HOMOTOPY THEORY OF DG SHEAVES

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ABSTRACT. In this note we study the local projective model structure on presheaves of complexes on a site, i. e. we describe its classes of cofibrations, fibrations and weak equivalences. In particular, we prove that the fibrant objects are those satisfying descent with respect to all hypercovers. We also describe cofibrant and fibrant replacement functors with pleasant properties.

CONTENTS

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1. Introduction

An important object in different fields of mathematics is the derived category of sheaves on some site. However, it is well-known that many constructions and proofs in this setting cannot be performed on the derived level but require recourse to a *model*. Our goal in this note is to describe in detail one specific homotopy-theoretic model for the unbounded derived category of sheaves on an arbitrary site. Although the model is well-known, there were several facts about it that we needed in our [5] but were not able to find in the literature, which is why we decided to write them up. To be useful in other contexts as well, we place ourselves in a more general setting, in particular we try to make as few assumptions as possible regarding the site.

Let us quickly give the definition of the model associated to a site $(\mathcal{C}, \tau)$. Start with the category of presheaves of unbounded complexes on $\mathcal{C}$ and declare weak equivalences and fibrations to be objectwise quasi-isomorphisms and epimorphisms, respectively. This yields the *projective model structure*. The *τ-local model structure* arises from it by a left Bousfield localization with respect to τ-local weak equivalences, i. e. morphisms inducing isomorphisms on all homology τ-sheaves. The resulting model category is our model for the derived category of τ-sheaves.

In §3 we recall the basic properties of the model category and describe the cofibrations. As an application we construct in §4 an explicit cofibrant replacement functor which resolves any presheaf of complexes by representables. The main theorem of §5 states that the τ-fibrant objects are precisely those presheaves satisfying descent with respect to τ-hypercovers. The analogous statement for simplicial presheaves is well-known, and our strategy is to reduce to this case via the Dold-Kan correspondence. We use the same strategy to prove a generalization of the Verdier hypercover theorem, expressing the hypercohomology of a complex of sheaves in terms of hypercovers. We also describe some modifications to our model and deduce some useful consequences from the main theorem. In the final section 6 we prove that the Godement resolution defines a fibrant replacement functor for our model.

We would like to remark that the model described in this note is not quite arbitrary but has a very satisfying universal property. To describe it, recall the easy fact from category theory that for a small category $\mathcal{C}$, the category of presheaves on $\mathcal{C}$ is its *universal (or free) cocompletion*. This means that any functor from $\mathcal{C}$ into a cocomplete category factors via a cocontinuous functor through the Yoneda embedding $\mathcal{C} \to \mathbf{PSh}(\mathcal{C})$ in an essentially unique way. This basic idea finds repercussions in the following two results:

- For a small dg category $\mathcal{D}$, the category of dg modules $[\mathcal{D}^{\mathrm{op}}, \mathbf{Cpl}]$ is its *universal dg cocompletion*.
- In [9], Dugger proves that any functor from $\mathcal{C}$ into a model category factors via a left Quillen functor through the category of simplicial presheaves on $\mathcal{C}$ with the projective model structure, in an essentially unique way. In other words, this is the *universal model category* associated to $\mathcal{C}$.

Combining these two examples we naturally arrive at the following guess: $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cpl}]$ with the projective model structure is the *universal model dg category* associated to the small category $\mathcal{C}$. We couldn't resist prepending a section (§2) in order to explain this result (in fact, a more general version where chain complexes are replaced by quite arbitrary enriching categories).

Such a statement invites us to view $\mathcal{C}$ as *generating* the dg category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cpl}]$, while the Bousfield localization yielding the local model structure plays the role of imposing *relations*. Namely, the localization stipulates that any object in $\mathcal{C}$ may be homotopically decomposed into the pieces of any cover. In a very precise sense then (cf. Corollary 5.14) our model for the derived category of τ-sheaves is the universal τ-local model dg category associated to $\mathcal{C}$.

Relation to other works in the literature. As mentioned above, our motivation for this note lies in [5]. Most importantly we needed there a description of the fibrant objects in the local projective model structure in terms of descent. At the time we were aware of such descriptions in the "non-linear" case of simplicial presheaves due to Dugger-Hollander-Isaksen ([10]), and in the "linear" case only under finiteness conditions not satisfied in our application ([21]). Only after the note had been written we came across the paper [11] by Hinich which establishes both the linear and the non-linear case without restrictions (and with a proof different from ours, cf. Remark 5.8). Even later we learned that Drew in his unpublished Ph.D. thesis established an essentially equivalent result (using the same idea as ours, cf. Remark 5.8). We believe that the other results presented here are probably known even if they haven't all appeared in print. The present note thus serves primarily as a reference for [5] but we hope it will be useful to other mathematicians as well.

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2. Universal enriched model categories

This section is very much inspired by Dugger's [9] where he proves the existence of a universal model category associated to a small category. Our goal is to establish an analogue of this result in the enriched setting.

"Monoidal" is an abbreviation for "unital monoidal"; the monoidal structure is always denoted by $\otimes$, the unit by $\mathbb{1}$. Fix a bicomplete closed symmetric monoidal category $\mathcal{V}$. We are first going to recall some basics in $\mathcal{V}$-enriched category theory, and for this we follow the terminology in [16]. In particular, $\mathcal{V}_0, \mathcal{M}_0$ etc. denote the ordinary categories underlying the $\mathcal{V}$-categories $\mathcal{V}, \mathcal{M}$ etc.

2.1. **Free enriched cocompletion.** Let $\mathcal{C}$ and $\mathcal{M}$ be $\mathcal{V}$-categories and assume that $\mathcal{C}$ is small. Recall ([16, §2]) that there is a $\mathcal{V}$-functor category $[\mathcal{C}, \mathcal{M}]$ whose underlying category is just the category of $\mathcal{V}$-functors $\mathcal{C} \to \mathcal{M}$ together with $\mathcal{V}$-natural transformations. Given such a $\mathcal{V}$-functor $\gamma : \mathcal{C} \to \mathcal{M}$ consider the $\mathcal{V}$-functor $\gamma_* : \mathcal{M} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ which takes m to $\mathcal{M}(\gamma(\bullet), m)$. In particular, if $\mathcal{C} = \mathcal{M}$ and γ is the identity then γ_* is the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. As in the classical case, the Yoneda embedding provides the free cocompletion as we are now going to explain (see [16, Thm. 4.51]).

Recall that a $\mathcal{V}$-category $\mathcal{M}$ is cocomplete if it has all small indexed colimits (sometimes also called weighted colimits). As a special case of an indexed colimit, recall that $\mathcal{M}$ is tensored if there exists a $\mathcal{V}$-bifunctor (called the tensor)

$$\bullet \odot \bullet : \mathcal{V} \otimes \mathcal{M} \to \mathcal{M}$$

together with, for each $v \in \mathcal{V}$ and each $m \in \mathcal{M}$, $\mathcal{V}$-natural isomorphisms

$$\mathcal{M}(v \odot m, \bullet) \cong \mathcal{V}(v, \mathcal{M}(m, \bullet)).$$

The dual notion is called a cotensor. If $\mathcal{M}$ is both tensored and cotensored then cocompleteness is equivalent to its underlying category being cocomplete in the ordinary sense. Accordingly, a $\mathcal{V}$-functor with cocomplete domain is cocontinuous if and only if it commutes with tensors and the underlying functor is cocontinuous. Dually one defines complete $\mathcal{V}$-categories and continuous $\mathcal{V}$-functors.

An example of a (complete and) cocomplete $\mathcal{V}$-category is $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ for a small $\mathcal{V}$-category $\mathcal{C}$. From now on, we denote it by $\mathbf{U}_{\mathcal{V}}\mathcal{C}$. In the special case that $\mathcal{C}$ is the free $\mathcal{V}$-category associated to an ordinary category (see §2.2 below), the tensor of $v \in \mathcal{V}$ and $f \in \mathbf{U}_{\mathcal{V}}\mathcal{C}$ is given explicitly by

$$v \odot f = v_{\mathrm{cst}} \otimes f,$$

where v_{cst} denotes the constant presheaf with value v, and $\otimes$ denotes the objectwise tensor product in $\mathcal{V}$.

Fact 2.1. *Let $\gamma : \mathcal{C} \to \mathcal{M}$ be a $\mathcal{V}$-functor and assume that $\mathcal{C}$ is small and $\mathcal{M}$ is cocomplete.*

(1) There is a $\mathcal{V}$-adjunction

$$(\gamma^*, \gamma_*) : \mathbf{U}_{\mathcal{V}}\mathcal{C} \to \mathcal{M},$$

where $\gamma^*(f)$ is given by the tensor product of f and γ, $f \odot_{\mathcal{C}} \gamma$.

(2) The association $\gamma \mapsto \gamma^*$ induces an equivalence of $\mathcal{V}$-categories

$$[\mathcal{C}, \mathcal{M}] \simeq [\mathbf{U}_{\mathcal{V}}\mathcal{C}, \mathcal{M}]_{\mathrm{coc}}$$

where $(\bullet)_{\mathrm{coc}}$ picks out the cocontinuous $\mathcal{V}$-functors.

(3) There is a canonical isomorphism $\gamma^* y \cong \gamma$.

The $\mathcal{V}$-functor γ^* is called the left $\mathcal{V}$-Kan extension *of γ along the Yoneda embedding.*

Here, the tensor product of the two $\mathcal{V}$-functors f and γ is the coend $\int^{c \in \mathcal{C}} f(c) \odot \gamma(c)$. Notice that part of the statement is the existence of $[\mathbf{U}_{\mathcal{V}}\mathcal{C}, \mathcal{M}]_{\mathrm{coc}}$ as a $\mathcal{V}$-category (this is not clear since $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ is not necessarily small).

If $\beta : \mathcal{D} \to \mathcal{C}$ is a $\mathcal{V}$-functor between small $\mathcal{V}$-categories, we denote $(y\beta)^*$ by β^* if no confusion is likely to arise. With this abuse of notation, there is a canonical isomorphism $(\gamma\beta)^* \cong \gamma^*\beta^*$. Similarly, if $\delta : \mathcal{M} \to \mathcal{N}$ is a cocontinous functor into another cocomplete $\mathcal{V}$-category $\mathcal{N}$, then $(\delta\gamma)^* \cong \delta\gamma^*$.

Assume now that $\mathcal{C}$ is a (symmetric) monoidal $\mathcal{V}$-category (this is the canonical translation of a (symmetric) monoidal structure to the enriched context; or see [7, p. 2f]). $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ inherits a (symmetric) monoidal structure called the (Day) convolution product ([7, Thm. 3.3 and 4.1]). Explicitly, the monoidal product of two presheaves f and g is given by

$$f \otimes g = \int^{c, c'} f(c) \otimes g(c') \otimes \mathcal{C}(\bullet, c \otimes c'),$$

and the unit by $y(\mathbb{1}) = \mathcal{C}(\bullet, \mathbb{1})$. It is clear that the Yoneda embedding $y : \mathcal{C} \to \mathbf{U}_{\mathcal{V}}\mathcal{C}$ is (symmetric) monoidal.

Lemma 2.2. *In the setting of Fact 2.1, assume in addition that γ is (lax) (symmetric) monoidal, and that the monoidal product in $\mathcal{M}$ commutes with indexed colimits. Then:*

(1) γ^* is (lax) (symmetric) monoidal.
(2) The canonical isomorphism $\gamma^* y \cong \gamma$ is monoidal.
(3) The association $\gamma \mapsto \gamma^*$ induces an equivalence of ordinary categories

$$\mathcal{V}\text{−Fun}_{\otimes}(\mathcal{C}, \mathcal{M}) \simeq \mathcal{V}\text{−Fun}_{\mathrm{coc},\otimes}(\mathbf{U}_{\mathcal{V}}\mathcal{C}, \mathcal{M})$$

of (lax) (symmetric) monoidal $\mathcal{V}$-functors.

Proof. Let $f, g \in \mathbf{U}_{\mathcal{V}}\mathcal{C}$. The (lax) monoidal structure on γ^* is defined as follows:

$$\begin{aligned}\left(\int f \odot \gamma\right) \otimes \left(\int g \odot \gamma\right) &\cong \int^{c,d} (f(c) \otimes g(d)) \odot (\gamma(c) \otimes \gamma(d)) \\ &\to \int^{c,d} (f(c) \otimes g(d)) \odot (\gamma(c \otimes d)) \\ &\cong \int^{e} \left(\int^{c,d} f(c) \otimes g(d) \otimes \mathcal{C}(e, c \otimes d)\right) \odot \gamma(e) \\ &\cong \int (f \otimes g) \odot \gamma\end{aligned}$$

and

$$\mathbb{1} \to \gamma(\mathbb{1}) \cong \gamma^*(\mathbb{1}).$$

We leave the details to the reader. □

In this sense, if $\mathcal{C}$ is (symmetric) monoidal then $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ is the free *(symmetric) monoidal* $\mathcal{V}$-cocompletion. Notice also that the "pseudo-functoriality" mentioned above, to wit $(\gamma\beta)^* \cong \gamma^*\beta^*$ and $(\delta\gamma)^* \cong \delta\gamma^*$, is compatible with monoidal structures.

