Infinite locally random graphs

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Abstract

Motivated by copying models of the web graph, Bonato and Janssen [3] introduced the following simple construction: given a graph $G$, for each vertex $x$ and each subset $X$ of its closed neighbourhood, add a new vertex $y$ whose neighbours are exactly $X$. Iterating this construction yields a limit graph $↑G$. Bonato and Janssen claimed that the limit graph is independent of $G$, and it is known as the infinite locally random graph. We show that this picture is incorrect: there are in fact infinitely many isomorphism classes of limit graph, and we give a classification. We also consider the inexhaustibility of these graphs.

1 Introduction

The Rado graph $\mathcal{R}$ is the unique graph with countably infinite vertex set such that, for any disjoint pair $X, Y$ of finite subsets of vertices, there is a vertex $z$ that is joined to every vertex in $X$ and no vertex in $Y$. If $0 < p < 1$, and $G$ is a random graph in $\mathcal{G}(\mathbb{N}, p)$, then with probability 1 we have $G \cong \mathcal{R}$. For this reason, the Rado graph is also known as the infinite random graph (see [5] for a survey).

The Rado graph can be obtained deterministically by beginning with any finite (or countably infinite) graph $G$ and iterating the following construction:

$[E1]$ For every finite subset $X$ of $V(G)$ add a vertex $y$ with neighbourhood $N(y) = X$.

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Here $N(x) = \{ y \in V(G) : xy \in E(G) \}$ is the \textit{neighbourhood} of $x$; we also write $N[x] = N(x) \cup \{x\}$ for the \textit{closed neighbourhood} of $x$.

Motivated by copying models of the web graph, Bonato and Janssen [3] (see also [1] and [4]) introduced the following interesting construction. For a finite graph $G$, the \textit{pure extension} $PE(G)$ of $G$ is obtained from $G$ by the following construction:

\[ \text{(E2) For every } x \in V(G) \text{ and every finite } X \subseteq N[x] \text{ add a vertex } y \text{ with neighbourhood } N(y) = X. \]

Iterating this construction, we obtain a limit graph, denoted by $\uparrow G$.

Bonato and Janssen ([3], Theorem 3.3) claimed that $\uparrow G \cong \uparrow H$ for every pair $G$, $H$ of finite graphs. The (claimed) unique limit graph, which has become known [1] as the \textit{infinite locally random graph} (see Proposition 1 below for the reason for this name). As we show below, Bonato and Janssen’s claim is incorrect. There are in fact infinitely many limit graphs $G$ (for instance, $\uparrow C_5$, $\uparrow C_6$, $\uparrow C_7$, … are all distinct), and we give a simple criterion that determines when $\uparrow G \cong \uparrow H$.

In the next section, we give a few simple properties of limit graphs $\uparrow G$; we prove our classification result in section 3. Finally, in section 4, we prove that for every finite $G$, $\uparrow G$ is inexhaustible, that is $(\uparrow G) \setminus x \cong \uparrow G$ for all $x \in V(\uparrow G)$. This corrects another result from [3].

\section{Simple properties of $\uparrow G$}

We begin with some notation. We shall refer to the vertices $y$ that are introduced in [E2] above with neighbourhoods contained in $N[x]$ as \textit{clones} of $x$. Thus a vertex of degree $d$ in $G$ has $2^{d+1}$ clones in $PE(G)$ (note that we take all subsets of the \textit{closed} neighbourhood $N[x]$), and $PE(G)$ contains $|G|$ isolated vertices, each one a clone of a different vertex from $G$. As indicated above, iterating construction [E2] gives a sequence of graphs $G \subseteq PE(G) \subseteq PE^2(G) \subseteq \cdots$, where $PE^n(G) = PE(PE^{n-1}(G))$; we write $\uparrow G$ for the limit of this sequence. We define the \textit{level} $L(x)$ of a vertex of $\uparrow G$ to be the least integer $k$ such that it is contained in $PE^k(G)$ (where $L(x) = 0$ for all $x \in V(G)$), and for a finite subset $X \subseteq V(\uparrow G)$, we write $L(X) = \max_{x \in X} L(x)$. We also write $L^{(k)}(\uparrow G)$ for the vertices of level $k$ in $\uparrow G$, and $L^{(\leq k)}(\uparrow G)$ for the vertices of level $k$ or less. Note that, by the construction, $L^{(k)}(\uparrow G)$ is an independent set for every $k \geq 1$. 

