ON LIMIT POINTS OF THE SEQUENCE OF NORMALIZED PRIME GAPS

WILLIAM D. BANKS, TRISTAN FREIBERG, AND JAMES MAYNARD

ABSTRACT. Let p_n denote the nth smallest prime number, and let $\boldsymbol{L}$ denote the set of limit points of the sequence $\{(p_{n+1}-p_n)/\log p_n\}_{n=1}^{\infty}$ of normalized differences between consecutive primes. We show that for any sequence of nine nonnegative real numbers $\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_9$, at least one of the numbers $\beta_j - \beta_i$ $(1 \leqslant i < j \leqslant 9)$ belongs to $\boldsymbol{L}$. It follows that at least 12.5% of all nonnegative real numbers belong to $\boldsymbol{L}$.

1. Introduction

Let $p_1 = 2 < p_2 = 3 < p_3 = 5 < \cdots$ be the sequence of all primes. The prime number theorem asserts that $p_n \sim n \log p_n$ as $n \to \infty$, hence the nth prime gap

$$d_n = p_{n+1} - p_n$$

is of length approximately $\log p_n$ on average. It is natural to ask how often the normalized nth prime gap $d_n/\log p_n$ lies between two given numbers α and β. For fixed $\beta > \alpha \geqslant 0$, heuristics based on Crámer's probabilistic model for primes lead to the conjecture that

$$N^{-1}\big|\{n \leqslant N : d_n/\log p_n \in (\alpha,\beta]\}\big| \sim \int_\alpha^\beta \mathrm{e}^{-t}\mathrm{d}t \qquad (N \to \infty). \tag{1.1}$$

Thus, one expects that the normalized prime gaps are distributed according to a Poisson process, and the probability that d_n is close to $t \log p_n$ is about e^{-t}. We refer the reader to the expository article [**26**] of Soundararajan for further discussion of these fascinating statistics.

Gallagher [**12**] has shown that (1.1) follows from the truth of a suitable uniform version of the Hardy–Littlewood prime k-tuple conjecture; however, such results must lie very deep. An approximation to (1.1) is the conjecture[1] of Erdős [**7**] that if $\boldsymbol{L}$ is the set of limit points in $[0,\infty]$ of the sequence $\{d_n/\log p_n\}_{n=1}^{\infty}$, then $\boldsymbol{L} = [0,\infty]$. It had already been established by Westzynthius [**28**] in 1931 that

$$\limsup_{n\to\infty} \frac{d_n}{\log p_n} = \infty.$$

In 2005, the groundbreaking work of Goldston–Pintz–Yıldırım [**14**] established for the first time that

$$\liminf_{n\to\infty} \frac{d_n}{\log p_n} = 0.$$

Hence, $0 \in \boldsymbol{L}$ and $\infty \in \boldsymbol{L}$, but no other limit point of $\boldsymbol{L}$ is known at present.

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[1] Erdős [**7**, p.4] wrote: "It seems certain that $d_n/\log n$ is everywhere dense in the interval $(0,\infty)$."

The prime number theorem implies the existence of a limit point in $\boldsymbol{L}$ that is less than or equal to 1. Erdős [**7**] and Ricci [**25**] were able to show that $\boldsymbol{L}$ has positive Lebesgue measure, but were unable to show that $\boldsymbol{L}$ contains a limit point greater than 1. Hildebrand and Maier [**15**] were the first to show that $\boldsymbol{L}$ contains a limit point greater than 1. Indeed, they showed that there is a positive constant c such that $\lambda([0,T]\cap\boldsymbol{L}) \geqslant cT$ holds for all sufficiently large T, where λ denotes the Lebesgue measure on $\mathbb{R}$, and hence that $\boldsymbol{L}$ contains arbitrarily large limit points. In fact, Hildebrand and Maier proved an m-dimensional analogue of this result for the limit points of "chains" of m consecutive gaps between primes (see Theorem 1.3 below).

Using the recent breakthrough work of Zhang [**29**] on bounded gaps between primes, Pintz [**20**] has shown that there is a small (ineffective) positive constant c such that $\boldsymbol{L} \supseteq [0,c]$. Most recently, Goldston and Ledoan [**13**] have shown that Erdős' method yields $\lambda([0,T]\cap\boldsymbol{L}) > (1/\mathcal{C})(1-1/T)$ for any $T>1$, where $\mathcal{C}$ is an overestimate in the sieve upper bound for the number of generalized twin primes (one can take $\mathcal{C}=7/2$ by work of Bombieri, Friedlander and Iwaniec [**3**]). Goldston and Ledoan [**13**] have also exhibited certain other intervals, for example $[1/2000, 3/4]$, that contain infinitely many limit points. In this paper, we prove the following.

Theorem 1.1. *Let $d_n = p_{n+1} - p_n$, where p_n denotes the nth smallest prime, and let $\boldsymbol{L}$ be the set of limit points of $\{d_n/\log p_n\}_{n=1}^{\infty}$. For any sequence of nine nonnegative real numbers $\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_9$, we have*

$$\big\{\beta_j - \beta_i : 1 \leqslant i < j \leqslant 9\big\} \cap \boldsymbol{L} \neq \varnothing. \tag{1.2}$$

We have the following corollary, which shows that at least 12.5% of nonnegative real numbers belong to $\boldsymbol{L}$.

Corollary 1.2. *Let $\boldsymbol{L}$ be as in Theorem 1.1, and let λ be the Lebesgue measure on $\mathbb{R}$. The following bound holds (with an ineffective $o(1)$):*

$$\lambda([0,T]\cap\boldsymbol{L}) \geqslant (1-o(1))T/8 \qquad (T\to\infty). \tag{1.3}$$

The following effective bound also holds:

$$\lambda([0,T]\cap\boldsymbol{L}) > T/22 \qquad (T>0). \tag{1.4}$$

Proof. We first observe that the set $\boldsymbol{L}$, being a countable number of unions and intersections of open balls, is Lebesgue measurable.

Now let $\kappa \geqslant 2$ be the smallest integer such that for any sequence of κ real numbers $\alpha_\kappa \geqslant \cdots \geqslant \alpha_1 \geqslant 0$, we have

$$\{\alpha_j - \alpha_i : 1 \leqslant i < j \leqslant \kappa\} \cap \boldsymbol{L} \neq \varnothing.$$

By Theorem 1.1 such a κ exists and is at most 9. If $\kappa = 2$ then $\boldsymbol{L} = [0,\infty]$. If $\kappa \geqslant 3$ then by minimality there is a sequence of real numbers $\hat\alpha_{\kappa-1} \geqslant \cdots \geqslant \hat\alpha_1 \geqslant 0$ such that

$$\{\hat\alpha_j - \hat\alpha_i : 1 \leqslant i < j \leqslant \kappa-1\} \cap \boldsymbol{L} = \varnothing.$$

Then for any number $\alpha \geqslant \hat{\alpha}_{\kappa-1}$, $\{\alpha - \hat{\alpha}_j : 1 \leqslant j \leqslant \kappa - 1\} \cap \boldsymbol{L} \neq \varnothing$, that is, for any $T_2 \geqslant T_1 \geqslant \hat{\alpha}_{\kappa-1}$,

$$[T_1, T_2] = \bigcup_{j=1}^{\kappa-1} \{\beta + \hat{\alpha}_j : \beta \in [T_1 - \hat{\alpha}_j, T_2 - \hat{\alpha}_j] \cap \boldsymbol{L}\}.$$

Thus, by subadditivity and translation invariance of Lebesgue measure,

$$T_2 - T_1 \leqslant \sum_{j=1}^{\kappa-1} \lambda([T_1 - \hat{\alpha}_j, T_2 - \hat{\alpha}_j] \cap \boldsymbol{L}) \leqslant (\kappa - 1)\lambda([0, T_2] \cap \boldsymbol{L}).$$

This gives (1.3).

With κ as above we have

$$\{\alpha, 2\alpha, \ldots, (\kappa - 1)\alpha\} \cap \boldsymbol{L} \neq \varnothing$$

for every real number $\alpha \geqslant 0$ (take $\hat{\alpha}_j = j\alpha$ for $1 \leqslant j \leqslant \kappa$). For any $T \geqslant 0$, by subadditivity and positive homogeneity of Lebesgue measure, we have

$$\begin{aligned}\lambda([0, T]) &\leqslant \sum_{j=1}^{\kappa-1} \lambda([0, T] \cap j^{-1}\boldsymbol{L}) = \sum_{j=1}^{\kappa-1} j^{-1}\lambda([0, jT] \cap \boldsymbol{L}) \\ &\leqslant \lambda([0, (\kappa - 1)T] \cap \boldsymbol{L}) \sum_{j=1}^{\kappa-1} j^{-1}.\end{aligned}$$

Replacing T by $(\kappa - 1)^{-1}T$ and recalling that $\kappa \leqslant 9$, this gives (1.4). □

We also prove the following result on "chains" of gaps between primes, for which Theorem 1.1 is a stronger version of the special case $m = 1$.

Theorem 1.3. *Let $d_n = p_{n+1} - p_n$, where p_n denotes the nth smallest prime. Fix an integer $m \geqslant 2$, and let $\boldsymbol{L}_m$ be the set of limit points in $[0, \infty]^m$ of*

$$\left\{\left(\tfrac{d_n}{\log p_n}, \ldots, \tfrac{d_{n+m-1}}{\log p_{n+m-1}}\right)\right\}_{n=1}^{\infty}.$$

Given $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^k$, let $S_m(\boldsymbol{\beta})$ be the set

$$\left\{\left(\beta_{J(2)} - \beta_{J(1)}, \ldots, \beta_{J(m+1)} - \beta_{J(m)}\right) : 1 \leqslant J(1) < \cdots < J(m+1) \leqslant k\right\}.$$

For any sequence of $k = 8m^2 + 16m$ nonnegative real numbers

$$\beta_1 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_{8m^2+16m},$$

we have

$$S_m(\boldsymbol{\beta}) \cap \boldsymbol{L}_m \neq \varnothing. \tag{1.5}$$

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2. Notation

The set of all primes is denoted by $\mathbb{P}$, the nth smallest prime by p_n, the nth difference $p_{n+1} - p_n$ in the sequence of primes by d_n, and p always stands for a prime. The indicator function for $\mathbb{P}$ is denoted $\mathbf{1}_{\mathbb{P}}$. The Euler, von Mangoldt and k-fold divisor functions are denoted by ϕ, Λ and τ_k, the prime counting functions by $\pi(x) = \sum_{n \leqslant x} \mathbf{1}_{\mathbb{P}}(n)$, $\psi(x) = \sum_{n \leqslant x} \Lambda(n)$,

$$\pi(N; q, a) = \sum_{\substack{n \leqslant N \\ n \equiv a \bmod q}} \mathbf{1}_{\mathbb{P}}(n), \quad \psi(N; q, a) = \sum_{\substack{n \leqslant N \\ n \equiv a \bmod q}} \Lambda(n).$$

A Dirichlet character to the modulus q is denoted $\chi \bmod q$ or simply χ, and the L-function associated with it is denoted $L(s,\chi)$.

The nth iterated logarithm is denoted by $\log_n x$ and defined recursively as follows: $\log_1 x = \max\{1, \log x\}$ and $\log_{n+1} x = \log_1(\log_n x)$ for $n \geqslant 1$.

The greatest prime factor of an integer q is denoted $P^+(q)$.

For any k-tuple of integers $\mathcal{H} = \{h_1, \ldots, h_k\}$ and an integer n, the translated k-tuple $\{n+h_1, \ldots, n+h_k\}$ is denoted by $\mathcal{H}(n)$.

Throughout, c denotes a positive constant that may differ at each occurrence. Expressions of the form $A = O(B)$, $A \ll B$ and $B \gg A$ signify that $|A| \leqslant c|B|$. If c depends on certain parameters this may be indicated by subscripts (as in $A \ll_\epsilon B$, etc.). The relation $A \ll B \ll A$ is denoted $A \asymp B$. Finally, $o(1)$ denotes a quantity that tends to 0 as a certain parameter (clear in context) tends to infinity.

3. Outline of the proof

For the sake of exposition we ignore (only in this section) the possibility of Siegel zeros[2] — accounting for this possibility introduces certain minor technical complications in parts of the proof.

The idea underlying our proof of Theorem 1.3 is to combine a construction of Erdős [**8**] and Rankin [**24**] with the recent theorem of Maynard [**17**] and[3] Tao.

