

Feedback Control Design using Sum of Squares Optimisation

Elias August¹ and Antonis Papachristodoulou²

Abstract—Applications in many engineering fields require nonlinear modelling and control. However, for nonlinear dynamical systems, designing a control law that makes the system's operating point globally asymptotically stable is challenging. In this paper, for systems whose dynamics can be modelled through polynomial vector fields, we provide a procedure to design a controller that guarantees global asymptotic stability and a certain cost. To achieve this goal, we use sum of squares programming. In particular, we further develop a previously proposed approach to provide such a control law also for cases where the previous approach failed. The new approach is less restrictive and requires less memory. We apply it to the van der Pol system, the Lorenz system, to a model of a hypersonic flight vehicle, and to satellite attitude control.

I. INTRODUCTION

Applications in many engineering fields, such as (but not limited to) robotic manipulators, prosthetics, and unmanned aerial and space vehicles require nonlinear modelling and control [1]–[3]. However, when system dynamics are nonlinear, designing a satisfactory feedback control law is rather challenging [1]. In this paper, for systems whose dynamics can be modelled through polynomial functions, we provide a procedure to design a controller that ensures global asymptotic stability and guarantees a cost.

In particular, for polynomial dynamical systems, we consider the following representation

$$\begin{aligned} \dot{x} &= A(x)x + Bu, \\ x(t) &\in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \\ A(x) &\in \mathbb{R}[x]^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \end{aligned} \quad (1)$$

where x is the system state, u the input, $A(x)$ the state-dependent polynomial system matrix representing the system dynamics, and B is the input matrix. Here, we denote the set of all polynomials by $\mathbb{R}[x]$. Then, $A(x) \in \mathbb{R}[x]^{n \times n}$ means that $A_{(i,j)} \in \mathbb{R}[x]$ for all i, j , where $A_{(i,j)}$ is the (i, j) -th entry to matrix $A(x)$ and $i, j \in \{1, 2, \dots, n\}$. Moreover, input matrix B can also be state-dependent, as for instance is the case in [4], however, for simplicity, we consider constant input matrices only.

Different approaches exist for control synthesis through the design of a polynomial state feedback control law given by $u(x) = K(x)x$, $K(x) \in \mathbb{R}[x]^{m \times n}$, such that the closed loop system is asymptotically stable and the following cost

is minimised

$$\int_0^\infty (z_1^T z_1 + z_2^T z_2) dt, \quad z_1 = C_1 x, \quad C_1 \in \mathbb{R}^{p \times n}, \quad z_2 = u. \quad (2)$$

The most straight-forward approach is to linearise Eq. (1) either by means of Jacobian linearisation or feedback linearisation and then design a Linear Quadratic Regulator (LQR) [5] by solving the Algebraic Riccati Equation (ARE). Another approach is to solve the State-Dependent Riccati Equation (SDRE) to obtain an optimal control law [6]. While these are powerful tools for controller design, most guarantee only local asymptotic stability, that is, in the vicinity of the operating point under consideration. Moreover, obtaining the control law by solving the SDRE is, often, very difficult.

For systems whose dynamics can be described using polynomial vector fields, a powerful approach to optimal controller design was developed in [4] and later applied, for example, to the control of satellite attitude in [7] (see also [8]). This approach, which is based on the Sum Of Squares Decomposition (SOSD), guarantees asymptotic stability of the closed loop system, while providing a suboptimal control law and is a relaxation of the work presented in [9] that requires solving nonlinear matrix inequalities (see [10] and the Appendix for more details on the SOSD). However, it does not scale well as the number of system states and the order of the vector field are increased. Since the optimisation problems presented in [4] require additional auxiliary state variables for the proposed feedback control design, which exponentially increases required memory and slows down computation, in this work, we present optimisation problems for the purpose of feedback control design that avoid introducing additional variables. The approach presented in this paper can succeed even in cases where the previous approach fails, as the addition of auxiliary state variables also restricts the optimisation problem that needs to be solved (see examples in Section IV).

The organisation of the paper is the following. In Section II, we introduce the problem of designing a stabilising controller for nonlinear systems. Particularly, we summarise the results of [4] and then present our novel approach that allows for a much wider set of problems to be solved. In Section III, we extend the results of the previous section to optimal control. We then apply our approach in Section IV, to an example from [4], the van der Pol system, the Lorenz system, to a simplified model of an Air-breathing Hypersonic Flight Vehicle (AHFV) [11]–[13] and to satellite control, since an important field for applying modern control techniques is Aerospace. Finally, we conclude the paper in Section V.

¹Elias August is with the Department of Engineering, Reykjavik University, Menntavegur 1, 102 Reykjavik, Iceland eliasaugust@ru.is

²Antonis Papachristodoulou is with the Department of Engineering Science, University of Oxford, Parks Road, Oxford, OX1 3PJ, United Kingdom antonis@eng.ox.ac.uk

II. STATE FEEDBACK SYNTHESIS: STABILITY

Consider the dynamical system described by Eq. (1). If $u = K(x)x$ and there exists a matrix $Q \succ 0$ such that

$$x^T Q(A(x) + BK(x))x < 0 \quad \forall x, x \neq 0, \quad (3)$$

then it follows from Lyapunov Theory [14] that the origin of the closed loop system is globally asymptotically stable. Note that we consider matrix Q to be constant to guarantee that the Lyapunov Function given by $V(x) = x^T Qx$ is radially unbounded, and therefore asymptotic stability holds globally.