2.2. **Enriched model categories.** We now discuss the interplay between basic enriched category theory as above and Quillen model structures. From now on we assume that the underlying category $\mathcal{V}_0$ is a symmetric monoidal model category in the sense of [13, Def. 4.2.6]. We also assume that this model structure is cofibrantly generated.

Fix a small ordinary category $\mathcal{C}$ and set $\mathcal{C}[\mathcal{V}]$ to be the associated free $\mathcal{V}$-category. It has the same objects as $\mathcal{C}$ and the $\mathcal{V}$-structure is given by

$$\mathcal{C}[\mathcal{V}](c, c') = \coprod_{\mathcal{C}(c,c')} \mathbb{1}$$

with the obvious composition. By definition, giving a $\mathcal{V}$-functor $\mathcal{C}[\mathcal{V}] \to \mathcal{M}$ into a $\mathcal{V}$-model category $\mathcal{M}$ is the same as giving an (ordinary) functor $\mathcal{C} \to \mathcal{M}_0$. In the sequel, we will often write abusively $\mathcal{C} \to \mathcal{M}$, sometimes thinking of the datum as a $\mathcal{V}$-functor, sometimes as an ordinary functor. We are positive that this will not lead to any confusion.

Thus the general small $\mathcal{V}$-category $\mathcal{C}$ in §2.1 will now always be of this special form. We impose this restriction because it simplifies most of the statements and proofs drastically, and because it is all we will need later on.

Since the underlying category of $\mathbf{U}_{\mathcal{V}}\mathcal{C} := \mathbf{U}_{\mathcal{V}}\mathcal{C}[\mathcal{V}]$ is just the category of presheaves on $\mathcal{C}$ with values in $\mathcal{V}$, the following result is well-known.

Fact 2.3.

1. *(1)* $(\mathbf{U}_{\mathcal{V}}\mathcal{C})_0$ *admits a cofibrantly generated model structure with weak equivalences and fibrations defined objectwise.*
2. *(2) If* $\mathcal{V}_0$ *is left (resp. right) proper then so is* $(\mathbf{U}_{\mathcal{V}}\mathcal{C})_0$*.*
3. *(3) If* $\mathcal{V}_0$ *is combinatorial (resp. tractable, cellular) then so is* $(\mathbf{U}_{\mathcal{V}}\mathcal{C})_0$*.*
4. *(4) If* $\mathcal{V}_0$ *is stable then so is* $(\mathbf{U}_{\mathcal{V}}\mathcal{C})_0$*.*

This is called the *projective model structure*. If not mentioned otherwise, we will consider $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ as endowed with the projective model structure from now on.

Proof. See [12, Thm. 11.6.1] for the first statement, [12, Thm. 13.1.14] for the second, [12, Pro. 12.1.5] and [4, Thm. 2.14] for the third. The last statement is obvious. □

Definition 2.4. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{V}$-categories with model structures on their underlying categories. A $\mathcal{V}$-adjunction $(\delta, \varepsilon) : \mathcal{M} \to \mathcal{N}$ is called a *Quillen $\mathcal{V}$-adjunction* if the underlying adjunction $(\delta_0, \varepsilon_0) : \mathcal{M}_0 \to \mathcal{N}_0$ is a Quillen adjunction. In that case δ is called a *left*, ε a *right Quillen $\mathcal{V}$-functor*.

We now come back to the situation of Fact 2.1. The question we should like to answer is: When is (γ^*, γ_*) a Quillen $\mathcal{V}$-adjunction?

Lemma 2.5. *Assume that* $\mathcal{M}_0$ *is endowed with a model structure. The following conditions are equivalent:*

1. *(1)* (γ^*, γ_*) *is a Quillen* $\mathcal{V}$*-adjunction.*
2. *(2) For each* $c \in \mathcal{C}$, $\mathcal{M}_0(\gamma(c), \bullet)$ *is a right Quillen functor.*
3. *(3) For each* $c \in \mathcal{C}$, $\bullet \odot \gamma(c)$ *is a left Quillen functor.*

Proof. The equivalence between the last two conditions is clear. The equivalence between the first two conditions follows from the description of γ_* given above and the fact that we imposed the projective model structure on $(\mathbf{U}_{\mathcal{V}}\mathcal{C})_0$. □

In particular, these equivalent conditions are satisfied if the image of γ consists of cofibrant objects, and the tensor on $\mathcal{M}$ is a "Quillen $\mathcal{V}$-adjunction of two variables", i. e. a $\mathcal{V}$-adjunction of two variables such that the underlying data form a Quillen adjunction of two variables in the sense of [13, Def. 4.2.1].

Definition 2.6. A *model $\mathcal{V}$-category* is a bicomplete $\mathcal{V}$-category $\mathcal{M}$ together with a model structure on $\mathcal{M}_0$ such that

- the tensor is a Quillen $\mathcal{V}$-adjunction of two variables;
- for any cofibrant object $m \in \mathcal{M}$, $\mathbb{1}_c \odot m \to \mathbb{1} \odot m$ is a weak equivalence, for a cofibrant replacement $\mathbb{1}_c \to \mathbb{1}$.

A *(symmetric) monoidal model $\mathcal{V}$-category* is a model $\mathcal{V}$-category $\mathcal{M}$ together with a Quillen $\mathcal{V}$-adjunction of two variables $\otimes : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ with a unit, and associativity (and symmetry) constraints satisfying the usual axioms.

These are equivalent to the definitions in [13, Def. 4.2.18, 4.2.20]. Also, it is a straight-forward generalization of the notion of a simplicial model category.

Example 2.7.

1. (1) If $\mathcal{V}$ is the category of simplicial sets with the standard model structure then we recover the notion of a simplicial model category.

(2) Our main example will be obtained by taking $\mathcal{V}$ to be the category of (unbounded) chain complexes of Λ-modules, Λ a (commutative unital) ring, with the projective model structure and the usual tensor product. A model $\mathcal{V}$-category will be called a model dg category. See §3.

Fact 2.8. $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ *is a model* $\mathcal{V}$*-category. Moreover, if* $\mathcal{C}$ *is (symmetric) monoidal and the unit in* $\mathcal{V}$ *cofibrant, then* $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ *is a (symmetric) monoidal model* $\mathcal{V}$*-category for the Day convolution product.*

Proof. The first statement is straightforward to check. The second is [15, Pro. 2.2.15]. □

Notice that if $\mathcal{C}$ is cartesian monoidal then the Day convolution product coincides with the objectwise monoidal product on $\mathbf{U}_{\mathcal{V}}\mathcal{C}$.

2.3. **Statement and proof.** Our goal is to establish $y : \mathcal{C} \to \mathbf{U}_{\mathcal{V}}\mathcal{C}$ (really, $\mathcal{C}[\mathcal{V}] \to \mathbf{U}_{\mathcal{V}}\mathcal{C}$) as the universal functor into a model $\mathcal{V}$-category. But first, we need to make precise what we mean by the universality in the statement. For this fix a model $\mathcal{V}$-category $\mathcal{M}$ and a functor $\gamma : \mathcal{C} \to \mathcal{M}$. Define a *factorization of* γ *through* y to be a pair (L, η) where $L : \mathbf{U}_{\mathcal{V}}\mathcal{C} \to \mathcal{M}$ is a left Quillen $\mathcal{V}$-functor, and $\eta : Ly \to \gamma$ a natural transformation which is objectwise a weak equivalence. A morphism of such factorizations $(L, \eta) \to (L', \eta')$ is a natural transformation $L \to L'$ compatible with η and η'. This clearly defines a category $\mathrm{Fact}(\gamma, y)$.

Proposition 2.9. *Assume that the unit in* $\mathcal{V}$ *is cofibrant. For any* γ*, the category* $\mathrm{Fact}(\gamma, y)$ *is contractible.*

Notice that in a homotopical context it is unreasonable to expect the category of choices to be a groupoid ("uniqueness up to unique isomorphism") and contractibility is usually the right thing to ask of this category.

Let $\mathrm{CofRep}(\gamma)$ be the category of cofibrant replacements of γ. Its objects are functors $\gamma' : \mathcal{C} \to \mathcal{M}$ together with a natural transformation $\gamma' \to \gamma$ which is objectwise a weak equivalence and such that the γ' takes values in cofibrant objects. The morphisms are the obvious ones.

Lemma 2.10. *Assume that the unit in* $\mathcal{V}$ *is cofibrant. There is a canonical equivalence of categories* $\mathrm{Fact}(\gamma, y) \simeq \mathrm{CofRep}(\gamma)$.

Proof. We give functors in both directions. That these are quasi-inverses to each other will then be seen to follow from the $\mathcal{V}$-equivalence of categories in Fact 2.1.

- Given $\gamma' \to \gamma$ on the right hand side, define $L = (\gamma')^*$ and choose the natural transformation $(\gamma')^* y \cong \gamma' \to \gamma$. Functoriality follows from the functoriality statement in Fact 2.1.
- Given $(L, Ly \to \gamma)$ on the left hand side, $Ly \to \gamma$ defines a cofibrant replacement since L is a left Quillen $\mathcal{V}$-functor and y takes values in cofibrant objects (here we use that the unit in $\mathcal{V}$ is cofibrant). Functoriality is obvious. □

Proof of Proposition 2.9. By the previous lemma, we need to show contractibility of $\mathrm{CofRep}(\gamma)$. Fix a cofibrant replacement functor F for the model structure on $\mathcal{M}_0$. Composing with γ we obtain an object $(F\gamma, F\gamma \to \gamma)$ of $\mathrm{CofRep}(\gamma)$. Given any other object $(\gamma', \gamma' \to \gamma)$, functoriality of F yields a commutative square

$$\begin{array}{ccc} F\gamma' & \longrightarrow & F\gamma \\ \downarrow & & \downarrow \\ \gamma' & \longrightarrow & \gamma \end{array}$$

and thus a zig-zag $\gamma' \leftarrow F\gamma' \to F\gamma$ in $\mathrm{CofRep}(\gamma)$. Moreover, this zig-zag is natural in γ' hence this construction provides a zig-zag of homotopies between the identity functor on $\mathrm{CofRep}(\gamma)$ and the constant functor $(F\gamma, F\gamma \to \gamma)$. □

For the reader's convenience we reformulate our main result.

Corollary 2.11. *Let $\mathcal{C}$ be a small category, and $\mathcal{V}$ a cofibrantly generated symmetric monoidal model category whose unit is cofibrant. There exists a functor $y : \mathcal{C} \to \mathbf{U}_{\mathcal{V}}\mathcal{C}$ into a model $\mathcal{V}$-category, universal in the sense that for any solid diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{y} & \mathbf{U}_{\mathcal{V}}\mathcal{C} \\ & \searrow^{\gamma} & \vdots\, L \\ & & \mathcal{M} \end{array}$$

with $\mathcal{M}$ a model $\mathcal{V}$-category, there exists a left Quillen $\mathcal{V}$-functor L as indicated by the dotted arrow, unique up to a contractible choice, making the diagram commutative up to a weak equivalence $Ly \to \gamma$.

Remark 2.12. One can dualize the discussion of this section in order to obtain universal model $\mathcal{V}$-categories for *right* Quillen $\mathcal{V}$-functors, as in [9, §4]. Unsurprisingly, one finds that this universal model $\mathcal{V}$-category associated to $\mathcal{C}$ is given by $[\mathcal{C}, \mathcal{V}]^{\mathrm{op}}$ with the opposite of the projective model structure. This can also be deduced from Corollary 2.11 applied to $\mathcal{C}^{\mathrm{op}}$.

3. Universal model dg categories

We now specialize to the case of dg categories. Fix a commutative unital ring Λ, denote by $\mathbf{Mod}(\Lambda)$ the category of Λ-modules, and by $\mathbf{Cpl}(\Lambda)$ the category of unbounded chain complexes of Λ-modules. Our conventions for chain complexes are that the differentials decrease the indices, and the shift operator satisfies $(A[p])_n = A_{p+n}$. The subobject of n-cycles (resp. n-boundaries) of A is denoted by $Z_n A$ (resp. $B_n A$). As usual, the nth homology is denoted by $\mathrm{H}_n A = Z_n A / B_n A$.