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Given a graph $H$, a graph $G$ is locally $H$ if, for every vertex $x$ of $G$, the graph induced by the neighbourhood $N(x)$ of $x$ is isomorphic to $H$.

Bonato and Janssen note the following property of the construction defined above.

**Proposition 1.** [3] For every finite graph $G$, $\uparrow G$ is locally $\mathcal{R}$

**Proof.** For every $x \in V(\uparrow G)$, and every $X$ and $Y$ finite disjoint subsets of $N(x)$, we want to find a vertex $z$ such that $z$ is adjacent to every vertex in $X$ and to none in $Y$. This is possible by the definition of $\uparrow G$ by taking a suitable vertex $z$ of level $L(X \cup Y) + 1$.

Since $\mathcal{R}$ is the (unique) infinite random graph, it therefore makes sense to refer to $\uparrow G$ as an infinite locally random graph.

**Corollary 2.** Let $G$ be a finite graph. Then $\uparrow G$ is $\aleph_0$-universal (that is, $\uparrow G$ contains every countable graph $H$ as an induced subgraph).

Another easy but important remark concerns the distance between vertices.

**Proposition 3.** Let $G$ be a finite graph and $x$ and $y$ two vertices of $PE^k(G)$, for some integer $k \geq 0$. Then the distance between $x$ and $y$ is the same in $PE^k(G)$ and in $\uparrow G$.

**Proof.** It is sufficient to note that the pure extension construction [E2] does not change the distance between vertices.

We also note the following simple property.

**Lemma 4.** Let $G$ be a finite graph and $x$ a vertex of $\uparrow G$. Let $X$ be a finite subset of $N(x)$. Then there exists a vertex $y$ with $L(y) \leq L(X)$ such that $X \subseteq N[y]$.

**Proof.** Let $x_0$ be a vertex of minimal level with $X \subseteq N[x_0]$. If $L(x_0) \leq L(X)$ then we can take $y = x_0$. Otherwise, $L(x_0) > L(X)$ and so $x_0 \not\in X$. But $x_0$ was constructed on level $L(x_0)$ as the clone of some vertex $x_1$ with $L(x_1) < L(x_0)$. In particular, $N(x_0) \cap L^{<L(x_0)}(\uparrow G) \subseteq N[x_1]$ and so $X \subseteq N[x_1]$, which contradicts the minimality of $L(x_0)$. 

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For \( x \in V(\uparrow G) \), we write

\[
N^-(x) = N(x) \cap L^{(<L(x))}(\uparrow G).
\]

Note that \( N^-(x) \) is the set of neighbours assigned to \( x \) at time \( L(x) \), when \( x \) is first introduced. We say that a subgraph \( G_1 \) of \( G \) is good if it is an induced subgraph of \( G \) and, for all \( x \) in \( V(G_1), N^-(x) \subseteq V(G_1) \). Equivalently, \( G_1 \) is an induced subgraph such that \( N(y) \cap V(G_1) \subseteq N^-(y) \) for all \( y \in V(G) \setminus V(G_1) \).

In this context, Lemma 4 gives the following result.

**Lemma 5.** Let \( G \) be a finite graph and suppose that \( H \) is a good subgraph of \( \uparrow G \). Then

\[
\forall x \in V(\uparrow G), \exists y \in V(H) \text{ such that } N(x) \cap V(H) \subseteq N[y] \cap V(H)
\]

**Proof.** We can assume that \( x \notin V(H) \). Let \( X = N(x) \cap V(H) \). Then \( X \subseteq N^-(x) \), and by Lemma 4 there exists \( y \) of level at most \( L(X) \) with \( X \subseteq N[y] \). If \( L(y) = L(X) \) then, since the levels are independent sets and \( X \subseteq N[y] \), \( y \) must belong to \( X \), and thus to \( H \). If \( L(y) < L(X) \), then \( y \) belongs to \( H \) as \( H \) is a good subgraph of \( \uparrow G \). \( \square \)

## 3 Classification

We now investigate when \( \uparrow G \) and \( \uparrow H \) are isomorphic. In [3], the authors claim that \( \uparrow G \cong \uparrow H \) for any pair of finite graphs \( G \) and \( H \) (this is their Theorem 3.3). Here we disprove this. Their proof seems to fail on page 209 at the end of the first paragraph: the equality \( H_{n+1} - S \cong G_1 \cup \overline{K_m} \) does not hold because these vertices can be linked by edges. Moreover, it is not clear why this equality would imply \( H - S \cong \uparrow(G_1 \cup \overline{K_m}) \) on the following line, as some vertices in \( H \) can be constructed by cloning elements in \( S \).

