem 7.3 and Corollary 3.6 provide an alternative proof of Corollary 5.11 (ii), in the separable case. We will see soon that there is a converse to Theorem 7.3.

PROPOSITION 7.4 Let A be a quasicentral, open C*-algebra with $0 < \mathrm{K(A)} < \infty$. Then for any $a \in \mathrm{A}$, and any $\epsilon > 0$, the set

$$\{G \in \text{Glimm A} : (\mathrm{K(A)} + \epsilon)\|D(a_G, \mathrm{A}/G)\| > \|a_G - \lambda(a_G)\|\}$$

is a dense, open subset of the open set

$$\{G \in \text{Glimm A} : a_G \neq \lambda(a_G)\}.$$

Proof: The openness of these two sets follows from the continuity of $\|a_G - \lambda(a_G)\|$, Proposition 7.2, and the lower semicontinuity of $\|D(a_G,\mathrm{A}/G)\|$, Lemma 7.1.

To prove density, let $a \in \mathrm{A}$ and suppose that $H \in$ Glimm A with $a_H \neq \lambda(a_H)$. Let W be any open neighbourhood of H. Let X be the open neighbourhood of H, contained in W, defined by

$$\mathrm{X} = \{G \in \mathrm{W} : \|a_G - \lambda(a_G)\| < \mathrm{K(A)} + \epsilon \; over \mathrm{K(A)} \|a_H - \lambda(a_H)\|.\}$$

Let $f \in C_0(\text{Glimm A})$ be a positive function of norm one with $f(H) = 1$ and $f(G) = 0$ for all $G \in$ Glimm A\X. Because A is quasicentral there is a $z \in \mathrm{Z(A)}$ such that $z_G = f(G)$ for all $G \in$ Glimm A, 1.8. Set $b = za$. Then

$$\mathrm{K(A)}\|D(b, \mathrm{A})\| \geq d(b, \mathrm{Z(A)})$$

$$= \sup\{\|b_G - \lambda(b_G)\| : G \in \text{Glimm A}\}$$

$$\geq \|b_H - \lambda(b_H)\|$$

$$= \|a_H - \lambda(a_H)\| \qquad \text{since } f(H) = 1.$$

But for each $G \in$ Glimm A $D(b_G,\mathrm{A}/G) = f(G)D(a_G,\mathrm{A}/G)$, so $D(b_G,\mathrm{A}/G)$ is zero for $G \in$ Glimm A\X. Hence there must be a Glimm ideal $L \in \mathrm{X}$ such that

$$\mathrm{K(A)}\|D(b_L, \mathrm{A}/L)\| = \mathrm{K(A)}f(L)\|D(a_L, \mathrm{A}/L)\| \geq \|a_H - \lambda(a_H)\|$$

$$> \frac{K(A)}{K(A)+\epsilon}\|a_L - \lambda(a_L)\| \qquad \text{by definition of X.}$$

Since $f(L) \leq 1$ it follows that $(K(A)+\epsilon)\|D(a_L, A/L)\| > \|a_L - \lambda(a_L)\|$. Q.E.D.

COROLLARY 7.5 Let A be a quasicentral, open C*-algebra with $K(A) < \infty$. For each $a \in A$ the set

$$\{G \in \text{Glimm A} : K(A)\|D(a_G, A/G)\| \geq \|a_G - \lambda(a_G)\|\}$$

is a dense G_δ in Glimm A.

Proof: The set in question is equal to the intersection of the countable collection of dense, open sets of the form

$$\{G \in \text{Glimm A} : \|a_G - \lambda(a_G)\| < \min\left(\frac{1}{n}, (K(A) + \frac{1}{n})\|D(a_G, A/G)\|\right)\}$$

for $n \in \mathbf{N}$. Since A is quasicentral (or since A is open) Glimm A is a locally compact Hausdorff space, hence a Baire space, and the result follows. Q.E.D.

THEOREM 7.6 Let A be a separable, quasicentral, open C*-algebra with $K(A) < \infty$. Then there is a dense subset X of Glimm A such that

(i) $Z(A/G)$ is trivial for all $G \in X$

(ii) $K(A) = \sup\{K(A/G) : G \in X\}$.