A. Previous Results

Let $v = Qx$, $P = Q^{-1}$, and $Y(x) = K(x)P$. Then, inequality (3) holds if and only if

$$\begin{aligned} v^T Q^{-1} Q(A(x) + BK(x)) Q^{-1} v \\ = v^T (A(x)P + BY(x))v < 0 \quad \forall x, x \neq 0, v = Qx. \end{aligned} \quad (4)$$

Now, finding a positive definite matrix P such that inequality (4) holds can be relaxed to the requirement that matrix $P \succ 0$ is such that

$$-v^T (A(x)P + BY(x) + \epsilon I)v \text{ is SOS } \forall x, x \neq 0, \quad (5)$$

where ϵ is a positive constant and SOS means Sum Of Squares. Importantly, Eq. (5) can be efficiently solved using SOSTOOLS [15], a MATLAB-based software tool [16]. A similar result was presented in [4].

B. Extension of Previous Results

Eq. (5) has free variables given by vectors v and x . Since, actually, $v = Qx$, Eq. (5) is more ‘‘conservative’’ than inequality (3). Thus, in this paper, we propose to reduce the number of free variables and present an approach that seeks to reduce the ‘‘conservativeness’’ in Eq. (5) and, also, requires less memory space and runtime.

Note that

$$Q(A(x) + BK(x)) + (A(x) + BK(x))^T Q \prec 0, \quad (6)$$

is a bilinear matrix inequality and, thus, presents as a non-convex problem. Solving inequality (6) is NP-hard [17]. Therefore, in this paper, we propose to constrain matrix Q such that we can transform inequality (6) into a linear matrix inequality. This implies that we cannot guarantee that we can always find a solution, even if inequality (6) has one.

Particularly, we search for a positive definite matrix Q and a, potentially, state-dependent matrix $\bar{K}(x)$, $\bar{K}(x) \in \mathbb{R}[x]^{m \times n}$, such that

$$QBK(x) = \bar{B}\bar{K}(x), \quad (7)$$

where matrix \bar{B} , $\bar{B} \in \mathbb{R}^{n \times m}$, must be chosen *a priori*. Under this condition, if there exists a positive constant ϵ such that

$$-x^T (QA(x) + \bar{B}\bar{K}(x) + \epsilon I)x \text{ is SOS}, \quad (8)$$

then

$$-x^T Q(A(x) + BK(x) + \epsilon I)x \text{ is SOS}. \quad (9)$$

That is, the zero equilibrium of the closed loop system is globally asymptotically stable.

Thus, in order to find the controller gain matrix $K(x)$ such that the origin of the system defined through Eq. (1) with the control law given by $u = K(x)$ is globally asymptotically stable

- 1) We need to choose \bar{B} ;
- 2) We need to solve (7)–(8) for $Q \succ 0$.

Note that Step 1 partly determines the structure of matrix Q (see examples in Section IV). In this paper, we make ‘natural’ choices for this matrix; either $\bar{B} = B$; or, if $B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$, $B_1 \in \mathbb{R}^{m \times m}$, with B_1 full rank,

$$\bar{B} = \begin{bmatrix} I_m \\ I_m \end{bmatrix}, \quad (10)$$

where I_m denotes the identity matrix of dimension m .

For Step 2, we use SOSTOOLS to search for a positive definite matrix Q and a, potentially, state-dependent matrix $\bar{K}(x)$ such that (7)–(8) hold. If we get a feasible solution and B has full rank then the controller matrix $K(x)$ is given by

$$K(x) = (B^T B)^{-1} B^T Q^{-1} \bar{B} \bar{K}(x). \quad (11)$$

III. STATE FEEDBACK SYNTHESIS: (SUB)OPTIMALITY

A. Previous Results

It is shown in [4] that if there exist positive constants ϵ , $\bar{\epsilon}$, and a positive definite matrix P such that $P \succeq \bar{\epsilon} I$ and

$$-v^T \begin{bmatrix} A(x)P + PA^T(x) - BB^T + \epsilon I & PC_1^T \\ C_1 P & -I \end{bmatrix} v \text{ is SOS}, \quad (12)$$

then $u(x) = -B^T P^{-1}x$ makes the origin of the system defined through Eq. (1) globally asymptotically stable. Moreover, for any initial condition x_0 , the following guarantee holds for the cost,

$$\int_0^\infty z_1^T z_1 + z_2^T z_2 \, dt < x_0^T P^{-1} x_0. \quad (13)$$

To see this, note that if condition (12) holds then, using the Schur complement,

$$\begin{aligned} A(x)P + PA^T(x) - BB^T + PC_1^T C_1 P &\preceq -\epsilon I \\ \Leftrightarrow A(x)P + PA^T(x) - 2BB^T &\preceq -\epsilon I - PC_1^T C_1 P - BB^T \\ \Leftrightarrow P^{-1}A(x) + A^T(x)P^{-1} - 2P^{-1}BB^T P^{-1} \\ &\preceq -\epsilon P^{-2} - C_1^T C_1 - P^{-1}BB^T P^{-1} \\ &\prec -C_1^T C_1 - P^{-1}BB^T P^{-1}. \end{aligned} \quad (14)$$

