

# Quantum Steenrod Squares, Related Operations, and their Properties



Nicholas Wilkins  
Green Templeton College  
University of Oxford

A thesis submitted for the degree of  
*Doctor of Philosophy*

Michaelmas 2018

Dedicated to my niece Rosemarie

## Acknowledgements

The biggest thanks goes to my supervisor Alex Ritter for his unceasing support, the amazing amount of time and all of the guidance with which he provided me. He has taught me so much, but more than this his relentless enthusiasm and passion to help me succeed has been nothing short of incredible.

Thank you to Paul Seidel, who suggested the topic of my thesis, and with whom I had very useful conversations about areas to explore and things to try. Thank you also for the amazing opportunity to work at MIT.

Thanks goes to Frances Kirwan (thrice), as well as Ulrike Tillman and Dominic Joyce for examining my confirmation of status and transfer of status and giving me very helpful feedback on my work to-date.

A big thank you goes to my fellow D.Phil. students and office mates. Kelli helped to teach me how to write a proper introduction, as well as striving to keep me sane. Aurelio kept me fed, as well as striving to keep me insane. They both helped to make the office the perfect place to spend these past three years: with them, it was like Christmas every day. I also thank Tom, my resident topology expert and my reason for knowing about many things I would not otherwise have known. I thank Filip for always being willing to take the time for a much-needed mid-afternoon walk or a trip to the pub. Both of my mathematical brothers have been great people to discuss symplectic geometry with, to travel to conferences with and to get to know, and welcome additions to the Symplectic Karate Dynasty.

My research was made possible by the generous funding received as an EPSRC grant, as well as the travel funds from the Mathematics department and Green Templeton college that allowed me to travel to many and varied conferences.

Thanks also goes to all of the friends who have been there for me: I will not list them all here, as too many have impacted on my life, whether

it was the regular meetups, weekend visits or the occasional supportive message online, but they should definitely know who they are. I will give a special mention to Dan, who kept me going with our weekly pizza and beer nights, Nikki who supported me through everything and Michaela with whom a weekly Skype call could make me happy even when I was at my lowest.

Last, and by no means least, I want to thank my family for everything that they have done. My parents have looked after me for my whole life and continue to do so in every way they can. They deserve much more space on this page than anything I could give them. My brother has always been willing to teach me in all manner of things, sometimes forcefully, and without his encouragement I would probably not be writing this. He and my sister most certainly helped to assuage my many fears about the D.Phil. and submission process. Whenever I visit home or my brother and sister-in-law, it has been a welcome relief from the stress of a D.Phil and I have been well looked-after. All of my grandparents have supported and encouraged me, those who are still with us and those who have passed away, and I get a lot of pride from my family's support.

# Abstract

In this thesis, we generalise the Steenrod square on the cohomology of a topological space to a quantum Steenrod square on the quantum cohomology of a symplectic manifold. We prove quantum versions of the Cartan and Adem relations, and use the former to calculate the quantum Steenrod square for  $\mathbb{CP}^n$ , as well as more generally giving the means to compute the operation for all monotone toric varieties. We calculate the quantum Steenrod squares for two examples of blowups along a subvariety, in which a quantum correction of the Steenrod square on the blowup is determined by the classical Steenrod square on the subvariety. We then relate the quantum Steenrod square to Seidel's equivariant pair-of-pants product for open convex weakly monotone symplectic manifolds, using an equivariant version of the PSS isomorphism. We proceed similarly for  $\mathbb{Z}/2$ -equivariant symplectic cohomology, using an equivariant version of the  $c^*$ -map. We prove a symplectic Cartan relation, pointing out the difficulties in stating it. We will give some chain level calculations of the equivariant pair-of-pants product for  $T^*S^n$ . We calculate the quantum Steenrod square for the negative line bundles  $\mathcal{O}(-1) \rightarrow \mathbb{CP}^m$ , and use this to calculate the equivariant pair-of-pants square by extending previous results due to Ritter. We finish with a brief sketch of further ideas, involving a  $\mathbb{Z}/2$ -equivariant version of the Chas-Sullivan product and the Viterbo isomorphism.

## Statement of Originality

I declare that the work contained in this thesis is, to the best of my knowledge, original and my own work, unless indicated otherwise. I also declare that the work contained in this thesis has not been submitted towards any other degree at this institution or at any other institution.

Chapter 3 is based on my arXiv preprint, [32].

Chapter 4 has been placed on the arXiv with minor modifications, [33].

Nicholas Wilkins

December 6, 2018

# Contents

|          |                                                                |           |
|----------|----------------------------------------------------------------|-----------|
| <b>1</b> | <b>Introduction</b>                                            | <b>1</b>  |
| 1.1      | Motivation . . . . .                                           | 1         |
| 1.2      | Summary of the thesis and major results . . . . .              | 3         |
| <b>2</b> | <b>Preliminaries</b>                                           | <b>9</b>  |
| 2.1      | Morse Theory . . . . .                                         | 9         |
| 2.2      | Floer Theory . . . . .                                         | 12        |
| 2.3      | The Steenrod Square . . . . .                                  | 15        |
| 2.3.1    | The Betz-Cohen Construction . . . . .                          | 15        |
| 2.4      | Gluing and compactness . . . . .                               | 19        |
| 2.4.1    | Gluing . . . . .                                               | 19        |
| 2.4.2    | Compactness . . . . .                                          | 21        |
| <b>3</b> | <b>The Quantum Steenrod Square</b>                             | <b>23</b> |
| 3.1      | Introduction . . . . .                                         | 23        |
| 3.2      | Preliminaries . . . . .                                        | 29        |
| 3.2.1    | Equivariant Cohomology . . . . .                               | 29        |
| 3.2.2    | The Steenrod Squares . . . . .                                 | 30        |
| 3.2.3    | The Quantum Cup Product . . . . .                              | 31        |
| 3.3      | Two constructions of the Steenrod Squares . . . . .            | 32        |
| 3.3.1    | Morse Steenrod square . . . . .                                | 33        |
| 3.3.1.1  | A note on transversality . . . . .                             | 35        |
| 3.3.2    | The Morse Steenrod square is the Steenrod square . . . . .     | 36        |
| 3.3.3    | The Cartan Relation . . . . .                                  | 37        |
| 3.3.4    | Steenrod Squares via intersections of cycles . . . . .         | 42        |
| 3.3.5    | Properties of the Steenrod Square . . . . .                    | 45        |
| 3.4      | Quantum Steenrod Square via Morse theory . . . . .             | 47        |
| 3.4.1    | Quantum Steenrod Squares via intersections of cycles . . . . . | 48        |

|          |                                                                           |           |
|----------|---------------------------------------------------------------------------|-----------|
| 3.4.2    | Quantum Stiefel Whitney Class . . . . .                                   | 49        |
| 3.5      | The Quantum Cartan relation . . . . .                                     | 50        |
| 3.6      | Computing the Quantum Steenrod Square for toric varieties . . . . .       | 59        |
| 3.6.1    | Quantum Steenrod squares for $\mathbb{CP}^n$ . . . . .                    | 59        |
| 3.6.2    | Fano Toric Varieties . . . . .                                            | 62        |
| 3.7      | The Quantum Adem Relations . . . . .                                      | 64        |
| 3.7.1    | Classical Adem Relations . . . . .                                        | 64        |
| 3.7.2    | Quantum Adem Relations . . . . .                                          | 67        |
| 3.8      | $QS$ for blow-ups . . . . .                                               | 71        |
| 3.8.1    | $\mathbb{CP}^3$ . . . . .                                                 | 71        |
| 3.8.2    | $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ . . . . .       | 75        |
| <b>4</b> | <b>The Quantum Steenrod Square and the Equivariant Pair-of-Pants</b>      | <b>76</b> |
| 4.1      | Introduction . . . . .                                                    | 76        |
| 4.2      | Preliminaries . . . . .                                                   | 80        |
| 4.2.1    | Equivariant Floer cohomology . . . . .                                    | 80        |
| 4.2.2    | The symplectic square and equivariant pair-of-pants product .             | 83        |
| 4.2.2.1  | The symplectic square is a map on cohomology . . .                        | 85        |
| 4.2.3    | Quantum cohomology . . . . .                                              | 86        |
| 4.2.4    | The quantum Steenrod square . . . . .                                     | 88        |
| 4.2.5    | The PSS isomorphism . . . . .                                             | 91        |
| 4.2.6    | The $c^*$ -map . . . . .                                                  | 93        |
| 4.2.7    | Equivariant gluing and compactness . . . . .                              | 93        |
| 4.2.7.1  | Equivariant gluing . . . . .                                              | 93        |
| 4.2.7.2  | Equivariant compactness . . . . .                                         | 95        |
| 4.3      | Equivariant continuation maps . . . . .                                   | 96        |
| 4.4      | The Equivariant PSS isomorphism . . . . .                                 | 99        |
| 4.5      | $QS$ and the Equivariant pair-of-pants for a $C^2$ -small Hamiltonian . . | 104       |
| 4.6      | $QS$ and the symplectic square for equivariant symplectic cohomology      | 108       |
| 4.6.1    | Equivariant Symplectic Cohomology . . . . .                               | 108       |
| 4.7      | The symplectic Cartan relation . . . . .                                  | 110       |
| 4.7.1    | The symplectic Cartan relation . . . . .                                  | 110       |
| 4.7.2    | A comparison to the quantum Cartan relation . . . . .                     | 117       |
| 4.7.3    | Calculations for $n$ -spheres . . . . .                                   | 120       |
| 4.8      | The Equivariant Pair-of-Pants for Negative Line Bundles . . . . .         | 121       |

|          |                                                                                                 |            |
|----------|-------------------------------------------------------------------------------------------------|------------|
| 4.8.1    | The relation between the quantum Steenrod square and the symplectic square . . . . .            | 122        |
| 4.8.2    | The symplectic square for $M = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{CP}^m)$ . . . . . | 125        |
| <b>5</b> | <b>The Equivariant Loop Space Isomorphism</b>                                                   | <b>131</b> |
| 5.1      | Introduction . . . . .                                                                          | 131        |
| 5.2      | Preliminaries . . . . .                                                                         | 132        |
| 5.2.1    | The Chas-Sullivan Product . . . . .                                                             | 132        |
| 5.2.2    | Viterbo Isomorphism . . . . .                                                                   | 134        |
| 5.3      | The Equivariant Chas-Sullivan Product . . . . .                                                 | 135        |
| 5.3.1    | The equivariant loop cohomology . . . . .                                                       | 135        |
| 5.3.2    | The equivariant Morse Chas-Sullivan square . . . . .                                            | 137        |
| 5.3.2.1  | The Morse equivariant Chas-Sullivan square . . . . .                                            | 138        |
| 5.3.2.2  | The equivariant Chas-Sullivan square . . . . .                                                  | 138        |
| 5.4      | The Equivariant Viterbo Isomorphism . . . . .                                                   | 139        |
| 5.4.1    | The equivariant Viterbo isomorphism . . . . .                                                   | 139        |
| 5.5      | Some calculations . . . . .                                                                     | 142        |
| 5.6      | Conclusions and Further Work . . . . .                                                          | 143        |

# List of Figures

|      |                                                                                                                                                                                |     |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 2.1  | The Morse cup product. . . . .                                                                                                                                                 | 11  |
| 2.2  | The data for the pair-of-pants product. . . . .                                                                                                                                | 14  |
| 2.3  | General broken flowlines, where $q_i \in \text{crit}(f)$ , and gluing solutions. . .                                                                                           | 19  |
| 3.1  | Morse flowline configurations for the Steenrod square. . . . .                                                                                                                 | 34  |
| 3.2  | Elements of $T^c$ . . . . .                                                                                                                                                    | 38  |
| 3.3  | Tree labelling for $\mathcal{M}_2$ . . . . .                                                                                                                                   | 40  |
| 3.4  | Flowline configurations for $Sq(x) \cup Sq(y)$ . . . . .                                                                                                                       | 41  |
| 3.5  | Flowline configurations for $Sq(x \cup y)$ . . . . .                                                                                                                           | 43  |
| 3.6  | $m_1, m_2 \in \overline{M}_{0,5}$ . . . . .                                                                                                                                    | 52  |
| 3.7  | Elements of $W_0$ . . . . .                                                                                                                                                    | 56  |
| 3.8  | Configurations for $q(W)(x^i, x)$ for $\mathbb{CP}^n$ . . . . .                                                                                                                | 61  |
| 3.9  | Configurations for $q(W)(b, x_p)$ for toric varieties, where $X = PD(x)$ ,<br>$B = PD(b)$ and $z$ is the output. Energy $k$ means $E(u) = c_1(u_*[S^2])/2N =$<br>$k$ . . . . . | 63  |
| 3.10 | Moduli space for the Adem relations. . . . .                                                                                                                                   | 70  |
| 3.11 | Configurations for $Q\mathcal{S}_{1,1}$ on $M$ and $Q\mathcal{S}_{1,2}$ on $E$ . . . . .                                                                                       | 72  |
| 3.12 | Lifting configurations of $Sq$ on $Y$ to $Q\mathcal{S}$ on $M$ . . . . .                                                                                                       | 73  |
| 3.13 | Configurations for contributions to $Q\mathcal{S}_{1,1}$ from lifts on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . .                                              | 75  |
| 4.1  | $J$ for the symplectic square. . . . .                                                                                                                                         | 84  |
| 4.2  | Spiked discs. . . . .                                                                                                                                                          | 93  |
| 4.3  | Reversed $i$ -equivariant spiked discs and gluing. . . . .                                                                                                                     | 103 |
| 4.4  | Gluing of spiked disks onto equivariant pair of pants. Double headed<br>arrows denote that the Hamiltonian loops are the same. . . . .                                         | 105 |
| 4.5  | Broken solutions for $P\mathcal{S} \circ \phi_{H,H'}$ and $\phi_{eq,H,H'} \circ P\mathcal{S}$ respectively. . . . .                                                            | 109 |
| 4.6  | The almost complex structure $J_z^w$ . The labels 0, 1, 2 next to the edges<br>indicate which $w_i$ is being used in that situation. . . . .                                   | 111 |
| 4.7  | Flowlines on $S^\infty$ . . . . .                                                                                                                                              | 112 |

|      |                                                                                                                                                                                                                                                                                                                                     |     |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 4.8  | The 1-dimensional moduli space linking $m_1$ and $m_\infty$ . The double-headed arrows denote $\mathbb{Z}/2$ symmetry. For simplicity we omit the $S^\infty$ data from this picture. . . . .                                                                                                                                        | 114 |
| 4.9  | Flowline setups used in $S^\infty$ . . . . .                                                                                                                                                                                                                                                                                        | 115 |
| 4.10 | Configurations for the coefficient of $a$ in the operation $PS'(x; y)$ . . . .                                                                                                                                                                                                                                                      | 116 |
| 4.11 | For the lower row, single arrows represent Floer trajectories for inputs from $SH^*(M)$ . Double arrows represent trajectories for inputs/outputs from $SH_{eq}^*(M)$ . Arrows split by an orthogonal line are broken trajectories. The upper row demonstrates the relevant flowlines on $S^\infty$ , labelled accordingly. . . . . | 118 |
| 4.12 | Setups for the correction term in calculating $QS$ for $M$ . . . . .                                                                                                                                                                                                                                                                | 126 |
| 5.1  | Setups counted for $\phi$ . . . . .                                                                                                                                                                                                                                                                                                 | 135 |
| 5.2  | A “proof by picture” that $\phi$ is a ring isomorphism. The lower half of the figure relates to $\phi(x \bullet y)$ and the upper half relates to $\phi x * \phi y$ . . .                                                                                                                                                           | 136 |

# Chapter 1

## Introduction

### 1.1 Motivation

The aim of this thesis is to generalise a classical cohomology operation, the Steenrod square, to an operation on the quantum cohomology of a symplectic manifold (when this quantum cohomology exists). This operation will be called the quantum Steenrod square. After defining the operation, we explore its various properties and use these properties to calculate the quantum Steenrod square in several examples.

The idea of constructing the quantum Steenrod square for this thesis was motivated by the equivariant pair-of-pants product by Seidel [28]. Specifically, we answer the question of whether an existing isomorphism, due to Piunikhin, Salamon and Schwarz [18], between quantum cohomology and Floer cohomology will extend to intertwine the quantum Steenrod square and the equivariant pair-of-pants. We prove the affirmative, and show how this isomorphism may be used to compute the equivariant pair-of-pants.

Here and throughout we will use  $\mathbb{Z}/2$  coefficients for all (co)homology unless otherwise stated. Recall for a topological space  $M$ , the Steenrod square itself is a unital ring homomorphism

$$Sq : H^*(M) \rightarrow H^{2*}(M \times \mathbb{RP}^\infty) = (H^\bullet(M)[h])^{2*},$$

where the latter group is the  $\mathbb{Z}/2$ -equivariant cohomology of  $M$ . We give the full construction and definition in Sections 2.3 and 3.2.2. The Steenrod square is comprised of a collection of other operations,  $\{Sq^i\}$ , which we call the Steenrod squares. They are related by  $Sq(x) = \sum_i Sq^i(x)h^{|x|-i}$  where  $Sq^i : H^*(M) \rightarrow H^{*+i}(M)$ . The Steenrod squares are defined by a set of axioms, which imply that

$$Sq(x) = x \cup x + \dots + xh^{|x|},$$

and that the Cartan Relation holds

$$Sq(x \cup y) = Sq(x) \cup Sq(y), \quad (1.1)$$

which is equivalent to saying that  $Sq$  is a ring homomorphism.

There are several important reasons why this operation is studied. The first reason is that the Steenrod squares and reduced powers (the latter of which we do not study in this thesis) generate the set of stable cohomology operations on the cohomology of a topological space. For reference see Hatcher [12, Proposition 4L.1] and [11, Theorem 1.32]. One useful relation here is the Adem relations

$$Sq^q Sq^p = \sum_{s=0}^{\lfloor q/2 \rfloor} \binom{p-s-1}{q-2s} Sq^{p+q-s} Sq^s \quad (1.2)$$

where  $\lfloor i/2 \rfloor$  is the integer part of  $i/2$ . This allows the compositions of the  $Sq^j$  to be expressed in terms of the *Serre-Cartan basis*, the definition of which we do not repeat here.

The Steenrod squares also allow one to prove results about vector bundles: one can use Steenrod squares to prove results about the maximum number of nonzero linearly independent vector fields on  $TS^n$ , see [12, Example 4L.5]. Also, suppose that  $N \subset M$  is an embedded submanifold of a smooth manifold representing a cohomology class  $\alpha$ . If  $Sq(\alpha) \neq \alpha h^{|\alpha|}$  then the normal bundle of  $N$  in  $M$  is nontrivial. The latter result is proved in Corollary 3.3.14. An interested reader is encouraged to learn more from, for example, [12, Section 4.L].

The links between the Steenrod square and topology suggest that a quantum version of the Steenrod square may be used to prove results for the symplectic topology of a symplectic manifold  $M$ . Some interesting phenomena appear in the quantum setting, in particular the quantum Steenrod square

$$QS : QH^*(M) \rightarrow QH^*(M)[h]$$

is not a ring homomorphism with the obvious ring structure. This means that the Cartan relation must be deformed to prove a quantum Cartan relation. This arises because in the space of five marked points on the sphere, the order of the points matters up to homotopy of order  $\geq 2$ . Indeed, it is linked to the fact that there is no  $E_\infty$ -structure on the quantum cohomology of  $M$ . We also prove a similar but more complicated deformation for the Adem relations.

In this thesis, the quantum Cartan relation is used to calculate the quantum Steenrod square iteratively for all monotone toric varieties, giving an explicit formula

for  $\mathbb{CP}^n$ . We can also calculate the quantum Steenrod square for the blowup  $M$  of  $X = \mathbb{CP}^3$  and  $X = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$  along a torus  $Y$ . We do this by showing that those quantum Steenrod squares on  $M$  that we cannot calculate using our knowledge of the quantum Steenrod square on  $X$  (which is a toric variety in both cases) are related to the classical Steenrod square on  $Y$ . We will also use the quantum Cartan relation to calculate the quantum Steenrod square for the total space of  $\mathcal{O}(-1) \rightarrow \mathbb{CP}^n$ . Subject to the isomorphism that we mentioned at the beginning of the section, and adapting a result by Ritter [22], this will allow us to calculate the equivariant pair-of-pants product for this total space.

Further work on this subject may involve work on a “quantum  $E_\infty$ -structure” on  $QH^*(M)$ , calculations of the equivariant pair-of-pants product for certain manifolds (which may give information about the symplectic cohomology of  $M$ ) or perhaps a quantum Stiefel-Whitney class that might allow us to study symplectic vector bundles. These possibilities are currently speculative, but this thesis lays the groundwork for further research.

## 1.2 Summary of the thesis and major results

Chapter 2 is a preliminary chapter. It will cover background material on Morse theory, Floer theory, and a summary of two related constructions of the Steenrod square due to Betz and Cohen [3] and Fukaya [10]. A brief description of the Floer gluing and Gromov compactness theorems will be given at the end of that chapter.

The proceeding chapters will then be structured as papers, as they are intended to be published, but in this thesis we expand on proofs and remarks. The final chapter is work in progress, and therefore not mathematically rigorous.

Chapter 3 will define the quantum Steenrod square, prove some properties and perform calculations. This definition uses a new, Morse theoretic definition of the Steenrod square adapted from work by Fukaya, Betz, Cohen, Norbury and Seidel in [10], [4], [3], [8], [28], which will be given in Section 3.3.

In Section 3.4 we will define the quantum Steenrod square in Definition 3.4.1, denoted by  $QS : QH^*(M) \rightarrow (QH^*(M)[h])^{2*}$ . There is also a quantum version of some of the classical properties of the Steenrod square: most notably there is a quantum Cartan relation, see Theorem 3.1.2, using a proof of the Cartan relation that the author believes to be novel, Theorem 3.3.7, which was based upon [8, Section 5.2]. We package Definition 3.4.1 and Theorem 3.1.2 into the following:

**Theorem.** *Let  $(M, \omega)$  be a closed, monotone symplectic manifold. There exists an additive unital homomorphism*

$$Q\mathcal{S} : QH^*(M) \rightarrow (QH^\bullet(M)[h])^{2*}$$

*satisfying the quantum Cartan relation*

$$Q\mathcal{S}(x * y) = Q\mathcal{S}(x) * Q\mathcal{S}(y) + \sum_{i,j} q_{i,j}(W_0 \times D^{i-2,+})(x, y) h^i$$

*for  $x, y \in QH^*(M)$ , where the notation  $q_{i,j}$  will be defined precisely in Section 3.5.*

The correction term  $\sum_{i,j} q_{i,j}(W_0 \times D^{i-2,+})(x, y) h^i$  will be defined fully in Section 3.5. Classically  $q(W_0 \times D^{i-2,+})(x, y)$  disappears, which is consistent with the fact that the Steenrod square is a homomorphism of rings. However, in general  $q(W_0 \times D^{i-2,+})$  does not vanish, as in the following example.

**Example.**

$$QH^*(\mathbb{CP}^1) = \frac{\mathbb{Z}}{2}[x, t]/(x^2 - t^2).$$

$$Q\mathcal{S}(x * x) = Q\mathcal{S}(t^2) = t^4.$$

$$Q\mathcal{S}(x) * Q\mathcal{S}(x) = (x * x + xh^2)^2 = t^4 + t^2h^4.$$

*It is defined that  $Q\mathcal{S}(t^2) = t^4$ , see the end of Definition 3.4.1. Further,  $Q\mathcal{S}(x) = x * x + xh^2$  for degree reasons: specifically  $|x| = 2$  implies that all other terms must necessarily vanish. Here the nontrivial correction term in the quantum Cartan relation is  $t^2h^4$ .*

Suppose that  $M$  is a Fano toric variety. For  $\mu \in H_2(M; \mathbb{Z})$  (which is a free  $\mathbb{Z}$ -module as  $M$  is simply connected), let  $\mu_2$  be the image of  $\mu$  under  $H_2(M; \mathbb{Z}) \rightarrow H_2(M; \mathbb{Z}/2)$ . For  $x, y \in QH^*(M)$ , denote by  $x *_{\mu,k} y$  the coefficient of  $t^{kN}$  in the quantum product  $x * y$ , using spheres representing  $\mu$ . Let  $N$  be the minimal Chern number, and  $|t| = 2$ . The quantum Cartan relation is used in Section 3.6 to calculate the quantum Steenrod squares for all closed Fano toric varieties. This is Theorem 3.1.3, i.e.:

**Theorem.** *Let  $M$  be a Fano toric manifold. For  $b, x \in H^*(M)$  and  $|x| = 2$ ,*

$$q_{i,j}(W_0 \times D^{i,+})(b, x) = \sum_{j \geq 1} \sum_{k=1}^j \sum_{c_1(\mu)=2kN} n(x, \mu_2) \cdot (Q\mathcal{S}_{|b|-i+2,j-k}(b) *_{\mu,k} x) \cdot t^{jN} \quad (1.3)$$

*summing over a basis of  $\mu \in H_2(M; \mathbb{Z})$ , so  $c_1(\mu) = 2kN \in \mathbb{Z}$  and  $\#(PD(x) \cap \mu_2) \in \mathbb{Z}/2$ . Let  $n(x, \mu_2) = \#(PD(x) \cap \mu_2)$ .*

In the case of  $M = \mathbb{CP}^n$ , we can use equation (1.3) to write the quantum Steenrod square in closed form.

In the penultimate section 3.7 of Chapter 3, we modify the classical Adem relations in Equation (1.2). These take the form in the theorem below, and we can see that they are a deformation of the classical case.

**Theorem** (Quantum Adem Relations). *For  $p, q > 0$ ,  $q < 2p$ ,  $\alpha \in QH^*(M)$ ,*

$$\sum_{b,d} \left( Q\mathcal{S}^{q,b} \circ Q\mathcal{S}^{p,d}(\alpha) - \sum_{s=0}^{q/2} \binom{p-s-1}{q-2s} Q\mathcal{S}^{p+q-s,b} \circ Q\mathcal{S}^{s,d}(\alpha) \right) = t^N \cdot Q(\alpha) \quad (1.4)$$

*for the correction term*

$$\begin{aligned} t^N \cdot Q(\alpha) &= q_{D_8}((gm_1 + g^2m_1) \otimes \Psi(e^{|\alpha|+p-q}\sigma_2^{|\alpha|-p}))(\alpha) \\ &- \sum_{s=0}^{[q/2]} \binom{p-s-1}{q-2s} q_{D_8}((gm_1 + g^2m_1) \times \Psi(e^{|\alpha|+2s-p-q}\sigma_2^{|\alpha|-s}))(\alpha). \end{aligned}$$

where the terms will be explained in Section 3.7. To finish Chapter 3, in Section 3.8 we will calculate some nontrivial quantum Steenrod squares for blowups  $M = Bl_Y X$  for  $X = \mathbb{CP}^3$  and  $X = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$  along a subvariety  $Y$ , where  $Y$  is a torus defined respectively by the intersection of two quadrics or two linear subspaces. We do this by noticing that  $J$ -holomorphic spheres in the blowup correspond to either:

- lifts of  $J$ -holomorphic spheres on the base space, or
- a fibre over a point in the torus.

In the case of

$$Q\mathcal{S}_{1,1}|_{H^3(M)} : H^3(M) \rightarrow H^3(M),$$

we can discount the former case and just work with the latter because of the Chern numbers of the spheres. We then show that counting setups for the Morse Steenrod square  $Sq_1|_{H^1(Y)}$  on the torus corresponds to counting setups for the Morse quantum Steenrod square  $Q\mathcal{S}_{1,1}|_{H^3(M)}$  on the blowup, which allows us to prove in Theorem 3.1.9 that:

**Theorem.**  $Q\mathcal{S}_{1,1}|_{H^3(M)}$  *is the identity map.*

Chapter 4 explores the relationship between the equivariant pair-of-pants due to Seidel in [28] and the quantum Steenrod square as in the Chapter 3. For a weakly

monotone convex symplectic manifold  $M$ , as in [23, Section 4.4] there is an isomorphism  $\Psi : QH^*(M) \rightarrow HF^*(H)$  for an appropriate choice of  $H : M \rightarrow \mathbb{R}$ , i.e. on the conical end of  $M$  it takes the form  $H = h(R)$ , where  $h$  is  $C^2$ -small and concave.

We will construct an analogous isomorphism

$$\Psi_{eq} : QH_{eq}^*(M) \xrightarrow{\cong} HF_{eq}^*(2 \cdot H),$$

where  $HF_{eq}^*(2 \cdot H)$  is the equivariant Floer cohomology for the Hamiltonian  $H$ , and  $QH_{eq}^*(M) := QH^*(M)[[h]]$ . The definition of  $HF_{eq}^*(2 \cdot H)$  is due to Seidel in [28] and is given for our purposes (i.e. a Hamiltonian symplectomorphism) in Section 4.2.1.

We will prove Theorem 4.1.1 in Section 4.5, i.e. that:

**Theorem.** *Diagram (1.5) commutes:*

$$\begin{array}{ccc} HF^*(H) & \xrightarrow{PS} & HF_{eq}^*(2 \cdot H) \\ \Psi \uparrow \cong & & \Psi_{eq} \uparrow \cong \\ QH^*(M) & \xrightarrow{QS} & QH_{eq}^*(M) \end{array} \quad (1.5)$$

where  $PS$  is the symplectic square, which is a specific case of the equivariant pair-of-pants in [28]. The definition is included for our case in Section 4.2.2.

In Diagram (1.5) above, we abusively denote  $QS$  for the composition

$$QH^*(M) \xrightarrow{QS} (QH^\bullet(M)[h])^{2*} \hookrightarrow QH_{eq}^{2*}(M).$$

By constructing a  $\mathbb{Z}/2$ -equivariant version of the continuation maps for Floer cohomology, we can define an equivariant version of symplectic cohomology,  $SH_{eq}^*(M)$ . For details of continuation maps and symplectic cohomology, see Section 4.3. The equivariant pair-of-pants commutes with the continuation maps, which means that there is an equivariant pair-of-pants induced on symplectic cohomology. There is also an equivariant version of the  $c^*$ -map, denoted  $c_{eq}^* : QH_{eq}^*(M) \rightarrow SH_{eq}^*(M)$ . This gives Corollary 4.1.3:

**Corollary.** *The following diagram commutes:*

$$\begin{array}{ccc} QH^*(M) & \xrightarrow{QS} & QH_{eq}^*(M) \\ c^* \downarrow & & c_{eq}^* \downarrow \\ SH^*(M) & \xrightarrow{PS} & SH_{eq}^*(M) \end{array} \quad (1.6)$$

A natural question, given the above results, is to ask whether there is a version of the Cartan relation for symplectic cohomology, and how it interacts with the equivariant maps constructed above. It turns out that for general symplectic manifolds there is no symplectic Cartan relation in the same way as the classical and quantum Cartan relations. This is because, unlike  $QH_{eq}^*(M) = QH^*(M)[[h]]$  and  $H^*(M)[h]$ , there is no obvious ring structure on  $SH_{eq}^*(M)$  (Section 4.7). There is, however, an  $SH^*(M)$ -module structure

$$PS' : SH^*(M) \otimes SH_{eq}^*(M) \rightarrow SH_{eq}^*(M).$$

We will prove in Section 4.7 that, for  $*$  being the pair-of-pants product on symplectic cohomology:

**Theorem.**

$$PS(x * y) = PS'(x; PS(y)).$$

We will sketch the possibility of using this to calculate nonvanishing of the equivariant pair-of-pants on  $T^*S^n$  and  $T^*\mathbb{CP}^n$  at the chain level, using the isomorphism

$$SH^*(T^*N) \cong H_{n-*}(\mathcal{LN}). \quad (1.7)$$

To finish Chapter 4, in Section 4.8 we will use Corollary 4.1.3, to extend the results due to Ritter in [22, Theorem 22, Sections 2-5] to yield Theorem 4.1.5, as in:

**Theorem.** *Let  $M = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{CP}^n)$  be the total space of the negative line bundle  $\mathcal{O}(-1) \rightarrow \mathbb{CP}^n$ . Then*

$$c_{eq}^* : QH_{eq}^*(M)/QS(x^n + T) \cdot \Lambda[[h]] \rightarrow SH_{eq}^*(M)$$

*is an isomorphism, where  $x \in H^2(\mathbb{CP}^n)$  is the generator, and  $T$  is the quantum variable with degree  $|T| = 2n$ .*

We can calculate the quantum Steenrod square for negative line bundles over  $\mathbb{CP}^n$ , and hence using the above theorem on (4.3) we can calculate  $PS$  using  $QS$ .

In the final Chapter 5, we will give a brief account of current and future plans to use the extended version of the isomorphism in equation (1.7) to an equivariant version of the Viterbo isomorphism. Specifically, one would need to use an equivariant version of the loop space homology,  $H_*^{eq}(\mathcal{LN})$ . This would involve defining an equivariant Chas-Sullivan product

$$CS : H_*(\mathcal{LN}) \rightarrow H_*^{eq}(\mathcal{LN}),$$

and an equivariant isomorphism

$$SH_{eq}^*(T^*N) \rightarrow H_{n-*}^{eq}(\mathcal{L}N)$$

such that:

**Theorem.** *Diagram (1.8) commutes:*

$$\begin{array}{ccc} SH^*(T^*N) & \xrightarrow{PS} & SH_{eq}^{2*}(T^*N) \\ \downarrow \cong & & \downarrow \cong \\ H_{n-*}(\mathcal{L}N) & \xrightarrow{CS} & H_{n-2*}^{eq}(\mathcal{L}N) \end{array} \quad (1.8)$$

This may allow us to calculate  $PS$  in certain cases by calculating  $CS$ .

# Chapter 2

## Preliminaries

### 2.1 Morse Theory

The origins of the approach to Morse theory detailed here appear to have become widely known with Floer and Witten. The relevant Morse theory is in Schwarz [25], and there are Part III Morse homology lecture notes by Ritter [20]. We will give the important details here.

Let  $X$  be a smooth compact Riemannian manifold, which in this section we assume to be closed (i.e. compact without boundary). A function  $f : X \rightarrow \mathbb{R}$  is Morse if all of its critical points are nondegenerate. This ensures that  $\text{crit}(f)$ , the set of critical points of  $f$ , is a discrete set. Observe that  $-\nabla f$  defines a one-parameter group of diffeomorphisms  $\phi_s : X \rightarrow X$  such that  $\phi_0 = \text{id}$  and  $\frac{d\phi_s}{ds} = -\nabla f$ . We use the metric  $g$  to define the gradient.

For  $x \in \text{crit}(f)$ , define

$$W^u(x, f) = \{y \in X : \lim_{s \rightarrow -\infty} \phi_s(y) = x\},$$

$$W^s(x, f) = \{y \in X : \lim_{s \rightarrow \infty} \phi_s(y) = x\},$$

respectively the unstable and stable manifolds of  $x$  under  $f$ . We will need a slightly stronger notion of function than Morse: in fact we need that  $(f, g)$  is Morse-Smale, which means that for every pair of critical points  $x_1, x_2 \in \text{crit}(f)$ , then  $W^u(x_1, f)$  meets  $W^s(x_2, f)$  transversely. A generic choice of  $(f, g)$  will be Morse-Smale, and for a fixed  $f$  a generic choice of  $g$  yields a Morse-Smale  $(f, g)$ , and vice versa.

The set of flowlines from  $x_1$  to  $x_2$ , i.e. the set of  $\gamma : \mathbb{R} \rightarrow X$  such that  $\dot{\gamma}(s) = -\nabla f(\gamma(s))$  with  $\lim_{s \rightarrow -\infty} \gamma(s) = x_1$  and  $\lim_{s \rightarrow \infty} \gamma(s) = x_2$ , is exactly  $W^u(x_1, f) \cap W^s(x_2, f)$ . As these intersect transversely, their intersection is a submanifold. There is an  $\mathbb{R}$ -action induced by translating  $\gamma$  in  $s$ , and we denote

$$\mathcal{M}(x_1, x_2) = W^u(x_1, f) \cap W^s(x_2, f) / \mathbb{R}.$$

The index of a critical point  $x \in \text{crit}(f)$  is the number of negative eigenvalues of the Hessian of  $f$  at  $x$ . We denote this by  $|x|$ . Then

$$\dim \mathcal{M}(x_1, x_2) = |x_1| - |x_2| - 1.$$

We discuss gluing and compactness theorems, which are necessary to rigorously define the Morse chain complex, in Section 2.4 at the end of this chapter. The idea behind the compactness result is that the space of flowlines is compactified by adding in broken flowlines, i.e. the limit of a sequence of flowlines  $(\gamma_k)_{k \geq 0} \subset \mathcal{M}(x, y)$  is a composite flowline  $\gamma_\infty \in \mathcal{M}(x, x_1) \times \mathcal{M}(x_1, x_2) \times \dots \times \mathcal{M}(x_r, y)$ .

**Definition 2.1.1.** Let  $CM^i(X, f) = \bigoplus_{x \in \text{crit}_i(f)} \langle x \rangle$ , where  $\text{crit}_i(f) \subset \text{crit}(f)$  consists of all those critical points of index  $i$ . Specifically,  $CM^*(X, f)$  is generated freely over  $\mathbb{Z}/2$  by the critical points of  $f$ .

$$d : CM^i(X, f) \rightarrow CM^{i+1}(X, f), \quad d(\langle x \rangle) = \sum_{y \in \text{crit}_{i+1}(f)} (\#\mathcal{M}(y, x)) \cdot \langle y \rangle.$$

As  $\mathcal{M}(y, x)$  is of dimension 0 for  $y \in \text{crit}_{|x|+1}(f)$  and the moduli space is compact, there is a well defined count. Recall that we work with  $\mathbb{Z}/2$  coefficients. One proves that  $d$  is a differential, i.e.  $d^2 = 0$ , using the compactification theorem.

There is an isomorphism  $\psi : HM^*(X, f) \rightarrow H^*(X)$  between Morse and singular cohomology, [26], which for  $x \in \text{crit}(f)$  one can think of intuitively (using the theory of pseudocycles) as

$$\langle x \rangle \mapsto \overline{W^s(x, f)}. \quad (2.1)$$

More specifically, one first defines the chain level map such that (for  $a$  in some fixed basis of  $H^*(X)$ ) if  $|a| = |x|$ , then the coefficient of  $a$  in  $\phi(x)$  will be the number of smooth maps  $u : \mathbb{R} \rightarrow X$  such that  $u$  is a  $-\nabla f$  flowline and  $u(0)$  lies in some fixed representative of  $PD(a)$ . We will not prove the isomorphism here, but we will use throughout that the two notions of cohomology are equivalent. Moreover, the Morse cohomology is independent of the choice of Morse function  $f$  up to a non-canonical isomorphism. This will imply that even though we may have to make some choices in the setup, such as a choice of Morse function, the operations resulting from two different choices will be intertwined by an isomorphism (which depends on a choice of interpolating Morse functions).

The final thing to discuss in Morse theory is the Morse theoretic version of the cup product. For this, we need to pick three  $s$ -dependent perturbations of the Morse

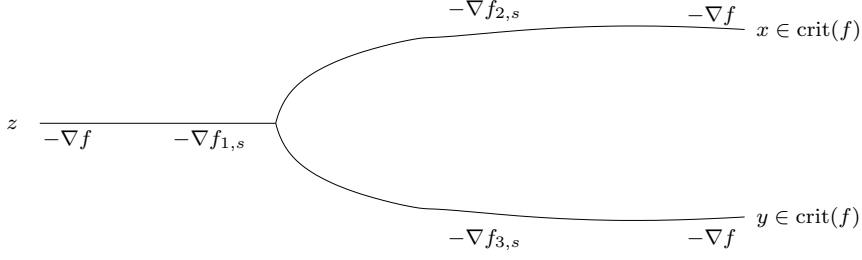


**Figure 2.1:** The Morse cup product.

function  $f$ , which we denote  $f_{1,s}, f_{2,s}, f_{3,s} : X \rightarrow \mathbb{R}$ , where  $f_{1,s}$  uses the coordinate  $s \in (-\infty, 0]$  and  $f_{2,s}, f_{3,s}$  use  $s \in [0, \infty)$ . We ensure that  $f_{i,s} = f$  for  $|s| \gg 0$  for  $i = 1, 2, 3$ . For a generic choice of  $f_{i,0}$ , the moduli space given below will be a smooth manifold.

With this in mind, given  $x, y, z \in \text{crit}(f)$  we define a moduli space by counting configurations of flowlines as in Figure 2.1, i.e.

$$\begin{aligned}
& u_1 : (-\infty, 0] \rightarrow X, \\
& u_2, u_3 : [0, \infty) \rightarrow X, \\
& \dot{u}_i(s) = -\nabla f_{i,s}(u_i(s)) \text{ for } i = 1, 2, 3, \\
& u_1(0) = u_2(0) = u_3(0), \\
& \lim_{s \rightarrow -\infty} u_1(s) = z, \\
& \lim_{s \rightarrow \infty} u_2(s) = x, \\
& \lim_{s \rightarrow \infty} u_3(s) = y.
\end{aligned} \tag{2.2}$$

Let  $\mathcal{M}(z; x, y) = \{(u_1, u_2, u_3) : \text{Equation (2.2)}\}$ . For a generic choice of the  $f_{i,s}$  this is a smooth manifold, and

$$\dim \mathcal{M}(z; x, y) = |z| - |x| - |y|. \tag{2.3}$$

**Definition 2.1.2.** *This allows us to define*

$$- \cup - : CM^i(M, f) \otimes CM^j(M, f) \rightarrow CM^{i+j}(M, f),$$

$$\langle x \rangle \cup \langle y \rangle = \sum_{z \in \text{crit}_{i+j}(f)} (\#\mathcal{M}(z; x, y)) \cdot \langle z \rangle.$$

*This is well defined using Equation (2.3). It descends to an additive map on cohomology, which again uses the compactification of the space of flowlines, and becomes the cup product on cohomology under the isomorphism in Equation (2.1).*

**Remark 2.1.3.** *Here and throughout, we will refer to diagrams in figures as “setups” or “configurations”. These are taken to mean (sometimes piecewise) smooth maps  $u$ ,*

whose codomain is  $M$  and whose domain is given in the diagram. This will often be a Riemann surface with cylindrical ends or a tree. The equations that  $u$  satisfy use data as given in the diagram: depending on context, this either refers to a perturbed  $J$ -holomorphic equation or a gradient flow equation, where respectively the almost complex structure  $J$  and Hamiltonian  $H$ , or the Morse function  $f$ , may be domain dependent, with the labels denoting this dependence. The asymptotic conditions for  $u$  are included when necessary.

Compare Figure 2.1 and Equation (2.2): in this context we could replace the triple  $(u_1, u_2, u_3)$  by  $u : \Gamma \rightarrow M$ , where  $\Gamma$  is the  $Y$ -shaped graph  $(-\infty, 0] \vee_0 [0, \infty) \vee_0 [0, \infty)$ . When we refer to an operation  $A$  “corresponding to” such setups, we mean that the coefficient of  $z$  in  $A(x_1, \dots, x_n)$  is the count of a moduli space of such setups, using the asymptotic conditions  $(z, x_1, \dots, x_n)$ .

## 2.2 Floer Theory

For a more complete introduction to Floer cohomology and symplectic cohomology, there are good descriptions in [14, Section 10], [24, Section 3] and [22, Section 2]. We will defer discussion of compactness and gluing theorems to Section 2.4, and we introduce the Novikov ring in Section 4.2.3.

There are two main types of manifold on which one historically defined Floer cohomology. The first is weakly monotone closed symplectic manifolds, but Floer homology in this case is actually the same as quantum cohomology. The other type is Liouville domains, which are exact. These cases are both examples of weakly monotone convex symplectic manifolds, whose symplectic cohomology was defined by Ritter in [19]. For simplicity in this section we will define the Floer cohomology for open exact convex symplectic manifolds.

Let  $(M, d\theta)$  be an open, exact, convex symplectic manifold, where convexity is as in Definition 4.2.12 in Chapter 4. We also choose an almost complex structure  $J$  that is compatible with the symplectic form. Then  $M$  consists of the union of a compact set  $C$  and a conical collar  $(\overline{M - C}, d\theta) \cong (\partial C \times [1, \infty), d(R^2\alpha))$  where  $R \in [1, \infty)$  is the radial coordinate of the conical collar, and  $(\partial C, \alpha)$  is a contact manifold. Let  $H : M \rightarrow \mathbb{R}$  be a Morse function, which we generally perturb to a  $t$ -dependent Hamiltonian  $H = H_t$  for  $t \in \mathbb{R}/\mathbb{Z}$ , such that  $H = \lambda_H \cdot R$  sufficiently far from  $C$ . We call such  $H$  “radial at infinity”. We can consider the set of Hamiltonian orbits of  $H$ . Let

$$\mathcal{L}_H = \{x : \mathbb{R}/\mathbb{Z} \rightarrow M | x \text{ is smooth and } \dot{x}(t) = X_H(x(t))\},$$

where  $X_H$  is the Hamiltonian vector field generated by  $H$ , defined by  $\omega(\cdot, X_H) = dH$ . We let our chain complex  $CF^*(H)$  be freely generated by the elements of  $\mathcal{L}_H$ , i.e. the generators are Hamiltonian orbits. If we define

$$\mathcal{L}M = \{x : \mathbb{R}/\mathbb{Z} \rightarrow M | x \text{ is smooth}\}$$

then we can define an “action functional”

$$\alpha_H : \mathcal{L}M \rightarrow \mathbb{R}, \quad \alpha_H(x) = - \int_{S^1} x^* \theta + \int_{t \in S^1} H_t(x(t)) dt.$$

The elements of  $\mathcal{L}_H$  are critical points of  $\alpha_H$ , comparing to the previous section.

There is a grading on this chain complex called the Conley-Zehnder index, which has a good description in [24, Section 2.4], originally by Conley and Zehnder in [9]. We will not go into details of its definition, but in the case where  $H$  is independent of  $t$  and  $x \in \mathcal{L}_H$  is a critical point of  $H$ , this recovers the Morse index. We denote the Conley-Zehnder index of  $x$  as  $|x|$ , as in the Morse case.

The differentials on the chain complex are analogous to the Morse theory case in the previous section, counting elements of a moduli space that will be described. Let  $x, y \in CF^*(H)$ . Define

$$\widetilde{\mathcal{M}}(y, x) = \left\{ u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow M \left| \begin{array}{l} \partial_s u + J(\partial_t u - X_H) = 0, \\ \lim_{s \rightarrow -\infty} u(s, t) = y(t), \\ \lim_{s \rightarrow \infty} u(s, t) = x(t) \end{array} \right. \right\}.$$

These “Floer cylinders” or “Floer trajectories” are analogues of the Morse trajectories in Section 2.1. For a generic choice of  $(H, J)$  one can compactify  $\widetilde{\mathcal{M}}(y, x)$  to be a manifold of dimension  $|y| - |x|$ . Observe that there is a  $\mathbb{R}$ -translation in  $s$ , so let  $\mathcal{M}(y, x) = \widetilde{\mathcal{M}}(y, x)/\mathbb{R}$ .

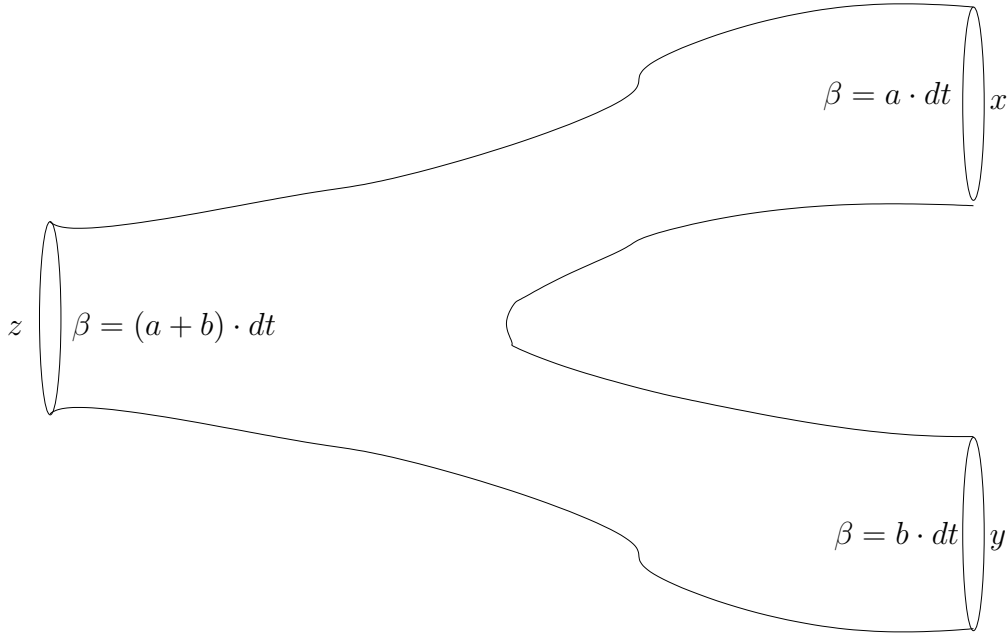
**Definition 2.2.1.**

$$d : CF^*(H) \rightarrow CF^{*+1}(H), \quad d(x) = \sum_{y \in \mathcal{L}_H : |y|=|x|+1} \# \mathcal{M}(y, x) \cdot y.$$

*It is a very nontrivial theorem that this is actually a differential on the given chain complex. See [24, Theorem 3.5]. Key ingredients are the compactness theorem due to Gromov, and the gluing theorem due to Floer. We will sketch this in Section 2.4.*

We end this section by describing the pair-of-pants operation (often called product) on Floer cohomology,

$$* : HF^*(aH) \otimes HF^*(bH) \rightarrow HF^*((a+b)H),$$



$(du - X_H \otimes \beta)^{0,1} = 0$ , where  $\beta$  is a 1-form on the pair-of-pants such that  $d\beta \leq 0$ .



**Figure 2.2:** The data for the pair-of-pants product.

for  $a, b \in \mathbb{Z}_{\geq 0}$ . To save on repeating technical details, we will draw analogy to the Morse cup product and the Floer trajectories above, and show in Figure 2.2 the relevant data for the setups we count for the coefficient of  $z$  in  $x * y$  (instead of  $u_1, u_2, u_3$  we count  $u : S \rightarrow M$ , where  $S$  is a genus 0 Riemann surface with a negative cylindrical end and two positive cylindrical ends, satisfying an appropriate equation and with asymptotic conditions).

**Remark 2.2.2.** 1. In constructions of Floer cohomology in more general symplectic manifolds, i.e. when the symplectic form is nonexact, we restrict our attention to contractible Hamiltonian orbits, and there is no longer a well defined action functional on  $\mathcal{LM}$ . Instead for simplicity we lift to a cover  $\widetilde{\mathcal{LM}}$ , which consists of pairs  $[x, u]$  where  $x$  is a Hamiltonian orbit and  $u$  is a filling disc. This is used for example in the case of closed symplectic manifolds. Further, the dimension of the moduli space  $\mathcal{M}(x, y)$  also has nontrivial contributions from spheres of nonzero energy. Many of the technical details, for example Gromov

compactness, are more difficult in the nonexact case.

2. Note that in general (i.e. for non-compact  $M$ ), this is not actually a product on Floer cohomology. This operation is called a “product” because it becomes a product when one takes the direct limit of Floer cohomologies to symplectic cohomology.

## 2.3 The Steenrod Square

The Steenrod square is axiomatically defined, but these axioms do not include an actual construction. We will describe two independent constructions in this preliminary section. The first is due to Betz in his thesis [4], although the actual outline that we give will come from a paper by Betz and Cohen, [3], and the second is given by Fukaya in [10] along with a note on how to extend to a quantum Steenrod square.

### 2.3.1 The Betz-Cohen Construction

The details are relevant for Section 3.3.2.

Fix a Morse-Smale function  $f$  on  $M$ , and pick a small convex neighbourhood  $U_f$  of  $f$  in  $C^\infty(M)$  consisting of Morse-Smale functions. Let  $\Gamma$  be the  $Y$ -shaped graph, oriented and parametrised as  $(-\infty, 0] \vee_0 [0, \infty) \vee_0 [0, \infty)$ . We denote  $S$  to be the set of triples  $\sigma = (f_{1,s}, f_{2,s}, f_{3,s})$  such that  $f_{1,s} \in U_f$  for each  $s \in (-\infty, 0]$  and  $f_{2,s}, f_{3,s} \in U_f$  for  $s \in [0, \infty)$ , subject to:

1.  $f_{1,0}, f_{2,0}, f_{3,0}$  are pairwise distinct.
2.  $f_{i,s} = \beta(|s|)f_{i,0} + (1 - \beta(|s|))f$ , where  $\beta : [0, \infty) \rightarrow [0, 1]$  is a fixed monotone bump function such that  $\beta(s) = 1$  for  $s \leq 1/2$  and  $\beta(s) = 0$  for  $s \geq 1$ .

We define  $\mathcal{M}_\sigma$  to be the set of  $\gamma : \Gamma \rightarrow M$  that are continuous and smooth on the edges, such that for each edge  $E_i$  of  $\Gamma$  we denote  $\gamma_i = \gamma|_{E_i}$ , and require

$$d\gamma_i/dt(s) + \nabla f_{i,s}(\gamma_i(s)) = 0.$$

This is actually slightly different to the construction in [3], in which the  $f_1, f_2, f_3$  were pairwise distinct and had no  $s$  dependence. The construction due to Betz-Cohen is equivalent to that given here, as we remark at the end of the section.

Let  $\mathcal{M}_{BC} = \sqcup_{\sigma \in S} \mathcal{M}_\sigma$ , topologised so that  $\mathcal{M}_{BC} \rightarrow S$  is continuous. Observe that there is a  $\mathbb{Z}/2$ -action  $\iota_S$  on  $S$ , induced by the permutation (23). This induces a

$\mathbb{Z}/2$ -action on  $\mathcal{M}_{BC}$ , via  $(\sigma, \gamma) \mapsto (\iota_S \circ \sigma, \gamma \circ R_\Gamma)$ . Here,  $R_\Gamma$  is the involution on the  $\Gamma$  that swaps the two positive half-lines and fixes the negative half-line.

For  $a_1, a_2, a_3 \in \text{crit}(f)$ , define  $\mathcal{M}_{BC}(a_1, a_2, a_3)$  to consist of equivalence classes of pairs  $[\sigma, \gamma] \in \mathcal{M}_{BC}/(\mathbb{Z}/2)$  such that

$$\lim_{i \rightarrow -\infty} \gamma_1(s) = a_1, \quad \lim_{i \rightarrow \infty} \gamma_2(s) = a_2, \quad \lim_{i \rightarrow \infty} \gamma_3(s) = a_3.$$

Observe that  $S$  is contractible (see Remark 2.3.1) and has a free  $\mathbb{Z}/2$ -action, so  $SB := S/(\mathbb{Z}/2) \cong \mathbb{RP}^\infty$ . Thus, there are representatives  $\delta_i$  of the nontrivial generator of  $H_i(SB) \cong \mathbb{Z}/2$  for each  $i$ . Strictly, we consider some  $\delta_i = \sum_j \tau_{i,j}$  where  $\tau_{i,j} : \Delta_i \rightarrow S$  is a simplex. For each  $i \geq 0$ , let

$$\mathcal{M}_{BC,i}(a_1, a_2, a_3) = \cup_j \tau_{i,j}^* \mathcal{M}_{BC}(a_1, a_2, a_3),$$

the union of the pullback of  $\mathcal{M}_{BC}(a_1, a_2, a_3)$  along  $\tau_{i,j} : \Delta_i \rightarrow S$ , glued along faces.

Recalling that in Morse theory, we may identify  $CM^*(M \times M, f+f)$  to  $CM^*(M, f) \otimes CM^*(M, f)$  via the Künneth isomorphism, induced in this case by the isomorphism on chains  $CM^*(M \times M, f+f) \cong CM^*(M, f) \otimes CM^*(M, f)$ , where

$$f+f : M \times M \rightarrow \mathbb{R}, \quad (f+f)(x, y) = f(x) + f(y).$$

This uses the correspondence between critical points of  $f+f$  and formal pairs of critical points of  $f$ , denoted  $a \otimes b$  for  $a, b \in \text{crit}(f)$ . The isomorphism between Morse and singular cohomology respects the involution that swaps the factors, and hence we may replace the equivariant cohomology of  $C^*(M) \otimes C^*(M)$ , denoted  $H_{\mathbb{Z}/2}^*(M \times M)$ , with the equivariant cohomology of  $CM^*(M, f) \otimes CM^*(M, f)$ . One can think of this in terms of equivariant Morse cohomology, as detailed in [29, Section 2], where for the given  $\mathbb{Z}/2$ -action all of the necessary transversality conditions are satisfied.

Using the fact that the equivariant chains are  $C_{\mathbb{Z}/2}^*(M \times M) = C^*(M \times M)[h]$ , as well as using the previous paragraph, elements of  $C_{\mathbb{Z}/2}^*(M \times M)$  may be written as a finite sum of  $(b \otimes c)h^j$  for some  $j \geq 0$  and  $b, c \in \text{crit}(f)$ . Let  $a \in \text{crit}(f)$  and  $\delta_i$  be the generator of  $H_i(SB) \cong \mathbb{Z}/2$  (because  $S$  is an  $E\mathbb{Z}/2$ ).

