CORNER SOLUTIONS OF THE LAPLACE-YOUNG EQUATION

by J. NORBURY

(*The Mathematical Institute, Oxford University, 24-29 St Giles, OX1 3LB, UK. email: norbury@maths.ox.ac.uk*)

G. C. SANDER *and* C. F. SCOTT

(*Department of Civil & Building Engineering, Loughborough University, Loughborough, LE11 3TU, UK. email: g.sander@lboro.ac.uk, c.f.scott@lboro.ac.uk*)

[Received 20 February 2004. Revise 24 May 2004]

Summary

The upper free surface $z = u(x, y)$ of a static fluid with gravity acting in the z direction, occupying a volume V, satisfies the Laplace-Young equation. The fluid wets the vertical boundaries of V so that the usual capillary contact conditions hold. This paper considers wedge shaped volumes V with corner angle 2α, that belong to the intermediate corner angle case of $\pi/2 - \gamma < \alpha < \pi/2$ where γ is the contact angle, and determines explicitly, a regular power series expansion for the height $u(r, \theta)$ of the fluid near the corner, $r = 0$, to all orders in r. Miersemann (1988) shows that it is possible to have logarithmic terms for a general corner expansion of the Laplace-Young equation, with appropriate boundary conditions. However, we suggest that the usual practical cases do not possess any singular terms near the corner, and we analytically and explicitly produce a non-singular series to any order in r, and propose that near the corner the far field effects are lost through any "interior or inner flat" region in exponentially small terms. We give computational solutions for these regular (energy minimising) cases based on a numerical finite volume method on an unstructured mesh, which fully support our assertions and our analytical series results, including the (minor) influence of the far field on local corner behaviour.

1. Introduction

The Laplace-Young equation has been the focus of considerable attention since Young's (**1**) and Laplace's (**2**) pioneering work in 1805 and 1806 on free surfaces of a static fluid under gravity. A detailed history on the development of this equation is provided in Chapter 1 of Finn (**3**) and references cited within, along with the equation's rigorous derivation based on minimum (gravitational, potential and capillary) energy considerations as proposed by Gauss (**4**). Exact, non-trivial solutions of the Laplace-Young equation have only been found in the cases of a fluid in a semi-infinite domain contact angle γ with a vertical plane wall, or for a fluid between two vertical parallel walls (Landau and Lifshitz, **5**). Thus it is not surprising to find a lot of work has been done on determining estimates or bounds on capillary surface heights for various shaped domains, e.g., Concus and Finn (**6**), Siegel (**7**) and Finn and Huang (**8**) to mention just a few.

Q. Jl Mech. Appl. Math. (2004) 60 (1), 1–16

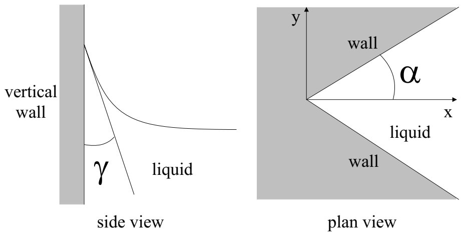

Fig. 1 Coordinate geometry.

When the domain contains a corner or wedge with angle 2α ($0 < \alpha < \pi$), particular interest has been in the development of asymptotic solutions for both the height rise of the fluid in the corner and the corresponding height along the wall (Korevaar (**9**), Miersemann (**10**), (**11**), Lancaster and Siegel (**12**), Fowkes and Hood (**13**), and King et al. (**14**)). These asymptotic results have considerable practical applications to industrial problems, such as the dipping of cuboid capacitors into a metallic paste and the subsequent determination of the wetted region on which any capacitor connection must occur, see King et al (**14**). An exact analytical solution to the linearised Laplace-Young equation for arbitrary γ and a range of wedge angles 2α, has been found and subsequently used to analyse this capacitor problem in Fowkes and Hood (**13**).

Using polar coordinates (r, θ) (where $x = r\cos\theta$, $y = r\sin\theta$) based upon the apex of the wedge being at $r = 0$ and the wetted vertical contact boundaries at $\theta = \pm\,\alpha$ as shown in Figure 1, previous work shows the following asymptotic results for the fluid rise in a corner:

(a) *Small corner angles* $0 < \alpha < \pi/2 - \gamma$
Here the height rise u is unbounded as $r \to 0$ and is given by (Concus and Finn, (**6**))

$$u \sim \frac{\cos\theta - (a^2 - \sin^2\theta)^{1/2}}{ar} + O(r^3), \tag{1.1}$$

where $a = \sin\alpha/\cos\gamma$. Note that in (1.1) and (1.2) below, both u and r are dimensionless variables with the appropriate scaling in terms of the capillary length scale given in section 2.

(b) *Intermediate corner angles* $\pi/2 - \gamma < \alpha < \pi/2 + \gamma$
Solutions are bounded (Concus and Finn, (**6**)) in this region (for higher orders when $\pi/2 - \gamma < \alpha < \pi 2$, see Miersemann, (**10**))

$$u \sim u_o - \frac{r\cos\theta}{(a^2 - 1)^{1/2}} + \ldots, \tag{1.2}$$

where u_o is the height in the corner.

(c) *Large corner angles* $\pi/2 + \gamma < \alpha < \pi$
While solutions in this region near the corner are still bounded (Concus and Finn, (**6**)), the local slope near $r = 0$ is not, and a complete asymptotic analysis requires information on the far field (p.90, King et al (**14**)). When the solution domain is symmetric about $\theta = 0$, King et al (**14**) demonstrate the existence of, and determine the structure of, interior layers which define regions in u, near the corner, on the basis

of the level lines of the surface. These layers are located along rays $\theta = \pi/2$ and $\theta = \alpha - \gamma$, and their structure is independent of far field conditions.

The asymptotic results mentioned above apply for arbitrary shaped far field domains which may influence the local behaviour in the corner in the next order term. Korevaar (**9**) showed that solutions exist for particular shaped domains where there is a jump discontinuity in u at a re-entrant corner (see p.381 and Theorem on p.384 of **9**). This is essentially an example of the effect of the far field when it is brought in very close to the corner. King et al (**14**) have made a systematic study of local corner behaviour for all wedge angles, and in addition have found possible approximate mathematical forms for the asymptotic structure of the two transition regimes $\alpha \simeq \pi/2 - \gamma$ and $\alpha \simeq \pi/2 + \gamma$ occurring between *(a)* – *(b)* and *(b)* – *(c)*. When the far field has little influence on corner behaviour, Miersemann (**11**) has shown that (1.1) is accurate to at least $O(r^3)$ and proceeds to show that a regular expansion exists for all orders in r that is symmetric in θ. For the intermediate angle subcase of $\pi/2 - \gamma < \alpha < \pi/2$, a complete expansion around $r = 0$ by Miersemann (**10**) shows that it is necessary to include logarithmic terms to account for all potential far field behaviour.