Since the origin is globally asymptotically stable,

$$\int_0^\infty \frac{d}{dt} x^T P^{-1} x \, dt = -x_0^T P^{-1} x_0.$$

Now,

$$\begin{aligned} \frac{d}{dt} x^T P^{-1} x &= \dot{x}^T P^{-1} x + x^T P^{-1} \dot{x} = \\ &= x^T (P^{-1}A(x) + A^T(x)P^{-1} - 2P^{-1}BB^T P^{-1})x \end{aligned}$$

and, thus,

$$\begin{aligned}
& \int_0^\infty z_1^T z_1 + z_2^T z_2 \, dt \\
&= \int_0^\infty x^T (C_1^T C_1 + P^{-1} B^T B P^{-1}) x \, dt \\
&< - \int_0^\infty x^T (P^{-1} A(x) + A^T(x) P^{-1}) x \, dt + \\
&\quad + \int_0^\infty 2P^{-1} B B^T P^{-1} \, dt \\
&= x_0^T P^{-1} x_0.
\end{aligned} \tag{15}$$

Finally, the full problem to solve becomes

$$\begin{aligned}
& \text{given } A(x), B, C_1, \epsilon, \bar{\epsilon} \\
& \min \quad \text{tr}(Z) \\
& \text{s.t. } P \succeq \bar{\epsilon} I \\
& \quad v^T \begin{bmatrix} Z & I \\ I & P \end{bmatrix} v \text{ is SOS} \\
& \quad -v^T M v \text{ is SOS}
\end{aligned} \tag{16}$$

where $\text{tr}(Z)$ denotes the trace of matrix Z that is of appropriate size, the second constraint means that $P^{-1} \preceq Z$, and

$$M = \begin{bmatrix} A(x)P + P A^T(x) - B B^T + \epsilon I & P C_1^T \\ C_1 P & -I \end{bmatrix}.$$

B. Extension of Previous Results

In the following, we provide suboptimal results for minimising the cost given by

$$\int_0^\infty z_1^T z_1 \, dt + \int_0^\infty z_2^T B^T Q B z_2 \, dt.$$

Let us assume that

$$\begin{aligned}
& \int_0^\infty x^T K^T(x) B^T Q B K(x) x \, dt \\
& \leq \int_0^\infty x^T K^T(x_0) B^T Q B K(x_0) x \, dt
\end{aligned}$$

and require that the following two inequalities hold,

$$x^T Q (A(x) + B K(x)) x \leq -\frac{1}{2} x^T C_1^T C_1 x$$

and

$$x^T \bar{K}^T(x_0) \bar{B}^T Q^{-1} \bar{B} \bar{K}(x_0) x \leq \gamma x^T x, \tag{17}$$

where γ is a positive real constant. Now, note that if Eq. (3) holds then there exists a positive constant ϵ such that

$$\frac{1}{2} \frac{d}{dt} x^T Q x = x^T Q \dot{x} = x^T Q (A(x) + B K(x)) x \leq -\epsilon x^T x \tag{18}$$

and the origin of the system defined through Eq. (1) is globally asymptotically stable. Thus, it follows that the following inequalities hold,

$$\begin{aligned}
& \int_0^\infty z_1^T z_1 \, dt = \int_0^\infty x^T C_1^T C_1 x \, dt \\
& \leq -2 \int_0^\infty x^T Q (A(x) + B K(x)) x \, dt \\
& = -2 \int_0^\infty x^T Q \dot{x} \, dt = x_0^T Q x_0
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
& \int_0^\infty z_2^T B^T Q B z_2 \, dt \\
&= \int_0^\infty x^T K^T(x) B^T Q B K(x) x \, dt \\
&\leq \int_0^\infty x^T K^T(x_0) B^T Q B K(x_0) x \, dt \\
&= \int_0^\infty x^T \bar{K}^T(x_0) \bar{B}^T Q^{-1} \bar{B} \bar{K}(x_0) x \, dt \\
&\leq \int_0^\infty \gamma x^T x \, dt \leq \frac{\gamma}{2\epsilon} x_0^T Q x_0.
\end{aligned} \tag{20}$$

