Hydrodynamic Limits Under Pressure



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To my family

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Abstract

We study the hydrodynamic limit for a one dimensional isothermal anharmonic finite chain in Lagrangian coordinates with hyperbolic space-time scaling. The temperature is kept constant by putting the chain in contact with a heath bath, realised via the addition of a stochastic momentum-preserving noise to the dynamics of the chain. The noise is designed to be large at the microscopic level, but vanishing in the hydrodynamic limit. Boundary conditions are also considered: one end of the chain is kept fixed, while a time-variable tension is applied to the other end. We show that the microscopic deformation and momentum converge (in an appropriate sense) to solutions of a system of hyperbolic conservation laws (isothermal Euler equations in Lagrangian coordinates) with boundary conditions. Since these solutions may develop shocks in a finite time, they are obtained in a weak sense. This is done by adapting the theory of compensated compactness to a stochastic setting. Finally, the external tension allows us to define thermodynamic transformations between equilibrium states. We use this to deduce the first and the second principle of Thermodynamics for our model.

Chapter 1 gives an overview of the problem. The context in which the problem is studied is explained, as well as how similar problems are typically dealt with. In Chapter 2 we give all the definitions we need in order to state our main theorem. At the end of the Chapter, a part of the proof of the main theorem is presented and it is explained what is the technical difficulty to overcome in order to complete the proof. In Chapter 3 we prove some important estimates concerning the relative entropy and the Dirichlet forms, the most important being a Logarithmic Sobolev inequality and a bound on the time integral of the Dirichlet forms. Chapter 4 is a technical chapter aimed to obtain one and two block estimates with explicit bounds: this is possible thanks to the estimates obtained on the

relative entropy and the Dirichlet forms. In Chapter 5 we discuss the stochastic version of the theory of compensated compactness. We define random Young measures, we establish sufficient conditions for a sequence of random Young measure to converge in law and we give conditions for the support of the limiting Young measure to be almost surely a point. This is done using div-curl lemma, the stochastic Murat lemma and a use of Lax entropy-entropy flux pairs for systems of conservation laws in one spatial dimension. Chapter 6 is devoted to verifying the assumptions of the stochastic Murat's lemma for our problem via the one and two block estimates. In Chapter 7 we complete the proof of the main theorem. Moreover, we discuss some thermodynamics consequences that the macroscopic system inherits from the microscopic dynamics. In particular, we define the relevant thermodynamic quantities and derive the first and the second law of Thermodynamics. In the final Chapter we summarise our result and we point out what possible directions can be pursued in the future.

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Chapter 1

Introduction

Physics is the science which studies natural phenomena (such as gravitation, electromagnetic or nuclear interactions, thermodynamics transformations, shock waves, etc...) at the most fundamental level.

Mathematics then comes in, because underneath each physical phenomena lies a mathematical model, and we typically all agree that that model "describes" the phenomenon under consideration. For example, if we look at the solar system as a whole, we may be happy with a model in which the sun is a sphere while all the planets (perhaps apart from Jupiter) are points.

This description however becomes less and less accurate the more we "zoom in": after zooming in for a while, we cannot just pretend that the Earth is a point!

Therefore, we can ask some very natural questions: at which *scale* is a model valid? How does the same system look like if we watch it at different scales?

It goes without saying that we can ask these questions for an uncountable number of models in many different areas of Physics and other Sciences. Even though many considerations we can make are applicable in other contexts, we shall focus on a specific and very interesting area of Physics, which is Hydrodynamics.

1.1 Fluids: Eulerian vs Lagrangian formulation

Hydrodynamics concerns the study of fluids, that is of gas and liquids. Suppose we live in $\mathbb{R}^d$, for some $d \geq 1$. A fluid is fully described by its *density* $\rho \in \mathbb{R}$, *velocity* $u \in \mathbb{R}^d$ and *internal energy density* $e \in \mathbb{R}$. Since the fluid may change in time (it may move, it may interact with some external force, etc), ρ, u and e will be functions of the time t. Moreover, we may have different densities, velocities and energies at different points of the fluid (for instance different points may have different temperatures, and so different densities), and so ρ, u and e will have a spatial dependence, too.

Accordingly to the point of view we take, we have to different pictures:

- **Lagrangian picture:** Suppose we identify each point on the fluid with a point in a subset $U \subseteq \mathbb{R}^d$. Then, for any $x \in U$, the density of the fluid at the point x at time t will be denoted by $\rho_L(t,x)$. Similarly, we define $u_L(t,x)$ and $e_L(t,x)$ (the subscript L stands for "Lagrangian" and it will be dropped shortly).

- **Eulerian picture:** Now take any $q \in \mathbb{R}^d$ and $t \in \mathbb{R}$. For $x \in U$ denote by $X(t,x)$ the position of the point on the fluid labelled by x at time t. Assume further that $X(t,\cdot)$ is injective. If there exists x such that $X(t,x) = q$, we set $\rho_E(t,q) = \rho_L(t,x)$. Otherwise, we set $\rho_E(t,q) = 0$. Similarly we can define $u_E(t,q)$ and $e_E(t,q)$.

The two descriptions are related by the following equation

$$u_E\left(t, X(t,x)\right) = \frac{\partial X}{\partial t}(t,x), \tag{1.1}$$

which tells us nothing more than the velocity must be identified with the time derivative of the position. We drop now the subscripts L and E: $\rho(t,x)$ stands for $\rho_L(t,x)$, while $\rho(t,q)$ stands for $\rho_E(t,q)$.

Now that we have these two descriptions for our fluid, a physical model is complete if we specify a set of equations that ρ, u and e solve. These equations of course depend on the properties of the fluid. In the case of a *inviscid* (i.e. ideal) fluid, one relevant example is given by the *(compressible) Euler equations*, which, in one dimension ($d = 1$) and in Eulerian coordinates read

$$\begin{cases} \partial_t \rho + \partial_q(\rho u) = 0 \\ \partial_t(\rho u) + \partial_q(\rho u^2 + P(\rho, e)) = 0 \\ \partial_t\left(\frac{1}{2}\rho u^2 + e\right) + \partial_q\left(\frac{1}{2}\rho u^3 + ue + uP(\rho,e)\right) = 0 \end{cases}, \tag{1.2}$$

where P is called the *pressure* and it is a known function of ρ and e which will, of course, depend on the fluid under consideration.

The three equations in (1.2) represent (from top to bottom) the conservation of mass, momentum and total energy.

If re-written in Lagrangian coordinates, the same system will become the following

$$\begin{cases} \partial_t v - \partial_x u = 0 \\ \partial_t u - \partial_x \tau(v,e) = 0 \\ \partial_t\left(\frac{1}{2}\rho u^2 + e\right) - \partial_x\left(u\tau(v,e)\right) = 0 \end{cases}, \tag{1.3}$$

where $v = 1/\rho$ is the specific volume and τ is related to the pressure by $\tau(v, e) = -P(1/v, e)$.

It is possible to show (see for example [54]) that the systems (1.2) and (1.3) are equivalent and so we can freely use one or the other.

Nevertheless, the system (1.3) in Lagrangian coordinates looks much simpler than its counterpart (1.2): for instance the conservation of momentum does not have a quadratic term in the velocities, and the conservation of energy does not have a cubic term.

Also, in Eulerian coordinates, there may be points where the density vanishes, that is we may see the *vacuum*. For these reasons it is typically much easier (and sometimes almost necessary) to work in the Lagrangian framework, and we shall do so for the rest of this thesis.

At this stage, one may ask why one believes Euler equations describe the space-time evolution of an inviscid fluid. Or, more appropriately, one may ask which models of which fluids lead to the Euler equations, and in which sense this happens. This is a fairly simple question to ask, but, as we shall see in the rest of this thesis, it is not obvious at all to address.

1.2 Hamiltonian systems: formally deriving the Euler equations

From now on we will focus on the Lagrangian picture, that is we have in mind the derivation of (1.3). The first thing we need to do is decide which *microscopic* system we expect will give us the Euler equations in Lagrangian coordinates. We choose to start from a Hamiltonian system, which is constructed as follows.

Let's take a system of $N+1$ particles in one dimension and let's label the particles by $i \in \{0, \dots, N\}$. Assume that all the particles have mass equal to 1. At time t, the particle labelled by i will be uniquely identified by its position $q_i(t)$ and its momentum (i.e. velocity, since the mass is one) $p_i(t)$. Then we ask that q_i and p_i evolve in time accordingly to the following system of ODEs

$$q_i'(t) = \frac{\partial \mathcal{H}_N}{\partial p_i}, \quad p_i'(t) = -\frac{\partial \mathcal{H}_N}{\partial q_i}, \tag{1.4}$$

where $\mathcal{H}_N \in C^1(\mathbb{R}^{2N})$ is a known function called the *Hamiltonian*. Many choices for the Hamiltonian are possible, but we shall take it of the form

$$\mathcal{H}_N(q_0, \dots, q_N, p_0, \dots, p_N) = \sum_{i=0}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N} V(q_i - q_{i-1}), \tag{1.5}$$

where $V \in C^1(\mathbb{R})$ is called the *potential*. V represents the interaction between particles and its gradient can be interpreted as a force. The Hamiltonian is designed in such a way that each particle interacts only with its nearest neighbour. This of course is one of the simplest situations we can have, and it's far from being general. It is therefore natural to work with a different set of variables. Namely, we define

$$r_i = q_i - q_{i-1}, \quad i \in \{1, \dots, N\}, \tag{1.6}$$

So that the Hamiltonian reads

$$\mathcal{H}_N(r_1, \dots, r_N, p_0, \dots, p_N) = \sum_{i=0}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N} V(r_i), \tag{1.7}$$

and the corresponding system of ODEs becomes

$$r_i'(t) = \frac{\partial \mathcal{H}_N}{\partial p_i} - \frac{\partial \mathcal{H}_N}{\partial p_{i-1}}, \quad p_i'(t) = -\frac{\partial \mathcal{H}_N}{\partial r_i} + \frac{\partial \mathcal{H}_N}{\partial r_{i+1}}, \tag{1.8}$$

or, equivalently,

$$r_i' = p_i - p_{i-1}, \quad p_i' = V'(r_{i+1}) - V'(r_i). \tag{1.9}$$

These two equations look like a discrete version of the first two equations in (1.3), once we "replace" finite differences with spatial derivatives. Moreover, if we define the energy of the i^{th} particle by

$$e_i(t) = \frac{p_i(t)^2}{2} + V(r_i(t)), \tag{1.10}$$

we obtain

$$e_i' = p_i V'(r_{i+1}) - V'(r_i) p_{i-1}, \tag{1.11}$$

which in turn looks like a discrete version of the third equation in (1.3). At this stage, therefore, one may be tempted to guess that the our Hamiltonian system converges to the system (1.3) where the role of τ is played by V'.

We can however argue the following: we have found three quantities (r_i, p_i and e_i) which formally lead to the Euler equations. But what if we missed some other quantities which lead to some other equations? In other words, how can we guarantee that (at least when the number of particles become infinite) the *only* conserved quantities are mass, momentum and energy?

1.3 Ergodicity and noisy dynamics

Informally speaking, we say that a system is *ergodic* if the only conserved quantities that *survive*, as $N \to +\infty$ are mass, momentum and energy. This of course is no definition, since, for instance, the energy may become infinity as $N \to +\infty$. One precise statement is the following (we refer to Chapter 2 for the definitions, [3] for a detailed discussion and [34] for further reference): a (infinite) system is ergodic if the only stationary translation invariant measures that are locally absolutely continuous with respect to the Lebesgue measure are given by Gibbs measures associated to the Hamiltonian of the system.

It is extremely easy to find examples of *non-ergodic* systems. For example, take the infinite harmonic chain ($V(r) = r^2/2$). This system has infinitely many conserved quantities: the energy of each particle is conserved.

One natural question at this point is the following: can we *design* interactions V for which the resulting infinite system is ergodic? To this day, the answer to this question is unknown, and the question of ergodicity is probably one of the hardest and more fascinating problems in Statistical Physics.

Since we do not know how to design ergodic deterministic interactions, we obtain ergodicity by another mean: we add a stochastic noise to the dynamics. If the noise is added properly, then we do have ergodicity and we are guaranteed that the only conservation laws we must expect macroscopically are conservations of mass, momentum and energy.

But we can do something more: we can choose the noise so it violates one of the three mentioned conservation laws. We may add a noise which does not conserve the energy or that maybe does not conserve the momentum. In this way, we may choose which conservation laws we expect to appear in the limit.

In our particular problem we introduce a noise which conserves mass and momentum but not the energy (see Chapter 2 for the details). Therefore, we expect to obtain a 2×2 Euler system of the form

$$\begin{cases} \partial_t v - \partial_x u = 0 \\ \partial_t u - \partial_x \tau(v) = 0 \end{cases} . \tag{1.12}$$

The introduction of a stochastic noise effectively changes our setting quite a bit. Now $r_i(t)$ and $p_i(t)$ are not deterministic solutions of a system of ODEs, but are rather stochastic processes which solve a system of SDEs (we refer once more to Chapter 2 for the details). Therefore, everything from now on will have to be taken in a stochastic setting.

1.4 Hydrodynamic limit: microscopic to macroscopic

So far we have not discussed at all what it means to obtain Euler equations as a limit of a microscopic dynamical system as $N \to +\infty$. This is the aim of this section.

We shall focus on the 2×2 system (1.12) only. The discussion for the 3×3 case will be analogous.

Suppose we have constructed an ergodic dynamics by adding noise to a Hamiltonian dynamics and suppose r_i and p_i are the corresponding solutions for the appropriate system of SDEs. r_i and p_i are stochastic processes on a probability space $(\Omega, \mathbb{P})$. We say that the hydrodynamic limit for our system is given by solutions of the Euler equations (1.12) provided

$$\begin{aligned} \lim_{N\to+\infty} \mathbb{P}\left\{ \left| \frac{1}{N}\sum_{i=1}^{N} \varphi\left(\frac{i}{N}\right) r_i(Nt) - \int_0^1 \varphi(x) r(t,x)dx \right| > \delta \right\} = 0 \\ \lim_{N\to+\infty} \mathbb{P}\left\{ \left| \frac{1}{N}\sum_{i=1}^{N} \varphi\left(\frac{i}{N}\right) p_i(Nt) - \int_0^1 \varphi(x) p(t,x)dx \right| > \delta \right\} = 0 \end{aligned} \tag{1.13}$$

for all $\delta > 0$ and all continuous $\varphi : [0,1] \to \mathbb{R}$, where $r, p : [0,\infty) \times [0,1] \to \mathbb{R}$ solve

$$\begin{cases} \partial_t r - \partial_x p = 0 \\ \partial_t p - \partial_x \tau(r) = 0 \end{cases} \tag{1.14}$$

for some suitable function τ which will depend on the interaction V.

Note that we have scaled space and time with the same factor N. This is a *hyperbolic* space-time scaling, and is not the only possible scaling we may perform. In fact, a different scaling would in general lead to a different set of macroscopic equations. The hyperbolic scaling is the one we need to do if we want to derive the hyperbolic system (1.14).

There is a major point to clarify, before moving on. We must specify what we mean for "solutions" of the system (1.14). In fact, hyperbolic systems develop shocks (i.e. discontinuities) in a finite time even if τ is smooth and given smooth initial data. Therefore, we may not hope to have differentiable (or even continuous) r and p satisfying (1.14) at all times in a classical sense.

In order to overcome this difficulty, two approaches are possible.

- Assume the first shock appears at time $t_* > 0$: $r(t,x)$ and $p(t,x)$ are *classical* (i.e. C^1) solutions of (1.14) for all $t \in [0, t_*)$.

- Take into account the formation of shocks: $r(t,x)$ and $p(t,x)$ are *weak* (i.e. possibly not even continuous) solutions of (1.14).

Within this thesis we shall take the second approach. The first will be discussed in the next section.

1.5 Smooth regime: the relative entropy method

Hydrodynamic limits in the smooth regime have been well studied and understood in the last 30 years. The first results go back to [41, 42, 52], while more recent results can be found for instance in [17, 32].

We describe how to obtain (1.13) in case of smooth solutions. First of all we assume that, for all $t > 0$ and $N \in \mathbb{N}$ there exists a probability measure μ_t^N on $\mathbb{R}^{2N}$ such that (1.13) is equivalent to the following (see Chapter 2 for further details):

$$\begin{array}{l} \lim_{N\to+\infty} \mu_t^N \left\{ \left| \frac{1}{N} \sum_{i=1}^N \varphi\left(\frac{i}{N}\right) r_i - \int_0^1 \varphi(x) r(t,x) dx \right| > \delta \right\} = 0 \\ \lim_{N\to+\infty} \mu_t^N \left\{ \left| \frac{1}{N} \sum_{i=1}^N \varphi\left(\frac{i}{N}\right) p_i - \int_0^1 \varphi(x) p(t,x) dx \right| > \delta \right\} = 0 \end{array} . \tag{1.15}$$

One then proves (1.15) via the following steps (recall that t_* is the time at which the first shock occurs):

i) We find $r(t,x)$ and $p(t,x)$ which solve (1.14) for all $x \in [0,1]$ and $t \in [0, t_*)$.

ii) Starting from the solutions $r(t,x)$ and $p(t,x)$ we construct, for $t \in [0,t_*)$ some measures ν_t^N such that

$$\begin{array}{l} \nu_t^N \left\{ \left| \frac{1}{N} \sum_{i=1}^N \varphi\left(\frac{i}{N}\right) r_i - \int_0^1 \varphi(x) r(t,x) dx \right| > \delta \right\} \leq e^{-C(\delta)N} \\ \nu_t^N \left\{ \left| \frac{1}{N} \sum_{i=1}^N \varphi\left(\frac{i}{N}\right) p_i - \int_0^1 \varphi(x) p(t,x) dx \right| > \delta \right\} \leq e^{-C(\delta)N} \end{array} \tag{1.16}$$

follows from the law of large numbers for ν_t^N. Here $C(\delta)$ is a positive constant independent of N.

iii) We denote by f_t^N and g_t^N the densities with respect to the Lebesgue measure of μ_t^N and ν_t^N, respectively. Then, we show that the *relative entropy*

$$H_N(t) = \int_{\mathbb{R}^{2N}} \log \frac{f_t^N}{g_t^N} d\mu_t^N \tag{1.17}$$

is such that

$$\lim_{N\to+\infty} \frac{H_N(t)}{N} = 0 \tag{1.18}$$

for all $t \in [0, t_*)$.

It is a standard result that (1.18) and (1.16) together imply (1.15).

Note that the first step in proving (1.15) requires us to *solve* (1.14) before proving the hydrodynamic limit. This will be one of the major differences between the relative entropy method and the approach we will use in the non-smooth regime.

The crucial conceptual step in the relative entropy method is to identify ν_t^N with some *local equilibrium* measure. Intuitively, even if the macroscopic system is not at equilibrium at time t and point x, it will look like it is, if we look at it from a microscopic point of view: in other words if we look near the particle i at time t, the equilibrium measure will be locally well approximated by a Gibbs measures with parameters related to $r(Nt, i/N)$ and $p(Nt, i/N)$ (see [17] for a more detailed discussion).

So far we have not specified the way in which τ is related to the potential V. The reason is that defining τ requires giving a few extra definitions, which we chose not to give here and can be found in Chapter 2. Several properties of τ will also be explored in Appendix A.

1.6 Shock regime: constructing weak solutions

While hydrodynamic limits in smooth regime have been well understood, the situation is rather different in the shock regime. Nevertheless, some techniques to attack the problem have been developed in [20–23]. In particular, this thesis extends the work in [22] to a finite chain with boundary conditions.

When a shock occurs, the functions $r(t, x)$ and $p(t, x)$, which are C^1 up to time t_*, become discontinuous. Therefore, they cannot solve (1.14) in a classical sense. However, it does make sense for them to solve the following system:

$$\begin{aligned} \int_0^\infty \int_0^1 (r(t,x)\partial_t\varphi(t,x) - p(t,x)\partial_x\varphi(t,x))\, dxdt = 0 \\ \int_0^\infty \int_0^1 (p(t,x)\partial_t\psi(t,x) - \tau(r(t,x))\partial_x\psi(t,x))\, dxdt = 0 \end{aligned} \tag{1.19}$$

where φ and ψ are compactly supported test functions. We say that r and p are *weak solutions* of the system (1.14) provided they solve (1.19) for all compactly supported test functions φ, ψ.

By performing an integration by parts it is immediate to see that classical solutions are also weak solutions. Of course the other way around is not necessarily true, as weak solutions do not need to be even continuous.

So far we have not mentioned initial data. The reason behind this is that initial data are typically added for uniqueness purposes, and uniqueness is an open and delicate question for weak solutions to hyperbolic systems of conservation laws such as (1.14). For a single conservation law, a uniqueness criterion can be given by the notion of *entropy solutions*, but this criterion fails in general for systems. For this reason, we shall never talk about uniqueness, and initial data will play a relevant role only when we will define thermodynamic transformations in Chapter 7.

The relative entropy method described in the previous section relied on constructing some local Gibbs measures ν_t^N starting from smooth solutions of (1.14). These measures required us to evaluate r and p at some specific points. Here we meet a first obstacle when dealing with weak solutions. Weak solutions do not need to be defined point-wise. In fact, changing a weak solution on a set of zero measure will still produce the same solution.

Moreover, the relative entropy method required us to *solve* the system (1.14) before proving the hydrodynamic limit. But since we do not have a uniqueness criterion for hyperbolic systems of conservation laws, we would not know *which* weak solution to select in order to prove the hydrodynamic limit.

We overcome both these difficulties at once by following the approach described in [20–23]:

i) For fixed N, we start from r_i and p_i to define some L^2-valued stochastic processes $r_N(t,x)$ and $p_N(t,x)$.

ii) We show that r_N and p_N solve the following approximate version of (1.19):

$$\begin{array}{l} \displaystyle\int_0^\infty \int_0^1 (r_N(t,x)\partial_t\varphi(t,x) - p_N(t,x)\partial_x\varphi(t,x))\,dxdt + R_N = 0 \\ \displaystyle\int_0^\infty \int_0^1 (p_N(t,x)\partial_t\psi(t,x) - \tau(r_N(t,x))\partial_x\psi(t,x))\,dxdt + Q_N = 0 \end{array}, \qquad (1.20)$$

where R_N and Q_N are error terms which go to zero in probability.

iii) We show that r_N and p_N converge weakly (in some sense to be specified) to some processes r, p. This is enough to obtain the first equation in (1.19) as the limit of the first equation in (1.20).

iv) We show that $\tau(r_N)$ converges weakly to $\tau(r)$, which leads to the second equation in (1.19).

The first two steps are explained in detail in Chapter 2, where everything is formulated with the addition of the boundary conditions.

Concerning the third step, r_N and p_N are stochastic processes, so it is not entirely clear what it means for them to converge weakly. This is a purely technical matter and it is extensively explained in Chapters 5 and 7.

The last step is the hardest to tackle, as weak convergence is in general not preserved under weak limits: a brief discussion about this will be given in next section, a detailed one is presented in Chapter 5.

1.7 Weak convergence: Young measures and compensated compactness

As we are dealing with stochastic processes, we need everything to be formulated in a stochastic setting. However, for the sake of clarity, here we will explain the problem in a deterministic setting. The stochastic extension is then a purely technical matter, which is discussed in Chapter 5.

For $U \subseteq \mathbb{R}^d$ and $1 \leq p < \infty$ we say that a sequence $(f_n) \subset L^p(U)$ converges weakly to $f \in L^p(U)$ (in symbols $f_n \rightharpoonup f$) provided

$$\lim_{n\to+\infty} \int_U \varphi f_n = \int_U \varphi f, \tag{1.21}$$

for all $\varphi \in L^q(U)$ where $1/p + 1/q = 1$ ($q = \infty$ if $p = 1$).

Moreover, we say that $(f_n) \subset L^\infty(U)$ converges weakly-$*$ to $f \in L^\infty(U)$ (in symbols $f_n \overset{*}{\rightharpoonup} f$) provided

$$\lim_{n\to+\infty} \int_U \varphi f_n = \int_U \varphi f, \tag{1.22}$$

for all $\varphi \in L^1(U)$.

Weak convergence does not commute with composition with continuous functions. Namely, if g is a continuous function and $f_n \rightharpoonup f$ it is not always true that $g(f_n) \rightharpoonup g(f)$. The same holds true if we replace weak convergence by weak-$*$ convergence. In fact, the typical example one gives is the following.

Let $U = (0, 1)$ and $f_n(x) = \sin(\pi n x)$. Clearly $f_n \in L^\infty(U)$ and, by the Riemann-Lebesgue lemma,

$$\lim_{n\to+\infty} \int_U \varphi(x) \sin(\pi n x) dx = 0, \tag{1.23}$$

for all $\varphi \in L^1(U)$, so that $f_n \overset{*}{\rightharpoonup} 0$. However, if $g(x) = x^2$ (which of course is a continuous function), we can take $\varphi = 1$ to obtain

$$\int_U \sin^2(\pi n x) dx = \frac{1}{2}, \tag{1.24}$$

so that $g(f_n)$ does not converge weakly-$*$ to $g(0) = 0$.

Of course no problem would arise if the function g is linear (or affine). Therefore the problem of the weak convergence of $g(f_n)$ is really due to the *nonlinearity* of g. In the case of (1.20), the function τ *will* be nonlinear, and so we must investigate under which conditions $\tau(r_N) \rightharpoonup \tau(r)$, provided $r_N \rightharpoonup r$.

The idea is to use Young measures (see [1]) and compensated compactness (see [35, 36, 49]), and it goes as follows. Suppose $(f_n) \subset L^p(U)$ converges weakly to f and let g be a continuous and bounded function. Under quite general assumptions, it is possible to find a probability measure ν_x on $\mathbb{R}$ for almost all $x \in U$ such that

$$\lim_{n \to +\infty} \int_U \varphi(x) g(f_n(x)) dx = \int_U \int_{\mathbb{R}} \varphi(x) g(\xi) d\nu_x(\xi) dx \tag{1.25}$$

for all $\varphi \in L^q(U)$, with $1/p + 1/q = 1$.

In our case, however, the nonlinear function τ will not be bounded. In order to overcome this, we can prove that if the sequence (f_n) is uniformly bounded in L^p, the limit (1.25) can be extended to functions $g \in L^r$ for all $r < p$, so that g does not need to be bounded (we refer to Chapter 5 for the details).

By looking at (1.25) we see that in order for $g(f_n)$ to converge weakly to $g(f)$ it is enough to show that the measure ν is a Dirac mass. In symbols, we want

$$\nu_x = \delta_{f(x)} \tag{1.26}$$

for almost all $x \in U$. Therefore, $g(f_n)$ converges weakly to $g(f)$ provided the measure ν_x is the Dirac mass $\delta_{f(x)}$ for almost all $x \in U$.

The ultimate goal is therefore to find conditions on ν_x so it reduces to a Dirac mass or, equivalently, so that the support of ν_x is a point for almost all x. The way this is done is explained in Chapter 5 and the main tools to attack the problem will be Lax entropy-entropy flux pairs, Tartar's commutation relations, the div-curl and the Murat's lemma.

1.8 Thermodynamics and entropy production

In Chapter 2 we shall introduce a microscopic dynamics on a anharmonic chain. The noise, added to provide ergodicity, is such that the temperature of the chain is kept

fixed and equal to β^{-1} at all times. Moreover, we introduce boundary conditions in the following way. One end of the chain (particle number 0) is kept fixed, while at the other end (particle number N) is applied a time-varying force $\bar{\tau}(t)$. $\bar{\tau}$ is designed so that it goes smoothly from τ_0 at time 0 to τ_1 as $t \to +\infty$. This corresponds to performing a thermodynamic transformation from the *equilibrium state* (β, τ_0) to the equilibrium state (β, τ_1).

From the microscopic dynamics, we can build two stochastic processes r_N and p_N and we can follow the procedure previously depicted to argue that, r_N and p_N converge to some weak solutions r, p, of the system (1.14) with appropriate boundary conditions (we refer to Chapter 2 for a precise statement).

The solutions r, p we deduce, however, are obtained from a *physical* microscopic model. Therefore we would expect them to carry some extra physical information. Indeed, in Chapter 7 we show that, with some extra assumptions, this is the case.

Starting from microscopic considerations, we define the *internal energy* U, we compute its difference between the two equilibrium states (β, τ_0) and (β, τ_1) and we show that it can be written as a sum $W + Q$, where W depends on the external tension $\bar{\tau}$, and we call it *work*, while Q depends on the noise, and we call it *heat*. In other words, we obtain

$$U(\beta, \tau_1) - U(\beta, \tau_0) = W + Q, \tag{1.27}$$

which is the first principle of Thermodynamics.

Then we define the *free energy* F and we prove that its difference between the equilibrium states (β, τ_0) and (β, τ_1) is never bigger than the work W. In formulae,

$$F(\beta, \tau^{-1}(\tau_1)) - F(\beta, \tau^{-1}(\tau_0)) \leq W, \tag{1.28}$$

where the equality occurs only for quasi-static (i.e. reversible, i.e. ideal) transformations.

This is known as *inequality of Clausius* and is a statement equivalent to the second principle of Thermodynamics that, for a isothermal transformation, reads

$$S(\beta, \tau_1) - S(\beta, \tau_0) \geq \beta Q, \tag{1.29}$$

where S is the *thermodynamic entropy*, defined as $S = \beta(U - F)$.

The first principle of Thermodynamics (1.27) is nothing but an energy balance: by doing some work on the system, part of the work goes into changing the internal energy of the system, and part will be converted into heat.

The second principle of Thermodynamics (1.29) tells us something different: if we do some work W on a system, the gain in free energy will never exceed the work done. In fact, it will be strictly smaller than the work done unless the transformation is reversible (which can never happen in any real-life situation). The second principle is therefore telling us something about dissipation: we may never "use" all the work we put in a system, but we always end up "losing" something. In other words, unless we do a reversible transformation, the free energy F is *not* a conserved quantity.

Thermodynamic consequences similar to the ones we consider have been studied in [39], where the fist and the second principle have been derived in a isothermal chain with *diffusive scaling* and with non-momentum-preserving noise. In that case, the resulting macroscopic deformation turns out to solve a *parabolic* equation (non-linear heat equation). Thanks to the parabolic regularity, weak solutions are classical solutions the inequality of Clausius can easily be seen at the *macroscopic* scale. This reflect the dissipative nature of parabolic equations.

We, on the other hand, derive a *hyperbolic* system and hyperbolic systems are *conservative* in the smooth regime. Moreover, we would have a neat *equality* of Clausius if we worked in the smooth regime. In order to deduce the inequality of Clausius and "see" the dissipation we must work in the shock regime and look at some intermediate scale between the microscopic and the microscopic. Only then we are able to deduce the desired inequality and realise that the noise is responsible for it.

Besides having a physical meaning, the inequality of Clausius is also relevant from a PDE point of view, as it can be read as an *entropy inequality* (from the Lax entropy-entropy-flux point of view) if we take the Lax entropy to be

$$\eta_*(r,p) = \frac{p^2}{2} + \int_0^r \tau(y)dy \tag{1.30}$$

and suitably include the boundary conditions (responsible for the work W). Therefore, besides proving the second principle of thermodynamics, we prove as a bonus that the solution we obtain is not only a weak solution, but is an *entropy solution* with respect to the convex entropy η_*.

Chapter 2

Microscopic dynamics and main result

2.1 The setting

We study a one-dimensional Hamiltonian system of $N+1 \in \mathbb{N}$ particles of unitary mass on the real line. The position of the i^{th} particle ($i \in \{0, 1, \dots, N\}$) at time $t \geq 0$ is denoted by $q_i(t) \in \mathbb{R}$ and its momentum by $p_i(t) \in \mathbb{R}$.

For $1 \leq i \leq N$, the particle i interacts with the particle $i-1$ via the potential $V(q_i - q_{i-1})$, where $V \in C^2(\mathbb{R})$ is a non-negative strictly convex function such that

$$c_1 \leq V''(x) \leq c_2, \quad \forall x \in \mathbb{R}, \tag{2.1}$$

for some $c_1, c_2 > 0$. Moreover, we require that V becomes quadratic exponentially fast at infinity. To be precise, there are some positive constants V''_+, V''_-, α and R such that

$$\begin{aligned} \left|V''(r) - V''_+\right| &\leq e^{-\alpha r}, \quad r > R \\ \left|V''(r) - V''_-\right| &\leq e^{\alpha r}, \quad r < -R. \end{aligned} \tag{2.2}$$

Remark. We will shortly define a function τ depending on V and, in Chapter 5, we will need it to satisfy several conditions. All of them apart from one ($\tau'' \neq 0$) follow from (2.1) and (2.2). To fulfil the condition $\tau'' \neq 0$ we shall in addition require V to take an explicit form. All the details are discussed in Appendix A.

We impose boundary conditions at the ends of the chain: the 0^{th} particle is kept fixed at the origin, that is $q_0(t) = p_0(t) = 0$ for all t, while the N^{th} particle is pulled by a time-dependent external force $\bar{\tau}(t)$. $\bar{\tau}$ is taken to be a bounded smooth function with bounded derivatives. Finally, we fix the temperature of the chain to be $\beta^{-1} > 0$.

2.2 Deterministic dynamics

The system described to far is a Hamiltonian system governed by the Hamiltonian

$$\mathcal{H}_N(q_0,\dots,q_N,p_0,\dots p_N) := \sum_{i=0}^{N}\left(\frac{p_i^2}{2}+V(q_i-q_{i-1})\right), \tag{2.3}$$

For $i \in \{1,2,\dots,N\}$ we define the variables

$$r_i := q_i - q_{i-1} \tag{2.4}$$

and, with this in mind, the dynamics is generated by the Liouville operator $L_N^{\bar\tau(t)}$, defined by

$$L_N^{\bar\tau(t)} f = \sum_{i=1}^{N}(p_i-p_{i-1})\frac{\partial f}{\partial r_i} + \sum_{i=1}^{N-1}\left(V'(r_{i+1})-V'(r_i)\right)\frac{\partial f}{\partial p_i} + (\bar\tau(t)-V'(r_N))\frac{\partial f}{\partial p_N}, \tag{2.5}$$

together with $p_0 = 0$, for all $f \in C^1(\mathbb{R}^{2N})$ and all $t \geq 0$.

Remark. From now on, when we say that a function f is defined on $\mathbb{R}^{2N}$, we mean that $f = f(\mathbf{r}_N, \mathbf{p}_N)$, where $\mathbf{r}_N := (r_1,\dots,r_N)$ and $\mathbf{p}_N := (p_1,\dots,p_N)$. We also shall refer to the r_i's as "positions" and to the p_i's as "momenta".

The next ingredient we need is the following family of Gibbs measures on $\mathbb{R}^{2N}$:

$$\lambda^N_{\beta,\pi,\gamma}(d\mathbf{r}_N, d\mathbf{p}_N) := \prod_{k=1}^{N}\exp\left(-\frac{\beta}{2}(p_k-\pi)^2 - \beta V(r_k) + \beta\gamma r_k - G(\beta,\gamma)\right)dr_k dp_k, \tag{2.6}$$

where $\pi, \gamma \in \mathbb{R}$ are the mean momentum and deformation, respectively and

$$\begin{aligned} G(\beta,\gamma) &:= \log\int_{-\infty}^{\infty}dr\int_{-\infty}^{\infty}dp\exp\left(-\frac{\beta}{2}(p-\pi)^2 - \beta V(r) + \beta\gamma r\right) \\ &= \log\int_{-\infty}^{+\infty}dr\exp\left(-\beta(V(r)-\gamma r)\right) + \log\sqrt{\frac{2\pi}{\beta}} \end{aligned} \tag{2.7}$$

is the one-particle partition function. $\lambda^N_{\beta,\pi,\gamma}$ has a relevant physical meaning, as $\lambda^N_{\beta,0,\tau^*}$ is the unique invariant measure (with respect to $L_N^{\tau^*}$) if τ^* is constant, meaning

$$\int (L_N^{\tau^*} f)\, d\lambda^N_{\beta,0,\tau^*} = 0 \quad \forall f \in C_b^1(\mathbb{R}^{2N}) \tag{2.8}$$

The positive parameter β is interpreted to be the reciprocal of the temperature, while τ^* is the tension of the chain.

2.3 The noise

At this point we introduce a stochastic noise which will provide ergodicity and bring the system at constant unit temperature. We want a noise which does *not* conserve the energy, but which does not spoil the conservation of mass and momentum. Namely, the generator $\mathcal{S}_N$ of the noise has to be such that

$$\mathcal{S}_N\left(\sum_{i=1}^N r_i\right) = \mathcal{S}_N\left(\sum_{i=1}^N p_i\right) = 0 \tag{2.9}$$

and

$$\mathcal{S}_N\left(\sum_{i=1}^N \left(\frac{p_i^2}{2} + V(r_i)\right)\right) \neq 0. \tag{2.10}$$

One suitable choice is given by $\mathcal{S}_N = S_N + \tilde{S}_N$, where, for any $f \in C^2(\mathbb{R}^{2N})$,

$$S_N f := -\sum_{i=1}^{N-1} D_i^* D_i f, \quad \tilde{S}_N f := -\sum_{i=1}^{N-1} \tilde{D}_i^* \tilde{D}_i f. \tag{2.11}$$

Here

$$D_i f := \frac{\partial f}{\partial p_{i+1}} - \frac{\partial f}{\partial p_i}, \qquad \tilde{D}_i f := \frac{\partial f}{\partial r_{i+1}} - \frac{\partial f}{\partial r_i} \tag{2.12}$$

and D_i^* ($\tilde{D}_i^*$) is the adjoint of D_i ($\tilde{D}_i$) with respect to $\lambda^N_{\beta,0,0}$. Explicitly:

$$D_i^* f = (p_{i+1} - p_i) f - \beta^{-1} D_i f, \qquad \tilde{D}_i^* f = (V'(r_{i+1}) - V'(r_i)) f - \beta^{-1} \tilde{D}_i f. \tag{2.13}$$

We want to study the hydrodynamic limit of the system, i.e. the behaviour of the system when $N \to +\infty$. In order to do so we need to properly rescale space and time with N. We choose a hyperbolic scaling, where both time and space are scaled by the same factor, namely N. However we let $\bar{\tau}$ to vary on a macroscopic time scale, that is we keep writing $\bar{\tau}(t)$ rather than $\bar{\tau}(Nt)$.

Therefore the complete generator of the dynamics reads

$$G_N^{\bar{\tau}(t)} := N L_N^{\bar{\tau}(t)} + N\sigma(N)(S_N + \tilde{S}_N), \tag{2.14}$$

where $\sigma : \mathbb{N} \to \mathbb{R}_+$ gives the strength of the noise. We need it to be big enough to provide ergodicity but small enough to disappear in the hydrodynamic limit. A possible choice is to take

$$\sigma(N) = N^{\alpha+1/2},$$

for some $\alpha \in (1/2, 1)$, so that

$$\lim_{N\to+\infty} \frac{\sigma(N)}{N} = \lim_{N\to+\infty} \frac{N}{\sigma(N)^2} = 0. \tag{2.15}$$

2.4 Full dynamics and transition probability

Given the generator $G_N^{\bar\tau(t)}$ and some initial conditions $p_0(t)$, $\{r_k(0), p_k(0)\}_{k=1}^N$, then $p_0(t)$ and $\{r_k(t), p_k(t)\}_{k=1}^N$ solve the following system of stochastic differential equations (SDEs):

$$\begin{cases} p_0 = 0 \\ dr_1 = Np_1 dt + N\sigma\left(V_2' - V_1'\right)dt - \sqrt{2\beta^{-1}N\sigma}d\widetilde{w}_1 \\ dr_i = N(p_i - p_{i-1})dt + N\sigma\left(V_{i+1}' + V_{i-1}' - 2V_i'\right)dt + \sqrt{2\beta^{-1}N\sigma}(d\widetilde{w}_{i-1} - d\widetilde{w}_i) \\ dr_N = N(p_N - p_{N-1})dt + N\sigma\left(V_{N-1}' - V_N'\right)dt + \sqrt{2\beta^{-1}N\sigma}d\widetilde{w}_{N-1} \\ dp_1 = N(V_2' - V_1')dt + N\sigma(N)\left(p_2 - p_1\right)dt - \sqrt{2\beta^{-1}N\sigma(N)}dw_1 \\ dp_i = N(V_{i+1}' - V_i')dt + N\sigma\left(p_{i+1} + p_{i-1} - 2p_i\right)dt + \sqrt{2\beta^{-1}N\sigma}(dw_{i-1} - dw_i) \\ dp_N = N(\bar\tau - V_N')dt + N\sigma\left(p_{N-1} - p_N\right)dt + \sqrt{2N\beta^{-1}\sigma}dw_{N-1}, \end{cases} \quad , \tag{2.16}$$

where $2 \le i \le N-1$. We have set $V_i' := V'(r_i)$ and taken $\{w_i\}_{i=1}^\infty$, $\{\widetilde{w}_i\}_{i=1}^\infty$ to be independent families of independent Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that, since $p_0(t)$ is identically zero for all t, we can (and we will) forget about it.

Remark. The existence of solutions for the above system of SDEs is standard and, from now on $\{r_k(t), p_k(t)\}_{k=1}^N$ will be such a solution. In particular, for $1 \le k \le N$, r_k and p_k are stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$.