In contrast to the results of King et al (**14**) and Miersemann (**10**), we continue our series for intermediate wedge angles ($\pi/2 - \gamma < \alpha < \pi/2$) in a regular manner to all orders in r, and suggest that the usual practical cases do not show any singular terms at the corner. Therefore only symmetric local solutions in the asymptotic construction are considered. We produce an explicit regular series in powers of r (based on an expansion for r small) where the coefficients are functions of θ, and produce simple explicit recursion formulae for the coefficients which satisfy inhomogeneous linear ordinary differential equations. We also show that these formulae can be continued and evaluated for all powers of r. The reason why a regular expansion is appropriate, is that when the far field is of sufficient distance from the corner such that an "interior flat" region develops, we believe the far field effects are lost (both in theory and in practical applications) by means of exponentially small terms in the flat region, resulting in a regular series as the (stable) energy minimizing solution.

If a fluid is bounded within a symmetric domain about the vertical plane $\theta = 0$ with the same contact conditions on all finite boundaries, then the uniqueness result of Finn and Hwang (**8**) implies symmetric solutions for u. King et al (**14**) show that for $\alpha > \pi/2$ various asymptotic solutions can exist in an infinite wedge which exhibit local asymmetry in u at the corner when far field boundary conditions are not symmetric. They suggest that symmetric local solutions are "non-generic" and asymmetric local solutions whose particular form depends on far field conditions, may be preferred. However, in contrast, we find that asymmetry effects, either from the far field domain shape or boundary conditions applied, can only produce asymmetry in the corner asymptotics when the "far field" is actually "close" to the corner. By "close" we mean that the shape of the resulting capillary surface has essentially no inner flat region, and the "far field" is within several capillary length scales from the origin.

We computationally verify our expansion through a series of test examples by comparing our explicit analytical approximations with a numerical solution based on a finite volume method with an unstructured mesh. The numerical code has been separately validated against both radially symmetric and plane wall solutions, together with internally consistent convergence. These numerical examples fully support our above assertions regarding the influence of the far field on local corner behaviour. We provide numerical solutions in a wedge with a finite "far field" at radii 1, 2, 4, 6, 8 and 10 capillary length scales from the origin, which show that for the "far field" boundary at distances greater than six capillary length scales, the corner solution is independent of the "far field". To reinforce this point we introduce asymmetry into the far field boundary conditions by a factor of ten to one for $\theta \geq 0$ compared to $\theta < 0$, again showing that the solution near the corner remains independent of this effect. Thus we suggest that usually our solution will hold for practical problems, and produce estimates for when it must be modified.

Our procedure depends on obtaining solutions to (see (3.9))

$$\mathcal{L}_n(u_n) = f_n(\theta)$$

where $u_n(\theta)$ is the coefficient of r^n, $\mathcal{L}_n$ is a second order linear differential operator with θ dependent coefficients which vary with n, and $f_n(\theta)$ is an $n-2$ degree polynomial in $\cos\theta$ containing only even (n even) or odd (n odd) powers. Our explicit expansion procedure can be carried out as suggested because of the cancellation of the terms with the highest powers of $\cos\theta$ in f_n, i.e., $\cos^n\theta$ and $\cos^{n-2}\theta$. This depends on establishing that the recurrence identities implied by the vanishing of (3.14) and (3.17) hold for arbitrary $n \geq 2$. Consequently we show that the solution $u_n(\theta)$ is an n^{th} degree polynomial in $\cos\theta$, and that the coefficients of $\cos^n\theta$ and $\cos^{n-2}\theta$ in $\mathcal{L}_n(u_n)$ also vanish for arbitrary $n \geq 2$. We show that this fortuitous cancellation must happen in both the homogeneous differential operator and the forcing term at each value of $n \geq 2$, for such a non-singular expansion to continue to all orders.

It was noted in Siegel (**7**) that the only nontrivial explicitly known solutions to the Laplace-Young equation, are the one-dimensional solutions in either a finite or semi-infinite domain (mentioned earlier). It seems that over the 20 year period since, this has not changed (Fowkes and Hood (**13**), King et al (**14**)).

2. Laplace-Young Equation and Series Expansion

The governing non-dimensional equation for the shape of the fluid surface around an arbitrary shaped wedge in polar coordinates is (Seigel (**7**), Miersemann (**10**))

$$\begin{aligned} \nabla.\left(\frac{\nabla u}{[1+|\nabla u|^2]^{1/2}}\right) &= \frac{1}{r}\left(\frac{r u_r}{[1+u_r^2+u_\theta^2/r^2]^{1/2}}\right)_r + \frac{1}{r^2}\left(\frac{u_\theta}{[1+u_r^2+u_\theta^2/r^2]^{1/2}}\right)_\theta \\ &= u, \quad -\alpha < \theta < \alpha \end{aligned} \tag{2.1}$$

with boundary conditions

$$\begin{aligned} u_\theta &= 0, \quad \theta = 0 \\ u_n &= \frac{1}{r}u_\theta = [1+u_r^2+u_\theta^2/r^2]^{1/2}\cos(\gamma)\,, \quad \theta = \alpha\ , \end{aligned} \tag{2.2}$$

where $u = u{*}\sqrt{\kappa}$, $r = r{*}\sqrt{\kappa}$, κ is the capillary constant and $u*$, $r*$ are the dimensional variables. Here the gravitational constant has been absorbed within κ so that r is a measure in capillary lengthscales from the wedge apex. Note that (2.2) can also be written as

$$\left(\frac{u_\theta}{r}\right)^2 = \frac{1+u_r^2}{\tan^2\gamma}, \quad \theta = \alpha. \tag{2.3}$$

