

A B S T R A C T

The thesis is concerned with groups acting on three dimensional spaces. The groups are assumed to have a compact simply-connected fundamental region. The action of the group is given partially by its action on the boundary of the fundamental region. This boundary is naturally split up by its intersection with transforms of the fundamental region. We assume that each such intersection is a single proper face of the boundary unless the transforming element is of order two, in which case there can be two faces. We also assume that any point of the boundary has only a finite number of transforms under the group which lie on the boundary. This enables one to give generators and defining relations for the group. The generators correspond to faces of the boundary inequivalent under the group, and defining relations to inequivalent lines.

In these circumstances two questions arise,

- (i) Is the three-dimensional space a manifold,
- (ii) Is the group finite?

If the space is not a manifold, then the group cannot be finite. So an answer to the first question gives some information about the second.

Page (ii)

line 6 :
of the boundary. Let

The thesis answers the first question as follows:-

THEOREM Given a group acting on a three-dimensional space with a fundamental region satisfying the conditions above, then the space is a manifold under the following condition.

Let X be any point which is the intersection of lines for the boundary. Let $\ell_1, \ell_2, \dots, \ell_n$ be a set of lines such that (i) given a line ℓ only one end of which is a point equivalent to X then exactly one ℓ_i is equivalent to ℓ , (ii) given a line both ends of which are points equivalent to X then exactly two ℓ_i are equivalent to ℓ . Let $r_1, r_2, \dots, r_n$ be the words (in terms of the generators and their inverses which correspond to the faces of the boundary) which are the shortest words which fix $\ell_1, \ell_2, \dots, \ell_n$ respectively (These are defined to within inversion.). If $r_i^{n_i} = 1$ are relations of the group, then the condition is for all X

$$\sum_{i=1}^n \frac{1}{n_i} > (n - 1) .$$

This theorem is proved by a detailed analysis of the neighbourhood of points X in the space.

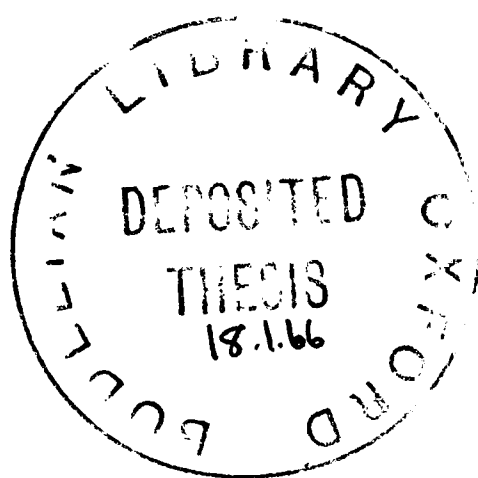
Another theorem which is a corollary of the methods used in proving the first theorem is:-

THEOREM Given a group acting on a three-dimensional space with a fundamental region satisfying the conditions above then the group has a soluble word problem.

P R O B L E M S I N G R O U P T H E O R Y

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INDEX TO TERMS USED

(i,j,k, r,s)	defines a tetrahedron	Pg. 8
$(\ell,m,n)^*$	the group whose generators and relations are $U^2=V^2=W^2=(UV)^{\ell}=(VW)^m=(WV)^n=1$	Pg. 3
(ℓ,m,n)	the "rotation" subgroup of the above	Pg. 3
(m,n)	the greatest common divisor of m and n	Pg.45
n_i	the powers of the words r_i	Pg. 9
p^i	the number of cells of dimension i	Pg.53
r_i	the word corresponding to a line	Pg. 9
(R,S)	the commutator of R and S	Pg. 5
R_i^E	R_i^1 or R_i^{-1}	Pg.18
W_i	the words corresponding to a line of the vertex graph	Pg.24
$\sim$	equivalent to	Pg.45
$\not\sim$	inequivalent to	Pg.45
$\sum_{x \sim x}$	summation over equivalence classes of x	Pg.54
$\cong$	isomorphic	Pg.29
polyhedral groups		Pg.37
tetrahedral groups		Pg. 8
vertex graph		Pg.18
vertex surface		Pg.20

Section

(1) General remarks about groups acting on spaces

Given a group G and generators R_α , then one can construct the Cayley graph of the group with respect to these generators. The points of the graph are in (1-1) correspondence with the elements of the group. The points g and h are connected by a directed line g towards h if and only if there exists a generator R_α such that $gR_\alpha = h$. The line is naturally labeled with the index α . Given a finite cycle in the graph then this corresponds to a relation in the group. For take a point of the cycle, say g_1 , then a line of the cycle joining g_1 is R_α or R_α^{-1} . This leads to g_2 i.e. $g_1 R_{\alpha_1}^{E_1} = g_2$. Similarly $g_2 R_{\alpha_2}^{E_2} = g_3 = g_1 R_{\alpha_1}^{E_1} R_{\alpha_2}^{E_2}$. Since the path in the graph is a cycle one has

$$g_1 R_{\alpha_1}^{E_1} R_{\alpha_2}^{E_2} \dots R_{\alpha_n}^{E_n} = g_{n+1} = g_1 \text{ for some } n.$$

$$\text{so } R_{\alpha_1}^{E_1} R_{\alpha_2}^{E_2} \dots R_{\alpha_n}^{E_n} = 1 \text{ is a relation.}$$

Conversely given a relation of the group, then there exists such cycles, one for each element of the group. Relations $r_\beta = 1$ define the group if it is possible to cut each of the corresponding cycles to reduce the graph to a tree (i.e. graph without cycles).

A group G is said to act on a space S if given an element of the group g and a point of the space s , sg is defined and

is a point of the space and $(sg)h = s(gh)$. For instance, a group G acts on the points of its Cayley graph. This action can be extended to the lines in a natural manner, if the graph itself is extended. Firstly, as with the points, we only need define the action of the generators. If the points g_1 and g_2 are joined by a line R_α i.e. $g_1 R_\alpha = g_2$ and $h_i R_\beta = g_i$ ($i = 1, 2$), then R_β acts on the line R_α by a homeomorphism taking it into a line joining h_1 to h_2 . This is labelled $R_\beta^{-1} R_\alpha R_\beta$, which may be a new line, in which case the Cayley graph must be extended. The homeomorphisms between the lines must clearly satisfy the condition that if $Q_1; l_1 \rightarrow l_2$ $Q_2; l_2 \rightarrow l_3$ and $Q_3; l_1 \rightarrow l_3$ then $Q_1 Q_2 = Q_3$.

The action of a group is said to be faithful if given an element $g \neq 1$, $g \in G$, there exists a point of the space s , such that $sg \neq s$. The elements of the group for which $sx = s$ for all points of the space forms a normal subgroup H . Then the action of the group gives a faithful action for G/H . In all the following, the actions are assumed to be faithful.

Given a group acting on a space S then one can construct a new space, called the orbit space by identifying the point s in the original space with all its images under the group i.e. all the sg , $g \in G$. This construction can also be made for any subset of the space not necessarily closed under the group.

A fundamental region for a group acting on a space S ,

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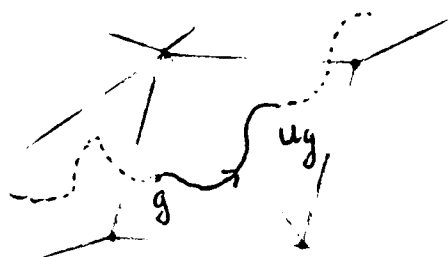
line 2:

an element g and a unique point r such that $sg=r \in R$.

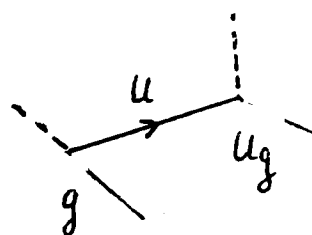
is a subset $R \subseteq S$ such that given a point $s \in S$ there exists a unique element g such that $sg \in R$.

In the following we consider groups acting on a manifold or a complex composed of cells of the same dimension. A classical example is given by the groups $(\ell, m, n)^*$.

Consider a triangle whose interior angles are π/ℓ , π/m , and π/n . This triangle is on the surface of a sphere if $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} > 1$, is in the Euclidean plane if $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} = 1$ and is in the hyperbolic plane otherwise. Let U , V and W be the reflections in the sides of this triangle. So $U^2 = V^2 = W^2 = (UV)^\ell = (VW)^m = (WU)^n = 1$. One can show that these are the only relations for the group; that is any other relation is an algebraic consequence of these. Firstly one notes that the triangle is a fundamental region for the group (with some suitable convention about its boundary). This triangle can be labelled with the identity element of the group. The transforms of this triangle can be labelled by the transforming element of the group. As the triangle is a fundamental region, these triangles cover the plane and do not intersect. A correspondence can now be set up between relations in the group and loops in the plane. This follows by noting that a path in the plane gives a path in the Cayley graph of the group with respect to the generators U , V and W , viz.



segment of loop

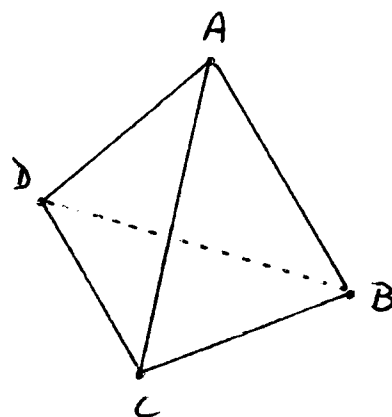
corresponding segment of
path in Cayley graph

Now a cycle in the Cayley graph corresponds to a relation in the way already described. Next we show that a loop in the plane can be contracted to a point by using the loops corresponding only to the relations $U^2 = V^2 = W^2 = (UV)^\ell = (VW)^m = (WU)^n = 1$. Firstly one can contract a path through an edge as this corresponds to one of the relations $U^2 = 1$, $V^2 = 1$, $W^2 = 1$. So the only remaining barrier to contracting the loop is the points. But a small loop around a point corresponds to one of the relations $(UV)^\ell = 1$, $(VW)^m = 1$, $(WU)^n = 1$. This shows that the above group is finite if and only if $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} > 1$.

Alternatively the plane could have been constructed by juxtaposition of triangles in a way which insures that the groups would act on the plane with the triangle as fundamental region and having the sole-relations as required. Unfortunately it is not then easy to see whether the plane obtained in this way is compact or not. Since the fundamental region was compact, the compactness of the plane would imply that

the group is finite and conversely. In the classical case above, the geometry of the plane enables one to determine whether the group is finite or not.

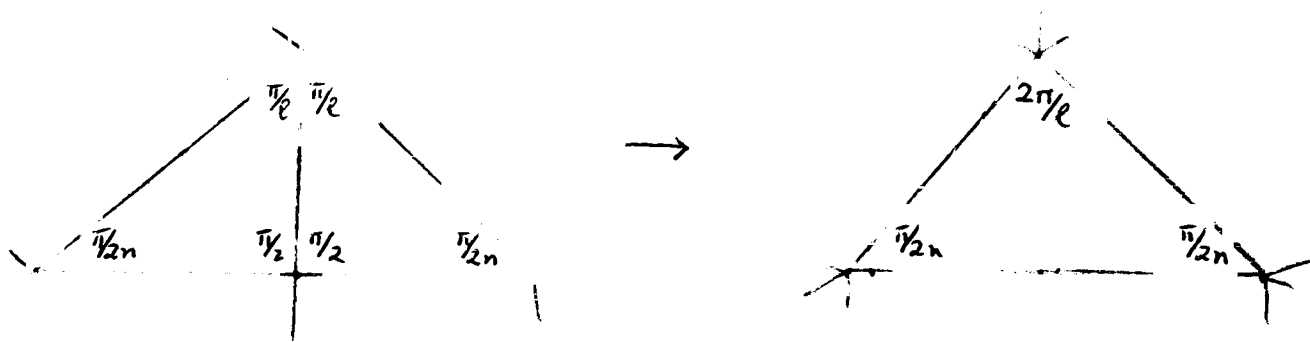
Here we shall be considering examples in three dimensions, one of which is due to Coxeter (3). Choose ℓ , m and n so that it is possible to find a spherical tetrahedron $ABCD$, the dihedral angles subtended by AB and CD being $2\pi/\ell$ and $2\pi/m$ and all the other angles being $\pi/2n$. Let R be a rotation about AB which takes C into D and S a rotation about DC which takes A into B . Then it is clear that the tetrahedron is a fundamental region for the group generated by R and S .



Also R and S satisfy $R^\ell = S^m = (R, S)^n = 1$. One can now show that these are the only relations, by the same method as for the groups $(\ell, m, n)^*$. One must show that one can contract a loop past any edge, which is similar to an edge of the fundamental region. But the edges of the fundamental region correspond to relations as follows. AB : $R^\ell = 1$, AC : $(RSR^{-1}S^{-1})^n = 1$, AD : $(SR^{-1}S^{-1}R)^n = 1$, BC : $(RS^{-1}R^{-1}S)^n = 1$, DC : $S^m = 1$. So ignoring conjugates the sole relations of the group are as stated.

The intersection of a spherical neighbourhood of A with

the fundamental region gives a spherical triangle whose angles are $2\pi/\ell$, $\pi/2n$, $\pi/2n$, so we require $\frac{1}{n} + \frac{2}{\ell} > 1$. Similarly $\frac{1}{n} + \frac{2}{m} > 1$. One can now check that spherical tetrahedra exist for all solutions of these inequalities ($\ell, m \neq 1$). The division of a neighbourhood of A into spherical triangles can be obtained from the one given by the group $(2, \ell, 2m)^*$ by joining adjacent triangles viz:



One could also consider this in the opposite manner. Given the group $R^\ell = S^m = (R, S)^n = 1$ can one make it act on a space so that the homotopy argument gives the required relations? The solution is clearly to construct a three dimensional space by joining tetrahedra in the manner dictated by the relations. What does this give when $\frac{1}{n} + \frac{2}{\ell} \leq 1$ or $\frac{1}{n} + \frac{2}{m} \leq 1$ or both? Although there are no geometric properties in this case, one can see that the division of a neighbourhood of A into "triangles" is similar to the one obtained from the group $(2, \ell, 2n)^*$ in the manner described above.

So when $\frac{1}{n} + \frac{2}{\ell} \leq 1$ there are infinitely many such triangles; that is the complex obtained by joining the tetrahedra is no

longer a manifold. This also implies that the group $R^\ell = S^m = (R, S)^n = 1$ is finite if and only if $\frac{1}{n} + \frac{2}{\ell} > 1$ and $\frac{1}{n} + \frac{2}{m} > 1$, as could have been seen by examining the subgroups generated by $R, S^{-1}R^{-1}S$ and $S, R^{-1}S^{-1}R$ respectively.

In the following we shall answer the question as to when a general complex in three dimensions upon which a group acts is a manifold.

In the above example, the four faces of the tetrahedron could be labelled with the label of the tetrahedron which is joined to this face. Then R , for instance, maps the face R^{-1} onto the face R , reversing the sense of description. So in general one must consider fundamental regions whose boundary contains an even number of faces with maps between pairs of faces suitably defined. This is the topic of section 3.

The map R can be defined by an integer r , $0 \leq r \leq i + j + \ell + 2$ such that C is displaced r steps in the sense $C \rightarrow D$ by R .

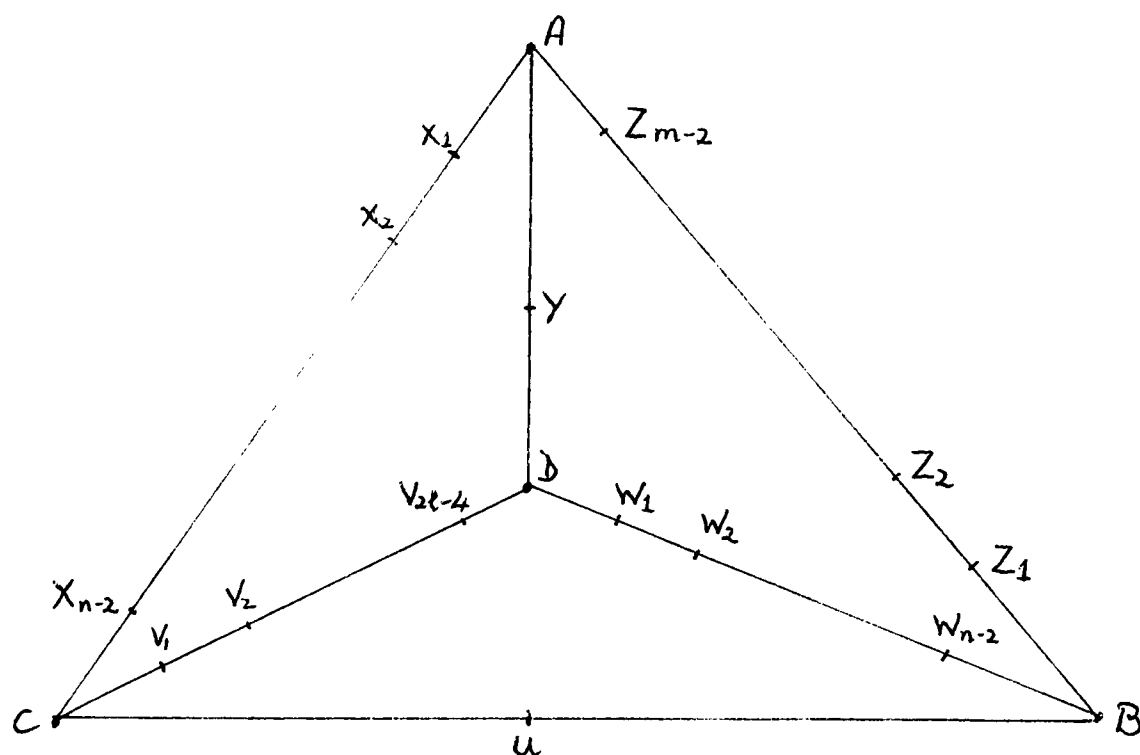
Similarly S can be defined by an integer s , $0 \leq s \leq i + j + k + 2$ such that B is displaced s steps along $B \rightarrow A$ by S .

Write this (i, j, k, ℓ, s, r)

Binary polyhedral groups

These are given by $R^\ell = S^m = (RS)^n$ ($\ell, m, n \geq 2$).