Therefore, in order to minimise cost, we resort to minimising $\text{tr}(Q) + \gamma$ subject to

$$\begin{bmatrix} \gamma I & \bar{K}^T(x_0) \bar{B}^T \\ \bar{B} \bar{K}(x_0) & Q \end{bmatrix} \succ 0, \tag{21}$$

and this constraint means that we require that the condition given by (17) holds. Finally, the full problem to solve becomes, where $\epsilon = 0$ if C_1 has full rank,

$$\begin{aligned}
& \text{given } A(x), \bar{B}, x_0, C_1, \epsilon, \bar{\epsilon} \\
& \min \quad \text{tr}(Q) + \gamma \\
& \text{s.t. } Q \succeq \bar{\epsilon} I \\
& \quad \begin{bmatrix} \gamma I & \bar{K}^T(x_0) \bar{B}^T \\ \bar{B} \bar{K}(x_0) & Q \end{bmatrix} \succ 0 \\
& \quad -x^T (Q A(x) + \bar{B} \bar{K}(x) + C_1^T C_1 + \epsilon I) x \text{ is SOS.}
\end{aligned} \tag{22}$$

IV. EXAMPLES

All problems are solved on a MacBook Pro with a 2.3 GHz Quad-Core Intel Core i5 processor. Moreover, for comparability, the cost is given by

$$\int_0^\infty z_1^T z_1 \, dt + \int_0^\infty z_2^T B^T B z_2 \, dt.$$

A. 2D Systems

1) *Example 1 in [4]:* This example is from [4] and represents the dynamics of a tunnel diode circuit given by

$$\begin{aligned}
\dot{x} &= A(x)x - B K(x)x, \\
A(x) &= \begin{bmatrix} -0.5g(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \\
g(x_1) &= 17.76 - 103.79x_1 + 229.62x_1^2 \\
&\quad - 226.31x_1^3 + 83.72x_1^4,
\end{aligned} \tag{23}$$

where x_1 is the voltage across the capacitor and x_2 the current through the inductor. Using `SOSTOOLS`, for $\epsilon = 0.001$ and $P \succ I$, solving problem (5) results in

$$K = \begin{bmatrix} 9 & 0.25 \end{bmatrix}.$$

Similarly, solving problem (8), for $\bar{B} = B$ and a diagonal matrix Q such that Eq. (7) holds, results in

$$K = \begin{bmatrix} 16.63 & 2.18 \end{bmatrix}.$$

Significantly, for both control laws, the origin of the controlled closed-loop system is globally asymptotically stable. Table I summarises the computation time for the two approaches.

TABLE I
COMPUTATION TIME FOR THE TWO APPROACHES.

Eq. (5) (ref. [4])	Eq. (8)
0.3 s	0.2 s

2) *Example 3 in [4]*: Consider the system given by

$$\begin{aligned} \dot{x} &= A(x)x - BK(x)x, \\ A(x) &= \begin{bmatrix} -1 + x_{(1)} - \frac{3}{2}x_{(1)}^2 - \frac{3}{4}x_{(2)}^2 & \frac{1}{4} - x_{(1)}^2 - \frac{1}{2}x_{(2)}^2 \\ 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = I, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \quad (24)$$

Using SOSTOOLS, for $\epsilon = 10$ and $\bar{\epsilon} = 0.001$, solving problem (16) results in

$$K = \begin{bmatrix} 0.249 & 0.768 \end{bmatrix}$$

and a cost of 1.28. Similarly, solving problem (22), for $\bar{B} = B$ and a diagonal matrix Q such that Eq. (7) holds, results in

$$K = \begin{bmatrix} -0.377x_{(2)}^2 + 0.216 & 0.015x_{(2)}^2 + 0.956 \end{bmatrix}$$

and a cost of 1.28. Significantly, for both control laws, the origin of the controlled closed-loop system is globally asymptotically stable.

Applying classical LQR after linearising the system given by Eq. (24) results in the following control law

$$K = \begin{bmatrix} 0.0618 & 1.015 \end{bmatrix}$$

and a cost of 1.29. However, this controlled system is only locally asymptotically stable. When solving the SDRE to obtain an optimal control law, we obtain a value for the cost that is 1.30 for

$$\begin{aligned} K_{(1,1)} &= \frac{1 - K_{(1,2)}^2}{2X_2}, \\ K_{(1,2)} &= \sqrt{X_1^2 + 2\sqrt{X_1^2 + X_2^2} + 1} - X_1, \\ X_1 &= \frac{3}{2}x_{(1)}^2 - x_{(1)} + \frac{3}{4}x_{(2)}^2 + 1, \\ X_2 &= x_{(1)}^2 + \frac{1}{2}x_{(2)}^2 - \frac{1}{4}. \end{aligned} \quad (25)$$

While for all control laws, for the chosen initial condition, cost and behaviour are similar (not shown), the one given by Eq. (25) is less favourable, for instance, it does not exist if $x_0 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$. Table II summarises the cost for the approaches and indicates computation time, where appropriate.

TABLE II
COST AND COMPUTATION TIME FOR DIFFERENT APPROACHES.