We define

$$q : H_i(SB) \otimes H_{\mathbb{Z}/2}^j(M \times M) \rightarrow H^{j-i}(M),$$

at the chain level, such that the coefficient of  $a$  in  $q(\delta_i \otimes (b \otimes c)h^k)$  is  $\# \mathcal{M}_{BC,i-k}(a, b, c)$ , when  $\mathcal{M}_{BC,i-k}(a, b, c)$  is a collection of points. Let  $q_i(b \otimes c) = q(\delta_i \otimes b \otimes c)$ . One defines

$$Sq^{|x|-i}(x) = q_i(x \otimes x).$$

**Remark 2.3.1.** We assume that  $M$  is compact, to show that  $S$  is contractible. Observe that using the map  $(f_1, f_2, f_3) \mapsto (f_1, f_2 - f_1, f_3 - f_1)$ , we see that

$$\text{Conf}_3(C^\infty(M)) \cong C^\infty(M) \times \text{Conf}_2(C^\infty(M) - \{0\}).$$

It is thus sufficient to show that  $\text{Conf}_2(C^\infty(M) - \{0\})$  is contractible, and we think of this intuitively as a bundle over  $C^\infty(M) - \{0\}$  where the fibre over  $g$  is  $C^\infty(M) - \{0, g\}$ . Pick  $m \in M$ . We can show that  $C^\infty(M) - \{0\}$  is contractible by observing that there is an injective, non-surjective map

$$T : C^\infty(M) - \{0\} \rightarrow C^\infty(M) - \{0\}, \quad f \mapsto \beta \cdot f,$$

where  $\beta : M \rightarrow \mathbb{R}$  is a bump function such that

$$\begin{aligned} \beta(M) &= [0, 1], \quad \beta(x) = 0 \text{ if and only if } x = m, \\ \text{and } \beta(x) &= 1 \text{ away from a small neighbourhood of } m. \end{aligned}$$

Then  $T$  is homotopic by linear interpolation to both the identity and the constant map with value 1, hence provides a deformation retraction to a point in  $C^\infty(M)$ .

Further, there is a homotopy equivalence from  $C^\infty(M) - \{0, g\}$  to  $C^\infty(M) - \{0\}$  that is continuous as we vary  $g$ . Just as we showed that  $C^\infty(M) - \{0\}$  is contractible in the previous paragraph, so can we prove that  $\overline{B(0, 1)} - \{0\}$ , the punctured closed ball around 0, is contractible. Then we apply this homotopy equivalence (scaled appropriately) to  $\overline{B(0, \|g\|/4)} - \{0\} \subset C^\infty(M) - \{0, g\}$  (extending this homotopy to be trivial outside of  $B(0, \|g\|/2)$ ) to obtain that  $C^\infty(M) - \{0, g\} \sim C^\infty(M) - \{g\}$ . Translating by  $-g$  yields a homotopy equivalence to  $C^\infty(M) - \{0\}$ , which we know is contractible. This induces a contraction of  $\text{Conf}_2(C^\infty(M) - \{0\})$  to a point.

**Remark 2.3.2.** Using the contraction of  $U_f$  to  $\{f\}$ , one can provide an isomorphism between the Morse cohomology  $HM^*(M, g)$  and  $HM^*(M, f)$  for each  $g \in U_f$ , induced by a bijection  $\text{crit}(f) \rightarrow \text{crit}(g)$ . This contraction is of the form

$$\mu : U_f \times [0, 1] \rightarrow U_f, \quad \mu(0, g) = g, \quad \mu(1, g) = f.$$

Consider the moduli space parametrised by  $\lambda \in [0, 1]$ , which for  $\sigma = (f_1, f_2, f_3) \in S$  we call  $\mathcal{M}_\sigma^L$ , consisting of  $\{(\lambda, \sigma, \gamma)\}$  such that  $\lambda \in [0, 1]$ , the maps  $f_i \in U_f$  are pairwise distinct and the continuous map  $\gamma : \Gamma \rightarrow M$  is such that  $\gamma_i$  is smooth and

$$d\gamma_i/dt = -\nabla (\beta(|s|)f_i + (1 - \beta(|s|))\mu(\lambda, f_i)).$$

Then in the case when  $\mathcal{M}_\sigma^L(a, b, c)$  is 1-dimensional, its boundary consists of the  $\lambda = 0, 1$  ends of the moduli space. The  $\lambda = 0$  is the setup as in [3], and the  $\lambda = 1$

is the setup given above. We may similarly define  $\mathcal{M}^L$ ,  $\mathcal{M}^L(a, b, c)$  and  $\mathcal{M}_i^L(a, b, c)$  parametrised by  $\lambda \in [0, 1]$  (where the latter is the pullback of  $\mathcal{M}^L(a, b, c)$  under the simplices as before). If  $|a| + i - |b| - |c| = 1$ , then  $\mathcal{M}_i^L(a, b, c)$  is a 1-cobordism between the moduli space due to Betz-Cohen and the moduli space above, so they have the same count.

**Remark 2.3.3** (Fukaya's Construction). We remark on Fukaya's construction, which is independent of the work by Betz-Cohen in this section, as background.

As stated by Fukaya in [10], this discussion of the Steenrod square is independent to. Fukaya further states a way to extend the Morse theoretic definition of the Steenrod square to quantum cohomology. In our case we work with an oriented manifold  $M$ . We define  $\overline{\mathcal{T}}_{0,3}$  to be the moduli space of metric trees, with three univalent vertices, no bivalent vertices and one trivalent vertex. It is immediate that this moduli space is contractible, and has a  $\mathbb{Z}/2$ -action. We now choose an arbitrary but fixed edge length, which we take to be 1, for the three edges of the unique graph  $\Gamma \in \overline{\mathcal{T}}_{0,3}$  ( $\Gamma$  is as above, the Y-shaped graph with one incoming and two outgoing edges).

Observe that embedding  $\Gamma$  into a disc of radius 1, the graph  $\Gamma$  separates the disc into three disjoint regions that we label 1, 2, 3. We construct a moduli space  $\mathcal{M}(M; f_1, f_2, f_3)$  which is the set of all piecewise smooth  $I : \Gamma \rightarrow M$  such that if  $I_i$  is the restriction of  $I$  to edge  $i$ , then  $\dot{I}_i = -\nabla(f_i - f_{i-1})$  (where the indices are cyclic modulo 3). There is an evaluation map

$$\pi_2 : \mathcal{M}(M; f_1, f_2, f_3) \rightarrow M \times M \times M$$

induced by evaluation at the three exterior vertices of  $\Gamma$ .

One lets  $\mathcal{M}_{0,3}$  be the union of  $\mathcal{M}(M; f_1, f_2, f_3)$  for all  $f_i$ . Then there is a map

$$\pi_1 : \mathcal{M}_{0,3} \rightarrow \overline{\mathcal{T}}_{0,3} \times (C^\infty(M))^3.$$

One reduces to a subspace  $\text{Conf}_3(C^\infty(M)) \subset (C^\infty(M))^3$  on which  $\mathbb{Z}/2$  acts freely. To see that this subspace is contractible, one proceeds as in Remark 2.3.1. Intuitively we may say that this holds because removing a closed subset of infinite codimension yields a space of the same homotopy type. Then  $\overline{\mathcal{T}}_{0,3} \times \text{Conf}_3(C^\infty(M))$  is an  $E\mathbb{Z}/2$ . For the moduli space  $\mathcal{M}_{0,3}$  above we restrict to  $(f_1, f_2, f_3) \in \text{Conf}_3(C^\infty(M))$ , denoting the new moduli space  $\overline{\mathcal{M}}_{0,3}$  but abusively using the same notation for the restricted maps  $\pi_1, \pi_2$ . Then if  $U$  is an  $(n - j)$ -cycle representing a cohomology class in  $H^j(M)$ , and  $[X_i]$  is the canonical generator of  $H_i(B\mathbb{Z}/2)$ , assuming everything is transverse we get a cycle

$$\pi_1^{-1} X_i \cap \pi_2^{-1} ((U \otimes U)/\mathbb{Z}/2).$$

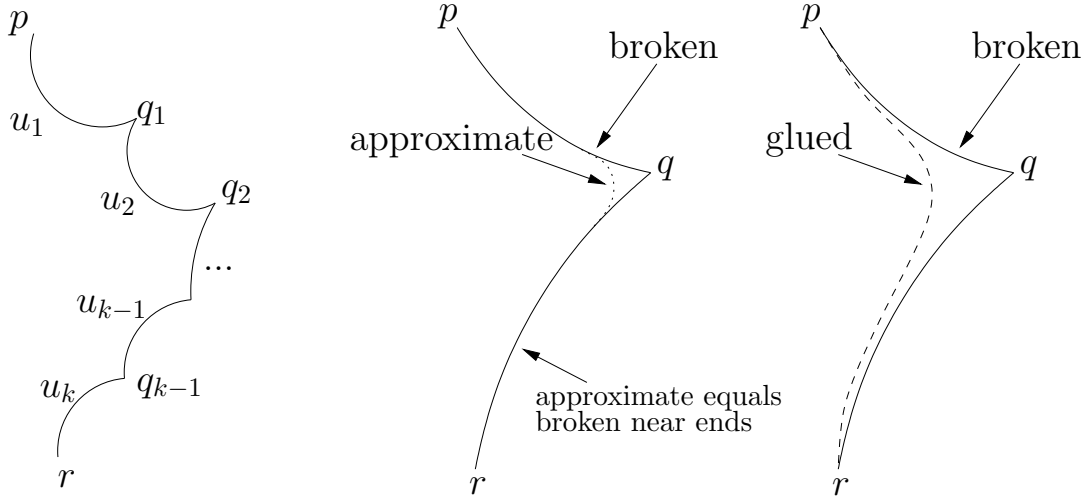


**Figure 2.3:** General broken flowlines, where  $q_i \in \text{crit}(f)$ , and gluing solutions.

*Fukaya also states that there is a universal family*

$$\text{Univ} \rightarrow \overline{\mathcal{T}}_{0,3} \times_{\mathbb{Z}/2} \text{Conf}_3(C^\infty(M))$$

*whose fibre is  $\Gamma$ , and contractibility of the tree allows us to choose a section  $s$ . One defines*

$$\pi_3 : \overline{\mathcal{M}}_{0,3} \rightarrow M, \quad \pi_3([I]) = I(s(I)).$$

*Then  $\pi_3(\pi_1^{-1}X_i \cap \pi_2^{-1}((U \otimes U)/\mathbb{Z}/2))$  represents  $Sq^{j-i}([U])$ .*

*The analogous construction of the quantum Steenrod square is not fully expanded on in the cited paper, but it would involve intersecting a  $J$ -holomorphic sphere with the three flowlines, instead of requiring that they intersect at the trivalent vertex of  $\Gamma$ . The construction of the quantum Steenrod square suggested by Fukaya is assumed to give the same answer as the one in Chapter 3, although this has not been proved. We will defer a description of Seidel's equivariant pair-of-pants and related constructions until Chapter 4.*

## 2.4 Gluing and compactness

This section will give an idea of the proofs of the technical problems that underpin Morse and Floer theory.

### 2.4.1 Gluing

In the following, a “solution”  $u$  is a smooth map  $u : R \rightarrow M$ , such that  $F(u) = 0$  (where  $F$  is as given below) and  $R$  is either  $\mathbb{R}$  (in the case of Morse theory) or  $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$

(in the case of Floer theory). In this context we define  $F$  by observing that a Morse flowline is a solution  $u$  to the equation

$$F(u) := \frac{du}{dt} - \nabla f(u(t)) = 0, \quad (2.4)$$

and the Floer trajectory is a solution  $u$  to the equation

$$F(u) := (du - (X_H \otimes dt) \circ u)^{0,1} = 0. \quad (2.5)$$

In the Floer case, locally near a Floer trajectory  $x$  we are actually using a function  $\mathcal{F}$  defined by  $F(u) = \Phi_x^{-1}(\exp(u))\mathcal{F}(\exp^{-1}u)$  where  $\exp$  is the exponential map, and  $\Phi_x(\xi)$  is parallel transport along the geodesic  $t \mapsto \exp_x(t\xi)$ .

To illustrate what is meant by a broken solution, consider the case of Morse theory. Given critical points  $p, q, r \in \text{crit}(f)$ , and a  $-\nabla f$  flowline  $u_1$  between  $p$  and  $q$ , and a  $-\nabla f$  flowline  $u_2$  between  $q$  and  $r$ , we call the pair  $(u_1, u_2)$  a “broken flowline” between  $p$  and  $r$ . In general a broken flowline consists of multiple flowlines  $(u_1, \dots, u_k)$  as in Figure 2.3. A gluing theorem answers the question: “if I am given a broken solution, is there an actual solution nearby?”

The idea is that given a broken solution we can first build an “approximate solution” nearby. For example, given a broken Floer trajectory  $(u, v)$  we may construct a smooth map  $u \#_R v : \mathbb{R} \times S^1 \rightarrow M$  that is not a Floer trajectory itself, but matches the broken Floer trajectory near to the ends. The  $R$  parameter tells us where we glue. See [20, Lecture 18, Section 5.5] for the Morse theoretic case, or [24, Section 3.3]. The next step of the argument is to show that near to any approximate solution there is an actual solution. This requires Fredholm theory, and it is the technical heart of the argument. The (vertical) derivative  $D$  of  $\mathcal{F}$  at 0 is Fredholm and surjective at the regular solutions  $u, v$  to  $F(u) = F(v) = 0$ . One needs to show that  $D$  also has a bounded right inverse  $Q_R$  at  $u \#_R v$ . This allows us to use an infinite dimensional implicit function theorem to locally write  $\mathcal{F}(\xi') = \mathcal{F}(\xi + Q_R \eta)$  for  $\xi \in \ker D$ . One then shows that the map  $\eta \mapsto \eta - \mathcal{F}(\xi + Q_R \eta)$  defines a contraction. Using the contraction mapping theorem shows that there is some  $\eta$  with  $\mathcal{F}(\xi + Q_R \eta) = 0$ , and hence  $\exp_{w_R}(\xi + Q_R \eta)$  is a solution.

We will be interested in a parametrized version of this gluing in this work. Note that the arguments follow similarly for the parametrized gluing, although this is something that one must check. We explore this more in Section 4.2.7.

**Remark 2.4.1** (Regularity). *Recalling from above that we took the (vertical) derivative  $D$  of  $\mathcal{F}$ . A regular solution  $u$  is one where  $D$  is surjective at  $u$ . Let  $\mathcal{E} \rightarrow \chi$  be a*

vector bundle with base  $\chi \subset C^\infty(\mathbb{R} \times S^1, M)$  (with appropriate conditions such that  $\chi$  is an infinite dimensional manifold with  $T_u\chi = C^\infty(u^*TM)$ ) and  $\mathcal{E}$  has the fibre at  $u$  being  $\Omega^{0,1}(u^*TM)$ . Recall that there is a section  $\bar{\partial}_J : \chi \rightarrow \mathcal{E}$ , which is the  $F$  from equation (2.5). Then  $u$  being  $J$ -holomorphic means that  $\bar{\partial}_Ju = 0$ . Hence the moduli space of  $J$ -holomorphic maps is  $\bar{\partial}_J^{-1}(0)$ . This moduli space is a manifold when  $\bar{\partial}_J$  is transversal to the 0-section. This is equivalent to the condition that  $D$  is surjective at all  $u$ , hence this is why regularity is important. This holds similarly for Hamiltonian perturbations of the Cauchy-Riemann equation, i.e. Equation (2.5). One can show that a generic choice of  $J$  ensures that all Floer trajectories are regular. See [14, Chapter 3].

## 2.4.2 Compactness

The compactness question is the converse of the previous, asking: “if I am given a family of solutions, is there a subfamily converging to a broken solution?” The idea here is to use the notion of “ $C^\infty$ -local convergence on compact subsets”. This convergence means exactly that  $u_n \rightarrow u$  exactly when  $u_n|_X \rightarrow u|_X$  uniformly in the  $C^\infty$  topology for every compact  $X \subset \mathbb{R}$ . Consider the Morse theoretic case: as in [20, Lecture 17], we have defined  $\mathcal{M}(p, q)$  to be the quotient of  $W(p, q) = W^u(p, f) \cap W^s(q, f)$  by reparametrising the flowlines by translation, i.e.  $\mathbb{R}$ . Hence a sequence  $[u_n] \rightarrow [u] \in W(p, q)/\mathbb{R}$  exactly when  $u_n(\cdot + s_n) \mapsto u(\cdot) \in W(p, q)$  for some sequence  $(s_n) \in \mathbb{R}$ . One can use this to show that there is a natural way to compactify the space of Morse flowlines  $\mathcal{M}(p, r)$  (when the index difference  $|p| - |r| = 2$ ) by adding in broken trajectories  $(u_1, u_2)$  as in the previous section. This is illustrated in Figure 2.3. As this is a well-trodden technical result, the details are omitted in this thesis. An important note for the Morse case is that for a general index difference we must add multiply broken trajectories  $(u_1, u_2, \dots, u_k)$ . To prove this, one shows first that a sequence  $u_n(\cdot)$  converges on compact subsets to some trajectory (by showing that  $W := \bigcup_{p, q \in \text{crit}(f)} W(p, q)$  is  $C_{\text{loc}}^\infty$ -compact using the Arzelà-Ascoli theorem). Hence  $(u_n) \in W(p_0, q_0)$  has some convergent subsequence, converging to  $u_\infty \in W(p_1, q_1)$  for some  $(p_1, q_1)$ . One shows that

$$f(q_0) \leq f(q_1) < f(p_1) \leq f(p_0).$$

Either  $q_0 = q_1$  and  $p_1 = p_0$ , or we can pick one of  $c \in (f(p_1), f(p_0))$  or  $c \in (f(q_0), f(q_1))$  and show there is a sequence  $(s_n) \in \mathbb{R}$  with  $u_n(s_n) \rightarrow c$ . The sequence  $u_n(\cdot + s_n)$  then converges in  $W$  to some limit  $u' \in W(p_2, q_2)$ , such that:

$$f(q_0) \leq f(q_2) < f(p_2) \leq f(q_1) \text{ or } f(p_1) \leq f(q_2) < f(p_2) \leq f(p_0).$$

We proceed to cover the interval  $[f(q_0), f(p_0)]$  in this way.

There is more subtlety in the case of Floer trajectories: see [24, Section 3.1]. There, we ask for what is known as “convergence modulo bubbling”. The idea here is that a sequence of Floer trajectories (with uniformly bounded energy) converges uniformly on compact subsets except perhaps at isolated points  $\{z_1, \dots, z_k\}$ , which are points where the “energy density” diverges, or rather points  $z \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$  such that there is  $\hbar > 0$  with

$$\liminf_{n \rightarrow \infty} E(u_n, B_\epsilon(z)) \geq \hbar$$

for all  $\epsilon > 0$ , where  $E(u, D)$  is the energy of the map  $u|_D$ . We can think of this as points where energy “accrues” in the limit. It makes sense that only finitely many such  $z_i$  exist, because the energy of the solutions is uniformly bounded and each of the  $z_i$  is an accruing point for an amount of energy bounded below by  $\hbar$ . There are different ways to compactify this space in general, but the idea of Gromov compactness is that at these points  $z_i$  we attach a (nodal)  $J$ -holomorphic sphere of total energy

$$\lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} E(u_n, B_\epsilon(z)).$$

The various conditions that we place on our spaces, for example monotone, weakly monotone or exact, are used to ensure that these  $J$ -holomorphic spheres must be constant (once we impose a constraint on the index of the Floer solutions) and therefore will not be a problem for our counting of moduli spaces.

# Chapter 3

## The Quantum Steenrod Square

### 3.1 Introduction

We begin with the background of the Steenrod squares. We will then mention quantum cohomology and the results in this chapter. The Steenrod squares are cohomology operations that are uniquely defined by a set of axioms, although this uniqueness does not include a construction. There are multiple ways of constructing the squares, one of which involves constructing the operations on  $H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$  for Eilenberg-MacLane spaces  $K(\mathbb{Z}/2, n)$ .

For a topological space  $M$ , the Steenrod squares are additive homomorphisms

$$Sq^i : H^n(M) \rightarrow H^{n+i}(M)$$

using  $\mathbb{Z}/2$  coefficients. They generalize the squaring operation on cohomology with respect to the cup product,  $x \mapsto x \cup x$ . The  $\mathbb{Z}/2$  coefficients ensure that the  $Sq^i$  are additive. These Steenrod squares together determine a degree doubling operation that we call the Steenrod square,

$$Sq : H^*(M) \rightarrow H^*(M)[h],$$

where

$$Sq(x) = \sum Sq^{|x|-i}(x) h^i.$$

Here  $h$  is a formal variable in degree 1 that represents the generator of

$$H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \frac{\mathbb{Z}}{2}[h].$$

The Steenrod square satisfies the Cartan relation

$$Sq(x \cup y) = Sq(x) \cup Sq(y) \tag{3.1}$$

which, for example, allows one to inductively compute the Steenrod squares for the cohomology of  $\mathbb{CP}^n$  (which we will review in example 3.2.4). The Steenrod square also satisfies the Adem relations, which are relations between compositions of the  $Sq^i$ . Namely, for all  $p, q > 0$  such that  $q < 2p$ ,

$$Sq^q Sq^p = \sum_{s=0}^{\lfloor q/2 \rfloor} \binom{p-s-1}{q-2s} Sq^{p+q-s} Sq^s \quad (3.2)$$

where  $\lfloor i/2 \rfloor$  is the integer part of  $i/2$ . The Adem relations are classically implied by the axioms.

This chapter begins in Section 3.2 with a preliminary section that explains in more detail the relevant background material.

We will then describe two different constructions of the Steenrod square in Section 3.3: the first construction uses Morse homology and the second uses intersections of cycles. The first construction is based on the definition for Floer theory by Seidel in [28], the origins of which are in the flowlines construction of Betz in [4], and Fukaya in [10], the former of which was extended to a more categorical definition by Betz, Cohen and Norbury in [3, 8]. The second construction we give will be isomorphic to the first construction, using the isomorphism between Morse and singular cohomology.

After considering these constructions of the Steenrod square, in Section 3.4 we extend them to define a quantum Steenrod square on the quantum cohomology of a closed monotone symplectic manifold  $(M, \omega)$ .

In Section 3.2.3 we will give details of the quantum cohomology  $QH^*(M)$ . Briefly,  $QH^*(M, \omega)$  is  $H^*(M)[[t]]$  as a vector space using a graded formal variable  $t$  of degree 2. However, the cup product is deformed by quantum contributions from counting 3-pointed genus zero Gromov-Witten invariants. That is, by counting certain  $J$ -holomorphic spheres in  $M$  where  $J$  is an almost complex structure on  $M$  compatible with  $\omega$ . We often abbreviate by  $T = t^N$  for  $N$  the minimal Chern number.

The quantum Steenrod square will be a degree doubling operation, denoted  $Q\mathcal{S}$ , where

$$Q\mathcal{S} : QH^*(M) = H^*(M)[[t]] \rightarrow H^*(M)[[t]][h] = QH^*(M)[h]. \quad (3.3)$$

As in the case of the classical Steenrod square,  $Q\mathcal{S}$  will be built using additive homomorphisms  $Q\mathcal{S}_{i,j} : QH^*(M) \rightarrow QH^{2*-i-2jN}(M)$ , so

$$Q\mathcal{S}(x) = \sum_{i,j \geq 0} Q\mathcal{S}_{i,j}(x) h^i T^j.$$

The quantum Steenrod square is not necessarily axiomatically defined, but a construction was first suggested by Fukaya in [10] based on his Morse homotopy theory.

Our construction is different from Fukaya's, and can be viewed as a Morse theory analogue of the work by Seidel in Floer theory in [28]. The first goal of this chapter is to solve an open problem posed by Fukaya in [10, Problem 2.11] as to whether the Adem and Cartan relations hold for quantum Steenrod squares and, if not, what their quantised versions should be. Our second goal is to explore consequences of the solution to this problem, specifically in computations for certain closed monotone symplectic manifolds.

In answer to the first part of Fukaya's problem, the immediate generalisation of the Cartan and Adem relations fail. In the case of the Cartan relation, this means that it is not in general true that  $Q\mathcal{S}(x * y) = Q\mathcal{S}(x) * Q\mathcal{S}(y)$ . We will show this in the following example.

**Example 3.1.1.** *In Definition 3.4.1 of the quantum Steenrod square,*

$$Q\mathcal{S}(aT) = Q\mathcal{S}(a)T^2 \tag{3.4}$$

*for any  $a \in QH^*(M)$ .*

*Let  $M = \mathbb{P}^1$ . Let  $x$  be the generator of  $H^2(M)$ . Recall that the quantum product is  $x * x = T$ , where  $T$  has degree 4. Then*

$$Q\mathcal{S}(x * x) = Q\mathcal{S}(T) = T^2,$$

*using (3.4). Using degree reasons, knowledge of the classical Steenrod square and the quantum cohomology ring, one can show that  $Q\mathcal{S}(x) = xh^2 + T$ . Then*

$$Q\mathcal{S}(x) * Q\mathcal{S}(x) = (xh^2 + T) * (xh^2 + T) = Th^4 + T^2.$$

*In this case  $Q\mathcal{S}(x) * Q\mathcal{S}(x) \neq Q\mathcal{S}(x * x)$ .*

In Section 3.5 we will prove why the Cartan relation does not immediately generalise and compute the actual quantum Cartan relation. Briefly, the quantum Cartan relation is deformed because the moduli space  $\overline{M}_{0,5}$  of genus zero stable curves with 5 marked points  $(z_0, z_1, z_2, z_3, z_4)$  has non-trivial  $\mathbb{Z}/2$ -equivariant cohomology, under the  $\mathbb{Z}/2$  action that transposes marked points via the permutation (12)(34). More precisely, the two configurations in Figure 3.6 that determine  $Q\mathcal{S}(x * y)$  and  $Q\mathcal{S}(x) * Q\mathcal{S}(y)$  are not connected by a  $\mathbb{Z}/2$ -invariant path in  $\overline{M}_{0,5}$ .

We will prove that a quantum deformation of the Cartan relation holds:

**Theorem 3.1.2** (Quantum Cartan relation).

$$Q\mathcal{S}(x * y) = Q\mathcal{S}(x) * Q\mathcal{S}(y) + \sum_{i,j} q_{i,j}(W_0 \times D^{i-2,+})(x, y)h^i$$

where the correction term is written in terms of linear homomorphisms

$$q_{i,j} : H_*^{\mathbb{Z}/2}(\overline{M_{0,5}}) \otimes QH^*(M) \otimes QH^*(M) \rightarrow QH^*(M)[h],$$

with all terms of positive  $h$ -power. The  $q_{i,j}$  will be defined precisely in Definition 3.5.2.

In the correction term,

$$W_0 \times D^{i,+} \subset \overline{M_{0,5}} \times_{\mathbb{Z}/2} S^\infty,$$

where  $W_0 \subset \overline{M_{0,5}} \simeq Bl_{\{(0,0),(1,1),(\infty,\infty)\}}(\mathbb{CP}^1 \times \mathbb{CP}^1)$  is the exceptional divisor over  $(0,0)$  (compare to Figure 3.7).

In Section 3.6 we use Theorem 3.1.2 to calculate the quantum Steenrod squares for a Fano toric variety  $M$ , as follows in Theorem 3.1.3. Here, for  $\mu \in H_2(M; \mathbb{Z})$  (which is a free  $\mathbb{Z}$ -module as  $M$  is simply connected), let  $\mu_2$  be the image of  $\mu$  under  $H_2(M; \mathbb{Z}) \rightarrow H_2(M; \mathbb{Z}/2)$ . Denote by  $x *_{\mu,k} y$  the coefficient of  $t^{kN}$  in the quantum product  $x * y$ , using spheres representing  $\mu$ . Let  $N$  be the minimal Chern number, and  $|t| = 2$ .

**Theorem 3.1.3.** *Let  $M$  be a Fano toric manifold. For  $b, x \in H^*(M)$  and  $|x| = 2$ ,*

$$q_{i,j}(W_0 \times D^{i,+})(b, x) = \sum_{j \geq 1} \sum_{k=1}^j \sum_{c_1(\mu)=2kN} n(x, \mu_2) \cdot (Q\mathcal{S}_{|b|-i+2,j-k}(b) *_{\mu,k} x) \cdot t^{jN} \quad (3.5)$$

summing over a basis of  $\mu \in H_2(M; \mathbb{Z})$ , so  $c_1(\mu) = 2kN \in \mathbb{Z}$  and  $n(x, \mu_2) := \#(PD(x) \cap \mu_2) \in \mathbb{Z}/2$ .

For example, if  $M = \mathbb{CP}^n$  then setting  $b = x^i$  for the generator  $x \in H^2(\mathbb{CP}^n)$ , we will show in Lemma 3.6.1 that:

$$q_{4i+2-2n,1}(W_0 \times D^{4i-2n,+})(x^i, x) = \binom{i}{n-i} T, \text{ else } q_{i,j} = 0.$$

Hence

$$Q\mathcal{S}(x^i) = \sum_{j=0}^i \left( \binom{i}{j} + \sum_{k=0}^{\lfloor n/2 \rfloor + 1} \binom{n-k}{k} \cdot \binom{i-(n+1-k)}{j-k} \right) x^{i+j} h^{2(i-j)}$$

for  $x^p$  the  $p$ -th quantum power of  $x$ . Omitting the inner summation would give the classical Steenrod square.

**Corollary 3.1.4.** *Let  $M$  be a Fano toric manifold. Then if we can compute  $QH^*(M)$  (over the Novikov ring as in [14, Section 9.2]) then we can compute  $QS$ .*

In Section 3.7 we extend the Adem relations from Equation (3.2) to the quantum Steenrod square. We use the following definition, where here upper indices for  $QS$  determine the change in index of the cohomological index and the coefficient of  $T$ .

**Definition 3.1.5.** *Define  $QS^{a,b}$  by*

$$QS(xT^i) = \sum_{a,b \in \mathbb{Z}} QS^{a,b}(xT^i) \cdot h^{|x|-2N(b+i)-a} \quad \text{where} \quad QS^{a,b}(xT^i) \in T^{b+i} H^{|x|+a}(M),$$

for any  $x \in H^*(M)$ .

Computing  $QS^{a,b}$  for  $\mathbb{CP}^2$ , we find that the naive generalisation of the Adem relation (Equation (3.2)), namely

$$\sum_{b,d} \left( QS^{q-2bN,b} \circ QS^{p-2dN,d}(\alpha) - \sum_{s=0}^{q/2} \binom{p-s-1}{q-2s} QS^{p+q-s-2bN,b} \circ QS^{s-2dN,d}(\alpha) \right) = 0,$$

does not hold, as in the example below.

**Example 3.1.6.** *Let  $M = \mathbb{CP}^2$ , so  $2N = 6$ . Then  $QS^{2-2N,1} \circ QS^{2-0N,0}(x) = T$ , but*

$$\sum_{s=0}^{s=1} \binom{1-s}{2-2s} QS^{2+2-s-2iN,i} \circ QS^{s-2jN,j}(x) = 0$$

for all  $i, j$ .

There is the technical Theorem 3.1.7, the terminology of which will be fully defined in Section 3.7.2. Much of this is similar to the previous terminology. There is a homomorphism

$$qq : H^*(M) \rightarrow H^*(M) \otimes H^*(BD_8)$$

where  $D_8$  is the dihedral group (compare to the quantum Steenrod square, which uses  $\mathbb{Z}/2$  instead of  $D_8$ ). This  $qq$  operation will correspond to the composition  $QS \circ QS$ . Then  $H^*(BD_8)$  has three generators (as a ring), labelled  $e, \sigma_1, \sigma_2$ , of which we only need to consider  $e$  and  $\sigma_2$ . We denote by  $qq_{i,j}(\alpha)$  the coefficient of  $e^i \sigma_2^j$  in  $qq(\alpha)$ , defined in Equation (3.26).

**Theorem 3.1.7.** *For  $M$  a closed monotone symplectic manifold, with  $\alpha \in QH^*(M)$ , and for  $p, q > 0$  such that  $q < 2p$ :*

$$qq_{|\alpha|+p-q, |\alpha|-p}(\alpha) = \sum_{s=0}^{q/2} \binom{p-s-1}{q-2s} qq_{|\alpha|+2s-p-q, |\alpha|-s}(\alpha).$$

This should be compared to equation (3.2). The above theorem leads to a Corollary in more familiar terms:

**Corollary 3.1.8** (Quantum Adem Relations). *For  $p, q > 0$  such that  $q < 2p$ , and  $\alpha \in QH^*(M)$ ,*

$$\sum_{b,d} \left( Q\mathcal{S}^{q,b} \circ Q\mathcal{S}^{p,d}(\alpha) - \sum_{s=0}^{q/2} \binom{p-s-1}{q-2s} Q\mathcal{S}^{p+q-s,b} \circ Q\mathcal{S}^{s,d}(\alpha) \right) = T \cdot Q(\alpha) \quad (3.6)$$

for the correction term

$$\begin{aligned} T \cdot Q(\alpha) &= q_{D_8}((gm_1 + g^2m_1) \otimes \Psi(e^{|\alpha|+p-q}\sigma_2^{|\alpha|-p}))(\alpha) \\ &\quad - \sum_{s=0}^{[q/2]} \binom{p-s-1}{q-2s} q_{D_8}((gm_1 + g^2m_1) \times \Psi(e^{|\alpha|+2s-p-q}\sigma_2^{|\alpha|-s}))(\alpha). \end{aligned}$$

In the above corollary, the dihedral group  $D_8 = \langle (12), (13)(24) \rangle$  acts on the four incoming marked points  $(z_1, z_2, z_3, z_4)$  by permutations. The operation  $q_{D_8}$  is a linear homomorphism determined by homology classes in  $\overline{M}_{0,5} \times_{D_8} ED_8$ , so  $q_{D_8}(A) : H^*(M) \rightarrow H^{4*-|A|}(M)$  for  $A \in H_*(\overline{M}_{0,5} \times_{D_8} ED_8)$ . It is analogous to the  $q_{i,j}$  in Theorem 3.1.2. Here  $m_1 \in \overline{M}_{0,5}$  (see Figure 3.6),  $e^i\sigma_2^j \in H^*(BD_8)$ ,  $\Psi : H^*(\overline{M}_{0,5} \times_{D_8} ED_8) \rightarrow H_*(\overline{M}_{0,5} \times_{D_8} ED_8)$  is the universal coefficients isomorphism and  $g = (123) \in S_4$  such that the cosets of  $D_8$  in  $S_4$  are  $D_8, gD_8, g^2D_8$ .

In Section 3.8, we calculate  $Q\mathcal{S}$  in the case of the blowups  $M = Bl_Y(\mathbb{CP}^3)$  and  $M = Bl_Y(\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1)$  where  $Y$  is respectively the intersection of two quadrics and the intersection of two linear hypersurfaces. The setup here is similar to Blaiher [5]. Most of the squares can be computed using the methods from Section 3.6. The new computation is of  $Q\mathcal{S}_{1,1}$ , which is given in the following theorem.

**Theorem 3.1.9.**

$$Q\mathcal{S}_{1,1} = id : H^3(Bl_Y(\mathbb{CP}^3)) \rightarrow H^3(Bl_Y(\mathbb{CP}^3)) \quad (3.7)$$

and

$$Q\mathcal{S}_{1,1} = id : H^3(Bl_Y(\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1)) \rightarrow H^3(Bl_Y(\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1)). \quad (3.8)$$

Observe that  $Q\mathcal{S}_{1,1}$  are quantum correction terms to the classical Steenrod square on the blowup  $M$ . They are determined by lifts of contributions to the classical Steenrod square on  $Y$ .

## 3.2 Preliminaries

Henceforth we always work with coefficients in  $\mathbb{Z}/2$ . For example  $H^*(M)$  means  $H^*(M; \mathbb{Z}/2)$ .

### 3.2.1 Equivariant Cohomology

We follow [28, Section 2].

**Definition 3.2.1** (Equivariant cohomology of a chain complex). *Let  $(C^\bullet, d)$  be a cochain complex over  $\mathbb{Z}/2$ . Suppose  $(C^\bullet, d)$  has a chain involution  $\iota$ , so  $\iota : C^\bullet \rightarrow C^\bullet$  is a chain map with  $\iota^2 = id_{C^\bullet}$ . Let  $h$  be a formal variable in grading 1. The equivariant chain complex is*

$$(C_{\mathbb{Z}/2}^\bullet, \delta) = (C^\bullet[[h]], d + h(id_C + \iota)).$$

*Define  $H_{\mathbb{Z}/2}^*(C) := H^*(C_{\mathbb{Z}/2}^\bullet, \delta)$ , the equivariant cohomology of  $(C, d, \iota)$ .*

**Definition 3.2.2** (Equivariant Cohomology of a manifold). *Let  $N$  be a topological space with a continuous involution  $\iota : N \rightarrow N$ . Let  $C = C^*(N)$  be the singular cochain complex of the topological space  $N$ . There is a  $\mathbb{Z}/2$  action on  $C^*(N)$  induced by  $\iota$ . As in Definition 3.2.1, define the equivariant cohomology of  $N$  as  $H_{\mathbb{Z}/2}^*(N) := H_{\mathbb{Z}/2}^*(C^*(N))$ .*

The important examples of this will be  $M^2$  with the involution swapping the factors and  $M$  with the trivial involution.

**Remark 3.2.3.** *There is another description of  $H_{\mathbb{Z}/2}^*(N)$  for a manifold  $N$  with a continuous involution. Recall that  $E\mathbb{Z}/2$  is the classifying space of  $\mathbb{Z}/2$ : a contractible space with a free  $\mathbb{Z}/2$  action, for example  $E\mathbb{Z}/2 = S^\infty$  with the involution being the antipodal map. Then*

$$H_{\mathbb{Z}/2}^*(N) := H^*(N \times_{\mathbb{Z}/2} E\mathbb{Z}/2).$$

*This definition is the same as the previous one. If we let  $N = \{pt\}$  then we obtain  $pt \times_{\mathbb{Z}/2} S^\infty = S^\infty / (\mathbb{Z}/2) = \mathbb{RP}^\infty$ , hence*

$$H_{\mathbb{Z}/2}^*(pt) = H^*(\mathbb{RP}^\infty) = \mathbb{Z}/2[h].$$

### 3.2.2 The Steenrod Squares

For a reference, see [12, Section 4.L].

The Steenrod square operations  $\{Sq^i\}$  are the unique collection of additive homomorphisms such that:

1.  $Sq^i : H^n(M) \rightarrow H^{n+i}(M)$  for each  $n \geq 0$  and topological space  $M$ ,
2. Each  $Sq^i$  is natural in  $M$ ,
3.  $Sq^0$  is the identity,
4.  $Sq^n$  acts as the cup square on  $H^n$ , so  $Sq^{|x|}(x) = x \cup x$ ,
5. If  $n > |x|$  or  $n < 0$  then  $Sq^n(x) = 0$ ,
6. (Cartan relation) For each  $n$ ,

$$Sq^n(x \cup y) = \sum_{i+j=n} Sq^i(x) \cup Sq^j(y).$$

Here  $|x|$  is the cohomological grading of  $x \in H^*(M)$ . Recall that we use  $\mathbb{Z}/2$  coefficients to ensure additivity:  $(x + y) \cup (x + y) = x \cup x + y \cup y$  modulo 2. These  $Sq^i$  together define a single operator  $Sq : H^*(M) \rightarrow (H^*(M)[h])^{2*}$ , where  $Sq^i$  is the coefficient of  $h^{n-i}$ , so  $Sq(x) = \sum_i Sq^i(x) \cdot h^i$ . The cup product on  $H^n(M)[h]$  is  $(a \cdot h^i) \cup (b \cdot h^j) = (a \cup b) \cdot h^{i+j}$ , so the Cartan relation becomes  $Sq(x \cup y) = Sq(x) \cup Sq(y)$  and thus  $Sq$  is a unital ring homomorphism.

One must note that although these axioms imply uniqueness, one also needs to prove existence by demonstrating a construction.

**Example 3.2.4** (The classical Steenrod square for  $\mathbb{CP}^n$ ).

$$H^*(\mathbb{CP}^n) \cong \frac{\mathbb{Z}}{2}[x]/(x^{n+1})$$

where  $|x| = 2$ . We see that  $Sq^0(x) = x$  and  $Sq^2(x) = x^2$  using axioms 3 and 4, and these are all of the nonzero terms by axiom 5. Hence  $Sq(x) = xh^2 + x^2$ . By the Cartan relation (axiom 6),

$$Sq(x^i) = Sq(x)^i = (xh^2 + x^2)^i = x^i \sum_{j=0}^i \binom{i}{j} x^j h^{2(i-j)}.$$

Looking at the coefficient of  $h^{2i-k}$ ,  $Sq^k(x) = 0$  for  $k$  odd and  $Sq^{2j}(x) = \binom{i}{j} x^{i+j}$ .

### 3.2.3 The Quantum Cup Product

For more details on the quantum cup product, see [14, Chapter 8]. Throughout this chapter,  $PD$  refers to the Poincaré duality operation over  $\mathbb{Z}/2$  coefficients for closed manifolds.

**Definition 3.2.5.** *A symplectic manifold  $M$  is monotone if there exists a constant  $\lambda > 0$  such that every  $J$ -holomorphic sphere  $u : S^2 \rightarrow M$  satisfies*

$$c_1(u_*([S^2])) = \lambda \cdot E(u)$$

where  $E(u) = \int_{S^2} u^* \omega \geq 0$  is the symplectic energy of  $u$  and  $c_1 = c_1(TM, J)$ .

Let  $(M, \omega)$  be a monotone symplectic manifold of dimension  $n$ , with a fixed almost complex structure  $J$  compatible with  $\omega$ . As an abelian group,  $QH^*(M) = H^*(M)[[t]]$  where  $t$  is a formal variable of degree 2. Let  $T = t^N$  for  $N \geq 0$  the minimal Chern number of  $M$  determined by  $c_1(\pi_2(M)) = N\mathbb{Z}$ . Throughout we assume  $\lambda = N$ .

As an important note, we define the quantum cochains

$$QC^*(M) := C^*(M) \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[[T]].$$

Then  $QH^*(M) = H^*(QC^*(M), d \otimes id)$ , where  $d$  is the differential on  $C^*(M)$ . Most of the operations are defined at the chain level, and then descend to maps on (co)homology.

We pick a basis  $\mathcal{B}$  for  $H^*(M)$  and a dual basis with respect to the nondegenerate cup product pairing  $(e, f) \mapsto \langle e \cup f, [M] \rangle$ . There is a dual basis  $\mathcal{B}^\vee$  with respect to this pairing. Let  $\alpha^\vee \in H^{n-|\alpha|}(M)$  denote the dual of the cohomology class  $\alpha \in H^{|\alpha|}(M)$ . Our operations on cohomology will not depend on this choice of basis, although they may affect the chain level description.

Given  $A \in H_2(M)$ , let  $\mathcal{M}_A(J)$  be the moduli space of  $J$ -holomorphic spheres  $u : S^2 \rightarrow M$  such that  $u_*([S^2]) = A$ , up to reparametrisation by  $PSL(2, \mathbb{C})$ . For a generic choice of  $J$ , this moduli space is a smooth manifold with

$$\dim \mathcal{M}_A(J) = 2c_1(A) + \dim(M).$$

For each  $z \in S^2$ , there is an evaluation map  $ev_{A,z} : \mathcal{M}_A(J) \rightarrow M$  with  $ev_{A,z}(u) = u(z)$ . Pick three distinct points,  $z_1, z_2, z_3 \in S^2$ . We use  $0, 1, \infty$  throughout.

**Definition 3.2.6** (Quantum Product). *For  $\alpha, \beta \in H^*(M) \subset QH^*(M)$ ,*

$$\alpha * \beta = \sum_{j \in \mathbb{Z}, \gamma \in \mathcal{B}: |\gamma| = |\beta| + |\alpha| - 2jN} n(\gamma, \alpha, \beta, j) \cdot \gamma \cdot T^j,$$

$$n(\gamma, \alpha, \beta, j) = \sum_{A \in H_2(M); c_1(A) = jN} \# ev_{A, z_1}^{-1}(PD(\alpha)) \cap ev_{A, z_2}^{-1}(PD(\beta)) \cap ev_{A, z_3}^{-1}(PD(\gamma^\vee)).$$

Here  $\#$  denotes the mod 2 count of a 0-dimensional compact manifold. Extending  $\mathbb{Z}/2[[t]]$ -linearly defines  $*$  on  $QH^*(M)$ .

In concrete terms we count the number of  $J$ -holomorphic spheres in  $M$  intersecting the cycles  $PD(\alpha)$ ,  $PD(\beta)$  and  $PD(\gamma^\vee)$ . The degree condition ensures that the intersection is a 0-dimensional compact submanifold of  $\mathcal{M}_A(J)$  (for generic  $J$ , the evaluation maps satisfy the necessary transversality conditions, [14]). Notice that  $*$  is compatible with the grading, using  $|T| = 2N$ . If  $A = 0$  (so  $E(u) = 0$  and  $u$  is constant), the intersection is a point and this recovers the classical intersection product.

### 3.3 Two constructions of the Steenrod Squares

The first construction will be based on that given in [28], [10] and [3]. The second construction is based on the first. In this section  $\Gamma$  is the Y-shaped graph with incoming edge  $e_1$  and outgoing edges  $e_2$  and  $e_3$ . Let  $e_1$  be parametrised by  $(-\infty, 0]$  and  $e_2, e_3$  by  $[0, \infty)$ .

Throughout this section,  $M$  will be a closed smooth manifold. We recall the Morse theoretic cup product: given a Morse function  $f$ , pick three generic perturbations  $f_s^1$  for  $s \in (-\infty, 0]$  and  $f_s^2, f_s^3$  for  $s \in [0, \infty)$ , transverse at 0 (genericity ensures the moduli space below is cut out transversely). There is an  $R > 0$  such that  $f_s^i = f$  if  $|s| \geq R$ . Denote by  $\text{crit}_k(f)$  the critical points of  $f$  of Morse index  $k$ . Write  $|x|$  for the Morse index of  $x \in \text{crit}(f)$ .

**Definition 3.3.1** (Morse cup product). *Let  $a_2, a_3$  be critical points of  $f$ , with respective Morse indices  $|a_2|, |a_3|$  and let  $k = |a_2| + |a_3|$ , then*

$$a_2 \cdot a_3 := \sum_{a_1 \in \text{crit}_k(f)} n_{a_1, a_2, a_3} a_1$$

where  $n_{a_1, a_2, a_3}$  is the number of solutions in the 0 dimensional moduli space  $\mathcal{M}(f_s^i, a_1, a_2, a_3)$  of continuous maps  $u : \Gamma \rightarrow M$ , smooth on the edges, such that:

1.  $d(u|_{e_i})/ds = -\nabla f_s^i$ ,
2.  $u|_{e_1}(x) \rightarrow a_1$  as  $x \rightarrow -\infty$ ,
3.  $u|_{e_i}(x) \rightarrow a_i$  as  $x \rightarrow \infty$  for  $i = 2, 3$ .

If  $a_2 = a_3$  then there is a  $\mathbb{Z}/2$  symmetry in maps  $u : \Gamma \rightarrow M$  with conditions as above, which one exhibits to define Steenrod squares as in the next section.

### 3.3.1 Morse Steenrod square

We refine the choice of  $f_s^i$  by picking a collection of smooth functions  $f_{v,s}^i : M \rightarrow \mathbb{R}$ , smoothly parametrised by  $v \in S^\infty$  and  $s \in (-\infty, 0]$  for  $i = 1$ , respectively  $s \in [0, \infty)$  for  $i = 2, 3$ , satisfying the following conditions:

1.  $f_{v,s}^2 = f_{-v,s}^3$ , and  $f_{v,s}^2$  is generic near  $s = 0$  so that the moduli space below is cut out transversely.
2. There is an  $R > 0$  such that  $f_{v,s}^i = f$  for  $|s| \geq R$ .
3.  $f_{v,s}^1 = f_{-v,s}^1$ .

For each  $v \in S^\infty$  we define  $\mathcal{M}'_v(a_1, a_2, a_3)$  to be the set of pairs  $(u : \Gamma \rightarrow M, v)$  such that:

1.  $d(u|_{e_i})/ds = -\nabla f_{v,s}^i$ .
2.  $u|_{e_1}(x) \rightarrow a_1$  as  $s \rightarrow -\infty$  and  $u|_{e_i}(x) \rightarrow a_i$  for  $i = 2, 3$  as  $s \rightarrow \infty$ .

In this section we assume  $a_2 = a_3$  as in figure 3.1, and we abbreviate  $\mathcal{M}'_v(a_1, a_2) := \mathcal{M}'_v(a_1, a_2, a_2)$ . We next choose a nested sequence  $S^0 \subset S^1 \subset \dots \subset S^\infty$  of equators in  $S^\infty$  that exhaust  $S^\infty$  and are preserved under the involution  $v \mapsto -v$ . Let

$$\mathcal{M}'_i(a_1, a_2) = \bigsqcup_{v \in S^i} \mathcal{M}'_v(a_1, a_2),$$

topologised as a subset of  $C(\Gamma, M) \times S^i$  (where  $C(\Gamma, M)$  is the space of maps from  $\Gamma$  to  $M$  that are continuous and smooth on the edges). The projection to  $S^i$  is then continuous for all  $i$ .

Let  $r : \Gamma \rightarrow \Gamma$  be the reflection that swaps  $e_2$  and  $e_3$  (preserving parametrisations) and fixes  $e_1$ . There is a free  $\mathbb{Z}/2$  action on the moduli space  $\mathcal{M}'_i(a_1, a_2)$ ,

$$(u, v) \mapsto (u \circ r, -v).$$

Let  $\mathcal{M}_i(a_1, a_2) = \mathcal{M}'_i(a_1, a_2)/(\mathbb{Z}/2)$ , the quotient by the  $\mathbb{Z}/2$  action. Consider the natural projection to  $\mathcal{M}_i(a_1, a_2) \rightarrow S^i$ . Over a generic  $v \in S^i$  there is a moduli space of degree  $|a_1| - 2|a_2|$ , so the dimension of the moduli space is

$$\dim \mathcal{M}_i(a_1, a_2) = |a_1| - 2|a_2| + i.$$

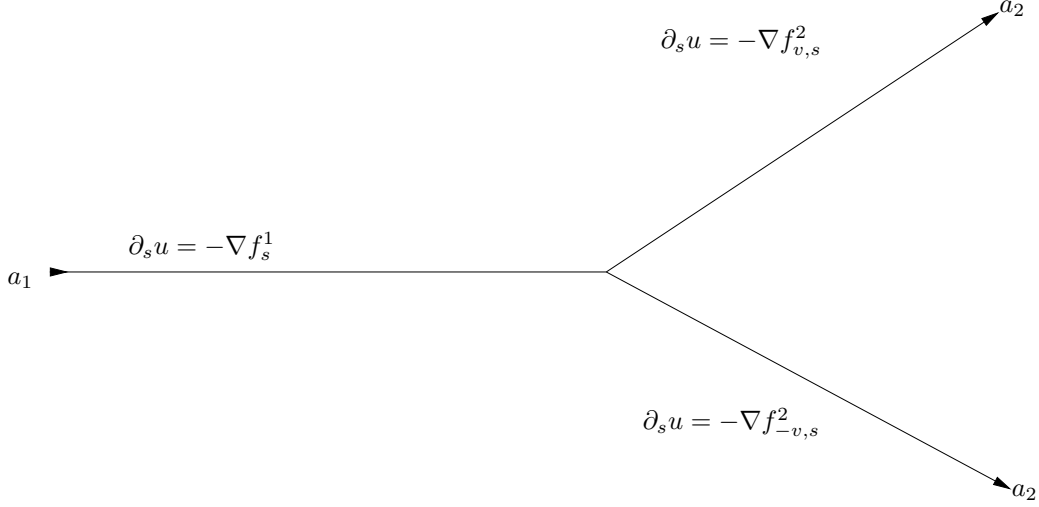


**Figure 3.1:** Morse flowline configurations for the Steenrod square.

**Definition 3.3.2** (The Morse Steenrod Square). *Let  $a \in H^{|a|}(M)$ . Define  $Sq : H^{|a|}(M) \rightarrow H_{\mathbb{Z}/2}^{2|a|}(M)$  by*

$$Sq(a) = \sum_{i \in \mathbb{Z}, b \in \text{crit}_{2|a|-i}(f)} n_{a,b,i} \cdot b \cdot h^i$$

where  $n_{a,b,i} = \#\mathcal{M}_i(b, a)$  for  $\#$  the number of points modulo 2. Denote the coefficient of  $h^{|a|-i}$  as  $Sq^i(a) \in H^{|a|+i}(M)$ .

**Lemma 3.3.3.**  *$Sq$  satisfies the axioms from Section 3.2.2.*

*Proof.* Some of these will be proved using the definition in the next section.

Additivity is immediate from the chain-level definition of the Morse Steenrod square.

*Axioms 1 and 2* are immediate from the definition of  $Sq^i$  and naturality is true for the same reason as for the Morse cup product. For *Axiom 4*,  $Sq^{|x|}(x) = x^2$  as this counts the moduli space  $M^0(y, x)$  for  $|y| = 2|x|$ , and the definition is the same as the Morse cup product.

For *Axiom 5(1)*,  $Sq^i(x) = 0$  for  $i > |x|$  as only non-negative powers of  $h$  are allowed:  $\mathbb{RP}^\infty$  has no cohomology in negative degrees so  $\mathcal{M}_{|x|-i}(\cdot, x) = \emptyset$ . For *Axiom 5(2)*,  $f_{v,s}$  is a perturbation of  $f$ . Assume the perturbation is small. For generic  $f$  there is no  $-\nabla f$  flowline from  $b$  to  $a$  if  $|b| < |a|$ . As  $f_{v,s}$  is close to  $f$ , this means that generically for any  $v$  there is no ‘flowline’ from  $b$  to  $a$  that has gradient  $-\nabla f$  for  $s < 0$  and  $-\nabla f_{v,s}$  for  $s > 0$ . Hence  $Sq^i(x) = 0$  for  $i < 0$ .

We prove axiom 3 in subsection 3.3.5 and axiom 6 in subsection 3.3.3.  $\square$

**Remark 3.3.4.** *It is not straightforward to prove  $Sq^0 = id$  without a specific choice of Morse functions. We prove it in section 3.3.4 using a different approach.*

### 3.3.1.1 A note on transversality

In the first of the conditions at the beginning of the section, we required that our moduli spaces were transverse. More specifically, denote by

$$W^s(a_2, f_{v,s}^2) = \{p \in M \mid \lim_{s \rightarrow \infty} \phi_{v,s}(p) = a_2\},$$

where  $\phi_{v,t}$  is defined for  $v \in S^i$  and  $t \geq 0$  by  $\frac{d\phi_{v,t}}{dt}(s) = -\nabla f_{v,s}^2$  and  $\phi_{v,0} = id$ . We consider

$$\mathcal{W}^{s,i}(a_2) := \sqcup_{v \in S^i} W^s(a_2, f_{v,s}^2) \times W^s(a_2, f_{-v,s}^2) \times \{v\}.$$

This is a smooth pseudocycle representative of a cycle in  $(M \times M) \times_{\mathbb{Z}/2} S^i$ , which we can see for example using the smooth  $\mathbb{Z}/2$ -equivariant map

$$\nu : M \times S^i \rightarrow M \times M \times S^i, (x, v) \mapsto (\phi_{v,-1}(x), \phi_{-v,-1}(x), v).$$

Here  $\phi_{-v,-1}$  is the diffeomorphism induced by backwards integrating along  $f_{v,s}^2$  from  $s = 1$  to  $s = 0$ . We see that after quotienting by  $\mathbb{Z}/2$  and descending to homology,  $\mathcal{W}^{s,i}(a_2)$  is the pseudocycle associated to the pushforward under  $\nu_*$  of the cycle associated to the pseudocycle  $W^s(a_2, f) \times \mathbb{RP}^i$ .

For transversality, we require that for

$$\zeta : M \times \mathbb{RP}^i \rightarrow (M \times M) \times_{\mathbb{Z}/2} S^i, (x, [v]) \mapsto [x, x, v],$$

the pseudocycle  $\mathcal{W}^{s,i}(a_2)$  is transverse to  $\zeta$  and

$$(W^u(a_1, f) \times \mathbb{RP}^i) \cap \zeta^{-1}(\mathcal{W}^{s,i}(a_2))$$

is a transverse intersection for all  $a_1, a_2$ . Indeed, this is what we count in Definition 3.3.2. Note that this transversality condition is a strictly weaker condition than requiring  $W^s(a_2, f_{v,s}^2) \cap W^s(a_2, f_{-v,s}^2)$  and  $(W^s(a_2, f_{v,s}^2) \cap W^s(a_2, f_{-v,s}^2)) \cap W^u(a_1, f)$  intersect transversely for all  $v$  (which is impossible in general with our given conditions).

To show that we can choose such an  $f_{v,s}$  (and indeed such a choice is generic in a reasonable sense), we first assume that the  $f_{v,s}$  are constrained to  $U_f$  from Section 2.3.1. We next, as in the same section, assume that  $f_{v,s} = \beta(s)f_{v,0} + (1 - \beta(s))f$  for the monotonic bump function  $\beta$ . It is then classical that a generic choice of  $f_{v,0}$  gives transversality.

### 3.3.2 The Morse Steenrod square is the Steenrod square

Recall from Section 2.3.1 the Steenrod square due to Betz and Cohen. Recall in the previous section the definition of the Morse Steenrod square. These two setups are very similar, and we claim that they are equivalent.

In the previous section we chose  $f_{v,s}^i$  for  $(v, s) \in S^\infty \times [0, \infty)$  and  $i = 1, 2, 3$ , such that  $f_{v,s}^2 = f_{-v,s}^3$ . We abbreviate  $f_{v,s} = f_{v,s}^2$  where appropriate, and observe we may choose  $f_{v,0}$  distinct from  $\{f_{-v,0}, f\}$  for each  $v$  (as  $\text{Conf}_3(C^\infty(M))$  is open and dense in  $(C^\infty(M))^3$ , hence the condition is generic). Observe  $S \xrightarrow{\cong} \text{Conf}_3(C^\infty(M))$  is a  $\mathbb{Z}/2$ -equivariant homotopy equivalence, using the map  $(f_{1,s}, f_{2,s}, f_{3,s}) \mapsto (f_{1,0}, f_{2,0}, f_{3,0})$ , with the obvious homotopy inverse. Henceforth, assume  $S = \text{Conf}_3(C^\infty(M))$ . Let  $SB = S/\langle(23)\rangle$ , where the transposition  $(23)$  acts on  $S$  by permutation of the components. By previous discussions (such as in Section 2.3.1), we know that  $S$  is contractible and  $(23)$  acts freely on  $S$ . Hence  $SB = S/\langle(23)\rangle$  is homotopy equivalent to  $\mathbb{RP}^\infty$ .