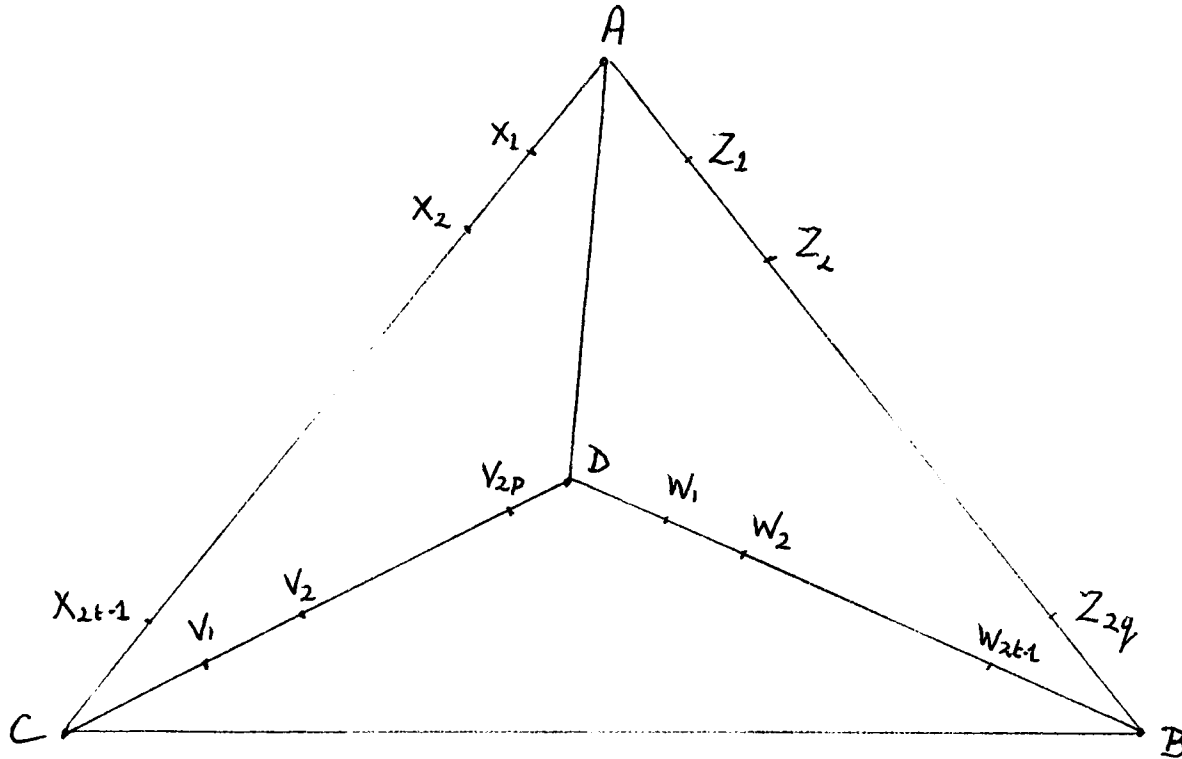
So consider $(1, n-2, m-2, 2\ell-4 \mid -1, 2)$



then UB gives the word $S^m R^{-\ell} = r_1$
and CU gives the word $(SR)^{n-1} SR^{-\ell+1} = r_2$

A group can be obtained whose relations are $r_1^{n_1} = r_2^{n_2} = 1$.
The fixed point free group of this family is given by
 $n_1 = n_2 = 1$ i.e. $R^\ell = S^m = (RS)^n$. One can easily see that
this group acts on a manifold for all values of ℓ, m and n .

A family of tetrahedral groups with (ℓ, m, n) groups as vertex subgroups



$$\text{i.e. } (0, 2t-1, 2q, 2p \mid 2, -2)$$

then the word corresponding to

$$\begin{aligned} CV_1 & \text{ is } (RS)^t R^{p+1} = r_1 \\ \text{to CB} & \text{ is } (RS)^t S^{-(q+2)} = r_2 \\ \text{to } CX_{2t-1} & \text{ is } S^{q+2} R^{p+1} = r_3 \end{aligned}$$

$$\therefore r_2 r_3 = r_1$$

So the group $r_1^\ell = r_2^m = r_3^n = 1$ has (ℓ, m, n) as the subgroup fixing C. The homomorphic image of this group given by

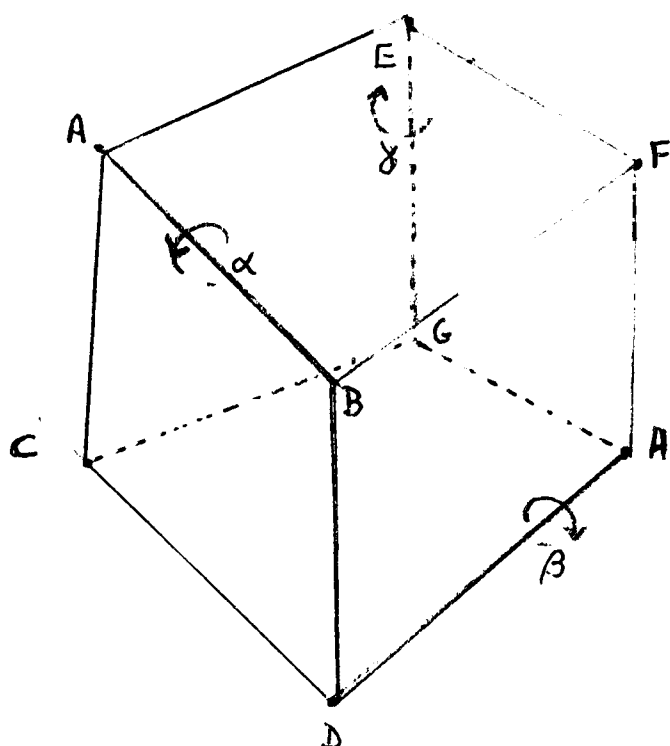
$r_1 = r_2 = r_3 = 1$ is the group

$$S^{q+r} = R^{-(p+1)} = (RS)^t$$

i.e. the binary polyhedral group $(q+2, -(p+1), t)$.

Two cubic groups

(due to Professor Higman)

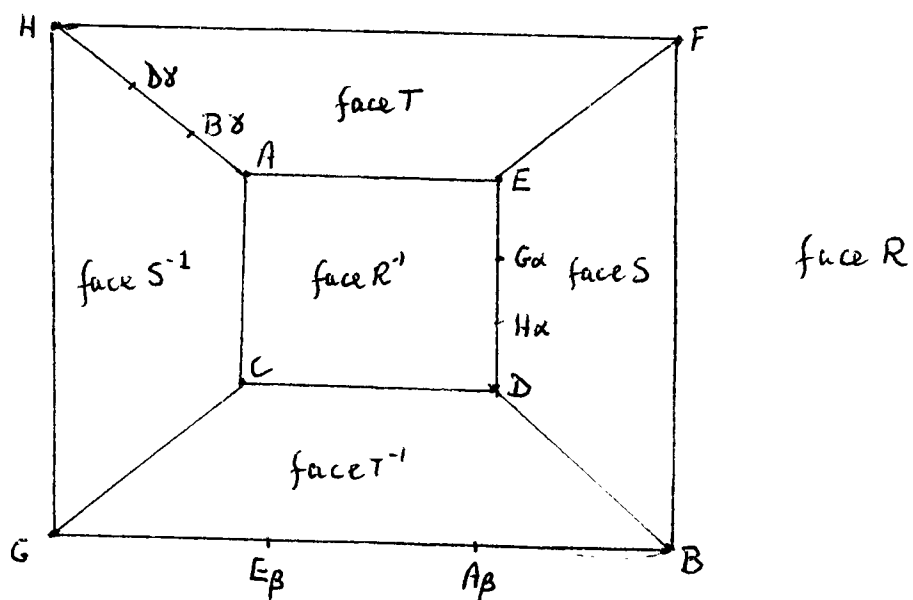


If α , β and γ are the "rotations" as shown, then the corresponding group has relations

$$\alpha^p = \beta^q = \gamma^r = (\alpha \gamma \alpha^{-1} \cdot \gamma^{-1} \beta^{-1} \gamma \cdot \beta \alpha^{-1} \beta^{-1})^s = 1$$

Consider the group $p = q = r = 2$, $s = 1$, which acts on a manifold. This has a homomorphism onto the Klein four group given by $\alpha \beta \gamma = 1$. Choosing $1, \alpha, \beta$, and γ as coset representatives, one can amalgamate the four corresponding cubes to give a fundamental region for the kernel of the above homomorphism. This is again a cubic group

If $R = \alpha \gamma \beta$, $S = \gamma \beta \alpha$, and $T = \beta \alpha \gamma$, then the faces are:-



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line 16:

So $\{R, S\}$ is of order 120, ...

R, S and T map like translations i.e. $R: CD \rightarrow E\beta, A\beta$

$$S: AC \rightarrow G\alpha, H\alpha$$

$$T: GC \rightarrow D\gamma, B\gamma$$

So the relations are $RST = 1$

$$TSR = RT$$

$$SRT = TS$$

$$RTS = SR$$

The first relation is redundant.

These relations are satisfied by $R = (12345)$ $S = (12453)$. If H is the subgroup fixing 1, then $1, R, R^2, R^3, R^4$ are coset representatives of this subgroup of index 5. By the Schreier method one can show that this subgroup is generated by $X = 5R^{-1}$ and $Y = R^2S$ and satisfy the relations $X^3Y^3 = 1$, $Y = XY^2X$, $X = YX^2Y$. The third relation is redundant, the first two relations in the form given is the tetrahedral group $(0,1,1,2 \mid 4,4)$ with $n_1 = n_2 = 1$. This is binary tetrahedral group of order 24. So R, S is of order 120, and being perfect must be the binary icosahedral group.

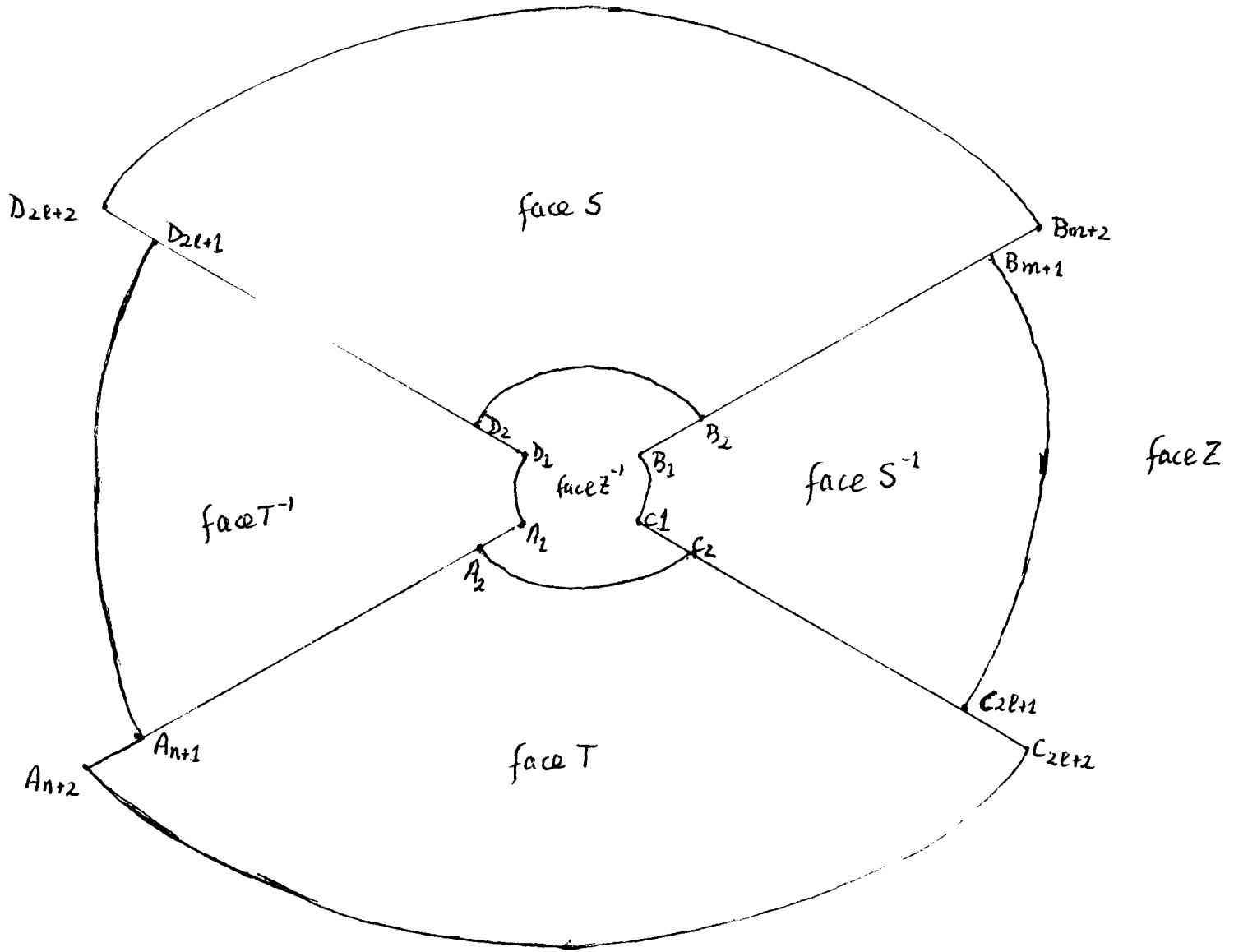
In fact the quaternions $R = \frac{1}{2}(t+j-t^{-1}k)$

$$S = \frac{1}{2}(t-j-t^{-1}k)$$

$$T = \frac{i}{2}(t+t^{-1}i + k) \quad \text{where } t^2-t-1=0$$

satisfy the relations and generate the binary icosahedral group.

Binary Polyhedral groups as cubic groups



Let the cube be as shown S , T , and Z mapping as follows

$$\begin{aligned}
 & (A_1 A_2 \dots A_n A_{n+1} D_{2l+1} D_{2l} \dots D_2 D_1) T \quad (B_1 B_2 \dots B_{m+1} C_{2l+1} C_{2l} \dots C_2 C_1) S \\
 & = (A_2 A_3 \dots A_{n+1} C_{2l+2} C_{2l+1} \dots C_3 C_2) = (B_2 B_3 \dots B_{m+2} D_{2l+2} D_{2l+1} \dots D_3 D_2) \\
 & \text{and } (D_1 D_2 B_2 B_1 C_1 C_2 A_2 A_1) Z \\
 & = (D_{2l+1} D_{2l+2} B_{m+2} B_{m+1} C_{2l+1} C_{2l+2} A_{n+2} A_{n+1})
 \end{aligned}$$

Then the relations for the fixed point free group are

$$T^n = S^m = (ST)^\ell = (TS)^\ell = Z \quad \text{and} \quad (T, Z) = (S, Z) = 1.$$

When this group is finite (i.e. $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} > 1$) one can find a fundamental region with the above properties for the

groups orthogonal action on S_3 . To do this one uses Cayley's correspondence between rotations on S_2 and Clifford translations on S_3 (4). The "natural" fundamental region for the group (ℓ, m, n) then corresponds to a torus in S_3 upon which Z acts. Cutting the torus in two in the appropriate manner gives the required fundamental region.

Section
(3) Polyhedral Groups

Description of the fundamental region

We assume that the fundamental region is compact and simply connected, so that its boundary is a sphere. The sphere is divided into faces by its intersection with neighbouring regions.

We assume that the intersection of two neighbouring regions is a simply connected 2 cell plus lines and points which may or may not be connected to the 2 cell, unless the generator corresponding to going from one region into the other is of order two, in which case two 2 cells are allowed, one formally corresponding to the generator, the other to its inverse. If there is only one 2 cell for such a generator then this 2 cell can be divided in two so that the generator maps one half into the other. The division of the sphere gives a finite planar graph with the following properties:-

- (i) It has an even number of faces, which are simply connected.
- (ii) The faces can be divided into pairs, each member of a pair having the same number of vertices.
- (iii) Each edge divides distinct faces.

(The faces being simply connected is equivalent to the graph being connected)

Given such a planar graph and a division of the faces

into pairs, label one member of the pair with a symbol R_i and the other with the symbol R_i^{-1} ($i = 1$ to K , there being $2K$ faces). Now define maps R_i which map $(\text{face } R_i^{-1}) \rightarrow (\text{face } R_i)$ homeomorphically so that the sense is reversed. Hence if faces R_i and R_i^{-1} intersect the sense of description along the intersection will be the same.

We now have the following:-

- (a) For any edge PQ of the graph there are exactly two maps T so that $(PQ)T$ is defined. This is an easy consequence of (iii).
- (b) For any edge PQ of the graph there exists a reduced word ω in the free group generated by the R_i which has no shorter conjugate such that $(PQ)\omega = PQ$. Moreover there exists no word ω so that $(PQ)\omega = QP$.

By (a) the maps R_i and their inverses induce cycles on the set of all edges. This shows there exists an ω so that $(PQ)\omega = PQ$. If ω had a shorter conjugate then $\omega = R_i \omega' R_i^{-1}$ or $\omega = \omega' R_i R_i^{-1} \omega''$. In the first case this implies that the two maps defined on PQ are R_i and R_i which contradicts (a). The second case follows with $P'Q' = (PQ)\omega' R_i$ instead of PQ .

We now have to show that there exists no ω so that $(PQ)\omega = QP$. Describe a positive sense along the faces R_i and a negative sense along the faces R_i^{-1} , so that the maps preserve this sense. As above ω is a cyclically reduced

word. But starting with $\overrightarrow{PQ}$, a change in sense is marked by a change of exponent. As the total number of changes in exponent in a cyclically reduced word is even the final sense is the same as the initial one, that is $\overrightarrow{PQ} \omega = \overrightarrow{PQ}$.

(c) If P_1P_2 and P_2P_3 are edges in the graph, then $(P_1P_2)\omega$ and $(P_2P_3)\omega$ intersect in a point if and only if they are both defined. This is obvious as the intersection is $(P_2)\omega$.

(d) The maps R_i^E induce a pairing on the set $\{PA_iQ\}$ where A_i ($i = 1$ to n) are the points of the planar graph of degree ≥ 3 and P and Q are points of the graph such that the vertices of a face (in cyclic order) are $P, A_i, Q, \dots$. In this set PA_iQ is distinct from QA_iP .

Proof. Consider an element of the set, PA_iQ . Since A_i has degree ≥ 3 in the planar graph, these three vertices lie in only one face such that the vertices of that face in cyclic order are $(PA_iQ \dots)$. Let this face be R_i^{ϵ} . Therefore $(PA_iQ)R_i^{-\epsilon}$ is defined and equals $P_1P_2P_3$ say. If P_2 has degree ≥ 3 , i.e. $P_2 = A_j$ the proof is complete with PA_iQ paired with $P_1A_jP_3$. Otherwise P_2 has degree 2 and so $P_1P_2P_3 \in (\text{face } R_i^{-\epsilon}) \cap (\text{face } R_j^{\epsilon'})$ so $(P_1P_2P_3)R_j^{-\epsilon'}$ is defined. A simple induction now gives the result.

(e) In (d) we can identify PA_iQ and QA_iP and still obtain a pairing. For this we must show that there exists no word ω such that $(PA_iQ)\omega = QA_iP$. This follows from exactly the same reasoning as the second part of (b).

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line 9:

there exists an w such

(f) If a point P is not an image of an A_i under any ω , then $\omega_{Q_1P} = \omega_{PQ_2}$ or $\omega_{Q_1P} = (\omega_{PQ_2})^{-1}$ (words defined in (b)) where Q_1 and Q_2 are the only neighbours of P . Such points are assumed to be omitted from the planar graph in the following discussion. This assumption is equivalent to taking the vertices of the graph to be the repeated intersection of the faces or the transforms of such points under words in the R_i^E .