The Erdős–Rankin construction produces long intervals $(n, n+z]$ containing only composite integers. This is accomplished by choosing a set of integers $\{a_p : p \leqslant y\}$, one for each prime $p \leqslant y < z$, so that for every integer $g \in (0,z]$, the congruence $g \equiv a_p \bmod p$ holds for at least one $p \leqslant y$. By the Chinese remainder theorem one can find an integer b, uniquely determined modulo $P(y) = \prod_{p \leqslant y} p$, such that $b \equiv -a_p \bmod p$ for every $p \leqslant y$. Now suppose $n \equiv b \bmod P(y)$ and $n > y$. For any $g \in (0,z]$ we have $g \equiv a_p \bmod p$ for some $p \mid P(y)$, and so $g + n \equiv a_p - a_p \equiv 0 \bmod p$; hence, $g+n$ is composite for each $g \in (0,z]$. In this situation we say that the progression $b \bmod P(y)$ *sieves out* intervals of the form $(n, n+z]$, where $n \equiv b \bmod P(y)$ and $n > y$. Noting that $\log P(y) \sim y$ by the prime number theorem, the goal is to maximize the ratio z/y.

The Maynard–Tao theorem establishes, for the first time, the existence of $(m+1)$-tuples of primes in k-tuples of integers the form

$$\mathcal{H}(n) = \{n+h_1, \ldots, n+h_k\},$$

whenever $\mathcal{H} = \{h_1, \ldots, h_k\}$ is an *admissible* k-tuple (see (4.1)) and k is large enough in terms of m, say $k \geqslant k_m$. The prime k-tuple conjecture asserts that one can take $k_m = m+1$, but since in the Maynard–Tao theorem k_m is exponential in m (and this seems to be the limit of the method of proof at present), no given admissible $(m+1)$-tuple $\mathcal{H} = \{h_1, \ldots, h_{m+1}\}$ is known to give $|\mathcal{H}(n) \cap \mathbb{P}| = m+1$ for infinitely many n.

[2]We are abusing terminology here. By a Siegel zero we mean a real, simple zero of a Dirichlet L-function (corresponding to a primitive character), in a region that we can show is otherwise zero free. In some cases this is a wider region than the classical one — see Lemma 4.1 below.

[3]Tao (unpublished) independently discovered the same method as Maynard around the same time.

It turns out that in the Maynard–Tao theorem one can restrict n to lie in an arithmetic progression — in fact this is a feature of its proof. Given a sufficiently large number N and a modulus $W = \prod_{p \leqslant w} p$, where w grows slowly with N, one can take $n \in (N, 2N]$ with $n \equiv b \bmod W$, provided that b is an integer for which $(b + h_i, W) = 1$ for each i. Choosing the progression $b \bmod W$ carefully, one can use it to sieve out all integers in intervals of the form $(n, n+z]$ with $n \equiv b \bmod W$ *except* for the integers in $\mathcal{H}(n)$. Used in this way, the Maynard–Tao theorem produces *consecutive* m-tuples of primes in intervals of *bounded length.*

In the present paper, we modify the above ideas to obtain consecutive primes in $\mathcal{H}(n) = \{n + h_1, \dots, n + h_k\}$, $n \in (N, 2N]$, with differences $h_j - h_i \asymp \log N$. To do this, we give a uniform version of the Maynard–Tao theorem in which the elements of the k-tuple $\mathcal{H} = \{h_1, \dots, h_k\}$ are allowed to grow with N, and in which w can be as large as $\epsilon \log N$ for a sufficiently small ϵ. This means that the modulus W is as large as a small power of N, and for reasons concerning level of distribution (see (4.2) *et seq.*), this extension of the Maynard–Tao theorem requires a modification of the Bombieri–Vinogradov theorem[4] that exploits the fact that the arithmetic progressions with which we are concerned have moduli that are all multiples of the smooth integer W.

To obtain stronger quantitative results, we use a further modification of the Maynard–Tao theorem, which might be of independent interest. We show that given a partition $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_{8m+1}$ of $\mathcal{H}$ into $8m+1$ equal sized subsets, there are infinitely many n such that $|\mathcal{H}_j(n) \cap \mathbb{P}| \geqslant 1$ for at least $m+1$ different values of $j \in \{1, \dots, 8m+1\}$, provided that the size of $\mathcal{H}$ is sufficiently large.

We use a slight modification of the Erdős–Rankin construction to find an arithmetic progression $b \bmod W$ that sieves out the integers in an interval $(0, z]$, *except* for precisely k integers $\mathcal{H} = \{h_1, \dots, h_k\} \subseteq (0, z]$ that constitute our admissible k-tuple. We want to choose $\mathcal{H}$ so that $h_j - h_i \sim (\beta_j - \beta_i) \log N$ for $1 \leqslant i < j \leqslant k$, where $\beta_k \geqslant \cdots \geqslant \beta_1 \geqslant 0$ are given.

As in the Erdős–Rankin construction, we select the integers $\{a_p : p \leqslant y\}$, $y \leqslant z$, in stages according to their size.[5] We take $0 < y_1 < y_2 < y < z$, say, where y_1 and y_2 are parameters to be chosen optimally later. First, we put $a_p = 0$ for primes $p \in (y_1, y_2]$. Next, we use a "greedy sieve" to choose the a_p optimally for the small primes $2 < p \leqslant y_1$, that is, we successively choose a_p so that $g \equiv a_p \bmod p$ for the maximum possible number of $g \in (0, z]$ that have remain "unsifted" thus far. Since we do not know the congruence classes $a_p \bmod p$ for the smallest primes, our approach does not work in general for all k-tuples $\mathcal{H} = \{h_1, \dots, h_k\}$; we find it convenient to select our k-tuple only after sieving by primes $p \leqslant y_2$. We choose the numbers h_i from among the primes in $(y, z]$. (This is why we do not use $p = 2$ "greedily" – if we had $a_2 = 1$ then only even integers would remain unsifted.) It is clear that each $h_i \not\equiv a_p \bmod p$ for all $p \in (y_1, y_2]$ since for those primes we have $a_p = 0$. We can also guarantee that $h_i \not\equiv a_p \bmod p$ for the small primes $p \leqslant y_1$ if we select primes h_i in a suitable arithmetic progression $b \bmod P_1$, where $P_1 = \prod_{2 < p \leqslant y_1} p$. We choose

[4]Putative Siegel zeros have an impact here, and any exceptional moduli must be taken into account.

[5]The effect of any Siegel zero here would mean that here we must actually select integers $\{a_p : p \leqslant y,\, p \notin \mathcal{Z}\}$ for a certain sparse set of primes $\mathcal{Z}$.

$y_1 = (\log y)^{1/4}$, so such primes exist by (Page's version of) the prime number theorem for arithmetic progressions.[6,7]

4. A uniform Maynard–Tao theorem

4.1. **Preliminaries.** A precise statement of the version of Maynard–Tao that we will use requires some notation, terminology and setting up.

We say that a given k-tuple of integers $\mathcal{H} = \{h_1, \ldots, h_k\}$ is *admissible* if

$$\left|\left\{n \bmod p : \textstyle\prod_{i=1}^k (n + h_i) \equiv 0 \bmod p\right\}\right| < p \qquad (p \in \mathbb{P}). \tag{4.1}$$

The *prime k-tuple conjecture* asserts that if $\mathcal{H}$ is admissible then there are infinitely many integers n for which $|\mathcal{H}(n) \cap \mathbb{P}| = k$.

Level of distribution concerns how evenly the primes are distributed among arithmetic progressions. We say that the primes have level of distribution θ if for any given $\epsilon \in (0, \theta)$ and $A > 0$ one has, for all $N > 2$, the bound

$$\sum_{q \leqslant N^{\theta-\epsilon}} \max_{(q,a)=1} \left| \psi(N; q, a) - \frac{\psi(N)}{\phi(q)} \right| \ll_{\epsilon,A} \frac{N}{(\log N)^A}. \tag{4.2}$$

The celebrated Bombieri–Vinogradov theorem [**2**, Théorème 17] implies that the primes have level of distribution $\theta = \frac{1}{2}$, and the *Elliott–Halberstam conjecture* [**6**, **11**] asserts that the primes have level of distribution $\theta = 1$.

Next, fix an integer $k \geqslant 2$ and a number $\eta \in [0, 1)$, and for any fixed compactly supported square-integrable function $F : [0, \infty+) \to \mathbb{R}$, define the functionals

$$I_k(F) = \int_{[0,\infty+)^k} F(t_1, \ldots, t_k)^2 \, \mathrm{d}t_1 \ldots \mathrm{d}t_k$$

and (for $i = 1, \ldots, k$),

$$J_{i,1-\eta}(F) = \int_{(1-\eta)\cdot\mathcal{R}_{k-1}} \left(\int_0^\infty F(t_1, \ldots, t_k) \, \mathrm{d}t_i \right)^2 \mathrm{d}t_1 \ldots \mathrm{d}t_{i-1} \mathrm{d}t_{i+1} \ldots \mathrm{d}t_k,$$

$(1 - \eta) \cdot \mathcal{R}_{k-1}$ being the simplex

$$(1 - \eta) \cdot \mathcal{R}_{k-1} = \{(t_1, \ldots, t_k) \in [0, 1]^{k-1} : t_1 + \cdots + t_k \leqslant 1 - \eta\}.$$

Define $M_{k,\eta}$ to be the supremum

$$M_{k,\eta} = \sup \frac{\sum_{i=1}^k J_{i,1-\eta}(F)}{I_k(F)} \tag{4.3}$$

over square-integrable functions F that are supported on the simplex

$$(1 + \eta) \cdot \mathcal{R}_k = \{(t_1, \ldots, t_k) \in [0, 1]^k : t_1 + \cdots + t_k \leqslant 1 + \eta\},$$

and are not identically zero. Maynard [**17**, Proposition 4.2] has shown that for any given m there are infinitely many n with $|\mathcal{H}(n) \cap \mathbb{P}| \geqslant m + 1$, provided $M_k = M_{k,0} > 2\theta^{-1} m$, where θ is an admissible level of distribution for $\mathbb{P}$. By [**17**,

[6]Again, this is assuming Siegel zeros do not exist. If they do, we need only discard at most one prime from the product defining P_1 to ensure it isn't a multiple of an "exceptional" modulus.

[7]The reader will note that $y_1 = (\log y)^{1/4}$ is smaller than the optimal choice for y_1 in the original Erdős–Rankin construction. By a more careful argument one may be able to take y_1 larger, but this is not necessary for our application, and so we satisfy ourselves with a smaller choice of y_1.

Proposition 4.3] one has $M_5 > 2$, $M_{105} > 4$, and that $M_k > \log k - \log\log k - 2$ for all sufficiently large k. A recent Polymath project [**23**, Theorem 3.9] has refined these bounds as follows:

$$M_{54} > 4, \quad M_k \geqslant \log k - c, \tag{4.4}$$

for some absolute constant c. Moreover, the Polymath project has refined the method of [**17**] slightly, allowing one to reduce the condition $M_k > 2\theta^{-1}m$ to $M_{k,\eta} > 2\theta^{-1}m$ for some $0 \leqslant \eta \leqslant \theta^{-1} - 1$. They have also [**23**, Theorem 3.13] produced the bound

$$M_{50,1/25} > 4. \tag{4.5}$$

Therefore, if $\mathcal{H}(x) = \{x + h_i\}_{i=1}^k$ is any admissible k-tuple, then for infinitely many n we have $|\mathcal{H}(n) \cap \mathbb{P}| \geqslant m + 1$, provided $k \geqslant 50$ in the case $m = 1$, and $k \geqslant \mathrm{e}^{4m+c}$ in general. On the Elliott–Halberstam conjecture this holds provided $k \geqslant 5$ in the case $m = 1$, and $k \geqslant \mathrm{e}^{2m+c}$ in general.

The key to extending Maynard–Tao in the way we require involves an extension of (4.2) in which the moduli q are all multiples of an integer q_0, which may be as large as a small power of N, but all of whose prime factors are relatively small. This extension of Bombieri–Vinogradov in turn requires a zero free region for the corresponding Dirichlet L-functions, given by the following lemma.