As discussed in the Introduction, we consider only intermediate corner angles, $\pi/2 - \gamma < \alpha < \pi/2$, where the far field has negligible influence on the corner behaviour. Following on from Miersemann (**10**),(**11**) we therefore write a regular power series solution for $u(r,\theta)$ as

$$u(r,\theta) = u_o + ru_1(\theta) + r^2u_2(\theta) + r^3u_3(\theta) \ldots = \sum_{n=0}^{\infty} r^n u_n(\theta), \tag{2.4}$$

where u_o is the height of the fluid at $r = 0$. In the following sections we proceed to determine the general form of the boundary value problem for each $u_n(\theta)$ along with their explicit solutions. We begin by giving the differential equation and boundary conditions satisfied by u_1, i.e.,

$$\begin{aligned} u_1 + u_1'' &= 0 \\ u_1' &= 0, \quad \theta = 0 \\ u_1'^2 &= \frac{1+u_1^2}{\tan^2\gamma}, \quad \theta = \alpha \end{aligned} \tag{2.5}$$

where $' = d/d\theta$. The solution is

$$u_1(\theta) = A\cos(\theta), \tag{2.6}$$

where

$$A = \frac{-\cos\gamma}{\sqrt{\sin^2\alpha - \cos^2\gamma}} = \frac{-1}{\sqrt{a^2-1}}, \tag{2.7}$$

and a defined in (1.1).

2.1 *Generator of the differential equation for $u_n(\theta)$*

By carrying out the differentiation in (2.1) we find

$$u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r + \frac{1}{r^2}u_\theta^2(u_{rr} + \frac{2}{r}u_r) + u_r^2(\frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) - \frac{2}{r^2}u_r u_\theta u_{r\theta}$$
$$= \left(1 + u_r^2 + \frac{1}{r^2}u_\theta^2\right)^{3/2} u \;. \tag{2.8}$$

Noting the following relations (from (2.4))

$$\begin{aligned} \left(\frac{u_\theta}{r}\right)^2 &= \left(\sum_{n=1}^{\infty} r^{n-1}u_n'\right)^2 = \sum_{n=0}^{\infty} r^n \sum_{j=1}^{n+1} u_j' u_{n+2-j}' \\ u_r^2 &= \left(\sum_{n=1}^{\infty} nr^{n-1}u_n\right)^2 = \sum_{n=0}^{\infty} r^n \sum_{j=1}^{n+1} j(n+2-j)u_j u_{n+2-j} \\ \frac{u_\theta}{r}\frac{u_{r\theta}}{r} &= \sum_{n=1}^{\infty} r^{n-1}u_n' \sum_{n=1}^{\infty} nr^{n-1}u_n' = \sum_{n=0}^{\infty} r^n \sum_{j=1}^{n+1} (n+2-j)u_j' u_{n+2-j}' \quad , \end{aligned} \tag{2.9}$$

and substituting into (2.8) (with $u_1'' + u_1 = 0$), then (2.8) becomes

$$
\begin{aligned}
&\sum_{n=2}^{\infty} r^{n-2}\left[u_n'' - 2(n-1)u_1u_1'u_n' + (n^2 - 2nu_1'^2)u_n\right] - 2u_1 \sum_{n=2}^{\infty} r^{n-2} \sum_{j=2}^{n} (n-j)u_j'u_{n+1-j}' \\
&+ \sum_{n=0}^{\infty} r^n \sum_{j=1}^{n+1} u_j'u_{n+2-j}' \bullet \sum_{n=2}^{\infty} (n^2+n)r^{n-2}u_n \\
&+ \sum_{n=0}^{\infty} r^n \sum_{j=1}^{n+1} j(n+2-j)u_ju_{n+2-j} \bullet \sum_{n=2}^{\infty} r^{n-2}(u_n'' + nu_n) \\
&-2\sum_{n=1}^{\infty} r^n \sum_{j=1}^{n+1} (n+2-j)u_j'u_{n+2-j}' \bullet \sum_{n=2}^{\infty} nr^{n-2}u_n \\
&= \left\{1 + u_1^2 + u_1'^2 + \sum_{n=1}^{\infty} r^n \sum_{j=1}^{n+1}\left[j(n+2-j)u_ju_{n+2-j} + u_j'u_{n+2-j}'\right]\right\}^{3/2} \sum_{n=0}^{\infty} r^n u_n.
\end{aligned}
\tag{2.10}
$$

If we replace the series multiplications in (2.10) with

$$
\begin{aligned}
&\sum_{n=0}^{\infty} r^n \sum_{j=1}^{n+1} u_j'u_{n+2-j}' \bullet \sum_{n=2}^{\infty} (n^2+n)r^{n-2}u_n \\
&= u_1'^2 \sum_{n=2}^{\infty} (n^2+n)r^{n-2}u_n + \sum_{n=1}^{\infty} r^n \sum_{k=2}^{n+1} (k^2+k)u_k \sum_{j=1}^{n-k+3} u_j'u_{n-k-j+4}' \\
&\sum_{n=0}^{\infty} r^n \sum_{j=1}^{n+1} j(n+2-j)u_ju_{n+2-j} \bullet \sum_{n=2}^{\infty} r^{n-2}(u_n'' + nu_n) \\
&= u_1^2 \sum_{n=2}^{\infty} r^{n-2}(u_n'' + nu_n) + \sum_{n=1}^{\infty} r^n \sum_{k=2}^{n+1} (u_k'' + ku_k) \sum_{j=1}^{n-k+3} j(n-k-j+4)u_ju_{n-k-j+4} \\
&\sum_{n=1}^{\infty} r^n \sum_{j=1}^{n+1} (n+2-j)u_j'u_{n+2-j}' \bullet \sum_{n=2}^{\infty} nr^{n-2}u_n \\
&= \sum_{n=1}^{\infty} r^n \sum_{k=2}^{n+1} ku_k \sum_{j=1}^{n-k+3} (n-k-j+4)u_j'u_{n-k-j+4}',
\end{aligned}
\tag{2.11}
$$

then (2.10) becomes

$$
\begin{aligned}
&\sum_{n=2}^{\infty} r^{n-2}\mathcal{L}_n(u_n) = 2u_1 \sum_{n=3}^{\infty} r^{n-2} \sum_{k=2}^{n-1} (n-k)u_k'u_{n+1-k}' \\
&- \sum_{n=3}^{\infty} r^{n-2} \sum_{k=2}^{n-1} (u_k'' + ku_k) \sum_{j=1}^{n-k+1} j(n-k-j+2)u_ju_{n-k-j+2} \\
&- \sum_{n=3}^{\infty} r^{n-2} \sum_{k=2}^{n-1} ku_k \sum_{j=1}^{n-k+1} (3k+2j-2n-3)u_j'u_{n-k-j+2}' \\
&+ \left\{1 + u_1^2 + u_1'^2 + \sum_{n=1}^{\infty} r^n \sum_{j=1}^{n+1}\left[j(n+2-j)u_ju_{n+2-j} + u_j'u_{n+2-j}'\right]\right\}^{3/2} \sum_{n=0}^{\infty} r^n u_n.
\end{aligned}
\tag{2.12}
$$

