

RESIDUAL IRREDUCIBILITY OF COMPATIBLE SYSTEMS

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ABSTRACT. We show that if $\{\rho_\ell\}$ is a compatible system of absolutely irreducible Galois representations of a number field then the residual representation $\bar{\rho}_\ell$ is absolutely irreducible for ℓ in a density 1 set of primes. The key technical result is the following theorem: the image of ρ_ℓ is an open subgroup of a hyperspecial maximal compact subgroup of its Zariski closure with bounded index (as ℓ varies). This result combines a theorem of Larsen on the semi-simple part of the image with an analogous result for the central torus that was recently proved by Barnet-Lamb, Gee, Geraghty, and Taylor, and for which we give a new proof.

CONTENTS

1. Introduction	1
2. The image in the torus	3
3. Irreducibility of residual representations	6
References	12

1. INTRODUCTION

The purpose of this paper is to prove the following theorem:

Theorem 1.1 (Informal version). *In a compatible system of absolutely irreducible Galois representations, a density 1 set of residual representations are absolutely irreducible.*

In the rest of the introduction, we state this theorem precisely, indicate the main idea of the proof, and discuss the connection to some previous work.

1.1. The main theorem. Let F and E be number fields. Let Γ_F be the absolute Galois group of F , let Σ be a set of places of E , and let $\{\rho_v\}_{v \in \Sigma}$ be a collection where $\rho_v: \Gamma_F \rightarrow \mathbf{GL}_n(E_v)$ is a continuous representation (for some n independent of v). We say that the family $\{\rho_v\}$ is a *compatible system* if there exists a finite set S of places of F and a finite multi-set I of integers such that the following conditions hold:

- Let $w \notin S$ be a place of F with residue characteristic different from that of $v \in \Sigma$. Then ρ_v is unramified at w , and the characteristic polynomial of $\rho_v(\text{Frob}_w)$ has coefficients in E and is independent of v .
- Let $w \notin S$ be a place of F with residue characteristic equal to that of $v \in \Sigma$. Then ρ_v is crystalline at w with Hodge–Tate weights in the multi-set I .

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We remark that often in the definition of a compatible system, a stronger condition is required on “labeled” Hodge–Tate weights, a refinement of the multi-set I into multi-sets indexed by embeddings $F \rightarrow \overline{E}$ (see [BLGGT, §5.1]); for us, however, the coarser condition just given suffices. For a set of rational prime numbers Π , let $\Sigma(\Pi)$ be the set of places of E above a prime in Σ . We will typically work with compatible systems indexed by sets of this form. For a representation ρ over a p -adic field, we write $\overline{\rho}$ for the residual representation (with respect to some lattice). Whenever we refer to the *density* of a set of places of a number field, we mean the Dirichlet density. We can now state the precise version of our theorem:

Theorem 1.2. *Let $\{\rho_v\}_{v \in \Sigma}$ be a compatible system of representations of Γ_F with coefficients in E , where $\Sigma = \Sigma(\Pi)$ for some set Π of rational primes of density 1. Suppose that each ρ_v is semi-simple, and let $\rho_v \otimes \overline{E}_v = \bigoplus_{i=1}^r \rho_{v,i}^{\oplus m_{v,i}}$ be its decomposition into irreducible representations over \overline{E}_v . (We assume $\rho_{v,i}$ and $\rho_{v,j}$ are non-isomorphic for $i \neq j$.) Then there exists a density 1 subset $\Pi' \subset \Pi$ such that for $v \in \Sigma' = \Sigma(\Pi')$ the representations $\overline{\rho}_{v,i}$ are irreducible and pairwise non-isomorphic (meaning $\overline{\rho}_{v,i}$ and $\overline{\rho}_{v,j}$ are non-isomorphic for $v \in \Sigma'$ and $i \neq j$).*

Corollary 1.3. *Suppose each ρ_v is absolutely irreducible. Then $\overline{\rho}_v$ is absolutely irreducible for all $v \in \Sigma'$.*

In the special case of Hodge–Tate regular compatible systems (see [BLGGT, §5.1]), this corollary is due to Barnet-Lamb, Gee, Geraghty, and Taylor (see [BLGGT, Proposition 5.3.2]), who have used it to powerful effect in establishing potential automorphy theorems for certain compatible systems of Galois representations over CM fields. As in their application, but now in greater generality, we obtain the following corollary:

Corollary 1.4. *Suppose each ρ_v is absolutely irreducible. Let v be a place in Σ' whose residue characteristic is at least $2(n+1)$. Then $H^1(\overline{\rho}_v(\Gamma_F), \text{Ad}^0(\overline{\rho}_v)) = 0$.*

Proof. Combine Corollary 1.3 with [GHTT, Appendix, Theorem 9]. \square

In the special case of abelian varieties—an example far removed from the case of regular compatible systems—our theorem yields by Faltings’ theorem:

Corollary 1.5. *Let A be an abelian variety over F with $\text{End}(A) = \mathbf{Z}$. Then the representation of Γ_F on $A[\ell]$ is absolutely irreducible for ℓ in a set of primes of density 1.*

This corollary is a weaker version of a result of Zarhin ([Z, Corollary 5.4.5]) that identifies $\text{End}(A[\ell])$ as $\text{End}(A) \otimes_{\mathbf{Z}} \mathbf{Z}/\ell\mathbf{Z}$ for all but finitely many primes ℓ . The corollary can be generalized in an obvious way to allow $\text{End}(A)$ to be an order in an arbitrary number field.

1.2. Algebraic monodromy groups. Let G be an algebraic group. We write G° for the identity component of G , which we assume to be reductive. We write G^{der} for the derived subgroup of G° , and G^{sc} for the universal cover of G^{der} , both of which are semi-simple. We let G^{tor} be the quotient G°/G^{der} , which is a torus. In a compatible system $\{\rho_v\}$, we write G_v for the Zariski closure of the image of ρ_v . If ρ_v is semi-simple then G_v° is reductive.

To prove Theorem 1.2, it is enough to show that the image of ρ_v in G_v is sufficiently large. We think of G_v as composed of three pieces: the component group $\pi_0(G_v)$, the semi-simple part G_v^{der} , and toral part G_v^{tor} . The image of ρ_v surjects onto $\pi_0(G_v)$, and Serre showed that this group is independent of ℓ . Larsen proved that the image of ρ_v hits a big piece of G_v^{der} . Our starting point is the observation that ρ_v hits enough of the torus; this has

been previously established in [BLGGT, Proposition 5.2.2], but we hope that our argument is somewhat easier to follow. We have also included our argument because it contains a finiteness observation (see the summary after Theorem 1.7) that may be of independent interest. Here is the technical result, proven in §2:

Theorem 1.6 (Compare [BLGGT], Proposition 5.2.2). *Let $\{\rho_\ell\}_{\ell \in \Sigma}$ be a compatible system of semi-simple representations of Γ_F with coefficients in \mathbf{Q} . Then there exists a positive integer N with the following property: the projection of $\mathrm{im}(\rho_\ell) \cap G_\ell^\circ$ to G_ℓ^{tor} has index at most N in the maximal compact subgroup of $G_\ell^{\mathrm{tor}}(\mathbf{Q}_\ell)$, for all $\ell \in \Sigma$.*