Proof. By similar reasoning to (d) there exists an ω such that $(Q_1PQ_2)\omega = Q_1PQ_2$. So $\omega = \omega_{Q_1P} = \omega_{PQ_2}$ (the inverse occurring if different maps are applied initially to Q_1P and PQ_2).

The vertex graph

The pairing discussed in (d), (e) and (f) gives a graph which is naturally embedded in a surface.

The points of the graph are in (1-1) correspondence with the A_i : that is the points of degree ≥ 3 of the planar graph. The lines are in (1-1) correspondence with the pairings; that is there are n lines joining A_i and A_j if and only if there exist n solutions of

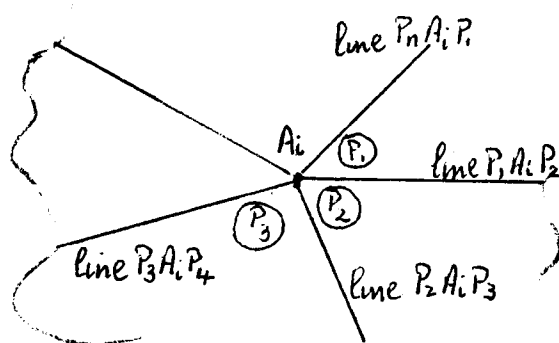
$$(P_1A_iQ_1)\omega = (P_2A_jQ_2)$$

(The P_i, Q_i and ω are the variables, interchanging the P 's and Q 's does not give another solution.)

We shall call this the vertex graph of the fundamental

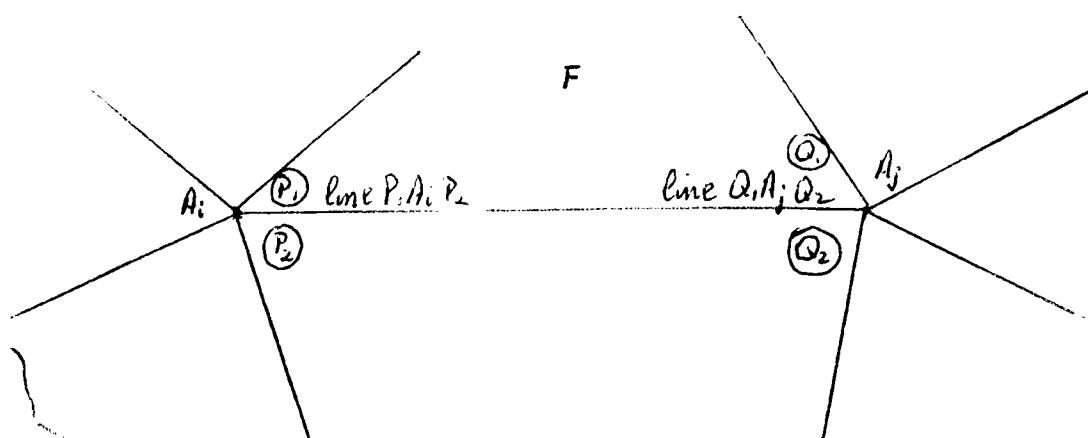
region.

Each connected component of the vertex graph is the one-skeleton of a surface in the following manner. If the neighbouring points to A_i are $P_1 \dots P_n$ and this is the cyclic order in which they occur in the planar graph, then the lines to A_i are embedded in a surface viz:

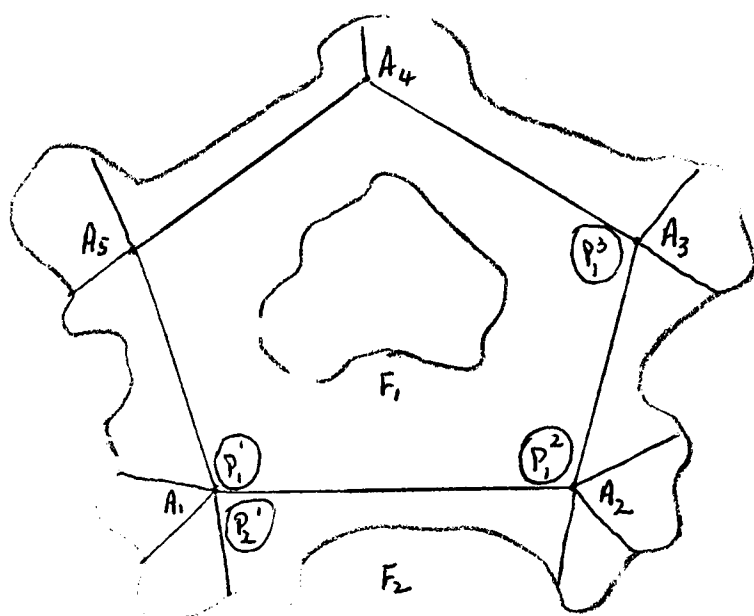


Hence the faces of the embedded vertex graph in the neighbourhood of A_i can be labelled with the P_k ($k = 1$ to n).

Further if $(P_1 A_i P_2) \omega = (Q_1 A_j Q_2)$ is a pairing, then the surface is:-



Let this be continued about the face F , so we have



(For convenience of the diagram assume that the A_i $i = 1$ to 5 are distinct.)

Let the polygon $A_1A_2A_3A_4A_5$ correspond to a face in the surface. This construction now gives a unique surface. Each connected component of the surface corresponds to a point in the orbit space of the planar graph, that is to a class of points A_i equivalent under the action of the R_i^E . The face above corresponds to the edge $A_1P_1^1$ which in turn corresponds to a word. An equivalence class of edges of the planar graph under the R_i^E gives two faces in the surface corresponding to each end of the edge. For instance, an edge QQ^1 may be equivalent under the R_i^E to $A_1P_1^1$ and $P_2^1A_1$ (in the above diagram). Then the corresponding faces would be F_1 and F_2 respectively.

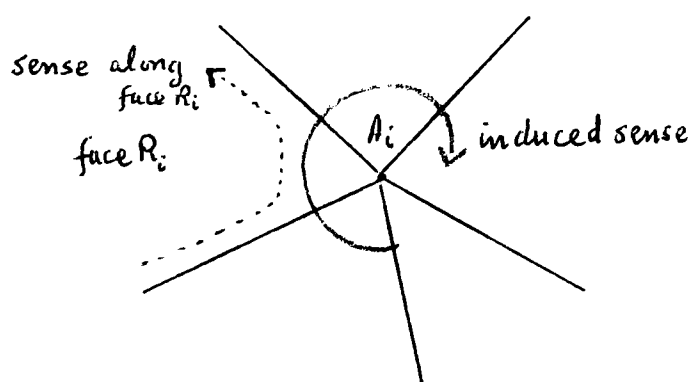
Call this embedded vertex graph the vertex surface.

A manifold can be constructed from the boundary of the fundamental region by identifying points equivalent under the group. The relationship between the above and this manifold constructed from the planar graph can now be seen.

Each face in the manifold gives two faces in the planar graph labelled R_i and R_i^{-1} . Each edge of the manifold corresponds to a cyclic word or its inverse in the free group generated by the R_i , which corresponds to two faces of the vertex surface.

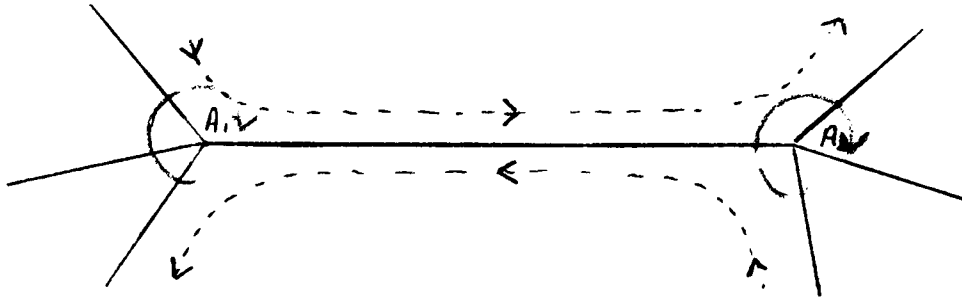
Sense restriction of pairings

Suppose, as in (b) above, that in the planar graph the faces labelled R_i are given a fixed sense and the faces R_i^{-1} the opposite sense. Then the maps preserve sense, and also this sense induces a sense of description about the vertices A_i .



Since the maps R_i preserve sense, the pairing reverses the induced sense. That is if $(P_1 A_1 Q_1) \omega = (P_2 A_2 Q_2)$ and $\overrightarrow{P_1 A_1 Q_1}$ then $\overleftarrow{P_2 A_2 Q_2}$.

This implies that a sense can be uniformly defined for the vertex surface. The sense about a point being the same as the induced sense for the planar graph; so we have



The induced sense along the faces of the vertex surface is shown with a dotted line.

This implies that each component of the vertex surface is orientable; but a graph can be embedded in an orientable surface so that no sense can be put on the points which is reversed by the lines in the manner above. (See page 33).

Section

(4) Analysis of the tetrahedral groups

We now consider the case where the fundamental region is a tetrahedron, using the concepts of the last section.

The vertex graph contains four points all of degree three, so that the connected constituents of the graph contain an even number of points. So under the action of the group the vertices A, B, C, and D are either divided into two classes, two in each class or they are all in one class.

We now calculate all possible vertex graphs, together with their embeddings into orientable surfaces, satisfying the sense restriction mentioned in the last section.

Two points

If the vertex graph has two constituents, then each one can be considered separately. As an abstract graph the constituent can have one of two forms, namely



Let the two points be A and B, and X_1, X_2, X_3 the neighbouring points to A in that cyclic order. Similarly let Y_1, Y_2, Y_3 be the neighbouring points to B.

(1) In this case, by renumbering the X's and Y's if necessary

but preserving their cyclic order we have

$$(X_1AX_2)\omega_1 = Y_2BY_1$$

$$(X_2AX_3)\omega_2 = X_1AX_3$$

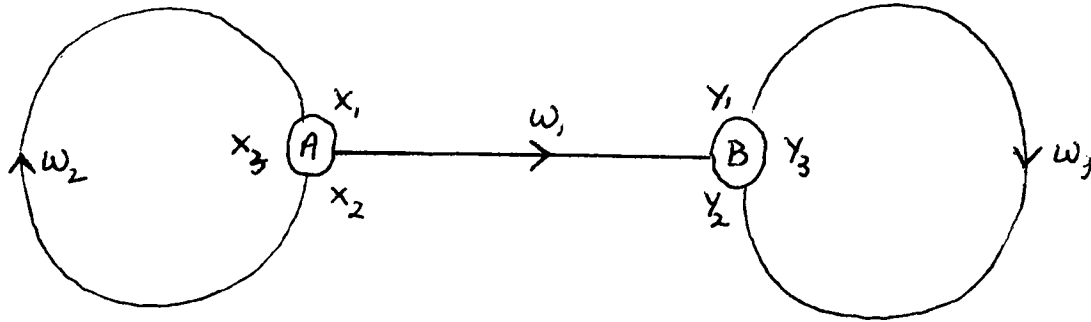
$$(Y_2BY_3)\omega_3 = Y_1BY_3$$

So the word corresponding to X_1A , in terms of the ω_i is $\omega_1\omega_3\omega_1^{-1}\omega_2$ as we have

$$X_1A \xrightarrow{\omega_1} Y_2B \xrightarrow{\omega_3} Y_1B \xrightarrow{\omega_1^{-1}} X_2A \xrightarrow{\omega_2} X_1A,$$

The other inequivalent edges are X_3A and Y_3B which give the words ω_2 and ω_3 respectively.

We now see that the above corresponds to the planar imbedding of the graph as first represented. Inserting the ω_i , A , B , X_i and Y_i we have



In the following calculation we omit the letters X and Y and use only the suffices.

(2) The vertex graph being 

One pairing can be chosen to be

$$1A2 \xrightarrow{\omega_1} 2B1$$

This leaves two choices for the remaining pairings, namely

$$2A3 \xrightarrow{\omega_2} 3B2$$

$$2A3 \xrightarrow{\omega_2} 1B3$$

or

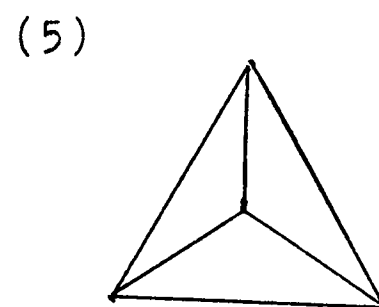
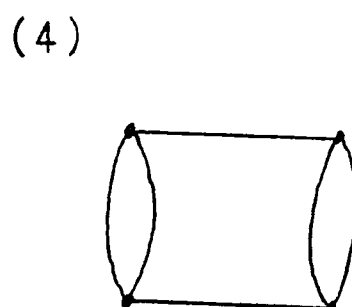
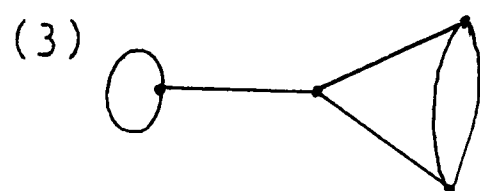
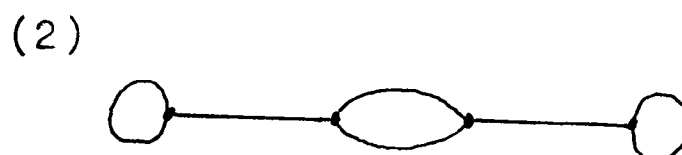
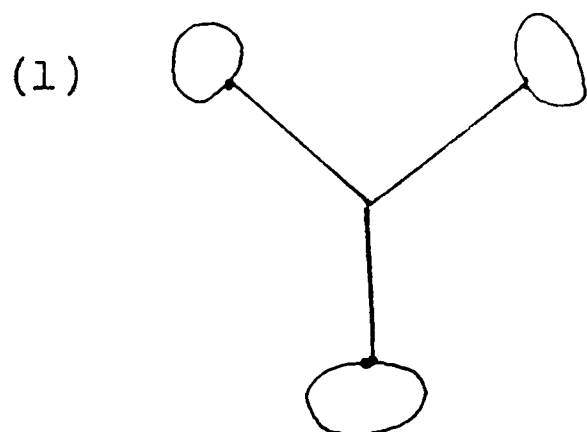
$$3A1 \xrightarrow{\omega_3} 1B3$$

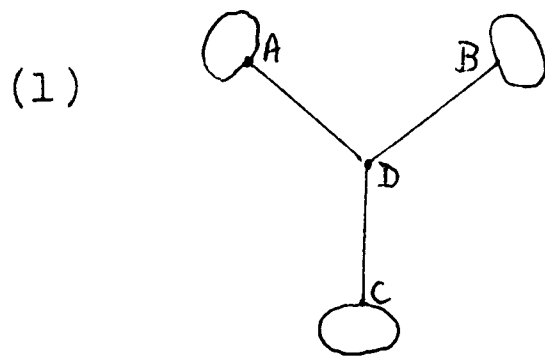
$$3A1 \xrightarrow{\omega_3} 3B2$$

First case:- $1A \xrightarrow{\omega_1} 2B \xrightarrow{\omega_2^{-1}} 3A \xrightarrow{\omega_3} 1B \xrightarrow{\omega_1^{-1}} 2A \xrightarrow{\omega_2} 3B \xrightarrow{\omega_3^{-1}},$

Second case:- $1A \xrightarrow{\omega_1} 1B \xrightarrow{\omega_3^{-1}}, \quad 2A \xrightarrow{\omega_2} 1B \xrightarrow{\omega_1^{-1}}, \quad 3A \xrightarrow{\omega_3} 3B \xrightarrow{\omega_2^{-1}},$

We now consider the case with all the vertices equivalent under R and S. The vertex graph is then a graph on four points, each of degree three. There are five such abstract graphs, namely





Remark. If a graph contains $\begin{array}{c} \bigcirc \\ \text{A} \end{array}$ then we can

Page 26

line 8:

By permuting B and C if necessary

2D1

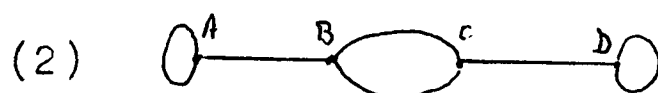
2B1

2C1

we have one case only

$$3A \xrightarrow{\omega_1}, \quad 3B \xrightarrow{\omega_2}, \quad 3C \xrightarrow{\omega_3},$$

$$1A \xrightarrow{\omega_1} 2A \xrightarrow{\omega_4} 1D \xrightarrow{\omega_6} 1C \xrightarrow{\omega_3} 2C \xrightarrow{\omega_6^{-1}} 3D \xrightarrow{\omega_5} 1B \xrightarrow{\omega_2} 2B \xrightarrow{\omega_5^{-1}} 2D \xrightarrow{\omega_4^{-1}},$$

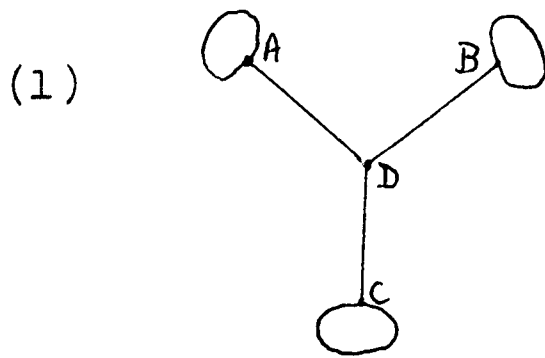


As before we have

$$3A1 \xrightarrow{\omega_1} 3A2$$

$$3D1 \xrightarrow{\omega_2} 3D2$$

By numbering B and C appropriately



Remark. If a graph contains $\bigcirc \xrightarrow{A}$, then we can assume that

$$3A1 \xrightarrow{\omega_1} 3A2,$$

In the above case we have

$$3A1 \xrightarrow{\omega_1} 3A2$$

$$3B1 \xrightarrow{\omega_2} 3B2$$

$$3C1 \xrightarrow{\omega_3} 3C2$$

By labelling on D appropriately $1A2 \xrightarrow{\omega_4} 2D1$

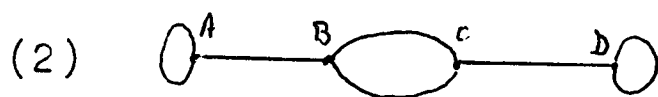
By permuting B and C is necessary $2D3 \xrightarrow{\omega_5} 2B1$

$$3D1 \xrightarrow{\omega_6} 2C1$$

So we have one case only

$$3A \xrightarrow{\omega_1}, \quad 3B \xrightarrow{\omega_2}, \quad 3C \xrightarrow{\omega_3},$$

$$1A \xrightarrow{\omega_1} 2A \xrightarrow{\omega_4} 1D \xrightarrow{\omega_6} 1C \xrightarrow{\omega_3} 2C \xrightarrow{\omega_6^{-1}} 3D \xrightarrow{\omega_5} 1B \xrightarrow{\omega_2} 2B \xrightarrow{\omega_5^{-1}} 2D \xrightarrow{\omega_4^{-1}},$$