$\mathbf{Cpl}(\Lambda)$ has a tensor product, defined by

$$(A \otimes B)_n = \oplus_{p+q=n} A_p \otimes B_q$$

with the Koszul sign convention for the differential. It also admits the "projective model structure" for which the weak equivalences are the quasi-isomorphisms, and the fibrations the epimorphisms (i. e. the degreewise surjections). In that way, $\mathbf{Cpl}(\Lambda)$ becomes a symmetric monoidal model category. In this section we always take $\mathcal{V}$ to be $\mathbf{Cpl}(\Lambda)$. The universal model category underlying a model dg category $(\mathbf{U}_{\mathrm{dg}}\mathcal{C})_0$ will now be denoted by $\mathbf{U}\mathcal{C}$. The complex of morphisms from K to K' in $\mathbf{U}_{\mathrm{dg}}\mathcal{C}$ is denoted by $\underline{\mathrm{hom}}_{\mathrm{dg}}(K, K') \in \mathbf{Cpl}(\Lambda)$. Recall that it is given explicitly by $\mathrm{Tot}^{\Pi}(\mathrm{hom}_{\mathbf{PSh}(\mathcal{C},\Lambda)}(K_{-p}, K'_q))_{p,q}$.

Our main goal in this section is to better understand the model structure on $\mathbf{U}\mathcal{C}$ (defined in Fact 2.3). In the last part we will also discuss a specific instance of a left dg Kan extension used in [5].

3.1. **Basic properties of the model category $\mathbf{U}\mathcal{C}$.** By Fact 2.8 we know that $\mathbf{U}_{\mathrm{dg}}\mathcal{C}$ is a model dg category, and a (symmetric) monoidal model dg category if $\mathcal{C}$ is (symmetric) monoidal. It follows from Fact 2.3 that the model category $\mathbf{U}\mathcal{C}$ is about as nice as it can get.

Corollary 3.1. $\mathbf{U}\mathcal{C}$ *is a*

(1) proper,
(2) stable,
(3) tractable (in particular combinatorial),
(4) cellular,

model category.

We will now describe explicitly sets of generating (trivial) cofibrations.

Definition 3.2. Let, for any presheaf F, S^nF be the complex of presheaves which has F in degree n and is 0 otherwise, and let D^nF be the complex of presheaves which has F in degree n and $n-1$, is 0 otherwise, and whose nontrivial differential is given by the identity on F. There exists a canonical morphism $S^{n-1}F \to D^nF$. Let I be the set of morphisms $S^{n-1}\Lambda(c) \to D^n\Lambda(c)$ for all $c \in \mathcal{C}$ and let J be the set of maps $0 \to D^n\Lambda(c)$. (Here $\Lambda : \mathcal{C} \to \mathbf{PSh}(\mathcal{C}, \Lambda)$ (abusively) denotes the Λ-enriched Yoneda embedding, i. e. $\Lambda(c)(d) = \coprod_{\mathcal{C}(d,c)} \Lambda$.)

Notice that there are adjunctions

$$(S^n, Z_n) \text{ and } (D^n, (\bullet)_n) : \mathbf{PSh}(\mathcal{C}, \Lambda) \to \mathbf{U}\mathcal{C}.$$

The same arguments as in [13, Pro. 2.3.4, 2.3.5] then establish the following result.

Fact 3.3. *A morphism in* $\mathbf{U}\mathcal{C}$ *is a fibration (resp. trivial fibration) if and only if it has the right lifting property with respect to J (resp. I).*

We will use another set of generating cofibrations later on.

Definition 3.4. Given a presheaf F of Λ-modules, let Δ^nF be the complex which has F in degree n and $F \oplus F$ in degree $n-1$, and zero otherwise, and whose only non-zero differential is given by $\mathrm{id} \times (-\mathrm{id}) : F \to F \oplus F$. Define also $\partial\Delta^nF$ to be the complex which has $F \oplus F$ in degree $n-1$ and 0 otherwise. Let I' be the set of morphisms $\partial\Delta^n\Lambda(c) \to \Delta^n\Lambda(c)$ which is the identity in degree n, for all $n \in \mathbb{Z}$ and $c \in \mathcal{C}$.

Lemma 3.5. *A morphism in* $\mathbf{U}\mathcal{C}$ *is a trivial fibration if and only if it has the right lifting property with respect to I'.*

Proof. Morphisms in I' are cofibrations by Fact 3.11. Conversely we will exhibit any morphism in I as a retract of some morphism in I'. Thus fix $c \in \mathcal{C}$ and $n \in \mathbb{Z}$, and consider the following diagram:

$$\begin{array}{ccccc} S^n\Lambda(c) & \xrightarrow{\mathrm{id}\times(-\mathrm{id})} & \partial\Delta^{n+1}\Lambda(c) & \xrightarrow{(\mathrm{id},0)} & S^n\Lambda(c) \\ \downarrow & & \downarrow & & \downarrow \\ D^{n+1}\Lambda(c) & \xrightarrow{r} & \Delta^{n+1}\Lambda(c) & \xrightarrow{s} & D^{n+1}\Lambda(c) \end{array}$$

Here, r in degree n is $\mathrm{id} \times (-\mathrm{id})$ and in degree $n+1$ is id, while s in degree n is the first projection and in degree $n+1$ the identity. It is easy to see that the diagram commutes and the compositions of each row are the identity morphism. □

3.2. **Projective cofibrations.** Since the fibrations and weak equivalences are given explicitly in $\mathbf{U}\mathcal{C}$ our goal is to better understand the cofibrations. They are called projective cofibrations. The discussion runs parallel to the description of projective cofibrations for the category of chain complexes (i. e. the case of $\mathcal{C}$ the terminal category), in [13, §2.3].

Lemma 3.6. *If $f : K \to K' \in \mathbf{U}\mathcal{C}$ is a trivial fibration then f induces a surjective morphism $f : Z_nK \to Z_nK'$ for all $n \in \mathbb{Z}$.*

Proof. Since f is degreewise surjective, it induces a surjective morphism on the boundaries $B_nK \to B_nK'$. Now consider the morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_nK & \longrightarrow & Z_nK & \longrightarrow & \mathrm{H}_nK & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_nK' & \longrightarrow & Z_nK' & \longrightarrow & \mathrm{H}_nK' & \longrightarrow & 0 \end{array}$$

The first and last vertical arrows are surjective, hence the middle one is too. □

Definition 3.7. A presheaf of Λ-modules $F \in \mathbf{PSh}(\mathcal{C}, \Lambda)$ is called *projective* if

$$\hom_{\mathbf{PSh}(\mathcal{C},\Lambda)}(F, \bullet) : \mathbf{PSh}(\mathcal{C}, \Lambda) \to \mathbf{Mod}(\Lambda)$$

is exact.

Example 3.8. For any $c \in \mathcal{C}$ the representable presheaf $\Lambda(c)$ is projective. Direct sums and direct summands of projectives are projective.

Lemma 3.9. *For any projective presheaf $F \in \mathbf{PSh}(\mathcal{C}, \Lambda)$, the complex S^0F is projective cofibrant.*

Proof. We have to prove that for any trivial fibration $f : K \to K' \in \mathbf{U}\mathcal{C}$, the induced morphism

$$\hom_{\mathbf{U}\mathcal{C}}(S^0F, K) \to \hom_{\mathbf{U}\mathcal{C}}(S^0F, K')$$

is surjective. But for any complex $L \in \mathbf{U}\mathcal{C}$, we have

$$\hom_{\mathbf{U}\mathcal{C}}(S^0F, L) = \hom_{\mathbf{Mod}(\Lambda)}(F, Z_0L).$$

Now the result follows from Lem. 3.6. □

Fact 3.10. *Let $K \in \mathbf{U}\mathcal{C}$. If K is projective cofibrant then each K_n is a projective presheaf. As a partial converse, if K is bounded below and each K_i is projective then K is projective cofibrant.*

Proof. The proof of [13, Lemma 2.3.6] applies. □

Fact 3.11. *A map $f : K \to K' \in \mathbf{U}\mathcal{C}$ is a projective cofibration if and only if f is a degreewise split injection and the cokernel of f is projective cofibrant.*

Proof. The proof of [13, Pro. 2.3.9] applies. □

Corollary 3.12. *Let $K = \varinjlim_{n\in\mathbb{N}} K^{(n)} \in \mathbf{U}\mathcal{C}$, such that $K^{(n)}$ is projective cofibrant and bounded below for each n, and such that the transition morphisms $K^{(n)} \to K^{(n+1)}$ are degreewise split injective. Then K is projective cofibrant.*

Proof. We use the fact that K is a sequential colimit of projective cofibrant objects with transition morphisms which are split injective in each degree hence the cokernel has projective objects in each degree. This implies together with boundedness and the previous lemma that the transition morphisms are projective cofibrations. Hence K is projective cofibrant. □

Independently of monoidal structures on $\mathcal{C}$, we can always define an objectwise tensor product on presheaves. The following lemma gives a necessary and sufficient condition for this product to be a Quillen bifunctor.

Lemma 3.13. *$\mathrm{U}\mathcal{C}$ is a symmetric monoidal model category for the objectwise tensor product if and only if for any pair of objects $c, d \in \mathcal{C}$, the presheaf of Λ-modules $\Lambda(c) \otimes \Lambda(d)$ is projective.*

Proof. Since representables are cofibrant (Fact 3.10) the condition is clearly necessary. For the converse, it suffices to prove the pushout-product $i \square j$ a (trivial) cofibration if i and j are generating cofibrations (and one of them a generating trivial cofibration). By Fact 3.3, i and j are of the form $i' \odot \Lambda(c)$ and $j' \odot \Lambda(d)$ for cofibrations i', j' of $\mathbf{Cpl}(\Lambda)$ (one of which is acylic), $c, d \in \mathcal{C}$. $i \square j$ can then be identified with $(i' \square j') \odot (\Lambda(c) \otimes \Lambda(d))$. $i' \square j'$ is a (trivial) cofibration since $\mathbf{Cpl}(\Lambda)$ is a symmetric monoidal model category. If $\Lambda(c) \otimes \Lambda(d)$ is projective then Lemma 3.9 together with Fact 2.8 yields what we want. □

3.3. **Dold-Kan correspondence.** Fix an abelian category $\mathcal{A}$. We start by recalling some basic constructions relating simplicial objects and connective chain complexes in $\mathcal{A}$.

Given a simplicial object $a_\bullet$ in $\mathcal{A}$, one can associate to it a connective chain complex (called the Moore complex, and usually still denoted by $a_\bullet$) which is a_n in degree n and whose differentials are given by

$$\sum_{i=0}^{n} (-1)^i d_i : a_n \to a_{n-1}.$$

This clearly defines a functor $\mathbf{\Delta}^{\mathrm{op}}\mathcal{A} \to \mathbf{Cpl}_{\geq 0}(\mathcal{A})$. Since every object in $\mathbf{\Delta}^{\mathrm{op}}\mathcal{A}$ is canonically split, we get a second functor $N : \mathbf{\Delta}^{\mathrm{op}}\mathcal{A} \to \mathbf{Cpl}_{\geq 0}(\mathcal{A})$, which associates to $a_\bullet$ the normalized chain complex:

$$N(a_\bullet)_n = \bigcap_{i=0}^{n-1} \ker(d_i : a_n \to a_{n-1}), \qquad (-1)^n d_n : N(a_\bullet)_n \to N(a_\bullet)_{n-1}.$$

Clearly, there is a canonical embedding $N(a_\bullet) \subset a_\bullet$ but more is true:

Fact 3.14.

(1) The inclusion $N(a_\bullet) \to a_\bullet$ is a natural chain homotopy equivalence.
(2) There is a functorial splitting $a_\bullet = N(a_\bullet) \oplus N'(a_\bullet)$ and N' takes values in acyclic complexes.
(3) N is an equivalence of categories. Let Γ denote any quasi-inverse.
(4) For any $n \in \mathbb{N}$, there is a natural isomorphism $\pi_n \Gamma \cong \mathrm{H}_n$.

In particular, we obtain a sequence of adjunctions

$$\mathbf{\Delta}^{\mathrm{op}}\mathrm{Set} \underset{}{\overset{\Lambda}{\rightleftarrows}} \mathbf{\Delta}^{\mathrm{op}}\mathbf{Mod}(\Lambda) \underset{\Gamma}{\overset{N}{\rightleftarrows}} \mathbf{Cpl}_{\geq 0}(\Lambda) \underset{\tau_{\geq 0}}{\rightleftarrows} \mathbf{Cpl}(\Lambda)\ , \tag{1}$$

where the first is the "free-forgetful" adjunction, and the last is the obvious adjunction between connective and unbounded chain complexes involving the good truncation functor $\tau_{\geq 0}$. Endow the category of simplicial sets with the Bousfield-Kan model structure for which cofibrations are levelwise injections and weak equivalences are weak homotopy equivalences, i. e. isomorphisms on the homotopy groups. By transfer along the forgetful functor this induces a model structure on simplicial Λ-modules, for which the Dold-Kan correspondence becomes a Quillen equivalence with the projective model structure on $\mathbf{Cpl}_{\geq 0}(\Lambda)$ (i. e. weak equivalences are quasi-isomorphisms, fibrations are surjections in strictly positive degrees). It is clear that the last adjunction in (1) is Quillen as well.

