

# Algebraic Geometry over $C^\infty$ -rings

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## Abstract

If  $X$  is a manifold then the  $\mathbb{R}$ -algebra  $C^\infty(X)$  of smooth functions  $c : X \rightarrow \mathbb{R}$  is a  $C^\infty$ -ring. That is, for each smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  there is an  $n$ -fold operation  $\Phi_f : C^\infty(X)^n \rightarrow C^\infty(X)$  acting by  $\Phi_f : (c_1, \dots, c_n) \mapsto f(c_1, \dots, c_n)$ , and these operations  $\Phi_f$  satisfy many natural identities. Thus,  $C^\infty(X)$  actually has a far richer structure than the obvious  $\mathbb{R}$ -algebra structure.

We explain the foundations of a version of algebraic geometry in which rings or algebras are replaced by  $C^\infty$ -rings. As schemes are the basic objects in algebraic geometry, the new basic objects are  $C^\infty$ -schemes, a category of geometric objects which generalize manifolds, and whose morphisms generalize smooth maps. We also study *quasicoherent sheaves* on  $C^\infty$ -schemes, and  $C^\infty$ -stacks, in particular *Deligne–Mumford  $C^\infty$ -stacks*, a 2-category of geometric objects generalizing orbifolds.

Many of these ideas are not new:  $C^\infty$ -rings and  $C^\infty$ -schemes have long been part of synthetic differential geometry. But we develop them in new directions. In [36–38], the author uses these tools to define *d-manifolds* and *d-orbifolds*, ‘derived’ versions of manifolds and orbifolds related to Spivak’s ‘derived manifolds’ [64].

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## CHAPTER 1

# Introduction

Let  $X$  be a smooth manifold, and write  $C^\infty(X)$  for the set of smooth functions  $c : X \rightarrow \mathbb{R}$ . Then  $C^\infty(X)$  is a commutative  $\mathbb{R}$ -algebra, with operations of addition, multiplication, and scalar multiplication defined pointwise. However,  $C^\infty(X)$  has much more structure than this. For example, if  $c : X \rightarrow \mathbb{R}$  is smooth then  $\exp(c) : X \rightarrow \mathbb{R}$  is smooth, and this defines an operation  $\exp : C^\infty(X) \rightarrow C^\infty(X)$  which cannot be expressed algebraically in terms of the  $\mathbb{R}$ -algebra structure. More generally, if  $n \geq 0$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth, define an  $n$ -fold operation  $\Phi_f : C^\infty(X)^n \rightarrow C^\infty(X)$  by

$$(\Phi_f(c_1, \dots, c_n))(x) = f(c_1(x), \dots, c_n(x)),$$

for all  $c_1, \dots, c_n \in C^\infty(X)$  and  $x \in X$ . These operations satisfy many identities: suppose  $m, n \geq 0$ , and  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $i = 1, \dots, m$  and  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  are smooth functions. Define a smooth function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$h(x_1, \dots, x_n) = g(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)),$$

for all  $(x_1, \dots, x_n) \in \mathbb{R}^n$ . Then for all  $c_1, \dots, c_n \in C^\infty(X)$  we have

$$\Phi_h(c_1, \dots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \dots, c_n), \dots, \Phi_{f_m}(c_1, \dots, c_n)). \quad (1.1)$$

A  $C^\infty$ -ring  $(\mathfrak{C}, (\Phi_f)_{f: \mathbb{R}^n \rightarrow \mathbb{R}})$  is a set  $\mathfrak{C}$  with operations  $\Phi_f : \mathfrak{C}^n \rightarrow \mathfrak{C}$  for all  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  smooth satisfying identities (1.1), and one other condition. For example  $C^\infty(X)$  is a  $C^\infty$ -ring for any manifold  $X$ , but there are also many  $C^\infty$ -rings which do not come from manifolds, and can be thought of as representing geometric objects which generalize manifolds.

The most basic objects in conventional algebraic geometry are commutative rings  $R$ , or commutative  $\mathbb{K}$ -algebras  $R$  for some field  $\mathbb{K}$ . The ‘spectrum’  $\text{Spec } R$  of  $R$  is an affine scheme, and  $R$  is interpreted as an algebra of functions on  $\text{Spec } R$ . More general kinds of spaces in algebraic geometry — schemes and stacks — are locally modelled on affine schemes  $\text{Spec } R$ . This book lays down the foundations of *Algebraic Geometry over  $C^\infty$ -rings*, in which we replace commutative rings in algebraic geometry by  $C^\infty$ -rings. It includes the study of  $C^\infty$ -schemes and *Deligne–Mumford  $C^\infty$ -stacks*, two classes of geometric spaces generalizing manifolds and orbifolds, respectively.

This is not a new idea, but was studied years ago as part of *synthetic differential geometry*, which grew out of ideas of Lawvere in the 1960s; see for instance Dubuc [23] on  $C^\infty$ -schemes, and the books by Moerdijk and Reyes [54] and Kock [44]. However, we have new things to say, as we are motivated by different problems (see below), and so are asking different questions.

Following Dubuc’s discussion of ‘models of synthetic differential geometry’ [21] and oversimplifying a bit, synthetic differential geometers are interested in  $C^\infty$ -schemes as they provide a category  $\mathbf{C}^\infty\mathbf{Sch}$  of geometric objects which includes

smooth manifolds and certain ‘infinitesimal’ objects, and all fibre products exist in  $\mathbf{C}^\infty\mathbf{Sch}$ , and  $\mathbf{C}^\infty\mathbf{Sch}$  has some other nice properties to do with open covers, and exponentials of infinitesimals.

Synthetic differential geometry concerns proving theorems about manifolds using synthetic reasoning involving ‘infinitesimals’. But one needs to check these methods of synthetic reasoning are valid. To do this you need a ‘model’, some category of geometric spaces including manifolds and infinitesimals, in which you can think of your synthetic arguments as happening. Once you know there exists at least one model with the properties you want, then as far as synthetic differential geometry is concerned the job is done. For this reason  $C^\infty$ -schemes have not been developed very far in synthetic differential geometry.

Recently,  $C^\infty$ -rings and  $C^\infty$ -ringed spaces appeared in a very different context, in the theory of *derived differential geometry*, the differential-geometric analogue of the derived algebraic geometry of Lurie [48] and Toën–Vezzosi [66, 67], which studies *derived smooth manifolds* and *derived smooth orbifolds*. This began with a short section in Lurie [48, §4.5], where he sketched how to define an  $\infty$ -category of *derived  $C^\infty$ -schemes*, including derived manifolds.

Lurie’s student David Spivak [64] worked out the details of this, defining an  $\infty$ -category of derived manifolds. Simplifications and extensions of Spivak’s theory were given by Borisov and Noel [9, 10] and the author [36–38]. An alternative approach to the foundations of derived differential geometry involving differential graded  $C^\infty$ -rings is proposed by Carchedi and Roytenberg [12, 13].

The author’s notion of derived manifolds [36–38] are called *d-manifolds*, and are built using our theory of  $C^\infty$ -schemes and quasicoherent sheaves upon them below. They form a 2-category. We also study orbifold versions, *d-orbifolds*, which are built using our theory of Deligne–Mumford  $C^\infty$ -stacks and their quasicoherent sheaves below.

Many areas of symplectic geometry involve studying moduli spaces of *J*-holomorphic curves in a symplectic manifold, which are made into *Kuranishi spaces* in the framework of Fukaya, Oh, Ohta and Ono [26, 27]. The author argues that *Kuranishi spaces are really derived orbifolds*, and has given a new definition [39, 41] of a 2-category of Kuranishi spaces  $\mathbf{Kur}$  which is equivalent to the 2-category of d-orbifolds  $\mathbf{dOrb}$  from [36–38]. Because of this, derived differential geometry will have important applications in symplectic geometry.

To set up our theory of d-manifolds and d-orbifolds requires a lot of preliminary work on  $C^\infty$ -schemes and  $C^\infty$ -stacks, and quasicoherent sheaves upon them. That is the purpose of this book. We have tried to present a complete, self-contained account which should be understandable to readers with a reasonable background in algebraic geometry, and we assume no familiarity with synthetic differential geometry. We expect this material may have other applications quite different to those the author has in mind in [36–38].

Section 2 explains  $C^\infty$ -rings. The archetypal examples of  $C^\infty$ -rings,  $C^\infty(X)$  for manifolds  $X$ , are discussed in §3. Section 4 studies  $C^\infty$ -schemes, and §5 modules over  $C^\infty$ -rings and sheaves of modules over  $C^\infty$ -schemes.

Sections 6–9 discuss  $C^\infty$ -stacks. Section 6 defines the 2-category  $\mathbf{C}^\infty\mathbf{Sta}$  of  *$C^\infty$ -stacks*, analogues of Artin stacks in algebraic geometry, and §7 the 2-subcategory  $\mathbf{DMC}^\infty\mathbf{Sta}$  of *Deligne–Mumford  $C^\infty$ -stacks*, which are  $C^\infty$ -stacks locally modelled on  $[\underline{U}/G]$  for  $\underline{U}$  an affine  $C^\infty$ -scheme and  $G$  a finite group acting



on  $\underline{U}$ , and are analogues of Deligne–Mumford stacks in algebraic geometry. We show that orbifolds **Orb** may be regarded as a 2-subcategory of **DMC<sup>∞</sup>Sta**. Section 8 studies quasicoherent sheaves on Deligne–Mumford  $C^\infty$ -stacks, generalizing §5, and §9 orbifold strata of Deligne–Mumford  $C^\infty$ -stacks.

Appendix A summarizes background on stacks from [3, 4, 29, 46, 49, 55], for use in §6–§9. Stacks are a very technical area, and §A is too terse to help a beginner learn the subject, it is intended only to establish notation and definitions for those already familiar with stacks. Readers with no experience of stacks are advised to first consult an introductory text such as Vistoli [68], Gomez [29], Laumon and Moret-Bailly [46], or the online ‘Stacks Project’ [34].

Much of §2–§4 is already understood in synthetic differential geometry, such as in the work of Dubuc [23] and Moerdijk and Reyes [54]. But we believe it is worthwhile giving a detailed and self-contained exposition, from our own point of view. Sections 5–9 are new, so far as the author knows, though §5–§8 are based on well known material in algebraic geometry.

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## CHAPTER 2

### $C^\infty$ -rings

We begin by explaining the basic objects out of which our theories are built,  $C^\infty$ -rings, or *smooth rings*. The archetypal example of a  $C^\infty$ -ring is the vector space  $C^\infty(X)$  of smooth functions  $c : X \rightarrow \mathbb{R}$  for a manifold  $X$ . Everything in this section is known to experts in synthetic differential geometry, and much of it can be found in Moerdijk and Reyes [54, Ch. I], Dubuc [21–24] or Kock [44, §III]. We introduce some new notation for brevity, in particular, our *fair*  $C^\infty$ -rings are known in the literature as ‘finitely generated and germ determined  $C^\infty$ -rings’.

#### 2.1. Two definitions of $C^\infty$ -ring

We first define  $C^\infty$ -rings in the style of classical algebra.

DEFINITION 2.1. A  $C^\infty$ -ring is a set  $\mathfrak{C}$  together with operations

$$\Phi_f : \mathfrak{C}^n = \mathfrak{C} \times \cdots \times \mathfrak{C} \longrightarrow \mathfrak{C}$$

$\uparrow \text{ } n \text{ copies } \uparrow$

for all  $n \geq 0$  and smooth maps  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , where by convention when  $n = 0$  we define  $\mathfrak{C}^0$  to be the single point  $\{\emptyset\}$ . These operations must satisfy the following relations: suppose  $m, n \geq 0$ , and  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $i = 1, \dots, m$  and  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  are smooth functions. Define a smooth function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$h(x_1, \dots, x_n) = g(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)),$$

for all  $(x_1, \dots, x_n) \in \mathbb{R}^n$ . Then for all  $(c_1, \dots, c_n) \in \mathfrak{C}^n$  we have

$$\Phi_h(c_1, \dots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \dots, c_n), \dots, \Phi_{f_m}(c_1, \dots, c_n)).$$

We also require that for all  $1 \leq j \leq n$ , defining  $\pi_j : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $\pi_j : (x_1, \dots, x_n) \mapsto x_j$ , we have  $\Phi_{\pi_j}(c_1, \dots, c_n) = c_j$  for all  $(c_1, \dots, c_n) \in \mathfrak{C}^n$ .

Usually we refer to  $\mathfrak{C}$  as the  $C^\infty$ -ring, leaving the operations  $\Phi_f$  implicit.

A *morphism* between  $C^\infty$ -rings  $(\mathfrak{C}, (\Phi_f)_{f: \mathbb{R}^n \rightarrow \mathbb{R}})$ ,  $(\mathfrak{D}, (\Psi_f)_{f: \mathbb{R}^n \rightarrow \mathbb{R}})$  is a map  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$  such that  $\Psi_f(\phi(c_1), \dots, \phi(c_n)) = \phi \circ \Phi_f(c_1, \dots, c_n)$  for all smooth  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $c_1, \dots, c_n \in \mathfrak{C}$ . We will write  **$C^\infty$ Rings** for the category of  $C^\infty$ -rings.

Here is the motivating example, which we will study at greater length in §3:

EXAMPLE 2.2. Let  $X$  be a manifold, which may be without boundary, or with boundary, or with corners. Write  $C^\infty(X)$  for the set of smooth functions  $c : X \rightarrow \mathbb{R}$ . For  $n \geq 0$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  smooth, define  $\Phi_f : C^\infty(X)^n \rightarrow C^\infty(X)$  by

$$(\Phi_f(c_1, \dots, c_n))(x) = f(c_1(x), \dots, c_n(x)), \tag{2.1}$$

for all  $c_1, \dots, c_n \in C^\infty(X)$  and  $x \in X$ . It is easy to see that  $C^\infty(X)$  and the operations  $\Phi_f$  form a  $C^\infty$ -ring.

EXAMPLE 2.3. Take  $X$  to be the point  $*$  in Example 2.2. Then  $C^\infty(*) = \mathbb{R}$ , with operations  $\Phi_f : \mathbb{R}^n \rightarrow \mathbb{R}$  given by  $\Phi_f(x_1, \dots, x_n) = f(x_1, \dots, x_n)$ . This makes  $\mathbb{R}$  into the simplest nonzero example of a  $C^\infty$ -ring, the initial object in  $\mathbf{C}^\infty\mathbf{Rings}$ .

Note that  $C^\infty$ -rings are far more general than those coming from manifolds. For example, if  $X$  is any topological space we could define a  $C^\infty$ -ring  $C^0(X)$  to be the set of *continuous*  $c : X \rightarrow \mathbb{R}$  with operations  $\Phi_f$  defined as in (2.1). For  $X$  a manifold with  $\dim X > 0$ , the  $C^\infty$ -rings  $C^\infty(X)$  and  $C^0(X)$  are different.

There is a more succinct definition of  $C^\infty$ -rings using category theory:

DEFINITION 2.4. Write  $\mathbf{Man}$  for the category of manifolds, and  $\mathbf{Euc}$  for the full subcategory of  $\mathbf{Man}$  with objects the Euclidean spaces  $\mathbb{R}^n$ . That is, the objects of  $\mathbf{Euc}$  are  $\mathbb{R}^n$  for  $n = 0, 1, 2, \dots$ , and the morphisms in  $\mathbf{Euc}$  are smooth maps  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Write  $\mathbf{Sets}$  for the category of sets. In both  $\mathbf{Euc}$  and  $\mathbf{Sets}$  we have notions of (finite) products of objects (that is,  $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ , and products  $S \times T$  of sets  $S, T$ ), and products of morphisms.

Define a (*category-theoretic*)  $C^\infty$ -ring to be a product-preserving functor  $F : \mathbf{Euc} \rightarrow \mathbf{Sets}$ . Here  $F$  should also preserve the empty product, that is, it maps  $\mathbb{R}^0$  in  $\mathbf{Euc}$  to the terminal object in  $\mathbf{Sets}$ , the point  $*$ .

$C^\infty$ -rings in this sense are an example of an *algebraic theory* in the sense of Adámek, Rosický and Vitale [1], and many of the basic categorical properties of  $C^\infty$ -rings follow from this.

Here is how this relates to Definition 2.1. Suppose  $F : \mathbf{Euc} \rightarrow \mathbf{Sets}$  is a product-preserving functor. Define  $\mathfrak{C} = F(\mathbb{R})$ . Then  $\mathfrak{C}$  is an object in  $\mathbf{Sets}$ , that is, a set. Suppose  $n \geq 0$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth. Then  $f$  is a morphism in  $\mathbf{Euc}$ , so  $F(f) : F(\mathbb{R}^n) \rightarrow F(\mathbb{R}) = \mathfrak{C}$  is a morphism in  $\mathbf{Sets}$ . Since  $F$  preserves products  $F(\mathbb{R}^n) = F(\mathbb{R}) \times \dots \times F(\mathbb{R}) = \mathfrak{C}^n$ , so  $F(f)$  maps  $\mathfrak{C}^n \rightarrow \mathfrak{C}$ . We define  $\Phi_f : \mathfrak{C}^n \rightarrow \mathfrak{C}$  by  $\Phi_f = F(f)$ . The fact that  $F$  is a functor implies that the  $\Phi_f$  satisfy the relations in Definition 2.1, so  $(\mathfrak{C}, (\Phi_f)_{f : \mathbb{R}^n \rightarrow \mathbb{R}})$  is a  $C^\infty$  ring.

Conversely, if  $(\mathfrak{C}, (\Phi_f)_{f : \mathbb{R}^n \rightarrow \mathbb{R}})$  is a  $C^\infty$ -ring then we define  $F : \mathbf{Euc} \rightarrow \mathbf{Sets}$  by  $F(\mathbb{R}^n) = \mathfrak{C}^n$ , and if  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is smooth then  $f = (f_1, \dots, f_m)$  for  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  smooth, and we define  $F(f) : \mathfrak{C}^n \rightarrow \mathfrak{C}^m$  by  $F(f) : (c_1, \dots, c_n) \mapsto (\Phi_{f_1}(c_1, \dots, c_n), \dots, \Phi_{f_m}(c_1, \dots, c_n))$ . Then  $F$  is a product-preserving functor. This defines a 1-1 correspondence between  $C^\infty$ -rings in the sense of Definition 2.1, and category-theoretic  $C^\infty$ -rings in the sense of Definition 2.4.

As in Moerdijk and Reyes [54, p. 21–22] we have:

PROPOSITION 2.5. *In the category  $\mathbf{C}^\infty\mathbf{Rings}$  of  $C^\infty$ -rings, all limits and all filtered colimits exist, and regarding  $C^\infty$ -rings as functors  $F : \mathbf{Euc} \rightarrow \mathbf{Sets}$  as in Definition 2.4, they may be computed objectwise in  $\mathbf{Euc}$  by taking the corresponding limits/filtered colimits in  $\mathbf{Sets}$ .*

*Also, all small colimits exist, though in general they are not computed objectwise in  $\mathbf{Euc}$  by taking colimits in  $\mathbf{Sets}$ . In particular, pushouts and all finite colimits exist in  $\mathbf{C}^\infty\mathbf{Rings}$ .*

We will write  $\mathfrak{D} \amalg_{\phi, \mathfrak{C}, \psi} \mathfrak{E}$  or  $\mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E}$  for the pushout of morphisms  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$ ,  $\psi : \mathfrak{C} \rightarrow \mathfrak{E}$  in  $\mathbf{C}^\infty\mathbf{Rings}$ . When  $\mathfrak{C} = \mathbb{R}$ , the initial object in  $\mathbf{C}^\infty\mathbf{Rings}$ , pushouts  $\mathfrak{D} \amalg_{\mathbb{R}} \mathfrak{E}$  are called *coproducts* and are usually written  $\mathfrak{D} \otimes_{\infty} \mathfrak{E}$ . For  $\mathbb{R}$ -algebras  $A, B$  the coproduct is the tensor product  $A \otimes B$ . But the coproduct  $\mathfrak{D} \otimes_{\infty} \mathfrak{E}$  of  $C^\infty$ -rings  $\mathfrak{D}, \mathfrak{E}$  is generally different from their coproduct  $\mathfrak{D} \otimes \mathfrak{E}$  as  $\mathbb{R}$ -algebras. For example

we have  $C^\infty(\mathbb{R}^m) \otimes_\infty C^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^{m+n})$ , which contains but is much larger than the tensor product  $C^\infty(\mathbb{R}^m) \otimes C^\infty(\mathbb{R}^n)$  for  $m, n > 0$ .

## 2.2. $C^\infty$ -rings as commutative $\mathbb{R}$ -algebras, and ideals

Every  $C^\infty$ -ring  $\mathfrak{C}$  has an underlying commutative  $\mathbb{R}$ -algebra:

DEFINITION 2.6. Let  $\mathfrak{C}$  be a  $C^\infty$ -ring. Then we may give  $\mathfrak{C}$  the structure of a *commutative  $\mathbb{R}$ -algebra*. Define addition ‘+’ on  $\mathfrak{C}$  by  $c + c' = \Phi_f(c, c')$  for  $c, c' \in \mathfrak{C}$ , where  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is  $f(x, y) = x + y$ . Define multiplication ‘ $\cdot$ ’ on  $\mathfrak{C}$  by  $c \cdot c' = \Phi_g(c, c')$ , where  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  is  $g(x, y) = xy$ . Define scalar multiplication by  $\lambda \in \mathbb{R}$  by  $\lambda c = \Phi_{\lambda'}(c)$ , where  $\lambda' : \mathbb{R} \rightarrow \mathbb{R}$  is  $\lambda'(x) = \lambda x$ . Define elements 0 and 1 in  $\mathfrak{C}$  by  $0 = \Phi_{0'}(\emptyset)$  and  $1 = \Phi_{1'}(\emptyset)$ , where  $0' : \mathbb{R}^0 \rightarrow \mathbb{R}$  and  $1' : \mathbb{R}^0 \rightarrow \mathbb{R}$  are the maps  $0' : \emptyset \mapsto 0$  and  $1' : \emptyset \mapsto 1$ . The relations on the  $\Phi_f$  imply that all the axioms of a commutative  $\mathbb{R}$ -algebra are satisfied. In Example 2.2, this yields the obvious  $\mathbb{R}$ -algebra structure on the smooth functions  $c : X \rightarrow \mathbb{R}$ .

Here is another way to say this. In an  $\mathbb{R}$ -algebra  $A$ , the  $n$ -fold ‘operations’  $\Phi : A^n \rightarrow A$ , that is, all the maps  $A^n \rightarrow A$  we can construct using only addition, multiplication, scalar multiplication, and the elements  $0, 1 \in A$ , correspond exactly to polynomials  $p : \mathbb{R}^n \rightarrow \mathbb{R}$ . Since polynomials are smooth, the operations of an  $\mathbb{R}$ -algebra are a subset of those of a  $C^\infty$ -ring, and we can truncate from  $C^\infty$ -rings to  $\mathbb{R}$ -algebras. As there are many more smooth functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  than there are polynomials, a  $C^\infty$ -ring has far more structure and operations than a commutative  $\mathbb{R}$ -algebra.

DEFINITION 2.7. An *ideal*  $I$  in  $\mathfrak{C}$  is an ideal  $I \subset \mathfrak{C}$  in  $\mathfrak{C}$  regarded as a commutative  $\mathbb{R}$ -algebra. Then we make the quotient  $\mathfrak{C}/I$  into a  $C^\infty$ -ring as follows. If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth, define  $\Phi_f^I : (\mathfrak{C}/I)^n \rightarrow \mathfrak{C}/I$  by

$$\Phi_f^I(c_1 + I, \dots, c_n + I) = \Phi_f(c_1, \dots, c_n) + I.$$

To show this is well-defined, we must show it is independent of the choice of representatives  $c_1, \dots, c_n$  in  $\mathfrak{C}$  for  $c_1 + I, \dots, c_n + I$  in  $\mathfrak{C}/I$ . By Hadamard’s Lemma there exist smooth functions  $g_i : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  for  $i = 1, \dots, n$  with

$$f(y_1, \dots, y_n) - f(x_1, \dots, x_n) = \sum_{i=1}^n (y_i - x_i) g_i(x_1, \dots, x_n, y_1, \dots, y_n)$$

for all  $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$ . If  $c'_1, \dots, c'_n$  are alternative choices for  $c_1, \dots, c_n$ , so that  $c'_i + I = c_i + I$  for  $i = 1, \dots, n$  and  $c'_i - c_i \in I$ , we have

$$\Phi_f(c'_1, \dots, c'_n) - \Phi_f(c_1, \dots, c_n) = \sum_{i=1}^n (c'_i - c_i) \Phi_{g_i}(c'_1, \dots, c'_n, c_1, \dots, c_n).$$

The second line lies in  $I$  as  $c'_i - c_i \in I$  and  $I$  is an ideal, so  $\Phi_f^I$  is well-defined, and clearly  $(\mathfrak{C}/I, (\Phi_f^I)_{f: \mathbb{R}^n \rightarrow \mathbb{R}})$  is a  $C^\infty$ -ring.

If  $\mathfrak{C}$  is a  $C^\infty$ -ring, we will use the notation  $(f_a : a \in A)$  to denote the ideal in  $\mathfrak{C}$  generated by a collection of elements  $f_a$ ,  $a \in A$  in  $\mathfrak{C}$ , in the sense of commutative  $\mathbb{R}$ -algebras. That is,

$$(f_a : a \in A) = \left\{ \sum_{i=1}^n f_{a_i} \cdot c_i : n \geq 0, a_1, \dots, a_n \in A, c_1, \dots, c_n \in \mathfrak{C} \right\}.$$

DEFINITION 2.8. A  $C^\infty$ -ring  $\mathfrak{C}$  is called *finitely generated* if there exist  $c_1, \dots, c_n$  in  $\mathfrak{C}$  which generate  $\mathfrak{C}$  over all  $C^\infty$ -operations. That is, for each  $c \in \mathfrak{C}$  there exists a smooth map  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $c = \Phi_f(c_1, \dots, c_n)$ . (This is a much weaker condition than  $\mathfrak{C}$  being finitely generated as a commutative  $\mathbb{R}$ -algebra.)

By Kock [44, Prop. III.5.1],  $C^\infty(\mathbb{R}^n)$  is the free  $C^\infty$ -ring with  $n$  generators. Given such  $\mathfrak{C}, c_1, \dots, c_n$ , define  $\phi : C^\infty(\mathbb{R}^n) \rightarrow \mathfrak{C}$  by  $\phi(f) = \Phi_f(c_1, \dots, c_n)$  for smooth  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , where  $C^\infty(\mathbb{R}^n)$  is as in Example 2.2 with  $X = \mathbb{R}^n$ . Then  $\phi$  is a surjective morphism of  $C^\infty$ -rings, so  $I = \text{Ker } \phi$  is an ideal in  $C^\infty(\mathbb{R}^n)$ , and  $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$  as a  $C^\infty$ -ring. Thus,  $\mathfrak{C}$  is finitely generated if and only if  $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$  for some  $n \geq 0$  and ideal  $I$  in  $C^\infty(\mathbb{R}^n)$ .

An ideal  $I$  in a  $C^\infty$ -ring  $\mathfrak{C}$  is called *finitely generated* if  $I$  is a finitely generated ideal of the underlying commutative  $\mathbb{R}$ -algebra of  $\mathfrak{C}$  in Definition 2.6, that is,  $I = (i_1, \dots, i_k)$  for some  $i_1, \dots, i_k \in \mathfrak{C}$ . A  $C^\infty$ -ring  $\mathfrak{C}$  is called *finitely presented* if  $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$  for some  $n \geq 0$ , where  $I$  is a finitely generated ideal in  $C^\infty(\mathbb{R}^n)$ .

A difference with conventional algebraic geometry is that  $C^\infty(\mathbb{R}^n)$  is not noetherian, so ideals in  $C^\infty(\mathbb{R}^n)$  may not be finitely generated, and  $\mathfrak{C}$  finitely generated does not imply  $\mathfrak{C}$  finitely presented.

Write  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fg}}$  and  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fp}}$  for the full subcategories of finitely generated and finitely presented  $C^\infty$ -rings in  $\mathbf{C}^\infty\mathbf{Rings}$ .

EXAMPLE 2.9. A *Weil algebra* [21, Def. 1.4] is a finite-dimensional commutative  $\mathbb{R}$ -algebra  $W$  which has a maximal ideal  $\mathfrak{m}$  with  $W/\mathfrak{m} \cong \mathbb{R}$  and  $\mathfrak{m}^n = 0$  for some  $n > 0$ . Then by Dubuc [21, Prop. 1.5] or Kock [44, Th. III.5.3], there is a unique way to make  $W$  into a  $C^\infty$ -ring compatible with the given underlying commutative  $\mathbb{R}$ -algebra. This  $C^\infty$ -ring is finitely presented [44, Prop. III.5.11].  $C^\infty$ -rings from Weil algebras are important in synthetic differential geometry, in arguments involving infinitesimals. See [11, §2] for a detailed study of this.

### 2.3. Local $C^\infty$ -rings, and localization

DEFINITION 2.10. A  $C^\infty$ -ring  $\mathfrak{C}$  is called *local* if regarded as an  $\mathbb{R}$ -algebra, as in Definition 2.6,  $\mathfrak{C}$  is a local  $\mathbb{R}$ -algebra with residue field  $\mathbb{R}$ . That is,  $\mathfrak{C}$  has a unique maximal ideal  $\mathfrak{m}_{\mathfrak{C}}$  with  $\mathfrak{C}/\mathfrak{m}_{\mathfrak{C}} \cong \mathbb{R}$ .

If  $\mathfrak{C}, \mathfrak{D}$  are local  $C^\infty$ -rings with maximal ideals  $\mathfrak{m}_{\mathfrak{C}}, \mathfrak{m}_{\mathfrak{D}}$ , and  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$  is a morphism of  $C^\infty$  rings, then using the fact that  $\mathfrak{C}/\mathfrak{m}_{\mathfrak{C}} \cong \mathbb{R} \cong \mathfrak{D}/\mathfrak{m}_{\mathfrak{D}}$  we see that  $\phi^{-1}(\mathfrak{m}_{\mathfrak{D}}) = \mathfrak{m}_{\mathfrak{C}}$ , that is,  $\phi$  is a *local* morphism of local  $C^\infty$ -rings. Thus, there is no difference between morphisms and local morphisms.

REMARK 2.11. We use the term ‘local  $C^\infty$ -ring’ following Dubuc [23, Def. 4]. They are also called  *$C^\infty$ -local rings* in Dubuc [22, Def. 2.13], *pointed local  $C^\infty$ -rings* in [54, §I.3] and *Archimedean local  $C^\infty$ -rings* in [52, §3].

Moerdijk and Reyes [52–54] use the term ‘local  $C^\infty$ -ring’ to mean a  $C^\infty$ -ring which is a local  $\mathbb{R}$ -algebra, but which need not have residue field  $\mathbb{R}$ .

The next example is taken from Moerdijk and Reyes [54, §I.3].

EXAMPLE 2.12. Write  $C^\infty(\mathbb{N})$  for the  $\mathbb{R}$ -algebra of all functions  $f : \mathbb{N} \rightarrow \mathbb{R}$ . It is a finitely generated  $C^\infty$ -ring isomorphic to  $C^\infty(\mathbb{R})/\{f \in C^\infty(\mathbb{R}) : f|_{\mathbb{N}} = 0\}$ . Let  $\mathcal{F}$  be a *non-principal ultrafilter* on  $\mathbb{N}$ , in the sense of Comfort and Negrepontis [16], and let  $I \subset \mathfrak{C}$  be the prime ideal of  $f : \mathbb{N} \rightarrow \mathbb{R}$  such that  $\{n \in \mathbb{N} : f(n) = 0\}$  lies in  $\mathcal{F}$ . Then  $\mathfrak{C} = C^\infty(\mathbb{N})/I$  is a finitely generated  $C^\infty$ -ring which is a local  $\mathbb{R}$ -algebra by [54, Ex. I.3.2], that is, it has a unique maximal ideal  $\mathfrak{m}_{\mathfrak{C}}$ , but its residue field is not  $\mathbb{R}$  by [54, Cor. I.3.4]. Hence  $\mathfrak{C}$  is a local  $C^\infty$ -ring in the sense of [52–54], but not in our sense.

*Localizations* of  $C^\infty$ -rings are studied in [22, 23, 52, 53], see [54, p. 23].

DEFINITION 2.13. Let  $\mathfrak{C}$  be a  $C^\infty$ -ring and  $S$  a subset of  $\mathfrak{C}$ . A *localization*  $\mathfrak{C}[s^{-1} : s \in S]$  of  $\mathfrak{C}$  at  $S$  is a  $C^\infty$ -ring  $\mathfrak{D} = \mathfrak{C}[s^{-1} : s \in S]$  and a morphism  $\pi : \mathfrak{C} \rightarrow \mathfrak{D}$  such that  $\pi(s)$  is invertible in  $\mathfrak{D}$  for all  $s \in S$ , with the universal property that if  $\mathfrak{E}$  is a  $C^\infty$ -ring and  $\phi : \mathfrak{C} \rightarrow \mathfrak{E}$  a morphism with  $\phi(s)$  invertible in  $\mathfrak{E}$  for all  $s \in S$ , then there is a unique morphism  $\psi : \mathfrak{D} \rightarrow \mathfrak{E}$  with  $\phi = \psi \circ \pi$ .

A localization  $\mathfrak{C}[s^{-1} : s \in S]$  always exists — it can be constructed by adjoining an extra generator  $s^{-1}$  and an extra relation  $s \cdot s^{-1} - 1 = 0$  for each  $s \in S$  — and is unique up to unique isomorphism. When  $S = \{c\}$  we have an exact sequence  $0 \rightarrow I \rightarrow \mathfrak{C} \otimes_\infty C^\infty(\mathbb{R}) \xrightarrow{\pi} \mathfrak{C}[c^{-1}] \rightarrow 0$ , where  $\mathfrak{C} \otimes_\infty C^\infty(\mathbb{R})$  is the coproduct of  $\mathfrak{C}, C^\infty(\mathbb{R})$  as in §2.1, with pushout morphisms  $\iota_1 : \mathfrak{C} \rightarrow \mathfrak{C} \otimes_\infty C^\infty(\mathbb{R})$ ,  $\iota_2 : C^\infty(\mathbb{R}) \rightarrow \mathfrak{C} \otimes_\infty C^\infty(\mathbb{R})$ , and  $I$  is the ideal in  $\mathfrak{C} \otimes_\infty C^\infty(\mathbb{R})$  generated by  $\iota_1(c) \cdot \iota_2(x) - 1$ , where  $x$  is the generator of  $C^\infty(\mathbb{R})$ .

An  $\mathbb{R}$ -point  $x$  of a  $C^\infty$ -ring  $\mathfrak{C}$  is a  $C^\infty$ -ring morphism  $x : \mathfrak{C} \rightarrow \mathbb{R}$ , where  $\mathbb{R}$  is regarded as a  $C^\infty$ -ring as in Example 2.3. By [54, Prop. I.3.6], a map  $x : \mathfrak{C} \rightarrow \mathbb{R}$  is a morphism of  $C^\infty$ -rings if and only if it is a morphism of the underlying  $\mathbb{R}$ -algebras, as in Definition 2.6. Define  $\mathfrak{C}_x$  to be the localization  $\mathfrak{C}_x = \mathfrak{C}[s^{-1} : s \in \mathfrak{C}, x(s) \neq 0]$ , with projection  $\pi_x : \mathfrak{C} \rightarrow \mathfrak{C}_x$ . Then  $\mathfrak{C}_x$  is a local  $C^\infty$ -ring by [53, Lem. 1.1]. The  $\mathbb{R}$ -points of  $C^\infty(\mathbb{R}^n)$  are just evaluation at points  $x \in \mathbb{R}^n$ . This also holds for  $C^\infty(X)$  for any manifold  $X$ .

In a new result, we can describe these local  $C^\infty$ -rings  $\mathfrak{C}_x$  explicitly. Note that the surjectivity of  $\pi_x : \mathfrak{C} \rightarrow \mathfrak{C}_x$  in the next proposition is surprising. It does not hold for general localizations of  $C^\infty$ -rings — for instance,  $\pi : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})[x^{-1}]$  is injective but not surjective, as  $x^{-1} \notin \text{Im } \pi$  — or for localizations  $\pi_x : A \rightarrow A_x$  of rings or  $\mathbb{K}$ -algebras in conventional algebraic geometry.

PROPOSITION 2.14. *Let  $\mathfrak{C}$  be a  $C^\infty$ -ring,  $x : \mathfrak{C} \rightarrow \mathbb{R}$  an  $\mathbb{R}$ -point of  $\mathfrak{C}$ , and  $\mathfrak{C}_x$  the localization, with projection  $\pi_x : \mathfrak{C} \rightarrow \mathfrak{C}_x$ . Then  $\pi_x$  is surjective with kernel an ideal  $I \subset \mathfrak{C}$ , so that  $\mathfrak{C}_x \cong \mathfrak{C}/I$ , where*

$$I = \{c \in \mathfrak{C} : \text{there exists } d \in \mathfrak{C} \text{ with } x(d) \neq 0 \text{ in } \mathbb{R} \text{ and } c \cdot d = 0 \text{ in } \mathfrak{C}\}. \quad (2.2)$$

PROOF. Clearly  $I$  in (2.2) is closed under multiplication by elements of  $\mathfrak{C}$ . Let  $c_1, c_2 \in I$ , so there exist  $d_1, d_2 \in \mathfrak{C}$  with  $x(d_1) \neq 0 \neq x(d_2)$  and  $c_1 d_1 = 0 = c_2 d_2$ . Then  $d_1 d_2 \in \mathfrak{C}$  with  $x(d_1 d_2) = x(d_1) x(d_2) \neq 0$ , and  $(c_1 + c_2)(d_1 \cdot d_2) = d_2(c_1 d_1) + d_1(c_2 d_2) = 0$ , so  $c_1 + c_2 \in I$ . Hence  $I$  is an ideal, and  $\mathfrak{C}/I$  a  $C^\infty$ -ring.

Suppose  $c \in I$ , so there exists  $d \in \mathfrak{C}$  with  $x(d) \neq 0$  and  $cd = 0$ . Then  $\pi_x(d)$  is invertible in  $\mathfrak{C}_x$  by definition. Thus

$$\pi_x(c) = \pi_x(c) \pi_x(d) \pi_x(d)^{-1} = \pi_x(cd) \pi_x(d)^{-1} = \pi_x(0) \pi_x(d)^{-1} = 0.$$

Therefore  $I \subseteq \text{Ker } \pi_x$ . So  $\pi_x : \mathfrak{C} \rightarrow \mathfrak{C}_x$  factorizes uniquely as  $\pi_x = \iota \circ \pi$ , where  $\pi : \mathfrak{C} \rightarrow \mathfrak{C}/I$  is the projection and  $\iota : \mathfrak{C}/I \rightarrow \mathfrak{C}_x$  is a  $C^\infty$ -ring morphism.

Suppose  $c \in \mathfrak{C}$  with  $x(c) \neq 0$ , and write  $\epsilon = \frac{1}{2}|x(c)|$ . Choose smooth functions  $\eta : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$ , so that  $\eta^{-1} : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$  is also smooth, such that  $\eta(t) = t$  for all  $t \in (x(c) - \epsilon, x(c) + \epsilon)$ , and  $\zeta : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\zeta(t) = 0$  for all  $t \in \mathbb{R} \setminus (x(c) - \epsilon, x(c) + \epsilon)$ , so that  $(\eta - \text{id}_{\mathbb{R}}) \cdot \zeta = 0$ , and  $\zeta(x(c)) = 1$ .

Set  $c_1 = \Phi_\eta(c)$ ,  $c_2 = \Phi_{\eta^{-1}}(c)$  and  $d = \Phi_\zeta(c)$  in  $\mathfrak{C}$ , using the  $C^\infty$ -ring operations from  $\eta, \eta^{-1}, \zeta$ . Then  $c_1 c_2 = 1$  in  $\mathfrak{C}$ , as  $\eta \cdot \eta^{-1} = 1$ , and  $x(d) = x(\Phi_\zeta(c)) = \zeta(x(c)) = 1$ , as  $x : \mathfrak{C} \rightarrow \mathbb{R}$  is a  $C^\infty$ -ring morphism. Also

$$(c_1 - c) \cdot d = (\Phi_\eta(c) - \Phi_{\text{id}_{\mathbb{R}}}(c)) \Phi_\zeta(c) = \Phi_{(\eta - \text{id}_{\mathbb{R}})\zeta}(c) = \Phi_0(c) = 0.$$

Hence  $c_1 - c \in I$  as  $x(d) \neq 0$ , so  $c + I = c_1 + I$ . But then  $(c + I)(c_2 + I) = (c_1 + I)(c_2 + I) = c_1 c_2 + I = 1 + I$  in  $\mathfrak{C}/I$ , so  $\pi(c) = c + I$  is invertible in  $\mathfrak{C}/I$ .

As this holds for all  $c \in \mathfrak{C}$  with  $x(c) \neq 0$ , by the universal property of  $\mathfrak{C}_x$  there exists a unique  $C^\infty$ -ring morphism  $j : \mathfrak{C}_x \rightarrow \mathfrak{C}/I$  with  $\pi = j \circ \pi_x$ . Since  $\pi_x, \pi$  are surjective,  $\pi_x = \iota \circ \pi$  and  $\pi = j \circ \pi_x$  imply that  $\iota : \mathfrak{C}/I \rightarrow \mathfrak{C}_x$  and  $j : \mathfrak{C}_x \rightarrow \mathfrak{C}/I$  are inverse, so both are isomorphisms.  $\square$

EXAMPLE 2.15. For  $n \geq 0$  and  $p \in \mathbb{R}^n$ , define  $C_p^\infty(\mathbb{R}^n)$  to be the set of germs of smooth functions  $c : \mathbb{R}^n \rightarrow \mathbb{R}$  at  $p \in \mathbb{R}^n$ , made into a  $C^\infty$ -ring in the obvious way. Then  $C_p^\infty(\mathbb{R}^n)$  is a local  $C^\infty$ -ring in the sense of Definition 2.10. Here are three different ways to define  $C_p^\infty(\mathbb{R}^n)$ , which yield isomorphic  $C^\infty$ -rings:

- (a) Defining  $C_p^\infty(\mathbb{R}^n)$  as the germs of functions of smooth functions at  $p$  means that points of  $C_p^\infty(\mathbb{R}^n)$  are  $\sim$ -equivalence classes  $[(U, c)]$  of pairs  $(U, c)$ , where  $U \subseteq \mathbb{R}^n$  is open with  $p \in U$  and  $c : U \rightarrow \mathbb{R}$  is smooth, and  $(U, c) \sim (U', c')$  if there exists  $V \subseteq U \cap U'$  open with  $c|_V \equiv c'|_V$ .
- (b) As the localization  $(C^\infty(\mathbb{R}^n))_p = C^\infty(\mathbb{R}^n)[g \in C^\infty(\mathbb{R}^n) : g(p) \neq 0]$ . Then points of  $(C^\infty(\mathbb{R}^n))_p$  are equivalence classes  $[f/g]$  of fractions  $f/g$  for  $f, g \in C^\infty(\mathbb{R}^n)$  with  $g(p) \neq 0$ , and fractions  $f/g, f'/g'$  are equivalent if there exists  $h \in C^\infty(\mathbb{R}^n)$  with  $h(p) \neq 0$  and  $h(fg' - f'g) \equiv 0$ .
- (c) As the quotient  $C^\infty(\mathbb{R}^n)/I$ , where  $I$  is the ideal of  $f \in C^\infty(\mathbb{R}^n)$  with  $f \equiv 0$  near  $p \in \mathbb{R}^n$ .

One can show (a)–(c) are isomorphic using the fact that if  $U$  is any open neighbourhood of  $p$  in  $\mathbb{R}^n$  then there exists smooth  $\eta : \mathbb{R}^n \rightarrow [0, 1]$  such that  $\eta \equiv 0$  on an open neighbourhood of  $\mathbb{R}^n \setminus U$  in  $\mathbb{R}^n$  and  $\eta \equiv 1$  on an open neighbourhood of  $p$  in  $U$ . By Moerdijk and Reyes [54, Prop. I.3.9], any finitely generated local  $C^\infty$ -ring is a quotient of some  $C_p^\infty(\mathbb{R}^n)$ .

## 2.4. Fair $C^\infty$ -rings

We now discuss an important class of  $C^\infty$ -rings, which we call *fair*  $C^\infty$ -rings, for brevity. Although our term ‘fair’ is new, we stress that the idea is already well-known, being originally introduced by Dubuc [22], [23, Def. 11], who first recognized their significance, under the name ‘ $C^\infty$ -rings of finite type presented by an ideal of local character’, and in more recent works would be referred to as ‘finitely generated and germ-determined  $C^\infty$ -rings’.

DEFINITION 2.16. An ideal  $I$  in  $C^\infty(\mathbb{R}^n)$  is called *fair* if for each  $f \in C^\infty(\mathbb{R}^n)$ ,  $f$  lies in  $I$  if and only if  $\pi_p(f)$  lies in  $\pi_p(I) \subseteq C_p^\infty(\mathbb{R}^n)$  for all  $p \in \mathbb{R}^n$ , where  $C_p^\infty(\mathbb{R}^n)$  is as in Example 2.15 and  $\pi_p : C^\infty(\mathbb{R}^n) \rightarrow C_p^\infty(\mathbb{R}^n)$  is the natural projection  $\pi_p : c \mapsto [( \mathbb{R}^n, c )]$ . A  $C^\infty$ -ring  $\mathfrak{C}$  is called *fair* if it is isomorphic to  $C^\infty(\mathbb{R}^n)/I$ , where  $I$  is a fair ideal. Equivalently,  $\mathfrak{C}$  is fair if it is finitely generated and whenever  $c \in \mathfrak{C}$  with  $\pi_p(c) = 0$  in  $\mathfrak{C}_p$  for all  $\mathbb{R}$ -points  $p : \mathfrak{C} \rightarrow \mathbb{R}$  then  $c = 0$ , using the notation of Definition 2.13.

Dubuc [22], [23, Def. 11] calls fair ideals *ideals of local character*, and Moerdijk and Reyes [54, I.4] call them *germ determined*, which has now become the accepted term. Fair  $C^\infty$ -rings are also sometimes called *germ determined  $C^\infty$ -rings*, a more descriptive term than ‘fair’, but the definition of germ determined  $C^\infty$ -rings  $\mathfrak{C}$  in [54, Def. I.4.1] does not require  $\mathfrak{C}$  finitely generated, so does not equate exactly to our fair  $C^\infty$ -rings. By Dubuc [22, Prop. 1.8], [23, Prop. 12] any finitely generated



ideal  $I$  is fair, so  $\mathfrak{C}$  finitely presented implies  $\mathfrak{C}$  fair. We write  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}}$  for the full subcategory of fair  $C^\infty$ -rings in  $\mathbf{C}^\infty\mathbf{Rings}$ .

**PROPOSITION 2.17.** *Suppose  $I \subset C^\infty(\mathbb{R}^m)$  and  $J \subset C^\infty(\mathbb{R}^n)$  are ideals with  $C^\infty(\mathbb{R}^m)/I \cong C^\infty(\mathbb{R}^n)/J$  as  $C^\infty$ -rings. Then  $I$  is finitely generated, or fair, if and only if  $J$  is finitely generated, or fair, respectively.*

**PROOF.** Write  $\phi : C^\infty(\mathbb{R}^m)/I \rightarrow C^\infty(\mathbb{R}^n)/J$  for the isomorphism, and  $x_1, \dots, x_m$  for the generators of  $C^\infty(\mathbb{R}^m)$ , and  $y_1, \dots, y_n$  for the generators of  $C^\infty(\mathbb{R}^n)$ . Since  $\phi$  is an isomorphism we can choose  $f_1, \dots, f_m \in C^\infty(\mathbb{R}^n)$  with  $\phi(x_i + I) = f_i + J$  for  $i = 1, \dots, m$  and  $g_1, \dots, g_n \in C^\infty(\mathbb{R}^m)$  with  $\phi(g_i + I) = y_i + J$  for  $i = 1, \dots, n$ . It is now easy to show that

$$I = (x_i - f_i(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m)), i = 1, \dots, m, \\ \text{and } h(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m)), h \in J).$$

Hence, if  $J$  is generated by  $h_1, \dots, h_k$  then  $I$  is generated by  $x_i - f_i(g_1, \dots, g_n)$  for  $i = 1, \dots, m$  and  $h_j(g_1, \dots, g_n)$  for  $j = 1, \dots, k$ , so  $J$  finitely generated implies  $I$  finitely generated. Applying the same argument to  $\phi^{-1} : C^\infty(\mathbb{R}^n)/J \rightarrow C^\infty(\mathbb{R}^m)/I$ , we see that  $I$  is finitely generated if and only if  $J$  is.

Suppose  $I$  is fair, and let  $f \in C^\infty(\mathbb{R}^n)$  with  $\pi_q(f) \in \pi_q(J) \subseteq C_q^\infty(\mathbb{R}^n)$  for all  $q \in \mathbb{R}^n$ . We will show that  $f \in J$ , so that  $J$  is fair. Consider the function  $f' = f(g_1, \dots, g_n) \in C^\infty(\mathbb{R}^m)$ . If  $p = (p_1, \dots, p_m)$  in  $\mathbb{R}^m$  and  $q = (q_1, \dots, q_n) = (g_1(p_1, \dots, p_m), \dots, g_n(p_1, \dots, p_m))$  then  $\phi : C^\infty(\mathbb{R}^m)/I \rightarrow C^\infty(\mathbb{R}^n)/J$  localizes to an isomorphism  $\phi_p : C_p^\infty(\mathbb{R}^m)/\pi_p(I) \rightarrow C_q^\infty(\mathbb{R}^n)/\pi_q(J)$  which maps  $\phi_p : \pi_p(f') + \pi_p(I) \mapsto \pi_q(f) + \pi_q(J)$ . Since  $\pi_q(f) \in \pi_q(J)$ , this gives  $\pi_p(f') \in \pi_p(I)$  for all  $p \in \mathbb{R}^m$ , so  $f' \in I$  as  $I$  is fair. But  $\phi(f' + I) = f + J$ , so  $f' \in I$  implies  $f \in J$ . Therefore  $J$  is fair. Conversely,  $J$  is fair implies  $I$  is fair.  $\square$

**EXAMPLE 2.18.** The local  $C^\infty$ -ring  $C_p^\infty(\mathbb{R}^n)$  of Example 2.15 is the quotient of  $C^\infty(\mathbb{R}^n)$  by the ideal  $I$  of functions  $f$  with  $f \equiv 0$  near  $p \in \mathbb{R}^n$ . For  $n > 0$  this  $I$  is fair, but not finitely generated. So  $C_p^\infty(\mathbb{R}^n)$  is fair, but not finitely presented, by Proposition 2.17.

The following example taken from Dubuc [24, Ex. 7.2] shows that localizations of fair  $C^\infty$ -rings need not be fair:

**EXAMPLE 2.19.** Let  $\mathfrak{C}$  be the local  $C^\infty$ -ring  $C_0^\infty(\mathbb{R})$ , as in Example 2.15. Then  $\mathfrak{C} \cong C^\infty(\mathbb{R})/I$ , where  $I$  is the ideal of all  $f \in C^\infty(\mathbb{R})$  with  $f \equiv 0$  near 0 in  $\mathbb{R}$ . This  $I$  is fair, so  $\mathfrak{C}$  is fair. Let  $c = [(x, \mathbb{R})] \in \mathfrak{C}$ . Then the localization  $\mathfrak{C}[c^{-1}]$  is the  $C^\infty$ -ring of germs at 0 in  $\mathbb{R}$  of smooth functions  $\mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ . Taking  $y = x^{-1}$  as a generator of  $\mathfrak{C}[c^{-1}]$ , we see that  $\mathfrak{C}[c^{-1}] \cong C^\infty(\mathbb{R})/J$ , where  $J$  is the ideal of compactly supported functions in  $C^\infty(\mathbb{R})$ . This  $J$  is not fair, so by Proposition 2.17,  $\mathfrak{C}[c^{-1}]$  is not fair.

Recall from category theory that if  $\mathcal{C}$  is a subcategory of a category  $\mathcal{D}$ , a *reflection*  $R : \mathcal{D} \rightarrow \mathcal{C}$  is a left adjoint to the inclusion  $\mathcal{C} \hookrightarrow \mathcal{D}$ . That is,  $R : \mathcal{D} \rightarrow \mathcal{C}$  is a functor with natural isomorphisms  $\text{Hom}_{\mathcal{C}}(R(D), C) \cong \text{Hom}_{\mathcal{D}}(D, C)$  for all  $C \in \mathcal{C}$  and  $D \in \mathcal{D}$ . We will define a reflection for  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}} \subset \mathbf{C}^\infty\mathbf{Rings}^{\text{fg}}$ , following Moerdijk and Reyes [54, p. 48–49] (see also Dubuc [23, Th. 13]).

**DEFINITION 2.20.** Let  $\mathfrak{C}$  be a finitely generated  $C^\infty$ -ring. Let  $I_{\mathfrak{C}}$  be the ideal of all  $c \in \mathfrak{C}$  such that  $\pi_p(c) = 0$  in  $\mathfrak{C}_p$  for all  $\mathbb{R}$ -points  $p : \mathfrak{C} \rightarrow \mathbb{R}$ . Then  $\mathfrak{C}/I_{\mathfrak{C}}$  is a

finitely generated  $C^\infty$ -ring, with projection  $\pi : \mathfrak{C} \rightarrow \mathfrak{C}/I_{\mathfrak{C}}$ . It has the same  $\mathbb{R}$ -points as  $\mathfrak{C}$ , that is, morphisms  $p : \mathfrak{C}/I_{\mathfrak{C}} \rightarrow \mathbb{R}$  are in 1-1 correspondence with morphisms  $p' : \mathfrak{C} \rightarrow \mathbb{R}$  by  $p' = p \circ \pi$ , and the local rings  $(\mathfrak{C}/I_{\mathfrak{C}})_p$  and  $\mathfrak{C}_{p'}$  are naturally isomorphic. It follows that  $\mathfrak{C}/I_{\mathfrak{C}}$  is fair. Define a functor  $R_{\text{fg}}^{\text{fa}} : \mathbf{C}^\infty\mathbf{Rings}^{\text{fg}} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\text{fa}}$  by  $R_{\text{fg}}^{\text{fa}}(\mathfrak{C}) = \mathfrak{C}/I_{\mathfrak{C}}$  on objects, and if  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$  is a morphism then  $\phi(I_{\mathfrak{C}}) \subseteq I_{\mathfrak{D}}$ , so  $\phi$  induces a morphism  $\phi_* : \mathfrak{C}/I_{\mathfrak{C}} \rightarrow \mathfrak{D}/I_{\mathfrak{D}}$ , and we set  $R_{\text{fg}}^{\text{fa}}(\phi) = \phi_*$ . It is easy to see  $R_{\text{fg}}^{\text{fa}}$  is a reflection.

If  $I$  is an ideal in  $C^\infty(\mathbb{R}^n)$ , write  $\bar{I}$  for the set of  $f \in C^\infty(\mathbb{R}^n)$  with  $\pi_p(f) \in \pi_p(I)$  for all  $p \in \mathbb{R}^n$ . Then  $\bar{I}$  is the smallest fair ideal in  $C^\infty(\mathbb{R}^n)$  containing  $I$ , the *germ-determined closure* of  $I$ , and  $R_{\text{fg}}^{\text{fa}}(C^\infty(\mathbb{R}^n)/I) \cong C^\infty(\mathbb{R}^n)/\bar{I}$ .

EXAMPLE 2.21. Let  $\eta : \mathbb{R} \rightarrow [0, \infty)$  be smooth with  $\eta(x) > 0$  for  $x \in (0, 1)$  and  $\eta(x) = 0$  for  $x \notin (0, 1)$ . Define  $I \subseteq C^\infty(\mathbb{R})$  by

$$I = \left\{ \sum_{a \in A} g_a(x) \eta(x - a) : A \subseteq \mathbb{Z} \text{ is finite, } g_a \in C^\infty(\mathbb{R}), a \in A \right\}.$$

Then  $I$  is an ideal in  $C^\infty(\mathbb{R})$ , so  $\mathfrak{C} = C^\infty(\mathbb{R})/I$  is a  $C^\infty$ -ring. The set of  $f \in C^\infty(\mathbb{R})$  such that  $\pi_p(f)$  lies in  $\pi_p(I) \subseteq C_p^\infty(\mathbb{R})$  for all  $p \in \mathbb{R}$  is

$$\bar{I} = \left\{ \sum_{a \in \mathbb{Z}} g_a(x) \eta(x - a) : g_a \in C^\infty(\mathbb{R}), a \in \mathbb{Z} \right\},$$

where the sum  $\sum_{a \in \mathbb{Z}} g_a(x) \eta(x - a)$  makes sense as at most one term is nonzero at any point  $x \in \mathbb{R}$ . Since  $\bar{I} \neq I$ , we see that  $I$  is *not fair*, so  $\mathfrak{C} = C^\infty(\mathbb{R})/I$  is *not a fair  $C^\infty$ -ring*. In fact  $\bar{I}$  is the smallest fair ideal containing  $I$ . We have  $I_{C^\infty(\mathbb{R})/I} = \bar{I}/I$ , and  $R_{\text{fg}}^{\text{fa}}(C^\infty(\mathbb{R})/I) = C^\infty(\mathbb{R})/\bar{I}$ .