Combined with a theorem of Larsen, we obtain (see the discussion after the proof of Proposition 3.4):

Theorem 1.7. *Let $\{\rho_\ell\}_{\ell \in \Sigma}$ be a compatible system of semi-simple representations of Γ_F with coefficients in \mathbf{Q} , where Σ is a set of primes of density 1. Then there exists a positive integer N and a subset $\Sigma' \subset \Sigma$, also of density 1, such that for $\ell \in \Sigma'$ we can find a (not necessarily connected) reductive group scheme¹ $\mathcal{G}_\ell/\mathbf{Z}_\ell$ with generic fiber G_ℓ such that $\mathcal{G}_\ell(\mathbf{Z}_\ell)$ contains $\mathrm{im}(\rho_\ell)$ as an open subgroup of index at most N .*

We now indicate the main idea in the proof Theorem 1.6. It suffices to treat the case where G_ℓ is connected for all ℓ . Suppose that ρ_ℓ decomposes as a direct sum of irreducible representations $\rho_\ell^1, \dots, \rho_\ell^{n(\ell)}$. The determinants of these summands control G_ℓ^{tor} . In particular, if one knew that these determinants formed compatible systems, then the theory of abelian Galois representations would show that their images are large enough to prove the theorem. Unfortunately, there seems to be no way at present to prove this compatibility.

Our main observation is that one can quite easily prove something very close to this compatibility, and which suffices for our purposes. Precisely, we notice that there are only finitely many possibilities for (the Hecke character associated to) $\det(\rho_\ell^i)$: indeed, its ramification is constrained, and there are only finitely many possibilities for its infinity type and its value on any particular Frobenius element. It follows that one can partition Σ into finitely many subsets $\Sigma_1, \dots, \Sigma_r$ such that $\{\det(\rho_\ell^i)\}_{\ell \in \Sigma_k}$ forms a compatible system for all i and k .

Remark 1.8. Larsen’s result ([La, Thm. 3.17]) is very general: it applies to compatible systems of representations of any profinite group possessing certain “Frobenius elements.” By contrast, Theorem 1.6 (or at least our proof of it) is specific to Galois groups of number fields, and make critical (though simple) use of ℓ -adic Hodge theory. We also note that Larsen’s theorem depends on the classification of finite simple groups. \square

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2. THE IMAGE IN THE TORUS

We now prove Theorem 1.6. Fix the d -dimensional compatible system $\{\rho_\ell\}_{\ell \in \Sigma}$. By Serre’s theorem, after replacing F with a finite Galois extension, we can assume that G_ℓ is connected for all ℓ . Let S be the set of places of ramification of the compatible system, and let I be

¹For this paper, a group scheme \mathcal{G}/S is *reductive* if there is a subgroup scheme \mathcal{G}° that is smooth affine over S with connected reductive geometric fibers and a finite étale scheme π_0/S such that \mathcal{G} is isomorphic, as an S -scheme, to $\mathcal{G}^\circ \times_S \pi_0$. This is an extremely restrictive (and rather unnatural) way to allow disconnectedness, but good enough for our purposes.

the finite multi-set of integers occurring as Hodge–Tate weights. The abelianization of the Galois group $\Gamma_{F,S}$ is topologically finitely generated. Let T be a finite set of places of F such that the Frobenius elements Frob_v with $v \in T$ topologically generate $\Gamma_{F,S}^{\text{ab}}$. Denote by $\overline{\mathbf{Q}}$ the algebraic closure of \mathbf{Q} in \mathbf{C} . Fix once and for all an isomorphism $\overline{\mathbf{Q}}_\ell \cong \mathbf{C}$ for each ℓ , and use these to regard $\overline{\mathbf{Q}}$ as a subfield of $\overline{\mathbf{Q}}_\ell$. For $v \in T$, let $\lambda_{v,1}, \dots, \lambda_{v,d} \in \overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_\ell$ be the eigenvalues of $\rho_\ell(\text{Frob}_v)$; notice that this set is independent of ℓ by compatibility. Let \mathcal{H} be the set of algebraic Hecke characters χ of F satisfying the following conditions: (1) χ is unramified away from S ; (2) the weights of χ belong to the set of finite sums of distinct elements of (the multi-set) I ; and (3) for $v \in T$, the number $\chi(\text{Frob}_v)$ is of the form $\lambda_{v,i_1} \cdots \lambda_{v,i_r}$ where the indices i_1, \dots, i_r are distinct.

Lemma 2.1. *The set \mathcal{H} is finite.*

Proof. By (1), there are only finitely many possibilities for the weights of an element of \mathcal{H} . If χ_1 and χ_2 are elements of \mathcal{H} with the same weight then $\chi_1 \chi_2^{-1}$ defines a character $\Gamma_{F,S}^{\text{ab}} \rightarrow \mathbf{C}^\times$. By (3), there are only finitely many possibilities for the values of this character on the Frobenius elements Frob_v with $v \in T$. Since these Frobenius elements generate $\Gamma_{F,S}^{\text{ab}}$, it follows that there are only finitely many possibilities for $\chi_1 \chi_2^{-1}$. \square

Via the fixed isomorphism $\overline{\mathbf{Q}}_\ell \cong \mathbf{C}$, we can associate to each Hecke character of F a de Rham character $\Gamma_F \rightarrow \overline{\mathbf{Q}}_\ell^\times$, and vice versa. The following lemma exposes the key property of the set \mathcal{H} :

Lemma 2.2. *Let σ be a summand of $\rho_\ell \otimes \overline{\mathbf{Q}}_\ell$. Then the Hecke character corresponding to $\det(\sigma)$ belongs to \mathcal{H} .*

Proof. Since ρ_ℓ is unramified (or crystalline) at the places outside of S , the same is true for $\det(\sigma)$, and so the corresponding Hecke character is unramified outside of S . The Hodge–Tate weights of σ are a subset of those of ρ_ℓ , and Hodge–Tate weights of $\det(\sigma)$ are sums of those of σ , and these coincide with the weights of the corresponding Hecke character. Finally, for $v \in T$, the eigenvalues of $\sigma(\text{Frob}_v)$ are $\lambda_{v,i_1}, \dots, \lambda_{v,i_r}$ for distinct indices i_1, \dots, i_r , and $\det(\sigma)(\text{Frob}_v)$ is the product of these. \square

The Galois group $\Gamma_{\mathbf{Q}}$ acts on \mathcal{H} by restricting its action on algebraic Hecke characters: for any algebraic Hecke character ψ , the finite part ψ_f can be defined over $\overline{\mathbf{Q}}$, and for any $\sigma \in \Gamma_{\mathbf{Q}}$, there is then a unique (still algebraic) Hecke character ${}^\sigma\psi$ whose finite part is isomorphic to $\psi_f \otimes_{\overline{\mathbf{Q}},\sigma} \overline{\mathbf{Q}}$. Let \mathcal{S} be a multi-set whose elements belong to \mathcal{H} and that is stable under $\Gamma_{\mathbf{Q}_\ell}$. There is then an associated representation $\tau_{\mathcal{S},\ell}: \Gamma_F \rightarrow \mathbf{GL}_n(\mathbf{Q}_\ell)$, with $n = \#\mathcal{S}$, such that $\tau_{\mathcal{S},\ell} \otimes \overline{\mathbf{Q}}_\ell$ is the sum of the one-dimensional representations given by elements of \mathcal{S} . Let $T_{\mathcal{S},\ell}/\mathbf{Q}_\ell$ be the Zariski closure of the image of $\tau_{\mathcal{S},\ell}$.