Proposition 3.15. *The sequence in (1) induces a Quillen adjunction*

$$(N\Lambda, \Gamma\tau_{\geq 0}) : \mathbf{\Delta}^{\mathrm{op}}\,\mathbf{PSh}(\mathcal{C}) \to \mathbf{U}\mathcal{C}.$$

Here both categories are equipped with the projective model structures.

Proof. Consider presheaves on $\mathcal{C}$ with values in the different categories appearing in (1). There is an induced sequence of adjunctions between these presheaf categories, similar to (1). If we endow each of them with the projective model structure, then each of the right adjoint preserves (trivial) fibrations by our discussion above. □

Lemma 3.16. *Let $K \in \mathbf{U}\mathcal{C}$ be cofibrant, $K' \in \mathbf{U}\mathcal{C}$ arbitrary. Then*

$$\Gamma\tau_{\geq 0}\,\underline{\mathrm{hom}}_{\mathrm{dg}}(K, K')$$

is a (left) homotopy function complex from K to K' (in the sense of [12, Def. 17.1.1]).

Proof. Since K is cofibrant, the functor $\bullet \odot K : \mathbf{Cpl}(\Lambda) \to \mathbf{U}\mathcal{C}$ is left Quillen, with right adjoint $\underline{\mathrm{hom}}_{\mathrm{dg}}(K, \bullet)$. We know that $\Delta^\bullet$ is a (the "standard") cosimplicial resolution of the terminal object in simplicial sets (cf. [12, §16.1]). By [12, Pro. 17.4.16], the left homotopy function complex from K to K' is then given by

$$\underline{\mathrm{hom}}_{\mathrm{dg}}(N\Lambda(\Delta^\bullet)\odot K, K') \cong \mathbf{\Delta}^{\mathrm{op}}\mathrm{Set}(\Delta^\bullet, \Gamma\tau_{\geq 0}\,\underline{\mathrm{hom}}_{\mathrm{dg}}(K, K')) \cong \Gamma\tau_{\geq 0}\,\underline{\mathrm{hom}}_{\mathrm{dg}}(K, K').$$

□

Corollary 3.17. *Let $K, K' \in \mathbf{U}\mathcal{C}$ and assume that K is cofibrant. For any $n \in \mathbb{Z}$, there is a natural isomorphism*

$$\mathrm{hom}_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(K, K'[n]) \cong \mathrm{H}_n\,\underline{\mathrm{hom}}_{\mathrm{dg}}(K, K'). \tag{2}$$

Proof. By [12, Pro. 17.7.1], $\pi_0\Gamma\tau_{\geq 0}\,\underline{\mathrm{hom}}_{\mathrm{dg}}(K, K'[n])$ is naturally isomorphic to the set of homotopy classes from K to $K'[n]$ which is equal to the left hand side of (2), by general properties of model categories. But

$$\begin{aligned}\pi_0\Gamma\tau_{\geq 0}\,\underline{\mathrm{hom}}_{\mathrm{dg}}(K, K'[n]) &\cong \mathrm{H}_0\,\underline{\mathrm{hom}}_{\mathrm{dg}}(K, K'[n])\\ &\cong \mathrm{H}_n\,\underline{\mathrm{hom}}_{\mathrm{dg}}(K, K').\end{aligned}$$

□

Lemma 3.18. *Let $K \in \mathbf{\Delta}^{\mathrm{op}}\mathbf{U}\mathcal{C}$ be a simplicial object in $\mathbf{U}\mathcal{C}$. Then the homotopy colimit $\mathrm{L}\,\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} K$ is given by*

$$\mathrm{Tot}^{\oplus}(K) \simeq \mathrm{Tot}^{\oplus}(NK).$$

Proof. The category $\mathbf{U}\mathcal{C}$ together with the class of quasi-isomorphisms and the functor $\mathrm{Tot}^{\oplus} : \mathbf{\Delta}^{\mathrm{op}}\mathbf{U}\mathcal{C} \to \mathbf{U}\mathcal{C}$ defines a "simplicial descent category" in the sense of [20, 19], see [19, §5.2]. The result for the first object now follows from [20, Thm. 5.1.i]. Since the Moore complex and the normalized complexes are homotopy equivalent (see Fact 3.14), the result for the second object follows from this (or see [19, Rem. 5.2.3]). □

The Moore and normalized complexes also induce functors from cosimplicial objects to coconnective chain complexes.

Lemma 3.19. *Let $K \in \mathbf{\Delta}\mathbf{U}\mathcal{C}$ be a cosimplicial object in $\mathbf{U}\mathcal{C}$. Then the homotopy limit $\mathrm{R}\lim_{\mathbf{\Delta}} K$ is given by*

$$\mathrm{Tot}^{\Pi}(K) \simeq \mathrm{Tot}^{\Pi}(NK).$$

Proof. This can be deduced from the proof of the previous lemma by passing to the opposite categories. □

Finally, the following result is often very useful (e. g. in [5]).

Lemma 3.20. *The derived category $\mathbf{Ho}(\mathbf{U}\mathcal{C})$ is compactly generated by the representable objects.*

Proof. If $\hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(\Lambda(c), K[n]) = 0$ for every $c \in \mathcal{C}$ and $n \in \mathbb{Z}$ then this means by Lemma 3.17 that K is objectwise acyclic and hence the zero object in the derived category.

Moreover, given a set $(K^{(i)})_{i\in I}$ of objects in $\mathbf{U}\mathcal{C}$ and $c \in \mathcal{C}$, the canonical morphism

$$\bigoplus_i \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(\Lambda(c), K^{(i)}) \to \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(\Lambda(c), \bigoplus_i K^{(i)})$$

is identified, again by Lemma 3.17, with

$$\bigoplus_i \mathrm{H}_0 K^{(i)}(c) \to \mathrm{H}_0 \bigoplus_i K^{(i)}(c),$$

which is invertible, thus the representable objects are also compact. □

3.4. **An example of a left dg Kan extension.** We would now like to give a more explicit description of the left dg Kan extension in a specific situation arising in [5]. The setup is as follows: Let $\mathcal{C}$ be a small ordinary category, and $\mathcal{B}$ a cocomplete Λ-linear category which is tensored over $\mathbf{Mod}(\Lambda)$. Finally, we are given a functor $\gamma : \mathcal{C} \to \mathbf{Cpl}(\mathcal{B})$.

First, notice that $\mathbf{Cpl}(\mathcal{B})$ is canonically a dg category, and the tensors on $\mathcal{B}$ induce a tensor operation of $\mathbf{Cpl}(\Lambda)$ on $\mathbf{Cpl}(\mathcal{B})$, by

$$(K \odot B)_n = \oplus_{p+q=n} K_p \odot B_q$$

with the usual differentials.

Notice that by considering a presheaf of Λ-modules as concentrated in degree 0, we can consider the restriction of γ^* to $\mathbf{PSh}(\mathcal{C}, \Lambda)$, still denoted by γ^*. The following lemma gives an alternative characterization of (the underlying functor of) such a left dg Kan extension.

Lemma 3.21.

(1) γ^* is the composition

$$\mathbf{U}\mathcal{C} \xrightarrow{\mathbf{Cpl}(\gamma^*)} \mathbf{Cpl}(\mathbf{Cpl}(\mathcal{B})) \xrightarrow{\mathrm{Tot}^{\oplus}} \mathbf{Cpl}(\mathcal{B}). \tag{3}$$

(2) Conversely, γ^* is characterized (up to natural isomorphism) by:
  (a) γ^* admits a factorization as in (3).
  (b) γ^* is cocontinuous.
  (c) $\gamma^* \circ \Lambda(\bullet) \cong \gamma$.

Proof.

(1) This follows easily from our definition of the tensor operation on $\mathbf{Cpl}(\mathcal{B})$ together with the fact that colimits in $\mathbf{Cpl}(\mathcal{B})$ are computed degreewise.

(2) We know that γ^* satisfies the three properties in the statement. Conversely, let us prove that they characterize a functor G completely (in terms of γ). By the first property we reduce to prove it for a presheaf K concentrated in degree 0. Then:

$$\begin{aligned} G(K) &\cong G\Big(\int^c K(c)_{\mathrm{cst}} \otimes \Lambda(c)\Big) && \text{by the Yoneda lemma} \\ &\cong \int^c G(K(c)_{\mathrm{cst}} \otimes \Lambda(c)) && \text{by cocontinuity.} \end{aligned}$$

We are thus reduced to show

$$G(K_{\mathrm{cst}} \otimes \Lambda(c)) \cong K \odot \gamma(c),$$

naturally in modules K and objects $c \in C$. For this we can take a functorial exact sequence

$$\oplus_{I_2}\Lambda \xrightarrow{\alpha} \oplus_{I_1}\Lambda \to K \to 0$$

of Λ-modules, by which we easily reduce to K free using the cocontinuity of G. Again by cocontinuity we further reduce to $K = \Lambda$ and then our contention follows from the third property. □

4. Cofibrant replacement

Our goal in this section is to resolve functorially any presheaf of complexes by a cofibrant object made up of representables. It is clear how to resolve a single presheaf of Λ-modules, and it is also not difficult to extend this to bounded below complexes of presheaves (essentially due to Fact 3.10). As the example in [13, 2.3.7] shows, not every complex of representables is cofibrant hence naively extending the procedure to the unbounded case might apriori run into problems. However, we will show that such problems do not occur.

4.1. **Preliminaries from homological algebra.** Recall the following basic facts in homological algebra.

Lemma 4.1. *Let $\mathcal{A}$ be a Grothendieck abelian category, and let $D, D' : \mathbb{N} \to \mathbf{Cpl}(\mathcal{A})$ be two diagrams of complexes in $\mathcal{A}$ ($\mathbb{N}$ considered as an ordered set). If $g : D \to D'$ is a morphism of diagrams of complexes which is objectwise a quasi-isomorphism, then also $\varinjlim g$ is a quasi-isomorphism.*

Proof. The functor $\varinjlim : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}$ is exact since $\mathcal{A}$ is a Grothendieck category. It follows that this functor commutes with homology and therefore preserves quasi-isomorphisms of chain complexes. □

Lemma 4.2. *Let $\mathcal{A}$ be an abelian Grothendieck category and let C, C' be two bounded below bicomplexes (i.e. $C_{\bullet,q} = 0$ for all $q \ll 0$) in $\mathcal{A}$, and let $f : C \to C'$ be a morphism of bicomplexes. If $f_{\bullet,q} : C_{\bullet,q} \to C'_{\bullet,q}$ is a quasi-isomorphism of complexes for all q, then $\mathrm{Tot}^{\oplus}(f)$ is a quasi-isomorphism.*

Proof. Without loss of generality, $C_{\bullet,q} = 0$ for all negative q. Let $C(n) = C_{\bullet,\leq n}$, $n \geq 0$, be the stupid truncation. In other words, $C(n)$ is the subbicomplex of C satisfying

$$C(n)_{p,q} := \begin{cases} C_{p,q} & : q \leq n \\ 0 & : q > n; \end{cases}$$

similarly for C' and f. We claim that $\mathrm{Tot}^{\oplus}(f(n))$ is a quasi-isomorphism for all n. This is proved by induction on n. For $n = 0$ it is true because of our assumption on f. For the induction step we use the short exact sequence

$$0 \to \mathrm{Tot}^{\oplus}(C(n-1)) \to \mathrm{Tot}^{\oplus}(C(n)) \to C_{\bullet,n}[-n] \to 0$$

of complexes in $\mathcal{A}$. f gives rise to a morphism of short exact sequences, where the induction hypothesis for $n-1$ together with our assumption on f show that the outer two arrows are quasi-isomorphisms. By the 5-lemma also the middle one, i. e. $\mathrm{Tot}^{\oplus}(f(n))$, is a quasi-isomorphism.

Now apply the previous lemma to $D_n = \mathrm{Tot}^{\oplus}(C(n))$, $D'_n = \mathrm{Tot}^{\oplus}(C'(n))$, and $g_n = \mathrm{Tot}^{\oplus}(f(n))$ to get the result. (One uses here that $\mathrm{Tot}^{\oplus}$ preserves colimits.) □

4.2. **Construction and proof.** Consider the functor category $\mathbf{PSh}(\mathcal{C}, \Lambda)$. It is a Grothendieck abelian category. We call an object of $\mathbf{PSh}(\mathcal{C}, \Lambda)$ *semi-representable* if it is a small coproduct of representables. An *SR-resolution* of an object $K \in \mathbf{PSh}(\mathcal{C}, \Lambda)$ is a complex $K_\bullet$ of semi-representables in $\mathbf{PSh}(\mathcal{C}, \Lambda)$ together with a quasi-isomorphism of complexes $K_\bullet \to S^0 K$. Similarly one defines SR-resolutions for complexes in $\mathbf{PSh}(\mathcal{C}, \Lambda)$. Note that a bounded below *SR-resolution* is a cofibrant replacement by Fact 3.10.