PROPOSITION 2.22. *Let  $\mathfrak{C}$  be a  $C^\infty$ -ring, and  $G$  a finite group acting on  $\mathfrak{C}$  by automorphisms. Then the fixed subset  $\mathfrak{C}^G$  of  $G$  in  $\mathfrak{C}$  has the structure of a  $C^\infty$ -ring in a unique way, such that the inclusion  $\mathfrak{C}^G \hookrightarrow \mathfrak{C}$  is a  $C^\infty$ -ring morphism. If  $\mathfrak{C}$  is fair, or finitely presented, then  $\mathfrak{C}^G$  is also fair, or finitely presented.*

PROOF. For the first part, let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be smooth, and  $c_1, \dots, c_n \in \mathfrak{C}^G$ . Then  $\gamma \cdot \Phi_f(c_1, \dots, c_n) = \Phi_f(\gamma \cdot c_1, \dots, \gamma \cdot c_n) = \Phi_f(c_1, \dots, c_n)$  for each  $\gamma \in G$ , so  $\Phi_f(c_1, \dots, c_n) \in \mathfrak{C}^G$ . Define  $\Phi_f^G : (\mathfrak{C}^G)^n \rightarrow \mathfrak{C}^G$  by  $\Phi_f^G = \Phi_f|_{(\mathfrak{C}^G)^n}$ . It is now trivial to check that the operations  $\Phi_f^G$  for smooth  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  make  $\mathfrak{C}^G$  into a  $C^\infty$ -ring, uniquely such that  $\mathfrak{C}^G \hookrightarrow \mathfrak{C}$  is a  $C^\infty$ -ring morphism.

Suppose now that  $\mathfrak{C}$  is finitely generated. Choose a finite set of generators for  $\mathfrak{C}$ , and by adding the images of these generators under  $G$ , extend to a set of (not necessarily distinct) generators  $x_1, \dots, x_n$  for  $\mathfrak{C}$ , on which  $G$  acts freely by permutation. This gives an exact sequence  $0 \hookrightarrow I \rightarrow C^\infty(\mathbb{R}^n) \rightarrow \mathfrak{C} \rightarrow 0$ , where  $C^\infty(\mathbb{R}^n)$  is freely generated by  $x_1, \dots, x_n$ . Here  $\mathbb{R}^n$  is a direct sum of copies of the regular representation of  $G$ , and  $C^\infty(\mathbb{R}^n) \rightarrow \mathfrak{C}$  is  $G$ -equivariant. Hence  $I$  is a  $G$ -invariant ideal in  $C^\infty(\mathbb{R}^n)$ , which is fair, or finitely generated, respectively. Taking  $G$ -invariant parts gives an exact sequence  $0 \hookrightarrow I^G \rightarrow C^\infty(\mathbb{R}^n)^G \xrightarrow{\pi} \mathfrak{C}^G \rightarrow 0$ , where  $C^\infty(\mathbb{R}^n)^G, \mathfrak{C}^G$  are clearly  $C^\infty$ -rings.

As  $G$  acts linearly on  $\mathbb{R}^n$  it acts by automorphisms on the polynomial ring  $\mathbb{R}[x_1, \dots, x_n]$ . By a classical theorem of Hilbert [70, p. 274],  $\mathbb{R}[x_1, \dots, x_n]^G$  is a finitely presented  $\mathbb{R}$ -algebra, so we choose generators  $p_1, \dots, p_l$  for  $\mathbb{R}[x_1, \dots, x_n]^G$ , which induce a surjective  $\mathbb{R}$ -algebra morphism  $\mathbb{R}[p_1, \dots, p_l] \rightarrow \mathbb{R}[x_1, \dots, x_n]^G$  with kernel generated by  $q_1, \dots, q_m \in \mathbb{R}[p_1, \dots, p_l]$ .

By results of Bierstone [6] for  $G$  a finite group and Schwarz [63] for  $G$  a compact Lie group, any  $G$ -invariant smooth function on  $\mathbb{R}^n$  may be written as a smooth

function of the generators  $p_1, \dots, p_l$  of  $\mathbb{R}[x_1, \dots, x_n]^G$ , giving a surjective morphism  $C^\infty(\mathbb{R}^l) \rightarrow C^\infty(\mathbb{R}^n)^G$ , whose kernel is the ideal in  $C^\infty(\mathbb{R}^l)$  generated by  $q_1, \dots, q_m$ . Thus  $C^\infty(\mathbb{R}^n)^G$  is finitely presented.

Also  $\mathfrak{C}^G$  is generated by  $\pi(p_1), \dots, \pi(p_l)$ , so  $\mathfrak{C}^G$  is finitely generated, and we have an exact sequence  $0 \hookrightarrow J \rightarrow C^\infty(\mathbb{R}^l) \xrightarrow{\pi} \mathfrak{C}^G \rightarrow 0$ , where  $J$  is the ideal in  $C^\infty(\mathbb{R}^l)$  generated by  $q_1, \dots, q_m$  and the lifts to  $C^\infty(\mathbb{R}^l)$  of a generating set for the ideal  $I^G$  in  $C^\infty(\mathbb{R}^n)^G \cong C^\infty(\mathbb{R}^l)/(q_1, \dots, q_m)$ .

Suppose now that  $I$  is fair. Then for  $f \in C^\infty(\mathbb{R}^n)^G$ ,  $f$  lies in  $I^G$  if and only if  $\pi_p(f) \in \pi_p(I) \subseteq C_p^\infty(\mathbb{R}^n)$  for all  $p \in \mathbb{R}^n$ . If  $H$  is the subgroup of  $G$  fixing  $p$  then  $H$  acts on  $C_p^\infty(\mathbb{R}^n)$ , and  $\pi_p(f)$  is  $H$ -invariant as  $f$  is  $G$ -invariant, and  $\pi_p(I)^H = \pi_p(I^G)$ . Thus we may rewrite the condition as  $f$  lies in  $I^G$  if and only if  $\pi_p(f) \in \pi_p(I^G) \subseteq C_p^\infty(\mathbb{R}^n)$  for all  $p \in \mathbb{R}^n$ . Projecting from  $\mathbb{R}^n$  to  $\mathbb{R}^n/G$ , this says that  $f$  lies in  $I^G$  if and only if  $\pi_p(f)$  lies in  $\pi_p(I^G) \subseteq (C^\infty(\mathbb{R}^n)^G)_p$  for all  $p \in \mathbb{R}^n/G$ . Since  $C^\infty(\mathbb{R}^n)^G$  is finitely presented, it follows as in [54, Cor. I.4.9] that  $J$  is fair, so  $\mathfrak{C}^G$  is fair.

Suppose  $I$  is finitely generated in  $C^\infty(\mathbb{R}^n)$ , with generators  $f_1, \dots, f_k$ . As  $\mathbb{R}^n$  is a sum of copies of the regular representation of  $G$ , so that every irreducible representation of  $G$  occurs as a summand of  $\mathbb{R}^n$ , one can show that  $I^G$  is generated as an ideal in  $C^\infty(\mathbb{R}^n/G)$  by the  $n(k+1)$  elements  $f_i^G$  and  $(f_i x_j)^G$  for  $i = 1, \dots, k$  and  $j = 1, \dots, n$ , where  $f^G = \frac{1}{|G|} \sum_{\gamma \in G} f \circ \gamma$  is the  $G$ -invariant part of  $f \in C^\infty(\mathbb{R}^n)$ . Therefore  $J$  is finitely generated by  $q_1, \dots, q_m$  and lifts of  $f_i^G, (f_i x_j)^G$ . Hence if  $\mathfrak{C}$  is finitely presented then  $\mathfrak{C}^G$  is finitely presented.  $\square$

## 2.5. Pushouts of $C^\infty$ -rings

Proposition 2.5 shows that pushouts of  $C^\infty$ -rings exist. For finitely generated  $C^\infty$ -rings, we can describe these pushouts explicitly.

EXAMPLE 2.23. Suppose the following is a pushout diagram of  $C^\infty$ -rings:

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\quad \beta \quad} & \mathfrak{E} \\ \downarrow \alpha & & \downarrow \delta \\ \mathfrak{D} & \xrightarrow{\quad \gamma \quad} & \mathfrak{F} \end{array}$$

so that  $\mathfrak{F} = \mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E}$ , with  $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$  finitely generated. Then we have exact sequences

$$\begin{aligned} 0 \rightarrow I \hookrightarrow C^\infty(\mathbb{R}^l) \xrightarrow{\phi} \mathfrak{C} \rightarrow 0, \quad 0 \rightarrow J \hookrightarrow C^\infty(\mathbb{R}^m) \xrightarrow{\psi} \mathfrak{D} \rightarrow 0, \\ \text{and } 0 \rightarrow K \hookrightarrow C^\infty(\mathbb{R}^n) \xrightarrow{\chi} \mathfrak{E} \rightarrow 0, \end{aligned} \quad (2.3)$$

where  $\phi, \psi, \chi$  are  $C^\infty$ -ring morphisms, and  $I, J, K$  are ideals in  $C^\infty(\mathbb{R}^l), C^\infty(\mathbb{R}^m), C^\infty(\mathbb{R}^n)$ . Write  $x_1, \dots, x_l$  and  $y_1, \dots, y_m$  and  $z_1, \dots, z_n$  for the generators of  $C^\infty(\mathbb{R}^l), C^\infty(\mathbb{R}^m), C^\infty(\mathbb{R}^n)$  respectively. Then  $\phi(x_1), \dots, \phi(x_l)$  generate  $\mathfrak{C}$ , and  $\alpha \circ \phi(x_1), \dots, \alpha \circ \phi(x_l)$  lie in  $\mathfrak{D}$ , so we may write  $\alpha \circ \phi(x_i) = \psi(f_i)$  for  $i = 1, \dots, l$  as  $\psi$  is surjective, where  $f_i : \mathbb{R}^m \rightarrow \mathbb{R}$  is smooth. Similarly  $\beta \circ \phi(x_1), \dots, \beta \circ \phi(x_l)$  lie in  $\mathfrak{E}$ , so we may write  $\beta \circ \phi(x_i) = \chi(g_i)$  for  $i = 1, \dots, l$ , where  $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth.

Then from the explicit construction of pushouts of  $C^\infty$ -rings we obtain an exact sequence with  $\xi$  a morphism of  $C^\infty$ -rings

$$0 \longrightarrow L \longrightarrow C^\infty(\mathbb{R}^{m+n}) \xrightarrow{\xi} \mathfrak{F} \longrightarrow 0, \quad (2.4)$$

where we write the generators of  $C^\infty(\mathbb{R}^{m+n})$  as  $y_1, \dots, y_m, z_1, \dots, z_n$ , and then  $L$  is the ideal in  $C^\infty(\mathbb{R}^{m+n})$  generated by the elements  $d(y_1, \dots, y_m)$  for  $d \in J \subseteq C^\infty(\mathbb{R}^m)$ , and  $e(z_1, \dots, z_n)$  for  $e \in K \subseteq C^\infty(\mathbb{R}^n)$ , and  $f_i(y_1, \dots, y_m) - g_i(z_1, \dots, z_n)$  for  $i = 1, \dots, l$ .

For the case of *coproducts*  $\mathfrak{D} \otimes_\infty \mathfrak{E}$ , with  $\mathfrak{C} = \mathbb{R}$ ,  $l = 0$  and  $I = \{0\}$ , we have

$$(C^\infty(\mathbb{R}^m)/J) \otimes_\infty (C^\infty(\mathbb{R}^n)/K) \cong C^\infty(\mathbb{R}^{m+n})/(J, K).$$

**PROPOSITION 2.24.** *The subcategories  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fg}}$  and  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fp}}$  are closed under pushouts and all finite colimits in  $\mathbf{C}^\infty\mathbf{Rings}$ .*

**PROOF.** First we show  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fg}}, \mathbf{C}^\infty\mathbf{Rings}^{\text{fp}}$  are closed under pushouts. Suppose  $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$  are finitely generated, and use the notation of Example 2.23. Then  $\mathfrak{F}$  is finitely generated with generators  $y_1, \dots, y_m, z_1, \dots, z_n$ , so  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fg}}$  is closed under pushouts. If  $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$  are finitely presented then we can take  $J = (d_1, \dots, d_j)$  and  $K = (e_1, \dots, e_k)$ , and then Example 2.23 gives

$$\begin{aligned} L = & (d_p(y_1, \dots, y_m), p = 1, \dots, j, e_p(z_1, \dots, z_n), p = 1, \dots, k, \\ & f_p(y_1, \dots, y_m) - g_p(z_1, \dots, z_n), p = 1, \dots, l). \end{aligned} \quad (2.5)$$

So  $L$  is finitely generated, and  $\mathfrak{F} \cong C^\infty(\mathbb{R}^{m+n})/L$  is finitely presented. Thus  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fp}}$  is closed under pushouts.

Now  $\mathbb{R}$  is an initial object in  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fg}}, \mathbf{C}^\infty\mathbf{Rings}^{\text{fp}} \subset \mathbf{C}^\infty\mathbf{Rings}$ , and all finite colimits may be constructed by repeated pushouts involving the initial object. Hence  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fg}}, \mathbf{C}^\infty\mathbf{Rings}^{\text{fp}}$  are closed under finite colimits.  $\square$

Here is an example from Dubuc [24, Ex. 7.1], Moerdijk and Reyes [54, p. 49].

**EXAMPLE 2.25.** Consider the coproduct  $C^\infty(\mathbb{R}) \otimes_\infty C_0^\infty(\mathbb{R})$ , where  $C_0^\infty(\mathbb{R})$  is the  $C^\infty$ -ring of germs of smooth functions at 0 in  $\mathbb{R}$  as in Example 2.15. Then  $C^\infty(\mathbb{R}), C_0^\infty(\mathbb{R})$  are fair  $C^\infty$ -rings, but  $C_0^\infty(\mathbb{R})$  is not finitely presented. By Example 2.23,  $C^\infty(\mathbb{R}) \otimes_\infty C_0^\infty(\mathbb{R}) = C^\infty(\mathbb{R}) \amalg_{\mathbb{R}} C_0^\infty(\mathbb{R}) \cong C^\infty(\mathbb{R}^2)/L$ , where  $L$  is the ideal in  $C^\infty(\mathbb{R}^2)$  generated by functions  $f(x, y) = g(y)$  for  $g \in C^\infty(\mathbb{R})$  with  $g \equiv 0$  near  $0 \in \mathbb{R}$ . This ideal  $L$  is not fair, since for example one can find  $f \in C^\infty(\mathbb{R}^2)$  with  $f(x, y) = 0$  if and only if  $|xy| \leq 1$ , and then  $f \notin L$  but  $\pi_p(f) \in \pi_p(L) \subseteq C_p^\infty(\mathbb{R}^2)$  for all  $p \in \mathbb{R}^2$ . Hence  $C^\infty(\mathbb{R}) \otimes_\infty C_0^\infty(\mathbb{R})$  is not a fair  $C^\infty$ -ring, by Proposition 2.17, and pushouts of fair  $C^\infty$ -rings need not be fair.

Our next result is referred to in the last part of Dubuc [23, Th. 13].

**PROPOSITION 2.26.**  *$\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}}$  is not closed under pushouts in  $\mathbf{C}^\infty\mathbf{Rings}$ . Nonetheless, pushouts and all finite colimits exist in  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}}$ , although they may not coincide with pushouts and finite colimits in  $\mathbf{C}^\infty\mathbf{Rings}$ .*

**PROOF.** Example 2.25 shows that  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}}$  is not closed under pushouts in  $\mathbf{C}^\infty\mathbf{Rings}$ . To construct finite colimits in  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}}$ , we first take the colimit in  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fg}}$ , which exists by Propositions 2.5 and 2.24, and then apply the reflection functor  $R_{\text{fg}}^{\text{fa}}$ . By the universal properties of colimits and reflection functors, the result is a colimit in  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}}$ .  $\square$

## 2.6. Flat ideals

The following class of ideals in  $C^\infty(\mathbb{R}^n)$  is defined by Moerdijk and Reyes [54, p. 47, p. 49] (see also Dubuc [22, §1.7(a)]), who call them *flat ideals*:

**DEFINITION 2.27.** Let  $X$  be a closed subset of  $\mathbb{R}^n$ . Define  $\mathfrak{m}_X^\infty$  to be the ideal of all functions  $g \in C^\infty(\mathbb{R}^n)$  such that  $\partial^k g|_X \equiv 0$  for all  $k \geq 0$ , that is,  $g$  and all its derivatives vanish at each  $x \in X$ . If the interior  $X^\circ$  of  $X$  in  $\mathbb{R}^n$  is dense in  $X$ , that is  $\overline{X^\circ} = X$ , then  $\partial^k g|_X \equiv 0$  for all  $k \geq 0$  if and only if  $g|_X \equiv 0$ . In this case  $C^\infty(\mathbb{R}^n)/\mathfrak{m}_X^\infty \cong C^\infty(X) := \{f|_X : f \in C^\infty(\mathbb{R}^n)\}$ .

Flat ideals are always fair. Here is an example from [54, Th. I.1.3].

**EXAMPLE 2.28.** Take  $X$  to be the point  $\{0\}$ . If  $f, f' \in C^\infty(\mathbb{R}^n)$  then  $f - f'$  lies in  $\mathfrak{m}_{\{0\}}^\infty$  if and only if  $f, f'$  have the same Taylor series at 0. Thus  $C^\infty(\mathbb{R}^n)/\mathfrak{m}_{\{0\}}^\infty$  is the  $C^\infty$ -ring of Taylor series at 0 of  $f \in C^\infty(\mathbb{R}^n)$ . Since any formal power series in  $x_1, \dots, x_n$  is the Taylor series of some  $f \in C^\infty(\mathbb{R}^n)$ , we have  $C^\infty(\mathbb{R}^n)/\mathfrak{m}_{\{0\}}^\infty \cong \mathbb{R}[[x_1, \dots, x_n]]$ . Thus the  $\mathbb{R}$ -algebra of formal power series  $\mathbb{R}[[x_1, \dots, x_n]]$  can be made into a  $C^\infty$ -ring.

The following nontrivial result is proved by Reyes and van Quê [60, Th. 1], generalizing an unpublished result of A.P. Calderón in the case  $X = Y = \{0\}$ . It can also be found in Moerdijk and Reyes [54, Cor. I.4.12].

**PROPOSITION 2.29.** Let  $X \subseteq \mathbb{R}^m$  and  $Y \subseteq \mathbb{R}^n$  be closed. Then as ideals in  $C^\infty(\mathbb{R}^{m+n})$  we have  $(\mathfrak{m}_X^\infty, \mathfrak{m}_Y^\infty) = \mathfrak{m}_{X \times Y}^\infty$ .

Moerdijk and Reyes [54, Cor. I.4.19] prove:

**PROPOSITION 2.30.** Let  $X \subseteq \mathbb{R}^n$  be closed with  $X \neq \emptyset, \mathbb{R}^n$ . Then the ideal  $\mathfrak{m}_X^\infty$  in  $C^\infty(\mathbb{R}^n)$  is not countably generated.

We can use these to study  $C^\infty$ -rings of manifolds with corners.

**EXAMPLE 2.31.** Let  $0 < k \leq n$ , and consider the closed subset  $\mathbb{R}_k^n = [0, \infty)^k \times \mathbb{R}^{n-k}$  in  $\mathbb{R}^n$ , the local model for manifolds with corners. Write  $C^\infty(\mathbb{R}_k^n)$  for the  $C^\infty$ -ring  $\{f|_{\mathbb{R}_k^n} : f \in C^\infty(\mathbb{R}^n)\}$ . Since the interior  $(\mathbb{R}_k^n)^\circ = (0, \infty)^k \times \mathbb{R}^{n-k}$  of  $\mathbb{R}_k^n$  is dense in  $\mathbb{R}_k^n$ , as in Definition 2.27 we have  $C^\infty(\mathbb{R}_k^n) = C^\infty(\mathbb{R}^n)/\mathfrak{m}_{\mathbb{R}_k^n}^\infty$ . As  $\mathfrak{m}_{\mathbb{R}_k^n}^\infty$  is not countably generated by Proposition 2.30, it is not finitely generated, and thus  $C^\infty(\mathbb{R}_k^n)$  is not a finitely presented  $C^\infty$ -ring, by Proposition 2.17.

Consider the coproduct  $C^\infty(\mathbb{R}_k^m) \otimes_\infty C^\infty(\mathbb{R}_l^n)$  in  **$C^\infty$ Rings**, that is, the pushout  $C^\infty(\mathbb{R}_k^m) \amalg_\mathbb{R} C^\infty(\mathbb{R}_l^n)$  over the trivial  $C^\infty$ -ring  $\mathbb{R}$ . By Example 2.23 and Proposition 2.29 we have

$$\begin{aligned} C^\infty(\mathbb{R}_k^m) \otimes_\infty C^\infty(\mathbb{R}_l^n) &\cong C^\infty(\mathbb{R}^{m+n})/(\mathfrak{m}_{\mathbb{R}_k^m}^\infty, \mathfrak{m}_{\mathbb{R}_l^n}^\infty) = C^\infty(\mathbb{R}^{m+n})/\mathfrak{m}_{\mathbb{R}_k^m \times \mathbb{R}_l^n}^\infty \\ &= C^\infty(\mathbb{R}_k^m \times \mathbb{R}_l^n) \cong C^\infty(\mathbb{R}_{k+l}^{m+n}). \end{aligned}$$

This is an example of Theorem 3.5 below, with  $X = \mathbb{R}_k^m$ ,  $Y = \mathbb{R}_l^n$  and  $Z = *$ .



## CHAPTER 3

### The $C^\infty$ -ring $C^\infty(X)$ of a manifold $X$

We now study the  $C^\infty$ -rings  $C^\infty(X)$  of manifolds  $X$  defined in Example 2.2. We are interested in *manifolds without boundary* (locally modelled on  $\mathbb{R}^n$ ), and in *manifolds with boundary* (locally modelled on  $[0, \infty) \times \mathbb{R}^{n-1}$ ), and in *manifolds with corners* (locally modelled on  $[0, \infty)^k \times \mathbb{R}^{n-k}$ ). Manifolds with corners were considered by the author [35, 40], and we use the conventions of those papers.

The  $C^\infty$ -rings of manifolds with boundary are discussed by Reyes [59] and Kock [44, §III.9], but Kock appears to have been unaware of Proposition 2.29, which makes  $C^\infty$ -rings of manifolds with boundary easier to understand.

If  $X, Y$  are manifolds with corners of dimensions  $m, n$ , then [40, §2.1] defined  $f : X \rightarrow Y$  to be *weakly smooth* if  $f$  is continuous and whenever  $(U, \phi), (V, \psi)$  are charts on  $X, Y$  then  $\psi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}(\psi(V)) \rightarrow V$  is a smooth map from  $(f \circ \phi)^{-1}(\psi(V)) \subset \mathbb{R}^m$  to  $V \subset \mathbb{R}^n$ . A *smooth map* is a weakly smooth map  $f$  satisfying some extra conditions over  $\partial^k X, \partial^l Y$  in [40, §2.1]. If  $\partial Y = \emptyset$  these conditions are vacuous, so for manifolds without boundary, weakly smooth maps and smooth maps coincide. Write **Man**, **Man<sup>b</sup>**, **Man<sup>c</sup>** for the categories of manifolds without boundary, and with boundary, and with corners, respectively, with morphisms smooth maps.

**PROPOSITION 3.1. (a)** *If  $X$  is a manifold without boundary then the  $C^\infty$ -ring  $C^\infty(X)$  of Example 2.2 is finitely presented.*

**(b)** *If  $X$  is a manifold with boundary, or with corners, and  $\partial X \neq \emptyset$ , then the  $C^\infty$ -ring  $C^\infty(X)$  of Example 2.2 is fair, but is not finitely presented.*

**PROOF.** Part (a) is proved in Dubuc [23, p. 687] and Moerdijk and Reyes [54, Th. I.2.3] following an observation of Lawvere, that if  $X$  is a manifold without boundary then we can choose a closed embedding  $i : X \hookrightarrow \mathbb{R}^N$  for  $N \gg 0$ , and then  $X$  is a retract of an open neighbourhood  $U$  of  $i(X)$ , so we have an exact sequence  $0 \rightarrow I \rightarrow C^\infty(\mathbb{R}^N) \xrightarrow{i^*} C^\infty(X) \rightarrow 0$  in which the ideal  $I$  is finitely generated, and thus the  $C^\infty$ -ring  $C^\infty(X)$  is finitely presented.

For (b), if  $X$  is an  $n$ -manifold with boundary, or with corners, then roughly by gluing on a ‘collar’  $\partial X \times (-\epsilon, 0]$  to  $X$  along  $\partial X$  for small  $\epsilon > 0$ , we can embed  $X$  as a closed subset in an  $n$ -manifold  $X'$  without boundary, such that the inclusion  $X \hookrightarrow X'$  is locally modelled on the inclusion of  $\mathbb{R}_k^n = [0, \infty)^k \times \mathbb{R}^{n-k}$  in  $(-\epsilon, \infty)^k \times \mathbb{R}^{n-k}$  for  $k \leq n$ . Choose a closed embedding  $i : X' \hookrightarrow \mathbb{R}^N$  for  $N \gg 0$  as above, giving  $0 \rightarrow I' \rightarrow C^\infty(\mathbb{R}^N) \xrightarrow{i^*} C^\infty(X') \rightarrow 0$  with  $I'$  generated by  $f_1, \dots, f_k \in C^\infty(\mathbb{R}^N)$ . Let  $i(X') \subset T \subset \mathbb{R}^N$  be an open tubular neighbourhood of  $i(X')$  in  $\mathbb{R}^N$ , with projection  $\pi : T \rightarrow i(X')$ . Set  $U = \pi^{-1}(i(X^\circ)) \subset T \subset \mathbb{R}^N$ , where  $X^\circ$  is the interior of  $X$ . Then  $U$  is open in  $\mathbb{R}^N$  with  $i(X^\circ) = U \cap i(X')$ , and the closure  $\bar{U}$  of  $U$  in  $\mathbb{R}^N$  has  $i(X) = \bar{U} \cap i(X')$ .

Let  $I$  be the ideal  $(f_1, \dots, f_k, \mathfrak{m}_{\bar{U}}^\infty)$  in  $C^\infty(\mathbb{R}^N)$ . Then  $I$  is fair, as  $(f_1, \dots, f_k)$  and  $\mathfrak{m}_{\bar{U}}^\infty$  are fair. Since  $U$  is open in  $\mathbb{R}^N$  and dense in  $\bar{U}$ , as in Definition 2.27 we have  $g \in \mathfrak{m}_{\bar{U}}^\infty$  if and only if  $g|_{\bar{U}} \equiv 0$ . Therefore the isomorphism  $(i_*)_* : C^\infty(\mathbb{R}^N)/I' \rightarrow C^\infty(X')$  identifies the ideal  $I/I'$  in  $C^\infty(X')$  with the ideal of  $f \in C^\infty(X')$  such that  $f|_X \equiv 0$ , since  $X = i^{-1}(\bar{U})$ . Hence

$$C^\infty(\mathbb{R}^N)/I \cong C^\infty(X')/\{f \in C^\infty(X') : f|_X \equiv 0\} \cong \{f|_X : f \in C^\infty(X')\} \cong C^\infty(X).$$

As  $I$  is a fair ideal, this implies that  $C^\infty(X)$  is a fair  $C^\infty$ -ring. If  $\partial X \neq \emptyset$  then using Proposition 2.30 we can show  $I$  is not countably generated, so  $C^\infty(X)$  is not finitely presented by Proposition 2.17.  $\square$

Next we consider the transformation  $X \mapsto C^\infty(X)$  as a functor.

DEFINITION 3.2. Write  $\mathbf{C}^\infty\mathbf{Rings}^{\text{op}}$ ,  $(\mathbf{C}^\infty\mathbf{Rings}^{\text{fp}})^{\text{op}}$ ,  $(\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}})^{\text{op}}$  for the opposite categories of  $\mathbf{C}^\infty\mathbf{Rings}$ ,  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fp}}$ ,  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}}$  (i.e. directions of morphisms are reversed). Define functors

$$\begin{aligned} F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Rings}} : \mathbf{Man} &\longrightarrow (\mathbf{C}^\infty\mathbf{Rings}^{\text{fp}})^{\text{op}} \subset \mathbf{C}^\infty\mathbf{Rings}^{\text{op}}, \\ F_{\mathbf{Man}^{\mathbf{b}}}^{\mathbf{C}^\infty\mathbf{Rings}} : \mathbf{Man}^{\mathbf{b}} &\longrightarrow (\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}})^{\text{op}} \subset \mathbf{C}^\infty\mathbf{Rings}^{\text{op}}, \\ F_{\mathbf{Man}^{\mathbf{c}}}^{\mathbf{C}^\infty\mathbf{Rings}} : \mathbf{Man}^{\mathbf{c}} &\longrightarrow (\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}})^{\text{op}} \subset \mathbf{C}^\infty\mathbf{Rings}^{\text{op}} \end{aligned}$$

as follows. On objects the functors  $F_{\mathbf{Man}^*}^{\mathbf{C}^\infty\mathbf{Rings}}$  map  $X \mapsto C^\infty(X)$ , where  $C^\infty(X)$  is a  $C^\infty$ -ring as in Example 2.2. On morphisms, if  $f : X \rightarrow Y$  is a smooth map of manifolds then  $f^* : C^\infty(Y) \rightarrow C^\infty(X)$  mapping  $c \mapsto c \circ f$  is a morphism of  $C^\infty$ -rings, so that  $f^* : C^\infty(Y) \rightarrow C^\infty(X)$  is a morphism in  $\mathbf{C}^\infty\mathbf{Rings}$ , and  $f^* : C^\infty(X) \rightarrow C^\infty(Y)$  a morphism in  $\mathbf{C}^\infty\mathbf{Rings}^{\text{op}}$ , and  $F_{\mathbf{Man}^*}^{\mathbf{C}^\infty\mathbf{Rings}}$  maps  $f \mapsto f^*$ . Clearly  $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Rings}}$ ,  $F_{\mathbf{Man}^{\mathbf{b}}}^{\mathbf{C}^\infty\mathbf{Rings}}$ ,  $F_{\mathbf{Man}^{\mathbf{c}}}^{\mathbf{C}^\infty\mathbf{Rings}}$  are functors.

If  $f : X \rightarrow Y$  is only *weakly smooth* then  $f^* : C^\infty(Y) \rightarrow C^\infty(X)$  in Definition 3.2 is still a morphism of  $C^\infty$ -rings. From [54, Prop. I.1.5] we deduce:

PROPOSITION 3.3. *Let  $X, Y$  be manifolds with corners. Then the map  $f \mapsto f^*$  from weakly smooth maps  $f : X \rightarrow Y$  to morphisms of  $C^\infty$ -rings  $\phi : C^\infty(Y) \rightarrow C^\infty(X)$  is a 1-1 correspondence.*

In the category of manifolds  $\mathbf{Man}$ , the morphisms are weakly smooth maps. So  $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Rings}}$  is both injective on morphisms (faithful), and surjective on morphisms (full), as in Moerdijk and Reyes [54, Th. I.2.8]. But in  $\mathbf{Man}^{\mathbf{b}}$ ,  $\mathbf{Man}^{\mathbf{c}}$  the morphisms are smooth maps, a proper subset of weakly smooth maps, so the functors are injective but not surjective on morphisms. That is:

COROLLARY 3.4. *The functor  $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Rings}} : \mathbf{Man} \rightarrow (\mathbf{C}^\infty\mathbf{Rings}^{\text{fp}})^{\text{op}}$  is full and faithful. However, the functors  $F_{\mathbf{Man}^{\mathbf{b}}}^{\mathbf{C}^\infty\mathbf{Rings}} : \mathbf{Man}^{\mathbf{b}} \rightarrow (\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}})^{\text{op}}$  and  $F_{\mathbf{Man}^{\mathbf{c}}}^{\mathbf{C}^\infty\mathbf{Rings}} : \mathbf{Man}^{\mathbf{c}} \rightarrow (\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}})^{\text{op}}$  are faithful, but not full.*

Of course, if we defined  $\mathbf{Man}^{\mathbf{b}}$ ,  $\mathbf{Man}^{\mathbf{c}}$  to have morphisms weakly smooth maps, then  $F_{\mathbf{Man}^{\mathbf{b}}}^{\mathbf{C}^\infty\mathbf{Rings}}$ ,  $F_{\mathbf{Man}^{\mathbf{c}}}^{\mathbf{C}^\infty\mathbf{Rings}}$  would be full and faithful.

Let  $X, Y, Z$  be manifolds and  $f : X \rightarrow Z$ ,  $g : Y \rightarrow Z$  be smooth maps. If  $X, Y, Z$  are without boundary then  $f, g$  are called *transverse* if whenever  $x \in X$  and  $y \in Y$  with  $f(x) = g(y) = z \in Z$  we have  $T_z Z = \text{df}(T_x X) + \text{dg}(T_y Y)$ . If  $f, g$  are transverse then a fibre product  $X \times_Z Y$  exists in  $\mathbf{Man}$ .



For manifolds with boundary, or with corners, the situation is more complicated, as explained in [35, §6], [40, §4.3]. In the definition of *smooth*  $f : X \rightarrow Y$  we impose extra conditions over  $\partial^j X, \partial^k Y$ , and in the definition of *transverse*  $f, g$  we impose extra conditions over  $\partial^j X, \partial^k Y, \partial^l Z$ . With these more restrictive definitions of smooth and transverse maps, transverse fibre products exist in  $\mathbf{Man}^c$  by [35, Th. 6.3] (see also [40, Th. 4.27]). The naïve definition of transversality is not a sufficient condition for fibre products to exist. Note too that a fibre product of manifolds with boundary may be a manifold with corners, so fibre products work best in  $\mathbf{Man}$  or  $\mathbf{Man}^c$  rather than  $\mathbf{Man}^b$ .

Our next theorem is given in [23, Th. 16] and [54, Prop. I.2.6] for manifolds without boundary, and the special case of products  $\mathbf{Man} \times \mathbf{Man}^b \rightarrow \mathbf{Man}^b$  follows from Reyes [59, Th. 2.5], see also Kock [44, §III.9]. It can be proved by combining the usual proof in the without boundary case, the proof of [35, Th. 6.3], and Proposition 2.29.

**THEOREM 3.5.** *The functors  $F_{\mathbf{Man}}^{\mathbf{C}^\infty \mathbf{Rings}}, F_{\mathbf{Man}^c}^{\mathbf{C}^\infty \mathbf{Rings}}$  preserve transverse fibre products in  $\mathbf{Man}, \mathbf{Man}^c$ , in the sense of [35, §6]. That is, if the following is a Cartesian square of manifolds with  $g, h$  transverse*

$$\begin{array}{ccc} W & \xrightarrow{\quad f \quad} & Y \\ \downarrow e & & \downarrow h \\ X & \xrightarrow{\quad g \quad} & Z, \end{array} \quad (3.1)$$

so that  $W = X \times_{g, Z, h} Y$ , then we have a pushout square of  $C^\infty$ -rings

$$\begin{array}{ccc} C^\infty(Z) & \xrightarrow{\quad h^* \quad} & C^\infty(Y) \\ \downarrow g^* & & \downarrow f^* \\ C^\infty(X) & \xrightarrow{\quad e^* \quad} & C^\infty(W), \end{array} \quad (3.2)$$

so that  $C^\infty(W) = C^\infty(X) \amalg_{g^*, C^\infty(Z), h^*} C^\infty(Y)$ .



## CHAPTER 4

### $C^\infty$ -ringed spaces and $C^\infty$ -schemes

In algebraic geometry, if  $A$  is an affine scheme and  $R$  the ring of regular functions on  $A$ , then we can recover  $A$  as the spectrum of the ring  $R$ ,  $A \cong \operatorname{Spec} R$ . One of the ideas of synthetic differential geometry, as in [54, §I], is to regard a manifold  $X$  as the ‘spectrum’ of the  $C^\infty$ -ring  $C^\infty(X)$  in Example 2.2. So we can try to develop analogues of the tools of scheme theory for smooth manifolds, replacing rings by  $C^\infty$ -rings throughout. This was done by Dubuc [22, 23]. The analogues of the algebraic geometry notions [31, §II.2] of ringed spaces, locally ringed spaces, and schemes, are called  $C^\infty$ -ringed spaces, local  $C^\infty$ -ringed spaces and  $C^\infty$ -schemes. The material of §4.6–§4.9 is new.

#### 4.1. Some basic topology

Later we will use several properties of topological spaces, e.g. second countable, metrizable, Lindelöf, . . . , so we now recall their definitions and some relationships between them. Let  $X$  be a topological space, with topology  $\mathcal{T}$ . Then:

- A *basis* for  $\mathcal{T}$  is a family  $\mathcal{B} \subseteq \mathcal{T}$  such that every open set in  $X$  is a union of sets in  $\mathcal{B}$ . We call  $X$  *second countable* if  $\mathcal{T}$  has a countable basis.
- An open cover  $\{U_i : i \in I\}$  of  $X$  is *locally finite* if every  $x \in X$  has an open neighbourhood  $W$  with  $W \cap U_i \neq \emptyset$  for only finitely many  $i \in I$ .  
An open cover  $\{V_j : j \in J\}$  of  $X$  is a *refinement* of another open cover  $\{U_i : i \in I\}$  if for all  $j \in J$  there exists  $i \in I$  with  $V_j \subseteq U_i \subseteq X$ .  
We call  $X$  *paracompact* if every open cover  $\{U_i : i \in I\}$  of  $X$  admits a locally finite refinement  $\{V_j : j \in J\}$ .
- We call  $X$  *Hausdorff* if for all  $x, y \in X$  with  $x \neq y$  there exist open  $U, V \subseteq X$  with  $x \in U$ ,  $y \in V$  and  $U \cap V = \emptyset$ .
- We call  $X$  *metrizable* if there exists a metric on  $X$  inducing topology  $\mathcal{T}$ .
- We call  $X$  *regular* if for every closed subset  $C \subseteq X$  and each  $x \in X \setminus C$  there exist disjoint open sets  $U, V \subseteq X$  with  $C \subseteq U$  and  $x \in V$ .
- We call  $X$  *completely regular* if for every closed  $C \subseteq X$  and  $x \in X \setminus C$  there exists a continuous  $f : X \rightarrow [0, 1]$  with  $f|_C = 0$  and  $f(x) = 1$ .
- We call  $X$  *separable* if it has a countable dense subset  $S \subseteq X$ .
- We call  $X$  *locally compact* if for all  $x \in X$  there exist  $x \in U \subseteq C \subseteq X$  with  $U$  open and  $C$  compact.
- We call  $X$  *Lindelöf* if every open cover of  $X$  has a countable subcover.

By well known results in topology, including Urysohn’s metrization theorem, the following are equivalent:

- (i)  $X$  is Hausdorff, second countable and regular.
- (ii)  $X$  is second countable and metrizable.
- (iii)  $X$  is separable and metrizable.

Here are some useful implications:

- $X$  Hausdorff and locally compact imply  $X$  is regular.
- $X$  metrizable implies  $X$  is Hausdorff, paracompact, and regular.
- $X$  second countable implies  $X$  is Lindelöf.
- $X$  Lindelöf and regular imply  $X$  is paracompact.

#### 4.2. Sheaves on topological spaces

Sheaves are a fundamental concept in algebraic geometry. They are necessary even to define schemes, since a scheme is a topological space  $X$  equipped with a sheaf of rings  $\mathcal{O}_X$ . In this book, sheaves of  $C^\infty$ -rings, and sheaves of modules over a sheaf of  $C^\infty$ -rings, play a fundamental rôle.

We now summarize some basics of sheaf theory, following Hartshorne [31, §II.1]. A more detailed reference is Godement [28]. We concentrate on sheaves of abelian groups; to define sheaves of  $C^\infty$ -rings, etc., one replaces abelian groups with  $C^\infty$ -rings, etc., throughout. This is justified since limits in all these categories (including abelian groups) are computed at the level of underlying sets, because they are all algebras for algebraic theories.

**DEFINITION 4.1.** Let  $X$  be a topological space. A *presheaf of abelian groups*  $\mathcal{E}$  on  $X$  consists of the data of an abelian group  $\mathcal{E}(U)$  for every open set  $U \subseteq X$ , and a morphism of abelian groups  $\rho_{UV} : \mathcal{E}(U) \rightarrow \mathcal{E}(V)$  called the *restriction map* for every inclusion  $V \subseteq U \subseteq X$  of open sets, satisfying the conditions that

- (i)  $\mathcal{E}(\emptyset) = 0$ ;
- (ii)  $\rho_{UU} = \text{id}_{\mathcal{E}(U)} : \mathcal{E}(U) \rightarrow \mathcal{E}(U)$  for all open  $U \subseteq X$ ; and
- (iii)  $\rho_{UW} = \rho_{VW} \circ \rho_{UV} : \mathcal{E}(U) \rightarrow \mathcal{E}(W)$  for all open  $W \subseteq V \subseteq U \subseteq X$ .

That is, a presheaf is a functor  $\mathcal{E} : \mathbf{Open}(X)^{\text{op}} \rightarrow \mathbf{AbGp}$ , where  $\mathbf{Open}(X)$  is the category of open subsets of  $X$  with morphisms inclusions, and  $\mathbf{AbGp}$  is the category of abelian groups.

A presheaf of abelian groups  $\mathcal{E}$  on  $X$  is called a *sheaf* if it also satisfies

- (iv) If  $U \subseteq X$  is open,  $\{V_i : i \in I\}$  is an open cover of  $U$ , and  $s \in \mathcal{E}(U)$  has  $\rho_{UV_i}(s) = 0$  in  $\mathcal{E}(V_i)$  for all  $i \in I$ , then  $s = 0$  in  $\mathcal{E}(U)$ ; and
- (v) If  $U \subseteq X$  is open,  $\{V_i : i \in I\}$  is an open cover of  $U$ , and we are given elements  $s_i \in \mathcal{E}(V_i)$  for all  $i \in I$  such that  $\rho_{V_i(V_i \cap V_j)}(s_i) = \rho_{V_j(V_i \cap V_j)}(s_j)$  in  $\mathcal{E}(V_i \cap V_j)$  for all  $i, j \in I$ , then there exists  $s \in \mathcal{E}(U)$  with  $\rho_{UV_i}(s) = s_i$  for all  $i \in I$ . This  $s$  is unique by (iv).

Suppose  $\mathcal{E}, \mathcal{F}$  are presheaves or sheaves of abelian groups on  $X$ . A *morphism*  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  consists of a morphism of abelian groups  $\phi(U) : \mathcal{E}(U) \rightarrow \mathcal{F}(U)$  for all open  $U \subseteq X$ , such that the following diagram commutes for all open  $V \subseteq U \subseteq X$

$$\begin{array}{ccc} \mathcal{E}(U) & \xrightarrow{\phi(U)} & \mathcal{F}(U) \\ \downarrow \rho_{UV} & & \downarrow \rho'_{UV} \\ \mathcal{E}(V) & \xrightarrow{\phi(V)} & \mathcal{F}(V), \end{array}$$

where  $\rho_{UV}$  is the restriction map for  $\mathcal{E}$ , and  $\rho'_{UV}$  the restriction map for  $\mathcal{F}$ .

**DEFINITION 4.2.** Let  $\mathcal{E}$  be a presheaf of abelian groups on  $X$ . For each  $x \in X$ , the *stalk*  $\mathcal{E}_x$  is the direct limit of the groups  $\mathcal{E}(U)$  for all  $x \in U \subseteq X$ , via the restriction maps  $\rho_{UV}$ . It is an abelian group. A morphism  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  induces

morphisms  $\phi_x : \mathcal{E}_x \rightarrow \mathcal{F}_x$  for all  $x \in X$ . If  $\mathcal{E}, \mathcal{F}$  are sheaves then  $\phi$  is an isomorphism if and only if  $\phi_x$  is an isomorphism for all  $x \in X$ .

Sheaves of abelian groups on  $X$  form an *abelian category*  $\text{Sh}(X)$ . Thus we have (category-theoretic) notions of when a morphism  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  in  $\text{Sh}(X)$  is *injective* or *surjective* (*epimorphic*), and when a sequence  $\mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G}$  in  $\text{Sh}(X)$  is *exact*. It turns out that  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  is injective if and only if  $\phi(U) : \mathcal{E}(U) \rightarrow \mathcal{F}(U)$  is injective for all open  $U \subseteq X$ . However  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  surjective does not imply that  $\phi(U) : \mathcal{E}(U) \rightarrow \mathcal{F}(U)$  is surjective for all open  $U \subseteq X$ . Instead,  $\phi$  is surjective if and only if  $\phi_x : \mathcal{E}_x \rightarrow \mathcal{F}_x$  is surjective for all  $x \in X$ .

**DEFINITION 4.3.** Let  $\mathcal{E}$  be a presheaf of abelian groups on  $X$ . A *sheafification* of  $\mathcal{E}$  is a sheaf of abelian groups  $\hat{\mathcal{E}}$  on  $X$  and a morphism  $\pi : \mathcal{E} \rightarrow \hat{\mathcal{E}}$ , such that whenever  $\mathcal{F}$  is a sheaf of abelian groups on  $X$  and  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  is a morphism, there is a unique morphism  $\hat{\phi} : \hat{\mathcal{E}} \rightarrow \mathcal{F}$  with  $\phi = \hat{\phi} \circ \pi$ . As in [31, Prop. II.1.2], a sheafification always exists, and is unique up to canonical isomorphism; one can be constructed explicitly using the stalks  $\mathcal{E}_x$  of  $\mathcal{E}$ .

Next we discuss *pushforwards* and *pullbacks* of sheaves by continuous maps.

**DEFINITION 4.4.** Let  $f : X \rightarrow Y$  be a continuous map of topological spaces, and  $\mathcal{E}$  a sheaf of abelian groups on  $X$ . Define the *pushforward* (*direct image*) sheaf  $f_*(\mathcal{E})$  on  $Y$  by  $(f_*(\mathcal{E}))(U) = \mathcal{E}(f^{-1}(U))$  for all open  $U \subseteq Y$ , with restriction maps  $\rho'_{UV} = \rho_{f^{-1}(U)f^{-1}(V)} : (f_*(\mathcal{E}))(U) \rightarrow (f_*(\mathcal{E}))(V)$  for all open  $V \subseteq U \subseteq Y$ . Then  $f_*(\mathcal{E})$  is a sheaf of abelian groups on  $Y$ .

If  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  is a morphism in  $\text{Sh}(X)$  we define  $f_*(\phi) : f_*(\mathcal{E}) \rightarrow f_*(\mathcal{F})$  by  $(f_*(\phi))(u) = \phi(f^{-1}(U))$  for all open  $U \subseteq Y$ . Then  $f_*(\phi)$  is a morphism in  $\text{Sh}(Y)$ , and  $f_*$  is a functor  $\text{Sh}(X) \rightarrow \text{Sh}(Y)$ . It is a left exact functor between abelian categories, but in general is not exact. For continuous maps  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  we have  $(g \circ f)_* = g_* \circ f_*$ .

**DEFINITION 4.5.** Let  $f : X \rightarrow Y$  be a continuous map of topological spaces, and  $\mathcal{E}$  a sheaf of abelian groups on  $Y$ . Define a presheaf  $\mathcal{P}f^{-1}(\mathcal{E})$  on  $X$  by  $(\mathcal{P}f^{-1}(\mathcal{E}))(U) = \lim_{A \supseteq f(U)} \mathcal{E}(A)$  for open  $U \subseteq X$ , where the direct limit is taken over all open  $A \subseteq Y$  containing  $f(U)$ , using the restriction maps  $\rho_{AB}$  in  $\mathcal{E}$ . For open  $V \subseteq U \subseteq X$ , define  $\rho'_{UV} : (\mathcal{P}f^{-1}(\mathcal{E}))(U) \rightarrow (\mathcal{P}f^{-1}(\mathcal{E}))(V)$  as the direct limit of the morphisms  $\rho_{AB}$  in  $\mathcal{E}$  for  $B \subseteq A \subseteq Y$  with  $f(U) \subseteq A$  and  $f(V) \subseteq B$ . Then we define the *pullback* (*inverse image*)  $f^{-1}(\mathcal{E})$  to be the sheafification of the presheaf  $\mathcal{P}f^{-1}(\mathcal{E})$ .

Pullbacks  $f^{-1}(\mathcal{E})$  are only unique up to canonical isomorphism, rather than unique. By convention we choose once and for all a pullback  $f^{-1}(\mathcal{E})$  for all  $X, Y, f, \mathcal{E}$ , using the Axiom of Choice if necessary. If  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  is a morphism in  $\text{Sh}(Y)$ , one can define a pullback morphism  $f^{-1}(\phi) : f^{-1}(\mathcal{E}) \rightarrow f^{-1}(\mathcal{F})$ . Then  $f^{-1} : \text{Sh}(Y) \rightarrow \text{Sh}(X)$  is an exact functor between abelian categories.

We compare pushforwards and pullbacks:

**REMARK 4.6. (a)** There are two kinds of pullback, with slightly different notation. The first kind, written  $f^{-1}(\mathcal{E})$  as in Definition 4.5, is used for sheaves of abelian groups or  $C^\infty$ -rings. The second kind, written  $\underline{f}^*(\mathcal{E})$  or  $f^*(\mathcal{E})$  and discussed in §5.3 and §8.3, is used for sheaves of  $\mathcal{O}_Y$ -modules  $\mathcal{E}$ .

(b) The definition of pushforward sheaves  $f_*(\mathcal{E})$  is wholly elementary. In contrast, the definition of pullbacks  $f^{-1}(\mathcal{E})$  is complex, involving a direct limit followed by a sheafification, and includes arbitrary choices.

Pushforwards  $f_*$  are strictly functorial in the continuous map  $f : X \rightarrow Y$ , that is, for continuous  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  we have  $(g \circ f)_* = g_* \circ f_* : \text{Sh}(X) \rightarrow \text{Sh}(Z)$ . However, pullbacks  $f^{-1}$  are only weakly functorial in  $f$ : if  $\mathcal{E} \in \text{Sh}(Z)$  then we need not have  $(g \circ f)^{-1}(\mathcal{E}) = f^{-1}(g^{-1}(\mathcal{E}))$ . This is because pullbacks are only natural up to canonical isomorphism, and we make an arbitrary choice for each pullback. So although  $f^{-1}(g^{-1}(\mathcal{E}))$  is a possible pullback for  $\mathcal{E}$  by  $g \circ f$ , it may not be the one we chose.

Thus, there is a canonical isomorphism  $(g \circ f)^{-1}(\mathcal{E}) \cong f^{-1}(g^{-1}(\mathcal{E}))$ , which we will write as  $I_{f,g}(\mathcal{E}) : (g \circ f)^{-1}(\mathcal{E}) \rightarrow f^{-1}(g^{-1}(\mathcal{E}))$ . The  $I_{f,g}(\mathcal{E})$  for all  $\mathcal{E} \in \text{Sh}(Z)$  comprise a natural isomorphism of functors  $I_{f,g} : (g \circ f)^{-1} \Rightarrow f^{-1} \circ g^{-1}$ . Similarly, for  $\mathcal{E} \in \text{Sh}(X)$  we may not have  $\text{id}_X^{-1}(\mathcal{E}) = \mathcal{E}$ , but instead there are canonical isomorphisms  $\delta_X(\mathcal{E}) : \text{id}_X^{-1}(\mathcal{E}) \rightarrow \mathcal{E}$ , which make up a natural isomorphism  $\delta_X : \text{id}_X^{-1} \Rightarrow \text{id}_{\text{Sh}(X)}$ . Many authors ignore the natural isomorphisms  $I_{f,g}, \delta_X$  entirely.

(c) Let  $f : X \rightarrow Y$  be a continuous map of topological spaces. Then we have functors  $f_* : \text{Sh}(X) \rightarrow \text{Sh}(Y)$ , and  $f^{-1} : \text{Sh}(Y) \rightarrow \text{Sh}(X)$ . As in [31, Ex. II.1.18],  $f_*$  is right adjoint to  $f^{-1}$ . That is, there is a natural bijection

$$\text{Hom}_X(f^{-1}(\mathcal{E}), \mathcal{F}) \cong \text{Hom}_Y(\mathcal{E}, f_*(\mathcal{F})) \quad (4.1)$$

for all  $\mathcal{E} \in \text{Sh}(Y)$  and  $\mathcal{F} \in \text{Sh}(X)$ , with functorial properties.

We define *fine* sheaves, as in Godement [28, §II.3.7] or Voisin [69, Def. 4.35]. They will be important in §4.7 and §5.3.

**DEFINITION 4.7.** Let  $X$  be a topological space (usually paracompact), and  $\mathcal{E}$  a sheaf of abelian groups on  $X$ , or more generally a sheaf of rings, or  $C^\infty$ -rings, or  $\mathcal{O}_X$ -modules, or any other objects which are also abelian groups. We call  $\mathcal{E}$  *fine* if for any open cover  $\{U_i : i \in I\}$  of  $X$ , a subordinate locally finite partition of unity  $\{\zeta_i : i \in I\}$  exists in the sheaf  $\text{Hom}(\mathcal{E}, \mathcal{E})$ .

Here  $\zeta_i : \mathcal{E} \rightarrow \mathcal{E}$  is a morphism of sheaves of abelian groups (or rings,  $C^\infty$ -rings, ...) for each  $i \in I$ . For  $\{\zeta_i : i \in I\}$  to be *subordinate to*  $\{U_i : i \in I\}$  means that  $\zeta_i$  is supported in  $U_i$  for each  $i \in I$ , that is, there exists open  $V_i \subseteq U_i$  with  $\zeta_i|_{V_i} = 0$  and  $U_i \cup V_i = X$ . For  $\{\zeta_i : i \in I\}$  to be *locally finite* means that each  $x \in X$  has an open neighbourhood  $W$  with  $\zeta_i|_W \neq 0$  for only finitely many  $i \in I$ . For  $\{\zeta_i : i \in I\}$  to be a *partition of unity* means that  $\sum_{i \in I} \zeta_i = \text{id}_{\mathcal{E}}$ , where the sum makes sense as  $\{\zeta_i : i \in I\}$  is locally finite.

If  $\mathcal{E} = \mathcal{O}_X$  is a sheaf of commutative rings or  $C^\infty$ -rings, then writing  $\eta_i = \zeta_i(1)$  in  $\mathcal{O}_X(X)$ , we see that  $\zeta_i = \eta_i \cdot$  is multiplication by  $\eta_i$ . So we can regard the partition of unity as living in  $\mathcal{O}_X(X)$  rather than  $\text{Hom}(\mathcal{O}_X, \mathcal{O}_X)$ .

### 4.3. $C^\infty$ -ringed spaces and local $C^\infty$ -ringed spaces

**DEFINITION 4.8.** A  $C^\infty$ -ringed space  $\underline{X} = (X, \mathcal{O}_X)$  is a topological space  $X$  with a sheaf  $\mathcal{O}_X$  of  $C^\infty$ -rings on  $X$ . That is, for each open set  $U \subseteq X$  we are given a  $C^\infty$  ring  $\mathcal{O}_X(U)$ , and for each inclusion of open sets  $V \subseteq U \subseteq X$  we are given a morphism of  $C^\infty$ -rings  $\rho_{UV} : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$ , called the *restriction maps*, and all this data satisfies the sheaf axioms in Definition 4.1.

Equivalently,  $\mathcal{O}_X$  is a presheaf of  $C^\infty$ -rings on  $X$ , that is, a functor

$$\mathcal{O}_X : \mathbf{Open}(X)^{\text{op}} \longrightarrow \mathbf{C}^\infty\mathbf{Rings},$$

whose underlying presheaf of abelian groups, or of sets, is a sheaf. The sheaf axioms Definition 4.1(iv),(v) do not use the  $C^\infty$ -ring structure.

A *morphism*  $\underline{f} = (f, f^\sharp) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  of  $C^\infty$  ringed spaces is a continuous map  $f : X \rightarrow Y$  and a morphism  $f^\sharp : f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$  of sheaves of  $C^\infty$ -rings on  $X$ , for  $f^{-1}(\mathcal{O}_Y)$  as in Definition 4.5. Since  $f_*$  is right adjoint to  $f^{-1}$ , as in (4.1) there is a natural bijection

$$\text{Hom}_X(f^{-1}(\mathcal{O}_Y), \mathcal{O}_X) \cong \text{Hom}_Y(\mathcal{O}_Y, f_*(\mathcal{O}_X)). \quad (4.2)$$

Write  $f_\sharp : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$  for the morphism of sheaves of  $C^\infty$ -rings on  $Y$  corresponding to  $f^\sharp$  under (4.2), so that

$$f^\sharp : f^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X \quad \longleftrightarrow \quad f_\sharp : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X). \quad (4.3)$$

If  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  and  $\underline{g} : \underline{Y} \rightarrow \underline{Z}$  are  $C^\infty$ -scheme morphisms, the composition is

$$\underline{g} \circ \underline{f} = (g \circ f, (g \circ f)^\sharp) = (g \circ f, f^\sharp \circ f^{-1}(g^\sharp) \circ I_{f,g}(\mathcal{O}_Z)),$$

where  $I_{f,g}(\mathcal{O}_Z) : (g \circ f)^{-1}(\mathcal{O}_Z) \rightarrow f^{-1}(g^{-1}(\mathcal{O}_Z))$  is the canonical isomorphism from Remark 4.6(b). In terms of  $f_\sharp : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ , composition is

$$(g \circ f)_\sharp = g_*(f_\sharp) \circ g_\sharp : \mathcal{O}_Z \longrightarrow (g \circ f)_*(\mathcal{O}_X) = g_* \circ f_*(\mathcal{O}_X).$$

A *local  $C^\infty$ -ringed space*  $\underline{X} = (X, \mathcal{O}_X)$  is a  $C^\infty$ -ringed space for which the stalks  $\mathcal{O}_{X,x}$  of  $\mathcal{O}_X$  at  $x$  are local  $C^\infty$ -rings for all  $x \in X$ . As in Definition 2.10, since morphisms of local  $C^\infty$ -rings are automatically local morphisms, morphisms of local  $C^\infty$ -ringed spaces  $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$  are just morphisms of  $C^\infty$ -ringed spaces, without any additional locality condition. Moerdijk, van Qu   and Reyes [52,  3] call our local  $C^\infty$ -ringed spaces *Archimedean  $C^\infty$ -spaces*.