As before we have $3A1 \xrightarrow{\omega_1} 3A2$

$$3D1 \xrightarrow{\omega_2} 3D2$$

By numbering B and C appropriately

$$1A2 \xrightarrow{\omega_3} 2E1$$

$$1D2 \xrightarrow{\omega_4} 2C1$$

This leaves two cases for the last two pairings

$$2B3 \xrightarrow{\omega_5} 3C2$$

$$3B1 \xrightarrow{\omega_6} 1C3$$

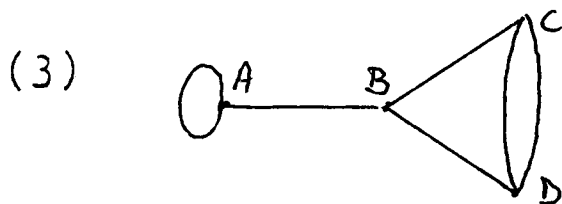
or

$$2B3 \xrightarrow{\omega_5} 1C3$$

$$3B1 \xrightarrow{\omega_6} 3C2$$

First case $3A \xrightarrow{\omega_1}, 3D \xrightarrow{\omega_2}, 1A \xrightarrow{\omega_1} 2A \xrightarrow{\omega_3} 1B \xrightarrow{\omega_6} 3C \xrightarrow{\omega_5^{-1}} 2B \xrightarrow{\omega_3^{-1}},$
 $1D \xrightarrow{\omega_2} 2D \xrightarrow{\omega_4} 1C \xrightarrow{\omega_6^{-1}} 3B \xrightarrow{\omega_5} 2C \xrightarrow{\omega_4^{-1}},$

Second case $3A \xrightarrow{\omega_1}, 3D \xrightarrow{\omega_2}, 3B \xrightarrow{\omega_5} 3C \xrightarrow{\omega_6^{-1}},$
 $1A \xrightarrow{\omega_1} 2A \xrightarrow{\omega_3} 1B \xrightarrow{\omega_6} 2C \xrightarrow{\omega_4^{-1}} 1D \xrightarrow{\omega_2} 2D \xrightarrow{\omega_4} 1C \xrightarrow{\omega_5^{-1}} 2B \xrightarrow{\omega_3^{-1}},$



Choose the numbering on B so that

$$1B2 \xrightarrow{\omega_1} 2A1$$

$$2B3 \xrightarrow{\omega_2} 2C1$$

$$3B1 \xrightarrow{\omega_3} 2D1$$

which implies $3A1 \xrightarrow{\omega_4} 3A2$

Two cases now arise

$$2C3 \xrightarrow{\omega_5} 3D2$$

$$3C1 \xrightarrow{\omega_6} 1D3$$

$$2C3 \xrightarrow{\omega_5} 1D3$$

$$3C1 \xrightarrow{\omega_6} 3D2$$

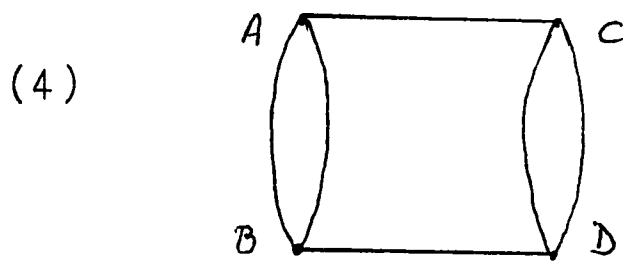
First case

$$1A \xrightarrow{\omega_1^{-1}} 2B \xrightarrow{\omega_2} 2C \xrightarrow{\omega_5} 3D \xrightarrow{\omega_6^{-1}} 1C \xrightarrow{\omega_2^{-1}} 3B \xrightarrow{\omega_3} 2D \xrightarrow{\omega_5^{-1}} 3C \xrightarrow{\omega_6} 1D \xrightarrow{\omega_3^{-1}} 1B \xrightarrow{\omega_1} 2A \xrightarrow{\omega_4^{-1}},$$

$$3A \xrightarrow{\omega_4},$$

Second case $1A \xrightarrow{\omega_1^{-1}} 2B \xrightarrow{\omega_2} 2C \xrightarrow{\omega_5} 1D \xrightarrow{\omega_3^{-1}} 1B \xrightarrow{\omega_1} 2A \xrightarrow{\omega_4^{-1}}, \quad 3A \xrightarrow{\omega_4},$

$$3B \xrightarrow{\omega_2} 1C \xrightarrow{\omega_6} 2D \xrightarrow{\omega_3^{-1}}, \quad 3C \xrightarrow{\omega_5} 3D \xrightarrow{\omega_6^{-1}}.$$



9 Choose the numbering so that

$$1A2 \xrightarrow{1} 2D1$$

$$1B2 \xrightarrow{2} 2C1$$

We now have four choices which we represent by two binary digits x_1 and x_2 .

$$x_1=0$$

$$2A3 \xrightarrow{\omega_3} 3B2$$

$$3A1 \xrightarrow{\omega_4} 1B3$$

$$x_1=1$$

$$2A3 \xrightarrow{\omega_3} 1B3$$

$$3A1 \xrightarrow{\omega_4} 3B2$$

$$\begin{array}{l} x_2=0 \\ 2D3 \xrightarrow{\omega_5} 3C2 \\ 3D1 \xrightarrow{\omega_6} 1C3 \end{array}$$

$$\begin{array}{l} x_2=1 \\ 2D3 \xrightarrow{\omega_5} 1C3 \\ 3D1 \xrightarrow{\omega_6} 3C2 \end{array}$$

By applying the permutation $(AD)(BC)$ we see that

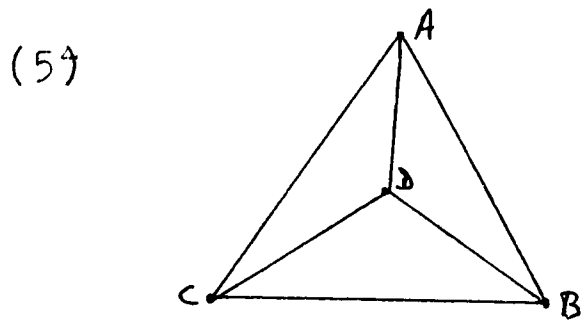
$$(x_1, x_2) \simeq (x_2, x_1)$$

so we have three cases, which can easily be seen to be non-isomorphic.

$$\begin{array}{l} (0,0) \quad 1A \xrightarrow{\omega_1} 2D \xrightarrow{\omega_5} 3C \xrightarrow{\omega_6^{-1}} 1D \xrightarrow{\omega_1^{-1}} 2A \xrightarrow{\omega_3} 3B \xrightarrow{\omega_4^{-1}} \\ \quad 3A \xrightarrow{\omega_3} 2B \xrightarrow{\omega_2} 1C \xrightarrow{\omega_6^{-1}} 3D \xrightarrow{\omega_5} 2C \xrightarrow{\omega_2^{-1}} 1B \xrightarrow{\omega_4^{-1}}. \end{array}$$

$$\begin{array}{l} (0,1) \quad 1A \xrightarrow{\omega_1} 2D \xrightarrow{\omega_5} 1C \xrightarrow{\omega_2^{-1}} 2B \xrightarrow{\omega_3^{-1}} 3A \xrightarrow{\omega_4} 1B \xrightarrow{\omega_2} 2C \xrightarrow{\omega_6^{-1}} 1D \xrightarrow{\omega_1^{-1}} 2A \xrightarrow{\omega_3} 3B \xrightarrow{\omega_4^{-1}}, \\ \quad 3D \xrightarrow{\omega_5} 3C \xrightarrow{\omega_6^{-1}}. \end{array}$$

$$\begin{array}{l} (1,1) \quad 1A \xrightarrow{\omega_1} 2D \xrightarrow{\omega_5} 1C \xrightarrow{\omega_2^{-1}} 2B \xrightarrow{\omega_4^{-1}}, \quad 2A \xrightarrow{\omega_1} 1D \xrightarrow{\omega_6} 2C \xrightarrow{\omega_2^{-1}} 1B \xrightarrow{\omega_3^{-1}}, \\ \quad 3A \xrightarrow{\omega_3} 3B \xrightarrow{\omega_4^{-1}}, \quad 3C \xrightarrow{\omega_6^{-1}} 3D \xrightarrow{\omega_5}, \end{array}$$



Number so that

$$1A2 \xrightarrow{\omega_1} 2B1$$

$$2A3 \xrightarrow{\omega_2} 2C1$$

$$3A1 \xrightarrow{\omega_3} 2D1$$

(i) Assume there are at least two triangular faces in the embedded surface. Without loss of generality A and B are the common vertices of these faces,

which implies $2B3 \xrightarrow{\omega_4} 1D3$

and $2C3 \xrightarrow{\omega_5} 1B3$

so that $2D3 \xrightarrow{\omega_6} 1C3$

i.e. $1A \xrightarrow{\omega_1} 2B \xrightarrow{\omega_4} 1D \xrightarrow{\omega_3^{-1}}, \quad 2A \xrightarrow{\omega_1} 1B \xrightarrow{\omega_5^{-1}} 2C \xrightarrow{\omega_2^{-1}}, \quad 3A \xrightarrow{\omega_3} 2D \xrightarrow{\omega_6} 1C \xrightarrow{\omega_2^{-1}},$
 $3B \xrightarrow{\omega_4} 3D \xrightarrow{\omega_6} 3C \xrightarrow{\omega_5},$

(ii) Exactly one triangular face

this gives $2B3 \xrightarrow{\omega_4} 1D3$ as before, so the last two pairings must be reversed, i.e. $2C3 \xrightarrow{\omega_5} 3D2$

$$3C1 \xrightarrow{\omega_6} 1B3$$

giving $1A \xrightarrow{\omega_1} 2B \xrightarrow{\omega_4} 1D \xrightarrow{\omega_3^{-1}},$

$$2A \xrightarrow{\omega_2} 2C \xrightarrow{\omega_5} 3D \xrightarrow{\omega_4^{-1}} 3B \xrightarrow{\omega_6^{-1}} 1C \xrightarrow{\omega_2^{-1}} 3A \xrightarrow{\omega_3} 2D \xrightarrow{\omega_5^{-1}} 3C \xrightarrow{\omega_6} 1B \xrightarrow{\omega_1^{-1}}.$$

(iii) No triangular faces.

This implies either $2B3 \xrightarrow{4} 3D2$ or $3B1 \xrightarrow{4} 1D3$

which gives $3B1 \xrightarrow{\omega_5} 1C3$ or $2D3 \xrightarrow{\omega_5} 3C2$

so by applying (CD) and interchanging 1 and 2 throughout one can assume the first case holds.

This gives $2C3 \xrightarrow{\omega_6} 1D3$

$$\begin{aligned} \text{i.e. } 1A \xrightarrow{\omega_1} 2B \xrightarrow{\omega_4} 3D \xrightarrow{\omega_6^{-1}} 3C \xrightarrow{\omega_5^{-1}} 1B \xrightarrow{\omega_1^{-1}} 2A \xrightarrow{\omega_2} 2C \xrightarrow{\omega_6} 1D \xrightarrow{\omega_3^{-1}}, \\ 3A \xrightarrow{\omega_2} 1C \xrightarrow{\omega_5^{-1}} 3B \xrightarrow{\omega_4} 2D \xrightarrow{\omega_3^{-1}}, \end{aligned}$$

Reduction in possible vertex surfaces

Consider the case when the vertex surface is connected, that is, all the points A, B, C, and D are equivalent under the maps R and S. Each end of an edge of the tetrahedron gives a face of the vertex surface. So there are an even number of faces to the vertex surface. This is already implied by the fact that the surface is orientable. However the two faces which correspond to the same edge also correspond to a cyclic word or its inverse in R and S. Hence the word length of words corresponding to these faces are the same. These words, however, are products of the ω_i which are reduced as written. It must be possible to pair the products of the ω_i so that each member of the pair can have the same word length.

For instance, in case (1) (page 26), let the word lengths of the ω_i be ℓ_i , so the faces correspond to words of length

$$l_1, \quad l_2, \quad l_3, \quad l_1 + l_2 + l_3 + 2(l_4 + l_5 + l_6).$$

As the length of the last word is greater than the others this case cannot arise.

By similar reasoning (2) second case, (3) both cases, (4) second case, (5) second and third cases; cannot arise.

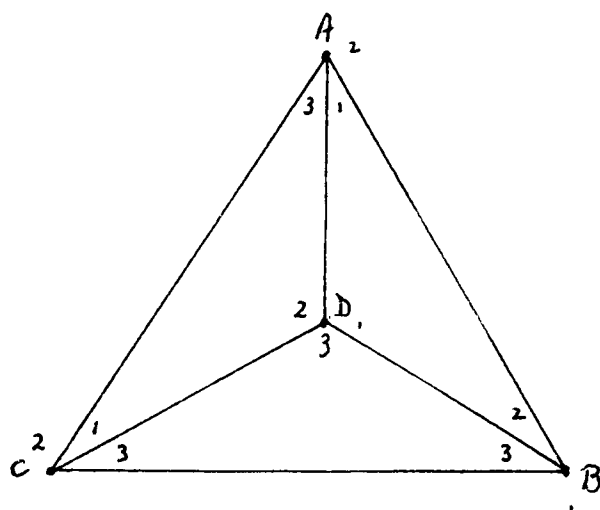
Diagrams of vertex surfaces

One can easily see by calculating the characteristic that the surface obtained from 2 and 4 points of degree three is either a torus or a sphere. We give examples of this, and an example of an embedding of one of the graphs into a torus which does not have the sense preserving property.

One requires six points for the abstract graph (all of degree three) to obtain a sphere with two handles. An example of this is given.