The way the system (2.16) is connected to the generator $G_N^{\bar\tau(t)}$ is also standard, and it is given by the following result.

For all $t \ge 0$, let μ_t^N be a probability measure on $\mathbb{R}^{2N}$ which is absolutely continuous with respect some reference measure, say $\lambda_{\beta,0,0}^N$ and such that the density $f_t^N := d\mu_t^N / d\lambda_{\beta,0,0}^N$ solves the Fokker-Plank equation

$$\frac{\partial f_t^N}{\partial t} = G_N^{\bar\tau(t)^*} f_t^N, \tag{2.17}$$

where $G_N^{\bar\tau(t)^*}$ is the adjoint of $G_N^{\bar\tau(t)}$ with respect to $\lambda_{\beta,0,0}^N$. Then, for any $t \ge 0$ and $g \in L^1(\mathbb{R}^{2N}; \mu_t^N)$ we have

$$\mathbb{E}\left[g(\mathbf{r}_N(t), \mathbf{p}_N(t))\right] = \int g d\mu_t^N. \tag{2.18}$$

This fact will be heavily (yet often implicitly) used through the rest of the thesis.

2.5 The main result

From now on we fix, once and for all, $\beta = 1$ and. we set $\lambda_{\pi,\gamma}^N = \lambda_{1,\pi,\gamma}^N$. Explicitly,

$$d\lambda_{\pi,\gamma}^N = \frac{1}{(2\pi)^{N/2}} \prod_{k=1}^N \exp\left(-\frac{(p_k - \pi)^2}{2} + \gamma r_k - V(r_k) - G(\gamma)\right) dr_k dp_k, \tag{2.19}$$

where

$$G(\gamma) := \log \int_{-\infty}^{+\infty} dr \exp\left(\gamma r - V(r)\right) \tag{2.20}$$

is the new partition function. We also set $\lambda^N = \lambda^N_{0,0}$.

Since the noise conserves length and momentum, we expect that, by taking the limit $N \to +\infty$, the variables (r_i) and (p_i) will converge (in some sense) to a deformation $r(t,x)$ and a velocity $p(t,x)$ that satisfy the isothermal Euler equation in Lagrangian coordinates with boundary conditions, namely the p-system

$$\begin{cases} \partial_t r(t,x) - \partial_x p(t,x) = 0 \\ \partial_t p(t,x) - \partial_x \tau(r(t,x)) = 0, \end{cases} \qquad p(t,0) = 0, \quad \tau(r(t,1)) = \bar{\tau}(t), \tag{2.21}$$

with $(t,x) \in [0,\infty) \times [0,1]$, for some function τ, which depends on the interaction V and will turn out to be nonlinear if V is not the harmonic potential.

As the (2.21) is a hyperbolic system of nonlinear partial differential equation, its solutions will, in general, develop shocks (that is discontinuities) in a finite time, even if smooth initial conditions are given. Therefore, we shall look for *weak* solutions to our equations, which are defined even if the functions have discontinuities.

Definition 2.1. *We say that* $(r,p) \in L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ *is a* weak solution *of the p-system* (2.21) *provided*

$$\int_0^\infty dt \int_0^1 dx \left(r(t,x)\partial_t\varphi(t,x) - p(t,x)\partial_x\varphi(t,x)\right) = 0 \tag{2.22}$$

$$\int_0^\infty dt \int_0^1 dx \left(p(t,x)\partial_t\psi(t,x) - \tau(r(t,x))\partial_x\psi(t,x)\right) + \int_0^\infty dt\psi(t,1)\bar{\tau}(t) = 0 \tag{2.23}$$

for all $C^2(\mathbb{R}^2)$ *test functions* φ, ψ *with the following properties:*

- $\varphi(\cdot,x)$ *and* $\varphi(\cdot,x)$ *are compactly supported in* $(0,\infty)$ *for all* $x \in [0,1]$*;*
- $\partial_t\varphi(t,\cdot), \partial_t\psi(t,\cdot), \partial_x\varphi(t,\cdot), \partial_x\psi(t,\cdot)$ *are compactly supported in* $(0,1)$ *for all* $t \geq 0$*;*
- $\varphi(t,1) = \psi(t,0) = 0$ *for all* $t \geq 0$.

Remark. The last two conditions ensure that $\varphi(t,x)$ stays zero near $x = 1$ and $\psi(t,x)$ stays zero near $x = 0$.

Before stating (and sketching the proof of) the main theorem we give some useful notation. Let, for any sequence $(f_i)_{i\in\mathbb{N}}$ and for any $k \in \mathbb{N}$

$$\nabla f_k := f_{k+1} - f_k, \quad \nabla^* f_k := f_{k-1} - f_k \tag{2.24}$$

and

$$\Delta f_k := -\nabla^*\nabla f_k = -\nabla\nabla^* f_k = (\nabla + \nabla^*) f_k = f_{k+1} + f_{k-1} - 2f_k. \tag{2.25}$$

We refer to ∇ and ∇^* as *discrete gradients* and to Δ as *discrete Laplacian*, since, if $f_k = \phi(k/N)$, for some $\phi \in C_b^3(\mathbb{R})$, then

$$\nabla f_k = \frac{1}{N}\phi'\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{N^2}\right) = -\nabla^* f_k \tag{2.26}$$

and

$$\Delta f_k = \frac{1}{N^2}\phi''\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{N^3}\right) \tag{2.27}$$

as $N \to +\infty$.

Remark. It is useful to rewrite the system (2.16) with $\beta = 1$ in terms of gradient and Laplacians, and the result is as follows.

$$\begin{cases} dr_1 = Np_1 dt + N\sigma\nabla V_1' dt - \sqrt{2N\sigma}d\widetilde{w}_1 \\ dr_i = -N\nabla^* p_i dt + N\sigma\Delta V_i' dt - \sqrt{2N\sigma}\nabla^* d\tilde{w}_i, \quad 2 \le i \le N-1 \\ dr_N = -N\nabla^* p_N dt - N\sigma\nabla^* V_N' dt + \sqrt{2N\sigma}d\widetilde{w}_{N-1} \\ dp_1 = N\nabla V_1' dt + N\sigma\nabla p_1 dt - \sqrt{2N\sigma}dw_1 \\ dp_i = N\nabla V_i' dt + N\sigma\Delta p_i dt - \sqrt{2N\sigma}\nabla^* dw_i, \quad 2 \le i \le N-1 \\ dp_N = N(\bar{\tau} - V_N')dt - N\sigma\nabla p_N dt + \sqrt{2N\sigma}dw_{N-1} \end{cases}. \tag{2.28}$$

We also have the following *summation by parts* formula: for any two sequences $(f_i)_{i\in\mathbb{N}}$, $(g_i)_{i\in\mathbb{N}}$ and any $m, n \in \mathbb{N}$, we have

$$\sum_{k=m}^{n} g_k \nabla^* f_k = \sum_{k=m}^{n} f_k \nabla g_k + g_m f_{m-1} - g_{n+1} f_n. \tag{2.29}$$

This explains the way we defined ∇^*: it is, in the sense of the summation by parts formula, the adjoint of ∇.

We are now ready to state our main result.

Theorem 2.1 (Main theorem)**.** *Let $t \ge 0$ and let $\{r_k(t), p_k(t)\}_{k=1}^N$ be a solution of the system (2.28) at time t and let μ_t^N, f_t^N as in (2.17), (2.18) such that*

$$\int f_0^N \log f_0^N d\lambda^N \le CN, \tag{2.30}$$

for some $C > 0$ independent of N. Moreover, let, for $1 \le k \le N$ and $t \ge 0$, $u_k(t) = (r_k(t), p_k(t))$ and let, for $T > 0$, $Q_T = [0, T] \times [0, 1]$ .Then we have the following

i) There exists a family $\{\hat{u}_N\}_{N\in\mathbb{N}}$ *of stochastic processes over* $\mathbb{R}^2$ *such that* $\hat{u}_N \in L^2(\Omega \times Q_T; \mathbb{R}^2)$ *for any* $T > 0$ *and such that*

$$\lim_{N\to+\infty} \mathbb{P}\left\{ \left| \frac{1}{N} \sum_{k=1}^{N} \Psi\left(\frac{k}{N}\right) \cdot u_k(t) - \int_0^1 \Psi(x) \cdot \hat{u}_N(t,x)dx \right| > \delta \right\} = 0, \quad (2.31)$$

for all $t \geq 0$, $\delta > 0$ *and all* $\Psi \in C_c^1((0,1); \mathbb{R}^2)$.

ii) There exists a a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *and some* $\tilde{u}_N, \tilde{u} \in L^2(\tilde{\Omega} \times Q_T; \mathbb{R}^2)$ *such that* $\tilde{u}_N$ *and* $\hat{u}_N$ *have the same law and* $\tilde{u}_N \to \tilde{u}$ $\tilde{\mathbb{P}}$*-almost surely as* $N \to +\infty$ *in* L^p*-strong for all* $1 \leq p < 2$, *where* $\tilde{u}$ *is almost surely a weak solution of the p-system* (2.21) *with*

$$\tau(r) = F'(r), \qquad F(r) = \sup_{\gamma\in\mathbb{R}} \{\gamma r - G(\gamma)\}. \quad (2.32)$$

and G *defined in* (2.20).

Sketch of the proof. The processes $\hat{u}_N$ are defined as follows:

$$\hat{u}_N(t,x) := \sum_{k=l}^{N-l+1} 1_{N,k}(x)\hat{u}_{l,k}(t),$$

where $1_{N,k}$ is the indicator function of the interval of centre k/N and diameter $1/N$, $l = l(N)$ is an integer and $\hat{u}_{l,k}$ is a weighted average of (u_i) over a block of size $2l-1$ and of centre u_k. We choose the dependence of l on N such that, among other things, $l/N \to 0$ as $N \to +\infty$.

The proof (2.31) will be presented in Chapter 7. It is a direct calculation and it relies on the smoothness of Ψ and the energy estimate

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\frac{1}{N} \sum_{k=1}^{N} \left(p_k^2 + r_k^2\right) \right] < +\infty, \quad (2.33)$$

which is obtained in Chapter 4 by a standard application of the relative entropy.

The proof of part *ii*) is conceptually more interesting and technically much more challenging. In order to prove (2.22) we start from the following identity, which

follows from (2.28) and Itô formula:

$$\begin{aligned}\int_0^\infty dt\frac{1}{N}\sum_{k=1}^{N}\partial_t\varphi\left(t,\frac{k}{N}\right)r_k(t)=\\ -\int_0^\infty\varphi_1\left(p_1dt+\sigma\nabla V_1'dt-\sqrt{2\frac{\sigma}{N}}d\tilde{w}_1\right)+\\ +\sum_{k=2}^{N-1}\int_0^\infty\varphi_k\left(\nabla^*p_kdt-\sigma\Delta V_k'dt+\sqrt{2\frac{\sigma}{N}}\nabla^*d\tilde{w}_k\right)+\\ +\int_0^\infty\varphi_N\left(\nabla^*p_Ndt+\sigma\nabla V_{N-1}'dt-\sqrt{2\frac{\sigma}{N}}d\tilde{w}_{N-1}\right),\end{aligned}\tag{2.34}$$

where $\varphi_k(t):=\varphi(t,k/N)$. After summing by parts we obtain

$$\begin{aligned}\int_0^\infty dt\frac{1}{N}\sum_{k=1}^{N}\partial_t\varphi\left(t,\frac{k}{N}\right)r_k(t)=-\int_0^\infty\varphi_1\left(p_1dt+\sigma\nabla V_1'dt-\sqrt{2\frac{\sigma}{N}}d\tilde{w}_1\right)+\\ +\int_0^\infty dt\left(\sum_{k=2}^{N-1}\nabla\varphi_kp_k+\varphi_2p_1-\varphi_Np_{N-1})\right)+\\ +\sigma\int_0^\infty dt\left(-\sum_{k=2}^{N-1}\Delta\varphi_kV_k'+\nabla\varphi_{N-1}V_N'-\nabla\varphi_1V_2'+\varphi_2\nabla V_1'-\varphi_N\nabla V_{N-1}'\right)\\ -\sqrt{2\frac{\sigma}{N}}\int_0^\infty\sum_{k=2}^{N-1}(\nabla\varphi_kd\tilde{w}_k+\varphi_2d\tilde{w}_1-\varphi_Nd\tilde{w}_{N-1}))+\\ +\int_0^\infty\varphi_N\left(\nabla^*p_Ndt+\sigma\nabla V_{N-1}'dt-\sqrt{2\frac{\sigma}{N}}d\tilde{w}_{N-1}\right)\end{aligned}\tag{2.35}$$

$$\begin{aligned}=\int_0^\infty\left(\frac{1}{N}\sum_{k=1}^{N-1}N\nabla\varphi_kp_kdt-\frac{\sigma}{N}\frac{1}{N}\sum_{k=2}^{N-1}N^2\Delta\varphi_kV_k'dt+\sqrt{2\frac{\sigma}{N}}\frac{1}{N}\sum_{k=1}^{N-1}N\nabla\varphi_kd\tilde{w}_k\right)+\\ -\int_0^\infty dt\left(\varphi_Np_N+\frac{\sigma}{N}(N\nabla\varphi_1V_1'+N\nabla^*\varphi_NV_N')\right).\end{aligned}\tag{2.36}$$

Since $N\nabla\varphi_k$, $N\nabla^*\varphi_k$ and $N^2\Delta\varphi_k$ are $\mathcal{O}(1)$ as $N\to+\infty$, since V' is linearly bounded, and thanks to the energy estimate, all the terms proportional to σ vanish in probability, as $\sigma/N\to 0$. The boundary term φ_Np_N is identically zero, since $\varphi_N(t)=\varphi(t,1)=0$. Therefore, using (2.31) and setting $\hat{u}_N=(\hat{r}_N,\hat{p}_N)$, we obtain

$$\lim_{N\to+\infty}\int_0^\infty\int_0^1 dtdx\left(\partial_t\varphi(t,x)\hat{r}_N(t,x)-\partial_x\varphi(t,x)\hat{p}_N(t,x)\right)=0,\tag{2.37}$$

in probability. This yields to (2.22), provided we can replace convergence in probability with almost surely convergence and provided $\hat{r}_N$ and $\hat{p}_N$ converge weakly almost

surely. By possibly taking a subsequence $N_j \to \infty$, we can assume we do have almost sure convergence. Concerning weak convergence, note that the energy estimate (2.33) implies that $\hat{r}_N$ and $\hat{p}_N$ are uniformly bounded in $L^2(\Omega \times Q_T)$ for any $T > 0$. However, this is not enough to ensure that they almost surely converge weakly in $L^2(Q_T)$. Nevertheless, as we shall see in Chapter 7, there exists a sequence of $\tilde{r}_N$ of random variables, defined on a possibly new probability space, such that the $\tilde{r}_N$ has the same law of $\hat{r}_N$, $\tilde{p}_N$ has the same law of $\hat{p}_N$, and $\tilde{r}_N$ and $\tilde{p}_N$ almost surely converge weakly in $L^2(Q_T)$. This completes the proof for (2.22).

In proving (2.23) we follow the same steps as above and, using the fact that $\psi_N(t) = \psi(t, 1)$ (which this time does not vanish), we obtain

$$\begin{aligned} &\int_0^\infty dt \frac{1}{N} \sum_{k=1}^{N} \partial_t \psi \left(t, \frac{k}{N} \right) p_k + \\ &= \int_0^\infty \left(\frac{1}{N} \sum_{k=1}^{N-1} N \nabla \psi_k V_k' dt - \psi(t,1) \bar{\tau}(t) dt + \frac{\sigma}{N} \frac{1}{N} \sum_{k=2}^{N-1} N^2 \Delta_k \psi p_k dt \right. \\ &\qquad \left. + \sqrt{2 \frac{\sigma}{N}} \frac{1}{N} \sum_{k=1}^{N-1} N \nabla \psi dw_k \right) + \\ &\quad + \int_0^\infty dt \left(\psi_1 V_1' - \frac{\sigma}{N} (N \nabla \psi_1 p_1 + N \nabla^* \psi_N p_N) \right). \end{aligned} \tag{2.38}$$

Once again, the terms proportional to σ vanish in probability as $N \to +\infty$. The boundary term $\psi_1 V_1'$ is identically zero for N large enough, since $\psi_1(t) = \psi(t, 1/N)$ and $\psi(t, \cdot)$ is compactly supported away from zero. The boundary term involving $\bar{\tau}$ is already what we were looking for. Therefore, the only term we have to concern about is the sum involving the nonlinear function V'. The idea here is to replace V' by its ensemble average, which is the function τ defined in the statement of the theorem. We do so as follows. Since V' is linearly bounded, as an immediate consequence of (2.31) we obtain

$$\lim_{N \to \infty} \left(\frac{1}{N} \int_0^\infty \sum_{k=1}^{N} N \nabla \psi_k V_k' dt - \int_0^\infty \frac{1}{N} \sum_{k=l}^{N-l+1} \partial_x \psi \left(t, \frac{k}{N} \right) \hat{V}_{l,k}' dt \right) = 0 \tag{2.39}$$

in probability, where $\hat{V}_{l,k}'$ is an average obtained from $(V'(r_i))$. Next, we can replace $\hat{V}_{l,k}'$ by $\tau(\hat{r}_{l,k})$ via the one block estimate (see Chapter 4)

$$\lim_{N \to \infty} \mathbb{E} \left[\frac{1}{N} \int_0^T \sum_{k=l}^{N-l+1} \left(\hat{V}_{l,k}' - \tau(\hat{r}_{l,k}) \right)^2 dt \right] = 0, \tag{2.40}$$

for any $T > 0$. Finally, we use part i) once more to replace the summation with an integral, so we get

$$\lim_{N\to\infty}\left(\frac{1}{N}\int_0^\infty \sum_{k=1}^{N} N\nabla\psi_k V_k' dt - \int_0^\infty\int_0^1 \partial_x\psi(t,x)\tau(\hat{r}_N(t,x))dtdx\right) = 0 \tag{2.41}$$

in probability. Putting everything together, we have obtained

$$\begin{aligned}\lim_{N\to+\infty}\int_0^\infty\int_0^1 dtdx\,(\partial_t\psi(t,x)\hat{p}_N(t,x) - \partial_x\psi(t,x)\tau(\hat{r}_N(t,x))) + \\ + \int_0^\infty dt\psi(t,1)\bar{\tau}(t) = 0.\end{aligned} \tag{2.42}$$

in probability. Therefore, after taking a subsequence, changing probability space and calling $\tilde{r}$ the weak limit of $\tilde{r}_N$, the problem is now to show that $\tau(\tilde{r}_N)$ converges weakly to $\tau(\tilde{r})$ almost surely.

This will be done using a stochastic version of Young measures and compensated compactness, which is discussed in detail in Chapters 5 and 7. These techniques ultimately allow us to deduce that $\tilde{r}_N$ almost surely converges to $\tilde{r}$ in L^p-strong for all $1 \leq p < 2$. This, together with the fact (proven in Appendix A) that τ grows less than quadratically at infinity implies that $\tau(\tilde{r}_N)$ almost surely converges to $\tau(\tilde{r})$.

□

Chapter 3

The relative entropy and the Dirichlet form

In this chapter we define the relative entropy and the Dirichlet forms. Even though we will not use the relative entropy method, the relative entropy and the Dirichlet forms will still be crucial in obtaining one and two block estimates, which we will heavily employ in the rest of this thesis.

3.1 Definitions

Definition 3.1. *Let μ and ν be finite measures on a measurable space (X, Σ). Let μ be absolutely continuous with respect to ν and let $f = d\mu/d\nu$ be the Radon-Nikodym derivative of μ with respect to ν. The* relative entropy *of μ with respect to ν is defined as*

$$H[\mu|\nu] := \int f \log f d\nu. \tag{3.1}$$

Remark. It is standard to show that the relative entropy is non-negative, sub-additive, convex and lower-semicontinuous. The relative entropy can be used in order to estimate expectation values. In fact it is also standard to show the following entropy inequality:

$$\int \varphi d\mu \leq \frac{1}{\alpha} \left(H[\mu|\nu] + \log \int e^{\alpha\varphi} d\nu \right), \tag{3.2}$$

for any $\alpha > 0$ and any $\varphi \in C_b(X)$. Therefore we can control expectation values with respect to μ provided we control the relative entropy and expectation values with respect to ν.

Definition 3.2. *Let S_N and $\tilde{S}_N$ as in* (2.11) *and let $f : \mathbb{R}^{2N} \to \mathbb{R}_+$ be differentiable. The* Dirichlet forms *of S_N and $\tilde{S}_N$ are defined as*

$$D_N(f) := \sum_{i=1}^{N-1} \int \left(D_i \sqrt{f}\right)^2 d\lambda^N = \sum_{i=1}^{N-1} \int \frac{1}{4f} \left(\frac{\partial f}{\partial p_{i+1}} - \frac{\partial f}{\partial p_i}\right)^2 d\lambda^N. \tag{3.3}$$

and

$$\tilde{D}_N(f) = \sum_{i=1}^{N-1} \int \left(\tilde{D}_i \sqrt{f}\right)^2 d\lambda^N = \sum_{i=1}^{N-1} \int \frac{1}{4f} \left(\frac{\partial f}{\partial r_{i+1}} - \frac{\partial f}{\partial r_i}\right)^2 d\lambda^N, \tag{3.4}$$

respectively.

3.2 A logarithmic Sobolev inequality

In this section we use a logarithmic Sobolev inequality to obtain explicit bounds on the relative entropy, which will in turn lead to explicit bounds for the one block estimate in next Chapter.

We start by defining conditional measures.

Definition 3.3. *Let $r \in \mathbb{N}$ and let $g : \mathbb{R}^2 \to \mathbb{R}^r$ be measurable. Set, for $1 \le m \le k \le N$,*

$$\bar{g}_{m,k} = \frac{1}{m} \sum_{j=0}^{m-1} g(r_{k-j}, p_{k-j}) \tag{3.5}$$

and, for any $y \in \mathbb{R}^r$,

$$\Sigma_{m,k}(y) = \{x \in \mathbb{R}^{2m} : \bar{g}_{m,k} = y\}, \tag{3.6}$$

Then, for any finite measure ν on $\mathbb{R}^{2N}$ we denote its the restriction to the σ-field generated by $\{(r_j, p_j) : k - m + 1 \le j \le k\}$ by $\nu_{m,k}$ and denote the projection *of the obtained measure on $\Sigma_{m,k}(y)$ by the positive measure $d\bar{\nu}_{m,k}(x, y)$ on $\Sigma_{m,k}(y)$ defined by*

$$\int_{\mathbb{R}^{2m}} F(\bar{g}_{m,k}) G(x) d\nu_{m,k}(x) = \int_{\mathbb{R}^r} F(y) \int_{\Sigma_{m,k}(y)} G(x) d\bar{\nu}_{m,k}(x, y), \tag{3.7}$$

for any bounded and measurable $F : \mathbb{R}^r \to \mathbb{R}$ and $G : \mathbb{R}^{2N} \to \mathbb{R}$. $\bar{\nu}$ is said to be the measure ν conditioned *to $\Sigma_{m,k}$.*

We shall consider $r = 2$ and $g(r, p) = (r, p)$, so that, for $\rho, \pi \in \mathbb{R}$ we have

$$\bar{p}_{m,k} := \frac{1}{m} \sum_{j=0}^{m-1} p_{k-j} = \pi, \quad \bar{r}_{m,k} := \frac{1}{m} \sum_{j=0}^{m-1} r_{k-j} = \rho. \tag{3.8}$$

on $\Sigma_{m,k}(\rho,\pi)$. Moreover we let $\bar{\lambda}^N_{m,k}$ be the measure λ^N conditioned to $\Sigma_{m,k}$. One can refer to $\bar{\lambda}^N_{m,k}$ as a *canonical measure*.

For fixed $\rho,\pi\in\mathbb{R}$ we write $d\bar{\lambda}^{\rho,\pi}_{m,k}$ for $d\bar{\lambda}_{m,k}(\cdot,\rho,\pi)$. Similarly, we denote by $\bar{\mu}^{\rho,\pi}_{m,k}$ the corresponding measure obtained from μ^N_t.

Since V is strictly convex, the density of λ^N with respect to the Lebesgue measure is log-concave. The same can be said about the density of the conditional measures $\bar{\lambda}^{\rho,\pi}_{m,k}$. Therefore we can apply the Bakry-Emery criterion and obtain the following classical result.

Proposition 3.1 (Logarithmic Sobolev inequality - LSI)**.** *There exists a universal constant C_{lsi} depending only on the interaction V such that*

$$\int g^2\log g^2\ d\bar{\lambda}^{\pi,\rho}_{m,k}\le C_{lsi}\sum_{i=1}^{m-1}\int\left(\left(\frac{\partial g}{\partial p_{k-i}}\right)^2+\left(\frac{\partial g}{\partial r_{k-i}}\right)^2\right)d\bar{\lambda}^{\pi,\rho}_{m,k},\tag{3.9}$$

for any $\rho,\pi\in\mathbb{R}$ and any non-vanishing $g\in C^\infty_b(\mathbb{R}^{2m})$.

Remark. One does *not* need V to be strictly convex in order to have a LSI. However, if we drop the strict convexity, then V has to be a *bounded* perturbation of the harmonic potential, and this is not compatible to the condition of genuine nonlinearity (see Chapter 5 and Appendix A) that τ must inherit from V.

As a consequence of the LSI we have the following

Corollary 3.2. *Let $\bar{f}^{\rho,\pi}_{m,k}$ be the derivative of $\bar{\mu}^{\rho,\pi}_{m,k}$ with respect to $\bar{\lambda}^{\rho,\pi}_{m,k}$. Then*

$$\sum_{k=m}^{N}\int_{\mathbb{R}^2}\int_{\Sigma_{m,k}(\rho,\pi)}\bar{f}^{\rho,\pi}_{m,k}(x)\log\bar{f}^{\rho,\pi}_{m,k}(x)d\bar{\lambda}^N_{m,k}(x,\rho,\pi)\le m^3C_{lsi}(D_N(f^N_t)+\tilde{D}_N(f^N_t)).\tag{3.10}$$

Proof. From (3.8) we can write

$$r_k=m\rho-\sum_{j=1}^{m-1}r_{k-j},\quad p_k=m\pi-\sum_{j=1}^{m-1}p_{k-j},\tag{3.11}$$

so that we regard $x_j=r_{k-j}$ for $1\le j\le m-1$ as independent variables so that

$$\frac{\partial r_k}{\partial x_j}=-1.\tag{3.12}$$

Then a change of variables and the Cauchy-Schwarz inequality yields, for any smooth g,

$$\begin{aligned}\sum_{i=1}^{m-1}\left(\frac{\partial g}{\partial x_i}\right)^2 &= \sum_{j=1}^{m-1}\left(\frac{\partial g}{\partial r_{k-j}}-\frac{\partial g}{\partial r_k}\right)^2 \le m^2\sum_{j=1}^{m-1}\left(\frac{\partial g}{\partial r_{k-j}}-\frac{\partial g}{\partial r_{k-j+1}}\right)^2 \\ &= m^2\sum_{j=1}^{m-1}(\tilde{D}_{k-j}g)^2.\end{aligned} \tag{3.13}$$

A similar result holds if we replace r by p. Therefore, choosing $g^2 = \bar{f}^{\rho,\pi}_{m,k}$ (3.9) together with (3.13) yield

$$\begin{aligned}&\int_{\Sigma_{m,k}(\rho,\pi)} \bar{f}^{\rho,\pi}_{m,k}\log\bar{f}^{\rho,\pi}_{m,k}\, d\bar{\lambda}^{\pi,\rho}_{m,k} \\ &\le C_{lsi}\sum_{j=1}^{m-1}\int_{\Sigma_{m,k}(\rho,\pi)}\frac{1}{4\bar{f}^{\rho,\pi}_{m,k}}\left(\left(\frac{\partial\bar{f}^{\rho,\pi}_{m,k}}{\partial p_{k-j}}\right)^2+\left(\frac{\partial\bar{f}^{\rho,\pi}_{m,k}}{\partial r_{k-j}}\right)^2\right)d\bar{\lambda}^{\pi,\rho}_{m,k}\end{aligned} \tag{3.14}$$

$$\le m^2 C_{lsi}\sum_{j=1}^{m-1}\int_{\Sigma_{m,k}(\rho,\pi)}\frac{1}{4\bar{f}^{\rho,\pi}_{m,k}}\left(\left(D_{k-j}\bar{f}^{\rho,\pi}_{m,k}\right)^2+\left(\tilde{D}_{k-j}\bar{f}^{\rho,\pi}_{m,k}\right)^2\right)d\bar{\lambda}^{\pi,\rho}_{m,k}. \tag{3.15}$$

Next, it immediately follows from the definition of conditional probability that, on $\Sigma_{m,k}(\rho,\pi)$, $\bar{f}^{\rho,\pi}_{m,k}(x)$ is λ^N-almost surely equal to $f^N_t(x)$, the derivative of μ^N_t with respect to λ^N. Therefore, after integrating over ρ and π, we obtain

$$\begin{aligned}&\sum_{k=m}^{N}\int_{\mathbb{R}^2}\int_{\Sigma_{m,k}(\rho,\pi)}\bar{f}^{\rho,\pi}_{m,k}(x)\log\bar{f}^{\rho,\pi}_{m,k}(x)d\bar{\lambda}^N_{m,k}(x,\rho,\pi) \\ &\le m^2 C\sum_{k=m}^{N}\sum_{j=1}^{m-1}\int_{\mathbb{R}^{2N}}\frac{1}{4f^N_t}\left(\left(D_{k-j}f^N_t\right)^2+\left(\tilde{D}_{k-j}f^N_t\right)^2\right)d\lambda^N.\end{aligned} \tag{3.16}$$

The last step is noting that, when summing over k, any of the terms $\left(D_{k-j}f^N_t\right)^2$ or $\left(\tilde{D}_{k-j}f^N_t\right)^2$ appears at most m times, and so

$$\begin{aligned}&\sum_{k=m}^{N}\int_{\mathbb{R}^2}\int_{\Sigma_{m,k}(\rho,\pi)}\bar{f}^{\rho,\pi}_{m,k}(x)\log\bar{f}^{\rho,\pi}_{m,k}(x)d\bar{\lambda}^N_{m,k}(x,\rho,\pi) \\ &\le m^3 C_{lsi}\sum_{k=1}^{N-1}\int_{\mathbb{R}^{2N}}\frac{1}{4f^N_t}\left(\left(D_k f^N_t\right)^2+\left(\tilde{D}_k f^N_t\right)^2\right)d\lambda^N\end{aligned} \tag{3.17}$$

$$= m^3 C_{lsi}\left(D_N(f^N_t)+\tilde{D}_N(f^N_t)\right). \tag{3.18}$$

□

We will use the LSI to estimate expectation values. Let $\phi : \mathbb{R}^{2m} \to \mathbb{R}$ be continuous and bounded and set
$\phi_{m,k} := \phi(r_{k-m+1}, p_{k-m+1}, \dots, r_{k-1}, p_{k-1}, r_k, p_k)$. Then we have, by (3.10) and the entropy inequality (3.2),

$$\begin{aligned}
\alpha \sum_{k=m}^{N} \int_{\mathbb{R}^{2N}} \phi_{m,k} d\mu_t^N &= \alpha \sum_{k=m}^{N} \int_{\mathbb{R}^2} \int_{\Sigma_{m,k}(\rho,\pi)} \phi_{m,k} d\bar{\mu}_{m,k}(x,\pi,\rho) \\
&\leq \sum_{k=m}^{N} \int_{\mathbb{R}^2} \int_{\Sigma_{m,k}(\rho,\pi)} \bar{f}_{m,k}^{\rho,\pi}(x) \log \bar{f}_{m,k}^{\rho,\pi}(x) d\bar{\lambda}_{m,k}^N(x,\rho,\pi) + \\
&\quad + \sum_{k=m}^{N} \log \int_{\mathbb{R}^2} \int_{\Sigma_{m,k}(\rho,\pi)} e^{\alpha\phi_{m,k}} d\bar{\lambda}_{m,k}^N(x,\rho,\pi) \qquad (3.19) \\
&\leq m^3 C(D_N(t) + \tilde{D}_N(t)) + \sum_{k=m}^{N} \log \int_{\mathbb{R}^2} \int_{\Sigma_{m,k}(\rho,\pi)} e^{\alpha\phi_{m,k}} d\bar{\lambda}_{m,k}^N(x,\rho,\pi). \qquad (3.20)
\end{aligned}$$

for any $\alpha > 0$.

Next, we need to estimate the expectation value with respect to the conditional measure. We do so by using the following result, whose proof can be found in [22].

Proposition 3.3. *Let $\lambda_{\pi,\gamma}^N$ as in (2.19). There exists a universal constant $C' > 0$ and an integer l_0 such that*

$$\int e^{\alpha\phi_{m,k}} d\bar{\lambda}_{m,k}^{\rho,\pi} \leq C' \int e^{\alpha\phi_{m,k}} d\lambda_{\pi,\tau(\rho)}^N, \qquad (3.21)$$

provided $N \geq m \geq l_0$. τ is the same function defined in (2.32).

Remark. It looks like we have one more step to do, since we still have to integrate over ρ and π. However, it will be the case that the integral at the right hand side of (3.21) will be estimated by a constant independent of ρ and π. Therefore integrating over those two variables will simply give another multiplicative constant, which we can absorb into C'.

In order to complete the estimate (3.20) we need a bound on the Dirichlet forms. We will obtain it, together with a bound on the relative entropy, via the following.

Proposition 3.4 (Bound on the relative entropy and the Dirichlet forms)**.** *Let D_N and $\tilde{D}_N$ as in (3.3), (3.4) and let*

$$H_N(f_t^N) := H[\mu_t^N | \lambda^N] = \int f_t^N \log f_t^N d\lambda^N. \qquad (3.22)$$

Assume $H(f_0^N) \leq CN$ for some $C_0 > 0$ independent of N. Then

$$H_N(f_t^N) + N\sigma \int_0^t (D_N(f_s^N) + \tilde{D}_N(f_s^N))ds \leq CN \tag{3.23}$$

for all $t \geq 0$ and some $C > 0$ independent of N.

Proof. The statement will follow by a Gronwall argument. We calculate

$$\frac{d}{dt}H_N(f_t^N) = \int (\partial_t f_t^N) \log f_t^N d\lambda^N + \int \partial_t f_t^N d\lambda^N = \int (\partial_t f_t^N) \log f_t^N d\lambda^N, \tag{3.24}$$

Since $f_t^N d\lambda^N = d\mu_t^N$ is a probability measure for all t. Using equation (2.17) we have

$$\begin{aligned} \frac{d}{dt}H_N(f_t^N) = \int (G_N^{\bar{\tau}(t)*} f_t^N) \log f_t^N d\lambda^N = \int f_t^N G_N^{\bar{\tau}(t)} \log f_t^N d\lambda^N + \\ = N \int f_t^N L_N^{\bar{\tau}(t)} \log f_t^N d\lambda^N \\ + N\sigma \int f_t^N S_N \log f_t^N d\lambda^N + N\sigma \int f_t^N \tilde{S}_N \log f_t^N d\lambda^N. \end{aligned} \tag{3.25}$$

Now we note that, since $L_N^{\bar{\tau}(t)}$ is a first order differential operator, we have

$$\int f_t^N L_N^{\bar{\tau}(t)} \log f_t^N d\lambda^N = \int L_N^{\bar{\tau}(t)} f_t^N d\lambda^N = \int (L_N^{\bar{\tau}(t)*} 1) f_t^N d\lambda^N. \tag{3.26}$$

In the absence boundary terms, the last term would be identically zero. However, because of the boundary, we get

$$\int f_t^N L_N^{\bar{\tau}(t)} \log f_t^N d\lambda^N = N\bar{\tau}(t) \int p_N f_t^N d\lambda^N. \tag{3.27}$$

Next, we estimate the second term in (3.25) (the third term will be analogous).

$$\int f_t^N S_N \log f_t^N d\lambda^N = -\sum_{k=1}^{N-1} \int f_t D_k^* D_k \log f_t^N d\lambda^N \tag{3.28}$$

$$= -\sum_{k=1}^{N-1} \int (D_k f_t^N)(D_k \log f_t^N) d\lambda^N = -\sum_{k=1}^{N-1} \int \frac{(D_k f_t^N)^2}{f_t^N} d\lambda^N = -4D_N(f_t^N). \tag{3.29}$$

Putting everything together, we obtain

$$\frac{d}{dt}H_N(f_t^N) = N\bar{\tau}(t) \int p_N f_t^N d\lambda^N - 4N\sigma(D_N(f_t^N) + \tilde{D}_N(f_t^N)) \tag{3.30}$$

which, after a time integration, becomes,

$$\begin{aligned} H_N(f_t^N) = H_N(f_0^N) + N \int_0^t ds\bar{\tau}(s) \int p_N f_s^N d\lambda^N + \\ -4N\sigma \int_0^t (D_N(f_s^N) + \tilde{D}_N(f_s^N))ds. \end{aligned} \tag{3.31}$$

We need to estimate the term involving p_N. We do so by writing

$$\begin{aligned} N\int p_N f_s^N d\lambda^N &= N\int (L_N^{\bar\tau(t)} q_N) f_s^N d\lambda^N \\ &= \int (G_N^{\bar\tau(s)} q_N) f_s^N d\lambda^N = \int q_N \partial_s f_s^N d\lambda^N, \end{aligned} \tag{3.32}$$

where we have used the *non* trivial identity

$$\tilde S_N q_N = \tilde S_N \left(\sum_{k=1}^N r_k \right) = 0. \tag{3.33}$$

Therefore we get

$$\begin{aligned} &N\int_0^t ds \bar\tau(s) \int p_N f_s^N d\lambda^N \\ &= \bar\tau(t)\int q_N f_t^N d\lambda^N - \bar\tau(0)\int q_N f_0^N d\lambda^N - \int_0^t ds \bar\tau'(s) \int q_N f_s^N d\lambda^N \end{aligned} \tag{3.34}$$

At this point we use the entropy inequality and estimate, for any $\alpha > 0$,

$$\begin{aligned} \int |q_N| f_t^N d\lambda^N &\le \frac{1}{\alpha} H_N(f_t^N) + \log \int e^{\alpha |q_N|} d\lambda^N \\ &\le \frac{1}{\alpha} H_N(f_t^N) + \log \int \prod_{k=1}^N e^{\alpha |r_k|} d\lambda^N \\ &= \frac{1}{\alpha} H_N(f_t^N) + \sum_{k=1}^N \log \int_{-\infty}^{+\infty} e^{\alpha |r_k| - V(r_k)} dr_k \\ &= \frac{1}{\alpha} H_N(f_t^N) + C(\alpha) N, \end{aligned} \tag{3.35}$$

for some constant $C(\alpha) < +\infty$ independent of N. This allows us to estimate

$$\begin{aligned} &N\left| \int_0^t ds \bar\tau(s) \int p_N f_s^N d\lambda^N \right| \\ &\le \frac{K_{\bar\tau}}{\alpha} \left(H_N(f_t^N) + H_N(f_0^N) + \int_0^t H_N(f_s^N) ds \right) + N(2+t) K_{\bar\tau} C(\alpha), \end{aligned} \tag{3.36}$$

where $K_{\bar\tau} = \sup_{t \ge 0} (|\bar\tau(t)| + |\bar\tau'(t)|)$. Putting everything together and choosing $\alpha = 2K_{\bar\tau}$ we obtain

$$H_N(f_t^N) \le 3H_N(f_0^N) + \int_0^t H_N(f_s^N) ds + C'N - 8N\sigma \int_0^t (D_N(f_s^N) + \tilde D_N(f_s^N)) ds, \tag{3.37}$$

where C' does not depend on N. Since D_N and $\tilde D_N$ are non-negative and since $H_N(f_0^N) \le C_0 N$, by Gronwall's inequality we obtain

$$H_N(f_t^N) \le C'' e^t N, \tag{3.38}$$

for some C'' independent of N. Using this, equation (3.37) becomes

$$N\sigma \int_0^t (D_N(f_s^N) + \tilde{D}_N(f_s^N))ds \leq C'''N, \tag{3.39}$$

for some C''' independent of N, which completes the proof. □

Remark. Note that we have not used the fact that σ depends on N. In fact, the Proposition stays true for any σ we choose.

Chapter 4

Block estimates

In this chapter we obtain one and two block estimate, which are performed on blocks of size l. These estimates will involve the Dirichlet forms, and so $\sigma = \sigma(N)$ will appear. Ultimately, l will depend on N, too, and it will grow at infinity in such a way that $l/N \to 0$. Nevertheless, all the estimates in this Chapter hold for any fixed $\sigma > 0$ and $1 \leq l \leq N$.