LQR	SDRE	Eq. (16) (ref. [4])	Eq. (22)
1.29	1.30	1.28 (0.3 s)	1.28 (0.3 s)

B. Van der Pol System

Consider the Van der Pol system given by

$$\dot{x} = A(x)x - BK(x)x, \quad A(x) = \begin{bmatrix} 0 & 1 \\ -1 & x_{(1)}^2 - 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = I. \quad (26)$$

Then, applying classical LQR after linearising the system given by Eq. (26) results in the following control law

$$K = \begin{bmatrix} 0.4142 & 0.6818 \end{bmatrix}.$$

However, this controlled system is only locally asymptotically stable. For instance, its behaviour diverges for the initial condition given by $x_0 = -\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Now, using SOSTOOLS, we solve Eq. (5) and obtain

$$K = \begin{bmatrix} 0 & 1.12x_{(1)}^2 \end{bmatrix}.$$

We also solve Eq. (8), for

$$\bar{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_2 \end{bmatrix},$$

which makes sure that Eq. (7) holds, and obtain

$$K = \begin{bmatrix} 0 & x_{(1)}^2 + 1.368 \end{bmatrix}.$$

Importantly, the origin of both controlled closed-loop system is globally asymptotically stable. Table III summarises the computation time for the approaches

TABLE III
COMPUTATION TIME FOR THE TWO APPROACHES.

Eq. (5) (ref. [4])	Eq. (8)
0.25 s	0.1 s

C. Lorenz System

Consider the Lorenz system given by

$$\begin{aligned} \dot{x} &= A(x)x - BK(x)x, \\ A(x) &= \begin{bmatrix} 10 & -10 & 0 \\ 28 & -1 & -x_{(1)} \\ x_{(2)} & 0 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ C_1 &= \sqrt{0.1}I, \quad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \end{aligned} \quad (27)$$

Problem (16) is infeasible. When we solve problem (22), for $\bar{B} = B$ and $Q \succ 0$, where $Q_{(2,2)} = Q_{(1,1)}$ and $Q_{(2,3)} = Q_{(1,3)}$ such that Eq. (7) holds, we obtain a cost of 60.5 for

$$K = \begin{bmatrix} 11.4 & 3.9 & 0 \\ 0 & 0 & 15.0 \end{bmatrix},$$

which compares well, while guaranteeing global asymptotic stability, to the cost obtained when solving the LQR problem for the linearised system, which is 57, for

$$K = \begin{bmatrix} 12 & 6 & 0 \\ 0 & 0 & 16 \end{bmatrix}.$$

The cost for the closed loop system, when applying the different control laws, is shown in Fig. 1 (see also, Table IV, where we show the computation time in brackets). Finally, note that, for this system, we could not solve the state-dependent Riccati equation.

TABLE IV

COST FOR DIFFERENT APPROACHES TO CONTROL THE LORENZ SYSTEM.

LQR	Eq. (22)
56.4	57.2 (0.6 s)

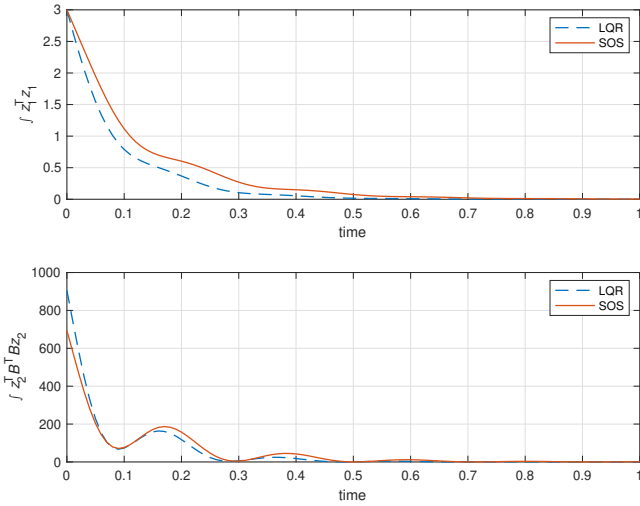


Fig. 1. The cost for the Lorenz system after closing the loop is shown for the different control laws.

D. Air-breathing Hypersonic Flight Vehicle

In this example, we use a simplified version of the model presented in [11] for the longitudinal dynamics of an air-breathing hypersonic flight vehicle given by

$$\dot{x} = A(x)x - BKx,$$

where $x_{(1)}$ is the velocity, $x_{(2)}$ the angle of attack, and we use the fact that for small angles $\sin(x_{(2)}) \approx x_{(2)}$. Furthermore, u is the thrust input, $A_{(1,1)} = 0$,

$$\begin{aligned} A_{(1,2)} &= \\ &-(0.645x_{(2)} + 0.01921)(0.1247x_{(1)} + 0.7370)^2, \\ A_{(2,1)} &= -0.0002706, \\ A_{(2,2)} &= -0.0574 - 0.009716x_{(1)}, \\ B &= \begin{bmatrix} 0.014 \\ 0 \end{bmatrix}, \quad C_1 = I, \quad x_0 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}. \end{aligned} \quad (28)$$

First, we partially linearise the AHFV system (28) using feedback such that $A_{(1,2)} = 0$. Next, solving the LQR problem results in

$$K = \begin{bmatrix} 1.000 & -0.0005 \end{bmatrix}$$

and a cost value of 0.8. A similar cost value is obtained when solving the SDRE. We could not solve problem (16). However, solving problem (22), for $\bar{B} = B$ and a diagonal matrix Q such that Eq. (7) holds, results in

$$K = \begin{bmatrix} 0.997 + 0.313x_{(2)}^2 & 0 \end{bmatrix},$$

and a cost value of 0.8. Importantly, with this control law, global asymptotic stability is guaranteed, while cost and behaviour are similar for all presented control laws (not shown) for the chosen initial condition. Table V summarises the cost for the different approaches and shows that computation time for solving Eq. (22).