For the different orders we therefore have the following differential equations

$$
\mathcal{L}_2(u_2) = u_o\left(1 + u_1^2 + u_1'^2\right)^{3/2}, \tag{2.13}
$$

and for $n = 3 \ldots \infty$

$$\begin{aligned}\mathcal{L}_n(u_n) &= 2u_1 \sum_{k=2}^{n-1}(n-k)u_k' u_{n+1-k}' \\ &- \sum_{k=2}^{n-1}(u_k'' + ku_k)\sum_{j=1}^{n-k+1} j(n-k-j+2)u_j u_{n-k-j+2} \\ &- \sum_{k=2}^{n-1} ku_k \sum_{j=1}^{n-k+1}(3k+2j-2n-3)u_j' u_{n-k-j+2}' + \left(1+u_1^2+u_1'^2\right)^{3/2} u_{n-2} \\ &+ \sum_{k=2}^{n-2} u_{n-k-3}\sum_{s=1}^{k-1} {}^{3/2}C_s \left(1+u_1^2+u_1'^2\right)^{3/2-s} b(s)_{k-1-s},\end{aligned} \tag{2.14}$$

where ${}^{3/2}C_s = (3/2)!/s!(3/2-s)!$ are the usual binomial coefficients and

$$b(s)_0 = a_0^s, \qquad b(s)_p = \frac{1}{pa_o}\sum_{i=1}^{p}(is+i-p)a_i b(s)_{p-i}$$

$$a_i = \sum_{j=1}^{i+2}\left[j(i+3-j)u_j u_{i+3-j} + u_j' u_{i+3-j}'\right],$$

and $\mathcal{L}_n(u_n)$ is defined by

$$\mathcal{L}_n(u_n) := (1+u_1^2)u_n'' - 2(n-1)u_1 u_1' u_n' + n[n + (n-1)u_1'^2 + u_1^2]u_n. \tag{2.15}$$

2.2 *Generator for Contact Boundary Condition on $\theta = \alpha$*

By substituting from (2.4) into (2.3) we obtain

$$\begin{aligned}&u_1'^2 + 2u_1'u_2'r + 2u_1'\sum_{n=2}^{\infty} r^n u_{n+1}' + \sum_{n=2}^{\infty} r^n \sum_{j=0}^{n-2} u_{2+j}' u_{n-j}' = \\ &\quad [u_1^2 + 2u_1u_2r + 2u_1\sum_{n=2}^{\infty}(n+1)r^n u_{n+1} + \sum_{n=2}^{\infty} r^n \sum_{j=0}^{n-2}(2+j)(n-j)u_{2+j}u_{n-j}]/\tan^2\gamma,\end{aligned} \tag{2.16}$$

resulting in the following different order contact boundary conditions

$n = 1$

$$u_2' = \frac{2u_1u_2}{u_1'\tan^2\gamma} \quad , \tag{2.17}$$

$n = 2 \ldots \infty$

$$u_{n+1}' = \frac{(n+1)u_1u_{n+1}}{u_1' \tan^2\gamma} + \sum_{j=0}^{n-2}\frac{(2+j)(n-j)u_{2+j}u_{n-j}}{2u_1' \tan^2\gamma} - \sum_{j=0}^{n-2}\frac{u_{2+j}'u_{n-j}'}{2u_1'} \quad , \tag{2.18}$$

or after changing counters

$$u_n' = \frac{nu_1u_n}{u_1' \tan^2\gamma} - \lambda_n(\alpha)\,, \qquad n = 3 \ldots \infty \quad , \tag{2.19}$$

where $\lambda_n(\alpha)$ is given by

$$\lambda_n(\alpha) = -\sum_{j=1}^{n-2}\frac{(1+j)(n-j)u_{1+j}u_{n-j}}{2u_1' \tan^2\gamma} + \sum_{j=1}^{n-2}\frac{u_{1+j}'u_{n-j}'}{2u_1'}. \tag{2.20}$$

3. Solutions for $u_n(\theta)$

The structure of the general solution for $u_n(\theta)$, is made explicit by examining the first few equations for $u_n(\theta)$. From (2.6), (2.13), (2.15) and (2.17), $u_2(\theta)$ satisfies

$$\begin{aligned} (1+A^2\cos^2\theta)u_2'' + 2A^2\cos\theta\sin\theta\, u_2' + 2(2+A^2)u_2 &= \left(1+A^2\right)^{3/2} u_o \\ u_2' = 0, \quad \theta &= 0 \\ u_2' = \frac{-2u_2}{\tan^2\gamma\,\tan\alpha}, \quad \theta &= \alpha, \end{aligned} \tag{3.1}$$

and has the solution, also given in Miersemann (**10**),

$$u_2(\theta) = \frac{u_o}{4}\sqrt{1+A^2}\left(1+A^2\cos^2\theta\right). \tag{3.2}$$

Next $u_3(\theta)$ is found from (2.6), (2.14), (2.19) and (3.2) to satisfy

$$\begin{aligned} (1+A^2\cos^2\theta)u_3'' + 4A^2\cos\theta\sin\theta\, u_3' + 3(3+2A^2-A^2\cos^2\theta)u_3 &= \\ A\left[(1+A^2)^{3/2} + u_o^2(1+A^2)^2\right]\cos\theta \\ u_3' = 0, \quad \theta = 0 \\ u_3' = \frac{-3u_3}{\tan^2\gamma\,\tan\alpha} - \frac{u_o^2}{8}A(1+A^2)\sin\alpha, \quad \theta = \alpha, \end{aligned} \tag{3.3}$$

and has the solution

$$u_3(\theta) = c_{31}\cos\theta + c_{33}\cos^3\theta, \tag{3.4}$$

where

$$\begin{aligned} 2(4+A^2)c_{31} + 6c_{33} &= A(1+A^2)^{3/2} + u_o^2 A(1+A^2)^2 \\ \left(1-2A^2\cos^2\alpha\right)c_{31} + 3c_{33}\cos^2\alpha &= \frac{\lambda_3(A^2\cos^2\alpha+1)}{\sin\alpha}. \end{aligned} \tag{3.5}$$