We begin with the following useful consequence of Lemma 5.

**Theorem 6.** Let \( G \) and \( H \) be finite graphs. Suppose that \( G_1 \supseteq G \) is a good subgraph of \( \uparrow G \) and \( H_1 \supseteq H \) is a good subgraph of \( \uparrow H \). If \( G_1 \cong H_1 \) then \( \uparrow G \cong \uparrow H \)

**Proof.** Let \( \phi : V(G_1) \rightarrow V(H_1) \) be an isomorphism (note that, as \( G_1 \) and \( H_1 \) are good, they are induced subgraphs of \( \uparrow G \) and \( \uparrow H \), respectively, so this is an isomorphism between induced subgraphs). Using a classical ‘back
and forth’ argument, we extend \( \phi \) one vertex at a time until, in the limit, we obtain an isomorphism between \( \uparrow G \) and \( \uparrow H \). Let \( x \in V(\uparrow G) \) be a vertex of minimal level with \( x \not\in V(G_1) \). By Lemma 5, there exists \( y \in V(G_1) \) such that \( N(x) \cap V(G_1) \subseteq N[y] \cap V(G_1) \). Let \( z \not\in V(H_1) \) be a clone of \( \phi(y) \) with

\[
N^-(z) = N(z) \cap V(H_1) = \phi(N(x) \cap V(G_1)).
\]

Such a clone is easily found: let \( k = L(V(H_1)) \), and take the clone of \( \phi(y) \) on level \( k + 1 \) with exactly this neighbourhood in \( L^{(\leq k)}(\uparrow H) \). Then \( V(H_1) \cup \{ z \} \) induces a good subgraph of \( \uparrow H \) and, by minimality of \( x \), \( V(G_1) \cup \{ x \} \) induces a good subgraph of \( \uparrow G \). We can therefore extend \( \phi \) by setting \( \phi(x) = z \). Repeating the construction in alternate directions we clearly obtain an isomorphism between \( \uparrow G \) and \( \uparrow H \).

We shall say that a vertex \( x \) of a graph \( G \) is \textit{inessential} if there exists \( y \in V(G) \), \( y \neq x \) such that \( N(x) \subseteq N[y] \). A graph is \textit{essential} if it contains no inessential vertices. Given a graph \( G \), a sequence of vertices \( x_1, \ldots, x_k \) is a \textit{maximal sequence of removals} if \( x_i \) is inessential in \( G \setminus \{ x_1, \ldots, x_{i-1} \} \) for each \( i \), and \( G \setminus \{ x_1, \ldots, x_k \} \) is an essential graph.

We shall show below that every maximal sequence of removals yields the same essential graph (up to isomorphism). However, we first prove a simple lemma. We say that two vertices \( x \) and \( y \) in a graph \( G \) are \textit{equivalent} if \( N(x) = N(y) \) or \( N(x) \subseteq N[y] \). Equivalently, \( N(x) \subseteq N[y] \) and \( N(y) \subseteq N[x] \). Clearly, if \( x \) and \( y \) are equivalent in \( G \) then \( G \setminus x \cong G \setminus y \), with the obvious isomorphism given by exchanging \( x \) for \( y \) and leaving the other vertices fixed.

Equivalent vertices play an important role in the removal of inessential vertices.

\textbf{Lemma 7.} Suppose that \( x \) and \( y \) are inessential in a graph \( G \), but \( x \) is not inessential in \( G \setminus y \). Then \( x \) and \( y \) are equivalent.

\textit{Proof.} Note first that since \( x \) and \( y \) are inessential in \( G \), there are \( x' \) and \( y' \) such that \( N(x) \subseteq N[x'] \) and \( N(y) \subseteq N[y'] \). If \( x' \neq y \) then considering the vertex \( x' \) in \( G \setminus y \) shows that \( x \) is inessential in \( G \setminus y \), a contradiction. So \( x' = y \), and \( N(x) \subseteq N[y] \).