Proposition 2.3. *There exists a positive integer M such that the image of $\tau_{\mathcal{S},\ell}$ in $T_{\mathcal{S},\ell}(\mathbf{Q}_\ell)$ has index at most M in the maximal compact, for any choice of ℓ and $\Gamma_{\mathbf{Q}_\ell}$ -stable \mathcal{S} .*

Proof. Let T_F denote the restriction of scalars torus $\text{Res}_{F/\mathbf{Q}}(\mathbf{G}_m)$, and let $T_{F,\ell}$ denote $T_F \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$. The representation $\tau_{\mathcal{S},\ell}$ is Hodge–Tate and abelian, hence locally algebraic ([Se, A7 Theorem 3]): there exists an algebraic morphism of tori $f_{\mathcal{S},\ell}: T_{F,\ell} \rightarrow T_{\mathcal{S},\ell}$ such that (leaving the class field theory identification implicit) $f_{\mathcal{S},\ell}|_U = \tau_{\mathcal{S},\ell}|_U$ for some open subgroup U of $(F \otimes_{\mathbf{Q}} \mathbf{Q}_\ell)^\times$. We claim that $f_{\mathcal{S},\ell}$ is surjective. Since $T_{\mathcal{S},\ell}(\mathbf{Q}_\ell)$ has a pro- ℓ open subgroup, and

$\tau_{S,\ell}(U)$ contains an open pro- ℓ subgroup of $\tau_{S,\ell}(\Gamma_F)$, $\tau_{S,\ell}(U)$ is open in $\tau_{S,\ell}(\Gamma_F)$. As $T_{S,\ell}$ is connected, $\tau_{S,\ell}(U) = f_{S,\ell}(U)$ is Zariski-dense in $T_{S,\ell}$, so $f_{S,\ell}$ is a surjection.

Next, we are allowed to discard a finite number of ℓ , so we now assume that the Hecke characters in \mathcal{H} are unramified at all primes above ℓ , and that F/\mathbf{Q} is unramified at ℓ . We may therefore assume $\tau_{S,\ell}$ is crystalline, and that the torus $T_{F,\ell}$, and therefore its quotient $T_{S,\ell}$, are unramified. These tori canonically extend to tori over \mathbf{Z}_ℓ , as does the morphism $f_{S,\ell}$ (all of which we continue to denote by the same symbols). Moreover, by [CCO, 3.9.7 Corollary], $\tau_{S,\ell}$ and $f_{S,\ell}$ in fact agree on all of $(\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_\ell)^\times$.

To find an upper bound on the index $[T_{S,\ell}(\mathbf{Z}_\ell) : \tau_{S,\ell}(\Gamma_F)]$, it therefore suffices to bound the cokernel of the induced map on \mathbf{Z}_ℓ -points, i.e. the index $[T_{S,\ell}(\mathbf{Z}_\ell) : f_{S,\ell}(T_{F,\ell}(\mathbf{Z}_\ell))]$. We begin by bounding the order of the torsion subgroup of $X^\bullet(T_{F,\ell})/f_{S,\ell}^*(X^\bullet(T_{S,\ell}))$, where we write X^\bullet for the character group. Denote by a the degree $[F : \mathbf{Q}]$. Each of the Hecke characters $\psi \in \mathcal{S} \subset \mathcal{H}$ has a collection of Hodge–Tate numbers, and there is an absolute bound, independent of ℓ and \mathcal{S} , for the greatest of these integers; let us call it N . We can then identify the image $f_{S,\ell}^*X^\bullet(T_{S,\ell})$ with the submodule of \mathbf{Z}^a spanned by a collection of integer vectors, all of whose coordinates have absolute value at most N . Next note that for any morphism of abelian groups $f : \mathbf{Z}^b \rightarrow \mathbf{Z}^a$ for which a set of generators of \mathbf{Z}^b map to vectors whose coordinates are bounded by N , the order of $\text{coker}(f)_{\text{tor}}$ is bounded by a function of a and N . Namely, $|\text{coker}(f)_{\text{tor}}|$ is equal to the greatest common divisor of the $r \times r$ minors, where r equals the rank of f (in our application, $r = b$). In general, then, the cokernel can have torsion order no greater than the largest determinant of an $a \times a$ integer matrix with entries bounded in absolute value by N .

We thus have a uniform bound for the torsion subgroup of $X^\bullet(T_{F,\ell})/f_{S,\ell}^*(X^\bullet(T_{S,\ell}))$. If we discard the finitely many ℓ below this bound, we see that $f_{S,\ell}$ is not only a surjection of tori, but is in fact a surjection of étale sheaves on $\text{Spec } \mathbf{Z}_\ell$ (we can check this over an unramified extension of \mathbf{Z}_ℓ that splits the tori, where by the invariant factor decomposition of the character groups it reduces to the assertion that multiplication by n on \mathbf{G}_m is a surjection of étale sheaves in characteristic not dividing n). We can therefore bound the index $[T_{S,\ell}(\mathbf{Z}_\ell) : f_{S,\ell}(T_{F,\ell}(\mathbf{Z}_\ell))]$ by²

$$\begin{aligned} H_{\text{ét}}^1(\text{Spec } \mathbf{Z}_\ell, \ker(f_{S,\ell})) &= H_{\text{ét}}^1(\text{Spec } \mathbf{F}_\ell, \ker(f_{S,\ell})) \\ &= H^1\left(\Gamma_{\mathbf{F}_\ell}, \text{Hom}(X^\bullet(T_{F,\ell})/f_{S,\ell}^*X^\bullet(T_{S,\ell}), \overline{\mathbf{F}}_\ell^\times)\right) \\ &= \text{Hom}(X^\bullet(T_{F,\ell})/f_{S,\ell}^*X^\bullet(T_{S,\ell}), \overline{\mathbf{F}}_\ell^\times)_{(\text{Fr}_\ell - 1)}. \end{aligned}$$

(The last subscript denotes $\text{Fr}_\ell - 1$ coinvariants.) Choose an integer m so that $\Gamma_{\mathbf{F}_{\ell^m}}$ acts trivially on $X^\bullet(T_{F,\ell})$. Then there is a surjection

$$\text{Hom}(X^\bullet(T_{F,\ell})/f_{S,\ell}^*X^\bullet(T_{S,\ell}), \overline{\mathbf{F}}_\ell^\times)_{(\text{Fr}_{\ell^m} - 1)} \rightarrow \text{Hom}(X^\bullet(T_{F,\ell})/f_{S,\ell}^*X^\bullet(T_{S,\ell}), \overline{\mathbf{F}}_\ell^\times)_{(\text{Fr}_\ell - 1)},$$

and the source of this map has order bounded by the order of the torsion subgroup of $X^\bullet(T_{F,\ell})/f_{S,\ell}^*X^\bullet(T_{S,\ell})$ (the torsion-free quotient has vanishing cohomology, by Hilbert 90), for which we have already obtained an independent of ℓ bound. The proposition follows. \square

Fix a prime $\ell \in \Sigma$ for the moment. Decompose $\rho_\ell \otimes \overline{\mathbf{Q}}_\ell$ as $\sigma_1 \oplus \cdots \oplus \sigma_r$ with σ_i irreducible of dimension d_i . Over $\overline{\mathbf{Q}}_\ell$, we can regard G_ℓ as a subgroup of $\mathbf{GL}_{d_1} \times \cdots \times \mathbf{GL}_{d_r} \subset \mathbf{GL}_d$.