Write  $\mathbf{C}^\infty\mathbf{RS}$  for the category of  $C^\infty$ -ringed spaces, and  $\mathbf{LC}^\infty\mathbf{RS}$  for the full subcategory of local  $C^\infty$ -ringed spaces.

For brevity, we will use the notation that underlined upper case letters  $\underline{X}, \underline{Y}, \underline{Z}, \dots$  represent  $C^\infty$ -ringed spaces  $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y), (Z, \mathcal{O}_Z), \dots$ , and underlined lower case letters  $\underline{f}, \underline{g}, \dots$  represent morphisms of  $C^\infty$ -ringed spaces  $(f, f^\sharp), (g, g^\sharp), \dots$ . When we write ' $x \in \underline{X}$ ' we mean that  $\underline{X} = (X, \mathcal{O}_X)$  and  $x \in X$ . When we write ' $\underline{U}$  is open in  $\underline{X}$ ' we mean that  $\underline{U} = (U, \mathcal{O}_U)$  and  $\underline{X} = (X, \mathcal{O}_X)$  with  $U \subseteq X$  an open set and  $\mathcal{O}_U = \mathcal{O}_X|_U$ .

REMARK 4.9. As above, there are two equivalent ways to write morphisms of  $C^\infty$ -ringed spaces  $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ , either using pullbacks as  $(f, f^\sharp)$  for  $f^\sharp : f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$ , or using pushforwards as  $(f, f_\sharp)$  for  $f_\sharp : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ . Each definition has advantages and disadvantages. We choose to regard  $f^\sharp : f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$  as the primary object, and so define morphisms of  $C^\infty$ -ringed spaces as  $(f, f^\sharp)$  rather than  $(f, f_\sharp)$ , although we will use  $f_\sharp$  in a few places. We can always switch between the two points of view using (4.3).

EXAMPLE 4.10. Let  $X$  be a manifold, which may have boundary or corners. Define a  $C^\infty$ -ringed space  $\underline{X} = (X, \mathcal{O}_X)$  to have topological space  $X$  and  $\mathcal{O}_X(U) = C^\infty(U)$  for each open subset  $U \subseteq X$ , where  $C^\infty(U)$  is the  $C^\infty$ -ring of smooth maps  $c : U \rightarrow \mathbb{R}$ , and if  $V \subseteq U \subseteq X$  are open we define  $\rho_{UV} : C^\infty(U) \rightarrow C^\infty(V)$  by  $\rho_{UV} : c \mapsto c|_V$ .

It is easy to verify that  $\mathcal{O}_X$  is a sheaf of  $C^\infty$ -rings on  $X$  (not just a presheaf), so  $\underline{X} = (X, \mathcal{O}_X)$  is a  $C^\infty$ -ringed space. For each  $x \in X$ , the stalk  $\mathcal{O}_{X,x}$  is the local  $C^\infty$ -ring of germs  $[(c, U)]$  of smooth functions  $c : X \rightarrow \mathbb{R}$  at  $x \in X$ , as in Example 2.15, with unique maximal ideal  $\mathfrak{m}_{X,x} = \{[(c, U)] \in \mathcal{O}_{X,x} : c(x) = 0\}$  and  $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \cong \mathbb{R}$ . Hence  $\underline{X}$  is a local  $C^\infty$ -ringed space.

Let  $X, Y$  be manifolds and  $f : X \rightarrow Y$  a weakly smooth map. Define  $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$  as above. For all open  $U \subseteq Y$  define  $f_\#(U) : \mathcal{O}_Y(U) = C^\infty(U) \rightarrow \mathcal{O}_X(f^{-1}(U)) = C^\infty(f^{-1}(U))$  by  $f_\#(U) : c \mapsto c \circ f$  for all  $c \in C^\infty(U)$ . Then  $f_\#(U)$  is a morphism of  $C^\infty$ -rings, and  $f_\# : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$  is a morphism of sheaves of  $C^\infty$ -rings on  $Y$ . Let  $f^\# : f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$  correspond to  $f_\#$  under (4.3). Then  $\underline{f} = (f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is a morphism of (local)  $C^\infty$ -ringed spaces.

As the category **Top** of topological spaces has all finite limits, and the construction of  **$C^\infty$ RS** involves **Top** in a covariant way and the category  **$C^\infty$ Rings** in a contravariant way, using Proposition 2.5 one may prove:

PROPOSITION 4.11. *All finite limits exist in the category  **$C^\infty$ RS**.*

Dubuc [23, Prop. 7] proves:

PROPOSITION 4.12. *The full subcategory **LC $^\infty$ RS** of local  $C^\infty$ -ringed spaces in  **$C^\infty$ RS** is closed under finite limits in  **$C^\infty$ RS**.*

#### 4.4. The spectrum functor

We now define a *spectrum functor*  $\text{Spec} : \mathbf{C}^\infty\mathbf{Rings}^{\text{op}} \rightarrow \mathbf{LC}^\infty\mathbf{RS}$ . It is equivalent to those constructed by Dubuc [22, 23] and Moerdijk, van Què and Reyes [52, §3], but our presentation is closer to that of Hartshorne [31, p. 70].

DEFINITION 4.13. Let  $\mathfrak{C}$  be a  $C^\infty$ -ring, and use the notation of Definition 2.13. Write  $X_{\mathfrak{C}}$  for the set of all  $\mathbb{R}$ -points  $x$  of  $\mathfrak{C}$ . Let  $\mathcal{T}_{\mathfrak{C}}$  be the topology on  $X_{\mathfrak{C}}$  generated by the basis of open sets  $U_c = \{x \in X_{\mathfrak{C}} : x(c) \neq 0\}$  for all  $c \in \mathfrak{C}$ .

For each  $c \in \mathfrak{C}$  define  $c_* : X_{\mathfrak{C}} \rightarrow \mathbb{R}$  to map  $c_* : x \mapsto x(c)$ .

EXAMPLE 4.14. Suppose  $\mathfrak{C}$  is a finitely generated  $C^\infty$ -ring, with exact sequence  $0 \rightarrow I \hookrightarrow C^\infty(\mathbb{R}^n) \xrightarrow{\phi} \mathfrak{C} \rightarrow 0$ . Define a map  $\phi_* : X_{\mathfrak{C}} \rightarrow \mathbb{R}^n$  by  $\phi_* : x \mapsto (x \circ \phi(x_1), \dots, x \circ \phi(x_n))$ , where  $x_1, \dots, x_n$  are the generators of  $C^\infty(\mathbb{R}^n)$ . Then  $\phi_*$  gives a homeomorphism

$$\phi_* : X_{\mathfrak{C}} \xrightarrow{\cong} X_{\mathfrak{C}}^\phi = \{(x_1, \dots, x_n) \in \mathbb{R}^n : f(x_1, \dots, x_n) = 0 \text{ for all } f \in I\}, \quad (4.4)$$

where the right hand side is a closed subset of  $\mathbb{R}^n$ . So the topological spaces  $(X_{\mathfrak{C}}, \mathcal{T}_{\mathfrak{C}})$  for finitely generated  $\mathfrak{C}$  are homeomorphic to closed subsets of  $\mathbb{R}^n$ .

Recall that a topological space  $X$  is *regular* if whenever  $S \subseteq X$  is closed and  $x \in X \setminus S$  then there exist open  $U, V \subseteq X$  with  $x \in U$ ,  $S \subseteq V$  and  $U \cap V = \emptyset$ .

LEMMA 4.15. *In Definition 4.13, the topology  $\mathcal{T}_{\mathfrak{C}}$  is also generated by the basis of open sets  $c_*^{-1}(V)$  for all  $c \in \mathfrak{C}$  and open  $V \subseteq \mathbb{R}$ . That is,  $\mathcal{T}_{\mathfrak{C}}$  is the weakest topology on  $X_{\mathfrak{C}}$  such that  $c_* : X_{\mathfrak{C}} \rightarrow \mathbb{R}$  is continuous for all  $c \in \mathfrak{C}$ . Also  $(X_{\mathfrak{C}}, \mathcal{T}_{\mathfrak{C}})$  is a Hausdorff, regular topological space.*

PROOF. Suppose  $c \in \mathfrak{C}$  and  $V \subseteq \mathbb{R}$  is open. Then there exists smooth  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $V = \{x \in \mathbb{R} : f(x) \neq 0\}$ . Set  $c' = \Phi_f(c)$ , using the  $C^\infty$ -ring operation  $\Phi_f : \mathfrak{C} \rightarrow \mathfrak{C}$ . Then  $c'_* = f \circ c_*$  as  $c : \mathfrak{C} \rightarrow \mathbb{R}$  is a  $C^\infty$ -ring morphism, so

$$U_{c'} = (c'_*)^{-1}(\mathbb{R} \setminus \{0\}) = (f \circ c_*)^{-1}(\mathbb{R} \setminus \{0\}) = c_*^{-1}[f^{-1}(0)] = c_*^{-1}(V).$$



So  $c_*^{-1}(V)$  is of the form  $U_{c'}$ . Conversely  $U_c = c_*^{-1}(V)$  for  $V = \mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$ . So the two given bases for  $\mathcal{T}_{\mathfrak{C}}$  are the same, proving the first part.

Let  $x, y$  be distinct points of  $X_{\mathfrak{C}}$ . Then there exists  $c \in \mathfrak{C}$  with  $x(c) \neq y(c)$ , as  $x \neq y$ . Set  $\epsilon = \frac{1}{2}|x(c) - y(c)| > 0$  and  $U = c_*^{-1}((x(c) - \epsilon, x(c) + \epsilon))$ ,  $V = c_*^{-1}((y(c) - \epsilon, y(c) + \epsilon))$ . Then  $U, V \subseteq X_{\mathfrak{C}}$  are disjoint open sets with  $x \in U$ ,  $y \in V$ , so  $(X_{\mathfrak{C}}, \mathcal{T}_{\mathfrak{C}})$  is Hausdorff.

Suppose  $S \subseteq X_{\mathfrak{C}}$  is closed, and  $x \in X \setminus S$ . Then there exists  $c \in \mathfrak{C}$  with  $x \in U_c \subseteq X_{\mathfrak{C}} \setminus S$ , since  $X_{\mathfrak{C}} \setminus S$  is open in  $X_{\mathfrak{C}}$  and the  $U_c$  are a basis for  $\mathcal{T}_{\mathfrak{C}}$ . Therefore  $c_*(x) \neq 0$  and  $c_*|_S = 0$ . Set  $\epsilon = \frac{1}{2}|c_*(x)| > 0$ ,  $U = c_*^{-1}((c_*(x) - \epsilon, c_*(x) + \epsilon))$  and  $V = c_*^{-1}((-\epsilon, \epsilon))$ . Then  $U, V \subseteq X_{\mathfrak{C}}$  are disjoint open sets with  $x \in U$ ,  $S \subseteq V$ , so  $(X_{\mathfrak{C}}, \mathcal{T}_{\mathfrak{C}})$  is regular.  $\square$

**DEFINITION 4.16.** Let  $\mathfrak{C}$  be a  $C^\infty$ -ring, and  $X_{\mathfrak{C}}$  the topological space from Definition 4.13. For each open  $U \subseteq X_{\mathfrak{C}}$ , define  $\mathcal{O}_{X_{\mathfrak{C}}}(U)$  to be the set of functions  $s : U \rightarrow \coprod_{x \in U} \mathfrak{C}_x$  with  $s(x) \in \mathfrak{C}_x$  for all  $x \in U$ , and such that  $U$  may be covered by open sets  $W \subseteq U \subseteq X_{\mathfrak{C}}$  for which there exist  $c \in \mathfrak{C}$  with  $s(x) = \pi_x(c) \in \mathfrak{C}_x$  for all  $x \in W$ . Define operations  $\Phi_f$  on  $\mathcal{O}_{X_{\mathfrak{C}}}(U)$  pointwise in  $x \in U$  using the operations  $\Phi_f$  on  $\mathfrak{C}_x$ . This makes  $\mathcal{O}_{X_{\mathfrak{C}}}(U)$  into a  $C^\infty$ -ring. If  $V \subseteq U \subseteq X_{\mathfrak{C}}$  are open, the restriction map  $\rho_{UV} : \mathcal{O}_{X_{\mathfrak{C}}}(U) \rightarrow \mathcal{O}_{X_{\mathfrak{C}}}(V)$  mapping  $\rho_{UV} : s \mapsto s|_V$  is a morphism of  $C^\infty$ -rings.

Clearly  $\mathcal{O}_{X_{\mathfrak{C}}}$  is a sheaf of  $C^\infty$ -rings on  $X_{\mathfrak{C}}$ . Lemma 4.18 shows that the stalk  $\mathcal{O}_{X_{\mathfrak{C}},x}$  at  $x \in X_{\mathfrak{C}}$  is  $\mathfrak{C}_x$ , which is a local  $C^\infty$ -ring. Hence  $(X_{\mathfrak{C}}, \mathcal{O}_{X_{\mathfrak{C}}})$  is a local  $C^\infty$ -ringed space, which we call the *spectrum* of  $\mathfrak{C}$ , and write as  $\text{Spec } \mathfrak{C}$ .

Now let  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$  be a morphism of  $C^\infty$ -rings. Define  $f_\phi : X_{\mathfrak{D}} \rightarrow X_{\mathfrak{C}}$  by  $f_\phi(x) = x \circ \phi$ . Then  $f_\phi$  is continuous. For  $U \subseteq X_{\mathfrak{C}}$  open define  $(f_\phi)_\#(U) : \mathcal{O}_{X_{\mathfrak{C}}}(U) \rightarrow \mathcal{O}_{X_{\mathfrak{D}}}(f_\phi^{-1}(U))$  by  $(f_\phi)_\#(U)s : x \mapsto \phi_x(s(f_\phi(x)))$ , where  $\phi_x : \mathfrak{C}_{f_\phi(x)} \rightarrow \mathfrak{D}_x$  is the induced morphism of local  $C^\infty$ -rings. Then  $(f_\phi)_\# : \mathcal{O}_{X_{\mathfrak{C}}} \rightarrow (f_\phi)_*(\mathcal{O}_{X_{\mathfrak{D}}})$  is a morphism of sheaves of  $C^\infty$ -rings on  $X_{\mathfrak{C}}$ . Let  $f_\phi^\# : f_\phi^{-1}(\mathcal{O}_{X_{\mathfrak{C}}}) \rightarrow \mathcal{O}_{X_{\mathfrak{D}}}$  be the corresponding morphism of sheaves of  $C^\infty$ -rings on  $X_{\mathfrak{D}}$  under (4.3). Then  $\underline{f}_\phi = (f_\phi, f_\phi^\#) : (X_{\mathfrak{D}}, \mathcal{O}_{X_{\mathfrak{D}}}) \rightarrow (X_{\mathfrak{C}}, \mathcal{O}_{X_{\mathfrak{C}}})$  is a morphism of local  $C^\infty$ -ringed spaces. Define  $\text{Spec } \phi : \text{Spec } \mathfrak{D} \rightarrow \text{Spec } \mathfrak{C}$  by  $\text{Spec } \phi = \underline{f}_\phi$ . Then  $\text{Spec}$  is a functor  $\mathbf{C}^\infty\mathbf{Rings}^{\text{op}} \rightarrow \mathbf{LC}^\infty\mathbf{RS}$ , the *spectrum functor*.

**EXAMPLE 4.17.** Let  $X$  be a manifold. Then it follows from Theorem 4.41 below that the local  $C^\infty$ -ringed space  $\underline{X}$  constructed in Example 4.10 is naturally isomorphic to  $\text{Spec } C^\infty(X)$ .

**LEMMA 4.18.** *In Definition 4.16, the stalk  $\mathcal{O}_{X_{\mathfrak{C}},x}$  of  $\mathcal{O}_{X_{\mathfrak{C}}}$  at  $x \in X_{\mathfrak{C}}$  is naturally isomorphic to  $\mathfrak{C}_x$ .*

**PROOF.** Elements of  $\mathcal{O}_{X_{\mathfrak{C}},x}$  are  $\sim$ -equivalence classes  $[U, s]$  of pairs  $(U, s)$ , where  $U$  is an open neighbourhood of  $x$  in  $X_{\mathfrak{C}}$  and  $s \in \mathcal{O}_{X_{\mathfrak{C}}}(U)$ , and  $(U, s) \sim (U', s')$  if there exists open  $x \in V \subseteq U \cap U'$  with  $s|_V = s'|_V$ . Define a  $C^\infty$ -ring morphism  $\Pi : \mathcal{O}_{X_{\mathfrak{C}},x} \rightarrow \mathfrak{C}_x$  by  $\Pi : [U, s] \mapsto s(x)$ .

Suppose  $c_x \in \mathfrak{C}_x$ . Then  $c_x = \pi_x(c)$  for some  $c \in \mathfrak{C}$  by Proposition 2.14. The map  $s : X_{\mathfrak{C}} \rightarrow \coprod_{x' \in X_{\mathfrak{C}}} \mathfrak{C}_{x'}$  mapping  $s : x' \mapsto \pi_{x'}(c)$  lies in  $\mathcal{O}_{X_{\mathfrak{C}}}(X_{\mathfrak{C}})$ , and  $\Pi : [X_{\mathfrak{C}}, s] \mapsto \pi_x(c) = c_x$ . Hence  $\Pi : \mathcal{O}_{X_{\mathfrak{C}},x} \rightarrow \mathfrak{C}_x$  is surjective.

Suppose  $[U, s] \in \mathcal{O}_{X_{\mathfrak{C}},x}$  with  $\Pi([U, s]) = 0 \in \mathfrak{C}_x$ . As  $s \in \mathcal{O}_{X_{\mathfrak{C}}}(U)$ , there exist open  $x \in V \subseteq U$  and  $c \in \mathfrak{C}$  with  $s(x') = \pi_{x'}(c) \in \mathfrak{C}_{x'}$  for all  $x' \in V$ . Then  $\pi_x(c) = s(x) = \Pi([U, s]) = 0$ , so  $c$  lies in the ideal  $I$  in (2.2) by Proposition 2.14. Thus there

exists  $d \in \mathfrak{C}$  with  $x(d) \neq 0$  in  $\mathbb{R}$  and  $cd = 0$  in  $\mathfrak{C}$ . Set  $W = \{x' \in V : x'(d) \neq 0\}$ , so that  $W$  is an open neighbourhood of  $x$  in  $U$ . If  $x' \in W$  then  $x'(d) \neq 0$ , so  $\pi_{x'}(d)$  is invertible in  $\mathfrak{C}_{x'}$ . Thus

$$s(x') = \pi_{x'}(c) = \pi_{x'}(c)\pi_{x'}(d)\pi_{x'}(d)^{-1} = \pi_{x'}(cd)\pi_{x'}(d)^{-1} = \pi_{x'}(0)\pi_{x'}(d)^{-1} = 0.$$

Hence  $[U, s] = [W, s|_W] = [W, 0] = 0$  in  $\mathcal{O}_{X_{\mathfrak{C}}, x}$ , so  $\Pi : \mathcal{O}_{X_{\mathfrak{C}}, x} \rightarrow \mathfrak{C}_x$  is injective. Thus  $\Pi : \mathcal{O}_{X_{\mathfrak{C}}, x} \rightarrow \mathfrak{C}_x$  is an isomorphism.  $\square$

**DEFINITION 4.19.** The *global sections functor*  $\Gamma : \mathbf{LC}^\infty\mathbf{RS} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\text{op}}$  acts on objects  $(X, \mathcal{O}_X)$  by  $\Gamma : (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$  and on morphisms  $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  by  $\Gamma : (f, f^\#) \mapsto f_\#(Y)$ , for  $f_\# : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$  as in (4.3).

Then  $\Gamma \circ \text{Spec}$  is a functor  $\mathbf{C}^\infty\mathbf{Rings}^{\text{op}} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\text{op}}$ , or equivalently a functor  $\mathbf{C}^\infty\mathbf{Rings} \rightarrow \mathbf{C}^\infty\mathbf{Rings}$ . For each  $C^\infty$ -ring  $\mathfrak{C}$  and  $c \in \mathfrak{C}$ , define  $\Psi_{\mathfrak{C}}(c) : X_{\mathfrak{C}} \rightarrow \coprod_{x \in X_{\mathfrak{C}}} \mathfrak{C}_x$  by  $\Psi_{\mathfrak{C}}(c) : x \mapsto \pi_x(c) \in \mathfrak{C}_x$ . Then  $\Psi_{\mathfrak{C}}(c) \in \mathcal{O}_{X_{\mathfrak{C}}}(X_{\mathfrak{C}}) = \Gamma \circ \text{Spec } \mathfrak{C}$  by Definition 4.16, so  $\Psi_{\mathfrak{C}} : \mathfrak{C} \rightarrow \Gamma \circ \text{Spec } \mathfrak{C}$  is a map. Since  $\pi_x : \mathfrak{C} \rightarrow \mathfrak{C}_x$  is a  $C^\infty$ -ring morphism and the  $C^\infty$ -ring operations on  $\mathcal{O}_{X_{\mathfrak{C}}}(X_{\mathfrak{C}})$  are defined pointwise in the  $\mathfrak{C}_x$ , this  $\Psi_{\mathfrak{C}}$  is a  $C^\infty$ -ring morphism. It is functorial in  $\mathfrak{C}$ , so that the  $\Psi_{\mathfrak{C}}$  for all  $\mathfrak{C}$  define a natural transformation  $\Psi : \text{id}_{\mathbf{C}^\infty\mathbf{Rings}} \Rightarrow \Gamma \circ \text{Spec}$  of functors  $\mathbf{C}^\infty\mathbf{Rings} \rightarrow \mathbf{C}^\infty\mathbf{Rings}$ .

**THEOREM 4.20.** *The functor  $\text{Spec} : \mathbf{C}^\infty\mathbf{Rings}^{\text{op}} \rightarrow \mathbf{LC}^\infty\mathbf{RS}$  is **right adjoint** to  $\Gamma : \mathbf{LC}^\infty\mathbf{RS} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\text{op}}$ . That is, for all  $\mathfrak{C} \in \mathbf{C}^\infty\mathbf{Rings}$  and  $\underline{X} \in \mathbf{LC}^\infty\mathbf{RS}$  there are inverse bijections*

$$\text{Hom}_{\mathbf{C}^\infty\mathbf{Rings}}(\mathfrak{C}, \Gamma(\underline{X})) \xrightleftharpoons[R_{\mathfrak{C}, \underline{X}}]{L_{\mathfrak{C}, \underline{X}}} \text{Hom}_{\mathbf{LC}^\infty\mathbf{RS}}(\underline{X}, \text{Spec } \mathfrak{C}), \quad (4.5)$$

which are functorial in the sense that if  $\lambda : \mathfrak{C} \rightarrow \mathfrak{D}$  is a morphism in  $\mathbf{C}^\infty\mathbf{Rings}$  and  $\underline{e} : \underline{X} \rightarrow \underline{Y}$  a morphism in  $\mathbf{LC}^\infty\mathbf{RS}$  then the following commutes:

$$\begin{array}{ccc} \text{Hom}_{\mathbf{C}^\infty\mathbf{Rings}}(\mathfrak{D}, \Gamma(\underline{Y})) & \xrightleftharpoons[R_{\mathfrak{D}, \underline{Y}}]{L_{\mathfrak{D}, \underline{Y}}} & \text{Hom}_{\mathbf{LC}^\infty\mathbf{RS}}(\underline{Y}, \text{Spec } \mathfrak{D}) \\ \downarrow \phi \mapsto \Gamma(\underline{e}) \circ \phi \circ \lambda & & \downarrow \underline{f} \mapsto \text{Spec } \lambda \circ \underline{f} \circ \underline{e} \\ \text{Hom}_{\mathbf{C}^\infty\mathbf{Rings}}(\mathfrak{C}, \Gamma(\underline{X})) & \xrightleftharpoons[R_{\mathfrak{C}, \underline{X}}]{L_{\mathfrak{C}, \underline{X}}} & \text{Hom}_{\mathbf{LC}^\infty\mathbf{RS}}(\underline{X}, \text{Spec } \mathfrak{C}). \end{array} \quad (4.6)$$

When  $\underline{X} = \text{Spec } \mathfrak{C}$  we have  $\Psi_{\mathfrak{C}} = R_{\mathfrak{C}, \underline{X}}(\text{id}_{\underline{X}})$ , so that  $\Psi_{\mathfrak{C}}$  is the unit of the adjunction between  $\Gamma$  and  $\text{Spec}$ .

**PROOF.** Let  $\mathfrak{C} \in \mathbf{C}^\infty\mathbf{Rings}$  and  $\underline{X} \in \mathbf{LC}^\infty\mathbf{RS}$ . Write  $\underline{Y} = (Y, \mathcal{O}_Y) = \text{Spec } \mathfrak{C}$ . Define  $R_{\mathfrak{C}, \underline{X}}$  in (4.5) by, for each morphism  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  in  $\mathbf{LC}^\infty\mathbf{RS}$ , taking  $R_{\mathfrak{C}, \underline{X}}(\underline{f}) : \mathfrak{C} \rightarrow \Gamma(\underline{X})$  to be the composition

$$\mathfrak{C} \xrightarrow{\Psi_{\mathfrak{C}}} \Gamma \circ \text{Spec } \mathfrak{C} = \Gamma(\underline{Y}) \xrightarrow{\Gamma(\underline{f})} \Gamma(\underline{X}). \quad (4.7)$$

For the last part, if  $\underline{X} = \text{Spec } \mathfrak{C}$  then  $\Psi_{\mathfrak{C}} = R_{\mathfrak{C}, \underline{X}}(\text{id}_{\underline{X}})$  as  $\Gamma(\text{id}_{\underline{X}}) = \text{id}_{\Gamma(\underline{X})}$ .

Let  $\phi : \mathfrak{C} \rightarrow \Gamma(\underline{X})$  be a morphism in  $\mathbf{C}^\infty\mathbf{Rings}$ . We will construct a morphism  $\underline{g} = (g, g^\#) : \underline{X} \rightarrow \underline{Y}$  in  $\mathbf{LC}^\infty\mathbf{RS}$ , and set  $L_{\mathfrak{C}, \underline{X}}(\phi) = \underline{g}$ . For any  $x \in X$  we have an  $\mathbb{R}$ -algebra morphism  $x_* : \Gamma(\underline{X}) \rightarrow \mathbb{R}$  by composing the stalk map  $\sigma_x : \Gamma(\underline{X}) \rightarrow \mathcal{O}_{X, x}$  with the unique morphism  $\pi : \mathcal{O}_{X, x} \rightarrow \mathbb{R}$ , as  $\mathcal{O}_{X, x}$  is a local  $C^\infty$ -ring. Then  $x_* \circ \phi : \mathfrak{C} \rightarrow \mathbb{R}$  is an  $\mathbb{R}$ -algebra morphism, and hence a point of  $Y$ . Define  $g : X \rightarrow Y$  by  $g(x) = x_* \circ \phi$ . If  $c \in \mathfrak{C}$  then  $U_c = \{y \in Y : y(c) \neq 0\}$  is open in  $Y$ , and  $g^{-1}(U_c) = \{x \in X : x_*(\phi(c)) \neq 0\}$  is open in  $X$ , as  $x \mapsto x_*(d)$  is a continuous map

$X \rightarrow \mathbb{R}$  for any  $d \in \Gamma(\underline{X})$ . Since the  $U_c$  for  $c \in \mathfrak{C}$  are a basis for the topology of  $Y$  by Definition 4.13,  $g : X \rightarrow Y$  is continuous.

Let  $x \in X$  with  $g(x) = y \in Y$ . Consider the diagram of  $C^\infty$ -rings

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\quad \phi \quad} & \Gamma(\underline{X}) \\ \downarrow \pi_y & & \downarrow \sigma_x \\ \mathfrak{C}_y \cong \mathcal{O}_{Y,y} & \xrightarrow{\quad \phi_x \quad} & \mathcal{O}_{X,x}. \end{array} \quad (4.8)$$

Here  $\mathfrak{C}_y \cong \mathcal{O}_{Y,y}$  by Lemma 4.18. If  $c \in \mathfrak{C}$  with  $y(c) \neq 0$  then  $\sigma_x \circ \phi(c) \in \mathcal{O}_{X,x}$  with  $\pi[\sigma_x \circ \phi(c)] \neq 0$ , so  $\sigma_x \circ \phi(c)$  is invertible in  $\mathcal{O}_{X,x}$  as  $\mathcal{O}_{X,x}$  is a local  $C^\infty$ -ring. Thus by the universal property of  $\pi_y : \mathfrak{C} \rightarrow \mathfrak{C}_y$  there is a unique morphism  $\phi_x : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$  making (4.8) commute.

For each open  $V \subseteq Y$  with  $U = g^{-1}(V) \subseteq X$ , define  $g_\#(V) : \mathcal{O}_Y(V) \rightarrow g_*(\mathcal{O}_X)(V) = \mathcal{O}_X(U)$  by  $g_\#(V)s : x \mapsto \phi_x(s(g(x)))$  for  $s \in \mathcal{O}_Y(V)$  and  $x \in U \subseteq X$ , so that  $g(x) \in V$ , and  $s(g(x)) \in \mathcal{O}_{Y,g(x)}$ , and  $\phi_x(s(g(x))) \in \mathcal{O}_{X,x}$ . Here as  $\mathcal{O}_X$  is a sheaf we may identify elements of  $\mathcal{O}_X(U)$  with maps  $t : U \rightarrow \prod_{x \in U} \mathcal{O}_{X,x}$  with  $t(x) \in \mathcal{O}_{X,x}$  for  $x \in U$ , such that  $t$  satisfies certain local conditions in  $U$ . If  $s \in \mathcal{O}_Y(V)$  and  $x \in U \subseteq X$  with  $g(x) = y \in V \subseteq Y$ , then by Definition 4.16 there is an open neighbourhood  $W_y$  of  $y$  in  $V$  and  $c \in \mathfrak{C}$  with  $s(y') = \pi_{y'}(c) \in \mathfrak{C}_{y'} \cong \mathcal{O}_{Y,y'}$  for all  $y' \in W_y$ . Therefore  $g_\#(V)s$  maps  $x' \mapsto \sigma_{x'}(\phi(c))$  for all  $x'$  in the open neighbourhood  $g^{-1}(W_y)$  of  $x$  in  $U$ , by (4.8). Since the open subsets  $g^{-1}(W_y)$  cover  $U$ ,  $g_\#(V)s$  is a section of  $\mathcal{O}_X|_U$ , and  $g_\#(V)$  is well defined.

As the  $\phi_x$  are  $C^\infty$ -ring morphisms, this defines a morphism  $g_\# : \mathcal{O}_Y \rightarrow g_*(\mathcal{O}_X)$  of sheaves of  $C^\infty$ -rings on  $Y$ . Let  $g^\# : g^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$  be the corresponding morphism of sheaves on  $X$  under (4.3). The stalk  $g_x^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$  of  $g^\#$  at  $x \in X$  with  $g(x) = y \in Y$  is  $g_x^\# = \phi_x$ . Then  $g = (g, g^\#)$  is a morphism in  $\mathbf{LC}^\infty\mathbf{RS}$ . Set  $L_{\mathfrak{C}, \underline{X}}(\phi) = g$ . This defines  $L_{\mathfrak{C}, \underline{X}}$  in (4.5).

For  $\phi, g$  as above,  $c \in \mathfrak{C}$ , and  $x \in X$  with  $g(x) = y = x_* \circ \phi \in Y$ , we have

$$\begin{aligned} \sigma_x[(R_{\mathfrak{C}, \underline{X}} \circ L_{\mathfrak{C}, \underline{X}}(\phi))(c)] &= \sigma_x[\Gamma(g) \circ \Psi_{\mathfrak{C}}(c)] = g_x^\# \circ \sigma_y[\Psi_{\mathfrak{C}}(c)] \\ &= \phi_x \circ \sigma_y[\Psi_{\mathfrak{C}}(c)] = \phi_x \circ \pi_y(c) = \sigma_x \circ \phi(c), \end{aligned}$$

using  $L_{\mathfrak{C}, \underline{X}}(\phi) = g$  and the definition (4.7) of  $R_{\mathfrak{C}, \underline{X}}(g)$  in the first step,  $\sigma_x \circ \Gamma(g) = g_x^\# \circ \sigma_y : \Gamma(\underline{Y}) \rightarrow \mathcal{O}_{X,x}$  in the second,  $g_x^\# = \phi_x$  in the third,  $\sigma_y \circ \Psi_{\mathfrak{C}} = \pi_y$  as maps  $\mathfrak{C} \rightarrow \mathcal{O}_{Y,y} \cong \mathfrak{C}_y$  in the fourth, and (4.8) in the fifth. As  $\prod_{x \in X} \sigma_x : \Gamma(\underline{X}) \rightarrow \prod_{x \in X} \mathcal{O}_{X,x}$  is injective, this implies that  $(R_{\mathfrak{C}, \underline{X}} \circ L_{\mathfrak{C}, \underline{X}}(\phi))(c) = \phi(c)$  for all  $c \in \mathfrak{C}$ , so  $R_{\mathfrak{C}, \underline{X}} \circ L_{\mathfrak{C}, \underline{X}}(\phi) = \phi$ , and  $R_{\mathfrak{C}, \underline{X}} \circ L_{\mathfrak{C}, \underline{X}} = \text{id}$ .

Suppose  $f : \underline{X} \rightarrow \underline{Y}$  is a morphism in  $\mathbf{LC}^\infty\mathbf{RS}$ , and set  $\phi = R_{\mathfrak{C}, \underline{X}}(f)$  and  $g = L_{\mathfrak{C}, \underline{X}}(\phi)$ . Let  $x \in X$  with  $f(x) = y \in Y$ . Then we have a commutative diagram in  $\mathbf{C}^\infty\mathbf{Rings}$

$$\begin{array}{ccccc} & & \phi & & \\ & & \curvearrowright & & \\ \mathfrak{C} & \xrightarrow{\quad \Psi_{\mathfrak{C}} \quad} & \Gamma \circ \text{Spec } \mathfrak{C} = \Gamma(\underline{Y}) & \xrightarrow{\quad \Gamma(f) \quad} & \Gamma(\underline{X}) \\ & \searrow y & \downarrow \sigma_y & \searrow x_* & \downarrow \sigma_x \\ \mathfrak{C}_y & \xrightarrow{\quad \cong \quad} & \mathcal{O}_{Y,y} & \xrightarrow{\quad f_x^\# \quad} & \mathcal{O}_{X,x} \\ & \searrow \pi & \downarrow \pi & \swarrow \pi & \\ & & \mathbb{R} & & \end{array} \quad (4.9)$$

where the isomorphism  $\mathfrak{C}_y \cong \mathcal{O}_{Y,y}$  comes from Lemma 4.18. Since  $g(x) = x_* \circ \phi : \mathfrak{C} \rightarrow \mathbb{R}$ , this proves that  $g(x) = y = f(x)$ , so  $f = g$ . Also by definition the stalk  $g_x^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$  is  $\phi_x$  in (4.8), so comparing (4.8) and (4.9) and using  $\pi_y : \mathfrak{C} \rightarrow \mathfrak{C}_y$  surjective by Proposition 2.14 shows that  $f_x^\# = g_x^\#$ . As this holds for all  $x \in X$  we have  $f^\# = g^\#$ , so  $\underline{f} = (f, f^\#) = (g, g^\#) = \underline{g}$ . Thus  $L_{\mathfrak{C}, \underline{X}} \circ R_{\mathfrak{C}, \underline{X}}(\underline{f}) = \underline{f}$  for all  $\underline{f} : \underline{X} \rightarrow \underline{Y}$ , so  $L_{\mathfrak{C}, \underline{X}} \circ R_{\mathfrak{C}, \underline{X}} = \text{id}$ . Therefore  $L_{\mathfrak{C}, \underline{X}}, R_{\mathfrak{C}, \underline{X}}$  in (4.5) are inverse bijections.

It is easy to see that the rectangle in (4.6) involving  $R_{\mathfrak{D}, \underline{Y}}, R_{\mathfrak{C}, \underline{X}}$  commutes using (4.7) and functoriality of the  $\Psi_{\mathfrak{C}}$  and  $\Gamma$ . Then the rectangle involving  $L_{\mathfrak{D}, \underline{Y}}, L_{\mathfrak{C}, \underline{X}}$  commutes as  $L_{\mathfrak{D}, \underline{Y}} = R_{\mathfrak{D}, \underline{Y}}^{-1}$  and  $L_{\mathfrak{C}, \underline{X}} = R_{\mathfrak{C}, \underline{X}}^{-1}$ . So (4.6) commutes. This completes the proof.  $\square$

REMARK 4.21. (a) The fact in Theorem 4.20 that  $\text{Spec} : \mathbf{C}^\infty\mathbf{Rings}^{\text{op}} \rightarrow \mathbf{LC}^\infty\mathbf{RS}$  is right adjoint to  $\Gamma : \mathbf{LC}^\infty\mathbf{RS} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\text{op}}$  determines  $\text{Spec}$  uniquely up to natural isomorphism, by properties of adjoint functors.

Dubuc [23] and Moerdijk, van Quê and Reyes [52, §3] both prove the existence of a right adjoint to  $\Gamma : \mathbf{LC}^\infty\mathbf{RS} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\text{op}}$ , which is therefore naturally isomorphic to our functor  $\text{Spec}$  in Definition 4.16. But they show  $\text{Spec}$  exists by category theory, without constructing it explicitly as we do.

Moerdijk et al. [52, §3] call our functor  $\text{Spec}$  the *Archimedean spectrum*. They also give a nonequivalent definition [52, §1] of the spectrum  $\text{Spec } \mathfrak{C}$ , in which the points are not  $\mathbb{R}$ -points, but ‘ $C^\infty$ -radical prime ideals’.

(b) Since  $\text{Spec}$  is a right adjoint functor, it preserves limits, as in [23, p. 687]. Equivalently,  $\text{Spec}$  takes colimits in  $\mathbf{C}^\infty\mathbf{Rings}$  to limits in  $\mathbf{LC}^\infty\mathbf{RS}$ . So, for example, a pushout  $\mathfrak{C} = \mathfrak{D} \amalg_{\mathfrak{F}} \mathfrak{E}$  of morphisms  $\phi : \mathfrak{F} \rightarrow \mathfrak{D}$ ,  $\psi : \mathfrak{F} \rightarrow \mathfrak{E}$  in  $\mathbf{C}^\infty\mathbf{Rings}$  is mapped to a fibre product  $\text{Spec } \mathfrak{C} \cong \text{Spec } \mathfrak{D} \times_{\text{Spec } \mathfrak{F}} \text{Spec } \mathfrak{E}$  of morphisms  $\text{Spec } \phi : \text{Spec } \mathfrak{D} \rightarrow \text{Spec } \mathfrak{F}$ ,  $\text{Spec } \psi : \text{Spec } \mathfrak{E} \rightarrow \text{Spec } \mathfrak{F}$  in  $\mathbf{LC}^\infty\mathbf{RS}$ .

Here are some properties of finitely generated and fair  $C^\infty$ -rings, due to Dubuc [23, Th. 13]. The reflection functor  $R_{\text{fg}}^{\text{fa}}$  is as in Definition 2.20.

THEOREM 4.22. (a) *If  $\mathfrak{C}$  is a finitely generated  $C^\infty$ -ring, there is a natural isomorphism  $\Gamma \circ \text{Spec } \mathfrak{C} \cong R_{\text{fg}}^{\text{fa}}(\mathfrak{C})$ , which identifies  $\Psi_{\mathfrak{C}} : \mathfrak{C} \rightarrow \Gamma(\text{Spec } \mathfrak{C})$  with the natural surjective projection  $\mathfrak{C} \rightarrow R_{\text{fg}}^{\text{fa}}(\mathfrak{C})$ .*

*These isomorphisms for all  $\mathfrak{C}$  form a natural isomorphism  $R_{\text{fg}}^{\text{fa}} \cong \Gamma \circ \text{Spec}$  of functors  $R_{\text{fg}}^{\text{fa}}, \Gamma \circ \text{Spec} : \mathbf{C}^\infty\mathbf{Rings}^{\text{fg}} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\text{fa}}$ .*

*Hence, if  $\mathfrak{C}$  is fair then  $\Psi_{\mathfrak{C}} : \mathfrak{C} \rightarrow \Gamma(\text{Spec } \mathfrak{C}) \cong R_{\text{fg}}^{\text{fa}}(\mathfrak{C})$  is an isomorphism.*

(b) *If  $\mathfrak{C}$  is finitely generated then  $\text{Spec } \Psi_{\mathfrak{C}} : \text{Spec } \mathfrak{C} \rightarrow \text{Spec } \Gamma(\text{Spec } \mathfrak{C}) \cong \text{Spec } R_{\text{fg}}^{\text{fa}}(\mathfrak{C})$  is an isomorphism in  $\mathbf{LC}^\infty\mathbf{RS}$ .*

(c) *The functor  $\text{Spec}|_{\dots} : (\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}})^{\text{op}} \rightarrow \mathbf{LC}^\infty\mathbf{RS}$  is full and faithful, and takes finite limits in  $(\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}})^{\text{op}}$  to finite limits in  $\mathbf{LC}^\infty\mathbf{RS}$ .*

To see that  $\text{Spec}$  is full and faithful on  $(\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}})^{\text{op}}$  in (c), let  $\mathfrak{C}, \mathfrak{D}$  be fair  $C^\infty$ -rings. Then putting  $\underline{X} = \text{Spec } \mathfrak{D}$  in (4.5) and using  $\mathfrak{D} \cong \Gamma \circ \text{Spec } \mathfrak{D}$  by (a) shows that the following is a bijection.

$$\text{Spec} : \text{Hom}_{\mathbf{C}^\infty\mathbf{Rings}}(\mathfrak{C}, \mathfrak{D}) \longrightarrow \text{Hom}_{\mathbf{LC}^\infty\mathbf{RS}}(\text{Spec } \mathfrak{D}, \text{Spec } \mathfrak{C}).$$

Note that  $\text{Spec}$  is neither full nor faithful on  $(\mathbf{C}^\infty\mathbf{Rings}^{\text{fg}})^{\text{op}}$  or  $\mathbf{C}^\infty\mathbf{Rings}^{\text{op}}$ . This is a contrast to conventional algebraic geometry, where  $\Gamma(\text{Spec } R) \cong R$  for arbitrary

rings  $R$ , as in [31, Prop. II.2.2], so that  $\text{Spec}$  is full and faithful. In §4.6 we will generalize Theorem 4.22 to non-finitely-generated  $C^\infty$ -rings.

#### 4.5. Affine $C^\infty$ -schemes and $C^\infty$ -schemes

As for the usual definitions of affine schemes and schemes, we define:

DEFINITION 4.23. A local  $C^\infty$ -ringed space  $\underline{X}$  is called an *affine  $C^\infty$ -scheme* if it is isomorphic in  $\mathbf{LC}^\infty\mathbf{RS}$  to  $\text{Spec } \mathfrak{C}$  for some  $C^\infty$ -ring  $\mathfrak{C}$ . We call  $\underline{X}$  a *finitely presented*, or *fair*, affine  $C^\infty$ -scheme if  $X \cong \text{Spec } \mathfrak{C}$  for  $\mathfrak{C}$  that kind of  $C^\infty$ -ring. Write  $\mathbf{AC}^\infty\mathbf{Sch}$ ,  $\mathbf{AC}^\infty\mathbf{Sch}^{\text{fp}}$ ,  $\mathbf{AC}^\infty\mathbf{Sch}^{\text{fa}}$  for the full subcategories of affine  $C^\infty$ -schemes and of finitely presented, and fair, affine  $C^\infty$ -schemes in  $\mathbf{LC}^\infty\mathbf{RS}$  respectively.

We do not define *finitely generated* affine  $C^\infty$ -schemes, because Theorem 4.22(b) implies that they coincide with fair affine  $C^\infty$ -schemes.

Let  $\underline{X} = (X, \mathcal{O}_X)$  be a local  $C^\infty$ -ringed space. We call  $\underline{X}$  a  $C^\infty$ -scheme if  $X$  can be covered by open sets  $U \subseteq X$  such that  $(U, \mathcal{O}_X|_U)$  is an affine  $C^\infty$ -scheme. We call a  $C^\infty$ -scheme  $\underline{X}$  *locally fair*, or *locally finitely presented*, if  $X$  can be covered by open  $U \subseteq X$  with  $(U, \mathcal{O}_X|_U)$  a fair, or finitely presented, affine  $C^\infty$ -scheme, respectively.

We call a  $C^\infty$ -scheme  $\underline{X}$  *Hausdorff*, *second countable*, *Lindelöf*, *compact*, *locally compact*, *paracompact*, *metrizable*, *regular*, or *separable*, if the topological space  $X$  is. Affine  $C^\infty$ -schemes are Hausdorff and regular by Lemma 4.15.

Write  $\mathbf{C}^\infty\mathbf{Sch}^{\text{lf}}$ ,  $\mathbf{C}^\infty\mathbf{Sch}^{\text{lf,fp}}$ ,  $\mathbf{C}^\infty\mathbf{Sch}$  for the full subcategories of locally fair  $C^\infty$ -schemes, locally finitely presented  $C^\infty$ -schemes, and all  $C^\infty$ -schemes, respectively, in  $\mathbf{LC}^\infty\mathbf{RS}$ .

REMARK 4.24. Ordinary schemes are a much larger class than ordinary affine schemes, and central examples such as  $\mathbb{CP}^n$  are not affine schemes. However, affine  $C^\infty$ -schemes are already general enough for many purposes. For example, all second countable, metrizable  $C^\infty$ -schemes are affine, as in §4.8, including manifolds and manifolds with corners. Affine  $C^\infty$ -schemes are Hausdorff and regular, so any non-Hausdorff or non-regular  $C^\infty$ -scheme is not affine.

For the next theorem, part (a) follows from Propositions 2.5, 2.24 and 2.26, Remark 4.21(b), and Theorem 4.22(c). Part (b) holds as finite limits in  $\mathbf{C}^\infty\mathbf{Sch}^{\text{lf,fp}}$ ,  $\mathbf{C}^\infty\mathbf{Sch}^{\text{lf}}$ ,  $\mathbf{C}^\infty\mathbf{Sch}$  are locally modelled on finite limits in  $\mathbf{AC}^\infty\mathbf{Sch}^{\text{fp}}$ ,  $\mathbf{AC}^\infty\mathbf{Sch}^{\text{fa}}$  and  $\mathbf{AC}^\infty\mathbf{Sch}$ .

THEOREM 4.25. (a) *The full subcategories  $\mathbf{AC}^\infty\mathbf{Sch}^{\text{fp}}$ ,  $\mathbf{AC}^\infty\mathbf{Sch}^{\text{fa}}$ ,  $\mathbf{AC}^\infty\mathbf{Sch}$  are closed under all finite limits in  $\mathbf{LC}^\infty\mathbf{RS}$ . Hence, fibre products and all finite limits exist in each of these subcategories.*

(b) *The full subcategories  $\mathbf{C}^\infty\mathbf{Sch}^{\text{lf,fp}}$ ,  $\mathbf{C}^\infty\mathbf{Sch}^{\text{lf}}$  and  $\mathbf{C}^\infty\mathbf{Sch}$  are closed under all finite limits in  $\mathbf{LC}^\infty\mathbf{RS}$ . Hence, fibre products and all finite limits exist in each of these subcategories.*

DEFINITION 4.26. Define functors

$$\begin{aligned} F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}} : \mathbf{Man} &\longrightarrow \mathbf{AC}^\infty\mathbf{Sch}^{\text{fp}} \subset \mathbf{AC}^\infty\mathbf{Sch}, \\ F_{\mathbf{Man}^{\text{b}}}^{\mathbf{C}^\infty\mathbf{Sch}} : \mathbf{Man}^{\text{b}} &\longrightarrow \mathbf{AC}^\infty\mathbf{Sch}^{\text{fa}} \subset \mathbf{AC}^\infty\mathbf{Sch}, \\ F_{\mathbf{Man}^{\text{c}}}^{\mathbf{C}^\infty\mathbf{Sch}} : \mathbf{Man}^{\text{c}} &\longrightarrow \mathbf{AC}^\infty\mathbf{Sch}^{\text{fa}} \subset \mathbf{AC}^\infty\mathbf{Sch}, \end{aligned}$$

by  $F_{\mathbf{Man}^*}^{\mathbf{C}^\infty\mathbf{Sch}} = \text{Spec} \circ F_{\mathbf{Man}^*}^{\mathbf{C}^\infty\mathbf{Rings}}$ , in the notation of Definitions 3.2 and 4.16.

By Example 4.17, if  $X$  is a manifold with corners then  $F_{\mathbf{Man}^c}^{C^\infty \text{Sch}}(X)$  is naturally isomorphic to the local  $C^\infty$ -ringed space  $\underline{X}$  in Example 4.10.

If  $X, Y, \dots$  are manifolds, or  $f, g, \dots$  are (weakly) smooth maps, we may use  $\underline{X}, \underline{Y}, \dots, \underline{f}, \underline{g}, \dots$  to denote the images of  $X, Y, \dots, f, g, \dots$  under  $F_{\mathbf{Man}^c}^{C^\infty \text{Sch}}$ . So for instance we will write  $\underline{\mathbb{R}}^n$  and  $\underline{[0, \infty)}$  for  $F_{\mathbf{Man}}^{C^\infty \text{Sch}}(\mathbb{R}^n)$  and  $F_{\mathbf{Man}^b}^{C^\infty \text{Sch}}([0, \infty))$ .

Our categories of spaces so far are related as follows:

$$\begin{array}{ccccccc}
 \mathbf{Man} & \xrightarrow{\quad \subset \quad} & \mathbf{Man}^b & \xrightarrow{\quad \subset \quad} & \mathbf{Man}^c & & \\
 \downarrow F_{\mathbf{Man}}^{C^\infty \text{Sch}} & & \downarrow F_{\mathbf{Man}^b}^{C^\infty \text{Sch}} & & \downarrow F_{\mathbf{Man}^c}^{C^\infty \text{Sch}} & & \\
 \mathbf{AC}^\infty \mathbf{Sch}^{\text{fp}} & \xrightarrow{\quad \subset \quad} & \mathbf{AC}^\infty \mathbf{Sch}^{\text{fa}} & \xrightarrow{\quad \subset \quad} & \mathbf{AC}^\infty \mathbf{Sch} & & \\
 \downarrow \subset & & \downarrow \subset & & \downarrow \subset & & \\
 \mathbf{C}^\infty \mathbf{Sch}^{\text{lf}} & \xrightarrow{\quad \subset \quad} & \mathbf{C}^\infty \mathbf{Sch}^{\text{lf}} & \xrightarrow{\quad \subset \quad} & \mathbf{C}^\infty \mathbf{Sch} & \xrightarrow{\quad \subset \quad} & \mathbf{LC}^\infty \mathbf{RS} \xrightarrow{\quad \subset \quad} \mathbf{C}^\infty \mathbf{RS}.
 \end{array}$$

By Corollary 3.4 and Theorems 3.5 and 4.22(c), we find as in [23, Th. 16]:

**COROLLARY 4.27.**  $F_{\mathbf{Man}}^{C^\infty \text{Sch}} : \mathbf{Man} \hookrightarrow \mathbf{AC}^\infty \mathbf{Sch}^{\text{fp}} \subset \mathbf{AC}^\infty \mathbf{Sch}$  is a full and faithful functor, and  $F_{\mathbf{Man}^b}^{C^\infty \text{Sch}} : \mathbf{Man}^b \rightarrow \mathbf{AC}^\infty \mathbf{Sch}^{\text{fa}} \subset \mathbf{AC}^\infty \mathbf{Sch}$ ,  $F_{\mathbf{Man}^c}^{C^\infty \text{Sch}} : \mathbf{Man}^c \rightarrow \mathbf{AC}^\infty \mathbf{Sch}^{\text{fa}} \subset \mathbf{AC}^\infty \mathbf{Sch}$  are both faithful functors, but are not full. Also these functors take transverse fibre products in  $\mathbf{Man}, \mathbf{Man}^c$  to fibre products in  $\mathbf{AC}^\infty \mathbf{Sch}^{\text{fp}}, \mathbf{AC}^\infty \mathbf{Sch}^{\text{fa}}$ .

We study open subspaces of  $C^\infty$ -schemes. The definition of  $\text{Spec } \mathfrak{C}$  implies:

**LEMMA 4.28.** Let  $\mathfrak{C}$  be a  $C^\infty$ -ring, and  $c \in \mathfrak{C}$ . Write  $\text{Spec } \mathfrak{C} = (X, \mathcal{O}_X)$  and  $U_c = \{x \in X : x(c) \neq 0\}$ . Then  $U_c \subseteq X$  is open with  $(U_c, \mathcal{O}_X|_{U_c}) \cong \text{Spec } \mathfrak{C}[c^{-1}]$ .

**COROLLARY 4.29.** Let  $\underline{X} = (X, \mathcal{O}_X)$  be a  $C^\infty$ -scheme and  $V \subseteq X$  be open. Then  $\underline{V} = (V, \mathcal{O}_X|_V)$  is also a  $C^\infty$ -scheme.

**PROOF.** Let  $x \in V$ . Then there exists an open  $x \in Y \subseteq X$  with  $Y \cong \text{Spec } \mathfrak{C}$  for some  $C^\infty$ -ring  $\mathfrak{C}$ , as  $\underline{X}$  as a  $C^\infty$ -scheme. Identify  $\underline{Y}$  with  $\text{Spec } \mathfrak{C}$ . As  $V \cap Y$  is open in  $Y = X_\mathfrak{C}$ , and the topology on  $X_\mathfrak{C}$  is generated by subsets  $U_c = \{\tilde{x} \in X_\mathfrak{C} : \tilde{x}(c) \neq 0\}$  for  $c \in \mathfrak{C}$ , there exists  $c \in \mathfrak{C}$  such that  $x \in U_c \subseteq V \cap Y$ . Then  $(U_c, \mathcal{O}_X|_{U_c}) \cong \text{Spec } \mathfrak{C}[c^{-1}]$  by Lemma 4.28. So every  $x \in \underline{V}$  has an affine open neighbourhood, and  $\underline{V}$  is a  $C^\infty$ -scheme.  $\square$

**LEMMA 4.30.** Let  $\mathfrak{C}$  be a finitely generated  $C^\infty$ -ring and  $(X, \mathcal{O}_X) = \text{Spec } \mathfrak{C}$ . Suppose  $V \subseteq X$  is open. Then there exists  $c \in \mathfrak{C}$  with  $V = \{x \in X : x(c) \neq 0\}$ . We call  $c$  a **characteristic function** for  $V$ .

**PROOF.** As  $\mathfrak{C}$  is a finitely generated  $C^\infty$ -ring it fits into an exact sequence  $0 \rightarrow I \hookrightarrow C^\infty(\mathbb{R}^n) \xrightarrow{\phi} \mathfrak{C} \rightarrow 0$ . Example 4.14 gives a homeomorphism  $\phi_* : X \rightarrow X_\mathfrak{C}^\phi$  with a closed subset  $X_\mathfrak{C}^\phi$  in  $\mathbb{R}^n$  given in (4.4). Then  $\phi_*(V)$  is open in  $X_\mathfrak{C}^\phi$ , so there exists an open  $U \subseteq \mathbb{R}^n$  with  $U \cap X_\mathfrak{C}^\phi = \phi_*(V)$ . By [54, Lem. I.1.4] there exists  $f \in C^\infty(\mathbb{R}^n)$  with  $U = \{x \in \mathbb{R}^n : f(x) \neq 0\}$ . Then  $c = \phi(f) \in \mathfrak{C}$  is a characteristic function for  $V$ .  $\square$

**EXAMPLE 4.31.** Let  $I$  be an infinite set, and write  $C^\infty(\mathbb{R}^I)$  for the free  $C^\infty$ -ring with generators  $x_i$  for  $i \in I$ . Then  $\underline{X} = \text{Spec } C^\infty(\mathbb{R}^I)$  has topological space  $X = \mathbb{R}^I$  with points  $(x_i)_{i \in I}$  for  $x_i \in \mathbb{R}$ . Elements of  $C^\infty(\mathbb{R}^I)$  are functions  $c : \mathbb{R}^I \rightarrow \mathbb{R}$

depending only on  $x_j$  for  $j$  in a *finite* subset  $J \subseteq I$ , and which are smooth functions of these  $x_j$ ,  $j \in J$ .

Let  $V = \mathbb{R}^I \setminus \{0\}$ . Then  $V$  is open in  $X$ . But no characteristic function  $c$  exists for  $V$  in  $C^\infty(\mathbb{R}^I)$ , since  $c$  would depend only on  $x_j$  for  $j$  in a finite subset  $J \subseteq I$ , but  $V$  depends on  $x_i$  for all  $i \in I$ . Thus, infinitely generated  $C^\infty$ -rings need not admit characteristic functions, in contrast to Lemma 4.30.

If  $\mathfrak{C}$  is a finitely generated (or finitely presented)  $C^\infty$ -ring and  $c \in \mathfrak{C}$  then  $\mathfrak{C}[c^{-1}]$  is also finitely generated (or finitely presented), since  $\mathfrak{C}[c^{-1}] \cong \mathfrak{C}[x]/(c \cdot x - 1)$  is the result of adding one extra generator and one extra relation to  $\mathfrak{C}$ . Thus from Lemmas 4.28 and 4.30 we deduce:

**COROLLARY 4.32.** (a) *Let  $(X, \mathcal{O}_X)$  be a fair (or finitely presented) affine  $C^\infty$ -scheme, and  $U \subseteq X$  be an open subset. Then  $(U, \mathcal{O}_X|_U)$  is also a fair (or finitely presented) affine  $C^\infty$ -scheme.*

(b) *Let  $(X, \mathcal{O}_X)$  be a locally fair (or locally finitely presented)  $C^\infty$ -scheme, and  $U \subseteq X$  be an open subset. Then  $(U, \mathcal{O}_X|_U)$  is also a locally fair (or locally finitely presented)  $C^\infty$ -scheme.*

Our next result describes the sheaf of  $C^\infty$ -rings  $\mathcal{O}_X$  in  $\text{Spec } \mathfrak{C}$  for  $\mathfrak{C}$  a finitely generated  $C^\infty$ -ring. It is a version of [31, Prop. I.2.2(b)] in algebraic geometry, and reduces to Moerdijk and Reyes [54, Prop. I.1.6] when  $\mathfrak{C} = C^\infty(\mathbb{R}^n)$ .