Lemma 4.1. *Let $T \geqslant 3$ and let $P \geqslant T^{1/\log_2 T}$. Among all primitive characters $\chi \bmod q$ to moduli q satisfying $q \leqslant T$ and $P^+(q) \leqslant P$, there is at most one for which $L(s,\chi)$ has a zero in the region*

$$\Re(s) > 1 - \frac{c}{\log P}, \quad |\Im(s)| \leqslant \exp\left(\log P/\sqrt{\log T}\right). \tag{4.6}$$

where c is a (sufficiently small) positive absolute constant. If such a character $\chi \bmod q$ exists, then χ is real, $L(s,\chi)$ has just one zero in the region (4.6), which is real and simple, and

$$P^+(q) \gg \log q \gg \log_2 T. \tag{4.7}$$

Proof outline. Lemma 4.1 follows from Chang's bound [**4**] for character sums to smooth moduli; the argument is somewhat standard and so we only give an outline of the proof.

If $\chi \bmod q$ is real and primitive then q is squarefree up to a factor of at most 4, so $\log q \ll \sum_{p\leqslant P^+(q)} \log p \ll P^+(q)$ by Chebyshev's bound. If β is any real zero of $L(s,\chi)$ then $1-\beta \gg 1/(\sqrt{q}(\log q)^2)$ [**5**, §14, (12)]. Hence (4.7).

If $\chi \bmod q$ is primitive and $\kappa\,(\log P^+(q) + \log q/\log_2 q) < \log u < \log q$, κ a sufficiently large absolute constant, then a result of Chang [**4**, Theorem 5] yields $\sum_{n\leqslant u}\chi(n) \ll u\mathrm{e}^{-\sqrt{\log u}}$. If $P^+(q) \leqslant P$, where $P \geqslant q^{1/\log_2 q}$, we can deduce that $L(\sigma+it,\chi) \ll (|t|+1)P^\eta \log P$ for $\sigma > 1 - \eta/(2\kappa)$, where $0 < \eta \leqslant 1/(2\sqrt{\log q})$. We do this by writing $L(s,\chi) = s\int_1^\infty u^{-s-1}(\sum_{n\leqslant u}\chi(n))\mathrm{d}u$, using Chang's bound for u in an applicable range, the Polya–Vinogradov bound for larger u and a trivial bound for smaller u.

Under the additional assumption $\eta \gg (\log_2 P)/\log P$, we can then show by standard calculations (see [**16**, Lemmas 10–12] for instance) that $L(\sigma+it,\chi)$ has no zeros for $\sigma > 1 - c_1\eta/\log((|t|+1)P^\eta)$ if χ is complex, and at most one zero in this region, necessarily real and simple, if χ is real. Moreover, we can show that for any distinct real primitive characters $\chi_1 \bmod q_1$ and $\chi_2 \bmod q_2$ (possibly $q_1 = q_2$), if $P^+(q) \leqslant P$, where $P \geqslant q^{1/\log_2 q}$ and $q = [q_1, q_2]$, and if β_1 and β_2 are real zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$, then $\min(\beta_1, \beta_2) \leqslant 1 - c_2/\log P$. (Here, c_1 and c_2 are a constants that are sufficiently small in terms of κ.) □

We fix an absolute constant c as in Lemma 4.1 and define

$$Z_T = P^+(q) \tag{4.8}$$

if such an exceptional modulus q exists, and $Z_T = 1$ if no such modulus exists. For clarity, we mention that Z_T actually depends (mildly) on $P \geqslant T^{1/\log_2 T}$ as well, but we do not indicate this dependence explicitly in our notation. For future reference, note that the bound (4.7) implies that, regardless of whether or not such a modulus exists, we have

$$\frac{Z_T}{\phi(Z_T)} = 1 + O\left(\frac{1}{\log_2 T}\right). \tag{4.9}$$

For the purposes of Theorem 4.2, for any function $\alpha : \mathbb{N} \to \mathbb{R}$ with finite support and any coprime pair of integers q and a, we define the "signed discrepancy"

$$\Delta(\alpha; q, a) = \sum_{n \equiv a \bmod q} \alpha(n) - \frac{1}{\phi(q)} \sum_n \alpha(n).$$

Thus, for instance,

$$\Delta(\Lambda \mathbf{1}_{(M,M+N]}; q, a) = \sum_{\substack{M < n \leqslant M+N \\ n \equiv a \bmod q}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{M < n \leqslant M+N} \Lambda(n).$$

Theorem 4.2 (Modified Bombieri–Vinogradov theorem). *Let $N > 2$. Fix any $C > 0$ and $\theta = 1/2 - \delta \in (0, 1/2)$. Fix $\epsilon > 0$ and suppose q_0 is a squarefree integer satisfying $q_0 < N^\epsilon$ and $P^+(q_0) < N^{\epsilon/\log_2 N}$. If $\epsilon = \epsilon(C, \delta, c)$ is sufficiently small in terms of C, δ and the constant c in Lemma 4.1, then with $Z_{N^{2\epsilon}}$ as in (4.8) we have*

$$\sum_{\substack{q < N^\theta \\ q_0 \mid q \\ (q, Z_{N^{2\epsilon}}) = 1}} \max_{0 \leqslant M \leqslant 3N} \max_{(q,a)=1} \left|\Delta(\Lambda \mathbf{1}_{(M,M+N]}; q, a)\right| \ll_{\delta,C} \frac{N}{\phi(q_0)(\log N)^C}. \tag{4.10}$$

The same result holds if, in (4.10), Λ is replaced by $\mathbf{1}_{\mathbb{P}}$ or by ϑ, where $\vartheta : \mathbb{N} \to \mathbb{R}$ is given by $\vartheta(n) = \mathbf{1}_{\mathbb{P}}(n) \log n$.

Proof. The result follows from standard zero density estimates combined with the zero free region for smooth moduli given in Lemma 4.1. We assume that $(q_0, Z_{N^{2\epsilon}}) = 1$, for otherwise the result is trivial. Let α be a placeholder for Λ, $\mathbf{1}_{\mathbb{P}}$ or ϑ. Let M be any number satisfying $0 \leqslant M \ll N$. First, we write

$$\sum_{\substack{M < n \leqslant M+N \\ n \equiv a \bmod q}} \alpha(n) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \sum_{M < n \leqslant M+N} \chi(n)\alpha(n).$$

Next, we replace $\sum_{M<n\leqslant M+N}\chi(n)\alpha(n)$ with $\sum_{M<n\leqslant M+N}\chi'(n)\alpha(n)$, where χ' is the primitive character that induces χ. Since $\alpha(n)\leqslant\Lambda(n)$ for all n, the error in making this change is at most

$$\sum_{q<N^\theta}\frac{1}{\phi(q)}\sum_{\chi \bmod q}\sum_{\substack{n\leqslant N\\(n,q)\neq 1}}\Lambda(n)\ll N^\theta(\log N)^2,$$

which is acceptable. Since $\sum_{M<n\leqslant M+N}\chi_0'(n)\alpha(n)=\sum_{M<n\leqslant M+N}\alpha(n)$ holds for the principal character χ_0 mod q, we need only bound

$$\sum_{\substack{q<N^\theta\\ q_0\,|\,q\\ (q,Z_{N^{2\epsilon}})=1}}\frac{1}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\Bigg|\sum_{M<n\leqslant M+N}\chi'(n)\alpha(n)\Bigg|. \tag{4.11}$$

For nonprincipal characters χ, the explicit formula [**5**, §19, (13)–(14)] yields

$$\begin{aligned}\Bigg|\sum_{M<n\leqslant M+N}\chi(n)\Lambda(n)\Bigg| &\leqslant \Bigg|\sum_{n\leqslant M+N}\chi(n)\Lambda(n)\Bigg|+\Bigg|\sum_{n\leqslant M}\chi(n)\Lambda(n)\Bigg|\\ &\ll \sum_{|\rho|<N^{1/2}}\frac{N^{\Re(\rho)}}{|\rho|}+O\big(N^{1/2}(\log qN)^2\big),\end{aligned}$$

where the sum is over nontrivial zeros of $L(s,\chi)$ with real part at least 1/2. (Here we have tacitly used our assumption that $M\ll N$.) The same bound holds if Λ is replaced by ϑ, because $\sum_{M<n\leqslant M+N}\chi(n)(\Lambda(n)-\vartheta(n))\ll N^{1/2}\log N$, and also if Λ is replaced by $\mathbf{1}_{\mathbb{P}}$, by partial summation. Thus, for nonprincipal characters χ and $M\ll N$, we have

$$\Bigg|\sum_{M<n\leqslant M+N}\chi(n)\alpha(n)\Bigg|\ll\sum_{|\rho|<N^{1/2}}\frac{N^{\Re(\rho)}}{|\rho|}+O\big(N^{1/2}(\log qN)^2\big), \tag{4.12}$$

where the sum is over nontrivial zeros of $L(s,\chi)$ with real part at least 1/2. The error term here makes a negligible contribution.

We substitute (4.12) into (4.11), and rewrite the summation in terms of the moduli q' of the primitive characters that are present. Thus, we need to bound

$$\begin{aligned}&\sum_{q'\leqslant N^\theta}\sideset{}{'}\sum_{\chi \bmod q'}\sum_{|\rho|<N^{1/2}}\frac{N^{\Re(\rho)}}{|\rho|}\sum_{\substack{q<N^\theta\\ [q_0,q']\,|\,q\\ (q,Z_{N^{2\epsilon}})=1}}\frac{1}{\phi(q)}\\ &\qquad\ll\frac{\log N}{\phi(q_0)}\sum_{a\,|\,q_0}\sum_{\substack{b<N^\theta/a\\ (b,Z_{N^{2\epsilon}})=1\\ p|(b,q_0)\Rightarrow p|a}}\frac{1}{\phi(b)}\sideset{}{'}\sum_{\chi \bmod ab}\sum_{|\rho|<N^{1/2}}\frac{N^{\Re(\rho)}}{|\rho|}.\end{aligned}$$

Here we have written $q'=ab$ with $a=(q',q_0)$; we use $\sum'$ to denote a sum restricted to primitive characters.

We cover the sum over a and b with $O((\log N)^2)$ dyadic ranges, and the sum over zeros with $O((\log N)^2)$ sums over zeros that satisfy

$$\Re(\rho)\in I_m=[1-m/\log N,1-(m-1)/\log N],\quad |\Im(\rho)|\in J_n=[n-1,2n],$$

where n runs over powers of 2. Hence, we are left to bound

$$\begin{aligned} \frac{(\log N)^5}{\phi(q_0)} \sup_{\substack{2m<\log N\\ 2n<N^{1/2}\\ A<q_0,\, AB<N^\theta}} \sum_{\substack{A\leqslant a<2A\\ B\leqslant b<2B\\ a\mid q_0,\, p\mid(b,q_0)\Rightarrow p\mid a\\ (b,Z_{N^{2\epsilon}})=1}} \frac{1}{\phi(b)} \sideset{}{'}\sum_{\chi \bmod ab} \sum_{\substack{\Re(\rho)\in I_m\\ |\Im(\rho)|\in J_n}} \frac{N^{\Re(\rho)}}{|\rho|} \\ \ll \frac{N(\log N)^6}{\phi(q_0)} \sup_{\substack{m<\log N\\ n<N^{1/2}\\ A<q_0,\, AB<N^\theta}} \frac{\mathrm{e}^{-m}}{nB} N^*\Big(1-\frac{m}{\log N}, A, B, n\Big), \end{aligned} \tag{4.13}$$

where

$$N^*(\sigma, A, B, T) = \sum_{\substack{A\leqslant a<2A\\ a\mid q_0}} \sum_{\substack{B\leqslant b<2B\\ (b,Z_{N^{2\epsilon}})=1\\ p\mid(b,q_0)\Rightarrow p\mid a}} \sideset{}{'}\sum_{\chi \bmod ab} \sum_{\substack{|\Im(\rho)|\leqslant T\\ \Re(\rho)\geqslant\sigma}} 1.$$

We first consider the range $m \geqslant C' \log_2 N$ where $C' = (C+15)/\delta$. Montgomery's estimate [**19**, Theorem 12.2] shows that

$$N^*(\sigma, A, B, T) \ll (A^2B^2T)^{3(1-\sigma)/(2-\sigma)}(\log(ABT))^9. \tag{4.14}$$

For $1/2 \leqslant \sigma \leqslant 1$, we have $1/(2-\sigma) \leqslant 1$, $6(1-\sigma)/(2-\sigma) \leqslant 1 + 2(1-\sigma)$ and $3(1-\sigma)/(2-\sigma) \leqslant 1$. For $4\epsilon \leqslant \delta$ we have $\log(A^6B^2) \leqslant \log N^{2\theta+4\epsilon} \leqslant (1-\delta)\log N$. Thus, (4.14) implies

$$N^*\Big(1-\frac{m}{\log N}, A, B, n\Big) \ll (\log N)^9 nB \exp(m(1-\delta)).$$

After using this bound, we see that the supremum in (4.13), when restricted to $m \geqslant C'\log_2 N$, occurs when $n = 1$, $m = C'\log_2 N$, $A = q_0$, $B = \log N$, and the overall contribution is $\ll \phi(q_0)^{-1}N(\log N)^{-C}$, as required.