4.1 Block averages

For any real valued $(\eta_k)_{k=1}^N$ and for $1 \leq l \leq N$, we define two types of block averages, namely

$$\bar{\eta}_{l,k} := \frac{1}{l} \sum_{j=0}^{l-1} \eta_{k-j}, \qquad l \leq k \leq N, \tag{4.1}$$

$$\hat{\eta}_{l,k} := \frac{1}{l} \sum_{|j|<l} \frac{l-|j|}{l} \eta_{k-j}, \qquad l \leq k \leq N-l+1. \tag{4.2}$$

Note that the average $\hat{\eta}_{l,k}$ is "smoother" than $\bar{\eta}_{l,k}$. They are related by the following

Lemma 4.1. *We have*

$$\hat{\eta}_{l,k+1} - \hat{\eta}_{l,k} = \frac{1}{l} \left(\bar{\eta}_{l,k+l} - \bar{\eta}_{l,k} \right), \tag{4.3}$$

for any real valued $(\eta_k)_{k=1}^N$, *and* $1 \leq l \leq k \leq N-l$.

Proof. We prove the statement by induction over l, for each fixed k. The statement for $l = 1$ is obvious, since both $\hat{\eta}_{1,k+1} - \hat{\eta}_{1,k}$ and $\bar{\eta}_{1,k+1} - \bar{\eta}_{1,k}$ are equal to $\eta_{k+1} - \eta_k$.

Assume now the statement is true for some $l \geq 1$, that is

$$\hat{\eta}_{l,k+1} - \hat{\eta}_{l,k} = \frac{1}{l} \left(\bar{\eta}_{l,k+l} - \bar{\eta}_{l,k} \right). \tag{4.4}$$

We prove it holds for $l+1$ as well. We have, in fact

$$\hat{\eta}_{l+1,k+1} - \hat{\eta}_{l+1,k} = \frac{1}{l+1} \sum_{|j|<l+1} \frac{l+1-|j|}{l+1} \eta_{k+1-j} - \frac{1}{l+1} \sum_{|j|<l+1} \frac{l+1-|j|}{l+1} \eta_{k-j} \quad (4.5)$$

$$= \frac{1}{(l+1)^2} \sum_{|j|<l+1} (l+1-|j|)(\eta_{k+1-j} - \eta_{k-j}) \quad (4.6)$$

$$= \frac{l^2}{(l+1)^2} \frac{1}{l} \sum_{|j|<l} \frac{l-|j|}{l} (\eta_{k+1-j} - \eta_{k-j}) + \frac{1}{(l+1)^2} \sum_{|j|<l+1} (\eta_{k+1-j} - \eta_{k-j}). \quad (4.7)$$

For the first summation we can use our inductive hypothesis, while the second summation is a telescopic one. Therefore we obtain

$$\hat{\eta}_{l+1,k+1} - \hat{\eta}_{l+1,k} = \frac{1}{(l+1)^2} \sum_{j=0}^{l-1} (\eta_{k+l-j} - \eta_{k-j}) + \frac{1}{(l+1)^2} (\eta_{k+l+1} - \eta_{k-l}) \quad (4.8)$$

$$= \frac{1}{(l+1)^2} \left(\sum_{j=1}^{l} \eta_{k+l+1-j} + \eta_{k+l+1} - \sum_{j=0}^{l-1} \eta_{k-j} - \eta_{k-l} \right) \quad (4.9)$$

$$= \frac{1}{(l+1)^2} \sum_{j=0}^{l} (\eta_{k+l+1-j} - \eta_{k-j}) \quad (4.10)$$

$$= \frac{1}{l+1} (\bar{\eta}_{l+1,k+l+1} - \bar{\eta}_{l+1,k}). \quad (4.11)$$

□

Remark. The lemma tells us that if we want to control the difference of two close "smooth" blocks we pay the price of having to control two far away blocks (two block estimates) but we gain a factor $1/l$. This is good if we keep in mind that, eventually, l is sent to infinity,

4.2 One and two block estimates

Recall that there exist $c_1, c_2 > 0$ such that

$$c_1 \leq V''(r) \leq c_2 \quad (4.12)$$

for all $r \in \mathbb{R}$. We shall assume, without loss of generality, that $V(0) = V'(0)$, as the general case is discussed analogously. This in particular means

$$\frac{c_1}{2} r^2 \leq V(r) \leq \frac{c_2}{2} r^2 \quad (4.13)$$

for all r.

Lemma 4.2 (Energy estimate)**.** *For any $t \geq 0$ we have*

$$\sum_{k=1}^{N} \int \left(r_k^2 + p_k^2\right) d\mu_t^N \leq C_1 N. \tag{4.14}$$

for some $C_1 > 0$ independent of N.

Proof. Let $\alpha > 0$. By the entropy inequality we have

$$\alpha \int \sum_{k=1}^{N} \left(r_k^2 + p_k^2\right) d\mu_t^N \leq H_N(f_t^N) + \log \int e^{\alpha \sum_{k=1}^{N}(p_k^2+r_k^2)} d\lambda^N, \tag{4.15}$$

Since $c_1 r^2/2 \leq V(r)$ we have

$$\int e^{\alpha \sum_{k=1}^{N}(p_k^2+r_k^2)} d\lambda^N = \left(\frac{1}{G(0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\alpha p^2 + \alpha r^2} e^{-\frac{p^2}{2} - V(r)} dr dp\right)^N \leq \tag{4.16}$$

$$\leq \left(\frac{1}{G(0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\alpha p^2 + \alpha r^2} e^{-\frac{p^2}{2} - \frac{c_1}{2} r^2} dr dp\right)^N \tag{4.17}$$

$$= \left(\frac{e}{G(0)} \sqrt{\frac{2\pi}{1-2\alpha}} \sqrt{\frac{2\pi}{c_1 - 2\alpha}}\right)^N, \tag{4.18}$$

provided $\alpha < \min\{1/2, c_1/2\}$. Therefore, since, by (3.23) $H(f_t^N) \leq CN$, choosing $\alpha = \min\{1/4, c_1/4\}$ gives

$$\sum_{k=1}^{N} \int \left(r_k^2 + p_k^2\right) d\mu_t^N \leq C_1 N, \tag{4.19}$$

for some constant C_1 independent of N. □

Remark. Since $V(r) \leq c_2 r^2/2$, for all $r \in \mathbb{R}$, the previous lemma implies that the total energy, properly normalised, is bounded, in the sense that

$$\mathbb{E}\left[\frac{1}{N} \sum_{k=1}^{N} \left(\frac{p_k^2}{2} + V(r_k)\right)\right] \leq C_1. \tag{4.20}$$

Lemma 4.3 (One block estimate)**.** *There exists $l_0 \in \mathbb{N}$ such that, for any $t \geq 0$ we have*

$$\sum_{k=l}^{N} \int_0^t \int \left(\bar{V}'_{l,k} - \tau\left(\bar{r}_{l,k}\right)\right)^2 d\mu_s^N ds \leq C_2 \left(\frac{Nt}{l} + \frac{l^2}{\sigma}\right), \tag{4.21}$$

for some constant C_2 independent of N and whenever $N \geq l > l_0$.

Proof. Let $\pi, \rho \in \mathbb{R}$. In order to get rid of the nonlinear dependence of τ on $\bar{r}_{l,k}$ we use the conditional measures $\bar{\lambda}^{\pi,\rho}_{l,k}$ as, in this case, we will have $\bar{r}_{l,k} = \rho$. For $\alpha > 0$ we have, by (3.20)

$$\begin{aligned}\sum_{k=l}^{N} \alpha \int_0^t \int \left(\bar{V}'_{l,k} - \tau\left(\bar{r}_{l,k}\right)\right)^2 d\mu_s^N \leq l^3 C_{lsi} \int_0^t (D_N(s) + \tilde{D}_N(s)) ds + \\ + t \sum_{k=l}^{N} \log \int_{\mathbb{R}^2} \int_{\Sigma(\rho,\pi)} \exp\left(\alpha \left(\bar{V}'_{l,k} - \tau(\rho)\right)^2\right) d\bar{\lambda}^N_{l,k}(x,\rho,\pi).\end{aligned} \tag{4.22}$$

Hence, using (3.23),

$$\begin{aligned}\sum_{k=l}^{N} \alpha \int_0^t \int \left(\bar{V}'_{l,k} - \tau\left(\bar{r}_{l,k}\right)\right)^2 d\mu_s^N \\ \leq t \sum_{k=l}^{N} \log \int_{\mathbb{R}^2} \int_{\Sigma(\rho,\pi)} \exp\left(\alpha \left(\bar{V}'_{l,k} - \tau(\rho)\right)^2\right) d\bar{\lambda}^N_{l,k}(x,\rho,\pi) + C\frac{l^3}{\sigma},\end{aligned} \tag{4.23}$$

for some $C > 0$ independent of N. We are left to estimate the canonical expectation value. In order to do so we introduce a normally distributed random variable ξ and we write

$$\begin{aligned}\int \exp\left(\alpha \left(\bar{V}'_{l,k} - \tau(\rho)\right)^2\right) d\bar{\lambda}^{\pi,\rho}_{l,k} \\ = \mathbb{E}_\xi \left[\int \exp\left(\xi\sqrt{2\alpha}\left(\bar{V}'_{l,k} - \tau(\rho)\right)\right) d\bar{\lambda}^{\pi,\rho}_{l,k}\right]\end{aligned} \tag{4.24}$$

$$\leq C' \mathbb{E}_\xi \left[\int \exp\left(\xi\sqrt{2\alpha}\left(\bar{V}'_{l,k} - \gamma\right)\right) d\lambda^N_{\pi,\gamma}\right] \tag{4.25}$$

$$= C' \mathbb{E}_\xi \left[e^{-\gamma\xi\sqrt{2\alpha}} \int \exp\left(\frac{\xi\sqrt{2\alpha}}{l} \sum_{j=0}^{l-1} V'(r_{k-j})\right) d\lambda^N_{\pi,\gamma}\right] \tag{4.26}$$

$$= C' \mathbb{E}_\xi \left[e^{-\gamma\xi\sqrt{2\alpha}} \left(\int \exp\left(\frac{\xi\sqrt{2\alpha}}{l} V'(r_1)\right) d\lambda^N_{\pi,\gamma}\right)^l\right], \tag{4.27}$$

were $\gamma = \tau(\rho)$ and where we have used (3.21). Therefore, since, by Lemma A.2,

$$\int \exp\left(\frac{\xi\sqrt{2\alpha}}{l} V'(r_1)\right) d\lambda^N_{\pi,\gamma} \leq \exp\left(\frac{c_2\alpha}{l^2}\xi^2 + \frac{\gamma\sqrt{2\alpha}}{l}\xi\right), \tag{4.28}$$

we obtain

$$\int \exp\left(\alpha \left(\bar{V}'_{l,k} - \tau(\rho)\right)^2\right) d\bar{\lambda}^{\pi,\rho}_{l,k} \leq C' \mathbb{E}_\xi \left[\exp\left(\frac{c_2\alpha}{l}\xi^2\right)\right], \tag{4.29}$$

and the right hand side is independent of π and γ. Putting everything together yields

$$\begin{aligned} &\alpha \sum_{k=l}^{N} \int_0^t \int \left(\bar{V}'_{l,k} - \tau\left(\bar{r}_{l,k}\right)\right)^2 d\mu_{s,N} \\ &\leq \frac{Cl^3}{\sigma} + (N-l+1)t \log\left(C' \mathbb{E}_\xi\left[\exp\left(\frac{c_2 \alpha}{l}\xi^2\right)\right]\right). \end{aligned} \tag{4.30}$$

The conclusion then follows by taking $\alpha = l/(4c_2)$. □

Remark. (4.21) can be read as the usual one block estimate, since it implies

$$\lim_{l\to+\infty} \lim_{N\to+\infty} \frac{1}{N} \sum_{k=l}^{N} \int_0^t \int \left(\bar{V}'_{l,k} - \tau\left(\bar{r}_{l,k}\right)\right)^2 d\mu_s^N ds = 0, \tag{4.31}$$

provided σ does not tend to zero too quickly as $N \to +\infty$. In this sense, the point of the proposition is to give *explicit bounds* for the one block estimate. This shall be crucial later on.

We will choose $l = l(N)$ to be the integer part of $N^{1/4}\sigma(N)^{1/2}$. Then, as $N \to +\infty$,

$$\frac{Nt}{l} + \frac{l^2}{\sigma} = \mathcal{O}\left(\frac{N^{3/4}}{\sigma^{1/2}} + \sqrt{N}\right) = \mathcal{O}\left(\sqrt{N}\left(1 + \left(\frac{N}{\sigma^2}\right)^{1/4}\right)\right) = \mathcal{O}(\sqrt{N}), \tag{4.32}$$

since $N/\sigma^2 \to 0$. This in turn implies

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{k=l}^{N} \int_0^t \int (\bar{V}'_{l,k} - \tau(\bar{r}_{l,k})^2 d\mu_s^N ds = 0. \tag{4.33}$$

Lemma 4.4 (Two block estimate)**.** *Let $t \geq 0$ and let l_0 as in Lemma 4.3. There exists C_3 independent of N such that, for $l_0 < l \leq m < N$,*

$$\sum_{k=l}^{N-m} \int_0^t \int \left(\bar{\eta}_{l,k+m} - \bar{\eta}_{l,k}\right)^2 d\mu_s^N ds \leq C_3\left(\frac{Nt}{l} + \frac{m^2}{\sigma}\right), \tag{4.34}$$

whenever $\bar{\eta}_{l,k} \in \{\bar{p}_{l,k}, \bar{V}'_{l,k}, \tau(\bar{r}_{l,k}), \bar{r}_{l,k}\}$.

Proof. We start with $\bar{\eta}_{l,k} = \bar{V}'_{l,k}$. We shall use the integration by parts formula

$$\begin{aligned} &\int \left(V'_{k+m} - V'_k\right) \varphi f_s^N d\lambda^N \\ &= \int \left(\tilde{\partial}_{k+m}\varphi - \tilde{\partial}_k \varphi\right) f_s^N d\lambda^N + \int \varphi\left(\tilde{\partial}_{k+m} f_s^N - \tilde{\partial}_k f_s^N\right) d\lambda^N, \end{aligned} \tag{4.35}$$

where $f_s^N = d\mu_s^N/d\lambda^N$. We have

$$\int \left(\bar{V}'_{l,k+m} - \bar{V}'_{l,k}\right)^2 d\mu_s^N = \frac{1}{l} \sum_{j=0}^{l-1} \int \left(V'_{k+m-j} - V'_{k-j}\right)\left(\bar{V}'_{l,k+m} - \bar{V}'_{l,k}\right) d\mu_s^N \tag{4.36}$$

$$= \frac{1}{l}\sum_{j=0}^{l-1}\int \left(\tilde{\partial}_{k+m-j}\left(\bar{V}'_{l,k+m}-\bar{V}'_{l,k}\right)-\tilde{\partial}_{k-j}\left(\bar{V}'_{l,k+m}-\bar{V}'_{l,k}\right)\right)d\mu_s^N + \\ + \int \left(\bar{V}'_{l,k+m}-\bar{V}'_{l,k}\right)\left(\bar{\tilde{\partial}}_{l,k+m}f_s^N-\bar{\tilde{\partial}}_{l,k}f_s^N\right)d\lambda^N. \tag{4.37}$$

Now,

$$\tilde{\partial}_{k+m-j}\left(\bar{V}'_{l,k+m}-\bar{V}'_{l,k}\right)-\tilde{\partial}_{k-j}\left(\bar{V}'_{l,k+m}-\bar{V}'_{l,k}\right) \\ = \frac{1}{l}\sum_{i=0}^{l-1}\left(\delta_{i,j}V''_{k+m-i}-\delta_{i,j-m}V''_{k-i}-\delta_{i,j+m}V''_{k+m-i}+\delta_{i,j}V''_{k-i}\right) \tag{4.38}$$

$$= \frac{1}{l}\left(V''_{k+m-j}+V''_{k-j}\right), \tag{4.39}$$

since $\delta_{i,j-m}=\delta_{i,j+m}=0$, as $0\le i,j<l$ and $m>l$. Moreover, by the Cauchy-Schwarz inequality we obtain

$$\int \left(\bar{V}'_{l,k+m}-\bar{V}'_{l,k}\right)\left(\bar{\tilde{\partial}}_{l,k+m}f_s^N-\bar{\tilde{\partial}}_{l,k}f_s^N\right)d\lambda^N \\ = \int 2\left(\bar{V}'_{l,k+m}-\bar{V}'_{l,k}\right)\sqrt{f}\left(\bar{\tilde{\partial}}_{l,k+m}\sqrt{f_s^N}-\bar{\tilde{\partial}}_{l,k}\sqrt{f_s^N}\right)d\lambda^N, \tag{4.40}$$

$$\le \frac{1}{2}\int (\bar{V}'_{l,k+m}-\bar{V}'_{l,k})^2 f_s^N d\lambda + 2\int \left(\bar{\tilde{\partial}}_{l,k+m}\sqrt{f_s^N}-\bar{\tilde{\partial}}_{l,k}\sqrt{f_s^N}\right)^2 d\lambda^N. \tag{4.41}$$

Therefore, putting everything together, we obtain

$$\int \left(\bar{V}'_{l,k+m}-\bar{V}'_{l,k}\right)^2 d\mu_s^N \le \frac{1}{l^2}\sum_{j=0}^{l-1}\int (V''_{k+m-j}+V''_{k-j})d\mu_s^N + \\ +\frac{1}{2}\int (\bar{V}'_{l,k+m}-\bar{V}'_{l,k})^2 f_s^N d\lambda + 2\int \left(\bar{\tilde{\partial}}_{l,k+m}\sqrt{f_s^N}-\bar{\tilde{\partial}}_{l,k}\sqrt{f_s^N}\right)^2 d\lambda^N, \tag{4.42}$$

which means

$$\int \left(\bar{V}'_{l,k+m}-\bar{V}'_{l,k}\right)^2 d\mu_s^N \le \frac{4\|V''\|_\infty}{l} + 4\int \left(\bar{\tilde{\partial}}_{l,k+m}\sqrt{f_s^N}-\bar{\tilde{\partial}}_{l,k}\sqrt{f_s^N}\right)^2 d\lambda^N. \tag{4.43}$$

Therefore we need an estimate for the last term. Using

$$\tilde{\partial}_b-\tilde{\partial}_a=\sum_{i=a}^{b-1}\tilde{D}_i, \quad \text{for } \ 1\le a<b\le N, \tag{4.44}$$

we have

$$\int \left(\bar{\tilde{\partial}}_{l,k+m}\sqrt{f_s^N}-\bar{\tilde{\partial}}_{l,k}\sqrt{f_s^N}\right)^2 d\lambda \\ = \frac{1}{l^2}\int \left(\sum_{j=0}^{l-1}\left(\tilde{\partial}_{k+m-j}\sqrt{f_s^N}-\tilde{\partial}_{k-j}\sqrt{f_s^N}\right)\right)^2 d\lambda \tag{4.45}$$

$$\leq \frac{1}{l}\sum_{j=0}^{l-1}\int\left(\tilde{\partial}_{k+m-j}\sqrt{f_s^N}-\tilde{\partial}_{k-j}\sqrt{f_s^N}\right)^2 d\lambda \tag{4.46}$$

$$= \frac{1}{l}\sum_{j=0}^{l-1}\int\left(\sum_{i=k-j}^{k+m-j-1}\tilde{D}_i\sqrt{f_s^N}\right)^2 d\lambda \tag{4.47}$$

$$\leq \frac{m}{l}\sum_{j=0}^{l-1}\sum_{i=k-j}^{k+m-j-1}\int\left(\tilde{D}_i\sqrt{f_s^N}\right)^2 d\lambda \tag{4.48}$$

$$\leq m\sum_{j=k-l+1}^{k+m-1}\int\left(\tilde{D}_j\sqrt{f_s^N}\right)^2 d\lambda \tag{4.49}$$

where we have used the fact that, for any positive integer a, and any real-valued sequence $(x_i)_{i\in\mathbb{N}}$,

$$\left(\sum_{i=1}^{a}x_i\right)^2 \leq a\sum_{i=1}^{a}x_i^2, \tag{4.50}$$

Putting everything together we obtain

$$\begin{aligned}&\sum_{k=l}^{N-m}\int_0^t\left(\bar{V}'_{l,k+m}-\bar{V}'_{l,k}\right)^2 d\mu_{s,N}ds\\ &\leq \frac{4\|V''\|_\infty(N-m-l+1)t}{l}+4m\sum_{k=l}^{N-m}\sum_{j=k-l+1}^{k+m-1}\int_0^t\int\left(\tilde{D}_j\sqrt{f_s^N}\right)^2 d\lambda^N ds\end{aligned} \tag{4.51}$$

$$\leq \frac{4\|V''\|_\infty Nt}{l}+4m(m+l-1)\sum_{k=1}^{N-1}\int_0^t\int\left(\tilde{D}_k\sqrt{f_s^N}\right)^2 d\lambda^N ds, \tag{4.52}$$

which leads to the conclusion, since $m+l-1\leq 2m$ and

$$\tilde{D}_N(s)=\sum_{k=1}^{N-1}\int\left(\tilde{D}_k\sqrt{f_s^N}\right)^2 d\lambda^N. \tag{4.53}$$

Thanks to the integration by parts formula

$$\begin{aligned}&\int\left(p_{k+m}-p_k\right)\varphi f_s^N d\lambda^N\\ &=\int\left(\partial_{k+m}\varphi-\partial_k\varphi\right)\varphi f_s^N d\lambda^N+\int\varphi\left(\partial_{k+m}\varphi f_s^N-\partial_k\varphi f_s^N\right)d\lambda^N,\end{aligned} \tag{4.54}$$

the case $\bar{\eta}_{l,k}=\bar{p}_{l,k}$ is analogous. Finally. we treat the cases $\bar{\eta}_{l,k}=\bar{r}_{l,k}$ and $\bar{\eta}_{l,k}=\tau(\bar{r}_{l,k})$ simultaneously, by noting that, since $\tau'(\rho)\geq C$, for some constant C, we have

$$C^2\left(\bar{r}_{l,k+l}-\bar{r}_{l,k}\right)^2\leq\left(\tau(\bar{r}_{l,k+l})-\tau(\bar{r}_{l,k})\right)^2 \tag{4.55}$$

$$\leq 3\left(\bar{V}'_{l,k+m}-\bar{V}'_{l,k}\right)^2+3\left(\tau(\bar{r}_{l,k})-\bar{V}'_{l,k}\right)^2+3\left(\tau(\bar{r}_{l,k+m})-\bar{V}'_{l,k+m}\right)^2, \tag{4.56}$$

which imply the conclusion by the first part of the proof and the one block estimate.

□

Corollary 4.5. *For $N \geq l > l_0$ and $\hat{\eta}_{l,k} \in \{\hat{p}_{l,k}, \hat{V}'_{l,k}, \tau(\hat{r}_{l,k}), \hat{r}_{l,k}\}$ we have*

$$\sum_{k=l}^{N-l}\int_0^t\int(\hat{\eta}_{l,k+1}-\hat{\eta}_{l,k})^2 d\mu_s^N ds \leq C_3\left(\frac{Nt}{l^3}+\frac{1}{\sigma}\right). \tag{4.57}$$

Proof. Let $\hat{\eta}_{l,k} \in \{\hat{p}_{l,k}, \hat{V}'_{l,k}, \hat{r}_{l,k}\}$. Then, from

$$\hat{\eta}_{l,k+1}-\hat{\eta}_{l,k}=\frac{1}{l}(\bar{\eta}_{l,k+l}-\bar{\eta}_{l,k}), \tag{4.58}$$

where $\bar{\eta}_{l,k}$ is defined as in the previous lemma, we have

$$(\hat{\eta}_{l,k+1}-\hat{\eta}_{l,k})^2=\frac{1}{l^2}(\bar{\eta}_{l,k+l}-\bar{\eta}_{l,k})^2, \tag{4.59}$$

and the conclusion follows from Lemma 4.4 with $m = l$.

For $\hat{\eta}_{l,k} = \tau(\hat{r}_{l,k})$ the conclusion follow once more from the lemma, since

$$(\tau(\hat{r}_{l,k+1})-\tau(\hat{r}_{l,k}))^2 \leq C(\hat{r}_{l,k+1}-\hat{r}_{l,k})^2. \tag{4.60}$$

□

Remark. Note that

$$\lim_{l\to+\infty}\lim_{N\to\infty}\sum_{k=l}^{N-l}\int_0^t\int(\hat{\eta}_{l,k+1}-\hat{\eta}_{l,k})^2 d\mu_s^N ds=+\infty \tag{4.61}$$

for any choice of σ. However, if $l = l(N)$ is the integer part of $N^{1/2}\sigma^{1/2}$ and if $\sigma \to +\infty$, we have

$$\lim_{N\to\infty}\sum_{k=l}^{N-l}\int_0^t\int(\hat{\eta}_{l,k+1}-\hat{\eta}_{l,k})^2 d\mu_s^N ds=0. \tag{4.62}$$

Moreover, since $N\sigma/l^3 \to = 0$, we have

$$\limsup_{N\to+\infty}\sigma\sum_{k=l}^{N-l}\int_0^t\int(\hat{\eta}_{l,k+1}-\hat{\eta}_{l,k})^2 d\mu_s^N ds<+\infty. \tag{4.63}$$

This is the analogous of

$$\limsup_{\varepsilon\to0}\varepsilon\int|\nabla u_\varepsilon|^2 dx<+\infty, \tag{4.64}$$

which is the standard estimate one encounters when applying the vanishing viscosity method to find weak solutions to hyperbolic PDEs from a parabolic approximation. In our context the role of ε is played by $\sigma(N)/N$, which vanishes as $N \to +\infty$.

Finally, we compare the averages $\bar{\eta}_{l,k}$ and $\hat{\eta}_{l,k}$.

Lemma 4.6 (Average comparison)**.** *Let $t \geq 0$ and l_0 as in Lemma 4.3. Then*

$$\sum_{k=l}^{N-l+1} \int_0^t \int (\hat{\eta}_{l,k} - \bar{\eta}_{l,k})^2 \, d\mu_s^N \leq C_4 \left(\frac{Nt}{l} + \frac{l^2}{\sigma} \right), \tag{4.65}$$

for $\eta_j \in \{r_j, p_j, V'(r_j)\}$ and whenever $N \geq l > l_0$.

Proof. We prove the statement for $\eta_j = V'(r_j)$, first. Integrating by parts we have

$$\begin{aligned} \int \left(\hat{V}'_{l,k} - \bar{V}'_{l,k} \right)^2 d\mu_s^N &= \int \left(\hat{V}'_{l,k} - \bar{V}'_{l,k} \right) \left(\hat{V}'_{l,k} - \bar{V}'_{l,k} \right) f_s^N d\lambda^N \\ = \int \left(\hat{\hat{\partial}}_{l,k} - \bar{\hat{\partial}}_{l,k} \right) \left(\hat{V}'_{l,k} - \bar{V}'_{l,k} \right) d\mu_s^N &+ \int \left(\hat{V}'_{l,k} - \bar{V}'_{l,k} \right) \left(\hat{\hat{\partial}}_{l,k} f_s^N - \bar{\hat{\partial}}_{l,k} f_s^N \right) d\lambda^N. \end{aligned} \tag{4.66}$$

We note that we can write

$$\hat{\eta}_{l,k} - \bar{\eta}_{l,k} = \frac{1}{l} \sum_{|j|<l} \frac{l - |j|}{l} \eta_{k-j} - \frac{1}{l} \sum_{j=0}^{l-1} \eta_{k-j} \tag{4.67}$$

$$= \frac{1}{l} \sum_{|j|<l} \left(\frac{l - |j|}{l} - a_j \right) \eta_{k-j}, \tag{4.68}$$

where $a_j = 1$ if $0 \leq j < l$ and $a_j = 0$ otherwise. If we define, for $|j| < l$,

$$c_j = \frac{l - |j|}{l} - a_j \tag{4.69}$$

we can then write

$$\hat{\eta}_{l,k} - \bar{\eta}_{l,k} = \frac{1}{l} \sum_{|j|<l} c_j \eta_{k-j}. \tag{4.70}$$

The numbers c_j have the following properties: they are equal to either $(l - |j|)/l$ or $-|j|/l$, so we have $c_j^2 \leq 1$, and they add up to zero. In fact

$$\sum_{|j|<l} c_j = 2 \sum_{j=1}^{l-1} \frac{l - j}{l} + 1 - l \tag{4.71}$$

$$= 2 \sum_{j=1}^{l-1} \frac{j}{l} + 1 - l \tag{4.72}$$

$$= \frac{2}{l} \frac{l(l-1)}{2} + 1 - l = 0. \tag{4.73}$$

Then we can estimate

$$\left(\hat{\hat{\partial}}_{l,k} - \bar{\hat{\partial}}_{l,k} \right) \left(\hat{V}'_{l,k} - \bar{V}'_{l,k} \right) = \frac{1}{l^2} \sum_{|i|<l} c_j \partial_{k-i} \sum_{|j|<l} c_j V'(r_{k-j}) \tag{4.74}$$

$$= \frac{1}{l^2} \sum_{|i|<l} \sum_{|j|<l} c_i c_j V''(r_{k-j}) \delta_{ij} \tag{4.75}$$

$$= \frac{1}{l^2} \sum_{|i|<l} c_i^2 V''(r_{k-j}) \leq \frac{3\|V''\|_\infty}{l}. \tag{4.76}$$

For the remaining term we use Cauchy-Schwarz, obtaining

$$\begin{aligned} &\int \left(\hat{V}'_{l,k} - \bar{V}'_{l,k}\right) \left(\hat{\bar{\partial}}_{l,k} f_s^N - \bar{\bar{\partial}}_{l,k} f_s^N\right) d\lambda^N \\ &\leq \frac{1}{2} \int \left(\hat{V}'_{l,k} - \bar{V}'_{l,k}\right)^2 d\mu_s^N + \frac{1}{2} \int \frac{1}{f_s^N} \left(\hat{\bar{\partial}}_{l,k} f_s^N - \bar{\bar{\partial}}_{l,k} f_s^N\right)^2 d\lambda^N. \end{aligned} \tag{4.77}$$

We evaluate the last term.

$$\hat{\bar{\partial}}_{l,k} f_s^N - \bar{\bar{\partial}}_{l,k} f_s^N = \frac{1}{l^2} \sum_{|j|<l} (l - |j|) \tilde{\partial}_{k-j} f_s^N - \frac{1}{l} \sum_{j=1}^{l} \tilde{\partial}_{k-j} f_s^N \tag{4.78}$$

$$= \frac{1}{l} \sum_{j=1}^{l} \frac{l-j}{l} \tilde{\partial}_{k-j} f_s^N - \frac{1}{l} \sum_{j=1}^{l} \tilde{\partial}_{k-j} f_s^N + \frac{1}{l} \sum_{j=1}^{l} \frac{l-j}{l} \tilde{\partial}_{k+j} f_s^N \tag{4.79}$$

$$= \frac{1}{l} \sum_{j=1}^{l} \left(1 - \frac{j}{l}\right) \tilde{\partial}_{k+j} f_s^N - \frac{1}{l} \sum_{j=1}^{l} \frac{j}{l} \tilde{\partial}_{k-j} f_s^N \tag{4.80}$$

$$= \frac{1}{l} \sum_{j=1}^{l} \frac{j}{l} \left(\tilde{\partial}_{k-j+l} f_s^N - \tilde{\partial}_{k-j} f_s^N\right). \tag{4.81}$$

Therefore

$$\begin{aligned} &\int \frac{1}{f_s^N} \left(\hat{\bar{\partial}}_{l,k} f_s^N - \bar{\bar{\partial}}_{l,k} f_s^N\right)^2 d\lambda^N \\ &= \frac{4}{l^2} \int \left(\sum_{j=1}^{l} \frac{j}{l} \left(\tilde{\partial}_{k-j+l} \sqrt{f_s^N} - \tilde{\partial}_{k-j} \sqrt{f_s^N}\right)\right)^2 d\lambda^N \end{aligned} \tag{4.82}$$

and this is estimated by the Dirichlet form as in the proof of the two block estimate, since $j/l < 1$, leading to the conclusion.

The statement for $\eta_j = p_j$ is analogous. We are left with the case $\eta_j = r_j$. Since we do not have an integration by parts formula involving r_j alone, we follow the proof of Lemma 4.3. Analogously to the proof of Lemma 4.3 we have

$$\begin{aligned} &\sum_{k=l}^{N} \alpha \int_0^t \int \left(\hat{r}_{l,k} - \bar{r}_{l,k}\right)^2 d\mu_s^N \\ &\leq t \sum_{k=l}^{N} \log \int_{\mathbb{R}^2} \int_{\Sigma(\rho,\pi)} \exp\left(\alpha \left(\hat{r}_{l,k} - \bar{r}_{l,k}\right)^2\right) d\bar{\lambda}_{2l-1,k+l-1}^N(x,\rho,\pi) + C \frac{(2l-1)^3}{\sigma}, \end{aligned} \tag{4.83}$$

for any $\alpha > 0$ and some constant C independent of N. Therefore we are left with estimating

$$\begin{aligned} &\int \exp\left(\alpha(\hat{r}_{l,k} - \bar{r}_{l,k})^2\right) d\bar{\lambda}^{\pi,\rho}_{2l-1,k+l-1} \\ &= \mathbb{E}_\xi \left[\int \exp\left(\sqrt{2\alpha}\xi(\hat{r}_{l,k} - \bar{r}_{l,k})\right) d\bar{\lambda}^{\pi,\rho}_{2l-1,k+l-1} \right] \end{aligned} \tag{4.84}$$

$$\leq C' \mathbb{E}_\xi \left[\int \exp\left(\sqrt{2\alpha}\xi(\hat{r}_{l,k} - \bar{r}_{l,k})\right) d\lambda^N_{\pi,\gamma} \right], \tag{4.85}$$

where ξ is a normally distributed random variable and $\gamma = \tau(\rho)$. In order to calculate the last integral, we write

$$\int \exp\left(\sqrt{2\alpha}\xi(\hat{r}_{l,k} - \bar{r}_{l,k})\right) d\lambda^N_{\pi,\gamma} = \int \exp\left(\frac{\sqrt{2\alpha}\xi}{l} \sum_{|j|<l} c_j r_{k-j}\right) d\lambda^N_{\pi,\gamma} \tag{4.86}$$

$$= \prod_{|j|<l} \int \exp\left(\frac{\sqrt{2\alpha}\xi}{l} c_j r_{k-j} + \gamma r_{k-j} - V(r_{k-j}) - G(\gamma)\right) dr_{k-j} \tag{4.87}$$

$$= \prod_{|j|<l} \exp\left(G\left(\gamma + \frac{\sqrt{2\alpha}\xi}{l} c_j\right) - G(\gamma)\right) \tag{4.88}$$

$$= \exp \sum_{|j|<l} \left(G'(\gamma)\frac{\sqrt{2\alpha}\xi}{l} c_j + G''(\tilde{\gamma})\frac{\alpha\xi^2}{l^2} c_j^2\right), \tag{4.89}$$

for some intermediate value $\tilde{\gamma}$. Since $\sum_j c_j = 0$ and $\sum_j c_j^2 \leq 1$, we have

$$\exp \sum_{|j|<l} \left(G'(\gamma)\frac{\sqrt{2\alpha}\xi}{l} c_j + G''(\tilde{\gamma})\frac{\alpha\xi^2}{l^2} c_j^2\right) \leq \exp\left(\frac{2\alpha(2l+1)\|G''\|_{L^\infty}}{l^2}\xi^2\right), \tag{4.90}$$

since G'' is bounded (see Appendix A). Therefore we have obtained

$$\int \exp\left(\alpha(\hat{r}_{l,k} - \bar{r}_{l,k})^2\right) d\bar{\lambda}^{\pi,\rho}_{2l-1,k+l-1} \leq C' \mathbb{E}_\xi \left[\exp\left(\frac{6\alpha\|G''\|_{L^\infty}}{l}\xi^2\right)\right], \tag{4.91}$$

which, again, is independent on π and γ. The conclusion follows as in the proof of the one block estimate. □

Corollary 4.7 (Another one block estimate)**.** *For $l > l_0$, we have*

$$\sum_{k=l}^{N-l+1} \int_0^t \int (\hat{V}'_{l,k} - \tau(\hat{r}_{l,k}))^2 d\mu^N_s ds \leq C\left(\frac{Nt}{l} + \frac{l^2}{\sigma}\right). \tag{4.92}$$

Proof. It follows immediately from the first one block estimate and the average comparison, since

$$(\hat{V}'_{l,k} - \tau(\hat{r}_{l,k}))^2 \leq 3(\hat{V}'_{l,k} - \bar{V}'_{l,k})^2 + 3(\bar{V}'_{l,k} - \tau(\bar{r}_{l,k}))^2 + 3(\tau(\bar{r}_{l,k}) - \tau(\hat{r}_{l,k}))^2 \tag{4.93}$$

and

$$(\tau(\bar{r}_{l,k}) - \tau(\hat{r}_{l,k}))^2 \leq C(\bar{r}_{l,k} - \hat{r}_{l,k})^2. \tag{4.94}$$

□

Chapter 5

Random Young measures and stochastic compensated compactness

This chapter is about use the notion of Young measure and the tool of compensated compactness. The usual framework involves L^∞ Young measures and everything is deterministic. We are, however, in a stochastic setting and L^∞ bounds are not available, so we need to use L^p-like *random* Young measures.

5.1 Strong and weak convergence

We begin defining weak convergence.

Definition 5.1. *Let $n, m \in \mathbb{N}$, let U be an open subset of $\mathbb{R}^m$ and let $u_N, u \in L^p(U;\mathbb{R}^n)$ for some $1 \leq p \leq \infty$ and all $N \in \mathbb{N}$. Finally, let $1 \leq q \leq \infty$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{5.1}$$

We say that

- *If $1 \leq p < \infty$, u_N converges* weakly in L^p *to u, and we write $u_N \rightharpoonup u$, provided*

$$\lim_{N\to+\infty} \int_U v \cdot u_N = \int_U v \cdot u, \quad \forall v \in L^q. \tag{5.2}$$

- *If $p = \infty$, u_N converges* weakly-* in L^∞ *to u, and we write $u_N \overset{*}{\rightharpoonup} u$, provided*

$$\lim_{N\to+\infty} \int_U v \cdot u_N = \int_U v \cdot u, \quad \forall v \in L^1, \tag{5.3}$$

where $\cdot$ denotes the scalar product on $\mathbb{R}^n$. Finally, we say that u_N converges strongly in L^p *(or simply that converges in L^p) if*

$$\lim_{N\to+\infty} \|u_N - u\|_p := \lim_{N\to+\infty} \left(\int_U |u_N - u|^p \right)^{1/p} = 0. \tag{5.4}$$

Strong converges implies weak convergence, while the converse is in general not true. Moreover, strong convergence can be translated in convergence almost everywhere, if we pay the price to pick a subsequence. To be precise we have that, if $u_N \to u$ in L^p, there exists a subsequence $N_j \to \infty$ such that

$$\lim_{j\to+\infty} u_{N_j}(x) \to u(x) \tag{5.5}$$

for almost all $x \in U$.

We have a very useful equivalent condition for weak convergence in L^p for $1 < p < \infty$ (the same result holds in L^∞ if we replace weak convergence by weak-* convergence). $u_N \rightharpoonup u$ (up to a subsequence) if and only if u_N in L^p is uniformly bounded, meaning

$$\sup_{N\in\mathbb{N}} \|u_N\|_p < +\infty. \tag{5.6}$$

From this thesis' point of view, this is the reason why we consider weak convergence in the first place. The energy estimate allows us to obtain only uniform bounds in L^2, therefore we only end up with weakly convergent sequences. Furthermore, we compose our weakly convergent sequence with a nonlinear function (τ, in our case), and so we must investigate how weak convergence behaves with respect to composition with nonlinear functions.

The answer is "badly". In fact, in general, if $u_N \rightharpoonup u$ and f is nonlinear $f(u_N)$ will *not* converge weakly to $f(u)$ (for a classical example, take $p = \infty$, $U = \mathbb{R}$, $u_N(x) = \sin(Nx)$ and $f(y) = y^2$).

Here is where the notion of Young measures comes into our help.

5.2 Random Young measures

We begin defining deterministic Young measures.

Definition 5.2 (Deterministic Young measure)**.** *Let $(Q, \mathcal{A}, \Lambda)$ be a probability space and E be a topological space endowed with the σ-algebra of Borel sets. Moreover, let $\mathcal{P}(E)$ be the set of probability measures on E, endowed with the topology induced by the weak convergence of measures, i.e. $\mu_N \to \mu$ in $\mathcal{P}(E)$ if*

$$\langle \mu_N, \varphi \rangle \to \langle \mu, \varphi \rangle, \qquad \forall \varphi \in C_b(E). \tag{5.7}$$

A Young measure *on Q with state space E is a measurable map $Q \to \mathcal{P}(E)$. The sets of all Young measures of Q is denoted by $\mathcal{Y}$.*

The following proposition characterises the set of Young measures.