There is a natural  $\mathbb{Z}/2$ -equivariant map  $i : S^\infty \hookrightarrow S$  induced by  $v \mapsto (f, f_{v,0}, f_{-v,0})$ , which descends to  $i : \mathbb{RP}^\infty \rightarrow SB$ .

**Lemma 3.3.5.**  $i_* : H_*(\mathbb{RP}^\infty) \rightarrow H_*(SB)$  is an isomorphism.

*Proof.* If  $i$  is a weak homotopy equivalence then it is a quasi-isomorphism, see [12, Proposition 4.21]. As the two spaces are both homotopy equivalent to  $\mathbb{RP}^\infty$ , it is sufficient to show that

$$i_* : \pi_1(\mathbb{RP}^\infty) \cong \mathbb{Z}/2 \rightarrow \pi_1(SB) \cong \mathbb{Z}/2$$

is nontrivial.

Parametrising  $S^1$  by  $\theta \in [0, 2\pi)$ , and denoting  $f_v = f_{v,0}$ , we wish to show that  $\theta \mapsto [(f, f_{\theta/2}, f_{-\theta/2})]$  determines a nontrivial loop, where  $[\cdot]$  denotes the  $\mathbb{Z}/2$ -equivalence class. For simplicity we reduce to showing that  $[f_{\theta/2}, f_{-\theta/2}]$  is a nontrivial loop in  $\text{Conf}_2(C^\infty(M))/(\mathbb{Z}/2)$ .

Suppose for a contradiction that this were a contractible loop. Then there exists  $H : \mathbb{D}^2 \rightarrow \text{Conf}_2(C^\infty(M))$  such that  $H(1, \theta) = [(f_{\theta/2}, f_{-\theta/2})]$ , where we parametrise the 2-disc by polar coordinates  $(r, \theta)$ . Let  $H(r, p(\xi)) = [(g_1(r, \xi), g_2(r, \xi))]$ , where  $\xi \in \mathbb{R}$  and  $p : \mathbb{R} \rightarrow \mathbb{R}/2\pi\mathbb{Z} \cong S^1$  is the projection. We use  $\xi \in \mathbb{R}$  because  $g_1, g_2$  may not necessarily be well defined on  $\mathbb{D}^2$ . Specifically for each  $(r, \theta)$ , exactly one of the following conditions holds:

1.  $g_1(r, \theta) = g_1(r, \theta + 2\pi)$  and  $g_2(r, \theta) = g_2(r, \theta + 2\pi)$

2.  $g_1(r, \theta) = g_2(r, \theta + 2\pi)$  and  $g_2(r, \theta) = g_1(r, \theta + 2\pi)$ .

Both cannot hold at the same time as  $(g_1(r, \theta), g_2(r, \theta)) \in \text{Conf}_2(C^\infty(M))$  for each  $(r, \theta)$ , hence must be distinct. However, these are mutually exclusive closed and open conditions (closed and open as subsets of  $\mathbb{D}^2$ ). As  $\mathbb{D}^2$  is connected only one can hold for all  $(r, \theta) \in \mathbb{D}^2$ . For  $r = 1$  the condition (2) holds. However,  $g_1(0, \theta)$  is independent of  $\theta$ , so condition (1) must hold, giving the contradiction.  $\square$

Consider diagram (3.9):

$$\begin{array}{ccc}
H_*(\mathbb{RP}^\infty) \otimes H^*(M) & \xrightarrow{MSq} & H^*(M) \\
\downarrow i_* \otimes id \cong & & \downarrow = \\
H_*(SB) \otimes H^*(M) & \xrightarrow{s} H_*(SB) \otimes H_{\mathbb{Z}/2}^*(M \times M) \xrightarrow{q} & H^*(M) \\
& \searrow Sq & 
\end{array} \tag{3.9}$$

Here  $s(A \otimes x) = A \otimes x \otimes x$ , and the map  $q$  is as in Section 2.3.1. We have reinterpreted the Morse Steenrod square from the previous section, here denoted  $MSq$ , to be a map  $MSq : H_*(\mathbb{RP}^\infty) \otimes H^*(M) \rightarrow H^*(M)$ , which we can do canonically as there is a unique graded basis of the homology of  $\mathbb{RP}^\infty$ . Observe that if we use the pushforward of the generator of  $H_i(\mathbb{RP}^\infty)$  as the generator of  $H_i(SB)$ , then it is immediate that the diagram (3.9) commutes. Hence, Definition 3.3.2 is the Steenrod square.

### 3.3.3 The Cartan Relation

Let  $T$  be a family of graphs as in figure 3.2, parametrised by  $t \in (0, \infty)$ . Edge  $e_1$  is a negative half-line and edges  $e_3, e_4, e_6, e_7$  are positive half-lines. Edges  $e_2, e_5$  are parametrised by  $[0, t]$ . Compactify  $T$  by adding the 0 and  $\infty$  as in the figure, so the compactification  $T^c \cong [0, 1]$ . Use edge labels as given in Figure 3.2. Fix a Morse function  $f$  on  $M$ . We always parametrise the edges with the coordinate  $s$ .

Pick 5 perturbations of  $f$  corresponding to the 5 tree edges in  $t = 0 \in T^c$  in figure 3.2. These are  $f_{v,s,0}^p$  for  $p$  the edge label,  $s \in \mathbb{R}^\pm$  and  $v \in S^\infty$ . For asymptotics,  $f^1 = f$  for all  $s, v$  and  $f^3, f^4, f^6, f^7 = f$  for  $s$  near  $+\infty$ . We ensure that  $f_{v,s,0}^3 = f_{-v,s,0}^4$  and  $f_{v,s,0}^6 = f_{-v,s,0}^7$  for all  $v, s$ . We choose  $f^p$  along with  $S_0$  such that  $f_{v,s,0}^p = f$  for  $s \geq S_0$ .

Choose 7 perturbations of  $f$  labelled  $f_{v,s,t}^p$  for  $p = 1, \dots, 7$  corresponding to the edge labels in Figure 3.2, where  $t \in T^c$ ,  $v \in S^\infty$  and  $s \in \mathbb{R}^+$  for  $p = 3, 4, 6, 7$ ,  $s \in \mathbb{R}^-$  for  $p = 1$  and  $s \in [0, t]$  for  $p = 2, 5$ . Choose  $f^1$  to be independent of  $s, v, t$  in this case. Choose Morse functions  $f_{s,2}^2, f_{s,2}^5$  for  $s \in [0, 2]$  such that  $f_s^p = f$  for  $s > 1$  and  $p = 2, 5$ .

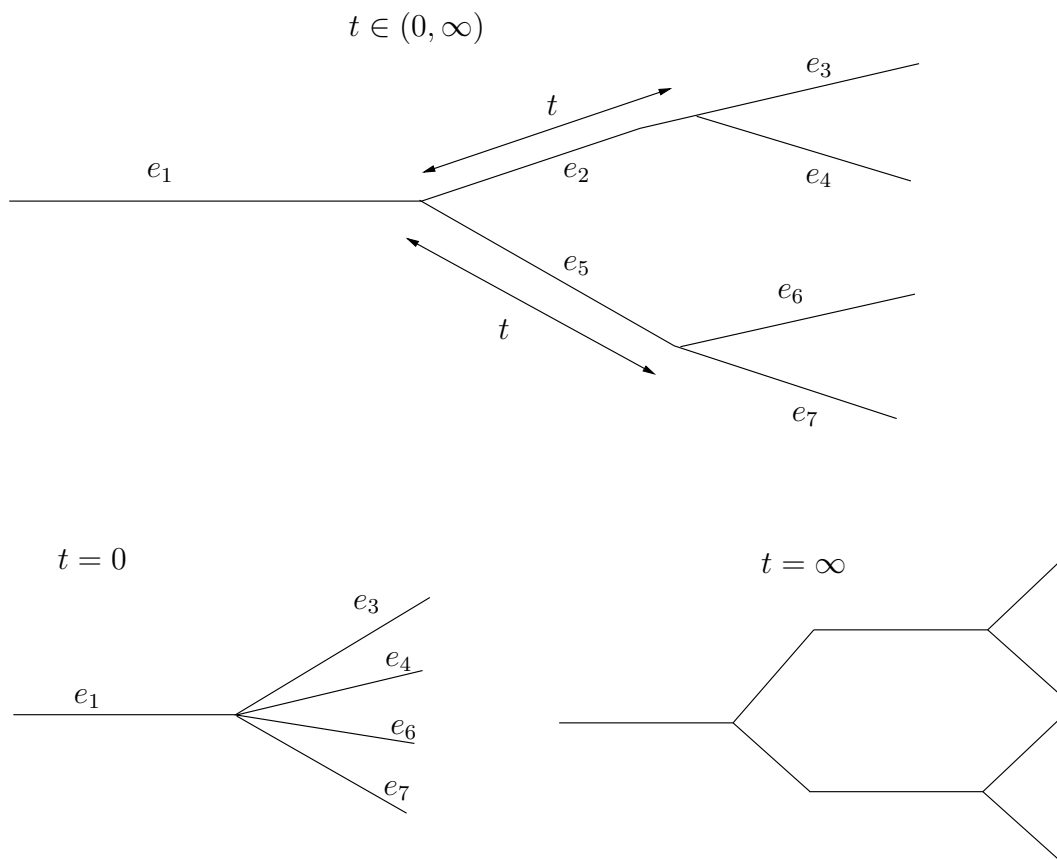


**Figure 3.2:** Elements of  $T^c$ .

The  $f^p$  must be chosen generically at each vertex of  $\Gamma$ , to ensure transversality of moduli spaces. The  $f_{v,s,t}^p$  satisfy the following conditions:

1.  $f_{v,s,t}^p = f_{v,s,0}^p$  as picked previously for  $p = 1, 3, 4, 6, 7$  and for all  $t$ .
2.  $f^2, f^5$  are independent of  $v$ .
3. For  $t \geq 2$  and  $p = 2, 5$ :  $\begin{cases} f_{s,t}^p = f_{s,2}^p \text{ for } s \leq 2, \\ f_{s,t}^p = f \text{ for } s \geq 2. \end{cases}$

Fix  $i \in \mathbb{N}$  and  $x, y \in \text{crit}(f)$ . Let  $\bar{T} \xrightarrow{\cong} T^c$  consist of pairs  $(|t|, t)$  where  $t \in T^c \cong [0, \infty]$  and  $|t|$  is the metric tree represented by  $t$  as a topological space. The metric structure for  $t \in [0, \infty)$  is that the outer edges are semi-infinite and parametrised by respectively  $(-\infty, 0]$  for the incoming edge and  $[0, \infty)$  for the outgoing edges. The inner edges are of length  $t$ , parametrised by  $[0, t]$ . For the  $t = \infty$  boundary, the metric structure on  $|\infty|$  is that the edges attached to bivalent vertices are semi-infinite with the infinite end at the bivalent vertex.

For  $z \in \text{crit}_{2|a|+2|b|-i}(f)$  consider the space  $\tilde{\mathcal{M}}_1(x, y, z)$  of triples  $(t, u, v)$  with  $t \in T^c$ ,  $u : |t| \rightarrow M$  a map and  $v \in S^{|x|+|y|-i}$ , such that  $u$  satisfies:

$$\partial_s u_{s,t} = -\nabla f_{v,s,t}^p$$

along edge  $p$ , with asymptotic conditions  $(z, x, x, y, y)$  on the exterior edges  $(1, 3, 4, 6, 7)$ .

For generic  $t \in T^c$  there is a 0-dimensional subset of pairs  $(u : |t| \rightarrow M, v \in S^i)$  satisfying the conditions. So  $\tilde{\mathcal{M}}_1(x, y, z)$  is 1-dimensional. Observe that  $\tilde{\mathcal{M}}_1(x, y, z)$  has a free  $\mathbb{Z}/2$  action,  $(t, u, v) \mapsto (t, u \circ \bar{r}, -v)$  for  $\bar{r}$  acting on  $|t|$  by the permutation of edges (34)(67). Let  $\mathcal{M}_1(x, y, z) = \tilde{\mathcal{M}}_1(x, y, z)/(\mathbb{Z}/2)$ , which is still 1-dimensional.

We also define a moduli space  $\tilde{\mathcal{M}}_2(x, y, z)$  by choosing another 7 Morse functions, labelled  $f_{v,s,t}^p$  as above, but now with the conditions:

1.  $f_{v,s,t}^p = f_{-v,s,t}^q$  for  $(p, q) = (3, 4), (6, 7), (2, 5)$ ,
2.  $f_{v,s,t}^p$  is independent of  $(v, t)$  for large enough  $t$  and for  $p = 1, 3, 4, 6, 7$ ,
3.  $f^p = f$  for  $p = 1, 3, 4, 6, 7$  and  $|s| \geq 1$ .
4. For large enough  $t$  and  $s \in [1, t]$ ,  $f_{v,s,t}^2 = f_{v,s,t}^5 = f$ .

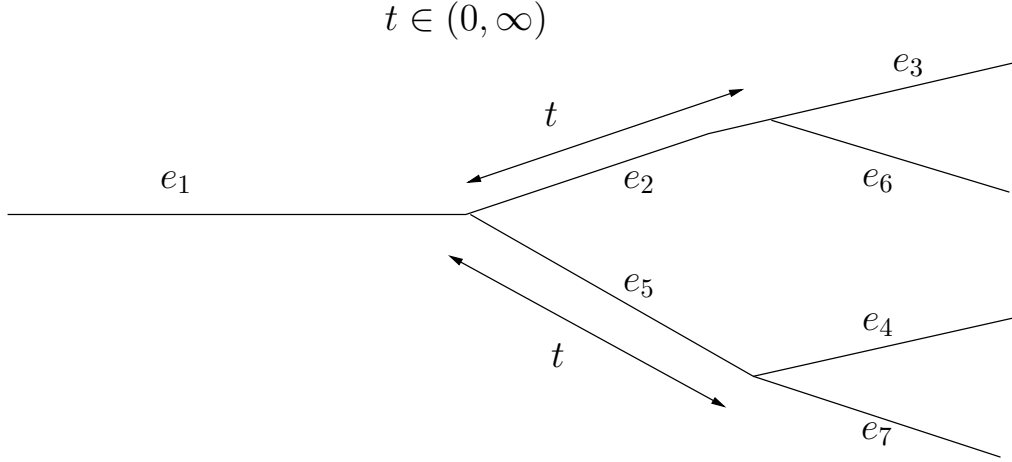


**Figure 3.3:** Tree labelling for  $\mathcal{M}_2$ .

In defining equations for pairs  $(t, u, v) \in \tilde{\mathcal{M}}_2(x, y, z)$ , use the edge labellings in figure 3.3, i.e. the edge labels 4 and 6 from figure 3.2 have been swapped. For each edge label the equations and asymptotic conditions are the same as in the  $\tilde{\mathcal{M}}_1$  case. Further, there is a free  $\mathbb{Z}/2$  action on  $\tilde{\mathcal{M}}_2(x, y, z)$  similarly to  $\tilde{\mathcal{M}}_1$  but with edge permutation (25)(36)(47). Quotienting, define  $\mathcal{M}_2(x, y, z) = \tilde{\mathcal{M}}_2(x, y, z)/(\mathbb{Z}/2)$ .

**Remark 3.3.6.** *Some of the conditions require a choice of  $f_{v,s,t}^p$  that becomes independent of  $v$  for large enough  $t$ . We can do this with moduli spaces remaining transverse because we keep the  $v$  dependence on some of the other  $f^q$ . Specifically, there is a tuple  $(f_{v,s,t}^1, \dots, f_{v,s,t}^7)$  for each  $v \in S^\infty$ , along with associated  $\mathbb{Z}/2$  action  $\iota$ . Our choices ensure that  $\iota : v \mapsto -v$  always acts freely on tuples, hence the union of tuples over all  $v \in S^\infty$  is an  $E\mathbb{Z}/2 \subset (C^\infty(M))^7$ .*

*Each pair  $(f_{v,s,t}^i, f_{v,s,t}^j)$  for  $(i, j) = (2, 5), (3, 4), (6, 7)$  also defines an  $E\mathbb{Z}/2$ . So as long as  $\iota$  acts freely on some pair, the set of these tuples is an  $E\mathbb{Z}/2$ . This is the fact that if  $E, F$  are both an  $E\mathbb{Z}/2$  then  $E \times F$  is an  $E\mathbb{Z}/2$ .*

The following theorem is classical, and the following proof is a modification of [3, Section 2, Example 2] for our definition of the Steenrod square.

**Theorem 3.3.7** (The Cartan Relation).

$$Sq^i(x \cup y) = \sum_{j+k=i} Sq^j(x) \cup Sq^k(y).$$

*Proof.* The moduli space  $\mathcal{M}_1(x, y, z)$  is a 1-dimensional cobordism, corresponding to  $[0, \infty]$ , so  $\#\partial\mathcal{M}_1(x, y, z) = 0$ . Similarly  $\#\partial\mathcal{M}_2(x, y, z) = 0$ . The boundary at  $t = \infty$

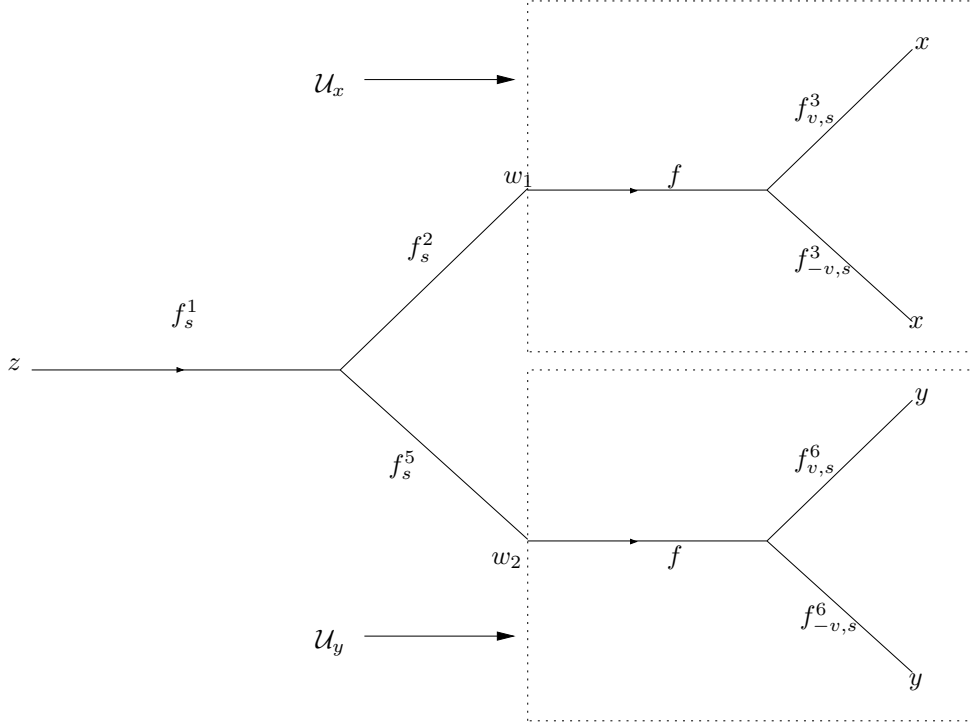


**Figure 3.4:** Flowline configurations for  $Sq(x) \cup Sq(y)$ .

is the count of the contribution of  $z$  in

$$\sum_{j+k=i} Sq^j(x) \cup Sq^k(y)$$

(see Figure 3.4 and Lemma 3.3.8). The boundary at  $t = 0$  for  $\mathcal{M}_2(x, y, z)$  is the same as for  $\mathcal{M}_1(x, y, z)$  (we are just using the same edge and asymptotic labels). Hence, the boundary count for  $t = \infty$  for  $\mathcal{M}_1(x, y, z)$  and  $\mathcal{M}_2(x, y, z)$  is the same in cohomology. The count for the  $t = \infty$  boundary component of  $\mathcal{M}_2(x, y, z)$  is the count of the contribution of  $z$  in  $Sq^i(x \cup y)$ , as in Lemma 3.3.9.  $\square$

**Lemma 3.3.8.** *Summing over all choices of  $w_1, w_2$ , counting  $[(u, v)]$  satisfying configurations as shown in Figure 3.4, gives  $\sum_{j+k=i} Sq^j(x) \cup Sq^k(y)$ .*

*Proof.*  $|w_1| + |w_2| = |z| = |x| + |y| + i$ . Hence if  $|w_1| = |x| + j$  and  $|w_2| = |y| + k$  then  $j + k = i$ . Throughout fix  $w_1, w_2$  both for the configuration and as outputs of  $Sq^j(x), Sq^k(y)$  respectively.

Restrict attention to the upper right-hand Y-shaped graph of Figure 3.4. If the  $v$  parameter space was  $\mathbb{RP}^{|x|-j}$ , then counting  $[(u, v)]$  satisfying the configuration

conditions would be exactly the count of the coefficient of  $w_1 \cdot h^{|x|-j}$  in  $Sq^j(x)$ , which we denote  $n_x$ . Letting  $v$  vary in  $\mathbb{RP}^{|x|+|y|-i}$ , we call the set of such pairs

$$\mathcal{U}_x = \{[v, u] | v \in \mathbb{RP}^{|x|+|y|-i} \text{ and } u : \Gamma \rightarrow M \text{ satisfies conditions as illustrated in Figure 3.4}\}.$$

Similarly for the lower right-hand branch there is a count of  $n_y$ , the coefficient of  $w_2 \cdot h^{|y|-k}$  in  $Sq^k(y)$ , for each  $\mathbb{RP}^{|y|-k} \subset \mathbb{RP}^{|x|+|y|-i}$ . Define similarly  $\mathcal{U}_y$ .

The contribution to  $z \cdot h^{|x|+|y|-i}$  in  $w_1 \cup w_2$  from  $Sq^j(x) \cup Sq^k(y)$  is  $n_x \cdot n_y \cdot n_z$  for  $n_z$  the coefficient of  $z$  in  $w_1 \cup w_2$ , i.e. the left hand Y-shaped graph of Figure 3.4. Let

$$\pi_x : \mathcal{U}_x \rightarrow \mathbb{RP}^{|x|+|y|-i}$$

be the second projection. Let  $u_x := \pi_{x*}[\mathcal{U}_x] \in H_{|x|-j}(\mathbb{RP}^{|x|+|y|-i})$ . By the discussion above,

$$u_x \bullet [\mathbb{RP}^{|x|-k}] = n_x,$$

where  $\bullet$  is the intersection product, hence  $u_x = n_x \cdot [\mathbb{RP}^{|y|-k}]$ . Similarly  $u_y = n_y \cdot [\mathbb{RP}^{|x|-j}]$ . The count of all solutions  $[(u, v)]$  satisfying the configuration in Figure 3.4 is

$$n_z \cdot (u_x \bullet u_y) = n_z \cdot n_x \cdot n_y. \quad \square$$

**Lemma 3.3.9.** *The count for the  $t = \infty$  boundary component of  $\mathcal{M}_2(x, y, z)$  is the count of the contribution of  $z$  in  $Sq^i(x \cup y)$ .*

*Proof.* The edge and asymptotic conditions are as shown in Figure 3.5. The edges attached to bivalent vertices are semi-infinite with the infinite end at the bivalent vertex, which is a critical point of  $f$ . For this operation, the  $t = \infty$  boundary, we may homotope the choice of perturbed Morse functions so that the two right-hand Y-shaped graphs use the same perturbations  $f^3, f^6$  of  $f$ . Specifically, we may assume that  $f^3$  and  $f^6$  are independent of  $v$ . We are then counting the contribution of  $z$  in  $Sq^i(x \cup y)$ .  $\square$

### 3.3.4 Steenrod Squares via intersections of cycles

Recall there are nested equators  $S^i \subset S^\infty$ , invariant under the antipodal action. Let  $a \in H^{|a|}(M)$ . Let  $\mathcal{B}$  be a basis of  $H^*(M)$ .

A representative of a homology class  $\alpha$  is a pair  $(A, \mu)$  where  $A$  is a smooth manifold and  $\mu : A \rightarrow M$  is smooth such that  $\mu_*[A] = \alpha$ . We recall that over  $\mathbb{Z}/2$ -coefficients every homology class is realisable (see e.g. [6, Theorem B]). The definition

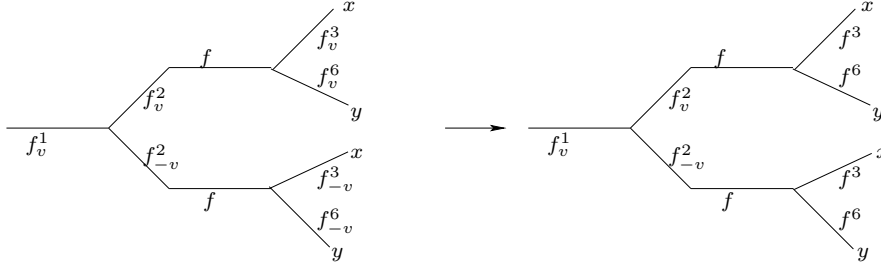


**Figure 3.5:** Flowline configurations for  $Sq(x \cup y)$ .

in this section will be used as a computational tool for our purposes, so for simplicity we will assume throughout this chapter that all of our homology classes in  $\mathcal{B}$  can be represented as embedded submanifolds. We will also interchangeably use  $\alpha$  to represent a homology class, the underlying cycle and the submanifold representative (where the usage is unambiguous). For  $b \in \mathcal{B}$ , we denote  $\beta = PD(b)$ . There is an involution

$$\iota = id_M \times (-id_{S^\infty}) : M \times S^\infty \rightarrow M \times S^\infty$$

Let  $A$  represent  $\alpha$ . Choose a sequence of embeddings  $\mu_i : \alpha_i := A \times S^i \rightarrow M \times S^i$  such that:

1. For  $\pi_2$  the second projection,  $\pi_2(\mu_i(a, v)) = v$  for all  $(a, v) \in A \times S^i$ .
2. The restriction  $\mu_i|_{\alpha_j} = \mu_j$  for  $j \leq i$ .
3. For  $\pi_1$  the first projection, for any  $v \in S^i$  then

$$\alpha_v := \pi_1(\alpha_i \cap \pi_2^{-1}(\{v\})) = \mu_i(A, v) \tag{3.10}$$

is a representative of  $\alpha$  in  $M$  (and is well defined by (2) above).

4. For  $b \in \mathcal{B}$ , the triple intersection

$$\mu_i(\alpha_i) \cap (\beta^\vee \times S^i) \cap \iota(\mu_i(\alpha_i)) \tag{3.11}$$

is transverse in  $M \times S^i$ .

The chains  $\alpha_i$  cannot be the “standard” representative of  $\alpha \times S^i$ , i.e.  $\{(p, v) | p \in \mathcal{A}, v \in S^i\}$  for some representative  $\mathcal{A} \subset M$  of  $\alpha$ , because then  $\alpha_i$  would not be transverse to  $\iota\alpha_i$ . In the next section we construct a family of admissible choices. We take  $\beta^\vee \times S^i$  to be such a standard representative.

All triple intersections in (3.11) come in pairs related by the free involution  $\iota$ . Provided  $|b| = 2|a| - i$ , define  $\mathcal{M}'_i(b, a)$  as the triple intersection (3.11): this moduli space is 0 dimensional. Quotienting by the action by  $\iota$  defines

$$\mathcal{M}_i(b, a) = \mathcal{M}'_i(b, a)/(\mathbb{Z}/2).$$

This moduli space is compact and 0 dimensional, so is a collection of points.

**Definition 3.3.10** (Steenrod Square). *Define*

$$Sq(a) = \sum_{i \in \mathbb{Z}, b \in \mathcal{B}, |b|=2|a|-i} \# \mathcal{M}_i(b, a) b h^i$$

**Remark 3.3.11.** *The Morse Steenrod square of Definitions 3.3.2 is the same as Definition 3.3.10 using the isomorphism  $HM^*(M, f) \cong H^*(M)$  that intertwines the Morse product and the cup product.*

*Recall we assume  $f_{v,s}^i = f$  for  $|s| \geq R$ . In the Morse case there are critical points  $a_1, a_2$ . Say  $a_2$  has a stable manifold  $W$  under  $f$ . By definition,  $W$  is the set of points  $p$  such that the flow of  $p$  under  $-\nabla f$  from time  $R$  to  $\infty$  is asymptotically  $a_2$ . For each  $v \in S^i$ , every  $w \in W$  is mapped bijectively to another point in  $M$  by backwards integrating for the vector field  $-\nabla f_{2,s,v}$  from  $R$  to 0: call this backwards operation  $\phi_v : W \rightarrow M$ .*

*This manifold  $\phi_v(W)$  is homotopy equivalent to the stable manifold  $W$ . If  $a_2$  represents a cohomology class then  $\overline{W}$ , hence  $\overline{\phi_v(W)}$ , represent  $PD(a_2)$ , viewed as a pseudocycle in  $M$ . In the Morse case we count the number of maps  $u : \Gamma \rightarrow M$  with  $u$  restricting to be a flowline on the edges.  $\bigsqcup_{v \in S^i} \phi_v(W) \times \{v\}$  represents  $PD(a_2) \times S^i$ , and*

$$\bigsqcup_{v \in S^i} \phi_{-v}(W) \times \{v\} = \iota \left( \bigsqcup_{v \in S^i} \phi_v(W) \times \{v\} \right)$$

*represents  $\iota(PD(a_2) \times S^i)$ . Further, these representatives of  $PD(a_2) \times S^i$  satisfy all of the conditions at the beginning of this section, so the Morse definition is a special case of Definition 3.3.10.*

*In general  $a_2 \in \text{crit}(f)$  is not a cycle, but some linear combination of critical points is: the union of the stable manifolds represents (as a pseudocycle) the Poincaré dual of the sum.*

### 3.3.5 Properties of the Steenrod Square

As promised in Section 3.3.1 we now check axiom 3 from Section 3.2.2.

**Lemma 3.3.12.**  $Sq^0(PD(pt)) = PD(pt)$ .

*Proof.* Let  $n = \dim(M)$ . Write  $a = PD(pt)$ . We construct a representative of  $\{pt\} \times S^n$  in  $M \times S^n$ :

The submanifold  $pt \subset M$  has trivialisable normal bundle, so the disc subbundle  $D(pt)$  of the normal bundle  $N(pt)$  embeds into  $M$  as a small disc around  $pt$ . Let  $S^n, D^n \subset \mathbb{R}^{n+1}$ , where  $D^n$  is the  $n$ -disc with the  $n+1^{th}$  coordinate 0. There is a natural flattening map  $\phi' : S^n \rightarrow D^n$ , where  $\phi'$  is projection of  $S^n$  onto the first  $n$  coordinates.  $\phi'$  is a double cover except on the equator, which bijects onto  $\partial D^n$ .

There is a diffeomorphism  $D^n \cong D(pt) \subset M$ . Composing  $\phi'$  with this diffeomorphism defines  $\phi : S^n \rightarrow M$ . The diffeomorphism  $\phi$  is homotopic to a constant map hence  $\bigsqcup_{v \in S^n} (\phi(v), v)$ , the graph of  $\phi$  in  $M \times S^n$ , is a representative of  $\{pt\} \times S^n$ . To calculate the coefficient of  $a$  in  $Sq^0(a)$ , count the number of solutions  $\phi(v) = \phi(-v)$  modulo  $\mathbb{Z}/2$ . The solution set is  $\{(0, \dots, 0, \pm 1)\} \subset S^n \subset \mathbb{R}^{n+1}$ . Taking this modulo the  $\mathbb{Z}/2$  action gives  $Sq^0(a) = a + \dots$ . The cycle  $\{a\}$  generates  $H^n(M)$  so there are no more contributions.  $\square$

An easy generalisation of the above proof shows:

**Lemma 3.3.13.**  $Sq^0 = id$ .

*Proof.* Proceed as in the previous lemma, but now  $a = PD(A)$  for some cycle  $A$ . It is convenient to assume that  $A$  is in a basis for the homology of  $M$ , with  $A^\vee$  the corresponding member of the dual basis under the intersection product. Similarly to above,  $A \cap A^\vee$  consists of a finite, odd number of points  $\{p_i\}$  (since  $A \cdot A^\vee = 1 \pmod{2}$ ). Each of these has a small neighbourhood  $U_i \subset M$  on which the normal bundle  $N(A)$  of  $A$  in  $M$  is trivial, with the  $U_i$  being pairwise disjoint. Pick a bump function  $\beta_i$  for each neighbourhood  $U_i$ . On the neighbourhood  $U_i$ , there is a diffeomorphism between the disc bundle and the trivial bundle  $D(U_i) \cong D^{n-\dim(A)} \times (U_i \cap A)$ . Using the tubular neighbourhood theorem,  $N(A)$  and hence  $D^{n-\dim(A)} \times (U_i \cap A)$  embeds into  $M$  via a map  $e$ .

Hence there is a map  $\phi : S^{n-\dim(A)} \times A \rightarrow M$ : if  $x$  is not in any  $U_i$ , then  $\phi(v, x) = x$ . Otherwise  $x$  is in exactly one  $U_i$  and we let  $\phi(v, x) = e(\beta_i(x)\phi'(v), x)$ , where  $\phi'$  is as in the previous lemma. This gives a representative of  $S^{n-\dim(A)} \times A$ , recalling that  $n - \dim(A) = |a|$ . Consider the intersections modulo  $\mathbb{Z}/2$ . The coefficient of  $ah^{|a|}$  is

obtained using output cycle  $S^{n-\dim(A)} \times A^\vee$ . By construction, such intersections only occur when the second coordinate is one of the  $p_i$ . At  $p_i$ , there is exactly one pair of solutions corresponding to the two solutions as in the previous claim: i.e.  $\phi'$  is 2 to 1 on a dense open subset.

Take the quotient by the  $\mathbb{Z}/2$  action to deduce that the number of contributions is an odd number (the number of  $p_i$ ) multiplied by an odd number (the number of solutions at each  $p_i$ ), hence is odd. Therefore  $Sq^0(a) = ah^{|a|} + \dots$ . To show that there are no more solutions, repeat this with output cycle as  $S^{n-\dim(A)} \times B$ , for another  $B$  in the dual basis of homology. As  $B \neq A^\vee$  is in the dual basis,  $B$  intersects  $A$  with an even number of points. We exactly as in the previous case, except the number of contributions is an even number (the number of  $B \cap A$ ) multiplied by an odd number. Hence the count is even and the contributions due to other  $B$  are 0.  $\square$

**Corollary 3.3.14.** *Let  $A$  be a closed submanifold of  $M$ , with trivialisable normal bundle. Then  $Sq^i(PD([A])) = 0$  for  $i \neq 0$ .*

*Proof.* Use the embedding  $e : A \times D^{n-\dim(A)} \rightarrow M$  by inclusion of the unit disc bundle of  $A$ , which exists because  $A$  has trivialisable normal bundle, to define  $A_v$  for  $v \in S^i$  for  $i > 0$ .  $\square$

**Remark 3.3.15.** *More generally, for any immersed submanifold  $A \rightarrow X$ , consider the homology class  $[A] \in H_*(X)$ . Then  $Sq^i(PD([A]))$  gives the Stiefel-Whitney class  $w_i(N_AX)$  (where  $N_AX$  is the normal bundle of  $A$  in  $X$ ).*

**Remark 3.3.16.** *The definition in this chapter can be extended more generally, i.e. when our homology classes are not represented by embedded submanifolds. For example, we replace  $\alpha_i$  by  $(A, \mu_i : A \times S^i \rightarrow M)$  and we let  $\alpha_v = \mu_i(A, v)$ . In this case we also replace  $\cap$  with the intersection product  $\bullet$ .*

**Remark 3.3.17.** *An intuitive and non-rigorous argument for the additivity for the Steenrod square (with Definition 3.3.10) is as follows.*

*Fix a choice of  $X_v$  representing  $PD(x)$ , and likewise  $Y_v$  representing  $PD(y)$  (using the notation from (3.10)). Assume that the  $X_v, Y_v$  are manifolds, and the pairs  $(X_v, Y_v)$  and  $(X_v, Y_{-v})$  intersect cleanly for all  $v$ . We then choose  $PD(x+y)_v = X_v \cup Y_v$ , representing the sum of the cycles, counted with multiplicity at intersections. Let  $c_{z,x+y}$  be the coefficient of  $z \cdot h^i$  in  $Sq(x+y)$ ,*

$$c_{z,x+y} = \# \bigsqcup_{v \in D^i, +} Z^\vee \cap (X_v \cup Y_v) \cap (X_{-v} \cup Y_{-v}) \quad (3.12)$$

for  $Z = PD(z)$  the output. Likewise  $c_{z,x}$  is the coefficient of  $z \cdot h^i$  in  $Sq(x)$ .

Expanding the right hand side of equation (3.12) gives

$$c_{z,x+y} = c_{z,x} + c_{z,y} + \# \bigsqcup_{v \in D^{i,+}} Z^\vee \cap X_v \cap Y_{-v} + \# \bigsqcup_{v \in D^{i,+}} Z^\vee \cap Y_v \cap X_{-v}.$$

The last two terms combine to  $\# \bigsqcup_{v \in S^i} Z \cap X_v \cap Y_{-v}$ .

For  $i \neq 0$ , suppose first that generically  $X \cap Y = \emptyset$ . Then as  $X_v$  is a perturbation of  $X$  and  $Y_v$  is a perturbation of  $Y$  for each  $V$ , we choose these perturbations to be small and thus we may assume that  $X_v \cap Y_{-v} = \emptyset$  for all  $v$ . Otherwise generically  $X$  and  $Y$  intersect in a submanifold  $W$ , a cycle. Indeed, by our assumptions there is a choice of  $X_v, Y_v$  so that  $X_v \cap Y_{-v}$  is a submanifold  $W$  for each  $v$ . Then given any neighbourhood  $v \in U \subset S^i$  there is  $v' \in U$  such that  $X_{v'} \cap Y_{-v'}$  is homotopy equivalent to  $W$  (hence nonempty). Thus if we count isolated solutions, there are none to be counted (as all of them are nonisolated).

For  $i = 0$ ,  $Sq$  is the cup product square in which case additivity holds.

### 3.4 Quantum Steenrod Square via Morse theory

Let  $M$  be a closed monotone symplectic manifold. The definition of the quantum Steenrod square uses a  $Y$ -shaped graph as with the Morse Steenrod square, but now allows for a  $J$ -holomorphic sphere at the trivalent vertex in the  $Y$ -shaped graph in the definition. This is a  $J$ -holomorphic sphere with  $2 + 1$  marked points, and 2 incoming and 1 outgoing Morse flowlines from the respective marked points.

Make a choice of  $f_{s,v}^p$  as in subsection 3.3.1, for  $p = 1, 2, 3$ . Let  $N$  be the minimal Chern number of  $M$ . Fix  $i, j \geq 0$  and  $a, b \in H^*(M)$  with

$$|b| - 2|a| + i + 2jN = 0.$$

Let  $\mathcal{M}'_{i,j}(b, a)$  be the moduli space of pairs  $(u, v)$ , for  $v \in S^i$  and  $u$  a  $J$ -holomorphic map  $u : \mathbb{CP}^1 \rightarrow M$  of Chern number  $2jN$ , such that the  $-\nabla f_{s,v}^1$  flowline from  $u(0)$  converges to  $b$  as  $s \rightarrow -\infty$  and the  $-\nabla f_{s,v}^p$  flowline from  $u(1), u(\infty)$  converge to  $a$  as  $s \rightarrow \infty$  for  $p = 2, 3$ . There is a  $\mathbb{Z}/2$  action on this moduli space:

$$\iota_{\mathcal{M}}(u, v) = (u \circ \mu, -v)$$

where  $\mu$  is the unique Möbius map in  $PSL(2, \mathbb{C})$  swapping 1 and  $\infty$  and fixing 0. Let

$$\mathcal{M}_{i,j}(a, b) = \mathcal{M}'_{i,j}(a, b) / \iota_{\mathcal{M}}.$$

**Definition 3.4.1** (Morse Quantum Steenrod Square). *Pick a basis  $\mathcal{B}$  of  $H^*(M)$ . Let  $a \in H^*(M)$ . For each  $i, j$ , let*

$$Q\mathcal{S}_{i,j}(a) = \sum_{b \in \mathcal{B}: |b|+i+2jN=2|a|} \#\mathcal{M}_{i,j}(b, a) \cdot b,$$

$$Q\mathcal{S}(a) = \sum_{i,j} Q\mathcal{S}_{i,j}(a) \cdot h^i T^j.$$

*Extend to a general element of  $QH^*(M)$  by  $Q\mathcal{S}(at^j) = Q\mathcal{S}(a)t^{2j}$ , and so that  $Q\mathcal{S}$  is an additive homomorphism.*

**Remark 3.4.2.** 1. For  $a \in H^*(M)$ ,

$$Q\mathcal{S}_{i,0}(a) = Sq^{|a|-i}(a)$$

*as it counts constant spheres. Further,*

$$\sum_{j \geq 0} Q\mathcal{S}_{0,j}(a) T^j = a * a$$

*is the usual quantum product.*

2. Observe that we do not need to use an almost complex structure  $J_v$  parametrised by  $S^\infty$ . See the remark before Theorem 3.3.7. A free  $\mathbb{Z}/2$ -action is only needed somewhere in the setup: here it is on the Morse flowlines via  $f_v$ . However, we could use a domain dependent  $J_v$  in the definition if we wished..

### 3.4.1 Quantum Steenrod Squares via intersections of cycles

Let  $a \in H^{|a|}(M)$ , and we pick a basis  $\mathcal{B}$  of  $H^*(M)$ . Denote  $\alpha = PD(a), \beta = PD(b)$  for  $b \in \mathcal{B}$ . We define a moduli space and evaluation maps analogously to Section 3.2.3: given  $j \geq 0$  consider  $\mathcal{M}_j(J) \times S^i$  consisting of pairs  $(u, v)$  where  $u$  is a  $J$ -holomorphic map of energy  $j$  and  $v \in S^i$ . The evaluation maps are  $ev_q \times id_{S^i} : \mathcal{M}_j(J) \times S^i \rightarrow M \times S^i$ , which we abusively call  $ev_q$ , for  $q \in \mathbb{CP}^1$ . Choose a sequence of  $(\alpha_i)_{i=0}^\infty$  in  $M \times S^\infty$  as in subsection 3.3.4, satisfying conditions 1, 2 and 3 but we modify 4:

4. For a choice of generators  $b$  of the cohomology of  $M$ , the triple intersection

$$\mathcal{M}'_{i,j}(b, a) := ev_1^{-1}(\alpha_i) \cap ev_0^{-1}(\beta^\vee \times S^i) \cap ev_\infty^{-1}(\iota(\alpha_i)) \quad (3.13)$$

is transverse in  $\mathcal{M}_j(J) \times S^i$ , where  $\iota = \text{id} \times -\text{id}$ .

All such intersections come in pairs, related by  $\iota_{\mathcal{M}} : (u, v) \mapsto (u \circ \mu, -v)$  as in the previous section. Define  $\mathcal{M}_{i,j}(b, a) = \mathcal{M}'_{i,j}(b, a)/\iota_{\mathcal{M}}$ . Given some  $i, j \in \mathbb{Z}_{\geq 0}$ , for  $|b| = 2|a| - i - 2jN$ , where  $N$  is the minimal Chern number of  $M$  and  $b$  a generator of  $H^{|b|}(M)$ , count the number of intersections as in equation (3.13) in the 0-dimensional components of  $\mathcal{M}_{i,j}(b, a)$ .

**Definition 3.4.3** (Quantum Steenrod Square). *For  $a \in H^*(M)$  define  $QS : QH^*(M) \rightarrow QH^*(M)[h]$  such that*

$$QS(a) := \sum_{i,j \in \mathbb{Z}_{\geq 0}, b \in \mathcal{B}, |b|=2|a|-i-2jN} \# \mathcal{M}_{i,j}(b, a) \cdot b T^j h^i$$

*with  $QS$  a linear homomorphism. Then extend  $QS$  linearly to  $QH^*$  by requiring that  $QS(at^k) = QS(a)t^{2k}$ . Also define  $QS_{i,j}(a)$  as previously.*

As in the classical case this is the same as Definition 3.4.1.

### 3.4.2 Quantum Stiefel Whitney Class

For a manifold  $M$ , the classical Stiefel-Whitney class of  $TM$ ,  $w(TM)$ , is constructed as in [8, Section 5.3], using a certain graph operation. We will not go into details.

Let  $x = PD(\{pt\})$ . If

$$a = n \cdot x \cdot h^i + \sum_{y \in \mathcal{B} - \{x\}} n_{i,y} \cdot y \cdot h^{j(y)} \in H_{\mathbb{Z}/2}^*(M)$$

for  $n, n_{i,y}$  coefficients, then  $\langle a, [M] \rangle := nh^i$ . A gluing theorem implies that:

**Lemma 3.4.4.**

$$w(TM) = \sum_{y \in \mathcal{B}} Sq(y) \cdot \langle Sq(y^\vee), [M] \rangle.$$

One can also prove this directly from definitions. If we let  $M$  be a closed monotone symplectic manifold.

**Definition 3.4.5** (Quantum Stiefel-Whitney Class). *The Quantum Stiefel-Whitney class is*

$$w_Q(TM) := \sum_{y \in \mathcal{B}} QS(y) \langle QS(y^\vee), [M] \rangle.$$

It follows from this definition and a grading argument that:

**Lemma 3.4.6.** *If the minimal Chern number  $N > \frac{1}{2} \dim M$  then  $w_Q(TM) = w(TM)$ .*

**Corollary 3.4.7.** *Let  $M = \mathbb{CP}^n$ . Then  $w_Q(TM) = w(TM)$ .*

*Proof.* The minimal Chern number for  $\mathbb{CP}^n$  is  $N = n + 1 > n$ . Now apply Lemma 3.4.6. □

### 3.5 The Quantum Cartan relation

We continue the discussion from Example 3.1.1. Consider the space  $M_{0,5}^\#$  of 5 distinct marked points on the 2-sphere, and let

$$M_{0,5} = M_{0,5}^\# / PSL(2, \mathbb{C})$$

where the Möbius group  $G = PSL(2, \mathbb{C})$  acts diagonally on the 5 marked points. There are two different representations of  $M_{0,5}$  that will be useful:

1.  $\{(z_0, z_1, z_2, z_3, z_4)\}/G$  of five distinct points modulo the action of  $G$ , reparametrising Möbius maps.
2.  $\{(0, 1, \infty, z_3, z_4)\}$  with  $z_3, z_4$  distinct from each other and from  $0, 1, \infty$ .

The former gives a simpler definition of the compactification, but the latter is more useful when describing homology classes. Letting  $z_3, z_4$  vary in the description (2) shows that

$$M_{0,5} \cong ((\mathbb{CP}^1 - \{0, 1, \infty\}) \times (\mathbb{CP}^1 - \{0, 1, \infty\})) - \Delta$$

for  $\Delta$  the diagonal. By compactifying, adding nodal curves as required (there are 10 copies of  $\mathbb{CP}^1 - \{0, 1, \infty\}$  and 15 points to add), there is a space

$$\overline{M}_{0,5} \simeq Bl_{\{(0,0),(1,1),(\infty,\infty)\}}(\mathbb{CP}^1 \times \mathbb{CP}^1).$$

We model  $\overline{M}_{0,5}$  as  $(\mathbb{CP}^1 \times \mathbb{CP}^1) \# 3(\overline{\mathbb{CP}^2})$ , which means:

$$H^*(\overline{M}_{0,5}) = \mathbb{F}_2[\delta_1, \delta_2, w_0, w_1, w_\infty] / (\delta_1^2, \delta_2^2, w_i^3, w_i^2 - \delta_1 \delta_2 \text{ for } i = 0, 1, \infty), \quad (3.14)$$

where  $w_i$  corresponds to the exceptional divisor at  $(i, i)$  and  $\delta_1, \delta_2$  correspond to the spheres  $\mathbb{CP}^1 \times \{pt\}$  and  $\{pt\} \times \mathbb{CP}^1$  respectively: thus all the generators have degree 2. A good treatment of this is [15, Section D.7]. Henceforth  $W_i = PD(w_i)$  and  $\Delta_i = PD(\delta_i)$ .

Let  $X, Y, Z$  be homology classes with Poincaré duals  $x, y, z$  respectively. Let  $\overline{M}_{0,5}^\#$  be the set of pairs  $m^\# = (m, \mu)$  where  $m \in \overline{M}_{0,5}$  and a  $\mu = (\mu_1, \dots, \mu_{k(m)})$  where  $\mu_i \in PSL(2, \mathbb{C})$  and  $k(m)$  is the number of spheres in the nodal curve configuration associated to  $m$ .

There is a natural  $\mathbb{Z}/2$  action on  $\overline{M}_{0,5}$ , and similarly  $\overline{M}_{0,5}^\#$ , induced by (12)(34). Specifically,

$$\iota : (z_0, z_1, z_2, z_3, z_4) \mapsto (z_0, z_2, z_1, z_4, z_3).$$

Then  $\iota \times -\text{id}$  defines a free diagonal  $\mathbb{Z}/2$  action on  $\overline{M}_{0,5} \times S^i$ . Define

$$P_i := (\overline{M}_{0,5} \times S^i)/(\iota \times -\text{id}) \text{ and } P_i^\# := (\overline{M}_{0,5}^\# \times S^i)/(\iota \times -\text{id}).$$

Pick representatives  $X_v, Y_v$  varying smoothly with  $v \in S^\infty$  as previously.  $\mathcal{M}_{i,j}(x, y, z)$  consists of equivalence classes  $[(u, m, v)]$ . Here  $m \in \overline{M}_{0,5}$ , we denote by  $|m|$  the underlying nodal curve configuration corresponding to  $m$ , and hence  $u : |m| \rightarrow M$  is a smooth (nodal) J-holomorphic curve representing a homology class of Chern number  $jN$ , and  $v \in S^i$ . The map  $u$  satisfies  $u(z_0) \in Z^\vee$ ,  $u(z_1) \in X_v$ ,  $u(z_2) \in X_{-v}$ ,  $u(z_3) \in Y_v$  and  $u(z_4) \in Y_{-v}$ . To make this well-defined, we fix a parametrisation of each  $m$  such that for  $m = [(z_0, z_1, z_2, z_3, z_4)] \in \overline{M}_{0,5}$ , we denote  $m = (0, 1, \infty, z_3, z_4)$ , and we proceed to fix parametrisations of the elements of  $\overline{M}_{0,5}$  in a coherent way. The equivalence class  $[(u, m, v)]$  is taken after quotienting by the  $\mathbb{Z}/2$ -action:

$$(u, m, v) \mapsto (u, \iota m, -v).$$

We observe that  $|\iota m|$  can be canonically identified with  $|m|$ .

There is a natural map

$$\pi : \mathcal{M}_{i,j}(x, y, z) \rightarrow P_i$$

obtained by ‘projecting’ onto its last two factors.

**Remark 3.5.1.**  $\mathcal{M}_{i,j}(x, y, z)$  is an inverse image under the evaluation map

$$ev : \mathcal{M}_j(J) \times_G P_i^\# \rightarrow M \times (M^4 \times_{\mathbb{Z}/2} S^i), \quad ev([u, [(p_0, p_1, \dots, p_4), v]]) = (u(p_0), [(u(p_1), \dots, u(p_4)), v])$$

where  $\mathcal{M}(A, J)$  is the set of all J-holomorphic maps  $u : \mathbb{CP}^1 \rightarrow M$  of energy  $j$ . Specifically,

$$\mathcal{M}_{i,j}(x, y, z) = ev^{-1} \left( Z' \times \left[ \bigcup_{v \in \mathbb{RP}^i} X_v \times X_{-v} \times Y_v \times Y_{-v} \right] \right).$$

This allows us to calculate  $\dim \mathcal{M}_{i,j}(x, y, z)$ .

Let  $Q$  be a closed submanifold of the parameter space  $P_i$ . Then  $Q$  represents a cycle in  $H_*(P_i)$ , such that:

$$\dim \pi^{-1}(Q) = |z| - 2|x| - 2|y| + \dim(Q) + 2jN. \quad (3.15)$$

In particular, for  $Q = P_i$ , using  $\dim(P_i) = 4 + i$ ,

$$\dim \mathcal{M}_{i,j}(x, y, z) = \dim \pi^{-1}(P_i) = |z| - 2|x| - 2|y| + i + 4 + 2jN.$$

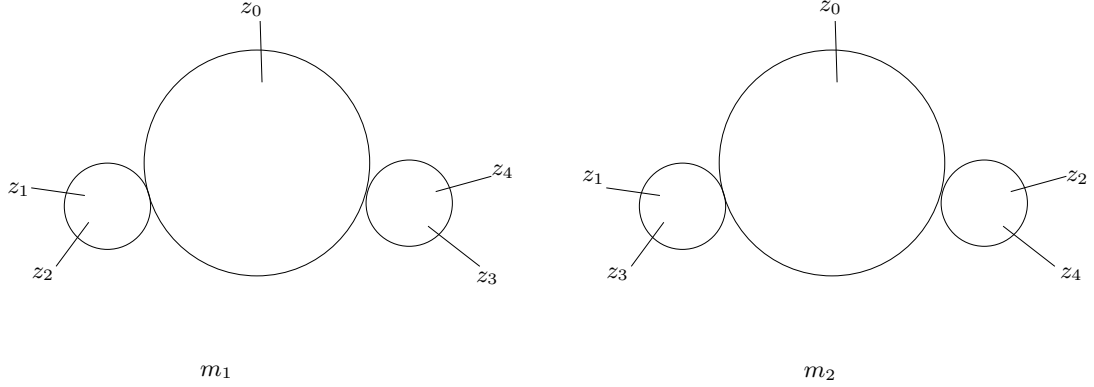


**Figure 3.6:**  $m_1, m_2 \in \overline{M}_{0,5}$

**Definition 3.5.2.** Let  $W$  be a cycle in  $H_*(P_i)$  with  $i, j$  fixed, represented by a union of submanifolds  $Q \subset P_i$  (counted with multiplicity at intersections). Define

$$q_{i,j}(W)(x, y) = \sum_{z: (3.15)=0} \#(\pi^{-1}(Q)) \cdot zT^j$$

where the sum is taken over a basis of  $z$  for  $H^{|z|}(M)$  such that equation (3.15) is 0. Explicitly, this defines a bilinear map

$$q_{i,j}(W) : QH^k(M) \otimes QH^l(M) \rightarrow QH^{k+l-\dim(Q)}(M).$$

Observe  $q_{i,j}(W + W') = q_{i,j}(W) + q_{i,j}(W')$ , as if  $W$  is represented by  $Q$  and  $W'$  by  $Q'$  then  $W + W'$  is represented by  $Q \cup Q'$  (counted with multiplicity at intersections).

**Remark 3.5.3.** More formally, we work with  $\rho = PD(Q) \in H^*(P_i)$ . Consider the cohomology class  $\pi^*\rho \in H^*(\mathcal{M}_{i,j}(x, y, z))$ . It is this cohomology class that defines the operation. However, for intuition we use homology.

In the following, use a cell decomposition for  $S^i$  with cells  $D^{i,\pm}$  in degree  $i$ , corresponding to the two hemispheres of dimension  $i$ . For  $d$  the differential on cellular chains,  $d(D^{i,\pm}) = D^{i-1,+} + D^{i-1,-}$ .

The class of cases we consider are when  $\dim(Q) = i$ . If  $m_1, m_2 \in \overline{M}_{0,5}$  are as given in figure 3.6, then  $m_1$  and  $m_2$  are invariants under the  $\mathbb{Z}/2$  action on  $\overline{M}_{0,5}$ . Hence  $\{m_1\} \times D^{i,+}$  and  $\{m_2\} \times D^{i,+}$  are well defined cycles in  $\overline{M}_{0,5} \times_{\mathbb{Z}/2} S^i$ . For  $p = 1, 2$  call these

$$Q_p^i := \{m_p\} \times D^{i,+}.$$

For  $i > 0$ , the chain “ $\{pt\} \times D^{i,+}$ ” is only a cycle when  $pt$  is a fixed point of the  $\mathbb{Z}/2$  action on  $\overline{M}_{0,5}$ . The space of fixed points  $(\overline{M}_{0,5})^{\mathbb{Z}/2}$  is the disjoint union

of a sphere containing  $m_2$  and the single point  $m_1$ . To see this, consider a general element of  $M_{0,5}$ : after reparametrising we represent this element by  $(0, 1, \infty, z_3, z_4)$ . The  $\mathbb{Z}/2$ -action takes

$$(0, 1, \infty, z_3, z_4) \mapsto (0, 1, \infty, \frac{z_4}{z_4-1}, \frac{z_3}{z_3-1}).$$

Such an element is fixed by  $\mathbb{Z}/2$  exactly when  $z_4 = \frac{z_3}{z_3-1}$ , which determines a 2-dimensional family of points, which is a copy of  $\mathbb{C} - \{0, 1\}$ . When  $z_3$  converges to 1, the point  $z_4$  converges to  $\infty$ , yielding the point  $m_2$ . Likewise when  $z_3$  converges to  $\infty$  add a similar point. When  $z_3$  converges to 0, so too does  $z_4$ . This provides a point in  $W_0$ . When we consider the fixed points in  $W_0$ , we see that there are two: there is  $m_1$  and the limit as  $z_3$  converges to 0 (corresponding to  $z = 2$  in  $W_0$ ). The point  $m_1$  cannot be that limit as  $z_3$  and  $z_4$  converge to 0 from opposite directions.

Hence, when we add the elements in  $\overline{M_{0,5}}$  fixed by  $\mathbb{Z}/2$ , we obtain the disjoint union of an isolated point with the compactification of the 2-dimensional family to a copy of  $\mathbb{CP}^1$ .