TABLE V

COST FOR DIFFERENT APPROACHES TO CONTROL THE AHFV SYSTEM USING FEEDBACK TO PARTIALLY LINEARISE THE SYSTEM.

LQR	SDRE	Eq. (22)
0.8	0.8	0.8 (0.3 s)

Finally, we compare results when feedback linearisation is not used. Jacobian linearisation of the AHFV system (28) leads to an LQR control law, with

$$K = \begin{bmatrix} 1.0028 & -0.1465 \end{bmatrix},$$

that results in a cost value of 0.338. Solving the SDRE leads to a cost value of 0.425. Solving problem (22), for $\bar{B} = B$ and a diagonal matrix Q , leads to a control law, with

$$K = \begin{bmatrix} 475.3x_{(1)}^4 + 317.1x_{(2)}^4 + 1.453 & -0.2405 \end{bmatrix},$$

that results in a cost value of 0.336, while global asymptotic stability of the origin is guaranteed. For instance, for

$$x_0 = \begin{bmatrix} -7 \\ 2.8 \end{bmatrix},$$

the LQR control law leads to unstable behaviour. Table VI summarises the cost for the different approaches and shows that computation time for solving Eq. (22).

TABLE VI

COST FOR DIFFERENT APPROACHES TO CONTROL THE AHFV SYSTEM.

LQR	SDRE	Eq. (22)
0.338	0.425	0.336 (0.4 s)

E. Satellite Attitude Control

In [7], the following system was used to model satellite attitude for control purposes,

$$\dot{x} = A(x)x - BK(x)x,$$

where $x_{(1)}$, $x_{(2)}$, and $x_{(3)}$ denote the three angular velocities. To control satellite attitude, one rather employs modified

Rodrigue parameters, represented here, by the variables $x_{(4)}$, $x_{(5)}$, and $x_{(6)}$. Matrix $A(x)$, matrix B , and matrix C_1 are given by:

$$A(x) = \begin{bmatrix} \bar{A}(x) & 0 \\ \bar{A}(x) & 0 \end{bmatrix},$$

$$\bar{A}(x) = \begin{bmatrix} 0 & 0.233x_{(3)} & 0 \\ 0 & 0 & -0.156x_{(1)} \\ -0.08x_{(2)} & 0 & 0 \end{bmatrix},$$

$$\bar{A}(x) = \frac{1}{4} \begin{bmatrix} 1+x_4^2-x_5^2-x_6^2 & 2(x_4x_5-x_6) & 2(x_4x_6+x_5) \\ 2(x_4x_5+x_6) & 1-x_4^2+x_5^2-x_6^2 & 2(x_5x_6-x_4) \\ 2(x_4x_6-x_5) & 2(x_5x_6+x_4) & 1-x_4^2-x_5^2+x_6^2 \end{bmatrix}$$

$$B = \begin{bmatrix} D \\ 0 \end{bmatrix}, \quad D = \text{diag} [\quad 0.0677 \quad 0.0625 \quad 0.0800 \quad],$$

$$C_1 = \begin{bmatrix} 0 & 0 \\ 0 & I_3 \end{bmatrix}.$$

We could solve neither the SDRE nor problem (16). However, solving problem (22), for

$$\bar{B} = \begin{bmatrix} I \\ I \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_1 & Q_1 \\ Q_1 & Q_2 \end{bmatrix},$$

which makes sure that Eq. (7) holds, where $Q_1 \in \mathbb{R}^{3 \times 3}$ and $Q_2 \in \mathbb{R}^{3 \times 3}$, takes only about 1.4 seconds. We obtain a cost of 3.60 for

$$\begin{aligned} x_0 &= [\quad 0 \quad 0 \quad 0 \quad 0.503 \quad -0.67 \quad 0.547 \quad]^T, \\ k_1^T x &= 27.25x_{(1)}x_{(4)}^2 + 27.16x_{(1)}x_{(5)}^2 + 27.16x_{(1)}x_{(6)}^2 \\ &\quad + 15.73x_{(1)} + 11.98x_{(4)}, \\ k_2^T x &= 29.48x_{(2)}x_{(4)}^2 + 29.38x_{(2)}x_{(5)}^2 + 29.38x_{(2)}x_{(6)}^2 \\ &\quad + 16.71x_{(2)} + 12.71x_{(5)}, \\ k_3^T x &= 23.02x_{(3)}x_{(4)}^2 + 22.94x_{(3)}x_{(5)}^2 + 22.94x_{(3)}x_{(6)}^2 \\ &\quad + 13.10x_{(3)} + 9.93x_{(6)}, \\ K &= \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}, \end{aligned}$$

while guaranteeing global asymptotic stability. This cost compares well to the one obtained when solving the LQR problem for the linearised system, which is 2.25. The behaviour of the closed loop system for the different control laws is shown in Fig. 2 (see also, Table VII for cost).