Proposition 5.1. *Let Q be a compact subset of $\mathbb{R}^s$ and E be a closed subset of $\mathbb{R}^m$, $m, s \in \mathbb{N}$. Let π be the projection $Q \times E \to Q$, ν be a measure on $Q \times E$ and π_* be the push-forward of ν by π, defined by $\pi_*\nu(A) = \nu(\pi^{-1}(A)))$ for all Borel $A \subseteq Q$. Then*

$$\mathcal{Y} = \{\nu \in \mathcal{P}(Q \times E) : \pi_*\nu = \Lambda\}. \tag{5.8}$$

Proof. Let ν be a Young measure on Q. We can extend it to a probability measure on $Q \times E$ by setting

$$\langle \nu, \psi \rangle = \int_Q \int_E \psi(z, \xi) d\nu_z(\xi) d\Lambda(z), \qquad \forall \psi \in C_b(Q \times E). \tag{5.9}$$

Therefore, if $\psi \in C_b(Q)$ (that is, if ψ is independent of $\xi \in E$), $\langle \nu, \psi \rangle = \langle \Lambda, \psi \rangle$, which means $\pi_*\nu = \Lambda$.

Conversely, let μ be a probability measure on $Q \times E$ such that $\pi_*\mu = \Lambda$. Then, by the Slicing Theorem, we have that, for Λ-a.e. $z \in Q$, there exists $\mu_z \in \mathcal{P}(E)$ such that $z \mapsto \langle \mu_z, \varphi \rangle$ is measurable from Q to $\mathbb{R}$ for every $\varphi \in C_b(E)$ and

$$\langle \mu, \psi \rangle = \int_Q \int_E \psi(z, \xi) d\mu_z(\xi) d\Lambda(z), \tag{5.10}$$

for all $\psi \in C_b(Q \times E)$. That is, μ is a Young measure on Q. □

Remark. From now on, we take Λ to be the Lebesgue measure on $\mathbb{R}^s$ and we denote $d\Lambda(z)$ by dz. Moreover, we have in mind a setting where $E = \mathbb{R}^2$ and Q is a compact subset of $(0, \infty) \times (0, 1)$.

Definition 5.3 (Random Young measure)**.** *A* random Young measure *defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a $\mathcal{Y}$-valued random variable.*

We regard $\mathcal{Y}$ as a metric space by defining the following.

Definition 5.4 (Prohorov's metric)**.** *Let (M, ρ) be a metric space with its Borel sigma algebra $\mathcal{B}(M)$. For any subset $A \subseteq M$, denote the ε-neighbourhood of A by*

$$A^\varepsilon := \bigcup_{\xi \in A} B_\varepsilon(\xi), \tag{5.11}$$

where $B_\varepsilon(\xi)$ is the open ball of radius ε centred at ξ. Then we define the Prokhorov's metric $d : \mathcal{P}(M) \times \mathcal{P}(M) \to [0, +\infty)$ by

$$d(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{B}(M)\}, \tag{5.12}$$

for all $\mu, \nu \in \mathcal{P}(M)$.

Before going on we shall give a few more definitions.

Definition 5.5. *Let X be a random variable with values in a metric space (M, ρ) and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The* law *of X is the probability measure $\mathcal{L}_X \in \mathcal{P}(M)$ defined by*

$$\mathcal{L}_X(A) = \mathbb{P}(X \in A), \tag{5.13}$$

for all $A \subseteq \mathcal{B}(M)$.

Remark. Let X and Y be two (M, ρ)-valued random variables defined on two possibly different probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Then we say that X and Y have the same law provided

$$\mathbb{E}[f(X)] = \tilde{\mathbb{E}}[f(Y)], \tag{5.14}$$

for all measurable functions $f : M \to \mathbb{R}$.

Definition 5.6. *Let X_N, X be (M, ρ)-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that X_N* converges in law *to X if $\mathcal{L}_{X_N}$ converges to $\mathcal{L}_X$ in $\mathcal{P}(M)$, namely if*

$$\lim_{N \to +\infty} \mathbb{E}[f(X_N)] = \mathbb{E}[f(X)] \tag{5.15}$$

for all continuous and bounded $f : M \to \mathbb{R}$.

Definition 5.7. *A subset $A \subseteq \mathcal{P}(M)$ is* tight *if, for all $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subseteq M$ such that*

$$\mu(M \setminus K_\varepsilon) \leq \varepsilon, \tag{5.16}$$

for all $\mu \in A$.

Tightness will be used to prove the following compactness criterion.

Proposition 5.2. *Let $h \in C(E; \mathbb{R}_+)$ satisfy the growth condition*

$$\lim_{|\xi| \to +\infty} h(\xi) = +\infty. \tag{5.17}$$

Then, for all $C > 0$, the set

$$K_C = \left\{ \nu \in \mathcal{Y} : \int_{Q \times E} h(\xi) d\nu_z(\xi) dz \leq C \right\} \tag{5.18}$$

is compact.

Proof. The set $\mathcal{Y}$ is closed in $\mathcal{P}(Q \times E)$, since the condition $\pi_* \nu = \Lambda$ is stable under weak convergence. Moreover, by Prokhorov's theorem, a subset of $\mathcal{Y}$ is relatively compact if and only if is tight. Therefore, K_C is compact provided it is closed and tight.

K_C is closed. In fact, if $(\nu^N) \subset K_C$ is such that $\nu^N \to \nu$, we have

$$\int_{Q\times E} h(\xi) d\nu(z,\xi) \leq \liminf_{N\to+\infty} \int_{Q\times E} h(\xi) d\nu^N(z,\xi) \leq C, \tag{5.19}$$

so that $\nu \in K_C$ as well.

K_C is tight. Let $\varepsilon > 0$. For $R \geq 0$, let

$$H(R) = \inf_{|p|>R} h(p), \tag{5.20}$$

so that $H(R) \to +\infty$ as $R \to +\infty$. We claim that the compact set $M_R = Q \times ([-R, R] \cap E)$ satisfies

$$\nu\left(M_R^c\right) \leq \varepsilon \tag{5.21}$$

for all $\nu \in K_C$ and provided R is large enough. Indeed,

$$H(R)\nu\left(M_R^c\right) \leq \int_{Q\times E} h(\xi) d\nu(z,\xi) \leq C, \tag{5.22}$$

from which the claim follows, as $H(R) \to +\infty$. □

Now that we have a compact subset of Young measures, we can use it to prove a sufficient criterion for convergence in law for sequences of random Young measures.

Proposition 5.3. *Let $h \in C(E; \mathbb{R}_+)$ satisfy the growth condition*

$$\lim_{|\xi|\to+\infty} h(\xi) = +\infty. \tag{5.23}$$

If $(\nu^N)_{N\in\mathbb{N}}$ is a sequence of random Young measures on Q satisfying the bound

$$\mathbb{E}\left[\int_{Q\times E} h(\xi) d\nu_z^N(\xi) dz\right] \leq M, \tag{5.24}$$

for some $M > 0$. Then, up to a subsequence, (ν^N) converges in law.

Proof. Let $\mathcal{L}_N$ be the law of ν^N. It is enough show that the family $(\mathcal{L}_N) \subset \mathcal{P}(\mathcal{Y})$ is tight. Let $\varepsilon > 0$ and let, for some $C > 0$, $K_C \subset \mathcal{Y}$ be the compact set defined in the previous proposition.

$$\sup_{N\in\mathbb{N}} \mathcal{L}_N(\mathcal{Y} \setminus K_C) \leq \varepsilon, \tag{5.25}$$

provided C is large enough First of all note that, if ν is a random Young measure,

$$\mathbb{P}(\nu \notin K_C) = \mathbb{P}\left(\int_{Q\times E} h(\xi)d\nu(z,\xi) > C\right) \leq \frac{1}{C}\mathbb{E}\int_{Q\times E} h(\xi)d\nu(z,\xi). \tag{5.26}$$

Then,

$$\sup_{N\in\mathbb{N}} \mathcal{L}_N(\mathcal{Y}\setminus K_C) = \sup_{N\in\mathbb{N}} \mathbb{P}(\nu^N \notin K_C) \leq \frac{M}{C} \leq \varepsilon, \tag{5.27}$$

if C is large enough. □

For our purposes we will need two more results.

Proposition 5.4. *Let $h, g \in C(E;\mathbb{R})$ such that $|g(\xi)| \leq h(\xi)$, h is bounded on bounded subsets of E and $g(\xi)/h(\xi) \to 0$ as $|\xi| \to +\infty$. Let $(\nu^N)_{N\in\mathbb{N}}$ be a sequence of random Young measures such that*

$$\sup_N \mathbb{E}\left[\int_{Q\times E} h(\xi)d\nu_z^N(\xi)dz\right] \leq C(h), \tag{5.28}$$

where $C(h)$ is a constant depending on h only, and assume that, almost surely, $\nu_N \to \nu$ in $\mathcal{Y}$. Then

$$\mathbb{E}\left[\int_{Q\times E} h(\xi)d\nu_z(\xi)dz\right] \leq C(h) \tag{5.29}$$

and

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\int_{Q\times E} \varphi(z)g(\xi)d\nu_z^N(\xi)dz - \int_{Q\times E} \varphi(z)g(\xi)d\nu_z(\xi)dz\right|\right] = 0 \tag{5.30}$$

for all $\varphi \in C(Q)$.

Proof. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a continuous, non-negative non-increasing function supported in $[0,2]$ which is identically equal to 1 on $[0,1]$. For $R > 1$ and $a \in \mathbb{R}$ define $\chi_R(a) := \chi\left(\frac{a}{R}\right)$. By the monotone convergence theorem,

$$\int_{Q\times E} h(\xi)d\nu_z(\xi)dz = \lim_{R\to+\infty} \int_{Q\times E} h(\xi)\chi_R(h(\xi))d\nu_z(\xi)dz. \tag{5.31}$$

Since now $h(\xi)\chi_R(h(\xi))$ is continuous and bounded, we have, almost surely,

$$\int_{Q\times E} h(\xi)\chi_R(h(\xi))d\nu_z(\xi)dz = \lim_{N\to+\infty} \int_{Q\times E} h(\xi)\chi_R(h(\xi))d\nu_z^N(\xi)dz. \tag{5.32}$$

Then, applying the Fatou lemma twice, we get

$$\mathbb{E}\left[\int_{Q\times E} h(\xi)d\nu_z(\xi)dz\right] \leq \liminf_{R\to+\infty} \mathbb{E}\left[\int_{Q\times E} h(\xi)\chi_R(h(\xi))d\nu_z(\xi)dz\right] \tag{5.33}$$

$$\leq \liminf_{R\to+\infty} \liminf_{N\to+\infty} \mathbb{E}\left[\int_{Q\times E} h(\xi)\chi_R(h(\xi))d\nu_z^N(\xi)dz\right] \leq C(h), \tag{5.34}$$

since $\chi_R \leq 1$.

In order to prove the second part of the proposition, let $\varphi \in C(Q)$ and define

$$X_N := \int_{Q\times E} \varphi(z)g(\xi)d\nu_z^N(\xi)dz, \quad X := \int_{Q\times E} \varphi(z)g(\xi)d\nu_z(\xi)dz. \tag{5.35}$$

Our aim is to prove that

$$\lim_{N\to+\infty} \mathbb{E}[|X_N - X|] = 0. \tag{5.36}$$

In order to do so we define

$$X_N^R := \int_{Q\times E} \varphi(z)g(\xi)\chi_R\left(\frac{h(\xi)}{1+|g(\xi)|}\right) d\nu_z^N(\xi)dz \tag{5.37}$$

and

$$X^R := \int_{Q\times E} \varphi(z)g(\xi)\chi_R\left(\frac{h(\xi)}{1+|g(\xi)|}\right) d\nu_z(\xi)dz, \tag{5.38}$$

and calculate

$$\mathbb{E}[|X_N - X|] \leq \mathbb{E}[|X_N - X_N^R|] + \mathbb{E}[|X_N^R - X^R|] + \mathbb{E}[|X^R - X|]. \tag{5.39}$$

The first term estimates as follows

$$\mathbb{E}[|X_N - X_N^R|] \leq \mathbb{E}\left[\int_{Q\times E} |\varphi(z)||g(\xi)|\left(1-\chi_R\left(\frac{h(\xi)}{1+|g(\xi)|}\right)\right) d\nu_z^N(\xi)dz\right]. \tag{5.40}$$

Since and $g(\xi)/h(\xi) \to 0$ as $|\xi| \to +\infty$ we have for any $R > 1$ that

$$1-\chi_R\left(\frac{h(\xi)}{1+|g(\xi)|}\right) \leq 1_{\frac{h(\xi)}{1+|g(\xi)|}>R} \leq \frac{h(\xi)}{R(1+|g(\xi)|)}, \tag{5.41}$$

which implies

$$\mathbb{E}[|X_N - X_N^R|] \leq \frac{\|\varphi\|_\infty}{R}\mathbb{E}\left[\int_{Q\times E} h(\xi)d\nu_z^N(\xi)dz\right] \leq \frac{C(h)\|\varphi\|_\infty}{R}. \tag{5.42}$$

For the previous part of the proof, we have as well

$$\mathbb{E}[|X - X^R|] \leq \frac{C(h)\|\varphi\|_\infty}{R}, \tag{5.43}$$

which yields

$$\mathbb{E}[|X_N - X|] \leq \frac{2C(h)\|\varphi\|_\infty}{R} + \mathbb{E}[|X_N^R - X^R|]. \tag{5.44}$$

In order to conclude the proof it is then enough to show that the last term goes to zero, as $N \to +\infty$ for all fixed R.

First of all we note that, since g may diverge only at infinity and $g(\xi)/h(\xi) \to 0$ as $|\xi| \to +\infty$, then $g(\xi)\chi_R\left(\frac{h(\xi)}{1+|g(\xi)|}\right)$ is continuous and bounded and therefore $X_N^R \to X^R$ almost surely. We however need something stronger than that. Therefore we note that, since

$$\mathbb{E}\left[|X_N^R|\right] \leq \|\varphi\|_\infty C(h), \tag{5.45}$$

the sequence $(X_N^R)_{N\in\mathbb{N}}$ is equi-integrable. Therefore, by the Vitali theorem, $X_N^R \to X^R$ in $L^1(\Omega)$, meaning

$$\lim_{N\to+\infty} \mathbb{E}[|X_N^R - X^R|] = 0, \tag{5.46}$$

which completes the proof. □

Remark. It follows from the previous proposition that, up to possibly taking a subsequence, we have, almost surely,

$$\lim_{N\to+\infty} \int_{Q\times E} \varphi(z)g(\xi)d\nu_z^N(\xi)dz = \int_{Q\times E} \varphi(z)g(\xi)d\nu_z(\xi)dz. \tag{5.47}$$

which is the type of result we will want to use.

We conclude this section by giving a definition and a result concerning random L^p-Dirac masses.

Definition 5.8. *Let $p \geq 1$ and let ν be a random Young measure. We say that ν is a* random L^p-Dirac mass *if there exists $u \in L^p(\Omega \times Q; E)$ such that, for $\mathbb{P}$-almost all $\omega \in \Omega$,*

$$\int_{Q\times E} \varphi(\xi, z)d\nu_z^\omega(\xi)dz = \int_Q \varphi\left(u^\omega(z), z\right) dz, \tag{5.48}$$

where the superscript ω denotes the dependence on ω.

Proposition 5.5. *Let $p \geq 1$, let ν be a random Young measure on the probability space $(\Omega, \mathcal{M}, \mathbb{P})$ and let $\tilde{\nu}$ be a random Young measure on the probability space $(\tilde{\Omega}, \tilde{\mathcal{M}}, \tilde{\mathbb{P}})$ such that ν and $\tilde{\nu}$ have the same law. Then ν is a L^p-random Dirac mass if and only if $\tilde{\nu}$ is.*

Proof. Assume ν is a L^p-random Dirac mass. Let $\psi : \mathbb{R}^m \to \mathbb{R}$ be a strictly convex function satisfying the growth condition

$$C_1|\xi|^p \leq |\psi(\xi)| \leq C_2(1 + |\xi|^p), \tag{5.49}$$

for some $C_1, C_2 > 0$. Since ν is a L^p-random Dirac mass, we have

$$\mathbb{E}\int_{Q\times E} \psi(\xi)d\nu_z(\xi)dz = \mathbb{E}\int_Q \psi\left(\int_E \xi d\nu_z(\xi)\right) dz \tag{5.50}$$

and both sides are finite. Since $\tilde{\nu}$ and ν have the same law, this implies

$$\tilde{\mathbb{E}} \int_{Q\times E} \psi(\xi) d\tilde{\nu}_z(\xi) dz = \tilde{\mathbb{E}} \int_Q \psi \left(\int_E \xi d\tilde{\nu}_z(\xi) \right) dz, \tag{5.51}$$

and both sides are finite. By Jensen's inequality we have

$$\int_E \psi(\xi) d\tilde{\nu}_z(\xi) \geq \psi \left(\int_E \xi d\nu_z(\xi) \right), \tag{5.52}$$

$\tilde{\mathbb{P}}$-almost surely and for almost all $z \in Q$. However (5.51) implies that (5.52) is actually an equality. But since ψ is strictly convex, this is possible only if $\tilde{\nu}_z$ is the Dirac mass $\tilde{\nu}_z = \delta_{\tilde{u}(z)}$, where

$$\tilde{u}(z) = \int_E \xi d\tilde{\nu}_z(\xi). \tag{5.53}$$

$\tilde{u}$ is measurable from $\tilde{\Omega} \times Q$ to E and, since

$$\tilde{\mathbb{E}} \int_Q |\tilde{u}(z)|^p dz \leq \tilde{\mathbb{E}} \int_Q \psi(\tilde{u}(z)) dz = \tilde{\mathbb{E}} \int_{Q\times E} \psi(\xi) d\tilde{\nu}_z(\xi) dz < +\infty \tag{5.54}$$

by (5.49) and (5.51), we have $u \in L^p(\Omega \times Q; E)$, $\tilde{u}$ is a random L^p-Dirac mass. □

5.3 Lax entropy pairs

In this section we will find conditions under which a random Young measure ν, limit of some sequence (ν^N), is supported at a single point and we will achieve this by using Tartar's commutation relation and a stochastic version of the div-curl Lemma. The fist ingredient we need is the Lax entropy pair.

Definition 5.9. *Let, for $d, n \in \mathbb{N}$, $U \subseteq \mathbb{R}_+ \times \mathbb{R}^d$, $u : U \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Let*

$$\partial_t u + \sum_{i=1}^{d} \partial_{x_i} f(u) = 0 \tag{5.55}$$

be a system of conservation laws. Two functions $\eta, q : \mathbb{R}^n \to \mathbb{R}^2$ form a Lax entropy-entropy flux pair *for* (5.55) *provided*

$$\partial_{u_i} q = \sum_{j=1}^{n} \partial_{u_i} f_j(u) \partial_{u_j} \eta, \quad 1 \leq i \leq n. \tag{5.56}$$

η is called an entropy*, while q is called its* entropy flux*.*

Remark. If u is a smooth solution of (5.55), then we obtain

$$\partial_t \eta(u(t,x)) = \sum_{i=1}^{n} \partial_{u_i}\eta \partial_t u_i = -\sum_{i=1}^{n} \partial_{u_i}\eta \sum_{k=1}^{d} \partial_{x_k} f_i(u) \tag{5.57}$$

$$= -\sum_{i=1}^{n} \partial_{u_i}\eta \sum_{k=1}^{d}\sum_{j=1}^{n} \partial_{u_j} f_i(u)\partial_{x_k} u_j = -\sum_{j=1}^{n}\sum_{k=1}^{d} \partial_{u_j} q \partial_{x_k} u_j = -\sum_{k=1}^{d} \partial_{x_k} q(u) \tag{5.58}$$

which gives us the new conservation law

$$\partial_t \eta(u) + \sum_{i=1}^{d} \partial_{x_i} q(u) = 0. \tag{5.59}$$

However, if we derive solutions to (5.55) from a microscopic *ergodic* dynamics, we do not expect to have conservation laws other than conservation of mass, momentum and energy. Therefore we expect equality (5.59) to be violated for most pairs The violation we expect, for reasons that shall be clear in the last Chapter, is the following *entropy inequality*

$$\partial_t \eta(u) + \sum_{i=1}^{d} \partial_{x_i} q(u) \leq 0. \tag{5.60}$$

Thus, we call u an *entropy solution* to (5.55) provided

$$\int_U dtdx \left(\partial_t \varphi \eta(u) + \sum_{i=1}^{d} \partial_{x_i} \varphi q(u) \right) \geq 0 \tag{5.61}$$

for all non-negative compactly supported test functions φ.

For a generic nonlinear system of conservation laws there are typically infinitely many weak solutions, even if initial conditions are given. In the case of a single conservation law, one way to eliminate the unphysical solutions and therefore to have uniqueness is indeed using entropy solutions, as they are unique (given initial conditions). However, it is unknown whether entropy solutions are unique for systems (including the p-system we are considering). Therefore we shall restrict ourselves in finding only weak solutions.

We recover the system we are considering

$$\begin{cases} \partial_t r - \partial_x p = 0 \\ \partial_t p - \partial_x \tau(r) = 0 \end{cases} \tag{5.62}$$

from (5.55) by setting $f(r,p) = (-p, -\tau(r))$. Thus, the entropy-entropy flux pair solves the following system

$$\begin{cases} \partial_r q = \partial_r f_1(r,p)\partial_r \eta + \partial_r f_2(r,p)\partial_p \eta \\ \partial_p q = \partial_p f_1(r,p)\partial_r \eta + \partial_p f_2(r,p)\partial_p \eta \end{cases} = \begin{cases} \partial_r q = -\tau'(r)\partial_p \eta \\ \partial_p q = -\partial_r \eta \end{cases} . \tag{5.63}$$

It turns out that (5.63) has infinitely many solutions (which is typical for entropy pairs), and we shall use this freedom in order to select special pairs which will, in the end, allow us to deduce that the limiting Young measure is supported at a single point.

Following the approach of Shearer in [47] and we require the following conditions on τ, which are proven in Appendix A.

i) Strict hyperbolicity: $\inf\{\tau'(\rho) : \rho \in \mathbb{R}\} > 0$;

ii) Genuine nonlinearity: $\tau''(\rho) \neq 0$ for all $\rho \in \mathbb{R}$.

iii) L^2 and L^∞ bounds: $\tau''(\rho)(\tau'(\rho))^{-5/4}$ and $\tau'''(\rho)(\tau'(\rho))^{-7/4}$ are in $L^2(\mathbb{R})$, while $\tau''(\rho)(\tau'(\rho))^{-3/4}$ and $\tau'''(\rho)(\tau'(\rho))^{-2}$ are bounded.

iv) Growth conditions: $(\tau'(\rho))^q \leq a + bF(\rho)$ for all $\rho \in \mathbb{R}$ and some constants $a, b > 0$ and $q > 1/2$, while $\tau(\rho)/F(\rho) \to 0$ as $|\rho| \to \infty$.

From the strict hyperbolicity we can define

$$z(r) := \int_0^r \sqrt{\tau'(s)}ds, \tag{5.64}$$

together with its inverse function $r(z)$. This define a smooth global transformation into a Riemann coordinate system

$$\begin{aligned} w_1(r,p) &:= p + z(r) \\ w_2(r,p) &:= p - z(r) \end{aligned}. \tag{5.65}$$

We also define two new functions Φ and Ψ of w_1 and w_2 by

$$\begin{aligned} \Phi &= (\tau')^{1/4}(\eta + q) \\ \Psi &= (\tau')^{1/4}(\eta - q) \end{aligned}. \tag{5.66}$$

Φ and Ψ then solve the following system:

$$\begin{aligned} \partial_{w_1}\Phi(w_1,w_2) &= a\left(\frac{w_1 - w_2}{2}\right)\Psi(w_1,w_2), \\ \partial_{w_2}\Psi(w_1,w_2) &= -a\left(\frac{w_1 - w_2}{2}\right)\Phi(w_1,w_2), \end{aligned} \tag{5.67}$$

where

$$a(w) := \frac{\tau''(r(w))}{8\left(\tau'(r(w))\right)^{3/2}}. \tag{5.68}$$

We fix a point $(\bar{w}_1, \bar{w}_2)$ and solve (5.67) with Goursat data given on the lines $w_1 = \bar{w}_1$ and $w_2 = \bar{w}_2$, namely we require

$$\begin{aligned} \Phi(\bar{w}_1, w_2) &= g(w_2), \\ \Psi(w_1, \bar{w}_2) &= h(w_1) \end{aligned} \tag{5.69}$$

for some $g, h \in W^{2,2} \cap W^{2,\infty}$. Note that, since $\bar{\Phi}(w_1, w_2) = \Psi(-w_2, -w_1)$ and $\bar{\Psi}(w_1, w_2) = \Phi(-w_2, -w_1)$ satisfy the same equations as (Φ, Ψ), we can set $h = 0$ without loss of generality.

We formally solve the system for (Φ, Ψ) first by integrating the second equation with respect to w_2 between $\bar{w}_2$ and w_2, obtaining

$$\Psi(w_1, w_2) = -\int_{\bar{w}_2}^{w_2} a(w_1 - y)\Phi(w_1, y)dy. \tag{5.70}$$

Then we integrate the first equation with respect to w_1 between $\bar{w}_1$ and w_1, getting

$$\Phi(w_1, w_2) = g(w_2) + \int_{\bar{w}_1}^{w_1} a(x - w_2)\Psi(x, w_2)dx. \tag{5.71}$$

Finally we substitute (5.71) in (5.70) and we iterate and obtaining the formal solution

$$\Phi(w_1, w_2) = g(w_2) + \sum_{n=1}^{\infty} (\mathcal{A}^n g)(w_1, w_2) \tag{5.72}$$

$$\Psi(w_1, w_2) = -\int_{\bar{w}_2}^{w_2} a(w_1 - y)\Phi(w_1, y)dy, \tag{5.73}$$

where

$$(\mathcal{A}f)(w_1, w_2) := -\int_{\bar{w}_1}^{w_1} \int_{\bar{w}_2}^{w_2} a(x - w_2)a(x - y)f(x, y)dxdy, \quad \forall f \in L^1_{loc}(\mathbb{R}^2).$$

At this stage one can prove that, if $a \in C^1$, then the above formal solutions are indeed classical solutions and the series is well behaved. To be precise, if we denote by

$$\mathrm{TV}[\log(\tau')](r) := \left| \int_0^r \frac{|\tau''(s)|}{\tau'(s)} ds \right|, \tag{5.74}$$

the total variation of $\log \tau'$, we have the following result, whose proof can be found in [47], Lemma 2.

Proposition 5.6. *Assume*

(i) $a \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$

(ii) $a' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$

(iii) $TV[\log\tau'](r) \le c + \log\tau'(r)$, *for some constant* c.

Then there are constants C_1, C_2 *depending respectively on the* $L^2\cap L^\infty$ *and* $W^{2,2}\cap W^{2,\infty}$ *norms of the Goursat datum* g *and also depending on* τ, $\bar{w}_1$, $\bar{w}_2$, *such that one has*

$$\begin{aligned} |\Phi| + |\Psi| &\le C_1\left(1 + (\tau'(r))^{\frac{1}{4}}\right), \\ |\mathscr{D}_w\Phi| + |\mathscr{D}_w^2\Phi| + |\mathscr{D}_w\Psi| + |\mathscr{D}_w^2\Psi| &\le C_2\left(1 + (\tau'(r))^{\frac{1}{4}}\right). \end{aligned} \tag{5.75}$$

Here $\mathscr{D}_w := (\partial_{w_1}, \partial_{w_2})$.

Remark. In our case τ' will be bounded from above, as well as from below. Therefore Ψ and Φ, together with their first and second derivatives, are bounded.

The same is then true for our original η and q. Namely, we obtain

$$\begin{aligned} |\eta| + |\partial_r\eta| + |\partial_p\eta| + |\partial^2_{rp}\eta| + |\partial^2_{rr}\eta| + |\partial^2_{pp}\eta| &\le C \\ |q| + |\partial_r 1| + |\partial_p q| + |\partial^2_{rp}q| + |\partial^2_{rr}q| + |\partial^2_{pp}q| &\le C \end{aligned}. \tag{5.76}$$

5.4 Div-curl and Stochastic Murat lemma

In this section we will present the (deterministic) div-curl lemma and a stochastic version of the Murat lemma, which shall be used in deriving Tartar equation, which in turn will imply the reduction of our limiting Young measure to a Dirac mass.

We shall need the following definitions.

Definition 5.10 (Sobolev spaces)**.** *Let* $U \subseteq \mathbb{R}^m$ *and* $1 \le p \le \infty$. *The* Sobolev space $W^{1,p}(U)$ *the set of all functions* $f : U \to \mathbb{R}$ *such that*

$$\|f\|_{W^{1,p}(U)} := \|f\|_{L^p(U)} + \sum_{i=1}^m \left\|\frac{\partial f}{\partial x_i}\right\|_{L^p(U)} < \infty, \tag{5.77}$$

where the derivatives are taken in the weak sense. We denote $W^{1,2}(U)$ *by* $H^1(U)$.

Definition 5.11 (Negative Sobolev spaces)**.** *Let* $U \subseteq \mathbb{R}^m$ *and let* $\mathcal{D}(U)$ *be the set of all test functions (smooth compactly supported functions) on* U. *For* $1 \le q < \infty$ *denote by* $W_0^{1,q}(U)$ *the closure of* $W^{1,q}(U)$ *in* $\mathcal{D}(U)$ *and equip it wit the* $W^{1,q}(U)$*-norm.*

For $1 < p \le \infty$, *we define* $W^{-1,p}(U)$ *to be the dual of* $W_0^{1,p'}(U)$. *where* $1/p + 1/p' = 1$, *and we equip it with the dual norm*

$$\|f\|_{W^{-1,p}(U)} := \sup\left\{|\langle f, g\rangle| : g \in W_0^{1,p'}(U), \|g\|_{W^{1,p}(U)} \le 1\right\}, \tag{5.78}$$

here $\langle\cdot,\cdot\rangle$ *denotes the duality pairing.*

Definition 5.12. *Let $\mathcal{M}(U)$ be the set of signed Radon (i.e. internally regular and locally finite) measures on U and let $C_0(U)$ be the set of continuous and compactly supported functions $\varphi : U \to \mathbb{R}$. The* total variation norm *of $\mu \in \mathcal{M}(U)$ is defined to be*

$$\|\mu\|_{\mathcal{M}(U)} := \sup\left\{ \int \varphi d\mu : \quad \varphi \in C_0(U), \quad \sup_{x\in U} |\varphi(x)| \leq 1 \right\}. \tag{5.79}$$

In this section, for $m \in \mathbb{N}$, $m \geq 2$, $U \subseteq \mathbb{R}^m$ is open and with C^∞ boundary ∂U. For $F = (F_1, \dots F_m) \in L^2(U; \mathbb{R}^m)$, we denote the curl of F by the $m \times m$ matrix having as (i, j) component

$$(\operatorname{curl} F)_{ij} := \partial_{x_i} F_j - \partial_{x_j} F_i \in H^{-1}(U). \tag{5.80}$$

Moreover, we denote the divergence of F by the scalar

$$\operatorname{div} F := \sum_{i=1}^{m} \partial_{x_i} F_i \in H^{-1}(U). \tag{5.81}$$

Remark. The case $m = 2$ is special, as $\operatorname{curl} F$ has only one distinct nonzero component, namely

$$(\operatorname{curl} F)_{12} = -(\operatorname{curl} F)_{21} = \partial_{x_1} F_2 - \partial_{x_2} F_1.$$

Therefore we shall set

$$\operatorname{curl} F := \partial_{x_1} F_2 - \partial_{x_2} F_1, \quad m = 2. \tag{5.82}$$

We have the following classical result, whose proof goes back to Murat [35].

Lemma 5.7 (Div-curl)**.** *Let $p, q > 1$ such that $1/p + 1/q = 1$ and let $G_N, G \in L^p(U; \mathbb{R}^m)$ and $H_N, H \in L^q(U; \mathbb{R}^m)$ with the following properties:*

i) $G_N \rightharpoonup G$ in $L^p(U; \mathbb{R}^m)$;

ii) $H_N \rightharpoonup H$ in $L^q(U; \mathbb{R}^m)$;

iii) $(\operatorname{div} G_N)_{N\in\mathbb{N}}$ lies in a compact subset of $H^{-1}(U)$;

iv) $(\operatorname{curl} H_N)_{N\in\mathbb{N}}$ lies in a compact subset of $H^{-1}(U; \mathbb{R}^{m\times m})$.

Then, $G_N \cdot H_N \to G \cdot H$ in the sense of distributions.

Conditions *iii*) and *iv*) are generally tricky to check. However, there is a sufficient condition for them to hold, and it is given by the Murat lemma, which we will state here in a stochastic setting and whose proof can be found in Appendix A of [18].

Lemma 5.8 (Stochastic Murat)**.** *Assume the following.*

(i) $(x_N)_{N>0}$ *is a stochastically bounded sequence in* $W^{-1,p}(U)$ *for some* $p > 2$*. That is, for each* $\delta > 0$ *there exists* $0 < C_\delta < +\infty$ *such that*

$$\sup_{N>0} \mathbb{P}\left(\|x_N\|_{W^{-1,p}} > C_\delta\right) < \delta. \tag{5.83}$$

(ii) $x_N = y_N + z_N$*.*

(iii) $(y_N)_{N\in\mathbb{N}}$ *is tight as a* $H^{-1}(U)$*-valued random variables.*

(iv) $(z_N)_{N\in\mathbb{N}}$*, as a signed Radon measure* $\mathcal{M}(U)$*-valued random variable, is stochastically bounded in the total variation norm* $\|\cdot\|_{\mathcal{M}(U)}$*. That is, for each* $\delta > 0$ *there exists* $0 < C_\delta < +\infty$ *such that*

$$\sup_{N>0} \mathbb{P}\left(\|z_N\|_{\mathcal{M}(U)} > C_\delta\right) < \delta. \tag{5.84}$$

Then $(x_N)_{N>0}$ *is tight as a sequence of* $H^{-1}(U)$*-valued random variables.*

Remark. In order to check assumptions *iii*) and *iv*) it is enough to show that there exist two deterministic sequences of real numbers a_N and b_N such that

$$\mathbb{E}\left|\int_U \varphi y_N\right| \leq a_N \|\varphi\|_{H^1(U)}, \qquad \mathbb{E}\left|\int_U \varphi z_N\right| \leq b_N \|\varphi\|_{L^\infty(U)}, \tag{5.85}$$

for all deterministic $\varphi \in H^1(U) \cap L^\infty(U)$, with

$$\lim_{N\to+\infty} a_N = 0, \qquad \limsup_{N\to+\infty} b_N < \infty. \tag{5.86}$$

5.5 Reduction of a limit Young measure

In this section we will show how to reduce the support of a limit Young to a point. In order to do so we first derive the Tartar's commutation relation and then we use it to prove that the limit Young measure is indeed supported at a single point.

Proposition 5.9 (Tartar's equation)**.** *Let* Q *be a compact subset of* $(0,\infty)\times(0,1)$ *and let* U *be its interior. Let* (u_N) *be a sequence of* $L^2(U)$*-valued stochastic processes on a probability space* $(\Omega,\mathbb{P})$ *and let* $\nu^N = \delta_{u_N}$ *be the random Young measure associated to* u_N*. Assume that, almost surely,* $\nu^N \to \nu$ *in* $\mathcal{Y}$*. Let* (η, q) *and* $(\tilde\eta, \tilde q)$ *be entropy pairs as in Section 5.3 and assume* $\partial_t\eta(u_N) + \partial_x q(u_N)$ *satisfy the hypothesis of Murat lemma. Then, almost surely and for a.e.* $(t,x) \in U$*,*

$$\langle\tilde\eta,\nu\rangle\langle q,\nu\rangle - \langle\eta,\nu\rangle\langle\tilde q,\nu\rangle = \langle\tilde\eta q - \eta\tilde q, \nu\rangle. \tag{5.87}$$

Proof. Since ν^N are Dirac masses, we have

$$\langle \tilde{\eta} q - \eta \tilde{q}, \nu^N \rangle = \tilde{\eta}(u_N) q(u_N) - \eta(u_N) \tilde{q}(u_N) = \langle \tilde{\eta}, \nu^N \rangle \langle q, \nu^N \rangle - \langle \eta, \nu^N \rangle \langle \tilde{q}, \nu^N \rangle. \quad (5.88)$$

Let $G_N, H_N \in L^2(U; \mathbb{R}^2)$ be defined by

$$G_N := (\tilde{\eta}(u_N), \tilde{q}(u_N)), \quad H_N := (q(u_N), -\eta(u_N)). \quad (5.89)$$

Since $\tilde{\eta}$ and $\tilde{q}$ are bounded, G_N is almost surely uniformly bounded in $L^2(U)$. Therefore, up to a subsequence, it almost surely converges weakly to some $G \in L^2(U; \mathbb{R}^2)$. But since $\nu^N \to \nu$ almost surely, we must have

$$G = (\langle \tilde{\eta}, \nu \rangle, \langle \tilde{q}, \nu \rangle). \quad (5.90)$$

Similarly, almost surely, $H_N \rightharpoonup H$ in $L^2(U; \mathbb{R}^2)$, where

$$H = (\langle q, \nu \rangle, -\langle \eta, \nu \rangle). \quad (5.91)$$

We would like to conclude that $G_N \cdot H_N \rightharpoonup G \cdot H$, but this is not true in general, as the weak limit of the product is not always equal to the product of the weak limits. Here is where the div-curl lemma comes in handy. Consider

$$\operatorname{div} G_N = \partial_t \tilde{\eta}(u_N) + \partial_x \tilde{q}(u_N), \quad \operatorname{curl} H_N = -(\partial_t \eta(u_N) + \partial_x q(u_N)), \quad (5.92)$$

so that both $\operatorname{div} G_N$ and $\operatorname{curl} H_N$ satisfy the hypothesis of Murat Lemma. Therefore, $\operatorname{div} G_N$ and $\operatorname{curl} H_N$ are tight in $H^{-1}(U)$ and so for any $\alpha > 0$ there exists $\Omega_\alpha \subset \Omega$ and a compact set $K_\alpha \subset H^{-1}(U)$ such that $\mathbb{P}(\Omega_\alpha) \geq 1 - \alpha$ and

$$\operatorname{div} G_N, \operatorname{curl} H_N \in K_\alpha \quad (5.93)$$

for all $\omega \in \Omega_\alpha$. Within Ω_α all the hypothesis of the div-curl lemma are satisfied and therefore we obtain

$$\langle \tilde{\eta}, \nu^N \rangle \langle q, \nu^N \rangle - \langle \eta, \nu^N \rangle \langle \tilde{q}, \nu^N \rangle = G_N \cdot H_N \rightharpoonup G \cdot H = \langle \tilde{\eta}, \nu \rangle \langle q, \nu \rangle - \langle \eta, \nu \rangle \langle \tilde{q}, \nu \rangle \quad (5.94)$$

in the sense of distributions. Next, we choose a sequence (α_n) which goes to zero as $n \to +\infty$ in order to obtain an increasing sequence (Ω_{α_n}) and a set

$$\hat{\Omega} = \bigcup_{n \in \mathbb{N}} \Omega_{\alpha_n}. \quad (5.95)$$

such that (5.94) holds on $\hat{\Omega}$. It follows that $\mathbb{P}(\hat{\Omega}) = 1$ and so (5.94) holds almost surely. Then the conclusion immediately follows from (5.88). □

The last step is proving that the limiting Young measure $\nu(d\xi, dz)$ is almost surely supported at a point (with respect to ξ) for almost all z. We do so by choosing choosing different entropy-flux pairs in (5.87) in order to constrain the support of ν and eventually prove it is a single point.

Proposition 5.10. *The support of the limiting Young measure ν is almost surely a point for almost every $(t, x) \in U$.*

Proof. Fix $\omega \in \hat{\Omega}$, so that (5.87) holds. At this point we are effectively in the deterministic case and we can follow step by step Lemmas 4-5-6-7 in [47] to obtain the conclusion. □

Chapter 6

Conditions of the Murat lemma

In this chapter we build a L^2-valued stochastic process such that the assumptions of the Murat lemma hold for entropy productions obtained from the entropy-entropy flux pairs obtained in Section 5.3.

6.1 Entropy production

Let, for $t \geq 0$, $\{r_k(t), p_k(t)\}_{k=1}^N$ be a solution of the system of SDEs (2.16). Let, for $N \in \mathbb{N}$,

$$l(N) = \left[N^{1/4}\sigma(N)^{1/2}\right], \tag{6.1}$$

so that

$$\lim_{N\to+\infty} \frac{l}{N} = \lim_{N\to+\infty} \frac{l^2}{N\sigma} = \lim_{N\to+\infty} \frac{N\sigma}{l^3} = 0. \tag{6.2}$$

Then, we define, for $t \geq 0$ and $x \in [0,1]$,

$$\begin{aligned} \hat{r}_N(t,x) &:= \sum_{k=l}^{N-l+1} 1_{N,k}(x)\hat{r}_{l(N),k}(t), \\ \hat{p}_N(t,x) &= \sum_{k=l}^{N-l+1} 1_{N,k}(x)\hat{p}_{l(N),k}(t), \end{aligned} \tag{6.3}$$

where $\hat{r}_{l,k}$ and $\hat{p}_{l,k}$ are defined as in (4.2) and $1_{N,k}$ is the characteristic function of the ball of center k/N and radius $1/(2N)$. Then we define

$$\hat{u}_N(t,x) = (\hat{r}_N(t,x), \hat{p}_N(t,x)). \tag{6.4}$$

Remark. From now on we shall simply write l instead of $l(N)$.