**PROPOSITION 4.33.** *Let  $\mathfrak{C}$  be a finitely generated  $C^\infty$ -ring, write  $(X, \mathcal{O}_X) = \text{Spec } \mathfrak{C}$ , and let  $U \subseteq X$  be open. By Lemma 4.30 we may choose a characteristic function  $c \in \mathfrak{C}$  for  $U$ . Then there is a canonical isomorphism  $\mathcal{O}_X(U) \cong R_{\text{fg}}^{\text{fa}}(\mathfrak{C}[c^{-1}])$ , in the notation of Definitions 2.13 and 2.20. If  $\mathfrak{C}$  is finitely presented then  $\mathcal{O}_X(U) \cong \mathfrak{C}[c^{-1}]$ .*

**PROOF.** We have morphisms of  $C^\infty$ -rings  $c_* : C^\infty(\mathbb{R}) \rightarrow \mathfrak{C}$  and  $i^* : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R} \setminus \{0\})$ , and  $C^\infty(\mathbb{R}), C^\infty(\mathbb{R} \setminus \{0\})$  are finitely presented  $C^\infty$ -rings by Proposition 3.1(a). So as  $\text{Spec}$  preserves limits in  $(\mathbf{C}^\infty\mathbf{Rings}^{\text{fg}})^{\text{op}}$  we have

$$\text{Spec}(\mathfrak{C} \amalg_{C^\infty(\mathbb{R}), i^*} C^\infty(\mathbb{R} \setminus \{0\})) \cong \text{Spec } \mathfrak{C} \times_{f, \mathbb{R}, i} \mathbb{R} \setminus \{0\} \cong (U, \mathcal{O}_X|_U).$$

But  $\mathfrak{C} \amalg_{C^\infty(\mathbb{R})} C^\infty(\mathbb{R} \setminus \{0\}) \cong \mathfrak{C}[c^{-1}]$  for formal reasons. Thus Theorem 4.22(a) gives  $\mathcal{O}_X(U) \cong \Gamma((U, \mathcal{O}_X|_U)) \cong R_{\text{fg}}^{\text{fa}}(\mathfrak{C}[c^{-1}])$ . If  $\mathfrak{C}$  is finitely presented then  $\mathfrak{C}[c^{-1}]$  is too, as in Corollary 4.32, so  $\mathfrak{C}[c^{-1}]$  is fair and  $R_{\text{fg}}^{\text{fa}}(\mathfrak{C}[c^{-1}]) = \mathfrak{C}[c^{-1}]$ , and therefore  $\mathcal{O}_X(U) \cong \mathfrak{C}[c^{-1}]$ .  $\square$

#### 4.6. Complete $C^\infty$ -rings

The material of this section appears to be new.

**PROPOSITION 4.34.** *Let  $\mathfrak{C}$  be a  $C^\infty$ -ring, and  $\Psi_{\mathfrak{C}}$  be as in Definition 4.19. Then  $\text{Spec } \Psi_{\mathfrak{C}} : \text{Spec } \circ \Gamma \circ \text{Spec } \mathfrak{C} \rightarrow \text{Spec } \mathfrak{C}$  is an isomorphism in  $\mathbf{LC}^\infty\mathbf{RS}$ .*

**PROOF.** Write  $\mathfrak{D} = \Gamma \circ \text{Spec } \mathfrak{C}$ ,  $\underline{X} = \text{Spec } \mathfrak{C}$ ,  $\underline{Y} = \text{Spec } \mathfrak{D}$ , and  $f = \text{Spec } \Psi_{\mathfrak{C}} : \underline{Y} \rightarrow \underline{X}$ . Let  $x \in X$ , and define  $y = \pi \circ \Pi_x : \mathfrak{D} \rightarrow \mathbb{R}$  to be the composition of the projection  $\Pi_x : \mathfrak{D} \rightarrow \mathfrak{C}_x$ , noting that  $\mathfrak{D} \subseteq \prod_{\bar{x} \in X} \mathfrak{C}_{\bar{x}}$  by Definition 4.19, and the unique morphism  $\pi : \mathfrak{C}_x \rightarrow \mathbb{R}$ , as  $\mathfrak{C}_x$  is a local  $C^\infty$ -ring. Then  $f(y) = \pi \circ \pi_x = x : \mathfrak{C} \rightarrow \mathbb{R}$  for  $\pi_x : \mathfrak{C} \rightarrow \mathfrak{C}_x$ , so  $f : Y \rightarrow X$  is surjective.

Suppose now that  $y \in Y$  with  $f(y) = x$ , so that  $y : \mathfrak{D} \rightarrow \mathbb{R}$  is an  $\mathbb{R}$ -algebra morphism. We will prove that  $y = \pi \circ \Pi_x$  as above. Let  $d \in \mathfrak{D}$ . By definition of

$\mathfrak{D} = \mathcal{O}_{X_{\mathfrak{C}}}(X_{\mathfrak{C}})$  there exist an open neighbourhood  $W$  of  $x$  in  $X$  and  $c_1 \in \mathfrak{C}$  such that  $d(\tilde{x}) = \pi_{\tilde{x}}(c_1)$  in  $\mathfrak{C}_{\tilde{x}}$  for all  $\tilde{x} \in W$ . By definition of the topology  $\mathcal{T}_{\mathfrak{C}}$ , there exists  $c_2 \in \mathfrak{C}$  such that  $U_{c_2} = \{\tilde{x} \in X : \tilde{x}(c_2) \neq 0\}$  is an open neighbourhood of  $x$  in  $W \subseteq X$ . Hence  $x(c_2) \neq 0$  and  $\tilde{x}(c_2) = 0$  for all  $\tilde{x} \in X \setminus W$ .

Choose smooth functions  $g, h : \mathbb{R} \rightarrow \mathbb{R}$  with  $g(x(c_2)) = 1$  and  $g = 0$  in an open neighbourhood  $(-\epsilon, \epsilon)$  of 0 in  $\mathbb{R}$ , and  $h(0) \neq 0$  and  $h = 0$  outside  $(-\epsilon, \epsilon)$ , so that  $g \cdot h = 0$ . Set  $c_3 = \Phi_g(c_2)$  and  $c_4 = \Phi_h(c_2)$ , with  $\Phi_g, \Phi_h : \mathfrak{C} \rightarrow \mathfrak{C}$  the  $C^\infty$ -ring operations. Then  $x(c_3) = 1$ , and  $\pi_{\tilde{x}}(c_3) = 0$  in  $\mathfrak{C}_{\tilde{x}}$  for all  $\tilde{x} \in X \setminus W$ , as

$$\pi_{\tilde{x}}(c_3) \cdot \pi_{\tilde{x}}(c_4) = \pi_{\tilde{x}}(\Phi_g(c_2) \cdot \Phi_h(c_2)) = \pi_{\tilde{x}} \circ \Phi_{gh}(c_2) = \pi_{\tilde{x}} \circ \Phi_0(c_2) = 0,$$

but  $\pi_{\tilde{x}}(c_4)$  is invertible in  $\mathfrak{C}_{\tilde{x}}$  as  $\tilde{x}(c_4) = h(\tilde{x}(c_2)) = h(0) \neq 0$ . Thus we have  $d \cdot \Psi_{\mathfrak{C}}(c_3) = \Psi_{\mathfrak{C}}(c_1) \cdot \Psi_{\mathfrak{C}}(c_3) = \Psi_{\mathfrak{C}}(c_1 \cdot c_3)$  in  $\mathfrak{D}$ , as  $d(\tilde{x}) = \Psi_{\mathfrak{C}}(c_1)\tilde{x}$  for all  $\tilde{x} \in W$ , and  $\Psi_{\mathfrak{C}}(c_3)\tilde{x} = 0$  for all  $\tilde{x} \in X \setminus W$ . Therefore

$$\begin{aligned} y(d) &= y(d) \cdot 1 = y(d) \cdot x(c_3) = y(d) \cdot y(\Psi_{\mathfrak{C}}(c_3)) = y(d \cdot \Psi_{\mathfrak{C}}(c_3)) \\ &= y(\Psi_{\mathfrak{C}}(c_1 \cdot c_3)) = x(c_1 \cdot c_3) = x(c_1) \cdot x(c_3) = (\pi \circ \Pi_x(d)) \cdot 1 = \pi \circ \Pi_x(d). \end{aligned}$$

As this holds for all  $d \in \mathfrak{D}$ , we see that  $y \in Y$  with  $f(y) = x$  implies that  $y = \pi \circ \Pi_x$ . Hence  $f : Y \rightarrow X$  is injective, and so bijective.

From above  $f : Y \rightarrow X$  is continuous. To show  $f^{-1} : X \rightarrow Y$  is continuous, note that the topology on  $Y$  is generated by the basis of open sets  $V_d = \{y \in Y : y(d) \neq 0\}$  for all  $d \in \mathfrak{D}$ . So it is enough to show that  $f(V_d) = \{x \in X : \pi \circ \Pi_x(d) \neq 0\}$  is open in  $X$  for all  $d$ . For fixed  $d$ , by definition we may cover  $X$  by open  $W \subseteq X$  for which there exist  $c \in \mathfrak{C}$  with  $d(x) = \pi_x(c) \in \mathfrak{C}_x$  for all  $x \in W$ . But then  $W \cap f(V_d) = W \cap U_c$ , where  $U_c = \{x \in X : x(c) \neq 0\}$  is open in  $X$ . So we can cover  $X$  by open  $W \subseteq X$  with  $W \cap f(V_d)$  open, and  $f(V_d)$  is open. Therefore  $f^{-1}$  is continuous, and  $f : Y \rightarrow X$  is a homeomorphism.

Let  $y \in Y$  with  $f(y) = x$ . Taking stalks of  $f^\sharp : f^{-1}(\mathcal{O}_X) \rightarrow \mathcal{O}_Y$  at  $y$  gives a morphism  $f_y^\sharp : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$ , where  $\mathcal{O}_{X,x} \cong \mathfrak{C}_x$  and  $\mathcal{O}_{Y,y} \cong \mathfrak{D}_y$  by Lemma 4.18, and we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Psi_{\mathfrak{C}}} & \mathfrak{D} \\ \downarrow \pi_x & \searrow \Pi_x & \downarrow \pi_y \\ \mathfrak{C}_x \cong \mathcal{O}_{X,x} & \xrightarrow{\Psi_{\mathfrak{C},x} \cong f_y^\sharp} & \mathcal{O}_{Y,y} \cong \mathfrak{D}_y \end{array} \quad (4.10)$$

Here the outer rectangle and top left triangle obviously commute. To see that the bottom right triangle commutes, we use that any  $d \in \mathfrak{D} = \mathcal{O}_{X_{\mathfrak{C}}}(X_{\mathfrak{C}})$  has  $d(\tilde{x}) = \Psi_{\mathfrak{C}}(c)\tilde{x}$  for some  $c \in \mathfrak{C}$  and all  $\tilde{x}$  in an open neighbourhood  $W$  of  $x$  in  $X$ . As in the first part of the proof, we can find  $c_3 \in \mathfrak{C}$  with  $x(c_3) = 1$  and  $\pi_{\tilde{x}}(c_3) = 0$  in  $\mathfrak{C}_{\tilde{x}}$  for all  $\tilde{x} \in X \setminus W$ . Then evaluating at  $\tilde{x} \in W$  and  $\tilde{x} \in X \setminus W$  we see that  $\Psi_{\mathfrak{C}}(c) \cdot \Psi_{\mathfrak{C}}(c_3) = d \cdot \Psi_{\mathfrak{C}}(c_3)$ , which forces  $\pi_y(d) = \pi_y(\Psi_{\mathfrak{C}}(c))$ , since  $\pi_y \circ \Psi_{\mathfrak{C}}(c_3)$  is invertible in  $\mathfrak{D}_y$  as  $\pi \circ \pi_y \circ \Psi_{\mathfrak{C}}(c_3) = x(c_3) = 1 > 0$ . Thus

$$\pi_y(d) = \pi_y \circ \Psi_{\mathfrak{C}}(c) = f_y^\sharp \circ \pi_x(c) = f_y^\sharp \circ \Pi_x \circ \Psi_{\mathfrak{C}}(c) = f_y^\sharp \circ \Pi_x(d).$$

Since  $\pi_y : \mathfrak{D} \rightarrow \mathfrak{D}_y$  is surjective by Proposition 2.14, the bottom right triangle in (4.10) implies that  $f_y^\sharp : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$  is surjective. Suppose  $c_x \in \mathcal{O}_{X,x}$  with  $f_y^\sharp(c_x) = 0$  in  $\mathcal{O}_{Y,y}$ . As  $\pi_x$  is surjective by Proposition 2.14 we may write  $c_x = \pi_x(c)$  for  $c \in \mathfrak{C}$ . Then  $\pi_y \circ \Psi_{\mathfrak{C}}(c) = f_y^\sharp \circ \pi_x(c) = f_y^\sharp(c_x) = 0$ , so  $\Psi_{\mathfrak{C}}(c) \in \text{Ker } \pi_y$ . Write  $I \subset \mathfrak{C}$  and  $J \subset \mathfrak{D}$  for the ideals in (2.2) for  $x, y$ . Then  $J = \text{Ker } \pi_y$ , so  $\Psi_{\mathfrak{C}}(c) \in J$ ,



and thus there exists  $d \in \mathfrak{D}$  with  $y(d) = \pi \circ \Pi_x(d) \neq 0$  in  $\mathbb{R}$  and  $\Psi_{\mathfrak{C}}(c) \cdot d = 0$  in  $\mathfrak{D}$ . Applying  $\Pi_x$  gives

$$c_x \cdot \Pi_x(d) = \pi_x(c) \cdot \Pi_x(d) = \Pi_x(\Psi_{\mathfrak{C}}(c)) \cdot \Pi_x(d) = \Pi_x(\Psi_{\mathfrak{C}}(c) \cdot d) = \Pi_x(0) = 0.$$

But  $\Pi_x(d)$  is invertible in  $\mathfrak{C}_x$  as  $\pi \circ \Pi_x(d) \neq 0$  in  $\mathbb{R}$ , so  $c_x = 0$ . Thus  $f_y^\sharp : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$  is injective, and so an isomorphism.

We have shown that  $f : Y \rightarrow X$  is a homeomorphism, and  $f_y^\sharp : \mathcal{O}_{X,f(y)} \rightarrow \mathcal{O}_{Y,y}$  is an isomorphism on stalks at all  $y \in Y$ . Hence  $\text{Spec } \Psi_{\mathfrak{C}} = (f, f^\sharp)$  is an isomorphism in  $\mathbf{LC}^\infty\mathbf{RS}$ , as we have to prove.  $\square$

**DEFINITION 4.35.** We call a  $C^\infty$ -ring  $\mathfrak{C}$  *complete* if the morphism  $\Psi_{\mathfrak{C}} : \mathfrak{C} \rightarrow \Gamma \circ \text{Spec } \mathfrak{C}$  in Definition 4.19 is an isomorphism. Write  $\mathbf{C}^\infty\mathbf{Rings}^{\text{co}}$  for the full subcategory of complete  $C^\infty$ -rings  $\mathfrak{C}$  in  $\mathbf{C}^\infty\mathbf{Rings}$ .

If  $\mathfrak{C}$  is any  $C^\infty$ -ring, applying  $\Gamma$  to  $\text{Spec } \Psi_{\mathfrak{C}}$  in Proposition 4.34 shows that

$$\Gamma \circ \text{Spec } \Psi_{\mathfrak{C}} = \Psi_{\Gamma \circ \text{Spec } \mathfrak{C}} : \Gamma \circ \text{Spec } \mathfrak{C} \longrightarrow \Gamma \circ \text{Spec}(\Gamma \circ \text{Spec } \mathfrak{C})$$

is an isomorphism in  $\mathbf{C}^\infty\mathbf{Rings}$ , where we check that  $\Gamma \circ \text{Spec } \Psi_{\mathfrak{C}} = \Psi_{\Gamma \circ \text{Spec } \mathfrak{C}}$  from Definitions 4.16 and 4.19. Hence  $\Gamma \circ \text{Spec } \mathfrak{C}$  is a complete  $C^\infty$ -ring. Define a functor  $R_{\text{all}}^{\text{co}} : \mathbf{C}^\infty\mathbf{Rings} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\text{co}}$  by  $R_{\text{all}}^{\text{co}} = \Gamma \circ \text{Spec}$ .

The next result extends Definition 2.20 and Theorem 4.22 from  $\mathbf{C}^\infty\mathbf{Rings}^{\text{fa}} \subset \mathbf{C}^\infty\mathbf{Rings}^{\text{fg}}$  to  $\mathbf{C}^\infty\mathbf{Rings}^{\text{co}} \subset \mathbf{C}^\infty\mathbf{Rings}$ .

**THEOREM 4.36.** (a) Let  $\underline{X}$  be an affine  $C^\infty$ -scheme. Then  $\underline{X} \cong \text{Spec } \mathcal{O}_X(X)$ , where  $\mathcal{O}_X(X)$  is a complete  $C^\infty$ -ring.

(b)  $\text{Spec}|_{(\mathbf{C}^\infty\mathbf{Rings}^{\text{co}})^{\text{op}}} : (\mathbf{C}^\infty\mathbf{Rings}^{\text{co}})^{\text{op}} \rightarrow \mathbf{LC}^\infty\mathbf{RS}$  is full and faithful, and an equivalence of categories  $\text{Spec}|_{\dots} : (\mathbf{C}^\infty\mathbf{Rings}^{\text{co}})^{\text{op}} \rightarrow \mathbf{AC}^\infty\mathbf{Sch}$ .

(c)  $R_{\text{all}}^{\text{co}} : \mathbf{C}^\infty\mathbf{Rings} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\text{co}}$  is left adjoint to the inclusion functor  $\text{inc} : \mathbf{C}^\infty\mathbf{Rings}^{\text{co}} \hookrightarrow \mathbf{C}^\infty\mathbf{Rings}$ . That is,  $R_{\text{all}}^{\text{co}}$  is a *reflection functor*.

(d) All small colimits exist in  $\mathbf{C}^\infty\mathbf{Rings}^{\text{co}}$ , although they may not coincide with the corresponding small colimits in  $\mathbf{C}^\infty\mathbf{Rings}$ .

(e)  $\text{Spec}|_{(\mathbf{C}^\infty\mathbf{Rings}^{\text{co}})^{\text{op}}} = \text{Spec} \circ \text{inc} : (\mathbf{C}^\infty\mathbf{Rings}^{\text{co}})^{\text{op}} \rightarrow \mathbf{LC}^\infty\mathbf{RS}$  is right adjoint to  $R_{\text{all}}^{\text{co}} \circ \Gamma : \mathbf{LC}^\infty\mathbf{RS} \rightarrow (\mathbf{C}^\infty\mathbf{Rings}^{\text{co}})^{\text{op}}$ . Thus  $\text{Spec}|_{\dots}$  takes limits in  $(\mathbf{C}^\infty\mathbf{Rings}^{\text{co}})^{\text{op}}$  (equivalently, colimits in  $\mathbf{C}^\infty\mathbf{Rings}^{\text{co}}$ ) to limits in  $\mathbf{LC}^\infty\mathbf{RS}$ .

**PROOF.** For (a), if  $\underline{X}$  is an affine  $C^\infty$ -scheme then  $\underline{X} \cong \text{Spec } \mathfrak{C}$  for some  $C^\infty$ -ring  $\mathfrak{C}$ , so  $\mathcal{O}_X(X) \cong \Gamma \circ \text{Spec } \mathfrak{C}$ , and thus  $\underline{X} \cong \text{Spec } \mathcal{O}_X(X)$  by Proposition 4.34. Also, applying  $\Gamma$  to  $\text{Spec } \Psi_{\mathfrak{C}}$  in Proposition 4.34 shows that

$$\Gamma \circ \text{Spec } \Psi_{\mathfrak{C}} = \Psi_{\Gamma \circ \text{Spec } \mathfrak{C}} : \Gamma \circ \text{Spec } \mathfrak{C} \longrightarrow \Gamma \circ \text{Spec}(\Gamma \circ \text{Spec } \mathfrak{C})$$

is an isomorphism in  $\mathbf{C}^\infty\mathbf{Rings}$ , where  $\Gamma \circ \text{Spec } \Psi_{\mathfrak{C}} = \Psi_{\Gamma \circ \text{Spec } \mathfrak{C}}$  follows from the definitions. Hence  $\Gamma \circ \text{Spec } \mathfrak{C} \cong \mathcal{O}_X(X)$  is complete, proving (a).

For (b), if  $\mathfrak{C}, \mathfrak{D}$  are complete  $C^\infty$ -rings then putting  $\underline{X} = \text{Spec } \mathfrak{D}$  in Theorem 4.20 and using  $\Gamma \circ \text{Spec } \mathfrak{D} \cong \mathfrak{D}$ , equation (4.5) shows that

$$\text{Spec} = L_{\mathfrak{C}, \underline{X}} : \text{Hom}_{\mathbf{C}^\infty\mathbf{Rings}}(\mathfrak{C}, \mathfrak{D}) \longrightarrow \text{Hom}_{\mathbf{LC}^\infty\mathbf{RS}}(\text{Spec } \mathfrak{D}, \text{Spec } \mathfrak{C})$$

is a bijection, where the definition of  $L_{\mathfrak{C}, \underline{X}}$  agrees with the definition of  $\text{Spec}$  on morphisms in this case. Thus  $\text{Spec}$  is full and faithful on complete  $C^\infty$ -rings. Therefore  $\text{Spec}|_{\dots} : (\mathbf{C}^\infty\mathbf{Rings}^{\text{co}})^{\text{op}} \rightarrow \mathbf{LC}^\infty\mathbf{RS}$  is an equivalence of categories from  $(\mathbf{C}^\infty\mathbf{Rings}^{\text{co}})^{\text{op}}$  to its essential image in  $\mathbf{LC}^\infty\mathbf{RS}$ , which is  $\mathbf{AC}^\infty\mathbf{Sch}$ .

For (c), let  $\mathfrak{C}, \mathfrak{D}$  be  $C^\infty$ -rings with  $\mathfrak{D}$  complete. Then we have bijections

$$\begin{aligned} \operatorname{Hom}_{\mathbf{C}^\infty\mathbf{Rings}^{\operatorname{co}}} (R_{\operatorname{all}}^{\operatorname{co}}(\mathfrak{C}), \mathfrak{D}) &\cong \operatorname{Hom}_{\mathbf{C}^\infty\mathbf{Rings}} (\Gamma \circ \operatorname{Spec} \mathfrak{C}, \Gamma \circ \operatorname{Spec} \mathfrak{D}) \\ &\cong \operatorname{Hom}_{\mathbf{LC}^\infty\mathbf{RS}} (\operatorname{Spec} \mathfrak{D}, \operatorname{Spec} \circ \Gamma \circ \operatorname{Spec} \mathfrak{C}) \cong \operatorname{Hom}_{\mathbf{LC}^\infty\mathbf{RS}} (\operatorname{Spec} \mathfrak{D}, \operatorname{Spec} \mathfrak{C}) \\ &\cong \operatorname{Hom}_{\mathbf{C}^\infty\mathbf{Rings}} (\mathfrak{C}, \Gamma \circ \operatorname{Spec} \mathfrak{D}) \cong \operatorname{Hom}_{\mathbf{C}^\infty\mathbf{Rings}} (\mathfrak{C}, \mathfrak{D}) \\ &= \operatorname{Hom}_{\mathbf{C}^\infty\mathbf{Rings}} (\mathfrak{C}, \operatorname{inc}(\mathfrak{D})), \end{aligned} \quad (4.11)$$

using  $\mathfrak{D} \cong \Gamma \circ \operatorname{Spec} \mathfrak{D}$  as  $\mathfrak{D}$  is complete in the first and fifth steps, Theorem 4.20 in the second and fourth, and Proposition 4.34 in the third. The bijections (4.11) are functorial in  $\mathfrak{C}, \mathfrak{D}$  as each step is. Hence  $R_{\operatorname{all}}^{\operatorname{co}}$  is left adjoint to  $\operatorname{inc}$ .

For (d), note that  $R_{\operatorname{all}}^{\operatorname{co}} : \mathbf{C}^\infty\mathbf{Rings} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\operatorname{co}}$  takes colimits to colimits, as it is a left adjoint functor by (a). So given a functor  $F : \mathcal{J} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\operatorname{co}}$  for  $\mathcal{J}$  a small category, we may take the colimit  $\mathfrak{C} = \operatorname{colim}_{\mathcal{J}} F$  in  $\mathbf{C}^\infty\mathbf{Rings}$ , which exists by Proposition 2.5, and then  $\mathfrak{D} = R_{\operatorname{all}}^{\operatorname{co}}(\mathfrak{C})$  is the colimit of  $R_{\operatorname{all}}^{\operatorname{co}} \circ F$  in  $\mathbf{C}^\infty\mathbf{Rings}^{\operatorname{co}}$ . But  $R_{\operatorname{all}}^{\operatorname{co}} \circ F \cong F$  as  $R_{\operatorname{all}}^{\operatorname{co}}|_{\mathbf{C}^\infty\mathbf{Rings}^{\operatorname{co}}} \cong \operatorname{id}$ . Hence  $\mathfrak{D} = \operatorname{colim}_{\mathcal{J}} F$  in  $\mathbf{C}^\infty\mathbf{Rings}^{\operatorname{co}}$ , and all small colimits exist in  $\mathbf{C}^\infty\mathbf{Rings}^{\operatorname{co}}$ . In Example 2.25, the colimits in  $\mathbf{C}^\infty\mathbf{Rings}^{\operatorname{co}}$  and  $\mathbf{C}^\infty\mathbf{Rings}$  are different.

The first part of (e) holds by composing (c) and Theorem 4.20, and the second part follows as right adjoint functors preserve limits. This completes the proof of Theorem 4.36.  $\square$

**REMARK 4.37.** Let  $\mathfrak{C}$  be a  $C^\infty$ -ring, so that  $\Psi_{\mathfrak{C}} : \mathfrak{C} \rightarrow R_{\operatorname{all}}^{\operatorname{co}}(\mathfrak{C})$  is a morphism of  $C^\infty$ -rings. If  $\mathfrak{C}$  is finitely generated then Theorem 4.22(a) gives an isomorphism  $R_{\operatorname{all}}^{\operatorname{co}}(\mathfrak{C}) \cong R_{\operatorname{fg}}^{\operatorname{fa}}(\mathfrak{C})$  identifying  $\Psi_{\mathfrak{C}}$  with the surjective projection  $\pi : \mathfrak{C} \rightarrow R_{\operatorname{fg}}^{\operatorname{fa}}(\mathfrak{C})$ , for  $R_{\operatorname{fg}}^{\operatorname{fa}}$  as in Definition 2.20. Thus  $\Psi_{\mathfrak{C}} : \mathfrak{C} \rightarrow R_{\operatorname{all}}^{\operatorname{co}}(\mathfrak{C})$  is surjective in this case, and  $R_{\operatorname{all}}^{\operatorname{co}}, R_{\operatorname{fg}}^{\operatorname{fa}}$  agree on finitely generated  $C^\infty$ -rings up to natural isomorphism.

For  $\mathfrak{C}$  infinitely generated,  $\Psi_{\mathfrak{C}} : \mathfrak{C} \rightarrow R_{\operatorname{all}}^{\operatorname{co}}(\mathfrak{C})$  need not be surjective, and  $R_{\operatorname{all}}^{\operatorname{co}}(\mathfrak{C})$  can be much larger than  $\mathfrak{C}$ . For example, if  $I$  is an infinite set and  $\mathfrak{C} = C^\infty(\mathbb{R}^I)$  is as in Example 4.31, then elements of  $\mathfrak{C}$  are functions  $c : \mathbb{R}^I \rightarrow \mathbb{R}$  which depend smoothly only on  $x_j$  for  $j$  in a finite subset  $J \subseteq I$ , but elements of  $R_{\operatorname{all}}^{\operatorname{co}}(\mathfrak{C})$  are functions  $c : \mathbb{R}^I \rightarrow \mathbb{R}$  which *locally in*  $\mathbb{R}^I$  depend smoothly only on  $x_j$  for  $j$  in a finite subset  $J \subseteq I$ , but globally may depend on  $x_i$  for infinitely many  $i \in I$ . So  $\Psi_{\mathfrak{C}} : \mathfrak{C} \rightarrow R_{\operatorname{all}}^{\operatorname{co}}(\mathfrak{C})$  is injective but not surjective.

#### 4.7. Partitions of unity

We now study the existence of smooth partitions on unity on  $C^\infty$ -schemes and local  $C^\infty$ -ringed spaces. We will need the next definition.

**DEFINITION 4.38.** Let  $\underline{X} = (X, \mathcal{O}_X)$  be a local  $C^\infty$ -ringed space. Then each  $c \in \mathcal{O}_X(X)$  defines a continuous map  $c_* : X \rightarrow \mathbb{R}$  mapping  $x \mapsto \pi \circ \pi_x(c)$ , for  $\pi_x : \mathcal{O}_X(X) \rightarrow \mathcal{O}_{X,x}$  and  $\pi : \mathcal{O}_{X,x} \rightarrow \mathbb{R}$  the natural  $C^\infty$ -ring morphisms. Thus  $U_c = \{x \in X : c_*(x) \neq 0\}$  is open in  $X$ . We say that *the topology on  $X$  is smoothly generated* if  $\{U_c : c \in \mathcal{O}_X(X)\}$  is a basis for the topology on  $X$ .

This implies  $X$  is a regular (and completely regular) topological space.

**EXAMPLE 4.39. (a)** Let  $X$  be a completely regular topological space, and define a sheaf of  $C^\infty$ -rings  $\mathcal{O}_X$  on  $X$  by taking  $\mathcal{O}_X(U) = C^0(U)$  to be the  $C^\infty$ -ring of continuous functions  $c : U \rightarrow \mathbb{R}$  for all open  $U \subseteq X$ . Then  $\underline{X} = (X, \mathcal{O}_X)$  is a local  $C^\infty$ -ringed space, and the topology on  $X$  is smoothly generated.

(b) Let  $\underline{X}$  be an affine  $C^\infty$ -scheme. Then  $\underline{X} \cong \text{Spec } \mathcal{O}_X(X)$  by Theorem 4.36(a). So the definition of the topology on  $X$  in Definition 4.13 implies that the topology on  $X$  is smoothly generated.

(c) Suppose  $\underline{X}$  is a regular  $C^\infty$ -scheme, and let  $T \subseteq X$  be open and  $x \in T$ . Then  $x$  has an affine open neighbourhood  $\underline{Y}$  in  $\underline{X}$ . Since  $X$  is regular, there exist disjoint open neighbourhoods  $V$  of  $x$  and  $W$  of  $X \setminus Y$  in  $X$ .

Then  $x \in T \cap V \subseteq Y$ , and the topology on  $Y$  is smoothly generated by (b), so there exists  $a \in \mathcal{O}_Y(Y)$  with  $x \in U_a^Y \subseteq T \cap V$ . Now  $a_*(x) \neq 0$  and  $a_*(y) = 0$  for all  $y \in Y \setminus U_a^Y$ , but this does not imply that  $a$  is supported in  $U_a^Y$ , as we could have  $\pi_y(a) \neq 0$  in  $\mathcal{O}_{Y,y}$  even though  $\pi \circ \pi_y(a) = 0$  in  $\mathbb{R}$ . Choose smooth  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $f(a_*(x)) \neq 0$  and  $f(t) = 0$  for  $t$  in an open neighbourhood of 0 in  $\mathbb{R}$ . Set  $b = \Phi_f(a)$ , for  $\Phi_f : \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_Y(Y)$  the  $C^\infty$ -ring operation.

Then  $b_*(x) \neq 0$ , and  $U_b^Y \subseteq U_a^Y \subseteq T$ , and  $b$  is supported in  $U_a^Y \subseteq V \subseteq Y$ . Since  $W$  is open in  $X$  with  $X \setminus Y \subseteq W \subseteq Y \setminus V$ , there exists a unique  $c \in \mathcal{O}_X(X)$  with  $c|_Y = b$  and  $c|_W = 0$ . We have  $x \in U_c^X = U_b^Y \subseteq T$ . Thus, for each open  $T \subseteq X$  and  $x \in T$  we can find  $c \in \mathcal{O}_X(X)$  with  $x \in U_c^X \subseteq T$ . So the topology on  $X$  is smoothly generated.

(d) Let  $X$  be an infinite-dimensional Banach space or Banach manifold, and make  $X$  into a local  $C^\infty$ -ringed space  $\underline{X} = (X, \mathcal{O}_X)$  as in Example 4.10. The question of when the topology of  $X$  is smoothly generated (framed in terms of the existence of ‘smooth bump functions’ on  $X$ ) is very well understood, as in Bonic and Frampton [10] and Deville, Godefroy and Zizler [18, §V]. For example, if  $X$  is a Hilbert manifold, or modelled on  $L^q(Y)$  or  $\ell^q$  for even  $q \geq 2$ , then the topology on  $X$  is smoothly generated, but if  $X$  is modelled on  $L^q(Y)$  or  $\ell^q$  for  $q \in [1, \infty]$  not even, the topology on  $X$  is not smoothly generated.

For the next theorem, §4.1 defined Lindelöf spaces, and explained their relation to other topological assumptions. Second countable implies Lindelöf, and Lindelöf and regular imply paracompact (note that  $X$  is regular as its topology is smoothly generated). It is easy to see that  $\mathcal{O}_X$  fine implies that the topology on  $X$  is smoothly generated.

The proof of Theorem 4.40 is based on the proof of the existence of smooth partitions on unity on suitable separable Banach manifolds in Bonic and Frampton [10, Th. 1] (see also Lang [45, §II.3] and Deville et al. [18, §VIII.3]).

Theorem 4.40 applies to a very large class of  $C^\infty$ -schemes, showing that partitions of unity exist on most interesting examples of  $C^\infty$ -schemes.

**THEOREM 4.40.** *Let  $\underline{X} = (X, \mathcal{O}_X)$  be a Lindelöf local  $C^\infty$ -ringed space, and suppose the topology on  $X$  is smoothly generated. Then  $\mathcal{O}_X$  is **fine**, as in Definition 4.7. That is, for every open cover  $\{V_i : i \in I\}$  of  $X$  there exists a subordinate locally finite partition of unity  $\{\eta_i : i \in I\}$  in  $\mathcal{O}_X(X)$ .*

**PROOF.** For  $c \in \mathcal{O}_X(X)$  and  $x \in X$  we have  $\pi_x(c) \in \mathcal{O}_{X,x}$  and  $c_*(x) = \pi \circ \pi_x(c) \in \mathbb{R}$ , where  $\pi_x : \mathcal{O}_X(X) \rightarrow \mathcal{O}_{X,x}$  and  $\pi : \mathcal{O}_{X,x} \rightarrow \mathbb{R}$  are the natural  $C^\infty$ -morphisms. Then  $c_* : X \rightarrow \mathbb{R}$  is continuous. Write  $U_c = \{x \in X : c_*(x) \neq 0\}$ , so that  $U_c$  is open in  $X$ . The *support* of  $c$  is  $\text{supp } c = \{x \in X : \pi_x(c) \neq 0\}$ .

Then  $\text{supp } c$  is closed in  $X$  with  $U_c \subseteq \text{supp } c$ , but  $\text{supp } c$  may be larger than the closure of  $U_c$ . Note that an infinite sum  $\sum_{j \in J} c_j$  in  $\mathcal{O}_X(X)$  is defined, as a section of the sheaf  $\mathcal{O}_X$ , if  $\{\text{supp } c_j : j \in J\}$  is locally finite (that is, each  $x \in X$  has an open neighbourhood  $W_x$  intersecting  $\text{supp } c_j$  for only finitely many  $j \in J$ ),

but may not make sense if only  $\{U_{c_j} : j \in J\}$  is locally finite. Because of this, we are careful to keep track of both  $U_{c_j}$  and  $\text{supp } c_j$  in the following proof.

Let  $\{V_i : i \in I\}$  be an open cover of  $X$ . Suppose  $i \in I$  and  $x \in V_i$ . As the topology on  $X$  is smoothly generated there exists  $c \in \mathcal{O}_X(X)$  with  $x \in U_c \subseteq V_i$ . So  $c_*(x) \neq 0$  and  $c_*|_{X \setminus V_i} = 0$ . We do not know that  $\text{supp } c \subseteq V_i$ , but we can correct this as follows. Choose smooth  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(c_*(x)) \neq 0$  and  $f = 0$  in a neighbourhood of 0 in  $\mathbb{R}$ . Set  $c' = \Phi_f(c)$ , where  $\Phi_f : \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(X)$  is the  $C^\infty$ -ring operation. Then  $x \in U_{c'} \subseteq \text{supp } c' \subseteq U_c \subseteq V_i \subseteq X$ .

Thus, we can choose a family  $\{c_j : j \in J\}$  such that  $c_j \in \mathcal{O}_X(X)$ , and  $U_{c_j} \subseteq \text{supp } c_j \subseteq V_{i_j} \subseteq X$  for each  $j \in J$  and some  $i_j \in I$ , and  $\{U_{c_j} : j \in J\}$  is an open cover of  $X$ . Since  $X$  is Lindelöf we can take  $J$  to be countable, and choose  $J = \mathbb{N}$ .

Replacing  $c_j$  by  $c_j^2$  we have  $(c_j)_* \geq 0$  on  $X$ . For each  $j \in \mathbb{N}$ , choose smooth  $f_j : \mathbb{R}^{j+1} \rightarrow \mathbb{R}$  such that  $f_j(t_0, t_1, \dots, t_j) > 0$  if  $t_i < 1/j$  for  $i = 0, 1, \dots, j-1$  and  $t_j > 0$ , and  $f_j(t_0, t_1, \dots, t_j) = 0$  otherwise. Define  $d_j = \Phi_{f_j}(c_0, c_1, \dots, c_j)$ , with  $\Phi_{f_j} : \mathcal{O}_X(X)^{j+1} \rightarrow \mathcal{O}_X(X)$  the  $C^\infty$ -ring operation. Then

$$\begin{aligned} U_{d_j} &= \{x \in X : (d_j)_*(x) \neq 0\} \\ &= \{x \in X : (c_i)_*(x) < 1/j, i = 0, \dots, j-1, (c_j)_*(x) \neq 0\} \subseteq V_{i_j}, \\ \text{supp } d_j &\subseteq \{x \in X : (c_i)_*(x) \leq 1/j, i = 1, \dots, j-1\} \cap \text{supp } c_j \subseteq V_{i_j}. \end{aligned} \quad (4.12)$$

Fix  $x \in X$ . Then  $x \in U_{c_j}$  for some  $j \in \mathbb{N}$  as  $\{U_{c_j} : j \in J\}$  covers  $X$ . Let  $j \in \mathbb{N}$  be least with  $x \in U_{c_j}$ . Then  $(c_j)_*(x) > 0$  and  $(c_i)_*(x) = 0$  for  $i = 0, 1, \dots, j-1$ . Thus  $x \in U_{d_j}$ , so  $\{U_{d_j} : j \in \mathbb{N}\}$  is an open cover of  $X$ . Define  $T_x = \{y \in X : (c_j)_*(y) > \frac{1}{2}(c_j)_*(x)\}$ . Then  $T_x$  is an open neighbourhood of  $x$  in  $X$ , and  $T_x \cap U_{d_k} = \emptyset = T_x \cap \text{supp } d_k$  provided  $k > \max(j, 2(c_j)_*(x)^{-1})$  by (4.12). Thus, both  $\{U_{d_j} : j \in \mathbb{N}\}$  and  $\{\text{supp } d_j : j \in \mathbb{N}\}$  are locally finite.

For each  $i \in I$ , define  $e_i = \sum_{j \in \mathbb{N} : i_j = i} d_j$  in  $\mathcal{O}_X(X)$ . This is well defined as  $\{\text{supp } d_j : j \in \mathbb{N}\}$  is locally finite. We have  $U_{e_i} \subseteq \text{supp } e_i \subseteq V_i$ , since  $U_{d_j} \subseteq \text{supp } d_j \subseteq V_i$  for each  $j \in \mathbb{N}$  with  $i_j = i$ . Both  $\{U_{e_i} : i \in I\}$  and  $\{\text{supp } e_i : i \in I\}$  are locally finite, as  $\{U_{d_j} : j \in \mathbb{N}\}$  and  $\{\text{supp } d_j : j \in \mathbb{N}\}$  are. Thus  $e = \sum_{i \in I} e_i$  is well defined in  $\mathcal{O}_X(X)$ . If  $x \in X$  then

$$e_*(x) = \sum_{i \in I} (e_i)_*(x) = \sum_{i \in I} \sum_{j \in \mathbb{N} : i_j = i} (d_j)_*(x) = \sum_{j \in \mathbb{N}} (d_j)_*(x) > 0,$$

where each sum has only finitely many nonzero terms, and  $\sum_{j \in \mathbb{N}} (d_j)_*(x) > 0$  as  $\{U_{d_j} : j \in \mathbb{N}\}$  covers  $X$  with  $(d_j)_* > 0$  on  $U_{d_j}$  and  $(d_j)_* = 0$  on  $X \setminus U_{d_j}$ . Since  $e_*$  is positive on  $X$ ,  $e$  is invertible in  $\mathcal{O}_X(X)$ . Set  $\eta_i = e^{-1} \cdot e_i$  for  $i \in I$ . Then  $\text{supp } \eta_i \subseteq V_i$ , as  $\text{supp } e_i \subseteq V_i$ , and  $\{\eta_i : i \in I\}$  is locally finite, as  $\{\text{supp } e_i : i \in I\}$  is, and  $\sum_{i \in I} \eta_i = \sum_{i \in I} e^{-1} \cdot e_i = e^{-1} \cdot e = 1$ . Hence  $\{\eta_i : i \in I\}$  is a locally finite partition of unity subordinate to  $\{V_i : i \in I\}$ , so  $\mathcal{O}_X$  is fine.  $\square$

#### 4.8. A criterion for affine $C^\infty$ -schemes

Here are sufficient conditions for a local  $C^\infty$ -ringed space  $\underline{X}$  to be an affine  $C^\infty$ -scheme. Note that affine  $C^\infty$ -schemes are Hausdorff with smoothly generated topology by Lemma 4.15 and Example 4.39(b), so Lindelöf is the only condition in the theorem which is not also necessary.

**THEOREM 4.41.** *Let  $\underline{X} = (X, \mathcal{O}_X)$  be a Hausdorff, Lindelöf, local  $C^\infty$ -ringed space, with smoothly generated topology. Then  $\underline{X}$  is an affine  $C^\infty$ -scheme.*

PROOF. Let  $\underline{X}$  be as in the theorem. Note that Theorem 4.40 shows that  $\mathcal{O}_X$  is fine. Write  $\mathfrak{C} = \mathcal{O}_X(X) = \Gamma(\underline{X})$ , and  $\underline{Y} = \text{Spec } \mathfrak{C}$ . Define a morphism  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  by  $\underline{f} = L_{\mathfrak{C}, \underline{X}}(\text{id}_{\mathfrak{C}})$ , using the notation of Theorem 4.20. We will show  $\underline{f}$  is an isomorphism, so that  $\underline{X} \cong \text{Spec } \mathfrak{C}$  is an affine  $C^\infty$ -scheme.

Points  $x \in X$  induce  $C^\infty$ -ring morphisms  $\pi \circ \pi_x : \mathfrak{C} = \mathcal{O}_X(X) \rightarrow \mathbb{R}$ , where  $\pi_x : \mathcal{O}_X(X) \rightarrow \mathcal{O}_{X,x}$  and  $\pi : \mathcal{O}_{X,x} \rightarrow \mathbb{R}$  are the natural projections. Points  $y \in Y$  are  $C^\infty$ -ring morphisms  $y : \mathfrak{C} \rightarrow \mathbb{R}$ , and  $f : X \rightarrow Y$  is  $f(x) = \pi \circ \pi_x$ .

Suppose  $x, x' \in X$  with  $x \neq x'$ , and set  $f(x) = y$  and  $f(x') = y'$ . Since  $X$  is Hausdorff there exists open  $U \subseteq X$  with  $x \in U$  and  $x' \notin U$ . As the topology on  $X$  is smoothly generated there exists  $c \in \mathcal{O}_X(X)$  with  $c_*(x) \neq 0$  and  $c_{*|X \setminus U} = 0$ , so that  $c_*(x') = 0$ . Then  $y(c) = c_*(x) \neq 0$  and  $y'(c) = c_*(x') = 0$ , so  $y \neq y'$ . Hence  $f : X \rightarrow Y$  is injective.

Suppose for a contradiction that  $y \in Y$ , but  $f(x) \neq y$  for all  $x \in X$ . Then for each  $x \in X$ , there exists  $a \in \mathfrak{C}$  with  $y(a) \neq \pi \circ \pi_x(a)$ . Choose smooth  $g : \mathbb{R} \rightarrow \mathbb{R}$  with  $g(y(a)) = 0$  and  $g = 1$  in an open neighbourhood of  $\pi \circ \pi_x(a)$  in  $\mathbb{R}$ . Set  $b = \Phi_g(a)$ , where  $\Phi_g : \mathfrak{C} \rightarrow \mathfrak{C}$  is the  $C^\infty$ -ring operation. Then  $y(b) = 0$  and  $\pi \circ \pi_x(b) = 1$  for  $\tilde{x}$  in an open neighbourhood  $V$  of  $x$  in  $X$ .

Thus we may choose a family of pairs  $\{(V_j, b_j) : j \in J\}$  such that for each  $j \in J$  we have  $V_j \subseteq X$  open and  $b_j \in \mathfrak{C}$  with  $y(b_j) = 0$  and  $\pi \circ \pi_x(b_j) = 1$  for  $x \in V_j$ , and  $\{V_j : j \in J\}$  is an open cover of  $X$ . As  $X$  is Lindelöf we can suppose  $J$  is countable, and so take  $J = \mathbb{N}$ . By Theorem 4.40 there exists a locally finite partition of unity  $\{\eta_j : j \in \mathbb{N}\}$  in  $\mathfrak{C}$  subordinate to  $\{V_j : j \in \mathbb{N}\}$ .

Set  $c = \sum_{j \in \mathbb{N}} j \cdot \eta_j \cdot b_j$  in  $\mathfrak{C} = \mathcal{O}_X(X)$ , which makes sense in global sections of  $\mathcal{O}_X$  as  $\{\eta_j : j \in \mathbb{N}\}$  is locally finite. Choose  $n \in \mathbb{N}$  with  $n > y(c)$ , and define  $d = c - y(c) \cdot 1_X + \sum_{j=0}^{n-1} (n-j) \cdot \eta_j \cdot b_j$  in  $\mathfrak{C}$ , where  $1_X \in \mathfrak{C}$  is the identity. Then

$$y(d) = y(c) - y(c) \cdot y(1_X) + \sum_{j=0}^{n-1} (n-j) \cdot y(\eta_j) \cdot y(b_j) = 0,$$

as  $y(1_X) = 1$  and  $y(b_j) = 0$ . And if  $x \in X$  then

$$\begin{aligned} \pi \circ \pi_x(d) &= \pi \circ \pi_x \left[ \sum_{j \in \mathbb{N}} j \cdot \eta_j \cdot b_j - y(c) \cdot \sum_{j \in \mathbb{N}} \eta_j + \sum_{j=0}^{n-1} (n-j) \cdot \eta_j \cdot b_j \right] \\ &= \sum_{j \in \mathbb{N}} (\max(j, n) - y(c)) \pi \circ \pi_x(\eta_j) > 0, \end{aligned}$$

where each sum has only finitely many nonzero terms, and we use  $\sum_{j \in \mathbb{N}} \eta_j = 1_X$ ,  $\pi \circ \pi_x(b_j) = 1$ , and  $\max(j, n) - y(c) > 0$ ,  $\pi \circ \pi_x(\eta_j) \geq 0$  for  $j \in \mathbb{N}$ .

Since  $\pi \circ \pi_x(d) > 0$  for all  $x \in X$ , we see that  $d$  is invertible in  $\mathfrak{C} = \mathcal{O}_X(X)$ , but this contradicts  $y(d) = 0$ . Hence each  $y \in Y$  has  $y = f(x)$  for some  $x \in X$ , and  $f$  is surjective, so  $f : X \rightarrow Y$  is a bijection. By definition of  $\underline{Y} = \text{Spec } \mathfrak{C}$ , the topology on  $Y$  is generated by the open sets  $U_c = \{y \in Y : y(c) \neq 0\}$  for all  $c \in \mathfrak{C}$ . As the topology on  $X$  is smoothly generated, it is generated by the open sets  $f^{-1}(U_c) = \{x \in X : c_*(x) \neq 0\}$  for  $c \in \mathfrak{C}$ . Therefore  $f : X \rightarrow Y$  is a bijection identifying bases for the topologies of  $X, Y$ , so  $f$  is a homeomorphism.

Let  $x \in X$  with  $f(x) = y \in Y$ . Taking stalks of  $f^\sharp : f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$  at  $x$  gives a morphism  $f_x^\sharp : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ . By the definition of  $\underline{f} = L_{\mathfrak{C}, \underline{X}}(\text{id}_{\mathfrak{C}})$  in the proof of Theorem 4.20,  $f_x^\sharp$  agrees with  $\phi_x$  in (4.8), and is the unique morphism making the

following commute, where  $\mathfrak{C}_y \cong \mathcal{O}_{Y,y}$  by Lemma 4.18:

$$\begin{array}{ccc}
 \mathfrak{C} & \xrightarrow{\quad\quad\quad} & \mathcal{O}_X(X) \\
 \downarrow \pi_y & \searrow y & \downarrow \pi_x \\
 \mathfrak{C}_y \cong \mathcal{O}_{Y,y} & \xrightarrow{f_x^\#} & \mathcal{O}_{X,x} \\
 & \searrow \pi & \swarrow \pi \\
 & \mathbb{R} &
 \end{array} \tag{4.13}$$

Suppose  $a_y \in \mathcal{O}_{Y,y}$  with  $f_x^\#(a_y) = 0$ . Then  $a_y = \pi_y(a)$  for some  $a \in \mathfrak{C} = \mathcal{O}_X(X)$ , as  $\pi_y$  is surjective by Proposition 2.14, and then  $\pi_x(a) = 0$  in  $\mathcal{O}_{X,x}$ , as (4.13) commutes. Hence there exists an open neighbourhood  $U$  of  $x$  in  $X$  with  $a|_U = 0$  in  $\mathcal{O}_X(U)$ . As the topology on  $X$  is smoothly generated, there exists  $b \in \mathcal{O}_X(X)$  with  $b_*(x) \neq 0$  and  $b_*|_{X \setminus U} = 0$ . Choose smooth  $g : \mathbb{R} \rightarrow \mathbb{R}$  with  $g(b_*(x)) \neq 0$  and  $g = 0$  near 0 in  $\mathbb{R}$ , and set  $c = \Phi_g(b)$ , where  $\Phi_g : \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(X)$  is the  $C^\infty$ -ring operation. Then  $y(c) = c_*(x) \neq 0$ , and  $c$  is supported in  $U$ . As  $a|_U = 0$  we see that  $a \cdot c = 0$  in  $\mathcal{O}_X(X)$ . Thus  $a$  lies in the ideal  $I$  in (2.2) which is the kernel of  $\pi_y : \mathfrak{C} \rightarrow \mathfrak{C}_y$ , by Proposition 2.14, and so  $a_y = \pi_y(a) = 0$ . Therefore  $f_x^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$  is injective.

Suppose  $a_x \in \mathcal{O}_{X,x}$ . Then by definition of  $\mathcal{O}_{X,x}$  there exists open  $x \in U \subseteq X$  and  $a \in \mathcal{O}_X(U)$  with  $\pi_x(a) = a_x$ . As the topology on  $X$  is smoothly generated there exists  $b \in \mathcal{O}_X(X)$  with  $b_*(x) \neq 0$  and  $b_*|_{X \setminus U} = 0$ . Choose smooth  $g : \mathbb{R} \rightarrow \mathbb{R}$  with  $g = 1$  near  $b_*(x)$  in  $\mathbb{R}$  and  $g = 0$  near 0 in  $\mathbb{R}$ . Set  $c = \Phi_g(b)$ , where  $\Phi_g : \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(X)$  is the  $C^\infty$ -ring operation. Then  $c$  is supported in  $U$ , and there exists an open neighbourhood  $V$  of  $x$  in  $U$  with  $c|_V = 1$ . Since  $c$  is supported in  $U$ , the section  $c|_U \cdot a \in \mathcal{O}_X(U)$  can be extended by zero over  $X \setminus U$  to give a unique  $d \in \mathcal{O}_X(X)$  supported in  $U$  with  $d|_U = c|_U \cdot a$ .

Then  $d|_V = c|_V \cdot a|_V = 1 \cdot a|_V = a|_V$ . Hence  $f_x^\# \circ \pi_y(d) = \pi_x(d) = a_x$ , so  $f_x^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$  is surjective, and an isomorphism. This proves that  $f^\# : f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$  is an isomorphism on stalks at every  $x \in X$ , so  $f^\#$  is an isomorphism. As  $f$  is a homeomorphism,  $\underline{f} = (f, f^\#) : \underline{X} \rightarrow \text{Spec } \mathfrak{C}$  is an isomorphism. This completes the proof of Theorem 4.41.  $\square$

**COROLLARY 4.42.** *Let  $\underline{X} = (X, \mathcal{O}_X)$  be a local  $C^\infty$ -ringed space. Then the following are equivalent:*

- (i)  *$X$  is Hausdorff and second countable, with smoothly generated topology.*
- (ii)  *$X$  is separable and metrizable, with smoothly generated topology.*
- (iii)  *$\underline{X}$  is a Hausdorff, second countable, regular  $C^\infty$ -scheme.*
- (iv)  *$\underline{X}$  is a separable, metrizable  $C^\infty$ -scheme.*
- (v)  *$\underline{X}$  is a second countable, affine  $C^\infty$ -scheme.*

*When these hold,  $X$  is regular, normal, and paracompact, and  $\mathcal{O}_X$  is fine.*

**PROOF.** Section 4.1 implies that (i),(ii) are equivalent (as  $X$  smoothly generated topology implies  $X$  regular), and (iii),(iv) are equivalent. Also (v) implies (iii) by Lemma 4.15, and (iii) implies (i) by Example 4.39(b), and (i) implies (v) by Theorem 4.41 (as second countable implies Lindelöf). Hence (i)–(v) are equivalent. The last part follows from §4.1 and Theorem 4.40.  $\square$

In comparison to Theorem 4.41, we have strengthened the Lindelöf assumption to second countable. The category of  $C^\infty$ -schemes in Corollary 4.42 is very large, and convenient to work in. They are closed under products, fibre products, and

arbitrary subspaces (Lindelöf spaces are none of these). They have partitions of unity, and as they are affine we can argue globally using  $C^\infty$ -rings.

EXAMPLE 4.43. Let  $\underline{X} = (X, \mathcal{O}_X)$  be a second countable, affine  $C^\infty$ -scheme, and let  $Y \subseteq X$  be *any* subset, not necessarily open or closed. Then  $\underline{Y} = (Y, \mathcal{O}_X|_Y)$  is also a second countable, affine  $C^\infty$ -scheme by Corollary 4.42, as being Hausdorff, second countable, and of smoothly generated topology, are all preserved under passing to subspaces, so  $\underline{Y}$  satisfies Corollary 4.42(i) as  $\underline{X}$  does.

EXAMPLE 4.44. Let  $X$  be a separable Banach manifold modelled locally on separable Banach spaces  $B$  which admit ‘smooth bump functions’ (that is, there exists a nonzero smooth function  $f : B \rightarrow \mathbb{R}$  with bounded support in  $B$ ). See Deville et al. [18, §V] for results on when a Banach space  $B$  has a smooth bump function, for example, every Hilbert space does.

Make  $X$  into a local  $C^\infty$ -ringed space  $\underline{X} = (X, \mathcal{O}_X)$  as in Example 4.10. Then the topology on  $X$  is smoothly generated as in Example 4.39(d), so  $\underline{X}$  is an affine  $C^\infty$ -scheme by Corollary 4.42(ii),(v).

#### 4.9. Quotients of $C^\infty$ -schemes by finite groups

Finally we discuss quotients of  $C^\infty$ -schemes by finite groups.

DEFINITION 4.45. Let  $\underline{X} = (X, \mathcal{O}_X)$  be a local  $C^\infty$ -ringed space,  $G$  a finite group, and  $r : G \rightarrow \text{Aut}(\underline{X})$  an action of  $G$  on  $\underline{X}$ . We will define a local  $C^\infty$ -ringed space  $\underline{Y} = \underline{X}/G$ .

Set  $Y = X/r(G)$  to be the quotient topological space. Open sets  $V \subseteq Y$  are of the form  $U/G$  for  $U \subseteq X$  open and  $G$ -invariant. Then  $\gamma \mapsto r^\#(\gamma)(U)$  gives an action of  $G$  on the  $C^\infty$ -ring  $\mathcal{O}_X(U)$ , so as in Proposition 2.22 we have a  $C^\infty$ -ring  $\mathcal{O}_X(U)^G$ , the  $G$ -invariant subspace in  $\mathcal{O}_X(U)$ . Define  $\mathcal{O}_Y(V) = \mathcal{O}_X(U)^G$ .