Lemma 4.3. *Objects in $\mathbf{PSh}(\mathcal{C}, \Lambda)$ possess a functorial SR-resolution; more precisely there exists a functor*

$$P : \mathbf{PSh}(\mathcal{C}, \Lambda) \to \mathbf{U}\mathcal{C}$$

together with a natural transformation $P \to S^0$ satisfying:

- *the components of $P \to S^0$ are all SR-resolutions concentrated in non-negative degrees;*
- *P maps the zero morphism to the zero morphism;*
- *P takes injective morphisms to degreewise split injective morphisms.*

Proof. Let K be an arbitrary object of $\mathbf{PSh}(\mathcal{C}, \Lambda)$. There is a canonical epimorphism

$$K_0 := \bigoplus_{K(c)\setminus 0} \Lambda(c) \to \operatorname*{colim}_{K(c)} \Lambda(c) \xrightarrow{\sim} K.$$

Taking the kernel and repeating this construction we get a complex $K_\bullet$ together with a quasi-isomorphism $K_\bullet \to S^0 K$.

Given $f : K \to K'$ and $x \in K(c)\setminus 0$ such that $f(x) = 0$, the component $\Lambda(c)$ corresponding to x is mapped to 0, otherwise it maps identically to $\Lambda(c)$ corresponding to $f(x)$. It is easily checked that this induces a morphism $\ker(K_0 \to$

$K) \to \ker(K'_0 \to K')$ hence repeating we obtain $P(f) : P(K) \to P(K')$. Functoriality is clear.

If f is injective then by this description $f_0 : K_0 \to K'_0$ is split injective, and the induced morphism $\ker(K_0 \to K) \to \ker(K'_0 \to K')$ is injective. Repeating this argument, we see that the induced morphism $P(f)$ is degreewise split injective. □

Proposition 4.4. *There exists an endofunctor $Q : \mathbf{U}\mathcal{C} \to \mathbf{U}\mathcal{C}$ together with a natural transformation $Q \to \mathrm{id}$ satisfying:*

- *the components of $Q \to \mathrm{id}$ are trivial fibrations;*
- *the image of Q consists of projective cofibrant complexes of semi-representables.*

In particular, Q is a cofibrant replacement functor.

Proof. Apply the functor P of the previous lemma in each degree, obtaining an SR-resolution $P(K_n)$ of K_n for each $n \in \mathbb{Z}$. We get maps $P(K_n) \to P(K_{n-1})$ of complexes and thus a bicomplex $P(K) := P_\bullet(K_\bullet)$ (since P takes 0 to 0) together with a map of bicomplexes $P(K) \to K$, the latter concentrated in horizontal degree 0. Taking the total complexes yields a morphism

$$Q(K) := \mathrm{Tot}^\oplus(P_\bullet(K_\bullet)) \to \mathrm{Tot}^\oplus(K_\bullet) = K. \tag{4}$$

Functoriality follows from functoriality in the previous lemma as well as functoriality of $\mathrm{Tot}^\oplus$. It remains to prove that (4) is a quasi-isomorphism with projective cofibrant domain.

For this let $\tau_{\geq n}K$ ($n \in \mathbb{Z}$) be the subcomplex of K satisfying

$$(\tau_{\geq n}K)_q = \begin{cases} K_q & : q > n \\ Z_nK & : q = n \\ 0 & : q < n. \end{cases}$$

Note that there are canonical morphisms $\tau_{\geq n}K \to \tau_{\geq n-1}K$ and the canonical morphism $\varinjlim_{n\in\mathbb{N}} \tau_{\geq -n}K \to K$ is an isomorphism. But also $\varinjlim_{n\in\mathbb{N}} P(\tau_{\geq -n}K) \to P(K)$ is an isomorphism of bicomplexes by construction. Since the total complex functor commutes with colimits we conclude that $\varinjlim_{n\in\mathbb{N}} Q(\tau_{\geq -n}K) \to Q(K)$ is an isomorphism.

By the previous lemma, $P(\tau_{\geq -n}K) \to P(\tau_{\geq -(n+1)}K)$ is a bidegreewise split injection hence $Q(\tau_{\geq -n}K) \to Q(\tau_{\geq -(n+1)}K)$ is a degreewise split injection. It follows from Corollary 3.12 that $Q(K)$ is projective cofibrant. Also by the previous lemma, $P(\tau_{\geq -n}K) \to \tau_{\geq -n}K$ is a quasi-isomorphism in each row. It follows from Lemma 4.2 that $Q(\tau_{\geq -n}K) \to \tau_{\geq -n}K$ is a quasi-isomorphism. (4) being the sequential colimit of these morphisms, Lemma 4.1 tells us that also (4) is a quasi-isomorphism. □

Remark 4.5. Even if this result is not very useful from a practical point of view, it does provide a conceptually satisfying method to compute the derived functor of a left dg Kan extension in the context of §3.4. Indeed, fix a functor $\gamma : \mathcal{C} \to \mathbf{Cpl}(\mathcal{B})$ for a $\mathbf{Mod}(\Lambda)$-cocomplete Λ-linear category $\mathcal{B}$, and assume that $\mathbf{Cpl}(\mathcal{B})$ comes with a model structure such that γ^* is a left Quillen functor. The image of any $K \in \mathbf{U}\mathcal{C}$ under $\mathrm{L}\gamma^*$ can be computed as follows:

1. Resolve K by a cofibrant complex QK of semi-representables.
2. Apply γ to each representable in QK obtaining a bicomplex $\gamma(QK)$ in $\mathcal{B}$.
3. Take the total complex $\mathrm{Tot}^\oplus(\gamma(QK))$.

In particular, this provides a more “elementary” description of the motivic realization constructed in [5, §7].

5. Local model structures

Having dealt with “generators” for universal enriched homotopy theories in §2 and for universal dg homotopy theories in more detail in the subsequent sections, we now turn to “relations”. The only sort of relations we will be interested in here are “topological”, i. e. induced by a Grothendieck topology on $\mathcal{C}$. Unfortunately we are not able to prove any substantial facts in the general enriched setting which is why we again restrict to the case of dg categories. Here, our main result is completely analogous to the main result of [10] where it is shown that a simplicial presheaf in the Jardine local model structure is fibrant if and only if it is injective fibrant and satisfies descent with respect to hypercovers.

Throughout this section we assume that $\mathcal{C}$ is endowed with a Grothendieck topology τ. Let $\mathbf{Sh}_\tau(\mathcal{C})$ (resp. $\mathbf{Sh}_\tau(\mathcal{C},\Lambda)$) denote the category of τ-sheaves (resp. of τ-sheaves of Λ-modules) on $\mathcal{C}$. The embedding $\mathbf{Sh}_\tau(\mathcal{C}) \to \mathbf{PSh}(\mathcal{C})$ (resp. $\mathbf{Sh}_\tau(\mathcal{C},\Lambda) \to \mathbf{PSh}(\mathcal{C},\Lambda)$) is right adjoint to the sheafification functor a_τ.

5.1. **Hypercovers and descent.** Recall ([10, §3]) that a morphism f of presheaves is a *generalized cover* if its sheafification $a_\tau(f)$ is an epimorphism.

Definition 5.1. For any object $c \in \mathcal{C}$ a *τ-hypercover of c* is a simplicial presheaf of sets $c_\bullet$ on $\mathcal{C}$ with an augmentation map $c_\bullet \to c =: c_{-1}$ such that

- c_n is a coproduct of representables for all $n \in \mathbb{N}$, and
- $c_n \to (\operatorname{cosk}_{n-1} c_\bullet)_n$ is a generalized cover for all $n \in \mathbb{N}$.

(To avoid any confusion, the cases $n = 0, 1$ of the second bullet point require $c_0 \to c$ and $d_0 \times d_1 : c_1 \to c_0 \times_c c_0$ to be generalized covers, respectively.) A *refinement* of a hypercover $c_\bullet \to c$ is a hypercover $c'_\bullet \to c$ together with a morphism of simplicial presheaves $c'_\bullet \to c_\bullet$ compatible with the augmentation by c. The class of all τ-hypercovers of c is denoted by $\mathcal{H}_{\tau,c}$. Also set $\mathcal{H}_\tau := \coprod_{c\in\mathcal{C}} \mathcal{H}_{\tau,c}$. A subclass $\mathcal{H}$ of $\mathcal{H}_\tau$ (resp. $\mathcal{H}_{\tau,c}$) is called *dense* if every τ-hypercover (resp. of c) admits a refinement by a hypercover in $\mathcal{H}$.

We refer to [10] for details about hypercovers. In particular, we recall without proof the following important fact.

Fact 5.2 ([10, Pro. 6.7])**.** *For every $c \in \mathcal{C}$, there exists a dense subset of $\mathcal{H}_{\tau,c}$. Therefore also $\mathcal{H}_\tau$ admits a dense subset.*

In the case of simplicial presheaves the τ-hypercovers provide the “topological” relations in that the hypercover $c_\bullet$ and the representable c are “identified”, and we want to translate these relations to the setting of presheaves of complexes. For this notice that given any hypercover $c_\bullet \to c$ we can use the Moore complex (cf. §3.3) to obtain an object $\Lambda(c_\bullet) \in \mathbf{U}\mathcal{C}$ together with a morphism $\Lambda(c_\bullet) \to \Lambda(c)$. Explicitly, $\Lambda(c_\bullet)$ is the complex

$$\cdots \to \Lambda(c_1) \to \Lambda(c_0) \to 0$$

with differentials given by the alternating sum of the face maps, and each $\Lambda(c_i)$ is semi-representable. It follows from Lemma 3.10 that $\Lambda(c_\bullet)$ is projective cofibrant.

Definition 5.3.

(1) Let $\mathcal{S}$ be a class of τ-hypercovers. A presheaf $K \in \mathbf{U}\mathcal{C}$ *satisfies $\mathcal{S}$-descent* if for any τ-hypercover $c_\bullet \to c$ in $\mathcal{S}$,

$$K(c) = \underline{\mathrm{hom}}_{\mathrm{dg}}(\Lambda(c), K) \to \underline{\mathrm{hom}}_{\mathrm{dg}}(\Lambda(c_\bullet), K) =: K(c_\bullet)$$

is a quasi-isomorphism of chain complexes.

(2) $K \in \mathbf{U}\mathcal{C}$ *satisfies τ-descent* if it satisfies $\mathcal{H}_\tau$-descent.

Explicitly, $K(c_\bullet)$ is given by the product total complex of the bicomplex

$$K(c_0) \to K(c_1) \to \cdots,$$

where $K(\coprod_{i\in I} d_i)$ for $d_i \in \mathcal{C}$ is defined as $\prod_{i\in I} K(d_i)$.

Remark 5.4. The condition of satisfying descent is homotopy invariant, i. e. given two quasi-isomorphic presheaves of complexes, one satisfies $\mathcal{S}$-descent if and only if the other does. Indeed, as we know from Fact 2.8, $\underline{\mathrm{hom}}_{\mathrm{dg}} : (\mathbf{U}\mathcal{C})^{\mathrm{op}} \times \mathbf{U}\mathcal{C} \to \mathbf{Cpl}(\Lambda)$ is part of a Quillen adjunction of two variables. And since every object in $\mathbf{U}\mathcal{C}$ is fibrant, and since both $\Lambda(c_\bullet)$ and $\Lambda(c)$ are cofibrant (by Fact 3.10), the condition on K to satisfy descent is that

$$\mathrm{R}\,\underline{\mathrm{hom}}_{\mathrm{dg}}(\Lambda(c), K) \to \mathrm{R}\,\underline{\mathrm{hom}}_{\mathrm{dg}}(\Lambda(c_\bullet), K)$$

be an isomorphism in the derived category of Λ. This is different from the situation of simplicial presheaves of sets where $c_\bullet$ is not necessarily projective cofibrant. Thus the interest in *split* hypercovers, cf. [9, Cor. 9.4].

We end this section with the following important result. In terminology to be introduced shortly it tells us that the augmentation morphism $\Lambda(c_\bullet) \to \Lambda(c)$ associated to any τ-hypercover is a τ-local equivalence.

Fact 5.5 ([1, Thm. V, 7.3.2])**.** *Any τ-hypercover $c_\bullet \to c$ induces identifications*

$$a_\tau \mathrm{H}_n \Lambda(c_\bullet) \cong \begin{cases} a_\tau \Lambda(c) & : n = 0 \\ 0 & : n \neq 0 \end{cases}$$

in $\mathbf{Sh}(\mathcal{C}, \Lambda)$.

5.2. **Localization.**

Definition 5.6. A morphism f in $\mathbf{U}\mathcal{C}$ is called a *τ-local equivalence* if the induced morphism of homology sheaves $a_\tau \mathrm{H}_n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.

The goal of this section is to prove the following theorem.