Definition 6.1. *Let $U = (0,\infty) \times (0,1)$ and let (η, q) be en entropy-entropy flux pair defined by* (5.62) *and satisfying* (5.76)*. Its corresponding* entropy production *is defined as*

$$X_N(\varphi,\eta) := -\int_0^\infty \int_0^1 \left(\partial_t\varphi(t,x)\eta(\hat{u}_N(t,x)) + \partial_x\varphi(t,x)q(\hat{u}_N(t,x))\right) dtdx, \tag{6.5}$$

for all $\varphi \in H^1(U)$.

We will need to check the conditions of the Murat lemma (Lemma 5.8) for $\partial_t\eta(\hat{u}_N) + \partial_x q(\hat{u}_N)$. The fist condition is immediately satisfied since η and q are bounded by Proposition 5.6. The remaining conditions are implied by the following

Proposition 6.1. *Let X_N be as in Definition 6.1. Then we have*

$$X_N(\varphi,\eta) = Y_N(\varphi,\eta) + Z_N(\varphi,\eta) \tag{6.6}$$

where, if $\varphi(t,x) = \phi(t,x)\psi(t,x)$, with ϕ compactly supported in U and $\psi \in H^1(U) \cap L^\infty(U)$, we have two deterministic functionals A_N and B_N depending on η only, such that

$$\mathbb{E}\left|Y_N(\psi\phi,\eta)\right| \le A_N(\phi)\left\|\psi\right\|_{H^1(U)}, \quad \mathbb{E}\left|Z_N(\psi\phi,\eta)\right| \le B_N(\phi)\left\|\psi\right\|_{L^\infty(U)} \tag{6.7}$$

with

$$\lim_{N\to+\infty} A_N(\phi) = 0, \quad \limsup_{N\to+\infty} B_N(\phi) < +\infty, \tag{6.8}$$

Remark. For the rest of this Chapter, ϕ will be supported in $[0,T]\times[a,b]$ for some $T > 0$ and $0 < a < b < 1$. Moreover, we will write H^1 and L^∞ instead of $H^1(U)$ and $L^\infty(U)$.

Lemma 6.2. *Let $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ be families of $L^2(\mathbb{R})$-valued random variables such that*

$$\mathbb{E}\left[\sum_{k=1}^N \int_0^t a_k(s)^2 ds\right] \mathbb{E}\left[\sum_{k=1}^N \int_0^t b_k(s)^2 ds\right] \tag{6.9}$$

is bounded, as $N \to +\infty$, for any $t > 0$.Let φ as in Proposition 6.1 and let

$$\varphi_k(t) = N\int_0^1 \varphi(t,x)1_{N,k}(x)dx, \tag{6.10}$$

where $1_{N,k}(x) = 1$ if $|x - k/N| \le 1/(2N)$ and $1_{N,k}(x) = 0$ otherwise. Then

$$\left|\sum_{k=1}^N \int_0^\infty \varphi_k a_k b_k dt\right| \le \|\psi\|_{L^\infty}\tilde{B}_N(\phi), \tag{6.11}$$

where $\tilde{B}_N$ is a random functional depending only on (a_k) and (b_k) such that

$$\limsup_{N\to+\infty} \mathbb{E}\left[\tilde{B}_N(\phi)\right] < +\infty. \tag{6.12}$$

Proof. First of all we note that

$$|\varphi_k| = \left| N \int_0^1 \varphi(t,x) 1_{N,k}(x) dx \right| \leq \|\varphi\|_{L^\infty} \leq c_\phi \|\psi\|_{L^\infty}, \tag{6.13}$$

where $c_\phi = \|\phi\|_{L^\infty}$ depends on ϕ only. Then, by the Cauchy-Schwarz inequality we have

$$\left| \sum_{k=1}^N \int_0^\infty \varphi_k a_k b_k dt \right| \leq \frac{\alpha}{2} c_\phi \|\psi\|_{L^\infty} \sum_{k=1}^N \int_0^T a_k^2 dt + \frac{1}{2\alpha} c_\phi \|\psi\|_{L^\infty} \sum_{k=1}^N \int_0^T b_k^2 dt, \tag{6.14}$$

for any $\alpha > 0$. Therefore, by taking the expectation value first and choosing a suitable value of α then, we obtain

$$\mathbb{E}\left[\left| \sum_{k=1}^N \int_0^\infty \varphi_k a_k b_k dt \right| \right] \leq c_\phi \|\psi\|_{L^\infty} \sqrt{\mathbb{E}\left[\sum_{k=1}^N \int_0^T a_k^2 dt \right] \mathbb{E}\left[\sum_{k=1}^N \int_0^T b_k^2 dt \right]}, \tag{6.15}$$

which leads to the conclusion. □

Lemma 6.3. *Let $(a_k)_{k\in\mathbb{N}}$ be a family of $L^2(\mathbb{R})$-valued random variables such that*

$$\lim_{N\to+\infty} \mathbb{E}\left[\frac{1}{N} \sum_{k=1}^N \int_0^t a_k(s)^2 ds \right] = 0 \tag{6.16}$$

for any $t \geq 0$. Then, for φ_k as in Lemma 6.2, we have

$$\sum_{k=1}^{N-1} \int_0^\infty a_k \nabla \varphi_k dt = Y_N^{(a)}(\varphi) + Z_N^{(a)}(\varphi), \tag{6.17}$$

where $Y_N^{(a)}$ and $Z_N^{(a)}$ depend on (a_k) only and are such that

$$|Y_N^{(a)}(\varphi)| \leq \|\psi\|_{H^1} A_N^{(a)}(\phi), \quad |Z_N^{(a)}(\varphi)| \leq \|\psi\|_{L^\infty} A_N^{(a)}(\phi), \tag{6.18}$$

where $A_N^{(a)}$ is random functional depending on (a_k) only and such that

$$\lim_{N\to+\infty} \mathbb{E}[A_N^{(a)}(\phi)] = 0. \tag{6.19}$$

The same result applies if we replace $\nabla\varphi_k$ by $\nabla^*\varphi_k$.

Proof. We consider only the case involving $\nabla\varphi_k$, as the other is analogous. Moreover, we prove the lemma only for φ smooth and compactly supported, since the result for $\varphi \in H^1 \cap L^\infty$ can be immediately recovered by approximating φ with smooth and compactly supported functions. By Cauchy-Schwarz we have

$$\left| \sum_{k=1}^{N-1} \int_0^\infty a_k \nabla \varphi_k dt \right| \leq \frac{\alpha}{2} \sum_{k=1}^{N-1} \int_0^\infty (\nabla \varphi_k)^2 dt + \frac{1}{2\alpha} \sum_{k=1}^{N-1} \int_0^T a_k^2 dt, \tag{6.20}$$

for any $\alpha > 0$. We estimate the first term. We have

$$\nabla\varphi_k = \varphi_{k+1} - \varphi_k = N\int_0^1 1_{N,k+1}(x)\varphi(t,x)dx - N\int_0^1 1_{N,k}(x)\varphi(t,x)dx \tag{6.21}$$

$$= N\int_0^1 1_{N,k}(x)\left(\varphi\left(t, x+\frac{1}{N}\right) - \varphi(t,x)\right)dx \tag{6.22}$$

$$= \int_0^1 1_{N,k}(x)\partial_x\varphi(t,x)dx + \frac{1}{2N}\int_0^1 1_{N,k}(x)\partial^2_{xx}\varphi(t,\xi)dx, \tag{6.23}$$

for some intermediate value ξ. Therefore we obtain

$$(\nabla\varphi_k)^2 \leq \frac{2}{N}\int_0^1 1_{N,k}(x)\partial_x\varphi(t,x)^2dx + \frac{1}{2N^3}\int_0^1 1_{N,k}(x)\partial^2_{xx}\varphi(t,\xi)dx. \tag{6.24}$$

By summing over k and noting that the term involving $\partial^2_{xx}\varphi$ vanishes, as $N \to +\infty$ more quickly than the term involving $\partial_x\varphi$ we have, for N large enough,

$$\int_0^\infty \sum_{k=1}^{N-1}(\nabla\varphi_k)^2dt \leq \frac{3}{N}\int_0^\infty\int_0^1 \partial_x\varphi(t,x)^2dxdt = \frac{3}{N}\|\partial_x\varphi\|_2^2. \tag{6.25}$$

Next, we note that

$$\|\partial_x\varphi\|_2^2 \leq 2\|\phi\|^2_{L^\infty}\|\partial_x\psi\|^2_{L^2} + 2\|\partial_x\phi\|_{L^2}\|\psi\|^2_{L^\infty} \leq C_\phi(\|\psi\|^2_{H^1} + \|\psi\|^2_{L^\infty}), \tag{6.26}$$

where $C_\phi = 2\max\{\|\phi\|^2_{L^\infty}, \|\partial_x\phi\|^2_{L^2}\}$ depends on ϕ only. Therefore, by choosing a suitable value of α in (6.20), we obtain

$$\left|\sum_{k=1}^{N-1}\int_0^\infty a_k\nabla\varphi_k dt\right| \leq A_N^{(a)}(\phi)\sqrt{\|\psi\|^2_{H^1} + \|\psi\|^2_{L^\infty}} \leq A_N^{(a)}(\phi)\left(\|\psi\|_{H^1} + \|\psi\|_{L^\infty}\right), \tag{6.27}$$

where

$$A_N^{(a)}(\phi) = \sqrt{\frac{C_\phi}{N}\sum_{k=1}^{N-1}\int_0^T a_k^2dt} \tag{6.28}$$

Therefore, we can find $-1 \leq \lambda \leq 1$ such that

$$\sum_{k=1}^{N-1}\int_0^\infty a_k\nabla\varphi_k dt = \lambda\|\psi\|_{H^1}A_N^{(a)}(\phi) + \lambda\|\psi\|_{L^\infty}A_N^{(a)}(\phi), \tag{6.29}$$

and the Lemma is proven, since

$$\lim_{N\to+\infty}\mathbb{E}[A_N^{(a)}(\phi)] = 0. \tag{6.30}$$

□

6.2 Evaluation of the entropy production

We are now read to prove Proposition 6.1 through a series of lemmas. The first is the following.

Lemma 6.4. *Let X_N be as in Definition 6.1. Then there exists $N_0 \in \mathbb{N}$ such that, for $N \geq N_0$,*

$$X_N = X_{a,N} + X_{s,N} + \tilde{X}_{s,N} + \mathcal{M}_N + \mathcal{N}_N, \tag{6.31}$$

where

$$\begin{aligned} X_{a,N}(\varphi,\eta) = &\int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \left(\partial_p \eta(\hat{u}_{l,k}) \nabla \hat{V}'_{l,k} - \partial_r \eta(\hat{u}_{l,k}) \nabla^* \hat{p}_{l,k}\right) dt + \\ &+ \int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \left(\partial_r q(\hat{u}_{l,k}) \nabla \hat{r}_{l,k} - \partial_p q(\hat{u}_{l,k}) \nabla^* \hat{p}_{l,k}\right) dt, \end{aligned} \tag{6.32}$$

$$\begin{aligned} X_{s,N}(\varphi,\eta) = &\sigma \int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \partial_p \eta(\hat{u}_{l,k}) \Delta \hat{p}_{l,k} dt + \\ &+ \sigma \int_0^\infty \sum_{k=l+1}^{N-l+1} \varphi_k \partial^2_{pp} \eta(\hat{u}_{l,k}) (\nabla^* d\hat{w}_{l,k})^2, \end{aligned} \tag{6.33}$$

$$\begin{aligned} \tilde{X}_{s,N}(\varphi,\eta) = &\sigma \int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \partial_r \eta(\hat{u}_{l,k}) \Delta \hat{V}'_{l,k} dt + \\ &+ \sigma \int_0^\infty \sum_{k=l+1}^{N-l+1} \varphi_k \partial^2_{rr} \eta(\hat{u}_{l,k}) (\nabla^* d\hat{\tilde{w}}_{l,k})^2, \end{aligned} \tag{6.34}$$

$$\mathcal{M}_N(\varphi,\eta) = -\sqrt{2\frac{\sigma}{N}} \int_0^\infty dt \sum_{k=l+1}^{N-l+1} \varphi_k \partial_p \eta(\hat{u}_{l,k}(s)) d\nabla^* \hat{w}_{l,k}, \tag{6.35}$$

$$\tilde{\mathcal{M}}_N(\varphi,\eta) = -\sqrt{2\frac{\sigma}{N}} \int_0^\infty dt \sum_{k=l+1}^{N-l+1} \varphi_k \partial_r \eta(\hat{u}_{l,k}(s)) d\nabla^* \hat{\tilde{w}}_{l,k} \tag{6.36}$$

and

$$\begin{aligned} \mathcal{N}_N(\varphi,\eta) = &- \int_0^\infty \int_0^1 \partial_x \varphi(t,x) q(\hat{u}_N(t,x)) dx dt + \\ &- \int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \left(\partial_r q(\hat{u}_{l,k}) \nabla \hat{r}_{l,k} - \partial_p q(\hat{u}_{l,k}) \nabla^* \hat{p}_{l,k}\right) dt. \end{aligned} \tag{6.37}$$

Proof. First of all, note that there is nothing to prove about (6.37), since the first line is one of the addends in (6.5) and the second line identically cancels the second line of (6.32).

By Itô formula we have

$$\begin{aligned} d\eta(\hat{r}_N, \hat{u}_N) &= \sum_{k=1}^{N} \left(\frac{\partial \eta}{\partial r_k} dr_k + \frac{\partial \eta}{\partial p_k} dp_k \right) + \\ &+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{\partial^2 \eta}{\partial r_i \partial r_j} dr_i dr_j + \frac{\partial^2 \eta}{\partial p_i \partial p_j} dp_i dp_j \right) \end{aligned} \tag{6.38}$$

$$= \sum_{k=1}^{N} \left(\partial_r \eta(\hat{u}_N) \frac{\partial \hat{r}_N}{\partial r_k} dr_k + \partial_p \eta(\hat{u}_N) \frac{\partial \hat{p}_N}{\partial p_k} dp_k \right) + \tag{6.39}$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\partial^2_{rr} \eta(\hat{u}_N) \frac{\partial \hat{r}_N}{\partial r_i} \frac{\partial \hat{r}_N}{\partial r_j} dr_i dr_j + \partial^2_{pp} \eta(\hat{u}_N) \frac{\partial \hat{p}_N}{\partial p_i} \frac{\partial \hat{p}_N}{\partial p_j} dp_i dp_j \right), \tag{6.40}$$

The above terms will be multiplied by φ, which is compactly supported in space. Let's then focus on the term

$$\varphi(t,x) \sum_{k=1}^{N} \partial_r \eta(\hat{u}_N) \frac{\partial \hat{r}_N}{\partial r_k} dr_k = \sum_{k=1}^{N} \partial_r \eta(\hat{u}_N) \sum_{h=l}^{N-l+1} \varphi(t,x) 1_{N,h}(x) \frac{\partial \hat{r}_{l,h}}{\partial r_k} dr_k. \tag{6.41}$$

Since $\varphi(t,\cdot)$ is supported in $[a,b]$, for some $0 < a < b < 1$, $1_{N,h}(x)\varphi(t,x)$ is non vanishing if and only if

$$Na - \frac{1}{2} \leq h \leq Nb + \frac{1}{2}. \tag{6.42}$$

Moreover, $\partial_r \eta(\hat{u}_N) 1_{N,h}(x) = \partial_r \eta(\hat{u}_{l,h}) 1_{N,h}(x)$, where $u_k = (r_k, p_k)$. Therefore we obtain

$$\varphi(t,x) \sum_{k=1}^{N} \partial_r \eta(\hat{u}_N) \frac{\partial \hat{r}_N}{\partial r_k} dr_k = \sum_{h=[Na]}^{[Nb]+1} \varphi(t,x) \partial_r \eta(\hat{u}_{l,h}) 1_{N,h}(x) \sum_{k=1}^{N} \frac{\partial \hat{r}_{l,h}}{\partial r_k} dr_k. \tag{6.43}$$

Let's now focus on the summation with respect to k. First of all we note that $\hat{r}_{l,h}$ depends only on the variables r_j with $h - l + 1 \leq j \leq h + l - 1$. Therefore, by taking into account the fact that h varies between $[Na]+1$ and $[Nb]+1$, we have that $\partial \hat{r}_{l,h}/\partial r_k$ may be non-zero only if $[Na] - l + 2 \leq k \leq [Nb] + l$. But, since $l/N \to 0$, as $N \to +\infty$, and since $0 < a < b < 1$, for N large enough $\partial \hat{r}_{l,h}/\partial r_k$ may be non zero only if $2 \leq k \leq N - 1$. For such values of k we have

$$dr_k = -N \nabla^* p_k dt + N\sigma \Delta V_k' dt - \sqrt{2N\sigma} \nabla^* d\tilde{w}_k, \tag{6.44}$$

which implies, for N large enough,

$$\sum_{k=1}^{N} \frac{\partial \hat{r}_{l,h}}{\partial r_k} dr_k = \sum_{k=2}^{N-1} \frac{\partial \hat{r}_{l,h}}{\partial r_k} dr_k = \frac{1}{l^2} \sum_{k=2}^{N-1} \sum_{|j|<l} (l - |j|) \delta_{h-j,k} dr_k = \frac{1}{l^2} \sum_{|j|<l} (l - |j|) dr_{h-j} \tag{6.45}$$

$$= -N\nabla^*\hat{p}_{l,h}dt + N\sigma\Delta\hat{V}'_{l,h}dt - \sqrt{2N\sigma}\nabla^* d\hat{\tilde{w}}_{l,k}, \tag{6.46}$$

where

$$\hat{V}'_{l,h} = \frac{1}{l^2}\sum_{|j|<l}(l-|j|)V'(r_{h-j}), \qquad d\hat{\tilde{w}}_{l,k} = \frac{1}{l^2}\sum_{|j|<l}(l-|j|)d\tilde{w}_{h-j}. \tag{6.47}$$

Applying the same procedure to the terms $\partial\hat{p}_{l,h}/\partial p_k$, using the rule

$$(dt)^2 = dtdw_k = dtd\tilde{w}_k = 0 \tag{6.48}$$

and putting everything together, we obtain

$$\begin{aligned}
\varphi d\eta = N\sum_{k=[Na]}^{[Nb]+1}\varphi(t,x)1_{N,k}(x)\left(-\partial_r\eta(\hat{u}_{l,k})\nabla^*\hat{p}_{l,k} + \partial_p\eta(\hat{u}_{l,k})\nabla\hat{V}'_{l,k}\right)dt + \\
+N\sigma\sum_{k=[Na]}^{[Nb]+1}\varphi(t,x)1_{N,k}(x)\partial_r\eta(\hat{u}_{l,k})\Delta\hat{V}'_{l,k}dt + \\
+N\sigma\sum_{k=[Na]}^{[Nb]+1}\varphi(t,x)1_{N,k}(x)\partial^2_{rr}\eta(\hat{u}_{l,k})(\nabla^* d\hat{\tilde{w}}_{l,k})^2 + \\
+N\sum_{k=[Na]}^{[Nb]+1}\varphi(t,x)1_{N,k}(x)\partial_p\eta(\hat{u}_{l,k})\Delta\hat{p}_{l,k}dt + \\
+N\sum_{k=[Na]}^{[Nb]+1}\varphi(t,x)1_{N,k}(x)\partial^2_{pp}\eta(\hat{u}_{l,k})(\nabla^* d\hat{w}_{l,k})^2 + \\
-\sqrt{2N\sigma}\sum_{k=[Na]}^{[Nb]+1}\varphi(t,x)1_{N,k}(x)\left(\partial_r\eta(\hat{u}_{l,k})\nabla^* d\hat{\tilde{w}}_{l,k} + \partial_p\eta(\hat{u}_{l,k})\nabla^* d\hat{w}_{l,k}\right).
\end{aligned} \tag{6.49}$$

The conclusion then follows once we use the support of φ once more to rewrite all the summations within the biggest range available. □

Remark. Through the rest of the chapter we will diffusively use summation by parts formulae like

$$\sum_{k=l+1}^{N-l}\varphi_k\nabla^*\hat{p}_{l,k} = \sum_{k=l+1}^{N-l}\hat{p}_{l,k}\nabla\varphi_k + \varphi_{l+1}\hat{p}_{l,l} - \varphi_{N-l+1}\hat{p}_{l,N-l} \tag{6.50}$$

The boundary terms are however identically zero, for N large enough. In fact the quantities

$$\varphi_{N-l+1}(t) = \varphi\left(t, 1-\frac{l-1}{N}\right), \quad \varphi_{l+1}(t) = \varphi\left(t, \frac{l+1}{N}\right), \tag{6.51}$$

are eventually zero since $l/N \to 0$ and $\varphi(t,\cdot)$ is compactly supported in $(0,1)$. Therefore, having always in mind that N is large enough, we omit the boundary terms altogether and simply write

$$\sum_{k=l+1}^{N-l} \varphi_k \nabla^* \hat{p}_{l,k} = \sum_{k=l+1}^{N-l} \hat{p}_{l,k} \nabla \varphi_k. \tag{6.52}$$

6.3 The antisymmetric component

We recall that

$$\begin{aligned} X_{a,N}(\varphi,\eta) = &\int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \left(\partial_p \eta(\hat{u}_{l,k}) \nabla \hat{V}'_{l,k} - \partial_r \eta(\hat{u}_{l,k}) \nabla^* \hat{p}_{l,k} \right) dt + \\ &+ \int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \left(\partial_r q(\hat{u}_{l,k}) \nabla \hat{r}_{l,k} - \partial_p q(\hat{u}_{l,k}) \nabla^* \hat{p}_{l,k} \right) dt, \end{aligned} \tag{6.53}$$

Lemma 6.5. *Let φ be as in Proposition 6.1 and let $X_{a,N}$ be as in (6.32). Then there exist two random functionals $A_{a,N}$, $B_{a,N}$ depending on η only such that the antisymmetric functional $X_{a,N}$ decomposes as $X_{a,N} = Y_{a,N} + Z_{a,N}$, where*

$$|Y_{a,N}(\varphi,\eta)| \le \|\psi\|_{H^1} A_{a,N}(\phi), \quad |Z_{a,N}(\varphi,\eta)| \le \|\psi\|_{L^\infty} B_{a,N}(\phi) \tag{6.54}$$

and

$$\lim_{N\to+\infty} \mathbb{E}[A_{a,N}(\phi)] = \lim_{N\to+\infty} \mathbb{E}[B_{a,N}(\phi)] = 0. \tag{6.55}$$

Proof. We can write

$$\begin{aligned} X_{a,N}(\varphi,\eta) = &- \int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \left(\partial_r \eta(\hat{u}_{l,k}) + \partial_p q(\hat{u}_{l,k}) \right) \nabla^* \hat{p}_{l,k} dt + \\ &+ \int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \left(\partial_p \eta(\hat{u}_{l,k}) \nabla \hat{V}'_{l,k} + \partial_r q(\hat{u}_{l,k}) \nabla \hat{r}_{l,k} \right) dt. \end{aligned} \tag{6.56}$$

We use the equations which define the entropy-entropy flux (η, q), namely

$$\begin{cases} \partial_r \eta + \partial_p q = 0 \\ \tau'(r) \partial_p \eta + \partial_r q = 0 \end{cases}. \tag{6.57}$$

to obtain

$$X_{a,N}(\varphi,\eta) = \sum_{k=l+1}^{N-l} \int_0^\infty \varphi_k \partial_p \eta(\hat{u}_{l,k}) \left(\nabla \hat{V}'_{l,k} - \tau'(\hat{r}_{l,k}) \nabla \hat{r}_{l,k} \right) dt. \tag{6.58}$$

$$= \sum_{k=l+1}^{N-l} \int_0^\infty \varphi_k \partial_p \eta(\hat{u}_{l,k}) \nabla(\hat{V}'_{l,k} - \tau(\hat{r}_{l,k}))dt + \tag{6.59}$$

$$+ \sum_{k=l+1}^{N-l} \int_0^\infty \varphi_k \partial_p \eta(\hat{u}_{l,k})(\nabla\tau(\hat{r}_{l,k}) - \tau'(\hat{r}_{l,k})\nabla\hat{r}_{l,k})dt. \tag{6.60}$$

Let's focus on (6.59). After a summation by parts we obtain

$$\sum_{k=l+1}^{N-l} \int_0^\infty \varphi_k \partial_p \eta(\hat{u}_{l,k}) \nabla(\hat{V}'_{l,k} - \tau(\hat{r}_{l,k}))dt = Q_{a,N}(\varphi, \eta) + Z_{a,N}(\varphi, \eta), \tag{6.61}$$

where

$$Q_{a,N}(\varphi, \eta) = \int_0^\infty \sum_{k=l+1}^{N-l} (\nabla^* \varphi_k) \partial_p \eta(\hat{u}_{l,k})(\hat{V}'_{l,k} - \tau(\hat{r}_{l,k}))dt \tag{6.62}$$

and

$$Z_{a1,N}(\varphi, \eta) = \int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k (\nabla^* \partial_p \eta(\hat{u}_{l,k}))(\hat{V}'_{l,k} - \tau(\hat{r}_{l,k}))dt. \tag{6.63}$$

Let's deal with $Q_{a,N}$. $\partial_p \eta$ is bounded; moreover, Lemma 4.3 implies

$$\lim_{N\to+\infty} \frac{1}{N} \int_0^T \sum_{k=l+1}^{N-l} \int (\hat{V}'_{l,k} - \tau(\hat{r}_{l,k}))^2 d\mu_t^N dt = 0, \tag{6.64}$$

for any $T > 0$. Therefore, Lemma 6.3 applies to $Q_{a,N}$, yielding

$$Q_{a,N}(\varphi, \eta) = Y_{a,N}(\varphi, \eta) + Z_{aQ,N}(\varphi, \eta), \tag{6.65}$$

where

$$|Y_{a,N}(\varphi, \eta)| \leq \|\psi\|_{H^1} A_{a,N}(\phi), \quad |Z_{Qa,N}(\varphi, \eta)| \leq \|\psi\|_{L^\infty} A_{a,N}(\phi), \tag{6.66}$$

for some random functions $A_{a,N}$ depending on η only and such that

$$\lim_{N\to+\infty} \mathbb{E}[A_{a,N}(\phi)] = 0. \tag{6.67}$$

Let's now consider $Z_{a1,N}$. Since the second derivatives of η are bounded, we have

$$(\nabla^* \partial_p \eta(\hat{u}_{l,k}))^2 \leq C((\nabla^* \hat{r}_{l,k})^2 + (\nabla^* \hat{p}_{l,k})^2), \tag{6.68}$$

for some $C > 0$. Furthermore, Lemma 4.4 implies

$$\left(\int_0^T \sum_{k=l+1}^{N-l} \int (\hat{r}_{l,k} - \hat{r}_{l,k-1})^2 d\mu_t^N dt\right)\left(\int_0^T \sum_{k=l+1}^{N-l} \int (\hat{V}'_{l,k} - \tau(\hat{r}_{l,k}))^2 d\mu_t^N dt\right) +$$
$$+ \left(\int_0^T \sum_{k=l+1}^{N-l} \int (\hat{r}_{l,k} - \hat{r}_{l,k-1})^2 d\mu_t^N dt\right)\left(\int_0^T \sum_{k=l+1}^{N-l} \int (\hat{V}'_{l,k} - \tau(\hat{r}_{l,k}))^2 d\mu_t^N dt\right) \tag{6.69}$$
$$\leq C\left(\frac{NT}{l^2} + \frac{l}{\sigma}\right)^2,$$

which vanishes as $N \to +\infty$ for any $T > 0$. Therefore, Lemma 6.2 applies to $Z_{a1,N}$, yielding

$$|Z_{a1,N}(\varphi, \eta)| \leq \|\psi\|_{L^\infty} B_{a1,N}(\phi), \tag{6.70}$$

where $B_{a1,N}$ is a functional depending only on η such that

$$\lim_{N \to +\infty} \mathbb{E}[B_{a1,N}(\phi)] = 0. \tag{6.71}$$

We now have to deal with (6.60), so we define

$$Z_{a2,N}(\varphi, \eta) = \sum_{k=l+1}^{N-l} \int_0^\infty \varphi_k \partial_p \eta(\hat{u}_{l,k})(\nabla \tau(\hat{r}_{l,k}) - \tau'(\hat{r}_{l,k}) \nabla \hat{r}_{l,k}) dt. \tag{6.72}$$

We write

$$\nabla \tau(\hat{r}_{l,k}) - \tau'(\hat{r}_{l,k}) \nabla \hat{r}_{l,k} = (\tau'(\tilde{r}_{l,k}) - \tau'(\hat{r}_{l,k})) \nabla \hat{r}_{l,k} = \tau''(\tilde{\tilde{r}}_{l,k})(\tilde{r}_{l,k} - \hat{r}_{l,k}) \nabla \hat{r}_{l,k}, \tag{6.73}$$

where $\tilde{r}_{l,k}$ is between $\hat{r}_{l,k+1}$ and $\hat{r}_{l,k}$. and $\tilde{\tilde{r}}_{l,k}$ is between $\hat{r}_{l,k}$ and $\tilde{r}_{l,k}$. With this in mind and using the fact (from Appendix A) that τ'' is bounded, we obtain

$$|\nabla \tau(\hat{r}_{l,k}) - \tau'(\hat{r}_{l,k}) \nabla \hat{r}_{l,k}| \leq \|\tau''\|_{L^\infty} |\nabla \hat{r}_{l,k}|^2. \tag{6.74}$$

Finally, since, for any $T > 0$,

$$\sum_{k=l+1}^{N-l} \int_0^T \int (\hat{r}_{l,k+1} - \hat{r}_{l,k})^2 d\mu_t^N dt \leq C \left(\frac{NT}{l^3} + \frac{1}{\sigma} \right) \to 0, \tag{6.75}$$

we obtain

$$|Z_{a2,N}(\varphi, \eta)| \leq \|\psi\|_{L^\infty} B_{a2,N}(\phi), \tag{6.76}$$

where $B_{a2,N}$ depends on η only and

$$\lim_{N \to +\infty} \mathbb{E}[B_{a2,N}(\phi)] = 0. \tag{6.77}$$

Putting everything together, we have obtained

$$X_{a,N} = Y_{a,N} + Z_{a,N}, \tag{6.78}$$

where

$$Z_{a,N} = Z_{Qa,N} + Z_{a1,N} + Z_{a2,N}, \tag{6.79}$$

and $Y_{a,N}$ and $Z_{a,N}$ have the claimed properties. □

6.4 The symmetric components

We discuss only the case of

$$\begin{aligned}\tilde{X}_{s,N}(\varphi,\eta) = \sigma\int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \partial_r\eta(\hat{u}_{l,k})\Delta\hat{V}'_{l,k}dt+\\ +\sigma\int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \partial^2_{rr}\eta(\hat{u}_{l,k})(\nabla^* d\hat{\tilde{w}}_{l,k})^2,\end{aligned} \tag{6.80}$$

as the statement for $X_{s,N}$ would follow from the same arguments once one replaces V' by p.

Lemma 6.6. *Let φ be as in Proposition 6.1 and let $\tilde{X}_{s,N}$ be as in* (6.34)*. Then there exists three random functionals $\tilde{A}_{s,N}$, $\tilde{B}_{s,N}$ and $\tilde{B}^*_{s,N}$, depending on η only such that the symmetric functional $\tilde{X}_{s,N}$ decomposes as $\tilde{X}_{s,N} = \tilde{Y}_{s,N} + \tilde{Z}_{s,N} + \tilde{Z}^*_{s,N}$, where*

$$\begin{gathered}|\tilde{Y}_{s,N}(\varphi,\eta)| \le \|\psi\|_{H^1}\tilde{A}_{s,N}(\phi), \quad |\tilde{Z}_{s,N}(\varphi,\eta)| \le \|\psi\|_{L^\infty}\tilde{B}_{s,N}(\phi),\\ |\tilde{Z}^*_{s,N}(\varphi,\eta)| \le \|\psi\|_{L^\infty}\tilde{B}^*_{s,N}(\phi)\end{gathered} \tag{6.81}$$

and

$$\lim_{N\to+\infty}\mathbb{E}[\tilde{A}_{s,N}(\phi)] = \lim_{N\to+\infty}\mathbb{E}[\tilde{B}_{s,N}(\phi)] = 0,\ \limsup_{N\to+\infty}\mathbb{E}[\tilde{B}^*_{s,N}(\phi)] < +\infty. \tag{6.82}$$

Proof. We look at the term involving V', first. We have

$$\begin{aligned}\sigma\int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \partial_r\eta(\hat{u}_{l,k})\Delta\hat{V}'_{l,k}dt &= -\sigma\int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \partial_r\eta(\hat{u}_{l,k})\nabla^*\nabla\hat{V}'_{l,k}dt\\ &= \tilde{Q}_{s,N}(\varphi,\eta) + \tilde{Z}^*_{s,N}(\varphi,\eta),\end{aligned} \tag{6.83}$$

where

$$\tilde{Q}_{s,N}(\varphi,\eta) = -\sigma\int_0^\infty \sum_{k=l+1}^{N-l} (\nabla^*\varphi_k)\partial_r\eta(\hat{u}_{l,k})\nabla\hat{V}'_{l,k}dt \tag{6.84}$$

and

$$\tilde{Z}^*_{s,N}(\varphi,\eta) = -\sigma\int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k(\nabla^*\partial_r\eta(\hat{u}_{l,k}))\nabla\hat{V}'_{l,k}dt. \tag{6.85}$$

Since $\partial_r\eta$ is bounded and

$$\lim_{N\to+\infty}\frac{\sigma^2}{N}\int_0^T \sum_{k=l+1}^{N-l}\int(\hat{V}'_{l,k+1} - \hat{V}'_{l,k})^2 d\mu^N_t dt \le C\lim_{N\to+\infty}\left(\frac{\sigma^2}{l^3}T + \frac{\sigma}{N}\right) = 0 \tag{6.86}$$

for any $T > 0$, Lemma 6.3 applies to $\tilde{Q}_{s,N}$, yielding

$$\tilde{Q}_{s,N}(\varphi,\eta) = \tilde{Y}_{s,N}(\varphi,\eta) + \tilde{Z}_{Qs,N}(\varphi,\eta), \tag{6.87}$$

where

$$|\tilde{Y}_{s,N}(\varphi,\eta)| \leq \|\psi\|_{H^1}\tilde{A}_{s,N}(\phi), \quad |\tilde{Z}_{sQ,N}(\varphi,\eta)| \leq \|\psi\|_{L^\infty}\tilde{A}_{s,N}(\phi) \tag{6.88}$$

with

$$\lim_{N\to+\infty}\mathbb{E}[\tilde{A}_{s,N}(\phi)] = 0. \tag{6.89}$$

From

$$(\nabla^*\partial_r\eta(\hat{u}_{l,k}))^2 \leq C((\nabla^*\hat{r}_{l,k})^2 + (\nabla^*\hat{p}_{l,k})^2) \tag{6.90}$$

and

$$\begin{aligned}\left(\sigma\int_0^T\sum_{k=l+1}^{N-l}\int(\hat{r}_{l,k}-\hat{r}_{l,k-1})^2d\mu_t^N dt\right)\left(\sigma\int_0^T\sum_{k=l+1}^{N-l}\int(\hat{V}'_{l,k+1}-\hat{V}'_{l,k})^2d\mu_t^N dt\right)+\\ +\left(\sigma\int_0^T\sum_{k=l+1}^{N-l}\int(\hat{p}_{l,k}-\hat{p}_{l,k-1})^2d\mu_t^N dt\right)\left(\sigma\int_0^T\sum_{k=l+1}^{N-l}\int(\hat{V}'_{l,k+1}-\hat{V}'_{l,k})^2d\mu_t^N dt\right)\\ \leq C\left(\frac{N\sigma}{l^3}T+1\right)^2,\end{aligned} \tag{6.91}$$

which stays bounded as $N \to +\infty$ for any $T > 0$, Lemma 6.2 applies to $\tilde{Z}^*_{s,N}$. Therefore, we have

$$|\tilde{Z}^*_{s,N}(\varphi,\eta)| \leq \|\psi\|_{L^\infty}\tilde{B}^*_{s,N}(\phi), \tag{6.92}$$

for some random functional $\tilde{B}^*_{s,N}$ depending on η only and such that

$$\limsup_{N\to+\infty}\mathbb{E}[\tilde{B}^*_{s,N}(\phi)] < +\infty. \tag{6.93}$$

We have now to estimate the term involving the Brownian motions, namely

$$\tilde{Z}_{s1,N}(\varphi,\eta) = \sigma\sum_{k=l+1}^{N-l}\int_0^\infty\varphi_k\partial^2_{rr}\eta(\hat{u}_{l,k})(\nabla^*d\hat{\tilde{w}}_{l,k})^2. \tag{6.94}$$

Since $\tilde{w}_k$ are independent Brownian motions, we have

$$(d\hat{\tilde{w}}_{l,k} - d\hat{\tilde{w}}_{l,k-1})^2 = \frac{1}{l^2}(d\bar{\tilde{w}}_{l,k-1+l} - d\bar{\tilde{w}}_{l,k-1})^2 \tag{6.95}$$

$$= \frac{1}{l^4}\left(\sum_{j=0}^{l-1}d\tilde{w}_{k-1+l-j} - \sum_{j=0}^{l-1}d\tilde{w}_{k-1-j}\right)^2 = \frac{2}{l^3}dt, \tag{6.96}$$

where we have used the rule $d\tilde{w}_i d\tilde{w}_j = \delta_{i,j} dt$. Therefore we obtain

$$|\tilde{Z}_{s1,N}(\varphi,\eta)| \leq C\frac{\sigma}{l^3}\int_0^\infty \sum_{k=l+1}^{N-l} |\varphi_k| dt \leq C\frac{N\sigma}{l^3}\|\psi\|_{L^\infty}. \tag{6.97}$$

Since $N\sigma/l^3 \to 0$, as $N \to +\infty$, the proof is completed if we set

$$\tilde{Z}_{s,N} = \tilde{Z}_{Qs,N} + \tilde{Z}_{s1,N}. \tag{6.98}$$

□

We end this section by stating the corresponding lemma for $X_{s,N}$, whose proof is exactly the same as the one we have just presented.

Lemma 6.7. *Let φ be as in Proposition 6.1 and let $X_{s,N}$ be as in* (6.33)*. Then there exists three random functionals $A_{s,N}$, $B_{s,N}$ and $B^*_{s,N}$, depending on η only such that the symmetric functional $X_{s,N}$ decomposes as $X_{s,N} = Y_{s,N} + Z_{s,N} + Z^*_{s,N}$, where*

$$\begin{gathered} |Y_{s,N}(\varphi,\eta)| \leq \|\psi\|_{H^1} A_{s,N}(\phi), \quad |Z_{s,N}(\varphi,\eta)| \leq \|\psi\|_{L^\infty} B_{s,N}(\phi), \\ |Z^*_{s,N}(\varphi,\eta)| \leq \|\psi\|_{L^\infty} B^*_{s,N}(\phi) \end{gathered} \tag{6.99}$$

and

$$\lim_{N\to+\infty} \mathbb{E}[A_{s,N}(\phi)] = \lim_{N\to+\infty} \mathbb{E}[B_{s,N}(\phi)] = 0, \quad \limsup_{N\to+\infty} \mathbb{E}[B^*_{s,N}(\phi)] < +\infty. \tag{6.100}$$

6.5 The martingales

As with the symmetric components, we discuss only the case

$$\mathcal{M}_N(\varphi,\eta) = -\sqrt{2\frac{\sigma}{N}}\int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \partial_p \eta(\hat{u}_{l,k}) \nabla^* d\hat{w}_{l,k}. \tag{6.101}$$

$\tilde{\mathcal{M}}_N$ is treated analogously and the corresponding result can be found at the end of this section..