We now consider the range $m \leqslant C'\log_2 N$. In this region (4.14) implies

$$N^*\Big(1-\frac{m}{\log N}, A, B, n\Big) \ll (\log N)^9 n^{1/2}B^{1/2} \exp\Big(6m\frac{\log A}{\log N}\Big).$$

After applying this bound, we see the supremum occurs at $m = 0$ (since $A^6 \leqslant N$), and then it is easy to see that if either $n \geqslant (\log N)^{2C'}$ or $B \geqslant (\log N)^{2C'}$ then the bound is acceptable. We therefore restrict our attention to $n, B < (\log N)^{2C'}$.

By Lemma 4.1, if χ mod q is primitive with $q < N^{2\epsilon}$ and $P^+(q) < N^{2\epsilon/\log_2 N^{2\epsilon}}$, then $L(s,\chi)$ has no zeros in the region

$$\Re(s) > 1 - c\frac{\log_2 N^{2\epsilon}}{\log N^{2\epsilon}}, \quad |\Im(s)| \leqslant \exp\left(\sqrt{\log N^{2\epsilon}}/\log_2 N^{2\epsilon}\right),$$

unless $(q, Z_{N^{2\epsilon}}) \neq 1$. If ϵ is sufficiently small in terms of C, δ and c, then this region covers the range $m \leqslant C'\log_2 N$, $n \leqslant (\log N)^{2C'}$. We are supposing that $q_0 < N^\epsilon$, $P^+(q_0) < N^{\epsilon/\log_2 N}$ and $B < (\log N)^{2C'}$, so for all remaining moduli $q' = ab$ we certainly have $q' < N^{2\epsilon}$ and $P^+(q') < N^{2\epsilon/\log_2 N^{2\epsilon}}$. Our assumptions also imply that $(q', Z_{N^{2\epsilon}}) = 1$. ☐

THEOREM 4.3. *Let m, k and $\epsilon = \epsilon(k)$ be fixed. If k is a sufficiently large multiple of $(8m+1)(8m^2+16m)$ and ϵ is sufficiently small, there is some $N(m,k,\epsilon)$ such* *that the following holds for all $N \geqslant N(m,k,\epsilon)$. With $Z_{N^{4\epsilon}}$ given by (4.8), let*

$$w = \epsilon \log N \quad \text{and} \quad W = \prod_{\substack{p \leqslant w \\ p \nmid Z_{N^{4\epsilon}}}} p. \tag{4.15}$$

Let $\mathcal{H} = \{h_1, \ldots, h_k\}$ be an admissible k-tuple such that

$$0 \leqslant h_1, \ldots, h_k \leqslant N \tag{4.16}$$

and

$$p \,\Big|\, \prod_{1 \leqslant i < j \leqslant k} (h_j - h_i) \implies p \leqslant w. \tag{4.17}$$

Let

$$\mathcal{H} = \mathcal{H}_1^{(1)} \cup \cdots \cup \mathcal{H}_{8m+1}^{(1)} = \mathcal{H}_1^{(2)} \cup \cdots \cup \mathcal{H}_{8m^2+16m}^{(2)} \tag{4.18}$$

be two partitions of $\mathcal{H}$ into $8m+1$ and $8m^2+16m$ sets of equal size respectively. Finally, let b be an integer such that

$$\Big(\prod_{i=1}^{k} (b + h_i), W \Big) = 1. \tag{4.19}$$

(i) There is some $n_1 \in (N, 2N]$ with $n_1 \equiv b \bmod W$, and some set of $m+1$ distinct indices $\{i_1^{(1)}, \ldots, i_{m+1}^{(1)}\} \subseteq \{1, \ldots, 8m+1\}$, such that

$$|\mathcal{H}_i^{(1)}(n_1) \cap \mathbb{P}| \geqslant 1 \quad \text{for all } i \in \{i_1^{(1)}, \ldots, i_{m+1}^{(1)}\}. \tag{4.20}$$

(ii) There is some $n_2 \in (N, 2N]$ with $n_2 \equiv b \bmod W$, and some set of $m+1$ indices $\{i_1^{(2)}, \ldots, i_{m+1}^{(2)}\} \subseteq \{1, \ldots, 8m^2+16m\}$, with $i_1^{(2)} < \cdots < i_{m+1}^{(2)}$, such that

$$\begin{aligned} |\mathcal{H}_i^{(2)}(n_2) \cap \mathbb{P}| &= 1 \quad \text{for all } i \in \{i_1^{(2)}, \ldots, i_{m+1}^{(2)}\}, \\ |\mathcal{H}_i^{(2)}(n_2) \cap \mathbb{P}| &\leqslant 1 \quad \text{for all } i_1^{(2)} < i < i_{m+1}^{(2)}. \end{aligned} \tag{4.21}$$

REMARK 4.4. In Theorem 4.3 (i) (respectively (ii)), k need only be a sufficiently large multiple of $8m+1$ (respectively $8m^2+16m$). We have combined the two hypotheses for conciseness. □

If we fix m, k and $\eta \in [0,1)$ with $M_{k,\eta} - 4m > 0$ (where $M_{k,\eta}$ is as in (4.3)), and if we assume the remaining hypotheses of Theorem 4.3 hold (disregarding (4.18)), then we can show, for all $N \geqslant N(m,k,\epsilon)$, that $|\mathcal{H}(n) \cap \mathbb{P}| \geqslant m+1$ for at least one $n \in (N, 2N]$ with $n \equiv b \bmod W$. This follows from an essentially identical argument to that presented in [**17**, **23**], although there are two differences in our setting that potentially affect the argument. Namely, w is considerably larger here than in [**17**] or [**23**] (we take $w = \epsilon \log N$ instead of $w = \log_3 N$), and the elements of $\mathcal{H}$ here may vary with N.

However, this actually only leads to weaker versions of Theorems 1.1 and 1.3, for instance (cf. (4.4), (4.5)) with $k = 50$ instead of $k = 9$ in our main theorem (Theorem 1.1), which is concerned with the case $m = 1$ of Theorem 4.3. The proof of Theorem 4.3, given in Section 4.2, does not require such refined estimates as in [**17**, **23**], but does require an additional sieve upper bound, whose use had been considered by the authors of [**23**].

We remark that with more significant modifications to the argument presented in [**17**, **23**], it is in principle possible to remove the requirement (4.17) from the statement of Theorem 4.3. We do not consider this here.

4.2. **Key estimates.** Throughout this section (including Lemmas 4.5 – 4.7): k is fixed; δ and ϵ are fixed and satisfy $0 < 24\epsilon < 12\delta < 1$; N is to be thought of as tending to infinity, hence is sufficiently large in terms of any fixed quantity; implicit constants may depend on any fixed quantity (though our notation will not indicate this explicitly); $Z_{N^{4\epsilon}}$ is given by (4.8); w, W, $\mathcal{H} = \{h_1, \dots, h_k\}$ and b are as in (4.15) – (4.19). Note for future reference that by our choice of δ and ϵ, we have $1/4 - 3\delta > 0$ and, since $W < N^{2\epsilon}$ by the prime number theorem, $N^{2\delta} W < N^{1/2-2\delta} W < N^{1/2-\delta}$.

Also, $\lambda_{d_1,\dots,d_k}$ are sieve weights given by

$$\lambda_{d_1,\dots,d_k} = \begin{cases} \big(\prod_{i=1}^k \mu(d_i)\big) \sum_{j=1}^J \prod_{\ell=1}^k F_{\ell,j}\big(\frac{\log d_\ell}{\log N}\big) & \text{if } (d_1 \cdots d_k, Z_{N^{4\epsilon}}) = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{4.22}$$

for some fixed J and fixed smooth nonnegative compactly supported functions $F_{\ell,j} : [0, \infty+) \to \mathbb{R}$ that are not identically 0 and that satisfy the support restriction

$$\sup\big\{ \textstyle\sum_{\ell=1}^k t_\ell : \prod_{\ell=1}^k F_{\ell,j}(t_\ell) \neq 0 \big\} \leqslant \delta,$$

for each $j \in \{1, \dots, J\}$. As we will see, the presence of the J-sum in (4.22) is imposed by the Stone–Weierstrass theorem. The above support condition implies $\lambda_{d_1,\dots,d_k}$ is supported on d_i with $\prod_{i=1}^k d_i \leqslant N^\delta$. The fact that J, $F_{\ell,j}$ are fixed means we have the bound $\lambda_{d_1,\dots,d_k} \ll 1$ uniformly in the d_i. To ease notation we put

$$F(t_1, \dots, t_k) = \sum_{j=1}^J \prod_{\ell=1}^k F'_{\ell,j}(t_\ell),$$

$F'_{\ell,j}$ denoting the derivative of $F_{\ell,j}$, and we assume that we have chosen the $F_{\ell,j}$ such that F is symmetric. Also, we put

$$B = \frac{\phi(W)}{W} \log N.$$

Lemma 4.5. *If $F_1, \dots, F_k, G_1, \dots, G_k : [0, \infty+) \to \mathbb{R}$ are fixed smooth compactly supported functions, then*

$$\sideset{}{'}\sum_{\substack{d_1,\dots,d_k \\ d'_1,\dots,d'_k}} \prod_{j=1}^k \frac{\mu(d_j)\mu(d'_j)}{[d_j, d'_j]} F_j\left(\frac{\log d_j}{\log N}\right) G_j\left(\frac{\log d'_j}{\log N}\right) = (c + o(1)) B^{-k},$$

where $\sum'$ denotes summation with the restriction that $[d_1, d'_1], \dots, [d_k, d'_k], W Z_{N^{4\epsilon}}$ are pairwise coprime, and

$$c = \prod_{j=1}^k \int_0^\infty F'_j(t_j) G'_j(t_j) \mathrm{d}t_j.$$

The same holds if the denominators $[d_j, d'_j]$ are replaced by $\phi([d_j, d'_j])$.