Theorem 5.7. *The left Bousfield localization $\mathbf{U}\mathcal{C}/\tau$ of $\mathbf{U}\mathcal{C}$ with respect to τ-local equivalences exists and satisfies:*

(1) The underlying category of $\mathbf{U}\mathcal{C}/\tau$ is the one of $\mathbf{U}\mathcal{C}$. The cofibrations are also the same. The weak equivalences are the τ-local equivalences.

(2) $\mathbf{U}\mathcal{C}/\tau$ is a proper, tractable, cellular, stable model category.

(3) The fibrations of $\mathbf{U}\mathcal{C}/\tau$ are the fibrations of $\mathbf{U}\mathcal{C}$ whose kernel satisfies τ-descent. In particular, the fibrant objects of $\mathbf{U}\mathcal{C}/\tau$ are the objects satisfying τ-descent.

The model structure on $\mathbf{U}\mathcal{C}/\tau$ is called the τ-local model structure.

Remark 5.8. This result was originally one of our main motivations to write the present note. The existence of this localization was known before, see [2, Def. 4.4.34], and we use this result in our proof. The main point of the theorem for us was part 3. The analogous description of the fibrant objects for simplicial sets instead of chain complexes is of course the main result of [10], and we deduce our result from theirs.

After having completed this note, we learned that also part 3 had appeared in the literature before, see [11]. His proof is different from ours in that he does not reduce to the case of simplicial sets nor uses the theory of Bousfield localizations but proves the axioms of a model structure "by hand". Even later it was brought to our attention that in his unpublished Ph.D. thesis, Drew establishes an essentially equivalent result (see [8, Prop. 1.3.12]), using the same technique of reducing to simplicial sets.

Let $\mathcal{S} \subset \mathcal{H}_\tau$ be some class of τ-hypercovers. We denote by $\Lambda(\mathcal{S})[\mathbb{Z}]$ the class

$$\{\Lambda(c_\bullet)[n] \to \Lambda(c)[n] \mid c_\bullet \to c \in \mathcal{S}, n \in \mathbb{Z}\}$$

of morphisms in $\mathbf{U}\mathcal{C}$.

Definition 5.9.

(1) Recall ([12, Def. 3.1.4]) that an object K in $\mathbf{U}\mathcal{C}$ is called *local* with respect to a class of morphisms $\mathcal{F}$ in $\mathbf{U}\mathcal{C}$ if for each $f \in \mathcal{F}$, the induced morphism of homotopy function complexes $\operatorname{Rmap}(f, K)$ is a weak homotopy equivalence of simplicial sets.

(2) Let $\mathcal{S}$ be a class of τ-hypercovers. We say that $K \in \mathbf{U}\mathcal{C}$ is $\mathcal{S}$-*local* if it is local with respect to $\Lambda(\mathcal{S})[\mathbb{Z}]$.

(3) We say that $K \in \mathbf{U}\mathcal{C}$ is τ-*local* if it is $\mathcal{H}_\tau$-local.

Lemma 5.10. *For a presheaf of complexes $K \in \mathbf{U}\mathcal{C}$ and a class $\mathcal{S}$ of τ-hypercovers the following two conditions are equivalent:*

(1) K is $\mathcal{S}$-local.

(2) K satisfies $\mathcal{S}$-descent.

In particular, the following two conditions are equivalent:

(1) K is τ-local.

(2) K satisfies τ-descent.

Proof. K is $\mathcal{S}$-local if and only if for any $c_\bullet \to c \in \mathcal{S}$, $n \in \mathbb{Z}$, the morphism of homotopy function complexes

$$\operatorname{Rmap}(\Lambda(c)[n], K) \to \operatorname{Rmap}(\Lambda(c_\bullet)[n], K) \tag{5}$$

is a weak equivalence of simplicial sets. But $\operatorname{Rmap}(A, B) \cong \Gamma\tau_{\geq 0}\mathbf{U}\mathcal{C}(A, B)$ by Lemma 3.16. So (5) is identified with

$$\Gamma\tau_{\geq 0}(K(c)[-n]) \to \Gamma\tau_{\geq 0}(K(c_\bullet)[-n]),$$

whose m-th homotopy group is thus

$$\mathrm{H}_{m-n}K(c) \to \mathrm{H}_{m-n}K(c_\bullet).$$

□

We will deduce Theorem 5.7 from the following (cf. [10, Thm. 6.2]).

Theorem 5.11. *Let $\mathcal{S}$ be a class of τ-hypercovers which contains a dense subset. Then the left Bousfield localization $\mathbf{U}\mathcal{C}/\mathcal{S}$ of $\mathbf{U}\mathcal{C}$ with respect to $\Lambda(\mathcal{S})[\mathbb{Z}]$ exists and coincides with $\mathbf{U}\mathcal{C}/\tau$.*

Proof of Theorem 5.7. Let $\mathcal{S}$ be the class of all τ-hypercovers. By Fact 5.2, $\mathcal{S}$ satisfies the assumption of Theorem 5.11. We know from Corollary 3.1 that $\mathbf{U}\mathcal{C}$ is left-proper, tractable, cellular. These are preserved by left Bousfield localizations by [12, Thm. 4.1.1] and [14, Pro. 4.3]. Since $\mathcal{S}$-local objects are closed under shifts by Lemma 5.10, $\mathbf{U}\mathcal{C}/\mathcal{S}$ and therefore $\mathbf{U}\mathcal{C}/\tau$ are stable model categories (see [3, Pro. 3.6]). Since $\mathbf{U}\mathcal{C}$ is a right proper model category so is $\mathbf{U}\mathcal{C}/\tau$ by [3, Pro. 3.7]. Since all objects are fibrant in $\mathbf{U}\mathcal{C}$, the fibrant objects of $\mathbf{U}\mathcal{C}/\mathcal{S}$ are the τ-local objects. We deduce from Lemma 5.10 and Theorem 5.11 that the fibrant objects of $\mathbf{U}\mathcal{C}/\tau$ are precisely the presheaves satisfying τ-descent. The description of the fibrations in $\mathbf{U}\mathcal{C}/\tau$ then follows from this and [3, Lem. 3.9]. Finally, that the weak equivalences of $\mathbf{U}\mathcal{C}/\tau$ are the τ-local equivalences is proven in [2, Pro. 4.4.32]. □

Assume for the moment that $\mathcal{S}$ in Theorem 5.11 is a set. In this case we know that the left Bousfield localization $\mathbf{U}\mathcal{C}/\mathcal{S}$ (resp. $\mathbf{\Delta}^{\mathrm{op}}\,\mathbf{PSh}(\mathcal{C})/\mathcal{S}$) with respect to $\Lambda(\mathcal{S})[\mathbb{Z}]$ (resp. $\mathcal{S}$) exists. Temporarily, we call these model structures the $\mathcal{S}$-local model structures, their fibrations are called $\mathcal{S}$-fibrations, their weak equivalences are called $\mathcal{S}$-equivalences.

Lemma 5.12. *The Dold-Kan correspondence (Proposition 3.15) induces a Quillen adjunction*

$$(N\Lambda, \Gamma\tau_{\geq 0}) : \mathbf{\Delta}^{\mathrm{op}}\,\mathbf{PSh}(\mathcal{C})/\mathcal{S} \longrightarrow \mathbf{U}\mathcal{C}/\mathcal{S}.$$

Moreover, $\Gamma\tau_{\geq 0}$ preserves τ-local equivalences.

Proof. Given $f : c_\bullet \to c \in \mathcal{S}$, the morphism $N\Lambda(f)$ factors as

$$N\Lambda(c_\bullet) \to \Lambda(c_\bullet) \xrightarrow{\Lambda(f)} \Lambda(c),$$

where the first arrow is a quasi-isomorphism by Fact 3.14, and the second arrow lies in $\Lambda(\mathcal{S})[\mathbb{Z}]$. Thus the first claim follows from the universal property of localizations. The second claim is also evident since the homotopy groups of $\Gamma\tau_{\geq 0}K$ are the homology groups of K in non-negative degrees, by Fact 3.14. □

Lemma 5.13. *Let $K, K' \in \mathbf{U}\mathcal{C}$ be $\mathcal{S}$-fibrant objects and let $f : K \to K'$ be an $\mathcal{S}$-fibration which is also a τ-local weak equivalence. Then f is a sectionwise trivial fibration, i. e. it is a trivial fibration in the projective model structure on $\mathbf{U}\mathcal{C}$.*

Proof. A morphism $f : K \to K' \in \mathbf{U}\mathcal{C}$ is a trivial fibration if and only if for all $c \in \mathcal{C}$ and all $n \in \mathbb{Z}$, f has the right lifting property with respect to $\partial\Delta^n\Lambda(c) \to \Delta^n\Lambda(c)$ (see Lemma 3.5).

Let $i : (\partial\Delta^1) \otimes c \to \Delta^1 \otimes c$ be the canonical cofibration of simplicial presheaves. Then $N\Lambda(i)$ is also a cofibration and $N\Lambda((\partial\Delta^1) \otimes c) = \partial\Delta^1\Lambda(c)$ and $N\Lambda(\Delta^1 \otimes c) = \Delta^1\Lambda(c)$. Also note that $\partial\Delta^n\Lambda(c) = \partial\Delta^1\Lambda(c)[-n+1]$, and similarly for $\Delta^n\Lambda(c)$. We want to show the existence of a lifting for every diagram of the following form

$$\begin{array}{ccc} \partial\Delta^n\Lambda(c) & \longrightarrow & K \\ \big\downarrow & & \big\downarrow \\ \Delta^n\Lambda(c) & \longrightarrow & K' \end{array}$$

Now using shifts this is the same as showing that

$$\begin{array}{ccc} \partial\Delta^1\Lambda(c) & \longrightarrow & K[n-1] \\ \downarrow & & \downarrow \\ \Delta^1\Lambda(c) & \longrightarrow & K'[n-1] \end{array}$$

has a lift. Notice that the right vertical arrow is still an $\mathcal{S}$-fibration.

But using the adjunction of Lemma 5.12 this is the same as showing that

$$\begin{array}{ccc} \partial\Delta^1 \otimes c & \longrightarrow & \Gamma\tau_{\geq 0}(K[n-1]) \\ \downarrow & & \downarrow \\ \Delta^1 \otimes c & \longrightarrow & \Gamma\tau_{\geq 0}(K'[n-1]) \end{array}$$

has a lift, where we know that the right vertical arrow is an $\mathcal{S}$-fibration and τ-local equivalence between $\mathcal{S}$-fibrant objects. Hence by [10, Lem. 6.5] it is a trivial fibration sectionwise. Now $i : (\partial\Delta^1) \otimes c \to \Delta^1 \otimes c$ is a projective cofibration, hence there is a lift in the last diagram above. This finishes the proof. □

Proof of Theorem 5.11. Let $\mathcal{S}$ be as in the theorem, and pick a dense subset $\mathcal{S}'$ of $\mathcal{S}$.

We claim that the $\mathcal{S}'$-local equivalences in $\mathbf{U}\mathcal{C}$ are precisely the τ-local equivalences. Indeed, by Fact 5.5, every $\mathcal{S}'$-local equivalence is a τ-local equivalence. For the converse, we may apply [10, Lem 6.4] together with Lemma 5.13 (we also use the existence of the τ-local model structure, see Remark 5.8). This proves the claim which in turn implies that $\mathbf{U}\mathcal{C}/\mathcal{S}' = \mathbf{U}\mathcal{C}/\tau$.

We deduce that every hypercover in $\mathcal{S}$ is an $\mathcal{S}'$-equivalence hence the localization of $\mathbf{U}\mathcal{C}$ with respect to $\Lambda(\mathcal{S})[\mathbb{Z}]$ exists and coincides with $\mathbf{U}\mathcal{C}/\mathcal{S}'$. □

Let us agree to call a model category $\mathcal{M}$ equipped with a functor $\gamma : \mathcal{C} \to \mathcal{M}$ *τ-local* if for every τ-hypercover $c_\bullet \to c$ in $\mathcal{C}$, $\operatorname{L}\operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \gamma(c_\bullet) \to \gamma(c)$ is an isomorphism in $\mathbf{Ho}(\mathcal{M})$. (Here, $\gamma(c_\bullet)$ is the simplicial object in $\mathcal{M}$ with n-simplices given by $\gamma^*(c_n)$.) In line with the viewpoint taken in §2 let us record the following corollary of Theorem 5.7. It asserts that $\mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau$ is the universal τ-local model dg category associated to $\mathcal{C}$.

Corollary 5.14. *Let $(\mathcal{C}, \tau)$ be a small site. Then there exists a functor $\Lambda : \mathcal{C} \to \mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau$ into a τ-local model dg category, universal in the sense that for any solid diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Lambda} & \mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau \\ & {\scriptstyle\gamma}\searrow & \vdots\, {\scriptstyle F} \\ & & \mathcal{M} \end{array}$$

with $\mathcal{M}$ a τ-local model dg category, there exists a left Quillen dg functor F as indicated by the dotted arrow, unique up to a contractible choice, making the diagram commutative up to a weak equivalence $F\Lambda \to \gamma$.