Lemma 6.8. *Let φ be as in Proposition 6.1 and let $\mathcal{M}_N$ be as in* (6.35)*. Then there exists two deterministic functionals $A_{\mathcal{M},N}$ and $B_{\mathcal{M},N}$ depending on η only such the martingale $\mathcal{M}_N$ decomposes as $\mathcal{M}_N = Y_{\mathcal{M},N} + Z_{\mathcal{M},N}$, where*

$$\mathbb{E}|Y_{\mathcal{M},N}(\varphi,\eta)| \leq \|\varphi\|_{H^1} A_{\mathcal{M},N}(\phi), \quad \mathbb{E}|Z_{\mathcal{M},N}(\varphi,\eta)| \leq \|\varphi\|_{L^\infty} B_{\mathcal{M},N}(\phi) \tag{6.102}$$

and

$$\lim_{N\to+\infty} A_{\mathcal{M},N}(\phi) = \lim_{N\to+\infty} B_{\mathcal{M},N}(\phi) = 0. \tag{6.103}$$

Proof. Recalling $\varphi = \phi\psi$ and summing by parts, we obtain $\mathcal{M}_N = Y_{\mathcal{M},n} + Z_{\mathcal{M}1,N} + Z_{\mathcal{M}1,N}$, where

$$Y_{\mathcal{M},N}(\varphi,\eta) = -\sqrt{2\frac{\sigma}{N}}\int_0^\infty \sum_{k=l+1}^{N-l} \phi_k(\nabla\psi_k)\partial_p\eta(\hat{u}_{l,k})d\hat{w}_{l,k}, \tag{6.104}$$

$$Z_{\mathcal{M}1,N}(\varphi,\eta) = -\sqrt{2\frac{\sigma}{N}}\int_0^\infty \sum_{k=l+1}^{N-l} \psi_k(\nabla\phi_k)\partial_p\eta(\hat{u}_{l,k})d\hat{w}_{l,k} \tag{6.105}$$

and

$$Z_{\mathcal{M}2,N}(\varphi,\eta) = -\sqrt{2\frac{\sigma}{N}}\int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k(\nabla\partial_p\eta(\hat{u}_{l,k}))d\hat{w}_{l,k}. \tag{6.106}$$

We have used the fact that $\varphi_k = \phi_k\psi_k + \|\psi\|_{L^\infty}\mathcal{O}(1/N^2)$. Recalling the definition

$$d\hat{w}_{l,k} = \frac{1}{l}\sum_{|j|<l}\frac{l-|j|}{l}dw_{k-j}, \tag{6.107}$$

we write

$$|Y_{\mathcal{M},N}(\varphi,\eta)| = \sqrt{2\frac{\sigma}{N}}\sqrt{\left(\int_0^\infty \sum_{k=l+1}^{N-l} \phi_k(\nabla\psi_k)\partial_p\eta(\hat{u}_{l,k})d\hat{w}_{l,k}\right)^2} \tag{6.108}$$

$$\leq \sqrt{2\frac{\sigma}{N}}\sqrt{N\frac{1}{l^2}\sum_{k=l+1}^{N-l}\left(\sum_{|j|<l}\int_0^\infty \phi_k(\nabla\psi_k)\partial_p\eta(\hat{u}_{l,k})\frac{l-|j|}{l}dw_{k-j}\right)^2}. \tag{6.109}$$

Now we write

$$\left(\sum_{|j|<l}\int_0^\infty \phi_k(\nabla\psi_k)\partial_p\eta(\hat{u}_{l,k})\frac{l-|j|}{l}dw_{k-j}\right)^2$$

$$= \sum_{|j|<l}\left(\int_0^\infty \phi_k(\nabla\psi_k)\partial_p\eta(\hat{u}_{l,k})\frac{l-|j|}{l}dw_{k-j}\right)^2 + \tag{6.110}$$

$$+\sum_{i\neq j}\left(\int_0^\infty \phi_k(\nabla\psi_k)\partial_p\eta(\hat{u}_{l,k})\frac{l-|i|}{l}dw_{k-i}\right)\left(\int_0^\infty \phi_k(\nabla\psi_k)\partial_p\eta(\hat{u}_{l,k})\frac{l-|j|}{l}dw_{k-j}\right).$$

This, together with Itô isometry (recall that (w_i) are independent Brownian motions)

$$\mathbb{E}\left[\left(\int_0^T X(t)dw_i(t)\right)\left(\int_0^T Y(t)dw_j(t)\right)\right] = \delta_{i,j}\mathbb{E}\left[\int_0^T X(t)Y(t)dt\right] \tag{6.111}$$

implies

$$\mathbb{E}|Y_{\mathcal{M},N}(\varphi,\eta)| \leq \sqrt{\frac{2\sigma}{l^2}\sum_{k=l+1}^{N-l}\sum_{|j|<l}\mathbb{E}\int_0^\infty \phi_k^2(\nabla\psi_k)^2(\partial_p\eta(\hat{u}_{l,k}))^2\left(\frac{l-|j|}{l}\right)^2 dt}.$$

Using the fact that $\partial_p \eta$ and ϕ are bounded we then obtain

$$\mathbb{E}|Y_{\mathcal{M},N}(\varphi,\eta)| \leq C_{\eta,\phi}\sqrt{\frac{\sigma}{Nl}\frac{1}{N}\sum_{k=l+1}^{N-l}\int_0^\infty N^2(\nabla\psi_k)^2 dt} \tag{6.112}$$

$$\leq C_{\eta,\phi}\sqrt{\frac{\sigma}{Nl}}\|\partial_x\psi\|_{L^2} \leq C_{\eta,\phi}\sqrt{\frac{\sigma}{Nl}}\|\psi\|_{H^1}, \tag{6.113}$$

and the coefficient of $\|\psi\|_{H^1}$ vanishes as $N \to +\infty$. Here and in the following, $C_{\eta,\phi}$ may denote different constants, but all depending only on η and ϕ. Similarly, we obtain

$$\mathbb{E}|Z_{\mathcal{M}1,N}(\varphi,\eta)| \leq C_{\eta,\phi}\sqrt{\frac{\sigma}{Nl}}\|\psi\|_{L^\infty}. \tag{6.114}$$

Finally, recalling that $\varphi(\cdot,x)$ is supported in $[0,T]$, we estimate

$$\mathbb{E}|Z_{\mathcal{M}1,N}(\varphi,\eta)| \leq \|\psi\|_{L^\infty}C_{\eta,\phi}\sqrt{\frac{\sigma}{l}\mathbb{E}\sum_{k=l+1}^{N-l}\int_0^T(\nabla\hat{p}_{l,k})^2 dt} \leq C_{\eta,\phi}\sqrt{\frac{1}{l}}. \tag{6.115}$$

Where the last inequality follows from the two block estimate and our choice of l (see equations (4.57) and (4.63)). □

Lemma 6.9. *Let φ be as in Proposition 6.1 and let $\tilde{\mathcal{M}}_N$ be as in* (6.36)*. Then there exists two deterministic functionals $\tilde{A}_{\mathcal{M},N}$ and $\tilde{B}_{\mathcal{M},N}$ depending on η only such the martingale $\tilde{\mathcal{M}}_N$ decomposes as $\tilde{\mathcal{M}}_N = \tilde{Y}_{\mathcal{M},N} + \tilde{Z}_{\mathcal{M},N}$, where*

$$\mathbb{E}|\tilde{Y}_{\mathcal{M},N}(\varphi,\eta)| \leq \|\varphi\|_{H^1}\tilde{A}_{\mathcal{M},N}(\phi), \quad \mathbb{E}|\tilde{Z}_{\mathcal{M},N}(\varphi,\eta)| \leq \|\varphi\|_{L^\infty}\tilde{B}_{\mathcal{M},N}(\phi) \tag{6.116}$$

and

$$\lim_{N\to+\infty}\tilde{A}_{\mathcal{M},N}(\phi) = \lim_{N\to+\infty}\tilde{B}_{\mathcal{M},N}(\phi) = 0. \tag{6.117}$$

6.6 The space derivative discretisation

We recall that

$$\begin{aligned}\mathcal{N}_N(\varphi,\eta) = &-\int_0^\infty\int_0^1 \partial_x\varphi(t,x)q(\hat{u}_N(t,x))dxdt + \\ &-\int_0^\infty\sum_{k=l+1}^{N-l}\varphi_k\left(\partial_r q(\hat{u}_{l,k})\nabla\hat{r}_{l,k} - \partial_p q(\hat{u}_{l,k})\nabla^*\hat{p}_{l,k}\right)dt.\end{aligned} \tag{6.118}$$

is the error we make when we integrate by parts and replace the x-derivative of q by its discrete version.

Lemma 6.10. *Let φ be as in Proposition 6.1 and let $\mathcal{N}_N$ be as in* (6.37)*. Then there exists two random functionals $A_{n,N}$ and $B_{n,N}$, depending on η only such that the space derivative discretisation $\mathcal{N}_N$ decomposes as $\mathcal{N}_N = Y_{n,N} + Z_{n,N}$, where*

$$|Y_{n,N}(\varphi,\eta)| \leq \|\psi\|_{H^1} A_{n,N}(\phi), \quad |Z_{n,N}(\varphi,\eta)| \leq \|\psi\|_{L^\infty} B_{n,N}(\phi), \tag{6.119}$$

and

$$\lim_{N\to+\infty} \mathbb{E}[A_{n,N}(\phi)] = \lim_{N\to+\infty} \mathbb{E}[B_{n,N}(\phi)] = 0. \tag{6.120}$$

Proof. As in the proof of Lemma 6.3, we prove the statement for φ smooth and compactly supported, and the general statement for $\varphi \in H^1 \cap L^\infty$ will follow by approximating φ with smooth and compactly supported functions.

First of all we write

$$-\int_0^\infty \int_0^1 \partial_x\varphi(t,x) q(\hat{u}_N(t,x))dxdt \tag{6.121}$$

$$= -\int_0^\infty \sum_{k=l+1}^{N-l} \left(\int_0^1 \partial_x\varphi(t,x) 1_{N,k}(x)dx \right) q(\hat{u}_{l,k})dt \tag{6.122}$$

$$= -\int_0^\infty \sum_{k=l+1}^{N-l} \left(\varphi\left(t, \frac{k}{N} + \frac{1}{2N}\right) - \varphi\left(t, \frac{k}{N} - \frac{1}{2N}\right) \right) q(\hat{u}_{l,k})dt \tag{6.123}$$

$$= \int_0^\infty \sum_{k=l+1}^{N-l} \varphi\left(t, \frac{k}{N} - \frac{1}{2N}\right) \nabla q(\hat{u}_{l,k})dt \tag{6.124}$$

$$= \int_0^\infty \sum_{k=N_a}^{N_b} \tilde{\varphi}_k \nabla q(\hat{u}_{l,k})dt \tag{6.125}$$

$$= \int_0^\infty \sum_{k=l+1}^{N-l} \tilde{\varphi}_k (\partial_r q(\tilde{u}_{l,k}) \nabla \hat{r}_{l,k} + \partial_p q(\tilde{u}_{l,k}) \nabla \hat{p}_{l,k}), \tag{6.126}$$

for some $\tilde{u}_{l,k}$ between $\hat{u}_{l,k}$ and $\hat{u}_{l,k+1}$ and where

$$\tilde{\varphi}_k(t) := \varphi\left(t, \frac{k}{N} - \frac{1}{2N}\right). \tag{6.127}$$

Putting everything together, we obtain

$$\begin{aligned} \mathcal{N}_N(\varphi,\eta) = &\int_0^\infty \sum_{k=l+1}^{N-l} \tilde{\varphi}_k (\partial_r q(\tilde{u}_{l,k}) \nabla \hat{r}_{l,k} + \partial_p q(\tilde{u}_{l,k}) \nabla \hat{p}_{l,k}) dt + \\ &- \int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \left(\partial_r q(\hat{u}_{l,k}) \nabla \hat{r}_{l,k} - \partial_p q(\hat{u}_{l,k}) \nabla^* \hat{p}_{l,k} \right) dt \end{aligned} \tag{6.128}$$

$$= \mathcal{N}_{r,N}(\varphi,\eta) + \mathcal{N}_{p,N}(\varphi,\eta) + \mathcal{N}_{1,N}(\varphi,\eta), \tag{6.129}$$

where

$$\mathcal{N}_{r,N}(\varphi,\eta) = \int_0^\infty \sum_{k=l+1}^{N-l} (\tilde{\varphi}_k \partial_r q(\tilde{u}_{l,k}) - \varphi_k \partial_r q(\hat{u}_{l,k})) \nabla \hat{r}_{l,k} dt, \tag{6.130}$$

$$\mathcal{N}_{p,N}(\varphi,\eta) = \int_0^\infty \sum_{k=l+1}^{N-l} (\tilde{\varphi}_k \partial_p q(\tilde{u}_{l,k}) - \varphi_k \partial_p q(\hat{u}_{l,k})) \nabla \hat{p}_{l,k} dt + \tag{6.131}$$

and

$$\mathcal{N}_{1,N}(\varphi,\eta) = \int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \partial_p q(\hat{u}_{l,k}) (\nabla \hat{p}_{l,k} + \nabla^* \hat{p}_{l,k}) dt. \tag{6.132}$$

Since

$$\nabla \hat{p}_{l,k} + \nabla^* \hat{p}_{l,k} = \hat{p}_{l,k+1} + \hat{p}_{l,k-1} - 2\hat{p}_{l,k} = \Delta \hat{p}_{l,k}, \tag{6.133}$$

the term $\mathcal{N}_{1,N}$ can be estimated exactly in the same way we estimated

$$\sigma \int_0^\infty \sum_{k=l+1}^{N-l} \varphi_k \partial_p \eta(\hat{u}_{l,k}) \Delta \hat{V}'_{l,k} dt \tag{6.134}$$

in Lemma 6.6. The main difference here is that in $\mathcal{N}_{1,N}$ does not have a σ in front, and therefore we obtain

$$\mathcal{N}_{1,N}(\varphi,\eta) = Y_{n1,N}(\varphi,\eta) + Z_{n1,N}(\varphi,\eta), \tag{6.135}$$

where

$$|Y_{n1,N}(\varphi,\eta)| \leq \|\psi\|_{H^1} A_{n1,N}(\phi), \quad |Z_{n1,N}(\varphi,\eta)| \leq \|\psi\|_{H^1} B_{n1,N}(\phi), \tag{6.136}$$

for some random functionals $A_{n1,N}$ and $B_{n1,N}$ depending on η only and such that

$$\lim_{N\to+\infty} \mathbb{E}[A_{n1,N}(\phi)] = \lim_{N\to+\infty} \mathbb{E}[B_{n1,N}(\phi)] = 0. \tag{6.137}$$

We are left with estimating $\mathcal{N}_{r,N}$ and $\mathcal{N}_{p,N}$. We only evaluate $\mathcal{N}_{r,N}$, as $\mathcal{N}_{p,N}$ is dealt with in a similar way. From (6.130), we evaluate

$$\tilde{\varphi}_k \partial_r q(\tilde{u}_{l,k}) - \varphi_k \partial_r q(\hat{u}_{l,k}) = (\tilde{\varphi}_k - \varphi_k) \partial_r q(\hat{u}_{l,k}) + \tilde{\varphi}_k (\partial_r q(\tilde{u}_{l,k}) - \partial_r q(\hat{u}_{l,k})), \tag{6.138}$$

which implies $\mathcal{N}_{r,N} = \mathcal{N}_{r1,N} + \mathcal{N}_{r2,N}$, where

$$\mathcal{N}_{r1,N}(\varphi,\eta) = \int_0^\infty \sum_{k=l+1}^{N-l} (\tilde{\varphi}_k - \varphi_k) \partial_r q(\hat{u}_{l,k}) \nabla \hat{r}_{l,k} dt \tag{6.139}$$

and

$$\mathcal{N}_{r2,N}(\varphi,\eta)=\int_0^\infty\sum_{k=l+1}^{N-l}\tilde{\varphi}_k(\partial_r q(\tilde{u}_{l,k})-\partial_r q(\hat{u}_{l,k}))\nabla\hat{r}_{l,k}dt. \tag{6.140}$$

Since $|\tilde{\varphi}_k|\leq C\|\psi\|_{L^\infty}$, doing estimates identical to the ones done in the proof of Lemma 6.6, we can write $\mathcal{N}_{r2,N}=Y_{r2,N}+Z_{r2,N}$, where

$$|Y_{r2,N}(\varphi,\eta)|\leq\|\psi\|_{H^1}A_{r2,N}(\phi),\quad |Z_{r2,N}(\varphi,\eta)|\leq\|\psi\|_{L^\infty}A_{r2,N}(\phi), \tag{6.141}$$

for some random functional $A_{r2,N}$ depending on η only and such that

$$\lim_{N\to+\infty}\mathbb{E}[A_{r2,N}(\phi)]=0. \tag{6.142}$$

Let's now estimate $\mathcal{N}_{r1,N}$, given by (6.139). We write

$$\tilde{\varphi}_k-\varphi_k=\varphi\left(t,\frac{k}{N}-\frac{1}{2N}\right)-N\int_0^1 1_{N,k}(x)\varphi(t,x)dx \tag{6.143}$$

$$=N\int_0^1 1_{N,k}(x)\left(\varphi\left(t,\frac{k}{N}-\frac{1}{2N}\right)-\varphi(t,x)\right)dx \tag{6.144}$$

$$\begin{aligned}&=N\int_0^1 1_{N,k}(x)\partial_x\varphi(t,x)\left(\frac{k}{N}-\frac{1}{2N}-x\right)dx+\\&+N\int_0^1 1_{N,k}(x)\partial^2_{xx}\varphi(t,\xi)\left(\frac{k}{N}-\frac{1}{2N}-x\right)^2dx,\end{aligned} \tag{6.145}$$

for some intermediate value ξ. Since, for $x\in[k/N-1/(2N),k/N+1/(2N)]$,

$$\left|\frac{k}{N}-\frac{1}{2N}-x\right|\leq\frac{1}{N}, \tag{6.146}$$

the term involving $\partial^2_{xx}\varphi$ will vanish faster than the term involving $\partial_x\varphi$ and we obtain, for N large enough,

$$\int_0^\infty\sum_{k=l+1}^{N-l}(\tilde{\varphi}_k-\varphi_k)^2dt\leq\frac{C}{N}\|\partial_x\varphi\|_2^2. \tag{6.147}$$

Moreover, since

$$\lim_{N\to+\infty}\frac{1}{N}\int_0^T\sum_{k=l+1}^{N-l}\int(\hat{r}_{l,k+1}-\hat{r}_{l,k})^2d\mu_t^Ndt=0, \tag{6.148}$$

for any $T>0$ and since the first derivatives of q are bounded, we can follow the proof of Lemma 6.3 and obtain $\mathcal{N}_{r1,N}=Y_{r1,N}+Z_{r1,N}$, where

$$|Y_{r1,N}(\varphi,\eta)|\leq\|\psi\|_{H^1}A_{r1,N}(\phi),\quad |Z_{r1,N}(\varphi,\eta)|\leq\|\psi\|_{L^\infty}A_{r1,N}(\phi), \tag{6.149}$$

for some random functional $A_{r1,N}$ depending on η only and such that

$$\lim_{N\to+\infty} \mathbb{E}[A_{r1,N}(\phi)] = 0. \tag{6.150}$$

The proof is concluded once we write $\mathcal{N}_N = Y_{n,N} + Z_{n,N}$, with $Y_{n,N} = Y_{n1,N} + Y_{r1,N} + Y_{r2,N}$ and $Z_{n,N} = Z_{n1,N} + Z_{r1,N} + Z_{r2,N}$

□

Chapter 7

Weak solutions and Thermodynamics

7.1 Completing the main proof

In this chapter we put together everything we have developed through the rest of the thesis and we prove

Theorem 2.1. *Let* $\{u_k(t)\}_{k=1}^N = \{(r_k(t), p_k(t))\}_{k=1}^N$ *be a solution of the system* (2.16) *at time* $t \geq 0$ *and let* μ_t^N, f_t^N *as in* (2.17) *such that*

$$\int f_0^N \log f_0^N d\lambda^N \leq CN, \tag{7.1}$$

for some $C > 0$ *independent of* N. *Moreover let, for* $T > 0$, $Q_T = [0,T] \times [0,1]$. *Then we have the following*

i) $(\hat{u}_N)_{N\in\mathbb{N}}$, *as defined in* (6.4) *is such that* $\hat{u}_N \in L^2(\Omega \times Q_T; \mathbb{R}^2)$ *for any* $N \in \mathbb{N}$ *and* $T > 0$. *Moreover,*

$$\lim_{N\to+\infty} \mathbb{P}\left\{ \left| \frac{1}{N} \sum_{k=1}^{N} \Psi\left(\frac{k}{N}\right) \cdot u_k(t) - \int_0^1 \Psi(x) \cdot \hat{u}_N(t,x) dx \right| > \delta \right\} = 0, \tag{7.2}$$

for all $t \geq 0$, $\delta > 0$ *and all* $\Psi \in C_c^1((0,1); \mathbb{R}^2)$.

ii) There exists a a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *and some* $L^2(\tilde{\Omega} \times Q_T; \mathbb{R}^2)$*-valued stochastic processes* $\{\tilde{u}_N\}_{N\in\mathbb{N}}$ *and* $\tilde{u}$ *such that* $\tilde{u}_N$ *and* $\hat{u}_N$ *have the same law for all* $N \in \mathbb{N}$ *and* $\tilde{u}_N \to \tilde{u}$ $\tilde{\mathbb{P}}$*-almost surely (as* $N \to +\infty$*) in* L^p*-strong for all* $1 \leq p < 2$, *where* $\tilde{u}$ *is almost surely a weak solution of the p-system* (2.21) *with*

$$\tau(r) = F'(r), \qquad F(r) = \sup_{\gamma\in\mathbb{R}} \{\gamma r - G(\gamma)\} \tag{7.3}$$

and with G *defined in* (2.20).

We shall prove the theorem by breaking down the proof into several lemmas and propositions.

We begin with proving part *i)* of Theorem 2.1. Recall that $l(N)$ is taken to be the integer part of $N^{1/4}\sigma(N)^{1/2}$.

Proposition 7.1. *For any $l \in \mathbb{N}$, let $(\alpha_{l,k})_{k\in\mathbb{Z}}$ be a real-valued sequence such that $0 \le \alpha_{l,k} \le 1$ and*

$$\frac{1}{l}\sum_{|k|<l} \alpha_{l,k} = 1. \tag{7.4}$$

Define, for $l = l(N)$,

$$u_N^\alpha(t,x) := \sum_{k=l}^{N-l+1} 1_{N,k}(x) u_{l,k}^\alpha(t), \qquad u_{l,k}^\alpha(t) := \frac{1}{l}\sum_{|j|<l} \alpha_{l,j} u_{k-j}(t). \tag{7.5}$$

Then

$$\lim_{N\to\infty} \mathbb{P}\left\{ \left| \frac{1}{N}\sum_{k=1}^{N} \psi\left(\frac{k}{N}\right)\cdot u_k - \int_0^1 \psi(x)\cdot u_N^\alpha(\omega_N, x)dx \right| > \delta \right\} = 0, \tag{7.6}$$

for all test functions ψ compactly supported in $(0,1)$, all $\delta > 0$ and all $t \ge 0$.

In order to prove the proposition we use the following lemma, which is a consequence of the energy estimate (4.14)

Lemma 7.2. *Let $u_{l,k}^\alpha$ be defined as above. Then, for any $T > 0$ there exists a constant $C > 0$ independent of N and α such that*

$$\mathbb{E}\left[\|u_N^\alpha\|_{L^2(Q_T)}^2 \right] \le C. \tag{7.7}$$

Proof. In order to simplify the notation, we focus only on one component of u, namely r. We have

$$(r_{l,k}^\alpha)^2 = \frac{1}{l^2}\left(\sum_{|j|<l} \alpha_j r_{k-j} \right)^2 \le \frac{2l-1}{l^2} \sum_{|j|<l} \alpha_j^2 r_{k-j}^2, \tag{7.8}$$

which implies

$$\sum_{k=l}^{N-l+1} (r_{l,k}^\alpha)^2 \le \sum_{k=l}^{N-l+1} \sum_{|j|<l} \frac{2l-1}{l^2} \alpha_j^2 r_{k-j}^2 = \sum_{|j|<l} \sum_{k=l-j}^{N-l+1-j} \frac{2l-1}{l^2} \alpha_j^2 r_k^2 \tag{7.9}$$

$$\le \sum_{|j|<l} \sum_{k=1}^{N} \frac{2l-1}{l^2} \alpha_j^2 r_k^2 \le \left(\frac{2l-1}{l}\right)^2 \sum_{k=1}^{N} r_k^2 \le 4 \sum_{k=1}^{N} r_k^2. \tag{7.10}$$

Therefore, by the energy estimate (4.14),

$$\mathbb{E}\|r_N^\alpha\|^2_{L^2(Q_T)} = \mathbb{E}\int_0^T\int_0^1 |r_N^\alpha(t,x)|^2 dxdt \tag{7.11}$$

$$= \mathbb{E}\int_0^1 \sum_{k=l}^{N-l+1} 1_{N,k}(x)dx \int_0^T |r_{l,k}^\alpha|^2 dt \tag{7.12}$$

$$= \mathbb{E}\frac{1}{N}\int_0^T \sum_{k=l}^{N-l+1} |r_{l,k}^\alpha|^2 dt \leq 4\int_0^T C_1(t)dt. \tag{7.13}$$

□

Proof of Proposition 7.1. By Markov's inequality it is enough to prove that

$$\lim_{N\to+\infty} \mathbb{E}\left|\frac{1}{N}\sum_{k=1}^N \psi\left(\frac{k}{N}\right)\cdot u_k - \int_0^1 \psi(x)\cdot u_N^\alpha(\omega_N,x)dx\right| = 0. \tag{7.14}$$

We do so in two consecutive steps. We write

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{k=1}^N \psi\left(\frac{k}{N}\right)\cdot u_k - \int_0^1 \psi(x)\cdot u_N^\alpha(\omega_N,x)dx\right| \\
&\leq \left|\frac{1}{N}\sum_{k=1}^N \psi\left(\frac{k}{N}\right)\cdot u_k - \frac{1}{N}\sum_{k=l}^{N-l+1} \psi\left(\frac{k}{N}\right)\cdot u_{l,k}^\alpha\right| + \\
&+ \left|\frac{1}{N}\sum_{k=l}^{N-l+1} \psi\left(\frac{k}{N}\right)\cdot u_{l,k}^\alpha - \int_0^1 \psi(x)\cdot u_N^\alpha(\omega_N,x)dx\right|
\end{aligned} \tag{7.15}$$

and we show that the limit of the expectation value of each term at the right hand side vanishes. To simplify the notation we focus once more on only one component of u_k, namely r_k, and we denote by φ the appropriate component of ψ. Since φ is compactly supported in $(0,1)$, for N large enough all the summations can be written between l and $N-l+1$, as in the previous chapter. Therefore we have

$$\frac{1}{N}\sum_{k=1}^N \varphi\left(\frac{k}{N}\right) r_k - \frac{1}{N}\sum_{k=l}^{N-l+1} \varphi\left(\frac{k}{N}\right) r_{l,k}^\alpha = \frac{1}{N}\sum_{k=l}^{N-l+1} \varphi\left(\frac{k}{N}\right)(r_k - r_{l,k}^\alpha) \tag{7.16}$$

$$= \frac{1}{lN}\sum_{k=l}^{N-l+1}\sum_{|j|<l} \varphi\left(\frac{k}{N}\right)\left(\frac{l}{2l-1}r_k - \alpha_{l,j}r_{k-j}\right). \tag{7.17}$$

Note that, for fixed $j \geq 0$, one has

$$\sum_{k=l}^{N-l+1} \varphi\left(\frac{k}{N}\right) r_{k-j} = \sum_{k=l-j}^{N-l+1-j} \varphi\left(\frac{k+j}{N}\right) r_k \tag{7.18}$$

$$= \sum_{k=l}^{N-l+1} \varphi\left(\frac{k+j}{N}\right) r_k + \sum_{k=l-j}^{l} \varphi\left(\frac{k+j}{N}\right) r_k - \sum_{k=N-l+1-j}^{N-l+1} \varphi\left(\frac{k+j}{N}\right) r_k. \tag{7.19}$$

Thanks to the support of φ, since $0 \leq j \leq l-1$, and since $l/N \to 0$, the second and the third summation are identically zero if N is large enough. The same also stands for $j < 0$. Therefore we obtain

$$\begin{aligned} &\frac{1}{lN} \sum_{k=l}^{N-l+1} \sum_{|j|<l} \varphi\left(\frac{k}{N}\right) \left(\frac{l}{2l-1} r_k - \alpha_{l,j} r_{k-j}\right) \\ &= \frac{1}{lN} \sum_{k=l}^{N-l+1} \sum_{|j|<l} \left(\frac{l}{2l-1} \varphi\left(\frac{k}{N}\right) - \alpha_{l,j} \varphi\left(\frac{k+j}{N}\right)\right) r_k \end{aligned} \tag{7.20}$$

For some intermediate ξ, we have

$$\begin{aligned} &\frac{1}{l} \sum_{|j|<l} \left(\frac{l}{2l-1} \varphi\left(\frac{k}{N}\right) - \alpha_j \varphi\left(\frac{k+j}{N}\right)\right) \\ &= \frac{1}{l} \sum_{|j|<l} \left(\frac{l}{2l-1} - \alpha_{l,j}\right) \varphi\left(\frac{k}{N}\right) - \frac{1}{l} \sum_{|j|<l} \alpha_{l,j} \varphi'(\xi) \frac{j}{N} \end{aligned} \tag{7.21}$$

$$= -\frac{1}{l} \sum_{|j|<l} \alpha_{l,j} \varphi'(\xi) \frac{j}{N}. \tag{7.22}$$

Putting everything together, we have obtained

$$\begin{aligned} &\mathbb{E}\left| \frac{1}{lN} \sum_{k=l}^{N-l+1} \sum_{|j|<l} \left(\frac{l}{2l-1} \varphi\left(\frac{k}{N}\right) - \alpha_j \varphi\left(\frac{k+j}{N}\right)\right) r_k \right| \\ &\leq C \frac{l}{N^2} \mathbb{E} \sum_{k=l}^{N-l+1} |r_k| \leq C \frac{l}{N^2} \mathbb{E} \sum_{k=l}^{N-l+1} (1 + r_k^2) \leq C \frac{l}{N}, \end{aligned} \tag{7.23}$$

which vanishes as $N \to +\infty$. Thus, we have proven

$$\lim_{N \to +\infty} \mathbb{E}\left| \frac{1}{N} \sum_{k=1}^{N} \psi\left(\frac{k}{N}\right) \cdot u_k - \frac{1}{N} \sum_{k=l}^{N-l+1} \psi\left(\frac{k}{N}\right) \cdot u_{l,k}^{\alpha} \right| = 0, \tag{7.24}$$

which concludes the first part of the proof thanks to inequality (7.15). In order to prove the second part, namely

$$\lim_{N \to +\infty} \mathbb{E}\left| \frac{1}{N} \sum_{k=l}^{N-l+1} \psi\left(\frac{k}{N}\right) \cdot u_{l,k}^{\alpha} - \int_0^1 \psi(x) \cdot u_N^{\alpha}(t,x) dx \right| = 0, \tag{7.25}$$

we write

$$\begin{aligned}&\frac{1}{N}\sum_{k=l}^{N-l+1}\varphi\left(\frac{k}{N}\right)r_{l,k}^{\alpha}-\int_0^1\varphi(x)r_N^{\alpha}(t,x)dx\\&=\frac{1}{N}\sum_{k=l}^{N-l+1}\varphi\left(\frac{k}{N}\right)r_{l,k}^{\alpha}-\frac{1}{N}\sum_{k=l}^{N-l+1}\varphi_k r_{l,k}^{\alpha},\end{aligned}\tag{7.26}$$

$$=\frac{1}{N}\sum_{k=l}^{N-l+1}\left(\varphi\left(\frac{k}{N}\right)-\varphi_k\right)r_{l,k}^{\alpha},\tag{7.27}$$

where

$$\varphi_k:=N\int_0^1\varphi(x)1_{N,k}(x)dx.\tag{7.28}$$

The term $\varphi(k/N)-\varphi_k$ is estimated exactly like the term $\tilde{\varphi}_k-\varphi_k$ in the proof of Lemma 6.10, and it yields

$$\left|\varphi\left(\frac{k}{N}\right)-\varphi_k\right|\leq\frac{C}{N},\tag{7.29}$$

for N large enough. We have therefore obtained

$$\mathbb{E}\left|\frac{1}{N}\sum_{k=l}^{N-l+1}\psi\left(\frac{k}{N}\right)\cdot u_{l,k}^{\alpha}(Nt)-\int_0^1\psi(x)\cdot u_N^{\alpha}(t,x)dx\right|\leq\frac{C}{N^2}\mathbb{E}\sum_{k=l}^{N-l+1}\int|r_{l,k}^{\alpha}|\tag{7.30}$$

$$\leq\frac{C}{N^2}\mathbb{E}\sum_{k=l}^{N-l+1}\int\left(1+(r_{l,k}^{\alpha})^2\right)\leq\frac{C}{N},\tag{7.31}$$

which concludes the proof. □

The previous proposition proves the first part of Theorem 2.1, as $\hat{u}_N$ is obtained from u_N^{α} by taking $\alpha_j=(l-|j|)/l$.

We make another intermediate step in proving Theorem 2.1, namely

Proposition 7.3. *Let, for $N\in\mathbb{N}$, $\nu^N=\delta_{\hat{u}_N}$ be the random Young measure associated to the $L^2(Q_T)$-valued random variable $\hat{u}_N$. Then there exists a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, some $\mathcal{Y}$-valued random variables $\{\tilde{\nu}^N\}_{N\in\mathbb{N}}$, $\tilde{\nu}$ and some $L^2(Q_T)$-valued random variables $\{\tilde{u}_N\}_{N\in\mathbb{N}}$, $\tilde{u}$ such that*

i) $\tilde{\nu}^N$ has the same law as ν^N and $\tilde{u}_N$ has the same law as $\hat{u}_N$;

ii) $\tilde{\mathbb{P}}$-almost surely, $\tilde{\nu}^N\to\tilde{\nu}$ in $\mathcal{Y}$ and $\tilde{u}_N\rightharpoonup\tilde{u}$ in $L^2(Q_T)$.

The proof of this proposition will be based on the following

Theorem 7.4 (Skorohod's representation)**.** *Let $(\mu_N)_{N\in\mathbb{N}}$ be a sequence of probability measures on a metric space (M,ρ) and assume that $\mu_N \to \mu$ in $\mathcal{P}(M)$ for some $\mu \in \mathcal{P}(M)$ with separable support. Then there exists a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and some M-valued random variables $\tilde{X}_N, \tilde{X}$ such that the law of $\tilde{X}_N$ is μ_N and, $\tilde{\mathbb{P}}$-almost surely, $\tilde{X}_N \to X$ in M.*

Proof of Proposition 7.3. In order to apply Skorohod's theorem to some appropriate M, to be determined, we need to a some weakly convergent sequence of probability measures (μ_N).

The uniform bound

$$\mathbb{E}\left[\|\hat{u}_N\|^2_{L^2(Q_T)}\right] \leq C \tag{7.32}$$

and Proposition 5.3 with $h(\xi) = |\xi|^2$ imply at once that, up to a subsequence, (ν^N) is convergent in law. That is, the sequence obtained by taking the laws of ν^N $(N \in \mathbb{N})$ is convergent in the sense of probability measures.

At this stage one could be tempted to replicate the proof of Proposition 5.2 and consider the set

$$K'_C = \left\{u \in L^2(Q_T) : \|u\|_{L^2(Q_T)} \leq C\right\} \tag{7.33}$$

to conclude that the laws of $\hat{u}_N$ are a tight family as probability measures on $L^2(Q_T)$. However this cannot be implied, as the set K_C is *not* a compact subset of $L^2(Q_T)$.

However, the set

$$K_C = \{x \in \mathbb{R} : |x| \leq C\} \tag{7.34}$$

is a compact subset of $\mathbb{R}$ and following the proof of Proposition 5.2 leads us to conclude that the sequence of the laws of the real-valued random variables $\|\hat{u}_N\|^2_{L^2(Q_T)}$ is weakly convergent.

Therefore we can apply Skorohod's theorem setting $M = \mathcal{Y}\times\mathbb{R}$ and $\mu_N \in \mathcal{P}(\mathcal{Y}\times\mathbb{R})$ to be the law of $(\nu^N, \|\hat{u}_N\|^2_{L^2(Q_T)})$. Hence, we can find a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, some random Young measures $\{\tilde{\nu}^N\}_{N\in\mathbb{N}}$, $\tilde{\nu}$ and some real-valued random variables $\{a_N\}_{N\in\mathbb{N}}$, a on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ such that, for all $N \in \mathbb{N}$, $\tilde{\nu}^N$ has the same law as ν^N, a_N has the same law as $\|\hat{u}_N\|^2_{L^2(Q_T)}$. Moreover, $\tilde{\mathbb{P}}$-almost surely, $\tilde{\nu}^N \to \tilde{\nu}$ in $\mathcal{Y}$ and $a_N \to a$ in $\mathbb{R}$.

By Proposition 5.5, $\tilde{\nu}^N$ is a L^2-random Dirac mass for all $N \in \mathbb{N}$. Therefore we have $\tilde{\nu}^N = \delta_{\tilde{u}_N}$ for some $L^2(Q_T)$-valued random variable $\tilde{u}_N$. But since ν^N and $\tilde{\nu}^N$ have the same law, $\hat{u}_N$ and $\tilde{u}_N$ have the same law, too. This implies that $\|\tilde{u}_N\|^2_{L^2(Q_T)}$ and a_N have the same law. Finally, (a_N) converges almost surely, so it is almost surely uniformly bounded. Therefore we can conclude the same for $\|\tilde{u}_N\|^2_{L^2(Q_T)}$ and $\tilde{u}_N$ is, up to a subsequence, almost surely weakly convergent to some $\tilde{u} \in L^2(Q_T)$. □

Proposition 7.5. *Let $\tilde{\nu}$, $\tilde{u}_N$ and $\tilde{u}$ be as in Proposition 7.3. Then $\tilde{\nu}_{t,x} = \delta_{\tilde{u}(t,x)}$ almost surely and almost everywhere and, almost surely, $\tilde{u}_N \to \tilde{u}$ strongly in $L^p(Q_T)$ for any $1 \leq p < 2$.*

Proof. We wish to apply the stochastic compensated compactness theory in order to conclude that the support of $\tilde{\nu}$ is almost surely a point. Therefore, we need to apply the stochastic Murat lemma 5.8 to the entropy production

$$\tilde{X}_N(\varphi, \eta) := -\int_0^\infty \int_0^1 (\partial_t \varphi(t,x)\eta(\tilde{u}_N(t,x)) + \partial_x \varphi(t,x) q(\tilde{u}_{t,x}))\, dtdx. \tag{7.35}$$

We have verified the hypothesis of Lemma 5.8 for the entropy production with $\tilde{u}_N$ replaced by $\hat{u}_N$. But since $\hat{u}_N$ and $\tilde{u}_N$ have the same law, the same estimates we obtained for the entropy production X_N will be true for $\tilde{X}_N$. This allows us to apply Lemma 5.8, and therefore to obtain $\tilde{\nu} = \delta_{u^*}$ almost surely, for some measurable process u^*.

Now we need to identify u^* with $\tilde{u}$ (the weak limit of $\tilde{u}_N$). We do so by taking, for any test function φ, the expression

$$\int_{Q_T} \varphi(t,x) \cdot \tilde{u}_N(t,x) dxdt. \tag{7.36}$$

Since, almost surely, $\tilde{u}_N$ converges weakly in $L^2(Q_T)$ to $\tilde{u}$ we have

$$\lim_{N \to +\infty} \int_{Q_T} \varphi(t,x) \cdot \tilde{u}_N(t,x) dxdt = \int_{Q_T} \varphi(t,x) \cdot \tilde{u}(t,x) dxdt \tag{7.37}$$

almost surely and up to a subsequence. On the other hand, we can apply Proposition 5.4 (and the following remark) with $g(r,p) = r$ and $g(r,p) = p$ and conclude

$$\lim_{N \to +\infty} \int_{Q_T} \varphi(t,x) \cdot \tilde{u}_N(t,x) dxdt = \lim_{N \to \infty} \int_{Q_T} \int_{\mathbb{R}^2} \varphi(t,x) \cdot \xi d\tilde{\nu}^N_{t,x}(\xi) dxdt \tag{7.38}$$

$$= \int_{Q_T} \int_{\mathbb{R}^2} \varphi(t,x) \cdot \xi d\tilde{\nu}_{t,x}(\xi) dxdt = \int_{Q_T} \varphi(t,x) \cdot u^*(t,x) dxdt \tag{7.39}$$

almost everywhere. This implies $\tilde{u} = u^*$ almost surely and almost everywhere, since φ was arbitrary.

Finally, since we have proven that, almost surely, $\tilde{u}_N$ converges weakly to $\tilde{u}$ and the limiting Young measure $\tilde{\nu}$ is almost surely equal to $\delta_{\tilde{u}}$, we can fix $\tilde{\omega} \in \tilde{\Omega}$ such that everything holds and invoke the deterministic result in order to conclude that, almost surely, $\tilde{u}_N \to \tilde{u}$ in $L^p(Q_T)$-strong for all $1 \leq p < 2$. □

Now we are in a position to identify $\tilde{u} = (\tilde{r}, \tilde{p})$ with a weak solution of our p-system.