Proof. This is [**23**, Lemma 4.1] combined with the fact that, by (4.9),

$$(Z_{N^{4\epsilon}}/\phi(Z_{N^{4\epsilon}}))^k = 1 + o(1)\,.$$

□

We may now prove the main estimates of the Maynard–Tao sieve method. To state the estimates we define

$$I_k(F) = \int_0^\infty \cdots \int_0^\infty F(t_1, \ldots, t_k)^2 \mathrm{d}t_1 \ldots \mathrm{d}t_k,$$

$$J_k(F) = \int_0^\infty \cdots \int_0^\infty \left(\int_0^\infty F(t_1, \ldots, t_k) \mathrm{d}t_k \right)^2 \mathrm{d}t_1 \ldots \mathrm{d}t_{k-1}$$

and

$$L_k(F) = \int_0^\infty \cdots \int_0^\infty \left(\int_0^\infty \int_0^\infty F(t_1, \ldots, t_k) \mathrm{d}t_{k-1} \mathrm{d}t_k \right)^2 \mathrm{d}t_1 \ldots \mathrm{d}t_{k-2}.$$

Lemma 4.6. *(i) We have*

$$\sum_{\substack{N<n\leqslant 2N \\ n\equiv b \bmod W}} \Bigg(\sum_{\substack{d_1,\ldots,d_k \\ d_i \mid n+h_i}} \lambda_{d_1,\ldots,d_k} \Bigg)^2 = (1 + o(1)) \frac{N}{W} B^{-k} I_k(F).$$

(ii) For each $j \in \{1, \ldots, k\}$, we have

$$\sum_{\substack{N<n\leqslant 2N \\ n\equiv b \bmod W}} \mathbf{1}_{\mathbb{P}}(n + h_j) \Bigg(\sum_{\substack{d_1,\ldots,d_k \\ d_i \mid n+h_i}} \lambda_{d_1,\ldots,d_k} \Bigg)^2 = (1 + o(1)) \frac{N}{W} B^{-k} J_k(F).$$

(iii) For each pair $j, \ell \in \{1, \ldots, k\}$, $j \neq \ell$, we have

$$\sum_{\substack{N<n\leqslant 2N \\ n\equiv b \bmod W}} \mathbf{1}_{\mathbb{P}}(n + h_j) \mathbf{1}_{\mathbb{P}}(n + h_\ell) \Bigg(\sum_{\substack{d_1,\ldots,d_k \\ d_i \mid n+h_i}} \lambda_{d_1,\ldots,d_k} \Bigg)^2 \leqslant (4 + O(\delta)) \frac{N}{W} B^{-k} L_k(F).$$

Proof. (i) We expand the square and swap the order of summation to obtain

$$\sum_{\substack{N<n\leqslant 2N \\ n\equiv b \bmod W}} \Bigg(\sum_{\substack{d_1,\ldots,d_k \\ d_i \mid n+h_i}} \lambda_{d_1,\ldots,d_k} \Bigg)^2 = \sum_{\substack{d_1,\ldots,d_k \\ d'_1,\ldots,d'_k}} \lambda_{d_1,\ldots,d_k} \lambda_{d'_1,\ldots,d'_k} \sum_{\substack{N<n\leqslant 2N \\ n\equiv b \bmod W \\ [d_i,d'_i] \mid n+h_i}} 1.$$

By our choice of b, there is no contribution to the inner sum unless all the d_i and d'_i are coprime to W. By the restriction on the support of $\lambda_{d_1,\ldots,d_k}$, there is no contribution unless all the d_i, d'_i are coprime to $Z_{N^{4\epsilon}}$. Since p does not divide $\prod_{i\neq j}(h_i - h_j)$ unless $p \leqslant w$, we see that there is no contribution unless all of $[d_1, d'_1], \ldots, [d_k, d'_k]$ are pairwise coprime. If all these conditions are satisfied then the inner sum is equal to

$$\frac{N}{W \prod_{i=1}^k [d_i, d'_i]} + O(1)\,.$$

Since $\lambda_{d_1,\ldots,d_k} \ll 1$ and is supported on $\prod_{i=1}^k d_i \leqslant N^\delta$, we see that the error term trivially contributes $O\big(N^{2\delta+o(1)}\big)$, which is negligible.

Expanding $\lambda_{d_1,\dots,d_k}$ using the definition (4.22), we see that the main term contributes

$$\frac{N}{W}\sum_{j=1}^{J}\sum_{j'=1}^{J}\ \sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ d_1',\dots,d_k'}}\ \prod_{\ell=1}^{k}\frac{\mu(d_\ell)\mu(d_\ell')}{[d_\ell,d_\ell']}F_{\ell,j}\left(\frac{\log d_\ell}{\log N}\right)F_{\ell,j'}\left(\frac{\log d_\ell'}{\log N}\right),$$

where $\sum'$ signifies pairwise coprimality of $[d_1,d_1'],\dots,[d_k,d_k'],WZ_{N^{4\epsilon}}$. The inner sum can be estimated by Lemma 4.5, which gives the result.

(ii) The argument here is similar. For ease of notation we will consider $j=k$, the other cases being entirely analogous. There is no contribution to the sum unless $d_k=1$. With this restriction, we expand the square and swap the order of summation to obtain

$$\sum_{\substack{N<n\leqslant 2N\\ n\equiv b \bmod W}}\mathbf{1}_{\mathbb{P}}(n+h_j)\Bigg(\sum_{\substack{d_1,\dots,d_{k-1}\\ d_i\mid n+h_i}}\lambda_{d_1,\dots,d_{k-1},1}\Bigg)^2$$
$$=\sum_{\substack{d_1,\dots,d_{k-1}\\ d_1',\dots,d_{k-1}'}}\lambda_{d_1,\dots,d_{k-1},1}\lambda_{d_1',\dots,d_{k-1}',1}\sum_{\substack{N<n\leqslant 2N\\ n\equiv b \bmod W\\ [d_i,d_i']\mid n+h_i}}\mathbf{1}_{\mathbb{P}}(n+h_j).$$

As in (i), we may assume pairwise coprimality of $[d_1,d_1'],\dots,[d_{k-1},d_{k'-1}],WZ_{N^{4\epsilon}}$, in which case the inner sum is equal to

$$\frac{\pi(2N+h_j)-\pi(N+h_j)}{\phi(W)\prod_{i=1}^{k-1}\phi([d_i,d_i'])}+O(E(N;[d_1,d_1']\cdots[d_{k-1},d_{k'-1}]W))\,,$$

where

$$E(N;q)=\max_{\substack{(a,q)=1\\ h\in\mathcal{H}}}\left|\pi(2N+h;q,a)-\pi(N+h;q,a)-\frac{\pi(2N+h)-\pi(N+h)}{\phi(q)}\right|.$$

By the bound $\lambda_{d_1,\dots,d_{k-1},1}\ll 1$, the trivial bound $E(N;q)\ll 1+N/\phi(q)$, the Cauchy–Schwarz inequality and Theorem 4.2, the error contributes

$$\sideset{}{'}\sum_{\substack{d_1,\dots,d_{k-1}\\ d_1',\dots,d_{k-1}'}}|\lambda_{d_1,\dots,d_{k-1},1}\lambda_{d_1',\dots,d_{k-1}',1}|E(N;W[d_1,d_1']\cdots[d_{k-1},d_{k'-1}])$$
$$\ll\sum_{\substack{r\leqslant N^{2\delta}\\ (r,WZ_{N^{4\epsilon}})=1}}\mu(r)^2\tau_{3k}(r)E(N;rW)$$
$$\ll\Bigg(\sum_{\substack{r\leqslant N^{2\delta}\\ (r,WZ_{N^{4\epsilon}})=1}}\mu(r)^2\tau_{3k}(r)^2(1+N/\phi(rW))\Bigg)^{1/2}\Bigg(\sum_{\substack{r\leqslant N^{2\delta}\\ (r,WZ_{N^{4\epsilon}})=1}}\mu(r)^2E(N;rW)\Bigg)^{1/2}$$
$$\ll\frac{N}{W(\log N)^{2k}}.$$

As in (i), expanding $\lambda_{d_1,\dots,d_k}$ using the definition (4.22) and applying Lemma 4.5 to the resulting sums shows that the main term contributes

$$(1+o(1))\frac{N}{W}B^{-k}\sum_{j=1}^{J}\sum_{j'=1}^{J}F_{k,j}(0)F_{k,j'}(0)\prod_{\ell=1}^{k-1}\int_0^\infty F_{\ell,j}'(t_\ell)F_{\ell,j'}'(t_\ell)\mathrm{d}t_\ell.$$

Noting that the double sum is $J_k(F)$ and that assumed symmetry of F means that the expression is independent of $j \in \{1, \dots, k\}$, this gives the result.

(iii) As in (ii), we see there is no contribution unless $d_j = d_\ell = 1$. We first impose this restriction, and then use the sieve upper bound

$$\mathbf{1}_{\mathbb{P}}(n+h_\ell) \leqslant \Big(\sum_{e|n+h_\ell} \mu(e) G\Big(\frac{\log e}{\log N}\Big)\Big)^2,$$

for a smooth function $G : [0, \infty+) \to \mathbb{R}$ supported on $[0, 1/4-3\delta]$, with $G(0) = 1$. (The use of such a bound was previously suggested in discussions of the Polymath 8b project.) Thus, we have

$$\sum_{\substack{N<n\leqslant 2N \\ n\equiv b \bmod W}} \mathbf{1}_{\mathbb{P}}(n+h_j)\mathbf{1}_{\mathbb{P}}(n+h_\ell)\Big(\sum_{\substack{d_1,\dots,d_k \\ d_i|n+h_i}} \lambda_{d_1,\dots,d_k}\Big)^2$$
$$\leqslant \sum_{\substack{N<n\leqslant 2N \\ n\equiv b \bmod W}} \mathbf{1}_{\mathbb{P}}(n+h_j)\Big(\sum_{e|n+h_\ell} \mu(e) G\Big(\frac{\log e}{\log N}\Big)\Big)^2\Big(\sum_{\substack{d_1,\dots,d_k \\ d_i|n+h_i \\ d_j=d_\ell=1}} \lambda_{d_1,\dots,d_k}\Big)^2.$$

The right-hand side of this expression is now of essentially an identical form to that of part (ii), with F replaced by $\tilde{F}$, where

$$\tilde{F}(t_1, \dots, t_k) = G'(t_\ell)\int_0^\infty F(t_1, \dots, t_{\ell-1}, u_\ell, t_{\ell+1}, \dots, t_k)\mathrm{d}u_\ell.$$

(The cases where $j \geqslant \ell - 1$ are analogous.) We note that $\tilde{F}$ is supported on $t_1, \dots, t_k$ such that $\sum_{i=1}^k t_i \leqslant 1/4 - 2\delta$, by the support of F and G. This means we can still apply Theorem 4.2 as in (ii) (since we may restrict to arithmetic progressions modulo rW, where $r = [d_1, d_1'] \cdots [d_k, d_k'][e_\ell, e_\ell'] \leqslant N^{1/2-2\delta}$). Therefore the same argument as in (ii) gives

$$\sum_{\substack{N<n\leqslant 2N \\ n\equiv b \bmod W}} \mathbf{1}_{\mathbb{P}}(n+h_j)\Big(\sum_{e|n+h_\ell} \mu(e) G\Big(\frac{\log e}{\log N}\Big)\Big)^2\Big(\sum_{\substack{d_1,\dots,d_k \\ d_i|n+h_i \\ d_j=d_\ell=1}} \lambda_{d_1,\dots,d_k}\Big)^2$$
$$= (1+o(1))\frac{N}{W}B^{-k}\int_0^\infty \cdots \int_0^\infty \Big(\int_0^\infty \tilde{F}(t_1, \dots, t_k)\mathrm{d}t_j\Big)^2 \mathrm{d}t_1 \dots \mathrm{d}t_{j-1}\mathrm{d}t_{j+1} \dots \mathrm{d}t_k$$
$$= (1+o(1))\frac{N}{W}B^{-k}L_k(F)\int_0^\infty G'(t_\ell)^2 \mathrm{d}t_\ell.$$

Finally, we take $G(t)$ to be a fixed smooth approximation to $1 - t/(1/4 - 3\delta)$ supported on $[0, 1/4 - 3\delta]$ with $G(0) = 1$ and $\int_0^\infty G'(t)^2\mathrm{d}t \leqslant 4 + O(\delta)$. This gives the result. □

LEMMA 4.7. *Let $0 < \rho < 1$. Then there is a fixed choice of J and $F_{\ell,j}$ for $\ell \in \{1,\dots,k\}$, $j \in \{1,\dots,J\}$, with the required properties such that*

$$J_k(F) \geqslant \big(1 + O\big((\log k)^{-1/2}\big)\big)\left(\frac{\rho\delta\log k}{k}\right)I_k(F),$$
$$L_k(F) \leqslant \big(1 + O\big((\log k)^{-1/2}\big)\big)\left(\frac{\rho\delta\log k}{k}\right)^2 I_k(F).$$

Proof. This follows from the method of [**17**, Proposition 4.3]. The result is trivial if k is bounded, so we assume that k is sufficiently large. Let $F_k = F_k(t_1,\dots,t_k)$ be defined by

$$F_k(t_1,\dots,t_k) = \begin{cases} \prod_{i=1}^k g(kt_i) & \text{if } \sum_{i=1}^k t_i \leqslant 1,\\ 0 & \text{otherwise,}\end{cases}$$
$$g(t) = \begin{cases} 1/(1+At) & \text{if } t \in [0,T],\\ 0 & \text{otherwise,}\end{cases}$$
$$A = \log k - 2\log\log k,$$
$$T = (\mathrm{e}^A - 1)/A.$$

The proof of [**17**, Proposition 4.3] shows that

$$J_k(F_k) \geqslant \big(1 + O\big((\log k)^{-1/2}\big)\big)\,(\log k) I_k(F_k)/k.$$