Finally u_4 is determined from

$$\begin{aligned} (1+A^2\cos^2\theta)u_4'' + 6A^2\cos\theta\sin\theta\, u_4' + 4(4+3A^2-2A^2\cos^2\theta)u_4 &= b_{40} + b_{42}\cos^2\theta \\ u_4' = 0, \quad \theta &= 0 \\ u_4' = \frac{-4u_4}{\tan^2\gamma\,\tan\alpha} - \frac{u_o}{2}\left[\frac{3\sqrt{1+A^2}u_3(\alpha)}{A\sin\alpha\,\tan^2\gamma} - \frac{u_o^2}{16}A^2(1+A^2)^{\frac{3}{2}}\sin(2\alpha)\right], \quad \theta &= \alpha, \end{aligned} \tag{3.6}$$

which has the solution

$$u_4(\theta) = c_{40} + c_{42}\cos^2\theta + c_{44}\cos^4\theta, \tag{3.7}$$

where

$$\begin{aligned} c_{44} &= \frac{b_{42}}{12} - \left(1+\frac{A^2}{6}\right)c_{42} + \frac{2}{3}A^2 c_{40} \\ c_{42} &= \frac{b_{40}}{2} - (6A^2+8)c_{40} \\ c_{40} &= \frac{c_{44}}{A^2}\cos^2\alpha + \frac{c_{42}}{2}\left(\frac{1}{A^2} - \cos^2\alpha\right) - \frac{\lambda_4}{4}\tan^2\gamma\,\tan\alpha\ , \end{aligned} \tag{3.8}$$

and $\lambda_4 = u_0/2[\dots]$ from the $\theta = \alpha$ boundary condition in (3.6).

It is quite straightforward to show that the above pattern continues to repeat itself for all n, leading to the boundary value problem for u_n as given by

$$\begin{aligned}
(1 + A^2\cos^2\theta)^n \left[(1 + A^2\cos^2\theta)^{1-n}u_n'\right]' + n[n + (n-1)A^2 - (n-2)A^2\cos^2\theta]u_n = f_n(\theta) \\
f_n(\theta) = \sum_{k=0}^{n/2-1} b_{n,2k}\cos^{2k}\theta \qquad n \text{ even} \\
f_n(\theta) = \sum_{k=1}^{(n-1)/2} b_{n,2k-1}\cos^{2k-1}\theta \qquad n \text{ odd} \\
u_n' = 0\ , \quad \theta = 0 \\
u_n' = \frac{-nu_n}{\tan^2\gamma\,\tan\alpha} - \lambda_n(\alpha), \quad \theta = \alpha,
\end{aligned} \tag{3.9}$$

whose solution is

$$\begin{aligned}
u_n(\theta) = \sum_{k=0}^{n/2} c_{n,2k}\cos^{2k}\theta \qquad n \text{ even} \\
u_n(\theta) = \sum_{k=1}^{(n+1)/2} c_{n,2k-1}\cos^{2k-1}\theta \qquad \text{odd},
\end{aligned} \tag{3.10}$$

where the $c_{n,j}$ and $b_{n,j}$ are constant coefficients related by identities that are discussed in the following sections. It is interesting to note that the right hand side of (2.14) for any n simplifies to f_n as given in (3.9).

3.1 *Homogeneous Equation*

While f_n in (3.9) contains terms up to $\cos^{n-2}\theta$, u_n contains terms up to $\cos^n\theta$. After substituting for (3.10) into (3.9), the three leading powers for $\cos\theta$ are $\cos^{n+2}\theta$, $\cos^n\theta$ and $\cos^{n-2}\theta$, hence to match the leading order term in f_n, the coefficients of $\cos^{n+2}\theta$, $\cos^n\theta$ must be zero for arbitrary n. To prove this we first rewrite (3.9) with $z = \cos\theta$ as the independent variable, so that we have

$$\begin{aligned}
(1 + A^2z^2)(1 - z^2)u_n'' - z[1 + 2(n-1)A^2 - (2n-3)A^2z^2]u_n' + \\
n[n + (n-1)A^2 - (n-2)A^2z^2]u_n = f_n(z)\ , \qquad \cos\alpha \le z \le 1\ .
\end{aligned} \tag{3.11}$$

If we substitute

$$u_n(z) = \ \dots\ +\ c_{n,n-2}z^{n-2} + c_{n,n}z^n \quad ,$$

into (3.11) and collect the coefficients of z^{n+2}, z^n we obtain

$$\begin{aligned}
&-c_{n,n}A^2[n(n-1) + n(3-2n) + n(n-2)]z^{n+2} \\
&+c_{n,n}[n(n-1)(A^2-1) - 2A^2n(n-1) - n + n^2 + n(n-1)A^2]z^n \\
&-c_{n,n-2}A^2[(n-2)(n-3) + (3-2n)(n-2) + n(n-2)]z^n \quad ,
\end{aligned} \tag{3.12}$$

which are identically zero for arbitrary n. In the next section we show that the equivalent powers on the right hand side in $f_n(z)$ also vanish for all n.

3.2 *Order of* $f_n(\theta)$

We suggest in (3.9) that the highest power of $\cos(\theta)$ in $f_n(\theta)$ is n-2, but since u_j goes like $\cos^j(\theta)$ or u_1^j, then the right hand side of (2.14) shows that there exist terms proportional to both u_1^{n+2} and u_1^n. Consequently their coefficients must be zero. We proceed to show this by establishing that the formulae (3.14) and (3.17) given below identically vanish. Consider only the highest two powers terms of any u_j and write these as

$$u_j(\theta) = C_j^j u_1^j + C_{j-2}^j u_1^{j-2} \quad \text{with } C_1^j = 1, \ \ C_{-1}^j = 0 \quad , \tag{3.13}$$

where $C_k^j = A^{-k} c_{j,k}$(note that if $j = k$ we will use the simpler notation C_j). Substituting this into the non-binomial terms in (2.14), and making use of the identities $u_1'' = -u_1$, $\ {u'}_1^2 = A^2 - u_1^2$, gives the following equation for the coefficient of u_1^{n+2}

$$\begin{aligned} &2\sum_{j=2}^{n-1} j(j-n)(n+1-j)C_j C_{n+1-j} \\ &\qquad + \ 2\sum_{k=2}^{n-1}\sum_{j=1}^{n-k+1} C_k C_j C_{n-k-j+2} kj(n-k-j+2)(2k+j-n-2) \ . \end{aligned} \tag{3.14}$$