Proposition 7.6. *$\tilde{u}$, as obtained in Proposition 7.3, is almost surely a weak solution of the p-system* (2.21)*, in the sense of Definition 2.1.*

Proof. We deal with (2.23), as (2.22) is easier, being linear. From (2.42) we have that

$$\lim_{N\to\infty}\int_0^\infty\int_0^1(\partial_t\psi(t,x)\hat{p}_N(t,x)-\partial_x\psi(t,x)\tau(\hat{r}_N(t,x)))dxdt+\int_0^\infty\psi(t,1)\bar{\tau}(t)dt=0 \tag{7.40}$$

in probability. Therefore, since $\hat{u}_N=(\hat{r}_N,\hat{p}_N)$ and $\tilde{u}_N=(\tilde{r}_N,\tilde{p}_N)$ have the same law, we obtain

$$\lim_{N\to\infty}\int_0^\infty\int_0^1(\partial_t\psi(t,x)\tilde{p}_N(t,x)-\partial_x\psi(t,x)\tau(\tilde{r}_N(t,x)))dxdt+\int_0^\infty\psi(t,1)\bar{\tau}(t)dt=0 \tag{7.41}$$

in probability. Hence, up to a subsequence, we have

$$\lim_{N\to\infty}\int_0^\infty\int_0^1(\partial_t\psi(t,x)\tilde{p}_N(t,x)-\partial_x\psi(t,x)\tau(\tilde{r}_N(t,x)))dxdt+\int_0^\infty\psi(t,1)\bar{\tau}(t)dt=0 \tag{7.42}$$

almost surely. Since $\tilde{p}_N$ converges weakly almost surely to $\tilde{p}$ we have

$$\lim_{N\to\infty}\int_0^\infty\partial_t\psi(t,x)\tilde{p}_N(t,x)dtdx=\int_0^\infty\int_0^1\partial_t\psi(t,x)\tilde{p}(t,x)dtdx \tag{7.43}$$

almost surely. Concerning the nonlinear term, we can apply all the results collected in the rest of the section: we write

$$\tau(\tilde{r}_N(t,x))=\int_{\mathbb{R}^2}\tau(\xi_1)d\tilde{\nu}^N_{t,x}(\xi), \tag{7.44}$$

where $\xi=(\xi_1,\xi_2)$. Since $\tilde{u}_N$ is uniformly bounded in $L^2(Q_T)$ and since $\tilde{\nu}^N\to\tilde{\nu}$ almost surely, we can pass to the limit in the sense of the Young measures to obtain, almost surely and possibly up to another subsequence,

$$\lim_{N\to\infty}\int_0^\infty\int_0^1\int_{\mathbb{R}^2}\partial_x\psi(t,x)\tau(\xi_1)d\nu^N_{t,x}(\xi)dtdx=\int_0^\infty\int_0^1\int_{\mathbb{R}^2}\partial_x\psi(t,x)\tau(\xi_1)d\nu_{t,x}(\xi)dtdx. \tag{7.45}$$

This concludes the proof, since, almost surely, $\tilde{\nu}=\delta_{\tilde{u}}$.

□

7.2 Thermodynamic consequences

In this final section we see that the solutions we have obtained lead to the first and the second principle of thermodynamics. This is a direct consequence of the physical framework we have used to obtain the solutions in the first place.

In this thesis we are working on a *isothermal system* where the temperature has been fixed to 1. However, in order to make things less mysterious, in this section we will denote the temperature by β^{-1}, for some $\beta > 0$.

Remark. We shall superintend the fact that we need to change probability space in order to converge to the solution $\tilde{u}(t,x)$. Therefore we will just we will simply write things like

$$\lim_{N\to\infty} \sum_{k=1}^{N} \varphi\left(\frac{k}{N}\right) r_k = \int_0^1 \varphi(x)\tilde{r}(t,x),$$

or

$$\lim_{N\to\infty} \int_0^\infty \int_0^1 \psi(t,x)\hat{u}_N(t,x)dxdt = \int_0^\infty \int_0^1 \psi(t,x)\tilde{u}(t,x)dxdt,$$

having in mind that, behind this, there is the theory we have developed in the rest of the thesis. Moreover, since, almost surely $\tilde{u}$ solves a deterministic system of PDEs, we will effectively forget about its random nature and regard it as a deterministic function.

Before proceeding, we fix $\tau_0, \tau_1, t_1 \in \mathbb{R}$ and we take $\bar{\tau}$ (the external force) to be a smooth function such that $\bar{\tau}(0) = \tau_0$ and $\tilde{\tau}(t) = \tau_1$ for all $t \geq t_1$. This corresponds to the following physical situation: at time 0 the system is at equilibrium, and the equilibrium state determined by (β, τ_0). Then we vary the external tension and we eventually bring the system to another equilibrium state (reached asymptotically as $t \to +\infty$), identified by (β, τ_1) (we are performing isothermal transformations, so the temperature does not change). That is the way we define a a thermodynamic isothermal transformation between two equilibrium states (β, τ_0) and (β, τ_1).

Through the rest of this chapter we will make some more or less strong assumptions concerning the solution $\tilde{u}$ and the convergence $\tilde{u}_N \to \tilde{u}$. Some are physically motivated, others are of a purely technical nature.

Before stating these assumptions, we re-write some definitions in order to make transparent the dependences on β. The logarithm G of the partition function is defined as

$$G(\beta,\gamma) = \log \int_{-\infty}^{+\infty} \exp(-\beta(V(r) - \gamma r))dr + \log\sqrt{\frac{2\pi}{\beta}}. \tag{7.46}$$

Moreover, the free energy F is defined as

$$F(\beta,\rho) = \sup_{\gamma\in\mathbb{R}}\{\gamma\rho - \beta^{-1}G(\beta,\gamma)\} \tag{7.47}$$

and the tension τ is given by

$$\tau(\beta,\rho) = \partial_\rho F(\beta,\rho). \tag{7.48}$$

Finally, the *internal energy* U is defined by

$$U(\beta,\gamma) = \beta^{-1}\gamma\partial_\gamma G(\beta,\gamma) - \partial_\beta G(\beta,\gamma). \tag{7.49}$$

We assume the following:

i) (Convergence of the energy)

$$\lim_{N\to\infty} E_N(t) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\left(\frac{p_i^2}{2} + V(r_i)\right) = \int_0^1 U(\beta,\tau(\tilde{r}(t,x))dx \tag{7.50}$$

and

$$\lim_{N\to\infty} E_N(0) = U(\beta,\tau_0). \tag{7.51}$$

Furthermore,

$$\lim_{N\to+\infty}\int_0^1\int_0^\infty \psi(t,x)f(\hat{u}_N(t,x))dxdt = \int_0^1\int_0^\infty \psi(t,x)f(\tilde{u}(t,x))dxdt \tag{7.52}$$

for all test functions ψ and all continuous f with quadratic growth.

ii) (Initial and asymptotic behaviour) $\tau(\beta,\tilde{r}(0,x)) = \tau_0$, $\tilde{p}(0,x) = 0$, while

$$\lim_{t\to+\infty}\tau(\beta,\tilde{r}(t,x)) = \tau_1, \quad \lim_{t\to+\infty}\tilde{p}(t,x) = 0 \tag{7.53}$$

for almost all $x \in [0,1]$.

iii) (Some smoothness) There exists $\varepsilon_0 > 0$ such that $\tilde{u}$ is smooth in $[0,\infty)\times([0,\varepsilon_0)\cup(1-\varepsilon_0,1])$. Moreover, the function

$$\mathcal{L}(t) = \int_0^1 \tilde{r}(t,x)dx \tag{7.54}$$

is differentiable for almost all $t > 0$.

iv) (Continuous approximation near the boundaries) Take ε_0 as in *ii*) and $0 < \varepsilon < \varepsilon_0$. Then $\hat{u}_N(t,x_N) \to \tilde{u}(t,\varepsilon)$ and $\hat{u}_N(t,1-x_N) \to \tilde{u}(t,1-\varepsilon)$, whenever $x_N \to \varepsilon$, for all $t \geq 0$.

v) (Higher regularity) The discrete Laplacian $\Delta\hat{r}_{l,k}$ satisfies the following condition:

$$\mathbb{E}\left[\nabla^*\left(a_k(r)\Delta\hat{r}_{l,k}\right)\right] = 0, \tag{7.55}$$

for all $t > 0$ and some functions $a_k(r) = a_k(r_1,\ldots,r_N)$ such that $\lambda \leq a_k(r) \leq \Lambda$ for all $r_1,\ldots,r_N$ and some positive constants λ and Λ.

Moreover, assume that

$$\mathbb{E}\int_0^t \sum_{k=l}^{N-l} (\nabla \hat{r}_{l,k})^4 ds \leq \frac{C}{\sigma^2 l^2}, \tag{7.56}$$

for all $t > 0$.

Remark. Assumption i) may seem quite strong. In fact in order to control the convergence of the energy we need to control moments higher than the second, and, for a generic V, the relative entropy does not help. However, (7.50) is true at least in one case, namely the harmonic one. In that case, in fact, the functions f_t^N are Gaussians, and we have control of any moment.

Assumptions iii) and iv) are motivated by physics: if we want to define the work done on (or by) the system, we need the system to be able to respond to external inputs. If we lose smoothness at the boundaries, however, an external input may not propagate to the system, and so the work turns out to be ill-defined.

We require assumption v) in order to have some extra control over the discrete Laplacian (to be used in proving the second principle of thermodynamics). To be precise, the first part is used to prove the following

Proposition 7.7 (Discrete Caccioppoli inequality)**.** *Let $(u_k)_{k=1}^N$ be a solution of*

$$\nabla^* \left(a_k(u) \nabla u_k\right) = 0, \tag{7.57}$$

where $a_k(u) = a_k(u_1, \ldots, u_N)$ and $\lambda \leq a_k(u) \leq \Lambda$ for some positive constants λ and Λ and all k. Then, there exists a positive constant C, depending on λ and Λ only, such that

$$\sum_{k=h}^{m} (\nabla u_k)^2 \leq C \left(\frac{1}{(n-m)^2} \sum_{k=h}^{n} u_k^2 - u_h \nabla u_h a_h(u) \right) \tag{7.58}$$

for any $N \geq n > m > h > 1$.

Proof. Let $(\eta_k)_{k=2}^N$ be non-negative, $\eta_k = 1$ if $h - 1 \leq k \leq m$, $\eta_k = 0$ if $h < k < m$ or $n \leq k \leq N$, and $\eta_k < \eta_{k-1}$ for $m < k \leq n$. Then, we have

$$\begin{aligned} \lambda \sum_{k=h}^{m} (\nabla u_k)^2 &\leq \sum_{k=h}^{n} \nabla u_k a_k(u) \nabla u_k \eta_k^2 \\ &= \sum_{k=h}^{n} u_k a_k(u) \nabla u_k (\eta_k + \eta_{k-1}) \nabla^* \eta_k - u_h \nabla u_h a_h(u) \end{aligned} \tag{7.59}$$

By Cauchy-Schwarz inequality we obtain, for any $\alpha > 0$,

$$\lambda \sum_{k=1}^{m} (\nabla u_k)^2 \leq \frac{\alpha\Lambda}{2} \sum_{k=h}^{n} (\nabla u_k)^2 (\eta_k + \eta_{k-1})^2 + \frac{\Lambda}{2\alpha} \sum_{k=h}^{n} u_k^2 (\nabla^* \eta_k)^2 - u_h \nabla u_h a_h(u). \quad (7.60)$$

In particular, using $0 \leq \eta_k \leq 1$ and choosing $\alpha = \lambda/(4\Lambda)$, we have

$$\frac{\lambda}{2} \sum_{k=1}^{m} (\nabla u_k)^2 \leq \frac{2\Lambda^2}{\lambda} \sum_{k=h}^{n} u_k^2 (\nabla^* \eta_k)^2 - u_h \nabla u_h a_h(u). \quad (7.61)$$

The conclusion then follows by noting that $|\nabla^* \eta_k| \leq 1/(n-m)$. □

The first law of thermodynamics is nothing but an energy balance which takes into account energy lost (or gain) via heat exchange. It reads as follows.

$$\Delta U = W + Q, \quad (7.62)$$

where ΔU is the difference of internal energy between two equilibrium states, W is the work done on the system (which depends on the external force $\bar{\tau}$) and Q is the heat exchanged (which depends on the noise, i.e. on σ). In order to obtain the first principle, we use the equations (2.16) to obtain, from a direct calculation,

$$E_N(t) - E_N(0) = \int_0^t dE_N(s) = W_N(t) + Q_N(t), \quad (7.63)$$

where

$$W_N(t) = \int_0^t \bar{\tau}(s) p_N(s) ds \quad (7.64)$$

and

$$\begin{aligned} Q_N(t) = &-\sigma \int_0^t \sum_{k=1}^{N-1} \left((\nabla p_k(s))^2 - 2\beta^{-1} \right) ds + \sqrt{2\frac{\sigma}{N}\beta^{-1}} \sum_{k=1}^{N-1} \int_0^t (\nabla p_k(s)) dw_k(s) + \\ &-\sigma \int_0^t \sum_{k=1}^{N-1} \left((\nabla V_k'(s))^2 - 2\beta^{-1} \right) ds + \sqrt{2\frac{\sigma}{N}\beta^{-1}} \sum_{k=1}^{N-1} \int_0^t (\nabla V_k'(s)) d\tilde{w}_k(s). \end{aligned} \quad (7.65)$$

W_N can be rewritten in the more expressive form

$$W_N(t) = \int_0^t \bar{\tau}(s) d\left(\frac{q_N(s)}{N} \right) = \int_0^t \bar{\tau}(s) d\left(\frac{1}{N} \sum_{i=1}^{N} r_k(s) \right), \quad (7.66)$$

so that we have

$$\lim_{N \to +\infty} W_N(t) = \int_0^t \bar{\tau}(s) d\mathcal{L}(s) = W(t), \quad (7.67)$$

with $\mathcal{L}$ as in equation (7.54). $W(t)$ is defined to be the work done by the external tension at time t.

By our assumption (7.52) about the convergence of the energy, we obtain that $E_N(t) - E_N(0)$ converges to the difference of internal energy, and so $Q_N(t)$ converges, as $N \to +\infty$ to the deterministic quantity

$$Q(t) = \int_0^1 (U(\beta, \tau(\tilde{r}(t,x))) - U(\beta, \tau_0))dx - W(t). \tag{7.68}$$

Therefore, taking the limit $t \to +\infty$ we obtain the first principle of thermodynamics

$$U(\beta, \tau_1) - U(\beta, \tau_0) = W + Q. \tag{7.69}$$

Let's move now to the second principle. It states that, during a thermodynamic transformation Γ_{AB} between two states A and B we have

$$\Delta S = S(B) - S(A) \geq \int_{\Gamma_{AB}} \beta dQ, \tag{7.70}$$

where S is the entropy and Q is the heat. The equality holds only for *reversible* (or quasistatic) transformations.

Two families of transformations allow to express the second principle of thermodynamics in a more catchy form.

- (Adiabatic transformation) There is no heat exchange, so dQ is identically zero and we have

  $$\Delta S \geq 0. \tag{7.71}$$

  Therefore, during an adiabatic transformation, the entropy is never decreasing, and it is actually increasing unless the transformation is reversible (which can never be the case in any real-life process).

- (Isothermal transformation) The temperature T is constant, so we have

  $$\Delta S \geq \beta Q. \tag{7.72}$$

  This is the case we are interested in.

Note that, mathematical convenience aside, adiabatic and isothermal transformation are of fundamental importance in thermodynamics, as they are the "components" of the Carnot cycle.

In order to get to the second principle we need of course to define the entropy S. We do so by defining

$$S = \beta(U - F). \tag{7.73}$$

where F is the free energy and U is the internal energy. We can combine the first principle (7.69) and (7.73) to have an equivalent formulation of the second principle (7.72) for an isothermal transformation. In fact, we have

$$\Delta F = \Delta U - T\Delta S = W + Q - \beta^{-1}\Delta S.$$

Therefore, since β is positive, the second principle is equivalent to the following *inequality of Clausius*

$$\Delta F \leq W. \tag{7.74}$$

We are going to prove (7.74) for our system, proving in turn the second principle of thermodynamics. As it will be clear in a few lines, the inequality of Clausius is precisely what we refer to when we write the "*entropy inequality*" in a PDE context. This also explains the reason why in PDEs we talk about "entropy *productions*" with apparently the wrong sign.

Define the free energy at time t by

$$\mathcal{F}(t) = \int_0^1 \left(\frac{\tilde{p}(t,x)^2}{2} + F(\beta, \tilde{r}(t,x)) \right) dx. \tag{7.75}$$

It follows from the initial and asymptotic conditions on $\tilde{r}$ and $\tilde{p}$ that

$$\mathcal{F}(0) = F(\beta, \tau^{-1}(\tau_0)), \quad \lim_{t\to+\infty} \mathcal{F}(t) = F(\beta, \tau^{-1}(\tau_1)). \tag{7.76}$$

We will show that

$$F(\beta, \tau^{-1}(\tau_1)) - F(\beta, \tau^{-1}(\tau_0)) \leq W, \tag{7.77}$$

where W is defined in (7.69). In order to make the following calculations more transparent, we omit to write the dependence on β.

Let ε_0 as in assumption *iii)* and fix $0 < \varepsilon < \varepsilon_0$. Let's evaluate

$$\begin{aligned}
&\mathcal{F}(t) - \mathcal{F}(0) \qquad (7.78)\\
&= \left(\int_0^\varepsilon + \int_{1-\varepsilon}^1 \right) \left(\frac{\tilde{p}(t,x)^2}{2} + F(\tilde{r}(t,x)) - \frac{\tilde{p}(0,x)^2}{2} - F(\tilde{r}(0,x)) \right) dx + \\
&\quad + \int_\varepsilon^{1-\varepsilon} \left(\frac{\tilde{p}(t,x)^2}{2} + F(\tilde{r}(t,x)) - \frac{\tilde{p}(0,x)^2}{2} - F(\tilde{r}(0,x)) \right) dx \\
&\quad = \int_0^t \frac{d}{ds} \left(\int_0^\varepsilon + \int_{1-\varepsilon}^1 \right) \left(\frac{\tilde{p}(s,x)^2}{2} + F(\tilde{r}(s,x)) \right) ds dx + \\
&+ \lim_{N\to+\infty} \int_\varepsilon^{1-\varepsilon} \left(\frac{\hat{p}_N(t,x)^2}{2} + F(\tilde{r}_N(t,x)) - \frac{\hat{p}_N(0,x)^2}{2} - F(\tilde{r}_N(0,x)) \right) dx
\end{aligned}$$

$$= \int_0^t \frac{d}{ds}\left(\int_0^\varepsilon + \int_{1-\varepsilon}^1\right)\left(\frac{\tilde{p}(s,x)^2}{2} + F(\tilde{r}(s,x))\right) dsdx \tag{7.79}$$

$$+ \lim_{N\to+\infty} \int_0^t \int_\varepsilon^{1-\varepsilon} d\left(\frac{\hat{p}_N(s,x)^2}{2} + F(\tilde{r}_N(s,x))\right) dx. \tag{7.80}$$

If $\tilde{u}$ is smooth in $[0,\infty)\times[a,b] \subseteq [0,\infty)\times[0,1]$, we have

$$\frac{d}{dt}\int_a^b \left(\frac{\tilde{p}(t,x)^2}{2} + F(\tilde{r}(t,x))\right) dx = \int_a^b (\tilde{p}\partial_t\tilde{p} + \tau(\tilde{r})\partial_t\tilde{r})dx \tag{7.81}$$

$$= \int_a^b (\tilde{p}\partial_x\tau(\tilde{r}) + \tau(\tilde{r})\partial_x\tilde{p})dx = \int_a^b \partial_x(\tilde{p}\tau(\tilde{r}))dx \tag{7.82}$$

$$= \tilde{p}(t,b)\tau(\tilde{r}(t,b)) - \tilde{p}(t,a)\tau(\tilde{r}(t,a)). \tag{7.83}$$

Therefore, (7.79) reads

$$\begin{aligned}&\int_0^t \frac{d}{ds}\left(\int_0^\varepsilon + \int_{1-\varepsilon}^1\right)\left(\frac{\tilde{p}(s,x)^2}{2} + F(\tilde{r}(s,x))\right) dsdx\\ &= \int_0^t (\tilde{p}(s,1)\tau(\tilde{r}(s,1)) - \tilde{p}(s,1-\varepsilon)\tau(\tilde{r}(s,1-\varepsilon)) +\\ &\qquad +\tilde{p}(s,\varepsilon)\tau(\tilde{r}(s,\varepsilon)) - \tilde{p}(s,0)\tau(\tilde{r}(s,0))ds\end{aligned} \tag{7.84}$$

$$= W(t) + \int_0^t (\tilde{p}(s,\varepsilon)\tau(\tilde{r}(s,\varepsilon)) - \tilde{p}(s,1-\varepsilon)\tau(\tilde{r}(s,1-\varepsilon)))ds, \tag{7.85}$$

where we have used $\tilde{p}(s,0) = 0$, $\tau(\tilde{r}(s,1)) = \bar{\tau}(s)$ and

$$\tilde{p}(s,1) = \frac{d}{ds}\int_0^1 \tilde{r}(s,x)dx. \tag{7.86}$$

We are left to evaluate (7.80). Recalling that $\partial_\rho F = \tau$, Itô formula yields, for N large enough,

$$\begin{aligned}&\int_\varepsilon^{1-\varepsilon} d\left(\frac{\hat{p}_N(s,x)^2}{2} + F(\hat{r}_N(s,x))\right) dx\\ &= \int_\varepsilon^{1-\varepsilon} \sum_{k=l}^{N-l+1} 1_{N,k}(x) d\left(\frac{\hat{p}_{l,k}^2}{2} + F(\hat{r}_{l,k})\right) dx\end{aligned} \tag{7.87}$$

$$= \frac{1}{N}\sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} d\left(\frac{\hat{p}_{l,k}^2}{2} + F(\hat{r}_{l,k})\right) \tag{7.88}$$

$$= \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} \left(\hat{p}_{l,k}\nabla\hat{V}'_{l,k} - \tau(\hat{r}_{l,k})\nabla^*\hat{p}_{l,k}\right) ds + \tag{7.89}$$

$$+ \sigma \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} \left(\hat{p}_{l,k}\Delta\hat{p}_{l,k} + \tau(\hat{r}_{l,k})\Delta\hat{V}'_{l,k}\right) ds + \tag{7.90}$$

$$+\sigma\beta^{-1}\sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]}\left((\nabla^* d\hat{w}_{l,k})^2+\tau'(\hat{r}_{l,k})(\nabla^* d\hat{\tilde{w}}_{l,k})^2\right)+ \tag{7.91}$$

$$+\sqrt{2\frac{\sigma}{N}\beta^{-1}}\sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]}\left(\hat{p}_{l,k}d\nabla^*\hat{w}_{l,k}+\tau(\hat{r}_{l,k})d\nabla^*\hat{\tilde{w}}_{l,k}\right). \tag{7.92}$$

Since τ' is bounded, (7.91) is estimated exactly like the term (6.94) in the previous chapter, and it vanishes in probability.

The time integrals of the martingales (7.92) also vanish in probability. This immediately follows from Lemmas 6.8 and 6.9 and the fact that τ' is bounded.

Let's evaluate (7.89). After a summation by parts we have

$$\begin{aligned}\sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]}\left(\hat{p}_{l,k}\nabla\hat{V}'_{l,k}-\tau(\hat{r}_{l,k})\nabla^*\hat{p}_{l,k}\right)ds=\sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]}\nabla^*\hat{p}_{l,k}\left(\hat{V}'_{l,k}-\tau(\hat{r}_{l,k})\right)ds+\\ +\hat{p}_{l,[N(1-\varepsilon)]}\hat{V}'_{l,[N(1-\varepsilon)]+1}-\hat{p}_{l,[N\varepsilon]-1}\hat{V}'_{l,[N(1-\varepsilon)]}.\end{aligned} \tag{7.93}$$

Thanks to the one and two block estimates, the time integral of the summation vanishes in the limit. Furthermore, thanks to our assumptions of continuous approximation near the boundaries and one more use of the one block estimate to replace V' by τ, the boundary terms converge to

$$\tilde{p}(s,1-\varepsilon)\tau(\tilde{r}(s,1-\varepsilon))-\tilde{p}(s,\varepsilon)\tau(\tilde{r}(s,\varepsilon)), \tag{7.94}$$

which will precisely cancel the boundary contribution in (7.84).

Note that, so far, every term we have considered vanishes in the limit. On the other hand, we have considered only terms whose sign we could not control. The sign in the inequality of Clausius comes, in fact from, (7.90), which will carry a sign and will not necessarily vanish in the limit (of course, it will vanish for smooth solutions, so we cannot expect it will be always non-vanishing). Let's start with the term

$$\sigma\sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]}\hat{p}_{l,k}\Delta\hat{p}_{l,k}=-\sigma\sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]}(\nabla\hat{p}_{l,k})^2+\sigma\hat{p}_{l,[N(1-\varepsilon)]+1}\nabla\hat{p}_{l,[N(1-\varepsilon)]}-\sigma\hat{p}_{l,[N\varepsilon]}\nabla\hat{p}_{l,[N\varepsilon]-1}. \tag{7.95}$$

The time integral of the summation is clearly negative and stays bounded as $N\to+\infty$ thanks to the two block estimate. The boundary terms, on the other hand, vanish. In fact,

$$\lim_{N\to+\infty}N\hat{p}_{l,[N\varepsilon]}\nabla\hat{p}_{l,[N\varepsilon]-1}=\tilde{p}(s,\varepsilon)\partial_x\tilde{p}(s,\varepsilon) \tag{7.96}$$

and $\sigma/N \to 0$. The other boundary term is analogous. Finally we consider

$$\sigma \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} \tau(\hat{r}_{l,k})\Delta\hat{V}'_{l,k} = -\sigma \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} \nabla\tau(\hat{r}_{l,k})\nabla\hat{V}'_{l,k} + \tag{7.97}$$

$$+ \sigma\tau(\hat{r}_{l,[N(1-\varepsilon)]+1})\nabla\hat{V}'_{l,[N(1-\varepsilon)]} - \sigma\tau(\hat{r}_{l,[N\varepsilon]})\nabla\hat{V}'_{l,[N\varepsilon]-1}. \tag{7.98}$$

Once more, the boundary terms vanish. The summation can be rewritten as follows:

$$-\sigma \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} \nabla\tau(\hat{r}_{l,k})\nabla\hat{V}'_{l,k} = -\sigma \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} (\nabla\tau(\hat{r}_{l,k}))^2 + \sigma \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} \nabla\tau(\hat{r}_{l,k})\nabla\left(\tau(\hat{r}_{l,k}) - \hat{V}'_{l,k}\right) \tag{7.99}$$

$$= -\sigma \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} (\nabla\tau(\hat{r}_{l,k}))^2 - \sigma \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} \Delta\tau(\hat{r}_{l,k})\left(\tau(\hat{r}_{l,k}) - \hat{V}'_{l,k}\right), \tag{7.100}$$

where the last equality is true up to boundary terms that vanish as $N \to \infty$. The first term in (7.100) is negative and bounded as $N \to +\infty$. The remaining term is evaluated as follows. First of all, we have the following identity:

$$\Delta\tau(\hat{r}_{l,k}) = \tau'(\hat{r}_{l,k})\Delta\hat{r}_{l,k} + \frac{1}{2}\tau''(\eta_k)(\nabla\hat{r}_{l,k})^2 + \frac{1}{2}\tau''(\xi^*_k)(\nabla^*\hat{r}_{l,k})^2, \tag{7.101}$$

for some intermediate points ξ_k, ξ^*_k, so that we need to evaluate the following terms:

$$T_1 = \sigma\mathbb{E}\int_0^t \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} (\nabla\hat{r}_{l,k})^2\left(\tau(\hat{r}_{l,k}) - \hat{V}'_{l,k}\right) ds, \tag{7.102}$$

$$T^*_1 = \sigma\mathbb{E}\int_0^t \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} (\nabla^*\hat{r}_{l,k})^2\left(\tau(\hat{r}_{l,k}) - \hat{V}'_{l,k}\right) ds, \tag{7.103}$$

$$T_2 = \sigma\mathbb{E}\int_0^t \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} \Delta\hat{r}_{l,k}\left(\tau(\hat{r}_{l,k}) - \hat{V}'_{l,k}\right) ds. \tag{7.104}$$

We estimate T_1 and T_2 (T^*_1 is estimate analogously to T_1). By Cauchy-Schwarz, we have, for any $\alpha > 0$,

$$|T_1| \le \frac{\sigma\alpha}{2}\mathbb{E}\int_0^t \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} (\nabla\hat{r}_{l,k})^4 ds + \frac{\sigma}{2\alpha}\mathbb{E}\int_0^t \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} \left(\tau(\hat{r}_{l,k}) - \hat{V}'_{l,k}\right)^2 ds. \tag{7.105}$$

$$\le C\frac{\alpha}{2}\frac{1}{\sigma l^2} + C\frac{1}{2\alpha}l^2, \tag{7.106}$$

where we have used the fact that, for N large enough, $Nt/l \le Cl^2/\sigma$. Therefore $|T_1| \to 0$, if we choose, say, $\alpha = l^2\sqrt{\sigma}$.

The term T_2 is estimated thanks to the discrete Caccioppoli inequality, used as follows. Assuming, without loss of generality, $\varepsilon < 1/2$, we write

$$\sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} (\Delta \hat{r}_{l,k})^2 = \sum_{k=[N\varepsilon]}^{[N/2]} (\Delta \hat{r}_{l,k})^2 + \sum_{k=[N/2]}^{[N(1-\varepsilon)]} (\Delta \hat{r}_{l,k})^2. \tag{7.107}$$

By (7.58), we have

$$\sum_{k=[N\varepsilon]}^{[N/2]} (\Delta \hat{r}_{l,k})^2 \leq C \left(\frac{1}{\left(N - l - \left[\frac{N}{2} \right] \right)^2} \sum_{k=[N\varepsilon]}^{N-l} (\nabla^* \hat{r}_{l,k})^2 - \nabla^* \hat{r}_{l,[N\varepsilon]} \Delta \hat{r}_{l,[N\varepsilon]} a_{[N\varepsilon]}(r) \right). \tag{7.108}$$

The boundary term can be handled as before and this time gives a term of order $1/N^3$, so we have

$$\mathbb{E} \int_0^t \sum_{k=[N\varepsilon]}^{[N/2]} (\Delta \hat{r}_{l,k})^2 \leq C \frac{1}{N^2 \sigma} + \mathcal{O}\left(\frac{1}{N^3} \right). \tag{7.109}$$

Adjusting the proof of (7.58) we obtain, by entirely similar arguments

$$\mathbb{E} \int_0^t \sum_{k=[N/2]}^{[N(1-\varepsilon)]} (\Delta \hat{r}_{l,k})^2 \leq C \frac{1}{N^2 \sigma} + \mathcal{O}\left(\frac{1}{N^3} \right). \tag{7.110}$$

Putting everything together, we obtain, after the usual Cauchy-Schwarz argument,

$$|T_2| \leq C \frac{\alpha}{2N^2} + C \frac{l^2}{2\alpha}, \tag{7.111}$$

which vanishes, in the limit, if we take, say, $\alpha = lN$.

Finally, putting everything back together, we have obtained

$$\mathcal{F}(t) - \mathcal{F}(0) = W(t) - \limsup_{N \to +\infty} \sigma \int_0^t \sum_{k=[N\varepsilon]}^{[N(1-\varepsilon)]} \left((\nabla \hat{p}_{l,k})^2 + (\nabla \tau(\hat{r}_{l,k}))^2 \right) ds. \tag{7.112}$$

The inequality of Clausius (7.74) then immediately follows by taking the limit $t \to +\infty$.

Remark. Note that the higher regularity assumptions comes in order to estimate the error we make when replacing V' with τ. In the linear case, i.e. for $V'(x) = x$, the higher regularity assumptions are not required. In fact in this case $\tau(\rho) = \rho$ and no replacement is needed.

Chapter 8

Conclusions and final remarks

8.1 Summary

In this thesis we have studied the hydrodynamic limit for a isothermal anharmonic chain with hyperbolic scaling and boundary conditions.

We started from a microscopic dynamics, and we derived a macroscopic hyperbolic system of conservation laws to which we have found solutions compatible with the boundary conditions imposed at the microscopic level.

Since hyperbolic systems may develop shocks in finite time even with smooth initial conditions, we took into account the possibility of having non-smooth solutions and have constructed a set of *weak* solutions using the method of compensated compactness, appropriately adapted to a stochastic setting.

Moreover, since the solutions we have constructed came from a physical model, we were able to extract extra physical some information. In particular we were able to define thermodynamic transformations and to deduce the first and second law of thermodynamics.

8.2 The assumptions

Throughout this thesis we had several assumptions, and it is natural to ask which of those were there for technical reasons and which served a perhaps deeper purpose.

- **Assumptions on the interaction V:** The interaction has been designed so that the tension τ has a certain set of properties (see Appendix A). It also gave us a logarithmic Sobolev inequality (LSI, see Chapter 3). In order to have a LSI we need V to be either strictly convex or to be a bounded perturbation of the harmonic (i.e. quadratic) potential. With the latter, however, we end up

losing the condition of genuine nonlinearity (i.e. $\tau'' \neq 0$). Therefore, we must stick to a strictly convex interaction.

The condition $\tau'' \neq 0$ of genuine nonlinearity is equivalent to ask τ to be either strictly concave or strictly convex, and it is not at all clear which general features allow an interaction V to produce a strictly convex or strictly concave tension τ.

This is the reason why we ultimately chose (refer, once more, to Appendix A) a specific form for V. Nevertheless, this is the only instance where we had to do that: all the other properties of τ follow from the general features of V, and most of them follow from strict convexity.

- **The noise:** The noise is used to fix the temperature and provide ergodicity, that is it prevents us from having, in the limit, conservation laws different from the conservation of mass, momentum and energy. It is also used to easily control the Dirichlet forms, which in turn allow us to perform the one and the two block estimates (see Chapter 4).

  However, the noise on the momenta (generated by S_N) alone is enough to provide ergodicity *and* the one block estimate. Therefore, the noise on the positions (generated by $\tilde{S}_N$) is really introduced only to have a bound on the Dirichlet form $\tilde{D}_N$ involving the positions, and such a bound is needed in order to perform the two block estimate and in order to exchange the averages $\bar{\eta}_{l,k}$ with $\hat{\eta}_{l,k}$.

- **All the assumptions in Section 7.2:** The assumptions of smoothness and continuous approximation near the boundaries are motivated by our wish to define thermodynamic transformations and some thermodynamic quantities such as the work: if we were to lose these extra smoothness and continuity features, we may apply an external tension whose effect would not transmit to the rest of the chain, and so we would not be able to define thermodynamic transformations.

  The assumption of convergence of the energy is of a technical nature. We would not need it provided we are able to control moments higher than the second. However, we encounter a hard obstacle here, as at the present time our best tool available to perform such estimates is the relative entropy, and that can at most control the Hamiltonian, that is the second moments.

  Finally, the higher integrability estimates are of a purely technical nature, and they are added so we obtain the inequality of Clausius. Nevertheless, note

that all the terms that required us to add those assumptions come from the noise on the positions, and they would not be there if we could afford not to add the generator $\tilde{S}_N$ to the dynamics. In other words, we can avoid the higher integrability assumptions if we can prove by other means bounds on the Dirichlet form on the positions.

8.3 Final remarks and open problems

We have used genuine nonlinearity (i.e. $\tau'' \neq 0$) so we could follow the work of Sheared in [47] and obtain a bounded Lax entropy-entropy flux pair with bounded derivatives. We may have chosen a slightly different approach, and followed the work of Serre and Shearer in [46], instead. We would have still obtained a bounded Lax entropy-entropy flux pair, but now we could have allowed τ'' to vanish at exactly one point. Even if it would have been much easier to produce a V which gave $\tau''(\rho) = 0$ for exactly one value of ρ, V would not have still been a bounded perturbation of the harmonic potential, and so we would have still needed it to be strictly convex, and we would have ended up in exactly the same situation described in this thesis.

As pointed out the previous section, the noise on the positions is problematic in deriving the inequality of Clausius and it required us to add further assumptions. Moreover, this kind of noise is *not* physically motivated, as the noise is typically models scattering among particles, and the momenta are the relevant quantities, in a scattering problem. Therefore, it would be good, both technically *and* physically, to get rid of the noise on the positions and estimate the Dirichlet form $\tilde{D}_N$ only via the noise on the momenta.

This has recently been achieved by S. Olla and V. Letizia in [32] and uses some hypocoercivity tools. One main point in which their setting is different from ours is that they perform a *diffusive* space-time scaling in which the coefficient of the noisy generator (the equivalent of our σ) does *not* depend on the number of particles and stays constant in the hydrodynamic limit. It is still not clear to us how to adapt their result in a hydrodynamic limit with hyperbolic scaling and with σ that goes to infinity with the number of particles.

We worked in Lagrangian coordinates, that is our spatial variable labels a point on the anharmonic chain, rather than a point in the whole space (at which one may or may not find the chain). It is of curse a natural question to address the problem in Eulerian coordinates, but this comes with more than a few obstacles. First of all, the conservation of momentum in Eulerian coordinates comes with a term quadratic

in the velocity. Therefore, in order to pass weak limits within that term, we need estimates in L^p for some $p > 2$, and the relative entropy cannot help with that. This problem was avoided in the infamous paper by Olla, Varadhan and Yau [42], where the kinetic energy is taken not to be $p^2/2$, but is cut-off and grows linearly, rather than quadratically.

In the theory of nonlinear PDEs, a well-studied system in Eulerian coordinates are the isentropic Euler equations, where the pressure (as a function of the density) is taken to be $P(\rho) = \rho^\gamma$, for some $\gamma > 1$ (see, for example, [7, 12, 14, 33]), and it would be good to find solutions to these equations from a microscopic dynamics. However, it is not at all clear how to set up a microscopic *isentropic* dynamics.

Nevertheless, many other interesting problems look more approachable: for instance one may study a *adiabatic* transformation instead of a isothermal one (see [40]), or one can add a temperature profile (i.e. each atom of the chain has its own temperature, which is kept fixed, see [32]). Also, we may add a noise which conserves the energy as well as the momentum, in order to get the full 3×3 Euler system in Lagrangian coordinates (see [17]).

There are many other interesting models worth studying even in the smooth regime, and the number of opportunities grows even more if one takes into account that the problem of obtaining non-smooth solutions from hydrodynamic limits has hardly been studied to date.

Appendix A

Properties of the tension

In this section we shall give some technical properties about the tension τ, defined by

$$\tau(\rho) = F'(\rho), \qquad F(\rho) = \sup_{\gamma \in \mathbb{R}} \{\gamma\rho - G(\gamma)\}, \tag{A.1}$$

with

$$G(\gamma) = \log \int_{-\infty}^{+\infty} e^{\gamma r - V(r)} dr. \tag{A.2}$$

We will in fact prove the following

Proposition A.1. *Let the potential $V \in C^2(\mathbb{R})$ be strictly convex, in the sense that there exist $c_1, c_2 \in \mathbb{R}$ such that*

$$0 < c_1 \leq V''(r) \leq c_2, \quad \forall r \in \mathbb{R}. \tag{A.3}$$

Moreover, assume there exist some positive constants V''_+, V''_-, α, R such that

$$\begin{aligned} \left|V''(r) - V''_+\right| &< e^{-\alpha r}, \quad \forall r > R \\ \left|V''(r) - V''_-\right| &< e^{\alpha r}, \quad \forall r < -R \end{aligned}. \tag{A.4}$$

Then the following properties hold true.

i) The p-system 2.21 is strictly hyperbolic, meaning $\tau'(\rho) \geq c_1 > 0$ for all $\rho \in \mathbb{R}$;

ii) $\tau''(\rho)(\tau'(\rho))^{-5/4}$ and $\tau'''(\rho)(\tau'(\rho))^{-7/4}$ are in $L^2(\mathbb{R})$, while $\tau''(\rho)(\tau'(\rho))^{-3/4}$ and $\tau'''(\rho)(\tau'(\rho))^{-2}$ are in $L^\infty(\mathbb{R})$.

iii) $\tau'(\rho) \leq c_2$ for all $\rho \in \mathbb{R}$. Moreover, $\tau(\rho)/F(\rho) \to 0$ as $|\rho| \to \infty$.

Finally, suppose we can write

$$V(x) = \frac{r^2}{2} + \theta U(r), \tag{A.5}$$

for some $\theta \in \mathbb{R}$, where U is a smooth and convex mollification of the function $\kappa r \max\{r, 0\}/2 - \kappa r^2/2$, with $0 < \kappa < 1/2$.

iv) There exists $\theta_\kappa > 0$ such that, for $0 < |\theta| < \theta_\kappa$, the p-system (2.21) is genuinely nonlinear, meaning $\tau''(\rho) \neq 0$ for all $\rho \in \mathbb{R}$.

We prove the previous propositions through a series of lemmas. Before starting, fix $\pi, \gamma \in \mathbb{R}$. We denote by $\lambda_{\pi,\gamma}$ the probability measure on $\mathbb{R}^2$ defined by

$$\int_{\mathbb{R}^2} f(r,p) d\lambda_{\pi,\gamma}(r,p) = \frac{e^{-G(\gamma)}}{\sqrt{2\pi}} \int_{\mathbb{R}^2} f(r,p) e^{-(p-\pi)^2/2+\gamma r - V(r)} dr dp, \tag{A.6}$$

for any measurable $f : \mathbb{R}^2 \to \mathbb{R}$.

The first lemma we state is used in the proof of the one block estimate.