**Lemma 3.5.4.**

$$\sum_{i,j} q_{i,j}(Q_1^i)(x, y)h^i = Q\mathcal{S}(x) * Q\mathcal{S}(y) \text{ and } \sum_{i,j} q_{i,j}(Q_2^i)(x, y)h^i = Q\mathcal{S}(x * y).$$

*Proof.* We prove for each fixed  $i, j$  that

$$q_{i,j}(Q_1^i)(x, y) = [Q\mathcal{S}(x) * Q\mathcal{S}(y)]_{i,j}T^j.$$

To do this we proceed as in Lemma 3.3.8, using 1-dimensional moduli spaces, the ends of which count e.g.  $\sum_{i,j} q_{i,j}(Q_1^i)(x, y) \cdot h^i$  and  $Q\mathcal{S}(x) * Q\mathcal{S}(y)$  respectively.

Fixing some  $i \in \mathbb{N}$  (the dimension of the sphere in which  $v$  will vary), we consider the 1-dimensional moduli spaces from Section 3.3.3, denoted  $\tilde{\mathcal{M}}_1(x, y, z)$  and  $\tilde{\mathcal{M}}_2(x, y, z)$ . Recall the spaces  $T^c \cong [0, \infty]$ , and  $\bar{T} \rightarrow T^c$ . For each  $t \in (0, \infty]$ , we define quantum analogues  $\tilde{\mathcal{M}}_{i,j}(x, y, z)$  of the  $\tilde{\mathcal{M}}_i(x, y, z)$ , where now each element of  $\tilde{\mathcal{M}}_{i,j}(x, y, z)$  is, rather than a pair  $(u, v)$  where  $u : (|t|, t) \rightarrow M$  is continuous, we require that  $u : (|t|_Q, t) \rightarrow M$  is continuous. Here,  $|t|_Q$  is obtained by taking the graph associated to  $|t|$ , and adding a sphere at each trivalent vertex (in such a way that the incoming edge of  $t$  is attached at 0 on the sphere, and the outgoing vertices are attached at  $1, \infty$  respectively). We then require that  $u$  is  $J$ -holomorphic on each sphere, satisfies the edge and asymptotic equations as in Section 3.3.3, and the sum of the Chern numbers of the three spheres is  $jN$ .

For the  $t = 0$  end of the moduli spaces, we instead consider for  $\tilde{\mathcal{M}}_{i,j}(x, y, z)$  continuous maps  $u : |m_i|' \rightarrow M$ . Here,  $m_i$  are the elements of  $\overline{M}_{0,5}$  as in Figure 3.6, and  $|m_i|'$  is obtained from the nodal sphere configuration  $|m_i|$  associated to  $m_i$ , by attaching to  $z_0$  the negative half-line  $(-\infty, 0]$  and to  $z_1, z_2, z_3, z_4$  the positive half-line  $[0, \infty)$ . We then require the same conditions on the edges, the asymptotics and the energy of the nodal spheres.

It is then immediate from the definition that counting the setups corresponding to the  $t = 0$  end of  $\mathcal{M}_{p,j}(x, y, z)$  is the coefficient of  $z$  in  $q_{i,j}(Q_p^i)(x, y)h^i$ , for each  $i$ . Hence, it remains to show that counting the  $t = \infty$  ends of  $\mathcal{M}_{p,j}(x, y, z)$  corresponds to the coefficient of  $Q\mathcal{S}(x) * Q\mathcal{S}(y)$  and  $Q\mathcal{S}(x * y)$  respectively for  $p = 1, 2$ . The proof of this is identical to Lemmas 3.3.8 and 3.3.9.  $\square$

Let  $A^i = Q_1^i - Q_2^i$ . Then, by the above Theorem 3.1.2 holds. Let  $W_0$  be the pullback under the blowdown

$$Bl_{(0,0),(1,1),(\infty,\infty)}(\mathbb{CP}^1 \times \mathbb{CP}^1) \rightarrow Bl_{(0,0)}(\mathbb{CP}^1 \times \mathbb{CP}^1),$$

of the exceptional  $\mathbb{CP}^1$  divisor in  $Bl_{(0,0)}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ .

The group  $H_r(P_i)$ , for  $r > 2, r \neq i + 2$ , is generated by the cycles (represented by submanifolds):

$$\{m_1\} \times D^{r,+}, \quad W_0 \times D^{r-2,+}, \quad \overline{M}_{0,5} \times D^{r-4,+}.$$

To see this, suppose that  $k \leq j \in \mathbb{Z}_{>0}$  and  $A = A_k \times D^{r-k,+} + \dots + A_j \times D^{r-j,+} \in H_r(P_i)$  is closed (where  $A_i \in C_i(\overline{M}_{0,5})$  and  $A_k, A_j \neq 0$ ). Recall that the differential  $d_{\mathbb{Z}/2}$  is defined by

$$d_{\mathbb{Z}/2}(A \times D^{i,+}) = (dA) \times D^{i,+} + A \times (D^{i-1,+} + D^{i-1,-}) = (dA) \times D^{i,+} + (A + \iota A) \times D^{i-1,+}.$$

Then in particular,  $dA_k = 0$ . Recall our submanifold representatives of the generators of  $H_*(\overline{M}_{0,5})$ , as in (3.14) and the proceeding discussion. Note that  $i_*[\Delta_1] = [\Delta_2]$  and  $i_*[W_1] = [W_\infty]$ . We note that:

- The fact that  $d_{\mathbb{Z}/2}(A) = 0$  implies (for  $r - k > 0$ ) that

$$A_k + \iota A_k = dA_{k+1},$$

and hence  $\iota_*[A_k] = [A_k]$  as elements of  $H_*(\overline{M}_{0,5})$ .

- Thus,  $k = 0, 2$  or  $4$  and  $[A_k]$  is in  $(H_*(\overline{M}_{0,5}))^{\mathbb{Z}/2}$ , which is generated by

$$[\{m_1\}], [\Delta_1 + \Delta_2], [W_0], [W_1 + W_\infty], \text{ and } [\overline{M}_{0,5}].$$

- If  $[A_k] = [B]$  then  $A_k = B + dC$  for some  $C \in C_{k+1}(\overline{M}_{0,5})$ . Hence

$$A_k \times D^{r-k,+} = (B + dC) \times D^{r-k,+} = B \times D^{r-k,+} + (C + \iota C) \times D^{r-k-1,+} + d_{\mathbb{Z}/2}(C \times D^{r-j,+}).$$

Hence without loss of generality we assume  $A_k = B$ , where  $B$  is one of the submanifold representatives above.

- Suppose that  $A_k = \Delta_1 + \Delta_2$  (hence  $k = 2$ ). If  $r \leq i + 1$  (hence  $r - 1 \leq i$ ), then  $[A] = [A - d(\Delta_1 \times D^{r-1,+})] = [A - (\Delta_1 + \Delta_2) \times D^{r-2,+}] \in H_r(P_i)$ . A similar argument holds for  $A_k = W_1 + W_\infty$ . Hence, we may assume that  $A_k = \{m_1\}, W_0$  or  $\overline{M}_{0,5}$ .
- As the  $\{m_1\}, W_0$  and  $\overline{M}_{0,5}$  are fixed sets under  $\iota$ , then  $A_k \times D^{r-k,+}$  is closed under  $d_{\mathbb{Z}/2}$ . We iterate with  $A$  replaced by  $A - A_k \times D^{r-k,+}$ .

The above bullet points imply that  $H_r(P_i)$  is indeed generated by the given cycles for  $2 < r < i + 2$ .

For  $r = i + 2$ , we need additionally  $(W_1 + W_\infty) \times D^{i,+}$  and  $(\Delta_1 + \Delta_2) \times D^{i,+}$ . For  $r \leq 2$ , we need additionally  $A \times D^{0,+}$  for any closed  $A$ , e.g.  $A = \Delta_i, W_1, W_\infty$ . The exceptional divisor  $W_0$  consists of sphere configurations as in Figure 3.7, removing reparametrization freedom by fixing the positions of  $z_0, z_1, z_2, z_3, z_4$  and letting  $z$  vary. To see that this is the case, observe that the exceptional divisor in the blowup over the point  $(0, 0)$  in  $\overline{M}_{0,5}$  (which we consider as the compactification of  $\{(0, 1, \infty, z_3, z_4)\}$  in this case) contains exactly the limiting configurations as  $z_3, z_4$  both coincide with  $z_0 = 0$  (the “relative velocity” of approach of  $z_3, z_4$ , i.e. the relative directions and relative “speed of approach” determines the element of  $W_0$  that we obtain). These are then the elements as given in the figure. We see that there are three special curves, when the free point  $z$  “collides” with each of the fixed points, giving rise to the three bubbling configurations as shown in the figure. To compute  $A^i$  we intersect chains, observing the following:

1. For the intersection theory in degree  $r$ , we may assume that  $r = i$ , so we work with  $\overline{M}_{0,5} \times_{\mathbb{Z}/2} S^i$  which is of dimension  $i + 4$ .

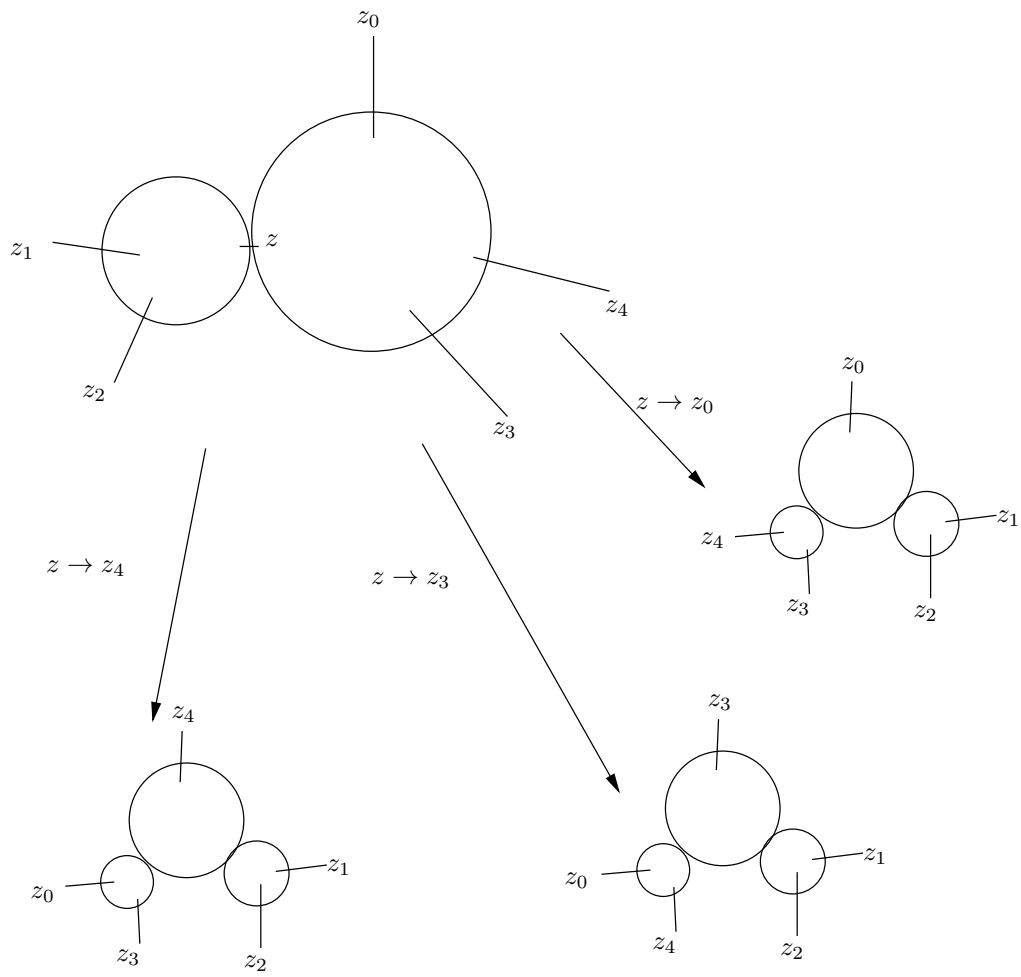


**Figure 3.7:** Elements of  $W_0$

2. So the elements of dual dimension to  $\{m_p\} \times D^{i,+} \in H^i(\overline{M}_{0,5} \times_{\mathbb{Z}/2} S^i)$  are in degree 4, and (for  $i > 2$ ) these are generated by  $\{m_1\} \times D^{4,+}$ ,  $W_0 \times D^{2,+}$  and  $\overline{M}_{0,5} \times D^{0,+}$ .
3. Hence if two homology classes in degree  $i$  have the same intersections with the three degree 4 elements listed above, then they must be the same by Poincaré duality.

**Lemma 3.5.5.**

$$[\{m_1\} \times D^{i,+}] = [\{m_2\} \times D^{i,+}] + [W_0 \times D^{i-2,+}]$$

in  $H_i(P_i)$ .

*Proof.* We tabulate the intersections of cycles (except for the  $i = 2$  case):

|                                     | $\{m_1\} \times D^{i,+}$ | $\{m_2\} \times D^{i,+}$ | $W_0 \times D^{i-2,+}$ |
|-------------------------------------|--------------------------|--------------------------|------------------------|
| $\{m_1\} \times D^{4,+}$            | 1                        | 0                        | 1                      |
| $W_0 \times D^{2,+}$                | 1                        | 0                        | 1                      |
| $\overline{M}_{0,5} \times D^{0,+}$ | 1                        | 1                        | 0                      |

For the bottom row, we use the standard representative

$$\{(m, v_0) | m \in \overline{M}_{0,5}, D^{0,+} = \{v_0\}\}$$

of  $\overline{M}_{0,5} \times D^{0,+}$ . Then  $(m_i, v_0)$  is the unique intersection for entries  $(3, 1)$ ,  $(3, 2)$  respectively. Using the projection  $\overline{M}_{0,5} \times_{\mathbb{Z}/2} S^i \rightarrow \mathbb{RP}^i$ , and considering intersections in  $\mathbb{RP}^i$ , we see that entry  $(3, 3)$  is 0. For entry  $(2, 3)$  we observe  $\mathbb{RP}^2$  and  $\mathbb{RP}^{i-2}$  (represented by  $D^{2,+}$  and  $D^{i-2,+}$  respectively) intersect at a point, and a generic choice of representative of  $W_0$  self-intersects at a point using (3.14). For the middle column we choose the standard representative of  $\{m_2\} \times D^{i,+}$  with  $m_2$  as in Figure 3.6, and observe that the intersection with  $W_0$  (and therefore also with  $m_1$ ) is trivial.

For entry  $(1, 1)$ , we pick a  $\mathbb{Z}/2$ -invariant neighbourhood  $U$  of  $m_1 \in \overline{M}_{0,5}$ , which is parametrised by a 4-ball  $\phi : D^{4,+} \xrightarrow{\cong} U$ . Let the representative of  $\{m_1\} \times D^{4,+}$  be  $\{(\phi(v), v) | v \in D^{i,+}\}$  and the representative of  $\{m_1\} \times D^{i,+}$  is  $\{(m_1, v) | v \in D^{i,+}\}$ . These intersect at the unique point  $(m_1, v)$  where  $\phi(v) = m_1$ .

For entry  $(1, 3)$  we recall  $\mathbb{RP}^{i-2} \cap \mathbb{RP}^4 = \mathbb{RP}^2$  in  $\mathbb{RP}^i$ . Recall for our choice of representative of  $\{m_1\} \times D^{4,+}$  given above, this neighbourhood  $U$  is the unit normal bundle and splits to give two rank 2 discs, one tangent and one normal to  $W_0$ . Given generic choices of  $\mathbb{RP}^{i-2}$  and  $\mathbb{RP}^4$ , we use the resulting  $\mathbb{RP}^2$  obtained by intersection

to give the 2-family  $\{\phi(v)|v \in D^{2,+}\} \subset \overline{M_{0,5}}$ , which generically is nowhere tangent to  $W_0$ . This ensures that this family intersects  $W_0$  transversely at the point  $m_1$ .

For entry  $(2, 1)$ , argue similarly to Lemma 3.3.13: consider the normal bundle of  $W_0$ , which is rank 2. There is a 2-dimensional neighbourhood  $V$  of  $m_1$  in  $W_0$  such that this normal bundle is trivial, i.e. locally homomorphich to  $V \times D^{2,+}$ . Ensure that on  $V$  the representative of  $W_0 \times D^{2,+}$  takes the standard form  $\{(x, v)|x \in V, v \in D^{2,+}\}$ . This intersects once with  $\{(m_1, v)|v \in D^{i,+}\}$  at  $(m_1, 0)$ .

For  $i = 2$ , the cohomology group in degree 4 has more generators to consider, meaning we must add more rows. However these extra generators have trivial intersection with each of the columns, and so all the extra rows will be 0.  $\square$

*Proof of Theorem 3.1.2.* By lemma 3.5.5,  $q_{i,j}(\{m_1\} \times D^{i,+}) = q_{i,j}(\{m_2\} \times D^{i,+}) + q_{i,j}(W_0 \times D^{i-2,+})$ . Multiply by  $h^i$  and sum over all  $i, j$  and apply Lemma 3.5.4.  $\square$

We will verify Theorem 3.1.2 in the case of  $\mathbb{CP}^1$ .

**Example 3.5.6** ( $\mathbb{CP}^1$ ). *Let  $x$  be the generator of 2-dimensional cohomology. We verify that*

$$[Q\mathcal{S}(x) * Q\mathcal{S}(x)]_{i,j} = Q\mathcal{S}_{i,j}(x * x) + q_{i,j}(W_0 \times D^{i-2,+})(x, x).$$

*This is trivial except for  $(i, j) = (4, 1)$ .*

$$[Q\mathcal{S}(x) * Q\mathcal{S}(x)]_{4,1} = [T^2 + h^4 T]_{4,1} = h^4 T$$

$$Q\mathcal{S}_{4,1}(x * x) = Q\mathcal{S}_{4,1}(T) = [T^2]_{4,1} = 0$$

*so it remains to calculate  $q_{4,1}(W_0 \times D^{2,+})(x, x)$ .*

*Pick representatives of  $PD(x) \times S^2$  as follows: let  $\phi : S^2 \rightarrow D^2$  be the “flattening map” of the sphere, e.g. if  $S^2 \subset \mathbb{R}^3$  it is projection onto  $\mathbb{R}^2 \subset \mathbb{R}^3$ . Pick two disjoint discs in  $\mathbb{CP}^1 = S^2$ , call them  $D$  and  $D'$ , and pick maps  $\eta$  and  $\eta'$  identifying  $D^2$  with  $D, D'$  respectively. Let  $\psi = \eta \circ \phi$ , and likewise  $\psi'$ . Then two representatives of  $PD(x)_v = \{pt\}_v$  are  $\psi(v)$  and  $\psi'(v)$  where  $v$  varies in  $S^2$ . Recall that elements of a standard choice for  $W_0$  are as in figure 3.7.*

*The sphere  $S^2$  has Chern number  $= N = 2$  so only one of the  $J$ -holomorphic spheres may be non-constant, and has degree 1. The sphere with three marked points must be trivial (if the other sphere were constant, then the solution cannot be rigid as  $z_4$  can vary freely), and the other sphere of  $u$  has degree 1. Then  $u(z_1) = \psi(v)$  and  $u(z_2) = \psi(-v)$  meet at the unique point on the sphere where  $\psi(v) = \psi(-v)$ . Hence there is one solution, and this solution gives the correction term  $h^4 T$ .*

**Lemma 3.5.7.**  $q(W)$  only contains terms of the form  $aT^ph^q$  for  $p > 0, q > 0$ .

*Proof.* The  $p = 0$  case corresponds to  $J$ -holomorphic maps that are constant. For such spheres, if there is a solution for some  $m \in W_0$  there is a solution for every element of  $W_0$ . So there are no isolated solutions. For  $q = 0$ , the  $h^q$  terms correspond to  $q(W \times D^{q-2,+})$  which vanishes for  $q < 2$ . Specifically  $S^\infty$  has no cells of negative dimension.  $\square$

**Remark 3.5.8** (Quantum Cartan in Classical Case). *Lemma 3.5.7 gives a sanity check that in the classical case,  $Sq(x) \cup Sq(y) = Sq(x \cup y)$ .*

## 3.6 Computing the Quantum Steenrod Square for toric varieties

### 3.6.1 Quantum Steenrod squares for $\mathbb{CP}^n$

Let  $x^i$  generate  $H^{2i}(\mathbb{CP}^n)$ . By the quantum Cartan relation, Theorem 3.1.2,

$$Q\mathcal{S}(x^{i+1}) = Q\mathcal{S}(x^i) * Q\mathcal{S}(x) + q(W)(x^i, x)$$

We can iteratively construct  $Q\mathcal{S}(x^{i+1})$  as long as we know  $q(W)(x^i, x)$ . For degree reasons,  $Q\mathcal{S}(x) = xh^2 + x * x$ .

**Lemma 3.6.1.** *For  $2i < n$ ,  $q(W)(x^i, x) = 0$ . For  $n \leq 2i \leq 2n$ ,*

$$q(W)(x^i, x) = \binom{i}{n-i} Th^{4i+2-2n}.$$

*Proof.* Observe that  $q(W)(x^i, x)$  has degree  $4i + 4$  so, by lemma 3.5.7, for  $i = 1, \dots, n$  we deduce that:

$$\begin{aligned} q(W)(x^i, x) &= \sum_{j=n-i}^i m_j^{i+1} x^{i+j-n} Th^{2(i+1)-2j} \\ &= m_{n-i}^{i+1} x^0 Th^{4i+2-2n} + m_{n-i-1}^{i+1} x^1 Th^{4i-2n} + \dots + m_i^{i+1} x^{2i-n} Th^2 \end{aligned} \quad (3.16)$$

where  $m_j^{i+1}$  are coefficients and the degrees are  $|x| = 2$ , so  $|x^i| + |x| = 2i + 2$  and  $|T| = 2(n + 1)$ . Equation (3.16) follows for grading reasons.

We claim that  $m_j^{i+1}$ , the coefficient of  $x^{i+j-n} Th^{2(i+1)-2j}$ , is the number of (unparametrised)  $J$ -holomorphic spheres that intersect both  $\mathbb{CP}^{i+j-n}$  and  $PD(Sq^{2j}(x^i))$ . This follows by three observations:

- i) Counting the component of  $x^{i+j-n}Th^{2(i+1)-2j}$  is the same as counting setups as in Figure 3.8(1) for  $v \in D^{2i-2j,+}$  (recall  $D^{2i-2j,+}$  corresponds to the  $h^{2i-2j+2}$  term when defining  $q(W)$  in Theorem 3.1.2).

Only  $T^1$  appears, so one of the holomorphic bubbles has degree 0 and the other degree 1. For the solutions to be rigid the sphere with the  $z_1, z_2$  must be constant, giving the setup in Figure 3.8(2).

- ii) The sphere is a  $\mathbb{CP}^1$  and  $\mathbb{CP}^1 \cap \mathbb{CP}_v^{n-1} = \{pt_v\}$  for each  $v$ . The sphere cannot be contained in  $\mathbb{CP}_v^{n-1}$  because if it were then under a generic setup it could not satisfy all of the other intersection conditions.
- iii) The intersection of  $\mathbb{CP}_v^{n-i}$  and  $\mathbb{CP}_{-v}^{n-i}$ , taken over all  $v \in D^{2i-2j,+}$ , is  $PD(Sq^{2j}(x^i))$ .

The output of  $Sq(x^i)$  corresponding to  $D^{2i-2j,+}$  is

$$Sq_{2(i-j)}(x^i) = Sq^{2j}(x^i) = \binom{i}{j} x^{i+j}$$

with  $PD(x^{i+j}) = \mathbb{CP}^{n-(i+j)}$ , for  $i+j \leq n$ . So  $m_j^{i+1}$  is:

$$\binom{i}{j} \cdot (\# \text{lines intersecting } \mathbb{CP}^{i+j-n} \text{ and } \mathbb{CP}^{n-(i+j)}).$$

As  $i+j-n \geq 0$  and  $n-(i+j) \geq 0$  the only nontrivial term arises for  $j = n-i$ , in which case there is precisely one line.  $\square$

**Theorem 3.6.2.** *For all  $i \geq 0$ ,*

$$Q\mathcal{S}(x^i) = \sum_{j=0}^i \left( \binom{i}{j} + \sum_{k=0}^{\lfloor n/2 \rfloor + 1} \binom{n-k}{k} \cdot \binom{i-(n+1-k)}{j-k} \right) x^{i+j} h^{2(i-j)}, \quad (3.17)$$

where  $x^{i+j}$  is the  $(i+j)$ -th quantum power of  $x$ .

Observe that if  $i+j \geq n$  then  $x^{i+j} = x^{i+j-n}T$ , as this is the quantum power.

Recall  $Q\mathcal{S}(x^i) = Sq(x^i) + \dots$  where by example 3.2.4:

$$Sq(x^i) = \sum_{j=0}^{n-i} \binom{i}{j} x^{i+j} h^{2i-2j}.$$

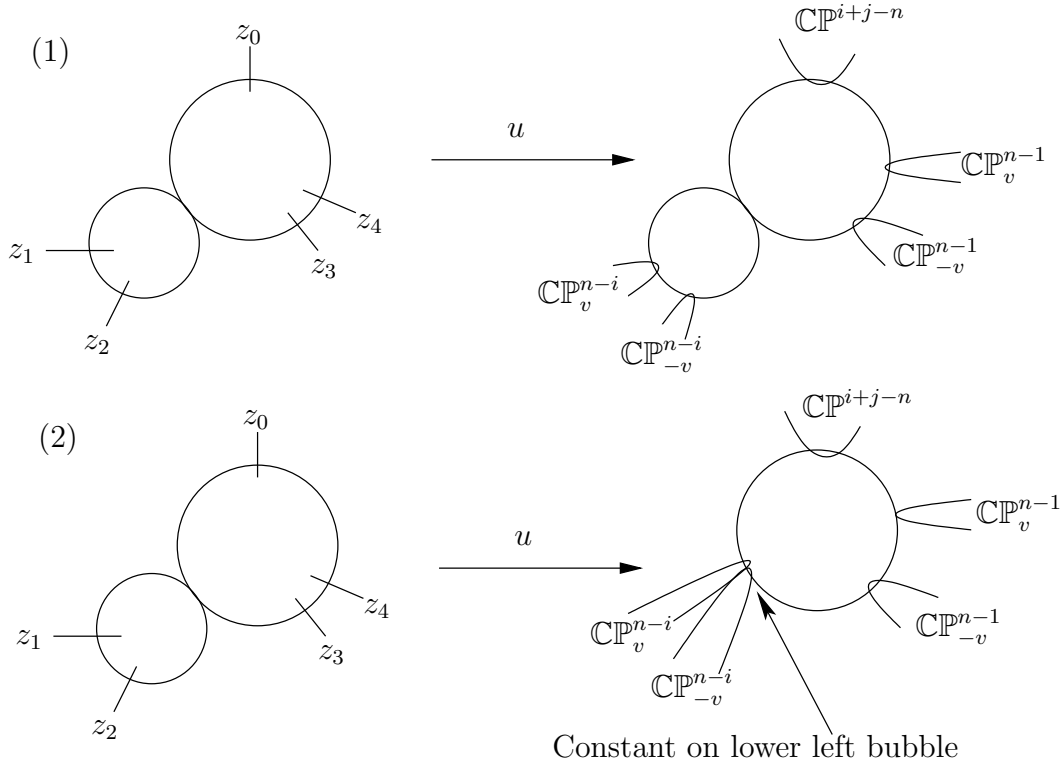


**Figure 3.8:** Configurations for  $q(W)(x^i, x)$  for  $\mathbb{CP}^n$

*Proof of theorem 3.6.2.* Since  $T = x^{n+1}$ , we can express the square as:

$$\begin{aligned}
 QS(x^i) &= \sum_{j=0}^i l_j^i x^{i+j} h^{2i-2j} \\
 &= l_0^i x^i h^{2i} + l_1^i x^{i+1} h^{2i-2} + \dots + l_i^i x^{2i} h^0
 \end{aligned}$$

for some  $l_j^i \in \mathbb{Z}/2$ .

By the Quantum Cartan relation, Theorem 3.1.2 and Lemma 3.6.1, the coefficients  $l_j^i$  satisfy  $l_j^{i+1} = l_j^i + l_{j-1}^i$  for  $j \neq n-i$  and  $l_{n-i}^{i+1} = l_{n-i-1}^i + l_{n-i}^i + \binom{i}{n-i}$  (the latter comes from the quantum correction). Using a Pascal Triangle and the iterative formula for the  $l_j^i$ , one can write down the closed form solution.  $\square$

In particular, truncating the sum in equation (3.17) to  $j \leq n-i$  recovers the classical Steenrod square formula for  $\mathbb{CP}^n$  from example 3.2.4. This is because if  $j \leq n-i$  then every term in the second summation in equation (3.17) vanishes because either

- $j - k < 0$
- or  $i - (n+1-k) = k + i - n - 1 \leq j + i - n - 1 \leq -1 < 0$ , using for the first inequality that  $j \geq k$  and in the second that  $j \leq n-i$ .

Explicit examples:

$$\mathbb{CP}^1 : q(W)(x, x) = \begin{pmatrix} 1 \\ 1-1 \end{pmatrix} Th^{4+2-2}$$

$$Q\mathcal{S}(x) = xh^2 + T$$

$$Q\mathcal{S}(T) = (xh^2 + T)^2 + \mathbf{T}h^4 = T^2.$$

$$\mathbb{CP}^2 : q(W)(x, x) = \begin{pmatrix} 1 \\ 2-1 \end{pmatrix} Th^{4+2-4}$$

$$Q\mathcal{S}(x) = xh^2 + x^2$$

$$Q\mathcal{S}(x^2) = (xh^2 + x^2)^2 + \mathbf{T}h^2 = x^2h^4 + Th^2 + xT.$$

$$\mathbb{CP}^3 : q(W)(x, x) = \begin{pmatrix} 1 \\ 3-1 \end{pmatrix} Th^{4+2-6} = 0 \text{ and } q(W)(x^2, x) = \begin{pmatrix} 2 \\ 3-2 \end{pmatrix} Th^{8+2-6} = 0$$

$$Q\mathcal{S}(x) = xh^2 + x^2$$

$$Q\mathcal{S}(x^2) = (xh^2 + x^2)^2 = x^2h^4 + T$$

$$Q\mathcal{S}(x^3) = (xh^2 + x^2)(x^2h^4 + T) = x^3h^6 + Th^4 + xTh^2 + x^2T.$$

**Remark 3.6.3.** Observe that, after one appeals to dimension reasons to rule out the other cases, the proof of Theorem 3.6.2 only uses  $GW(\mathbb{CP}^{n-1}, \{pt\}, \{pt'\})$ .

### 3.6.2 Fano Toric Varieties

Let  $M$  be a compact monotone toric manifold, with  $b \in H^{|b|}(M)$  and  $x \in H^2(M)$ . Then analogously to  $\mathbb{CP}^n$ , one proves Theorem 3.1.3.

*Proof of Theorem 3.1.3.* Consider setups as in Figure 3.9. We describe related setups, which we call *reduced setups*, where we neglect the cycle  $X_v$  and the marked point  $z_4$  corresponding to it. We still have a dimension 0 setup: removing  $z_4$  removes 2 dimensions, and removing the intersection with  $X_v$  adds 2 dimensions.

Setups and reduced setups are in a 1 to  $\#(X \cap \mu)$  correspondence. Strictly, for every reduced setup (so a pair consisting of a  $J$ -holomorphic map with evaluation conditions and some  $v_0 \in D^{i,+}$ ), there are exactly  $\#(X_{v_0} \cap \mu)$  setups. We must note that a generic point in  $S^2$  is an injective point of  $u$ , e.g. [14, Proposition 1.3.1] and hence fixing an intersection point in the image of  $u$  fixes the free point  $z_4$  in  $W_0$ . A generic choice of  $X_v$  ensures that  $\#(X_{v_0} \cap \mu) = \#(X \cap \mu)$  for each  $v_0$ . Hence it is sufficient to prove that reduced setups count

$$\sum_{2i=0}^{|b|} \sum_{j \geq 1} \sum_{k=1}^j \sum_{\mu \in H_2(M): E(\mu)=k} (Q\mathcal{S}_{2i,j-k}(b) *_{\mu,k} x) \cdot h^{|b|-2i+2} T^j.$$

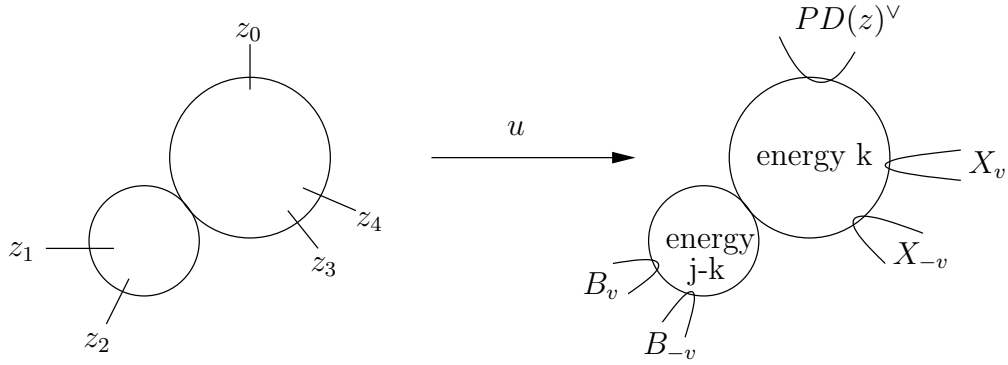


**Figure 3.9:** Configurations for  $q(W)(b, x_p)$  for toric varieties, where  $X = PD(x)$ ,  $B = PD(b)$  and  $z$  is the output. Energy  $k$  means  $E(u) = c_1(u_*[S^2])/2N = k$ .

However, considering reduced setups alone one may choose  $X_{-v}$  to be independent of  $v \in D^{i,+}$ . The result follows immediately from the definitions of  $QS$  and the quantum product.  $\square$

As the cohomology of a toric variety  $M$  is generated by  $H^2(M)$ , iterated application of (3.5) yields a general solution, i.e. one can calculate  $QS(x_{p_1}x_{p_2}\dots x_{p_r})$  assuming the base cases  $QS(x_{p_i})$  for a basis  $\{x_p\}$  of  $H^2(M)$  are known. Using a combination of degree reasons and the classical Steenrod square,  $QS(x_p) = x_p * x_p + x_p \cdot h^2$ .

**Corollary 3.6.4.** *For  $M$  a compact, monotone toric variety,  $QS$  is determined completely by  $QH^*(M)$ .*

*Proof.* We induct on degree. The base case  $|x| = 2$ ,  $QS(x) = xh^2 + x * x$  is determined by  $QH^*(M)$ . Given  $a \in H^*(M)$  for  $|a| > 2$ , write  $a = b * x$  for  $x \in H^2(M)$ . By Theorem 3.1.2,  $QS(a) = QS(b) * QS(x) + q(W)(b, x)$ . By induction  $QS(b)$  and  $QS(x)$  are determined by  $QH^*(M)$ , hence so is  $QS(b) * QS(x)$ . By Theorem 3.1.3,  $q(W)(b, x)$  is determined by  $QH^*(M)$ .  $\square$

Let  $\beta : QH^*(M) \rightarrow QH^*(M)$  be a ring homomorphism satisfying  $\beta(T) = T$ . In the notation of Theorem 3.1.3, we deduce that for  $a, b \in QH^*(M)$ , then  $\beta(a *_{0,0} b) = \beta(a) *_{0,0} \beta(b)$ . This is because  $\mu = 0$  is the only possible element of  $H_2(M, \mathbb{Z})$  of Chern number 0 when  $M$  is monotone. Indeed, Theorem 3.1.3 simplifies to state that if  $|x| = 2$ , then

$$QS(b * x) = QS(b) * QS(x) + (QS(b) * x - QS(b) *_{0,0} x).$$

Thus, any ring homomorphism  $\beta$  with the given constraint is compatible with the quantum Steenrod square.

**Example 3.6.5** ( $\mathbb{CP}^1 \times \mathbb{CP}^1$ ). We let  $x, y$  be the generators of  $H^2(\mathbb{CP}^1 \times \mathbb{CP}^1)$ , with  $PD(x) = \{pt\} \times \mathbb{CP}^1$  and  $PD(y) = \mathbb{CP}^1 \times \{pt'\}$ . Here  $q(W_0 \times D^{i-2,+})(x, y) = 0$  hence  $QS(x) * QS(y) = QS(x * y)$ . Indeed by equation (3.5),

$$q(W_0 \times D^{i-2,+})(x, y) = \sum_{2i=0}^2 \sum_{j \geq 1} \sum_{k=1}^j k \cdot QS_{2i,j-k}(x) *_{\mu,k} y h^{4-2i} T^j.$$

Working from definitions,  $QS(x) = xh^2 + T$ . Then  $\alpha *_{\mu,k} y \neq 0 \implies k = 1, \alpha = y$ . There are no  $i, j, k$  such that  $QS_{2i,j-k}(x) = y$ . Hence the sum on the right hand side is 0.

## 3.7 The Quantum Adem Relations

### 3.7.1 Classical Adem Relations

We begin with a discussion of the group cohomology of  $S_4$  and  $D_8$ . This will involve adding details to the argument alluded to by Cohen-Norbury to prove the classical relations in [8, Section 5.2].

It is proved in [2, Sections IV.1, VI.1] that

$$H^*(BD_8) = \mathbb{Z}/2[e, \sigma_1, \sigma_2]/(e\sigma_1) \quad (3.18)$$

where  $e, \sigma_1$  are of degree 1 and  $\sigma_2$  is of degree 2, and

$$H^*(BS_4) = \mathbb{Z}/2[n_1, n_2, c_3]/(n_1 c_3), \quad (3.19)$$

where again subscripts denote the degree of the elements. Considering

$$D_8 = \langle (12), (34), (13)(24) \rangle \subset S_4,$$

there are subgroups

$$\mathbb{Z}/2 = \langle (13)(24) \rangle, \quad \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle (12), (34) \rangle.$$

Then

$$H^*(B\mathbb{Z}/2) = \mathbb{Z}/2[e], \quad H^*(B(\mathbb{Z}/2 \times \mathbb{Z}/2)) = \mathbb{Z}/2[x, y].$$

Consider the commutative diagram (3.20) induced by the various inclusion maps of groups. As in [2], one shows that:

$$\begin{array}{ccc}
H^*(B\mathbb{Z}/2) & & \\
\uparrow i_1 & \nwarrow j_1 & \\
H^*(BD_8) & \xleftarrow{\pi^*} & H^*(BS_4) \\
\downarrow i_2 & \swarrow j_2 & \\
H^*(B(\mathbb{Z}/2 \times \mathbb{Z}/2)) & & 
\end{array} \tag{3.20}$$

$$i_1(e) = e$$

$$i_2(\sigma_1) = x + y \quad i_2(\sigma_2) = xy$$

$$j_1(n_2) = e^2$$

$$j_2(n_1) = x + y \quad j_2(n_2) = xy$$

All of the other generators map to 0 via the  $i, j$  maps. From this, and the fact that  $\pi^*$  is injective, we deduce that

$$\pi^*(n_1) = \sigma_1 \quad \pi^*(n_2) = \sigma_2 + e^2 \quad \pi^*(c_3) = e\sigma_2.$$

By Cohen-Norbury, [8], there is a commutative diagram, namely diagram (3.21), where  $qq$  satisfies

$$qq(\alpha) = \sum_{p,q} Sq^q \circ Sq^p(\alpha) e^{|\alpha|+p-q} \sigma_2^{|\alpha|-p}.$$

$$\begin{array}{ccc}
H^*(M) & \xrightarrow{q_{S_4}} & H^*(M) \otimes H^*(BS_4) \\
\downarrow = & & \downarrow id_{H^*(M)} \otimes \pi^* \\
H^*(M) & \xrightarrow{qq} & H^*(M) \otimes H^*(BD_8)
\end{array} \tag{3.21}$$

We do not in general know a closed form definition of  $q_{S_4}$ , e.g. in terms of compositions of Steenrod squares, but the Adem relation is a purely algebraic relation, only using the fact that  $qq$  lifts to a function  $q_{S_4}$  (and not any information about the homomorphism itself).

**Fact 1.** *By Theorem 19 (Invariance) in [8], the diagram (3.21) commutes. This implies that the image of  $qq$  lies in the image of  $id_{H^*(M)} \times \pi^*$ . Hence there are constraints on the image. Specifically,  $e^{2i}\sigma_2^j$  may only appear in  $qq(\alpha)$  if it arises from some  $(e\sigma_2)^{2k}(e^2 + \sigma_2)^{i+j-3k}$  for  $k = 0, 1, \dots$ , with coefficient  $\binom{i+j-3k}{i-k}$ . This will be proven as a special case of Lemma 3.7.8.*

**Lemma 3.7.1.** *For any  $s, m$ ,*

$$\binom{3s+m}{s+m} = \sum_{l=0}^{\infty} \binom{m+l-1}{2l} \binom{3s+m}{s-l} \tag{3.22}$$

*modulo 2.*

*Proof.* We prove this by induction. Let  $c(m, s) = \binom{m+3s}{m+s}$ . Then modulo 2,

$$c(m+2, s) = c(m, s) + c(m+3, s-1).$$

Define  $S(m, s) = \sum_l \binom{m-1+l}{2l} \binom{3s+m}{s-l}$ . Check that  $S(m, s) = c(m, s)$  for  $s = 0, 1$  and  $m = 1, 2$ . These are the base cases. Hence if  $S(m+2, s) = S(m, s) + S(m+3, s-1)$  for all  $m, s$  then the lemma holds by induction. This is an exercise in binomial coefficient algebra modulo  $\mathbb{Z}/2$ .  $\square$

**Theorem 3.7.2** (Classical Adem Relations). *Given  $\alpha \in H^*(M)$  and  $q, p > 0$  such that  $q < 2p$ ,*

$$Sq^q Sq^p(\alpha) = \sum_{k=0}^{\lfloor q/2 \rfloor} \binom{p-k-1}{q-2k} Sq^{p+q-k} Sq^k(\alpha).$$

*Proof.* Suppose  $q$  is even. Let  $l = |\alpha| - p$ ,  $m = p - q/2$ ,  $n = q/2 - k$ , thus

$$\binom{p-k-1}{q-2k} = \binom{m+n-1}{2n}.$$

Assume  $l = 2r$ . The cases for  $q$  or  $l$  odd are similar. Throughout let  $\text{cff}(E)$  be the coefficient of  $E$  in  $qq(\alpha)$ . By definition of  $qq$

$$Sq^q \circ Sq^p(\alpha) = \text{cff}(e^{l+2m}\sigma_2^l) \quad \text{and} \quad Sq^{p+1-k} \circ Sq^k(\alpha) = \text{cff}(e^{l-2n}\sigma_2^{l+m-n}).$$

By Fact 1,

$$\text{cff}(e^{l+2m}\sigma_2^l) = \sum_i \binom{3r+m-3i}{r+m-i} \cdot \text{cff}((e\sigma_2)^{2i}(e^2 + \sigma_2)^{3r+m-3i})$$

and

$$\text{cff}(e^{l-2n}\sigma_2^{l+m-n}) = \sum_i \binom{3r+m-3i}{r-n-i} \cdot \text{cff}((e\sigma_2)^{2i}(e^2 + \sigma_2)^{3r+m-3i}).$$

The claim now follows since by lemma 3.7.1,

$$\binom{3r+m-3i}{r+m-i} = \sum_{n=0}^{\infty} \binom{m+n-1}{2n} \binom{3r+m-3i}{r-n-i}.$$

The terms with  $n > q/2$  will not appear in the final statement because  $n > q/2$  implies  $k < 0$ , and  $Sq^k(\alpha) = 0$  for  $k < 0$ .  $\square$

### 3.7.2 Quantum Adem Relations

Recall in Section 3.5 we constructed homomorphisms  $q(W)$  for  $W \in H_{\mathbb{Z}/2}^*(\overline{M}_{0,5})$ . We will define a similar construction for elements of  $H_{D_8}^*(\overline{M}_{0,5})$  and  $H_{S_4}^*(\overline{M}_{0,5})$ , where

$$D_8 = \langle (12), (34), (13)(24) \rangle \subset S_4$$

acts by permutations on the indices of  $[z_0, z_1, z_2, z_3, z_4] \in \mathcal{M}_{0,5}$ . We will abbreviate  $P_{D_8} = \overline{M}_{0,5} \times_{D_8} ES_4$  and  $P_{S_4} = \overline{M}_{0,5} \times_{S_4} ES_4$ . We note that for any  $M$  with  $H^*(M)$  finitely generated in all degrees, there is a map  $\Psi : H^*(M) \rightarrow H_*(M)$  which is an isomorphism via universal coefficients (as usual working over  $\mathbb{Z}/2$ ): explicitly, one picks a dual basis under the pairing  $\langle \alpha, a \rangle \mapsto \alpha(a)$  given by evaluation of cocycles.  $P_{D_8}$  and  $P_{S_4}$  satisfy this finite generation condition, which one can see using the Cartan Leray spectral sequence.

Given  $\alpha \in H^*(M)$ , choose  $A_v$  for  $v \in ES_4$  such that  $[A_v] = PD(a)$ , with invariance and genericity conditions: importantly,  $A_v = A_{(23)v} = A_{(24)v}$ .

**Definition 3.7.3.** For  $d \in H_{D_8}^*(\overline{M}_{0,5})$ , we pick a representative  $D$  of  $\Psi(d) \in H_*(P_{D_8})$ . Then

$$q_{D_8}(D)(\alpha) := \sum_{z \in H^*(M), j \geq 0} n_{z, \alpha, b, j} \cdot z \cdot T^j$$

where  $n_{z, \alpha, b, j}$  counts the number of  $D_8$  equivalence classes of triples  $(m, u, v)$  with  $[m, v] \in D \subset P_{D_8}$  and  $u : m \rightarrow M$  a  $J$ -holomorphic curve of energy  $j$  such that

$$u(z_0) \in PD(z^\vee), \quad u(z_1) \in A_v, \quad u(z_2) \in A_{(12)v}, \quad u(z_3) \in A_{(13)v}, \quad u(z_4) \in A_{(14)v}.$$

On cohomology the operation will be independent of the representative  $D$  of  $\Psi(d)$ .

The definition of  $q_{S_4}(D)$  is analogous.

We now pick a basis  $\mathcal{B}$  for  $H^*(BS_4)$ , for example as in (3.19). As in the case of the quantum Cartan relation, we would like to consider cycles in  $H_*(P_{S_4})$  parametrised by  $\mathcal{B}$ . Compare this to the proof of the quantum Cartan relations, where  $\{m_1\} \times D^{i,+} \in H_*(P_{\mathbb{Z}/2})$  was parametrized by  $D^{i,+} \in H_*(B\mathbb{Z}/2) = H_*(\mathbb{RP}^\infty)$ . Further, we will show later that

$$QS \circ QS(\alpha) = \sum_{i,j} q_{D_8}(\{m_1\} \otimes \Psi(e^i \sigma_2^j))(\alpha) \cdot e^i \sigma_2^j. \quad (3.23)$$

Hence, ideally we would like the basis of  $H_*(P_{S_4})$  to be of the form  $\{m_1\} \otimes B$  for  $B \in \mathcal{B}$ . This will not work because  $m_1$  is not  $S_4$ -invariant. However, the cycle  $m_1 + gm_1 + g^2m_1 \in H_*(\overline{M}_{0,5})$  is  $S_4$  invariant, where  $g = (123)$  generates the cosets of  $D_8$  in  $S_4$  (note that  $gm_1 = m_2$ ).

**Definition 3.7.4.**

$$q_{S_4} : H^*(M) \rightarrow QH^*(M) \otimes H^*(BS_4),$$

$$q_{S_4}(\alpha) = \sum_{b \in \mathcal{B}} q_{S_4}((m_1 + gm_1 + g^2m_1) \otimes \Psi(b))(\alpha) \cdot b.$$

Recall that there are quotient maps  $\pi_* : H_*(BD_8) \rightarrow H_*(BS_4)$  and  $\pi^* : H^*(BS_4) \rightarrow H^*(BD_8)$ . There are also

$$i_* : H_*(BS_4) \rightarrow H_*(BD_8), \quad i_*(D) = D + gD + g^2D$$

and

$$i^* : H^*(BD_8) \rightarrow H^*(BS_4), \quad i^*(d) = d + gd + g^2d.$$

As we work over  $\mathbb{Z}/2$  we see that  $\pi_* \circ i_* = id$  and  $i^* \circ \pi^* = id$ , which also shows that  $\pi^*$  is injective. As  $\pi^*$  is injective,  $\pi^*\mathcal{B}$  is linearly independent in  $H^*(BD_8)$ . We extend this to a basis  $\hat{\mathcal{B}} = \pi^*\mathcal{B} \cup \mathcal{B}'$ .

**Definition 3.7.5.**

$$qq : H^*(M) \rightarrow QH^*(M) \otimes H^*(BD_8),$$

$$qq(\alpha) := \sum_{\hat{b} \in \hat{\mathcal{B}}} q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(\hat{b}))(\alpha) \cdot \hat{b}.$$

**Lemma 3.7.6.**

$$q_{S_4}((m_1 + gm_1 + g^2m_1) \otimes \pi_*\Psi(b)) = q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(b)).$$

*Proof.* This is immediate from the definitions. A representative of  $(m_1 + gm_1 + g^2m_1) \otimes \pi_*\Psi(b)$  is in 3 : 1 correspondence with a representative of  $(m_1 + gm_1 + g^2m_1) \otimes \Psi(b)$ . Quotient by  $S_4$ , and as we work over  $\mathbb{Z}/2$  the count of the number of solutions is the same.  $\square$

**Lemma 3.7.7.** For  $b' \in \mathcal{B}' = \hat{\mathcal{B}} - \pi^*\mathcal{B}$ .

$$q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(b')) = 0.$$

*Proof.* By Lemma 3.7.6,

$$q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(b')) = q_{S_4}((m_1 + gm_1 + g^2m_1) \otimes \pi_*\Psi(b')).$$

If  $B' = \Psi(b')$  then for all  $b \in \mathcal{B}$ ,

$$\langle b, \pi_*\Psi(b') \rangle = \langle b, \pi_*B' \rangle = \langle \pi^*b, B' \rangle = \langle \pi^*b, \Psi(b') \rangle = 0$$

by definition of the dualising isomorphism  $\Psi$ . Hence  $\pi_*\Psi(b') = 0$ .  $\square$

This implies that

$$qq(\alpha) := \sum_{\pi^*b \in \pi^*\mathcal{B}} q_{D_8}((m_1 + gm_1 + g^2m_1) \otimes \Psi(\pi^*b))(\alpha) \cdot \pi^*b. \quad (3.24)$$

**Lemma 3.7.8.** *The following diagram commutes:*

$$\begin{array}{ccc} H^*(M) & \xrightarrow{q_{S_4}} & QH^*(M) \otimes H^*(BS_4) \\ \downarrow = & & \downarrow id_{H^*(M)} \otimes \pi^* \\ H^*(M) & \xrightarrow{qq} & QH^*(M) \otimes H^*(BD_8) \end{array} \quad (3.25)$$

*Proof.*

$$(id \otimes \pi^*)q_{S_4}(\alpha) = \sum_{\pi^*b \in \pi^*\mathcal{B}} q_{S_4}((m_1 + gm_1 + g^2m_1) \otimes \Psi(b))(\alpha) \cdot \pi^*b.$$

$$qq(\alpha) = \sum_{\pi^*b \in \pi^*\mathcal{B}} q_{S_4}((m_1 + gm_1 + g^2m_1) \otimes \pi_*\Psi(\pi^*b))(\alpha) \cdot \pi^*b$$

using Equation (3.24) and Lemma 3.7.6. For  $b \in \mathcal{B}$ , let  $D = \Psi(\pi^*b)$ . Then  $\langle \pi^*b, D \rangle = 1$  and  $\langle \hat{b}, D \rangle = 0$  for all  $\hat{b} \in \hat{\mathcal{B}} - \pi^*b$ , specifically for  $\hat{b} = \pi^*d$  with  $d \in \mathcal{B} - b$ . Hence  $\langle d, \pi_*D \rangle = 0$  for  $d \in \mathcal{B} - b$  and  $\langle b, \pi_*D \rangle = 1$ , so  $\pi_*\Psi(\pi^*b) = \Psi(b)$  by definition of  $\Psi$ .  $\square$

Observe that using monomials in  $n_1, n_2, c_3$  for our basis  $\mathcal{B}$  of  $H^*(BS_4)$  (see the beginning of Section 3.7.1),  $\pi^*\mathcal{B}$  consists of products of  $e^2 + \sigma_2$ ,  $e\sigma_2$  and  $\sigma_1$ . We may partition the basis  $\mathcal{B} = \mathcal{B}_0 \sqcup \mathcal{B}_1$ , where each basis element (which are all monomials) in  $\mathcal{B}_1$  contains  $n_1^k$  for some  $k > 0$ , whereas  $\mathcal{B}_0$  is generated by  $n_2, c_3$ . Then  $\pi^*B_0$  extends to a linear basis for the subalgebra of  $H^*(BD_8)$  that does not involve  $\sigma_1$ . This allows us to work only with this subalgebra generated by  $\pi^*B_0$ . Recall from (3.18) that there is also a standard basis of  $H^*(BD_8)$  consisting of elements of the form  $e^i\sigma_2^j\sigma_1^k$  (again, see Section 3.7.1 to recall the notation).

Let

$$qq_{i,j}(\alpha) := q_{D_8}((m_1 + gm_1 + gm_1) \otimes \Psi(e^i\sigma_2^j)), \quad (3.26)$$

the coefficient of  $e^i\sigma_2^j$  in  $qq(\alpha)$ .

*Proof of Theorem 3.1.7, The Adem Relations.* The theorem follows by Lemma 3.7.8 and the combinatorial argument in Theorem 3.7.2.  $\square$

To relate this to a composition of quantum Steenrod squares:

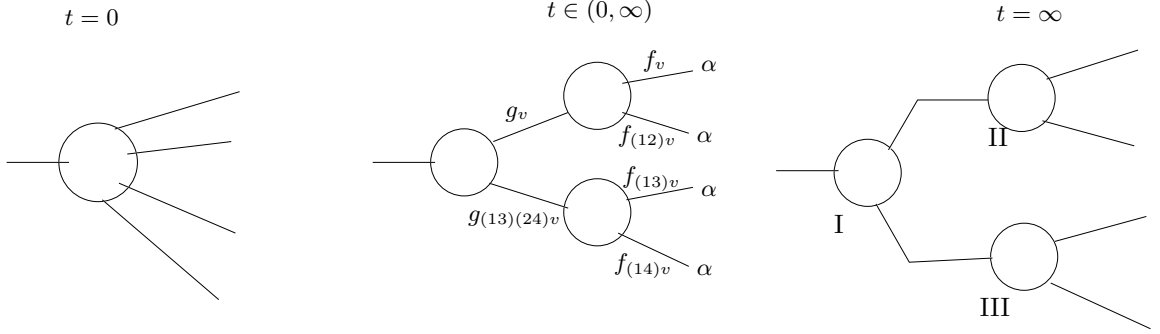


**Figure 3.10:** Moduli space for the Adem relations.

*Proof of Corollary 3.1.8.* To show that Equation (3.23) holds, we use a similar idea to proving the Cartan relation. Specifically, we use an interpolating moduli space as in Figure 3.10, here using Morse flowlines where for all  $v$ :

1.  $f_{(23)v,t} = f_{(24)v,t} = f_{v,t}$  for all  $t$ .
2.  $g_{(12)v,t} = g_{(34)v,t} = g_{v,t}$  for all  $t$ .

We see that the  $t = 0$  boundary is  $\sum_{i,j} q_{D_8}(\{m_1\} \otimes \Psi(e^i \sigma_2^j))(\alpha) \cdot e^i \sigma_2^j$ . The  $t = \infty$  boundary is split into regions I, II, III. Consider the contractible space  $S^\infty \times (S^\infty \times S^\infty)$ , along with an action of  $D_8 = \langle r, a | r^2 = a^4 = 1, r \cdot a = a^3 \cdot r \rangle$  as follows:

- $r \cdot (z, (z_1, z_2)) = (-z, (z_2, z_1))$ .
- $a \cdot (z, (z_1, z_2)) = (z, (-z_2, z_1))$ .

This is a free action of  $D_8$  on the contractible space  $S^\infty \times (S^\infty \times S^\infty)$ , hence it is an  $ED_8$ . If  $D^{j,+}$  is the upper  $i$ -dimensional hemisphere as usual, then  $D^{i,+} \times D^{j,+} \times D^{j,+}$  represents  $\Psi(e^i \sigma_2^j)$ . Using this model, and restricting attention to each of the regions I for  $(Q\mathcal{S}_{i,\cdot} \circ \dots)$  and II, III for  $(\dots \circ Q\mathcal{S}_{j,\cdot})$  respectively, we see that the  $t = \infty$  boundary counts coefficients of  $Q\mathcal{S} \circ Q\mathcal{S}(\alpha)$ .  $\square$

Observe that in the classical case, i.e. constant spheres, the operations

$$q_{D_8}(gm_1 \times \Psi b) = q_{D_8}(g^2 m_1 \times \Psi b),$$

and so the quantum case reduces to the classical case, since the solutions corresponding to those two operations cancel in pairs mod 2.

**Remark 3.7.9.**  $qq_{j,0}(\alpha)$  is the coefficient of  $h^j$  in  $Q\mathcal{S}(\alpha) * Q\mathcal{S}(\alpha)$ . This is one of the correction terms that can be computed, e.g. the  $p = |\alpha|$  term in Corollary 3.1.8.

## 3.8 $QS$ for blow-ups

We will demonstrate calculations of  $QS_{1,1}$  in two cases. The setup in both cases will be similar to the setup in [5, Section 8], where Blavier computes the quantum Massey product.