Proof. $\mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau$ as a dg category is just $\mathbf{U}_{\mathrm{dg}}\mathcal{C}$ and the cofibrations are the same hence to prove that $\mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau$ is a model dg category, it suffices to see that the pushout-product $i \square f$ is a τ-weak equivalence for every cofibration i in $\mathbf{Cpl}(\Lambda)$ and every τ-acyclic cofibration $f \in \mathbf{U}\mathcal{C}$. This can be established exactly as in the proof of [4, Thm. 4.46]. (For this step it is not necessary to assume as is done in loc. cit. that the localization is with respect to a set but only that it exists.) The essential point is that $\mathbf{U}\mathcal{C}$ is a tractable model category (by Proposition 3.1).

Next we claim that $\mathrm{L}\operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \Lambda(c_\bullet) \to \Lambda(c)$ is an isomorphism in $\mathbf{Ho}(\mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau)$. But by Lemma 3.18, this morphism can be identified with $\Lambda(c_\bullet) \to \Lambda(c)$ hence the claim follows from Fact 5.5.

Given a solid diagram as in the statement of the corollary we know by Corollary 2.11 the existence of a left Quillen dg functor $F : \mathbf{U}_{\mathrm{dg}}\mathcal{C} \to \mathcal{M}$, unique up to contractible choice, making the triangle commutative up to a weak equivalence $Fy \to \gamma$. By the universal property of the localization of model categories together with Theorem 5.11, it now suffices to prove that the left derived functor $\mathrm{L}F$ takes $\Lambda(\mathcal{H}_\tau)[\mathbb{Z}]$ to isomorphisms in $\mathbf{Ho}(\mathcal{M})$. Thus let $c_\bullet \to c \in \mathcal{H}_\tau$ and $n \in \mathbb{Z}$. First notice that F "commutes with shifts" in the sense that

$$F(\bullet[n]) \cong F(S^n \odot \bullet) \cong S^n \odot F(\bullet),$$

and since $\mathcal{M}$ is a model dg category, $S^n \odot \bullet$ preserves weak equivalences. We thus reduce to the case $n = 0$.

Now, again by Lemma 3.18, $\Lambda(c_\bullet)$ can be identified with the homotopy colimit of $\Lambda(c_\bullet)$. Since F is a left Quillen dg functor it will commute with homotopy colimits in the homotopy category. Thus we want the upper row in the following commutative square to be invertible in $\mathbf{Ho}(\mathcal{M})$.

$$\begin{array}{ccc} \mathrm{L}\operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} F\Lambda(c_\bullet) & \longrightarrow & F\Lambda(c) \\ \downarrow & & \downarrow \\ \mathrm{L}\operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \gamma(c_\bullet) & \longrightarrow & \gamma(c) \end{array}$$

Our assumptions tell us that the vertical arrows as well as the bottom arrow are isomorphisms so we conclude. □

5.3. **Smaller models.** Having described explicitly generators and relations for the model dg category $\mathbf{U}\mathcal{C}/\tau$ associated to a small site $(\mathcal{C}, \tau)$, we give in this section two methods to modify the model $\mathbf{U}\mathcal{C}/\tau$ up to Quillen equivalence which are useful in practice. The first consists in replacing presheaves by sheaves, the second allows to reduce the "number" of generators. In both cases therefore we obtain "smaller" models with the same homotopy category. Both modifications are straightforward and have been employed before in the literature.

The category of τ-sheaves of complexes on $\mathcal{C}$, $\mathbf{Sh}_\tau(\mathcal{C}, \mathbf{Cpl}(\Lambda))$, admits the τ-local model structure, obtained by transfer along the right adjoint $\mathbf{Sh}_\tau(\mathcal{C}, \mathbf{Cpl}(\Lambda)) \to \mathbf{PSh}(\mathcal{C}, \mathbf{Cpl}(\Lambda))$ (cf. [2, Cor. 4.4.43]). Since the morphism $K \to a_\tau K$ is a τ-local equivalence for every $K \in \mathbf{U}\mathcal{C}$, the following statement is immediate.

Fact 5.15.

$$\mathbf{U}\mathcal{C}/\tau \underset{\longleftarrow}{\overset{a_\tau}{\longrightarrow}} \mathbf{Sh}_\tau(\mathcal{C}, \mathbf{Cpl}(\Lambda))/\tau$$

defines a Quillen equivalence. Their homotopy categories are the derived category of τ-sheaves on $\mathcal{C}$.

It happens frequently that every object $c \in \mathcal{C}$ can be covered by objects belonging to a distinguished strict subcategory $\mathcal{C}'$. Certainly one then expects the model dg categories generated by $\mathcal{C}$ and $\mathcal{C}'$ with the topological relations to be "the same". The following result makes this precise.

Corollary 5.16. *Let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$, and endow it with the topology τ' induced from τ. Assume that every object $c \in \mathcal{C}$ can be covered by objects belonging to $\mathcal{C}'$. Then the (functor underlying the) canonical dg functor*

$$\mathbf{U}\mathcal{C}'/\tau' \longrightarrow \mathbf{U}\mathcal{C}/\tau$$

defines a Quillen equivalence.

Proof. The composition $\mathcal{C}' \xrightarrow{u} \mathcal{C} \to \mathbf{U}\mathcal{C}/\tau$ induces the left Quillen dg functor $u_!$ in the statement by the universal property of $\mathbf{U}\mathcal{C}'/\tau'$ (Corollary 5.14), left-adjoint to the restriction functor u^*. Consider the square of Quillen right functors:

$$\begin{array}{ccc} \mathbf{Sh}_\tau(\mathcal{C}, \mathbf{Cpl}(\Lambda))/\tau & \xrightarrow{u^*} & \mathbf{Sh}_{\tau'}(\mathcal{C}', \mathbf{Cpl}(\Lambda))/\tau' \\ \downarrow & & \downarrow \\ \mathbf{U}\mathcal{C}/\tau & \xrightarrow[u^*]{} & \mathbf{U}\mathcal{C}'/\tau' \end{array}$$

Clearly, it commutes. By the previous fact, the vertical arrows are part of a Quillen equivalence, and the homotopy categories in the top row are the derived categories of τ-sheaves (resp. τ'-sheaves) on $\mathcal{C}$ (resp. $\mathcal{C}'$). By [1, Thm. III.4.1], the top arrow is an equivalence of the underlying categories hence so is the induced functor on their derived categories. □

5.4. **Hypercohomology.** One might hope that the results obtained so far in this section allow to describe a τ-fibrant replacement directly in terms of hypercovers. In particular, this would lead to an expression for the hypercohomology of complexes of sheaves using hypercovers alone. We have not been able to provide such a fibrant replacement but, as we will now show, the hypercohomology does indeed admit such an expected description. This result should be compared to Verdier's hypercover theorem in [1, Thm. V, 7.4.1]. Our proof once again proceeds by reducing to the case of simplicial (pre)sheaves of sets in [10]. (In the following, we write H^n for H_{-n}.)

Proposition 5.17. *Assume that every τ-hypercover can be refined by a split one. Let $K \in \mathbf{U}\mathcal{C}$ be a presheaf of complexes on $\mathcal{C}$, $c \in \mathcal{C}$, and $n \in \mathbb{Z}$. Then there is a canonical isomorphism of Λ-modules*

$$\mathbb{H}^n_\tau(c, a_\tau K) \cong \operatorname*{colim}_{c_\bullet \to c} \mathrm{H}^n K(c_\bullet),$$

where the left hand side denotes hypercohomology of the complex of τ-sheaves $a_\tau K$ on $\mathcal{C}/c$, and the colimit on the right hand side is over the opposite category of τ-hypercovers of c up to simplicial homotopy (cf. [1, § V.7.3]).

Proof. This follows from the following sequence of isomorphisms:

$$\begin{aligned}
\mathbb{H}^n_\tau(c, a_\tau K) &\cong \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C}/\tau)}(\Lambda(c), a_\tau K[-n]) && \text{Corollary 3.17}\\
&\cong \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C}/\tau)}(\Lambda(c), K[-n]) && K \to a_\tau K \ \tau\text{-local equivalence}\\
&\cong \hom_{\mathbf{Ho}(\mathbf{\Delta}^{\mathrm{op}}\,\mathbf{PSh}(\mathcal{C})/\tau)}(c, \Gamma\tau_{\geq 0}K[-n]) && \text{Lemma 5.12}\\
&\cong \operatorname*{colim}_{c_\bullet \to c} \pi(c_\bullet, \Gamma\tau_{\geq 0}K[-n]) && \text{[10, Thm. 7.6(b)]}\\
&\cong \operatorname*{colim}_{c_\bullet \to c \text{ split}} \pi(c_\bullet, \Gamma\tau_{\geq 0}K[-n]) && \text{assumption}\\
&\cong \operatorname*{colim}_{c_\bullet \to c \text{ split}} \hom_{\mathbf{Ho}(\mathbf{\Delta}^{\mathrm{op}}\,\mathbf{PSh}(\mathcal{C}))}(c_\bullet, \Gamma\tau_{\geq 0}K[-n]) && \text{split hypercovers cofibrant}\\
&\cong \operatorname*{colim}_{c_\bullet \to c \text{ split}} \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(N\Lambda(c_\bullet), K[-n]) && \text{Lemma 3.15}\\
&\cong \operatorname*{colim}_{c_\bullet \to c \text{ split}} \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(\Lambda(c_\bullet), K[-n]) && \text{Lemma 3.14}\\
&\cong \operatorname*{colim}_{c_\bullet \to c \text{ split}} \mathrm{H}^n K(c_\bullet) && \text{Corollary 3.17}\\
&\cong \operatorname*{colim}_{c_\bullet \to c} \mathrm{H}^n K(c_\bullet) && \text{assumption}
\end{aligned}$$

□

Remark 5.18. The hypothesis of the Proposition, i. e. that every hypercover admits a split refinement, is satisfied in many cases, e. g. when $(\mathcal{C}, \tau)$ is a Verdier site, see [10, Thm. 8.6].

Moreover, in these cases the proposition represents another approach to Theorem 5.7. Indeed, the essential point, as we mentioned in Remark 5.8, is the description of the τ-fibrant objects in $\mathbf{U}\mathcal{C}/\tau$. Since $\Lambda(c_\bullet) \to \Lambda(c)$ is a τ-local equivalence for each τ-hypercover $c_\bullet \to c$ (Fact 5.5) it is clear that τ-fibrant objects satisfy τ-descent. Conversely, suppose $K \in \mathbf{U}\mathcal{C}$ satisfies τ-descent and choose a τ-fibrant replacement $f : K \to K'$. Using the previous proposition we will prove that f is a quasi-isomorphism.

Fix $c \in \mathcal{C}$ and $n \in \mathbb{Z}$. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{colim}_{c_\bullet \to c}\, \mathrm{H}^n K(c_\bullet) & \xrightarrow{\ \sim\ } & \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C}/\tau)}(\Lambda(c), K[-n]) \\
\uparrow & & \downarrow f \\
\mathrm{H}^n K(c) & \xrightarrow[f]{} & \mathrm{H}^n K'(c)
\end{array}$$

The left vertical arrow is an isomorphism since K satisfies τ-descent. The right vertical arrow is an isomorphism since K' is τ-fibrant. Thus the claim.

5.5. **Complements.** In this last paragraph we discuss two further aspects of the local dg homotopy theory: monoidal structures, and closure of fibrant objects under certain operations.

Proposition 5.19. *Assume that either of the following conditions is satisfied:*

(1) $\mathcal{C}$ is cartesian monoidal.
(2) For any objects $c, d \in \mathcal{C}$, $\Lambda(c) \otimes \Lambda(d)$ is projective, and $(\mathcal{C}, \tau)$ has enough points.

Then $\mathbf{U}\mathcal{C}/\tau$ is a symmetric monoidal model category for the objectwise tensor product.

Proof.