If  $V_2 \subseteq V_1 \subseteq Y$  are open then  $V_1 = U_1/G$ ,  $V_2 = U_2/G$  for  $U_2 \subseteq U_1 \subseteq X$  open and  $G$ -invariant. The restriction morphism  $\rho_{U_1 U_2} : \mathcal{O}_X(U_1) \rightarrow \mathcal{O}_X(U_2)$  in  $\mathcal{O}_X$  is  $G$ -equivariant, and so restricts to  $\rho_{U_1 U_2}|_{\mathcal{O}_X(U_1)^G} : \mathcal{O}_X(U_1)^G \rightarrow \mathcal{O}_X(U_2)^G$ . Set  $\rho_{V_1 V_2} = \rho_{U_1 U_2}|_{\mathcal{O}_X(U_1)^G} : \mathcal{O}_Y(V_1) \rightarrow \mathcal{O}_Y(V_2)$ . It is now easy to check that  $\mathcal{O}_Y$  is a sheaf of  $C^\infty$ -rings on  $Y$ , so  $\underline{Y} = (Y, \mathcal{O}_Y)$  is a  $C^\infty$ -ringed space.

If  $x \in X$  and  $y = xG \in Y$ , the stalk  $\mathcal{O}_{Y,y}$  of  $\mathcal{O}_Y$  at  $y$  is  $(\mathcal{O}_{X,x})^H$ , where  $\mathcal{O}_{X,x}$  is a local  $C^\infty$ -ring, and  $H = \{\gamma \in G : \gamma(x) = x\}$  is the stabilizer group of  $x$  in  $G$ , which acts on  $\mathcal{O}_{X,x}$  in the obvious way. As  $\mathcal{O}_{X,x}$  is local there is an  $\mathbb{R}$ -algebra morphism  $\pi : \mathcal{O}_{X,x} \rightarrow \mathbb{R}$ , such that  $c \in \mathcal{O}_{X,x}$  is invertible if and only if  $\pi(c) \neq 0$ . Thus  $\pi|_{(\mathcal{O}_{X,x})^H} : (\mathcal{O}_{X,x})^H \rightarrow \mathbb{R}$  is an  $\mathbb{R}$ -algebra morphism, and  $c \in (\mathcal{O}_{X,x})^H$  is invertible in  $\mathcal{O}_{X,x}$  if and only if  $\pi(c) \neq 0$ . But if  $c \in (\mathcal{O}_{X,x})^H$  is invertible in  $\mathcal{O}_{X,x}$  then  $c^{-1}$  is  $H$ -invariant, so  $c$  is invertible in  $(\mathcal{O}_{X,x})^H$ . Therefore  $\mathcal{O}_{Y,y} \cong (\mathcal{O}_{X,x})^H$  is a local  $C^\infty$ -ring, and  $\underline{Y}$  is a local  $C^\infty$ -ringed space. Write  $\underline{X}/G = \underline{Y}$ .

Define  $\pi : X \rightarrow X/G$  to be the natural projection. Define a morphism  $\pi_\# : \mathcal{O}_Y \rightarrow \pi_*(\mathcal{O}_X)$  of sheaves of  $C^\infty$ -rings on  $Y = X/G$  by

$$\pi_\#(V) = \text{inc} : \mathcal{O}_Y(V) = \mathcal{O}_X(U)^G \longrightarrow \mathcal{O}_X(U) = \pi_*(\mathcal{O}_X)(V)$$

for all open  $V = U/G \subseteq Y = X/G$ , where  $\text{inc} : \mathcal{O}_X(U)^G \hookrightarrow \mathcal{O}_X(U)$  is the inclusion. Let  $\pi^\# : \pi^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$  be the morphism of sheaves of  $C^\infty$ -rings on  $X$  corresponding to  $\pi_\#$  under (4.3). Then  $\underline{\pi} = (\pi, \pi^\#) : \underline{X} \rightarrow \underline{X}/G$  is a morphism of local  $C^\infty$ -ringed spaces.

It is easy to see that  $\underline{X}/G, \pi$  have the universal property that if  $\underline{f} : \underline{X} \rightarrow \underline{Z}$  is a morphism in  $\mathbf{LC}^\infty\mathbf{RS}$  with  $\underline{f} \circ \underline{r}(\gamma) = \underline{f}$  for all  $\gamma \in G$  then  $\underline{f} = \underline{g} \circ \pi$  for a unique morphism  $\underline{g} : \underline{X}/G \rightarrow \underline{Z}$  in  $\mathbf{LC}^\infty\mathbf{RS}$ .

**PROPOSITION 4.46.** *Let  $\underline{X} = (X, \mathcal{O}_X)$  be an affine  $C^\infty$ -scheme,  $G$  a finite group, and  $\underline{r} : G \rightarrow \text{Aut}(\underline{X})$  an action of  $G$  on  $\underline{X}$ . Suppose  $X$  is Lindelöf.*

*Then  $\underline{X} = \text{Spec } \mathfrak{C}$  for  $\mathfrak{C} = \mathcal{O}_X(X)$  a complete  $C^\infty$ -ring, and  $\underline{r} = \text{Spec } s$  for  $s : G \rightarrow \text{Aut}(\mathfrak{C})$  a unique action of  $G$  on  $\mathfrak{C}$ . Form the  $G$ -invariant  $C^\infty$ -ring  $\mathfrak{C}^G \subseteq \mathfrak{C}$  as in Proposition 2.22. Then  $\mathfrak{C}^G$  is complete, and there is a canonical isomorphism  $\underline{X}/G \cong \text{Spec } \mathfrak{C}^G$  in  $\mathbf{LC}^\infty\mathbf{RS}$ .*

**PROOF.** Theorem 4.36(a) shows that  $\underline{X} \cong \text{Spec } \mathfrak{C}$ , where  $\mathfrak{C} = \mathcal{O}_X(X)$  is a complete  $C^\infty$ -ring. As  $\text{Spec}$  is full and faithful on complete  $C^\infty$ -rings by Theorem 4.36(b),  $\text{Spec} : \text{Aut}(\mathfrak{C}) \rightarrow \text{Aut}(\underline{X})$  is an isomorphism, so there is a unique action  $s : G \rightarrow \text{Aut}(\mathfrak{C})$  with  $\underline{r} = \text{Spec } s$ .

Let  $\underline{Y} = \underline{X}/G$  be as in Definition 4.45. Then  $Y = X/G$  is Hausdorff, as  $X$  is Hausdorff and  $G$  is finite. Suppose  $\{V_i : i \in I\}$  is an open cover of  $Y$ . Then  $V_i = U_i/G$  for  $\{U_i : i \in I\}$  an open cover of  $X$ . As  $X$  is Lindelöf there exists a subcover  $\{U_i : i \in S\}$  for countable  $S \subseteq I$ , and then  $\{V_i : i \in S\}$  is a countable subcover of  $\{V_i : i \in I\}$ . Hence  $Y$  is Lindelöf.

Suppose  $V \subseteq Y$  is open and  $y \in V$ . Then  $V = U/G$  and  $y = xG$  for  $G$ -invariant open  $U \subseteq X$  with  $x \in U$ . As the topology on  $X$  is smoothly generated, there exists  $c \in \mathfrak{C}$  with  $c_*(x) \neq 0$  and  $c_*(x') = 0$  for all  $x' \in X \setminus U$ . Define  $d = \sum_{\gamma \in G} \gamma^*(c^2)$  in  $\mathfrak{C}$ . Then  $d$  is  $G$ -invariant with  $d_*(x) > 0$  and  $d_*(x') = 0$  for all  $x' \in X \setminus U$ . Hence  $d \in \mathcal{O}_Y(Y) = \mathcal{O}_X(X)^G = \mathfrak{C}^G$ , with  $d_*(y) > 0$  and  $d_*(y') = 0$  for all  $y' \in Y \setminus V$ . Thus the topology of  $Y$  is smoothly generated.

Theorem 4.41 now implies that  $\underline{Y} = \underline{X}/G$  is an affine  $C^\infty$ -scheme, and Theorem 4.36(a) gives a canonical isomorphism  $\underline{X}/G \cong \text{Spec } \mathcal{O}_Y(Y) = \text{Spec } \mathfrak{C}^G$ , where  $\mathfrak{C}^G$  is complete.  $\square$

**PROPOSITION 4.47.** *Suppose  $\underline{X}$  is a Hausdorff, second countable  $C^\infty$ -scheme,  $G$  a finite group, and  $\underline{r} : G \rightarrow \text{Aut}(\underline{X})$  an action of  $G$  on  $\underline{X}$ . Then the quotient  $\underline{X}/G$  is also a Hausdorff, second countable  $C^\infty$ -scheme. If  $\underline{X}$  is locally fair, or locally finitely presented, then so is  $\underline{X}/G$ .*

**PROOF.** Let  $x \in X$ , and write  $H = \{\gamma \in G : \gamma(x) = x\}$ . Then the  $G$ -orbit  $xG$  is  $|G|/|H|$  points. Since  $X$  is Hausdorff and  $G$  is finite, we can find an open neighbourhood  $R$  of  $x$  in  $X$  such that  $R$  is  $H$ -invariant and  $R \cap \gamma \cdot R = \emptyset$  for all  $\gamma \in G \setminus H$ . As  $\underline{X}$  is a  $C^\infty$ -scheme, there is an open neighbourhood  $S$  of  $x$  in  $R$  with  $(S, \mathcal{O}_X|_S)$  an affine  $C^\infty$ -scheme. Then  $T = \bigcap_{\gamma \in H} \gamma \cdot S$  is an  $H$ -invariant open neighbourhood of  $x$  in  $S$ . Choose an open neighbourhood  $U$  of  $x$  in  $T$  with  $(U, \mathcal{O}_X|_U)$  an affine  $C^\infty$ -scheme.

Define  $V = \bigcap_{\gamma \in H} \gamma \cdot U$ . Then  $V$  is an  $H$ -invariant open neighbourhood of  $x$  in  $U \subseteq T \subseteq S \subseteq X$ . It is the intersection of the  $|H|$  affine  $C^\infty$ -subschemes  $(\gamma \cdot U, \mathcal{O}_X|_{\gamma \cdot U})$  for  $\gamma \in H$  inside the affine  $C^\infty$ -scheme  $(S, \mathcal{O}_X|_S)$ . Finite intersections of affine  $C^\infty$ -subschemes in an affine  $C^\infty$ -scheme are affine, as such intersections are fibre products and  $\text{Spec} : \mathbf{C}^\infty\mathbf{Rings}^{\text{op}} \rightarrow \mathbf{LC}^\infty\mathbf{RS}$  preserves limits by Remark 4.21(b). Thus  $(V, \mathcal{O}_X|_V)$  is an affine  $C^\infty$ -scheme.

Set  $W = \bigcup_{\gamma \in G/H} \gamma \cdot V$ . Then  $W$  is a  $G$ -invariant open neighbourhood of  $x$  in  $X$ , and  $(W, \mathcal{O}_X|_W)$  is the disjoint union of  $|G|/|H|$  affine  $C^\infty$ -schemes isomorphic



to  $(V, \mathcal{O}_X|_V)$ , so it is affine. We have shown that every  $x \in X$  has a  $G$ -invariant open neighbourhood  $W \subseteq X$  with  $\underline{W} = (W, \mathcal{O}_X|_W)$  affine. Then  $\underline{W}/G$  is an open neighbourhood of  $xG$  in  $\underline{X}/G$ . As  $\underline{X}$  is second countable,  $\underline{W}$  is second countable and so Lindelöf. Thus  $\underline{W}/G$  is an affine  $C^\infty$ -scheme by Proposition 4.46. As we can cover  $\underline{X}/G$  by such open  $\underline{W}/G$ , it is a  $C^\infty$ -scheme.

If  $\underline{X}$  is locally fair, or locally finitely presented, we can do the argument above with  $\underline{S}, \underline{U}, \underline{V}, \underline{W}, \underline{W}/G$  fair, or finitely presented, using Proposition 2.22 for  $\underline{W}/G$ , so  $\underline{X}/G$  is also locally fair, or locally finitely presented.  $\square$



## CHAPTER 5

### Modules over $C^\infty$ -rings and $C^\infty$ -schemes

Next we discuss modules over  $C^\infty$ -rings, and sheaves of modules on  $C^\infty$ -schemes. The author knows of no previous work on these, so all this section may be new, although much of it is a straightforward generalization of well known facts.

#### 5.1. Modules over $C^\infty$ -rings

**DEFINITION 5.1.** Let  $\mathfrak{C}$  be a  $C^\infty$ -ring. A *module  $M$  over  $\mathfrak{C}$* , or  *$\mathfrak{C}$ -module*, is a module over  $\mathfrak{C}$  regarded as a commutative  $\mathbb{R}$ -algebra as in Definition 2.6, and morphisms of  $\mathfrak{C}$ -modules are morphisms of  $\mathbb{R}$ -algebra modules. We will write  $\mu_M : \mathfrak{C} \times M \rightarrow M$  for the multiplication map, and also write  $\mu_M(c, m) = c \cdot m$  for  $c \in \mathfrak{C}$  and  $m \in M$ . Then  $\mathfrak{C}$ -modules form an abelian category, which we write as  $\mathfrak{C}\text{-mod}$ .

The action of  $\mathfrak{C}$  on itself by multiplication makes  $\mathfrak{C}$  into a  $\mathfrak{C}$ -module, and more generally  $\mathfrak{C} \otimes_{\mathbb{R}} V$  is a  $\mathfrak{C}$ -module for any  $\mathbb{R}$ -vector space  $V$ . A  $\mathfrak{C}$ -module  $M$  is *finitely generated* if it fits into an exact sequence  $\mathfrak{C} \otimes \mathbb{R}^n \rightarrow M \rightarrow 0$  in  $\mathfrak{C}\text{-mod}$ , and *finitely presented* if it fits into an exact sequence  $\mathfrak{C} \otimes \mathbb{R}^m \rightarrow \mathfrak{C} \otimes \mathbb{R}^n \rightarrow M \rightarrow 0$ .

Because  $C^\infty$ -rings such as  $C^\infty(\mathbb{R}^n)$  are not noetherian, finitely generated  $\mathfrak{C}$ -modules generally need not be finitely presented.

Now let  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$  be a morphism of  $C^\infty$ -rings. If  $M$  is a  $\mathfrak{C}$ -module then  $\phi_*(M) = M \otimes_{\mathfrak{C}} \mathfrak{D}$  is a  $\mathfrak{D}$ -module, and this induces a functor  $\phi_* : \mathfrak{C}\text{-mod} \rightarrow \mathfrak{D}\text{-mod}$ . Also, any  $\mathfrak{D}$ -module  $N$  may be regarded as a  $\mathfrak{C}$ -module  $\phi^*(N) = N$  with  $\mathfrak{C}$ -action  $\mu_{\phi^*(N)}(c, n) = \mu_N(\phi(c), n)$ , and this defines a functor  $\phi^* : \mathfrak{D}\text{-mod} \rightarrow \mathfrak{C}\text{-mod}$ . Note that  $\phi_* : \mathfrak{C}\text{-mod} \rightarrow \mathfrak{D}\text{-mod}$  takes finitely generated (or finitely presented)  $\mathfrak{C}$ -modules to finitely generated (or finitely presented)  $\mathfrak{D}$ -modules, but  $\phi^* : \mathfrak{D}\text{-mod} \rightarrow \mathfrak{C}\text{-mod}$  generally does not.

Vector bundles  $E$  over manifolds  $X$  give examples of modules over  $C^\infty(X)$ .

**EXAMPLE 5.2.** Let  $X$  be a manifold and  $E \rightarrow X$  be a vector bundle, and write  $\Gamma^\infty(E)$  for the vector space of smooth sections  $e$  of  $E$ . This is a module over the  $C^\infty$ -ring  $C^\infty(X)$ , multiplying functions on  $X$  by sections of  $E$ .

Let  $E, F \rightarrow X$  be vector bundles over  $X$  and  $\lambda : E \rightarrow F$  a morphism of vector bundles. Then  $\lambda_* : \Gamma^\infty(E) \rightarrow \Gamma^\infty(F)$  defined by  $\lambda_* : e \mapsto \lambda \circ e$  is a morphism of  $C^\infty(X)$ -modules.

Now let  $X, Y$  be manifolds and  $f : X \rightarrow Y$  a (weakly) smooth map. Then  $f^* : C^\infty(Y) \rightarrow C^\infty(X)$  is a morphism of  $C^\infty$ -rings. If  $E \rightarrow Y$  is a vector bundle over  $Y$ , then  $f^*(E)$  is a vector bundle over  $X$ . Under the functor  $(f^*)_* : C^\infty(Y)\text{-mod} \rightarrow C^\infty(X)\text{-mod}$  of Definition 5.1, we see that  $(f^*)_*(\Gamma^\infty(E)) = \Gamma^\infty(E) \otimes_{C^\infty(Y)} C^\infty(X)$  is isomorphic as a  $C^\infty(X)$ -module to  $\Gamma^\infty(f^*(E))$ .

If  $E \rightarrow X$  is any vector bundle over a manifold  $X$  then by choosing sections  $e_1, \dots, e_n \in \Gamma^\infty(E)$  for  $n \gg 0$  such that  $e_1|_x, \dots, e_n|_x$  span  $E|_x$  for all  $x \in X$  we

obtain a surjective morphism of vector bundles  $\psi : X \times \mathbb{R}^n \rightarrow E$ , whose kernel is another vector bundle  $F$ . By choosing another surjective morphism  $\phi : X \times \mathbb{R}^m \rightarrow F$  we obtain an exact sequence of vector bundles

$$X \times \mathbb{R}^m \xrightarrow{\phi} X \times \mathbb{R}^n \xrightarrow{\psi} E \longrightarrow 0,$$

which induces an exact sequence of  $C^\infty(X)$ -modules

$$C^\infty(X) \otimes_{\mathbb{R}} \mathbb{R}^m \xrightarrow{\phi_*} C^\infty(X) \otimes_{\mathbb{R}} \mathbb{R}^n \xrightarrow{\psi_*} \Gamma^\infty(E) \longrightarrow 0.$$

Thus  $\Gamma^\infty(E)$  is a finitely presented  $C^\infty(X)$ -module.

### 5.2. Cotangent modules of $C^\infty$ -rings

Given a  $C^\infty$ -ring  $\mathfrak{C}$ , we will define the *cotangent module*  $\Omega_{\mathfrak{C}}$  of  $\mathfrak{C}$ . Although our definition of  $\mathfrak{C}$ -module only used the commutative  $\mathbb{R}$ -algebra underlying the  $C^\infty$ -ring  $\mathfrak{C}$ , our definition of the particular  $\mathfrak{C}$ -module  $\Omega_{\mathfrak{C}}$  does use the  $C^\infty$ -ring structure in a nontrivial way. It is a  $C^\infty$ -ring version of the *module of relative differential forms* or *Kähler differentials* in Hartshorne [31, p. 172], and is an example of a construction for Fermat theories by Dubuc and Kock [25].

DEFINITION 5.3. Suppose  $\mathfrak{C}$  is a  $C^\infty$ -ring, and  $M$  a  $\mathfrak{C}$ -module. A  $C^\infty$ -derivation is an  $\mathbb{R}$ -linear map  $d : \mathfrak{C} \rightarrow M$  such that whenever  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth map and  $c_1, \dots, c_n \in \mathfrak{C}$ , we have

$$d\Phi_f(c_1, \dots, c_n) = \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \dots, c_n) \cdot dc_i. \quad (5.1)$$

Note that  $d$  is *not* a morphism of  $\mathfrak{C}$ -modules. We call such a pair  $M, d$  a *cotangent module* for  $\mathfrak{C}$  if it has the universal property that for any  $C^\infty$ -derivation  $d' : \mathfrak{C} \rightarrow M'$ , there exists a unique morphism of  $\mathfrak{C}$ -modules  $\lambda : M \rightarrow M'$  with  $d' = \lambda \circ d$ .

There is a natural construction for a cotangent module: we take  $M$  to be the quotient of the free  $\mathfrak{C}$ -module with basis of symbols  $dc$  for  $c \in \mathfrak{C}$  by the  $\mathfrak{C}$ -submodule spanned by all expressions of the form  $d\Phi_f(c_1, \dots, c_n) - \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \dots, c_n) \cdot dc_i$  for  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  smooth and  $c_1, \dots, c_n \in \mathfrak{C}$ . Thus cotangent modules exist, and are unique up to unique isomorphism. When we speak of ‘the’ cotangent module, we mean that constructed above. We write  $d_{\mathfrak{C}} : \mathfrak{C} \rightarrow \Omega_{\mathfrak{C}}$  for the cotangent module of  $\mathfrak{C}$ .

Let  $\mathfrak{C}, \mathfrak{D}$  be  $C^\infty$ -rings with cotangent modules  $\Omega_{\mathfrak{C}}, d_{\mathfrak{C}}, \Omega_{\mathfrak{D}}, d_{\mathfrak{D}}$ , and  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$  be a morphism of  $C^\infty$ -rings. Then we may regard  $\Omega_{\mathfrak{D}} = \phi^*(\Omega_{\mathfrak{D}})$  as a  $\mathfrak{C}$ -module, and  $d_{\mathfrak{D}} \circ \phi : \mathfrak{C} \rightarrow \Omega_{\mathfrak{D}}$  as a  $C^\infty$ -derivation. Thus by the universal property of  $\Omega_{\mathfrak{C}}$ , there exists a unique morphism of  $\mathfrak{C}$ -modules  $\Omega_{\phi} : \Omega_{\mathfrak{C}} \rightarrow \Omega_{\mathfrak{D}}$  with  $d_{\mathfrak{D}} \circ \phi = \Omega_{\phi} \circ d_{\mathfrak{C}}$ . This then induces a morphism of  $\mathfrak{D}$ -modules  $(\Omega_{\phi})_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{D} \rightarrow \Omega_{\mathfrak{D}}$ . If  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$ ,  $\psi : \mathfrak{D} \rightarrow \mathfrak{E}$  are morphisms of  $C^\infty$ -rings then  $\Omega_{\psi \circ \phi} = \Omega_{\psi} \circ \Omega_{\phi} : \Omega_{\mathfrak{C}} \rightarrow \Omega_{\mathfrak{E}}$ .

EXAMPLE 5.4. Let  $X$  be a manifold. Then the cotangent bundle  $T^*X$  is a vector bundle over  $X$ , so as in Example 5.2 it yields a  $C^\infty(X)$ -module  $\Gamma^\infty(T^*X)$ . The exterior derivative  $d : C^\infty(X) \rightarrow \Gamma^\infty(T^*X)$ ,  $d : c \mapsto dc$  is then a  $C^\infty$ -derivation, since equation (5.1) follows from

$$d(f(c_1, \dots, c_n)) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(c_1, \dots, c_n) dc_i$$

for  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  smooth and  $c_1, \dots, c_n \in C^\infty(X)$ , which holds by the chain rule. It is easy to show that  $\Gamma^\infty(T^*X), d$  have the universal property in Definition 5.3, and so form a cotangent module for  $C^\infty(X)$ .

Now let  $X, Y$  be manifolds, and  $f : X \rightarrow Y$  a smooth map. Then  $f^*(T^*Y)$ ,  $T^*X$  are vector bundles over  $X$ , and the derivative of  $f$  gives a vector bundle morphism  $df : f^*(T^*Y) \rightarrow T^*X$ . This induces a morphism of  $C^\infty(X)$ -modules  $(df)_* : \Gamma^\infty(f^*(T^*Y)) \rightarrow \Gamma^\infty(T^*X)$ . This  $(df)_*$  is identified with  $(\Omega_{f^*})_*$  under the natural isomorphism  $\Gamma^\infty(f^*(T^*Y)) \cong \Gamma^\infty(T^*Y) \otimes_{C^\infty(Y)} C^\infty(X)$ , where we identify  $C^\infty(Y), C^\infty(X), f^*$  with  $\mathfrak{C}, \mathfrak{D}, \phi$  in Definition 5.3.

The importance of Definition 5.3 is that it abstracts the notion of cotangent bundle of a manifold in a way that makes sense for any  $C^\infty$ -ring.

REMARK 5.5. There is a second way to define a cotangent-type module for a  $C^\infty$ -ring  $\mathfrak{C}$ , namely the module  $\text{Kd}_{\mathfrak{C}}$  of *Kähler differentials* of the underlying  $\mathbb{R}$ -algebra of  $\mathfrak{C}$ . This is defined as for  $\Omega_{\mathfrak{C}}$ , but requiring (5.1) to hold only when  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a polynomial. Since we impose many fewer relations,  $\text{Kd}_{\mathfrak{C}}$  is generally much larger than  $\Omega_{\mathfrak{C}}$ , so that  $\text{Kd}_{C^\infty(\mathbb{R}^n)}$  is not a finitely generated  $C^\infty(\mathbb{R}^n)$ -module for  $n > 0$ , for instance.

PROPOSITION 5.6. *If  $\mathfrak{C}$  is a finitely generated  $C^\infty$ -ring then  $\Omega_{\mathfrak{C}}$  is a finitely generated  $\mathfrak{C}$ -module. If  $\mathfrak{C}$  is finitely presented, then  $\Omega_{\mathfrak{C}}$  is finitely presented.*

PROOF. If  $\mathfrak{C}$  is finitely generated we have an exact sequence

$$0 \longrightarrow I \longrightarrow C^\infty(\mathbb{R}^n) \xrightarrow{\phi} \mathfrak{C} \longrightarrow 0. \quad (5.2)$$

Write  $x_1, \dots, x_n$  for the generators of  $C^\infty(\mathbb{R}^n)$ . Then any  $c \in \mathfrak{C}$  may be written as  $\phi(f)$  for some  $f \in C^\infty(\mathbb{R}^n)$ , and (5.1) implies that

$$dc = d\Phi_f(\phi(x_1), \dots, \phi(x_n)) = \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(\phi(x_1), \dots, \phi(x_n)) \cdot d \circ \phi(x_i).$$

Hence the generators  $dc$  of  $\Omega_{\mathfrak{C}}$  for  $c \in \mathfrak{C}$  are  $\mathfrak{C}$ -linear combinations of  $d \circ \phi(x_i)$ ,  $i = 1, \dots, n$ , so  $\Omega_{\mathfrak{C}}$  is spanned by the  $d \circ \phi(x_i)$ , and is finitely generated.

Suppose  $\mathfrak{C}$  is finitely presented. Then we have an exact sequence (5.2) with ideal  $I = (f_1, \dots, f_m)$ . We will define an exact sequence of  $\mathfrak{C}$ -modules

$$\mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^m \xrightarrow{\alpha} \mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^n \xrightarrow{\beta} \Omega_{\mathfrak{C}} \longrightarrow 0. \quad (5.3)$$

Write  $(a_1, \dots, a_m), (b_1, \dots, b_n)$  for bases of  $\mathbb{R}^m, \mathbb{R}^n$ . As  $\mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^m, \mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^n$  are free  $\mathfrak{C}$ -modules, the  $\mathfrak{C}$ -module morphisms  $\alpha, \beta$  are specified uniquely by giving  $\alpha(a_i)$  for  $i = 1, \dots, m$  and  $\beta(b_j)$  for  $j = 1, \dots, n$ , which we define to be

$$\alpha : a_i \mapsto \sum_{j=1}^n \Phi_{\frac{\partial f_i}{\partial x_j}}(\phi(x_1), \dots, \phi(x_n)) \cdot b_j \quad \text{and} \quad \beta : b_j \mapsto d_{\mathfrak{C}}(\phi(x_j)).$$

Then for  $i = 1, \dots, m$  we have

$$\begin{aligned} \beta \circ \alpha(a_i) &= \sum_{j=1}^n \Phi_{\frac{\partial f_i}{\partial x_j}}(\phi(x_1), \dots, \phi(x_n)) \cdot d_{\mathfrak{C}}(\phi(x_j)) \\ &= d_{\mathfrak{C}}(\Phi_{f_i}(\phi(x_1), \dots, \phi(x_n))) \\ &= d_{\mathfrak{C}} \circ \phi(\Phi_{f_i}(x_1, \dots, x_n)) = d_{\mathfrak{C}} \circ \phi(f_i(x_1, \dots, x_n)) = d_{\mathfrak{C}}(0) = 0, \end{aligned}$$

using (5.1) in the second step,  $\phi$  a morphism of  $C^\infty$ -rings in the third, the definition of  $C^\infty(\mathbb{R}^n)$  as a  $C^\infty$ -ring in the fourth, and  $f_i(x_1, \dots, x_n) \in I = \text{Ker } \phi$  in the fifth. Hence  $\beta \circ \alpha = 0$ , and (5.3) is a complex.

Thus  $\beta$  induces  $\beta_* : (\mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^n) / \alpha(\mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^m) \rightarrow \Omega_{\mathfrak{C}}$ . We will show  $\beta_*$  is an isomorphism, so that (5.3) is exact. Define  $d : \mathfrak{C} \rightarrow (\mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^n) / \alpha(\mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^m)$  by

$$d(\phi(h)) = \sum_{j=1}^n \Phi_{\frac{\partial h}{\partial x_j}}(\phi(x_1), \dots, \phi(x_n)) \cdot b_j + \alpha(\mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^m). \quad (5.4)$$

Here every  $c \in \mathfrak{C}$  may be written as  $\phi(h)$  for some  $h \in C^\infty(\mathbb{R}^n)$  as  $\phi$  is surjective. To show (5.4) is well-defined we must show the right hand side is independent of the choice of  $h$  with  $\phi(h) = c$ , that is, we must show that the right hand side is zero if  $h \in I$ . It is enough to check this when  $h$  is a generator  $f_1, \dots, f_m$  of  $I$ , and this holds by definition of  $\alpha$ . Hence  $d$  in (5.4) is well-defined.

It is easy to see that  $d$  is a  $C^\infty$ -derivation, and that  $\beta_* \circ d = d_{\mathfrak{C}}$ . So by the universal property of  $\Omega_{\mathfrak{C}}$ , there is a unique  $\mathfrak{C}$ -module morphism  $\psi : \Omega_{\mathfrak{C}} \rightarrow (\mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^n) / \alpha(\mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^n)$  with  $d = \psi \circ d_{\mathfrak{C}}$ . Thus  $\beta_* \circ \psi \circ d_{\mathfrak{C}} = \beta_* \circ d = d_{\mathfrak{C}} = \text{id}_{\Omega_{\mathfrak{C}}} \circ d_{\mathfrak{C}}$ , so as  $\text{Im } d_{\mathfrak{C}}$  generates  $\Omega_{\mathfrak{C}}$  as an  $\mathfrak{C}$ -module we see that  $\beta_* \circ \psi = \text{id}_{\Omega_{\mathfrak{C}}}$ . Similarly  $\psi \circ \beta_*$  is the identity, so  $\psi, \beta_*$  are inverse, and  $\beta_*$  is an isomorphism. Therefore (5.3) is exact, and  $\Omega_{\mathfrak{C}}$  is finitely presented.  $\square$

Cotangent modules behave well under localization.

**PROPOSITION 5.7.** *Let  $\mathfrak{C}$  be a  $C^\infty$ -ring,  $S \subseteq \mathfrak{C}$ , and  $\mathfrak{D} = \mathfrak{C}[s^{-1} : s \in S]$  be the localization of  $\mathfrak{C}$  at  $S$  with projection  $\pi : \mathfrak{C} \rightarrow \mathfrak{D}$ , as in Definition 2.13. Then  $(\Omega_\pi)_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{D} \rightarrow \Omega_{\mathfrak{D}}$  is an isomorphism of  $\mathfrak{D}$ -modules.*

**PROOF.** Let  $\Omega_{\mathfrak{C}}, \Omega_{\mathfrak{D}}$  be constructed as in Definition 5.3. As  $\mathfrak{D} = \mathfrak{C}[s^{-1} : s \in S]$  is  $\mathfrak{C}$  together with an extra generator  $s^{-1}$  and an extra relation  $s \cdot s^{-1} = 1$  for each  $s \in S$ , we see that the  $\mathfrak{D}$ -module  $\Omega_{\mathfrak{D}}$  may be constructed from  $\Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{D}$  by adding an extra generator  $d(s^{-1})$  and an extra relation  $d(s \cdot s^{-1} - 1) = 0$  for each  $s \in S$ . But using (5.1) and  $s \cdot s^{-1} = 1$  in  $\mathfrak{D}$ , we see that this extra relation is equivalent to  $d(s^{-1}) = -(s^{-1})^2 ds$ . Thus the extra relations exactly cancel the effect of adding the extra generators, so  $(\Omega_\pi)_*$  is an isomorphism.  $\square$

Here is a useful exactness property of cotangent modules.

**THEOREM 5.8.** *Suppose we are given a pushout diagram of  $C^\infty$ -rings:*

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\beta} & \mathfrak{E} \\ \downarrow \alpha & & \downarrow \delta \\ \mathfrak{D} & \xrightarrow{\gamma} & \mathfrak{F}, \end{array} \quad (5.5)$$

so that  $\mathfrak{F} = \mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E}$ . Then the following sequence of  $\mathfrak{F}$ -modules is exact:

$$\Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}, \gamma \circ \alpha} \mathfrak{F} \xrightarrow{(\Omega_\alpha)_* \oplus -(\Omega_\beta)_*} \Omega_{\mathfrak{D}} \otimes_{\mathfrak{D}, \gamma} \mathfrak{F} \oplus \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}, \delta} \mathfrak{F} \xrightarrow{(\Omega_\gamma)_* \oplus (\Omega_\delta)_*} \Omega_{\mathfrak{F}} \longrightarrow 0. \quad (5.6)$$

Here  $(\Omega_\alpha)_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}, \gamma \circ \alpha} \mathfrak{F} \rightarrow \Omega_{\mathfrak{D}} \otimes_{\mathfrak{D}, \gamma} \mathfrak{F}$  is induced by  $\Omega_\alpha : \Omega_{\mathfrak{C}} \rightarrow \Omega_{\mathfrak{D}}$ , and so on. Note the sign of  $-(\Omega_\beta)_*$  in (5.6).

**PROOF.** By  $\Omega_{\psi \circ \phi} = \Omega_\psi \circ \Omega_\phi$  in Definition 5.3 and commutativity of (5.5) we have  $\Omega_\gamma \circ \Omega_\alpha = \Omega_{\gamma \circ \alpha} = \Omega_{\delta \circ \beta} = \Omega_\delta \circ \Omega_\beta : \Omega_{\mathfrak{C}} \rightarrow \Omega_{\mathfrak{F}}$ . Tensoring with  $\mathfrak{F}$  then gives  $(\Omega_\gamma)_* \circ (\Omega_\alpha)_* = (\Omega_\delta)_* \circ (\Omega_\beta)_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{F} \rightarrow \Omega_{\mathfrak{F}}$ . As the composition of morphisms in (5.6) is  $(\Omega_\gamma)_* \circ (\Omega_\alpha)_* - (\Omega_\delta)_* \circ (\Omega_\beta)_*$ , this implies (5.6) is a complex.

For simplicity, first suppose  $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}$  are finitely presented. Use the notation of Example 2.23 and the proof of Proposition 2.24, with exact sequences (2.3) and (2.4), where  $I = (h_1, \dots, h_i) \subset C^\infty(\mathbb{R}^l)$ ,  $J = (d_1, \dots, d_j) \subset C^\infty(\mathbb{R}^m)$  and  $K = (e_1, \dots, e_k) \subset C^\infty(\mathbb{R}^n)$ . Then  $L$  is given by (2.5). Applying the proof of

Proposition 5.6 to (2.3)–(2.4) yields exact sequences of  $\mathfrak{F}$ -modules

$$\mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^i \xrightarrow{\epsilon_1} \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^l \xrightarrow{\zeta_1} \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{F} \longrightarrow 0, \quad (5.7)$$

$$\mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^j \xrightarrow{\epsilon_2} \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^m \xrightarrow{\zeta_2} \Omega_{\mathfrak{D}} \otimes_{\mathfrak{D}} \mathfrak{F} \longrightarrow 0, \quad (5.8)$$

$$\mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^k \xrightarrow{\epsilon_3} \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^n \xrightarrow{\zeta_3} \Omega_{\mathfrak{E}} \otimes_{\mathfrak{E}} \mathfrak{F} \longrightarrow 0, \quad (5.9)$$

$$\mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^{j+k+l} \xrightarrow{\epsilon_4} \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^{m+n} = \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^m \oplus \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^n \xrightarrow{\zeta_4} \Omega_{\mathfrak{F}} \longrightarrow 0, \quad (5.10)$$

where for (5.7)–(5.9) we have tensored (5.3) for  $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$  with  $\mathfrak{F}$ .

Define  $\mathfrak{F}$ -module morphisms  $\theta_1 : \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^l \rightarrow \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^m$ ,  $\theta_2 : \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^l \rightarrow \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^n$  by  $\theta_1(a_1, \dots, a_l) = (b_1, \dots, b_m)$ ,  $\theta_2(a_1, \dots, a_l) = (c_1, \dots, c_n)$  with

$$b_q = \sum_{p=1}^l \Phi_{\frac{\partial f_p}{\partial y_q}}(\xi(y_1), \dots, \xi(y_m)) \cdot a_p, \quad c_r = \sum_{p=1}^l \Phi_{\frac{\partial g_p}{\partial y_r}}(\xi(z_1), \dots, \xi(z_n)) \cdot a_p,$$

for  $a_p, b_q, c_r \in \mathfrak{F}$ . Now consider the diagram

$$\begin{array}{ccccc} \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^j \oplus & & \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^m \oplus & & \Omega_{\mathfrak{F}} \longrightarrow 0 \\ \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^k \oplus & \xrightarrow{\epsilon_4 = \begin{pmatrix} \epsilon_2 & 0 & \theta_1 \\ 0 & \epsilon_3 & -\theta_2 \end{pmatrix}} & \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^n & \xrightarrow{\zeta_4} & \parallel \text{id}_{\Omega_{\mathfrak{F}}} \\ \mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^l & & & & \Omega_{\mathfrak{F}} \longrightarrow 0 \\ \downarrow (0 \ 0 \ \zeta_1) & & \downarrow \begin{pmatrix} \zeta_2 & 0 \\ 0 & \zeta_3 \end{pmatrix} & & \\ \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{F} & \xrightarrow{\begin{pmatrix} (\Omega_{\alpha})_* \\ -(\Omega_{\beta})_* \end{pmatrix}} & \Omega_{\mathfrak{D}} \otimes_{\mathfrak{D}} \mathfrak{F} \oplus \Omega_{\mathfrak{E}} \otimes_{\mathfrak{E}} \mathfrak{F} & \xrightarrow{((\Omega_{\gamma})_* \ (\Omega_{\delta})_*)} & \Omega_{\mathfrak{F}} \longrightarrow 0 \end{array} \quad (5.11)$$

using matrix notation. The top line is the exact sequence (5.10), where the sign in  $-\theta_2$  comes from the sign of  $g_p$  in the generators  $f_p(y_1, \dots, y_m) - g_p(z_1, \dots, z_n)$  of  $L$  in (2.5). The bottom line is the complex (5.6).

The left hand square commutes as  $\zeta_2 \circ \epsilon_2 = \zeta_3 \circ \epsilon_3 = 0$  by exactness of (5.8)–(5.9) and  $\zeta_2 \circ \theta_1 = (\Omega_{\alpha})_* \circ \zeta_1$  follows from  $\alpha \circ \phi(x_p) = \psi(f_p)$ , and  $\zeta_3 \circ \theta_2 = (\Omega_{\beta})_* \circ \zeta_1$  follows from  $\beta \circ \phi(x_p) = \chi(g_p)$ . The right hand square commutes as  $\zeta_4$  and  $(\Omega_{\gamma})_* \circ \zeta_2$  act on  $\mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^m$  by  $(a_1, \dots, a_m) \mapsto \sum_{q=1}^m a_q d_{\mathfrak{F}} \circ \xi(y_q)$ , and  $\zeta_4$  and  $(\Omega_{\delta})_* \circ \zeta_3$  act on  $\mathfrak{F} \otimes_{\mathbb{R}} \mathbb{R}^n$  by  $(b_1, \dots, b_n) \mapsto \sum_{r=1}^n b_r d_{\mathfrak{F}} \circ \xi(z_r)$ . Hence (5.11) is commutative. The columns are surjective since  $\zeta_1, \zeta_2, \zeta_3$  are surjective as (5.7)–(5.9) are exact and identities are surjective.

The bottom right morphism  $((\Omega_{\gamma})_* \ (\Omega_{\delta})_*)$  in (5.11) is surjective as  $\zeta_4$  is and the right hand square commutes. Also surjectivity of the middle column implies that it maps  $\text{Ker } \zeta_4$  surjectively onto  $\text{Ker}((\Omega_{\gamma})_* \ (\Omega_{\delta})_*)$ . But  $\text{Ker } \zeta_4 = \text{Im } \epsilon_4$  as the top row is exact, so as the left hand square commutes we see that  $((\Omega_{\alpha})_* - (\Omega_{\beta})_*)^T$  surjects onto  $\text{Ker}((\Omega_{\gamma})_* \ (\Omega_{\delta})_*)$ , and the bottom row of (5.11) is exact. This proves the theorem for  $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}$  finitely presented. For the general case we can use the same proof, but allowing  $i, j, k, l, m, n$  infinite.  $\square$

Here is an example of the situation of Theorem 5.8 for manifolds.

EXAMPLE 5.9. Let  $W, X, Y, Z, e, f, g, h$  be as in Theorem 3.5, so that (3.1) is a Cartesian square of manifolds and (3.2) a pushout square of  $C^\infty$ -rings. We have the following sequence of morphisms of vector bundles on  $W$ :

$$0 \rightarrow (g \circ e)^*(T^*Z) \xrightarrow{e^*(dg^*) \oplus -f^*(dh^*)} e^*(T^*X) \oplus f^*(T^*Y) \xrightarrow{de^* \oplus df^*} T^*W \rightarrow 0. \quad (5.12)$$

Here  $dg : TX \rightarrow g^*(TZ)$  is a morphism of vector bundles over  $X$ , and  $dg^* : g^*(T^*Z) \rightarrow T^*X$  is the dual morphism, and  $e^*(dg^*) : (g \circ e)^*(T^*Z) \rightarrow e^*(T^*X)$  is the pullback of this dual morphism to  $W$ .

Since  $g \circ e = h \circ f$ , we have  $de^* \circ e^*(dg^*) = df^* \circ f^*(dh^*)$ , and so (5.12) is a complex. As  $g, h$  are transverse and (3.1) is Cartesian, (5.12) is exact. So passing to smooth sections in (5.12) we get an exact sequence of  $C^\infty(W)$ -modules:

$$0 \longrightarrow \Gamma^\infty((g \circ e)^*(T^*Z)) \xrightarrow{\begin{pmatrix} (e^*(dg^*)) \oplus \\ -f^*(dh^*) \end{pmatrix}^*} \Gamma^\infty(e^*(T^*X) \oplus f^*(T^*Y)) \xrightarrow{\begin{pmatrix} (de^*) \oplus \\ df^* \end{pmatrix}^*} \Gamma^\infty(T^*W) \longrightarrow 0.$$

The final four terms are the exact sequence (5.6) for the pushout diagram (3.2).

### 5.3. Sheaves of $\mathcal{O}_X$ -modules on a $C^\infty$ -ringed space $(X, \mathcal{O}_X)$

We define sheaves of  $\mathcal{O}_X$ -modules on a  $C^\infty$ -ringed space, following [31, §II.5].

DEFINITION 5.10. Let  $(X, \mathcal{O}_X)$  be a  $C^\infty$ -ringed space. A *sheaf of  $\mathcal{O}_X$ -modules*, or simply an  *$\mathcal{O}_X$ -module*,  $\mathcal{E}$  on  $X$  assigns a module  $\mathcal{E}(U)$  over the  $C^\infty$ -ring  $\mathcal{O}_X(U)$  for each open set  $U \subseteq X$ , and a linear map  $\mathcal{E}_{UV} : \mathcal{E}(U) \rightarrow \mathcal{E}(V)$  for each inclusion of open sets  $V \subseteq U \subseteq X$ , such that the following commutes

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{E}(U) & \xrightarrow{\mu_{\mathcal{E}(U)}} & \mathcal{E}(U) \\ \downarrow \rho_{UV} \times \mathcal{E}_{UV} & & \mathcal{E}_{UV} \downarrow \\ \mathcal{O}_X(V) \times \mathcal{E}(V) & \xrightarrow{\mu_{\mathcal{E}(V)}} & \mathcal{E}(V), \end{array} \quad (5.13)$$

and all this data  $\mathcal{E}(U), \mathcal{E}_{UV}$  satisfies the sheaf axioms in Definition 4.1.

A *morphism of sheaves of  $\mathcal{O}_X$ -modules*  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  assigns a morphism of  $\mathcal{O}_X(U)$ -modules  $\phi(U) : \mathcal{E}(U) \rightarrow \mathcal{F}(U)$  for each open set  $U \subseteq X$ , such that  $\phi(V) \circ \mathcal{E}_{UV} = \mathcal{F}_{UV} \circ \phi(U)$  for each inclusion of open sets  $V \subseteq U \subseteq X$ . Then  $\mathcal{O}_X$ -modules form an abelian category, which we write as  $\mathcal{O}_X\text{-mod}$ .

An  $\mathcal{O}_X$ -module  $\mathcal{E}$  is called a *vector bundle of rank  $n$*  if we may cover  $X$  by open  $U \subseteq X$  with  $\mathcal{E}|_U \cong \mathcal{O}_X|_U \otimes_{\mathbb{R}} \mathbb{R}^n$ .

In Definition 4.7 we defined *fine* sheaves  $\mathcal{E}$  on a topological space  $X$ . In §4.7 we gave sufficient conditions for when a  $C^\infty$ -ringed space  $\underline{X} = (X, \mathcal{O}_X)$  has  $\mathcal{O}_X$  fine, which hold if  $\underline{X}$  is an affine  $C^\infty$ -scheme with  $X$  Lindelöf. Now if  $\mathcal{O}_X$  is fine, then any  $\mathcal{O}_X$ -module  $\mathcal{E}$  is also fine, since partitions of unity in  $\mathcal{O}_X$  induce partitions of unity in  $\text{Hom}(\mathcal{E}, \mathcal{E})$ .

As in Voisin [69, Prop. 4.36], a fundamental property of fine sheaves  $\mathcal{E}$  is that their cohomology groups  $H^i(\mathcal{E})$  are zero for all  $i > 0$ . This means that  $H^0$  is an exact functor on fine sheaves, rather than just left exact, since  $H^1$  measures the failure of  $H^0$  to be right exact. If  $X$  is second countable then  $(U, \mathcal{O}_X|_U)$  is a Lindelöf affine  $C^\infty$ -scheme for all open  $U \subseteq X$ . Thus we deduce:

PROPOSITION 5.11. *Let  $(X, \mathcal{O}_X)$  be an affine  $C^\infty$ -scheme with  $X$  Lindelöf, and*

$$\dots \longrightarrow \mathcal{E}^i \xrightarrow{\phi^i} \mathcal{E}^{i+1} \xrightarrow{\phi^{i+1}} \mathcal{E}^{i+2} \longrightarrow \dots$$

*be an exact sequence in  $\mathcal{O}_X\text{-mod}$ . Then*

$$\dots \longrightarrow \mathcal{E}^i(X) \xrightarrow{\phi^i(X)} \mathcal{E}^{i+1}(X) \xrightarrow{\phi^{i+1}(X)} \mathcal{E}^{i+2}(X) \longrightarrow \dots$$



is an exact sequence of  $\mathcal{O}_X(X)$ -modules. If  $X$  is also second countable then the following is an exact sequence of  $\mathcal{O}_X(U)$ -modules for all open  $U \subseteq X$ :

$$\cdots \longrightarrow \mathcal{E}^i(U) \xrightarrow{\phi^i(U)} \mathcal{E}^{i+1}(U) \xrightarrow{\phi^{i+1}(U)} \mathcal{E}^{i+2}(U) \longrightarrow \cdots.$$

REMARK 5.12. Recall that a  $C^\infty$ -ring  $\mathfrak{C}$  has an underlying commutative  $\mathbb{R}$ -algebra, and a module over  $\mathfrak{C}$  is a module over this  $\mathbb{R}$ -algebra, by Definitions 2.6 and 5.1. Thus, by truncating the  $C^\infty$ -rings  $\mathcal{O}_X(U)$  to commutative  $\mathbb{R}$ -algebras, regarded as rings, a  $C^\infty$ -ringed space  $(X, \mathcal{O}_X)$  has an underlying ringed space in the usual sense of algebraic geometry [31, p. 72], [30, §0.4]. Our definition of  $\mathcal{O}_X$ -modules are simply  $\mathcal{O}_X$ -modules on this underlying ringed space [31, §II.5], [30, §0.4.1]. Thus we can apply results from algebraic geometry without change, for instance that  $\mathcal{O}_X\text{-mod}$  is an abelian category, as in [31, p. 202].

DEFINITION 5.13. Let  $f = (f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  be a morphism of  $C^\infty$ -ringed spaces, and  $\mathcal{E}$  be an  $\mathcal{O}_Y$ -module. Define the *pullback*  $f^*(\mathcal{E})$  by  $f^*(\mathcal{E}) = f^{-1}(\mathcal{E}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$ , where  $f^{-1}(\mathcal{E})$  is as in Definition 4.5, a sheaf of modules over the sheaf of  $C^\infty$ -rings  $f^{-1}(\mathcal{O}_Y)$  on  $X$ , and the tensor product uses the morphism  $f^\# : f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$ . If  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  is a morphism of  $\mathcal{O}_Y$ -modules we have a morphism of  $\mathcal{O}_X$ -modules  $f^*(\phi) = f^{-1}(\phi) \otimes \text{id}_{\mathcal{O}_X} : f^*(\mathcal{E}) \rightarrow f^*(\mathcal{F})$ .

REMARK 5.14. Pullbacks  $f^*(\mathcal{E})$  are a kind of fibre product, and may be characterized by a universal property in  $\mathcal{O}_X\text{-mod}$ . So they should be regarded as being *unique up to canonical isomorphism*, rather than unique. One can give an explicit construction for pullbacks, or use the Axiom of Choice to choose  $f^*(\mathcal{E})$  for all  $f, \mathcal{E}$ , and so speak of ‘the’ pullback  $f^*(\mathcal{E})$ . However, it may not be possible to make these choices strictly functorial in  $f$ .

That is, if  $f : \underline{X} \rightarrow \underline{Y}$ ,  $g : \underline{Y} \rightarrow \underline{Z}$  are morphisms and  $\mathcal{E} \in \mathcal{O}_Z\text{-mod}$  then  $(g \circ f)^*(\mathcal{E})$ ,  $f^*(g^*(\mathcal{E}))$  are canonically isomorphic in  $\mathcal{O}_X\text{-mod}$ , but may not be equal. We will write  $I_{f,g}(\mathcal{E}) : (g \circ f)^*(\mathcal{E}) \rightarrow f^*(g^*(\mathcal{E}))$  for these canonical isomorphisms, as in Remark 4.6(b). Then  $I_{f,g} : (g \circ f)^* \Rightarrow f^* \circ g^*$  is a natural isomorphism of functors. It is common to ignore this point and identify  $(g \circ f)^*$  with  $f^* \circ g^*$ . Vistoli [68] makes careful use of natural isomorphisms  $(g \circ f)^* \Rightarrow f^* \circ g^*$  in his treatment of descent theory.

When  $f$  is the identity  $\text{id}_X : X \rightarrow X$  and  $\mathcal{E} \in \mathcal{O}_X\text{-mod}$  we do not require  $\text{id}_X^*(\mathcal{E}) = \mathcal{E}$ , but as  $\mathcal{E}$  is a possible pullback for  $\text{id}_X^*(\mathcal{E})$  there is a canonical isomorphism  $\delta_X(\mathcal{E}) : \text{id}_X^*(\mathcal{E}) \rightarrow \mathcal{E}$ , and then  $\delta_X : \text{id}_X^* \Rightarrow \text{id}_{\mathcal{O}_X\text{-mod}}$  is a natural isomorphism of functors.

By Grothendieck [30, §0.4.3.1] we have:

PROPOSITION 5.15. *Let  $\underline{X}, \underline{Y}$  be  $C^\infty$ -ringed spaces and  $f : \underline{X} \rightarrow \underline{Y}$  a morphism. Then pullback  $f^* : \mathcal{O}_Y\text{-mod} \rightarrow \mathcal{O}_X\text{-mod}$  is a **right exact functor** between abelian categories. That is, if  $\mathcal{E} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \rightarrow 0$  is exact in  $\mathcal{O}_Y\text{-mod}$  then  $f^*(\mathcal{E}) \xrightarrow{f^*(\phi)} f^*(\mathcal{F}) \xrightarrow{f^*(\psi)} f^*(\mathcal{G}) \rightarrow 0$  is exact in  $\mathcal{O}_X\text{-mod}$ .*

In general  $f^*$  is not exact, or left exact, unless  $f : \underline{X} \rightarrow \underline{Y}$  is flat.

#### 5.4. Sheaves on affine $C^\infty$ -schemes, $\text{MSpec}$ and $\Gamma$

In §4.4 we defined  $\text{Spec} : \mathbf{C}^\infty\mathbf{Rings}^{\text{op}} \rightarrow \mathbf{LC}^\infty\mathbf{RS}$ . In a similar way, if  $\mathfrak{C}$  is a  $C^\infty$ -ring and  $(X, \mathcal{O}_X) = \text{Spec } \mathfrak{C}$  we can define  $\text{MSpec} : \mathfrak{C}\text{-mod} \rightarrow \mathcal{O}_X\text{-mod}$ , a spectrum functor for modules.

DEFINITION 5.16. Let  $(X, \mathcal{O}_X) = \text{Spec } \mathfrak{C}$  for some  $C^\infty$ -ring  $\mathfrak{C}$  and  $M$  be a  $\mathfrak{C}$ -module. We will define an  $\mathcal{O}_X$ -module  $\mathcal{E} = \text{MSpec } M$ . For each open  $U \subseteq X$ , define  $\mathcal{E}(U)$  to be the  $\mathbb{R}$ -vector space of functions  $e : U \rightarrow \prod_{x \in U} (M \otimes_{\mathfrak{C}} \mathfrak{C}_x)$  with  $e(x) \in M \otimes_{\mathfrak{C}} \mathfrak{C}_x$  for all  $x \in U$ , and such that  $U$  may be covered by open sets  $W \subseteq U \subseteq X$  for which there exist  $m \in M$  with  $e(x) = m \otimes 1 \in M \otimes_{\mathfrak{C}} \mathfrak{C}_x$  for all  $x \in W$ . Here the  $\mathfrak{C}_x$ -module  $M \otimes_{\mathfrak{C}} \mathfrak{C}_x$  is defined using the  $\mathfrak{C}$ -module structure on  $M$  and the projection  $\pi_x : \mathfrak{C} \rightarrow \mathfrak{C}_x$ .

Definition 4.16 defines  $\mathcal{O}_X(U)$  as a set of functions  $U \rightarrow \prod_{x \in U} \mathfrak{C}_x$ . Define an  $\mathcal{O}_X(U)$ -module structure  $\mu_{\mathcal{E}(U)} : \mathcal{O}_X(U) \times \mathcal{E}(U) \rightarrow \mathcal{E}(U)$  on  $\mathcal{E}(U)$  by

$$\mu_{\mathcal{E}(U)}(s, e) : x \mapsto s(x) \cdot e(x),$$

for all  $s \in \mathcal{O}_X(U)$ ,  $e \in \mathcal{E}(U)$  and  $x \in U$ . For open  $V \subseteq U \subseteq X$ , define  $\mathcal{E}_{UV} : \mathcal{E}(U) \rightarrow \mathcal{E}(V)$  by  $\mathcal{E}_{UV} : e \mapsto e|_V$ . It is now easy to check that  $\mathcal{E}$  is a sheaf of  $\mathcal{O}_X$ -modules on  $X$ . Define  $\text{MSpec } M = \mathcal{E}$  in  $\mathcal{O}_X\text{-mod}$ .

An equivalent way to define  $\text{MSpec } M$  is as the sheafification of the presheaf  $U \mapsto M \otimes_{\mathfrak{C}} \mathcal{O}_X(U)$ . The definition above performs the sheafification explicitly.

Now let  $\alpha : M \rightarrow N$  be a morphism in  $\mathfrak{C}\text{-mod}$ , and set  $\mathcal{E} = \text{MSpec } M$  and  $\mathcal{F} = \text{MSpec } N$ . For each open  $U \subseteq X$ , define  $\lambda(U) : \mathcal{E}(U) \rightarrow \mathcal{F}(U)$  by

$$\lambda(U)(e) : x \mapsto (\alpha \otimes \text{id})(e(x)) \quad \text{for } x \in U,$$

where  $\alpha \otimes \text{id}$  maps  $M \otimes_{\mathfrak{C}} \mathfrak{C}_x \rightarrow N \otimes_{\mathfrak{C}} \mathfrak{C}_x$ . It is easy to check that  $\lambda(U)$  is an  $\mathcal{O}_X(U)$ -module morphism and  $\lambda(V) \circ \mathcal{E}_{UV} = \mathcal{F}_{UV} \circ \lambda(U) : \mathcal{E}(U) \rightarrow \mathcal{F}(V)$  for all open  $V \subseteq U \subseteq X$ . Hence  $\lambda : \mathcal{E} \rightarrow \mathcal{F}$  is a morphism in  $\mathcal{O}_X\text{-mod}$ . Define  $\text{MSpec } \alpha = \lambda$ , so that  $\text{MSpec } \alpha : \text{MSpec } M \rightarrow \text{MSpec } N$ . This defines a functor  $\text{MSpec} : \mathfrak{C}\text{-mod} \rightarrow \mathcal{O}_X\text{-mod}$ . It is an exact functor of abelian categories, since  $M \mapsto M \otimes_{\mathfrak{C}} \mathfrak{C}_x$  is an exact functor  $\mathfrak{C}\text{-mod} \rightarrow \mathfrak{C}_x\text{-mod}$  for each  $x \in X$ , as the localization  $\pi_x : \mathfrak{C} \rightarrow \mathfrak{C}_x$  is a flat morphism of  $\mathbb{R}$ -algebras.