### 3.8.1 $\mathbb{CP}^3$

Fix two generic quadric hypersurfaces in  $X = \mathbb{CP}^3$ . Let  $Y$  be their intersection.  $Y$  is an elliptic curve, hence a torus. We let  $M = Bl_Y X$ . Recall that there is a  $\mathbb{CP}^1$ -bundle  $\pi : E \rightarrow Y$  over the torus and an inclusion  $i : E \rightarrow M$  of the exceptional divisor  $E$ . Specifically  $E$  is the projectivisation of the normal bundle of  $Y \subset X$ .

Consider the 3-disc bundle  $DY \rightarrow Y$  such that  $E \hookrightarrow DY$  is an inclusion of the subbundle  $E$ , with the maps of fibres being inclusion of the boundary  $S^2 \hookrightarrow D^3$ . One can use the Mayer-Vietoris sequence, by observing that  $M \cup_E DY \simeq X$ , to write down the *long exact sequence of a blow-up*,

$$\dots \longrightarrow H_*(E) \longrightarrow H_*(M) \oplus H_*(Y) \longrightarrow H_*(X) \longrightarrow H_{*-1}(E) \longrightarrow \dots \quad (3.27)$$

One can also use the homological Gysin sequence in this case to get an exact sequence:

$$\dots \longrightarrow H_*(E) \longrightarrow H_*(Y) \longrightarrow H_{*-3}(Y) \xrightarrow{\phi} H_{*-1}(E) \longrightarrow \dots \quad (3.28)$$

Putting these together, we can prove that:

$$H_2(M) \cong H_2(X) \oplus H_2(E)/H_2(Y) \quad (3.29)$$

$$i_* : H_3(E) \xrightarrow{\cong} H_3(M) \quad (3.30)$$

$$\phi : H_1(Y) \xrightarrow{\cong} H_3(E). \quad (3.31)$$

In particular  $\dim H_3(E) = 2$ . Moreover, the two generators of  $H_2(M)$  are: the spheres lifted from  $\mathbb{P}^3$  to  $M$  and the spheres representing a fibre  $\pi^{-1}$  over a point  $y \in Y$ .

We calculate the first Chern class of  $M$ . Observe that we may embed  $j : M \rightarrow \mathbb{P}^3 \times \mathbb{P}^1$  as a complex hypersurface of degree  $(2, 1)$ , where this degree is defined to be the standard definition of algebraic degree on each factor. Then functoriality and the Whitney sum formula imply that

$$c_1(TM) + c_1(vM) = c_1(T(\mathbb{P}^3 \times \mathbb{P}^1)|_M)$$

where  $vM$  is the normal bundle of  $M$  in  $\mathbb{P}^3 \times \mathbb{P}^1$ . It is known that  $c_1(T(\mathbb{P}^3 \times \mathbb{P}^1)|_M) = (4, 2)$ , and  $c_1(vM) = (2, 1)$  because it is the same as the degree (here we note that the

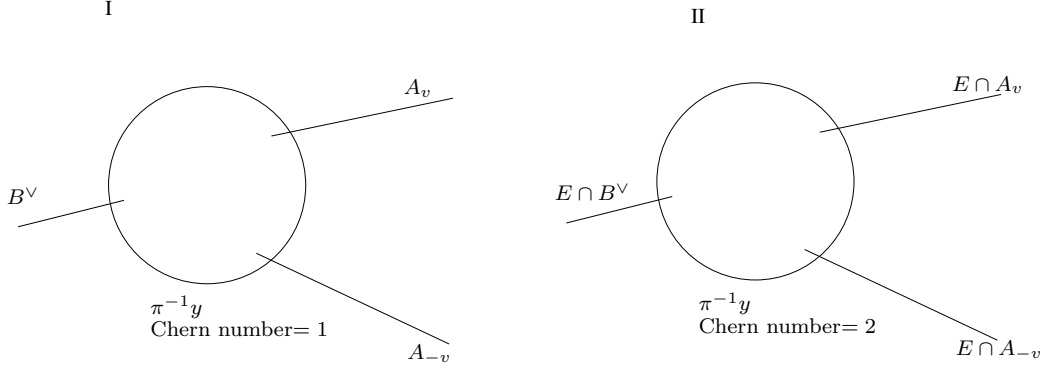


**Figure 3.11:** Configurations for  $QS_{1,1}$  on  $M$  and  $QS_{1,2}$  on  $E$ .

Euler class can be reinterpreted as  $j^*PD([M])$ , represented by the self intersection of  $M$ , where  $[M] \in H_6(\mathbb{P}^3 \times \mathbb{P}^1)$ . Hence  $c_1(TM) = (2, 1)$ . Therefore, when calculating  $QS_{1,1}$  we only need to consider the spheres in the fibre class of  $M$  as these are the only  $J$ -holomorphic spheres of Chern number 1, which are confined to be in  $E$ .

$QS_{1,1} : H^3(M) \rightarrow H^3(M)$ . We will show that  $QS_{1,1}|_{H^3(M)} = id$ . First we show that we may reduce to  $QS_{1,2} : H^3(E) \rightarrow H^1(E)$ . We use  $i^!$  on cohomology to mean  $PD \circ i_* \circ PD^{-1}$ , and let  $i_! = PD^{-1} \circ i^* \circ PD$  on homology.

**Lemma 3.8.1.** *Fix  $a \in H^3(M)$ . Let  $A = PD(a)$ .*

$$QS_{1,2} \circ i^*(a) = i^! \circ QS_{1,1}(a)$$

for  $a \in H^3(M)$ , i.e. (3.32) commutes:

$$\begin{array}{ccc} H^3(M) & \xrightarrow{QS_{1,1}} & H^3(M) \\ i^* \downarrow & & \uparrow i^! \\ H^3(E) & \xrightarrow{QS_{1,2}} & H^1(E) \end{array} \quad (3.32)$$

*Proof.* Given a generator  $a \in H^3(M)$  whose Poincaré dual is represented by  $A$ , the generator  $i^*a$  is represented by  $E \cap A$  (for a generic choice of  $A$ ). So if we calculated  $QS_{1,1}(a)$  using some  $A_v$ , then we may use  $E \cap A_v$  to calculate  $QS_{1,2}(i^*a)$ . Using the choice of basis induced from  $H_1(Y)$  by Equations (3.30) and (3.31), by direct computation we may show for any  $b \in H^3(M)$ , where  $B = PD(b)$ ,

$$i_! B^\vee = (i_*^{-1} B)^\vee. \quad (3.33)$$

Given a setup as in Figure (3.11)I, which calculates the coefficient of  $b$  in  $QS_{1,1}(a)$ , we note that as all of the  $J$ -holomorphic spheres of relevant Chern number are fibres in  $E$ ,

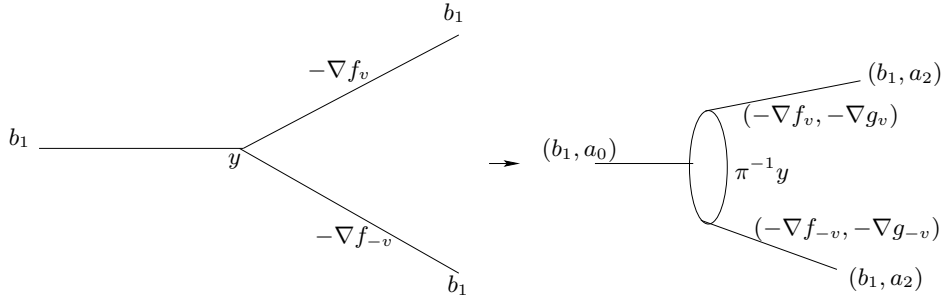


**Figure 3.12:** Lifting configurations of  $Sq$  on  $Y$  to  $QS$  on  $M$ .

by intersecting the setup I with  $E$  (and using (3.33) to show  $E \cap B^\vee = i_! B^\vee = (i_*^{-1} B)^\vee$ ) we get a setup II from Figure (3.11), counting the coefficient of  $PD i_*^{-1} B = (i^!)^{-1} b$  in  $QS_{1,2}(i^* a)$ . Conversely, as  $i^*$  is an isomorphism, given  $a' \in H^3(E)$  there is a unique  $a \in H^3(M)$  such that  $a' = i^* a$ . Hence we may calculate  $QS_{1,2}(a')$  using  $E \cap A_v$  to represent  $a'$ . Thus a setup II yields a setup I.  $\square$

Note the change in degree of 2, which comes from the fact that  $i^!$  changes cohomological degree by 2. We now use the Morse theoretic definition of the quantum square.  $E = Y \times \mathbb{P}^1$  so we may pick the Morse function on  $E$  to be  $f + g$  where  $f : Y \rightarrow \mathbb{R}$  and  $g : \mathbb{P}^1 \rightarrow \mathbb{R}$ , such that  $g$  has two critical points of index 0, 2, which we call  $a_0, a_2$ , and  $f$  has critical points  $b_0, b_1, b'_1, b_2$  (whose indices are the subscripts).

**Lemma 3.8.2.**  $QS_{1,2} = \pi^* \circ Sq_1 \circ \pi^!$ .

*Proof.* Given a setup to calculate  $QS_{1,2}$ , Figure (3.11)[II], project down to  $Y$  to get a setup for calculating  $Sq$ : a fibre sphere in  $E$  lives above a point  $y \in Y$ , hence if three flowlines meet a fibre sphere then this projects down to three flowlines coinciding at a point  $y \in Y$ . Input elements of  $H^3(E)$  correspond to  $(b_1, a_2)$  or  $(b'_1, a_2)$ , which project down to  $b_1$  or  $b'_1$  respectively. Output elements of  $H^1(E)$  correspond to  $(b_1, a_0)$  or  $(b'_1, a_0)$ .

For the opposite direction, consider a configuration for  $Sq$  in  $Y$  with the three flowlines meeting at  $y$ . If the input elements are of the form  $b_1$ , then they lift to  $(b_1, a_2)$ . The flowline  $l$  incoming to  $b_1$  in  $Y$  lifts uniquely to an incoming flowline to  $(b_1, a_2)$  in  $E$  because  $a_2$  is the maximum of  $g$ , and hence we may flow back uniquely along  $(-\nabla f, -\nabla g)$  (subject to the condition that the projection of this flowline to  $Y$  recovers  $l$ ). See Figure 3.12. Likewise the output critical point  $b_1$  lifts to  $(b_1, a_0)$ , with the output flowline on  $Y$  lifting uniquely to  $E$ .

Moreover, because this setup lifts from  $Y$  we know that the three flowlines will all intersect the  $J$ -holomorphic sphere  $\pi^{-1}y$ . Because this is a blow-up, we also know

where each lifted flowline intersects  $J$ -holomorphic sphere  $\pi^{-1}y$ , as this is determined by the gradient of  $f + g$  on each of the flowlines at  $s = 0$ . Hence there is a unique  $J$ -holomorphic sphere that fits into the lifted setup, giving a unique configuration on  $E$  corresponding to the configuration on  $Y$ .  $\square$

*Proof of Equation (3.7) in Theorem 3.1.9.*  $Sq : H^1(Y) \rightarrow H^1(Y)$  is the identity for degree reasons. Lemmas 3.8.1 and 3.8.2 imply that Diagram (3.34) commutes.

$$\begin{array}{ccc}
H^3(M) & \xrightarrow{Q\mathcal{S}_{1,1}} & H^3(M) \\
i^* \downarrow & & \uparrow i^! \\
H^3(E) & \xrightarrow{Q\mathcal{S}_{1,2}} & H^1(E) \\
\pi^! \downarrow & & \uparrow \pi^* \\
H^1(Y) & \xrightarrow{Sq_1} & H^1(Y)
\end{array} \tag{3.34}$$

$Sq_1 = id$ . In Morse cohomology,

$$(\pi^!)^{-1}(b_1) = (b_1, a_0) \text{ and } \pi^*(b_1) = (b_1, a_2)$$

so

$$Q\mathcal{S}_{1,2}(b_1, a_0) = (b_1, a_2). \tag{3.35}$$

$H^i(E)$  is generated by  $\{(b_1, a_{i-1}), (b'_1, a_{i-1})\}$  for  $i = 1, 3$ . Identify the cycles  $B_1 \in H_1(E)$  to  $(b_1, a_2) \in H^3(E)$  and  $B_3 \in H_3(E)$  to  $(b_1, a_0) \in H^1(E)$ , using Poincaré duality and the isomorphism between Morse and classical cohomology (likewise  $B'_i$  for  $(b'_1, a_{3-i})$ ). Note that

$$B_3^\vee = B'_1, \tag{3.36}$$

and so on. In this notation  $Q\mathcal{S}_{1,2}(B_1) = B_3$ . Observe that  $B_3 \cap B_3 = \emptyset$  so  $i_* B_3 \cap i_* B_3 = \emptyset$  generically. As  $H_3(M)$  is generated by  $i_* B_3$  and  $i_* B'_3$ , this implies

$$(i_* B_3)^\vee = i_* B'_3. \tag{3.37}$$

By Equation (3.37),

$$i_* \circ Q\mathcal{S}_{1,2} \circ i_!((i_* B_3)^\vee) = i_* \circ Q\mathcal{S}_{1,2} \circ i_!(i_* B'_3).$$

By Equation (3.33), then Equations (3.35) and (3.36),

$$i_* \circ Q\mathcal{S}_{1,2} \circ i_!((i_* B_3)^\vee) = i_* \circ Q\mathcal{S}_{1,2} B_3^\vee = i_* B'_3.$$

From this and similar calculations, and Diagram (3.34), we deduce that

$$Q\mathcal{S}_{1,1} = i_* \circ Q\mathcal{S}_{1,2} \circ i^* = id.$$

$\square$

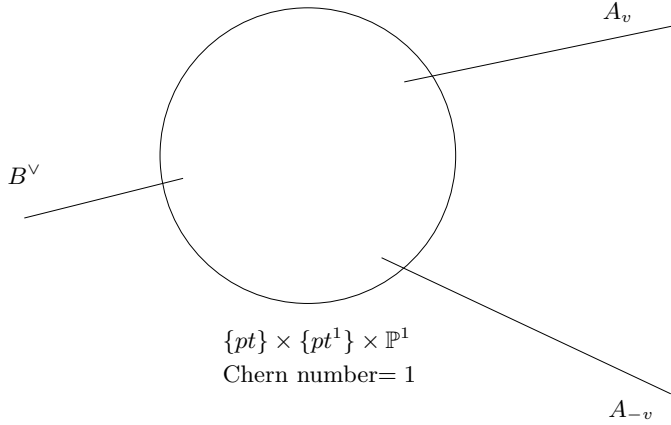


**Figure 3.13:** Configurations for contributions to  $Q\mathcal{S}_{1,1}$  from lifts on  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ .

### 3.8.2 $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$

Now let  $X = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ , with  $Y \subset X$  a torus defined by the intersection of two generic linear hypersurfaces, and  $M = Bl_Y X$ . Using a similar method to the  $\mathbb{CP}^3$  case, we can show that the Chern class of  $M$  is  $(1, 1, 1, 1)$ . Hence, when calculating  $Q\mathcal{S}_{1,1} : H^3(M) \rightarrow H^3(M)$  there are contributions from the fibre direction plus those from  $J$ -holomorphic spheres in  $X$  that have been lifted to  $M$ . The fibrewise contributions are calculated in exactly the same way as for  $\mathbb{CP}^3$ , so we turn our attention to the spheres lifted from  $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ .

*Proof of Equation (3.8) in Theorem 3.1.9.* Suppose the defining linear equations for  $Y$  are  $P_1(x, y, z)$  and  $P_2(x, y, z)$  in local coordinates on  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . Fixing  $x, y$ , there is at most one solution  $z$  such that  $P_1(x, y, z) = P_2(x, y, z) = 0$ . Hence, let  $S = \{x\} \times \{y\} \times \mathbb{P}^1$  be a  $J$ -holomorphic curve in  $X$ , and  $\tilde{S}$  its lift to  $M$ . By the previous,  $\tilde{S} \cap E$  is at most 1 point. If  $A \in H_3(M)$  then recall from (3.30), we may assume that  $A = i_* A_E$  for some  $A_E \in H_3(E)$ . Recall that to calculate  $Q\mathcal{S}_{1,1}(a)$ , where  $A = PD(a)$ , we need to choose some  $A_v$  satisfying transversality. We may pick representatives of  $A_v$  some  $D_v \times \mathbb{P}^1$ , where  $D_v$  is a representative of  $D \in H_1(Y)$ . Then assuming  $S$  is not contained in  $Y$ , there are no solutions to Figure 3.13. For such a solution, we would need that  $\tilde{S}$  intersects  $E$  in at least 2 points (as  $A_v \subset E$  for all  $v$ ), which we know is impossible. This setup is transverse, because it is empty.

The case  $S \subset Y$  is not possible, because there is no degree 1 holomorphic map  $\mathbb{P}^1 \rightarrow Y$ .  $\square$

# Chapter 4

## The Quantum Steenrod Square and the Equivariant Pair-of-Pants

### 4.1 Introduction

Throughout, the term “equivariant” will refer to  $\mathbb{Z}/2$ -equivariance. All cohomology will have  $\mathbb{Z}/2$ -coefficients. We will also interchangeably use Morse cohomology and singular cohomology in all of our constructions.

In this chapter we discuss the relationship between the construction of the quantum Steenrod square in the previous chapter and the equivariant pair-of-pants product due to Seidel. They are both generalisations of the Steenrod square on a topological space  $M$ , which is an additive homomorphism

$$Sq : H^n(M) \rightarrow H_{eq}^{2n}(M) := (H^\bullet(M)[[h]])^{2n},$$

where  $h$  is a variable of degree 1. The quantum Steenrod square was defined for closed monotone symplectic manifolds in Chapter 3, but its definition holds any time quantum cohomology is well defined, i.e. weakly monotone symplectic manifolds. The equivariant pair-of-pants product was defined for exact symplectomorphisms of Liouville domains by Seidel in [28], but here we use the identity symplectomorphism and therefore may use a more general symplectic manifold, i.e. open convex symplectic manifolds as in Definition 4.2.12.

We begin with a preliminary section, which brings together the important background material that will be used. We will define the quantum cohomology in Section 4.2.3, recalling that for a weakly monotone symplectic manifold  $M$  the quantum cohomology  $QH^*(M, \omega) = H^*(M, \Lambda)$  as a vector space, using a Novikov ring  $\Lambda$ . The cup product is deformed by quantum contributions from counting 3-pointed genus zero Gromov-Witten invariants, i.e. counting certain  $J$ -holomorphic spheres in  $M$

where  $J$  is an almost complex structure on  $M$  compatible with  $\omega$ . We will denote this product  $*$ . We introduce in Section 4.2.4 the quantum Steenrod square, which is an operation

$$QS : QH^*(M) = H^*(M)[t] \rightarrow H^*(M)[t][[h]] = QH^*(M)[[h]].$$

Throughout we will denote  $QH_{eq}^*(M) := (QH^\bullet(M)[[h]])^*$ , to adhere to the notation of [28]. The quantum Steenrod square differs from the classical Steenrod square because it is not a ring morphism: specifically, the obvious analogue of the Cartan relation

$$Sq(x \cup y) = Sq(x) \cup Sq(y) \quad (4.1)$$

does not hold. However, there is a correction term that can be computed, yielding a quantum Cartan relation

$$QS(x * y) = QS(x) * QS(y) + \sum_{i,j} q_{i,j}(W_0 \times D^{i,+})(x, y)h^i,$$

for a correction term  $q_{i,j}$  as in Theorem 3.1.2. We will also recall in Section 4.2.1 the notion of equivariant Floer cohomology  $HF_{eq}^*(H)$  for a Hamiltonian  $H : M \rightarrow \mathbb{R}$ , [28], which is a way of generalising the  $\mathbb{Z}/2$ -equivariant cohomology to Floer theory. In the same paper Seidel defined a generalisation of the Steenrod square, which throughout we call the symplectic square:

$$PS : HF^*(H) \rightarrow HF_{eq}^*(2 \cdot H).$$

We describe this in Section 4.2.2. In Section 4.2.7 we give a sketch of how the standard gluing and compactness arguments generalise to the equivariant case.

In Section 4.3, we will define the equivariant continuation maps  $\phi_{eq,H,H'}$ , which are homomorphisms  $HF_{eq}^*(2 \cdot H) \rightarrow HF_{eq}^*(2 \cdot H')$ . Our convex symplectic manifolds  $M$  are split into a compact symplectic manifold  $C$ , with  $\dim C = \dim M$  such that  $\partial C$  is contact, and a collar neighbourhood symplectomorphic to  $\partial C \times [1, \infty)$ . To be radial at infinity is to say that there is some  $R \geq 1$  such that  $H(z, r) = \lambda_H \cdot r$  for  $(z, r) \in \partial C \times [1, \infty)$  and  $r \geq R$ . The equivariant continuation maps are defined when  $H, H'$  are radial at infinity with  $\lambda_H \leq \lambda_{H'}$ .

We recall in Section 4.2.5 the ring isomorphism due to Piunikhin-Salamon-Schwarz [18], denoted  $\Psi : QH^*(M) \xrightarrow{\cong} HF^*(H)$  is a homomorphism from quantum cohomology to Floer cohomology. The PSS isomorphism  $\Psi$  is defined for a small Hamiltonian  $H$ , where “small” will be made precise in the referenced section. In Section 4.4 we will construct an equivariant version of the PSS-isomorphism,

$$\Psi_{eq} : QH_{eq}^*(M) \rightarrow HF_{eq}^*(2 \cdot H),$$

where  $2 \cdot H$  is “small”. In Section 4.5 we will prove the main result of this chapter:

**Theorem 4.1.1.** *Let  $M$  be a convex symplectic manifold, and  $H : M \rightarrow \mathbb{R}$  a  $C^2$ -small Hamiltonian. Then the following diagram commutes:*

$$\begin{array}{ccc} QH^*(M) & \xrightarrow{QS} & QH_{eq}^*(M) \\ \Psi \downarrow \cong & & \Psi_{eq} \downarrow \cong \\ HF^*(H) & \xrightarrow{PS} & HF_{eq}^*(2 \cdot H) \end{array} \quad (4.2)$$

In Section 4.6, we will use the continuation maps from Section 4.3 to define the equivariant symplectic cohomology  $SH_{eq}^*(M)$  as the direct limit of  $HF_{eq}^*(2 \cdot H)$  over all  $H$  that are radial at infinity, using the maps  $\phi_{eq,H,H'}$ . Composing the equivariant PSS isomorphism  $QH_{eq}^*(M) \xrightarrow{\cong} HF_{eq}^*(2 \cdot H)$  for a  $C^2$ -small Morse  $H$  with the map  $HF_{eq}^*(2 \cdot H) \rightarrow SH_{eq}^*(M)$  to the direct limit, there is a map  $c_{eq}^* : QH_{eq}^*(M) \rightarrow SH_{eq}^*(M)$ . We observe that the symplectic square commutes with the equivariant continuation maps:

**Lemma 4.1.2.**

$$\phi_{eq,H,H'} \circ PS = PS \circ \phi_{H,H'}.$$

This lemma allows us to define a symplectic square  $PS : SH^*(M) \rightarrow SH_{eq}^*(M)$ , and from the previous results obtain the following corollary:

**Corollary 4.1.3.** *The following diagram commutes:*

$$\begin{array}{ccc} QH^*(M) & \xrightarrow{QS} & QH_{eq}^*(M) \\ c^* \downarrow & & c_{eq}^* \downarrow \\ SH^*(M) & \xrightarrow{PS} & SH_{eq}^*(M) \end{array} \quad (4.3)$$

In Section 4.7 we discuss an attempt to construct a symplectic version of the Cartan relation. At the beginning of the referenced section we see that an immediate generalisation of the Cartan relation does not make sense because there is no obvious pair-of-pants type product on  $SH_{eq}^*(M)$ . This is because the pair-of-pants is too rigid to have a holomorphic involution that rotates each of the three cylindrical ends halfway. This is disappointing because the classical version of the Cartan relation is a very useful computational tool, as is the quantum Cartan relation. We are partially saved because while  $SH_{eq}^*(M)$  is not a ring, it is a module over  $SH^*(M)$ , which is encoded in an operation

$$PS' : SH^*(M) \otimes SH_{eq}^*(M) \rightarrow SH_{eq}^*(M),$$

and we can prove:

**Theorem 4.1.4** (Symplectic Cartan relation).

$$PS(x * y) = PS'(x; PS(y)).$$

Here  $*$  denotes the pair-of-pants product on  $SH^*(M)$ . We can see that this is of a similar form to the classical Cartan relation (in fact, one could rephrase the classical Cartan relation in terms of Theorem 4.1.4). Indeed, this shows that in fact  $PS$  is completely determined by  $PS(1)$  and the operation  $PS'$ , using that  $PS(x * y) = PS'(x * y, PS(1))$ . In practise computing  $PS'$  is as difficult as computing  $PS$ .

In Section 4.7.3 we will demonstrate the nonvanishing of  $PS(a)$  for half of the  $a \in SH^*(T^*S^n)$ .

We finish this chapter in Section 4.8 by considering  $M$  being the total space of a negative line bundle over a closed symplectic manifold. The specific case will be  $M = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{CP}^m)$ . We use the work of Ritter in [22] and modify to the case of equivariant cohomology. In the given case, Ritter proved that the  $c^*$ -map induces an isomorphism  $c^* : QH^*(M)/\ker r^k \xrightarrow{\cong} SH^*(M)$  for a particular linear homomorphism  $r : QH^*(M) \rightarrow QH^*(M)$ . We will use Diagram (4.3) to show that  $c_{eq}^*$  descends to a map

$$c_{eq}^* : QH_{eq}^*(M)/(Q\mathcal{S}(\ker r^k)) \rightarrow SH_{eq}^*(M)$$

in this case. It is immediate from Corollary 4.1.3 that

$$PS = c_{eq}^* \circ Q\mathcal{S} \circ (c^*)^{-1}, \quad (4.4)$$

and we will prove (where  $\Lambda$  is an appropriate Novikov field) that

**Theorem 4.1.5.** *Let  $M = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{CP}^m)$ , and  $x$  generates  $QH^*(M)$ , where  $T$  is the quantum variable with  $|T| = 2m$ . Then:*

$$c_{eq}^* : QH_{eq}^*(M)/Q\mathcal{S}(x^m + T) \cdot \Lambda[[h]] \rightarrow SH_{eq}^*(M)$$

*is an isomorphism.*

Together with Equation 4.3, this yields the following commutative diagram:

$$\begin{array}{ccc} QH^*(M)/(x^m + T) \cdot \Lambda & \xrightarrow{Q\mathcal{S}} & QH_{eq}^*(M)/Q\mathcal{S}(x^m + T) \cdot \Lambda[[h]] \\ c^* \downarrow \cong & & c_{eq}^* \downarrow \cong \\ SH^*(M) & \xrightarrow{PS} & SH_{eq}^*(M) \end{array} \quad (4.5)$$

This will allow us to calculate the symplectic square in terms of the quantum Steenrod square. For  $M = \text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{CP}^m)$  we can use the quantum Cartan relation, Theorem 3.1.2, to calculate the quantum Steenrod squares for  $M$  similarly to in Section 3.6.1. Using Equations (4.4) and the calculation of the quantum Steenrod square, we can then calculate the symplectic square.

## 4.2 Preliminaries

### 4.2.1 Equivariant Floer cohomology

A full treatment of this due to Seidel can be found in [28, Section (4b)]. The cited paper focuses on the case where  $M$  is a Liouville domain, with [28, Section 7] discussing the nonexact case. We will be considering the specific case where our symplectic manifold is weakly monotone (we use this to deal with bubbling). We are only interested in the cases where the symplectomorphism is Hamiltonian (we will use the identity). For a Hamiltonian symplectomorphism  $\phi^H$ , we rephrase the definition in terms of Hamiltonian rather than fixed point Floer cohomology. Using an unwrapping isomorphism one can show that the fixed-point and Hamiltonian Floer cohomologies are the same,  $HF^*(\phi_1^H) \cong HF^*(H)$ . Specifically, given a Hamiltonian  $H$ , let  $\phi_t = \phi_t^H$  be the time  $t$  flow of  $H_t$ . If  $u(s, t)$  is a Hamiltonian Floer trajectory with respect to  $H$  i.e.

$$\frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} - X_H \right) = 0 \quad (4.6)$$

then define  $v(s, t) = \phi_t u(s, t)$ , so  $v$  is a  $\phi_1$ -twisted  $J$ -holomorphic strip. Then  $v$  is  $J$ -holomorphic and  $\phi_1 v(s, t - 1) = v(s, t)$ . Hence the two Floer cohomologies are isomorphic.

Let  $H$  be a Hamiltonian. Let

$$\mathcal{L} = \{x : \mathbb{R}/2\mathbb{Z} \rightarrow M | \dot{x} = X_H\}$$

be the space of 2-periodic Hamiltonian loops with respect to the Hamiltonian  $H$ . Recall that  $x(t)$  is a 2-periodic Hamiltonian loop for  $H$  if and only if  $x(2t)$  is a 1-periodic Hamiltonian loop for  $2 \cdot H$ . We think of the generators of  $CF^*(2 \cdot H)$  (working over the Novikov ring  $\Lambda$ ) as being 2-periodic Hamiltonian loops for  $H$ , as this will be more useful for the definition of the symplectic square in Section 4.2.2. Define

$$\mathcal{J}_\epsilon = \{(J_t) : (J_t) \text{ is an almost complex structure on } M \text{ with } J_{t+\epsilon} = J_t\}.$$

There is a  $\mathbb{Z}/2$  action  $\rho_*$  on  $\mathcal{J}_2$ , with  $(\rho_* J)_t = J_{t+1}$ . The fixed set of  $\rho_*$  is  $\mathcal{J}_1 \subset \mathcal{J}_2$ . The map  $\rho_*$  is induced by a map  $\rho$  on  $\mathcal{L}$ , where  $(\rho x)(t) = x(t+1)$ . This induces an involution on the abelian group  $CF^*(2 \cdot H)$  by action on the basis elements, i.e. 2-periodic Hamiltonian loops with respect to  $H$ . However, this involution is not compatible with the Floer differential.

More concretely, transversality may fail if we chose  $J \in \mathcal{J}_1$ , and hence we cannot ensure that  $\rho_*$  fixes  $J$ . In the Floer differential we count Floer trajectories, which are smooth maps satisfying Equation (4.6); observe this equation depends upon a choice of  $J$ . So our differential  $d = d_J$  depends upon our almost complex structure, and hence is not fixed by  $\rho_*$ . When  $\rho_*$  acts on the abelian groups  $CF^*(2 \cdot H)$ , the almost complex structure  $J \in \mathcal{J}_2$  changes to  $\rho_* J$ . So  $\rho$  induces an isomorphism of chain complexes

$$\rho : (CF^*(2 \cdot H), d_{\rho_* J}) \cong (CF^*(2 \cdot H), d_J)$$

but  $\rho$  is not a chain involution itself. Precomposing this with a continuation map

$$\Phi : (CF^*(2 \cdot H), d_J) \rightarrow (CF^*(2 \cdot H), d_{\rho_* J})$$

yields a chain map

$$\rho \circ \Phi : (CF^*(2 \cdot H), d_J) \rightarrow (CF^*(2 \cdot H), d_J).$$

We notice that  $\rho \circ \Phi$  is a chain map, but not an involution. However, it induces an involution on homology. This gives a partial reason behind the difficulties in Section 4.7. We contrast with the quantum equivariant cohomology case, where the  $\mathbb{Z}/2$ -action on chains is trivial, hence  $QH_{eq}^*(M) = QH^*(M)[[h]]$ .

We now construct  $HF_{eq}^*(2 \cdot H)$ . Given coordinates  $x_i$  for  $i = 0, 1, 2, \dots$  on  $S^\infty$ , so

$$S^\infty = \left\{ (x_0, x_1, \dots) \left| \sum x_i^2 = 1 \text{ and only finitely many } x_i \text{ are nonzero} \right. \right\},$$

define a Morse function

$$g : S^\infty \rightarrow \mathbb{R}, \quad g(x) = \sum_k k x_k^2 \tag{4.7}$$

on  $S^\infty$  using the round metric, with critical points  $v^{i,\pm}$  of index  $i$ . The  $\{v^{i,+}\}$  are the standard Euclidean basis in  $\mathbb{R}^\infty$  and  $v^{i,-} = -v^{i,+}$ . Fix an almost complex structure  $J \in \mathcal{J}_2$  on  $M$ . For each  $v \in S^\infty$  pick  $J_{eq,v} \in \mathcal{J}_2$  such that:

1.  $J_{eq,-v} = \rho_* J_{eq,v}$

2. In a neighbourhood of  $v^{i,+}$  for any  $i$ , the almost complex structure  $J_{eq,v}$  is independent of  $v$  and equal to  $J$ .

3.  $J_{eq,\tau(v)} = J_{eq,v}$  for  $\tau : S^\infty \rightarrow S^\infty$  the shift map, i.e.  $\tau(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$ .

Given a negative gradient flowline  $w : \mathbb{R} \rightarrow S^\infty$  of  $g$  above, with  $w(-\infty) = v^{i,\pm}$  and  $w(\infty) = v^{0,+}$ , we define a domain dependent almost complex structure on  $M$ , parametrised by  $(s, t) \in \mathbb{R} \times \mathbb{R}/2\mathbb{Z}$ ,

$$J_{s,t}^w = J_{eq,w(s),t}. \quad (4.8)$$

Using  $J_{s,t}^w$  we define a Cauchy Riemann Equation, for  $x, y$  being 2-periodic Hamiltonian loops with respect to  $H$ :

$$\left\{ \begin{array}{l} u : \mathbb{R} \times (\mathbb{R}/2\mathbb{Z}) \rightarrow M \\ \partial_s u + J_{s,t}^w \partial_t u = -\nabla H \\ \lim_{s \rightarrow -\infty} u(s, t) = y(t) \text{ if } w(-\infty) = v^{i,+} \\ \lim_{s \rightarrow -\infty} u(s, t) = y(t+1) \text{ if } w(-\infty) = v^{i,-} \\ \lim_{s \rightarrow +\infty} u(s, t) = x(t) \end{array} \right. \quad (4.9)$$

There is an  $\mathbb{R}$  action simultaneously translating the  $s$  for both  $w$  and  $u$ . Quotienting by this gives a moduli space of ( $\mathbb{R}$ -equivalence classes of) pairs,

$$\mathcal{M}_{eq}^{i,\sigma}(y, x) = \{[w, u] : (4.9), w(-\infty) = v^{i,\sigma}\}$$

**Definition 4.2.1.** *Define the equivariant differential*

$$d_{eq} = d_J + \sum_{i \geq 1} h^i (d_{eq}^{i,+} + d_{eq}^{i,-})$$

where  $d_J$  is the differential on Hamiltonian Floer Cohomology (with almost complex structure  $J$ ), and

$$d_{eq}^{i,\sigma}(x) = \sum_y \# \mathcal{M}_{eq}^{i,\sigma}(y, x) y$$

Here  $\#$  is the count of isolated solutions in the parametrised moduli space.

It is shown in [28, Definition 4.4] that  $d_{eq}$  is a differential on

$$CF_{eq}^*(2 \cdot H) := CF^*(2 \cdot H)[[h]].$$

Let  $HF_{eq}^*(2 \cdot H)$  be the cohomology of  $(CF_{eq}^*(2 \cdot H), d_{eq})$ . Consider  $HF_{eq}^*(2 \cdot H)$  for  $2 \cdot H$  a  $C^2$ -small, time independent Morse function. The elements of  $\mathcal{M}_{eq}^{i,\sigma}(y, x)$  become negative gradient flowlines of  $H$ , and  $HF_{eq}^*(2 \cdot H)$  becomes  $HM_{\mathbb{Z}/2}^*(M, H)$ , the equivariant Morse cohomology of  $M$ .

- Remark 4.2.2.** 1. In practise we need a time dependent perturbation  $H = H_t$ , although we choose this such that  $H_t = H_{t+1}$  (even though the Hamiltonian loops we consider are 2-periodic). In this case  $(2 \cdot H)_t := 2H_{2t}$ .
2. We cannot choose  $J \in \mathcal{J}_1$  because our moduli spaces are not guaranteed to be regular.
3. We will discuss compactification and gluing in Section 4.2.7.

## 4.2.2 The symplectic square and equivariant pair-of-pants product

The construction of the symplectic square (which is a specific case of the equivariant pair-of-pants product due to Seidel) is as in [28, Section (4c)], although we work in the case of a Hamiltonian symplectomorphism. The symplectic square will be a homomorphism:

$$PS : HF^*(H) \rightarrow HF_{eq}^{2*}(2 \cdot H).$$

Let  $S$  be a 2-to-1 branched cover of the cylinder  $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ , with covering map  $\pi$ . Specifically, there is a left-hand region parametrised by

$$\epsilon^+ : (-\infty, -1] \times \mathbb{R}/2\mathbb{Z} \rightarrow S$$

and two right hand regions parametrised by

$$\delta^\pm : [1, \infty) \times \mathbb{R}/\mathbb{Z} \rightarrow S.$$

Let  $\epsilon^-(s, t) = \epsilon^+(s, t + 1)$ . The covering involution  $\gamma$  on  $S$  swaps  $\epsilon^\pm$ , and swaps  $\delta^\pm$ .

Choose  $J_{\text{left}, v, s, t} \in \mathcal{J}_2$  for  $s < 1$ ,  $v \in S^\infty$  such that:

1.  $J_{\text{left}, -v, s, t} = J_{\text{left}, v, s, t+1}$
2.  $J_{\text{left}, v, s, t} \in \mathcal{J}_1$  for  $s \geq -1$
3.  $J_{\text{left}, v, s, t} = J_{\text{eq}, v, t}$  for  $s \leq -2$
4.  $J_{\text{left}, \tau v, s, t} = J_{\text{left}, v, s, t}$

Further, choose  $J_{\text{right}, v, s, t}^\pm \in \mathcal{J}_1$  for  $s > -1$  such that:

1.  $J_{\text{right}, -v, s, t}^\pm = J_{\text{right}, v, s, t}^\mp$
2.  $J_{\text{right}, v, s, t}^\pm$  is independent of  $v, s$  for  $s \geq 2$

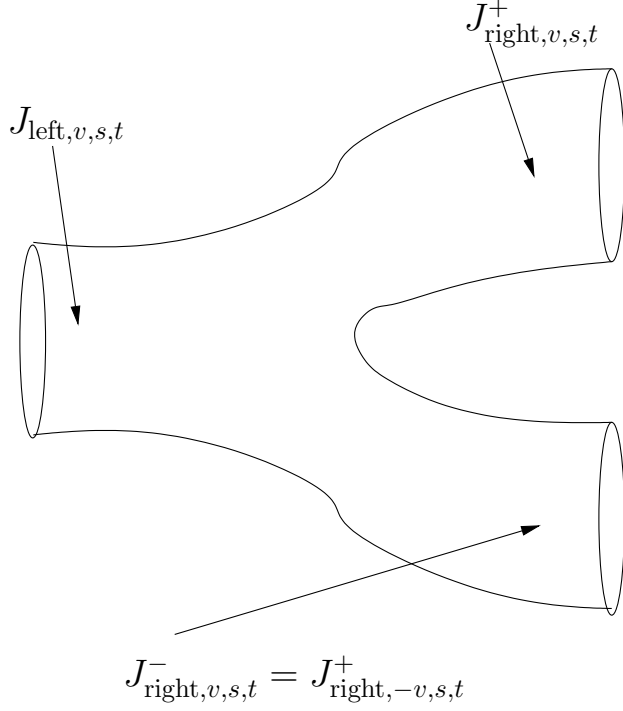


**Figure 4.1:**  $J$  for the symplectic square.

$$3. J_{\text{right},v,s,t}^{\pm} = J_{\text{left},v,s,t} \text{ for } s \leq 1$$

$$4. J_{\text{right},\tau v,s,t}^{\pm} = J_{\text{right},v,s,t}^{\pm}$$

Let  $w : \mathbb{R} \rightarrow S^{\infty}$  be a negative gradient flowline of  $f$ . Define  $J_z^w$  for  $z \in S$ , with  $J_z^w$  illustrated in Figure 4.1.

$$1. J_z^w = J_{\text{left},w(s),s,t} = J_{\text{right},w(s),s,t}^{\pm} \text{ if } \pi(z) = (s, t) \text{ with } -1 \leq s \leq 1$$

$$2. \text{ if } z = \epsilon^+(s, t) \text{ then } J_z^w = J_{\text{left},w(s),s,t}$$

$$3. \text{ if } z = \delta^{\pm}(s, t) \text{ then } J_z^w = J_{\text{right},w(s),s,t}^{\pm}$$

For a fixed complex structure  $j$  on  $S$ , given a family of almost complex structures  $J_z^w$  as above,  $x$  being a 1-periodic Hamiltonian loop for  $H$  and  $y$  a 2-periodic Hamiltonian loop for  $H$ , consider the pair-of-pants equation:

$$\begin{cases} u : S \rightarrow M \\ (du - X_H \otimes \beta + Y) \circ j = J_z^w(du - X_H \otimes \beta + Y) \\ \lim_{s \rightarrow -\infty} u(\epsilon^+(s, t)) = \begin{cases} y(t) & \text{if } w(-\infty) = v^{i,+} \\ y(t+1) & \text{if } w(-\infty) = v^{i,-} \end{cases} \\ \lim_{s \rightarrow +\infty} u(\delta^{\pm}(s, t)) = x(t) \end{cases} \quad (4.10)$$

where  $X_H$  is the Hamiltonian vector field of  $H$ , and  $\beta$  is a 1-form on  $S$  with  $d\beta = 0$  such that  $\beta = dt$  near the cylindrical ends of  $S$ . The holomorphicity equation in (4.10) is perturbed by a small Hamiltonian perturbation  $Y$ , e.g. with support restricted to  $\delta^\pm((1, 2) \times \mathbb{R}/\mathbb{Z})$ , where  $Y$  is invariant under  $\gamma$ , to ensure that there are no solutions that are constant at Hamiltonian loops of  $H$ .

The moduli space  $\mathcal{M}_{prod}^{i,\sigma}(y, x)$  consists of rigid pairs  $(w, u)$  with  $w : \mathbb{R} \rightarrow S^\infty$  a negative gradient flowline of  $g$  such that  $w(-\infty) = v^{i,\sigma}$  and  $w(\infty) = v^{0,+}$ , and  $u$  being a solution to equation (4.10).

**Definition 4.2.3** (Symplectic square).

$$PS^{i,\sigma} : CH^*(H) \rightarrow (CF^\bullet(2 \cdot H))^{2*-i}, \quad PS^{i,\sigma}(x) = \sum_{y \in \mathcal{L}} \# \mathcal{M}_{prod}^{i,\sigma}(y, x) y$$

where  $\#$  counts isolated solutions.

$$PS(x) := \sum_{i \geq 0} (PS^{i,+}(x) + PS^{i,-}(x)) h^i,$$

which descends to a map on cohomology as shown in Section 4.2.2.1.

#### 4.2.2.1 The symplectic square is a map on cohomology

Recall that given a chain complex  $(C, d)$  with a chain involution  $\iota$ , the  $\mathbb{Z}/2$ -equivariant cohomology of  $(C, d, \iota)$  is

$$H_{\mathbb{Z}/2}^*(C) := H^*(C_{\mathbb{Z}/2}, d_{\mathbb{Z}/2}),$$

where

$$C_{\mathbb{Z}/2} = C[[h]], \quad d_{\mathbb{Z}/2} = d + (id + \iota)h.$$

In particular, for  $C = CF^*(H) \otimes CF^*(H)$  with  $\iota$  defined on generators by  $\iota(x \otimes y) = y \otimes x$ , we obtain  $d_{\mathbb{Z}/2}(x \otimes y) = d(x \otimes y) + (x \otimes y + y \otimes x)h$ , where  $d$  is the Floer differential. In [28], Seidel constructs a map, the equivariant pair-of-pants product, defining:

$$\mathcal{P}_{i,\sigma} : (CF^*(H) \otimes CF^*(H))[[h]] \rightarrow CF_{eq}^*(2 \cdot H),$$

the definition of which is identical to that of  $PS$  in Definition 4.2.3 except that to calculate the coefficient of  $y$  in  $\mathcal{P}_{i,\sigma}(x_+(t) \otimes x_-(t))$ , we replace  $x(t)$  by  $x_\pm(t)$  in the last line of Equation (4.10). Seidel shows in [28, Equation (4.100)] that  $\mathcal{P}$  satisfies:

$$\sum_{j+k=i} d_{eq}^j \circ \mathcal{P}_k(x \otimes y) = \mathcal{P}_i(d(x \otimes y)) + \mathcal{P}_{i-1}(x \otimes y + y \otimes x), \quad (4.11)$$

where  $d$  is the standard Floer differential,  $d_{eq}^i = d_{eq}^{i,+} + d_{eq}^{i,-}$ ,  $\mathcal{P}_i = \mathcal{P}_{i,+} + \mathcal{P}_{i,-}$  and  $\mathcal{P} = \sum_{i \geq 0} h^i \mathcal{P}_i$  (this uses the compactification in 4.3. Hence  $\mathcal{P}$  is a chain map. One can also define the doubling map

$$\eta : CF^*(H) \rightarrow (CF^*(H) \otimes CF^*(H))[[h]], \quad \alpha \mapsto \alpha \otimes \alpha,$$

and it is immediate that  $PS = \mathcal{P} \circ \eta$ . It is also immediate that  $\eta$  descends to a map  $\eta : HF^*(H) \rightarrow H_{\mathbb{Z}/2}^{2*}(CF^*(H) \otimes CF^*(H))$  using the involution that swaps the factors on the right hand side. Hence  $PS$  is well defined on cohomology.

**Remark 4.2.4.** *The operation  $\mathcal{P}$  will contain more information than  $PS$  exactly in the instances that  $\mathcal{P}$  does not vanish on the complement of  $\text{Im}(\eta)$ .*

**Remark 4.2.5.** *If  $w : \mathbb{R} \rightarrow S^\infty$  is a negative gradient flowline for  $g$ , with  $w(-\infty) = v^{i,\sigma'}$  and  $w(\infty) = v^{j,\sigma}$  with  $j > 0$  and  $\sigma, \sigma' \in \{\pm\}$ , then  $w$  corresponds to  $\sigma\tau^j w'$ , where  $w'(-\infty) = v^{i-j,\sigma\sigma'}$  and  $w'(\infty) = v^{0,+}$ . It is important that the  $J^w$  satisfy the conditions at the beginning of Section 4.2.2, because they mean that:*

1.  $J_z^{\tau w} = J_z^w$ ,
2.  $J_z^{-w} = J_{\gamma z}^w$ .

*This is used when compactifying the moduli spaces, which is discussed in Section 4.2.7: given any negative gradient flowline  $w$  of  $g$  as above,  $(w, u)$  satisfies the equivariant Floer equation (4.10) exactly when*

$$\begin{cases} (\tau^j w, u) \text{ does, if } \sigma = + \\ (-\tau^j w, u \circ \gamma) \text{ does, if } \sigma = - \end{cases}$$

*It is necessary here that  $\gamma$  is holomorphic, as repeated at the beginning of Section 4.7, so that  $u \circ \gamma$  is holomorphic.*

### 4.2.3 Quantum cohomology

For more details on the quantum cup product, see [14, Chapter 9.2] whose exposition we follow. Throughout,  $PD$  refers to the Poincaré duality operation over  $\mathbb{Z}/2$  coefficients. In the case of an open manifold  $X$ , Poincaré duality states that  $PD : H^*(X) \cong H_{\dim X - *}^{\text{lf}}(X)$ . The open manifolds we will consider have a conical end, so they deformation retract along the fibres of the conical end to a manifold with boundary  $(C, \partial C)$  of the same dimension. We interchangeably use  $PD : H^*(C) \cong H_{\dim X - *}(C, \partial C)$ . The notation  $A^\vee$  denotes the intersection dual of  $A$  (after choosing

a basis) with respect to either the pairing  $H_*(M) \otimes H_*^{\text{lf}}(M) \rightarrow \mathbb{Z}/2$  (for an open manifold  $M$ ) or  $H_*(C) \otimes H_*(C, \partial C) \rightarrow \mathbb{Z}/2$  (for a compact manifold with boundary  $C$ ).

**Definition 4.2.6.** *A symplectic manifold  $M$  is weakly monotone if for every spherical  $A \in H_2(M, \mathbb{Z})$  (i.e.  $A$  in the image of the Hurewicz homomorphism) such that  $\omega(A) > 0$  and  $c_1(A) < 0$ , then  $c_1(A) < 2 - n$ , where  $\dim(M) = 2n$ .*

Let  $(M, \omega)$  be a weakly monotone symplectic manifold of dimension  $n$ , with a fixed almost complex structure  $J$  compatible with  $\omega$ . As an abelian group,  $QH^*(M) = H^*(M) \otimes \Lambda$ , where  $\Lambda$  is the Novikov ring associated to  $\omega$  as in [14, Chapter 9.2]. Specifically, if  $\Gamma$  is the image of the Hurewicz homomorphism then  $\omega : \Gamma \rightarrow \mathbb{Z}$  is a homomorphism, and

$$\Lambda = \left\{ \lambda = \sum_{A \in \Gamma} \lambda_A \cdot q^A \mid \begin{array}{l} \lambda_A \in \mathbb{Z}/2, \text{ such that for all } c > 0 \text{ there are} \\ \text{only finitely many } \lambda_A \neq 0 \text{ with } \omega(A) \leq c \end{array} \right\}.$$

There is a natural grading given by  $|q^A| = 2c_1(A)$ .

As an important note, we denote the quantum cochains

$$QC^*(M) := C^*(M) \otimes_{\mathbb{Z}/2} \Lambda.$$

Then  $QH^*(M) = H^*(QC^*(M), d \otimes id_\Lambda)$ , where  $d$  is the differential on  $C^*(M)$ . Most of the operations are defined at the chain level, and then descend to maps on (co)homology.

Given  $x \in H^k(M)$ ,  $y \in H^l(M)$  and  $A \in \Gamma$ , for  $Z \in H_{k+l-2c_1(A)}^{\text{lf}}(M)$  we define the 0-dimensional moduli space  $\mathcal{M}_A(X, Y, Z)$  to be the set of  $J$ -holomorphic  $u : S^2 \rightarrow M$  such that:

1.  $u_*[S^2] = A$ ,
2.  $u(0) \in PD(x)$ ,  $u(1) \in PD(y)$ ,  $u(\infty) \in Z$ .

**Definition 4.2.7** (Quantum Product). *For  $a, b \in H^*(M)$ ,*

$$a * b := \sum_{A \in H_2(M)} (a * b)_A q^A$$

where  $(a * b)_A$  is defined by  $\int_Z (a * b)_A = \#\mathcal{M}_A(PD(x), PD(y), Z)$  for all  $Z \in H_{k+l-2c_1(A)}^{\text{lf}}(M)$ . This descends to a well defined map on homology,  $H^*(M) \otimes H^*(M) \rightarrow QH^*(M)$ . Extending  $\Lambda$ -bilinearly defines  $*$  on  $QH^*(M)$ .

Notice that  $*$  is compatible with the grading, using  $|q^A| = 2c_1(A)$ . If  $A = 0$ , the intersection is a point and this recovers the classical intersection product. We will not recap the technical issues of bubbling, multiply covered curves and so on that are covered in [14, Sections 3-6]. We interchangeably use Morse theory for  $(C, \partial C)$ , which is elaborated on in the next section.

#### 4.2.4 The quantum Steenrod square

We give two definitions: the first will use Morse theory and the second will involve intersections of chains. These are equivalent, as shown in Chapter 3. Recall that  $S^\infty$  has subsets  $S^i$  such that each  $S^i$  is an antipodally invariant equator, and  $\bigcup_{i \geq 0} S^i = S^\infty$ . Let  $(M, \omega)$  be a weakly monotone symplectic manifold, with the property that:

- there is some compact submanifold  $C \subset M$  with boundary, such that  $\dim C = \dim M = m$
- $M$  deformation retracts onto  $C$
- $\omega$  is exact outside of  $C$ .

We fix an almost complex structure  $J$  on  $M$  compatible with  $\omega$ .

We will use Morse theory for  $(C, \partial C)$ , as detailed in [13]. Choosing a Morse function  $f$  such that  $\nabla f \pitchfork \partial C$  and  $f$  is both minimised and constant on  $\partial C$ , we obtain a Morse complex whose homology  $HM_*(C, \partial C)$  recovers  $H_*(C, \partial C)$  in the same way as Morse homology for closed manifolds. Suppose that we pick a choice of perturbations  $f_{v,s}$  for  $v \in S^\infty$  and  $s \in [0, \infty)$  such that:

1.  $f_{v,s} = f$  for  $s \geq 1$ .
2.  $f_{v,0}$  are generic, to make all the relevant moduli spaces transverse.
3.  $f_{\tau v,s} = f_{v,s}$  for all  $v, s$ .

We ensure that  $f_{v,s}$  are close to  $f$  for all  $v, s$ : as the Morse-Smale condition is open and dense in the space of smooth functions, we can assume the  $f_{v,s}$  are all Morse-Smale, and we fix that  $f_{v,s}|_{\partial C}$  is independent of  $v, s$ . Then there are two equivalent definitions of the Morse quantum Steenrod square, depending on whether we use singular or Morse homology on our parameter space  $\mathbb{RP}^\infty = S^\infty/(\mathbb{Z}/2)$ . Specifically, recall  $g : S^\infty \rightarrow \mathbb{R}$  in Equation (4.7), and let  $\mathcal{Y}$  be  $S^2$  with three half-lines attached. There is one incoming edge  $e_1$  parametrized by  $(-\infty, 0]$  and two outgoing edges  $e_2, e_3$  parametrized by  $[0, \infty)$ . The edges are attached at  $0, 1, \infty$  respectively. Let:

- $\mathcal{M}_{\text{Sing},i,A}(x,y)$  consists of pairs  $(v,u)$  where  $v \in S^i$  and  $u : \mathcal{Y} \rightarrow M$  such that  $\frac{d}{dt}(u|_{e_1})(s) = -\nabla f(u|_{e_1}(s))$ ,  $\frac{d}{dt}(u|_{e_2})(s) = -\nabla f_{v,s}(u|_{e_2}(s))$  and  $\frac{d}{dt}(u|_{e_3})(s) = -\nabla f_{-v,s}(u|_{e_3}(s))$ . The maps  $u|_{S^2}$  are  $J$ -holomorphic and represent the homology class  $A \in H_2(M)$ . Asymptotic conditions are given in point (2) below.
- $\mathcal{M}_{\text{Morse},i,A}(x,y)$  consists of pairs  $(w,u)$  where  $w : \mathbb{R} \rightarrow S^i$ ,  $\dot{w} = -\nabla g$  and  $u : \mathcal{Y} \rightarrow M$  such that  $\frac{d}{dt}(u|_{e_1})(s) = -\nabla f(u|_{e_1}(s))$ ,  $\frac{d}{dt}(u|_{e_2})(s) = -\nabla f_{w(s),s}(u|_{e_2}(s))$  and  $\frac{d}{dt}(u|_{e_3})(s) = -\nabla f_{-w(s),s}(u|_{e_3}(s))$ . The maps  $u|_{S^2}$  are  $J$ -holomorphic and represent the homology class  $A \in H_2(M)$ . Asymptotic conditions are given in points (1) and (2) below.

1.  $w(-\infty) = v^{i,\sigma_1}$ ,  $w(\infty) = v^{0,\sigma_2}$  for  $\sigma_1, \sigma_2 \in \{\pm\}$ .

2.  $u|_{e_i}(s) \rightarrow \begin{cases} y & \text{for } i = 1 \text{ and } s \rightarrow -\infty \\ x & \text{for } i = 2, 3 \text{ and } s \rightarrow \infty \end{cases}$

**Definition 4.2.8** (The Morse Quantum Steenrod Square). *For  $i \in \mathbb{Z}$ ,  $A \in H_2(M)$  and  $x, y \in \text{crit}(f)$  such that  $|y| = 2|x| - i - 2c_1(A)$ , the coefficient of  $y \cdot h^i \cdot q^A$  in  $Q\mathcal{S}(x) \in QH_{eq}^*(M)$  is  $\#\mathcal{M}_{\bullet,i,A}(x,y)/(\mathbb{Z}/2)$ , where  $\bullet = \text{Morse or Sing}$  and the  $\mathbb{Z}/2$  action is  $-1 \times r^*$ , where  $r^*u = u \circ r$ , and  $r : \mathcal{Y} \rightarrow \mathcal{Y}$  acts by fixing  $e_1$ , swapping  $e_2$  and  $e_3$  (without changing the parametrisation) and acting by  $z \mapsto 1/z$  on  $S^2$ .*

Now recall that for a manifold with boundary  $(C, \partial C)$ , a homology class  $\alpha \in H_*(C, \partial C)$  is realizable if there is a manifold with boundary  $A$  and  $\mu : (A, \partial A) \rightarrow (C, \partial C)$  with  $\mu_*([A]) = \alpha$ . Thom proved in [30] (for  $\partial C = \emptyset$ ) that all homology classes over  $\mathbb{Z}/2$  coefficients are realizable. A slight modification of the constructive proof by Buoncrisiano-Hacon in [6, Theorem B] yields that all elements of  $H_*(C, \partial C)$  are realisable. Hence, we associate a homology class  $\alpha$  interchangeably with a pair  $(A, \mu)$ . The conditions we placed on  $M$  allow us to pick representatives of  $x \in H_*^{\text{lf}}(M)$  to be contained in  $H_*(C, \partial C)$ .

We fix  $x \in H^*(M)$ . To calculate  $Q\mathcal{S}(x)$ , we begin by choosing a sequence of pairs  $(\chi, \mu_i : \chi_i \rightarrow M \times S^i)_{i=0}^\infty$  where  $\chi$  is a smooth manifold,  $\chi_i = \chi \times S^i$  and  $\mu_i$  is a smooth map. The  $\mu_i : \chi_i \rightarrow M \times S^i$  satisfy:

1. For  $\pi_2 : M \times S^i \rightarrow S^i$  the second projection,  $\pi_2(\mu_i(x, v)) = v$  for all  $(x, v) \in \chi_i$ .
2. The restriction  $\mu_i|_{\chi_j} = \mu_j$  for  $j \leq i$ .