Lemma A.2. *Let $\alpha \in \mathbb{R}$. Then*

$$\int e^{\alpha V'(r)} d\lambda_{\pi,\gamma} \leq \exp\left(\alpha\gamma + \frac{c_2}{2}\alpha^2\right). \tag{A.7}$$

Proof. Let

$$A(\alpha,\gamma) = \log \int e^{\alpha V'(r)} d\lambda_{\pi,\gamma}. \tag{A.8}$$

Then, integrating by parts, we have

$$\partial_\alpha A(\alpha,\gamma) = \int_{-\infty}^{\infty} V'(r) \exp\left(\alpha V'(r) + \gamma r - V(r) - G(\gamma) - A(\alpha,\gamma)\right) dr \tag{A.9}$$

$$= -\int_{-\infty}^{\infty} \left(\frac{\partial}{\partial r} e^{-V(r)}\right) \exp\left(\alpha V'(r) + \gamma r - G(\gamma) - A(\alpha,\gamma)\right) dr \tag{A.10}$$

$$= \int_{-\infty}^{\infty} (\alpha V''(r) + \gamma) \exp\left(\alpha V'(r) + \gamma r - V(r) - G(\gamma) - A(\alpha,\gamma)\right) dr. \tag{A.11}$$

Since

$$c_1 \leq V''(r) \leq c_2, \quad \forall r \in \mathbb{R}, \tag{A.12}$$

if $\alpha > 0$ we obtain

$$\partial_\alpha A(\alpha,\gamma) \leq c_2\alpha + \gamma, \tag{A.13}$$

while, if $\alpha \leq 0$

$$\partial_\alpha A(\alpha,\gamma) \geq c_2\alpha + \gamma. \tag{A.14}$$

(A.13), together with (A.14) and $A(0,\gamma) = 0$ imply

$$A(\alpha,\gamma) \leq \alpha\gamma + \frac{c_2}{2}\alpha^2, \tag{A.15}$$

for all $\alpha \in \mathbb{R}$, from which the claim follows. □

Lemma A.3. *Let τ and G as in* (A.1) *and* (A.2). *Moreover, let c_1 and c_2 be as in* (A.3). *Then $c_2^{-1} \leq G''(\gamma) \leq c_1^{-1}$ for all $\gamma \in \mathbb{R}$. Moreover, $c_1 \leq \tau'(\rho) \leq c_2$ for all $\rho \in \mathbb{R}$.*

Proof. Let $f(r,p) = r$ and let $\rho(\gamma)$ be the expectation value of f with respect to $\lambda_{\pi,\gamma}$. We have

$$\rho(\gamma) = \int r \exp\left(\gamma r - V(r) - G(\gamma)\right) dr = G'(\gamma) \tag{A.16}$$

and so

$$G''(\gamma) = \int r^2 \exp\left(\gamma r - V(r) - G(\gamma)\right) dr - \int r G'(\gamma) \exp\left(\gamma r - V(r) - G(\gamma)\right) dr$$

$$= \int r^2 d\lambda_{\pi,\gamma} - \left(\int r d\lambda_{\pi,\gamma}\right)^2 \tag{A.17}$$

$$= \int (r^2 - \rho(\gamma)^2) d\lambda_{\pi,\gamma} \tag{A.18}$$

$$= \int (r - \rho(\gamma))^2 d\lambda_{\pi,\gamma} + 2\rho \int (r - \rho(\gamma)) d\lambda_{\pi,\gamma} \tag{A.19}$$

$$= \int (r - \rho(\gamma))^2 d\lambda_{\pi,\gamma} > 0. \tag{A.20}$$

Therefore G is smooth and convex on $\mathbb{R}$, and so is its Legendre transform F. Then, integrating by parts yields

$$1 = \int (r - \rho(\gamma)) \left(V'(r) - \gamma\right) d\lambda_{\pi,\gamma} \tag{A.21}$$

$$= \int (r - \rho(\gamma)) \left(V'(r) - V'(\rho(\gamma))\right) d\lambda_{\pi,\gamma} = \int (r - \rho(\gamma))^2 V''(\tilde{r}) d\lambda_{\pi,\gamma}, \tag{A.22}$$

where $\tilde{r}$ is between r and $\rho(\gamma)$. Recalling that $c_1 \leq V''(r) \leq c_2$, this implies

$$c_2^{-1} \leq \int (r - \rho(\gamma))^2 d\lambda_{\pi,\gamma} \leq c_1^{-1}, \tag{A.23}$$

that is

$$c_2^{-1} \leq G''(\gamma) \leq c_1^{-1}. \tag{A.24}$$

Finally, since G is smooth, the supremum in (A.1) is attained when $G'(\gamma) = \rho$. But since we have proven that G' is invertible (G'' is strictly positive), the equation $G'(\gamma) = \rho$ has exactly one solution for any $\rho \in \mathbb{R}$. We claim that this solution is precisely $\tau(\rho)$, as defined in (A.1). In fact, let $\rho \in \mathbb{R}$ and let $\gamma = \gamma(\rho)$ solve $G'(\gamma) = \rho$. We have

$$F(\rho) = \rho\gamma - G(\gamma). \tag{A.25}$$

This implies

$$\tau(\rho) = F'(\rho) = \gamma + \rho\gamma'(\rho) - G'(\gamma)\gamma'(\rho) = \gamma \tag{A.26}$$

and, in turn,

$$\tau'(\rho) = \gamma'(\rho) = \frac{1}{G''(\gamma)}. \tag{A.27}$$

Therefore we have the desired bound on τ' and the proof is complete. □

Remark. Since there is a 1:1 correspondence between γ and ρ via the equation $\rho = G'(\gamma)$, we can always express γ as a function of ρ and vice versa. For this reason, we shall adopt the following notation. When writing a chain of equalities (or inequalities), the object at the far left tells us which, between γ and ρ is the independent variable. To be precise, the writing

$$f(\gamma) = g(\gamma, \rho)$$

stands for

$$f(\gamma) = g(\gamma, G'(\gamma))$$

while

$$f(\rho) = g(\gamma, \rho)$$

stands for

$$f(\rho) = g(\tau(\rho), \rho).$$

Corollary A.4. *Let τ and F be defined by* (A.1). *Moreover, let c_2 be as in* (A.3). *Then $\tau'(\rho) \leq c_2$ for all $\rho \in \mathbb{R}$ and $\tau(\rho)/F(\rho) \to 0$ as $|\rho| \to +\infty$.*

Proof. τ' is bounded from above thanks to (A.3). Since $\tau = F'$, it is enough to show that $F(\rho)$ grows at least quadratically and $F'(\rho)$ grows at most linearly in ρ. We consider $\rho \to +\infty$, as $\rho \to -\infty$ will be analogous. Since $\tau = F'$, we have $c_1 \leq F''(\rho) \leq c_2$. Integrating this twice we obtain

$$F'(0) + c_1\rho \leq F'(\rho) \leq F'(0) + c_2\rho. \tag{A.28}$$

and

$$F(0) + F'(0)\rho + \frac{c_1}{2}\rho^2 \leq F(\rho) \leq F(0) + F'(0)\rho + \frac{c_2}{2}\rho^2. \tag{A.29}$$

Therefore, since $c_1, c_2 > 0$, F grows at least quadratically and F' at most linearly, and the conclusion follows. □

Since we have shown that τ' is bounded from below, the L^∞ bounds in part *iii)* of Proposition A.1 follow from the following

Lemma A.5. *Let τ be as in A.1. Then both τ'' and τ''' are bounded.*

Proof. First of all let's note that

$$\tau''(\rho) = -\frac{G'''(\gamma)\tau'(\rho)}{G''(\gamma)^2} = -G'''(\gamma)\tau'(\rho)^3 \tag{A.30}$$

and

$$\tau'''(\rho) = -G^{(iv)}(\gamma)\tau'(\rho)^4 - 3G'''(\gamma)\tau'(\rho)^2\tau''(\rho) \tag{A.31}$$

$$= 3\frac{\tau''(\rho)^2}{\tau'(\rho)} - G^{(iv)}(\gamma)\tau'(\rho)^4. \tag{A.32}$$

Therefore it is enough to prove that G''' and $G^{(iv)}$ are bounded. We have

$$G'''(\gamma) = \frac{d}{d\gamma}\int (r-\rho)^2 d\lambda_{\pi,\gamma} \tag{A.33}$$

$$= \int (r-\rho)^2(r - G'(\gamma))d\lambda_{\pi,\gamma} - 2\rho'(\gamma)\int (r-\rho)d\lambda_{\pi,\gamma} \tag{A.34}$$

$$= \int (r-\rho)^3 d\lambda_{\pi,\gamma} \tag{A.35}$$

and

$$G^{(iv)}(\gamma) = \int (r-\rho)^4 d\lambda_{\pi,\gamma} - 3\rho'(\gamma)\int (r-\rho)^2 d\lambda_{\pi,\gamma} \tag{A.36}$$

$$= \int (r-\rho)^4 d\lambda_{\pi,\gamma} - 3\left(\int (r-\rho)^2 d\lambda_{\pi,\gamma}\right)^2 \tag{A.37}$$

$$= \int (r-\rho)^4 d\lambda_{\pi,\gamma} - 3G''(\gamma)^2 \tag{A.38}$$

Moreover, since

$$\int |r-\rho|^3 d\lambda_{\pi,\gamma} \leq \frac{1}{2}\int (r-\rho)^2 d\lambda_{\pi,\gamma} + \frac{1}{2}\int (r-\rho)^4 d\lambda_{\pi,\gamma}, \tag{A.39}$$

it is sufficient to show that $\int (r-\rho)^4 d\lambda_{\pi,\gamma}$ is a bounded function of γ. In order to do so, let $\delta \in \mathbb{R}$ be the point at which the function $r \mapsto \gamma r - V(r)$ attains its maximum (note that δ will depend on γ). Since V is strictly convex, δ is the unique root the equation $V'(\delta) = \gamma$. We claim that $\int (r-\rho)^4 d\lambda_{\pi,\gamma}$ is bounded provided $\int (r-\delta)^2 d\lambda_{\pi,\gamma}$ and $|\rho - \delta|$ are. In fact, from

$$\int (r-\rho)^4 d\lambda_{\pi,\gamma} \leq 8\int (r-\delta)^4 d\lambda_{\pi,\gamma} + 8\int (\rho-\delta)^4 d\lambda_{\pi,\gamma} \tag{A.40}$$

and

$$3\int (r-\delta)^2 d\lambda_{\pi,\gamma} = \int (r-\delta)^3 \left(V'(r) - \gamma\right) d\lambda_{\pi,\gamma} \tag{A.41}$$

$$= \int (r-\delta)^3 (V'(r)-V'(\delta)) d\lambda_{\pi,\gamma} = \int V''(\tilde{r})(r-\delta)^4 d\lambda_{\pi,\gamma} \geq c_1 \int (r-\delta)^4 d\lambda_{\pi,\gamma}, \quad \text{(A.42)}$$

for some $\tilde{r}$ between r and δ, we obtain

$$\int (r-\rho)^4 d\lambda_{\pi,\gamma} \leq \frac{24}{c_1} \int (r-\delta)^2 d\lambda_{\pi,\gamma} + 8(\rho-\delta)^4. \quad \text{(A.43)}$$

The boundedness of $\int (r-\delta)^2 d\lambda_{\pi,\gamma}$ is in turn given by

$$1 = \int (r-\delta)(V'(r)-\gamma) d\lambda_{\pi,\gamma} = \int V''(\tilde{r})(r-\delta)^2 d\lambda_{\pi,\gamma}. \quad \text{(A.44)}$$

Finally, a bound for $|\rho - \delta|$ follows from

$$\int (r-\delta)^2 d\lambda_{\pi,\gamma} = \int (r-\rho)^2 d\lambda_{\pi,\gamma} + (\rho-\delta)^2 + 2(\rho-\delta) \int (r-\rho) d\lambda_{\pi,\gamma} \quad \text{(A.45)}$$

$$= G''(\gamma) + (\rho-\delta)^2. \quad \text{(A.46)}$$

□

Let's now prove the L^2 bounds in part *iii*) of Proposition A.1. By

$$\tau''(\rho) = -\tau'(\rho)^3 G'''(\gamma), \quad \text{(A.47)}$$

$$\tau'''(\rho) = \frac{3}{\tau'(\rho)} \tau''(\rho)^2 - \tau'(\rho)^4 G^{(iv)}(\gamma), \quad \text{(A.48)}$$

and the fact that τ' is bounded away from zero, τ'' is in L^2 if and only if $G'''(\tau)$ is. Moreover τ''' is in L^2 provided both $(\tau'')^2$ and $G^{(iv)}(\tau)$ are. But $\tau'' \in L^\infty \cap L^2$ implies $\tau'' \in L^p$ for all $p \geq 2$: in particular $\tau'' \in L^4$, and so $(\tau'')^2 \in L^2$ Finally, via the substitution $\gamma = \tau(\rho)$ (or, equivalently, $\rho = G'(\gamma)$), for $f \in \{G''', G^{(iv)}\}$ we have

$$\int_{-\infty}^{+\infty} f(\tau(\rho))^2 d\rho = \int_{-\infty}^{+\infty} \frac{f(\gamma)^2}{\tau'(G'(\gamma))} d\gamma, \quad \text{(A.49)}$$

Therefore using once more the boundedness from below of τ', the L^2 bounds in part *iii*) of Proposition A.1 follow from the following

Lemma A.6. *Let G be defined by A.2. Then both G''' and $G^{(iv)}$ are in $L^2(\mathbb{R})$.*

Proof. We observe that, since G''' and $G^{(iv)}$ are bounded, it is enough to prove that they vanish quickly enough at infinity. We shall prove that they indeed vanish and the decay is exponential. Since

$$G'''(\gamma) = \int (r-\rho)^3 d\lambda_{\pi,\gamma} \quad \text{(A.50)}$$

and

$$G^{(iv)}(\gamma) = \int (r-\rho)^4 d\lambda_{\pi,\gamma} - 3\left(\int (r-\rho)^2 d\lambda_{\pi,\gamma}\right)^2, \tag{A.51}$$

we need to estimate the quantities

$$\int (r-\rho)^m d\lambda_{\pi,\gamma} = \int_{-\infty}^{\infty} (r-\rho)^m \exp(\gamma r - V(r) - G(\gamma)) dr \tag{A.52}$$

for integers $2 \leq m \leq 4$, as well as

$$\rho = \int r d\lambda_{\pi,\gamma} = \int_{-\infty}^{\infty} r \exp(\gamma r - V(r) - G(\gamma)) dr, \tag{A.53}$$

as $\gamma \to \infty$. We will deal with $\gamma \to +\infty$, as the case $\gamma \to -\infty$ is analogous. Recalling that δ is such that $\gamma = V'(\delta)$, we can write

$$\exp(\gamma r - V(r) - G(\gamma)) = \exp\left(\gamma\delta - V(\delta) - G(\gamma) - \frac{V''(\tilde{r})}{2}(r-\delta)^2\right), \tag{A.54}$$

for some $\tilde{r}$ between r and δ. Also, integrating $c_1 \leq V''(r) \leq c_2$ between 0 and δ gives

$$c_2^{-1}\gamma - c_2^{-1}V'(0) \leq \delta \leq c_1^{-1}\gamma - c_1^{-1}V'(0), \tag{A.55}$$

and so $\delta \to +\infty$ as $\gamma \to +\infty$.

Next, we show the following:

$$\left|\int_{-\infty}^{\infty} (r-\delta)^m \exp\left(-\frac{V''(\tilde{r})}{2}(r-\delta)^2\right) dr - \int_{-\infty}^{\infty} r^m \exp\left(-\frac{V_+''}{2}r^2\right) dr\right| \leq e^{-\tilde{\alpha}\gamma} \tag{A.56}$$

for integers $0 \leq m \leq 4$, some $0 < \tilde{\alpha} \leq \alpha$ and γ large enough. The constant V_+'' is defined in (A.4). Let $a \in (0, \gamma)$ be a multiple of γ. We divide the domain of integration as follows: $(-\infty, +\infty) = (-\infty, \delta - a) \cup (\delta - a, \delta + a) \cup (\delta + a, +\infty)$. The integrals over the unbounded domains vanish exponentially fast as $\gamma \to \infty$. In fact, since there exists $0 < \beta < c_1$ such that, for γ large enough,

$$|r-\delta|^m \exp\left(-\frac{V''(\tilde{r})}{2}(r-\delta)^2\right) \leq \exp\left(-\frac{\beta}{2}(r-\delta)^2\right), \tag{A.57}$$

we have

$$\int_{\delta+a}^{\infty} |r-\delta|^m \exp\left(-\frac{V''(\tilde{r})}{2}(r-\delta)^2\right) dr \leq \int_{\delta+a}^{\infty} \exp\left(-\frac{\beta}{2}(r-\delta)^2\right) dr \tag{A.58}$$

$$= \int_0^{\infty} \exp\left(-\frac{\beta}{2}(r+a)^2\right) dr \leq \exp\left(-\frac{\beta}{2}a^2\right) \int_0^{\infty} \exp\left(-\frac{\beta}{2}r^2\right) dr, \tag{A.59}$$

which vanishes exponentially fast since a is a multiple of γ. The case of $(-\infty, \delta - a)$ is analogous. From similar calculations we obtain also that

$$\int_{-\infty}^{\infty} r^m \exp\left(-\frac{V_+''}{2} r^2\right) dr = \int_{-\infty}^{\infty} (r-\delta)^m \exp\left(-\frac{V_+''}{2}(r-\delta)^2\right) dr \tag{A.60}$$

$$= \int_{\delta-a}^{\delta+a} (r-\delta)^m \exp\left(-\frac{V_+''}{2}(r-\delta)^2\right) dr + R(\gamma), \tag{A.61}$$

with $R(\gamma)$ vanishing exponentially fast in γ. Therefore (A.56) follows provided

$$\left| \int_{\delta-a}^{\delta+a} (r-\delta)^m \left(\exp\left(-\frac{V''(\tilde{r})}{2}(r-\delta)^2\right) - \exp\left(-\frac{V_+''}{2}(r-\delta)^2\right)\right) dr \right| \leq e^{-\tilde{\alpha}\gamma}. \tag{A.62}$$

Recall that $\tilde{r}$ is between r and δ, and therefore is in $(\delta - a, \delta + a)$. We recall that $\delta \to +\infty$ as $\gamma \to +\infty$. Moreover, $\delta - a$ goes to $+\infty$ as well, provided $a < c_2^{-1}\gamma$. For such a choice of a, $\tilde{r}$ goes to $+\infty$ as $\gamma \to +\infty$ and so, because of the asymptotic normality of V, for γ large enough we have

$$\frac{1}{2}|V''(\tilde{r}) - V_+''|(r-\delta)^2 \leq e^{-\tilde{\alpha}\gamma}, \tag{A.63}$$

for some $\tilde{\alpha}$ possibly smaller than α. This implies

$$\left| \int_{\delta-a}^{\delta+a} (r-\delta)^m \exp\left(-\frac{V''(\tilde{r})}{2}(r-\delta)^2\right) dr - \int_{\delta-a}^{\delta+a} (r-\delta)^m \exp\left(-\frac{V_+''}{2}(r-\delta)^2\right) dr \right|$$

$$\leq \int_{\delta-a}^{\delta+a} |r-\delta|^m \exp\left(-\frac{V_+''}{2}(r-\delta)^2\right) \left| \exp\left(-\frac{1}{2}(V''(\tilde{r}) - V_+'')(r-\delta)^2\right) - 1 \right| dr \tag{A.64}$$

$$\leq 2a^m e^{-\tilde{\alpha}\gamma} \int_{\delta-a}^{\delta+a} \exp\left(-\frac{V_+''}{2}(r-\delta)^2\right) dr \tag{A.65}$$

$$\leq 2\sqrt{\frac{2\pi}{V_+''}} a^m e^{-\tilde{\alpha}\gamma} \leq e^{-\tilde{\alpha}\gamma} \tag{A.66}$$

for γ large enough and a possibly different choice of $\tilde{\alpha}$, so that (A.56) is proven.

We have now all we need to prove the actual theorem. Note the following:

$$1 = \int_{-\infty}^{\infty} \exp(\gamma r - V(r) - G(\gamma)) dr \tag{A.67}$$

$$= \int_{-\infty}^{\infty} \exp\left(\gamma\delta - V(\delta) - G(\gamma) - \frac{V''(\tilde{r})}{2}(r-\delta)^2\right) dr, \tag{A.68}$$

implies

$$\exp(G(\gamma) - \gamma\delta + V(\delta)) = \int_{-\infty}^{\infty} \exp\left(-\frac{V''(\tilde{r})}{2}(r-\delta)^2\right) dr, \tag{A.69}$$

the left hand side is bounded away from zero, as the right hand side is. This, together with (A.56) for $m = 0$ tells us that $\exp(\gamma\delta - V(\delta) - G(\gamma))$ converges to $\sqrt{V_+''/(2\pi)}$ exponentially fast. In particular, $\exp(\gamma\delta - V(\delta) - G(\gamma))$ is bounded.

Next, we show that $\rho - \delta \to 0$ exponentially fast. We write

$$\rho - \delta = \int_{-\infty}^{\infty} (r-\delta) \exp\left(\gamma\delta - V(\delta) - G(\gamma) - \frac{V''(\tilde{r})}{2}(r-\delta)^2\right) dr \tag{A.70}$$

$$= \exp\left(\gamma\delta - V(\delta) - G(\gamma)\right) \int_{-\infty}^{\infty} (r-\delta) \exp\left(-\frac{V''(\tilde{r})}{2}(r-\delta)^2\right) dr, \tag{A.71}$$

which converges to zero exponentially fast, as $\exp\left(\gamma\delta - V(\delta) - G(\gamma)\right)$ is bounded and the integral converges to

$$\int_{-\infty}^{\infty} r \exp\left(-\frac{V_+''}{2} r^2\right) dr = 0 \tag{A.72}$$

exponentially fast. Next, we write

$$G''(\gamma) = \int (r-\rho)^2 d\lambda_{\pi,\gamma} = \int (r^2 - \rho^2) d\lambda_{\pi,\gamma} \tag{A.73}$$

$$= \int_{-\infty}^{\infty} (r-\delta)^2 \exp\left(\gamma\delta - V(\delta) - G(\gamma) - \frac{V''(\tilde{r})}{2}(r-\delta)^2\right) dr - (\rho-\delta)^2. \tag{A.74}$$

The term $(\rho - \delta)^2$ goes to zero, while the integral, and so $G''(\gamma)$, converges to

$$\sqrt{\frac{V_+''}{2\pi}} \int_{-\infty}^{\infty} r^2 \exp\left(-\frac{V_+''}{2} r^2\right) dr = \frac{1}{V_+''} \tag{A.75}$$

exponentially fast. $G'''(\gamma)$ goes to zero exponentially fast. In fact from

$$(r-\rho)^3 = (r-\delta)^3 + (\delta-\rho)\left((r-\rho)^2 + (r-\rho)(r-\delta) + (r-\delta)^2\right) \tag{A.76}$$

and after integration, the first term vanishes, in the limit, by symmetry. Moreover, all the other terms vanish, after integration, as they are bounded terms multiplied by $\delta - \rho$.

Finally, we deal with $G^{(iv)}$, which will vanish exponentially fast as well. This time, though, we have the difference of two non-vanishing terms, so we do need to pay some extra attention. The negative term $-3G''(\gamma)^2$ converges to $-3/(V_+'')^2$. On the other hand, the quartic term decomposes as

$$(r-\rho)^4 = (r-\delta)^4 + (\delta-\rho)(2r-\rho-\delta)\left((r-\rho)^2 + (r-\delta)^2\right) \tag{A.77}$$

$$= (r-\delta)^4 + (\delta-\rho)\left(2(r-\rho)^3 + 2(r-\delta)^3 + (\rho-\delta)(r-\rho)^2 + (\delta-\rho)(r-\delta)^2\right). \tag{A.78}$$

Again, all the terms that multiply $\delta - \rho$ are, after integration, bounded, and therefore the only term which survives is $(r-\delta)^4$, whose integral converges to

$$\sqrt{\frac{V_+''}{2\pi}} \int_{-\infty}^{\infty} r^4 \exp\left(-\frac{V_+''}{2} r^2\right) dr = \frac{3}{(V_+'')^2}. \tag{A.79}$$

Putting everything together we obtain that $G^{(iv)}(\gamma)$ converges exponentially fast to zero, and the proof is complete. □

We now deal with the question of genuine nonlinearity. First of all we make the following remark.

Remark. Suppose $V(r) = r^2/2 + \theta U(r)$. Then, if U is an even function, the function τ'' vanishes at the origin for all $\theta \in \mathbb{R}$. In particular we can never have genuine nonlinearity.

In order to see this, it is enough to show that F is even: in fact in this case its third derivative, τ'', is odd and so $\tau''(0) = 0$. F is indeed even:

$$F(-\rho) = \sup_{\gamma \in \mathbb{R}} \{-\gamma\rho - G(\gamma)\} = \sup_{\gamma \in \mathbb{R}} \{\gamma\rho - G(-\gamma)\} = F(\rho), \tag{A.80}$$

since

$$F(-\gamma) = \log \int_{-\infty}^{\infty} \exp\left(-\gamma r - V(r)\right) dr \tag{A.81}$$

$$= \log \int_{-\infty}^{\infty} \exp\left(\gamma r - V(-r)\right) dr = G(\gamma). \tag{A.82}$$

Thanks to the remark, in order to have genuine nonlinearity we must look among non-symmetric interaction. Nevertheless, apart from that, it is not known to us which features a potential must possess in order to ensure $\tau'' \neq 0$. Therefore, we shall only give one family of examples which work.

Proposition A.7. *Suppose we can write*

$$V(r) = \frac{r^2}{2} + \theta U(r), \tag{A.83}$$

where $U(r)$ is a smooth and convex mollification of the function $\kappa r \max\{r, 0\}/2 - \kappa r^2/2$, with $0 < \kappa < 1/2$. Then there exist $\theta_\kappa > 0$, depending on κ, such that, for $0 < |\theta| < \theta_\kappa$, τ is either strictly convex or strictly concave, meaning that either $\tau''(\rho) > 0$ or $\tau''(\rho) < 0$ for all $\rho \in \mathbb{R}$.

Proof. Since

$$\tau''(\rho) = -\tau'(\rho)G'''(\gamma), \tag{A.84}$$

with $\tau' > 0$, the sign of τ'' is the same as the sign of $-G'''$. Therefore we need to study

$$-G'''(\gamma) = -\int (r-\rho)^3 d\lambda_{\pi,\gamma}, \tag{A.85}$$

with $\gamma \in \mathbb{R}$. Let $|r|_+ = \max\{r, 0\}$. In order to make things slightly less technical, we use directly $U(r) = \kappa r|r|_+/2 - \kappa r^2/2$ instead of its smooth mollification (note that U is already everywhere differentiable and the mollification only adds the existence of U'' at the origin).

$$V(r) = \frac{r^2}{2} + \theta\left(\frac{\kappa}{2}r|r|_+ - \frac{\kappa}{2}r^2\right) = \frac{a}{2}r^2 + \theta W(r), \tag{A.86}$$

where $a = 1-\theta\kappa$ and $W(r) = \kappa r|r|_+/2$. Then we notice that, by the usual integration by parts trick, we have

$$\int (r-\rho)^2(V'(r)-\gamma)d\lambda_{\pi,\gamma} = 2\int (r-\rho)d\lambda_{\pi,\gamma} = 0. \tag{A.87}$$

Therefore we write

$$a\int (r-\rho)^3 d\lambda_{\pi,\gamma} = \int (r-\rho)^2(ar - a\rho - V'(r) + \gamma)d\lambda_{\pi,\gamma} \tag{A.88}$$

$$= \int (r-\rho)^2(\gamma - a\rho - \theta W'(r))d\lambda_{\pi,\gamma}, \tag{A.89}$$

from which, together with $W'(r) = \kappa|r|_+$, follows

$$-aG'''(\gamma) = \theta\int (r-\rho)^2 W'(r)d\lambda_{\pi,\gamma} - (\gamma - a\rho)G''(\gamma) \tag{A.90}$$

$$= \theta\kappa\int |r|_+(r-\rho)^2 d\lambda_{\pi,\gamma} + (a\rho - \gamma)G''(\gamma). \tag{A.91}$$

We evaluate

$$\int |r|_+(r-\rho)^2 d\lambda_{\pi,\gamma} = e^{-G(\gamma)}\int_0^\infty r(r-\rho)^2 e^{\gamma r - \frac{r^2}{2}}dr = \tag{A.92}$$

$$= e^{-G(\gamma)}\int_0^\infty r^3 e^{\gamma r - \frac{r^2}{2}}dr - 2\rho e^{-G(\gamma)}\int_0^\infty r^2 e^{\gamma r - \frac{r^2}{2}}dr + \rho^2 e^{-G(\gamma)}\int_0^\infty r e^{\gamma r - \frac{r^2}{2}}dr. \tag{A.93}$$

Now, for $m \in \mathbb{N}$, $m \geq 1$,

$$\int_0^\infty r^m e^{\gamma r - \frac{r^2}{2}}dr = \frac{d}{d\gamma}\int_0^\infty r^{m-1}e^{\gamma r - \frac{r^2}{2}}dr, \tag{A.94}$$

with

$$\int_0^\infty e^{\gamma r - \frac{r^2}{2}} dr = e^{\frac{\gamma^2}{2}} \int_{-\gamma}^\infty e^{-\frac{r^2}{2}} dr. \tag{A.95}$$

Therefore, setting $\Phi(\gamma) := \int_{-\gamma}^\infty e^{-\frac{r^2}{2}} dr$ and noting that $\Phi'(\gamma) = e^{-\frac{\gamma^2}{2}}$ yields

$$\int_0^\infty r e^{\gamma r - \frac{r^2}{2}} dr = \frac{d}{d\gamma}\left(e^{\frac{\gamma^2}{2}}\Phi(\gamma)\right) = \gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma) + 1, \tag{A.96}$$

$$\int_0^\infty r^2 e^{\gamma r - \frac{r^2}{2}} dr = \frac{d}{d\gamma}\int_0^\infty r e^{\gamma r - \frac{r^2}{2}} dr = \gamma^2 e^{\frac{\gamma^2}{2}}\Phi(\gamma) + e^{\frac{\gamma^2}{2}}\Phi(\gamma) + \gamma \tag{A.97}$$

$$\int_0^\infty r^3 e^{\gamma r - \frac{r^2}{2}} dr = \frac{d}{d\gamma}\int_0^\infty r^2 e^{\gamma r - \frac{r^2}{2}} dr \tag{A.98}$$

$$= \gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma) + 1 + 2\gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma) + \gamma^3 e^{\frac{\gamma^2}{2}}\Phi(\gamma) + \gamma^2 + 1 \tag{A.99}$$

$$= \gamma^3 e^{\frac{\gamma^2}{2}}\Phi(\gamma) + 3\gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma) + \gamma^2 + 2. \tag{A.100}$$

Putting everything together we obtain

$$e^{G(\gamma)} \int |r|_+ (r-\rho)^2 d\lambda_{\pi,\gamma} = \gamma^3 e^{\frac{\gamma^2}{2}}\Phi(\gamma) + 3\gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma) + 2 + \tag{A.101}$$

$$- 2\rho(\gamma^2 e^{\frac{\gamma^2}{2}}\Phi(\gamma) + e^{\frac{\gamma^2}{2}}\Phi(\gamma) + \gamma) + \rho^2(\gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma) + 1) \tag{A.102}$$

$$= (2 + \gamma^2 - 2\rho\gamma + \rho^2) + (\gamma^3 - 2\rho\gamma^2 + (3+\rho^2)\gamma - 2\rho)e^{\frac{\gamma^2}{2}}\Phi(\gamma) \tag{A.103}$$

$$= (2 + (\gamma-\rho)^2) + \left(3\gamma - 2\rho + \gamma(\gamma-\rho)^2\right) e^{\frac{\gamma^2}{2}}\Phi(\gamma). \tag{A.104}$$

Next, we have to evaluate the term proportional to $G''(\gamma)$ in (A.91), which can be written as follows

$$a\rho - \gamma = a\rho - \int V'(r) d\lambda_{\pi,\gamma} \tag{A.105}$$

$$= a\rho - \int (ar + \theta W'(r)) d\lambda_{\pi,\gamma} \tag{A.106}$$

$$= -\theta \int W'(r) d\lambda_{\pi,\gamma} \tag{A.107}$$

$$= -\theta\kappa e^{-G(\gamma)} \int_0^\infty r e^{\gamma r - \frac{r^2}{2}} dr \tag{A.108}$$

$$= -\theta\kappa e^{-G(\gamma)}(1 + \gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma)). \tag{A.109}$$

We discuss the case $\theta > 0$, which will give $\tau'' > 0$. The case $\theta < 0$ is analogous and gives $\tau'' < 0$. Note that $1 + \gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma) > 0$ for $\gamma \geq 0$. On the other hand, for $\gamma < 0$,

$$\int_{-\gamma}^\infty e^{-\frac{r^2}{2}} dr < \frac{e^{-\frac{\gamma^2}{2}}}{-\gamma}, \tag{A.110}$$

which implies

$$\gamma\Phi(\gamma) > -e^{-\frac{\gamma^2}{2}}. \tag{A.111}$$

Therefore we have $1 + \gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma) > 0$ also for $\gamma < 0$ and, as a consequence

$$\gamma > a\rho = (1 - \theta\kappa)\rho, \quad \forall\gamma \in \mathbb{R}. \tag{A.112}$$

Putting everything together we obtain

$$\begin{aligned} -\frac{ae^{G(\gamma)}}{\theta\kappa}G'''(\gamma) &= \left(2 - G''(\gamma) + (\gamma - \rho)^2\right) + \\ &+ \left(\gamma(3 - G''(\gamma)) - 2\rho + \gamma(\gamma - \rho)^2\right) e^{\frac{\gamma^2}{2}}\Phi(\gamma). \end{aligned} \tag{A.113}$$

Since

$$V''(r) = \begin{cases} 1, & r > 0 \\ 1 - \theta\kappa, & r < 0 \end{cases}, \tag{A.114}$$

in order to have $V'' > 0$ we need $\theta < 1/\kappa$, and, since $0 < k < 1/2$, we are allowed to fix $\theta < 1$. Under this assumption we have $a = 1 - \theta\kappa > 0$ and

$$1 \leq G''(\gamma) \leq \frac{1}{1 - \theta\kappa} < 2. \tag{A.115}$$

We show that $-G'''$ (and therefore τ'') is positive fo $\gamma \leq 0$. Using $1 \leq G''(\gamma) < 2$ yields

$$-\frac{ae^{G(\gamma)}}{\theta\kappa}G'''(\gamma) > (2\gamma - 2\rho + \gamma(\gamma - \rho)^2)e^{\frac{\gamma^2}{2}}\Phi(\gamma). \tag{A.116}$$

Moreover, using $\gamma > (1 - \theta\kappa)\rho$ and $\gamma\Phi(\gamma) > -e^{-\frac{\gamma^2}{2}}$ gives

$$-\frac{ae^{G(\gamma)}}{\theta\kappa}G'''(\gamma) > -\frac{2\theta\kappa}{1 - \theta\kappa}\gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma) + (\gamma - \rho)^2\gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma) \tag{A.117}$$

$$> -\frac{2\theta\kappa}{1 - \theta\kappa}\gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma) \geq 0. \tag{A.118}$$

Therefore $\tau''(\rho) > 0$ if $\gamma \leq 0$ and $0 < \theta < 1$.

For $\gamma > 0$ we have to be more careful. First of all we note that

$$a\rho - \gamma = -\theta\kappa e^{-G(\gamma)}\left(1 + \gamma e^{\frac{\gamma^2}{2}}\Phi(\gamma)\right) \tag{A.119}$$

implies

$$\gamma - \rho = \frac{\theta\kappa}{1 - \theta\kappa}e^{-G(\gamma)}\left(\gamma\left(e^{\frac{\gamma^2}{2}}\Phi(\gamma) - e^{G(\gamma)}\right) + 1\right). \tag{A.120}$$

This, together with

$$e^{G(\gamma)} = \int_{-\infty}^{\infty} e^{\gamma r - V(r)}dr \tag{A.121}$$

$$= \int_0^\infty e^{\gamma r - \frac{r^2}{2}} dr + \int_{-\infty}^0 e^{\gamma r - \frac{1-\theta\kappa}{2} r^2} dr \tag{A.122}$$

$$= e^{\frac{\gamma^2}{2}} \Phi(\gamma) + \frac{1}{\sqrt{1-\theta\kappa}} e^{\frac{\gamma^2}{2(1-\theta\kappa)}} \Phi\left(-\frac{\gamma}{\sqrt{1-\theta\kappa}}\right) \tag{A.123}$$

and $\gamma > 0$ gives

$$\gamma - \rho = \frac{\theta\kappa}{1-\theta\kappa} e^{-G(\gamma)} \left(1 + \frac{-\gamma}{\sqrt{1-\theta\kappa}} e^{\frac{\gamma^2}{2(1-\theta\kappa)}} \Phi\left(-\frac{\gamma}{\sqrt{1-\theta\kappa}}\right)\right) \tag{A.124}$$

$$> e^{-G(\gamma)} \frac{\theta\kappa}{1-\theta\kappa} \left(1 - e^{\frac{\gamma^2}{2(1-\theta\kappa)}} e^{-\frac{\gamma^2}{2(1-\theta\kappa)}}\right) = 0. \tag{A.125}$$

Therefore $\gamma - \rho > 0$ if $\gamma > 0$ (note here that $\gamma - \rho$ is trivially positive for $\gamma \leq 0$, too). From this we get

$$-\frac{a e^{G(\gamma)}}{\theta\kappa} G'''(\gamma) = (2 - G''(\gamma) + (\gamma - \rho)^2) + (\gamma(1 - G''(\gamma)) + 2(\gamma - \rho) + \\ + \gamma(\gamma - \rho)^2) e^{\frac{\gamma^2}{2}} \Phi(\gamma) \tag{A.126}$$

$$> (2 - G''(\gamma)) + (1 - G''(\gamma)) \gamma e^{\frac{\gamma^2}{2}} \Phi(\gamma). \tag{A.127}$$

$2 - G''(\gamma)$ is positive, while $1 - G''(\gamma)$ is negative, so we need to perform a careful estimate. First of all, the estimate $G''(\gamma) < 2$ is too blunt, and will be replaced by $G''(\gamma) \leq 1/(1-\theta\kappa)$, so that

$$2 - G''(\gamma) \geq 2 - \frac{1}{1-\theta\kappa} = 1 - \frac{\theta\kappa}{1-\theta\kappa}, \tag{A.128}$$

which is positive, since $\theta\kappa < 1/2$. In order to estimate $1 - G''(\gamma)$ we calculate

$$a G''(\gamma) = 1 + \theta\kappa\rho e^{-G(\gamma)} (1 - \gamma e^{\frac{\gamma^2}{2}} \Phi(\gamma)) - \theta\kappa e^{-G(\gamma)} (e^{\frac{\gamma^2}{2}} \Phi(\gamma) + \gamma^2 e^{\frac{\gamma^2}{2}} \Phi(\gamma) + \gamma) \tag{A.129}$$

$$= 1 + \rho(\gamma - a\rho) - \theta\kappa e^{-G(\gamma)} e^{\frac{\gamma^2}{2}} \Phi(\gamma) - \gamma(\gamma - a\rho) \tag{A.130}$$

$$= 1 - (\gamma - a\rho)(\gamma - \rho) - \theta\kappa e^{-G(\gamma)} e^{\frac{\gamma^2}{2}} \Phi(\gamma). \tag{A.131}$$

Therefore

$$1 - G''(\gamma) = 1 - \frac{1}{a} + \frac{1}{a}(\gamma - a\rho)(\gamma - \rho) + \frac{\theta\kappa}{a} e^{-G(\gamma)} e^{\frac{\gamma^2}{2}} \Phi(\gamma) \tag{A.132}$$

$$> -\frac{\theta\kappa}{1-\theta\kappa} + \frac{\theta\kappa}{1-\theta\kappa} e^{-G(\gamma)} e^{\frac{\gamma^2}{2}} \Phi(\gamma), \tag{A.133}$$

which implies

$$(1 - G''(\gamma)) \gamma e^{\frac{\gamma^2}{2}} \Phi(\gamma) > \frac{\theta\kappa}{1-\theta\kappa} \frac{e^{\frac{\gamma^2}{2}} \Phi(\gamma)}{e^{G(\gamma)}} \gamma \left(e^{\frac{\gamma^2}{2}} \Phi(\gamma) - e^{G(\gamma)}\right) \tag{A.134}$$

$$= \frac{\theta\kappa}{1-\theta\kappa}\frac{e^{\frac{\gamma^2}{2}}\Phi(\gamma)}{e^{G(\gamma)}}\frac{-\gamma}{\sqrt{1-\theta\kappa}}e^{\frac{\gamma^2}{2(1-\theta\kappa)}}\Phi\left(-\frac{\gamma}{\sqrt{1-\theta\kappa}}\right) \tag{A.135}$$

$$> -\frac{\theta\kappa}{1-\theta\kappa}\frac{e^{\frac{\gamma^2}{2}}\Phi(\gamma)}{e^{G(\gamma)}} > -\frac{\theta\kappa}{1-\theta\kappa}, \tag{A.136}$$

since $e^{\frac{\gamma^2}{2}}\Phi(\gamma) < e^{G(\gamma)}$. Putting everything together we obtain

$$-\frac{ae^{G(\gamma)}}{\theta\kappa}G'''(\gamma) > 1 - \frac{2\theta\kappa}{1-\theta\kappa}, \tag{A.137}$$

and the right hand side is positive provided θ is small enough (say $\theta < 1/2$). □

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