We now show that (3.14) identically vanishes for any C_j, j = 2, ..., n-1. Changing the order of the summation in the second term of (3.14) results in

$$\begin{aligned} &2\sum_{k=2}^{n-1}\sum_{j=1}^{n-k+1} C_k C_j C_{n-k-j+2} kj(n-k-j+2)(2k+j-n-2) \\ &\quad = \ 2\sum_{k=2}^{n-1} C_k C_{n+1-k} k(n-k+1)(2k-n-1) \\ &\qquad + 2\sum_{j=2}^{n-1}\sum_{k=2}^{n-j+1} C_k C_j C_{n-k-j+2}\left[k^2(n-k-j+2) - k(n-k-j+2)^2\right] j \quad . \end{aligned} \tag{3.15}$$

As k goes from 2 to $n-1$, $n-k$+1 goes from $n-1$ to 2 while $2k-n-1$ goes from $3-n$ to $n-3$, hence the first term on the right hand side of (3.15) is zero. Noting that due to symmetry the square bracketed term is also zero for $k = 2..n-j$, then the only nonzero term arises for $k = n-j+1$ thus

$$\begin{aligned} &2\sum_{k=2}^{n-1}\sum_{j=1}^{n-k+1} C_k C_j C_{n-k-j+2} kj(n-k-j+2)(2k+j-n-2) \\ &\qquad = \ -2\sum_{j=2}^{n-1} C_{n+1-j} C_j (n+1-j)(j-n)j \end{aligned} \quad , \tag{3.16}$$

and hence the second term in (3.14) cancels the first. This completes the proof that the coefficient of u_1^{n+2} is zero. We now proceed to prove that the coefficient of u_1^n is also zero.

The coefficient for the u_1^n term is given by

$$2\sum_{j=2}^{n-1}(n-j)\left[\begin{array}{l} A^2 j(n+1-j)C_jC_{n+1-j} \\ -(j-2)(n+1-j)C_{j-2}^{j}C_{n+1-j} - j(n-1-j)C_jC_{n-1-j}^{n+1-j} \end{array}\right] +$$
$$2\sum_{k=2}^{n-1}\sum_{j=1}^{n-k+1}\left\{\begin{array}{l} -A^2C_kC_jC_{n-k-j+2}\,kj(n-k-j+2)(2k+j-n-2) \\ +C_kC_jC_{n-k-j}^{n-k-j+2}\,kj(n-k-j+1)(2k+j-n-1) \\ +C_kC_{j-2}^{j}C_{n-k-j+2}\,k(n-k-j+2)(2kj-3k-nj+j^2-4j+2n+3) \\ +C_{k-2}^{k}C_jC_{n-k-j+2}\,j(n-k-j+2)(2k^2+kj-nk-4k+2) \end{array}\right\}, \tag{3.17}$$

which we now show is zero for any A and C_j^k, subject to $C_1^j = 1$ and $C_{-1}^k = 0$. First note that the terms containing A in (3.17) are equivalent to (3.14) and therefore cancel. Secondly by changing the counter for the final term in the single summation to $j = n{+}1{-}s$ and to $j = n-k-s{+}2$ for the third term in the double summation, (3.17) can be written as

$$\begin{aligned} &2\sum_{j=2}^{n-1}(n-1)(2-j)(n+1-j)C_{j-2}^{j}C_{n+1-j} \\ &+\ 2\sum_{k=2}^{n-1}\sum_{j=1}^{n-k+1}C_kC_{j-2}^{j}C_{n-k-j+2}\,k(n-k-j+2)(3kj-4k-nj-2j+2n+2) \\ &+\ 2\sum_{j=2}^{n-1}\sum_{k=1}^{n-j+1}C_kC_{j-2}^{j}C_{n-k-j+2}\,k(n-k-j+2)(2j^2+kj-nj-4j+2)\ , \end{aligned} \tag{3.18}$$

after swapping the counters in the final term of (3.18). Next take out $k = 1$ from the third term in (3.18) and combine it with the single summation. Then j =1 in the second term is redundant since it gives coefficients containing C_{-1}^{j}which are by definition zero, thus the second and third terms can be combined and following a change in the order of this summation (3.18) simplifies to

$$\begin{aligned} &4\sum_{j=2}^{n-1}(j-1)(j-n)(n+1-j)C_{j-2}^{j}C_{n+1-j}\quad + \\ &4\sum_{k=2}^{n-1}(j-1)C_{j-2}^{j}\sum_{k=2}^{n-j+1}C_kC_{n-k-j+2}\left[k^2(n-k-j+2)-k(n-k-j+2)^2\right]\ . \end{aligned} \tag{3.19}$$

The term in the square brackets is always zero due to symmetry, except for $k = n - j{+}1$ when it then cancels the single summation, hence (3.17) is also zero.

Considering now the binomial term in (2.14), it is clear that if we expand this out then as long as $j(m+2-j)u_ju_{m+2-j} + u_j'u_{m+2-j}'$ is zero for the highest order terms in u_1, then overall the binomial terms only contribute to terms in cosθ of at most u_1^{n-2}. From (2.14) and (3.13) we then have

$$\begin{aligned} &j(m+2-j)u_ju_{m+2-j} + u_j'u_{m+2-j}' \\ &\quad = j(m+2-j)C_jC_{m+2-j}\,u_1^{m+2} - j(m+2-j)C_jC_{m+2-j}\,u_1^{m+2} = 0\ . \end{aligned} \tag{3.20}$$

3.3 *Recurrence Relations Between c and b*

In this section we develop the recurrence relations between the coefficients $c_{n,j}$ in the solution $u_n(z)(z = \cos\theta)$ and the constants $b_{n,j}$ from $f_n(z)$ in (3.9). The details are only given for n even since similar calculations yield the corresponding recurrence relations for n odd.