3. For  $\pi_1 : M \times S^i \rightarrow M$  the first projection, define for any  $v \in S^i$

$$\mu_v : \chi \rightarrow M, \mu_v(x) = \pi_1 \mu_i(v, x),$$

(which is well defined by (2) above) and

$$\chi_v := \pi_1(\mu_i(\chi_i) \cap \pi_2^{-1}(\{v\})) = \mu_v(\chi_i)$$

is a representative of  $PD(x)$  in  $M$ .

4. For a choice of generators  $b$  of the cohomology of  $M$ , with  $\beta = PD(b)$ , the triple intersection

$$ev_1^{-1}(\mu_i(\chi_i)) \cap ev_0^{-1}(\beta^\vee \times S^i) \cap ev_\infty^{-1}(\iota(\mu_i(\chi_i))) \quad (4.12)$$

is transverse in  $\mathcal{M}_A(J) \times S^i$ , where  $\mathcal{M}_A(J)$  is the space of all  $J$ -holomorphic spheres representing  $A$ ,  $^\vee$  is intersection duality,  $\iota = \text{id} \times -\text{id}$  and  $ev_z : \mathcal{M}_A(J) \times S^i \rightarrow M \times S^i$  is defined by  $ev_z(u, v) = (u(z), v)$ .

For our purposes we will assume that  $\mu_v$  is an embedding of a submanifold for all  $v \in S^\infty$ . This amounts to choosing  $X_v = \mu_v(X) \subset C$  for each  $v \in S^\infty$  such that  $[X_v] = PD(x) \in H_{n-*}(C, \partial C)$ , while ensuring conditions so that the moduli space in Definition 4.2.9 is transverse. The results hold more generally when  $\mu_v$  is not necessarily an embedding for all  $v$ .

To show that it is possible to choose such  $X_v$ , we relate this to the Morse quantum Steenrod square. The isomorphism between Morse homology and singular cohomology identifies

$$x \in \text{crit}(f) \mapsto \overline{W^u(x, f)},$$

where  $W^u(x, f)$  is the unstable manifold of  $x$  under  $f$ .

We use the  $\mathcal{M}_{\text{Sing}, i, A}$  formalism from Definition 4.2.8. For each  $v$  there is a smooth map  $\Phi_v : C \rightarrow C$  obtained by backwards integrating along the flowline  $-\nabla f_{v,s}$  from  $s = 1$  to  $s = 0$ . The map  $\Phi_v$  is well defined for each  $v$  because  $f_{v,s}$  is minimised and constant on  $\partial C$  for all  $(v, s)$ . Then  $\overline{\Phi_v W^u(x, f)}$  is homotopic to  $\overline{W^u(x, f)}$ , so let  $X_v = \overline{\Phi_v W^u(x, f)}$ . The problem of finding appropriate  $X_v$  reduces to a generic condition on  $f_{v,0}$ , as we showed in Section 3.3.1.1. Fixing  $i \geq 0$ ,  $A \in \Gamma$ ,  $x \in H^*(C)$ , and  $Y \in H_*(C, \partial C)$ , we choose  $X_v$  as above, and define a moduli space:

**Definition 4.2.9.**

$$\mathcal{M}_{i,A}(x, Y) = \left\{ (v, u) \left| \begin{array}{l} v \in S^i, u : S^2 \rightarrow C, u_*[S^2] = A, \\ u \text{ is } J\text{-holomorphic, } u(0) \in Y, u(1) \in X_v, u(\infty) \in X_{-v} \end{array} \right. \right\}.$$

If  $|Y| = 2|x| - i - 2c_1(A)$  then  $\mathcal{M}_{i,A}(x, Y)$  is a 0-dimensional manifold (this is just a  $v$ -parametrised version of the moduli space in the previous section, with the parameter  $v \in S^i$ ). Define  $Q\mathcal{S}_{i,A}(x) \in H^{2|x|-i-2c_1(A)}(C)$  by  $\int_Y Q\mathcal{S}_{i,A}(x) = \#\mathcal{M}_{i,A}(x, Y)$  for all  $Y \in H_{2|x|-i-2c_1(A)}(C, \partial C)$ .

**Definition 4.2.10** (The Quantum Steenrod Square).

$$Q\mathcal{S} : H^*(C) \rightarrow QH^{2*}(C)[[h]], \quad Q\mathcal{S}(x) = \sum_{i \geq 0, A \in \Gamma} Q\mathcal{S}_{i,A}(x) h^i q^A.$$

We extend this over  $\Lambda$  by requiring

$$Q\mathcal{S} \left( \sum_{A \in \Gamma} x_A q^A \right) = \sum_{A \in \Gamma} Q\mathcal{S}(x_A) q^{2A}.$$

For  $a \in H^*(M)$ ,

$$Q\mathcal{S}_{i,0}(a) = Sq^{|a|-i}(a) \text{ and } \sum_{A \in H_2(M)} Q\mathcal{S}_{0,A}(a) q^A = a * a. \quad (4.13)$$

**Remark 4.2.11.** In Definition 4.2.8, we could use either of the two moduli spaces listed because these correspond to singular and Morse cohomology of  $\mathbb{RP}^\infty$  respectively. Using an appropriate Morse function  $g$ , these two definitions are equivalent in the same way as for finite dimensional manifolds, and we will use them interchangeably.

## 4.2.5 The PSS isomorphism

Henceforth we will consider symplectic manifolds  $(M, \omega)$  that are weakly monotone, open and convex at infinity. The convex at infinity condition, which we will shorten to *convex*, implies among other things that  $M$  satisfies the conditions of the previous section. Such symplectic manifolds, specifically their symplectic cohomology, were first studied by Ritter in [19, Section 3], where the technical machinery is proven in depth. The definition we use is from [22, Section 3.1].

**Definition 4.2.12.** A symplectic manifold  $(M, \omega)$  is convex if

$$M = C \cup_{\partial C} (\partial C \times [1, \infty)), \quad (4.14)$$

for some compact symplectic manifold  $C$  with contact-type boundary  $(\partial C, \theta)$ , such that  $(M - C, \omega|_{M-C}) \cong (\partial C \times [1, \infty), d(e^r \theta))$ , where  $r \in [0, \infty)$  denotes the radial coordinate.

Examples of such symplectic manifolds are (the completion of) Liouville domains, and closed symplectic manifolds. One can think of a closed symplectic manifold to be as in (4.14), but with  $\partial C = \emptyset$ .

Recall that for some  $H : M \rightarrow \mathbb{R}$  where  $H$  is  $C^2$ -small and Morse, there is an isomorphism of rings due to Piunikhin, Salamon and Schwarz [18, Example 3.3],

$$\Psi : QH^*(M) \rightarrow HF^*(H). \quad (4.15)$$

We pick an almost complex structure  $J$  on  $M$  and an interpolation  $H_s$  for  $s \in \mathbb{R}$ , such that  $H_s = H$  for  $s$  near  $-\infty$  and  $H_s = 0$  for  $s \geq 0$ . Consider  $\mathbb{C}$  as a disc with a cylindrical end: we begin with  $\mathbb{C}^*$  parametrised as the cylinder with logarithmic coordinates,

$$(s, t) \in \mathbb{R} \times S^1 \rightarrow e^{-2\pi(s+it)} \in \mathbb{C}^*.$$

Suppose  $u : \mathbb{C}^* \rightarrow M$  satisfies (4.16) (using logarithmic coordinates). We define the *geometric energy* of  $u$  to be  $E(u) = \int_{\mathbb{C}^*} |\partial_s u|_J^2 ds dt$ , where  $|\cdot|_J$  is defined using the metric  $\omega(-, J-)$  induced by the symplectic form  $\omega$  of  $M$  and the compatible almost-complex structure  $J$ . Using removal of singularities,  $u$  is finite energy exactly when it extends to a smooth map  $u : \mathbb{C} \rightarrow M$  satisfying a perturbed  $J$ -holomorphic equation (4.16), such that  $\lim_{s \rightarrow -\infty} u(s, t)$  is a Hamiltonian loop, and  $u$  is  $J$ -holomorphic on the unit disc.

Let

$$\mathcal{L}_1 = \{y : \mathbb{R}/\mathbb{Z} \rightarrow M | \dot{y} = X_H(y)\}.$$

If  $y \in \mathcal{L}_1$  then the coefficient of  $y$  in  $\Psi(x)$  is the number of “spiked discs” asymptoting to  $y$  and intersecting  $x$ . These spiked discs are smooth maps

$$u : \mathbb{C} \rightarrow M$$

where we parametrise  $\mathbb{C}$  as above, satisfying

$$\partial_s u + J(\partial_t u - X_{H_s}) = 0 \quad (4.16)$$

and  $u$  is  $J$ -holomorphic on the unit disc, with  $\lim_{s \rightarrow -\infty} u(s, t) = y(t)$ , and  $u(0) \in PD(x)$ : the notation means that we have fixed a generic chain representative  $X$  of  $PD(x)$  and require  $u(0) \in X$ . See Figure 4.2.

**Definition 4.2.13** (The PSS Map). *The PSS map  $\Psi : QC^*(M) \rightarrow CF^*(H, J)$  is defined by*

$$\Psi(x) = \sum_{y \in \mathcal{L}_1} n_{y,x} \cdot y$$

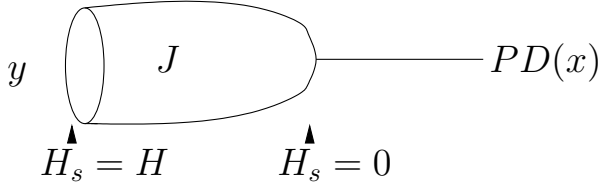


**Figure 4.2:** Spiked discs.

where  $n_{y,x}$  is the count of “spiked discs” as above, with fixed parametrisation. This map induces a well-defined map on cohomology, and there is an inverse map (counting  $J$ -holomorphic discs parametrised by  $(s, t) \mapsto e^{2\pi(s+it)}$ ).

**Remark 4.2.14.** For Liouville domains, exactness implies  $QH^*(M) \cong H^*(M)$ .

**Remark 4.2.15.** The Hamiltonian boundary condition in the definition of spiked discs can be thought of as an infinite dimensional analogue to the result in Morse theory, that (on a closed manifold), given a Morse flowline  $u : \mathbb{R} \rightarrow M$ , the limits of  $u(s)$  as  $s \rightarrow \pm\infty$  are Morse critical points.

## 4.2.6 The $c^*$ -map

A Hamiltonian  $H$  is radial at infinity if on the conical end (i.e. the region  $\partial C \times [1, \infty)$ ), there is some  $R$  such that  $H(z, r) = \lambda_H \cdot r$  for  $(z, r)$  coordinates on the conical end, and  $r > R$ . Let  $H, H'$  be Hamiltonians that are radial at infinity with  $\lambda_H < \lambda_{H'}$ . There exist continuation maps  $\Phi_{H, H'} : HF^*(H) \rightarrow HF^*(H')$ , as in Section 4.3 (nonequivariant continuation maps in this section correspond to the  $i = 0$ -term). We define

$$SH^*(M) := \varinjlim_H HF^*(2 \cdot H),$$

where the direct limit is taken over Hamiltonians that are radial at infinity. For a  $C^2$ -small  $H_0$ , we define the PSS isomorphism from Definition 4.2.13,  $\Psi : QH^*(M) \rightarrow HF^*(H_0)$ . There is also a natural map  $\xi : HF^*(H_0) \rightarrow SH^*(M)$  to the direct limit.

**Definition 4.2.16.**

$$c^* := \xi \circ \Psi : QH^*(M) \rightarrow SH^*(M).$$

## 4.2.7 Equivariant gluing and compactness

### 4.2.7.1 Equivariant gluing

We will not repeat the gluing argument, as pertinent details are given in Section 2.4.1. Instead we give the main modification for the equivariant case. The equivariant

gluing theorem uses two gluing theorems working in tandem: Morse theory for  $-\nabla g$  flowlines in  $S^i$  (for each fixed  $i$ ) and Floer theory for trajectories in  $M$ . The operators  $w \mapsto \partial_s w - \nabla g(w)$  and  $u \mapsto \bar{\partial}_J(u) = (du - X_H \times dt)^{0,1}$  combine to give

$$(w, u) \mapsto (\partial_s w - \nabla g(w), \bar{\partial}_{J^w}(u)).$$

The additional component of having a  $w$ -dependent almost complex structure  $J^w$  does not affect the gluing theorem because if  $A$  and  $B$  are linear operators with smooth right inverses then the “block-diagonal matrix” of operators

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix},$$

has a smooth right inverse for any  $C$ . This holds without need for extra comment because we only ever consider moduli spaces parametrised by the finite dimensional  $S^i \subset S^\infty$ , so  $w$  only ever varies in a compact space.

An unparametrised gluing theorem is used to show that the equivariant Floer differential is well defined, such as in [24, Section 3.3]. A parametrised gluing theorem shows that various operations are well defined on cohomology, such as in [24, Section 3.4]. We also use a parametrised gluing theorem when composing two parametrised operations, noting that we need to make our data (i.e. almost complex structure and/or Hamiltonian) dependent on our gluing parameter (because in these instances both of the objects we want to glue are parametrised operations).

Further, in the proofs of Theorem 4.1.1 and Lemma 4.1.2, we perform a parametrised gluing theorem on the two legs of the pair-of-pants simultaneously which (drawing an analogue with Floer cylinders) corresponds to

$$(w, u_1, u_2) \mapsto (\partial_s w - \nabla g(w), \bar{\partial}_{J^w}(u_1), \bar{\partial}_{J^{-w}}(u_2)).$$

This again is a gluing theorem of a familiar form.

We observe that in the cases we consider there is always a way to define an  $s$ -coordinate ( $s \in \mathbb{R}$ ), for example equivariant Floer cylinders in Section 4.2.1 and the equivariant pair-of-pants in Section 4.2.2. For the rest of Section 4.2.7.1 we ignore all of the domain dependence except the  $s$ -coordinate.

In practise, consider a broken solution  $((w_1, u_1), (w_2, u_2))$  to

$$\partial_s w_i(s) - \nabla g(w_i(s)), \bar{\partial}_{J_{i,s}^{w_i, H_{i,s}}} u_i = 0,$$

where for  $s \geq S_0$ ,

$$J_{1,s}^{w_1} = J_{2,-s}^{w_2} = J, \quad H_{1,s} = H_{2,-s} = H$$

are independent of  $s$ . For a sufficiently large gluing parameter  $\lambda > S_0$ , there is a glued solution  $(w, u)$  to

$$\partial_s w(s) - \nabla g(w(s)), \quad \bar{\partial}_{J_s^w, H_s} u = 0,$$

where

$$(J_s^w, H_s) = \begin{cases} (J_{1,s+\lambda}^{w_1}, H_{1,s+\lambda}) & \text{for } s \leq 0 \\ (J_{2,s-\lambda}^{w_2}, H_{2,s-\lambda}) & \text{for } s \geq 0 \end{cases}.$$

#### 4.2.7.2 Equivariant compactness

Refer to [28, Section 3, Section (4b) after Addendum 4.6] for a more detailed description of the parameter space and the various breakings of Morse flowlines on  $S^\infty$ .

We denote by  $\mathcal{P}^{i,\sigma}$  the space of  $w : \mathbb{R} \rightarrow S^\infty$  such that  $w(-\infty) = v^{i,\sigma}$  and  $w(\infty) = v^{0,+}$ . These spaces may be compactified to  $\bar{\mathcal{P}}^{i,\sigma}$  by adding broken Morse trajectories, i.e. tuples  $([w_0], \dots, [w_{j-1}], w_j, [w_{j+1}], \dots, [w_l])$  where  $w_n : \mathbb{R} \rightarrow S^\infty$  for  $n = 0, \dots, l$ , and  $w_n(\infty) = w_{n+1}(-\infty)$  and  $[w_n] = w_n/\mathbb{R} \in \mathcal{Q}^{i,\sigma} := \mathcal{P}^{i,\sigma}/\mathbb{R}$  for  $n \neq j$  are unparametrised trajectories obtained by quotienting by the translation in  $\mathbb{R}$ . We denote this  $w_0 \# w_1 \# w_2 \# \dots \# w_l$ , where it is understood that only one  $w_i$  is parametrised. Observe that the  $\mathcal{Q}^{i,\sigma}$  have compactifications  $\bar{\mathcal{Q}}^{i,\sigma}$  where all of the limiting flowlines are unparametrised. In general, given a sequence  $(w^k, u^k) \in \mathcal{M}_{\text{prod}}^{i,\sigma}(y, x)$  (in which case we are using parametrized flowlines on  $S^\infty$ , hence use parameter spaces  $\mathcal{P}^{i,\sigma}$ ) we may need to add limit points in the following way:

- some subsequence of the  $w^k$  converges to the limit  $w^\infty = w_0 \# w_2 \# w_3 \# \dots \# w_l$  on compact subsets, where the  $w_n$  are Morse trajectories of  $g$ . Exactly one of the  $w_j$  is parametrised: this is due to the compactification of the space of parametrised flowlines. The other flowlines are unparametrised. It is possible for one of the  $w_i$  to be constant at a critical point of  $g$ , assuming that this is the parametrised flowline. This is a standard compactification of  $\mathcal{P}^{i,\sigma}$ .
- some subsequence of the  $u^k$  converges to a broken Floer trajectory with respect to a family of complex structures,  $u^\infty = u_0 \# \dots \# u_l$  with  $u_n$  corresponding to (i.e. satisfying Equation (4.8) with respect to) the  $w_n$ . There may also be legitimate (i.e. nonequivariant) Floer trajectories between  $u_n$  and  $u_{n+1}$  for some  $n$ .

We can do similarly for a sequence  $[w^k, u^k] \in \mathcal{M}_{\text{eq}}^{i,\sigma}(y, x)$ , although in this case because we have quotiented by an  $\mathbb{R}$ -action we will need to base the compactification

of  $\mathcal{M}_{\text{eq}}^{i,\sigma}(y, x)$  on a compactification of  $\mathcal{Q}^{i,\pm}$ . Concretely, we want the forgetful maps to respect the compactification

$$\mathcal{M}_{\text{eq}}^{i,\sigma}(y, x) \rightarrow \mathcal{Q}^{i,\sigma}, \quad \mathcal{M}_{\text{prod}}^{i,\sigma}(y, x) \rightarrow \mathcal{P}^{i,\sigma}.$$

This is important in two cases: for the isolated points of  $\mathcal{M}_{\text{prod}}^{i,\sigma}(y, x)$ , there is a finite count. In the one-dimensional case, the limit points may only occur in the following ways:

- the sequence of pairs converges in  $\mathcal{M}_{\text{prod}}^{i,\sigma}(y, x)$ , so no new limit points are added.
- there is convergence in  $\mathcal{P}^{i,\sigma}$  of the  $w^k$  component (under the forgetful map), so a new Floer trajectory appears.
- there is convergence in  $\overline{\mathcal{P}}^{i,\sigma}$ , and for degree reasons  $l = 2$  above and no new Floer trajectories appear.

This shows that  $\overline{P}^{i,\sigma}$  is a chain map: cf Equation (4.11). Likewise, the compactification of  $\mathcal{M}_{\text{eq}}^{i,\sigma}(y, x)$  yields that

$$\sum_{j+k=i} d_{eq}^j \circ d_{eq}^k = 0,$$

where  $d_{eq}^i = d_{eq}^{i,+} + d_{eq}^{i,-}$ , and hence  $d_{eq}$  is a differential.

### 4.3 Equivariant continuation maps

Recall that for  $H \leq H'$  being radial at infinity, there exist continuation maps  $\Phi_{H,H'} : HF^*(H) \rightarrow HF^*(H')$ . These satisfy three conditions:

1. A generic homotopy of the data relative to the endpoints induces a chain homotopy on  $CF^*(H)$ .
2.  $\Phi_{H,H'} = \Phi_{H'',H'} \circ \Phi_{H,H''}$  for any  $H \leq H'' \leq H'$ .
3.  $\Phi_{H,H} = id$ .

These continuation maps are constructed by counting  $s$ -perturbed Floer trajectories. The  $s$ -perturbation changes the dimension of the moduli space that we consider (there is no longer an  $\mathbb{R}$ -action by translating  $s$ ). It is important that  $H \leq H'$  for these continuation maps to exist in general, although for closed monotone symplectic manifolds there are continuation maps between any two Hamiltonians.

Let  $H \leq H'$  be radial at infinity. Pick  $H_s$  for  $s \in (-\infty, \infty)$  and  $S_0 \in \mathbb{R}_{>0}$  such that  $H_s = H'$  for  $s \leq -S_0$  and  $H_s = H$  for  $s \geq S_0$ , where there is some  $R \geq 1$  such that  $\partial_s H_s(r, m) \leq 0$  for  $r \geq R$  on the conical collar of  $M$ . Let  $i \in \mathbb{Z}_{\geq 0}$ , and  $x \in CF^*(2 \cdot H)$  and  $y \in CF^*(2 \cdot H')$ . Define  $\mathcal{M}_{\text{cont}}^{i,\sigma}(y, x)$  to be the set of pairs  $(w, u)$  where  $w$  is a  $-\nabla g$  flowline on  $S^\infty$  with  $w(-\infty) = v^{i,\sigma}$  and  $w(\infty) = v^{0,+}$  and  $u : \mathbb{R} \times \mathbb{R}/2\mathbb{Z} \rightarrow M$  satisfying

$$\partial_s u + J_{eq,w(s),t} \partial_t u = -\nabla H_s,$$

with  $u(s, t) \rightarrow x(t)$  as  $s \rightarrow \infty$  and

$$u(s, t) \rightarrow \begin{cases} y(t) & \text{if } \sigma = + \\ y(t+1) & \text{if } \sigma = - \end{cases} \quad \text{as } s \rightarrow -\infty,$$

**Definition 4.3.1** (Equivariant continuation maps).

$$\Phi_{eq,H,H'}^{i,\sigma} : HF_{eq}^*(2 \cdot H) \rightarrow HF_{eq}^{*-i}(2 \cdot H'),$$

$$\Phi_{eq,H,H'}^{i,\sigma} \left( \sum_{j \geq 0} x_j h^j \right) = \sum_{j \geq 0} \sum_{y \in \mathcal{L}} \# \mathcal{M}_{\text{cont}}^{i,\sigma}(x_j, y) \cdot y \cdot h^j.$$

*Yielding*

$$\Phi_{eq,H,H'} : HF_{eq}^*(2 \cdot H) \rightarrow HF_{eq}^*(2 \cdot H'), \quad \Phi_{eq,H,H'} = \sum_{i,\sigma} h^i \Phi_{eq,H,H'}^{i,\sigma}.$$

**Lemma 4.3.2.** *As with nonequivariant continuation maps:*

1. *A generic homotopy of the data relative to the endpoints induces a chain homotopy on  $CF_{eq}^*(2 \cdot H)$ .*
2.  $\Phi_{eq,H,H'} = \Phi_{eq,H'',H'} \circ \Phi_{eq,H,H''}$  for any  $H \leq H'' \leq H'$ .
3.  $\Phi_{eq,H,H} = id$ .

*Proof.* The proof of (1) is the same as in the nonequivariant case: see [24, Lemma 3.12]. For (2) we use equivariant gluing for the right hand side of the equality, as in Section 4.2.7, using as our data associated to a sufficiently large gluing parameter  $\lambda > S_0$  being (for  $v \in S^\infty$  and  $(s, t) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ ):

- $J_{v,s,t} = J_{eq,v,t}$ .

- $H_{s,\lambda} = H_{s+\lambda}^1$  for  $s \leq 0$  and  $H_{s,\lambda} = H_{s-\lambda}^2$  for  $s \geq 0$ , where  $H_s^1$  is the  $s$ -dependent Hamiltonian used for  $\Phi_{eq,H'',H'}$  and  $H_s^2$  is the  $s$ -dependent Hamiltonian used for  $\Phi_{eq,H,H''}$ .

This is well defined and  $H_{s,\lambda}$  is smooth for  $\lambda > S_0$ . We then appeal to (1) to allow a different choice of equivariant data on the left hand side. For (3) we use (1) and note that we may choose  $H_s \equiv H$ , and then solutions that are not constant at a Hamiltonian loop come in an  $\mathbb{R}$ -family by translating in  $s$ , hence are nonisolated.  $\square$

**Remark 4.3.3** (Equivariant Auxiliary Data). *When constructing quantum cohomology, there is auxiliary data in the form of the almost complex structure  $J$  and the (perturbed) Morse functions  $f_s^p$  for  $p = 1, 2, 3$ . When constructing Floer cohomology the auxiliary data comprises of an almost complex structure  $J$  and the Hamiltonian  $H$ , along with a small perturbation of  $H$ . To make these concepts equivariant, we need to introduce a dependency of this data on  $v \in S^\infty$ . However, this dependency need not be applied to all of the ancillary data. Specifically:*

- *for the quantum Steenrod square we used a  $v$ -dependent Morse function  $f_v$  and a  $v$ -independent almost complex structure  $J$ ,*
- *for the symplectic square we used a  $v$ -dependent almost complex structure  $J_v$  and a  $v$ -independent Hamiltonian  $H$ .*

*The space of auxiliary data with equivariant conditions (excluding transversality) is connected, which is a useful exercise, and this induces a homotopy between different choices of such data as in the nonequivariant case, by choosing a path in the space of data. We will also want this space of auxiliary data to be contractible, using an argument as in Remark 2.3.1. More than this, if we assume that we have made a choice of auxiliary data such that all of the moduli spaces are transverse, and if we introduce a generic nontrivial  $v$  dependence on some of the auxiliary data (which may not already be  $v$ -dependent) then the setup remains transverse. By this method, we may define a different way of constructing the quantum Steenrod square: we begin with a  $v$ -dependent Morse function  $f_v$ , and we introduce a  $\mathbb{Z}/2$ -equivariant (with respect to  $z \mapsto 1/z$ )  $v$ -dependence to the almost complex structure  $J$ . Then, assuming we can do this in a way that our moduli spaces remain transverse, we may remove the  $v$ -dependence from  $f$ . There is a minor issue in this case that if we let  $f_v$  be independent of  $v$ , then for constant spheres the moduli space is no longer transverse: with this in mind, we must add a Hamiltonian perturbation and ask that our  $u$  satisfy a Floer-type equation, i.e. Equation (4.6), as opposed to being strictly  $J$ -holomorphic.*

The key point is that the order in which we add or remove  $\mathbb{Z}/2$ -equivariance matters: see Lemma 4.5.1. The  $v$ -dependent ancillary data is a model for  $E\mathbb{Z}/2$ , and the fact that it generally does not matter where the  $v$ -dependence appears in the data is the fact that if  $E, F$  are models for  $E\mathbb{Z}/2$  then so is  $E \times F$ . In general this requires us to prove that this space of auxiliary data is contractible. We considered the Morse theory case in Remark 2.3.1.

## 4.4 The Equivariant PSS isomorphism

The PSS isomorphism  $\Psi$  of Definition 4.2.13 extends to an isomorphism

$$\Psi_{eq} : QH^*(M)[[h]] \rightarrow HF_{eq}^*(2 \cdot H)$$

when  $H$  is  $C^2$ -small. Recall the spiked discs used in the definition of  $\Psi$ . We will replace these with “ $i$ -equivariant spiked discs”. We parametrize  $\mathbb{C}$  away from 0 by logarithmic coordinates  $e^{-\pi(2s+it)}$  for  $(s, t) \in (-\infty, \infty) \times \mathbb{R}/2\mathbb{Z}$  as in Section 4.2.5 (note we use  $\pi it$  because our cylinder has circumference 2). Fixing an almost complex structure  $J$  on  $M$ , for  $(v, z) \in S^\infty \times \mathbb{C}$ , pick almost complex structures  $J_{v,z}^\Psi$  on  $M$  with the conditions that:

1.  $J_{v,s,t}^\Psi = J_{v,s,t+2}^\Psi$ .
2.  $J_{v,s,t}^\Psi = J$  for  $s \geq 0$ .
3.  $J_{-v,s,t}^\Psi = J_{v,s,t+1}^\Psi$ .
4. There is  $S_0 > 0$  such that  $J_{v,s,t}^\Psi = J_{eq,v,t}$  for  $s \leq -S_0$ .

Given a flowline  $w$  of  $-\nabla g$  on  $S^\infty$ , we define  $J_{w(s),s,t}^w = J_{w(s),s,t}^\Psi$ . Pick an interpolation  $H_s$  for  $s \in \mathbb{R}$  such that  $H_s = H$  for  $s$  near  $-\infty$  and  $H_s = 0$  for  $s \geq 0$ . As in the discussion in Remark 4.3.3, we need not put any  $v$  dependence on the  $H_s$  assuming that the  $J_v$  are generic.

**Definition 4.4.1** ( $i$ -equivariant spiked discs). *Given  $x \in C^*(M)$  and  $y \in \mathcal{L}$ , an  $i$ -equivariant spiked disc from  $x$  to  $y$  is a pair  $(w, u)$  where  $w : \mathbb{R} \rightarrow S^\infty$  is a  $-\nabla g$  flowline from  $v^{i,\sigma}$  to  $v^{0,+}$  on  $S^\infty$  and*

$$u : \mathbb{C} \rightarrow M$$

*is smooth and satisfies*

$$\partial_s u + J_{w(s),s,t}^w (\partial_t u - X_{H_s}) = 0 \text{ on } \mathbb{C},$$

*with*

- $\lim_{s \rightarrow -\infty} u(s, t) = \begin{cases} y(t) & \text{if } \sigma = + \\ y(t+1) & \text{if } \sigma = - \end{cases}$
- $u(0) \in PD(x)$ .

**Definition 4.4.2.**  $\Psi_{eq,i} : QC^*(M)[[h]] \rightarrow CF^*(2 \cdot H)[[h]]$  is defined by

$$\Psi_{eq,i} \left( \sum_{j \geq 0} x_j h^j \right) = \sum_{j \geq 0} \sum_{y \in \mathcal{L}} n_{y,x_j,i} \cdot y \cdot h^j$$

where  $n_{y,x_j,i}$  counts isolated  $i$ -equivariant spiked discs from  $x$  to  $y$  as above, with fixed parametrisation. Define

$$\Psi_{eq}(x) = \sum_{i \geq 0} \Psi_{eq,i}(x) \cdot h^i.$$

We also give a Morse theoretic definition of the  $i$ -equivariant spiked discs. For the Morse theoretic case, in the definition of  $i$ -equivariant spiked discs the second bullet point is replaced by:

- there exists  $\alpha : [0, \infty) \rightarrow M$  such that  $\dot{\alpha}(t) = -\nabla f(\alpha(t))$  and  $u(0) = \alpha(0)$ , and  $\alpha(\infty) = x$  (where we abusively identify  $x \in CM^*(M, f)$  with  $x \in C^*(M)$ ).

Using gluing and compactness arguments as in Section 4.2.7 and [28, End of Section 4] yields the equations

$$\sum_{j+k=i} d_{eq}^k \circ \Psi_{eq,j}(x) = \Psi_{eq,i}(dx)$$

for isolated solutions, which shows that  $\Psi_{eq}$  descends to a map on equivariant cohomology.

Recall a classical lemma:

**Lemma 4.4.3.** *Let  $\Lambda$  be a ring. An element  $\sum_{i \geq 0} \lambda_i h^i$  of  $\Lambda[[h]]$  is invertible when  $\lambda_0$  is invertible.*

*Proof.* We write

$$\sum_{i \geq 0} \lambda_i h^i = \lambda_0 \left( 1 + \sum_{i \geq 1} \lambda_0^{-1} \lambda_i h^i \right),$$

and a Taylor expansion of  $(1+b)^{-1}$  exists, where  $b = \sum_{i \geq 1} \lambda_0^{-1} \lambda_i h^i$ . □

**Lemma 4.4.4.**  $\Psi_{eq}$  is an isomorphism.

*Proof.* Note  $\Psi_{eq,0} = \Psi$  which is an isomorphism. Use Lemma 4.4.3. □

One can also construct the inverse map  $\Psi_{eq}^{-1}$  directly, using reversed spiked discs as in Figure 4.3(I). Specifically, we require almost complex structures  $J_{v,z}^{R\Psi} \in \mathcal{J}_2$  on  $M$  with the conditions that:

1.  $J_{v,s,t}^{R\Psi} = J$  for  $s \leq 0$ .
2.  $J_{-v,s,t}^{R\Psi} = J_{v,s,t+1}^{R\Psi}$ .
3. There is  $S_0 > 0$  such that for  $s \geq S_0$ ,  $J_{v,s,t}^{R\Psi} = J_{eq,v,t}$ .

We can likewise define  $J_{R\Psi,s,t}^w = J_{w(s),s,t}^{R\Psi}$ , and the definition of the reversed  $i$ -equivariant spiked discs is analogous to Definition 4.4.2, i.e. for  $x \in \mathcal{L}$  and  $y \in C^*(M)$ , a reversed  $i$ -equivariant spiked disc is a pair  $(w, u)$  where  $w : \mathbb{R} \rightarrow S^\infty$  is a  $-\nabla g$  flowline from  $v^{i,\pm}$  to  $v^{0,+}$  and  $u : \mathbb{C} \rightarrow M$ . Now we parametrise  $\mathbb{C} - \{0\}$  as  $(s, t) \in \mathbb{R} \times \mathbb{R}/2\mathbb{Z} \mapsto e^{\pi(2s+it)}$ , and  $u$  satisfies:

- $\partial_s u + J_{R\Psi,s,t}^w(\partial_t u - X_{H_s}) = 0$  on  $\mathbb{C}$ ,
- $u(0) \in PD(y)^\vee$ ,
- $\lim_{s \rightarrow \infty} u(s, t) = \begin{cases} x(t) & \text{if } \sigma = + \\ x(t+1) & \text{if } \sigma = - \end{cases}$

To prove that the map  $A$  counting reversed  $i$ -equivariant spiked discs as in Figure 4.3(I) gives an inverse for  $\Psi_{eq}$ , use equivariant gluing to give setups as in Figure 4.3(II)-(III) and argue as in the nonequivariant case.

**Lemma 4.4.5.**  $A = \Psi_{eq}^{-1}$ .

*Proof.* We argue as in [18, Theorem 4.1]. After equivariant gluing,  $A \circ \Psi_{eq}$  is counted by setups as in Figure 4.3(II). A dimension argument, as in the nonequivariant case for weakly monotone symplectic manifolds, shows that there are not any nontrivial spheres. Specifically, for the coefficient of  $y$  in the image of  $x$  under the operation in (II), we count perturbed  $J$ -holomorphic spheres (i.e. the almost complex structure  $J = J_{v,z}$  depends on  $v \in S^i$  and  $z \in S^2$ , and the equation that  $u$  satisfies is perturbed by a Hamiltonian). In particular, we count the number of perturbed holomorphic spheres intersecting representatives of  $PD(x)$  and  $PD(y)^\vee$ . However, after gluing we may homotope the  $J_{v,z}$  (for  $z \in S^2$ ) so that  $J$  is independent of  $v$ . Neglecting transversality concerns momentarily, now we decouple the Floer theory on  $M$  and the Morse theory on  $S^\infty$ . By the dimension argument as in [14, 8.1.4(ii)], we see that

there are no such spheres (and as the set is empty, this data is trivially transverse even in the equivariant case).

After gluing,  $\Psi_{eq} \circ A$  is counted by setups as in Figure 4.3(III). This is the equivariant continuation map from  $H_0$  to  $H_0$ , hence the identity. The figures in (III) correspond to:

- taking a homotopy from the finite length Morse trajectory to a point,
- gluing the two discs to give a cylinder,
- homotoping the data so that the glued almost complex structure becomes  $J_{v,s,t} = J_{eq,t}$  for all  $s$ .

For the homotopy in the third bullet point, we want to choose  $J_{v,s,t,\eta}$  for  $\eta \in [0, 1]$  such that  $J_{v,s,t,0} = J_{v,s,t}$  and  $J_{v,s,t,1} = (J_{eq,v})_t$  such that  $J_{-v,s,t+1,\eta} = J_{v,s,t,\eta}$  for all  $\eta$ . We prove that we can do this, in the same way as Lemma 4.5.1. Specifically, let  $\mathcal{J}_1 \subset \mathcal{J}$  be the space of such allowable homotopies inside the space of all homotopies from  $J_{v,s,t}$  to  $J_{eq,v}$ . If  $\rho_*$  is the action on  $\mathcal{J}$  induced from  $\mathcal{J}_2$  then as  $\mathcal{J}_1$  and  $\rho_*\mathcal{J}_1$  are open and dense in  $\mathcal{J}$ , so is their intersection. Choose a homotopy from this intersection.  $\square$

**Remark 4.4.6.** *The argument in [14, 8.1.4(ii)], as referenced in 4.4.5 for the gluing in (II), is a rigorous proof of the intuitive argument that “a pseudoholomorphic sphere with two marked points is unstable, even for a domain dependent almost complex structure”.*

**Remark 4.4.7.** *In this remark, we use  $\phi$  and  $\psi$  independently to their usage in the rest of the thesis.*

To choose a  $J_{v,z}^\Psi$  as above, we proceed as follows. Recall that  $\mathcal{J}_2$  is a Banach manifold, and for a fixed time independent almost complex structure  $J$  on  $M$  it is immediate that  $J \in \mathcal{J}_2$ . There is a contractible neighbourhood  $U$  of  $J$  homeomorphic via some map  $\phi$  to a Banach space  $E$  (such that  $\phi(J) = 0$ ). The map  $\rho_*$  acts on  $\mathcal{J}$ , and  $U \cap \rho_*U$  is a  $\rho_*$ -invariant open neighbourhood of  $J$ . Indeed, it is homeomorphic under  $\phi$  to an open subset  $V$  of  $E$ . Pick a small ball  $0 \in B_\epsilon \subset V$ . We may replace the standard metric on  $V$  with a strongly equivalent,  $\rho_*$ -invariant metric, to assume  $B_\epsilon$  is  $\rho_*$ -invariant. There is an embedding  $\psi : S^\infty \times (0, \epsilon] \rightarrow B_\epsilon - \{0\}$ . Let  $(J'_{v,s})_t = (\phi^{-1}\psi(v, \epsilon - s))_t$  for  $s \in [0, \epsilon/2]$ , and perturb this so that it is smooth in all of its parameters. Use a  $\mathbb{Z}/2$ -equivariant homotopy (see the end of Lemma 4.4.5) from  $J'_{v,0,t}$  to  $J_{eq,v,t}$  to define  $J'_{v,s,t}$  in the region  $(s, t) \in [-S_0, \epsilon/2] \times \mathbb{R}/2\mathbb{Z}$ , and extend smoothly to define  $J_{v,z}^\psi$  for all  $z$ .

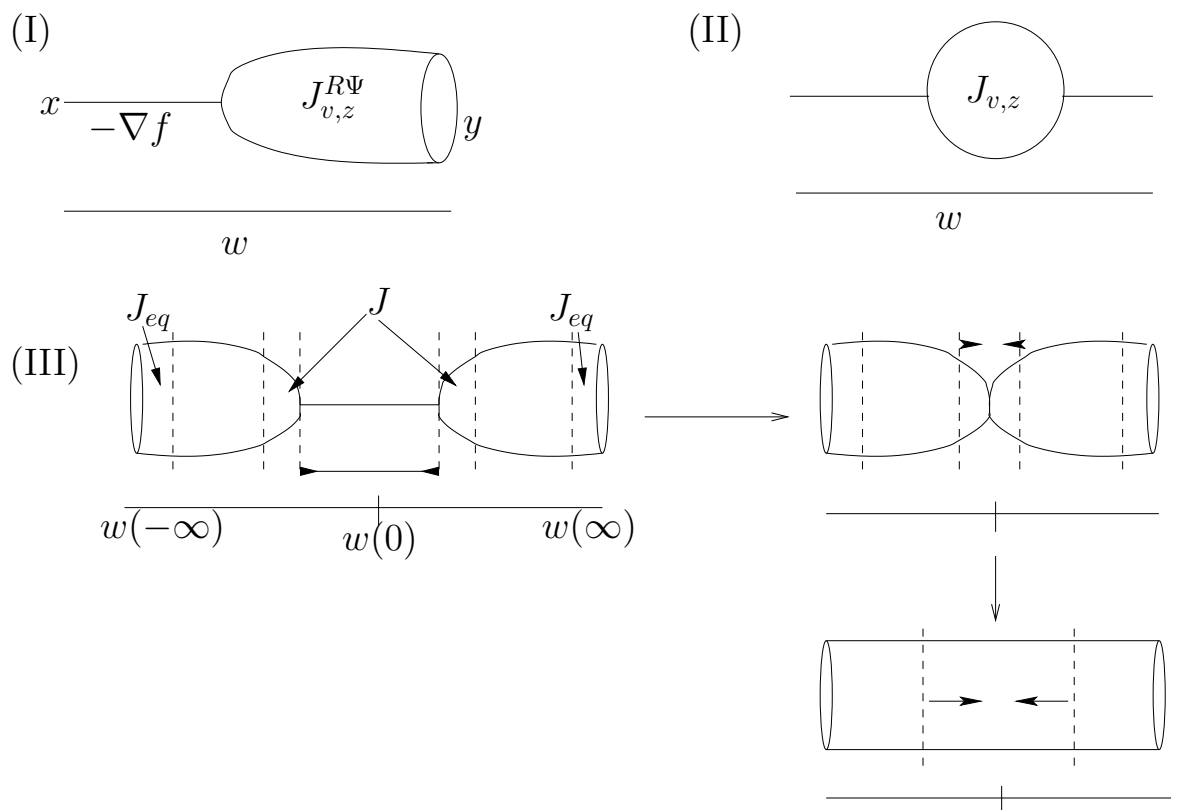


**Figure 4.3:** Reversed  $i$ -equivariant spiked discs and gluing.

## 4.5 $Q\mathcal{S}$ and the Equivariant pair-of-pants for a $C^2$ -small Hamiltonian

*Proof of Theorem 4.1.1.*  $\Psi_{eq}^{-1} \circ P\mathcal{S} \circ \Psi$  is calculated using setups as in the top of Figure 4.4. Specifically, for each  $i \geq 0$  we require sextuples  $(w_L, w_M, u_L, u_M, u_{R,+}, u_{L,+})$  such that:

- $w_L$  is a  $-\nabla g$  flowline from  $v^{i,\sigma_1}$  to  $v^{j,\sigma_2}$ , for some  $0 \leq j \leq i$  and  $\sigma_1, \sigma_2 \in \{\pm 1\}$ .
- $w_M$  is a  $-\nabla g$  flowline from  $v^{j,\sigma_2}$  to  $v^{0,+}$ ,
- $u_L$  is a reversed  $(i-j)$ -equivariant spiked disc, as in Section 4.4,
- $u_M$  satisfies Equation (4.10) with respect to  $w_M$ ,
- $u_{R,+}$  and  $u_{R,-}$  are spiked discs, using the same  $J, H_s$  and representative of  $PD(x)$  (or the same choice of  $f_s$  perturbing the Morse function  $f$ , if we work with Morse cohomology).

We use the equivariant gluing theorem, as in Section 4.2.7, on the right hand side: observe that near the right hand cylindrical ends of the pair of pants used for  $u_M$ , we have that  $J_{right,v,s,t}^+ = J_{right,v,s,t}^- = J_t$  for some  $J_t$ , and on the left hand cylindrical ends of the 0-equivariant spiked discs  $u_{R,\pm}$ , we use the data  $J_{s,t}^w = J_t$  for the same  $J_t$ . This is true for a sufficiently large gluing parameter. After gluing, we see that  $\Psi_{eq}^{-1} \circ P\mathcal{S} \circ \Psi$  is calculated using domains as in the middle figure of 4.4. We may also use the equivariant gluing theorem for the left hand broken equivariant Floer trajectory. Hence  $\Psi_{eq}^{-1} \circ P\mathcal{S} \circ \Psi$  is calculated by counting objects as in the bottom of Figure 4.4: more specifically, consider the sphere  $\tilde{S}$  as the pair-of-pants with finite ends, with three capping discs attached, one for each cylindrical end.

Concretely, the sphere  $\tilde{S}$  consists of a pair-of-pants (with finite ends) truncated to the region  $s \in [-\lambda, \lambda]$ , with a  $\frac{1}{\pi} \log(2\lambda)$  radius reversed spiked disc  $D_L$  glued on the left and two  $\frac{1}{\pi} \log(2\lambda)$  radius spiked discs  $D_R^\pm$  glued on the right hand ends. For  $\lambda > 0$  sufficiently large, and for  $w : \mathbb{R} \rightarrow S^\infty$  a  $-\nabla g$  flowline, we may define  $J_z^w$  for  $z \in \tilde{S}$  as:

- $J_z^w = J_{R\Psi,s',t'}^w$  for  $z \in D_L$  parametrised using logarithmic coordinates on the left capping disc  $(s', t') \mapsto \exp(2\pi(s' + it'))$ ,
- $J_z^w$  satisfies the conditions in Section 4.2.2 for  $z$  in the region  $s \in [-\lambda, \lambda]$ ,

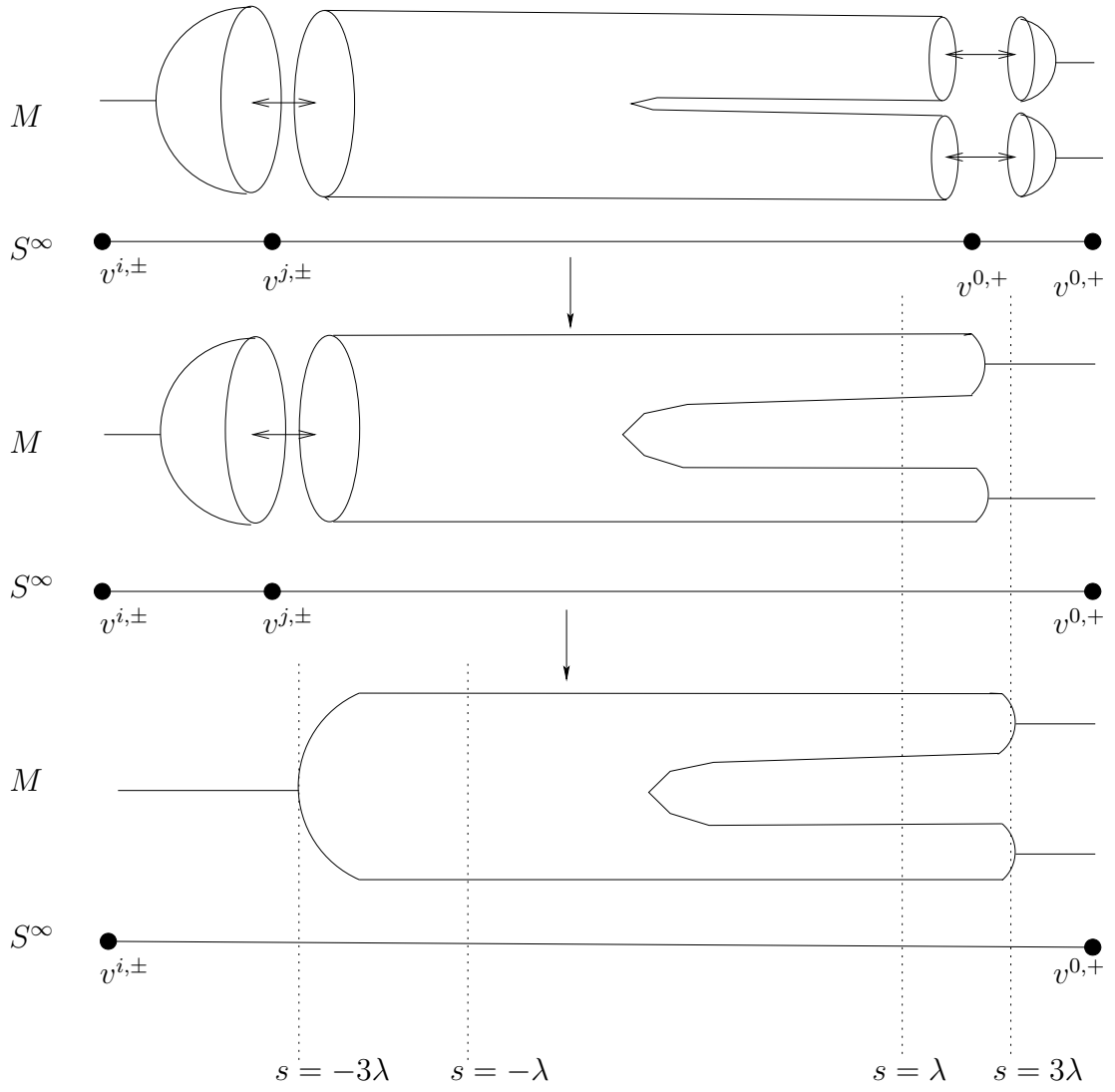


**Figure 4.4:** Gluing of spiked disks onto equivariant pair of pants. Double headed arrows denote that the Hamiltonian loops are the same.

- $J_z^w = J_{\Psi, s', t'}^w$  for  $z \in D_R^\pm$  using logarithmic coordinates on the right hand capping discs  $(s', t') \mapsto \exp(-2\pi(s' + it'))$ .

If the gluing parameter  $\lambda$  is sufficiently large then this is well defined, i.e.  $J_z^w = J_t$  in some neighbourhood of  $s = \pm\lambda$ . We conclude that the coefficient of  $y \in H^*(M)$  in  $\Psi_{eq, i-k}^{-1} \circ P\mathcal{S}_k \circ \Psi(x)$  for  $x \in H^*(M)$  is the number of isolated pairs  $(w, u)$  such that:

- $w$  is a  $-\nabla g$  flowline on  $S^\infty$  from  $v^{i, \pm}$  to  $v^{0, +}$ ,
- $u : \tilde{S} \rightarrow M$ ,
- $(du - X_H \otimes \beta + Y) \circ j = J_z^w(du - X_H \otimes \beta + Y)$ ,
- $u(0_L) \in PD(y)^\vee$ ,  $u(0_{R, \pm}) \in PD(x)$ , where  $0_L$  is the 0-point in the left capping disc, and  $0_{R, \pm}$  are the 0-points in the right capping discs. We use the same representative of  $PD(x)$  for  $0_{R, \pm}$ .

Recalling Section 4.2.4, we replace the above intersection condition with Morse flowlines

$$\alpha_L : (-\infty, 3\lambda] \rightarrow M, \quad \alpha_{R, \pm} : [3\lambda, \infty) \rightarrow M,$$

such that

$$\begin{aligned} \alpha_L(0) &= u(0_L) \text{ and } \alpha_{R, \pm}(0) = u(0_{R, \pm}), \\ \dot{\alpha}_L(s) &= -\nabla f(\alpha_L(s)) \text{ and } \dot{\alpha}_{R, \pm}(s) = -\nabla f_s^2(\alpha_{R, \pm}(s)), \end{aligned}$$

for a choice of  $f_s^2$  with  $s \in [3\lambda, \infty)$  such that  $f_s^2 = f$  for  $s \gg 0$ . Thus, under the identification of Morse and singular cohomology, the intersections with cycles become intersections with the finite endpoints  $(\pm 3\lambda)$  on Morse flowlines. This is a transverse setup for us to use the same Morse function  $f_s^2$  for both of the right-hand Morse flows as we do not allow constant spheres: we chose  $Y$  for the equivariant Floer equations, (4.10) so that there are no constant solutions.

**Lemma 4.5.1.** *The count of these glued solutions calculates  $QS$ .*

*Proof.* Choose  $f_{v, s}^2$  as for the Morse quantum Steenrod square, see Section 4.2.4. We seek a homotopy  $f_{v, s, \eta}^2$  for  $\eta \in [0, 1]$  with  $f_{v, s, 0}^2 = f^2$  and  $f_{v, s, 1}^2 = f_{v, s}^2$ . Let  $f_{v, s, \eta}^3 := f_{-v, s, \eta}^2$ . Given a homotopy  $f_{v, s, \eta}^2$ , it is not immediate that generically if  $f_{v, s, \eta}^2$  is an allowable homotopy from  $f^2$  to  $f_{v, s}^2$  then  $f_{-v, s, \eta}^2$  is an allowable perturbation from  $f^2$  to  $f_{-v, s}^2$ .

Let  $F_{v, s}^{\lambda, i} = (1-\lambda)f_s^i + \lambda f_{v, s}^i$  for  $i = 2, 3$ . This is some homotopy between the original choice of  $f^i$  and  $f_{s, v}^i$ , which is  $\mathbb{Z}/2$ -equivariant. In general we may need to perturb this

homotopy for transversality. Let  $\mathcal{B}$  be the space of allowable perturbations of  $F^\lambda = F_{v,s}^{\lambda,2}$ . Let  $\mathcal{B}_i$  be allowable perturbations of the Morse functions for the upper/lower leg, for  $i = 2, 3$  respectively, of the setup as in the bottom of figure 4.4. These  $\mathcal{B}_i$  are Baire sets in  $\mathcal{B}$ . There is an involution  $\iota$  of  $\mathcal{B}$  induced by  $v \mapsto -v$ . As  $\mathcal{B}_i$  is Baire for  $i = 2, 3$ , so too is  $\mathcal{B}_2 \cap \iota(\mathcal{B}_3)$ . Choose any  $F'_{v,s}^\lambda$  as the homotopy for  $f^2$  and  $F'_{-v,s}^\lambda = \iota F'_{v,s}^\lambda$  as the homotopy for  $f^3$ : this is an appropriate perturbation as  $\mathcal{B}_2 \cap \iota(\mathcal{B}_3)$  is Baire.

As discussed in Remark 4.3.3, we homotope the  $f_s^2$  to be  $v$ -dependent using  $F_{v,s}^{\lambda,2}$  for  $\alpha_{R,+}$  and  $F_{-v,s}^{\lambda,2}$  for  $\alpha_{R,-}$ . That is, we require after homotopy that the respective flowlines on the right hand side satisfy  $\dot{\alpha}_{R,\pm}(t) = -\nabla f_{s,\pm w(s)}(\alpha_{R,\pm}(t))$  for  $s \in [3\lambda, \infty)$  (and similarly on the left hand side). We then homotope  $Y_z$  and  $\beta$  to 0 and remove the  $v$ -dependency on  $J$ . We do this  $\mathbb{Z}/2$ -equivariantly, so desire  $J_{z,\eta}^w$  for  $\eta \in [0, 1]$  such that  $J_{z,0}^w = J_z^w$  and  $J_{z,1}^w = J$  independent of  $w$  and  $z$ . We require that  $J_{z,\eta}^w = J_{\gamma z,\eta}^{-w}$  for all  $\eta \in [0, 1]$ , where  $\gamma$  is the holomorphic involution of the sphere induced by the involution on the pair-of-pants. We can do this by the same argument as for  $f_{v,s}^2$  above. After translating in  $s$  we may assume  $\alpha_{R,\pm} : [0, \infty) \rightarrow M$  and  $\alpha_L : (\infty, 0] \rightarrow M$ , along with the relevant  $s$ -translation in  $f_{v,s}^2$ . This is Definition 4.2.8 (using  $\mathcal{M}_{\text{Morse},i,A}$ ) of the quantum Steenrod square.  $\square$

This concludes the proof of Theorem 4.1.1.  $\square$

- Remark 4.5.2.** 1. *There is a  $\mathbb{Z}/2$  action on the broken solutions, induced from the action of  $\gamma$  on the pair-of-pants and the capping discs. This action induces an action on the glued solutions. Moreover it induces the same  $\mathbb{Z}/2$  action as in the definition of  $QS$ , by triple transitivity of  $PSL(2, \mathbb{C})$ . The induced  $\mathbb{Z}/2$  action on glued solutions must be the same action as in the  $QS$  case, as it is a biholomorphism and its action on three points is known. The  $\mathbb{Z}/2$  action in the symplectic square case is not explicitly stated in [28], but can be deduced from addendum 4.12 on page 41. It is that solutions  $(w, u)$  biject with  $(-w, u \circ \gamma)$ , where  $\gamma : S \rightarrow S$  is the covering involution of  $S \rightarrow \mathbb{R}/\mathbb{Z}$ .*
2. *Our sphere  $\tilde{S}$  in the proof of Theorem 4.1.1 can be thought of as a conformal rescaling of the sphere  $S^2$ , in an annulus near each of the three points. Specifically, pick three points  $z_1, z_2, z_3 \in S^2$ , and  $0 < \epsilon < \epsilon'$  such that the  $\epsilon'$  balls around the  $z_i$  are pairwise distinct. We can think of the finite length cylindrical edges in  $\tilde{S}$  as taking logarithmic coordinates on the annulus  $B_{\epsilon'}(z_i) - \mathring{B}_\epsilon(z_i)$ .*

## 4.6 $QS$ and the symplectic square for equivariant symplectic cohomology

### 4.6.1 Equivariant Symplectic Cohomology

Symplectic cohomology is formed as the direct limit of Floer cohomologies using continuation maps  $\phi_{H,H'} : HF^*(H) \rightarrow HF^*(H')$  where  $H, H'$  are radial at infinity with  $\lambda_H \leq \lambda_{H'}$ . Specifically,

$$SH^*(M) := \varinjlim_H HF^*(H),$$

where the direct limit is taken over all Hamiltonians  $H$  that are radial at infinity.