²The following calculation comes from [BLGJT, Lemma A.1.6], although that lemma is stated somewhat imprecisely: multiplication by 2 on \mathbf{G}_m over \mathbf{Z}_2 is a counter-example, unless “surjection” is read as “surjection of étale sheaves.”

Consider the diagram

$$\begin{array}{ccc} G_\ell & \longrightarrow & \mathbf{GL}_{d_1} \times \cdots \times \mathbf{GL}_{d_r} \\ \downarrow & & \downarrow \\ G_\ell^{\text{tor}} & \xrightarrow{\delta} & \mathbf{G}_m^r \end{array}$$

where the right vertical maps are determinants, and the map δ is induced by the universal property of G_ℓ^{tor} (every map from G to a torus factors through G^{tor}). The central torus Z of G_ℓ maps to the central \mathbf{G}_m in each \mathbf{GL}_{d_i} . The map $\mathbf{G}_m^r \subset \mathbf{GL}_{d_1} \times \cdots \times \mathbf{GL}_{d_r} \xrightarrow{\det} \mathbf{G}_m^r$ has kernel of size $d_1 \cdots d_r$. It follows that the kernel of the determinant map $Z \rightarrow \mathbf{G}_m^r$ is also bounded by this, and so is the kernel of δ . The composite $\Gamma_F \rightarrow G_\ell^{\text{tor}}(\overline{\mathbf{Q}}_\ell) \rightarrow \mathbf{G}_m^r(\overline{\mathbf{Q}}_\ell)$ is given explicitly by $(\det(\sigma_1), \dots, \det(\sigma_r))$. Let $\mathcal{S} = \{\det(\sigma_i)\} \subset \mathcal{H}$. This set is clearly $\Gamma_{\mathbf{Q}_\ell}$ -stable. Theorem 1.6 now follows from Proposition 2.3 and the following lemma.

Lemma 2.4. *There is an isogeny of tori $G_\ell^{\text{tor}} \rightarrow T_{\mathcal{S},\ell}$ (defined over \mathbf{Q}_ℓ), the degree of which is bounded independent of ℓ , such that the diagram*

$$\begin{array}{ccc} \Gamma_F & \longrightarrow & G_\ell^{\text{tor}}(\mathbf{Q}_\ell) \\ & \searrow \tau_{\mathcal{S},\ell} & \downarrow \\ & & T_{\mathcal{S},\ell}(\mathbf{Q}_\ell) \end{array}$$

commutes.

Proof. The composite of the above homomorphism $\Gamma_F \rightarrow \mathbf{G}_m^r(\overline{\mathbf{Q}}_\ell)$ with the direct sum map $\mathbf{G}_m^r \rightarrow \mathbf{GL}_r$ is $\tau_{\mathcal{S},\ell} \otimes \overline{\mathbf{Q}}_\ell$. By the universal property of G_ℓ^{tor} , there is a unique homomorphism $\pi_{\mathcal{S},\ell}: G_\ell^{\text{tor}} \otimes \overline{\mathbf{Q}}_\ell \rightarrow T_{\mathcal{S},\ell} \otimes \overline{\mathbf{Q}}_\ell$ such that the diagram

$$\begin{array}{ccc} \Gamma_F & \longrightarrow & G_\ell^{\text{tor}}(\overline{\mathbf{Q}}_\ell) \\ & \searrow \tau_{\mathcal{S},\ell} & \downarrow \pi_{\mathcal{S},\ell} \\ & & T_{\mathcal{S},\ell}(\overline{\mathbf{Q}}_\ell) \end{array}$$

commutes. Conjugating $\pi_{\mathcal{S},\ell}$ by any element of $\Gamma_{\mathbf{Q}_\ell}$ yields another such diagram, since the image of Γ_F in both G_ℓ^{tor} and $T_{\mathcal{S},\ell}$ lies in the \mathbf{Q}_ℓ -points. By uniqueness, we conclude that $\pi_{\mathcal{S},\ell}$ is $\Gamma_{\mathbf{Q}_\ell}$ -invariant, hence descends to a morphism $\pi_{\mathcal{S},\ell}: G_\ell^{\text{tor}} \rightarrow T_{\mathcal{S},\ell}$. It is a surjection since $T_{\mathcal{S},\ell}$ is by definition the Zariski-closure of the image of $\tau_{\mathcal{S},\ell}$; and its kernel is finite, bounded independently of ℓ by $d_1 \cdots d_r$. \square

3. IRREDUCIBILITY OF RESIDUAL REPRESENTATIONS

3.1. Extending automorphisms of group schemes.

Lemma 3.1. *Let \mathcal{O} be a complete DVR with fraction field K and residue field k , let X/\mathcal{O} be a smooth affine scheme, and let $f: X_K \rightarrow X_K$ be a morphism of K -schemes such that f carries $X(\mathcal{O}_L)$ into itself for every finite unramified extension L/K . Then f extends uniquely to a morphism of \mathcal{O} -schemes $X \rightarrow X$.*

Proof. Write $X = \text{Spec}(A)$, with A a flat \mathcal{O} -algebra of finite type, so that f corresponds to a map of K -algebras $f^*: A_K \rightarrow A_K$. We must show that f^* carries A into itself. Let φ be an element of A , and let π be a uniformizer of \mathcal{O} . Let $n \geq 0$ be minimal so that

$\pi^n f^*(\varphi) \in A$. We claim $n = 0$, which will prove the lemma. Suppose not. Let x be an ℓ -point of X for some finite separable extension ℓ/k , and let L be the unramified extension of K corresponding to ℓ . Since X is smooth and \mathcal{O}_L is complete, x lifts to a \mathcal{O}_L -point \tilde{x} of X . We have $f^*(\varphi)(\tilde{x}) = \varphi(f(\tilde{x})) \in \mathcal{O}_L$ since $f(\tilde{x}) \in X(\mathcal{O}_L)$. Since $n > 0$, it follows that $(\pi^n f^*(\varphi))(x) = 0$. Thus $\pi^n f^*(\varphi)$ vanishes on all k^s -points of the special fiber, and thus vanishes on the special fiber since the special fiber is reduced. That is, $\pi^n f^*(\varphi) = 0$ in $A/\pi A$, and so $\pi^n f^*(\varphi) \in \pi A$. But this implies $\pi^{n-1} f^*(\varphi) \in A$ since A is flat over \mathcal{O} , a contradiction. \square

Lemma 3.2. *Let K/\mathbf{Q}_ℓ be a finite extension, let $\mathcal{O} = \mathcal{O}_K$, let \mathcal{G}/\mathcal{O} be a simply connected semi-simple group, and let $f: \mathcal{G}_K \rightarrow \mathcal{G}_K$ be an automorphism of the generic fiber such that f carries $\mathcal{G}(\mathcal{O})$ into itself. Then f extends uniquely to an automorphism $\mathcal{G} \rightarrow \mathcal{G}$ of \mathcal{O} -groups.*