We now consider $L_k(F_k)$. By Cauchy-Schwarz, we have for $0 \leqslant x \leqslant \log k$

$$\left(\int_{t_1+t_2\leqslant x} g(t_1)g(t_2)\mathrm{d}t_1\mathrm{d}t_2\right)^2 \leqslant \frac{x^2}{2}\int_{t_1+t_2\leqslant x} g(t_1)^2 g(t_2)^2 \mathrm{d}t_1\mathrm{d}t_2$$
$$\leqslant (\log k)^2 \int_{t_1+t_2\leqslant x} g(t_1)^2 g(t_2)^2 \mathrm{d}t_1\mathrm{d}t_2. \qquad (4.23)$$

If instead $x \geqslant \log k$, then putting $y = \min(x,T)$ and noting $\log(1+Ay) \leqslant A$, we have

$$\int_{t_1+t_2\leqslant x} g(t_1)^2 g(t_2)^2 \mathrm{d}t_1\mathrm{d}t_2 \geqslant \int_{t_1+t_2\leqslant y} g(t_1)^2 g(t_2)^2 \mathrm{d}t_1\mathrm{d}t_2$$
$$= \frac{1}{A^2} - \frac{2}{A^2(2+Ay)} - \frac{2\log(1+Ay)}{A^2(2+Ay)^2}$$
$$\geqslant \frac{1}{A^2} - \frac{4}{A^3 y}$$
$$\geqslant \frac{1}{(\log k)^2}$$

if k is sufficiently large. Since the integral of $g(t)$ over $[0,\infty]$ is 1, we therefore see that (4.23) holds for all $x \geqslant 0$. Hence

$$\begin{aligned}
L_k(F_k) &= \underset{\sum_{i=1}^{k-2} t_i \leqslant 1}{\int \cdots \int} \Big(\prod_{i=1}^{k-2} g(kt_i)^2 \Big) \\
&\quad \times \Big(\int_{t_{k-1}+t_k \leqslant 1-\sum_{i=1}^{k-2} t_i} g(kt_{k-1}) g(kt_k) \mathrm{d}t_{k-1} \mathrm{d}t_k \Big)^2 \mathrm{d}t_1 \dots \mathrm{d}t_{k-2} \\
&\leqslant \Big(\frac{\log k}{k} \Big)^2 \underset{\sum_{i=1}^{k-2} t_i \leqslant 1}{\int \cdots \int} \Big(\prod_{i=1}^{k-2} g(kt_i)^2 \Big) \\
&\quad \times \Big(\int_{t_{k-1}+t_k \leqslant 1-\sum_{i=1}^{k-2} t_i} g(kt_{k-1})^2 g(kt_k)^2 \mathrm{d}t_{k-1} \mathrm{d}t_k \Big) \mathrm{d}t_1 \dots \mathrm{d}t_{k-2} \\
&= \Big(\frac{\log k}{k} \Big)^2 I_k(F_k).
\end{aligned}$$

By the Stone–Weierstrass theorem we can take $F(t_1, \dots, t_k)$ to be a smooth approximation to $F_k(t_1/(\rho\delta), \dots, t_k/(\rho\delta))$ such that

$$\begin{aligned}
I_k(F) &= (\delta\rho)^k \left(1 + O\big((\log k)^{-1/2}\big)\right) I_k(F_k), \\
J_k(F) &= (\delta\rho)^{k+1} \left(1 + O\big((\log k)^{-1/2}\big)\right) J_k(F_k)
\end{aligned}$$

and

$$L_k(F) = (\delta\rho)^{k+2} \left(1 + O\big((\log k)^{-1/2}\big)\right) L_k(F_k).$$

This gives the result. □

Deduction of Theorem 4.3. We first consider part (i). We suppose k is a multiple of $8m+1$ and

$$\mathcal{H} = \mathcal{H}_1^{(1)} \cup \cdots \cup \mathcal{H}_{8m+1}^{(1)}$$

is a partition of $\mathcal{H}$ into $8m+1$ sets each of size $k/(8m+1)$. We consider

$$\begin{aligned}
S = \sum_{\substack{N < n \leqslant 2N \\ n \equiv b \bmod W}} \Big(\sum_{i=1}^{k} \mathbf{1}_{\mathbb{P}}(n+h_i) - m - \frac{1}{2} \sum_{j=1}^{8m+1} \sum_{\substack{h,h' \in \mathcal{H}_j^{(1)} \\ h \neq h'}} \mathbf{1}_{\mathbb{P}}(n+h) \mathbf{1}_{\mathbb{P}}(n+h') \Big) \\
\times \Big(\sum_{\substack{d_1,\dots,d_k \\ d_i | n+h_i \, \forall i}} \lambda_{d_1,\dots,d_k} \Big)^2 .
\end{aligned}$$

We note that if $S > 0$ then there must be at least one n that makes a positive contribution to the sum, and this occurs only when there exists $m+1$ elements $h_1', \dots, h_{m+1}'$ of $\mathcal{H}$ each in different subsets $\mathcal{H}_i^{(1)}$ such that $n + h_j'$ is prime for all $1 \leqslant j \leqslant m+1$. By Lemmas 4.6 and 4.7, we see that for $k > k_0(m, \delta)$, by choosing $\rho < 1$ such that $\delta\rho \log k = 2m$ there exists a choice of F such that

$$S = \frac{N}{W} B^{-k} I_k(F) \Bigg(\sum_{i=1}^{k} \frac{2m}{k} - m - 2 \sum_{j=1}^{8m+1} \sum_{\substack{h,h' \in \mathcal{H}_j^{(1)} \\ h \neq h'}} \frac{(2m)^2}{k^2} + O(\delta) \Bigg)$$
$$= \frac{N}{W} B^{-k} I_k(F) \Bigg(\frac{m}{8m+1} + \frac{8m^2}{k} + O(\delta) \Bigg).$$

Thus, $S > 0$ for δ sufficiently small, as required.

Part (ii) follows from an essentially identical argument. Given a partition

$$\mathcal{H} = \mathcal{H}_1^{(2)} \cup \cdots \cup \mathcal{H}_{8m^2+16m}^{(2)}$$

of $\mathcal{H}$ into equally sized sets, we consider

$$S' = \sum_{\substack{N < n \leqslant 2N \\ n \equiv b \bmod W}} \Bigg(\sum_{i=1}^{k} \mathbf{1}_{\mathbb{P}}(n+h_i) - m - \frac{(m+2)}{2} \sum_{j=1}^{8m^2+16m} \sum_{\substack{h,h' \in \mathcal{H}_j^{(2)} \\ h \neq h'}} \mathbf{1}_{\mathbb{P}}(n+h) \mathbf{1}_{\mathbb{P}}(n+h') \Bigg) \times \Bigg(\sum_{\substack{d_1,\dots,d_k \\ d_i | n+h_i \, \forall i}} \lambda_{d_1,\dots,d_k} \Bigg)^2.$$

If n makes a positive contribution to S' then we must have that the number of indices j for which $|\mathcal{H}_j^{(2)}(n) \cap \mathbb{P}| = 1$ is at least $m + 1 + mr$, where r is the number of indices i for which $|\mathcal{H}_i^{(2)}(n) \cap \mathbb{P}| > 1$. Thus in particular, there must be some set of $m+1$ indices $i_1 < \cdots < i_{m+1}$ for which $|\mathcal{H}_i^{(2)}(n) \cap \mathbb{P}| = 1$ for $i = i_1, \dots, i_{m+1}$, and $|\mathcal{H}_i^{(2)}(n) \cap \mathbb{P}| = 0$ for $i_1 < i < i_{m+1}$ and $i \neq i_1, \dots, i_{m+1}$. Applying Lemmas 4.6 and 4.7 and choosing $\delta\rho \log k = 2m$ as above, we find that $S' > 0$ for δ sufficiently small and N sufficiently large, so such an n must exist. ∎

5. An Erdős–Rankin type construction

We give our Erdős–Rankin type construction in Lemma 5.2. We need the following elementary lemma.

Lemma 5.1. *Let $\{h_1, \dots, h_k\}$ be an admissible k-tuple, let S be a set of integers, and let $\mathscr{P}$ be a set of primes, such that for some $x \geqslant 2$,*

$$\{h_1, \dots, h_k\} \subseteq S \subseteq [0, x^2] \qquad \text{and} \qquad |\{p \in \mathscr{P} : p > x\}| > |S| + k.$$

There is a set of integers $\{a_p : p \in \mathscr{P}\}$ with the property that

$$\{h_1, \dots, h_k\} = S \setminus \textstyle\bigcup_{p \in \mathscr{P}} \{g : g \equiv a_p \bmod p\}.$$

Proof. First, we observe the following. Let $\{h_1, \dots, h_k\}$ be an admissible k-tuple, let $\mathscr{P}_0 \subseteq \mathscr{P}$ be sets of primes, and let $\{a_p : p \in \mathscr{P}_0\}$ be a set of integers. If

$$\{h_1, \dots, h_k\} = S \setminus \textstyle\bigcup_{p \in \mathscr{P}_0} \{g : g \equiv a_p \bmod p\},$$

then we can add integers to $\{a_p : p \in \mathscr{P}_0\}$ to form a set $\{a_p : p \in \mathscr{P}\}$ such that

$$\{h_1, \ldots, h_k\} = S \backslash \bigcup_{p \in \mathscr{P}} \{g : g \equiv a_p \bmod p\}.$$

Indeed, since $\{h_1, \ldots, h_k\}$ is admissible, for every prime p there is a congruence class $b_p \bmod p$ for which $\prod_{i=1}^{k}(b_p - h_i) \not\equiv 0 \bmod p$, so for any $p \in \mathscr{P} \backslash \mathscr{P}_0$ we can choose a_p with $a_p \equiv b_p \bmod p$.

Second, we observe that for any given integer n, if

$$\prod_{i=1}^{k}(n - h_i) \equiv 0 \bmod p$$

for every prime p in a set $\mathscr{P}_0$ of $k + 1$ or more distinct primes, then

$$n - h_i \equiv 0 \bmod pp'$$

for some $h_i \in \{h_1, \ldots, h_k\}$ and $p, p' \in \mathscr{P}_0$, so either $n - h_i = 0$ or $|n - h_i| \geqslant pp'$. Therefore, if $0 \leqslant n, h_1, \ldots, h_k \leqslant x^2$, $n \notin \{h_1, \ldots, h_k\}$, and $\mathscr{P}_0$ is any set of primes at least $k + 1$ of which are greater than x, there must be a prime $p \in \mathscr{P}_0$ such that

$$\prod_{i=1}^{k}(n - h_i) \not\equiv 0 \bmod p.$$

Now, let $\{h_1, \ldots, h_k\}$ be an admissible k-tuple contained in $S \subseteq [0, x^2]$, and let $\mathscr{P}$ be any set of primes such that $|\{p \in \mathscr{P} : p > x\}| \geqslant |S| + k + 1$. By our first observation it suffices to show that

$$\{h_1, \ldots, h_k\} = S \backslash \bigcup_{p \in \mathscr{P}_0} \{g : g \equiv a_p \bmod p\},$$

for some $\mathscr{P}_0 \subseteq \mathscr{P}$. Suppose $n \in S \backslash \{h_1, \ldots, h_k\}$. By our second observation we may choose a prime $p \in \mathscr{P}$ such that $\prod_{i=1}^{k}(n - h_i) \not\equiv 0 \bmod p$. Choose any such prime p and choose any a_p with $a_p \equiv n \bmod p$. Let $S_1 = S \backslash \{g : g \equiv a_p \bmod p\}$, so that $n \notin S_1$, and let $\mathscr{P}_1 = \mathscr{P} \backslash \{p\}$. If $S_1 = \{h_1, \ldots, h_k\}$ then we're done. Otherwise, we have $\{h_1, \ldots, h_k\} \subsetneq S_1$ and $|\mathscr{P}_1| = |\mathscr{P}| - 1 \geqslant |S| + k \geqslant |S_1| + k + 1$. We repeat the above argument as many times as necessary. □

To prove Lemma 5.2 we also need some standard estimates. First, we use Mertens' theorem in the following forms (see [**27**, Theorems I.5.9 and I.6.11]). For $x \geqslant 2$,

$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + c_0 + O\big((\log x)^{-1}\big)\,, \tag{5.1}$$

where $c_0 = 0.2614\ldots$, and

$$\prod_{p \leqslant x}\left(1 - \frac{1}{p}\right) = \frac{\mathrm{e}^{-\gamma}}{\log x}\left(1 + O\!\left(\frac{1}{\log x}\right)\right), \tag{5.2}$$

where $\gamma = 0.5772\ldots$ is the Euler–Mascheroni constant. Second, we use a bound for the number of y-smooth numbers less than or equal to x, that is for

$$\Psi(x, y) = |\{n \leqslant x : p \mid n \implies p \leqslant y\}|.$$

Namely, as a consequence of [**27**, Theorem III.5.1], we have

$$\Psi(x, y) \ll x(\log x)^{-1} \qquad (1 \leqslant 2\log y \leqslant (\log x)(\log_2 x)^{-1}). \tag{5.3}$$

Third, we use the prime number theorem for arithmetic progressions in the following form due to Page (see [**5**, §20, (13)] and also the proof of (4.7) above). Let

c be any positive constant. There is a positive constant c', which is determined by c, such that

$$\sum_{\substack{x<p\leqslant x+y\\ p\equiv a \bmod q}} \log p = \frac{y}{\phi(q)} + O\Big(x\exp(-c'\sqrt{\log x})\Big) \tag{5.4}$$

uniformly for $2 \leqslant y \leqslant x$, $q \leqslant \exp\big(c\sqrt{\log x}\big)$ and $(q,a)=1$, except possibly if q is a multiple of a certain integer q_1 depending on x which, if it exists, satisfies $P^+(q_1) \gg \log_2 x$ (the implicit constant also determined by c).