We want to modify this to define  $SH_{eq}^*(M)$ . There are nonequivalent ways of doing this in general, but we work as suggested in [28, Equation (2.50)]. Specifically we use the equivariant continuation maps we defined in Section 4.3 to define the direct limit of the directed set  $\{HF_{eq}^*(2 \cdot H)\}$ , where the  $H$  are radial at infinity, so

$$SH_{eq}^*(M) := \varinjlim_H HF_{eq}^*(2 \cdot H).$$

We will define a symplectic square

$$PS : SH^*(M) \rightarrow SH_{eq}^{2*}(M) \tag{4.17}$$

induced by the symplectic square on Floer cohomology. This requires Lemma 4.1.2 to be well defined:

*Proof of Lemma 4.1.2.* Using Figure 4.5, apply the equivariant gluing theorem on the left hand cylinder of the lower setup, and the two right hand legs of the upper setup. The proof follows by invariance under  $\mathbb{Z}/2$ -equivariant changes of auxiliary data. The flowline from  $v^{i,\sigma}$  to  $v^{j,\sigma'}$  (for  $j \leq i$ ) in the lower part of Figure 4.5 could be rephrased as a solution for asymptotics  $v^{i-j,\sigma\sigma'}$  and  $v^{0,+}$ , as in Remark 4.2.5.  $\square$

The proof of Lemma 4.1.2 allows us to define  $PS$  in (4.17) as the direct limit of the Hamiltonian Floer case. Let  $\xi_{eq} : HF_{eq}^*(2 \cdot H) \rightarrow SH_{eq}^*(M)$  be the natural map to the direct limit.

**Definition 4.6.1** (The equivariant  $c^*$ -map).  $c_{eq}^* := \xi_{eq} \circ \Psi_{eq}$ .

*Proof of Corollary 4.1.3.* Use Theorem 4.1.1 and apply the direct limit.  $\square$

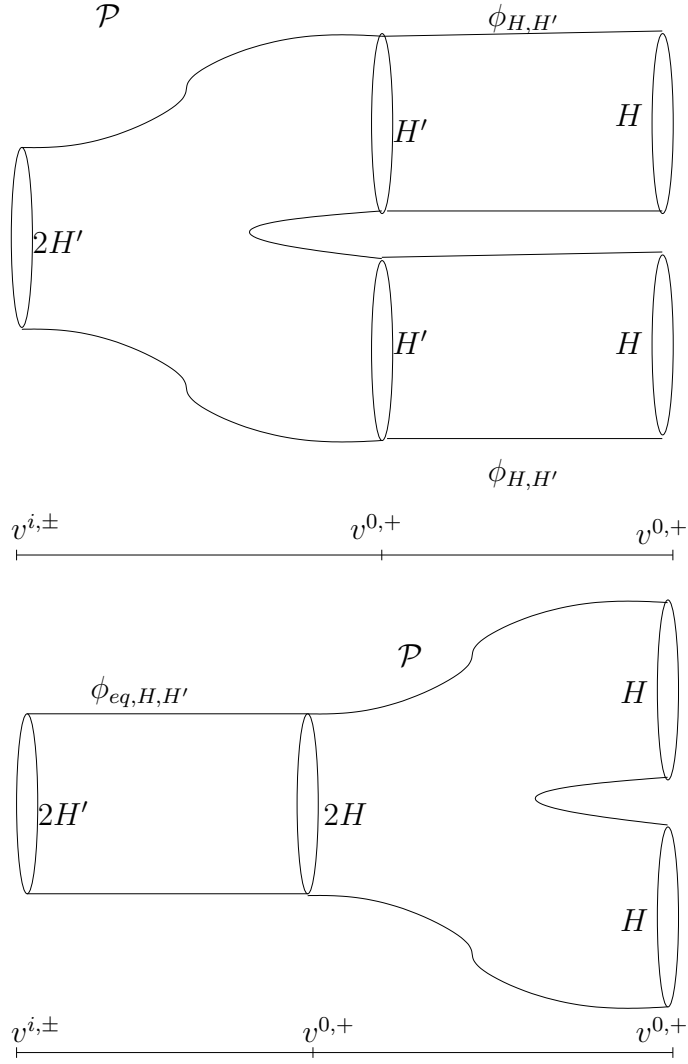


**Figure 4.5:** Broken solutions for  $PS \circ \phi_{H,H'}$  and  $\phi_{eq,H,H'} \circ PS$  respectively.

## 4.7 The symplectic Cartan relation

There is no immediate analogue of the (quantum) Cartan relation for the symplectic square. This is because there is no obvious product  $*_{eq}$  on  $SH_{eq}^*(M)$ . A symplectic Cartan relation might have been expected to take the form

$$PS(x * y) = PS(x) *_{eq} PS(y) + q(x, y),$$

for an appropriate product  $*_{eq}$  on  $SH_{eq}^*(M)$  and a correction term  $q : SH^*(M) \otimes SH^*(M) \rightarrow SH_{eq}^*(M)$  as in Theorem 3.1.2. A product is not an issue in the quantum and classical cases because for example  $QH_{eq}^*(M) = QH^*(M)[[h]]$ , which has an obvious product. This is not true for  $SH_{eq}^*(M)$ .

Recall in Remark 4.2.5 that it was important for defining the symplectic square that we use an almost complex structure  $J_z^w$  such that  $J_z^w = J_{\gamma z}^{-w}$ , where  $\gamma$  is an involutory biholomorphism of the pair-of-pants  $S$ . Specifically, we require that if  $u : S \rightarrow M$  is  $J^w$ -holomorphic then  $u \circ \gamma : S \rightarrow M$  is  $J^{-w}$ -holomorphic, and the two solutions are related by a  $\mathbb{Z}/2$  action. Attempting to do the same for  $*_{eq}$  fails because we would need to choose  $J_z^w$  for  $z \in S$  such that  $J_z^w = (J_{eq, w(s)})_t$  for  $z = (s, t)$  in cylindrical coordinates near the ends of the cylindrical ends. This is to ensure that we are constructing a chain map. Suppose for a contradiction that  $J_z^{-w} = J_{\gamma' z}^w$  for some involutory biholomorphic  $\gamma' : S \rightarrow S$  then  $\gamma'$  would have to be a half-rotation near each of the three cylindrical ends. As we may view the pair-of-pants as a three punctured sphere (using logarithmic coordinates), we may consider  $\gamma'$  to be a biholomorphism of the 3-punctured sphere. The condition that  $\gamma'$  must be a half-rotation near the cylindrical ends means that  $\gamma'$  is bounded near each of the punctures, so it extends to a biholomorphism  $\gamma' : S^2 \rightarrow S^2$ , which fixes each of the three points that were removable singularities. A biholomorphism of  $S^2$  that fixes three points is the identity, giving a contradiction, so no such  $\gamma'$  exists.

### 4.7.1 The symplectic Cartan relation

In this section we will define some almost complex structures  $J_{v,s,t}^p$  where  $(s, t)$  parametrises a half-cylinder as given below,  $v \in S^\infty$  and  $p$  is an edge label. We give some important terminology:

1.  $J_{v,s,t}^p$  is incoming if  $s \in (-\infty, 0]$  and outgoing if  $s \in [0, \infty)$ .
2.  $J_{v,s,t}^p$  is equivariant if  $t \in \mathbb{R}/2\mathbb{Z}$  and nonequivariant if  $t \in \mathbb{R}/\mathbb{Z}$ .

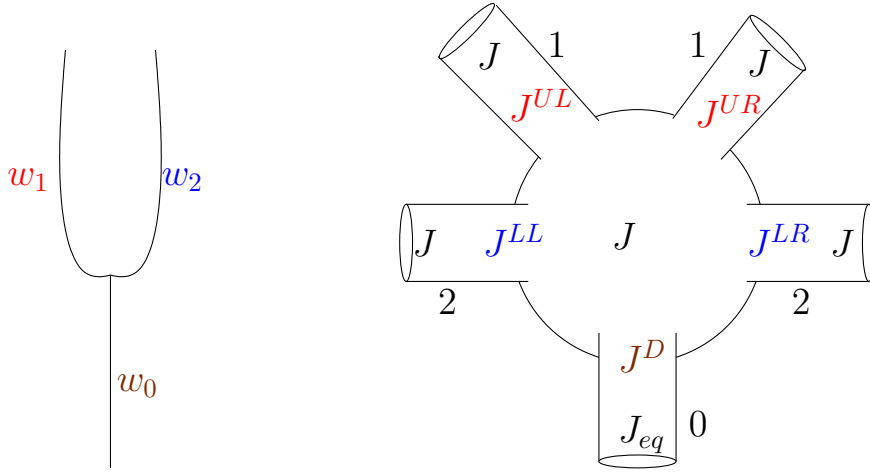


**Figure 4.6:** The almost complex structure  $J_z^w$ . The labels 0, 1, 2 next to the edges indicate which  $w_i$  is being used in that situation.

3.  $(J_{v,s,t}^p, J_{v,s,t}^q)$  are symmetric if  $J_{v,s,t}^p = J_{-v,s,t}^q$ .

Further:

1. If  $J_{v,s,t}^p$  is equivariant then it satisfies  $J_{v,s,t}^p = J_{-v,s,t+1}^p$  for all  $(v, s, t)$ .
2. If  $J_{v,s,t}^p$  is equivariant then  $J_{v,s,t}^p = J_{eq,v,t}$  for  $|s| \gg 1$ .
3. If  $J_{v,s,t}^p$  is nonequivariant then  $J_{v,s,t}^p = J_t$  for  $|s| \ll 1$ , where  $J_t = J_{right,v,2,t}^\pm$  as in Section 4.2.2.
4. For all  $p$ , the  $J_{v,s,t}^p = J$  for  $0 \leq |s| \leq 1$  for some fixed almost complex structure  $J$  on  $M$ .

We choose almost complex structures:

- $(J_{v,s,t}^{LL}, J_{v,s,t}^{LR})$  to be outgoing, nonequivariant and symmetric.
- $(J_{v,s,t}^{UL}, J_{v,s,t}^{UR})$  to be outgoing, nonequivariant and symmetric.
- $J_{v,s,t}^D$  to be incoming and equivariant.

As we work with cohomology, “outgoing” ends correspond to inputs and “incoming” ends correspond to outputs. For each  $R \in (1, \infty)$  and  $p \in \{LL, LR, UL, UR, D\}$ , let  $Z_p \in S^2$  such that:

$$Z_{LL} = 0, \quad Z_{LR} = \infty, \quad Z_D = 1, \quad Z_{UL} = -1/R, \quad Z_{UR} = -R.$$

$$\frac{dw_0}{dt}(s) = -\nabla g_{0,s} \quad \begin{array}{l} \frac{dw_1}{dt}(s) = -\nabla g_{1,s} \\ \frac{dw_2}{dt}(s) = -\nabla g_{2,s} \end{array}$$

**Figure 4.7:** Flowlines on  $S^\infty$ .

For  $p \in \{LL, LR, UL, UR, D\}$  pick  $\theta_p(R) \in ST_{Z_p}S^2$ , the unit circle bundle of  $S^2$  at  $Z_p$ , such that  $\theta_{LL} = i^*\theta_{LR}$  and  $\theta_{UL} = i^*\theta_{UR}$ , where  $i(z) = 1/z$ . Pick smooth functions  $r_p : (1, \infty) \rightarrow \mathbb{R}_{>0}$  such that the discs  $B_{r_p(R)}(Z_p)$  are pairwise distinct for each  $R$ , and  $r_p(R) = r_{p'}(R)$  for  $(p, p') = (UL, UR), (LL, LR)$ . We let  $m_R$  be the associated Riemann surface using logarithmic coordinates on  $B_{r_p(R)}(Z_p) - \{Z_p\}$ . The parametrisation of the cylindrical ends for each  $p$  are determined by the relevant descriptors “outgoing, incoming” etc for the corresponding  $J^p$  above.

Let  $w = (w_0, w_1, w_2)$  be flowlines on  $S^\infty$  with respect to perturbations  $g_{0,s}, g_{1,s}, g_{2,s}$  of the Morse function  $g$ , defined for  $s \in (-\infty, 0]$  for  $g_{0,s}$  and  $s \in [0, \infty)$  for  $g_{1,s}, g_{2,s}$ . This is illustrated in Figure 4.7. These  $g$  must be chosen to satisfy:

- Each  $g_{q,s} = g$  for  $|s| \gg 0$ ,
- the  $g_{q,0}$  are generic, to ensure transverse moduli spaces,
- $g_{q,s}(\tau x) = g_{q,s}(x) + \text{constant}$ , so that  $\tau^i$  induces a bijection of flowlines with endpoints  $v^{i,+}$  and  $v^{0,+}$  as in Remark 4.2.5,
- $g_{q,s}(-x) = g_{q,s}(x)$ , so that they descend to functions on  $\mathbb{RP}^\infty$ .

One should think of these  $g_{q,s}$  as being used analogously to calculating the cup product on  $\mathbb{RP}^\infty$  using Morse theory (and is equal to this when restricting to each  $\mathbb{RP}^i$ ). An example of such  $g_{i,s}$  is:

- $g_{0,s} = g_{1,s} = g$  for all  $s$ ,
- Let  $\bar{g}(x_0, x_1, \dots) = g(x_0, x_1, \dots) + \epsilon \sum_{j \geq 0} x_{2j} x_{2j+1}$  for some small  $\epsilon$ . Then let  $g_{2,s}(v) = g(v) + \epsilon(1 - \beta(s))\bar{g}(\mathcal{R}v)$  where  $\beta : [0, \infty) \rightarrow \mathbb{R}$  is smooth, nonincreasing, and  $\beta(s) = 1$  for  $s \geq 1$  and  $\beta(s) = 0$  for  $s \leq 1/2$ . This satisfies all of the conditions above: crucially, it ensures the moduli spaces that we count will be transverse.

We define  $J_z^w$  for  $z \in m_R$  such that

$$J_{s,t}^w = J_{p,w_q(s),s,t} \quad (4.18)$$

for  $z = (s, t)$  on the cylindrical end  $e_p$  associated to  $Z_p$ , and  $J_z = J$  away from the cylindrical ends, and for pairs

$$(p, q) = (D, 0), (UL, 1), (UR, 1), (LL, 2), (LR, 2).$$

We see that  $J_z^w = J_{1/z}^{-w}$ .

Given  $x, y, z \in SC^*(M)$ ,  $i \in \mathbb{Z}$  and  $R \in (1, \infty)$ , let  $\mathcal{M}_{i,R}(z; x, y)$  consist of pairs  $(w, u)$  such that  $w$  is a triple of flowlines as in Figure 4.7, with  $w_0(-\infty) = v^{i,\sigma}$ ,  $w_1(\infty) = v^{0,+}$ ,  $w_2(\infty) = v^{0,\pm}$  for  $\sigma \in \{\pm\}$ , and  $u : m_R \rightarrow M$  such that  $\lim_{s \rightarrow \infty} u|_{e_p}(s, t) = \alpha(t)$  for

$$(p, \alpha) = (D, z), (UL, y), (UR, y), (LL, x), (LR, x),$$

for  $\sigma = +$ . For  $\sigma = -$ , replace  $z(t)$  by  $z(t+1)$ . Then  $u$  satisfies

$$(du - X_H \otimes \beta + Y) \circ j = J_z^w \circ (du - X_H \otimes \beta + Y) \quad (4.19)$$

as in the standard Floer equation where  $X_H$  is the Hamiltonian vector field associated to  $H$  and  $\beta$  is a 1-form on  $m_R$  with  $d\beta = 0$ , and  $\beta = b_1 dt$  on  $e_{LL}, e_{LR}$  where  $x \in CF^*(b_1 H)$ , similarly  $\beta = b_2 dt$  on  $e_{UL}, e_{UR}$  and  $\beta = (b_1 + b_2) dt$  on  $e_D$ , where  $b_1, b_2 \in \mathbb{Z}_{>0}$  (where  $\beta$  varies smoothly with  $R$ ). We generally have to perturb using a Hamiltonian  $Y$ , invariant under  $z \mapsto 1/z$  and supported only on a compact subset of the cylindrical ends.

**Definition 4.7.1.** For generic  $R \in (1, \infty)$ ,  $i \geq 0$

$$q(R)^i : SC^*(M) \otimes SC^*(M) \rightarrow SC^*(M), \quad q(R)^i(x, y) = \sum_{z \in \mathcal{L}} \# \mathcal{M}_{i,R}(z; x, y) \cdot z$$

where  $\#$  counts isolated solutions. Then

$$x \otimes y \mapsto \sum_{i \geq 0} h^i q(R)^i(x, y)$$

descends to a well defined map  $q(R) : SH^*(M) \otimes SH^*(M) \rightarrow SH_{eq}^*(M)$ , with a similar proof as in Section 4.2.2.1.

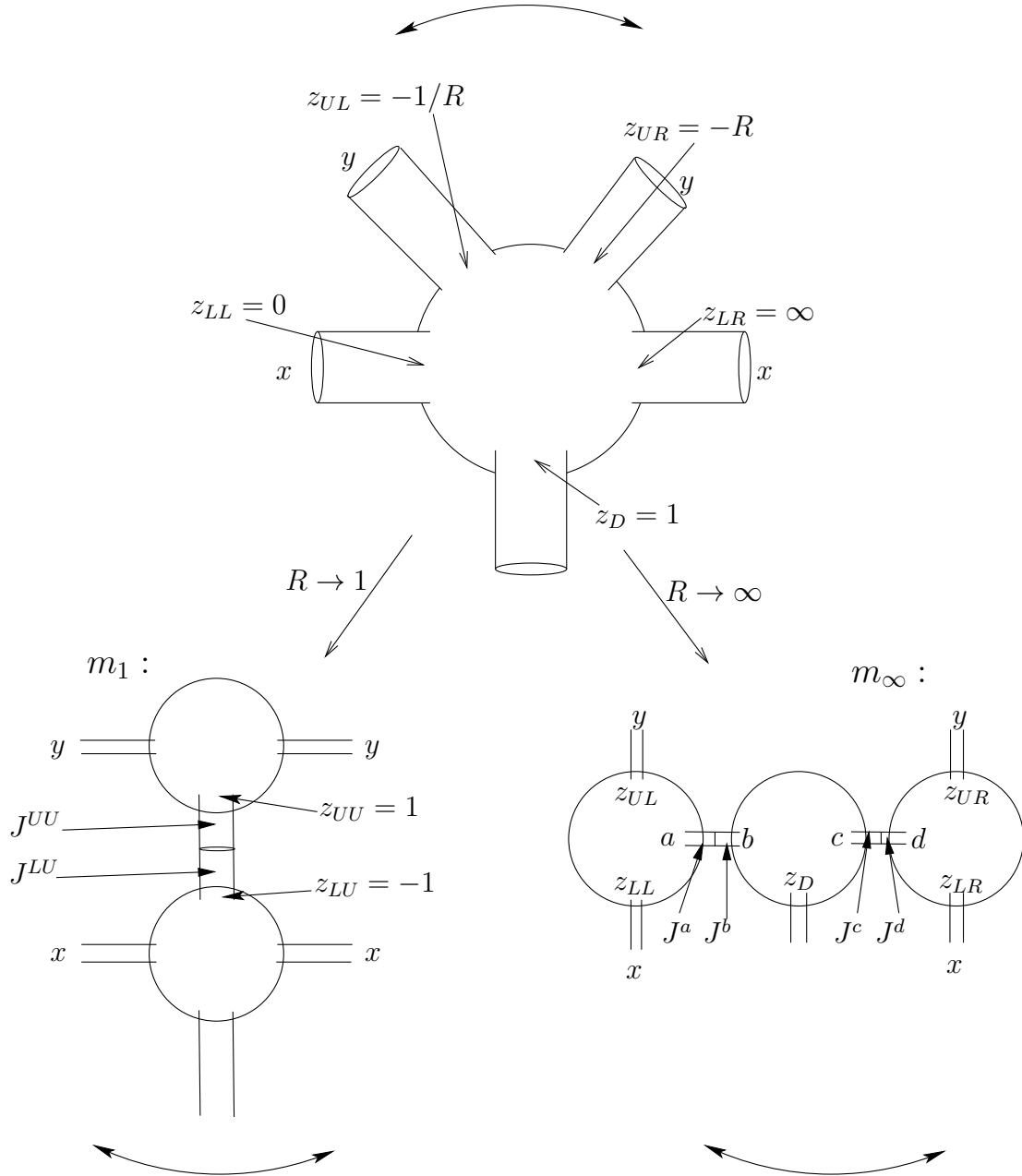


**Figure 4.8:** The 1-dimensional moduli space linking  $m_1$  and  $m_\infty$ . The double-headed arrows denote  $\mathbb{Z}/2$  symmetry. For simplicity we omit the  $S^\infty$  data from this picture.

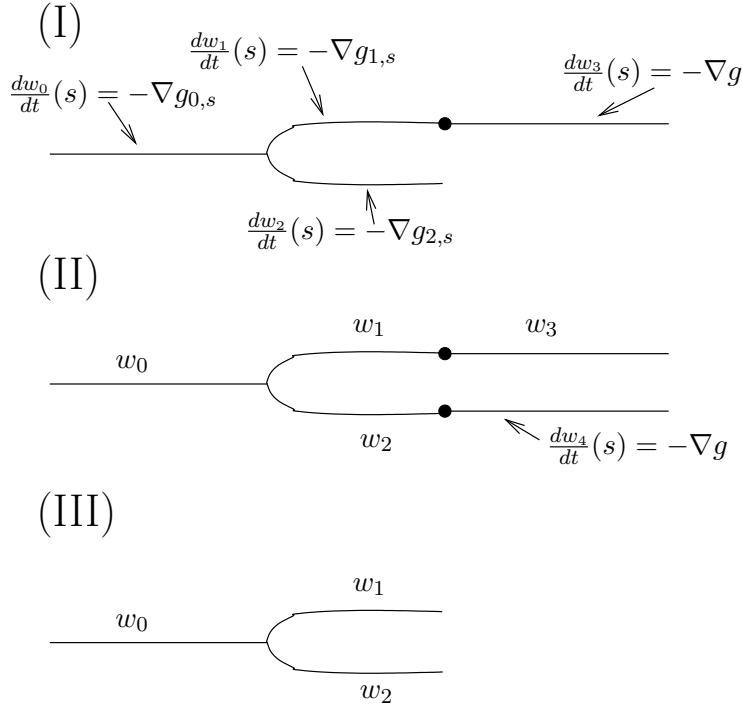


**Figure 4.9:** Flowline setups used in  $S^\infty$ .

The 1-cobordism associated to  $R \in (1, \infty)$  is illustrated in Figure 4.8. Its boundary consists of operations corresponding to  $m_1$  and  $m_\infty$  (given in Figure 4.8).

For  $m_1$ , two new points are introduced,  $Z_{UU}, Z_{LU}$ . These points correspond to cylindrical ends, and one must make a consistent choice of  $\theta$  and radius for each end. We choose  $J_{v,s,t}^{UU}$  to be incoming and equivariant and  $J_{v,s,t}^{LU}$  to be outgoing and equivariant. In this case, for isolated solutions we use quartets of flowlines on  $S^\infty$  denoted  $w = (w_0, w_1, w_2, w_3)$  as in Figure 4.9(I), and our almost complex structure  $J_z$  for  $z \in m_1$  satisfies Equation (4.18) for

$$(p, q) = (D, 0), (LU, 1), (LR, 2), (LL, 2), (UU, 3), (UL, 3), (UR, 3).$$

For  $m_\infty$ , there are four new edges that appear, which are labelled  $a, b, c, d$  as in Figure 4.8. We choose almost complex structures such that:

- $(J_{v,s,t}^a, J_{v,s,t}^d)$  are incoming, nonequivariant and symmetric.
- $(J_{v,s,t}^b, J_{v,s,t}^c)$  are outgoing, nonequivariant and symmetric.

In this case we use quintets of flowlines on  $S^\infty$ , denoted  $w = (w_1, w_2, w_3, w_4, w_5)$  as in Figure 4.9(II). After homotoping the data as in Lemma 4.7.2, the only isolated solutions will be for  $w_3$  and  $w_4$  being constant flowlines at  $v^{0,+}$  and  $v^{0,\pm}$ . So this really reduces to the setup of Figure 4.9(III).

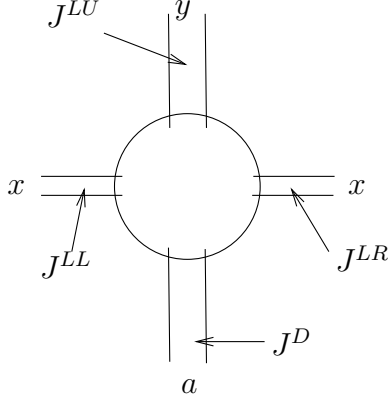


**Figure 4.10:** Configurations for the coefficient of  $a$  in the operation  $PS'(x; y)$ .

The points in  $\mathcal{M}_{i,1}(z; x, y) \sqcup \mathcal{M}_{i,\infty}(z; x, y)$  form the boundary of the 1-dimensional cobordism  $\bigsqcup_{R \in [1, \infty]} \mathcal{M}_{i,R}(z; x, y)$ , hence  $\#\mathcal{M}_{i,1}(z; x, y) = \#\mathcal{M}_{i,\infty}(z; x, y)$  and so  $q(1)^i = q(\infty)^i$ .

**Lemma 4.7.2.**  $\sum_i q(\infty)^i(x, y)h^i = PS(x * y)$  where  $*$  is the pair-of-pants product.

*Proof.* Consider Figure 4.8, specifically the  $m_\infty$  endpoint. Homotope the data (i.e. the almost complex structures) on the left and right spheres so that they are independent of  $v$ . It is then immediate that  $q(\infty)^i(x, y) = PS_i(x * y)$ .  $\square$

Let  $\tilde{S}$  be the domain  $m_1$ , but removing the sphere with edges  $Z_{UL}, Z_{UU}, Z_{UR}$ , see Figure 4.10. Thus,  $\tilde{S}$  is a Riemann surface with  $1 + 3$  cylindrical ends. Keep all almost complex structures and other conditions the same as for  $m_1$ .

**Definition 4.7.3.** Using the same edge labels and almost complex structures for  $\tilde{S}$  as for  $m$ , define

$$PS'_i : CF^j(b_1 H) \otimes CF_{eq}^k(2 \cdot b_2 H) \rightarrow CF_{eq}^{2j+k-i}(2 \cdot (b_1 + b_2) H), \quad PS'_i(x; y h^l) = \sum_a n_{x,y,a,l,i} \cdot a$$

where  $n_{x,y,a,l,i}$  counts the number of pairs  $(w, u)$  where  $w$  is a setup as in Figure 4.9(III) with  $w_0(-\infty) = v^{i+l, \sigma_0}$ ,  $w_1(\infty) = v^{l, \sigma_1}$ ,  $w_2(\infty) = v^{0, +}$  and  $u : \tilde{S} \rightarrow M$  satisfies Equation (4.19) on  $\tilde{S}$  with:

$$\begin{aligned} \lim_{s \rightarrow -\infty} u|_{e_D} &= \begin{cases} a(t) & \text{if } \sigma_0 = + \\ a(t+1) & \text{if } \sigma_0 = - \end{cases} \\ \lim_{s \rightarrow \infty} u|_{e_{LU}} &= \begin{cases} y(t) & \text{if } \sigma_1 = + \\ y(t+1) & \text{if } \sigma_1 = - \end{cases} \\ \lim_{s \rightarrow \infty} u|_{e_k} &= x(t) \text{ for } k = LR, LL. \end{aligned} \tag{4.20}$$

$$PS'(x; yh^j) := \sum_{i \geq 0} PS'_i(x; yh^j) h^{j+i}.$$

Note that  $PS'$  descends to a map on cohomology as in Section 4.2.2.1: first we construct

$$\mathcal{P}' : (CF^*(b_1H) \otimes CF^*(b_1H)) \otimes CF_{eq}^*(2 \cdot b_2H) \rightarrow CF_{eq}^*(2 \cdot (b_1 + b_2)H),$$

where for  $\mathcal{P}'(x_{LL}, x_{LR}, y)$  we consider  $u$  satisfying Equations (4.19) and (4.20), changing  $x(t)$  to  $x_k(t)$  in (4.20). We extend  $h$ -linearly to

$$\mathcal{P}' : C_{\mathbb{Z}/2}^*(CF^*(b_1H) \otimes CF^*(b_1H)) \otimes CF_{eq}^*(2 \cdot b_2H) \rightarrow CF_{eq}^*(2 \cdot (b_1 + b_2)H).$$

We then analyse the different possible ways in which the cylindrical ends may break. This shows that  $\mathcal{P}'$  is a chain map, hence  $PS' = \mathcal{P}' \circ (\eta \otimes id)$  is a well-defined map on cohomology.

It is immediate that  $PS'_0(x; yh^i)$  is a chain representative of  $x^2yh^i$  and  $PS'_i(x; 1)$  is a chain representative of  $PS_i(x)$ .

**Remark 4.7.4.** *Using the 1-dimensional cobordism illustrated in Figure 4.11 for  $R \in [1, \infty]$ , arguing as in Lemma 4.7.2 it can be shown that*

$$PS'(x * z; y) = PS'(x; PS'(z, y))$$

*or rather that  $PS'$  defines a  $SH^*(M)$ -module structure on  $SH_{eq}^*(M)$ . This is the best we can do, considering there is no obvious product structure on  $SH_{eq}^*(M)$ .*

**Lemma 4.7.5.**

$$\sum_i q(1)^i(x, y) h^i = PS'(x; PS(y)).$$

*Proof.* This is immediate from the configurations used in each definition. □

*Proof of Theorem 4.1.4.* Use Lemmas 4.7.5 and Lemma 4.7.2. □

## 4.7.2 A comparison to the quantum Cartan relation

Recall from Section 3.5 how we proved the quantum Cartan relation. The operations

$$(x, y) \mapsto \begin{cases} QS(x) * QS(y) \\ QS(x * y) \end{cases}$$

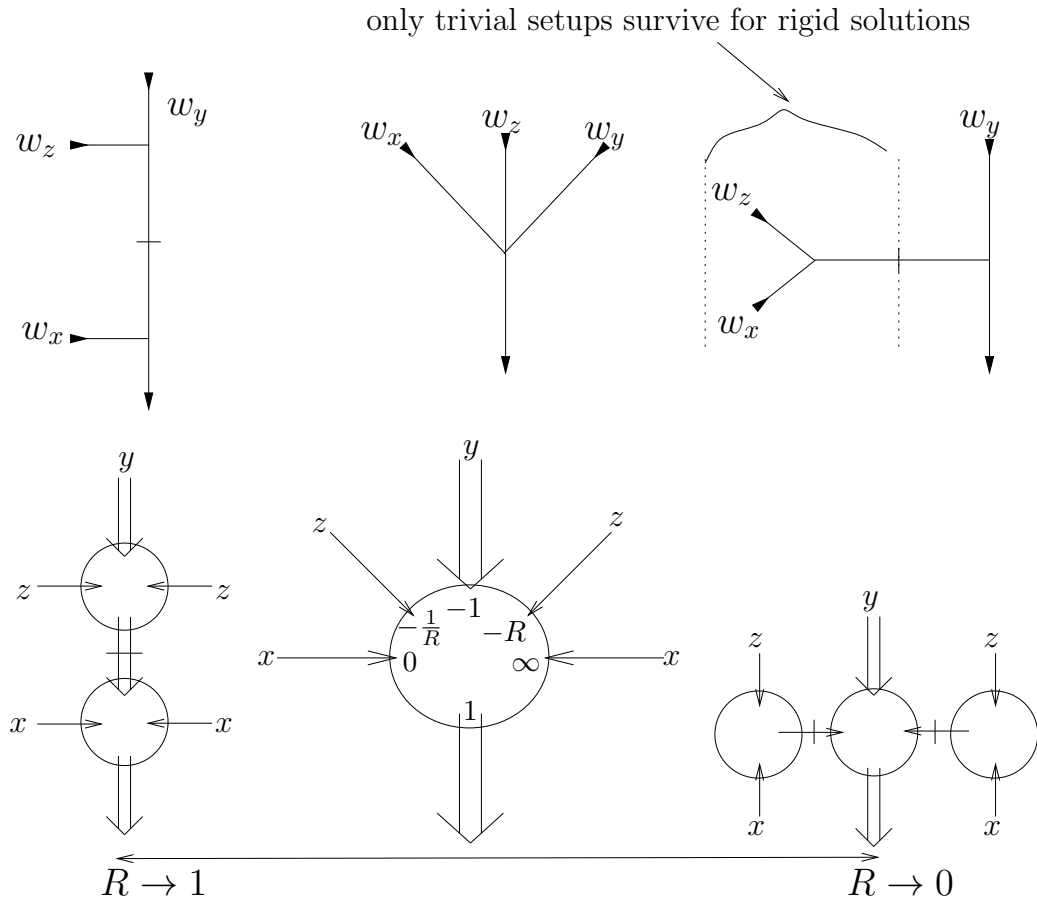


**Figure 4.11:** For the lower row, single arrows represent Floer trajectories for inputs from  $SH^*(M)$ . Double arrows represent trajectories for inputs/outputs from  $SH_{eq}^*(M)$ . Arrows split by an orthogonal line are broken trajectories. The upper row demonstrates the relevant flowlines on  $S^\infty$ , labelled accordingly.

correspond to two different configurations of five marked points on nodal spheres. To identify these two operations would have required a  $\mathbb{Z}/2$ -invariant homotopy between these two point configurations, and such a homotopy cannot exist. Here, the  $\mathbb{Z}/2$ -action acts by permuting the marked points.

In the referenced section, these configurations were embedded as points in a four-dimensional space of domains, and it was observed that while there was no  $\mathbb{Z}/2$ -invariant homotopy connecting these configurations, we could do the next best thing and find a  $\mathbb{Z}/2$ -equivariant cycle (i.e. an element of the  $\mathbb{Z}/2$ -equivariant homology of the space of domains) that allowed us to construct a correction term in the quantum Cartan relation. We sketch an attempt to repeat this for the symplectic Cartan relation. Here the domains are Riemann surfaces with  $1+4$  cylindrical ends: alternatively, this is a choice of 5 marked points on  $S^2$ , along with a radius and an orientation of the unit circle at each of the points, allowing us to define logarithmic coordinates.

As we noted at the beginning of Section 4.7 there is no product on  $SH_{eq}^*(M)$ , hence instead we consider the setup corresponding to  $PS(x * y)$  (identically  $PS'(x; PS(y))$ ) using Theorem 4.1.4). The space of domains has more than four dimensions in this case, and is too unwieldy to analyse in its entirety. Suppose instead that we consider a 2-family of domains,  $\mathcal{M}$ , consisting of nodal curves  $S^2 \vee S^2$  attached along  $-1 \sim 1$ , with  $3 + 2$  marked points at  $((0, z, \infty), (0, \infty))$  respectively (where  $z \in S^2$  is free to vary). There is a  $\mathbb{Z}/2$ -action on  $\mathcal{M}$ , via  $z \mapsto 1/z$ . We associate to each marked point  $p$ , as well as the points where the two spheres join, a radius  $r = r_p(z)$  depending on  $z$  such that the  $B_{r_p(z)}(p)$  are distinct for each  $z$ , along with a symmetry condition. We also make a choice of asymptotic point  $\theta_p$  for  $p \neq z$ . For  $z \neq -1$  we make a coherent choice of  $\theta_z$ , using stereographic projection from  $-1$ . Notice that as  $z \rightarrow -1$  there is an ambiguity in the choice of  $\theta_z$  depending on the direction from which  $z$  approaches  $-1$ .

Hence, to compactify this space of domains we must add a circle at infinity. Observe that the configuration associated to a  $z$  on this circle at infinity is exactly the configuration that we would hope for if we were able to define  $*_{eq}$  as we tried at the beginning of Section 4.7. This ambiguity in the choice of  $\theta_z$  is one part of why the product cannot be constructed. The  $\mathbb{Z}/2$ -action extends to this circle at infinity, acting by half-rotation.

The configuration associated to  $z = 1$  corresponds to  $PS'(x; PS(y))$ , and observe that  $z = 1$  is the only fixed point of the  $\mathbb{Z}/2$ -action on the space of domains induced by  $((0, z, \infty), (0, \infty)) \mapsto ((0, 1/z, \infty), (0, \infty))$ . Hence  $\mathcal{M}$  deformation retracts  $\mathbb{Z}/2$ -equivariantly onto  $z = 1$  and the only equivariant operation that we can define in

this way is  $PS'(x; PS(y))$ . We could increase the size of the moduli space by varying the choice of marked points, but the same problem arises: we replace a fixed point (denoted by  $m_1$  in the case of Section 3.5) with a fixed circle of points in our current setting, hence the  $\mathbb{Z}/2$ -equivariant homology of  $\mathcal{M}$  is less interesting than the similar case in the quantum setting.

**Remark 4.7.6.** *We may think of the  $3 + 2$  marked points in this section as corresponding to  $Z_{LL}, Z_D, Z_{LR}, Z_{UL}, Z_{UR}$  respectively, linking this to the previous section.*

### 4.7.3 Calculations for $n$ -spheres

Seeliger in [27] calculates  $\mathbb{H}_*(\mathcal{L}S^n; \mathbb{Z}) := H_{*+n}(\mathcal{L}S^n; \mathbb{Z})$  using the Serre fibration  $\Omega S^n \hookrightarrow \mathcal{L}S^n \rightarrow S^n$ , where  $\Omega S^n$  is the based loop space of  $S^n$  and  $\mathcal{L}S^n = C^\infty(S^1, S^n)$  is the free loop space of  $S^n$ . We can take a tensor product of the spectral sequence with  $\mathbb{Z}/2$  to show that with  $\mathbb{Z}/2$  coefficients:

$$\mathbb{H}_*(\mathcal{L}S^n) \cong \mathbb{H}_*(S^n) \otimes H_*(\Omega S^n)$$

as rings, where  $\mathbb{H}_*(S^n) = H_{*+n}(S^n)$ . The ring structure uses the Chas-Sullivan loop product, the intersection product and the Pontrjagin product respectively. Suppose that  $n$  is odd. Then  $\mathbb{H}_*(\mathcal{L}S^n) \cong E(x) \otimes \frac{\mathbb{Z}}{2}[y]$  where  $|x| = -n$  and  $|y| = n - 1$  and  $E(x)$  is the exterior algebra on  $x$ .

The Viterbo isomorphism states that  $SH^*(T^*S^n) \cong \mathbb{H}_*(\mathcal{L}S^n)$  as rings, see [31], so these  $x, y \in \mathbb{H}_*(\mathcal{L}S^n)$  correspond (abusively) to  $|x| = n$  and  $|y| = 1 - n$  in  $SH^*(T^*S^n)$ . For example we may use as representatives of the generators of  $H_*(\mathcal{L}S^n)$  embedded submanifolds as given by Oancea [17, Section 7], for  $n > 1$  and odd.

By Theorem 4.1.4, to calculate  $PS(xy^i)$  we may calculate:

$$\sum_{k=0}^r PS'_{r-k}(y^i; PS_k(x)) \text{ for all } r.$$

By considering  $\mathbb{H}_*(\mathcal{L}S^n)$ , we see that  $x = c^*(x_n)$  where  $x_n \in H^n(S^n)$  is the generator. Using Corollary 4.1.3,

$$PS \circ c^*(x_n) = c_{eq}^* \circ Sq(x_n)$$

and classically  $Sq(x_n) = x_n h^n$ . Recall that  $c_{eq}^* = c^* + h(\dots)$ , so  $PS(x) = x h^n + h^{n+1}(\dots)$ . Specifically there is a representative of  $PS(x)$  such that  $PS_n(x)$  represents  $x$  and  $PS_r(x) = 0$  for  $r < n$ . This implies that for  $r = n$ :

$$PS_n(xy^i) = \sum_{k=0}^n PS'_{n-k}(y^i; PS_k(x)) = PS'_0(y^i; x) = xy^{2i},$$

and for  $r < n$ ,

$$PS_r(xy^i) = \sum_{k=0}^r PS'_{r-k}(y^i; PS_k(x)) = 0.$$

This is as much as we may prove for these generators (in fact, we have not even shown that  $PS(xy^i) \neq 0$ , because we have not ruled out  $xy^i h^n + h^{n+1}(\dots)$  being exact).

The other generators of  $SH^*(T^*S^n)$  correspond to  $y^i$  (for  $i \in \mathbb{N}$ ). Recall that  $PS_0(y^i)$  is a chain representative of  $y^i * y^i = y^{2i}$ . Hence  $PS(y^i) = y^{2i} + h(\dots)$ . We will abusively denote by  $y^{2i}$  our chain level representative of  $y^{2i}$ . From this we can prove slightly more, specifically that:

**Lemma 4.7.7.**  $PS(y^i) \neq 0$ .

*Proof.* Suppose for a contradiction that  $PS(y^i) = y^{2i} + h(\dots)$  is a boundary, so  $y^{2i} + h(\dots) = d_{eq}(A_0 + A_1 h^1 + \dots)$ . This implies that  $y^{2i} = d(A_0)$  ( $d$  here being the nonequivariant differential), hence  $y^{2i}$  is exact. This is a contradiction from the ring structure of  $SH^*(T^*S^n)$ .  $\square$

**Remark 4.7.8.** *It must be reiterated that the calculations of the  $PS_i$  and  $PS'_j$  are defined on the chain level, even though  $PS$  and  $PS'$  are well defined on homology.*

## 4.8 The Equivariant Pair-of-Pants for Negative Line Bundles

In this section we extend the work by Ritter in [22] to the  $\mathbb{Z}/2$ -equivariant setting. Specifically, we are interested in the total space of a negative line bundle  $E$  over a closed symplectic manifold  $B$ , which we denote  $M = \text{Tot}(E \rightarrow B)$ . We are interested in the case where  $M$  is weakly monotone, and we use a Novikov field

$$\Lambda = \left\{ \sum_{j=0}^{\infty} n_j \gamma_j \mid n_j \in \mathbb{Z}/2, \gamma_j \in H_2(M) \text{ is spherical, } \lim_{j \rightarrow \infty} \omega(\gamma_j) = \infty \right\},$$

as in Section 4.2.3, following [22, Section 2.6]. If  $M$  is monotone we will use the Novikov field  $\Lambda = \mathbb{Z}/2((T))$ , where  $|T| = 2N$ . The minimal Chern number  $N$  in such a case is defined by  $c_1(\pi_2(M)) = N\mathbb{Z}$  for  $N > 0$ .

### 4.8.1 The relation between the quantum Steenrod square and the symplectic square

We recall that Ritter proved in [22, Theorem 1, see Section 1.3], with  $M$  as above, that there is a particular linear homomorphism  $r : QH^*(M) \rightarrow QH^{*+2}(M)$  such that

$$QH^*(M)/(\ker r^k) \xrightarrow{\cong} SH^*(M), \quad (4.21)$$

for  $k \gg 0$ , with the isomorphism being induced by the  $c^*$  map. Indeed,  $\ker r^k = \ker c^*$  and we denote  $K = \ker r^k$ .

For the case of  $\mathbb{Z}/2$ -equivariant Floer theory, starting with the commuting square in Equation (4.3), we may write down the following commutative diagram:

$$\begin{array}{ccc} QH^*(M)/K \cdot \Lambda & \xrightarrow{QS} & QH_{eq}^*(M)/QS(K) \cdot \Lambda[[h]] \\ c^* \downarrow \cong & & c_{eq}^* \downarrow \\ SH^*(M) & \xrightarrow{PS} & SH_{eq}^*(M) \end{array} \quad (4.22)$$

The fact that  $QS$  descends to the top horizontal map of Equation (4.22) is by linear algebra. The fact that  $c_{eq}^*$  descends to the right hand vertical map of Equation (4.22) comes from the commutativity of (4.3), i.e.

$$c_{eq}^*(QS(k)) = PS(c^*(k)) = 0, \quad (4.23)$$

for any  $k \in K$ . In general we do not know any more about the homomorphism  $c_{eq}^*$ , but we will discuss the setup of the problem. In the next section we will prove that  $c_{eq}^*$  is an isomorphism for  $\text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{CP}^n)$ .

As in [22], we choose a loop of Hamiltonian symplectomorphisms  $g_t : M \rightarrow M$  for  $t \in \mathbb{R}/\mathbb{Z}$  such that  $g_t$  is multiplication by  $e^{2\pi it}$  (rotation of the fibre  $\mathbb{C}$  of  $E$ ). The  $g_t$  are generated by the Hamiltonian  $H_1 = H_1(R) = (1 + \epsilon)R$  for some  $0 < \epsilon \ll 1$  depending on the choice of symplectic form. Here  $R$  is the radial coordinate of the line bundle. Let  $H_k(R) = k(1 + \epsilon)R$ . Define for  $t \in \mathbb{R}/2\mathbb{Z}$

$$g^*H_k = H_k \circ g_t - H_1 \circ g_t = H_{k-1}, \quad (4.24)$$

as  $g_t$  preserves  $R$ , and

$$(g^*J_{eq,v})_t = d(g_t)^{-1} \circ J_{eq,v,t} \circ d(g_t), \quad (4.25)$$

which still satisfies the conditions of Section 4.2.1, remaining regular as in [22, Theorem 18]. Notice that  $g_t$  is 1-periodic, so for  $t \in \mathbb{R}/2\mathbb{Z}$  it is twice wrapped. In

particular, (4.25) ensures that  $(g^*J_{eq,-v})_{t+1} = (g^*J_{eq,v})_t$ . There is a choice of lift of  $g_t$  to  $\tilde{g}$ , an action on the cover  $\widetilde{\mathcal{L}_0 M} \rightarrow \mathcal{L}_0 M$ , where  $\mathcal{L}_0 M$  consists of contractible free loops on  $M$  and  $\widetilde{\mathcal{L}_0 M}$  consists of pairs  $(v, x)$  where  $x \in \mathcal{L}_0 M$  and  $v : D^2 \rightarrow M$  with  $\partial v = x$ , up to a relation  $\sim$  where  $(v_1, x_1) \sim (v_2, x_2)$  exactly when  $x_1 = x_2$  and  $\omega, c_1$  both vanish on  $v_1 \# \overline{v_2}$ . In the case of this specific  $g_t$  we choose  $\tilde{g}_t$  as in [22, Section 7.8], such that  $\tilde{g} \cdot (c_x, x) = (c_x, x)$  where  $x \in \mathcal{L}_0 M$  is a constant loop in the zero section of  $M$  and  $c_x : D^2 \rightarrow M$  is the constant map at  $x$ .

We define

$$S_{\tilde{g}}^{eq} : CF_{eq}^*(2 \cdot H, J_{eq}) \rightarrow CF_{eq}^{*+4}(2 \cdot g^*H, g^*J_{eq}), \quad c \mapsto \tilde{g}^{-1}c.$$

Specifically, let  $c = (x, \tilde{x}) \in CF^*(2 \cdot H)$ , where we consider  $x$  as being 2-periodic with respect to  $H$  (Section 4.2.1) and  $\tilde{x}$  is a filling disc for  $x$ . Then  $\tilde{g}^{-1}(x, \tilde{x}) = (y, \tilde{y})$  where  $\tilde{y}(t) = \tilde{g}_t^{-1}(\tilde{x}(r, t))$ , using polar coordinates on the disc for  $t \in \mathbb{R}/2\mathbb{Z}$ . The grading shift  $* \mapsto * + 4$  comes from the Maslov index of  $(g^2, \tilde{g}^2)$ : see [22, Section 3.1 and Lemma 48]. Further, this commutes with the equivariant continuation maps using standard equivariant gluing arguments, so

$$S_{\tilde{g}}^{eq} \circ \Phi_{eq, H, H'} = \Phi_{eq, g^*H, g^*H'} \circ S_{\tilde{g}}^{eq}. \quad (4.26)$$

Observe that, for  $r_{eq}$  defined by the composition

$$r_{eq} = \Psi_{eq}^{-1} \circ \Phi_{eq, H_{-1}, H_0} \circ S_{\tilde{g}}^{eq} \circ \Psi_{eq},$$

the following diagram commutes:

$$\begin{array}{ccccccc} SH_{eq}^*(M) & = \varinjlim_H & (HF_{eq}^*(H_0) & \xrightarrow{\Phi_{eq, H_0, H_1}} & HF_{eq}^*(H_1) & \xrightarrow{\Phi_{eq, H_1, H_2}} & HF_{eq}^*(H_2) \xrightarrow{\Phi_{eq, H_2, H_3}} \dots) \\ & \cong \uparrow \Psi_{eq} & & & \cong \uparrow (S_{\tilde{g}}^{eq})^{-1} \circ \Psi_{eq} & & \cong \uparrow (S_{\tilde{g}}^{eq})^{-2} \circ \Psi_{eq} \\ & = \varinjlim_H & (QH_{eq}^*(M) & \xrightarrow{r_{eq}} & QH_{eq}^{*+4}(M) & \xrightarrow{r_{eq}} & QH_{eq}^{*+8}(M) \xrightarrow{r_{eq}} \dots) \end{array} \quad (4.27)$$

Hence we may determine  $SH_{eq}^*(M)$  using the lower row. Compare to [22, Section 3, Section 4.2], where the map

$$S_{\tilde{g}} : HF^*(H, J) \rightarrow HF^{*+2}(g^*H, g^*J), \quad c \mapsto g^{-1}c,$$

was similarly defined, where  $c$  is a 1-periodic Hamiltonian loop for  $H$  and  $(gc)(t) = g_t \cdot c(t)$ . As 1-periodic loops were used in [22], the index change is 2, the Maslov

index of  $(g, \tilde{g})$ . Likewise the map  $r : QH^*(M) \rightarrow QH^{*+2}(M)$  was defined by the composition

$$r = \Psi^{-1} \circ \Phi_{H_{-1}, H_0} \circ S_{\tilde{g}} \circ \Psi. \quad (4.28)$$

Ritter then showed that there is some  $k > 0$  such that  $r|_{\text{Im } r^k} : \text{Im } r^k \rightarrow \text{Im } r^{k+1}$  is an isomorphism, hence  $c^* : QH^*(M)/\ker r^k \xrightarrow{\cong} SH^*(M)$ .

**Lemma 4.8.1.**  $r_{eq} \circ Q\mathcal{S} = Q\mathcal{S} \circ r$ .

*Proof.*

$$\begin{aligned} Q\mathcal{S} \circ r &= Q\mathcal{S} \circ \Psi^{-1} \circ \Phi_{H_{-1}, H_0} \circ S_{\tilde{g}} \circ \Psi \\ &= \Psi_{eq}^{-1} \circ P\mathcal{S} \circ \Phi_{H_{-1}, H_0} \circ S_{\tilde{g}} \circ \Psi \\ &= \Psi_{eq}^{-1} \circ \Phi_{eq, H_{-1}, H_0} \circ P\mathcal{S} \circ S_{\tilde{g}} \circ \Psi \end{aligned} \quad (4.29)$$

using respectively Theorem 4.1.1 and Lemma 4.3.2. To see that

$$P\mathcal{S} \circ S_{\tilde{g}} = S_{\tilde{g}}^{eq} \circ P\mathcal{S},$$

observe that if we count the coefficient of  $y$  in  $S_{\tilde{g}}^{eq} \circ P\mathcal{S}(x)$  then we count  $u : S \rightarrow M$  satisfying (4.10) except that we replace the conditions  $y(t)$  and  $y(t+1)$  by  $g_t \cdot y(t)$  and  $g_t \cdot y(t+1)$  respectively. Similarly for  $x(t)$  being replaced by  $g_t x(t)$  in (4.10) for  $P\mathcal{S} \circ S_{\tilde{g}}(x)$ .

Let  $(g^*u)(z) = g_{\pi_2 z}^{-1} u(z)$ , where the 2 to 1 branched cover  $S \rightarrow \mathbb{R} \times \mathbb{R}/\mathbb{Z}$  uses the projection map  $\pi = \pi_1 \times \pi_2$ . Then there is a bijective correspondence between solutions  $(w, u)$  counted for the coefficient of  $y$  in  $S_{\tilde{g}}^{eq} \circ P\mathcal{S}(x)$  and solutions  $(w, g^*u)$  counted for the coefficient of  $y$  in  $P\mathcal{S} \circ S_{\tilde{g}}(x)$ . Here, if we use the data  $(H, J_z^w, Y)$  for  $S_{\tilde{g}}^{eq} \circ P\mathcal{S}$  then we use the data  $(g^*H, g^*J_z^w, g^*Y)$  for  $P\mathcal{S} \circ S_{\tilde{g}}^{eq}$ , where

$$g^*J_z^w := d(g_{\pi_2 z})^{-1} \circ J_z^w \circ dg_{\pi_2 z}$$

satisfies the relevant conditions in Section 4.2.2. □

**Remark 4.8.2.** *The Maslov index above uses  $g^2$ , because  $g_t$  wraps twice for  $t \in \mathbb{R}/2\mathbb{Z}$ . We recall that in the space of based loops in a Lie group, the product induced by the group action is homotopic to the product induced by composition of loops.*

**Remark 4.8.3.** *By Lemma 4.8.1 above, we see that*

$$Q\mathcal{S}(\ker r^k) \subset \ker r_{eq}^k.$$

*This gives another proof that (4.23) holds.*

### 4.8.2 The symplectic square for $M = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{CP}^m)$

Observe first that  $M = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{CP}^m)$  deformation retracts onto  $\mathbb{CP}^m$ , hence they have the same cohomology. Therefore, all that is strictly different between the quantum cohomologies of  $M$  and  $\mathbb{CP}^m$  is the interaction with  $J$ -holomorphic spheres. As  $M$  is monotone in this case, we use the smaller Novikov field  $\mathbb{Z}/2((T))$ , where here  $|T| = 2m$ .

We state but do not reprove from [22, Theorem 61] that

$$QH^*(M) = \Lambda[\omega_Q]/(\omega_Q^{m+1} + T \cdot \omega_Q), \quad (4.30)$$

where  $\omega_Q$  is the symplectic form on  $\mathbb{CP}^m$ . We will distinguish by  $\omega^i$  and  $\omega_Q^i$  taking  $\omega \cup \dots \cup \omega$  and  $\omega * \dots * \omega$ , respectively the  $i$ -th power for the cup and the quantum cup products. Hence in this notation  $\omega = \omega_Q$ .

We will give a way to iteratively compute  $Q\mathcal{S}$ , in the same way as was used in Section 3.6.1. We refer the reader to the aforementioned section for more details. We use the quantum Cartan relation from Theorem 3.1.2 to show that

$$Q\mathcal{S}(\omega_Q^{i+1}) = Q\mathcal{S}(\omega_Q^i) * Q\mathcal{S}(\omega_Q) + \sum_{j,k} q_{j,k}(W_0 \times D^{j-2,+})(\omega_Q^i, \omega_Q) h^j.$$

Using that  $Q\mathcal{S}(\omega_Q) = \omega_Q * \omega_Q + \omega_Q h^2$  for degree reasons, it is sufficient to calculate  $q_{j,k}(W_0 \times D^{j-2,+})(\omega_Q^i, \omega_Q)$  for each  $i, j, k$  such that  $0 \leq i \leq m$ , and use the quantum Cartan relation.

Recall from Lemma 3.5.7 that  $q_{j,0} = 0$ . For degree reasons  $q_{j,k} = 0$  when both  $k > 1$  and  $i < m$ . Hence, for  $i < m$  we only need to consider  $k = 1$ , i.e. a degree 1  $J$ -holomorphic sphere. This corresponds to setups of the form of Figure 4.12 (see Section 3.5 for justification). One can show that because  $\text{codim } PD(\omega_Q) = 2$  which is the dimension of the  $J$ -holomorphic sphere, the intersection conditions with  $\omega_Q$  are unnecessary except for fixing marked points. Thus, the calculation of the coefficient of  $\omega_Q^l \cdot T$  in  $q_{j,1}(W_0 \times D^{j-2,+})(\omega_Q^i, \omega_Q)$  is counting how many degree 1  $J$ -holomorphic spheres there are intersecting  $PD(Sq^j(\omega_Q^i)) \in H_*(D, \partial D)$  and  $PD(\omega_Q^l)^\vee \in H_*(D)$ , where  $\vee$  is the intersection duality. For degree reasons there can only be solutions for  $(j, l) = (2(m-i), 2), (2(m-1-i), 0)$ . By moving the  $\mathbb{CP}^0$  in the  $l = 0$  case, i.e. the point  $PD(\omega_Q^l)^\vee$ , to infinity (see [22, Theorem 61]) we see the only possibly non-zero term occurs for  $(j, l) = (2(m-i), 2)$ . Hence

$$\sum_{j,k} q_{j,k}(W_0 \times D^{j-2,+})(\omega_Q^i, \omega_Q) h^j = \binom{i}{m-i} \omega_Q T h^{2+4i-2m}, \quad (4.31)$$

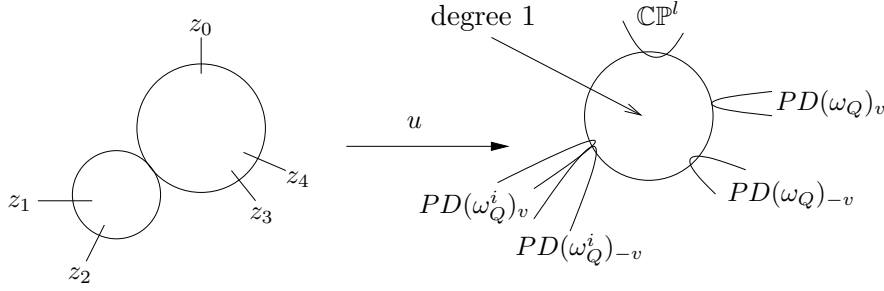


**Figure 4.12:** Setups for the correction term in calculating  $QS$  for  $M$ .

where  $\binom{i}{m-i}$  is the coefficient of  $\omega_Q^m$  in  $Sq^{m-i}(\omega_Q^i)$  (the classical Steenrod square is a homotopy invariant, and we know the answer for  $\mathbb{CP}^m$ ). For  $i = m$ , all of the previous holds for  $k = 1$ , but there is also the possibility of  $k = 2$ . If  $k = 2$  then  $j = 2$ , and recall that  $j = 2$  corresponds to  $D^{0,+}$ . Hence this is the  $T^2$  term in the standard quantum product  $\omega_Q^m * \omega_Q^m * \omega_Q h^2$  (it is not  $\omega_Q^m * \omega_Q^m * \omega_Q * \omega_Q$ , because the final intersection with  $\omega_Q$  fixes the marked point associated to varying our domains in  $W_0$ ), giving that

$$\sum_{j,k} q_{j,k}(W_0 \times D^{j-2,+})(\omega_Q^m, \omega_Q) h^j = \omega_Q T h^{2+2m} + \omega_Q T^2 h^2.$$

These can be used to calculate all of the quantum Steenrod squares for any  $m$ .