Proof. By [SW, Lemma 6.2], f carries $\mathcal{G}(\mathcal{O}_L)$ into itself for any tamely ramified finite extension L/K . Therefore, by the previous lemma, f uniquely extends to a map $\mathcal{G} \rightarrow \mathcal{G}$ of \mathcal{O} -schemes, which is necessarily a group automorphism by uniqueness. \square

Remark 3.3. In fact, Lemma 3.2 is true when \mathcal{G} is connected reductive. One can deduce the general result from the simply connected case fairly easily. \square

3.2. A result on hyperspecial groups. Let K/\mathbf{Q}_ℓ be a finite extension and let $\mathcal{O} = \mathcal{O}_K$ be the ring of integers. Let G/K be a reductive group with G° unramified and let $\Gamma \subset G(K)$ be a Zariski dense profinite group. We use the following notation:

$$\begin{array}{ccc} G^{\text{sc}} & \xrightarrow{\pi} & G^\circ \\ & \searrow \sigma & \downarrow \tau \\ & & G^{\text{ad}} \end{array}$$

(Note that π factors as $G^{\text{sc}} \rightarrow G^{\text{der}} \subset G^\circ$.) We let $\Gamma^\circ = \Gamma \cap G^\circ(K)$.

Proposition 3.4. *Assume $\sigma^{-1}(\tau(\Gamma^\circ))$ is a hyperspecial subgroup of $G^{\text{sc}}(K)$ and that Γ^{tor} is open in $G^{\text{tor}}(K)$. Also assume that the residue characteristic ℓ is sufficiently large compared to $\dim(G)$. Then there exists a reductive group \mathcal{G}/\mathcal{O} with generic fiber G such that Γ is an open subgroup of $\mathcal{G}(\mathcal{O})$. Moreover, the index of Γ in $\mathcal{G}(\mathcal{O})$ can be bounded in terms of the dimension of G and the index of Γ^{tor} in the maximal compact of $G^{\text{tor}}(K)$.*

Proof. Let $\mathcal{G}^{\text{sc}}/\mathcal{O}$ be the given semi-simple group with generic fiber G^{sc} such that $\mathcal{G}^{\text{sc}}(\mathcal{O}) = \sigma^{-1}(\tau(\Gamma^\circ))$. Let \mathcal{G}° be the unique extension of G° to \mathcal{O} for which the map $G^{\text{sc}} \rightarrow G^\circ$ extends. One can construct this extension as follows. Let T be the central torus in G° , and let \mathcal{T}/\mathcal{O} be the unique torus with generic fiber T (note that T is unramified by our assumption that G° is unramified). Let \mathcal{Z}/\mathcal{O} be the center of \mathcal{G}^{sc} , which is finite étale (due to our assumption on ℓ), and let $Z = \mathcal{Z}_K$. Let H be the kernel of the map $T \times G^{\text{sc}} \rightarrow G^\circ$, which is contained in $T \times Z$. The order n of H can be bounded in terms of $\dim(G)$, and is therefore small compared to ℓ . Let \mathcal{H} be the closure of H in $\mathcal{T} \times \mathcal{Z}$. Since H is contained in the finite étale group scheme $\mathcal{T}[n] \times \mathcal{Z}$ it follows that \mathcal{H} is finite étale. The group \mathcal{G}° is then the quotient of $\mathcal{T} \times \mathcal{G}^{\text{sc}}$ by \mathcal{H} , which exists by general theory. We do not prove uniqueness of \mathcal{G}° , as it is not needed in the proof.

Let γ be an element of Γ . Then conjugation by γ induces an automorphism of G° which descends to G^{ad} and then lifts (uniquely) to G^{sc} . Denote this automorphism of G^{sc} by f'_γ .

Let $g \in \mathcal{G}^{\text{sc}}(\mathcal{O})$. Then $\sigma(g) = \tau(\alpha)$ for some $\alpha \in \Gamma^\circ$. We have

$$\sigma(f'_\gamma(g)) = \gamma\sigma(g)\gamma^{-1} = \gamma\tau(\alpha)\gamma^{-1} = \tau(\beta)$$

where $\beta = \gamma\alpha\gamma^{-1}$. Since $\beta \in \Gamma^\circ$, it follows that $f'_\gamma(g) \in \sigma^{-1}(\tau(\Gamma^\circ)) = \mathcal{G}^{\text{sc}}(\mathcal{O})$. Thus f'_γ maps $\mathcal{G}^{\text{sc}}(\mathcal{O})$ to itself, and therefore, by Lemma 3.2, extends uniquely to an automorphism of \mathcal{G}^{sc} , still denoted f'_γ .

It is clear that conjugation by γ carries T into itself. It therefore extends uniquely to an automorphism f''_γ of \mathcal{T} . (Reason: if T is split then $\text{Aut}(T) = \text{Aut}(\mathcal{T})$. In general, we can reduce to the split case by passing to a finite unramified extension of \mathcal{O} . Since the extension is unique, it will descend to \mathcal{O} .)

Now consider the quotient map $\varphi: \mathcal{T} \times \mathcal{G}^{\text{sc}} \rightarrow \mathcal{G}^\circ$. Let $t \in \mathcal{T}(\overline{K})$ and $h \in \mathcal{G}^{\text{sc}}(\overline{K})$. Then

$$\varphi(f''_\gamma(t), f'_\gamma(h)) = f''_\gamma(t)\pi(f'_\gamma(h)) = (\gamma t \gamma^{-1})(\gamma \pi(h) \gamma^{-1}) = \gamma \varphi(t, h) \gamma^{-1}.$$

It follows that (f''_γ, f'_γ) maps $\mathcal{H}_{\overline{K}}$, and therefore \mathcal{H} , to itself, and thus descends to an automorphism f_γ of \mathcal{G}° . Moreover, it follows that f_γ induces conjugation by γ on the generic fiber of \mathcal{G}° .

We thus see that Γ normalizes $\mathcal{G}^\circ(\mathcal{O}) \subset G(K)$. In particular, Γ° is a compact subgroup of $G^\circ(K)$ normalizing $\mathcal{G}^\circ(\mathcal{O})$, and must therefore be contained in $\mathcal{G}^\circ(\mathcal{O})$, as $\mathcal{G}^\circ(\mathcal{O})$ is maximal compact.

We now define \mathcal{G} to be the pushout of $\Gamma^\circ \rightarrow \Gamma$ along the map $\Gamma^\circ \rightarrow \mathcal{G}^\circ$. Explicitly, write $\Gamma = \coprod_{i=1}^r \gamma_i \Gamma^\circ$ with $\gamma_i \in \Gamma$. Then $\mathcal{G} = \coprod_{i=1}^r \gamma_i \mathcal{G}^\circ$. Suppose $\gamma_i \gamma_j = \gamma_k \gamma'$ with $\gamma' \in \Gamma^\circ$. Then we define a map

$$\gamma_i \mathcal{G}^\circ \times \gamma_j \mathcal{G}^\circ \rightarrow \gamma_k \mathcal{G}^\circ$$

by mapping $(\gamma_i g, \gamma_j h)$ to $\gamma_k \gamma' f_{\gamma_j^{-1}}(g) h$. Note that $(g, h) \mapsto \gamma' f_{\gamma_j^{-1}}(g) h$ is a well-defined map of \mathcal{O} -schemes $\mathcal{G}^\circ \times \mathcal{G}^\circ \rightarrow \mathcal{G}^\circ$. Combining these maps for all i and j gives the multiplication on \mathcal{G} .