Now consider \( y' \). If \( y' \neq x \) then \( N(x) \subseteq N[y] = \{ y \} \cup N(y) \subseteq \{ y \} \cup N[y'] \) implies that \( N(x) \setminus y \subseteq N[y'] \), and so \( y' \) shows that \( x \) is inessential in \( G \setminus y \), a contradiction. Thus we have \( y' = x \), and so \( N(y) \subseteq N[x] \). It follows that \( x \) and \( y \) are equivalent. \( \square \)
We now show that maximal sequences of removals define a unique graph up to isomorphism.

**Theorem 8.** Suppose that \( x_1, \ldots, x_k \) and \( y_1, \ldots, y_l \) are two maximal sequences of removals in a finite graph \( G \). Then \( G \setminus \{x_1, \ldots, x_k\} \cong G \setminus \{y_1, \ldots, y_l\} \).

**Proof.** We claim that we can modify the sequence \( \{y_1, \ldots, y_l\} \) to obtain the sequence \( \{x_1, \ldots, x_k\} \) without changing the isomorphism type of the resulting essential graph \( G \setminus \{y_1, \ldots, y_l\} \).

Suppose first that \( x_1 \notin \{y_1, \ldots, y_l\} \). Then (by maximality) \( x_1 \) is inessential in \( G \) but not in \( G \setminus \{y_1, \ldots, y_l\} \). Let \( i \) be maximal such that \( x_1 \) is inessential in \( G \setminus \{y_1, \ldots, y_i\} \). Then, by Lemma 7, \( x_1 \) and \( y_{i+1} \) are equivalent in \( G \setminus \{y_1, \ldots, y_i\} \), and so we can replace \( y_{i+1} \) by \( x_1 \) in the sequence \( y_1, \ldots, y_l \), without effecting the isomorphism type of \( G \setminus \{y_1, \ldots, y_l\} \) (the isomorphism is given by exchanging \( x_1 \) and \( y_{i+1} \)). We may therefore assume that \( x_1 \in \{y_1, \ldots, y_l\} \).

We now show that we can modify \( y_1, \ldots, y_l \) so that \( y_1 = x_1 \). Suppose that \( x_1 = y_{i+1} \) for some \( i \geq 1 \). If there exists some \( 0 \leq j < i - 1 \) such that \( x_1 \) is inessential in \( G \setminus \{y_1, \ldots, y_j\} \) and not in \( G \setminus \{y_1, \ldots, y_{j+1}\} \), Lemma 7 implies that \( x_1 \) and \( y_{j+1} \) are equivalent in \( G \setminus \{y_1, \ldots, y_j\} \). Therefore we can exchange them in the sequence. We can repeat this operation as long as such an integer \( j \) exists, and thus we can assume that \( x_1 = y_{i+1} \) is inessential in \( G \setminus \{y_1, \ldots, y_j\} \) for all \( j \leq i \). Now, if \( y_i \) is not inessential in \( G \setminus \{y_1, \ldots, y_{i-1}, x_1\} \) then (as it is inessential in \( G \setminus \{y_1, \ldots, y_{i-1}\} \)) Lemma 7 shows that \( x_1 \) and \( y_i \) are equivalent in \( G \setminus \{y_1, \ldots, y_{i-1}\} \). It is clear that we may therefore exchange \( y_i \) and \( y_{i+1} = x_1 \) in the sequence \( y_1, \ldots, y_l \). Repeating this argument, we move \( x_1 \) forward in the sequence \( y_1, \ldots, y_l \) until \( x_1 = y_1 \).

Finally, if \( x_1 = y_1 \), we can work instead with the graph \( G \setminus x_1 \) and the sequences \( x_2, \ldots, x_k \) and \( y_2, \ldots, y_l \), continuing until one (and hence both) of the sequences is exhausted. \( \square \)

We shall denote the (isomorphism type of the) subgraph of \( G \) obtained by deleting a maximal sequence of removals \( \downarrow G \). For instance, \( \downarrow K_n = \downarrow C_4 = K_1 \), but \( \downarrow C_k = C_k \) for all \( k \geq 5 \).

We next show that inessential vertices have no effect on limit graphs.