From (2.4) we write

$$u_n(z) = \sum_{k=0}^{n/2} c_{n,2k} z^{2k}, \quad u_n'(z) = \sum_{k=1}^{n/2} 2k\ c_{n,2k} z^{2k-1}\ , \quad u_n''(z) = \sum_{k=1}^{n/2} 2k(2k-1)\ c_{n,2k} z^{2k-2} \quad , \tag{3.21}$$

and combining this with (3.9) results in

$$\begin{aligned} &\sum_{k=0}^{n/2} (2k+2-n)(n-2k)A^2 c_{n,2k} z^{2k+2} \\ &+ \sum_{k=0}^{n/2} \left\{(n-2k)[n+2k+(n-2k-1)]A^2\right\} c_{n,2k} z^{2k} \\ &+ \sum_{k=1}^{n/2} 2k(2k-1) c_{n,2k} z^{2k-2} = \sum_{k=0}^{n/2-1} b_{n,2k} z^{2k} \quad , \end{aligned} \tag{3.22}$$

or, after changing counters,

$$\begin{aligned} &\sum_{k=1}^{n/2-1} (2k-n)(n+2-2k)A^2 c_{n,2k-2} z^{2k} \\ &+ \sum_{k=0}^{n/2-1} \left\{(n-2k)[n+2k+(n-2k-1)]A^2\right\} c_{n,2k} z^{2k} \\ &+ \sum_{k=0}^{n/2-1} (2k+2)(2k+1) c_{n,2k+2} z^{2k} = \sum_{k=0}^{n/2-1} b_{n,2k} z^{2k} \quad . \end{aligned} \tag{3.23}$$

Taking out the $k = 0$ term results in

$$c_{n,2} = \frac{b_{n,0}}{2} - [n^2 + n(n-1)A^2] \frac{c_{n,0}}{2} \quad , \tag{3.24}$$

leaving the following recurrence relation for k =1... n/2-1

$$\begin{aligned} &(2k+2)(2k+1) c_{n,2k+2} + \left\{(n-2k)[n+2k+(n-2k-1)]A^2\right\} c_{n,2k} \\ &+A^2(2k-n)(n+2-2k) c_{n,2k-2} = b_{n,2k} \quad . \end{aligned} \tag{3.25}$$

Finally the boundary condition on $\theta = \alpha$ in (3.9) becomes

$$-\sin\alpha \frac{du_n}{dz} + \frac{n u_n(z)}{\tan\alpha \tan^2\gamma} + \lambda_n(\alpha) = 0, \tag{3.26}$$

giving $c_{n,0}$ as

$$c_{n,0} = \sum_{k=1}^{n/2} \cos^{2k-2}\alpha \left[\frac{2k}{nA^2} - \left(1 - \frac{2k}{n}\right)\cos^2\alpha\right] c_{n,2k} - \frac{\lambda_n}{n} \tan\alpha\ \tan^2\gamma, \tag{3.27}$$

since $\tan^2\alpha\ \tan^2\gamma - 1 = (A^2\cos^2\alpha)^{-1}$.

By going through a similar process, the recurrence relations for n odd are found as

$$c_{n,3} = \frac{b_{n,1}}{6} - \frac{n-1}{6}[n+1+(n-2)A^2] c_{n,1} \quad , \tag{3.28}$$

with k =2 ... (n-1)/2

$$\begin{array}{ll} 2k(2k+1)c_{n,2k+1} + (1+n-2k)\left[(n+2k-1)+(n-2k)A^2\right]c_{n,2k-1} & \\ +A^2(n+3-2k)(2k-n-1)c_{n,2k-3} = b_{n,2k-1} & , \end{array} \tag{3.29}$$

and

$$\left[1+(1-n)A^2\cos^2\alpha\right]c_{n,1} + \sum_{k=2}^{(n+1)/2} \cos^{2k-2}\alpha\left[2k-1+(2k-1-n)A^2\cos^2\alpha\right]c_{n,2k-1} = \frac{\lambda_n(1+A^2\cos^2\alpha)}{\sin\alpha}. \tag{3.30}$$

4. Numerical Results

The usefulness of the expansion (2.4) together with our explicitly found $u_1(\theta)$, $u_2(\theta)$, ..., is now shown by comparing it with the numerical solution of (2.1) and (2.2) from Scott et al (**15**), based on a finite volume method with an unstructured mesh on an arbitrary shaped domain. Scott et al (**15**) have shown that their scheme is second order and can obtain solutions accurate to 4 significant figures when matched against the plane wall case and radially symmetric solutions of (2.1).

In order to investigate our assertion that the far field behaviour has negligible effects on the height rise in the corner when there is an "interior flat region", we numerically solve (2.1) on the domain $0 \le r \le R$, subject to the following boundary conditions

$$\begin{array}{ll} \frac{1}{r}u_\theta = \pm[1+u_r^2+u_\theta^2/r^2]^{1/2}\cos(\gamma)\,, & \theta=\pm\alpha \\ u_r = -\beta^+ ru\,, \quad r=R\,, \quad \theta \ge 0 & \\ u_r = -\beta^- ru\,, \quad r=R\,, \quad \theta < 0 & . \end{array} \tag{4.1}$$

The role of β^+ and β^- is to force asymmetry in the solution, about $\theta = 0$, and then to observe its effect on the behaviour of the fluid surface in the corner. Figure (2) is a plot of the height of the capillary surface as a function of $\theta r/|\theta|$ for $\theta = \pm\alpha$, for various sized domains R as indicated, with β^+ =1 and $\beta^- = 10$. It is clear from figure (2) that the far field's behaviour on the corner is indeed negligible, and that asymmetry effects from the boundary conditions are also lost (through exponentially small terms) once an interior flat region is formed. The appearance of an interior flat region seems to occur at around $2-3$ capillary lengthscales from the origin, which is in agreement with the linearised solutions of Fowkes and Hood (**13**). The extent of the loss of asymmetry effects on the corner asymptotics is further demonstrated by noting that three significant figure agreement between the heights on the $\theta = \pm\alpha$ walls occurs for $r < 0.5$, $R = 6$, $r < 2.2$, $R = 8$ and $r < 5.7$ when $R = 10$.