It is clear that \mathcal{G} is a reductive group over \mathcal{O} with generic fiber G , and that Γ is contained in $\mathcal{G}(\mathcal{O})$. We now show that Γ is open. It suffices to show Γ° is open. Since $\sigma^{-1}(\tau(\Gamma^\circ))$ is open in $G^{\text{sc}}(K)$, it follows that $\tau(\Gamma^\circ)$ is open. The image of Γ° in $\tau(\Gamma^\circ) \times \Gamma^{\text{tor}}$ is open, and thus the image of Γ° under the map $G^\circ(K) \rightarrow G^{\text{ad}}(K) \times G^{\text{tor}}(K)$ is open. It follows that Γ° is open in $G^\circ(K)$.

Finally, we bound the index of Γ in $\mathcal{G}(\mathcal{O})$. Since Γ meets every connected component, it suffices to bound the index of Γ° in $\mathcal{G}^\circ(\mathcal{O})$. The kernel of the map $\varphi: \mathcal{G}^\circ(\mathcal{O}) \rightarrow \mathcal{G}^{\text{ad}}(\mathcal{O}) \times \mathcal{G}^{\text{tor}}(\mathcal{O})$ is finite, with order bounded in terms of $\dim(G)$, so it suffices to bound the index of $\varphi(\Gamma^\circ)$ in the target. The group $\varphi(\Gamma^\circ)$ is contained in $\tau(\Gamma^\circ) \times \Gamma^{\text{tor}}$, and the index can be bounded in terms of $\dim(G)$. It thus suffices to bound the index of $\tau(\Gamma^\circ)$ in $\mathcal{G}^{\text{ad}}(\mathcal{O})$. We have $\tau(\Gamma^\circ) = \sigma(\mathcal{G}^{\text{sc}}(\mathcal{O}))$, so we must control the index of the image of the map $\mathcal{G}^{\text{sc}} \rightarrow \mathcal{G}^{\text{ad}}$ on \mathcal{O} points. Since this map is a surjection of group schemes with finite étale kernel \mathcal{Z} , it follows that the index is bounded by the order of $H_{\text{et}}^1(\text{Spec}(\mathcal{O}), \mathcal{Z})$. This can be bounded in terms of the order of \mathcal{Z} , and thus in terms of $\dim(G)$. \square

Proof of Theorem 1.7. The theorem follows by combining Proposition 3.4 with Larsen's theorem on hyperspecial image ([La, Thm. 3.17]) and Theorem 1.6: indeed, Larsen's theorem exactly states that the first hypothesis of Proposition 3.4 holds (taking $\Gamma = \text{im}(\rho_\ell)$), while Theorem 1.6 ensures the second (and furthermore, in the notation of Proposition 3.4, bounds the index of Γ^{tor} in the maximal compact of $G^{\text{tor}}(K)$ uniformly in ℓ). \square

3.3. Some representation theory. Let k be a field, let G/k be a connected semi-simple group, and let V be an algebraic representation of $G_{\bar{k}}$. Pick a maximal torus T in $G_{\bar{k}}$. For a weight λ of T , we define $\|\lambda\|$ to be the maximum of $|\langle \lambda, \alpha^\vee \rangle|$ as α varies over the roots of $G_{\bar{k}}$. We define $\|V\|$ to be the maximum of $\|\lambda\|$ over those weights appearing in V . This is independent of the choice of T . If k has characteristic 0 then (for fixed $\dim(G)$) one can bound $\dim(V)$ in terms of $\|V\|$, and vice versa.

Now suppose that G/k is a torus and λ is a character of $G_{\bar{k}}$. Define $\|\lambda\|$ to be $\#\pi_0(\ker(\lambda))$. For a representation V of $G_{\bar{k}}$ define $\|V\|$ to be the maximum of $\|\lambda\|$ over characters λ appearing in V .

Finally, suppose that G/k is a reductive group and V is a representation of $G_{\bar{k}}$. We define $\|V\|_{\text{der}}$ to be the norm of V restricted to G^{der} and $\|V\|_{\text{cen}}$ to be the norm of V restricted to the central torus of G° .

Proposition 3.5. *Let \mathcal{O} be a complete DVR with fraction field K and residue field k , let \mathcal{G}/\mathcal{O} be a reductive group (as defined in the introduction), and let \mathcal{V} be a representation of \mathcal{G} over \mathcal{O} . Suppose that $\dim(\mathcal{G})$, $\dim(\mathcal{V})$, and $\#\pi_0(G)$ are small compared to $\text{char}(k)$. Then the natural map $(\mathcal{V}^{\mathcal{G}})_k \rightarrow \mathcal{V}_k^{\mathcal{G}_k}$ is an isomorphism.*

Proof. The map in question is injective, and we must prove it is surjective. That is, given a \mathcal{G}_k -invariant vector v_1 in \mathcal{V}_k , we must produce a lift to a \mathcal{G} -invariant vector in \mathcal{V} . Suppose that v_n is a lift of v_1 to a \mathcal{G} -invariant vector in $\mathcal{V}/\pi^n \mathcal{V}$, where π is a uniformizer of \mathcal{O} . Let X be the fiber of the reduction map $\mathcal{V}/\pi^{n+1} \mathcal{V} \rightarrow \mathcal{V}/\pi^n \mathcal{V}$ over v_n . Since v_n is invariant, X is stable for the action of \mathcal{G} , and is naturally a torsor for \mathcal{V}_k . Thus X is classified by an element of $H^1(\mathcal{G}_k, \mathcal{V}_k)$. (This cohomology is algebraic group cohomology. Explicitly, let \tilde{v} be a lift of v_n . Then $g \mapsto g\tilde{v} - \tilde{v} \in \pi^n \mathcal{V}/\pi^{n+1} \mathcal{V} \cong \mathcal{V}_k$ is a representing cocycle.) We claim that this cohomology group vanishes. First, the representation \mathcal{V}_k of \mathcal{G}_k° has small norm (see, for instance, [SW, Prop. 3.5]). An element of $H^1(\mathcal{G}_k^\circ, \mathcal{V}_k)$ is represented by an extension of \mathcal{V}_k by the trivial representation. However, such an extension still has small norm, and is therefore semi-simple (see, for instance, [SW, Prop. 3.3]). Thus $H^1(\mathcal{G}_k^\circ, \mathcal{V}_k)$ vanishes. The map

$$H^1(\mathcal{G}_k, \mathcal{V}_k) \rightarrow H^1(\mathcal{G}_k^\circ, \mathcal{V}_k)$$

is injective, since $\#\pi_0(G)$ is coprime to $\text{char}(k)$, and so $H^1(\mathcal{G}_k, \mathcal{V}_k)$ vanishes. Thus X is a trivial torsor, and therefore has a fixed point v_{n+1} . We thereby obtain a compatible sequence $(v_n)_{n \geq 0}$ of \mathcal{G} -invariant elements which defines the requisite \mathcal{G} -invariant element in \mathcal{V} . \square