**Corollary 9.** Let \( G \) be a finite graph and \( x \) an inessential vertex of \( G \). Then \( \uparrow G \cong \uparrow (G \setminus x) \)
Proof. Let $H = G \setminus x$. Since $x$ is inessential, there exists $y$ in $G$ such that $N(x) \subseteq N[y]$ in $G$. In $\uparrow H$, $y$ has a clone $x'$ such that $N^{-1}(x') = N(x) \cap V(G)$. Clearly $G_1 = G$ is a good subgraph of $\uparrow G$ and $V(H) \cup \{x'\}$ induces a good subgraph $H_1$ of $\uparrow H$. Thus it suffices to apply Theorem 6 to $G_1$ and $H_1$. \qed

Corollary 9 implies the following theorem.

**Theorem 10.** Let $G$ be a finite graph. Then $\uparrow G \cong \uparrow (\downarrow G)$

If $H$ is an induced subgraph of $\uparrow G$, then we define two kinds of transformations on this subgraph, called reductions.

(i) Delete an inessential vertex of $H$.

(ii) For a pair of vertices $x \in V(H)$ and $y \notin V(H)$ with $N(x) \cap V(H) \subseteq N(y) \cap V(H)$, replace $H$ by the subgraph of $\uparrow G$ induced by $(V(H) \setminus x) \cup \{y\}$.

**Lemma 11.** If $H$ is a finite induced subgraph of $\uparrow G$, it is possible to apply a sequence of reductions to transform $H$ into a subgraph of $G$.

**Proof.** Define the weight $w(H')$ of an induced subgraph of $\uparrow G$ by

$$w(H') = \sum_{v \in V(H')} L(v).$$

If $w(H) = 0$ then $H$ is a subgraph of $G$. If $w(H) > 0$, then we look for a reduction that decreases the weight or the number of vertices. If $H$ contains an inessential vertex, then delete it (this can occur at most $|H| - 1$ times). Otherwise, let $x \in V(H)$ be a vertex of highest level. Then $N(x) \cap V(H) = N(x) \cap V(H) \cap L^{<L(x)}(\uparrow G)$, as $L(L(x))(\uparrow G)$ is an independent set. Since $x$ was built at level $L(x)$ as the clone of some vertex $y$ that satisfies $N(x) \cap V(H) \cap L^{<L(x)}(\uparrow G) \subseteq N[y] \cap V(H)$ and $L(y) < L(x)$, we can replace $x$ by $y$, to obtain $H'$ with $w(H') < w(H)$. Repeating this process, we eventually obtain an induced subgraph of $\uparrow G$ with weight 0 which, as already noted, is a subgraph of $G$. \qed

We are now ready to prove our main result.

**Theorem 12.** Let $G$ and $H$ be finite graphs. Then $\uparrow G \cong \uparrow H \iff \downarrow G \cong \downarrow H$
Proof. By Theorem 10, we may assume that $G$ and $H$ do not contain any inessential vertices, that is $\downarrow G = G$ and $\downarrow H = H$. Suppose that $\uparrow G \cong \uparrow H$, and fix an isomorphism.

Let $\{1, 2, \ldots, n\}$ be the vertices of $G$. We partition the vertices of $\uparrow G$ into $n$ classes in the following way. For $i = 1, \ldots, n$, let $A_{i,0} = \{i\}$, and for $j \geq 1$, let $A_{i,j}$ be the vertices of $\uparrow G$ which are clones of vertices in $A_{i,j-1}$. We then define $A_i = \bigcup_{j=0}^{\infty} A_{i,j}$. Thus $A_i$ is the smallest set of vertices containing $i$ and closed under taking clones. It is easy to see that, for $i \neq k$, there is an edge between class $A_i$ and $A_k$ if and only if there is an edge between $i$ and $k$ (as creating a clone cannot create adjacencies between a new pair of classes). We shall say that edges between classes respect $G$.

Now consider an isomorphic embedding $\phi$ of $G$ into $\uparrow G$. We say that $\phi$ is good if $\phi(i) \in A_i$ for every $i \in V(G)$. Suppose that $\phi$ is good and let $G'$ be the image of $G$ under $\phi$. If we apply a type (ii) reduction to some vertex of $G'$, say $v_i := \phi(i)$, then it is replaced by a vertex $x$ such that $N(x) \cap V(G') \supseteq N(v_i) \cap V(G')$. Let $A_j$ be the class containing $x$. Since $\phi$ is good, there is an edge between $A_j$ and $A_k$ whenever $k \in N(i)$. Since edges between classes respect $G$, this implies $N[j] \supseteq N(i)$. But since we assumed that $G$ contains no inessential vertices, this is possible only if $i = j$. Indeed, $N(x) \cap V(G') = N(v_i) \cap V(G')$, or else we would introduce edges between new pairs of classes. It follows that we obtain a good embedding $\phi'$ of $G$ by setting $\phi'(i) = x$ and $\phi'(j) = \phi(j)$ otherwise. This remains true for any sequence of reductions starting from a good embedding. In particular, any sequence of reductions starting from $G$ produces an induced copy of $G$ (note that reductions of type (i) are not possible at any stage).