Proof: Let $(a^n)_{n \geq 1}$ be a countable, dense subset of A. The set

$$X = \{G \in \text{Glimm A} : \text{for all } n \in \mathbf{N}\ K(A)\|D(a^n_G, A/G\| \geq \|a^n_G - \lambda(a^n_G)\|\}$$

is dense in Glimm A, by Corollary 7.5 and the fact that Glimm A is a Baire space. A simple approximation argument now shows that if $G \in X$ then for all $a_G \in A/G$ $K(A)\|D(a_G, A)/G\| \geq \|a_G - \lambda(a_G)\|$. If $a_G \in Z(A/G)$ then $D(a_G, A/G) = 0$, so it follows that $a_G = \lambda(a_G)$. This proves (i), and (ii) follows immediately from Theorem 7.3. Q.E.D.

Theorem 7.6 does not hold without the separability condition. For example, let A be the C*-algebra of [Dixmier 1961; Proposition 4], mentioned after Corollary 3.6. Let B equal A+R where R is the centre of the multiplier algebra of A. Then it can be checked that B is a unital, quasi-standard C*-algebra, so that K(B)=K_s(B)=$\frac{1}{2}$ by Corollary 5.11 (ii). If $G \in$ Glimm B then B/G is *-isomorphic to (LC(H)⊕LC(H))+C1, so that K(B/G)=K_s(B/G)=1, by Theorems 5.8 and 6.8. Thus (i) of Theorem 7.6 holds, with X = Glimm B, but (ii) does not. On the other hand, if we assume Martin's Axiom then Theorem 7.6 can be extended to cover C*-algebras A which have a dense subset of cardinality less than $2^{\aleph_0}$ and such that Glimm A satifies the countable chain condition, see [Dales and Woodin 1987; Definition 5.14].

Theorem 7.6 raises the interesting question of whether every C*-algebra A has a dense subset X of Glimm A such that the centre of A/G is trivial for all $G \in$ X.

Finally, we consider the possibility that for each $a \in$ A the map $G \mapsto \|D(a_G,A/G)\|$ is continuous on Glimm A.

LEMMA 7.7 Let A be a quasicentral, open C*-algebra with K(A) $< \infty$. If for all $a \in$ A the function $G \mapsto \|D(a_G,A/G)\|$ ($G \in$ Glimm A) is continuous on Glimm A then

(i) Z(A/G) is trivial for all $G \in$ Glimm A

(ii) K(A/G) $\leq$ K(A) for all $G \in$ Glimm A.

Proof: By Corollary 7.5, the hypotheses imply that for all $a \in$ A and for all $G \in$ Glimm A $K(A)\|D(a_G,A/G\| \geq \|a_G - \lambda(a_G)\|$. The method of proof of Theorem 7.6 now shows that (i) and (ii) hold. Q.E.D.

PROPOSITION 7.8 Let A be a quasicentral, quasi-standard C*-algebra. Then the following are equivalent:

(i) for all $a \in$ A the function $G \mapsto \|D(a_G, \mathrm{A}/G)\|$ ($G \in$ Glimm A) is continuous on Glimm A

(ii) every Glimm ideal of A is prime.

Proof: By Corollary 5.11 (ii) K(A)$\leq \frac{1}{2}$, so it follows from Lemma 7.7 and the remark after Corollary 5.11 that (i) implies (ii). Conversely, if every Glimm ideal of A is prime then for each $G \in$ Glimm A $\|D(a_G, \mathrm{A}/G)\| = \|a_G - \lambda(a_G)\|$, by Corollary 5.11 (i), so that the function is continuous by Proposition 7.2. Q.E.D.

REFERENCES

AKEMANN, C.A., and PEDERSEN, G.K.

Ideal perturbations of elements in C*-algebras, *Math. Scand.*, 41 (1977) 117-139.

ALBRECHT, E., and DALES, H.G.

Continuity of homomorphisms from C*-algebras and other Banach algebras, *Proceedings, conference on radical Banach algebras and automatic continuity, Long Beach, California, July 1981*, Lecture Notes in Mathematics 975 (Springer 1983).

ANDERSON, J., and PASCHKE, W.