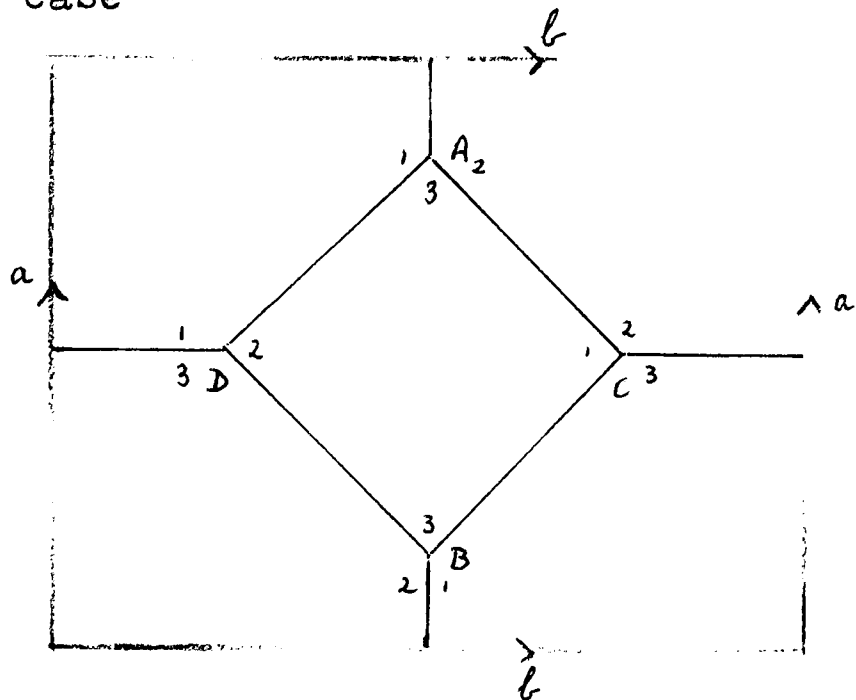
Embedding in a plane

(5) first case

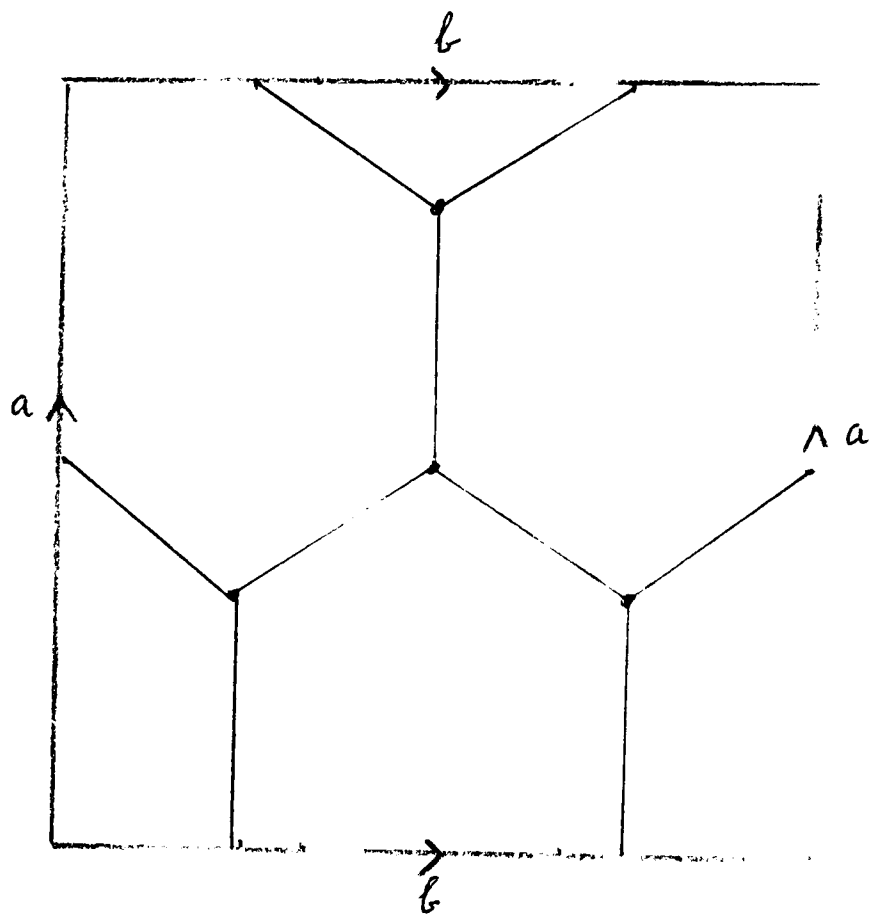


Embedding in a torus

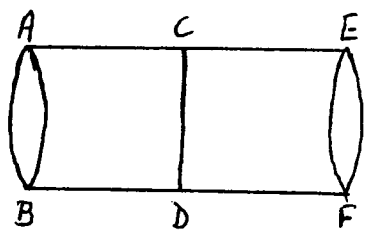
(5) last case



Embedding in a torus which is not sense preserving

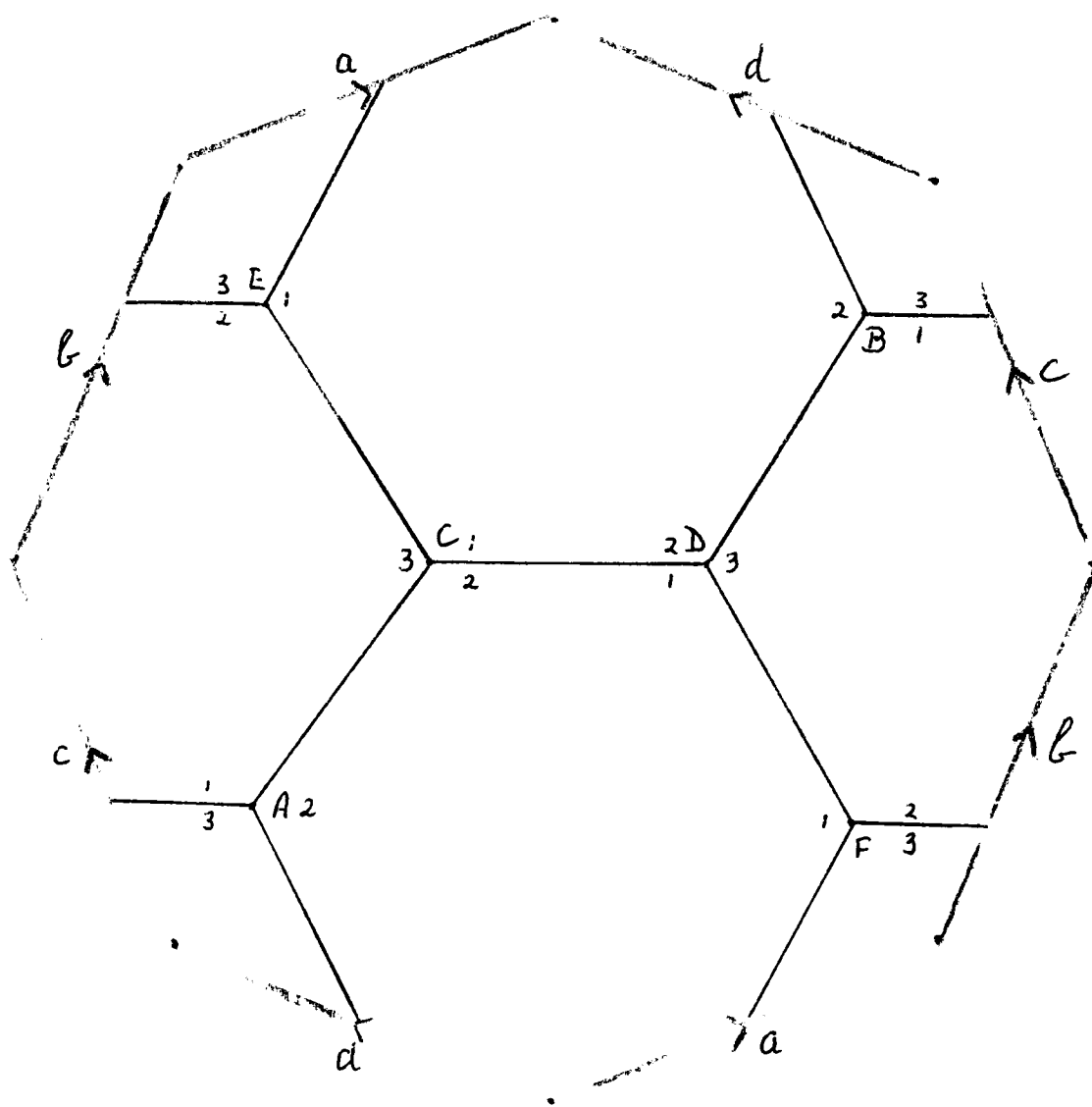


The graph



can be embedded in a sense

preserving manner in a sphere with two handles.



List of vertex surfaces for a tetrahedral fundamental region

<u>Orbit space</u>	<u>Type No.</u>	<u>Letter sequence describing vertex surface</u>
	1*	DCDCDC, ABABAB;
	2	ACACAC, BDBDBD;
	3	DDCC, BBAA; D,B; A,C;
	4*	AACC, BBDD; A,B; C,D;
	5	AACC, BBDD; A,D; B,C;
	6	DC,AB; DC,AB; DC,AB;
	7*	AC,BD; AC,BD; AC,BD;
	8*	DCDCDC, BBAA; A,B;
	9	ACACAC, BBDD; B,D;
	10*	DCDCDC, BA; BA,BA;
	11	ACACAC, BD; BD,BD;
	12*	DC,DC; DC,AB; AB,AB;
	13	AC,AC; AC,BD; BD,BD;
	14*	D,C; DDCC, BA; BA,BA;
	15	A,C; AACC, BD; BD,BD;
	16*	D,C; DDCC, BBAA; B,A;
	17	A,C; AACC, BBDD; B,D;
	18*	ABC, ABD; ADC, BDC;
	19	ABC, ADC; ABD, BDC;
	20*	ABDC, ABDC; AC,BD;
	21	ABDC, ABDC; DC,AB;
	22	ADBC, AD BC; AD,BC;
	23*	AACDC, BBDCD; B,A;
	24	DDCAC, BBACA; D,B;
	25	DDACA, BBCAC; D,B;
	26*	ABACDC, BABDCD;
	27	DBDCAC, BDBACA;

* known to occur, an example for each such type is given below

Types known to occur

<u>Type</u>	<u>Value of parameters</u>	<u>Words</u>
1	0,0,1,1 2,2	(R^2, S^2)
4	0,1,0,0 0,0	$R, S, R(S, R)S^{-1}$
7	0,1,0,0 2,2	R^2S, RS^2, RS^{-1}
8	0,0,0,1 0,2	$S, (R^2, S)$
10	0,1,0,2 3,3	$(R^2, S), R^2S^{-2}$
12	1,1,0,1 4,1	$R^3S^{-1}, (R, S), RS^2$
14	0,1,0,0 3,0	$R, (R, S), RS^2$
16	0,0,0,0 0,0	$R, S, (R, S)$
18	0,0,0,0 2,1	R^2S^{-1}, RS^2
20	0,0,0,0 2,2	R^2S^2, RS^{-1}
23	0,0,0,0 2,0	$S, R^2SR^{-1}S$
26	0,0,0,1 2,2	$S^2R^2S^2R^{-1}$

Section
(5) Vertex diagrams

We now return to a general fundamental region as given on page 15. For every equivalence class of edges, we take an integer n_j . If r_j are the words derived from a set of representative edges one from each class, then the group G generated by the R_i ($i=1$ to k) and subject only to the relations $r_j^{n_j} = 1$ ($j=1$ to n) is defined to be a polyhedral group.

We now construct a three dimensional space upon which G acts. Take a copy of the fundamental region $\mathcal{R}$ for every group element $g \in G$. Call this copy $\mathcal{R}g$. Now make the following identifications. Identify the (face R_i) of $\mathcal{R}g$ with the (face R_i^{-1}) of $\mathcal{R}R_i g$ by identifying the corresponding images under the map R_i^{-1} .

We have now constructed a space S upon which G acts, in such a way that if $\mathcal{R}1$ is chosen as a fundamental region for G , then the "homotopy" argument shows that the only relations of G are $r_j^{n_j} = 1$, since paths about each of the edges of $\mathcal{R}1$ give these relations.

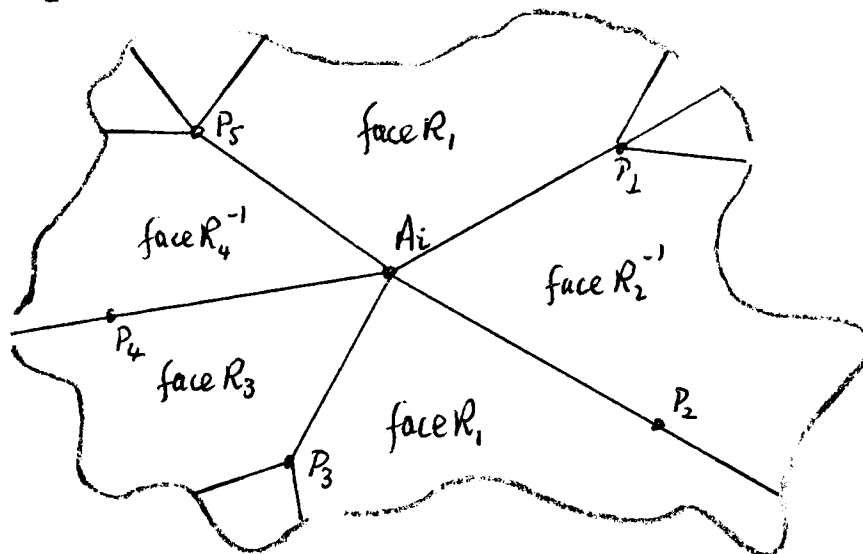
If we can give an effective construction of this space, then we can solve the word problem in the groups G and conversely if the word problem is soluble, then the construction given above is effective. This is equivalent to giving an effective construction of the neighbourhood of any point, for the nature of the identification round any line can be

determined from this.

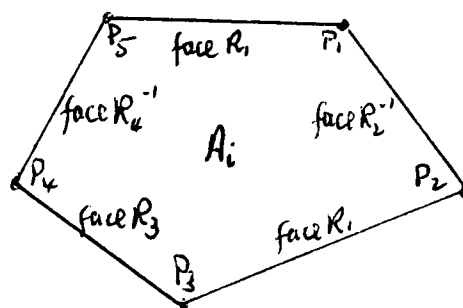
We now give a construction of the neighbourhood of a vertex of the fundamental region. Then we show that this gives an effective construction in the case of the tetrahedral groups, by using the analysis in the last section.

Construction of the Vertex diagrams

We require to give a construction of the vertex diagram (i.e. the diagram homeomorphic with a neighbourhood of a point) for the vertex $A=A_1$. Let A_i be the vertices equivalent to A under the R_i^E . For each A_i we take a countable number of copies of the intersection of a neighbourhood of the A_i with the fundamental region. That is, if A_i in the planar graph is



then the intersection of the vertex A_i with the fundamental region is



So this is an n -gon if and only if the degree of A_i in the planar graph is n . Each vertex of the n -gon is labelled with the neighbouring vertices of the A_i in the planar graph in the same cyclic order, and the edge P_1P_2 is labelled R_i^E if and only if $P_1A_iP_2$ are adjacent vertices of the face R_i^E . (In the example given $P_1A_iP_2$ are vertices belonging to the face R_1 , but not adjacent vertices of the face.) It is possible for $P_i = P_j$ ($i \neq j$). If $P_1 = P_2$ in the diagram above, then the face R_2^{-1} is a digon.

We now give an inductive construction of the diagram.

We start with the polygon labelled A (and marked).

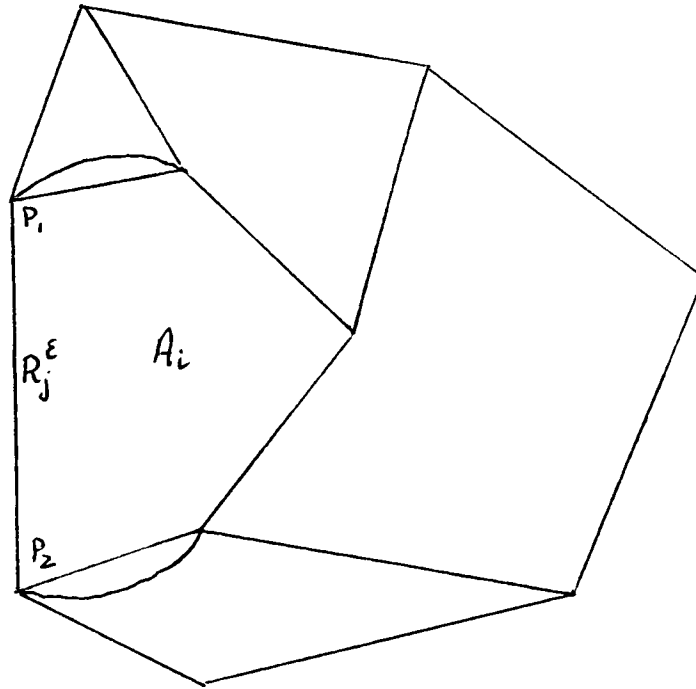
At the end of the n th stage, the diagram is either a number of polygons joined so as to give a simply-connected compact subset of E_2 , or it is a sphere, in which case the construction is finished.

We prove this by induction on the number of stages.

The first polygon which is labelled A (but not necessarily the only one so labelled) is also marked. For any polygon B , define the distance between A and B to be equal to the least number k , of polygons one need pass through in order to go from A to B .

Of all polygons on the boundary of the region constructed at the n 'th stage, choose one whose distance from the first polygon is minimal. Then choose any edge of this polygon which lies on the boundary. Let us say that this bounding

polygon is labelled A_i , the chosen edge with the generator R_j^E and each end of this edge with the vertices P_1 and P_2



Now adjoin a copy of the polygon labelled $(A_i)R_j^{-E}$. This is defined as A_i belongs to the face R_j^E upon which the map R_j^{-E} is defined. Moreover $P_1 A_i P_2$ are consecutive vertices of the face R_j^E , so $(P_1)R_j^{-E}$, $(A_i)R_j^{-E}$, $(P_2)R_j^{-E}$ are consecutive vertices of the face R_j^{-E} . Hence there is a unique edge of the polygon labelled $(A_i)R_j^{-E}$, which is labelled R_j^{-E} having $(P_1)R_j^{-E}$, $(P_2)R_j^{-E}$ as end points. So identify this edge with the chosen one viz:-

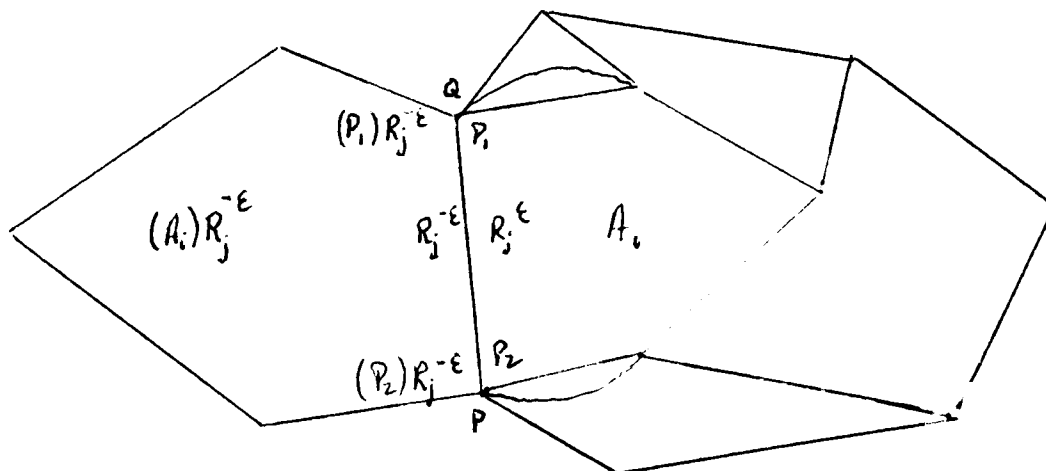
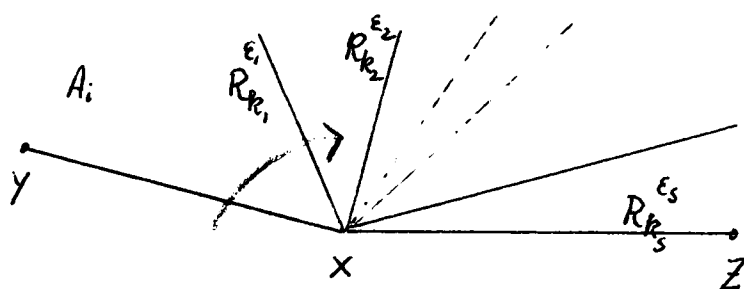


Fig 1

We now make a series of identifications to complete this stage. These identifications are all of the same nature, so we describe it in general.

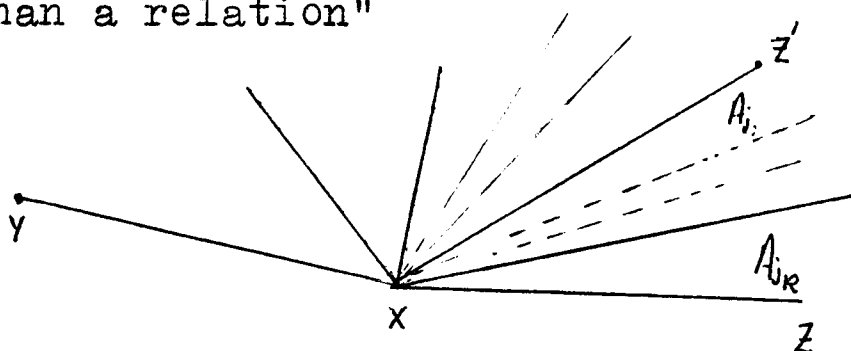
Let X be a point on the boundary



As one moves round X (in a fixed direction) one can construct a word applicable to $A_i P_1$ namely $R_{k_1}^{E_1} R_{k_2}^{E_2} \dots R_{k_s}^{E_s} = \omega$. But there is a word applicable to $A_i P_1$ which is a conjugate of r_j or r_j^{-1} for some j . Therefore one of three cases occurs (i) $\omega = (\omega' r_j^{E_j} \omega'^{-1})^{n_j}$ (ii) $\omega = (\omega' r_j^{E_j} \omega'^{-1})^{n_j} \omega''$ or (iii) $\omega \omega'' = (\omega' r_j^{E_j} \omega'^{-1})^{n_j}$ where the words outside the brackets are reduced as written and $\omega'' \neq 1$.

In case (i) the two external edges leading to X are identified. The point X is now internal, and the word obtained by moving round it is a conjugate of a relation (in the free group).

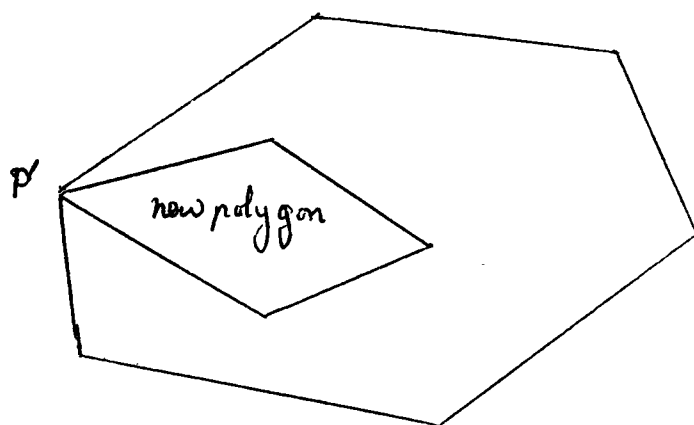
In case (ii) the word corresponding to going round X is "more than a relation"



In this case we pick the line XZ' such that the identification would follow as in type (i). Now cut out all polygons $A_{j_1} \dots A_{j_k}$ and identify YX and $Z'X$.