By Lemma 11, any induced subgraph of $\uparrow H$ isomorphic to $G$ can be reduced to a subgraph of $H$. It follows that $G$ must be isomorphic to a subgraph of $H$. Arguing similarly the other way round, we see that $H$ is isomorphic to a subgraph of $G$, and so $G \cong H$. \hfill \Box

Now it is clear that $\uparrow G$ is not independent of $G$: it suffices to consider two circuits of different length (larger than 4). In fact, Theorem 12 immediately gives the following classification of possible limit graphs.

**Corollary 13.** The isomorphism classes of limit graphs $\uparrow G$ of finite graphs $G$ are in bijective correspondence with the class of essential finite graphs.
4 Inexhaustibility

A graph $G$ is *inexhaustible* if $G \setminus x \cong G$ for every vertex $x \in V(G)$. For instance, the infinite complete graph $K_\omega$ and its complement are trivially inexhaustible; the Rado graph $\mathcal{R}$ is also inexhaustible. On the other hand, the infinite two-way path is not inexhaustible, as deleting any vertex increases the number of components. For results on inexhaustible graphs, see Pouzet [7], El-Zahar and Sauer [6] and Bonato and Delić [2].

Bonato and Janssen [3] consider the inexhaustibility of infinite graphs satisfying various properties, and claim a rather general result. Let us define two properties of (infinite) graphs as follows. We say that a graph $G$ has Property A if it satisfies the following condition.

(A) For every vertex $x$ of $G$, every finite $X \subseteq N[x]$, and every finite $Y \subseteq V(G) \setminus X$, there is a vertex $z \notin X \cup Y$ such that $X \subseteq N(z)$ and $Y \cap N(z) = \emptyset$,

and $G$ has Property B if it satisfies the following.

(B) For every vertex $x$ of $G$, every finite $X \subseteq N(x)$, and every finite $Y \subseteq V(G) \setminus X$, there is a vertex $z \notin X \cup Y$ such that $X \subseteq N(z)$ and $Y \cap N(z) = \emptyset$.

Note that the only difference between (A) and (B) is that (A) is concerned with closed neighbourhoods, while (B) is only concerned with neighbourhoods. Clearly Property A implies Property B; furthermore, for any finite $G$, it is clear from the constructive step [E2] that $\uparrow G$ has Property A (and therefore Property B).

Bonato and Janssen ([3], Theorem 4.1) claim that every graph with Property B is inexhaustible. However, there is a simple counterexample to this assertion: let $G$ be the Rado graph $\mathcal{R}$ with an additional isolated vertex $x$. Since the Rado graph is connected, and $G$ is not, it is clear that $G \setminus x \not\cong G$. (The proof of Bonato and Janssen in [3] appears to fail with the definition of their sets $S_i$.)

In fact, even the stronger Property A does not imply that a graph is inexhaustible. Consider the graph $G$ defined by starting from the path $x_1x_2x_3x_4$ of length 3, and alternating the pure extension construction [E2] with the following step.

**[E3]** For every pair of vertices $\{x, y\} \neq \{x_1, x_4\}$, add a vertex $z$ with $N(z) = \{x, y\}$. 


Note that $x_1$ and $x_4$ are at distance 3 in the initial graph. The pure extension step [E2] does not change the distance between vertices, while [E3] does not create a path of length 2 from $x_1$ to $x_4$. Thus $x_1$ and $x_4$ are at distance 3 in the limit graph. On the other hand, there are infinitely many paths of length 2 between any other pair of vertices. Thus $G \setminus \{x_1, x_4\} \not\cong G$, and so $G$ cannot be inexhaustible (if $G$ is inexhaustible, then clearly $G \setminus X \cong G$ for every finite $X \subseteq V(G)$).

On the positive side, we can show that for any finite $G$, the limit graph $\uparrow G$ is actually inexhaustible.