Lemma 5.2. *Fix an integer $k \geqslant 1$ and real numbers $\beta_k \geqslant \cdots \geqslant \beta_1 \geqslant 0$. There is a number $y(\boldsymbol{\beta},k)$, depending only on $\beta_1,\dots,\beta_k$ and k, such that the following holds. Let x,y,z be any numbers satisfying $x \geqslant 1$, $y \geqslant y(\boldsymbol{\beta},k)$ and*

$$2y(1+(1+\beta_k)x) \leqslant 8z \leqslant y(\log_2 y)(\log_3 y)^{-1}. \tag{5.5}$$

Let $\mathcal{Z}$ be any (possibly empty) set of primes such that for all $p' \in \mathcal{Z}$,

$$\sum_{\substack{p\in\mathcal{Z}\\ p\geqslant p'}} \frac{1}{p} \ll \frac{1}{p'} \ll \frac{1}{\log z}. \tag{5.6}$$

There is a set $\{a_p : p \leqslant y,\, p \notin \mathcal{Z}\}$ and an admissible k-tuple $\{h_1,\dots,h_k\}$ such that

$$\{h_1,\dots,h_k\} = ((0,z]\cap\mathbb{Z})\setminus \textstyle\bigcup_{p\leqslant y,\, p\notin\mathcal{Z}}\{g : g\equiv a_p \bmod p\}. \tag{5.7}$$

Moreover, for $1 \leqslant i < j \leqslant k$,

$$p \mid (h_j - h_i) \implies p \leqslant y, \tag{5.8}$$

and for $1 \leqslant i \leqslant k$,

$$h_i = \beta_i xy + y + O\big(y\mathrm{e}^{-(\log y)^{1/4}}\big). \tag{5.9}$$

Proof. Let y_1, y_2, y and z be numbers such that

$$2 < y_1 < y_2 < y < z < y_1y_2 \quad \text{and} \quad 2\log y_1 \leqslant (\log z)(\log_2 z)^{-1}. \tag{5.10}$$

Let $\mathcal{Z}$ be any set of primes satisfying (5.6). We assume that $2 \notin \mathcal{Z}$ (which follows from (5.6) if y [and hence z] is large enough). Let

$$P_1 = \prod_{\substack{2<p\leqslant y_1\\ p\notin\mathcal{Z},\, p\neq\ell}} p, \quad P_2 = \prod_{\substack{y_1<p\leqslant y_2\\ p\notin\mathcal{Z}}} p, \quad P_3 = \prod_{\substack{y_2<p\leqslant y\\ p\notin\mathcal{Z}}} p,$$

where in the definition of P_1, ℓ is a prime satisfying $\ell \gg \log y_1$. (We will eventually specify ℓ according to (5.4), but for the time being it can be treated as arbitrary.) It is important to note that $2 \nmid P_1$.

We record three bounds related to $\mathcal{Z}$, which all follow from (5.6). First, using the notation $(n,\mathcal{Z}) \neq 1$ to indicate that $p \mid n$ for some $p \in \mathcal{Z}$, we have

$$\sum_{\substack{n\leqslant z\\ (n,\mathcal{Z})\neq 1}} 1 \leqslant \sum_{p\in\mathcal{Z}} \left[\frac{z}{p}\right] \ll \frac{z}{\log z}. \tag{5.11}$$

Second, we have

$$\sum_{p\in\mathcal{Z}} \log\left(\frac{p}{p-1}\right) \leqslant \sum_{p\in\mathcal{Z}} \frac{1}{p-1} \ll \frac{1}{\log z},$$

hence (upon exponentiation),

$$\prod_{p\in\mathcal{Z}} \left(1-\frac{1}{p}\right)^{-1} = 1 + O\left(\frac{1}{\log z}\right). \tag{5.12}$$

Third, since $\sum_{p\in\mathcal{Z},\, p\geqslant p'} 1/p \ll 1/p'$ for all $p' \in \mathcal{Z}$, we have

$$\sum_{\substack{p\in\mathcal{Z}\\ p\leqslant y_0}} 1 \ll \log y_0 \tag{5.13}$$

for all $y_0 \geqslant 1$. (Write $1 = p/p$ and sum dyadically.)

For $p \mid P_2$ we choose $a_p = 0$. Thus, letting

$$\mathcal{N}_1 = ((0,z] \cap \mathbb{Z}) \setminus \textstyle\bigcup_{p \mid P_2} \{g : g \equiv a_p \bmod p\} = \{h \in (0,z] : (h, P_2) = 1\},$$

it is clear that $h \in \mathcal{N}_1$ only if at least one of the following holds:

(i) $(h, \mathcal{Z}) \neq 1$;
(ii) h is y_1-smooth;
(iii) $h = pm$ for some prime $p > y_2$ and positive integer $m \leqslant z/p$.

In case (iii), the prime p is uniquely determined since $z < y_1 y_2 < y_2^2$. Therefore, by (5.11), the smooth number bound (5.3) and Mertens' theorem (5.1),

$$|\mathcal{N}_1| \leqslant \sum_{\substack{h\leqslant z\\ (h,\mathcal{Z})\neq 1}} 1 + \Psi(z, y_1) + \sum_{y_2<p\leqslant z} \left[\frac{z}{p}\right] = z \log\left(\frac{\log z}{\log y_2}\right) + O\left(\frac{z}{\log y_2}\right).$$

Taking into account that

$$\log\left(\frac{\log z}{\log y_2}\right) = \log\left(1 + \frac{\log(z/y_2)}{\log y_2}\right) \leqslant \frac{\log(z/y_2)}{\log y_2},$$

it follows that

$$|\mathcal{N}_1| \leqslant \frac{z}{\log y_2} \left(\log(z/y_2) + O(1)\right). \tag{5.14}$$

For $p \mid P_1$ we choose a_p "greedily" as follows. For any finite set S of integers and any prime p,

$$|S| = \sum_{a \bmod p} \sum_{\substack{g\in S\\ g\equiv a \bmod p}} 1,$$

so there exists an integer a_p such that $|\{g \in S : g \equiv a_p \bmod p\}| \geqslant |S|/p$. We select a prime $p \mid P_1$ and choose a_p so that this holds with $\mathcal{N}_1$ in place of S. Repeating this process one prime at a time, with p varying over the prime divisors of P_1, we eventually obtain a set

$$\mathcal{N}_2 = \mathcal{N}_1 \setminus \textstyle\bigcup_{p \mid P_1} \{g : g \equiv a_p \bmod p\}$$

whose cardinality satisfies the bound

$$|\mathcal{N}_2| \leqslant |\mathcal{N}_1| \prod_{p \mid P_1} \left(1 - \frac{1}{p}\right) \leqslant 2\mathrm{e}^{-\gamma} \frac{z\,(\log(z/y_2) + O(1))}{(\log y_1)(\log y_2)}. \tag{5.15}$$

The last bound follows by combining Mertens' theorem (5.2), (5.12) and (5.14). (Recall that $2 \nmid P_1$, $\ell \nmid P_1$, $\ell \gg \log y_1$ and $\log(z/y_2) < \log y_1$.)

Now, by the prime number theorem,

$$\begin{aligned}\pi(y) - \pi(y_2) &= \frac{y}{\log y} + O\biggl(\frac{y}{(\log y)^2} + \frac{y_2}{\log y_2}\biggr)\\ &\geqslant \frac{y}{\log y_2} + O\biggl(\frac{y_2}{\log y_2} + \frac{y}{(\log y_2)(\log y)}\biggr).\end{aligned}$$

Combining this with (5.13) and (5.15), we obtain

$$\begin{aligned}|\{p \in (y_2, y] : p \notin \mathcal{Z}\}| - |\mathcal{N}_2| \geqslant \frac{y}{\log y_2}&\biggl(1 - 2\mathrm{e}^{-\gamma}\frac{z\log(z/y_2)}{y\log y_1}\biggr)\\ &+ O\biggl(\frac{y_2}{\log y_2} + \frac{z}{(\log y_1)(\log y_2)}\biggr).\end{aligned} \tag{5.16}$$

We will presently require that $y_1 \leqslant c\sqrt{\log y}$, so we now assume that

$$y_1 = (\log y)^{1/4}, \quad y_2 = y(\log_3 y)^{-1}, \quad y < 8z \leqslant y(\log_2 y)(\log_3 y)^{-1}.$$

Then by (5.16) we have

$$|\{p \in (y_2, y] : p \notin \mathcal{Z}\}| - |\mathcal{N}_2| \geqslant \frac{y}{\log y}\bigl(1 - \mathrm{e}^{-\gamma}\bigr) + O\biggl(\frac{y}{(\log y)(\log_3 y)}\biggr).$$

The right-hand side tends to infinity with y, and so

$$|\{p \in (y_2, y] : p \notin \mathcal{Z}\}| > |\mathcal{N}_2| + k$$

if y is sufficiently large in terms of k, as we now assume.

Applying Lemma 5.1 we see that if $\{h_1, \dots, h_k\}$ is an arbitrary admissible k-tuple contained in $\mathcal{N}_2$, then there are integers $\{a_p : p \mid 2\ell P_3\}$ such that

$$\{h_1, \dots, h_k\} = \mathcal{N}_2 \setminus \textstyle\bigcup_{p \mid 2\ell P_3}\{g : g \equiv a_p \bmod p\}.$$

Therefore, since $\{p \leqslant y : p \notin \mathcal{Z}\} = \{p \leqslant y : p \mid 2\ell P_1 P_2 P_3\}$, to complete the proof it suffices to show that there is an admissible k-tuple $\{h_1, \dots, h_k\} \subseteq \mathcal{N}_2$ satisfying (5.8) and (5.9).

To this end, let $A \bmod P_1$ be the arithmetic progression modulo P_1 such that for all $p \mid P_1$,

$$A \equiv \begin{cases} -1 & \text{if } a_p \equiv 1 \bmod p,\\ 1 & \text{if } a_p \not\equiv 1 \bmod p.\end{cases}$$

(Recall that $2 \nmid P_1$, so $-1 \not\equiv 1 \bmod p$ for all $p \mid P_1$.) Then $(A, P_1) = 1$ and the primes $h \in (y, z]$ with $h \equiv A \bmod P_1$ all lie in $\mathcal{N}_2$. (If $h \in (y, z]$ is prime then $(h, P_2) = 1$, hence $h \in \mathcal{N}_1$.) We choose the elements of our k-tuple from among those primes. We note that by the prime number theorem and (5.13), $P_1 = \mathrm{e}^{(1+o(1))y_1}$ as y (and hence y_1) tends to infinity. Thus, if h and $h' < h$ are any two such primes then

$$p \mid h - h' \quad\Longrightarrow\quad p \mid P_1 \text{ or } p \mid (h - h')/P_1 \quad\Longrightarrow\quad p \leqslant \max\{y_1, z/P_1\} < y$$

if y is large enough, as we assume, so any k-tuple of primes $\{h_1, \dots, h_k\}$ chosen in this way satisfies (5.8). Moreover, such a k-tuple of primes is admissible since $\min\{h_1, \dots, h_k\} > k$ (we assume that $y > k$).