DEFINITION 5.17. Let  $\mathfrak{C}$  be a  $C^\infty$ -ring, and  $(X, \mathcal{O}_X) = \text{Spec } \mathfrak{C}$ . If  $\mathcal{E}$  is an  $\mathcal{O}_X$ -module then  $\mathcal{E}(X)$  is a module over  $\mathcal{O}_X(X)$ , so using  $\Psi_{\mathfrak{C}} : \mathfrak{C} \rightarrow \Gamma(\text{Spec } \mathfrak{C}) = \mathcal{O}_X(X)$  we may regard  $\mathcal{E}(X)$  as a  $\mathfrak{C}$ -module. Define  $\Gamma(\mathcal{E})$  to be the  $\mathfrak{C}$ -module  $\mathcal{E}(X)$ . If  $\alpha : \mathcal{E} \rightarrow \mathcal{F}$  is a morphism of  $\mathcal{O}_X$ -modules then  $\Gamma(\alpha) := \alpha(X) : \mathcal{E}(X) \rightarrow \mathcal{F}(X)$  is a morphism  $\Gamma(\alpha) : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{F})$  in  $\mathfrak{C}\text{-mod}$ . This defines the *global sections functor*  $\Gamma : \mathcal{O}_X\text{-mod} \rightarrow \mathfrak{C}\text{-mod}$ .

In general  $\Gamma$  is a left exact functor of abelian categories, but may not be right exact. However, if  $X$  is Lindelöf (for example, if  $\mathfrak{C}$  is finitely or countably generated) then Proposition 5.11 shows that  $\Gamma$  is an exact functor.

Now  $\Gamma \circ \text{MSpec}$  is a functor  $\mathfrak{C}\text{-mod} \rightarrow \mathfrak{C}\text{-mod}$ . For each  $\mathfrak{C}$ -module  $M$  and  $m \in M$ , define  $\Psi_M(m) : X \rightarrow \prod_{x \in X} M \otimes_{\mathfrak{C}} \mathfrak{C}_x$  by  $\Psi_M(m) : x \mapsto m \otimes 1_{\mathfrak{C}_x} \in M \otimes_{\mathfrak{C}} \mathfrak{C}_x$ . Then  $\Psi_M(m) \in \text{MSpec } M(X) = \Gamma \circ \text{MSpec } M$  by Definition 5.16, so  $\Psi_M : M \rightarrow \Gamma \circ \text{MSpec } M$  is a linear map, and in fact a  $\mathfrak{C}$ -module morphism.

It is functorial in  $M$ , so that the  $\Psi_M$  for all  $M$  define a natural transformation  $\Psi : \text{id}_{\mathfrak{C}\text{-mod}} \Rightarrow \Gamma \circ \text{MSpec}$  of functors  $\text{id}_{\mathfrak{C}\text{-mod}}, \Gamma \circ \text{MSpec} : \mathfrak{C}\text{-mod} \rightarrow \mathfrak{C}\text{-mod}$ .

Here are the analogues of Lemma 4.18 and Theorem 4.20:

LEMMA 5.18. *In Definition 5.16, the stalk  $(\text{MSpec } M)_x = \mathcal{E}_x$  of  $\text{MSpec } M$  at  $x \in X$  is naturally isomorphic to  $M \otimes_{\mathfrak{C}} \mathfrak{C}_x$ , as modules over  $\mathfrak{C}_x \cong \mathcal{O}_{X,x}$ .*

PROOF. Elements of  $\mathcal{E}_x$  are  $\sim$ -equivalence classes  $[U, e]$  of pairs  $(U, e)$ , where  $U$  is an open neighbourhood of  $x$  in  $X$  and  $e \in \mathcal{E}(U)$ , and  $(U, e) \sim (U', e')$  if there

exists open  $x \in V \subseteq U \cap U'$  with  $e|_V = e'|_V$ . Define a  $\mathfrak{C}_x$ -module morphism  $\Pi : \mathcal{E}_x \rightarrow M \otimes_{\mathfrak{C}} \mathfrak{C}_x$  by  $\Pi : [U, e] \mapsto e(x)$ .

Proposition 2.14 shows that  $\mathfrak{C}_x \cong \mathfrak{C}/I$  for  $I$  the ideal in (2.2). Hence  $M \otimes_{\mathfrak{C}} \mathfrak{C}_x \cong M/(I \cdot M)$ , and thus every element of  $M \otimes_{\mathfrak{C}} \mathfrak{C}_x$  is of the form  $m \otimes 1_{\mathfrak{C}_x}$  for some  $m \in M$ . But  $\Psi_M(m) \in \mathcal{E}(X)$ , so that  $[X, \Psi_M(m)] \in \mathcal{E}_x$ , with  $\Pi : [X, \Psi_M(m)] \mapsto m \otimes 1_{\mathfrak{C}_x}$ . Hence  $\Pi : \mathcal{E}_x \rightarrow M \otimes_{\mathfrak{C}} \mathfrak{C}_x$  is surjective.

Suppose  $[U, e] \in \mathcal{E}_x$  with  $\Pi([U, e]) = 0 \in M \otimes_{\mathfrak{C}} \mathfrak{C}_x$ . As  $e \in \mathcal{E}(U)$ , there exist open  $x \in V \subseteq U$  and  $m \in M$  with  $e(x') = m \otimes 1_{\mathfrak{C}_{x'}}$  for all  $x' \in V$ . Then  $m \otimes 1_{\mathfrak{C}_x} = e(x) = \Pi([U, e]) = 0$  in  $M \otimes_{\mathfrak{C}} \mathfrak{C}_x$ , so  $m \in I \cdot M \subseteq M$ , and we may write  $m = \sum_{a=1}^k i_a \cdot m_a$  for  $i_a \in I$  and  $m_a \in M$ . By (2.2) we may choose  $d_1, \dots, d_k \in \mathfrak{C}$  with  $x(d_a) \neq 0$  and  $i_a \cdot d_a = 0$  in  $\mathfrak{C}$  for  $a = 1, \dots, k$ .

Set  $W = \{x' \in V : x'(d_a) \neq 0, a = 1, \dots, k\}$ , so that  $W$  is an open neighbourhood of  $x$  in  $U$ . If  $x' \in W$  then  $x'(d_a) \neq 0$ , so  $\pi_{x'}(d_a)$  is invertible in  $\mathfrak{C}_{x'}$ . But  $i_a \cdot d_a = 0$ , so  $\pi_{x'}(i_a) = 0$  in  $\mathfrak{C}_{x'}$  for  $a = 1, \dots, k$ . As  $m = \sum_{a=1}^k i_a \cdot m_a$  it follows that  $e(x') = m \otimes 1_{\mathfrak{C}_{x'}} = 0$  in  $M \otimes_{\mathfrak{C}} \mathfrak{C}_{x'}$  for all  $x' \in W$ . Thus  $e|_W = 0$  in  $\mathcal{E}(W)$ , so  $[U, e] = [W, e|_W] = 0$  in  $\mathcal{E}_x$ . Therefore  $\Pi : \mathcal{E}_x \rightarrow M \otimes_{\mathfrak{C}} \mathfrak{C}_x$  is injective, and so an isomorphism.  $\square$

**THEOREM 5.19.** *Let  $\mathfrak{C}$  be a  $C^\infty$ -ring, and  $(X, \mathcal{O}_X) = \text{Spec } \mathfrak{C}$ . Then  $\Gamma : \mathcal{O}_X\text{-mod} \rightarrow \mathfrak{C}\text{-mod}$  is **right adjoint** to  $\text{MSpec} : \mathfrak{C}\text{-mod} \rightarrow \mathcal{O}_X\text{-mod}$ . That is, for all  $M \in \mathfrak{C}\text{-mod}$  and  $\mathcal{E} \in \mathcal{O}_X\text{-mod}$  there are inverse bijections*

$$\text{Hom}_{\mathfrak{C}\text{-mod}}(M, \Gamma(\mathcal{E})) \xrightleftharpoons[R_{M, \mathcal{E}}]{L_{M, \mathcal{E}}} \text{Hom}_{\mathcal{O}_X\text{-mod}}(\text{MSpec } M, \mathcal{E}), \quad (5.14)$$

which are functorial in  $M, \mathcal{E}$ . When  $\mathcal{E} = \text{MSpec } M$  we have  $\Psi_M = R_{M, \mathcal{E}}(\text{id}_{\mathcal{E}})$ , so that  $\Psi_M$  is the unit of the adjunction between  $\Gamma$  and  $\text{MSpec}$ .

**PROOF.** Let  $M \in \mathfrak{C}\text{-mod}$  and  $\mathcal{E} \in \mathcal{O}_X\text{-mod}$ , and set  $\mathcal{D} = \text{MSpec } M$ . Define  $R_{M, \mathcal{E}}$  in (5.14) by, for each morphism  $\alpha : \mathcal{D} \rightarrow \mathcal{E}$  in  $\mathcal{O}_X\text{-mod}$ , taking  $R_{M, \mathcal{E}}(\alpha) : M \rightarrow \Gamma(\mathcal{E})$  to be the composition

$$M \xrightarrow{\Psi_M} \Gamma \circ \text{MSpec } M = \Gamma(\mathcal{D}) \xrightarrow{\Gamma(\alpha)} \Gamma(\mathcal{E}).$$

For the last part, if  $\mathcal{E} = \text{MSpec } M$  then  $\Psi_M = R_{M, \mathcal{E}}(\text{id}_{\mathcal{E}})$  as  $\Gamma(\text{id}_{\mathcal{E}}) = \text{id}_{\Gamma(\mathcal{E})}$ .

Let  $\beta : M \rightarrow \Gamma(\mathcal{E})$  be a morphism in  $\mathfrak{C}\text{-mod}$ . We will construct a morphism  $\lambda : \mathcal{D} \rightarrow \mathcal{E}$  in  $\mathcal{O}_X\text{-mod}$ , and set  $L_{M, \mathcal{E}}(\beta) = \lambda$ . Let  $x \in X$ . Consider the diagram

$$\begin{array}{ccc} M \otimes_{\mathfrak{C}} \mathfrak{C} = M & \xrightarrow{\beta} & \Gamma(\mathcal{E}) \\ \downarrow \text{id} \otimes \pi_x & & \downarrow \sigma_x \\ M \otimes_{\mathfrak{C}} \mathfrak{C}_x \cong \mathcal{D}_x & \xrightarrow{\beta_x} & \mathcal{E}_x \end{array} \quad (5.15)$$

in  $\mathfrak{C}\text{-mod}$ , where the isomorphism  $M \otimes_{\mathfrak{C}} \mathfrak{C}_x \cong \mathcal{D}_x$  comes from Lemma 5.18. Here  $\mathcal{E}_x$  is the stalk of  $\mathcal{E}$  at  $x$ , and  $\sigma_x : \Gamma(\mathcal{E}) = \mathcal{E}(X) \rightarrow \mathcal{E}_x$  takes stalks at  $x$ . The  $\mathfrak{C}$ -action on  $\Gamma(\mathcal{E})$  factors via  $\mathfrak{C} \xrightarrow{\Psi_{\mathfrak{C}}} \mathcal{O}_X(X)$ , and the  $\mathfrak{C}$ -action on  $\mathcal{E}_x$  factors via  $\mathfrak{C} \xrightarrow{\Psi_{\mathfrak{C}}} \mathcal{O}_X(X) \xrightarrow{\pi_x} \mathcal{O}_{X, x}$ , and  $\beta, \sigma_x$  are both  $\mathfrak{C}$ -module morphisms. But  $\mathcal{O}_{X, x} \cong \mathfrak{C}_x$  by Lemma 4.18, so  $\sigma_x \circ \beta : M \rightarrow \mathcal{E}_x$  is a  $\mathfrak{C}$ -module morphism, where the  $\mathfrak{C}$ -action on  $\mathcal{E}_x$  factors via  $\mathfrak{C} \xrightarrow{\pi_x} \mathfrak{C}_x$ . Hence there is a unique  $\mathcal{O}_{X, x}$ -module morphism  $\beta_x : \mathcal{D}_x \rightarrow \mathcal{E}_x$  making (5.15) commute.

For each open  $U \subseteq X$ , define  $\lambda(U) : \mathcal{D}(U) \rightarrow \mathcal{E}(U)$  by  $\lambda(U)d : x \mapsto \beta_x(d(x))$  for  $d \in \mathcal{D}(U)$  and  $x \in U \subseteq X$ , and  $d(x) \in \mathcal{D}_x$ , and  $\beta_x(d(x)) \in \mathcal{E}_x$ . Here as  $\mathcal{E}$  is a sheaf

we may identify elements of  $\mathcal{E}(U)$  with maps  $e : U \rightarrow \prod_{x \in U} \mathcal{E}_x$  with  $e(x) \in \mathcal{E}_x$  for  $x \in U$ , such that  $e$  satisfies certain local conditions in  $U$ .

If  $d \in \mathcal{D}(U) = \text{MSpec } M(U)$  and  $x \in U$  then by Definition 5.16 we may cover  $U$  by open  $W \subseteq U$  for which there exist  $m \in M$  with  $d(x) = m \otimes 1_{\mathfrak{C}_x}$  in  $M \otimes_{\mathfrak{C}} \mathfrak{C}_x$  for all  $x \in W$ . Therefore  $\lambda(U)d$  maps  $x \mapsto \sigma_x(\beta(m))$  for all  $x \in W$  by (5.15), so  $\lambda(U)d$  is a section  $\beta(m)|_W$  of  $\mathcal{E}$  on  $W$ . Hence  $\lambda(U)d$  is a section of  $\mathcal{E}|_U$ , as such  $W$  cover  $U$ , and  $\lambda(U) : \mathcal{D}(U) \rightarrow \mathcal{E}(U)$  is well defined.

As  $\beta_x$  is an  $\mathcal{O}_{X,x}$ -module morphism for all  $x \in U$ ,  $\lambda(U) : \mathcal{D}(U) \rightarrow \mathcal{E}(U)$  is an  $\mathcal{O}_X(U)$ -module morphism. The definition of  $\lambda(U)$  is clearly compatible with restriction to open  $V \subseteq U \subseteq X$ . Thus the  $\lambda(U)$  for all open  $U \subseteq X$  define a sheaf morphism  $\lambda : \mathcal{D} \rightarrow \mathcal{E}$  in  $\mathcal{O}_X\text{-mod}$ . Set  $L_{M,\mathcal{E}}(\beta) = \lambda$ . This defines  $L_{M,\mathcal{E}}$  in (5.14). A very similar proof to that of Theorem 4.20 shows that  $L_{M,\mathcal{E}}, R_{M,\mathcal{E}}$  are inverse maps, so they are bijections, and that they are functorial in  $M, \mathcal{E}$ .  $\square$

We show that  $\Gamma$  is a right inverse for  $\text{MSpec}$ :

**PROPOSITION 5.20.** *Let  $\mathfrak{C}$  be a  $C^\infty$ -ring, and  $(X, \mathcal{O}_X) = \text{Spec } \mathfrak{C}$ , and  $\mathcal{E}$  be an  $\mathcal{O}_X$ -module. Set  $M = \Gamma(\mathcal{E})$  in  $\mathfrak{C}\text{-mod}$ , and write  $\Psi_{\mathcal{E}} = L_{M,\mathcal{E}}(\text{id}_M)$ . Then  $\Psi_{\mathcal{E}} : \text{MSpec } \mathfrak{O}(\mathcal{E}) \rightarrow \mathcal{E}$  is an isomorphism in  $\mathcal{O}_X\text{-mod}$ , for any  $\mathcal{E}$ .*

*These isomorphisms  $\Psi_{\mathcal{E}}$  are functorial in  $\mathcal{E}$ , and so define a natural isomorphism  $\Psi : \text{MSpec } \mathfrak{O} \Rightarrow \text{id}_{\mathcal{O}_X\text{-mod}}$  of functors  $\mathcal{O}_X\text{-mod} \rightarrow \mathcal{O}_X\text{-mod}$ .*

**PROOF.** Set  $\mathcal{D} = \text{MSpec } M = \text{MSpec } \mathfrak{O}(\mathcal{E})$ , and let  $x \in X$ . Then by definition of  $\Psi_{\mathcal{E}} = L_{M,\mathcal{E}}(\text{id}_M) : \mathcal{D} \rightarrow \mathcal{E}$  in the proof of Theorem 5.19, as in (5.15) the stalk map  $\Psi_{\mathcal{E},x} : \mathcal{D}_x \rightarrow \mathcal{E}_x$  is the unique morphism of modules over  $\mathfrak{C}_x \cong \mathcal{O}_{X,x}$  making the following diagram of  $\mathfrak{C}$ -modules commute:

$$\begin{array}{ccc} M \otimes_{\mathfrak{C}} \mathfrak{C} = M & \xrightarrow{\text{id}_M} & M = \Gamma(\mathcal{E}) \\ \downarrow \text{id} \otimes \pi_x & & \downarrow \sigma_x \\ M \otimes_{\mathfrak{C}} \mathfrak{C}_x \cong \mathcal{D}_x & \xrightarrow{\Psi_{\mathcal{E},x}} & \mathcal{E}_x. \end{array} \quad (5.16)$$

Let  $[U, e] \in \mathcal{E}_x$ , so that  $x \in U \subseteq X$  is open and  $e \in \mathcal{E}(U)$ . By Definition 4.13 there exists  $c \in \mathfrak{C}$  such that  $x(c) \neq 0$  and  $y(c) = 0$  for all  $y \in X \setminus U$ . Choose smooth  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f = 0$  near 0 in  $\mathbb{R}$  and  $f = 1$  near  $x(c)$  in  $\mathbb{R}$ . Set  $c' = \Phi_f(c)$ , where  $\Phi_f : \mathfrak{C} \rightarrow \mathfrak{C}$  is the  $C^\infty$ -ring operation. Then  $\eta = \Psi_{\mathfrak{C}}(c') \in \mathcal{O}_X(X)$ , and there exist open neighbourhoods  $V$  of  $X \setminus U$  and  $W$  of  $x$  in  $X$  with  $\eta|_V = 0$  and  $\eta|_W = 1$ . Clearly  $V \cap W = \emptyset$ , so  $x \in W \subseteq U$ . We have  $\eta|_U \cdot e \in \mathcal{E}(U)$ , with  $(\eta|_U \cdot e)|_{U \cap V} = 0$  and  $(\eta|_U \cdot e)|_W = e|_W$ .

Since  $\{U, V\}$  is an open cover of  $X$  and  $(\eta|_U \cdot e)|_{U \cap V} = 0 = 0|_{U \cap V}$ , by the sheaf property of  $\mathcal{E}$  there is a unique  $e' \in \mathcal{E}(X)$  with  $e'|_U = \eta|_U \cdot e$  and  $e'|_V = 0$ . Then  $e'|_W = (\eta|_U \cdot e)|_W = e|_W$ . Thus

$$\sigma_x(e') = [X, e'] = [W, e'|_W] = [W, e|_W] = [U, e]$$

in  $\mathcal{E}_x$ . Hence  $\sigma_x : \Gamma(\mathcal{E}) \rightarrow \mathcal{E}_x$  is surjective, so  $\Psi_{\mathcal{E},x} : \mathcal{D}_x \rightarrow \mathcal{E}_x$  is surjective by (5.16), as  $\pi_x : \mathfrak{C} \rightarrow \mathfrak{C}_x$  is surjective by Proposition 2.14.

Suppose  $d \in \mathcal{D}_x$  with  $\Psi_{\mathcal{E},x}(d) = 0$ . We may write  $m \otimes 1_{\mathfrak{C}_x} \cong d$  under the isomorphism  $M \otimes_{\mathfrak{C}} \mathfrak{C}_x \cong \mathcal{D}_x$  for some  $m \in M$ , and then (5.16) gives  $\sigma_x(m) = \Psi_{\mathcal{E},x}(d) = 0$ . Hence there exists open  $x \in U \subseteq X$  with  $m|_U = 0$ . As above we may construct  $\eta \in \mathcal{O}_X(X)$  and open  $V, W \subseteq X$  with  $X \setminus U \subseteq V$ ,  $x \in W \subseteq U$ ,  $\eta|_V = 0$  and  $\eta|_W = 1$ . Then  $\eta \cdot m = 0$  in  $M$  as  $m|_U = 0$ ,  $\eta|_V = 0$  with  $U \cup V = X$ , and

$\pi_x(\eta) = 1_{\mathfrak{C}_x}$  in  $\mathfrak{C}_x$  as  $\eta = 1$  near  $x$  in  $X$ . Hence

$$m \otimes 1_{\mathfrak{C}_x} = 1_{\mathfrak{C}_x} \cdot (m \otimes 1_{\mathfrak{C}_x}) = \pi_x(\eta) \cdot (m \otimes 1_{\mathfrak{C}_x}) = (\eta \cdot m) \otimes 1_{\mathfrak{C}_x} = 0 \otimes 1_{\mathfrak{C}_x} = 0$$

in  $M \otimes_{\mathfrak{C}} \mathfrak{C}_x$ . Therefore  $d = 0$  in  $\mathfrak{D}_x$ , and  $\Psi_{\mathcal{E},x} : \mathcal{D}_x \rightarrow \mathcal{E}_x$  is injective, and so an isomorphism. As this holds for all  $x \in X$ ,  $\Psi_{\mathcal{E}} : \mathcal{D} \rightarrow \mathcal{E}$  is an isomorphism, proving the first part of the proposition. The second part follows from  $L_{M,\mathcal{E}}$  functorial in  $M, \mathcal{E}$  in Theorem 5.19.  $\square$

As for quasicoherent sheaves in conventional algebraic geometry, we define:

**DEFINITION 5.21.** Let  $\underline{X} = (X, \mathcal{O}_X)$  be a  $C^\infty$ -scheme, and  $\mathcal{E}$  be an  $\mathcal{O}_X$ -module. We call  $\mathcal{E}$  *quasicoherent* if we may cover  $X$  with open  $U \subseteq X$  such that  $(U, \mathcal{O}_X|_U) \cong \mathrm{Spec} \mathfrak{C}$  and  $\mathcal{E}|_U \cong \mathrm{MSpec} M$  for some  $C^\infty$ -ring  $\mathfrak{C}$  and  $\mathfrak{C}$ -module  $M$ .

We write  $\mathrm{qcoh}(\underline{X})$  for the category of quasicoherent sheaves on  $\underline{X}$ .

If  $(X, \mathcal{O}_X)$  is a  $C^\infty$ -scheme and  $\mathcal{E}$  an  $\mathcal{O}_X$ -module, we can cover  $X$  by open  $U \subseteq X$  with  $(U, \mathcal{O}_X|_U) \cong \mathrm{Spec} \mathfrak{C}$  affine, and then Proposition 5.20 shows that  $\mathcal{E}|_U \cong \mathrm{MSpec} M$  for  $M = \mathcal{E}(U)$ . Thus we have:

**COROLLARY 5.22.** Let  $\underline{X} = (X, \mathcal{O}_X)$  be a  $C^\infty$ -scheme. Then every  $\mathcal{O}_X$ -module  $\mathcal{E}$  is quasicoherent, so that  $\mathrm{qcoh}(\underline{X}) = \mathcal{O}_X\text{-mod}$ .

**REMARK 5.23. (a)** In conventional algebraic geometry, as in Hartshorne [31, §II.5], if  $R$  is a ring and  $(X, \mathcal{O}_X) = \mathrm{Spec} R$  the corresponding affine scheme, we also have functors  $\mathrm{MSpec} : R\text{-mod} \rightarrow \mathcal{O}_X\text{-mod}$  and  $\Gamma : \mathcal{O}_X\text{-mod} \rightarrow R\text{-mod}$ . In  $C^\infty$ -algebraic geometry, as in Proposition 5.20,  $\Gamma$  is a right inverse for  $\mathrm{MSpec}$ , but may not be a left inverse. But in algebraic geometry the opposite happens, as  $\Gamma$  is a left inverse for  $\mathrm{MSpec}$  [31, Cor. II.5.5], but may not be a right inverse.

The fact that  $\Gamma$  is a right inverse for  $\mathrm{MSpec}$  in  $C^\infty$ -algebraic geometry means that all  $\mathcal{O}_X$ -modules on a  $C^\infty$ -scheme  $(X, \mathcal{O}_X)$  are quasicoherent, so quasicoherence is not a very useful idea. But in algebraic geometry, as  $\Gamma$  is not a right inverse for  $\mathrm{MSpec}$ , this is *false*: there are many examples of schemes  $(X, \mathcal{O}_X)$  and  $\mathcal{O}_X$ -modules  $\mathcal{E}$  which are not quasicoherent. For instance, we may take  $X = \mathbb{A}^1$  and  $\mathcal{E}(U) = 0$  if  $0 \in U$ ,  $\mathcal{E}(U) = \mathcal{O}_X(U)$  if  $0 \notin U$  for all open  $U \subseteq X$ .

In §5.5 we will define a module  $M$  over a  $C^\infty$  ring  $\mathfrak{C}$  to be *complete* if  $M \cong \Gamma \circ \mathrm{MSpec} M$ . Then  $\Gamma$  is a left inverse for  $\mathrm{MSpec}$  on the subcategory  $\mathfrak{C}\text{-mod}^{\mathrm{co}} \subset \mathfrak{C}\text{-mod}$  of complete  $\mathfrak{C}$ -modules. In general  $\mathfrak{C}$ -modules need not be complete. But in conventional algebraic geometry, as  $\Gamma$  is a left inverse for  $\mathrm{MSpec}$  all  $R$ -modules are complete, so completeness is not a useful idea.

**(b)** In conventional algebraic geometry one defines *coherent sheaves* [31, §II.5] to be quasicoherent sheaves  $\mathcal{E}$  locally modelled on  $\mathrm{MSpec} M$  for  $M$  a finitely generated  $\mathfrak{C}$ -module. However, coherent sheaves are only well behaved on *noetherian* schemes, and most interesting  $C^\infty$ -rings, such as  $C^\infty(\mathbb{R}^n)$  for  $n > 0$ , are not noetherian  $\mathbb{R}$ -algebras. Because of this, coherent sheaves do not seem to be a useful idea in  $C^\infty$ -algebraic geometry (for instance,  $\mathrm{coh}(\underline{X})$  is not closed under kernels in  $\mathrm{qcoh}(\underline{X})$ , and is not an abelian category), and we do not discuss them.

We can understand the pullback functor  $f^*$  in Definition 5.13 explicitly in terms of modules over the corresponding  $C^\infty$ -rings:

**PROPOSITION 5.24.** Let  $\mathfrak{C}, \mathfrak{D}$  be  $C^\infty$ -rings,  $\phi : \mathfrak{D} \rightarrow \mathfrak{C}$  a morphism,  $M, N$  be  $\mathfrak{D}$ -modules, and  $\alpha : M \rightarrow N$  a morphism of  $\mathfrak{D}$ -modules. Write  $\underline{X} = \mathrm{Spec} \mathfrak{C}$ ,

$\underline{Y} = \text{Spec } \mathfrak{D}$ ,  $\underline{f} = \text{Spec } \phi : \underline{X} \rightarrow \underline{Y}$ , and  $\mathcal{E} = \text{MSpec } M$ ,  $\mathcal{F} = \text{MSpec } N$  in  $\text{qcoh}(\underline{Y})$ . Then there are natural isomorphisms  $\underline{f}^*(\mathcal{E}) \cong \text{MSpec}(M \otimes_{\mathfrak{D}} \mathfrak{C})$  and  $\underline{f}^*(\mathcal{F}) \cong \text{MSpec}(N \otimes_{\mathfrak{D}} \mathfrak{C})$  in  $\text{qcoh}(\underline{X})$ . These identify  $\text{MSpec}(\alpha \otimes \text{id}_{\mathfrak{C}}) : \text{MSpec}(M \otimes_{\mathfrak{D}} \mathfrak{C}) \rightarrow \text{MSpec}(N \otimes_{\mathfrak{D}} \mathfrak{C})$  with  $\underline{f}^*(\text{MSpec } \alpha) : \underline{f}^*(\mathcal{E}) \rightarrow \underline{f}^*(\mathcal{F})$ .

PROOF. Write  $\underline{X} = (X, \mathcal{O}_X)$ ,  $\underline{Y} = (Y, \mathcal{O}_Y)$  and  $\underline{f} = (f, f^\#)$ . Then  $\mathcal{E}$  is the sheafification of the presheaf  $V \mapsto M \otimes_{\mathfrak{D}} \mathcal{O}_Y(V)$ , and  $\underline{f}^{-1}(\mathcal{E})$  is the sheafification of the presheaf  $U \mapsto \lim_{V \supseteq f(U)} \mathcal{E}(V)$ , and  $\underline{f}^{-1}(\mathcal{O}_Y)$  is the sheafification of the presheaf  $U \mapsto \lim_{V \supseteq f(U)} \mathcal{O}_Y(V)$ . In  $\underline{f}^*(\mathcal{E}) = \underline{f}^{-1}(\mathcal{E}) \otimes_{\underline{f}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$ , these three sheafifications combine into one, so  $\underline{f}^*(\mathcal{E})$  is the sheafification of the presheaf  $U \mapsto \lim_{V \supseteq f(U)} (M \otimes_{\mathfrak{D}} \mathcal{O}_Y(V)) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U)$ . But

$$(M \otimes_{\mathfrak{D}} \mathcal{O}_Y(V)) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U) \cong M \otimes_{\mathfrak{D}} \mathcal{O}_X(U) \cong (M \otimes_{\mathfrak{D}} \mathfrak{C}) \otimes_{\mathfrak{C}} \mathcal{O}_X(U),$$

so this is canonically isomorphic to the presheaf  $U \mapsto (M \otimes_{\mathfrak{D}} \mathfrak{C}) \otimes_{\mathfrak{C}} \mathcal{O}_X(U)$  whose sheafification is  $\text{MSpec}(M \otimes_{\mathfrak{D}} \mathfrak{C})$ . This gives a natural isomorphism  $\underline{f}^*(\mathcal{E}) \cong \text{MSpec}(M \otimes_{\mathfrak{D}} \mathfrak{C})$ . The same holds for  $N$ . The identification of  $\text{MSpec}(\alpha \otimes \text{id}_{\mathfrak{C}})$  and  $\underline{f}^*(\text{MSpec } \alpha)$  follows by passing from morphisms of presheaves to morphisms of the associated sheaves.  $\square$

### 5.5. Complete modules over $C^\infty$ -rings

Here are the module analogues of Definition 4.35 and Theorem 4.36(b),(c).

DEFINITION 5.25. Let  $\mathfrak{C}$  be a  $C^\infty$ -ring, and  $M$  a  $\mathfrak{C}$ -module. We call  $M$  *complete* if  $\Psi_M : M \rightarrow \Gamma \circ \text{MSpec } M$  in Definition 5.17 is an isomorphism.

Write  $\mathfrak{C}\text{-mod}^{\text{co}}$  for the full subcategory of complete  $\mathfrak{C}$ -modules in  $\mathfrak{C}\text{-mod}$ .

If  $M$  is a  $\mathfrak{C}$ -module then applying  $\Gamma$  to Proposition 5.20 shows that

$$\Gamma(\Psi_{\text{MSpec } M}) : \Gamma \circ \text{MSpec}(\Gamma \circ \text{MSpec } M) \longrightarrow \Gamma \circ \text{MSpec } M$$

is an isomorphism. The definitions imply that  $\Psi_{\Gamma \circ \text{MSpec } M} = \Gamma(\Psi_{\text{MSpec } M})^{-1}$ . Thus  $\Gamma \circ \text{MSpec } M$  is complete, for any  $\mathfrak{C}$ -module  $M$ . Define a functor  $R_{\text{all}}^{\text{co}} = \Gamma \circ \text{MSpec} : \mathfrak{C}\text{-mod} \rightarrow \mathfrak{C}\text{-mod}^{\text{co}}$ .

THEOREM 5.26. Let  $\mathfrak{C}$  be a  $C^\infty$ -ring, and  $\underline{X} = (X, \mathcal{O}_X) = \text{Spec } \mathfrak{C}$ . Then

- (a)  $\text{MSpec}|_{\mathfrak{C}\text{-mod}^{\text{co}}} : \mathfrak{C}\text{-mod}^{\text{co}} \rightarrow \text{qcoh}(\underline{X})$  is an equivalence of categories.
- (b)  $R_{\text{all}}^{\text{co}} : \mathfrak{C}\text{-mod} \rightarrow \mathfrak{C}\text{-mod}^{\text{co}}$  is left adjoint to the inclusion functor  $\text{inc} : \mathfrak{C}\text{-mod}^{\text{co}} \hookrightarrow \mathfrak{C}\text{-mod}$ . That is,  $R_{\text{all}}^{\text{co}}$  is a **reflection functor**.

PROOF. For (a), if  $M, N$  are complete  $\mathfrak{C}$ -modules then putting  $\mathcal{E} = \text{MSpec } N$  in Theorem 5.19 and using  $\Gamma \circ \text{MSpec } N \cong N$ , equation (5.14) shows that

$$\text{MSpec} = L_{M, \mathcal{E}} : \text{Hom}_{\mathfrak{C}\text{-mod}^{\text{co}}}(M, N) \longrightarrow \text{Hom}_{\mathcal{O}_X\text{-mod}}(\text{MSpec } M, \text{MSpec } N)$$

is a bijection, where the definition of  $L_{M, \mathcal{E}}$  agrees with the definition of  $\text{MSpec}$  on morphisms in this case. Thus  $\text{MSpec}$  is full and faithful on complete  $\mathfrak{C}$ -modules.

If  $\mathcal{E} \in \mathcal{O}_X\text{-mod} = \text{qcoh}(\underline{X})$  then  $\mathcal{E} \cong \text{MSpec} \circ \Gamma(\mathcal{E})$  by Proposition 5.20. Thus  $\Gamma(\mathcal{E}) \cong \Gamma \circ \text{MSpec} \circ \Gamma(\mathcal{E})$ , so  $\Gamma(\mathcal{E})$  is complete by Definition 5.25. Hence  $\mathcal{E} \cong \text{MSpec}|_{\mathfrak{C}\text{-mod}^{\text{co}}}[\Gamma(\mathcal{E})]$ , and the essential image of  $\text{MSpec}|_{\mathfrak{C}\text{-mod}^{\text{co}}}$  is  $\text{qcoh}(\underline{X})$ . Therefore  $\text{MSpec}|_{\mathfrak{C}\text{-mod}^{\text{co}}}$  is an equivalence of categories.

For (b), let  $M, N$  be  $\mathfrak{C}$ -modules with  $N$  complete. Then we have bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{C}\text{-mod}^{\mathrm{co}}}(R_{\mathrm{all}}^{\mathrm{co}}(M), N) &\cong \mathrm{Hom}_{\mathfrak{C}\text{-mod}}(\Gamma \circ \mathrm{MSpec} M, \Gamma \circ \mathrm{MSpec} N) \\
&\cong \mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(\mathrm{MSpec} \circ \Gamma \circ \mathrm{MSpec} M, \mathrm{MSpec} N) \\
&\cong \mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(\mathrm{MSpec} M, \mathrm{MSpec} N) \\
&\cong \mathrm{Hom}_{\mathfrak{C}\text{-mod}}(M, \Gamma \circ \mathrm{MSpec} N) \cong \mathrm{Hom}_{\mathfrak{C}\text{-mod}}(M, N) = \mathrm{Hom}_{\mathfrak{C}\text{-mod}}(M, \mathrm{inc}(N)),
\end{aligned} \tag{5.17}$$

using  $N \cong \Gamma \circ \mathrm{MSpec} N$  as  $N$  is complete in the first and fifth steps, Theorem 5.19 in the second and fourth, and Proposition 5.20 in the third. The bijections (5.17) are functorial in  $M, N$  as each step is. Hence  $R_{\mathrm{all}}^{\mathrm{co}}$  is left adjoint to  $\mathrm{inc}$ .  $\square$

**PROPOSITION 5.27.** *Let  $\mathfrak{C}$  be a  $C^\infty$ -ring and  $(X, \mathcal{O}_X) = \mathrm{Spec} \mathfrak{C}$ , and suppose  $X$  is Lindelöf. Then  $\mathfrak{C}\text{-mod}^{\mathrm{co}}$  is closed under kernels, cokernels and extensions in  $\mathfrak{C}\text{-mod}$ , that is,  $\mathfrak{C}\text{-mod}^{\mathrm{co}}$  is an abelian subcategory of  $\mathfrak{C}\text{-mod}$ .*

**PROOF.** As in §5.4,  $\mathrm{MSpec} : \mathfrak{C}\text{-mod} \rightarrow \mathcal{O}_X\text{-mod}$  is an exact functor, and as  $X$  is Lindelöf  $\Gamma : \mathcal{O}_X\text{-mod} \rightarrow \mathfrak{C}\text{-mod}$  is also exact by Proposition 5.11. Hence  $R_{\mathrm{all}}^{\mathrm{co}} = \Gamma \circ \mathrm{MSpec} : \mathfrak{C}\text{-mod} \rightarrow \mathfrak{C}\text{-mod}$  is an exact functor. Let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3$  be exact in  $\mathfrak{C}\text{-mod}$  with  $M_2, M_3$  complete. Then we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 \\
& & \downarrow \Psi_{M_1} & & \cong \downarrow \Psi_{M_2} & & \cong \downarrow \Psi_{M_3} \\
0 & \longrightarrow & R_{\mathrm{all}}^{\mathrm{co}}(M_1) & \longrightarrow & R_{\mathrm{all}}^{\mathrm{co}}(M_2) & \longrightarrow & R_{\mathrm{all}}^{\mathrm{co}}(M_3)
\end{array}$$

in  $\mathfrak{C}\text{-mod}$ , where both rows are exact as  $R_{\mathrm{all}}^{\mathrm{co}}$  is an exact functor, and the second and third columns are isomorphisms. Hence the first column is also an isomorphism, and  $M_1$  is complete, so  $\mathfrak{C}\text{-mod}^{\mathrm{co}}$  is closed under kernels in  $\mathfrak{C}\text{-mod}$ . It is closed under cokernels and extensions by very similar arguments.  $\square$

**EXAMPLE 5.28.** Let  $\mathfrak{C}$  be a  $C^\infty$ -ring with  $(X, \mathcal{O}_X) = \mathrm{Spec} \mathfrak{C}$ . Then:

- (a) Considering  $\mathfrak{C}$  as a  $\mathfrak{C}$ -module, we have  $\Gamma \circ \mathrm{MSpec} \mathfrak{C} = \Gamma \circ \mathrm{Spec} \mathfrak{C} = \mathcal{O}_X(X)$ , and  $\Psi_{\mathfrak{C}} : \mathfrak{C} \rightarrow \mathcal{O}_X(X)$  in Definitions 4.19 and 5.17 coincide. Hence  $\mathfrak{C}$  is complete as a  $\mathfrak{C}$ -module if and only if it is complete as a  $C^\infty$ -ring, in the sense of §4.6. So, if  $\mathfrak{C}$  is a finitely generated but not fair  $C^\infty$ -ring, as in Examples 2.19 and 2.21, then  $\mathfrak{C}$  is a non-complete  $\mathfrak{C}$ -module.
- (b) Suppose  $\mathfrak{C}$  is complete and  $X$  is Lindelöf. Let  $M$  be a finitely presented  $\mathfrak{C}$ -module, so we have an exact sequence  $\mathfrak{C} \otimes \mathbb{R}^m \rightarrow \mathfrak{C} \otimes \mathbb{R}^n \rightarrow M \rightarrow 0$  in  $\mathfrak{C}\text{-mod}$ . Here  $\mathfrak{C} \otimes \mathbb{R}^m, \mathfrak{C} \otimes \mathbb{R}^n$  are complete as  $\mathfrak{C}$  is by (a), so  $M$  is complete by Proposition 5.27 as  $\mathfrak{C}\text{-mod}$  is closed under cokernels.
- (c) Suppose  $\mathfrak{C}$  is complete,  $X$  is Lindelöf, and  $I \subseteq \mathfrak{C}$  is a finitely generated ideal. Choose generators  $i_1, \dots, i_n$  for  $I$ . Then we have an exact sequence  $\mathfrak{C} \otimes \mathbb{R}^n \rightarrow \mathfrak{C} \rightarrow \mathfrak{C}/I \rightarrow 0$  in  $\mathfrak{C}\text{-mod}$  with  $\mathfrak{C} \otimes \mathbb{R}^n, \mathfrak{C}$  complete, so  $\mathfrak{C}/I$  is a complete  $\mathfrak{C}$ -module by Proposition 5.27. Also we have an exact sequence  $0 \rightarrow I \rightarrow \mathfrak{C} \rightarrow \mathfrak{C}/I$  with  $\mathfrak{C}, \mathfrak{C}/I$  complete, so  $I$  is a complete  $\mathfrak{C}$ -module.
- (d) Let  $\mathfrak{C}$  be complete and  $V$  be an infinite-dimensional  $\mathbb{R}$ -vector space. One can show that  $\mathfrak{C} \otimes_{\mathbb{R}} V$  is a complete  $\mathfrak{C}$ -module if and only if  $X$  is compact.

## 5.6. Cotangent sheaves of $C^\infty$ -schemes

We now define *cotangent sheaves*, the sheaf version of cotangent modules in §5.2.

DEFINITION 5.29. Let  $\underline{X} = (X, \mathcal{O}_X)$  be a  $C^\infty$ -ringed space. Define  $\mathcal{PT}^*\underline{X}$  to associate to each open  $U \subseteq X$  the cotangent module  $\Omega_{\mathcal{O}_X(U)}$  of Definition 5.3, regarded as a module over the  $C^\infty$ -ring  $\mathcal{O}_X(U)$ , and to each inclusion of open sets  $V \subseteq U \subseteq X$  the morphism of  $\mathcal{O}_X(U)$ -modules  $\Omega_{\rho_{UV}} : \Omega_{\mathcal{O}_X(U)} \rightarrow \Omega_{\mathcal{O}_X(V)}$  associated to the morphism of  $C^\infty$ -rings  $\rho_{UV} : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$ . Then as we want for (5.13) the following commutes:

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \Omega_{\mathcal{O}_X(U)} & \xrightarrow{\mu_{\mathcal{O}_X(U)}} & \Omega_{\mathcal{O}_X(U)} \\ \downarrow \rho_{UV} \times \Omega_{\rho_{UV}} & & \downarrow \Omega_{\rho_{UV}} \\ \mathcal{O}_X(V) \times \Omega_{\mathcal{O}_X(V)} & \xrightarrow{\mu_{\mathcal{O}_X(V)}} & \Omega_{\mathcal{O}_X(V)}. \end{array}$$

Using this and functoriality of cotangent modules  $\Omega_{\psi \circ \phi} = \Omega_\psi \circ \Omega_\phi$  in Definition 5.3, we see that  $\mathcal{PT}^*\underline{X}$  is a presheaf of  $\mathcal{O}_X$ -modules on  $\underline{X}$ . Define the *cotangent sheaf*  $T^*\underline{X}$  of  $\underline{X}$  to be the sheaf of  $\mathcal{O}_X$ -modules associated to  $\mathcal{PT}^*\underline{X}$ .

If  $U \subseteq X$  is open then we have an equality of sheaves of  $\mathcal{O}_X|_U$ -modules

$$T^*(U, \mathcal{O}_X|_U) = T^*\underline{X}|_U.$$

As in Example 5.4, if  $f : X \rightarrow Y$  is a smooth map of manifolds we have a morphism  $df : f^*(T^*Y) \rightarrow T^*X$  of vector bundles over  $X$ . Here is an analogue for  $C^\infty$ -ringed spaces. Let  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  be a morphism of  $C^\infty$ -ringed spaces. Then by Definition 5.13,  $\underline{f}^*(T^*\underline{Y}) = f^{-1}(T^*\underline{Y}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$ , where  $T^*\underline{Y}$  is the sheafification of the presheaf  $V \mapsto \Omega_{\mathcal{O}_Y(V)}$ , and  $f^{-1}(T^*\underline{Y})$  the sheafification of the presheaf  $U \mapsto \lim_{V \supseteq f(U)} (T^*\underline{Y})(V)$ , and  $f^{-1}(\mathcal{O}_Y)$  the sheafification of the presheaf  $U \mapsto \lim_{V \supseteq f(U)} \mathcal{O}_Y(V)$ . These three sheafifications combine into one, so that  $\underline{f}^*(T^*\underline{Y})$  is the sheafification of the presheaf  $\mathcal{P}(\underline{f}^*(T^*\underline{Y}))$  acting by

$$U \mapsto \mathcal{P}(\underline{f}^*(T^*\underline{Y}))(U) = \lim_{V \supseteq f(U)} \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U).$$

Define a morphism of presheaves  $\mathcal{P}\Omega_{\underline{f}} : \mathcal{P}(\underline{f}^*(T^*\underline{Y})) \rightarrow \mathcal{PT}^*\underline{X}$  on  $X$  by

$$(\mathcal{P}\Omega_{\underline{f}})(U) = \lim_{V \supseteq f(U)} (\Omega_{\rho_{f^{-1}(V)U}} \circ f_{\#}(V))^*,$$

where  $(\Omega_{\rho_{f^{-1}(V)U}} \circ f_{\#}(V))^* : \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U) \rightarrow \Omega_{\mathcal{O}_X(U)} = (\mathcal{PT}^*\underline{X})(U)$  is constructed as in Definition 5.3 from the  $C^\infty$ -ring morphisms  $f_{\#}(V) : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(f^{-1}(V))$  from  $f_{\#} : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$  corresponding to  $f^{\#}$  in  $\underline{f}$  as in (4.3), and  $\rho_{f^{-1}(V)U} : \mathcal{O}_X(f^{-1}(V)) \rightarrow \mathcal{O}_X(U)$  in  $\mathcal{O}_X$ . Define  $\Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \rightarrow T^*\underline{X}$  to be the induced morphism of the associated sheaves.

REMARK 5.30. There is an alternative definition of the cotangent sheaf  $T^*\underline{X}$  following Hartshorne [31, p. 175]. We can form the product  $\underline{X} \times \underline{X}$  in  $\mathbf{C}^\infty\mathbf{RS}$ , and there is a natural diagonal morphism  $\underline{\Delta}_X : \underline{X} \rightarrow \underline{X} \times \underline{X}$ . Write  $\mathcal{I}_X$  for the sheaf of ideals in  $\mathcal{O}_{X \times X}$  vanishing on the closed  $C^\infty$ -ringed subspace  $\underline{\Delta}_X$ . Then  $T^*\underline{X} \cong \underline{\Delta}_X^*(\mathcal{I}_X/\mathcal{I}_X^2)$ . This can be proved using the equivalence of two definitions of cotangent module in [31, Prop. II.8.1A]. An affine version of this also appears in Dubuc and Kock [25].

PROPOSITION 5.31. Let  $\mathfrak{C}$  be a  $C^\infty$ -ring and  $\underline{X} = \text{Spec } \mathfrak{C}$ . Then there is a canonical isomorphism  $T^*\underline{X} \cong \text{MSpec } \Omega_{\mathfrak{C}}$ .

PROOF. By Definitions 5.16 and 5.29,  $\text{MSpec } \Omega_{\mathfrak{C}}$  and  $T^*\underline{X}$  are sheafifications of presheaves  $\mathcal{P}\text{MSpec } \Omega_{\mathfrak{C}}, \mathcal{PT}^*\underline{X}$ , where for open  $U \subseteq X$  we have

$$\mathcal{P}\text{MSpec } \Omega_{\mathfrak{C}}(U) = \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathcal{O}_X(U) \quad \text{and} \quad \mathcal{PT}^*\underline{X}(U) = \Omega_{\mathcal{O}_X(U)}.$$



We have  $C^\infty$ -ring morphisms  $\Psi_{\mathfrak{C}} : \mathfrak{C} \rightarrow \mathcal{O}_X(X)$  from Definition 4.19 and restriction  $\rho_{XU} : \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U)$  from  $\mathcal{O}_X$ , and so as in Definition 5.3 a morphism of  $\mathcal{O}_X(U)$ -modules  $\mathcal{P}\rho(U) := (\rho_{XU} \circ \Psi_{\mathfrak{C}})_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathcal{O}_X(U) \rightarrow \Omega_{\mathcal{O}_X(U)}$ . This defines a morphism of presheaves  $\mathcal{P}\rho : \mathcal{P}\text{MSpec } \Omega_{\mathfrak{C}} \rightarrow \mathcal{P}T^*\underline{X}$ , and so sheafifying induces a morphism  $\rho : \text{MSpec } \Omega_{\mathfrak{C}} \rightarrow T^*\underline{X}$ .

The induced morphism on stalks at  $x \in X$  is  $\rho_x = (\pi_x)_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{C}_x \rightarrow \Omega_{\mathfrak{C}_x}$ , where  $\pi_x : \mathfrak{C} \rightarrow \mathfrak{C}_x$  is projection to the local  $C^\infty$ -ring  $\mathfrak{C}_x$ , noting that  $\mathcal{O}_{X,x} \cong \mathfrak{C}_x$ . But  $\mathfrak{C}_x$  is the localization  $\mathfrak{C}[c^{-1} : c \in \mathfrak{C}, c(x) \neq 0]$ , so Proposition 5.7 implies that  $(\pi_x)_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{C}_x \rightarrow \Omega_{\mathfrak{C}_x}$  is an isomorphism. Hence  $\rho : \text{MSpec } \Omega_{\mathfrak{C}} \rightarrow T^*\underline{X}$  is a sheaf morphism which induces isomorphisms on stalks at all  $x \in X$ , so  $\rho$  is an isomorphism.  $\square$

Here are some properties of the morphisms  $\Omega_{\underline{f}}$  in Definition 5.29. Equation (5.20) is an analogue of (5.6) and (5.12).

**THEOREM 5.32. (a)** *Let  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  and  $\underline{g} : \underline{Y} \rightarrow \underline{Z}$  be morphisms of  $C^\infty$ -schemes. Then*

$$\Omega_{\underline{g} \circ \underline{f}} = \Omega_{\underline{f}} \circ \underline{f}^*(\Omega_{\underline{g}}) \circ I_{\underline{f}, \underline{g}}(T^*\underline{Z}) \quad (5.18)$$

as morphisms  $(\underline{g} \circ \underline{f})^*(T^*\underline{Z}) \rightarrow T^*\underline{X}$  in  $\text{qcoh}(\underline{X})$ . Here  $\Omega_{\underline{g}} : \underline{g}^*(T^*\underline{Z}) \rightarrow T^*\underline{Y}$  is a morphism in  $\text{qcoh}(\underline{Y})$ , so applying  $\underline{f}^*$  gives  $\underline{f}^*(\Omega_{\underline{g}}) : \underline{f}^*(\underline{g}^*(T^*\underline{Z})) \rightarrow \underline{f}^*(T^*\underline{Y})$  in  $\text{qcoh}(\underline{X})$ , and  $I_{\underline{f}, \underline{g}}(T^*\underline{Z}) : (\underline{g} \circ \underline{f})^*(T^*\underline{Z}) \rightarrow \underline{f}^*(\underline{g}^*(T^*\underline{Z}))$  is as in Remark 5.14.

**(b)** *Suppose we are given a Cartesian square in  $\mathbf{C}^\infty\text{Sch}$*

$$\begin{array}{ccc} \underline{W} & \xrightarrow{\quad \quad} & \underline{Y} \\ \downarrow \underline{e} & \begin{array}{c} \underline{f} \\ \underline{g} \end{array} & \downarrow \underline{h} \\ \underline{X} & \xrightarrow{\quad \quad} & \underline{Z}, \end{array} \quad (5.19)$$

so that  $\underline{W} = \underline{X} \times_{\underline{Z}} \underline{Y}$ . Then the following is exact in  $\text{qcoh}(\underline{W})$ :

$$(g \circ e)^*(T^*\underline{Z}) \xrightarrow{\begin{array}{c} \underline{e}^*(\Omega_{\underline{g}}) \circ I_{\underline{e}, \underline{g}}(T^*\underline{Z}) \oplus \\ -\underline{f}^*(\Omega_{\underline{h}}) \circ I_{\underline{f}, \underline{h}}(T^*\underline{Z}) \end{array}} \oplus \underline{f}^*(T^*\underline{Y}) \xrightarrow{\Omega_{\underline{e}} \oplus \Omega_{\underline{f}}} T^*\underline{W} \rightarrow 0. \quad (5.20)$$

**PROOF.** Combining several sheafifications into one as in the proof of Proposition 5.24, we see that the sheaves  $T^*\underline{X}$ ,  $\underline{f}^*(T^*\underline{Y})$ ,  $\underline{f}^*(\underline{g}^*(T^*\underline{Z}))$  and  $(\underline{g} \circ \underline{f})^*(T^*\underline{Z})$  on  $\underline{X}$  are isomorphic to the sheafifications of the following presheaves:

$$T^*\underline{X} \quad \rightsquigarrow \quad U \mapsto \Omega_{\mathcal{O}_X(U)}, \quad (5.21)$$

$$\underline{f}^*(T^*\underline{Y}) \quad \rightsquigarrow \quad U \mapsto \lim_{V \supseteq f(U)} \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U), \quad (5.22)$$

$$\underline{f}^*(\underline{g}^*(T^*\underline{Z})) \quad \rightsquigarrow \quad U \mapsto \lim_{V \supseteq f(U)} \lim_{W \supseteq g(V)} (\Omega_{\mathcal{O}_Z(W)} \otimes_{\mathcal{O}_Z(W)} \mathcal{O}_Y(V)) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U), \quad (5.23)$$

$$(\underline{g} \circ \underline{f})^*(T^*\underline{Z}) \quad \rightsquigarrow \quad U \mapsto \lim_{W \supseteq g \circ f(U)} \Omega_{\mathcal{O}_Z(W)} \otimes_{\mathcal{O}_Z(W)} \mathcal{O}_X(U). \quad (5.24)$$

Then  $\Omega_{\underline{f}}, \Omega_{g \circ \underline{f}}, \underline{f}^*(\Omega_g), I_{\underline{f}, g}(T^*\underline{Z})$  are the morphisms of sheaves associated to the following morphisms of the presheaves in (5.21)–(5.24):

$$\Omega_{\underline{f}} \rightsquigarrow U \longmapsto \lim_{V \supseteq f(U)} (\Omega_{\rho_{f^{-1}(V)} U \circ f_\#(V)})_*, \quad (5.25)$$

$$\Omega_{g \circ \underline{f}} \rightsquigarrow U \longmapsto \lim_{W \supseteq g \circ f(U)} (\Omega_{\rho_{(g \circ f)^{-1}(W)} U \circ (g \circ f)_\#(W)})_*, \quad (5.26)$$

$$\underline{f}^*(\Omega_g) \rightsquigarrow U \longmapsto \lim_{V \supseteq f(U)} \lim_{W \supseteq g(V)} (\Omega_{\rho_{g^{-1}(W)} V \circ g_\#(W)})_*, \quad (5.27)$$

$$I_{\underline{f}, g}(T^*\underline{Z}) \rightsquigarrow U \longmapsto \lim_{V \supseteq f(U)} \lim_{W \supseteq g(V)} I_{UVW}, \quad (5.28)$$

by Definition 5.29, where  $I_{UVW} : \Omega_{\mathcal{O}_Z(W)} \otimes_{\mathcal{O}_Z(W)} \mathcal{O}_X(U) \rightarrow (\Omega_{\mathcal{O}_Z(W)} \otimes_{\mathcal{O}_Z(W)} \mathcal{O}_Y(V)) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U)$  is the natural isomorphism.

Now if  $U \subseteq X$ ,  $V \subseteq Y$ ,  $W \subseteq Z$  are open with  $V \supseteq f(U)$ ,  $W \supseteq g(V)$  then

$$\rho_{(g \circ f)^{-1}(W)} U \circ (g \circ f)_\#(W) = [\rho_{f^{-1}(V)} U \circ f_\#(V)] \circ [\rho_{g^{-1}(W)} V \circ g_\#(W)]$$

as morphisms  $\mathcal{O}_Z(W) \rightarrow \mathcal{O}_X(U)$ , so  $\Omega_{\phi \circ \psi} = \Omega_\phi \circ \Omega_\psi$  in Definition 5.3 implies

$$(\Omega_{\rho_{(g \circ f)^{-1}(W)} U \circ (g \circ f)_\#(W)})_* = (\Omega_{\rho_{f^{-1}(V)} U \circ f_\#(V)})_* \circ (\Omega_{\rho_{g^{-1}(W)} V \circ g_\#(W)})_* \circ I_{UVW}.$$

Taking limits  $\lim_{V \supseteq f(U)} \lim_{W \supseteq g(V)}$  implies that the morphisms of presheaves in (5.25)–(5.28) satisfy the analogue of (5.18), so passing to sheaves proves (a).

For (b), first observe that as (5.19) is commutative, by (a) we have

$$\Omega_{\underline{e}} \circ \underline{e}^*(\Omega_g) \circ I_{\underline{e}, g}(T^*\underline{Z}) = \Omega_{g \circ \underline{e}} = \Omega_{h \circ \underline{f}} = \Omega_{\underline{f}} \circ \underline{f}^*(\Omega_h) \circ I_{\underline{f}, h}(T^*\underline{Z}),$$

$$\text{so } \Omega_{\underline{e}} \circ (\underline{e}^*(\Omega_g) \circ I_{\underline{e}, g}(T^*\underline{Z})) - \Omega_{\underline{f}} \circ (\underline{f}^*(\Omega_h) \circ I_{\underline{f}, h}(T^*\underline{Z})) = 0,$$

and (5.20) is a complex. To show it is exact, note that as in the first part of the proof, (5.20) is the sheafification of a complex of presheaves, and the presheaves are defined as direct limits. Let  $S \subseteq W$  be open. Then the complex of presheaves corresponding to (5.20) evaluated at  $S \subseteq W$  is the direct limit over all open  $T \subseteq X$ ,  $U \subseteq Y$ ,  $V \subseteq Z$  with  $e(S) \subseteq T$ ,  $f(S) \subseteq U$ ,  $g(T) \subseteq V$ ,  $h(U) \subseteq V$  of equation (5.6) with  $\mathcal{O}_Z(V), \mathcal{O}_X(T), \mathcal{O}_Y(U), \mathcal{O}_W(S)$  in place of  $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}$ .