In case (iii) no identification is made, so the process finished when either there are no boundary points, or all the boundary points are of type (iii).

Initially only points P and Q Fig. 1 can be of type (i) or (ii), in fact P and Q are of type (i) or (iii). After each identification only one new point can be of type (i) or (ii) namely $Y = Z$ in the above. Call these new points P' and Q' , P' derived from P and Q' from Q , but only so label them if they are of type (i) or (ii). If P' is of type (ii) then Q' is not defined, for it is only when the new polygon becomes internal that a point of type (ii) can be formed viz:-



In which case P' and Q' have either just been identified or Q' was undefined at an earlier stage.

We now have to show that if the end of the n 'th stage results in no boundary points then the polygons form a sphere.

Before the last identification when there are only two bounding points they are of types (i) and (i), (i) and (iii) or (ii) and (iii). In the last case cutting out polygons A_j, A_{jk} reduces the point of type (ii) to type (i), and as there are still only two boundary points the other one is clearly of type (iii). The first case clearly gives a sphere, with all the points of the diagram corresponding to relations. The second case, the final identification again gives a sphere but the point which was of type (iii) corresponds to "less than a relation", and the group is said to collapse. All the vertex diagrams for a polyhedral group must be drawn before it is known if any collapse occurs.

The process of identification is clearly finite as the number of edges is strictly reduced at each stage.

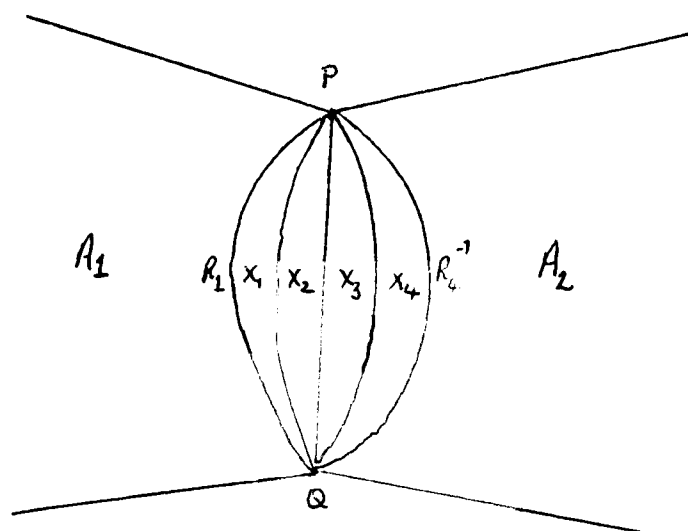
This construction will be effective if for every n there exists an N such that no identification beyond the N -th stage of construction involves any polygons at a distance $\leq n$ from the initial polygon.

The contracted vertex diagrams

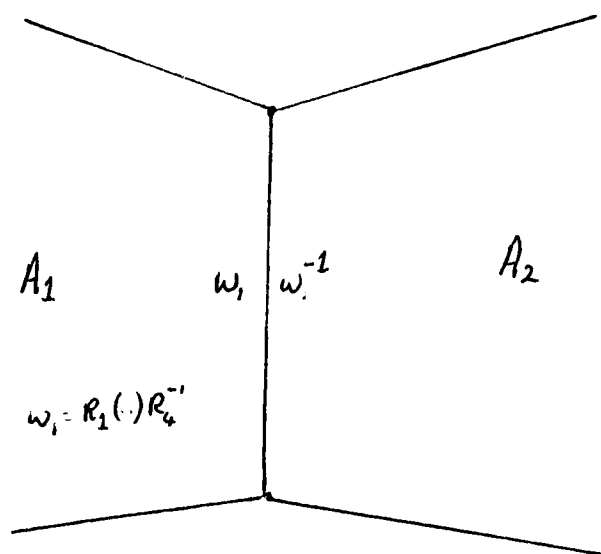
The construction of vertex diagrams can be simplified by omitting any digons which may occur. We describe this in more detail.

Let A_i be the points equivalent to A ($=A_1$) under the R_i^E , such that the degree of A_i in the planar graph is ≥ 3 . Let

X_i be the points equivalent to A of degree 2 in the planar graph.



Vertex diagram



Contracted Vertex diagram

This process is illustrated above. Given the adjacent vertices of the face R_1 say PA_1Q , then there exists a reduced word ω such that $(PA_1Q)\omega$ is defined but this holds for no longer reduced word. Then $A_1\omega = A_i$ for some i (say 2). The contracted diagram can now be drawn, consisting entirely of n -gons, $n \geq 3$, the edges being labelled with the ω_i defined on page 17. instead of the R_i^E .

It is now clear that the contracted vertex diagram is determined uniquely by the vertex surface and the integers n_i . (It is not necessary to know which faces in the surface correspond to the same relation, merely the value of n_i for each face.)

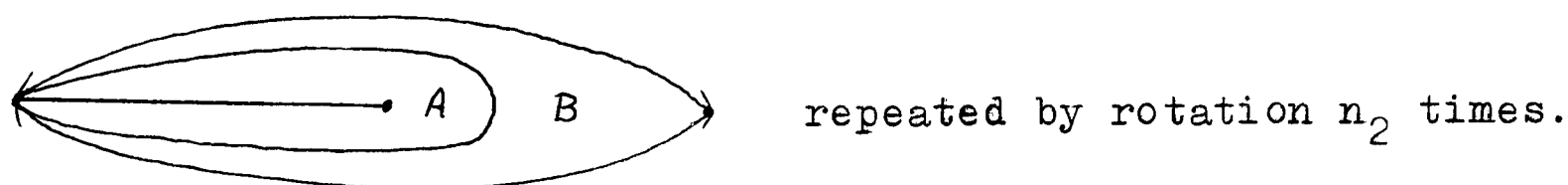
Section(6) Construction of contracted vertex diagrams for the tetrahedral groupsCases with $A \sim B \not\sim C$ or D

(1) Constituent vertex surface gives letter sequence

$(A)^{n_1}, (B)^{n_2}, (AABB)^{n_3}$ n_i be the powers of the respective relations.

If $n_1=1$ then the vertex diagram collapses if $n_2 \neq n_3$. The collapse is to $A, B^d, (AABB)^d$ where $d = (n_2, n_3)$

The diagram with $n_2 = n_3$ is

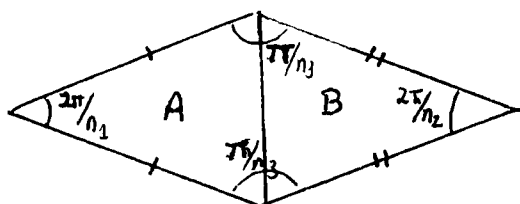


Similarly if $n_3 = 1$, then the vertex diagram collapses if $n_1 \neq n_2$ to $A^d, B^d, AABB$, where $d = (n_1, n_2)$

The diagram is then



Otherwise there exists a quadrilateral



angles as marked and sides equal in length in pairs as shown.

This is on the surface of a sphere when $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} > 1$, is

Euclidean when $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$ and hyperbolic otherwise.

The quadrilateral can then be repeated in the obvious way to cover the corresponding plane (viz angles equal to $2\pi/n_1$, π/n_3 next to each other, with sides of equal length juxtaposed) to give the vertex diagram.

(2) $(ABABAB)^{n_1}$

Consider a triangle whose angles are $\pi/3n_1$, $\pi/3n_1$, $\pi/3n_1$ (Euclidean when $n_1 = 1$, hyperbolic otherwise). This triangle can be regarded as the fundamental region for the group; in Coxeter's notation $(3n_1, 3n_1, 3n_1)^*$. If, in the division of the plane given by this, alternate triangles are labelled A and B we obtain the vertex diagram required.

(3) $(AB)^{n_1}$, $(AB)^{n_2}$, $(AB)^{n_3}$.

Consider the triangles whose angles are π/n_1 , π/n_2 , π/n_3 , that is the natural fundamental region for the group $(n_1, n_2, n_3)^*$. If alternate triangles in the plane are labelled A and B one obtains the required vertex diagram. If $n_1 = 1$ and $n_2 = n_3$ (or a permutation of this) then there is partial collapse to $(1, d, d)^*$ where $d = (n_2, n_3)$.

This is on a sphere if $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} > 1$ Euclidean when $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$ and hyperbolic otherwise.

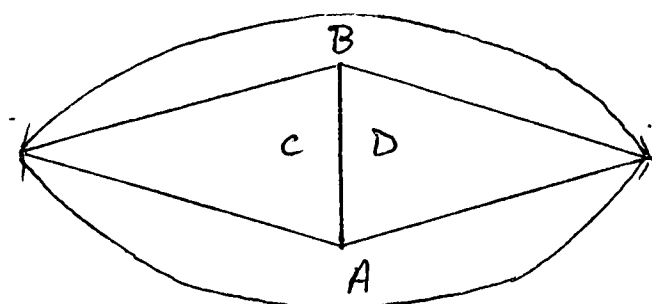
Cases with $A \sim B \sim C \sim D$

(1) $(ABC)^{n_1}$, $(ABD)^{n_1}$, $(ACD)^{n_2}$, $(BCD)^{n_2}$.

This and the following cases depend on a somewhat less obvious division of the plane, in general one requires an existence theorem for certain hexagons.

(i) n_1 or $n_2 = 1$, without loss of generality assume $n_2 = 1$.

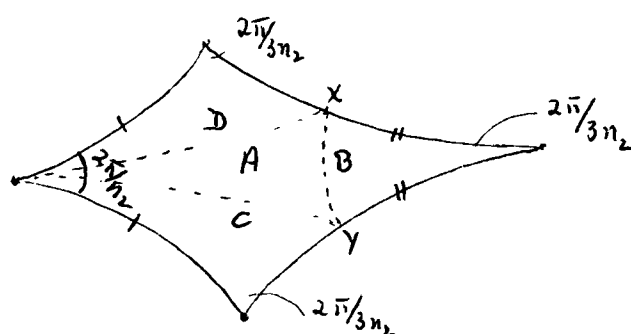
Then the vertex diagram is



repeated by rotation
 n_1 times

(ii) n_1 or $n_2 = 2$, without loss of generality assume $n_1 = 2, n_2 \geq 2$.

There exists a triangle whose angles are $2\pi/3n_2, \pi/3n_2, \pi/n_2$, which is Euclidean for $n_2 = 2$ and hyperbolic otherwise. Reflection in the side opposite the angle $2\pi/3n_2$ shows that there exists a quadrilateral



The sides similarly marked being of equal length.

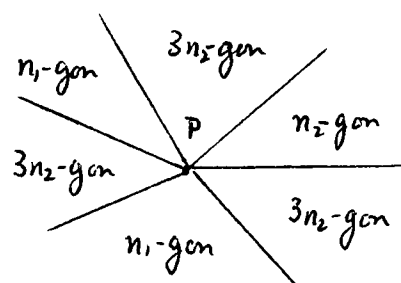
Copies of this quadrilateral can cover the plane by juxtaposition of the sides of equal length. By dividing the quadrilateral as shown, X and Y being the midpoints of their

respective sides one obtains the diagram for this case.

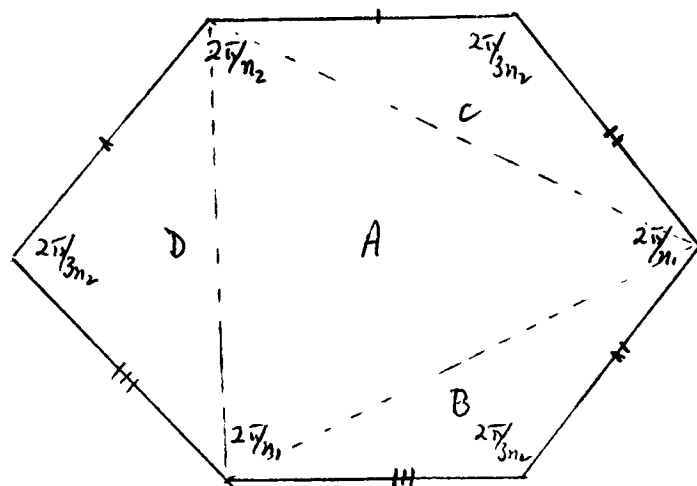
(iii) We can now assume $n_1, n_2 \geq 3$.

In the hyperbolic plane there exists exactly one regular n -gon having a given length for its side. Further as this length approaches infinity the internal angle between the sides approaches zero.

Construct a regular n_2 -gon with side of length x . Join this to a regular $3n_2$ -gon so they share a side of the same length x . About one of the ends of the common side joined to the $3n_2$ -gon construct a n_1 -gon, next another $3n_2$ -gon, another n_1 -gon and finally yet another $3n_2$ -gon. The internal angle subtended by all these n -gons varies from $6\pi - 4\pi/n_1 - 4\pi/n_2 > 2\pi$ to zero as x increases. Choose the x so that the angle equals 2π .



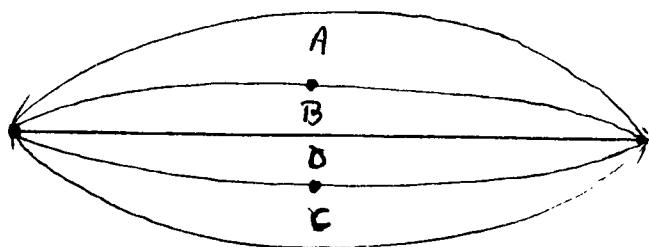
Now obtain a hexagon by joining the centres of the n -gons in cyclic order



This hexagon has angles as shown and sides similarly marked being of equal length. Dividing the hexagon as shown and covering the plane with these hexagons in the obvious way gives the diagram in this case.

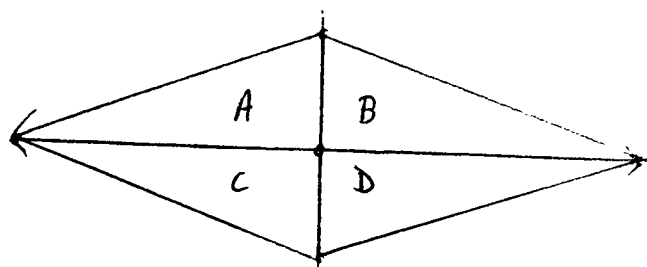
$$(2) \quad (AB)^{n_1}, \quad (CD)^{n_1}, \quad (ACDB)^{n_2}, \quad (ACDB)^{n_2}$$

(i) $n_1 = 1$, the vertex diagram is



repeated by rotation n_2 times

(ii) $n_2 = 1$, the vertex diagram is

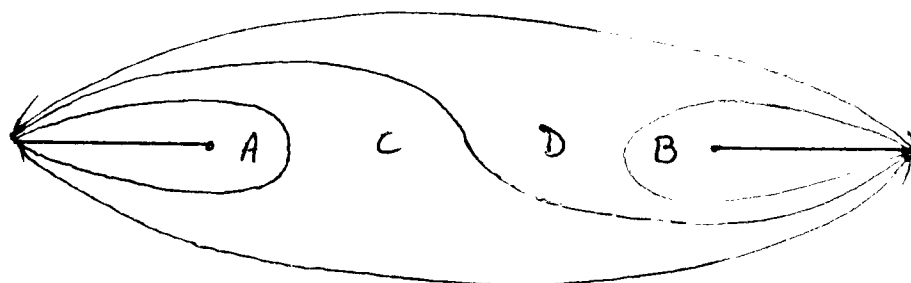


repeated by rotation n_1 times

(iii) Otherwise the triangle A can be taken to be the natural fundamental region for the group $(n_1, 2n_2, 2n_2)^*$, B, C and D being identical.

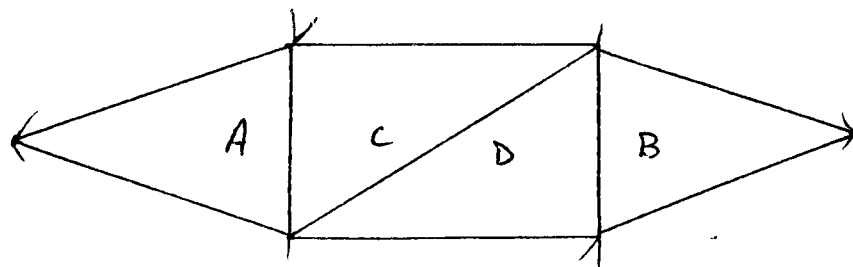
(3) $A^{n_1}, B^{n_1}, (AACDC)^{n_2}, (BBD CD)^{n_2}$

(i) For $n_1 = 1$ the vertex diagram is

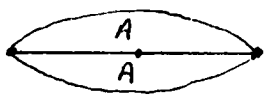


repeated by rotation n_2 times

(ii) For $n_2 = 1$ the vertex diagram is

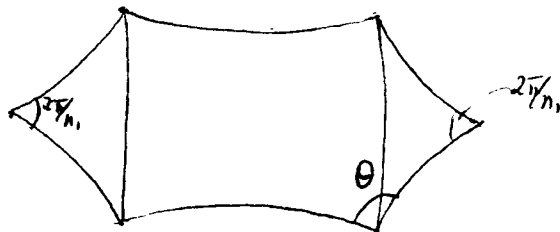


repeated by rotation n_1 times

(iii) When $n_1 = 2$ the A and B triangles appear in pairs viz:  . Ignoring these pairs one has a vertex diagram with $3n_2$ triangles about each point. This is finite for $n_2 = 1$, Euclidean when $n_2 = 2$ and hyperbolic otherwise.

(iv) Other cases

Consider a hyperbolic square on a given base length x . Construct two isosceles triangles on opposite sides having vertex angle $2\pi/n_1$



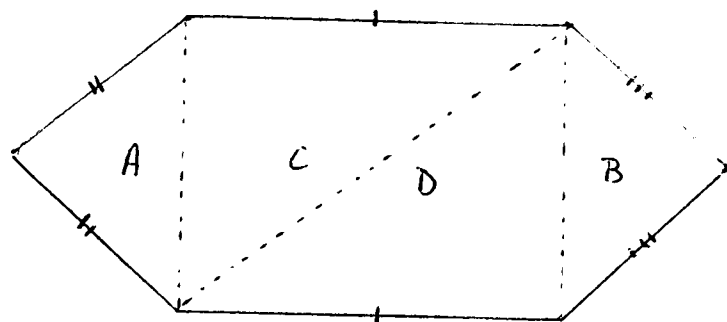
As x varies from 0 to ∞ , θ varies from $\pi - \pi/n_1$ to 0.