TABLE VII

COST FOR DIFFERENT APPROACHES TO CONTROL SATELLITE ATTITUDE.

LQR	Eq. (22)
2.25	3.60 (1.4 s)

V. DISCUSSION AND CONCLUSIONS

In this paper, we developed a new approach to feedback controller design for systems whose dynamics are described by polynomial functions, that provides global asymptotic stability results. We presented examples, for which this approach guarantees asymptotic stability of the closed loop system and performs as well as all other approaches and

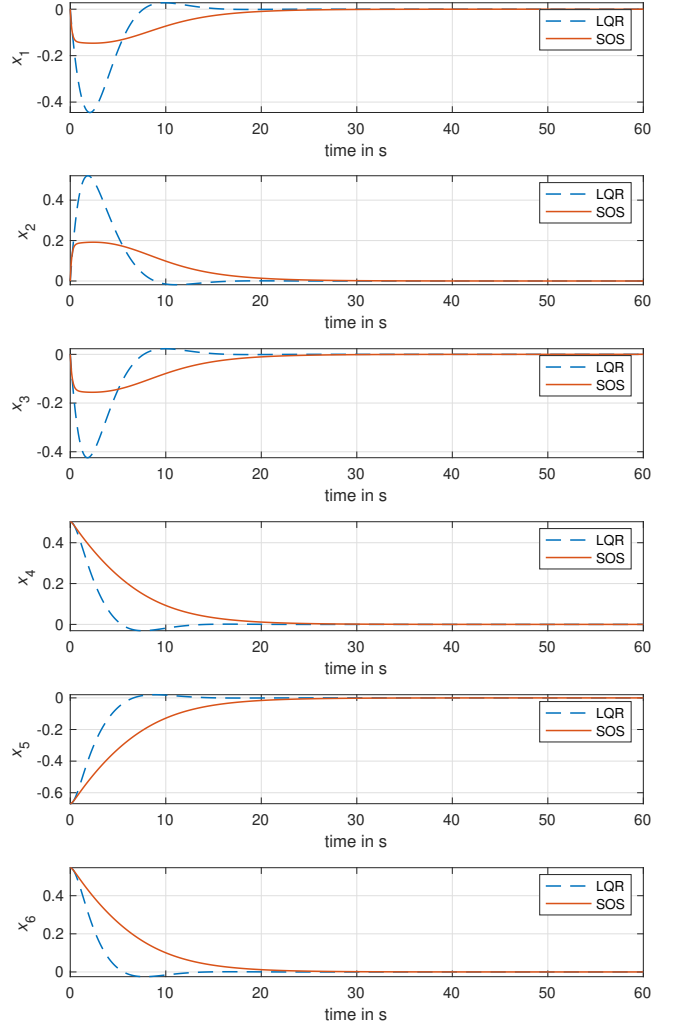


Fig. 2. The behaviour of the satellite attitude model after closing the loop is shown for the different control laws.

better in terms of providing a solution to finding a desired control law. Its importance lies in the fact that, in many applications, nonlinear effects might quickly make results obtained from a linear analysis invalid and lead to inadequate control synthesis that drives the system to undesired states (see, for example, [18], [19]). Finally, a related problem is the effect of parameter uncertainties on control quality. Fortunately, SOS techniques are also suitable to deal with this problem, which we will address in the future.

APPENDIX

Sum of Squares Decomposition

Consider the real-valued polynomial function $F(x)$ of degree $2d$, $x \in \mathbb{R}^n$. Testing for non-negativity of $F(x)$ is NP-hard [20]. However, a sufficient condition for $F(x)$ to be nonnegative is that it can be decomposed into a SOS [21]:

$$F(x) = \sum_i f_i^2(x) \geq 0, \quad (29)$$

where f_i are polynomial functions. Now, $F(x)$ is a SOS if and only if there exists a positive semidefinite matrix R and

$$F(x) = \sum_i f_i^2(x) = \chi^T R \chi, \quad \chi^T = \begin{bmatrix} 1 & x_{(1)} & \dots & x_{(n)} & x_{(1)}x_{(2)} & \dots & x_{(n)}^d \end{bmatrix}. \quad (30)$$

The entries of vector χ consist of all monomial combinations of the elements of vector x up to degree d (including $x_{(i)}^0 = 1$) and, thus, its length is $\ell = \binom{n+d}{d}$. Note that R is not necessarily unique. However, Eq. (30) poses certain constraints on R of the form $\text{tr}(A_j R) = c_j$, where A_j and c_j are appropriate matrices and constants respectively. As illustration, in the following, we reproduce Example 3.5 of [21].