The rotation algebra, preprint, MSRI.

APOSTOL, C., and ZSIDO, L.

Ideals in W*-algebras and the function η of of A. Brown and C. Pearcy, *Rev. Roum. Math. Pures Appl.*, 18 (1973) 1151-1170.

ARCHBOLD, R.J.

Prime C*-algebras and antilattices, *Proc. London Math. Soc.*, 24 (1972) 669-680.

Density theorems for the centre of a C*-algebra, *J. London Math. Soc.*, 10 (1975) 189-197.

On the norm of an inner derivation of a C*-algebra, *Math. Proc. Camb. Phil. Soc.*, 84 (1978) 273-291.

Topologies for primal ideals, *J. London Math. Soc.*, 36 (1987) 524-542.

ARCHBOLD, R.J., and BATTY, C.J.K.

On factorial states of operator algebras, III, *J. Operator Theory*, 15 (1986) 53-81.

ARCHBOLD, R.J., and SHULTZ, F.W.

Characterizations of C*-algebras with continuous trace by properties of their pure states, *Pacific J. Math.*, 136 (1989) 1-13.

ARCHBOLD, R.J., and SOMERSET, D.W.B.

Quasi-standard C*-algebras, *Math. Proc. Camb. Phil. Soc.*, to appear.

BAKER, C.W.

The Pedersen ideal and the representation of C*-algebras, *Rocky Mountain J. Math.*, 13 (1983) 699-707.

BATTY, C.J.K.

Derivations of tensor products of C*-algebras, *J. London Math. Soc.*, 17 (1978) 129-140.

Irreducible representations of inseparable C*-algebras, *Rocky Mountain J. Math.*, 14 (1984) 721-727.

BECKER, T.

A few remarks on the Dauns-Hofmann theorems for C*-algebras, *Arch. Math. (Basel)*, 43 (1984) 265-269.

BERBERIAN, S.K.

*Baer *-rings* (Springer Verlag 1972)

CHING, W.M.

The structure of standard C*-algebras and their representations, *Pacific J. Math.*, 67 (1976) 131-153.

DALES, H.G., and WOODIN, W.H.
An introduction to independence for analysts, London Math. Soc. Lecture Note Series 115, (C.U.P. 1987)
DAUNS, J.
Multiplier rings and primitive ideals, *Trans. AMS*, 145 (1969) 125-158.
The primitive ideal space of a C*-algebra, *Can. J. Math.*, 26 (1974) 42-49.
DAUNS, J., and HOFMANN, K.H.
Representations of rings by continuous sections, Memoirs AMS, No. 83, 1968.
Spectral theory of algebras and adjunction of identity, *Math. Ann.*, 179 (1969) 175-202.
DELAROCHE, C.
Sur les centres des C*-algèbres, *Bull. Sci. Math.*, 91 (1967) 105-112.
Sur les centres des C*-algèbres, II, *Bull. Sci. Math.*, 92 (1968) 111-128.
DIXMIER, J.
Points séparé dans le spectre d'une C*-algèbre, *Acta Sci. Math. Szeged*, 22 (1961) 115-128.
C-algebras* (North-Holland 1977).
Von Neumann algebras (North-Holland 1981)
DORAN, R.S., and BELFI, V.A.
Characterizations of C-algebras* (Marcel Dekker 1986).
DUPRÉ, M.J., and GILLETTE, R.M.
Banach bundles, Banach modules and automorphisms of C-algebras*, Pitman Research Notes, No. 92, 1983.
EFFROS, E.G.
Order ideals in a C*-algebra and its dual, *Duke Math. J.*, 30 (1963) 392-412.
ELLIOTT, G.A
On derivations of AW*-algebras, *Tohoku Math. J.*, 30 (1978) 263-276.
ELLIOTT, G.A., and ZSIDO, L.
Almost uniformly continuous automorphism groups of operator algebras, *J. Operator Theory*, 8 (1982) 227-277.
FELDMAN, J.
The uniformly closed ideals in an AW*-algebra, *Bull. AMS*, 62 (1958) 245-246.
FELL, J.M.G.
The structure of algebras of operator fields, *Acta Math.*, 106 (1961) 233-280.
FELL, J.M.G., and DORAN, R.S.
*Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles, Vol. 1*, (Academic Press 1988)
GAJENDRAGADKAR, P.,
Norm of a derivation of a von Neumann algebra, *Trans. AMS*, 170 (1972) 165-170.
GILLMAN, L., and JERISON, M.
Rings of continuous functions (Van Nostrand 1960).
GLIMM, J.
A Stone-Weierstrass theorem for C*-algebras, *Ann. of Math.*, 72 (1960) 216-244.
GODEMENT, R.,