Remark 3.6. B. Conrad pointed out that this proposition is false if one allows a very permissive definition of “reductive” for non-connected groups. \square

Proposition 3.7. *Let k be a finite field, let G/k be a reductive group, and let $\Gamma \subset G(k)$ be a subgroup of small index (compared to $\text{char}(k)$) that meets every geometric component. Let V be a representation of G over \bar{k} . Assume that $\dim(G)$, $\|V\|_{\text{der}}$, and $\|V\|_{\text{cen}}$ are small compared to $\text{char}(k)$. Then $V^G = V^\Gamma$.*

Proof. It suffices to treat the case where G is connected. We can treat separately the cases where G is a torus and G is semi-simple, and we first treat the semi-simple case, for which it suffices to assume that G is simply connected. Let $\bar{\Gamma}$ denote the image of Γ in the quotient group $PG(k) = \text{im}(G(k) \rightarrow G^{\text{ad}}(k))$. If G is simple and simply-connected, then $PG(k)$ is (for $\text{char}(k) \gg 0$) a finite simple group, and then consideration of the permutation representation

of $PG(k)$ on $PG(k)/\bar{\Gamma}$ implies that either $\bar{\Gamma} = PG(k)$ or

$$|PG(k)| \leq [PG(k) : \bar{\Gamma}]! \leq ([G(k) : \Gamma] \cdot |\ker(G(k) \rightarrow G^{\text{ad}}(k))|)!.$$

By assumption, the right-hand side of this inequality is uniformly bounded, while the left-hand side tends to infinity with $\text{char}(k)$. We conclude that when G is simple, and for $\text{char}(k)$ sufficiently large, we must have $\bar{\Gamma} = PG(k)$. Now, for any simply-connected G , Goursat's Lemma (and again the assumption that $\Gamma \subset G(k)$ has small index) then implies that $\bar{\Gamma} = PG(k)$. The (central) kernel of $G(k) \rightarrow G^{\text{ad}}(k)$ and Γ therefore generate $G(k)$, and the action of $G(k)$ on V^Γ decomposes into a sum of characters; moreover, since $\dim(G)$ and $\|V\|_{\text{der}}$ are small compared to $\text{char}(k)$, the action on V^Γ is algebraic (see, for instance, [SW, Prop. 3.6]). But G is simply-connected, so its one-dimensional representations are trivial, hence $V^\Gamma = V^{G(k)}$. (In fact, $\Gamma = G(k)$, but we do not need this.) The result then follows from the fact that $V^G = V^{G(k)}$, which again relies on $\dim(G)$ and $\|V\|_{\text{der}}$ being small compared to $\text{char}(k)$ (using [SW, Prop. 3.6]).

We now treat the case where G is a torus. It suffices to treat the case where V is one-dimensional. Let λ be the character giving the action of G . It suffices to treat the case where λ is non-trivial, and show that $\lambda|_\Gamma$ is non-trivial. Note that λ factors as $G_{\bar{k}} \xrightarrow{p} \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m$ where p is the projection onto a direct factor and $|n| = \|\lambda\|$ is small by the assumption on $\|V\|_{\text{cen}}$. Since $p(\Gamma)$ has small index in $\mathbf{G}_m(\bar{k})$, so does $p(\Gamma)^n$, and so λ is non-trivial on Γ . \square

3.4. A general irreducibility result. We return to the setting of §3.2.

Theorem 3.8. *Let G/K be a reductive group with G° unramified, and let $\Gamma \subset G(K)$ be a Zariski dense profinite group such that $\sigma^{-1}(\tau(\Gamma^\circ))$ is hyperspecial in $G^{\text{sc}}(K)$. Assume that Γ^{tor} is open in $G^{\text{tor}}(K)$ and its index in the maximal compact is small compared to $\text{char}(k)$. Also assume that $\dim(G)$ and $\#\pi_0(G)$ are small compared to $\text{char}(k)$.*

Let (ρ, V) be a representation of G over \bar{K} with $\dim(V)$ small compared to $\text{char}(k)$. Then there exists a Γ -stable lattice \mathcal{V} in V such that the natural map

$$\text{End}_\Gamma(\mathcal{V}) \otimes \bar{k} \rightarrow \text{End}_\Gamma(\mathcal{V}_{\bar{k}})$$

is an isomorphism.

Corollary 3.9. *Let ρ be as in Theorem 3.8. Suppose that ρ decomposes as $\bigoplus_{i=1}^r \rho_i^{\oplus m_i}$ where the ρ_i are pairwise non-isomorphic irreducible representations. Then the $\bar{\rho}_i$ are irreducible and pairwise non-isomorphic and (for an appropriate choice of lattice) $\bar{\rho}$ decomposes as $\bigoplus_{i=1}^r \bar{\rho}_i^{\oplus m_i}$.*

Proof of theorem. By Proposition 3.4, we can find a reductive group \mathcal{G}/\mathcal{O} with generic fiber G such that Γ is a Zariski-dense open subgroup of $\mathcal{G}(\mathcal{O})$ with small index. Descend V to a representation V' over a finite extension K'/K , and let \mathcal{V} be a \mathcal{G} -stable lattice in V' (see [La, §1.12] for a proof of existence). Let $\mathcal{E} = \text{End}(\mathcal{V})$, which is also a representation of small norm (again by [SW, Prop. 3.5]). We have

$$(\mathcal{E}^\Gamma)_{\bar{k}} = (\mathcal{E}^\mathcal{G})_{\bar{k}} = (\mathcal{E}_{\bar{k}})^{\mathcal{G}_{\bar{k}}} = (\mathcal{E}_{\bar{k}})^\Gamma$$

The first equality follows from the fact that Γ and \mathcal{G} have the same Zariski closure. The second is Proposition 3.5 (working over $\mathcal{O}_{K'}^{\text{un}}$), while the third is Proposition 3.7. \square

3.5. Proof of the Theorem 1.2. Let $\{\rho_v\}_{v \in \Sigma}$ be a d -dimensional compatible system of representations of Γ_F with coefficients in E , where $\Sigma = \Sigma(\Pi)$. For $\ell \in \Pi$ define $\rho_\ell = \bigoplus_{v|\ell} \rho_v$. We regard ρ_ℓ as a representation of Γ_F over the field \mathbf{Q}_ℓ of dimension $d \cdot [E : \mathbf{Q}_\ell]$. As such, $\{\rho_\ell\}_{\ell \in \Pi}$ forms a compatible system with coefficients in \mathbf{Q} . Let $\Pi' \subset \Pi$ be a density one subset for which Larsen's theorem on hyperspecial image holds, and fix $\ell \in \Pi'$. Let G be the Zariski closure of the image of ρ_ℓ , and let $\Gamma = \rho_\ell(\Gamma_F)$. Then we can regard ρ_v as an algebraic representation of G over the field E_v . The result now follows from Theorem 3.8, which we may apply since by Larsen's theorem G° is unramified and $\sigma^{-1}(\tau(\Gamma^0))$ is hyperspecial, by Theorem 1.6 the image of Γ^{tor} in the maximal compact of $G^{\text{tor}}(\mathbf{Q}_\ell)$ is bounded independently of $\text{char}(k)$, and by Serre's theorem the component group $\pi_0(G_\ell)$ is independent of ℓ . \square