**Theorem 14.** For every finite graph $G$, $\uparrow G$ is inexhaustible.

**Proof.** Let $v$ be any vertex of $\uparrow G$. We shall show that $\uparrow G \cong (\uparrow G) \setminus v$. Note that since $\uparrow G \cong \uparrow PE^L(v)(G)$, we can replace $G$ by $PE^L(v)(G)$, and so we may assume that $v \in V(G)$.

On the first level above $G$, $v$ has a clone $v'$ with $N(v) \cap G = N(v') \cap G$. Thus we have an isomorphism between $G_1 = G$ and $G_2 = G \setminus v \cup \{v'\}$. It is clear that $G_1$ and $G_2 \cup \{v\}$ are good subgraphs. We will extend this isomorphism by a ‘back and forth’ argument.

Suppose we are given a partial isomorphism $\phi$ between two subgraphs $G_1$ and $G_2$ of $\uparrow G$, with the following properties:

1. $G_1$ and $G_2 \cup \{v\}$ are good subgraphs of $\uparrow G$

2. $V(G) \subseteq V(G_1)$, $V(G) \setminus v \subseteq V(G_2)$ and $v \not\in V(G_2)$

3. There is a vertex $\bar{v} \in V(G_2)$ such that $N(v) \cap V(G_2) \subseteq N(\bar{v}) \cap V(G_2)$

The vertex $\bar{v}$ (in the third property) will change at each step of our construction. We begin by setting $\bar{v} = v'$, and note that our initial $G_1$ and $G_2$ satisfy the conditions above.

Let $x \in V(\uparrow G)$ be a vertex of minimal level with $x \not\in V(G_1)$. This property implies that $N^-(x) \subseteq V(G_1)$ and so $G_1 \cup \{x\}$ is still a good graph. By Lemma 5, there exists $y \in V(G_1)$ such that $N(x) \cap V(G_1) \subseteq N[y] \cap V(G_1)$ and we can define $\phi(x)$ by taking a clone of $\phi(y)$ of level greater than $L(V(G_1) \cup V(G_2))$ such that

$$N^-(\phi(x)) = N(\phi(x)) \cap V(G_2) = \phi(N(x) \cap V(G_1)).$$

This extends the isomorphism, implies that $G_2 \cup \{\phi(x), v\}$ is still a good graph and that the vertex $\bar{v}$ still satisfies the desired property.
We now go in the opposite direction. Let $z$ be a vertex of minimal level with $z \notin V(G_2) \cup \{v\}$: we attempt to define $\phi^{-1}(z)$.

We distinguish two cases:

- $zv \notin E(\uparrow G)$, or $zv \in E(\uparrow G)$ and $z\tilde{v} \in E(\uparrow G)$.

As before, we can apply Lemma 5 to get $y \in V(G_2) \cup \{v\}$ such that $N(z) \cap V(G_2) \subseteq N[y] \cap V(G_2)$. If $y = v$, we can instead choose $y = \tilde{v}$. We can then define $\phi^{-1}(z)$ as previously to be a suitable clone of $\phi^{-1}(y)$.

- $zv \in E(\uparrow G)$ and $z\tilde{v} \notin E(\uparrow G)$.

In this case we will have to change $\tilde{v}$, because we want the condition $N(v) \cap V(G_2) \subseteq N(\tilde{v}) \cap V(G_2)$ to hold after adding $z$ to $G_2$. Let $w$ be a clone of $v$ such that $L(w) > L(V(G_1) \cup V(G_2))$ and $N^-(w) = (N(v) \cap V(G_2)) \cup \{z\}$. Such a vertex exists, since $z$ is a neighbour of $v$.

The only reason why the subgraph induced by $V(G_2) \cup \{v, w\}$ might not be a good graph is the edge $zw$. We therefore extend the isomorphism to $G_2 \cup \{z, v\}$. Since $G_2$ is a good graph, we can use Lemma 5 as before to first extend the isomorphism to $z$. Since, by minimality of $z$, the subgraph induced by $V(G_2) \cup \{z, v\}$ is also a good graph, we can use Lemma 5 again to extend the isomorphism to $w$. Finally, the definition of $w$ implies that $G_2 \cup \{z, w, v\}$ is a good graph, and we can choose the new $\tilde{v}$ to be $w$, as it satisfies the desired property.

Repeating the argument gives, in the limit, an isomorphism between $\uparrow G$ and $(\uparrow G) \setminus v$. \hfill \Box

### References