Figure (3) gives a comparison of the power series solution (2.4) with the numerical solution along the wall for $r \le 0.5$ when $R \ge 6$, as figure (2) shows graphically there is no difference between the numerical solutions in this region. We choose $\theta = \alpha$ for the comparison since the r-derivatives are largest along the wall and the θ-derivatives are greatest where the fluid surface comes off the wall. Thus, $\theta = \alpha$ is the most demanding case numerically. At least the same level of accuracy is found locally for all θ around $r = 0$. It is again clear that the power series solution provides an accurate description of the correct corner behaviour, and these results also show that while logarithmic terms may be needed

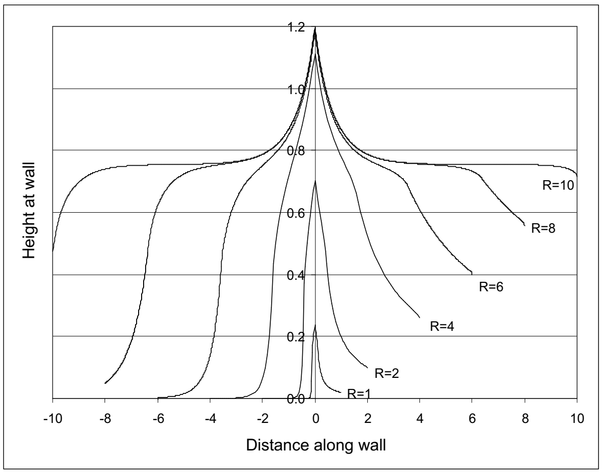

Fig. 2 Fluid height along each wall for different size solution domains $0 \le r \le R$, $-\alpha \le \theta \le \alpha$. All curves are for wedge angle $\alpha = \pi/3$, contact angle $\gamma = \pi/4$, with significantly nonsymmetric outer boundary conditions at $r = R$ being $u_r = Ru$ on $\theta \ge 0$ and $u_r = -10Ru$ on $\theta < 0$.

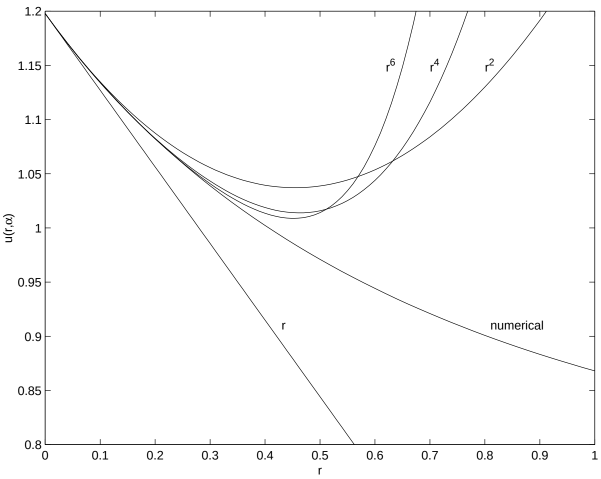

Fig. 3 Comparison of regular power series and numerical solutions near the apex $r = 0$ for $R = 6$ with $2\alpha = \pi/3$ and $\gamma = \pi/4$. Terms up to order n in the expansion have been used for curves labelled r^n.

to cope with every possible corner asymptotic expansion as shown by Miersemann (**10**), they are not needed for usual practical applications that possess an interior flat region.

Finally, for the linearised approximation to (2.1) and (2.2) given by $|\nabla u|^2 << 1$, Fowkes and Hood (**13**) give the following simple globally valid solution when $\alpha = \pi/4$

$$u(r,\theta) = \cos\gamma\left(e^{-r\sin(\pi/4-\theta)} + e^{-r\sin(\pi/4+\theta)}\right) \quad . \tag{4.2}$$

This solution applies for contact angles near $\pi/2$ and can be expanded around $r = 0$ to give

$$\frac{u(r,\theta)}{\cos\gamma} = 2 - \sqrt{2}\cos\theta\, r + \frac{r^2}{2} - \frac{r^3}{3!\sqrt{2}}\left(3\cos\theta - 2\cos^3\theta\right) + O(r^4). \tag{4.3}$$

It is straightforward to show that when $\alpha = \pi/4$ and γ near $\pi/2$, our series solution (2.4), for the full nonlinear equations, reproduces (4.3).

In summary then, we have shown that a straightforward symmetric power series solution is sufficient to describe more accurate (higher order) behaviour for the fluid rise in the corner of a wedge with static fluid acted on by gravity. The solution applies for intermediate wedge angles, when an interior flat region exists, even if the outer (far field) boundary conditions are nowhere near symmetric about $\theta = 0$. We derive such a series solution for all orders in r, and determine simple explicit recurrence relations for the coefficient functions which satisfy nonhomogeneous linear second order ordinary differential equations in θ with Neumann and Robin boundary conditions.

References

1. T. Young, An essay on the cohesion of fluids. *Philos. Trans. Roy. Soc. London* **95** (1805) 65-87.

2. P.S. Laplace, Traité de méchanique céleste; suppléments au Livre X, (1805 and 1806 resp.). In Euvres Complete Vol **4**. Gauthier-Villars, Paris.

3. R. Finn, *Equilibrium Capillary Surfaces* (Springer, Berlin, 1986).

4. C.F. Gauss, Principia Generalia Theoriae Figurae Fluidorum (1830). Comment. Soc. Regia Scient. Gottingensis Rec 7. Reprinted as *"Grundlagen einer Theorie der Gestalt von Flüssigkeiten im Zustand des Gleichgewichtes"* in Ostwald's *Klassiker der exakten Wissenschaften*, vol **135**. W. Engelmann, Leipzig, 1903. C. Vii, 4, 15, 219.

5. L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, Permagon Press, Oxford (1987).

6. P. Concus, and R. Finn, On a class of capillary surfaces, *J. Analyse Math.* **23** (1970) 65-70.

7. D. Siegel, Height estimates for capillary surfaces. *Pacific J. Maths.*, **88** (1980) 471-515.

8. R. Finn and J.-F. Hwang, On the caparison principle for capillary surfaces. *J. Fac. Sci. Univ. Tokyo* **36** (1989) 131-134.

9. N.J. Korevaar, On the behaviour of a capillary surface at a re-entrant corner. *Pacific J. Maths.* **88** (1980) 379-385.

10. E. Miersemann, Asymptotic expansion at a corner for the capillary problem. *Pacific J. Maths.* **134** (1988) 99-311.

11. E. Miersemann, Asymptotic expansion at a corner for the capillary problem: The singular case. *Pacific J. Maths.* **157** (1993) 95-107.

12. K.E. Lancaster and D. Siegel, Existence and behaviour of the radial limits of a bounded capillary surface at a corner. *Pacific J. Maths.* **176** (1996) 165-195.
13. N.D. Fowkes and M.J. Hood, , Surface tension effects in a wedge. *Q. J. Mech. Appl. Maths.* **51** (1998) 553-561.
14. J.R. King, J.R. Ockendon and H. Ockendon, The Laplace-Young equation near a corner. *Q. J. Mech. Appl. Mech.* **52** (1999) 73-97.
15. C.F. Scott, G.C. Sander and J. Norbury, Computation of capillary surfaces for the Laplace-Young equation. (Submitted to *Q. Jl Mech. Appl. Math.* (2004))