Page 51

line 1:

an x such that $\theta = 2\pi/n_2$

So there exists an x such that $\theta = \pi/n_2$



Covering the plane by juxtaposition of such hexagons, joining sides similarly marked, gives the vertex diagram for these cases.

$$(4) \quad (ABACDC)^{n_1}, \quad (BABDCD)^{n_1}$$

A, B, C, and D are similar triangles having angles $\pi/3n_1$, $\pi/3n_1$, $\pi/3n_1$, that is, the natural fundamental region for the group $(3n_1, 3n_1, 3n_1)^*$. This is Euclidean for $n_1 = 1$ and hyperbolic otherwise.

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last line:

of edges which are joined to the points P .

Section(7) Theorems about tetrahedral groups

It is now clear that the construction given on page 45 for the contracted vertex diagrams gives an effective construction for the space upon which the tetrahedral groups act. From page 37 this gives us

Theorem 1 The tetrahedral groups defined in (2) have a computable word problem which can be solved by constructing the space S described on page 8 . A tetrahedral group collapses if and only if the relations corresponding to edges in the orbit space meeting a point P in the orbit space are $r_1^{n_1} = 1, \dots, r_i^{n_i} = 1$ and all the $n_i = 1$ except two which are not equal. If n_1 and n_2 are the $n_i \neq 1$, then the group collapses to the group $r_1^d = r_2^d = r_3 \dots = r_i$ where $d = (n_1, n_2)$.

Also the condition for the space S to be a manifold is clear.

Theorem 2 The tetrahedral groups act on a manifold if and only if $\sum_P \frac{1}{n_i} > 1$ for every equivalence class of points P , where n_i is the power of the word which is a relation and the summation is over those n_i corresponding to equivalence class of edges which is joined to the points P .

Section(8) Tetrahedral groups and the characteristic of the space

For a finite tetrahedral group, the space upon which the group acts is necessarily a manifold. Since this manifold is simply connected and is constructed by amalgamating a finite number of tetrahedra, the Euler-Poincaré characteristic is zero.

Calculation of Characteristic

Number of 3-cells = p^3 = number of tetrahedra = order of group = g

Number of 2-cells = p^2 . There are four faces to each tetrahedron but each face is common to two tetrahedra, so there are $2g$ faces

Number of one-cells = p^1 . Given an edge XY , whose corresponding word is r_i and the relation $r_i^{n_i} = 1$, then there are $n_i \ell_i$ tetrahedra about XY (ℓ_i being the length of the word r_i). Hence $p^1 = g(\sum_{XY} \frac{1}{n_i \ell_i})$ where the summation is over all edges of the tetrahedron. Let ω_i $i = 0$ to $(\ell_i - 1)$ be the left partial words of r_i . Hence $(XY)\omega_i$ is defined and is an edge which gives the same contribution to the sum as XY . Grouping these together we have

$$p^1 = g(\sum \frac{1}{n_i})$$

where the summation is over equivalence classes of edges (i.e. relations).

Number of zero-cells = p^0 . In the vertex diagram for A

let p_A be the number of faces. Let n_A be the number of vertices equivalent to A. Then in a manner similar to the calculation of p^1 we have

$$p^0 = g\left(\sum_{\sim A} \frac{n_A}{p_A}\right) \quad \text{where the summation}$$

is over inequivalent vertices.

We now have

$$\chi = p^3 - p^2 + p^1 - p^0 = g\left(1 - 2 + \sum_i \frac{1}{n_i} - \sum_{\sim A} \frac{n_A}{p_A}\right) = 0$$

$$\text{i.e.} \quad \sum_i \frac{1}{n_i} = 1 + \sum_{\sim A} \frac{n_A}{p_A} \quad (\text{I})$$

Since this equation does not involve the order of the group, the question arises as to whether this is satisfied when the group acts on a manifold i.e. $p_A < \infty$ but the group has infinite order. With the convention that $\frac{n_A}{p_A} = 0$ if $p_A = \infty$ equation (I) can be examined in general.

Examination of Equation I

For each type listed on page 35 we can calculate n_A and p_A for different values of n_i to see if equation I is satisfied. Types which differ only by a permutation of the letters A, B, C and D are identical as far as this calculation is concerned.

Types 1 and 2

For all the groups $p_A = p_C = \infty$

$$\text{so } \frac{1}{n_i} = \frac{1}{n_1} \quad \text{and} \quad 1 + \sum \frac{n_A}{p_A} = 1$$

Therefore equation I is satisfied if and only if $n_i = 1$, that is, when the contracted vertex diagram is realisable in the Euclidean plane.

$$\text{Otherwise } \sum \frac{1}{n_i} < 1 + \sum \frac{n_A}{p_A}$$

Types 3, 4 and 5

The relations, ω_i , and letters are,

$$(A \xrightarrow{\omega_2})^{n_1}, \quad (C \xrightarrow{\omega_5})^{n_1}; \quad (B \xrightarrow{\omega_3})^{n_2}, \quad (D \xrightarrow{\omega_6})^{n_2};$$

$$(A \xrightarrow{\omega_2} A \xrightarrow{\omega_1} B \xrightarrow{\omega_3} B \xrightarrow{\omega_1^{-1}})^{n_3}, \quad (C \xrightarrow{\omega_5} C \xrightarrow{\omega_4} D \xrightarrow{\omega_6} D \xrightarrow{\omega_4^{-1}})^{n_3};$$

Calculation of n_A

If ℓ_i is the length of the word ω_i , then

$$n_A = \sum_i (\ell_i - 1) + n_A^1,$$

where the summation is over all pairings between vertices equivalent to A and n_A^1 is the number of the A, B, C and D equivalent to A.

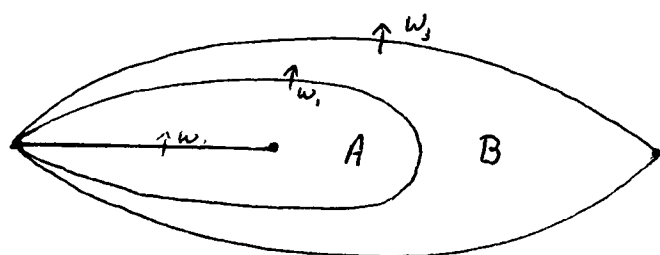
$$\text{So } n_A = \ell_1 + \ell_2 + \ell_3 - 1$$

$$n_C = \ell_4 + \ell_5 + \ell_6 - 1$$

Calculation of p_A

In the contracted vertex diagrams on page 45 each line corresponds to a word ω_i and hence the uncontracted diagram would have $(\ell_i - 1)$ digons instead of the line. So $p_A = p_{A'} + \sum_{\ell} (\ell_i - 1)$ where $p_{A'}$ is the number of faces in the contracted diagram and the summation is over all lines of the contracted diagram.

For instance, $n_1 = 1$ and $n_2 = n_3$ we have



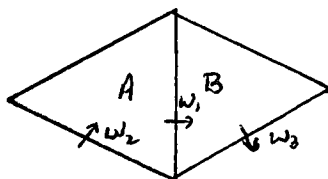
repeated by rotation
 n_2 times

$$\text{so } p_A = (2 + \ell_1 + \ell_2 + \ell_3 - 3)n_2$$

$$\text{Hence } \frac{n_A}{p_A} = \frac{1}{n_2} = \frac{n_c}{p_c} \quad \therefore 1 + \sum \frac{n_A}{p_A} = 1 + \frac{2}{n_2} = \sum \frac{1}{n_i}$$

Similarly when $n_3 = 1$ and $n_1 = n_2$

Otherwise the quadrilateral



is repeated $\frac{2}{(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} - 1)}$ times so $\frac{n_A}{p_A} = \frac{n_c}{p_c} = \frac{1}{2}(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} - 1)$

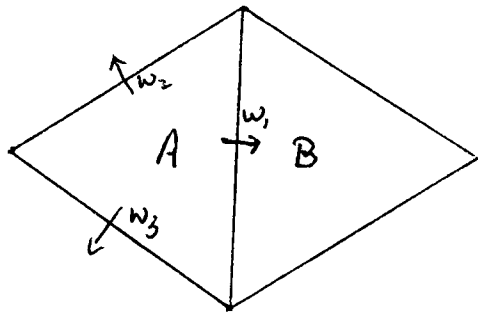
and $1 + \sum_{\sim A} \frac{n_A}{p_A} = \sum \frac{1}{n_i}$. With $p_A = \infty$ equation I is

satisfied only if $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$

Types 6 and 7

$$\begin{aligned}
 & (A \xrightarrow{\omega_1} B \xrightarrow{\omega_3^{-1}})^{n_1}, (C \xrightarrow{\omega_4} D \xrightarrow{\omega_6^{-1}})^{n_1}; \quad (A \xrightarrow{\omega_2} B \xrightarrow{\omega_1^{-1}})^{n_2}, (C \xrightarrow{\omega_5} D \xrightarrow{\omega_6^{-1}})^{n_2}; \\
 & (A \xrightarrow{\omega_3} B \xrightarrow{\omega_2^{-1}})^{n_3}, (C \xrightarrow{\omega_6} D \xrightarrow{\omega_5^{-1}})^{n_3};
 \end{aligned}$$

When these groups act on a manifold i.e. when $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} > 1$, the vertex diagram is $2k$ repetitions of



$$\text{where } k = \frac{1}{(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} - 1)}$$

$$\text{Hence } p_A = 2k (\ell_1 + \ell_2 + \ell_3 - 1)$$

$$1 + \sum_{\gamma A} \frac{n_A}{p_A} = 1 + \frac{\ell_1 + \ell_2 + \ell_3 - 1}{2k(\ell_1 + \ell_2 + \ell_3 - 1)} + \frac{\ell_4 + \ell_5 + \ell_6 - 1}{2k(\ell_4 + \ell_5 + \ell_6 - 1)} = 1 + 1/k = \sum \frac{1}{n_i}$$

If $p_A = \infty$, then equation (I) is satisfied if and only if $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$

Types 8 and 9

$$(AABB)^{n_1}, (CDCDCD)^{n_1}; \quad (A)^{n_2}, (B)^{n_2};$$

In this case the values of p_A and p_C have already been calculated. One can easily see that equation (I) is satisfied only when $n_1 = 1$, that is when the vertex diagrams for C and D are Euclidean

$$\text{Otherwise } \sum \frac{1}{n_i} < 1 + \sum_{\gamma A} \frac{n_A}{p_A}$$

Types 10 and 11

$$(AB)^{n_1}, (CDCDCD)^{n_1}; (AB)^{n_2}, (AB)^{n_2};$$

As in the last two cases p_A and p_C have already been calculated. Again equation (I) is satisfied only when $n_1 = 1$, that is when the vertex diagrams for C and D are Euclidean.

$$\text{Otherwise } \sum \frac{1}{n_i} < 1 + \sum_A \frac{n_A}{p_A}$$

Types 12 and 13

$$(DC)^{n_1}, (DC)^{n_1}; (DC)^{n_2}, (AB)^{n_2}; (AB)^{n_3}, (AB)^{n_3};$$

The groups act on a manifold when $\frac{1}{n_2} + \frac{2}{n_1} > 1$ and

$$\frac{1}{n_2} + \frac{2}{n_3} > 1. \quad \text{In which case } \frac{n_A}{p_A} = \frac{1}{2}(\frac{1}{n_2} + \frac{2}{n_3} - 1)$$

$$\frac{n_C}{p_C} = \frac{1}{2}(\frac{1}{n_2} + \frac{2}{n_1} - 1)$$

$$\text{so } 1 + \sum \frac{n_A}{p_A} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \sum \frac{1}{n_i}$$

Similarly equation (I) holds if the vertex diagrams for A or C or both are Euclidean, otherwise $\sum \frac{1}{n_i} < 1 + \sum \frac{n_A}{p_A}$.

Types 14 and 15

$$(D)^{n_1}, (C)^{n_1}; (DDCC)^{n_2}, (BA)^{n_2}; (AB)^{n_3}, (AB)^{n_3};$$

Types 16 and 17

$$(D)^{n_1}, (C)^{n_1}; (DDCC)^{n_2}, (BBAA)^{n_2}; (B)^{n_3}, (A)^{n_3};$$

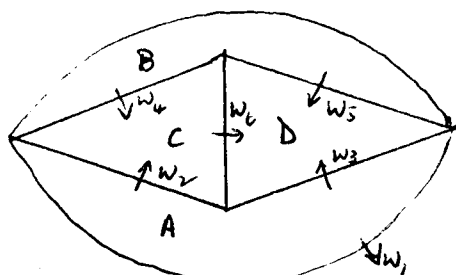
These four cases follow in exactly the same manner as Types 12 and 13 as the values of n_A , n_C , p_A and p_C are the

same (for all values of n_i).

Types 18 and 19

$$\begin{aligned} & (A \xrightarrow{\omega_1} B \xrightarrow{\omega_4} C \xrightarrow{\omega_2^{-1}})^{n_1}, \quad (A \xrightarrow{\omega_1} B \xrightarrow{\omega_5} D \xrightarrow{\omega_3^{-1}})^{n_1}; \\ & (A \xrightarrow{\omega_2} C \xrightarrow{\omega_6} D \xrightarrow{\omega_3^{-1}})^{n_2}, \quad (B \xrightarrow{\omega_4} C \xrightarrow{\omega_6} D \xrightarrow{\omega_5^{-1}})^{n_2}; \end{aligned}$$

For $p_A < \infty$ we have n_1 or $n_2 = 1$, so assume without loss of generality $n_2 = 1$.



repeated by rotation n_1 times

$$n_A = \sum_{i=1}^6 (l_i - 1) + 4 = l_1 + l_2 + l_3 + l_4 + l_5 + l_6 - 2$$

From the above diagram $p_A = n_1 (\sum l_i - 6 + 4) = n_1 n_A$

$$\text{so } 1 + \sum \frac{n_A}{p_A} = 1 + \frac{1}{n_1} = \sum \frac{1}{n_i}$$

The only Euclidean case is when $n_1 = n_2 = 2$

$$\text{when } \sum \frac{1}{n_i} = 1 = 1 + \sum \frac{n_A}{p_A}$$

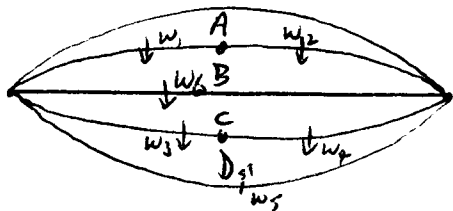
$$\text{Otherwise } \sum \frac{1}{n_i} < 1 = 1 + \sum \frac{n_A}{p_A}$$

Types 20, 21 and 22

$$\begin{aligned} & (A \xrightarrow{\omega_1} B \xrightarrow{\omega_2^{-1}})^{n_1}, \quad (C \xrightarrow{\omega_4} D \xrightarrow{\omega_3^{-1}})^{n_1}; \\ & (A \xrightarrow{\omega_1} B \xrightarrow{\omega_6} C \xrightarrow{\omega_3} D \xrightarrow{\omega_5^{-1}})^{n_2}, \quad (A \xrightarrow{\omega_2} B \xrightarrow{\omega_6} C \xrightarrow{\omega_4} D \xrightarrow{\omega_5^{-1}})^{n_2}; \end{aligned}$$

Again $p_A < \infty$ if and only if n_1 or $n_2 = 1$

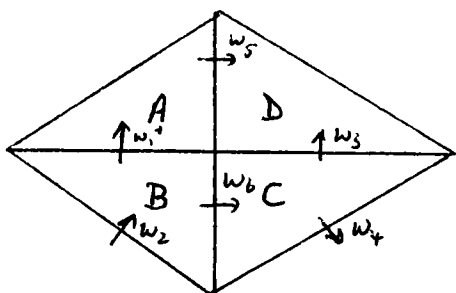
$$\underline{n_1 = 1}$$



repeated by rotation n_2 times

As above $p_A = n_2(\sum \ell_i - 6 + 4) = n_2 n_A$
giving the same result.

$$\underline{n_2 = 1}$$



repeated by rotation n_1 times

Again $p_A = n_1 n_A$ giving the required result.

The Euclidean case $n_1 = n_2 = 2$ follows as before.

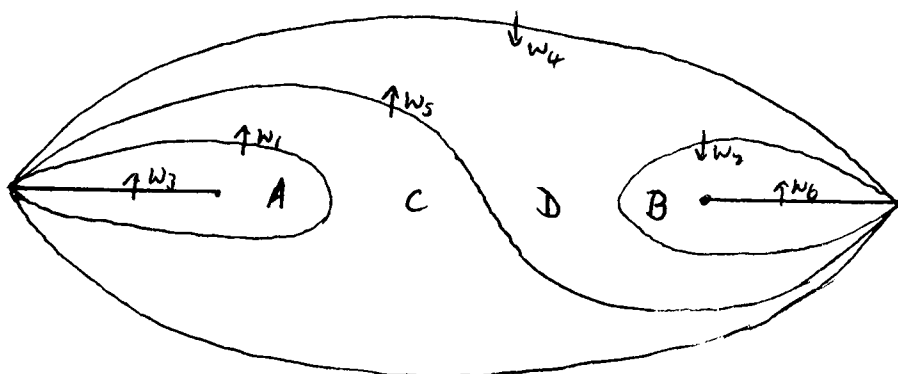
Types 23, 24 and 25

$$(A \xrightarrow{\omega_3})^{n_1}, (B \xrightarrow{\omega_6})^{n_1};$$

$$(A \xrightarrow{\omega_3} A \xrightarrow{\omega_1} C \xrightarrow{\omega_5} D \xrightarrow{\omega_4^{-1}} C \xrightarrow{\omega_1^{-1}})^{n_2},$$

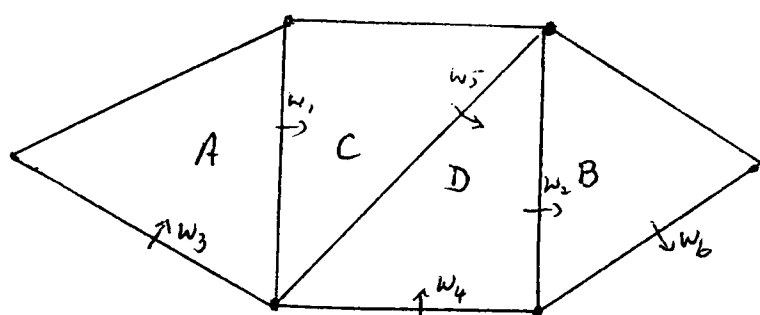
$$(B \xrightarrow{\omega_6} B \xrightarrow{\omega_2^{-1}} D \xrightarrow{\omega_4^{-1}} C \xrightarrow{\omega_5} D \xrightarrow{\omega_2})^{n_2};$$

for $n_1 = 1$ one has the diagram



repeated by rotation
 n_2 times

for $n_2 = 1$ one has the diagram



repeated by rotation
 n_1 times

One can now see that the values of p_A are the same as in the last cases.