(1) If $\mathcal{C}$ is cartesian monoidal, we may adapt the proof of [4, Thm. 4.58]. By [4, Pro. 4.47], it suffices to prove that for each $d \in \mathcal{C}$, and each τ-local $K \in \mathbf{U}\mathcal{C}$, the internal hom object $[\Lambda(d), K]$ is τ-local. Thus let $c_\bullet \to c$ be a τ-hypercover. Using the commutative diagram

$$\begin{array}{ccc}
\underline{\mathrm{hom}}_{\mathrm{dg}}(\Lambda(c), [\Lambda(d), K]) & \longrightarrow & \underline{\mathrm{hom}}_{\mathrm{dg}}(\Lambda(c_\bullet), [\Lambda(d), K)] \\
\downarrow \sim & & \downarrow \sim \\
\underline{\mathrm{hom}}_{\mathrm{dg}}(\Lambda(c) \otimes \Lambda(d), K) & \longrightarrow & \underline{\mathrm{hom}}_{\mathrm{dg}}(\Lambda(c_\bullet) \otimes \Lambda(d), K) \\
\downarrow \sim & & \downarrow \sim \\
\underline{\mathrm{hom}}_{\mathrm{dg}}(\Lambda(c \times d), K) & \longrightarrow & \underline{\mathrm{hom}}_{\mathrm{dg}}(\Lambda(c_\bullet \times d), K)
\end{array}$$

we reduce to showing that $c_\bullet \times d \to c \times d$ is a τ-local equivalence of simplicial presheaves. This follows from the fact that homotopy groups and sheafification commute with finite products.

(2) By Lemma 3.13, $\mathbf{U}\mathcal{C}$ is a symmetric monoidal model category. The result then follows from the proof of [2, Pro. 4.4.63] (whereas the statement in loc. cit. misses the first hypothesis above). □

Our description of τ-fibrant objects in Theorem 5.7 allows one to prove easily that these are closed under various operations. In the following lemmas we discuss two examples.

Lemma 5.20. *Let $K_\bullet$ be a bounded complex of τ-fibrant objects in $\mathbf{U}\mathcal{C}$. Then $\mathrm{Tot}^{\oplus} K_\bullet \in \mathbf{U}\mathcal{C}$ is τ-fibrant.*

Proof. Let $c_\bullet \to c$ be a τ-hypercover. We know that for any $l \in \mathbb{Z}$, $K_l(c) \to K_l(c_\bullet)$ is a quasi-isomorphism. Since $K_\bullet$ is bounded below, it follows from Lemma 4.2 that also

$$\mathrm{Tot}^{\oplus}(K_\bullet(c)) \to \mathrm{Tot}^{\oplus}(K_\bullet(c_\bullet))$$

is a quasi-isomorphism. Since $K_\bullet$ is bounded (hence $\mathrm{Tot}^{\oplus}$ and Tot^{Π} agree), one easily checks that this morphism can be identified with

$$(\mathrm{Tot}^{\oplus} K_\bullet)(c) \to (\mathrm{Tot}^{\oplus} K_\bullet)(c_\bullet).$$ □

Let κ be a regular cardinal. We say that the site $(\mathcal{C}, \tau)$ is *κ-noetherian* if every cover $\{c_i \to c\}_{i \in I}$ has a subcover $\{c_i \to c\}_{i \in J \subset I}$ with $|J| < \kappa$. An $\aleph_0$-noetherian site is called simply *noetherian*, as in [17, §III.3]. Also, recall the notion of Verdier sites from [10, Def. 8.1].

Lemma 5.21. *Let $(\mathcal{C}, \tau)$ be a κ-noetherian Verdier site, $\kappa > \aleph_0$. Then τ-fibrant objects in $\mathbf{U}\mathcal{C}$ are closed under κ-filtered colimits.*

Proof. By [10, Rem. 8.7], there is a dense set of τ-hypercovers $\mathcal{S}$ such that for each $c_\bullet \to c \in \mathcal{S}$ and each $n \in \mathbb{N}$, c_n is a coproduct $c_n \cong \coprod_{i \in I_n} c_{n,i}$ with $c_{n,i}$ representable and $|I_n| < \kappa$. By Theorem 5.11 and Lemma 5.10, being τ-fibrant is equivalent to satisfying $\mathcal{S}$-descent. Now let $K : J \to \mathbf{U}\mathcal{C}$ be a κ-filtered diagram of

τ-fibrant objects, and $c_\bullet \to c \in \mathcal{S}$. The claim then follows from the isomorphism

$$\begin{aligned}(\operatorname*{colim}_j K(j))(c_\bullet) &\cong \mathrm{Tot}^\Pi(\operatorname*{colim}_j K(j)_p(c_q))_{p,q} \\ &\cong \mathrm{Tot}^\Pi(\prod_{i\in I_q} \operatorname*{colim}_j K(j)_p(c_{q,i}))_{p,q} \\ &\cong \operatorname*{colim}_j \mathrm{Tot}^\Pi(\prod_{i\in I_q} K(j)_p(c_{q,i}))_{p,q} \\ &\cong \operatorname*{colim}_j (K_j(c_\bullet)),\end{aligned}$$

as κ-filtered colimits commute with products indexed by cardinals smaller than κ. □

Lemma 5.22. *Let $(\mathcal{C}, \tau)$ be a noetherian Verdier site. Any filtered colimit of bounded above τ-fibrant objects in $\mathbf{U}\mathcal{C}$ is τ-fibrant.*

Proof. The proof is essentially the same as in the previous lemma. We must assume bounded above objects so that the product totalization involves only finitely many factors in each degree hence commutes with filtered colimits. □

6. Fibrant replacement

In this section we would like to give an "explicit" fibrant replacement functor in $\mathbf{U}\mathcal{C}/\tau$ using the Godement resolution. It is a direct translation of the analogous construction for simplicial (pre)sheaves in [18, p. 66ff], with, again, the only problem created by the unboundedness of our complexes. We first establish the tools to overcome this difficulty.

6.1. **Local model structure and truncation.** Consider the functor $\Gamma\tau_{\geq 0} : \mathbf{U}\mathcal{C} \to \mathbf{\Delta}^{\mathrm{op}}\,\mathbf{PSh}(\mathcal{C})$. Applying it objectwise, this generalizes to a functor defined on diagrams with values in $\mathbf{U}\mathcal{C}$ which we still denote by $\Gamma\tau_{\geq 0}$.

Lemma 6.1. *The canonical arrow*

$$\mathrm{R}\lim_{\mathbf{\Delta}} \Gamma\tau_{\geq 0} K \to \Gamma\tau_{\geq 0} \mathrm{R}\lim_{\mathbf{\Delta}} K$$

is a weak homotopy equivalence for every $K \in (\mathbf{U}\mathcal{C})^{\mathbf{\Delta}}$.

Proof. One way to see this is as follows. $\Gamma\tau_{\geq 0}$ is a right Quillen functor for the projective model structures on $\mathcal{M} := \mathbf{U}\mathcal{C}$ and $\mathcal{N} := \mathbf{PSh}(\mathcal{C}, \mathbf{\Delta}^{\mathrm{op}}\mathrm{Set})$. It follows that the induced morphism of derivators $\mathbb{D}_{\mathcal{M}} \to \mathbb{D}_{\mathcal{N}}$ is continuous (see [6, Pro. 6.12]), in particular it commutes with homotopy limits. The claim now follows from the fact that $\Gamma\tau_{\geq 0}$ takes quasi-isomorphisms to weak homotopy equivalences hence doesn't need to be derived. □

Proposition 6.2.

(1) *For a morphism $f : K \to K'$ in $\mathbf{U}\mathcal{C}$ the following are equivalent:*
   - (a) *f is a τ-local equivalence.*
   - (b) *$\Gamma\tau_{\geq 0} f[n]$ is a τ-local equivalence for all $n \in \mathbb{Z}$.*
   - (c) *$\Gamma\tau_{\geq 0} f[n]$ is a τ-local equivalence for $n \ll 0$.*

(2) *For $K \in \mathbf{U}\mathcal{C}$ the following are equivalent:*
   - (a) *K is τ-fibrant.*
   - (b) *$\Gamma\tau_{\geq 0} K[n]$ is τ-fibrant for all $n \in \mathbb{Z}$.*

(c) $\Gamma\tau_{\geq 0}K[n]$ *is* τ*-fibrant for* $n \ll 0$.

Proof.

(1) This is obvious since τ-local equivalences are defined via (the sheafification of) the homology groups which coincide with the homotopy groups after applying Γ.

(2) The implication "(a)⇒(b)" follows from Lemma 5.12. The implication "(b)⇒(c)" is trivial. For the implication "(c)⇒(a)" let $f : K \to K'$ be a τ-fibrant replacement. Again by Lemma 5.12, $\Gamma\tau_{\geq 0}(f[n])$ is a τ-local equivalence between τ-fibrant objects hence it is a sectionwise weak equivalence. It follows that $\tau_{\geq 0}(f[n])$ is a sectionwise weak equivalence. As f is the filtered colimit of $\tau_{\geq 0}(f[n])$, f is a sectionwise weak equivalence. □

6.2. **Godement resolution.** Now suppose that $(\mathcal{C}, \tau)$ has enough points. This means that there is a set $\mathcal{P}$ of morphisms of sites $p : \mathrm{Set} \to (\mathcal{C}, \tau)$ such that a morphism f of sheaves of sets on $\mathcal{C}$ is an isomorphism if and only if p^*f is an isomorphism for all $p \in \mathcal{P}$. There is an induced morphism of sites $\mathrm{Set}^{\mathcal{P}} \to (\mathcal{C}, \tau)$, and we denote by $(a^*, a_*) : \mathbf{U}\mathcal{C} \to \mathbf{Cpl}(\Lambda)^{\mathcal{P}}$ the induced adjunction. The associated comonad induces functorially for each $K \in \mathbf{U}\mathcal{C}$ a coaugmented cosimplicial object $K \to G^\bullet(K)$, where $G^n(K) = (a_*a^*)^{n+1}(K) \in \mathbf{U}\mathcal{C}$. The *Godement resolution* of K is defined to be

$$\mathcal{G}(K) := \mathrm{Tot}^{\Pi}(G^\bullet(K))$$

which according to Lemma 3.19 is a model for $\mathrm{R}\lim_{\mathbf{\Delta}} G^\bullet(K)$.

Recall [18, Def. 1.31] that the site $(\mathcal{C}, \tau)$ is said to be of finite type if "Postnikov towers converge".

Theorem 6.3. *There is a functor* $\mathcal{G} : \mathbf{U}\mathcal{C} \to \mathbf{U}\mathcal{C}$ *and a natural transformation* $\mathrm{id} \to \mathcal{G}$ *satisfying:*

(1) $\mathcal{G}$ *is an exact functor of abelian categories.*

(2) $\mathcal{G}$ *takes each presheaf of complexes to a* τ*-fibrant sheaf of complexes.*

(3) $\mathcal{G}$ *takes fibrations (i. e. degreewise surjections) to* τ*-fibrations.*

(4) If $(\mathcal{C}, \tau)$ *is a finite type site, then* $K \to \mathcal{G}(K)$ *is a* τ*-local equivalence for any* K*.*

Proof.

(1) $\mathcal{G}$ is the composition of exact functors thus exact.

(2) We use Proposition 6.2 to check that $\mathcal{G}(K)$ is τ-fibrant. Thus let $n \in \mathbb{Z}$, and $c_\bullet \to c$ a τ-hypercover. We need to check that

$$\Gamma\tau_{\geq 0}\mathcal{G}(K)(c)[n] \to \mathrm{R}\lim_{\mathbf{\Delta}} \Gamma\tau_{\geq 0}\mathcal{G}(K)(c_\bullet)[n]$$

is a weak homotopy equivalence. This will follow from [18, Pro. 1.59] if we can prove that the canonical arrow

$$\mathcal{G}(\Gamma\tau_{\geq 0}L) \to \Gamma\tau_{\geq 0}\mathcal{G}(L)$$

is an objectwise weak homotopy equivalence for any L, where the left hand side denotes the Godement resolution for simplicial (pre)sheaves as defined in [18, p. 66], analogous to our construction above. By Lemma 6.1, we see that $\Gamma\tau_{\geq 0}$ commutes with $\mathrm{R}\lim_{\mathbf{\Delta}}$ up to objectwise weak equivalence, so we reduce to show that it also commutes with a_*a^* up to objectwise weak equivalence.

a_*a^* is applied degreewise and is a composition of left-exact functors hence clearly commutes with $\tau_{\geq 0}$. It is also clear that a_*a^* commutes with the Moore complex functor therefore the same holds for the quasi-inverse Γ. Finally, a_*a^* commutes with the forgetful functor $\mathbf{Mod}(\Lambda) \to \mathrm{Set}$.

(3) Let f be an epimorphism with kernel K in $\mathbf{U}\mathcal{C}$. By part 1, $\mathcal{G}(f)$ is an epimorphism with kernel $\mathcal{G}(K)$, which is τ-fibrant by part 2. $\mathcal{G}(f)$ is thus a τ-fibration by Theorem 5.7.

(4) Again, by Proposition 6.2, we need to check that

$$\Gamma\tau_{\geq 0}K[n] \to \Gamma\tau_{\geq 0}\mathcal{G}(K)[n]$$

is a τ-local equivalence for all $n \in \mathbb{Z}$. But by the same reasoning as in part 2, the target of this morphism is identified (up to sectionwise weak homotopy equivalence) with $\mathcal{G}(\Gamma\tau_{\geq 0}K[n])$ hence the claim follows from [18, Pro. 1.65].

□

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