Since (5.6) is exact by Theorem 5.8 and direct limits are exact, the complex of presheaves whose sheafification is (5.20) is exact when evaluated on each open  $S \subseteq W$ , so it is exact. As sheafification is an exact functor, this implies that equation (5.20) is exact. This completes the proof.  $\square$

## CHAPTER 6

### $C^\infty$ -stacks

We now discuss  $C^\infty$ -stacks, that is, geometric stacks over the site  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$  of  $C^\infty$ -schemes with the open cover topology. The author knows of no previous work on these. For the rest of the book, we will assume the reader has some familiarity with stacks in algebraic geometry. Appendix A summarizes the main definitions and results on stacks that we will use, but it is too brief to help someone learn about stacks for the first time. Readers with little experience of stacks are advised to first consult an introductory text such as Olsson [57], Vistoli [68], Gomez [29], Laumon and Moret-Bailly [46], or the online ‘Stacks Project’ [34].

The author found Metzler [49] and Noohi [55] useful in writing this section.

#### 6.1. $C^\infty$ -stacks

We use the material of §A.2–§A.5.

**DEFINITION 6.1.** Define a Grothendieck pretopology  $\mathcal{PJ}$  on the category of  $C^\infty$ -schemes  $\mathbf{C}^\infty\mathbf{Sch}$  to have coverings  $\{\underline{i}_a : \underline{U}_a \rightarrow \underline{U}\}_{a \in A}$  where  $V_a = i_a(U_a)$  is open in  $U$  with  $\underline{i}_a : \underline{U}_a \rightarrow (V_a, \mathcal{O}_U|_{V_a})$  and isomorphism for all  $a \in A$ , and  $U = \bigcup_{a \in A} V_a$ . Using Corollary 4.29 we see that up to isomorphisms of the  $\underline{U}_a$ , the coverings  $\{\underline{i}_a : \underline{U}_a \rightarrow \underline{U}\}_{a \in A}$  of  $\underline{U}$  correspond exactly to open covers  $\{V_a : a \in A\}$  of  $U$ . Write  $\mathcal{J}$  for the associated Grothendieck topology.

It is a straightforward exercise in sheaf theory to prove:

**PROPOSITION 6.2.** *The site  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$  has descent for objects and morphisms, in the sense of §A.3. Thus it is subcanonical.*

The point here is that since coverings of  $\underline{U}$  in  $\mathcal{J}$  are just open covers of the underlying topological space  $U$ , rather than something more complicated like étale covers in algebraic geometry, proving descent is easy: for objects, we glue the topological spaces  $X_a$  of  $\underline{X}_a$  together in the usual way to get a topological space  $X$ , then we glue the  $\mathcal{O}_{X_a}$  together to get a presheaf of  $C^\infty$ -rings  $\tilde{\mathcal{O}}_X$  on  $X$  isomorphic to  $\mathcal{O}_{X_a}$  on  $X_a \subseteq X$  for all  $a \in A$ , and finally we sheafify  $\tilde{\mathcal{O}}_X$  to a sheaf of  $C^\infty$ -rings  $\mathcal{O}_X$  on  $X$ , which is still isomorphic to  $\mathcal{O}_{X_a}$  on  $X_a \subseteq X$ .

**DEFINITION 6.3.** A  $C^\infty$ -stack  $\mathcal{X}$  is a geometric stack on the site  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$ . Write  $\mathbf{C}^\infty\mathbf{Sta}$  for the 2-category of  $C^\infty$ -stacks,  $\mathbf{C}^\infty\mathbf{Sta} = \mathbf{GSta}_{(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})}$ .

As in Definition A.13, we will very often use the notation that if  $\underline{X}$  is a  $C^\infty$ -scheme then  $\bar{X}$  is the associated  $C^\infty$ -stack, and if  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  is a morphism of  $C^\infty$ -schemes then  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  is the associated 1-morphism of  $C^\infty$ -stacks. Write  $\bar{\mathbf{C}}^\infty\mathbf{Sch}^{\text{lfp}}, \bar{\mathbf{C}}^\infty\mathbf{Sch}^{\text{lf}}, \bar{\mathbf{C}}^\infty\mathbf{Sch}$  for the full 2-subcategories of  $C^\infty$ -stacks  $\mathcal{X}$  in  $\mathbf{C}^\infty\mathbf{Sta}$  which are equivalent to  $\bar{X}$  for  $\underline{X}$  in  $\mathbf{C}^\infty\mathbf{Sch}^{\text{lfp}}, \mathbf{C}^\infty\mathbf{Sch}^{\text{lf}}$  or

$\mathbf{C}^\infty\mathbf{Sch}$ , respectively. When we say that a  $C^\infty$ -stack  $\mathcal{X}$  is a  $C^\infty$ -scheme, we mean that  $\mathcal{X} \in \bar{\mathbf{C}}^\infty\mathbf{Sch}$ .

Since  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{I})$  is a subcanonical site, the embedding  $\mathbf{C}^\infty\mathbf{Sch} \rightarrow \mathbf{C}^\infty\mathbf{Sta}$  taking  $\underline{X} \mapsto \bar{\underline{X}}$ ,  $\underline{f} \mapsto \bar{\underline{f}}$  is fully faithful. We write this as a full and faithful functor  $F_{\mathbf{C}^\infty\mathbf{Sch}}^{\mathbf{C}^\infty\mathbf{Sta}} : \mathbf{C}^\infty\mathbf{Sch} \rightarrow \mathbf{C}^\infty\mathbf{Sta}$  mapping  $F_{\mathbf{C}^\infty\mathbf{Sch}}^{\mathbf{C}^\infty\mathbf{Sta}} : \underline{X} \mapsto \bar{\underline{X}}$  on objects and  $F_{\mathbf{C}^\infty\mathbf{Sch}}^{\mathbf{C}^\infty\mathbf{Sta}} : \underline{f} \mapsto \bar{\underline{f}}$  on (1-)morphisms. Hence  $\mathbf{C}^\infty\mathbf{Sch}^{\text{lfp}}, \bar{\mathbf{C}}^\infty\mathbf{Sch}^{\text{lfp}}, \mathbf{C}^\infty\mathbf{Sch}$  are equivalent to  $\mathbf{C}^\infty\mathbf{Sch}^{\text{lfp}}, \mathbf{C}^\infty\mathbf{Sch}^{\text{lf}}, \mathbf{C}^\infty\mathbf{Sch}$ , considered as 2-categories with only identity 2-morphisms. In practice one often does not distinguish between schemes and stacks which are equivalent to schemes, that is, one identifies  $\mathbf{C}^\infty\mathbf{Sch}^{\text{lfp}}, \dots, \mathbf{C}^\infty\mathbf{Sch}$  and  $\bar{\mathbf{C}}^\infty\mathbf{Sch}^{\text{lfp}}, \dots, \bar{\mathbf{C}}^\infty\mathbf{Sch}$ .

REMARK 6.4. Behrend and Xu [5, Def. 2.15] use ‘ $C^\infty$ -stack’ to mean something different, a stack  $X$  over the site  $(\mathbf{Man}, \mathcal{I}_{\mathbf{Man}})$  of manifolds with Grothendieck topology  $\mathcal{I}_{\mathbf{Man}}$  associated to the Grothendieck pretopology  $\mathcal{P}\mathcal{I}_{\mathbf{Man}}$  given by open covers, such that there exists a surjective representable submersion  $\pi : \bar{U} \rightarrow X$  from some manifold  $U$ . These are also called ‘smooth stacks’ or ‘differentiable stacks’ in [5, 32, 49, 55]. The quotient  $[V/G]$  of a manifold  $V$  by a Lie group  $G$  is an example of a differentiable stack. By Zung’s linearization theorem [71, Th. 2.3], a differentiable stack  $\mathcal{X}$  with proper diagonal is Zariski locally equivalent to such a quotient  $[V/G]$  with  $G$  compact. Our  $C^\infty$ -stacks are a far larger class of more singular objects than the differentiable stacks of [5, 32, 49, 55].

Theorems 4.25(b) and A.23, Corollary A.26 and Proposition 6.2 imply:

THEOREM 6.5. *Let  $\mathcal{X}$  be a  $C^\infty$ -stack. Then  $\mathcal{X}$  is equivalent to the stack  $[\underline{V} \rightrightarrows \underline{U}]$  associated to a groupoid  $(\underline{U}, \underline{V}, s, t, u, i, \underline{m})$  in  $\mathbf{C}^\infty\mathbf{Sch}$ . Conversely, any groupoid in  $\mathbf{C}^\infty\mathbf{Sch}$  defines a  $C^\infty$ -stack  $[\underline{V} \rightrightarrows \underline{U}]$ . All fibre products exist in the 2-category  $\mathbf{C}^\infty\mathbf{Sta}$ .*

Quotient  $C^\infty$ -stacks  $[\underline{X}/\underline{G}]$  are a special class of  $C^\infty$ -stacks.

DEFINITION 6.6. A  $C^\infty$ -group  $\underline{G}$  is a group object in  $\mathbf{C}^\infty\mathbf{Sch}$ , that is, a  $C^\infty$ -scheme  $\underline{G} = (G, \mathcal{O}_G)$  equipped with an identity element  $1 \in G$  and multiplication and inverse morphisms  $\underline{m} : \underline{G} \times \underline{G} \rightarrow \underline{G}$ ,  $\underline{i} : \underline{G} \rightarrow \underline{G}$  in  $\mathbf{C}^\infty\mathbf{Sch}$  such that  $(\ast, \underline{G}, \pi, \pi, 1, \underline{i}, \underline{m})$  is a groupoid in  $\mathbf{C}^\infty\mathbf{Sch}$ . Here  $\ast = \text{Spec } \mathbb{R}$  is a point, and  $\pi : \underline{G} \rightarrow \ast$  is the projection, and we regard  $1 \in G$  as a morphism  $1 : \ast \rightarrow \underline{G}$ .

Let  $\underline{G}$  be a  $C^\infty$ -group, and  $\underline{X}$  a  $C^\infty$ -scheme. A (left) action of  $\underline{G}$  on  $\underline{X}$  is a morphism  $\underline{\mu} : \underline{G} \times \underline{X} \rightarrow \underline{X}$  such that

$$(\underline{X}, \underline{G} \times \underline{X}, \pi_{\underline{X}}, \underline{\mu}, 1 \times \text{id}_{\underline{X}}, (\underline{i} \circ \pi_{\underline{G}}) \times \underline{\mu}, (\underline{m} \circ ((\pi_{\underline{G}} \circ \pi_1) \times (\pi_{\underline{G}} \circ \pi_2))) \times (\pi_{\underline{X}} \circ \pi_2)) \quad (6.1)$$

is a groupoid object in  $\mathbf{C}^\infty\mathbf{Sch}$ , where in the final morphism  $\pi_1, \pi_2$  are the projections from  $(\underline{G} \times \underline{X}) \times_{\pi_{\underline{X}}, \underline{X}, \underline{\mu}} (\underline{G} \times \underline{X})$  to the first and second factors  $\underline{G} \times \underline{X}$ . Then define the quotient  $C^\infty$ -stack  $[\underline{X}/\underline{G}]$  to be the stack  $[\underline{G} \times \underline{X} \rightrightarrows \underline{X}]$  associated to the groupoid (6.1). It is a  $C^\infty$ -stack.

If  $\underline{G} = (G, \mathcal{O}_G)$  is a  $C^\infty$ -group then the underlying space  $G$  is a topological group, and is in particular a group, and if  $\underline{G} = (G, \mathcal{O}_G)$  acts on  $\underline{X} = (X, \mathcal{O}_X)$  then  $G$  acts continuously on  $X$ .

If  $G$  is a Lie group then  $\underline{G} = F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}(G)$  is a  $C^\infty$ -group in a natural way, by applying  $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}$  to the smooth multiplication and inverse maps  $m : G \times G \rightarrow G$  and  $i : G \rightarrow G$ . If a Lie group  $G$  acts smoothly on a manifold  $X$  with action  $\mu : G \times X \rightarrow X$  then the  $C^\infty$ -group  $\underline{G} = F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}(G)$  acts on the  $C^\infty$ -scheme

$\underline{X} = F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}(X)$  with action  $\underline{\mu} = F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}(\mu) : \underline{G} \times \underline{X} \rightarrow \underline{X}$ , so we can form the quotient  $C^\infty$ -stack  $[\underline{X}/\underline{G}]$ .

EXAMPLE 6.7. Let  $\underline{G}$  be a  $C^\infty$ -group, and  $\underline{X} = *$  be the point in  $\mathbf{C}^\infty\mathbf{Sch}$ , with trivial  $\underline{G}$ -action. The quotient  $C^\infty$ -stack  $[\ast/\underline{G}]$  is known as  $B\underline{G}$ , the classifying stack for principal  $\underline{G}$ -bundles on  $C^\infty$ -schemes.

If  $\underline{S}$  is a  $C^\infty$ -scheme, a *principal  $\underline{G}$ -bundle*  $(\underline{P}, \underline{\pi}, \underline{\mu})$  over  $\underline{S}$  is a  $C^\infty$ -scheme  $\underline{P}$ , a morphism  $\underline{\pi} : \underline{P} \rightarrow \underline{S}$ , and a  $\underline{G}$ -action  $\underline{\mu} : \underline{G} \times \underline{P} \rightarrow \underline{P}$  of  $\underline{G}$  on  $\underline{P}$ , such that  $\underline{\pi}$  is  $\underline{G}$ -invariant, and  $\underline{S}$  may be covered by open  $C^\infty$ -subschemes  $\underline{U} \subseteq \underline{S}$  such that there exists an isomorphism  $\underline{\pi}^{-1}(\underline{U}) \cong \underline{G} \times \underline{U}$  which identifies the  $\underline{G}$ -action on  $\underline{\pi}^{-1}(\underline{U}) \subseteq \underline{P}$  with the product of the left  $\underline{G}$ -action on  $\underline{G}$  and the trivial  $\underline{G}$ -action on  $\underline{U}$ , and identifies  $\underline{\pi}|_{\dots} : \underline{\pi}^{-1}(\underline{U}) \rightarrow \underline{U}$  with  $\underline{\pi}_{\underline{U}} : \underline{G} \times \underline{U} \rightarrow \underline{U}$ . Often we write  $\underline{P}$  as the principal bundle, leaving  $\underline{\pi}, \underline{\mu}$  implicit.

One well known way to write  $B\underline{G}$  explicitly as a category fibred in groupoids  $p_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{C}^\infty\mathbf{Sch}$ , as in §A.2, is to define  $\mathcal{X}$  to be the category with objects pairs  $(\underline{S}, \underline{P})$  of a  $C^\infty$ -scheme  $\underline{S}$  and  $\underline{P}$  a principal  $\underline{G}$ -bundle over  $\underline{S}$ , and morphisms  $(\underline{f}, \underline{u}) : (\underline{S}, \underline{P}) \rightarrow (\underline{T}, \underline{Q})$  consisting of  $C^\infty$ -scheme morphisms  $\underline{f} : \underline{S} \rightarrow \underline{T}$  and  $\underline{u} : \underline{P} \rightarrow \underline{Q}$ , such that  $\underline{u}$  is  $\underline{G}$ -equivariant and

$$\begin{array}{ccc} \underline{P} & \xrightarrow{\quad \underline{u} \quad} & \underline{Q} \\ \downarrow \underline{\pi} & & \downarrow \underline{\pi} \\ \underline{S} & \xrightarrow{\quad \underline{f} \quad} & \underline{T} \end{array} \quad (6.2)$$

is a Cartesian square in  $\mathbf{C}^\infty\mathbf{Sch}$ , which implies that  $\underline{P}$  is canonically isomorphic to the pullback principal  $\underline{G}$ -bundle  $\underline{f}^*(\underline{Q})$ . Composition of morphisms is  $(\underline{g}, \underline{v}) \circ (\underline{f}, \underline{u}) = (\underline{g} \circ \underline{f}, \underline{v} \circ \underline{u})$ , and identity morphisms are  $\text{id}_{(\underline{S}, \underline{P})} = (\underline{\text{id}}_{\underline{S}}, \underline{\text{id}}_{\underline{P}})$ . The functor  $p_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{C}^\infty\mathbf{Sch}$  maps  $p_{\mathcal{X}} : (\underline{S}, \underline{P}) \mapsto \underline{S}$  on objects and  $p_{\mathcal{X}} : (\underline{f}, \underline{u}) \mapsto \underline{f}$  on morphisms.

In §7.1 we will give a more detailed treatment of quotient  $C^\infty$ -stacks  $[\underline{X}/\underline{G}]$  of a  $C^\infty$ -scheme  $\underline{X}$  by a finite group  $\underline{G}$ .

## 6.2. Properties of 1-morphisms of $C^\infty$ -stacks

We use the material of §A.4. We define some classes of  $C^\infty$ -scheme morphisms.

DEFINITION 6.8. Let  $\underline{f} = (f, f^\#) : \underline{X} = (X, \mathcal{O}_X) \rightarrow \underline{Y} = (Y, \mathcal{O}_Y)$  be a morphism in  $\mathbf{C}^\infty\mathbf{Sch}$ . Then:

- We call  $\underline{f}$  an *open embedding* if  $V = f(X)$  is an open subset in  $Y$  and  $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (V, \mathcal{O}_Y|_V)$  is an isomorphism.
- We call  $\underline{f}$  a *closed embedding* if  $f : X \rightarrow Y$  is a homeomorphism with a closed subset of  $Y$ , and  $f^\# : f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$  is a surjective morphism of sheaves of  $C^\infty$ -rings. Equivalently,  $\underline{f}$  is an isomorphism with a closed  $C^\infty$ -subscheme of  $\underline{Y}$ . Over affine open subsets  $\underline{U} \cong \text{Spec } \mathfrak{C}$  in  $\underline{Y}$ ,  $\underline{f}$  is modelled on the natural morphism  $\text{Spec}(\mathfrak{C}/I) \hookrightarrow \text{Spec } \mathfrak{C}$  for some ideal  $I$  in  $\mathfrak{C}$ .
- We call  $\underline{f}$  an *embedding* if we may write  $\underline{f} = \underline{g} \circ \underline{h}$  where  $\underline{h}$  is an open embedding and  $\underline{g}$  is a closed embedding.
- We call  $\underline{f}$  *étale* if each  $x \in X$  has an open neighbourhood  $U$  in  $X$  such that  $V = f(U)$  is open in  $Y$  and  $(f|_U, f^\#|_U) : (U, \mathcal{O}_X|_U) \rightarrow (V, \mathcal{O}_Y|_V)$  is an isomorphism. That is,  $\underline{f}$  is a local isomorphism.

- We call  $\underline{f}$  *proper* if  $f : X \rightarrow Y$  is a proper map of topological spaces, that is, if  $S \subseteq Y$  is compact then  $f^{-1}(S) \subseteq X$  is compact.
- We say that  $\underline{f}$  *has finite fibres* if  $f : X \rightarrow Y$  is a finite map, that is,  $f^{-1}(y)$  is a finite subset of  $X$  for all  $y \in Y$ .
- We call  $\underline{f}$  *separated* if  $f : X \rightarrow Y$  is a separated map of topological spaces, that is,  $\Delta_X = \{(x, x) : x \in X\}$  is a closed subset of the topological fibre product  $X \times_{f,Y,f} X = \{(x, x') \in X \times X : f(x) = f(x')\}$ .
- We call  $\underline{f}$  *closed* if  $f : X \rightarrow Y$  is a closed map of topological spaces, that is,  $S \subseteq X$  closed implies  $f(S) \subseteq Y$  closed.
- We call  $\underline{f}$  *universally closed* if whenever  $\underline{g} : \underline{W} \rightarrow \underline{Y}$  is a morphism then  $\pi_{\underline{W}} : \underline{X} \times_{\underline{f}, \underline{Y}, \underline{g}} \underline{W} \rightarrow \underline{W}$  is closed.
- We call  $\underline{f}$  a *submersion* if for all  $x \in X$  with  $f(x) = y$ , there exists an open neighbourhood  $U$  of  $y$  in  $Y$  and a morphism  $\underline{g} = (g, g^\sharp) : (U, \mathcal{O}_Y|_U) \rightarrow (X, \mathcal{O}_X)$  with  $g(y) = x$  and  $\underline{f} \circ \underline{g} = \text{id}_{(U, \mathcal{O}_Y|_U)}$ .
- We call  $\underline{f}$  *locally fair*, or *locally finitely presented*, if whenever  $\underline{U}$  is a locally fair, or locally finitely presented  $C^\infty$ -scheme, respectively, and  $\underline{g} : \underline{U} \rightarrow \underline{Y}$  is a morphism then  $\underline{X} \times_{\underline{f}, \underline{Y}, \underline{g}} \underline{U}$  is locally fair, or locally finitely presented, respectively.

REMARK 6.9. These are mostly analogues of standard concepts in algebraic geometry, as in Hartshorne [31] for instance. But because the topology on  $C^\infty$ -schemes is finer than the Zariski topology in algebraic geometry — for example, affine  $C^\infty$ -schemes are Hausdorff — our definitions of étale and proper are simpler than in algebraic geometry. (Open or closed) embeddings correspond to (open or closed) immersions in algebraic geometry, but we prefer the word ‘embedding’, as immersion has a different meaning in differential geometry. Closed morphisms are not invariant under base change, which is why we define universally closed. If  $X, Y$  are manifolds and  $\underline{X}, \underline{Y} = F_{\text{Man}}^{C^\infty \text{Sch}}(X, Y)$ , then  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  is a submersion of  $C^\infty$ -schemes if and only if  $\underline{f} = F_{\text{Man}}^{C^\infty \text{Sch}}(f)$  for  $f : X \rightarrow Y$  a submersion of manifolds.

DEFINITION 6.10. Let  $\mathbf{P}$  be a property of morphisms in  $\mathbf{C}^\infty \text{Sch}$ . We say that  $\mathbf{P}$  is *stable under open embedding* if whenever  $\underline{f} : \underline{U} \rightarrow \underline{V}$  is  $\mathbf{P}$  and  $\underline{i} : \underline{V} \rightarrow \underline{W}$  is an open embedding, then  $\underline{i} \circ \underline{f} : \underline{U} \rightarrow \underline{W}$  is  $\mathbf{P}$ .

The next proposition is elementary. See Laumon and Bailly [46, §3.10] and Noohi [55, Ex. 4.6] for similar lists for the étale and topological sites.

PROPOSITION 6.11. *The following properties of morphisms in  $\mathbf{C}^\infty \text{Sch}$  are invariant under base change and local in the target in the site  $(\mathbf{C}^\infty \text{Sch}, \mathcal{J})$ , in the sense of §A.4: open embedding, closed embedding, embedding, étale, proper, has finite fibres, separated, universally closed, submersion, locally fair, locally finitely presented. The following properties are also stable under open embedding, in the sense of Definition 6.10: open embedding, embedding, étale, has finite fibres, separated, submersion, locally fair, locally finitely presented.*

As in §A.4, this implies that these properties are also defined for representable 1-morphisms in  $\mathbf{C}^\infty \text{Sta}$ . In particular, if  $\mathcal{X}$  is a  $C^\infty$ -stack then  $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is representable, and if  $\Pi : \underline{\mathcal{U}} \rightarrow \mathcal{X}$  is an atlas then  $\Pi$  is representable, so we can require that  $\Delta_{\mathcal{X}}$  or  $\Pi$  has some of these properties.

DEFINITION 6.12. Let  $\mathcal{X}$  be a  $C^\infty$ -stack. Following [46, Def. 7.6], we say that  $\mathcal{X}$  is *separated* if the diagonal 1-morphism  $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is universally closed. If  $\mathcal{X} = \bar{X}$  for some  $C^\infty$ -scheme  $\bar{X} = (X, \mathcal{O}_X)$  then  $\mathcal{X}$  is separated if and only if  $\Delta_X : X \rightarrow X \times X$  is closed, that is, if and only if  $X$  is Hausdorff.

PROPOSITION 6.13. Let  $\mathcal{W} = \mathcal{X} \times_{f, \mathcal{Z}, g} \mathcal{Y}$  be a fibre product of  $C^\infty$ -stacks with  $\mathcal{X}, \mathcal{Y}$  separated. Then  $\mathcal{W}$  is separated.

PROOF. We have a 2-commutative diagram with both squares 2-Cartesian:

$$\begin{array}{ccccc}
 & \mathcal{W} & \xrightarrow{\Delta_{\mathcal{W}}} & \mathcal{W} \times \mathcal{W} & \\
 & \searrow \pi_1 & & \searrow \pi_2 & \\
 \mathcal{Z} & & \mathcal{X} \times_{f \circ \Delta_{\mathcal{Z}}, \mathcal{Z} \times \mathcal{Z}, g \circ \Delta_{\mathcal{Y}}} \mathcal{Y} & & \mathcal{X} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{Y} \\
 & \searrow j_{\mathcal{Z}} & & \searrow \Delta_{\mathcal{X}} \times \Delta_{\mathcal{Y}} & \\
 & I_{\mathcal{Z}} & & \mathcal{X} \times \mathcal{Y} & 
 \end{array} \quad (6.3)$$

Let  $[V \rightrightarrows U]$  be a groupoid presentation of  $\mathcal{Z}$ , and consider the fourth 2-Cartesian diagram of (A.12), with surjective rows. The left hand morphism  $\bar{u} \times \bar{u}_U$  has a left inverse  $\pi_U$ , and so is automatically universally closed. Hence  $j_{\mathcal{Z}}$  is universally closed by Propositions A.18(c) and 6.11, so  $\pi_1$  in (6.3) is universally closed by Propositions A.18(a) and 6.11. Also  $\Delta_{\mathcal{X}}, \Delta_{\mathcal{Y}}$  are universally closed as  $\mathcal{X}, \mathcal{Y}$  are separated, so  $\Delta_{\mathcal{X}} \times \Delta_{\mathcal{Y}}$  in (6.3) is universally closed, and  $\pi_2$  is universally closed. Thus  $\Delta_{\mathcal{W}} \cong \pi_2 \circ \pi_1$  is universally closed, and  $\mathcal{W}$  is separated.  $\square$

### 6.3. Open $C^\infty$ -substacks and open covers

DEFINITION 6.14. Let  $\mathcal{X}$  be a  $C^\infty$ -stack. A  $C^\infty$ -substack  $\mathcal{Y}$  in  $\mathcal{X}$  is a substack of  $\mathcal{X}$ , in the sense of Definition A.7, which is also a  $C^\infty$ -stack. It has a natural inclusion 1-morphism  $i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{X}$ . We call  $\mathcal{Y}$  an *open  $C^\infty$ -substack* of  $\mathcal{X}$  if  $i_{\mathcal{Y}}$  is a representable open embedding, a *closed  $C^\infty$ -substack* of  $\mathcal{X}$  if  $i_{\mathcal{Y}}$  is a representable closed embedding, and a *locally closed  $C^\infty$ -substack* of  $\mathcal{X}$  if  $i_{\mathcal{Y}}$  is a representable embedding.

An *open cover*  $\{\mathcal{U}_a : a \in A\}$  of  $\mathcal{X}$  is a family of open  $C^\infty$ -substacks  $\mathcal{U}_a$  in  $\mathcal{X}$  with  $\coprod_{a \in A} i_{\mathcal{U}_a} : \coprod_{a \in A} \mathcal{U}_a \rightarrow \mathcal{X}$  surjective. We write  $\mathcal{U} \subseteq \mathcal{X}$  when  $\mathcal{U}$  is an open  $C^\infty$ -substack of  $\mathcal{X}$ , and  $\bigcup_{a \in A} \mathcal{U} = \mathcal{X}$  to mean that  $\coprod_{a \in A} i_{\mathcal{U}_a}$  is surjective.

Some properties of  $\Delta_{\mathcal{X}}, \iota_{\mathcal{X}}, j_{\mathcal{X}}$  and atlases for  $\mathcal{X}$  can be tested on the elements of an open cover. The proof is elementary.

PROPOSITION 6.15. Let  $\mathcal{X}$  be a  $C^\infty$ -stack, and  $\{\mathcal{U}_a : a \in A\}$  an open cover of  $\mathcal{X}$ . Suppose  $\mathbf{P}$  and  $\mathbf{Q}$  are properties of morphisms in  $\mathbf{C}^\infty\mathbf{Sch}$  which are invariant under base change and local in the target in  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{I})$ , and that  $\mathbf{P}$  is stable under open embedding. Then:

- (a) Let  $\Pi_a : \bar{U}_a \rightarrow \mathcal{U}_a$  be an atlas for  $\mathcal{U}_a$  for  $a \in A$ . Set  $\bar{U} = \coprod_{a \in A} \bar{U}_a$  and  $\Pi = \coprod_{a \in A} i_{\mathcal{U}_a} \circ \Pi_a : \bar{U} \rightarrow \mathcal{X}$ . Then  $\Pi$  is an atlas for  $\mathcal{X}$ , and  $\Pi$  is  $\mathbf{P}$  if and only if  $\Pi_a$  is  $\mathbf{P}$  for all  $a \in A$ .
- (b)  $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is  $\mathbf{P}$  if and only if  $\Delta_{\mathcal{U}_a} : \mathcal{U}_a \rightarrow \mathcal{U}_a \times \mathcal{U}_a$  is  $\mathbf{P}$  for all  $a \in A$ .
- (c)  $\iota_{\mathcal{X}} : I_{\mathcal{X}} \rightarrow \mathcal{X}$  is  $\mathbf{Q}$  if and only if  $\iota_{\mathcal{U}_a} : I_{\mathcal{U}_a} \rightarrow \mathcal{U}_a$  is  $\mathbf{Q}$  for all  $a \in A$ .
- (d)  $j_{\mathcal{X}} : \mathcal{X} \rightarrow I_{\mathcal{X}}$  is  $\mathbf{Q}$  if and only if  $j_{\mathcal{U}_a} : \mathcal{U}_a \rightarrow I_{\mathcal{U}_a}$  is  $\mathbf{Q}$  for all  $a \in A$ .

If  $\mathcal{X} = \bar{U}$  for some  $C^\infty$ -scheme  $\bar{U} = (U, \mathcal{O}_U)$ , then the open  $C^\infty$ -substacks of  $\mathcal{X}$  are precisely those subsheaves of the form  $(V, \mathcal{O}_{U|V})$  for all open  $V \subseteq U$ , that

is, they are the images in  $\mathbf{C}^\infty\mathbf{Sta}$  of the open  $C^\infty$ -subschemes of  $U$ . We can also describe the open substacks of stacks  $[V \rightrightarrows U]$  associated to groupoids:

**PROPOSITION 6.16.** *Let  $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$  be a groupoid in  $\mathbf{C}^\infty\mathbf{Sch}$  and  $\mathcal{X} = [V \rightrightarrows U]$  the associated  $C^\infty$ -stack, and write  $\underline{U} = (U, \mathcal{O}_U)$ , and so on. Then open  $C^\infty$ -substacks  $\mathcal{X}'$  of  $\mathcal{X}$  are naturally in 1-1 correspondence with open subsets  $U' \subseteq U$  with  $s^{-1}(U') = t^{-1}(U')$ , where  $\mathcal{X}' = [V' \rightrightarrows U']$  for  $\underline{U}' = (U', \mathcal{O}_{U|U'})$  and  $\underline{V}' = (s^{-1}(U'), \mathcal{O}_{V|s^{-1}(U')})$ . If  $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$  is as in (6.1), so that  $\mathcal{X}$  is a quotient  $C^\infty$ -stack  $[\underline{U}/\underline{G}]$ , then open  $C^\infty$ -substacks  $\mathcal{X}'$  of  $\mathcal{X}$  correspond to  $G$ -invariant open subsets  $U' \subseteq U$ .*

**PROOF.** From Theorem A.23, as  $\mathcal{X} = [V \rightrightarrows U]$  we have a natural surjective, representable 1-morphism  $\Pi : \underline{U} \rightarrow \mathcal{X}$ . If  $\mathcal{X}'$  is an open  $C^\infty$ -substack of  $\mathcal{X}$  then  $\underline{U} \times_{\Pi, \mathcal{X}, i_{\mathcal{X}'}} \mathcal{X}'$  is an open  $C^\infty$ -substack of  $\underline{U}$ , and so is of the form  $(U', \mathcal{O}_{U|U'})$  for some open  $U' \subseteq U$ . We have natural equivalences

$$\begin{aligned} \overline{(s^{-1}(U'), \mathcal{O}_{V|s^{-1}(U')})} &\simeq \bar{U}' \times_{i_{\bar{U}'}, \bar{U}, \bar{s}} \bar{V} \simeq \mathcal{X}' \times_{\mathcal{X}} (\bar{U} \times_{i_{\bar{U}}, \bar{U}, \bar{s}} \bar{V}) \simeq \mathcal{X}' \times_{i'_{\mathcal{X}'}, \mathcal{X}, \pi_{\mathcal{X}}} \bar{V} \\ &\simeq \mathcal{X}' \times_{\mathcal{X}} (\bar{U} \times_{i_{\bar{U}}, \bar{U}, \bar{t}} \bar{V}) \simeq \bar{U}' \times_{i_{\bar{U}'}, \bar{U}, \bar{t}} \bar{V} \simeq \overline{(t^{-1}(U'), \mathcal{O}_{V|t^{-1}(U')})}, \end{aligned}$$

by associativity properties of fibre products in 2-categories, which implies that  $s^{-1}(U') = t^{-1}(U')$ . Conversely, if  $s^{-1}(U') = t^{-1}(U')$  then defining  $\underline{U}', \underline{V}'$  as in the proposition, we get a  $C^\infty$ -stack  $\mathcal{X}' = [V' \rightrightarrows U']$  which is naturally an open  $C^\infty$ -substack of  $\mathcal{X}$ . When  $\mathcal{X} = [\underline{U}/\underline{G}]$ , we see that  $s^{-1}(U') = t^{-1}(U')$  if and only if  $U'$  is  $G$ -invariant.  $\square$

#### 6.4. The underlying topological space of a $C^\infty$ -stack

Following Noohi [55, §4.3, §11] in the case of topological stacks, we associate a topological space  $\mathcal{X}_{\text{top}}$  to a  $C^\infty$ -stack  $\mathcal{X}$ . In §7.4, if  $\mathcal{X}$  is a Deligne–Mumford  $C^\infty$ -stack, we will also give  $\mathcal{X}_{\text{top}}$  the structure of a  $C^\infty$ -scheme.

**DEFINITION 6.17.** Let  $\mathcal{X}$  be a  $C^\infty$ -stack. Write  $\underline{x}$  for the point  $\text{Spec } \mathbb{R}$  in  $\mathbf{C}^\infty\mathbf{Sch}$ , and  $\bar{x}$  for the associated point in  $\mathbf{C}^\infty\mathbf{Sta}$ . Define  $\mathcal{X}_{\text{top}}$  to be the set of 2-isomorphism classes  $[x]$  of 1-morphisms  $x : \bar{x} \rightarrow \mathcal{X}$ .

Suppose  $\mathcal{U} \subseteq \mathcal{X}$  is an open  $C^\infty$ -substack. Since  $\mathcal{U}$  is a subcategory of  $\mathcal{X}$ , any 1-morphism  $u : \bar{x} \rightarrow \mathcal{U}$ , regarded as a functor from the category  $\bar{x}$  to the category  $\mathcal{U}$ , is also a 1-morphism  $u : \bar{x} \rightarrow \mathcal{X}$ . Also, as  $\mathcal{U}$  is a strictly full subcategory of  $\mathcal{X}$ , if  $x : \bar{x} \rightarrow \mathcal{X}$  is a 1-morphism and  $\eta : u \Rightarrow x$  a 2-morphism of 1-morphisms  $\bar{x} \rightarrow \mathcal{X}$ , then  $x$  is also a 1-morphism  $u : \bar{x} \rightarrow \mathcal{U}$ , and  $\eta$  is also a 2-morphism of 1-morphisms  $\bar{x} \rightarrow \mathcal{U}$ . This implies that  $\mathcal{U}_{\text{top}}$  is a subset of  $\mathcal{X}_{\text{top}}$ .

Define  $\mathcal{T}_{\mathcal{X}_{\text{top}}} = \{\mathcal{U}_{\text{top}} : \mathcal{U} \subseteq \mathcal{X} \text{ is an open } C^\infty\text{-substack in } \mathcal{X}\}$ , a set of subsets of  $\mathcal{X}_{\text{top}}$ . We claim that  $\mathcal{T}_{\mathcal{X}_{\text{top}}}$  is a topology on  $\mathcal{X}_{\text{top}}$ . To see this, note that taking  $\mathcal{U}$  to be  $\mathcal{X}$  or the empty  $C^\infty$ -substack gives  $\mathcal{X}_{\text{top}}, \emptyset \in \mathcal{T}_{\mathcal{X}_{\text{top}}}$ . If  $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$  are open  $C^\infty$ -substacks of  $\mathcal{X}$  then the intersection of subcategories  $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$  is an open  $C^\infty$ -substack of  $\mathcal{X}$  equivalent to the fibre product  $\mathcal{U} \times_{i_{\mathcal{U}}, \mathcal{X}, i_{\mathcal{V}}} \mathcal{V}$ , with  $\mathcal{W}_{\text{top}} = \mathcal{U}_{\text{top}} \cap \mathcal{V}_{\text{top}}$ , so  $\mathcal{T}_{\mathcal{X}_{\text{top}}}$  is closed under finite intersections.

If  $\{\mathcal{U}_a : a \in A\}$  is a family of open  $C^\infty$ -substacks in  $\mathcal{X}$ , define  $\mathcal{V}$  to be the unique smallest strictly full subcategory of  $\mathcal{X}$  which contains  $\mathcal{U}_a$  for each  $a \in A$  and is closed under the stack axiom (A.9) in Definition A.6. Then  $\mathcal{V}$  is an open  $C^\infty$ -substack of  $\mathcal{X}$ , which we write as  $\mathcal{V} = \bigcup_{a \in A} \mathcal{U}_a$ , and  $\mathcal{V}_{\text{top}} = \bigcup_{a \in A} \mathcal{U}_{a, \text{top}}$ . So  $\mathcal{T}_{\mathcal{X}_{\text{top}}}$  is closed under arbitrary unions.



Thus  $(\mathcal{X}_{\text{top}}, \mathcal{T}_{\mathcal{X}_{\text{top}}})$  is a topological space, which we call the *underlying topological space* of  $\mathcal{X}$ , and usually write as  $\mathcal{X}_{\text{top}}$ . It has the following properties. If  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a 1-morphism of  $C^\infty$ -stacks then there is a natural continuous map  $f_{\text{top}} : \mathcal{X}_{\text{top}} \rightarrow \mathcal{Y}_{\text{top}}$  defined by  $f_{\text{top}}([x]) = [f \circ x]$ . If  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  are 1-morphisms and  $\eta : f \Rightarrow g$  is a 2-isomorphism then  $f_{\text{top}} = g_{\text{top}}$ . Mapping  $\mathcal{X} \mapsto \mathcal{X}_{\text{top}}$ ,  $f \mapsto f_{\text{top}}$  and 2-morphisms to identities defines a 2-functor  $F_{\mathbf{C}^\infty \mathbf{Sta}}^{\mathbf{Top}} : \mathbf{C}^\infty \mathbf{Sta} \rightarrow \mathbf{Top}$ , where the category of topological spaces  $\mathbf{Top}$  is regarded as a 2-category with only identity 2-morphisms.

If  $\underline{X} = (X, \mathcal{O}_X)$  is a  $C^\infty$ -scheme, so that  $\bar{\underline{X}}$  is a  $C^\infty$ -stack, then  $\bar{\underline{X}}_{\text{top}}$  is naturally homeomorphic to  $X$ , and we will identify  $\bar{\underline{X}}_{\text{top}}$  with  $X$ . If  $\underline{f} = (f, f^\#) : \underline{X} = (X, \mathcal{O}_X) \rightarrow \underline{Y} = (Y, \mathcal{O}_Y)$  is a morphism of  $C^\infty$ -schemes, so that  $\bar{\underline{f}} : \bar{\underline{X}} \rightarrow \bar{\underline{Y}}$  is a 1-morphism of  $C^\infty$ -stacks, then  $\bar{\underline{f}}_{\text{top}} : \bar{\underline{X}}_{\text{top}} \rightarrow \bar{\underline{Y}}_{\text{top}}$  is  $f : X \rightarrow Y$ .

For a  $C^\infty$ -stack  $\mathcal{X}$ , we can characterize  $\mathcal{X}_{\text{top}}$  by the following universal property. We are given a topological space  $\mathcal{X}_{\text{top}}$  and for every 1-morphism  $f : \bar{\underline{U}} \rightarrow \mathcal{X}$  for a  $C^\infty$ -scheme  $\bar{\underline{U}} = (U, \mathcal{O}_U)$  we are given a continuous map  $f_{\text{top}} : U \rightarrow \mathcal{X}_{\text{top}}$ , such that if  $f$  is 2-isomorphic to  $h \circ \bar{g}$  for some morphism  $\bar{g} = (g, g^\#) : \bar{\underline{U}} \rightarrow \bar{\underline{V}}$  and 1-morphism  $h : \bar{\underline{V}} \rightarrow \mathcal{X}$  then  $f_{\text{top}} = h_{\text{top}} \circ g$ . If  $\mathcal{X}'_{\text{top}}, f'_{\text{top}}$  are alternative choices of data with these properties then there is a unique continuous map  $j : \mathcal{X}_{\text{top}} \rightarrow \mathcal{X}'_{\text{top}}$  with  $f'_{\text{top}} = j \circ f_{\text{top}}$  for all  $f$ .

We can also make  $\mathcal{X}_{\text{top}}$  into a  $C^\infty$ -ringed space  $\underline{\mathcal{X}}_{\text{top}}$ :

**DEFINITION 6.18.** Let  $\mathcal{X}$  be a  $C^\infty$ -stack. Define a sheaf of  $C^\infty$ -rings  $\mathcal{O}_{\mathcal{X}_{\text{top}}}$  on  $\mathcal{X}_{\text{top}}$  as follows: each open set in  $\mathcal{X}_{\text{top}}$  is  $\mathcal{U}_{\text{top}}$  for some unique open  $C^\infty$ -substack  $\mathcal{U} \subseteq \mathcal{X}$ . Define  $\mathcal{O}_{\mathcal{X}_{\text{top}}}(\mathcal{U}_{\text{top}})$  to be the set of 2-isomorphism classes  $[c]$  of 1-morphisms  $c : \mathcal{U} \rightarrow \bar{\mathbb{R}}$ . If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth and  $[c_1], \dots, [c_n] \in \mathcal{O}_{\mathcal{X}_{\text{top}}}(\mathcal{U}_{\text{top}})$ , define  $\Phi_{f, \bar{\mathbb{R}}}([c_1], \dots, [c_n]) = [\bar{f} \circ (c_1 \times \dots \times c_n)]$ , using the composition  $\mathcal{U} \xrightarrow{c_1 \times \dots \times c_n} \bar{\mathbb{R}} \times \dots \times \bar{\mathbb{R}} \xrightarrow{\bar{f}} \bar{\mathbb{R}}$ . Then  $\mathcal{O}_{\mathcal{X}_{\text{top}}}(\mathcal{U}_{\text{top}})$  is a  $C^\infty$ -ring.

If  $\mathcal{V}_{\text{top}} \subseteq \mathcal{U}_{\text{top}} \subseteq \mathcal{X}_{\text{top}}$  are open, so that  $\mathcal{V} \subseteq \mathcal{U} \subseteq \mathcal{X}$ , define a  $C^\infty$ -ring morphism  $\rho_{\mathcal{U}\mathcal{V}} : \mathcal{O}_{\mathcal{X}_{\text{top}}}(\mathcal{U}_{\text{top}}) \rightarrow \mathcal{O}_{\mathcal{X}_{\text{top}}}(\mathcal{V}_{\text{top}})$  by  $\rho_{\mathcal{U}\mathcal{V}} : [c] \mapsto [c|_{\mathcal{V}}]$ . It is now easy to check that  $\mathcal{O}_{\mathcal{X}_{\text{top}}}$  is a presheaf of  $C^\infty$ -rings on  $\mathcal{X}_{\text{top}}$ , but it is less obvious that it is a sheaf. To see this, note that by general properties of stacks,  $\mathcal{U} \mapsto \mathbf{Hom}(\mathcal{U}, \bar{\mathbb{R}})$  is a 2-sheaf (stack) of groupoids on the topological space  $\mathcal{X}_{\text{top}}$ , where  $\mathbf{Hom}(\mathcal{U}, \bar{\mathbb{R}})$  is the groupoid of 1- and 2-morphisms  $\mathcal{U} \rightarrow \bar{\mathbb{R}}$ , and  $\mathcal{O}_{\mathcal{X}_{\text{top}}}(\mathcal{U}_{\text{top}})$  is its set of isomorphism classes.

Starting with a 2-sheaf and taking sets of isomorphism classes generally yields only a presheaf of sets, not a sheaf. But as  $\bar{\mathbb{R}}$  is a  $C^\infty$ -scheme the groupoids  $\mathbf{Hom}(\mathcal{U}, \bar{\mathbb{R}})$  are discrete (have no nontrivial automorphisms), so taking isomorphism classes loses no information, and the 2-sheaf property implies that  $\mathcal{O}_{\mathcal{X}_{\text{top}}}$  is a sheaf of sets, and so of  $C^\infty$ -rings. Thus  $\underline{\mathcal{X}}_{\text{top}} = (\mathcal{X}_{\text{top}}, \mathcal{O}_{\mathcal{X}_{\text{top}}})$  is a  $C^\infty$ -ringed space, the *underlying  $C^\infty$ -ringed space* of  $\mathcal{X}$ .

For general  $\mathcal{X}$  this  $\underline{\mathcal{X}}_{\text{top}}$  need not be a  $C^\infty$ -scheme. If it is, we call  $\underline{\mathcal{X}}_{\text{top}}$  the *coarse moduli  $C^\infty$ -scheme* of  $\mathcal{X}$ . Coarse moduli  $C^\infty$ -schemes have the following universal property: there is a 1-morphism  $\pi : \mathcal{X} \rightarrow \underline{\mathcal{X}}_{\text{top}}$  called the *structural morphism*, such that if  $f : \mathcal{X} \rightarrow \bar{\underline{Y}}$  is a 1-morphism for any  $C^\infty$ -scheme  $\bar{\underline{Y}}$  then  $f$  is 2-isomorphic to  $\bar{g} \circ \pi$  for some unique  $C^\infty$ -scheme morphism  $\bar{g} : \underline{\mathcal{X}}_{\text{top}} \rightarrow \bar{\underline{Y}}$ .

We can think of a  $C^\infty$ -stack  $\mathcal{X}$  as being a topological space  $\mathcal{X}_{\text{top}}$  equipped with some complicated extra geometrical structure, just as manifolds and orbifolds are

usually thought of as topological spaces equipped with extra structure coming from an atlas of charts. As in Noohi [55, Ex. 4.13], it is easy to describe  $\mathcal{X}_{\text{top}}$  using a groupoid presentation  $[V \rightrightarrows U]$  of  $\mathcal{X}$ :

**PROPOSITION 6.19.** *Let  $\mathcal{X}$  be equivalent to the  $C^\infty$ -stack  $[V \rightrightarrows U]$  associated to a groupoid  $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$  in  $\mathbf{C}^\infty\mathbf{Sch}$ , where  $\underline{U} = (U, \mathcal{O}_U)$ ,  $\underline{s} = (s, s^\sharp)$ , and so on. Define  $\sim$  on  $U$  by  $p \sim p'$  if there exists  $q \in V$  with  $s(q) = p$  and  $t(q) = p'$ . Then  $\sim$  is an equivalence relation on  $U$ , so we can form the quotient  $U/\sim$ , with the quotient topology. There is a natural homeomorphism  $\mathcal{X}_{\text{top}} \cong U/\sim$ .*

*For a quotient  $C^\infty$ -stack  $\mathcal{X} \simeq [\underline{U}/\underline{G}]$  we have  $\mathcal{X}_{\text{top}} \cong U/G$ .*

Using this we can deduce properties of  $\mathcal{X}_{\text{top}}$  from properties of  $\mathcal{X}$  expressed in terms of  $\underline{V} \rightrightarrows \underline{U}$ . For instance, if  $\mathcal{X}$  is separated then  $s \times t : V \rightarrow U \times U$  is (universally) closed, and we can take  $U$  Hausdorff. But the quotient of a Hausdorff topological space by a closed equivalence relation is Hausdorff, yielding:

**LEMMA 6.20.** *Let  $\mathcal{X}$  be a separated  $C^\infty$ -stack. Then the underlying topological space  $\mathcal{X}_{\text{top}}$  is Hausdorff.*

Next we discuss *isotropy groups* of  $C^\infty$ -stacks.

**DEFINITION 6.21.** Let  $\mathcal{X}$  be a  $C^\infty$ -stack, and  $[x] \in \mathcal{X}_{\text{top}}$ . Pick a representative  $x$  for  $[x]$ , so that  $x : \bar{x} \rightarrow \mathcal{X}$  is a 1-morphism. Then there exists a  $C^\infty$ -scheme  $\underline{G} = (G, \mathcal{O}_G)$ , unique up to isomorphism, with  $\underline{G} = \bar{x} \times_{x, \mathcal{X}, x} \bar{x}$ . Applying the construction of the groupoid in Definition A.21 with  $\Pi : U \rightarrow \mathcal{X}$  replaced by  $x : \bar{x} \rightarrow \mathcal{X}$ , we give  $\underline{G}$  the structure of a  $C^\infty$ -group. The underlying group  $G$  is canonically isomorphic to the group of 2-morphisms  $\eta : x \Rightarrow x$ .

With  $[x]$  fixed, this  $C^\infty$ -group  $\underline{G}$  is independent of choices up to noncanonical isomorphism; roughly,  $\underline{G}$  is canonical up to conjugation in  $\underline{G}$ . We define the *isotropy group* (or *orbifold group*, or *stabilizer group*)  $\text{Iso}_{\mathcal{X}}([x])$  or  $\text{Iso}([x])$  of  $[x]$  to be this  $C^\infty$ -group  $\underline{G}$ , regarded as a  $C^\infty$ -group up to noncanonical isomorphism.

If  $\mathcal{X} = [V \rightrightarrows U]$  is associated to a groupoid  $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$  then  $x : \bar{x} \rightarrow \mathcal{X}$  factors through  $\bar{w} : \bar{x} \rightarrow \underline{U}$  up to 2-isomorphism for some point  $w \in \underline{U}$ , and then  $\underline{G}$  is isomorphic to the  $C^\infty$ -subscheme  $\underline{G}' = \bar{s}^{-1}(w) \cap \bar{t}^{-1}(w)$  in  $\underline{V}$ , with identity  $\underline{u}|_w : * \rightarrow \underline{G}'$ , inverse  $\underline{i}|_{\underline{G}'} : \underline{G}' \rightarrow \underline{G}'$ , and multiplication  $\underline{m}|_{\underline{G}' \times \underline{G}'} : \underline{G}' \times \underline{G}' \rightarrow \underline{G}'$ .

If  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a 1-morphism of  $C^\infty$ -stacks and  $[x] \in \mathcal{X}_{\text{top}}$  with  $f_{\text{top}}([x]) = [y] \in \mathcal{Y}_{\text{top}}$ , for  $y = f \circ x$ , then at the level of sets we define  $f_* : \text{Iso}_{\mathcal{X}}([x]) \rightarrow \text{Iso}_{\mathcal{Y}}([y])$  by  $f_*(\eta) = \text{id}_f * \eta$ . This is a group morphism, by compatibility of horizontal and vertical composition in 2-categories. We can extend  $f_*$  naturally to a morphism  $\underline{f}_* : \text{Iso}_{\mathcal{X}}([x]) \rightarrow \text{Iso}_{\mathcal{Y}}([y])$  of  $C^\infty$ -groups, such that

$$\underline{f}_* : \overline{\text{Iso}_{\mathcal{X}}([x])} = \bar{x} \times_{x, \mathcal{X}, x} \bar{x} \longrightarrow \bar{x} \times_{f \circ x, \mathcal{Y}, f \circ x} \bar{x} = \overline{\text{Iso}_{\mathcal{Y}}([y])}$$

is induced from  $f : \mathcal{X} \rightarrow \mathcal{Y}$  by the universal property of fibre products. Then  $f_*, \underline{f}_*$  are independent of the choice of  $x \in [x]$  up to conjugation in  $\text{Iso}_{\mathcal{Y}}([y])$ .

### 6.5. Gluing $C^\infty$ -stacks by equivalences

Here are two propositions on gluing  $C^\infty$ -stacks by equivalences. They are exercises in stack theory, with no special  $C^\infty$  issues, and also hold for other classes of stacks. See Rydh [61, Th. C] for stronger results for algebraic stacks.

PROPOSITION 6.22. *Suppose  $\mathcal{X}, \mathcal{Y}$  are  $C^\infty$ -stacks,  $\mathcal{U} \subseteq \mathcal{X}$ ,  $\mathcal{V} \subseteq \mathcal{Y}$  are open  $C^\infty$ -substacks, and  $f : \mathcal{U} \rightarrow \mathcal{Y}$  is an equivalence in  $\mathbf{C}^\infty\mathbf{Sta}$ . Then there exist a  $C^\infty$ -stack  $\mathcal{Z}$ , open  $C^\infty$ -substacks  $\hat{\mathcal{X}}, \hat{\mathcal{Y}}$  in  $\mathcal{Z}$  with  $\mathcal{Z} = \hat{\mathcal{X}} \cup \hat{\mathcal{Y}}$ , equivalences  $g : \mathcal{X} \rightarrow \hat{\mathcal{X}}$  and  $h : \mathcal{Y} \rightarrow \hat{\mathcal{Y}}$  such that  $g|_{\mathcal{U}}$  and  $h|_{\mathcal{V}}$  are both equivalences with  $\hat{\mathcal{X}} \cap \hat{\mathcal{Y}}$ , and a 2-morphism  $\eta : g|_{\mathcal{U}} \Rightarrow h \circ f : \mathcal{U} \rightarrow \hat{\mathcal{X}} \cap \hat{\mathcal{Y}}$  in  $\mathbf{C}^\infty\mathbf{Sta}$ . Furthermore,  $\mathcal{Z}$  is independent of choices up to equivalence.*

PROPOSITION 6.23. *Suppose  $\mathcal{X}, \mathcal{Y}$  are  $C^\infty$ -stacks,  $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$  are open  $C^\infty$ -substacks with  $\mathcal{X} = \mathcal{U} \cup \mathcal{V}$ ,  $f : \mathcal{U} \rightarrow \mathcal{Y}$  and  $g : \mathcal{V} \rightarrow \mathcal{Y}$  are 1-morphisms, and  $\eta : f|_{\mathcal{U} \cap \mathcal{V}} \Rightarrow g|_{\mathcal{U} \cap \mathcal{V}}$  is a 2-morphism in  $\mathbf{C}^\infty\mathbf{Sta}$ . Then there exists a 1-morphism  $h : \mathcal{X} \rightarrow \mathcal{Y}$  and 2-morphisms  $\zeta : h|_{\mathcal{U}} \Rightarrow f$ ,  $\theta : h|_{\mathcal{V}} \Rightarrow g$  such that  $\theta|_{\mathcal{U} \cap \mathcal{V}} = \eta \odot \zeta|_{\mathcal{U} \cap \mathcal{V}} : h|_{\mathcal{U} \cap \mathcal{V}} \Rightarrow g|_{\mathcal{U} \cap \mathcal{V}}$ . This  $h$  is unique up to 2-isomorphism.*

*In general,  $h$  is **not** independent up to 2-isomorphism of the choice of  $\eta$ .*

Here is an example in which  $h$  is not independent of  $\eta$  up to 2-isomorphism in the last part of Proposition 6.23.

EXAMPLE 6.24. Let  $\mathcal{X}$  be the  $C^\infty$ -stack associated to the circle  $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ , and  $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$  the substacks associated to the open sets  $U = \{(x, y) \in X : x > -\frac{1}{2}\}$  and  $V = \{(x, y) \in X : x < \frac{1}{2}\}$ . Let  $\mathcal{Y}$  be the quotient  $C^\infty$ -stack  $[*/\mathbb{Z}_2]$ . Then 1-morphisms  $f : \mathcal{X} \rightarrow \mathcal{Y}$  correspond to principal  $\mathbb{Z}_2$ -bundles  $P_f \rightarrow X$ , and for 1-morphisms  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  with principal  $\mathbb{Z}_2$ -bundles  $P_f, P_g \rightarrow X$ , a 2-morphism  $\eta : f \Rightarrow g$  corresponds to an isomorphism of principal  $\mathbb{Z}_2$ -bundles  $P_f \cong P_g$ . The same holds for 1-morphisms  $\mathcal{U}, \mathcal{V}, \mathcal{U} \cup \mathcal{V} \rightarrow \mathcal{Y}$  and their 2-morphisms.

Let  $f : \mathcal{U} \rightarrow \mathcal{Y}$  and  $g : \mathcal{V} \rightarrow \mathcal{Y}$  be the 1-morphisms corresponding to the trivial  $\mathbb{Z}_2$ -bundles  $P_f = \mathbb{Z}_2 \times U \rightarrow U$ ,  $P_g = \mathbb{Z}_2 \times V \rightarrow V$ . Then 2-morphisms  $\eta : f|_{\mathcal{U} \cap \mathcal{V}} \Rightarrow g|_{\mathcal{U} \cap \mathcal{V}}$  correspond to automorphisms of the trivial  $\mathbb{Z}_2$ -bundle  $\mathbb{Z}_2 \times (U \cap V) \rightarrow U \cap V$ , that is, to continuous maps  $U \cap V \rightarrow \mathbb{Z}_2$ . Note that  $U \cap V$  has two connected components  $\{(x, y) \in X : -\frac{1}{2} < x < \frac{1}{2}, y > 0\}$  and  $\{(x, y) \in X : -\frac{1}{2} < x < \frac{1}{2}, y < 0\}$ .