Types 26 and 27

These two cases $p_A = \infty$ always, so that $\frac{n_A}{p_A} = 0$. So equation (I) is satisfied only when $n_1 = 1$, that is the vertex diagram is Euclidean, otherwise $\sum \frac{1}{n_i} < 1 + \sum \frac{n_A}{p_A}$

Theorem 3 The contracted vertex diagram of every point is finite or Euclidean if and only if $\sum \frac{1}{n_i} = 1 + \sum \frac{n_A}{p_A}$, and in the other cases $\sum \frac{1}{n_i} < 1 + \sum \frac{n_A}{p_A}$ where we have the convention that if $p_A = \infty$, $\frac{n_A}{p_A} = 0$.

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line 14:

For define $A \sim B$ if Ac is paired

Section(9) General vertex diagram

From the vertex surface and the integers n_i , we can construct a general vertex diagram.

Firstly cut the surface to make a disc avoiding the points of the embedded graph, so that the graph embedded in the disc is a connected tree. From this disc obtain a polygon as follows. The boundary of the disc contains points of the lines of the graph. Interchange points and lines on this boundary to obtain the polygon. This polygon has the following property. It has $2n$ sides which are paired; the identification of paired sides in opposing sense giving the original surface. This splits the vertices of the polygon into equivalence classes. For, define $A \sim B$ if AC is paired with BD where the polygon is $AC \dots BD \dots$, which can be extended to an equivalence relation by transitivity. Then each equivalence class corresponds to a face in the vertex surface, which in turn corresponds to a word r_i and a relation $r_i^{n_i} = 1$. Note that n is the number of cuts required to make the vertex graph into a connected tree. So n does not depend on the embedding.

We now show that the vertex diagram can be constructed by juxtaposition of copies of the polygon described above.

To construct the vertex diagram from the polygons one

proceeds in the same manner as Section 5. Rules must merely be given as to how neighbouring polygons are joined and how the polygons fit around around a point. To the side of a polygon already constructed one joins another copy along its paired side in the opposite sense (so the equivalent vertices on the paired sides are identified). The number of polygons about a vertex X is $n_i m$ where $r_i^{n_i} = 1$ is the relation corresponding to X and m is the number of vertices equivalent to X . If the diagram collapses, then one vertex will have fewer polygons about it than this rule gives.

We now give the polygon a geometric structure by showing the existence of a polygon with straight sides having the required properties on the sphere, Euclidean or hyperbolic planes.

The properties we need are that paired sides must have the same length and the sum of the interior angles at the vertices equivalent to X equals $2\pi/n_i$ where $r_i^{n_i} = 1$ is the relation corresponding to X .

We can assume that no angle equals 2π , for one can identify the sides leading to the vertex and obtain a polygon with two fewer sides. If there was only two sides, and the other angle was not 2π , then the diagram clearly collapses.

If there are m vertices equivalent to X , then we take the interior angle at each of these vertices to be $2\pi/mn_i$. So that if we can show there exists a polygon whose angles

are $2\pi/m_1, 2\pi/m_2, \dots, 2\pi/m_{2n}$ in that cyclic order and that the length of the side, the interior angles at the end of which are $2\pi/\ell$ and $2\pi/m$ depends only on ℓ and m (symetrically) for this polygon, then the result would follow.

Spherical case

Here we must have
$$\sum_i 2\pi/n_i > (2n-2)\pi$$

The summation is over equivalence classes of vertices, $2\pi/n_i$ being the sum of the interior angles of the i th equivalence class. $2n$ is the number of sides to the polygon.

As above let the angles of the polygon be $2\pi/m_i$ ($i = 1, \dots, 2n$). On a side of length x construct a regular m_1 -gon. Join to this a regular m_2 -gon on a common side. Continue this so that about a point P we have regular ℓ -gons, joined to each other in order $\ell = m_1, \dots, m_{2n}$. The length of the side x can be varied between zero and $2\pi/(\max m_i)$ if $\max m_i \neq 2$ (length of the whole line is 2π). If all the angles equal π , then the solution is obvious.

One can easily see that the maximum angle at P subtended by all the polygons is greater than 2π unless only two of the $m_i \neq 2$ and they are not equal. As the minimum angle at P is $2n\pi - \sum_i \frac{2\pi}{m_i} < 2\pi$ we see that with one exception there exists an x such that the interior angle at P is 2π .

The required polygon is now constructed by joining the

Page 64

line 1

about P in cyclic order.

centres of the regular m_i -gons about P is cyclic order.

The Euclidean case follows easily by the same method only any value will suffice for the length of the side on which to construct the polygons. An identical argument gives the hyperbolic case, the minimum value for the angle at P being zero (when x is infinite), the other bound following from the inequality $\sum 2\pi/n_i < (2n-2)\pi$.

So we have

Theorem 4 Given a polyhedral group and a vertex $X(=X_0)$. Then if X_i ($i=1$ to k) are the vertices equivalent to X , the neighbourhood of the point X is a polygon repeated a number of times. The polygon is formed by joining one copy of each of the intersections of a neighbourhood of the X_i with the fundamental region, so that every vertex is on the boundary of the polygon. The vertices X_i are equivalent to a point in the orbit space of the polyhedral group in its action on the boundary of the fundamental region. If r_i ($i = 1$ to n) and the words corresponding to each line which meets this point and $r_i^{n_i} = 1$ are the relations, then the vertex diagram for X collapses if and only if $n_i=1$ except for two values which are not equal, so the polygon has all angles except two equal to π or 2π . Here n is the number of cuts required to make the vertex graph for X into a connected tree.

The polygon is realisable on the sphere when $\sum \frac{1}{n_i} > (n-1)$

in which case the number of repetitions of it is

$2 / \left[\sum \frac{1}{n_i} - (n-1) \right]$. The polygon is realisable in the Euclidean plane when $\sum \frac{1}{n_i} = (n-1)$ and on the hyperbolic plane otherwise.

Condition on the vertex graph for the vertex diagram to be finite

By Theorem 4 we require $\sum \frac{1}{n_i} > (n-1)$ (1)

Let p^2 be the number of faces of the vertex surface
(connected constituent)

" p^1 " " edges " " "

" p^0 " " points " " "

Then (1) becomes $\sum_{p^2} \frac{1}{n_i} > (n-1)$

$$p^2 > (n-1) = p^1 - p^0$$

$$\text{i.e. } p^2 - p^1 + p^0 > 0$$

Since the surface is orientable $p^2 - p^1 + p^0 = 2$, that is the vertex surface is a plane. So there are $(n+1)$ terms in the summation $\sum \frac{1}{n_i}$

Note: In the Euclidean case we have $p^2 - p^1 + p^0 \geq 0$, so that the graph is either planar or on a torus. In the latter case all the $n_i = 1$. An example of this is the case where the pattern is ABABAB for the tetrahedral groups, see page 25.

Characteristic of space

Assume that a given polyhedral group has finite order g .

Let p^2 = number of faces to the fundamental region

" p^1 = " edges " " "

" p^0 = " points " " "

Then g = number of 3-cells in the space

" $\frac{p^2}{2}$ = " 2-cells " "

$\left(\sum_{p^1} \frac{1}{\ell}\right)g$ = " edges " "

where ℓ = number of regions sharing an edge

= $\left(\sum_{\neq p^1} \frac{1}{n_i}\right)g$ where $r_i^{n_i} = 1$ are the relations, and the

summation is over equivalence classes of edges.

number of points = $\left(\sum_A \frac{1}{p_A}\right)g$ p_A = number of regions meeting A

= $\left(\sum_{\neq A} \frac{n_A}{p_A}\right)g$ n_A = number of points equivalent to A

summation is over equivalent classes of points.

As, by construction, the characteristic of the space is zero,

$$1 - p^2_2 + \sum_{\neq p^1} \frac{1}{n_i} - \sum_{\neq A} \frac{n_A}{p_A} = 0 \quad (1)$$

But $\frac{n_A}{p_A}$ is equal to the inverse of the number of repetitions

of the polygon defined in Theorem 4,

$$\text{so } \frac{n_A}{p_A} = \frac{1}{2} \left(\sum_1^{n+1} \frac{1}{n_i} - (n-1) \right)$$

Now equation (1) becomes

$$1 - p_2^2 + \frac{1}{2} \sum_{\sim A} (n_{\sim A} - 2) = 0 \quad (2)$$

where $n_{\sim A}$ is the degree of A in the orbit space of the boundary of the fundamental region.

Let $\bar{p}^2$ be the number of faces in the orbit space of the boundary

" $\bar{p}^1$	"	edges	"	"	"
" $\bar{p}^0$	"	points	"	"	"

Then $p_2^2 = \bar{p}^2 =$ number of generators of the polyhedral group.

$$\text{Also } \frac{1}{2} \sum_{\sim A} (n_{\sim A} - 2) = \frac{1}{2} \left(\sum_{\sim A} n_{\sim A} - \sum_{\sim A} 2 \right) = \bar{p}^1 - \bar{p}^0$$

So (2) becomes

$$\bar{p}^2 - \bar{p}^1 + \bar{p}^0 = 1$$

That is, the quotient space of the boundary is the projective plane.

Theorem 5 For a finite polyhedral group, the quotient space of the boundary is the projective plane.

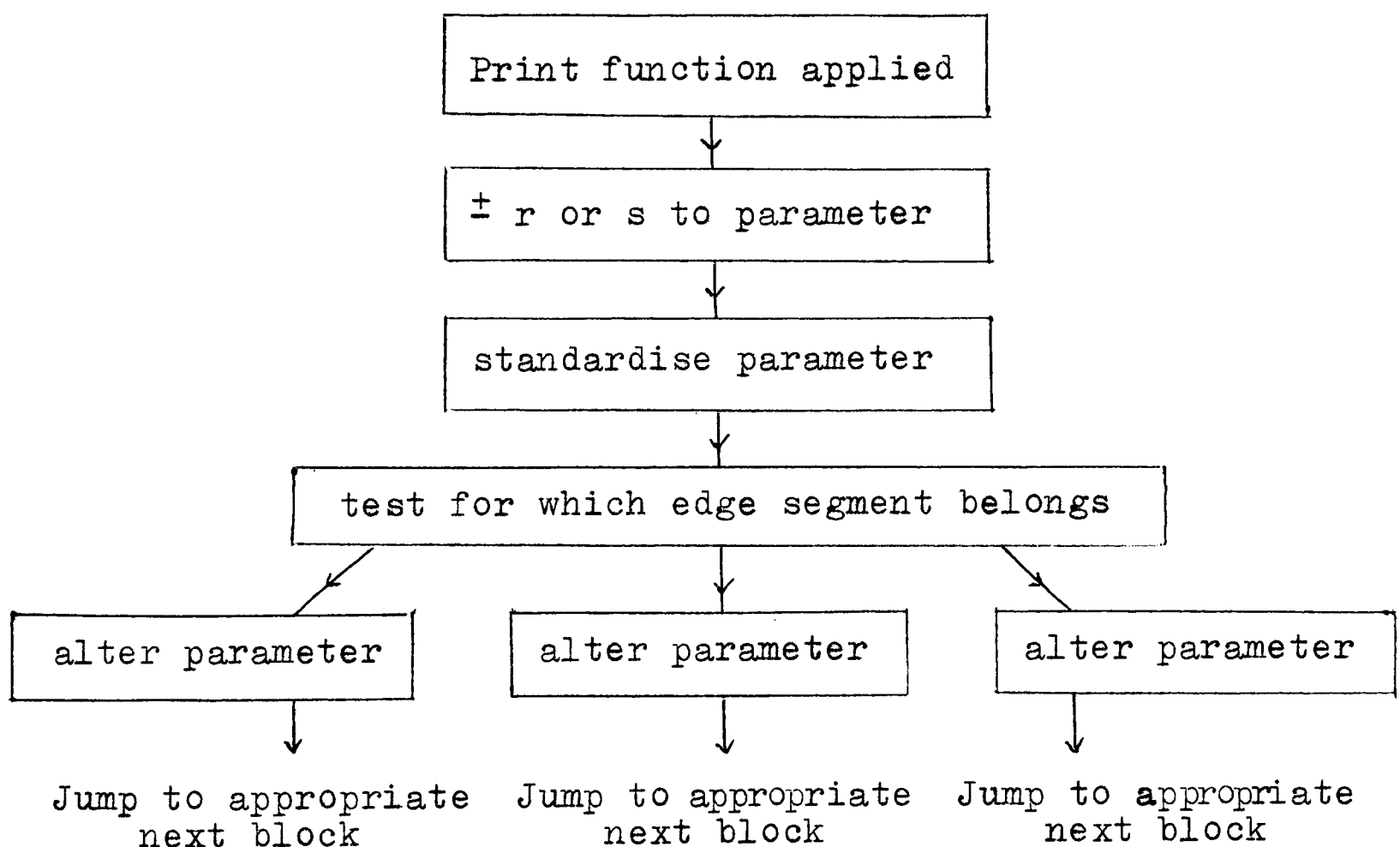
An examination of the reasoning gives:-

Theorem 6 Equation (1) holds if the neighbourhood of any point is Euclidean or spherical in the sense of Theorem 4,

and the vertex surface is a number of planes; and if as well the quotient space of the boundary of the fundamental region is the projective plane.

be standardised, i.e. if it is greater than $I+J+L$ then $I+J+L+1$ must be subtracted from it. One now tests to see along which edge the segment lies, and then alters the parameter to the one on the other side. One is now in a position to apply one of the functions R , R^{-1} or S^{-1} .

Consequently the programme consists of four blocks all of which are of the form



Since the teleprinter cannot write exponents, the functions R , R^{-1} , S and S^{-1} are represented respectively by the symbols $(,)$, $+$, $-$, A s it stands the programme will cycle indefinitely so a check must be introduced to stop the programme after one complete cycle.

Now the programme will calculate one relation corresponding to any segment. This was extended to enumerate all relations, with repetitions by calculating the relations corresponding to every segment along CD, BD and BC. The relation $S^R=1$ is the only one that could be missed by this process, so a special test is applied to see if $s = 0$ to correct this possible omission.*

In the final form, the programme cycled through small values for I, J, K and L. A test was applied to remove isomorphic cases arising through interchanging R and S, R and R^{-1} etc.

Sample output from programme (1)

0 1 0 0

0 3

+

(+(

((+

(-)+

(+)-

The first row gives:-

number of points along BC (and AD)
 number of points along AC (and BD)
 number of points along AB
 number of points along CD.

* it is possible for S^k to be the word corresponding to a segment along AB, but this could not arise for the values under consideration.

The second row gives:-

value of s, value of r
 In the notation on page 8 this is i, j, k, l,
 s, r

So the relations for these groups are:

$$S^{n_1} = (R^2 S)^{n_2} = (R, S)^{n_3} = 1$$

(2) The relations and patterns

In programme (1) is I, J, R and S are even, L and K odd, alternate points on the tetrahedron are not images of A, B, C or D under the maps R and S. In other words, every segment has a neighbouring segment which gives the same relation. One can now use programme (1) to follow this pair of segments under maps R and S (and their inverses). If at any stage the first member of the pair has X (X = A, B, C or D) as an end point one can print out X, and similarly for the second member one can print out * X. The sequence of letters X, Y, Z.. and *X*Y*Z... give the two patterns corresponding to the relation being calculated.

Sample output from programme (2)

The same tetrahedral family of groups given as an example for programme (1) would give:-

```

0      1      0      0
0      3
+ A * B
( * C + D ( C * D
( C * D ( * C + D
( D * A - * A ) C * B + * B
( A + A * D ) B - B * C

```

(3) Search for different patterns

After running programme (2) it was found that only a few of the possible patterns actually occurred for small values of the parameters. So a programme to make an extended search was written. As the speed of punching output is the main limit to computer time, this programme calculates the pattern tests to see if it is one of the known types going on to the next case if it is and printing out the relation and pattern otherwise.

As the testing of the patterns presents a few difficulties this is described in more detail.

The types which were known to occur at this stage were 1, 4, 7, 8, 10, 12, 14, 16, 18, 20, 23 and 26. The testing is considerably simplified by noticing that for these cases all the A + B's (and C + D's) occur in a single block in each cyclic word of the pattern. So one counts the changes from A or B to C or D and vice-versa with a parameter N.

To describe such patterns one uses two extra parameters A and C, whose initial value is 1. Instead of printing A as in programme (2) one increases A by 4 and with B one multiplies A by 3. Similarly with C. Tests are now performed on the final values of N, A and C to see which type was involved. In the first instance one calculates the pattern corresponding to $Z = 0$. If this does not determine the type involved one goes on to $Z = 1$, $I+J+L$, $I+J+L-1$, $I+J+L-R$, $I+J+L-R-1$, as required. The stage of the calculation is recorded by using variables B and D.

No new types were discovered this way, although most of the types were first noted when programme (2) was run for the first time.

The cubic groups

In this instance there are three essentially different cases according to the three ways in which the maps R, S and T can map the faces. The map can be between two neighbouring faces like a rotation, or opposite faces like a translation. The three cases are three translations, two rotations and one rotation and finally three rotations.

A programme has been written for each case, the twelve parameters to describe the cube are printed out first together with the values of r, s and t. The relations and letter sequences are calculated in the same manner as in programme

(2) for the tetrahedral groups. Also repetitions of relations which have already been calculated are avoided.

Output

The eight vertices are labelled A,B,C,D,E,F,G and H.
The functions $R, R^{-1}, S, S^{-1}, T, T^{-1}$ are represented by

(,), +, -, =, $\neq$.

Sample Output

CUBIC GROUPS THREE TRANSLATIONS

1 1 1 0 0 0 0 0 0 1 1 1

0 0 0

(E * F + * G) D * C - A

(F $\neq$ * E + H) * D = C - * B

(* G $\neq$ E * H) A = E * C

+ D * H = * G - B * F $\neq$ A

So the relations of this group are

$$(R,S)^{n_1} = (RT^{-1}SR^{-1}TS^{-1})^{n_2} = (R,T)^{n_3} = (S,T)^{n_4} = 1$$

CUBIC GROUPS THREE ROTATIONS

1 1 1 0 0 0 0 0 0 0 0 0

3 3 3

(F * E - H * D = * G = B * C (A

(E * A + H + F * G $\neq$ C * D (* B

(B * F $\neq$ * C $\neq$ H * G - E - D * A

So the relations are

$$(R^2 S^2 T^{-1})^{n_1} = (S^2 T^2 R^{-1})^{n_2} = (T^2 R^2 S^{-1})^{n_3} = 1$$

CUBIC GROUPS TWO ROTATIONS AND ONE TRANSLATION

0 0 0 0 0 0 0 0 0 0 0 0

2 2 1

(F * B - C * G - H * D (A * E

(B * A = E * H

(E * F ≠ B * C - G * H ≠ D * A

+ G * F ≠ D * C

So the relations are

$$(R^2 S^{-2})^{n_1} = (RT)^{n_2} = (RT^{-1} S^{-1} T^{-1})^{n_3} = (ST^{-1})^{n_4} = 1.$$

R e f e r e n c e s

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The groups determined by the relations $S^l = T^m = (S^{-1}T^{-1}ST)^P = 1$
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379 1940.

C o n t e n t s

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