**Example 4.8.4.** We let  $x = \omega_Q$ , and  $M = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{CP}^4)$ .

$$\begin{aligned} QS(x) &= x^2 + xh^2, \\ QS(x^2) &= QS(x) * QS(x) + \binom{1}{3} xTh^{-2} \\ &= (x^2 + xh^2)^2 = x^4 + x^2h^4, \\ QS(x^3) &= QS(x^2) * QS(x) + \binom{2}{2} xTh^2 \\ &= (x^2 + xh^2) * (x^4 + x^2h^4) + xTh^2 \\ &= x^2T + x^4h^4 + x^3h^6, \\ QS(x^4) &= QS(x^3) * QS(x) + xTh^6 \\ &= x^4T + x^3Th^2 + x^2Th^4 + xTh^6 + x^4h^8, \\ QS(x^5) &= QS(x^4) * QS(x) + xTh^{10} + xT^2h^2 \\ &= (x^2 + xh^2)T^2 = QS(xT), \end{aligned}$$

remembering that in this case there is the extra correction term  $xT^2h^2$ .

For this choice of  $M$ , the linear map  $r$  is multiplication by the symplectic form  $x$  (see e.g. [22, Lemma 60]). In general it is multiplication by the Seidel element. Then  $\ker r = x^n + T$ , and  $r|_{\text{Im } r} : \text{Im } r \rightarrow \text{Im } r^2$  is an isomorphism. We denote

$$r_{eq} = \sum_{i \geq 0} h^{2i} r_i, \quad (4.32)$$

where  $r_i : QH^*(M) \rightarrow QH^{*+4-2i}(M)$ . Observe that  $r_0$  uses constant flowlines on  $S^\infty$ , hence  $r_0 = r^2$  where  $r$  is as in (4.28).

We note that:

- we have calculated the correction terms of the quantum Cartan relation above,
- we can show that  $Q\mathcal{S}(x^m) = x^m h^{2m} + \dots$  (lower order in  $h$ ), because  $Q\mathcal{S}(x) = xh^2 + x^2$ , then we use the quantum Cartan relation and induction to show

$$Q\mathcal{S}(x^i) = x^i h^{2i} + \dots \text{ (lower order in } h), \quad (4.33)$$

for each  $i = 0, \dots, m$ .

- from equation (4.30), we know that  $x^{m+1} = xT$
- hence  $Q\mathcal{S}(x^{m+1}) = Q\mathcal{S}(xT) = (x^2 + xh^2)T^2$ ,

we can deduce, using the quantum Cartan relation, the points above and dividing  $Q\mathcal{S}(x^{m+1}) + xTh^{2m+2} + xT^2h^2$  by  $Q\mathcal{S}(x) = x^2 + xh^2$ , that

$$Q\mathcal{S}(x^m) = x^m h^{2m} + T \left( \sum_{i=1}^m x^i h^{2m-2i} \right).$$

*Proof of Theorem 4.1.5.* We proceed by lemmas.

**Lemma 4.8.5.**

$$r_{eq}|_{\text{Im } r_{eq}} : \text{Im } r_{eq} \rightarrow \text{Im } r_{eq}^2$$

is an isomorphism.

*Proof.* The  $\Lambda$  vector space  $QH^*(M)$  is generated by  $\{x, \dots, x^m, x^m + T\}$ . Here  $\{x, \dots, x^m\}$  generate  $r(QH^*(M)) = x * QH^*(M)$ , the image of  $r$ , and  $x^m + T$  generates  $K = \ker r$ . We will prove that  $r_i(QH^*(M)) \subset \text{Im } r$  for all  $i$ . Observe that for  $x \in H^*(M)$ , we have  $r_i(x) = \sum_{j \geq 0} r_{i,j}(x)T^j$ , where  $r_{i,j}(x) \in H^{*+4-2i-2jN}(M)$ . As the cohomology is bounded below degree 0 we see that  $r_i = 0$  for  $2i > \dim M + 4$ , hence only a finite number of  $r_i$  are nonzero.

There are five classes of cases to check (all other  $r_i$  must land in  $\text{Im } r$  for degree reasons). These are:

1.  $r_0(x^{m-2})$ ,
2.  $r_2(1)$ ,
3.  $r_{i+2}(x^i)$  for  $i = 1, \dots, m$ ,

4.  $r_2(x^m)$ ,

5.  $r_1(x^{m-1})$ .

Case (1) is immediate, as  $r_0 = r^2$ . For Case (2), Lemma 4.8.1 implies that

$$r_{eq}(1) = r_{eq}(Q\mathcal{S}(1)) = Q\mathcal{S}(r(1)) = Q\mathcal{S}(x) = x^2 + xh^2.$$

Hence  $r_2(1) = 0$ . For Case (3), we use downwards induction. For the base case, Lemma 4.8.1 implies that

$$r_{eq}(Q\mathcal{S}(x^m + T)) = Q\mathcal{S}(r(x^m + T)) = 0. \quad (4.34)$$

The  $h^{4m+4}$  term of the left hand side of (4.34) is  $r_{m+2}(x^m)$ , and of the right hand side is 0, hence

$$r_{m+2}(x^m) = 0.$$

For the induction step we proceed similarly. Lemma 4.8.1 implies that

$$r_{eq}(Q\mathcal{S}(x^i)) = Q\mathcal{S}(x^{i+1}). \quad (4.35)$$

We rewrite Equation (4.33) as:

$$Q\mathcal{S}(x^i) = x^i h^{2i} + \sum_{j=1}^{m-i} \epsilon_j x^{i+j} h^{2i-2j} + \delta T h^{4i-2m} + \sum_{j \geq 1} \eta_j T x^j h^{4i-2m-2j}, \quad (4.36)$$

for some  $\epsilon_j, \delta, \eta_j \in \mathbb{Z}/2$ . Hence the  $h^{4i+4}$  term on the left hand side is

$$r_{i+2}(x^i) + \sum_{j=1}^{m-i} \epsilon_j r_{i+j+2}(x^{i+j}) + \delta T r_{m+2}(1) + \sum_{j \geq 1} T r_{m+2+j}(x^j).$$

Using (4.32), as  $m \geq 1$ , for degree reasons  $r_{m+2+j}(x^j) = 0$  for all  $j \geq 0$ . By the induction hypothesis, we assume that  $r_{i+j+2}(x^{i+j}) = 0$  for all  $1 \leq j \leq m-i$ . We also note that the  $h^{4i+4}$  term on the right hand side of (4.35) is 0 using Equation (4.36). Hence by induction  $r_{i+2}(x^i) = 0$ .

For Case (4), the  $h^{2m+4}$  term of (4.34) implies that

$$r_2(x^m) + T \sum_{i=1}^m r_{i+2}(x^i) + T^2 r_{m+2}(1) = 0. \quad (4.37)$$

We know from the previous cases that all of the terms except  $r_2(x^m)$  on the left hand side of (4.37) vanish, hence  $r_2(x^m) = 0$ .

For the final case, considering the  $h^4$  term of (4.34) implies that

$$T(r_2(x^m + T) + r_1(x^{m-1}) + r_0(x^{m-2})) = 0.$$

We know that  $r_2(x^m + T) = 0$  from cases (2) and (4), hence

$$r_1(x^{m-1}) = r_0(x^{m-2}) = r^2(x^{m-2}) = x^m.$$

Knowing that  $r|_{\text{Im } r} : \text{Im } r \xrightarrow{\cong} \text{Im } r^2 \subset \text{Im } r$  is an isomorphism, there exists  $a : \text{Im } r^2 = \text{Im } r \rightarrow \text{Im } r$  such that  $ar^2 = r$ . Use an argument as in Lemma 4.4.3 to iteratively construct some  $a_{eq} : r(QH^*(M))[[h]] \rightarrow r(QH^*(M))[[h]]$ , of the form

$$a_{eq} = a^2 + h^2 a^2 r_1 a^2 + \sum_{i \geq 2} h^{2i} a_i,$$

such that the

$$a_i : \text{Im } r^2 = \text{Im } r \rightarrow \text{Im } r$$

are written as compositions of  $a$  and the  $r_i$ , and  $a_{eq} r_{eq}^2 = r_{eq}$ . We required that  $r_i(QH^*(M)) \subset r(QH^*(M)) = r^2(QH^*(M))$  for all  $i$  so that  $a_{eq}$  is well defined (specifically, it ensures that  $a^k r^k r_i = r_i$ ).  $\square$

**Lemma 4.8.6.**

$$\ker r_{eq} = Q\mathcal{S}(x^n + T) \cdot \Lambda[[h]].$$

*Proof.* Suppose  $x = \sum_{i \geq 0} x_i h^i \neq 0$  is some element of  $\ker r_{eq}$ . Let  $j$  be minimal such that  $x_j \neq 0$ . Observe  $r^2(x_j) = 0$  as  $r_{eq}(x) = 0$ , hence

$$x_j = \lambda_j \cdot (x^n + T),$$

for  $\lambda_j \in \Lambda$ . Replace  $x$  by  $x - h^j \lambda_j T^{-1} Q\mathcal{S}(x^n + T)$  and note that the coefficient of  $h^i$  in  $x - h^j \lambda_j T^{-1} Q\mathcal{S}(x^n + T)$  is zero for  $i \leq j$ . We then iterate. This yields that

$$x = \left( T^{-1} \sum_{i \geq 0} \lambda_i h^i \right) Q\mathcal{S}(x^n + T),$$

for some  $\lambda_i \in \Lambda$ .  $\square$

Using an argument as in [22, Section 4.2] and the diagram 4.27, by Lemma 4.8.5 we deduce that

$$c_{eq}^* : QH_{eq}^*(M) / \ker r_{eq} \xrightarrow{\cong} SH_{eq}^*(M).$$

The theorem then follows by Lemma 4.8.6.  $\square$

We finish by calculating  $PS$  for the same case as example 4.8.4, using Equation (4.22), and observing by the work earlier in this subsection that we can do a similar calculation for all  $\text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{CP}^n)$ . Recall that in general  $SH_{eq}^*(M)$  does not have a ring structure, but in this case it inherits one from  $QH_{eq}^*(M)$ .

**Example 4.8.7.** *We let  $x = \omega_Q$ , and  $M = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{CP}^4)$ , using Theorem 4.1.5, we know that*

$$SH_{eq}^*(M) \cong \Lambda[x][[h]]/QS(x^4 + T) \cdot \Lambda[[h]].$$

*Then using Example 4.8.4 we calculate:*

$$\begin{aligned} PS(x) &= x^2 + xh^2, \\ PS(x^2) &= x^4 + x^2h^4 \\ &= T + x^3h^2 + xh^6 + \sum_{i \geq 0} h^{8i} T^{-i} (T + x^3h^2 + x^2h^4 + xh^6), \\ PS(x^3) &= x^2T + x^4h^4 + x^3h^6 \\ &= x^2T + Th^4 + x^2h^8 + xh^{10} + \sum_{i \geq 0} h^{4+8i} T^{-i} (T + x^3h^2 + x^2h^4 + xh^6), \\ PS(x^4) &= x^4T + x^3Th^2 + x^2Th^4 + xTh^6 + x^4h^8 \\ &= T^2 = PS(T), \end{aligned}$$

**Remark 4.8.8.** *Lemma 4.8.6 works more generally for*

$$M = \text{Tot}(\mathcal{O}(-k) \rightarrow \mathbb{CP}^m),$$

*when  $k \leq (m+1)/2$  is odd. That is because in this case,*

$$QH^*(M) \cong \Lambda[x]/(x^{m+1} + Tx^k),$$

*$r$  is quantum product with  $x$  and*

$$(x^{m-k+1} + T) = \ker r^k.$$

*For the given range of  $k$ , we have that  $(x^{m-k+1} + T)^2 = T(x^{m-k+1} + T)$ , hence it generates  $\ker r^k$ . That is the key idea in the proof of the lemma.*

*It should also be reasonable that Lemma 4.8.5 works for the given range of  $k$ , although this has not been proven. More generally, there is no guarantee that either lemma holds for the total space of a negative line bundle over a closed symplectic manifold, or even for odd  $k > (m+1)/2$ . For even  $k$  the symplectic cohomology is trivial.*

# Chapter 5

## The Equivariant Loop Space Isomorphism

### 5.1 Introduction

Unlike the previous chapters, the work in this chapter is still in progress. Hence all of the proofs are sketch proofs, although enough intuition has been gained that the sketch proofs should become real proofs once the details are filled in. With this in mind, the following chapter will aim to illuminate the more recent work and future goals, while perhaps acting as a way of demonstrating the sort of directions towards which the techniques in this thesis could be extended. As in all previous chapters, all homology will be taken with  $\mathbb{Z}/2$ -coefficients. If further details are omitted, the term “equivariant” refers to  $\mathbb{Z}/2$ -equivariance.

The work in this chapter aims to extend the previous work. In Chapter 3 we extended the Steenrod square to quantum cohomology and in Chapter 4 we linked this to the equivariant pair-of-pants by Seidel. In a similar way to the aforementioned  $\mathbb{Z}/2$ -equivariant Steenrod square type operations, one can feasibly consider other types of  $\mathbb{Z}/2$ -equivariant products for geometric multiplicative structures. In this section we will discuss a case where this works (i.e. the Chas-Sullivan product).

This equivariant Chas-Sullivan product ties in well with the work of Chapter 4 using the Viterbo theorem. Specifically, for a closed manifold  $M$  there is an isomorphism due to Viterbo [31],

$$\phi : SH^*(T^*M) \cong H_{m-*}(\mathcal{L}M)$$

where  $\mathcal{L}M = C^\infty(\mathbb{R}/2\mathbb{Z}, M)$  is the free loop space of  $M$  and  $m = \dim M$ . Indeed, this is an isomorphism of rings, where the product on the left is the pair-of-pants and the product on the right is the Chas-Sullivan product.

The natural question to ask is whether there is an equivariant version of the Viterbo isomorphism to suit our purposes. Such a map should exist, and this chapter will sketch a definition and consider the ramifications. We denote this  $H_*^{eq}(\mathcal{L}M)$ , to adhere to the notation for the equivariant symplectic cohomology in the previous chapter. There is then a  $\mathbb{Z}/2$ -equivariant version of the Chas-Sullivan product, more precisely an equivariant Chas-Sullivan square  $CS : H_*(\mathcal{L}M) \rightarrow H_{2*-m}^{eq}(\mathcal{L}M)$ . Denoting by  $\phi_{eq}$  the equivariant Viterbo isomorphism, the following diagram commutes:

$$\begin{array}{ccc} SH_{eq}^{2*}(T^*M) & \xrightarrow[\cong]{\phi_{eq}} & H_{m-2*}^{eq}(\mathcal{L}M) \\ \uparrow PS & & \uparrow CS \\ SH^*(T^*M) & \xrightarrow[\cong]{\phi} & H_{m-*}(\mathcal{L}M) \end{array} \quad (5.1)$$

This result, once proved, may then allow us to calculate some of the equivariant pair-of-pants for the symplectic cohomology of a cotangent bundle, by calculating the equivariant Chas-Sullivan square. Considering that it is in general difficult to calculate the homology of a loop space without using the spectral sequence associated to the fibration

$$\Omega M \hookrightarrow \mathcal{L}M \rightarrow M,$$

(which is not  $\mathbb{Z}/2$ -equivariant under the involution that we use) one tactic might be to try to find an appropriate spectral sequence in the equivariant setting: for example

$$\Omega M \times \Omega M \hookrightarrow \mathcal{L}M \xrightarrow{ev_0 \times ev_1} M \times M,$$

where  $ev_i$  is evaluation  $i \in \mathbb{R}/2\mathbb{Z}$ . Another tactic might be to look at the work on closed geodesics by Oancea [17], and see if it is possible to use this in the case of  $\mathbb{Z}/2$ -equivariant loop space homology. It seems reasonable that for the case of  $M = S^n$  we can calculate  $H_*^{eq}(\mathcal{L}M)$  and  $CS$  by hand. It may also be that  $CS$  splits into a  $\mathbb{Z}/2$ -operation on  $\Omega M$  and the Steenrod square on  $M$  (once we have made rigorous the meaning of the term “splits” in this context).

## 5.2 Preliminaries

### 5.2.1 The Chas-Sullivan Product

Given a closed manifold  $M$  of dimension  $m$ , we define its free loop space  $\mathcal{L}M := C^\infty(S^1, M)$ . Fixing some  $0 \in S^1$ , there is an evaluation map

$$ev : \mathcal{L}M \rightarrow M, \quad ev(\gamma) = \gamma(0).$$

Given some  $m \in M$ , we use the terminology  $\Omega_m M := ev^{-1}(\{m\})$ . The Pontrjagin product on  $\Omega M$  is composition of loops,

$$\cdot : \Omega M \times \Omega M \rightarrow \Omega M$$

which induces a product

$$\cdot : H_*(\Omega M) \times H_*(\Omega M) \rightarrow H_*(\Omega M).$$

Define  $\mathbb{H}_*(\mathcal{L}M) := H_{*+m}(\mathcal{L}M)$  and  $\mathbb{H}_*(M) := H_{*+m}(M)$ . There is a fibration  $\Omega M \hookrightarrow \mathcal{L}M \rightarrow M$ . We will construct the Chas-Sullivan product,  $\bullet$ , as a product on  $\mathbb{H}_*(\mathcal{L}M)$ . There is a map  $\cap \Omega : \mathbb{H}_*(\mathcal{L}M) \rightarrow H_*(\Omega M)$  induced by intersection with the fibre. This map intertwines the Chas-Sullivan and the Pontrjagin product.

The sketch-definition of the Chas Sullivan product we follow is from [21]. For a more precise definition see [7]. If  $\sigma_1 : \Delta^{a_1} \rightarrow \mathcal{L}M$  and  $\sigma_2 : \Delta^{a_2} \rightarrow \mathcal{L}M$  are two singular cochains, then let  $E \subset \Delta^{a_1} \times \Delta^{a_2}$  be the pullback:

$$\begin{array}{ccc} E = \Delta^{a_1} \times_M \Delta^{a_2} & \xrightarrow{\pi_1} & \Delta^{a_1} \\ \downarrow \pi_2 & & \downarrow ev \circ \sigma_1 \\ \Delta^{a_2} & \xrightarrow{ev \circ \sigma_2} & M \end{array} \quad (5.2)$$

There are projections  $\pi_i : E \rightarrow \Delta^{a_i}$ . Let  $j : E \rightarrow \mathcal{L}M$  where  $j(e)$  is the loop in  $\mathcal{L}M$  associated to the concatenated loop  $\pi_1(e) \cdot \pi_2(e)$ . Making generic choices of the  $\sigma_i$  ensures that  $E$  is a manifold with corners.

**Definition 5.2.1** (Chas-Sullivan Product). *Define:*

$$\sigma_1 \bullet \sigma_2 := j_*[E] \in C_{a_1+a_2-\dim M}(\mathcal{L}M).$$

*To define  $A \bullet B$  for  $A, B \in C_*(\mathcal{L}M)$ , extend the above linearly, so*

$$\left( \sum_i \sigma_i \right) \bullet \left( \sum_j \sigma_j \right) = \sum_{i,j} \sigma_i \bullet \sigma_j.$$

One then proves that this operation commutes with the differential (and hence descends to a map on homology). Assuming that this is true, it defines a product

$$\bullet : \mathbb{H}_*(\mathcal{L}M) \otimes \mathbb{H}_*(\mathcal{L}M) \rightarrow \mathbb{H}_*(\mathcal{L}M),$$

which is additive on degrees.

### 5.2.2 Viterbo Isomorphism

Abbondandolo-Schwarz proved in [1, Theorems A,B] that Viterbo's isomorphism [31] is a homomorphism of rings

$$\Phi : (\mathbb{H}_*(\mathcal{LM}), \bullet) \xrightarrow{\cong} (SH^*(T^*M), *)$$

where  $SH^*(T^*M)$  is the symplectic cohomology of the symplectic manifold  $T^*M$  and  $*$  is the pair-of-pants product. We will sketch their argument following [21], as this will be the most easily extendable to the equivariant case.

One first chooses  $L : TM \rightarrow \mathbb{R}$  that is quadratic in the fibres. Define

$$\epsilon(\gamma) := \int_0^1 L(\gamma, \dot{\gamma}) dt$$

for  $\gamma \in \mathcal{LM}$ . Identify  $H_*(\mathcal{LM})$  with  $HM_*(\mathcal{LM}, \epsilon)$ . We actually use  $HM_*(W^{1,2}(S^1, M), \epsilon)$ , observing that  $W^{1,2}(S^1, M)$  is homotopy equivalent to  $\mathcal{LM}$ .

Associated to  $L$  is the Hamiltonian

$$H : T^*M \rightarrow \mathbb{R},$$

defined by  $H(q, p) = \max_{\alpha \in T_q M} (p \cdot \alpha - L(q, \alpha))$ . The Legendre Transform  $\mathfrak{L} : TM \rightarrow T^*M$  defined by

$$\mathfrak{L}(q, \alpha) := (q, dL(q, \alpha)|_{T_{(q, \alpha)}^{vert}(TM) \cong T_q M})$$

relates the Floer cohomology of  $H$  on  $T^*M$  to the Morse theory of  $\epsilon$  on  $\mathcal{LM}$ . Specifically,  $\gamma \in \text{crit}(\epsilon)$  if and only if  $\mathfrak{L}(\gamma, \dot{\gamma}) \in \text{crit}(\mathbb{A}_H)$  is a 1-periodic orbit of  $X_H$  in  $T^*M$ , i.e. a critical point of the action functional. This induces  $\phi : CM_*(\epsilon) \rightarrow SC^{n-*}(H)$  where

$$\phi(\gamma) = \sum_{a \# b \in \mathcal{M}^+(\gamma, x) \text{ isolated}} x.$$

$\mathcal{M}^+(\gamma, x)$  consists of pairs  $(a, b)$  where  $a : (-\infty, 0] \rightarrow \mathcal{LM}$  is a  $-\nabla \epsilon$  flowline converging to  $\gamma \in \text{crit}(\epsilon)$  as  $s \rightarrow -\infty$  and  $b : (-\infty, 0] \times S^1 \rightarrow T^*M$  is a Floer trajectory with respect to  $H$ , so

$$\partial_s u + J \partial_t u + \nabla H = 0$$

such that  $b$  converges to  $x$  as  $s \rightarrow -\infty$ , and  $\pi \circ b(0, t) = a(0)(t)$  where  $\pi : T^*M \rightarrow M$  is the projection. These setups are illustrated in Figure 5.1.

To show that this isomorphism is in fact an isomorphism of rings, intertwining the Chas-Sullivan and pair-of-pants products, one uses the two 1-dimensional moduli spaces as in Figure 5.2.

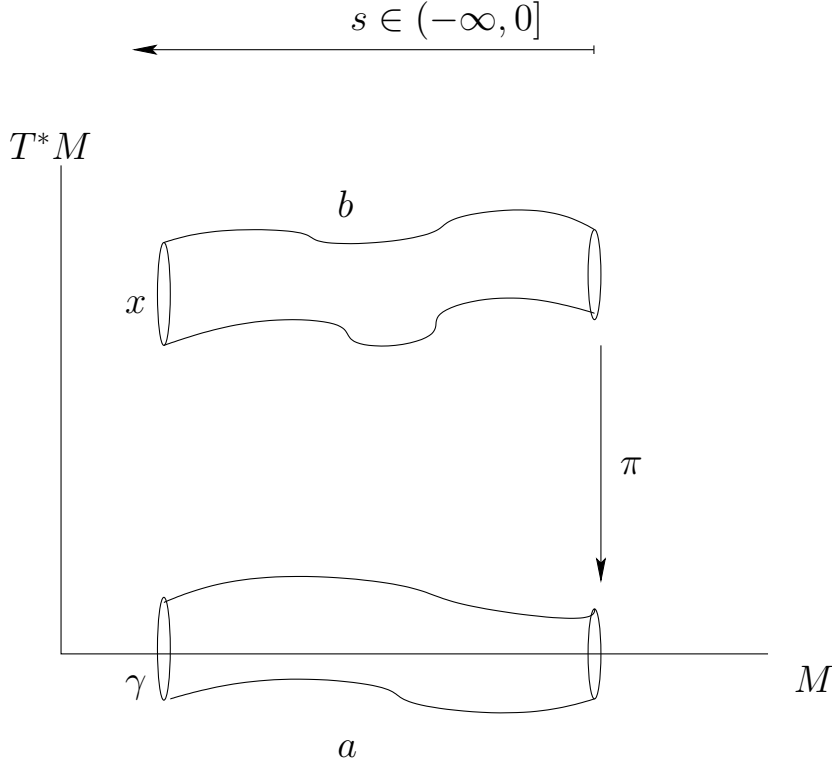


Figure 5.1: Setups counted for  $\phi$ .

## 5.3 The Equivariant Chas-Sullivan Product

### 5.3.1 The equivariant loop cohomology

Because the Chas-Sullivan product is commutative, we may construct an equivariant version in a similar way to the Steenrod square or the equivariant pair-of-pants. We first define the equivariant loop homology. For the purposes of equivariant cohomology we model  $\mathcal{LM} = W^{1,2}(\mathbb{R}/2\mathbb{Z}, M)$ , the  $(1, 2)$ -Sobolev space of such maps.

There is a  $\mathbb{Z}/2$ -action  $\iota : \mathcal{LM} \rightarrow \mathcal{LM}$ , where  $\iota(\gamma)(t) = \gamma(t + 1)$ . This induces a  $\mathbb{Z}/2$ -action on  $CM_*(\mathcal{LM}, \epsilon)$ . Recall from Section 5.2.2 there is a function  $L = L(t, q, \alpha) = L(t + 1, q, \alpha)$ . We pick a generic  $v$ -dependent perturbation  $L_v$  of our function  $L$  for  $v \in S^\infty$  such that:

1.  $L_v = L$  for  $v$  near  $v^{i,+}$ .
2.  $L_{\tau v} = L_v$  for  $\tau : S^\infty \rightarrow S^\infty$ ,  $\tau(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$ .

This induces

$$\epsilon_v(\gamma) = \int_0^2 L_v(t, \gamma(t), \dot{\gamma}(t)) dt$$

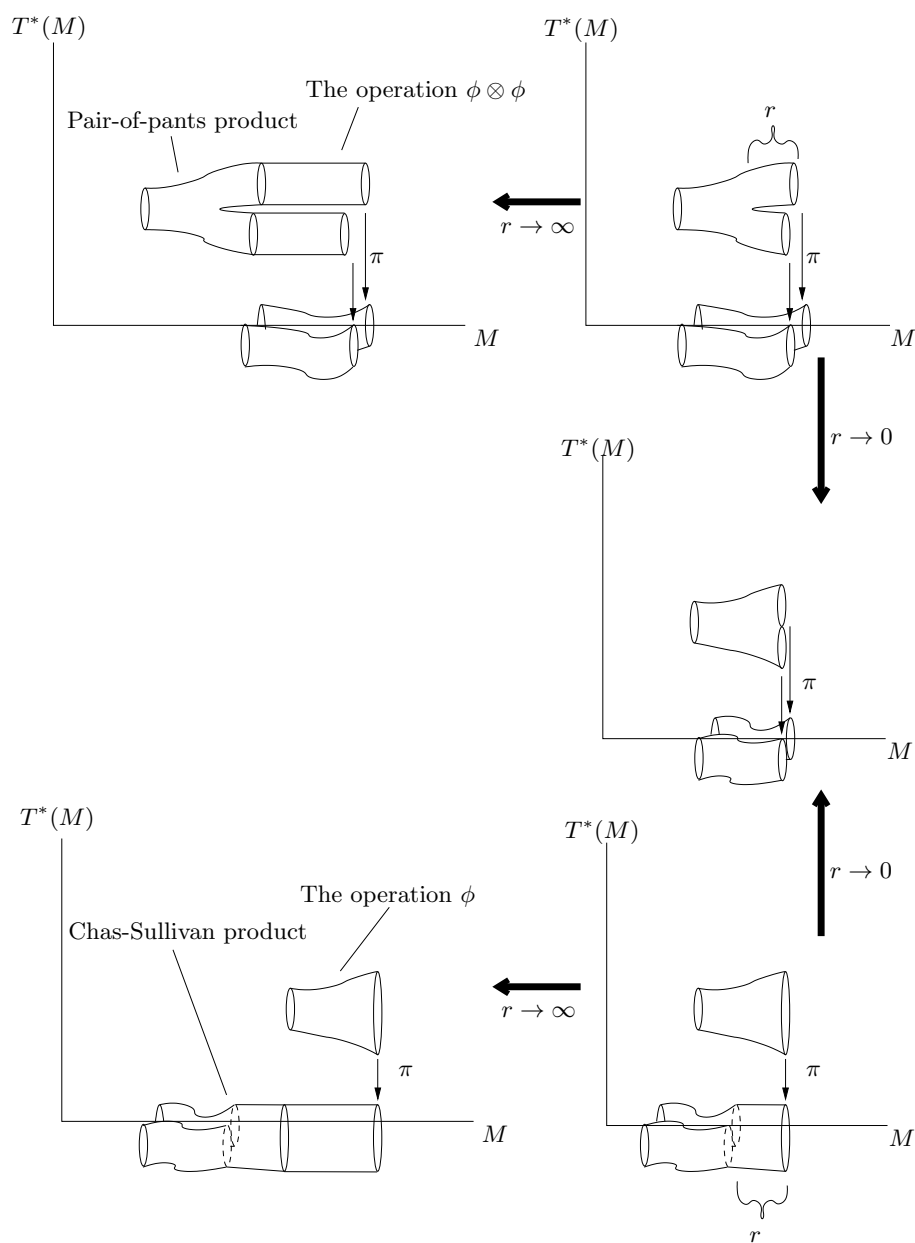


**Figure 5.2:** A “proof by picture” that  $\phi$  is a ring isomorphism. The lower half of the figure relates to  $\phi(x \bullet y)$  and the upper half relates to  $\phi x * \phi y$ .

and we see that  $\iota_* \epsilon_{-v} = \epsilon_v$ . Hence there is an  $\iota$  action on  $\text{crit}(\epsilon)$ . Let  $CH_*^{eq}(\mathcal{L}M, \epsilon) = CM_*(\mathcal{L}M, \epsilon)[h^{-1}]$ , where  $|h^{-1}| = -1$ .

Define equivariant differentials  $d_i : CM_*^{eq}(\mathcal{L}M, \epsilon) \rightarrow CM_{*+i-1}^{eq}(\mathcal{L}M, \epsilon)$  by

$$d_i(x) = \sum_y n_{x,y} y$$

where  $n_{x,y}$  is the number of isolated  $\mathbb{R}$ -equivalence classes of pairs  $[w, u]$  where  $w : S^\infty \rightarrow S^\infty$  is a negative gradient flowline of  $g$  with  $w(-\infty) = v^{i,\pm}$  and  $w(\infty) = v^{0,+}$  and  $u : \mathbb{R} \rightarrow \mathcal{L}M$  is such that  $\partial_s u = -\nabla \epsilon_{w(s)}$ , with:

- $u(-\infty)(t) = x(t)$  at  $\infty$ .
- $u(\infty)(t) = y(t)$  if  $w(-\infty) = v^{i,+}$ .
- $u(\infty)(t) = (\iota y)(t)$  if  $w(-\infty) = v^{i,-}$ .

The  $\mathbb{R}$ -action comes from simultaneously reparametrising the  $s$ -coordinate of  $u$  and  $w$ , as is standard.

$$d(x) = \sum_{i \geq 0} d_i(x) h^{-i}.$$

**Definition 5.3.1** (Equivariant Loop Homology).

$$H_*^{eq}(\mathcal{L}M) := H_*(C_*^{eq}(\mathcal{L}M, \epsilon), d).$$

**Remark 5.3.2.** *As the choice of perturbation  $L_v$  does not matter, we may choose  $L$  to be independent of  $v$ . In this case, the only isolated solutions occur for  $d_0$  and  $d_1$ , where  $d_0$  is the standard Morse differential on  $CM_*(\mathcal{L})$  and  $d_1 = id + \iota$ . All other differentials must be 0, hence  $d = d_0 + (id + \iota)h^{-1}$ , which is of a similar form to the equivariant cohomology of a space with involution.*

### 5.3.2 The equivariant Morse Chas-Sullivan square

The equivariant Chas-Sullivan square will be a homomorphism  $CS : H_*(\mathcal{L}M) \rightarrow H_{2*-\dim M}^{eq}(\mathcal{L}M)$ . We present it in two different ways: using Morse theory, and using singular chains. Note that this is not the Steenrod square, and should be compared with the fact that the Chas-Sullivan loop product is not the cup product.

### 5.3.2.1 The Morse equivariant Chas-Sullivan square

We choose a perturbation  $L_{v,s}$  of  $L$  for  $v \in S^\infty$  and  $s \in [0, \infty)$  such that  $L_{v,s} = L$  for  $|s| \gg 0$ . Define  $\epsilon_{v,s} = \int_0^2 L_{v,s}(t, \gamma(t), \dot{\gamma}(t)) dt$ . For each  $x, y \in \text{crit}(\epsilon)$ ,  $i \in \mathbb{Z}_{\geq 0}$  and  $w : \mathbb{R} \rightarrow S^\infty$  a negative flowline of  $g$  such that  $w(-\infty) = v^{i,\pm}$ ,  $w(\infty) = v^{0,+}$ , consider triples  $(u_0, u_1, u_2)$  such that:

$$\left\{ \begin{array}{l} u_0 : [0, \infty) \rightarrow \mathcal{L}M \\ u_1, u_2 : (-\infty, 0] \rightarrow \mathcal{L}M \\ u_0(0) = u_1(0) \cdot u_2(0) \\ \partial_s u_0 = -\nabla \epsilon \\ \partial_s u_1 = -\nabla \epsilon_{w(s),s} \\ \partial_s u_2 = -\nabla \epsilon_{-w(s),s} \\ u_0(\infty)(t) = \begin{cases} y(t) & \text{if } w(-\infty) = v^{i,+} \\ (\iota y)(t) & \text{if } w(-\infty) = v^{i,-} \end{cases} \\ u_1(-\infty)(t) = u_2(-\infty)(t) = x(t). \end{array} \right. \quad (5.3)$$

$\mathcal{M}_i(x, y) := \{(w, u_0, u_1, u_2) | w(-\infty) = v^{i,\pm}, w(\infty) = v^{0,+}, (u_0, u_1, u_2) \text{ satisfy (5.3)}\}$ .

**Definition 5.3.3.** *Let*

$$CS_i : CM_*(\mathcal{L}M) \rightarrow CM_*(\mathcal{L}M), \quad CS_{eq,i}(x) = \sum_y \# \mathcal{M}_i(x, y) y$$

where  $\#$  counts isolated solutions. *Let*

$$CS = \sum_{i \geq 0} h^{-i} \cdot CS_i.$$

This descends to a map on homology (to see this is the case, one needs the breaking conditions for the parameter space  $\mathcal{P}$  as in [28, Equation (4.100)]).

### 5.3.2.2 The equivariant Chas-Sullivan square

For  $X \in H_*(\mathcal{L}M)$  and  $v \in D^{i,+}$ , where  $D^{i,+}$  is the closed upper  $i$ -dimensional hemisphere in  $S^\infty$  representing the generator of  $H_i(\mathbb{RP}^\infty)$ , pick a family of cycles  $X_v$  representing  $X$ .

Consider the cycle  $X_v \bullet X_{-v}$  for each  $v$ .

**Definition 5.3.4.** *Define*

$$CS_{eq,i}(X) = \bigsqcup_{v \in D^{i,+}} X_v \bullet X_{-v} \in C_{2* - \dim M + i}(\mathcal{L}M).$$

Let

$$C\mathcal{S}_{eq} = \sum_{i \geq 0} h^{-i} C\mathcal{S}_{eq,i}.$$

Using  $d = d_0 + (id + \iota)h^{-1}$ , one can see that  $C\mathcal{S}$  descends to a map on homology,

$$C\mathcal{S} : \mathbb{H}_*(\mathcal{L}M) \rightarrow \mathbb{H}_{2*}^{eq}(\mathcal{L}M).$$

This definition relates to the previous definition, where to the critical point  $x \in \text{crit}(\epsilon)$  we assign the pseudo-cycle  $X = W^u(x, \epsilon)$ , the unstable manifold of  $x$  with respect to  $\epsilon$ . We can see that for a  $v$ -dependent  $L_{v,s}$  we obtain a  $v$ -dependent  $X_v$  as above. The flowline  $w : \mathbb{R} \rightarrow S^\infty$  with  $w(-\infty) = v^{i,\pm}$  and  $w(\infty) = v^{0,+}$  relates to the choice of  $v \in D^{i,+}$  in the same way, by observing that  $D^{i,\pm}$  is the stable manifold for  $v^{0,+}$ .

## 5.4 The Equivariant Viterbo Isomorphism

We wish to prove an equivariant version of the Viterbo isomorphism: specifically, we define  $\phi_{eq} : SH_{eq}^*(T^*M) \rightarrow \mathbb{H}_{-*}^{eq}(\mathcal{L}M)$  such that diagrams (5.1) and (5.4) commute.

$$\begin{array}{ccc} SH_{eq}^*(T^*M) & \xleftarrow{\phi_{eq}} & \mathbb{H}_{-*}^{eq}(\mathcal{L}M) \\ \uparrow c_{eq}^* & & \uparrow c_{eq}^* \\ H^*(T^*M)[h] & & \\ \uparrow & & \\ H^*(M)[h] & \xleftarrow{PD} & \mathbb{H}_{-*}(M)[h^{-1}] \end{array} \quad (5.4)$$

Here, the homomorphism  $c_{eq}^* : H^*(T^*M)[h] \rightarrow SH_{eq}^*(T^*M)$  was made precise in Section 4.6.1 and the homomorphism  $c_{eq}^* : H_{n-*}(M)[h^{-1}] \rightarrow \mathbb{H}_{n-*}^{eq}(\mathcal{L}M)$  is the inclusion of constant loops, linear over  $\mathbb{Z}/2[h^{-1}]$ . We define  $PD(xh^i) := PD(x)h^{-i}$  for all  $x \in H^*(M)$  and  $i \geq 0$ , where  $PD : H^*(M) \rightarrow H_{n-*}(M)$  is Poincaré duality for  $M$  over  $\mathbb{Z}/2$ .

### 5.4.1 The equivariant Viterbo isomorphism

We first choose  $L$  as in Section 5.2.2, and associated  $H$  and  $\epsilon$ . Recall that in Section 4.2.1, we had picked an equivariant almost complex structure  $J_{eq,v}$ . We parametrise  $S^\infty$  by  $(x_0, x_1, \dots)$  such that  $\sum_{i \geq 0} x_i^2 = 1$  and only finitely many  $x_n$  are nonzero. Given  $i, j \in \mathbb{Z}_{\geq 0}$ ,  $\gamma \in \text{crit}(\epsilon)$  and  $x \in CF^*(H)$ , the moduli space  $\mathcal{M}_i^+(\gamma, x)$  consists of triples  $(w, (a, b))$  where:

- $w : \mathbb{R} \rightarrow S^\infty$  satisfies  $\partial_s w = -\nabla g$  and  $w(-\infty) = v^{i,\sigma}$  and  $w(\infty)v^{0,+}$ .
- $a : (-\infty, 0] \rightarrow \mathcal{L}M$  satisfies  $\partial_s a = -\nabla \epsilon$  such that  $\lim_{s \rightarrow -\infty} u(s, t) = \gamma(t)$ .
- $b : (-\infty, 0] \times \mathbb{R}/2\mathbb{Z} \rightarrow T^*M$  satisfies

$$\partial_s u + J_{eq, w(s)} \partial_t u - \nabla H = 0,$$

and  $\lim_{s \rightarrow -\infty} u(s, t) = x(t)$  if  $\sigma = +$ , and  $\lim_{s \rightarrow -\infty} u(s, t) = x(t+1)$  if  $\sigma = -$ .

- $\pi \circ b(0, t) = a(0)(t)$  if  $\sigma = +$  and  $\pi \circ b(0, t) = a(0)(t+1)$  if  $\sigma = -$ , where  $\pi : T^*M \rightarrow M$  is the projection.

**Definition 5.4.1.**

$$\phi_{eq,i} : CM_*^{eq}(\mathcal{L}M, \epsilon) \rightarrow SC_{eq}^{n-* - i}(M)$$

such that for a generator  $\gamma \in CM_*^{eq}(\mathcal{L}M, \epsilon)$ ,

$$\phi_{eq,i}(\gamma) = \sum_{(w, a \# b) \in \mathcal{M}_i^+(\gamma, x) \text{ isolated}} x.$$

We define  $\phi_{eq,i}(\gamma \cdot h^{-k}) = \phi_{eq,i}(\gamma) \cdot h^k$ . Let

$$\phi_{eq} : CM_*^{eq}(\mathcal{L}M) \rightarrow SH_{eq}^{n-*}(T^*M), \quad \phi_{eq}(\gamma) := \sum_{i \geq 0} \phi_{eq,i}(\gamma) \cdot h^i.$$

Then  $\phi_{eq}$  descends to a map

$$\phi_{eq} : \mathbb{H}_{-*}^{eq}(\mathcal{L}M) \rightarrow SH_{eq}^*(T^*M)$$

in a similar way to how Definition 5.3.3 is a chain map: specifically, by classifying the different ways in which an equivariant Floer or Morse trajectory may break.

There are four possible types of breaking (for a 1-dimensional moduli space): either  $a$  breaks,  $(w, a)$  break,  $b$  breaks or  $(w, b)$  break. To link in to the breaking of flowlines on  $S^\infty$  as in Section 4.2.7.2, we note that if  $w$  breaks (as a parametrised flowline on  $S^\infty$ ) broken trajectories on  $S^\infty$  must look like a  $w$  setup as before, with an unparametrized trajectory  $w'$  attached either such that  $w'(\infty) = w(-\infty)$  or such that  $w'(\infty) = w(-\infty)$  respectively, and this cannot occur alone in dimension 1. A breaking of  $a$  corresponds to  $\phi_{eq} \circ d_M$ , where  $d_M$  is the Morse differential on  $\mathcal{L}M$ . The breaking of  $w$  gives that  $w$  becomes a broken flowline  $(w_1, w_2)$  with  $w_1(-\infty) = v^{i,\sigma}$ ,  $w_2(\infty) = v^{0,+}$  and either:

1.  $w_1(\infty) = w_2(-\infty) = v^{1,+}$ , or

$$2. w_1(\infty) = w_2(-\infty) = v^{1,-}.$$

The first case corresponds to  $\phi_{eq}(h^{-1} \cdot \gamma)$ . The second case corresponds to  $\phi_{eq}(h^{-1} \cdot \iota \gamma)$ . The  $(w, b)$  breaking corresponds to  $d_{eq} \circ \phi_{eq}$ . Taking all of these contributions together,  $\phi_{eq}$  is a chain map. Respectively (i.e. with respect to the four types of breakings listed above), this means that

$$\phi_{eq,i}(dc) + \phi_{eq,i-1}(h^{-1}(c + \iota c)) + d\phi_{eq,i}(c) + \sum_{j+k=i} \phi_{eq,k}(c),$$

where  $d$  refers to both the standard (nonequivariant) differential on  $C_*(\mathcal{L}M)$  and  $SC^*(T^*M)$ .

**Lemma 5.4.2.**  *$\phi_{eq}$  is an chain isomorphism.*

*Sketch of proof.* Observe that  $\phi_{eq,0} : CM_{n-*}(\mathcal{L}M) \rightarrow SC^*(M)$  is an isomorphism (being the nonequivariant Viterbo isomorphism). By Lemma 4.4.3,  $\phi_{eq}$  is an isomorphism.  $\square$

**Lemma 5.4.3.** *Diagram (5.1) commutes.*

*Sketch of proof.* The proof of this is the same as in the nonequivariant case, using a 1-dimensional moduli space as in Figure 5.2. In this case we also keep track of the Morse theory on  $S^\infty$ , using the flowline  $w$ , which may or may not also break.  $\square$

**Lemma 5.4.4.** *Diagram (5.4) commutes.*

*Sketch of proof.* For our purposes, for  $r \in [0, 1]$  we parametrise the half-cylinder  $(s, t) \in (-\infty, -\log(r)] \times \mathbb{R}/2\mathbb{Z}$  by logarithmic coordinates on  $\mathbb{C} - \mathbb{D}_r$ , where  $\mathbb{D}_r$  is the open disc of radius  $r$  in  $\mathbb{C}$ . Specifically, we use a chart of the form

$$(s, t) \in (-\infty, -\log(r)] \times \mathbb{R}/2\mathbb{Z} \quad \mapsto \quad e^{-s+\pi it} \in \mathbb{C} - \mathbb{D}_r.$$

Given a Morse function  $f$  on  $M$ , and  $y \in \text{crit}(f)$ , let  $W^u(y, f)$  be the unstable manifold of  $y$  under  $f$ . The restriction of  $\epsilon$  to constant loops in  $\mathcal{L}M$  gives a Morse function on  $M$ , which we call  $f$ . For  $s \in \mathbb{R}$ , define  $H_s$  such that  $H_s = H$  for  $s \geq 2$ , and  $H_s = 0$  for  $s \leq 1$ . Similarly, define  $J_{v,s,t}$  such that  $J_{v,s,t} = J_{eq,v,t}$  for  $s \geq 2$ ,  $J_{-v,s,t} = J_{v,s,t+1}$  and  $J_{v,s,t}$  is independent of  $v, s, t$  for  $s \leq 1$ . Given  $r \in [0, 1]$ ,  $i \in \mathbb{Z}_{\geq 0}$  and  $y \in \text{crit}(f)$ , the moduli space  $\mathcal{M}_{r,i}^+(y, x)$  consists of pairs  $(w, a)$  where:

- $w : \mathbb{R} \rightarrow S^\infty$ , where  $\partial_s w = -\nabla g$ ,  $w(-\infty) = v^{i,\sigma}$ , and  $w(\infty) = v^{0,+}$ , where  $\sigma \in \{+, -\}$ .

- $a : \mathbb{C} - \mathbb{D}_r \rightarrow T^*M$  satisfies

$$\partial_s a + J_{eq,w(s)} \partial_t a + \nabla H = 0,$$

and  $\lim_{s \rightarrow -\infty} u(s, t) = x(t)$  if  $\sigma_b = +$ , and  $\lim_{s \rightarrow -\infty} u(s, t) = x(t + 1)$  if  $\sigma = -$ .

•

$$\pi \circ b(r, t) \in W^u(y, f). \quad (5.5)$$

We note that if  $y \in \text{crit}(f)$  represents an element of  $HM_*(M, f)$  then using the Poincaré duality theorem (see Section 4.2.3),  $PD^{-1}\pi^*PD(y)$  is represented by  $\pi^{-1}W^u(y, f)$ . Then the condition in Equation (5.5) is exactly an intersection condition of a disc with  $PD^{-1}(\pi^*PD(y))$  at 0. This is the actual intersection condition used in defining  $c_{eq}^*(\pi^*PD(y))$  for  $SH^*(T^*M)$ . This moduli space then gives a 1-cobordism that induces a chain homotopy between  $\phi_{eq} \circ c_{eq}^*$  (when  $r = 1$ ), and the operation  $c_{eq}^* \circ \pi^* \circ PD$  (when  $r = 0$ ).  $\square$

**Remark 5.4.5.**  $\mathbb{H}_{-*}^{eq}(\mathcal{L}M)$  is much simpler to compute than  $SH_{eq}^*(T^*M)$ . This is because for  $SH_{eq}^*(M)$  we need to worry about the regularity of our solutions, and hence we (generically) choose an almost complex structure  $J_{eq,v}$  such that  $J_{eq,v,t+2} = J_{eq,v,t}$ . Lemma 5.4.2 will tell us that regularity is actually not an issue, i.e.  $SH_{eq}^*(T^*M) \cong \mathbb{H}_{-*}^{eq}(\mathcal{L}M)$ .

## 5.5 Some calculations

We will calculate a nontrivial equivariant pair-of-pants product for  $T^*S^2$ . In general, the  $\mathbb{Z}/2$  homology of the loop spaces is not well known. For the case of the loop spaces of  $M = S^n, \mathbb{CP}^n$  we may use the knowledge of the  $\mathbb{Z}$ -homology of these spaces to construct the  $\mathbb{Z}/2$  homology ring.

Using the fibration  $\Omega M \hookrightarrow \mathcal{L}M \rightarrow M$ , one constructs a Serre spectral sequence for the above cases. In all of these cases the spectral sequence collapses, and Lennert Meier in [16] proves that the product on  $\mathbb{H}_*(\mathcal{L}M)$  is induced by the product on  $E_{p,q}^2 = \mathbb{H}_p(M; H_q(\Omega M))$  via the intersection and Pontrjagin products.

Seeliger in [27] states the  $\mathbb{Z}$  homology rings of the listed spaces. Using this and the homologies of  $M$  and  $\Omega M$ , we can work out the  $\mathbb{Z}$ -differentials of the Serre spectral sequence in this case. Tensoring with  $\mathbb{Z}/2$  allows us to calculate the spectral sequence and differentials for these cases, and therefore the homology of  $\mathcal{L}M$ . With this in mind, there is a generator of  $\mathbb{H}_{-1}(\mathcal{L}S^2)$  corresponding to the generator

$x \bullet y \in \mathbb{H}_{-2}(S^2; H_1(\Omega S^2))$ . Letting  $x \bullet y^2$  being the generator of  $\mathbb{H}_{-2}(S^2; H_2(\Omega S^2))$ , we see that  $x \bullet y^2 h^{-2}$  is an element of  $\mathbb{H}_{-2}^{eq}(\mathcal{L}M)$ .

**Claim.**  $CS_2(x \bullet y) = x \bullet y^2$ .

*Sketch of proof.* A representative of  $x \bullet y$  is  $\{pt\} \times y$ , where locally  $\mathcal{L}M$  is  $U \times \Omega M$  for some  $pt \in U \subset M$ . Here  $y$  represents the generator of  $H_1(\Omega M)$ .

For each  $v \in S^2$  we pick  $\{pt_v\} \times y_v$ . As we do this in a contractible way, we use a flattening map  $\phi : S^2 \rightarrow D^2 \rightarrow M$  where  $D^2$  is a 2-disc and  $\phi$  maps this 2-disc onto a small circular neighborhood of  $pt$ . Then we see that generically there is a unique  $v \in D^{2,+} \subset S^2$  such that  $pt_v = pt_{-v}$  and hence the output is  $\{pt_v\} \times (y_v \cdot y_{-v})$ . This represents  $x \bullet y^2$  as required.  $\square$

**Remark 5.5.1.** *We note that this is a chain level map. The maps  $CS_i$  are, as the  $PS_i$  in the previous chapter, only well defined on chains (although their sum is well defined on homology).*

## 5.6 Conclusions and Further Work

The claim is that the techniques used throughout this thesis are not specific just to the cases that we considered. Indeed, given a commutative product with some choice of auxiliary data  $\mathcal{D}$ , where the  $\mathcal{D}$  live in some infinite dimensional locally contractible space (e.g. a Banach space), we could play the same game by embedding a copy of  $S^\infty \subset \mathcal{D}$ . In Chapter 3, for a smooth manifold  $M$  this  $\mathcal{D}$  was a choice of smooth functions  $f_{v,s} \subset C^\infty(M)$ . For the equivariant pair-of-pants on a symplectic manifold,  $\mathcal{D}$  was a choice of almost complex structures  $J_{v,z} \subset \mathcal{J}$ . For the Chas-Sullivan product, the setup is morally the same as the Morse Steenrod square, so we have a choice of  $L_{v,s}$ .

There may be a way to calculate some of these  $H_*^{eq}(\mathcal{L}M)$  groups and the equivariant Chas-Sullivan product  $CS$ , either by using a spectral sequence similar to that for the fibration  $\Omega M \hookrightarrow \mathcal{L}M \rightarrow M$ , or by direct computation in certain cases. The latter is not unreasonable for example in the case of  $\mathcal{L}S^n$  and  $\mathcal{L}\mathbb{CP}^n$ , where Oancea in [17, Section 7] proved that elements of the homology of these loop spaces are represented by embedded submanifolds. Calculating these equivariant Chas-Sullivan products would then, by the equivariant Viterbo isomorphism, allow us to calculate the equivariant pair-of-pants in these settings. Whether this will allow us to say anything useful about the cotangent bundles of  $S^n$  or  $\mathbb{CP}^n$  remains to be seen.

Moving on from this, there is more work to be done with these sort of invariants: for example, quantum/symplectic reduced Steenrod powers are currently being studied. This quantum Steenrod square may also allow us to give some sort of quantum  $E_\infty$ -structure for quantum cohomology, which would be a more general result.

# Bibliography

- [1] Alberto Abbondandolo and Matthias Schwarz. Floer homology of cotangent bundles and the loop product. *Geom. Topol.*, 14(3):1569–1722, 2010.
- [2] Alejandro Adem and R. James Milgram. *Cohomology of finite groups*, volume 309 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004.
- [3] Martin Betz and Ralph L. Cohen. Graph moduli spaces and cohomology operations. *Turkish J. Math.*, 18(1):23–41, 1994.
- [4] Martin Scott Betz. *Operad representations in Morse theory and Floer homology*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–Stanford University.
- [5] Netanel S. Blaier. *The quantum Johnson homomorphism and symplectomorphism of 3-folds*. ProQuest LLC, Ann Arbor, MI, 2016. Thesis (Ph.D.)–Massachusetts Institute of Technology.
- [6] Sandro Buoncristiano and Derek Hacon. An elementary geometric proof of two theorems of Thom. *Topology*, 20(1):97–99, 1981.
- [7] Moira Chas and Dennis Sullivan. String topology. preprint arXiv:math/9911159, 1999.
- [8] Ralph Cohen and Paul Norbury. Morse field theory. *Asian J. Math.*, 16(4):661–711, 2012.
- [9] C. C. Conley and E. Zehnder. The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnol’ d. *Invent. Math.*, 73(1):33–49, 1983.
- [10] Kenji Fukaya. Morse homotopy and its quantization. In *Geometric topology (Athens, GA, 1993)*, volume 2 of *AMS/IP Stud. Adv. Math.*, pages 409–440. Amer. Math. Soc., Providence, RI, 1997.

- [11] Allen Hatcher. *Spectral Sequences in Algebraic Topology*, last modified 05/01/2004, available at <http://www.math.cornell.edu/hatcher/>.
- [12] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [13] François Laudenbach. A Morse complex on manifolds with boundary. *Geom. Dedicata*, 153:47–57, 2011.
- [14] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and quantum cohomology*, volume 6 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1994.
- [15] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, second edition, 2012.
- [16] Lennart Meier. Spectral sequences in string topology. *Algebr. Geom. Topol.*, 11(5):2829–2860, 2011.
- [17] Alexandru Oancea. Morse theory, closed geodesics, and the homology of free loop spaces. In *Free loop spaces in geometry and topology*, volume 24 of *IRMA Lect. Math. Theor. Phys.*, pages 67–109. Eur. Math. Soc., Zürich, 2015. With an appendix by Umberto Hryniewicz.
- [18] S. Piunikhin, D. Salamon, and M. Schwarz. Symplectic Floer-Donaldson theory and quantum cohomology. In *Contact and symplectic geometry (Cambridge, 1994)*, volume 8 of *Publ. Newton Inst.*, pages 171–200. Cambridge Univ. Press, Cambridge, 1996.
- [19] Alexander F. Ritter. Deformations of symplectic cohomology and exact Lagrangians in ALE spaces. *Geom. Funct. Anal.*, 20(3):779–816, 2010.
- [20] Alexander F. Ritter. University of Cambridge lecture notes in Part III Morse Theory, Lent 2011.
- [21] Alexander F. Ritter. Topological quantum field theory structure on symplectic cohomology. *J. Topol.*, 6(2):391–489, 2013.
- [22] Alexander F. Ritter. Floer theory for negative line bundles via Gromov-Witten invariants. *Adv. Math.*, 262:1035–1106, 2014.

- [23] Alexander F. Ritter and Ivan Smith. The monotone wrapped Fukaya category and the open-closed string map. *Selecta Math. (N.S.)*, 23(1):533–642, 2017.
- [24] Dietmar Salamon. Lectures on Floer homology. In *Symplectic geometry and topology (Park City, UT, 1997)*, volume 7 of *IAS/Park City Math. Ser.*, pages 143–229. Amer. Math. Soc., Providence, RI, 1999.
- [25] Matthias Schwarz. *Morse homology*, volume 111 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1993.
- [26] Matthias Schwarz. Equivalences for Morse homology. In *Geometry and topology in dynamics (Winston-Salem, NC, 1998/San Antonio, TX, 1999)*, volume 246 of *Contemp. Math.*, pages 197–216. Amer. Math. Soc., Providence, RI, 1999.
- [27] Nora Seeliger. Loop homology of spheres and complex projective spaces. *Forum Math.*, 26(4):967–981, 2014.
- [28] Paul Seidel. The equivariant pair-of-pants product in fixed point Floer cohomology. *Geom. Funct. Anal.*, 25(3):942–1007, 2015.
- [29] Paul Seidel and Ivan Smith. Localization for involutions in Floer cohomology. *Geom. Funct. Anal.*, 20(6):1464–1501, 2010.
- [30] René Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86, 1954.
- [31] Claude Viterbo. Functors and computations in Floer homology with applications. I. *Geom. Funct. Anal.*, 9(5):985–1033, 1999.
- [32] Nicholas Wilkins. A Construction of the Quantum Steenrod Squares and Their Algebraic Relations. arXiv:1805.02438, 2018.
- [33] Nicholas Wilkins. Quantum Steenrod Squares and the Equivariant Pair-of-Pants in Symplectic Cohomology. arXiv:1810.02738, 2018.
- [34] Jingyu Zhao. *Periodic Symplectic Cohomologies and Obstructions to Exact Lagrangian Immersions*. ProQuest LLC, Ann Arbor, MI, 2016. Thesis (Ph.D.)—Columbia University.