Consider the following polynomial function,

$$\begin{aligned} F(x) &= 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4 \\ &= \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix} \\ &= q_{11}x_1^4 + q_{22}x_2^4 + (q_{33} + 2q_{12})x_1^2x_2^2 \\ &\quad + 2q_{13}x_1^3x_2 + 2q_{23}x_1x_2^3. \end{aligned}$$

For $\chi_1 = x_1^2$, $\chi_2 = x_2^2$, and $\chi_3 = x_1x_2$, this leads to the following set of linear equalities:

$$q_{11} = 2, \quad q_{22} = 5, \quad q_{33} + 2q_{12} = -1, \quad 2q_{13} = 2, \quad 2q_{23} = 0.$$

Thus, $j = 1, \dots, 5$, and, for example, $c_1 = 2$, $c_3 = -1$,

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In general, in order to find R , we solve the optimisation problem associated with the following semidefinite programme:

$$\begin{aligned} \min \quad & \text{tr}(A_0 R) \\ \text{s.t.} \quad & \text{tr}(A_j R) = c_j, \quad j = 1, \dots, m \\ & R = R^T \succeq 0. \end{aligned} \quad (31)$$

In this paper, to formulate the semidefinite programmes, we use SOSTOOLS [15].

REFERENCES

- [1] J. Iqbal, M. Ullah, S. G. Khan, B. Khelifa, and S. Ćuković, "Nonlinear control systems - a brief overview of historical and recent advances," *Nonlinear Engineering*, vol. 6, no. 4, pp. 301–312, 2017.
- [2] E. Lavretsky and K. Wise, *Robust and Adaptive Control: With Aerospace Applications*. London, United Kingdom: Springer-Verlag London, 2013.
- [3] J. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, NJ, USA: Prentice Hall, 1991.
- [4] S. Prajna, A. Papachristodoulou, and F. Wu, "Nonlinear control synthesis by sum of squares optimization: A lyapunov-based approach," in *2004 5th Asian Control Conference*, 2004, pp. 157–165.
- [5] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, New Jersey: Prentice-Hall, 1996.
- [6] J. Cloutier, "State-dependent riccati equation techniques: an overview," in *Proceedings of the 1997 American Control Conference (Cat. No. 97CH36041)*, 1997, pp. 932–936.
- [7] F. Meng, D. Wang, P. Yang, and G. Xie, "Application of Sum of Squares Method in Nonlinear H_∞ Control for Satellite Attitude Maneuvers," *Complexity*, vol. 2019, 2019, article ID 5124108.
- [8] A. Pang, Z. He, M. Zhao, G. Wang, Q. Wu, and Z. Li, "Sum of Squares Approach for Nonlinear H_∞ Control," *Complexity*, vol. 2018, 2018, article ID 8325609.
- [9] W. Lu and J. Doyle, "A state-space approach to robustness analysis and synthesis of nonlinear uncertain systems," Caltech, Tech. Rep. CIT-CDS-94-010, 1994.
- [10] J. Anderson and A. Papachristodoulou, "Advances in computational lyapunov analysis using sum-of-squares programming," *Discrete and Continuous Dynamical Systems - B*, vol. 20, no. 8, pp. 2361–2381, 2015.
- [11] O. U. Rehman, B. Fidan, and I. R. Petersen, "Robust minimax optimal control of nonlinear uncertain systems using feedback linearization with application to hypersonic flight vehicles," in *Proceedings of the 48th IEEE Conference on Decision and Control (CDC) held jointly with the 28th Chinese Control Conference*, 2009, pp. 720–726.
- [12] F. Ma, Y. Wu, X. Liu, B. Li, and X. Liu, "Adaptive sliding mode backstepping control for hypersonic flight vehicle with state constraints and unmodeled disturbances," in *2020 IEEE 16th International Conference on Control Automation (ICCA)*, 2020, pp. 1547–1552.
- [13] A. Ataei and Q. Wang, "Non-linear control of an uncertain hypersonic aircraft model using robust sum-of-squares method," *Control Theory & Applications, IET*, vol. 6, pp. 203–215, 2012.
- [14] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, New Jersey: Prentice-Hall, 2000.
- [15] A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, and P. A. Parrilo, *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, 2013, available from <http://www.eng.ox.ac.uk/control/sostools>, and <http://www.cds.caltech.edu/sostools> and <http://www.mit.edu/~parrilo/sostools>.
- [16] MATLAB, 9.8.0.1538580 (R2020a). Natick, Massachusetts: The MathWorks Inc., 2020.
- [17] O. Toker and H. Ozbay, "On the np-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback," in *Proceedings of 1995 American Control Conference - ACC'95*, vol. 4, 1995, pp. 2525–2526.
- [18] J. Shamma and M. Athans, "Gain scheduling: potential hazards and possible remedies," *IEEE Control Systems Magazine*, vol. 12, no. 3, pp. 101–107, 1992.
- [19] D. Nguyen, M. Lowenberg, and S. Neild, "Identifying limits of linear control design validity in nonlinear systems: a continuation-based approach," *Nonlinear Dynamics*, vol. 104, pp. 901–921, 2021.
- [20] K. G. Murty and S. N. Kabadi, "Some NP-complete problems in quadratic and nonlinear programming," *Math. Program.*, vol. 39, pp. 117–129, 1987.
- [21] P. A. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Math. Program., Ser. B*, vol. 96, pp. 293–320, 2003.