Théorie générale des sommes continues d'espaces de Banach, *C.R. Acad. Sci. Paris*, 228 (1949) 1321-1323.
HALL, A.A.
Ph.D. Thesis (University of Newcastle upon Tyne, 1972)
HALPERN, H.
The maximal GCR ideal in an AW*-algebra, *Proc. AMS*, 17 (1966) 906-914.
Commutators in properly infinite von Neumann algebras, *Trans. AMS*, 139 (1969) 55-74.
Primitive ideals in von Neumann algebras, *Proc. AMS*, 39 (1973) 563-566.
HANDELMAN, D., HIGGS, D., and LAWRENCE, J.
Directed abelian groups, countably continuous rings and Rickart C*-algebras, *J. London Math. Soc.*, 21 (1980) 193-202.
JOHNSON, B.E.
Norms of derivations on L(X), *Pacific J. Math.*, 38 (1971) 465-469.
KADISON, R.V., LANCE, E.C., and RINGROSE, J.R.
Derivations and automorphisms of operator algebras, II, *J. Functional Analysis*, 1 (1967) 204-221.
KAPLANSKY, I.
The structure of certain operator algebras, *Trans. AMS*, 70 (1951) 219-255.
KYLE, J.
Norms of derivations, *J. London Math Soc.*, 16 (1977) 297-312.
LAURSEN, K.B.
Central factorization in C*-algebras and continuity of homomorphisms, *J. London Math. Soc.*, 28 (1983) 123-130.
LEE, Ru-Ying
On the C*-algebras of operator fields, *Indiana Univ. Math. J.*, 25 (1976) 303-314.
Full algebras of operator fields trivial except at one point, *Indiana Univ. Math. J.*, 26 (1977) 351-372.
MISONOU, Y.
On a weakly central operator algebra, *Tohoku Math. J.*, 4 (1952) 194-202.
PEDERSEN, G.K.
C-algebras and their automorphism groups* (Academic Press 1979).
Multipliers of AW*-algebras, *Math. Z.*, 187 (1984) 23-24.
RICKART, C.E.
General theory of Banach algebras (Van Nostrand 1960).
RIEFFEL, M.A.
Continuous fields of C*-algebras coming from group cocycles and actions, *Math. Ann.*, 283 (1989) 631-643.
RINGROSE, J.R.
Derivations of quotients of von Neumann algebras, *Proc. London Math. Soc.*, 36 (1978) 1-26.
RUDIN, W.
Real and complex analysis (McGraw-Hill 1966).
SINCLAIR, A.M.

Automatic continuity of linear operators, London Math. Soc. Lecture Note Series 21, (C.U.P 1976).

STAMPFLI, J.G.

The norm of a derivation, *Pacific J. Math.*, 33 (1970) 737-748.

STRATILA, S., and ZSIDO, L.

Lectures on von Neumann algebras (Abacus Press 1979).

TOMIYAMA, J.,

Topological representations of C*-algebras, *Tohoku Math. J.*, 14 (1962) 187-204.

VESTERSTRØM, J.

On the homomorphic image of the centre of a C*-algebra, *Math. Scand.*, 29 (1971) 134-136.

On open maps, compact convex sets, and operator algebras, *J. London Math. Soc.*, 6 (1973) 289-297.

WILS, W.I.M.

Two-sided ideals in W*-algebras, *J. Reine Angew. Math.*, 244 (1970) 55-68.

WRIGHT, F.B.

A reduction theory for algebras of finite type, *Ann. Math.*, 60 (1954) 560-570.

ZSIDO, L.

The norm of a derivation of a W*-algebra, *Proc. AMS*, 38 (1973) 147-150.