3.6. Further results. In [BLGGT], a somewhat stronger result is needed, and proven, for application to potential automorphy theorems for Hodge–Tate regular compatible systems. We can prove the general version of this as well:

Corollary 3.10. *Let $\{\rho_v\}_{v \in \Sigma}$ be a compatible system of representations of Γ_F with coefficients in a number field E , where $\Sigma = \Sigma(\Pi)$ for some set Π of rational primes of density 1. Suppose that each ρ_v is absolutely irreducible. Then there exists a density 1 subset $\Pi' \subset \Pi$ such that for all $v \in \Sigma' = \Sigma(\Pi')$, lying above $\ell \in \Pi'$, the restriction $\bar{\rho}_v|_{\Gamma_F(\zeta_\ell)}$ is absolutely irreducible.*

Proof. This time we consider the compatible system of Γ_F -representations with \mathbf{Q} -coefficients $\rho'_\ell = \{\kappa_\ell \oplus \bigoplus_{v|\ell} \rho_v\}_\ell$, where κ_ℓ denotes the ℓ -adic cyclotomic character. By Frobenius reciprocity, we must show that $\text{Hom}_{\Gamma_F}(\bar{\rho}_v, \bar{\rho}_v(i)) = 0$ for all $i = 1, \dots, \ell - 2$, for all v lying above some density one set of rational primes ℓ . Again let $\Pi' \subset \Pi$ be a density one subset for which Larsen's theorem on hyperspecial image holds, fix $\ell \in \Pi'$, and let $G' \subset \mathbf{G}_m \times G$ be the Zariski closure of ρ'_ℓ , with G continuing to denote the Zariski closure of $\rho_\ell = \bigoplus_{v|\ell} \rho_v$. Both κ_ℓ and ρ_v are algebraic representations of G' over E_v , and so $\rho_v(i)$ is as well, for any integer i . We then follow the proof of Theorem 3.8, but without the hypothesis that Γ'^{tor} has ‘small’ index in the maximal compact subgroup of $G'^{\text{tor}}(\mathbf{Q}_\ell)$ (Γ' here is taken to be $\rho'_\ell(\Gamma_F)$). Proposition 3.4 produces a reductive group $\mathcal{G}'/\mathbf{Z}_\ell$ with generic fiber G' such that Γ' is an open subgroup of $\mathcal{G}'(\mathbf{Z}_\ell)$. Let \mathcal{V} be a \mathcal{G}' -stable lattice in ρ_v , and consider the representation $\mathcal{E}_i = \text{Hom}(\mathcal{V}, \mathcal{V}(i))$. Exactly as in the proof of Theorem 3.8, we have

$$(\mathcal{E}_i^{\Gamma'})_{\bar{k}} = (\mathcal{E}_i^{\mathcal{G}'})_{\bar{k}} = (\mathcal{E}_{i,\bar{k}})^{\mathcal{G}'_{\bar{k}}}.$$

We now discard from Π' the finitely many rational primes such that $[F(\zeta_\ell) : F]$ is smaller than $\ell - 1$, and claim that we can also deduce that $(\mathcal{E}_{i,\bar{k}})^{\mathcal{G}'_{\bar{k}}} = (\mathcal{E}_{i,\bar{k}})^{\Gamma'}$ for $i = 1, \dots, \ell - 2$. Indeed, all the hypotheses of Proposition 3.7 are still satisfied, with the possible exception that $\|\mathcal{E}_i\|_{\text{cen}}$ may not be small compared to $\text{char}(k)$. Following the proof of Proposition 3.7, however, we need only check that upon restricting $\mathcal{E}_{i,\bar{k}}$ to the maximal central torus Z of $\mathcal{G}'_{\bar{k}}$, any irreducible (one-dimensional) constituent on which Z acts non-trivially is also acted on non-trivially by the image of Γ' in Z . Since Z acts trivially on $\bar{\rho}_v^* \otimes \bar{\rho}_v$ ($\bar{\rho}_v$ being irreducible, by the main theorem), we are reduced to the assertion that $\bar{\kappa}^i$ is a non-trivial Γ_F -representation for $i = 1, \dots, \ell - 2$, which holds by our restriction on ℓ .

The corollary now results from the fact that $\text{Hom}_{\Gamma_F}(\rho_v, \rho_v(i)) = 0$ for any $i \neq 0$. \square

We likewise deduce the following application:

Corollary 3.11. *Let $\{\rho_v\}_{v \in \Sigma}$ be a compatible system of representations with coefficients in E , where $\Sigma = \Sigma(\Pi)$ for Π a set of rational primes of density 1. Suppose that each ρ_v is absolutely Lie-irreducible, in the sense that for all finite extensions L/F , the restriction $\rho_v|_{\Gamma_L}$ remains absolutely irreducible. Then for any integer d , there is a density one subset $\Pi_d \subset \Pi$ such that for all $v \in \Sigma(\Pi_d)$, and all extensions L/F of degree at most d , the restriction $\bar{\rho}_v|_{\Gamma_L}$ is absolutely irreducible.*

Proof. As above, let $\{\rho_\ell = \bigoplus_{v|\ell} \rho_v\}_{\ell \in \Pi}$ be the associated compatible system with coefficients in \mathbf{Q} . By Corollary 1.7, there is a positive integer N , a density one subset $\Pi' \subset \Pi$, and for each $\ell \in \Pi'$ a reductive group scheme $\mathcal{G}_\ell/\mathbf{Z}_\ell$ with generic fiber G_ℓ such that $\mathcal{G}_\ell(\mathbf{Z}_\ell)$ contains $\Gamma = \rho_\ell(\Gamma_F)$ as an open subgroup of index at most N . For any L/F satisfying $[L:F] \leq d$, it follows that $\mathcal{G}_\ell(\mathbf{Z}_\ell)$ contains $\rho_\ell(\Gamma_L)$ with index at most Nd , and that it contains $\rho_\ell(\Gamma_L) \cap \Gamma^0$ with index at most $Nd\pi_0$, where π_0 is the order of the component group (independent of ℓ by Serre's theorem). Consider the representation ρ_v over E_v of the group G_ℓ , and as before let \mathcal{V} be a \mathcal{G}_ℓ -stable lattice, and let $\mathcal{E} = \text{End}(\mathcal{V})$. Finally let \bar{k}_v be an algebraic closure of the residue field of E_v . Provided ℓ is sufficiently large, we then have

$$(\mathcal{E}^{\rho_\ell(\Gamma_L) \cap \Gamma^0})_{\bar{k}_v} = (\mathcal{E}^{\mathcal{G}_\ell})_{\bar{k}_v} = (\mathcal{E}_{\bar{k}_v})_{\ell, \bar{k}_v}^{\mathcal{G}_\ell} = (\mathcal{E}_{\bar{k}_v})^{\rho_\ell(\Gamma_L) \cap \Gamma^0}.$$

The first equality holds because \mathcal{V} is absolutely Lie-irreducible, so the invariants are in both cases just the scalars. The second is again Proposition 3.5, and the third is again Proposition 3.7, invoking the uniform index bound. \square

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