By Chebyshev's bound we have $\sum_{p\leqslant y_1}\log p < 2y_1$, whence $P_1 < \mathrm{e}^{2(\log y)^{1/4}}$. Thus, by (5.4) we have

$$\sum_{\substack{u<p\leqslant u+\Delta\\ p\equiv A \bmod P_1}} \log p = \frac{\Delta}{\phi(P_1)} + O\Big(y\exp\left(-c'\sqrt{\log y}\right)\Big),$$

uniformly for $2\leqslant \Delta\leqslant y\leqslant u\leqslant z$, where c' is an absolute constant, except possibly if P_1 is a multiple of a certain integer q_1 whose greatest prime factor satisfies $P^+(q_1)\gg \log_2 y\gg \log y_1$. We now specify ℓ accordingly so that this possibility cannot arise.[8] We let $\Delta = y\mathrm{e}^{-(\log y)^{1/4}}$. Thus,

$$\sum_{\substack{u<p\leqslant u+\Delta\\ p\equiv A \bmod P_1}} \log p \gg y\exp\left(-3(\log y)^{1/4}\right)$$

uniformly for $y\leqslant u\leqslant z$, and the left-hand side is a sum over at least k primes if y is sufficiently large in terms of k, as we now assume.

Recall that $\beta_k\geqslant\cdots\geqslant\beta_1\geqslant 0$ are given real numbers. We now assume that y is large enough in terms of β_k so that

$$2(1+(1+\beta_k))\leqslant(\log_2 y)(\log_3 y)^{-1},$$

and we let x be any number such that $x\geqslant 1$ and

$$2y(1+(1+\beta_k)x)\leqslant 8z\leqslant y(\log_2 y)(\log_3 y)^{-1}.$$

For $1\leqslant i\leqslant k$, let

$$u_i = \beta_i xy + y,$$

so that the intervals $(u_i, u_i+\Delta]$ are all contained in $(y,z]$. For each $1\leqslant i\leqslant k$ in turn, we choose a prime $h_i\in(u_i,u_i+\Delta]$ with $h_i\equiv A \bmod P_1$ a nd $h_i\neq h_j$ for any $j\leqslant i$. This is possible since each interval contains at least k primes that are congruent to $A \bmod P_1$. We see that the resulting set $\{h_1,\ldots,h_k\}$ is admissible since no element is congruent to $a_p \bmod p$ for any prime $p\leqslant k$. Moreover, $h_i = u_i + O(\Delta)$, which gives (5.9). □

6. Deduction of Theorems 1.1 and 1.3

Deduction of Theorem 1.3. Fix $k\geqslant m\geqslant 2$ and $\epsilon=\epsilon(k,m)\in(0,1)$, with k a sufficiently large multiple of $(8m+1)(8m^2+16m)$, and ϵ sufficiently small, in the sense of Theorem 4.3. (In fact, it suffices to take k to be a sufficiently large multiple of $8m^2+16m$ here [see Remark 4.4].)

Fix real numbers $\beta_{8m^2+16m}\geqslant\cdots\geqslant\beta_1\geqslant 0$. Let $\boldsymbol{\beta}\in\mathbb{R}^k$ be given by

$$\boldsymbol{\beta} = (\beta_1,\ldots,\beta_1,\beta_2,\ldots,\beta_2,\ldots,\beta_{8m^2+16m},\ldots,\beta_{8m^2+16m}),$$

where there are $k/(8m^2+16m)$ consecutive copies of each β_i appearing in $\boldsymbol{\beta}$. Let $N\geqslant N(k,m,\epsilon)$ (as in Theorem 4.3) and put

$$x=\epsilon^{-1},\quad y=w=\epsilon\log N,\quad z=y(\log_2 y)(2\log_3 y)^{-1}.$$

[8] If q_1 does not exist we can either let $\ell=1$ or choose any $\ell\gg\log y_1$. Indeed, we could remove any set $\mathcal{Z}_1$ of primes from P_1 such that $\sum_{\ell\in\mathcal{Z}'}1/\ell\ll 1/\log y_1$, without affecting the proof.

If $N \geqslant N(\boldsymbol{\beta}, k, m, \epsilon)$ is large enough in terms of $\boldsymbol{\beta}$ and k, then with $y(\boldsymbol{\beta}, k)$ as in Lemma 5.2 we have

$$x > 1, \quad y \geqslant y(\boldsymbol{\beta}, k), \quad 2y(1 + (1 + \beta_{8m^2+16m})x) \leqslant 8z \leqslant y(\log_2 y)(\log_3 y)^{-1}.$$

Let $Z_{N^{4\epsilon}}$ be given by (4.8) and let $W = \prod_{p \leqslant w,\, p \nmid Z_{N^{4\epsilon}}} p$. Let us define $\mathcal{Z}$ by putting $\mathcal{Z} = \varnothing$ if $Z_{N^{4\epsilon}} = 1$ and $\mathcal{Z} = \{Z_{N^{4\epsilon}}\}$ if $Z_{N^{4\epsilon}} \neq 1$. Then (4.7) implies that the condition (5.6) is satisfied since $\log z \ll \log_2 N^{\epsilon}$.

The hypotheses of Lemma 5.2 being verified, we conclude that there exists a set $\{a_p : p \leqslant y,\, p \notin \mathcal{Z}\}$ and an admissible k-tuple $\{h_1, \dots, h_k\}$ such that

$$\{h_1, \dots, h_k\} = ((0, z] \cap \mathbb{Z}) \setminus \bigcup_{p \leqslant y,\, p \notin \mathcal{Z}} \{g : g \equiv a_p \bmod p\}. \tag{6.1}$$

Moreover, for $1 \leqslant i < j \leqslant k$,

$$p \mid (h_j - h_i) \implies p \leqslant y = w \tag{6.2}$$

and we have the partition

$$\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_{8m^2+16m} \tag{6.3}$$

such that for each $j \in \{1, \dots, 8m^2 + 16m\}$ we have

$$h = (\beta_j + \epsilon + o(1)) \log N \quad \text{for all } h \in \mathcal{H}_j. \tag{6.4}$$

We let b be an integer satisfying

$$b \equiv -a_p \bmod p \tag{6.5}$$

for all $p \leqslant y$, $p \notin \mathcal{Z}$.

We now wish to apply part (ii) of Theorem 4.3. We have $0 < h_i \leqslant z < N$ for each i, so (4.16) is satisfied. We see (6.2) and (6.3) give the conditions (4.17) and (4.18). Finally, by (6.1) and (6.5), we have $(\prod_{i=1}^{k}(b + h_i), W) = 1$, and so (4.19) also holds. We conclude that there exists some $n \in (N, 2N]$ with $n \equiv b \bmod W$, and some indices $i_1 < \cdots < i_{m+1}$, such that

$$|\mathcal{H}_i(n) \cap \mathbb{P}| = 1 \quad \text{for all } i \in \{i_1, \dots, i_{m+1}\},$$
$$|\mathcal{H}_i(n) \cap \mathbb{P}| \leqslant 1 \quad \text{for all } i_1 < i < i_{m+1}.$$

For any $n > y$ such that $n \equiv b \bmod W$, (6.1) implies that

$$(n, n + z] \cap \mathbb{P} = \mathcal{H}(n) \cap \mathbb{P},$$

because if $g \in (0, z]$ and $g \notin \{h_1, \dots, h_k\}$, we have $g + n \equiv a_p - a_p \equiv 0 \bmod p$ for some $p \leqslant w$ with $p \notin \mathcal{Z}$. The primes in $\mathcal{H}(n)$ are therefore consecutive primes. Therefore there are indices $J(1) < \cdots < J(m+1)$ for which $|\mathcal{H}_{J(i)}(n) \cap \mathbb{P}| = 1$ and the primes counted here form a sequence of $m+1$ consecutive primes. Thus, by (6.4), and since $N \leqslant n + h_i \leqslant 3N$, we have for some r that

$$\frac{p_{r+i+1} - p_{r+i}}{\log p_{r+i}} = \beta_{J(i+1)} - \beta_{J(i)} + o(1)\,, \tag{6.6}$$

for $1 \leqslant i \leqslant m$.

Letting N tend to infinity, we see that for infinitely many r there exists some $1 \leqslant J(1) < \cdots < J(m+1) \leqslant 8m^2 + 16m$ such that (6.6) holds. Since there are at most $O_k(1)$ distinct ways to choose the indices $J(i)$, at least one pattern of indices occurs infinitely often. For that pattern we have (6.6) for infinitely many r, and so $(\beta_{J(2)} - \beta_{J(1)}, \dots, \beta_{J(m+1)} - \beta_{J(m)}) \in \boldsymbol{L}_m$. □

Deduction of Theorem 1.1. The argument is essentially the same as that for Theorem 1.3, but uses part (i) of Theorem 4.3 instead of part (ii). We take k to be a sufficiently large multiple of 24×9. (In fact, it suffices to take k to be a sufficiently large multiple of 9 here [see Remark 4.4].) Given $\beta_9 \geqslant \cdots \geqslant \beta_1 \geqslant 0$, we construct $\mathcal{H}$ as before and form a partition $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_9$, so that each $\mathcal{H}_i$ has size $k/9$ and all elements of $\mathcal{H}_i$ have size $(\beta_i + \epsilon + o(1)) \log N$. Applying part (i) of Theorem 4.3 (with $m = 1$) then shows that there is an $n \in (N, 2N]$ such that $|\mathcal{H}_i(n) \cap \mathbb{P}| \geqslant 1$, $|\mathcal{H}_j(n) \cap \mathbb{P}| \geqslant 1$ for some $1 \leqslant i < j \leqslant 9$. As before, our construction shows that there are no other primes in $[n, n+z]$, and so there must be two consecutive primes p_r, p_{r+1} of the form $n+h, n+h'$ with h, h' in different sets $\mathcal{H}_i$. But then we have

$$\frac{p_{r+1} - p_r}{\log p_r} = \beta_j - \beta_i + o(1)\,,$$

for some $i < j$. Since this occurs for every large N, we obtain the result. □

7. Concluding remarks

As a note added in press we mention that Pintz [**22**] has shown that in fact, given any sequence of *five* nonnegative real numbers $\beta_1 \leqslant \cdots \leqslant \beta_5$,

$$\{\beta_j - \beta_i : 1 \leqslant i < j \leqslant 5\} \cap \boldsymbol{L} \neq \varnothing. \tag{7.1}$$

By the proof of Corollary 1.2 we therefore have $\lambda([0,T] \cap \boldsymbol{L}) \geqslant 3T/25$ for $T \geqslant 0$ and $\lambda([0,T] \cap \boldsymbol{L}) \geqslant (1 - o(1))T/4$ as $T \to \infty$. Thus, at least 25% of nonnegative real numbers are limit points of $\{d_n/\log p_n\}_{n=1}^{\infty}$.

We may also consider limit points $\boldsymbol{L}_f$ of $\{d_n/f(p_n)\}_{n \geqslant n_0}$ for any function f that satisfies some mild technical conditions – let us call such a function "reasonable" – and tends to infinity sufficiently slowly. Ford, Green, Konyagin and Tao [**9**] and Maynard [**18**] have broken ground on the notorious "Erdős–Rankin problem" by showing that $\infty \in \boldsymbol{L}_R$, where

$$R(p_n) = \frac{\log p_n \log_2 p_n \log_4 p_n}{(\log_3 p_n)^2}.$$

Using the work of [**9**], Pintz [**21, 22**] has shown that one can replace $\boldsymbol{L}$ by $\boldsymbol{L}_f$ in (7.1), for any "reasonable" function f satisfying $f(p_n) \ll R(p_n)$ for all sufficiently large p_n. In fact, Ford, Green, Konyagin, Maynard and Tao [**10**] have together shown that $d_n \gg R(p_n) \log_3 p_n$ for infinitely many n. Using this work, Baker and Freiberg [**1**] have shown that one can replace $\boldsymbol{L}$ by $\boldsymbol{L}_f$ in (7.1), for any "reasonable" function f satisfying $f(p_n) = o(R(p_n) \log_3 p_n)$ as $p_n \to \infty$.

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Department of Mathematics, University of Missouri, Columbia MO, USA.
E-mail address: bankswd@missouri.edu

Department of Pure Mathematics, University of Waterloo, Waterloo ON, Canada.
E-mail address: tfreiberg@uwaterloo.ca

Magdalen College, Oxford, UK.
E-mail address: maynard@maths.ox.ac.uk