Define 2-morphisms  $\eta_1, \eta_2 : f|_{\mathcal{U} \cap \mathcal{V}} \Rightarrow g|_{\mathcal{U} \cap \mathcal{V}}$  such that  $\eta_1$  corresponds to the map  $1 : (U \cap V) \rightarrow \mathbb{Z}_2 = \{\pm 1\}$ , and  $\eta_2$  corresponds to the map  $\text{sign}(y) : (U \cap V) \rightarrow \mathbb{Z}_2 = \{\pm 1\}$ . Then Proposition 6.23 gives 1-morphisms  $h_1, h_2 : \mathcal{X} \rightarrow \mathcal{Y}$  from  $\eta_1, \eta_2$ . The associated principal  $\mathbb{Z}_2$ -bundles  $P_{h_1}, P_{h_2}$  over  $X$  come from gluing  $P_f, P_g$  over  $U, V$  using the transition functions  $1, \text{sign}(y)$ . Therefore  $P_{h_1}$  is the trivial  $\mathbb{Z}_2$ -bundle over  $X = \mathcal{S}^1$ , and  $P_{h_2}$  the nontrivial  $\mathbb{Z}_2$ -bundle. Hence  $P_{h_1}, P_{h_2}$  are not isomorphic as principal  $\mathbb{Z}_2$ -bundles, and  $h_1, h_2$  are not 2-isomorphic. Hence in this example,  $h$  is not independent up to 2-isomorphism of the choice of  $\eta$ .



## CHAPTER 7

### Deligne–Mumford $C^\infty$ -stacks

We now introduce *Deligne–Mumford  $C^\infty$ -stacks*, which are  $C^\infty$ -stacks locally modelled on quotients  $[\underline{U}/G]$  for  $\underline{U}$  an affine  $C^\infty$ -scheme and  $G$  a finite group. As we explain in §7.6, orbifolds may be defined as a 2-subcategory of Deligne–Mumford  $C^\infty$ -stacks.

#### 7.1. Quotient $C^\infty$ -stacks, 1-morphisms, and 2-morphisms

When a  $C^\infty$ -group  $G$  acts on a  $C^\infty$ -scheme  $\underline{X}$ , Definition 6.6 gives the quotient  $C^\infty$ -stack  $[\underline{X}/G]$ . It is a stack associated to a groupoid  $[G \times \underline{X} \rightrightarrows \underline{X}]$  from Definition A.22, which is the stackification of a certain prestack. By Proposition A.9, stackifications always exist, and are unique up to equivalence. Thus, Definition 6.6 actually only specifies  $[\underline{X}/G]$  up to equivalence in  $\mathbf{C}^\infty\mathbf{Sta}$ .

When a finite group  $G$  acts on a  $C^\infty$ -scheme  $\underline{X}$ , we will now define an explicit  $C^\infty$ -stack  $[\underline{X}/G]$ , which is in the equivalence class of  $[\underline{X}/G]$  in Definition 6.6 for  $G = F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}(G)$ . These quotient  $C^\infty$ -stacks  $[\underline{X}/G]$  (for  $\underline{X}$  Hausdorff) will be our local models for defining Deligne–Mumford  $C^\infty$ -stacks in §7.2.

We will also define *quotient 1-morphisms*  $[f, \rho] : [\underline{X}/G] \rightarrow [\underline{Y}/H]$  of quotient  $C^\infty$ -stacks  $[\underline{X}/G], [\underline{Y}/H]$  when  $\rho : G \rightarrow H$  is a group morphism and  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  a  $\rho$ -equivariant  $C^\infty$ -morphism, and *quotient 2-morphisms*  $[\delta] : [f, \rho] \Rightarrow [g, \sigma]$  for quotient 1-morphisms  $[f, \rho], [g, \sigma] : [\underline{X}/G] \rightarrow [\underline{Y}/H]$ , when  $\delta \in H$  with  $\sigma(\gamma) = \delta \rho(\gamma) \delta^{-1}$  for all  $\gamma \in G$ , and  $\underline{g} = \delta \cdot \underline{f}$ . We will see in §7.4 that all 1- and 2-morphisms of Deligne–Mumford  $C^\infty$ -stacks are locally modelled on quotient 1- and 2-morphisms.

**DEFINITION 7.1.** Let  $\underline{X}$  be a  $C^\infty$ -scheme,  $G$  a finite group, and  $r : G \rightarrow \text{Aut}(\underline{X})$  an action of  $G$  on  $\underline{X}$  by isomorphisms. We will define the *quotient  $C^\infty$ -stack*  $\mathcal{X} = [\underline{X}/G]$ , generalizing the description of  $[\ast/G]$  in Example 6.24. It is a well known construction, as in Behrend et al. [4, Ex. 2.6] and Noohi [55, Ex. 12.4].

Define a category  $\mathcal{X}$  to have objects triples  $(\underline{S}, \underline{P}, p)$  where  $\underline{S}$  is a  $C^\infty$ -scheme, and  $\underline{P}$  is a principal  $G$ -bundle over  $\underline{S}$  in the sense of Example 6.7 (or  $(\underline{P}, \pi, \mu)$  rather than  $\underline{P}$ , but we leave  $\pi : \underline{P} \rightarrow \underline{S}$  and the  $G$ -action  $\mu$  implicit), and  $p : \underline{P} \rightarrow \underline{X}$  is a  $G$ -equivariant morphism. Define morphisms  $(\underline{m}, \underline{u}) : (\underline{S}, \underline{P}, p) \rightarrow (\underline{T}, \underline{Q}, q)$  in  $\mathcal{X}$  to be  $C^\infty$ -scheme morphisms  $\underline{m} : \underline{S} \rightarrow \underline{T}$  and  $\underline{u} : \underline{P} \rightarrow \underline{Q}$ , such that  $\underline{u}$  is  $G$ -equivariant, and (6.2) is a Cartesian square in  $\mathbf{C}^\infty\mathbf{Sch}$ , and  $\underline{p} = q \circ \underline{u} : \underline{P} \rightarrow \underline{X}$ . Composition of morphisms is  $(\underline{n}, \underline{v}) \circ (\underline{m}, \underline{u}) = (\underline{n} \circ \underline{m}, \underline{v} \circ \underline{u})$ , and identity morphisms are  $\text{id}_{(\underline{S}, \underline{P}, p)} = (\text{id}_{\underline{S}}, \text{id}_{\underline{P}})$ . The functor  $p_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{C}^\infty\mathbf{Sch}$  maps  $p_{\mathcal{X}} : (\underline{S}, \underline{P}, p) \mapsto \underline{S}$  on objects and  $p_{\mathcal{X}} : (\underline{m}, \underline{u}) \mapsto \underline{m}$  on morphisms.

Then  $\mathcal{X}$  is a  $C^\infty$ -stack, which we write as  $[\underline{X}/G]$ . It is equivalent in  $\mathbf{C}^\infty\mathbf{Sta}$  to the quotient  $C^\infty$ -stack  $[\underline{X}/G]$  in Definition 6.6 for  $G = F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}(G)$ . To see this, note that by Definition A.22  $[\underline{X}/G]$  is the stackification of a prestack  $p_{\mathcal{X}'} :$

$\mathcal{X}' \rightarrow \mathbf{C}^\infty\mathbf{Sch}$ , where  $\mathcal{X}'$  may be written as the category whose objects are pairs  $(\underline{S}, \underline{p}')$  of a  $C^\infty$ -scheme  $\underline{S}$  and a morphism  $\underline{p}' : \underline{S} \rightarrow \underline{X}$ , and whose morphisms  $(\underline{m}, \underline{u}') : (\underline{S}, \underline{p}') \rightarrow (\underline{T}, \underline{q}')$  consist of morphisms  $\underline{m} : \underline{S} \rightarrow \underline{T}$  and  $\underline{u}' : \underline{S} \rightarrow \underline{G}$  with  $\underline{p}' = \underline{q}' \circ \underline{m}$ , with composition  $(\underline{n}, \underline{v}') \circ (\underline{m}, \underline{u}') = (\underline{n} \circ \underline{m}, \underline{u}' \cdot (\underline{v}' \circ \underline{m}))$ , and  $p_{\mathcal{X}'}$  maps  $p_{\mathcal{X}'} : (\underline{S}, \underline{p}') \mapsto \underline{S}$  and  $p_{\mathcal{X}'} : (\underline{m}, \underline{u}') \mapsto \underline{u}'$ .

We may identify  $\mathcal{X}'$  with the full subcategory of  $\mathcal{X}$  with objects  $(\underline{S}, G \times \underline{S}, \underline{p})$  in which  $\underline{P}$  is the trivial principal  $G$ -bundle  $G \times \underline{S} \rightarrow \underline{S}$ , where  $(\underline{S}, \underline{p}')$  in  $\mathcal{X}'$  is identified with  $(\underline{S}, G \times \underline{S}, \underline{p})$  in  $\mathcal{X}$  for  $\underline{p}' = \underline{p}|_{\underline{S} \times \{1\}} : \underline{S} \cong \underline{S} \times \{1\} \rightarrow \underline{X}$ , and  $(\underline{m}, \underline{u}') : (\underline{S}, \underline{p}') \rightarrow (\underline{T}, \underline{q}')$  in  $\mathcal{X}'$  is identified with  $(\underline{m}, \underline{u}) : (\underline{S}, G \times \underline{S}, \underline{p}) \rightarrow (\underline{T}, G \times \underline{T}, \underline{q})$  in  $\mathcal{X}$ , where  $\underline{u} : G \times \underline{S} \rightarrow G \times \underline{T}$  maps  $(\gamma, s) \mapsto (\gamma \cdot \underline{u}'(s), \underline{m}(s))$ . Stackifying  $\mathcal{X}'$  enlarges from trivial principal  $G$ -bundles to all principal  $G$ -bundles.

Define a functor  $\pi_{[\underline{X}/G]} : \bar{\underline{X}} \rightarrow [\underline{X}/G]$  by  $\pi_{[\underline{X}/G]} : (\underline{S}, \underline{p}') \mapsto (\underline{S}, G \times \underline{S}, \underline{p})$  on objects, where  $\underline{p} : G \times \underline{S} \rightarrow \underline{X}$  is the unique  $G$ -equivariant morphism with  $\underline{p}' = \underline{p}|_{\underline{S} \times \{1\}} : \underline{S} \cong \underline{S} \times \{1\} \rightarrow \underline{X}$ , and  $\pi_{[\underline{X}/G]} : \underline{m} \mapsto (\underline{m}, \text{id}_G \times \underline{m})$  on morphisms. Then  $\pi_{[\underline{X}/G]} : \bar{\underline{X}} \rightarrow [\underline{X}/G]$  is a representable 1-morphism, and makes  $\bar{\underline{X}}$  into a principal  $G$ -bundle over  $[\underline{X}/G]$ .

**DEFINITION 7.2.** Let  $\underline{X}, \underline{Y}$  be  $C^\infty$ -schemes acted on by finite groups  $G, H$  with actions  $r : G \rightarrow \text{Aut}(\underline{X})$ ,  $s : H \rightarrow \text{Aut}(\underline{Y})$ , so that we have quotient  $C^\infty$ -stacks  $\mathcal{X} = [\underline{X}/G]$  and  $\mathcal{Y} = [\underline{Y}/H]$  as in Definition 7.1. Suppose we have morphisms  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  of  $C^\infty$ -schemes and  $\rho : G \rightarrow H$  of groups, with  $\underline{f} \circ r(\gamma) = s(\rho(\gamma)) \circ \underline{f}$  for all  $\gamma \in G$ . We will define a *quotient 1-morphism*  $[\underline{f}, \rho] : \mathcal{X} \rightarrow \mathcal{Y}$ .

Define a functor  $[\underline{f}, \rho] : \mathcal{X} \rightarrow \mathcal{Y}$  by  $[\underline{f}, \rho] : (\underline{S}, \underline{P}, \underline{p}) \mapsto (\underline{S}, \tilde{\underline{P}}, \tilde{\underline{p}})$  on objects  $(\underline{S}, \underline{P}, \underline{p})$  in  $\mathcal{X}$ , where  $\tilde{\underline{P}} = (H \times \underline{P})/\rho G$  is the principal  $H$ -bundle on  $\underline{S}$  constructed from  $\underline{P}$  and  $\rho : G \rightarrow H$ , and  $\tilde{\underline{p}} : \tilde{\underline{P}} \rightarrow \underline{Y}$  is the  $H$ -equivariant  $C^\infty$ -scheme morphism induced from the  $\rho$ -equivariant morphism  $\underline{f} \circ \underline{p} : \underline{P} \rightarrow \underline{Y}$ , which acts on points by  $\tilde{\underline{p}} : (h, p)G \mapsto h \cdot \underline{f} \circ \underline{p}(p)$ . Define  $[\underline{f}, \rho] : (\underline{m}, \underline{u}) \mapsto (\underline{m}, \tilde{\underline{u}})$  on morphisms  $(\underline{m}, \underline{u}) : (\underline{S}, \underline{P}, \underline{p}) \rightarrow (\underline{T}, \underline{Q}, \underline{q})$  in  $\mathcal{X}$ , where  $\tilde{\underline{u}} : (H \times \underline{P})/\rho G \rightarrow (H \times \underline{Q})/\rho G$  is induced by  $\text{id}_H \times \underline{u} : H \times \underline{P} \rightarrow H \times \underline{Q}$ . Then  $[\underline{f}, \rho] : \mathcal{X} \rightarrow \mathcal{Y}$  is a functor, with  $p_{\mathcal{X}} = p_{\mathcal{Y}} \circ [\underline{f}, \rho]$ , so  $[\underline{f}, \rho]$  is a 1-morphism of  $C^\infty$ -stacks, which we write as  $[\underline{f}, \rho] : [\underline{X}/G] \rightarrow [\underline{Y}/H]$ .

It is easy to check that  $[\underline{f}, \rho] \circ \pi_{[\underline{X}/G]} \cong \pi_{[\underline{Y}/H]} \circ \bar{\underline{f}}$ , and if  $[\underline{f}, \rho] : [\underline{X}/G] \rightarrow [\underline{Y}/H]$ ,  $[\underline{g}, \sigma] : [\underline{Y}/H] \rightarrow [\underline{Z}/I]$  are quotient 1-morphisms then there is a canonical 2-isomorphism  $[\underline{g}, \sigma] \circ [\underline{f}, \rho] \cong [\underline{g} \circ \underline{f}, \sigma \circ \rho]$  coming from the canonical  $C^\infty$ -scheme isomorphisms  $(I \times ((H \times \underline{P})/\rho G))/\sigma H \cong (I \times \underline{P})/\sigma \rho G$ .

**DEFINITION 7.3.** Let  $[\underline{f}, \rho] : [\underline{X}/G] \rightarrow [\underline{Y}/H]$  and  $[\underline{g}, \sigma] : [\underline{X}/G] \rightarrow [\underline{Y}/H]$  be quotient 1-morphisms, so that  $\underline{f}, \underline{g} : \underline{X} \rightarrow \underline{Y}$  and  $\rho, \sigma : G \rightarrow H$  are morphisms. Suppose  $\delta \in H$  satisfies  $\sigma(\gamma) = \delta \rho(\gamma) \delta^{-1}$  for all  $\gamma \in G$ , and  $\underline{g} = s(\delta) \circ \underline{f}$ . We will define a 2-morphism  $[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$ , which we call a *quotient 2-morphism*.

Here  $[\delta]$  must be a natural isomorphism of functors  $[\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$ . Let  $(\underline{S}, \underline{P}, \underline{p})$  be an object in  $[\underline{X}/G]$ . Define an isomorphism in  $[\underline{Y}/H]$ :

$$\begin{aligned} [\delta]((\underline{S}, \underline{P}, \underline{p})) &= (\text{id}_{\underline{S}}, (r_{\delta^{-1}} \times \text{id}_{\underline{P}})_*) : [\underline{f}, \rho]((\underline{S}, \underline{P}, \underline{p})) = \\ &(\underline{S}, (H \times \underline{P})/\rho G, \tilde{\underline{p}}) \longrightarrow [\underline{g}, \sigma]((\underline{S}, \underline{P}, \underline{p})) = (\underline{S}, (H \times \underline{P})/\sigma G, \tilde{\underline{p}}), \end{aligned}$$

where  $r_{\delta^{-1}} : H \rightarrow H$  maps  $\epsilon \mapsto \epsilon \delta^{-1}$ , and  $r_{\delta^{-1}} \times \text{id}_{\underline{P}} : H \times \underline{P} \rightarrow H \times \underline{P}$  is equivariant under the actions of  $G$  on  $H \times \underline{P}$  induced by  $\rho$  on the domain and  $\sigma$  on the target, so that it descends to an isomorphism  $(r_{\delta^{-1}} \times \text{id}_{\underline{P}})_* : (H \times \underline{P})/\rho G \rightarrow (H \times \underline{P})/\sigma G$ . It is

now easy to check that  $[\delta]((S, P, p))$  is an isomorphism in  $[Y/H]$ , and  $[\delta]$  is a natural isomorphism of functors, and a 2-morphism  $[\delta] : [f, \rho] \Rightarrow [g, \sigma]$  in  $\mathbf{C}^\infty\mathbf{Sta}$ . Quotient 2-morphisms have functorial properties under horizontal and vertical composition. For instance, if  $[f, \rho], [g, \sigma], [h, \tau] : [X/G] \rightarrow [Y/H]$  are quotient 1-morphisms and  $[\delta] : [f, \rho] \Rightarrow [g, \sigma], [\epsilon] : [g, \sigma] \Rightarrow [h, \tau]$  are quotient 2-morphisms then  $[\epsilon] \odot [\delta] = [\epsilon\delta] : [f, \rho] \Rightarrow [h, \tau]$ .

REMARK 7.4. Studying  $C^\infty$ -stacks  $[X/G]$  and their 1- and 2-morphisms is a good way to develop geometric intuition about Deligne–Mumford  $C^\infty$ -stacks (including orbifolds) and their 1- and 2-morphisms. If  $[X/G], [Y/H]$  are quotient  $C^\infty$ -stacks, then general 1-morphisms  $f : [X/G] \rightarrow [Y/H]$  in  $\mathbf{C}^\infty\mathbf{Sta}$  need not be quotient 1-morphisms  $[f, \rho]$ , or even 2-isomorphic to  $[f, \rho]$ . But Theorem 7.18(b) says that  $f \cong [f, \rho]$  locally in  $[X/G]$ . If  $[f, \rho], [g, \sigma] : [X/G] \rightarrow [Y/H]$  are quotient 1-morphisms, and  $[X/G]$  is connected, then Proposition 7.19 says that all 2-morphisms  $\eta : [f, \rho] \Rightarrow [g, \sigma]$  are quotient 2-morphisms  $[\delta] : [f, \rho] \Rightarrow [g, \sigma]$ .

## 7.2. Deligne–Mumford $C^\infty$ -stacks

*Deligne–Mumford stacks* in algebraic geometry were introduced in [17] to study moduli spaces of algebraic curves. As in [46, Th. 6.2], Deligne–Mumford stacks are locally modelled (in the étale topology, at least, but with isomorphisms of isotropy groups) on quotient  $C^\infty$ -stacks  $[X/G]$  for  $X$  an affine scheme and  $G$  a finite group. This motivates:

DEFINITION 7.5. A *Deligne–Mumford  $C^\infty$ -stack* is a  $C^\infty$ -stack  $\mathcal{X}$  which admits a (Zariski) open cover  $\{\mathcal{U}_a : a \in A\}$ , as in Definition 6.14, with each  $\mathcal{U}_a$  equivalent to a quotient  $C^\infty$ -stack  $[\underline{U}_a/G_a]$  in Definition 7.1 for  $\underline{U}_a$  an affine  $C^\infty$ -scheme and  $G_a$  a finite group. We call  $\mathcal{X}$  *locally fair*, or *locally finitely presented*, or *locally Lindelöf*, if it admits such an open cover with each  $\underline{U}_a$  a fair, or finitely presented, or Lindelöf, affine  $C^\infty$ -scheme, respectively. We call  $\mathcal{X}$  *second countable* if the underlying topological space  $\mathcal{X}_{\text{top}}$  is second countable.

Write  $\mathbf{DMC}^\infty\mathbf{Sta}^{\text{lf}}, \mathbf{DMC}^\infty\mathbf{Sta}^{\text{lfp}}$  and  $\mathbf{DMC}^\infty\mathbf{Sta}$  for the full 2-subcategories of locally fair, locally finitely presented, and all, Deligne–Mumford  $C^\infty$ -stacks in  $\mathbf{C}^\infty\mathbf{Sta}$ , respectively.

Now  $F_{\mathbf{C}^\infty\mathbf{Sch}}^{\mathbf{C}^\infty\mathbf{Sta}} : \mathbf{C}^\infty\mathbf{Sch} \rightarrow \mathbf{C}^\infty\mathbf{Sta}$  in Definition 6.3 maps into  $\mathbf{DMC}^\infty\mathbf{Sta} \subset \mathbf{C}^\infty\mathbf{Sta}$ , so the 2-categories  $\bar{\mathbf{C}}^\infty\mathbf{Sch}^{\text{lf}}, \bar{\mathbf{C}}^\infty\mathbf{Sch}^{\text{lfp}}, \bar{\mathbf{C}}^\infty\mathbf{Sch}$  are 2-subcategories of  $\mathbf{DMC}^\infty\mathbf{Sta}^{\text{lf}}, \mathbf{DMC}^\infty\mathbf{Sta}^{\text{lfp}}, \mathbf{DMC}^\infty\mathbf{Sta}$ , respectively. If a  $C^\infty$ -stack  $\mathcal{X}$  is a  $C^\infty$ -scheme, then it is a Deligne–Mumford  $C^\infty$ -stack.

PROPOSITION 7.6.  $\mathbf{DMC}^\infty\mathbf{Sta}^{\text{lf}}, \mathbf{DMC}^\infty\mathbf{Sta}^{\text{lfp}}, \mathbf{DMC}^\infty\mathbf{Sta}$  are closed under taking open  $C^\infty$ -substacks in  $\mathbf{C}^\infty\mathbf{Sta}$ .

PROOF. Let  $\mathcal{X}$  lie in one of these 2-categories, and  $\mathcal{X}'$  be an open  $C^\infty$ -substack of  $\mathcal{X}$ . Then  $\mathcal{X}$  admits an open cover  $\{\mathcal{U}_a : a \in A\}$  with  $\mathcal{U}_a \simeq [\underline{U}_a/G_a]$  with  $\underline{U}_a$  affine and  $G_a$  finite, and  $\{\mathcal{U}'_a : a \in A\}$  is an open cover of  $\mathcal{X}'$ , where  $\mathcal{U}'_a = \mathcal{U}_a \times_{\mathcal{X}} \mathcal{X}'$  is an open  $C^\infty$ -substack of  $\mathcal{U}_a$ . Thus  $\mathcal{U}'_a \simeq [\underline{U}'_a/G_a]$  by Proposition 6.16, where  $\underline{U}'_a$  is a  $G_a$ -invariant open  $C^\infty$ -subscheme of  $\underline{U}_a$ . If the  $\underline{U}_a$  are fair, or finitely presented then the  $\underline{U}'_a$  are too by Corollary 4.32(a). Thus  $\mathbf{DMC}^\infty\mathbf{Sta}^{\text{lf}}, \mathbf{DMC}^\infty\mathbf{Sta}^{\text{lfp}}$  are closed under open subsets.

For  $\mathbf{DMC}^\infty\mathbf{Sta}$ , as open subsets of affine  $C^\infty$ -schemes need not be affine, the  $\underline{U}'_a$  need not be affine. We will show that we can cover  $\underline{U}'_a$  by  $G_a$ -invariant open

affine  $C^\infty$ -subschemes  $\underline{U}'_{au}$ . Write  $\underline{U}'_a = (U'_a, \mathcal{O}_{\underline{U}'_a})$  and  $G_a = (G_a, \mathcal{O}_{G_a})$ . Then the finite group  $G_a$  acts continuously on  $U'_a$ . Let  $u \in U'_a$ , and  $H_u = \{\gamma \in G_a : \gamma u = u\}$  be the stabilizer of  $u$  in  $G_a$ . Then the orbit  $\{\gamma u : \gamma \in G\} \cong G_a/H_u$  of  $u$  is a finite set, so as  $U'_a$  is Hausdorff we can choose affine open neighbourhoods  $V_{\gamma u}$  of  $\gamma u$  for each point in the orbit such that  $V_{\gamma u} \cap V_{\gamma' u} = \emptyset$  if  $\gamma u \neq \gamma' u$ . Define  $W_u = \bigcap_{\gamma \in G} \gamma^{-1} V_{\gamma u}$ . Then  $W_u$  is an  $H_u$ -invariant open neighbourhood of  $u$  in  $U'_a$ , and if  $\gamma \in G_a \setminus H_u$  then  $\gamma W_u \cap W_u = \emptyset$ .

As in Corollary 4.29 we can choose an affine open neighbourhood  $W'_u$  of  $u$  in  $W_u$ . Define  $W''_u = \bigcap_{\gamma \in H_u} W'_u$ , an  $H_u$ -invariant open neighbourhood of  $u$  in  $W_u$ . This is a finite intersection of affine open  $C^\infty$ -subschemes  $\underline{W}'_u$  in the affine  $C^\infty$ -scheme  $\underline{U}_u$ , and so is affine, since intersection is a kind of fibre product, and **AC $^\infty$ Sch** is closed under fibre products by Theorem 4.25(a). Define  $\underline{U}'_{au} = \bigcup_{\gamma \in G_a} W''_u$ . Then  $\underline{U}'_{au}$  is a  $G_a$ -invariant open neighbourhood of  $u$  in  $\underline{U}'_a$ . Since  $W''_u$  is  $H_u$ -invariant and  $\gamma W''_u \cap W''_u = \emptyset$  if  $\gamma \in G_a \setminus H_u$ , we see that  $\underline{U}'_{au}$  is isomorphic to the disjoint union of  $|G_a|/|H_u|$  copies of  $W''_u$ . Hence  $\underline{U}'_{au} = (U'_{au}, \mathcal{O}_{\underline{U}'_a}|_{\underline{U}'_{au}})$  is a finite disjoint union of affine  $C^\infty$ -schemes, and is an affine  $C^\infty$ -scheme. Therefore we may cover  $\underline{U}'_a$  by  $G_a$ -invariant open affine  $C^\infty$ -subschemes  $\underline{U}'_{au}$ . Using these we obtain an open cover  $\{\underline{U}'_{au} : a \in A, u \in U_a\}$  of  $\mathcal{X}'$  with  $\underline{U}'_{au} \simeq [\underline{U}'_{au}/G_a]$ , so  $\mathcal{X}'$  is Deligne–Mumford.  $\square$

The proof of Proposition 7.6 only uses  $\underline{U}_a = (U_a, \mathcal{O}_{U_a})$  a  $C^\infty$ -scheme and  $U_a$  Hausdorff, it does not need  $\underline{U}_a$  to be affine. So the same proof yields:

**PROPOSITION 7.7.** *Any  $C^\infty$ -stack of the form  $[\underline{X}/G]$  in §7.1 with  $\underline{X}$  a Hausdorff  $C^\infty$ -scheme and  $G$  finite is a separated Deligne–Mumford  $C^\infty$ -stack.*

However, if  $\underline{X}$  is not Hausdorff then  $[\underline{X}/G]$  need not be Deligne–Mumford:

**EXAMPLE 7.8.** Let  $\underline{X}$  be the non-Hausdorff  $C^\infty$ -scheme  $(\mathbb{R} \amalg \mathbb{R})/\sim$ , where  $\sim$  is the equivalence relation which identifies the two copies of  $\mathbb{R}$  on  $(0, \infty)$ . Let  $G = \mathbb{Z}_2$  act on  $\underline{X}$  by exchanging the two copies of  $\mathbb{R}$ . Let  $\mathcal{X}$  be the quotient  $C^\infty$ -stack  $[\underline{X}/G]$ . We can think of  $\mathcal{X}$  as a like copy of  $\mathbb{R}$ , where the stabilizer group of  $x \in \mathbb{R}$  is  $\{1\}$  if  $x \in (-\infty, 0]$  and  $\mathbb{Z}_2$  if  $x \in (0, \infty)$ . Using the obvious atlas  $\Pi : \mathbb{R} \rightarrow \mathcal{X}$ , the third diagram of (A.12) yields a 2-Cartesian square

$$\begin{array}{ccc} \mathbb{R} \amalg \overline{(0, \infty)} & \xrightarrow{\quad} & I_{\mathcal{X}} \\ \downarrow & \nearrow & \downarrow \iota_{\mathcal{X}} \\ \mathbb{R} & \xrightarrow{\quad \Pi \quad} & \mathcal{X}. \end{array}$$

As the left hand column is not proper,  $\iota_{\mathcal{X}}$  is not proper, so  $\mathcal{X} = [\underline{X}/G]$  is not Deligne–Mumford by Corollary 7.14 below.

We show that the 2-subcategory of quotient  $C^\infty$ -stacks  $[\underline{X}/G]$  in **C $^\infty$ Sta** is closed under fibre products:

**PROPOSITION 7.9.** *Suppose  $g : [\underline{X}/F] \rightarrow [\underline{Z}/H]$ ,  $h : [\underline{Y}/G] \rightarrow [\underline{Z}/H]$  are 1-morphisms of quotient  $C^\infty$ -stacks, where  $\underline{X}, \underline{Y}, \underline{Z}$  are  $C^\infty$ -schemes and  $F, G, H$  are finite groups. Then we have a 2-Cartesian square*

$$\begin{array}{ccc} [\underline{W}/(F \times G)] & \xrightarrow{\quad f \quad} & [\underline{Y}/G] \\ \downarrow e & \nearrow & \downarrow h \\ [\underline{X}/F] & \xrightarrow{\quad g \quad} & [\underline{Z}/H], \end{array} \tag{7.1}$$



where  $\Pi_{\underline{X}} : \bar{\underline{X}} \rightarrow [\underline{X}/F]$ ,  $\Pi_{\underline{Y}} : \bar{\underline{Y}} \rightarrow [\underline{Y}/G]$  are the natural atlases and  $\bar{\underline{W}} = \bar{\underline{X}} \times_{g \circ \Pi_{\underline{X}}, [\underline{Z}/H], h \circ \Pi_{\underline{Y}}} \bar{\underline{Y}}$ . If  $\underline{X}, \underline{Y}, \underline{Z}$  are Hausdorff, or locally fair, or locally finitely presented, then  $\underline{W}$  is too.

PROOF. Write  $\mathcal{W} = [\underline{X}/F] \times_{[\underline{Z}/H]} [\underline{Y}/G]$ . Then from the atlases  $\Pi_{\underline{X}}, \Pi_{\underline{Y}}$ , Example A.25 constructs an atlas  $\Pi_{\underline{W}} : \bar{\underline{W}} \rightarrow \mathcal{W}$  for  $\mathcal{W}$ . Since  $[\underline{X}/F] \simeq [F \times \underline{X} \rightrightarrows \underline{X}]$  and  $[\underline{Y}/G] \simeq [G \times \underline{Y} \rightrightarrows \underline{Y}]$  it follows from (A.14) that  $\mathcal{W}$  is equivalent to the stack associated to the groupoid  $[(F \times G) \times \underline{W} \rightrightarrows \underline{W}]$  for a natural action of  $F \times G$  on  $\underline{W}$ . This proves (7.1).

If  $\underline{X}, \underline{Y}, \underline{Z}$  are Hausdorff then  $[\underline{Z}/H]$  is Deligne–Mumford by Proposition 7.7, so  $\Delta_{[\underline{Z}/H]}$  is separated by Corollary 7.14 below, and thus  $\underline{W}$  is Hausdorff as  $\underline{X}, \underline{Y}$  are and  $\bar{\underline{W}} \cong (\bar{\underline{X}} \times \bar{\underline{Y}}) \times_{[\underline{Z}/H] \times [\underline{Z}/H], \Delta_{[\underline{Z}/H]}} [\underline{Z}/H]$ . Form the diagram

$$\begin{array}{ccccc}
 & \bar{\underline{W}}' & \xrightarrow{\quad} & \bar{\underline{Y}}' & \\
 \pi_{\bar{\underline{W}}} \swarrow & & \searrow \pi_{\bar{\underline{Y}}} & & \searrow \pi_{\bar{\underline{Z}}} \\
 \bar{\underline{W}} & \xrightarrow{\quad} & \bar{\underline{Y}} & \xrightarrow{\quad} & \bar{\underline{Z}} \\
 & \searrow \pi_{\bar{\underline{X}}} & & \searrow h \circ \Pi_{\bar{\underline{Y}}} & \\
 & \bar{\underline{X}}' & \xrightarrow{\quad} & \bar{\underline{Z}} & \\
 \pi_{\underline{X}} \swarrow & & \searrow g \circ \Pi_{\underline{X}} & & \searrow \Pi_{\underline{Z}} \\
 \underline{X} & \xrightarrow{\quad} & [\underline{Z}/H] & \xrightarrow{\quad} & \underline{Z}
 \end{array}$$

with squares 2-Cartesian, where  $\bar{\underline{W}}', \bar{\underline{X}}', \bar{\underline{Y}}'$  are  $C^\infty$ -schemes. Then  $\pi_{\bar{\underline{W}}}, \pi_{\bar{\underline{X}}}, \pi_{\bar{\underline{Y}}}$  are étale and surjective, as  $\Pi_{\bar{\underline{Z}}}$  is. If  $\underline{X}, \underline{Y}, \underline{Z}$  are locally fair, then  $\bar{\underline{X}}', \bar{\underline{Y}}'$  are locally fair as  $\underline{X}, \underline{Y}$  are and  $\pi_{\bar{\underline{X}}}, \pi_{\bar{\underline{Y}}}$  are étale, so  $\bar{\underline{W}}' \cong \bar{\underline{X}}' \times_{\bar{\underline{Z}}} \bar{\underline{Y}}'$  is locally fair by Theorem 4.25(b), and thus  $\bar{\underline{W}}$  is locally fair as  $\pi_{\bar{\underline{W}}} : \bar{\underline{W}}' \rightarrow \bar{\underline{W}}$  is étale and surjective. The proof for locally finitely presented is the same.  $\square$

Using this we prove:

**THEOREM 7.10.**  $\mathbf{DMC}^\infty\mathbf{Sta}, \mathbf{DMC}^\infty\mathbf{Sta}^{\text{lf}}$  and  $\mathbf{DMC}^\infty\mathbf{Sta}^{\text{lfp}}$  are closed under fibre products in  $\mathbf{C}^\infty\mathbf{Sta}$ .

PROOF. Let  $\mathcal{W} = \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$  be a fibre product in  $\mathbf{C}^\infty\mathbf{Sta}$  of Deligne–Mumford  $C^\infty$ -stacks  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ . We must show  $\mathcal{W}$  is Deligne–Mumford. Now  $\mathcal{Z}$  admits an open cover  $\{\mathcal{Z}_c : c \in C\}$  with  $\mathcal{Z}_c \simeq [\underline{Z}_c/H_c]$  for  $\underline{Z}_c$  an affine  $C^\infty$ -scheme and  $H_c$  finite. For  $c \in C$  define  $\mathcal{X}_c = \mathcal{X} \times_{\mathcal{Z}} \mathcal{Z}_c$  and  $\mathcal{Y}_c = \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Z}_c$ , which are open  $C^\infty$ -substacks of  $\mathcal{X}, \mathcal{Y}$ , and so are Deligne–Mumford by Proposition 7.6. Then  $\{\mathcal{X}_c \times_{\mathcal{Z}_c} \mathcal{Y}_c : c \in C\}$  is an open cover of  $\mathcal{W}$ , so it is enough to prove  $\mathcal{X}_c \times_{\mathcal{Z}_c} \mathcal{Y}_c$  is Deligne–Mumford. That is, we may replace  $\mathcal{Z}$  by  $\mathcal{Z}_c \simeq [\underline{Z}_c/H_c]$ .

Similarly, by choosing open covers of  $\mathcal{X}_c, \mathcal{Y}_c$  by substacks equivalent to  $[\underline{X}/F]$ ,  $[\underline{Y}/G]$ , we reduce the problem to showing  $[\underline{X}/F] \times_{[\underline{Z}/H]} [\underline{Y}/G]$  is Deligne–Mumford, for  $\underline{X}, \underline{Y}, \underline{Z}$  affine  $C^\infty$ -schemes and  $F, G, H$  finite groups. This follows from Propositions 7.7 and 7.9, noting that  $\underline{X}, \underline{Y}, \underline{Z}$  are Hausdorff as they are affine, so  $\bar{\underline{W}}$  is Hausdorff in Proposition 7.9. This shows  $\mathbf{DMC}^\infty\mathbf{Sta}$  is closed under fibre products. For  $\mathbf{DMC}^\infty\mathbf{Sta}^{\text{lf}}, \mathbf{DMC}^\infty\mathbf{Sta}^{\text{lfp}}$  we use the same argument with  $\underline{Z}_c, \underline{Z}, \underline{X}, \underline{Y}, \bar{\underline{W}}$  locally fair, or locally finitely presented.  $\square$

Under weak conditions Deligne–Mumford  $C^\infty$ -stacks have coarse moduli  $C^\infty$ -schemes, in the sense of §6.4.

**THEOREM 7.11.** Let  $\mathcal{X}$  be a locally Lindelöf Deligne–Mumford  $C^\infty$ -stack. Then the  $C^\infty$ -ringed space  $\underline{\mathcal{X}}_{\text{top}}$  in Definition 6.18 is a  $C^\infty$ -scheme. If  $\mathcal{X}$  is locally fair, or locally finitely presented, then  $\underline{\mathcal{X}}_{\text{top}}$  is too.

PROOF. By definition  $\mathcal{X}$  can be covered by open  $C^\infty$ -substacks  $\mathcal{U}$  equivalent to  $[Y/G]$  for  $Y$  a Lindelöf affine  $C^\infty$ -scheme. Then the  $C^\infty$ -ringed space  $\underline{\mathcal{U}}_{\text{top}}$  is isomorphic to  $Y/G$  in Definition 4.45, so Proposition 4.46 shows that  $\underline{\mathcal{U}}_{\text{top}}$  is an affine  $C^\infty$ -scheme. Hence  $\underline{\mathcal{X}}_{\text{top}}$  can be covered by open affine  $\underline{\mathcal{U}}_{\text{top}} \subseteq \underline{\mathcal{X}}_{\text{top}}$ , so  $\underline{\mathcal{X}}_{\text{top}}$  is a  $C^\infty$ -scheme. If  $\mathcal{X}$  is locally fair (or locally finitely presented) the same argument works with  $Y$  fair (or finitely presented), and then  $\underline{\mathcal{U}}_{\text{top}} \cong Y/G$  is fair (or finitely presented), and  $\underline{\mathcal{X}}_{\text{top}}$  is locally fair (or locally finitely presented).  $\square$

REMARK 7.12. In §4.7 we discussed *partitions of unity* on  $C^\infty$ -schemes. We can use Theorem 7.11 to extend these ideas to Deligne–Mumford  $C^\infty$ -stacks.

Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack, and suppose  $\mathcal{X}_{\text{top}}$  is regular and Lindelöf. Then  $\underline{\mathcal{X}}_{\text{top}}$  is a  $C^\infty$ -scheme by Theorem 7.11, and the topology on  $\underline{\mathcal{X}}_{\text{top}}$  is smoothly generated by Example 4.39(c) as  $\mathcal{X}_{\text{top}}$  is regular. Hence Theorem 4.40 shows that  $\mathcal{O}_{\mathcal{X}_{\text{top}}}$  is fine. Suppose  $\{\mathcal{U}_a : a \in A\}$  is a (Zariski) open cover of  $\mathcal{X}$ . Then  $\{\underline{\mathcal{U}}_{a,\text{top}} : a \in A\}$  is an open cover of  $\underline{\mathcal{X}}_{\text{top}}$ , so there exists a partition of unity  $\{\eta_a : a \in A\}$  on  $\underline{\mathcal{X}}_{\text{top}}$  subordinate to  $\{\underline{\mathcal{U}}_{a,\text{top}} : a \in A\}$ . Therefore  $\{\pi^*(\eta_a) : a \in A\}$  is (in a suitable sense) a partition of unity on  $\mathcal{X}$  subordinate to  $\{\mathcal{U}_a : a \in A\}$ , where  $\pi : \mathcal{X} \rightarrow \underline{\mathcal{X}}_{\text{top}}$  is the structural morphism.

### 7.3. Characterizing Deligne–Mumford $C^\infty$ -stacks

We now explore ways to characterize when a  $C^\infty$ -stack  $\mathcal{X}$  is Deligne–Mumford.

PROPOSITION 7.13. *Let  $\mathcal{X}$  be a quotient  $C^\infty$ -stack  $[\underline{U}/G]$  for  $\underline{U}$  affine and  $G$  finite. Then the natural 1-morphism  $\Pi : \bar{\underline{U}} \rightarrow \mathcal{X}$  is an étale atlas, and  $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ ,  $\iota_{\mathcal{X}} : I_{\mathcal{X}} \rightarrow \mathcal{X}$  are universally closed, proper, and separated, with finite fibres, and  $j_{\mathcal{X}} : \mathcal{X} \rightarrow I_{\mathcal{X}}$  is an open and closed embedding.*

PROOF. As in (A.12) we have 2-Cartesian diagrams with surjective rows:

$$\begin{array}{ccc}
 \bar{G} \times \bar{\underline{U}} & \xrightarrow{\quad \bar{\mu} \quad} & \bar{\underline{U}} \\
 \downarrow \bar{\pi}_{\underline{U}} & \uparrow & \downarrow \Pi \\
 \bar{\underline{U}} & \xrightarrow{\quad \Pi \quad} & \mathcal{X}
 \end{array}
 \quad
 \begin{array}{ccc}
 \bar{G} \times \bar{\underline{U}} & \xrightarrow{\quad \Pi \circ \bar{\pi}_{\underline{U}} \quad} & \mathcal{X} \\
 \downarrow \bar{\pi}_{\underline{U}} \times \bar{\mu} & \uparrow & \downarrow \Delta_{\mathcal{X}} \\
 \bar{\underline{U}} \times \bar{\underline{U}} & \xrightarrow{\quad \Pi \times \Pi \quad} & \mathcal{X} \times \mathcal{X}
 \end{array}$$

$$\begin{array}{ccc}
 (\bar{G} \times \bar{\underline{U}}) \times_{\bar{\underline{U}} \times \bar{\underline{U}}} \bar{\underline{U}} & \xrightarrow{\quad \bar{\mu} \quad} & I_{\mathcal{X}} \\
 \downarrow \bar{\pi}_{\underline{U}} & \uparrow & \downarrow \iota_{\mathcal{X}} \\
 \bar{\underline{U}} & \xrightarrow{\quad \Pi \quad} & \mathcal{X}
 \end{array}
 \quad
 \begin{array}{ccc}
 \bar{\underline{U}} & \xrightarrow{\quad \Pi \quad} & \mathcal{X} \\
 \downarrow (1 \times \text{id}_{\underline{U}}) \times \text{id}_{\underline{U}} & \uparrow & \downarrow j_{\mathcal{X}} \\
 (\bar{G} \times \bar{\underline{U}}) \times_{\bar{\underline{U}} \times \bar{\underline{U}}} \bar{\underline{U}} & \xrightarrow{\quad \bar{\mu} \quad} & I_{\mathcal{X}}
 \end{array}$$

The left column  $\bar{\pi}_{\underline{U}}$  in the first diagram is étale. The left columns in the second and third diagrams are both universally closed, proper, and separated, with finite fibres, since  $G$  is finite with the discrete topology, and  $U$  is Hausdorff as  $\underline{U}$  is affine. This left column in the fourth is an open and closed embedding. The result now follows from Propositions 6.11 and A.18(c).  $\square$

Propositions 6.11, 6.15 and 7.13 now imply:

COROLLARY 7.14. *Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack. Then  $\mathcal{X}$  has an étale atlas  $\Pi : \bar{\underline{U}} \rightarrow \mathcal{X}$ , the diagonal  $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is separated with finite fibres, and the inertia morphism  $\iota_{\mathcal{X}} : I_{\mathcal{X}} \rightarrow \mathcal{X}$  is universally closed, proper, and separated, with finite fibres, and  $j_{\mathcal{X}} : \mathcal{X} \rightarrow I_{\mathcal{X}}$  is an open and closed embedding. If  $\mathcal{X}$  is separated then  $\Delta_{\mathcal{X}}$  is also universally closed and proper.*

The last part holds as then  $\Delta_{\mathcal{X}}$  is universally closed with finite fibres, which implies  $\Delta_{\mathcal{X}}$  is proper. Note that for  $\mathcal{X}$  not separated we cannot conclude from Proposition 7.13 that  $\Delta_{\mathcal{X}}$  is universally closed or proper, since these properties are not stable under open embedding. Some of the conclusions of Corollary 7.14 are sufficient for  $\mathcal{X}$  to be separated and Deligne–Mumford.

**THEOREM 7.15.** *Let  $\mathcal{X}$  be a  $C^\infty$ -stack, and suppose  $\mathcal{X}$  has an étale atlas  $\Pi : \bar{U} \rightarrow \mathcal{X}$ , and the diagonal  $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is universally closed and separated. Then  $\mathcal{X}$  is a separated Deligne–Mumford  $C^\infty$ -stack.*

**PROOF.** Let  $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$  be the groupoid in  $\mathbf{C}^\infty\mathbf{Sch}$  constructed from  $\Pi : \bar{U} \rightarrow \mathcal{X}$  as in §A.5, so that  $\mathcal{X} \simeq [\underline{V} \rightrightarrows \underline{U}]$ . Then (A.12) gives 2-Cartesian diagrams with surjective rows. From the first and Propositions A.18(a) and 6.11 we see that  $\underline{s}, \underline{t}$  are étale, since  $\Pi$  is. From the second  $\underline{s} \times \underline{t} : \underline{V} \rightarrow \underline{U} \times \underline{U}$  is universally closed and separated, as  $\Delta_{\mathcal{X}}$  is. Let  $p \in U$ . Define

$$H = \{q \in V : s(q) = t(q) = p\} \subseteq s^{-1}(\{p\}).$$

It has the discrete topology, as  $\underline{s}, \underline{t}$  are étale.

Suppose for a contradiction that  $H$  is infinite. Define a  $C^\infty$ -ring

$$\mathfrak{C} = \{c : H \amalg \{\infty\} \rightarrow \mathbb{R} : c(q) = c(\infty) \text{ for all but finitely many } q \in H\},$$

with  $C^\infty$  operations defined pointwise in  $H \amalg \{\infty\}$ . Then  $\text{Spec } \mathfrak{C}$  has underlying topological space the one point compactification  $H \amalg \{\infty\}$  of the discrete topological space  $H$ , since  $\mathfrak{C} = C^0(H \amalg \{\infty\})$  is the set of continuous maps  $H \amalg \{\infty\} \rightarrow \mathbb{R}$ , with the natural  $C^\infty$ -ring structure. Define  $g : \text{Spec } \mathfrak{C} \rightarrow \underline{U} \times \underline{U}$  to project  $\text{Spec } \mathfrak{C}$  to the point  $(p, p)$ . Then the morphism

$$\pi_{\text{Spec } \mathfrak{C}} : \underline{V} \times_{\underline{s} \times \underline{t}, \underline{U} \times \underline{U}, g} \text{Spec } \mathfrak{C} \longrightarrow \text{Spec } \mathfrak{C} \quad (7.2)$$

is the projection  $H \times (H \amalg \{\infty\}) \rightarrow H \amalg \{\infty\}$ . The diagonal in  $H$  is closed in  $H \times (H \amalg \{\infty\})$ , but its image is  $H$ , which is not closed in  $H \amalg \{\infty\}$ . Hence (7.2) is not a closed morphism, contradicting  $\underline{s} \times \underline{t}$  universally closed. So  $H$  is finite.

As  $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$  is a groupoid,  $H$  is a *finite group*, with identity  $u(p)$ , inverse map  $i|_H$ , and multiplication  $m_H = m|_{H \times H}$ . Since  $\underline{s}, \underline{t}$  are étale, we can choose small open neighbourhoods  $Z_q$  of  $q$  in  $V$  for all  $q \in H$  such that  $\underline{s}|_{Z_q}, \underline{t}|_{Z_q}$  are isomorphisms with open subsets of  $\underline{U}$ . As  $\underline{s} \times \underline{t}$  is separated,  $\{(v, v) : v \in V\}$  is closed in  $\{(v, v') \in V \times V : s(v) = s(v'), t(v) = t(v')\}$ , which has the subspace topology from  $V \times V$ . If  $q \neq q' \in H$  then  $(q, q')$  lies in  $\{(v, v') \in V \times V : s(v) = s(v'), t(v) = t(v')\}$  but not in  $\{(v, v) : v \in V\}$ , so  $(q, q')$  has an open neighbourhood in  $V \times V$  which does not intersect  $\{(v, v) : v \in V\}$ . Making  $Z_q, Z_{q'}$  smaller if necessary, we can take this open neighbourhood to be  $Z_q \times Z_{q'}$ , and then  $Z_q \cap Z_{q'} = \emptyset$ . Thus, we can choose these open neighbourhoods  $Z_q$  for  $q \in H$  to be *disjoint*.

Define  $Y = \bigcap_{q \in H} s(Z_q)$  and  $\underline{Y} = (Y, \mathcal{O}_Y|_Y)$ . Then  $Y$  is a small open neighbourhood of  $p$  in  $U$ . Making  $Y$  smaller if necessary we can suppose it is contained in an affine open neighbourhood of  $p$  in  $U$ , and so is Hausdorff. Replace  $Z_q$  by  $Z_q \cap s^{-1}(Y)$  for all  $q \in H$ . Then  $\underline{s}|_{Z_q} : (Z_q, \mathcal{O}_V|_{Z_q}) \rightarrow \underline{Y}$  is an isomorphism for  $q \in H$ . Set  $Z = \bigcup_{q \in H} Z_q$ , noting the union is disjoint, and  $\underline{Z} = (Z, \mathcal{O}_V|_Z)$ . Then we have an isomorphism  $\underline{\phi} = (\phi, \phi^\#) : H \times \underline{Y} \rightarrow \underline{Z}$ , such that  $\underline{s} \circ \underline{\phi} = \underline{id}_Y$  and  $\phi(q \times Y) = Z_q$  for  $q \in H$ .

Now  $\underline{Z}$  is open in  $\underline{V}$ , so  $\underline{Z} \times_{s,U,t} \underline{Z}$  is open in  $\underline{V} \times_{s,U,t} \underline{V}$ , and we can restrict the morphism  $\underline{m} : \underline{V} \times_{s,U,t} \underline{V} \rightarrow \underline{V}$  to  $\underline{m}|_{\underline{Z} \times_U \underline{Z}} : \underline{Z} \times_{s,U,t} \underline{Z} \rightarrow \underline{V}$ . But

$$\begin{aligned} \underline{Z} \times_{s,U,t} \underline{Z} &\cong (H \times \underline{Y}) \times_{i_Y \circ \pi_Y, U, t} \underline{Z} \\ &\cong H \times (Z \cap t^{-1}(Y), \mathcal{O}_V|_{Z \cap t^{-1}(Y)}) \subseteq H \times \underline{Z} \cong H \times H \times \underline{Y}, \end{aligned}$$

using  $\phi$  an isomorphism and  $s \circ \phi = \text{id}_Y$ . Write  $\Phi : \underline{Z} \times_{s,U,t} \underline{Z} \hookrightarrow H \times H \times \underline{Y}$  for the induced open embedding. Define a second morphism  $\underline{m}' : \underline{Z} \times_{s,U,t} \underline{Z} \rightarrow \underline{V}$  by  $\underline{m}' = \phi \circ (\underline{m}_H \times \text{id}_Y) \circ \Phi$ , where  $\underline{m}_H : H \times H \rightarrow H$  is the group multiplication  $m_H : H \times H \rightarrow H$ , regarded as a morphism of  $C^\infty$ -schemes.

Following the definitions we find that  $s \circ (\underline{m}|_{\underline{Z} \times_U \underline{Z}}) = s \circ \underline{m}' : \underline{Z} \times_{s,U,t} \underline{Z} \rightarrow \underline{Y} \subset \underline{U}$ . Also  $H \subset Z$ , and the definition of  $\underline{m}_H$  from  $\underline{m}$  implies that  $\underline{m}|_{\underline{Z} \times_U \underline{Z}}$  and  $\underline{m}'$  coincide on the finite set  $H \times_U H$  in  $Z \times_U Z$ . Since  $s$  is étale, this implies that  $\underline{m}|_{\underline{Z} \times_U \underline{Z}}$  and  $\underline{m}'$  must coincide near the finite set  $H \times_U H$  in  $Z \times_U Z$ . Therefore by making the open neighbourhood  $Y$  of  $p$  in  $U$  smaller, and hence making  $W_q, W, Z$  smaller too, we can assume that  $\underline{m}|_{\underline{Z} \times_U \underline{Z}} = \underline{m}'$ .

Let us summarize what we have done so far. We have constructed a finite group  $H$ , a Hausdorff open neighbourhood  $Y$  of  $p$  in  $\underline{U}$ , an open and closed subset  $\underline{Z}$  of  $s^{-1}(Y)$  in  $\underline{Z}$  which contains  $s^{-1}(p) \cap t^{-1}(p)$ , and an isomorphism  $\phi : H \times Y \rightarrow \underline{Z}$  with  $s \circ \phi = \pi_Y$  which identifies the groupoid multiplication  $\underline{m}|_{\underline{Z} \times_U \underline{Z}}$  with the restriction to  $\underline{Z} \times_U \underline{Z}$  of the morphism  $\underline{m}_H \times \text{id}_Y : H \times H \times Y \rightarrow Y$  from multiplication in the finite group  $H$ .

Consider the morphism  $t \circ \phi : H \times Y \rightarrow \underline{U} \supset Y$ . Roughly speaking,  $t \circ \phi$  is an  $H$ -action on  $Y$ . More accurately, there should be an  $H$ -action on some open subset of  $\underline{U}$  containing  $Y$ , but  $Y$  may not be  $H$ -invariant, so that  $t \circ \phi$  need not map  $H \times Y$  to  $Y$ . Replace  $Y$  by  $Y' = \bigcap_{q \in H} t(Z_q)$ , which is an open subset of  $Y$  since when  $q$  is the identity  $u(p)$  in  $H$  we have  $t(Z_{u(p)}) = s(Z_{u(p)}) = Y$ , and  $p \in Y'$  as  $p = t(q) \in t(Z_q)$  for  $q \in H$ . Replace  $Z_q$  by  $Z'_q = Z_q \cap s^{-1}(Y)$  and  $Z$  by  $Z' = \bigcup_{q \in H} Z'_q$ . Then using  $\underline{m}|_{\underline{Z} \times_U \underline{Z}} = \underline{m}'$  we can show that  $s(Z'_q) = t(Z'_q) = Y'$  for all  $q \in H$ , so  $Y'$  is an  $H$ -invariant open set, and  $t \circ \phi$  maps  $H \times Y' \rightarrow Y'$ . Restricting the groupoid axioms shows that  $t \circ \phi$  gives an action of  $H$  on  $Y'$ .

Now consider the morphism

$$\underline{s} \times \underline{t} : s^{-1}(Y') \cap t^{-1}(Y') : (s^{-1}(Y') \cap t^{-1}(Y'), \mathcal{O}_V|_{s^{-1}(Y') \cap t^{-1}(Y')}) \longrightarrow Y' \times Y'.$$

This is closed, as  $\underline{s} \times \underline{t}$  is universally closed. Since  $Z'$  is open and closed in  $s^{-1}(Y') \cap t^{-1}(Y')$ , its complement is closed, so its image  $\{(s(v), t(v)) \in Y' \times Y' : v \in V \setminus Z'\}$  is closed in  $Y'$ . But  $(p, p)$  does not lie in this image, since  $s^{-1}(p) \cap t^{-1}(p) \subseteq Z'$ . Thus, by making the  $H$ -invariant open neighbourhood  $Y'$  of  $p$  in  $U$  smaller if necessary, we can suppose that  $s^{-1}(Y') \cap t^{-1}(Y') = Z'$ .

The quotient  $C^\infty$ -stack  $[Y'/H]$  is Deligne–Mumford by Proposition 7.7, since  $Y'$  is Hausdorff. Thus there exists an open embedding  $\mathcal{Y}_p \hookrightarrow [Y'/H]$  with  $\mathcal{Y}_p \simeq [\underline{U}_p/G_p]$  for  $\underline{U}_p$  affine and  $G_p$  finite, which includes  $p$  in its image. The inclusion morphisms  $\underline{Y}' \hookrightarrow \underline{U}$ ,  $\underline{Z}' \hookrightarrow \underline{V}$  induce a 1-morphism  $[\underline{Z}' \rightrightarrows \underline{Y}'] \hookrightarrow [\underline{V} \rightrightarrows \underline{U}]$ , which is an open embedding as  $\underline{Y}'$  is open in  $\underline{U}$ ,  $\underline{Z}'$  is open in  $\underline{V}$  and  $s^{-1}(Y') \cap t^{-1}(Y') = Z'$  in  $V$ . Let  $i_{\mathcal{Y}_p} : \mathcal{Y}_p \rightarrow \mathcal{X}$  be the composition  $\mathcal{Y}_p \hookrightarrow [Y'/H] \simeq [\underline{Z}' \rightrightarrows \underline{Y}'] \hookrightarrow [\underline{V} \rightrightarrows \underline{U}] \simeq \mathcal{X}$ . Then  $i_{\mathcal{Y}_p}$  is an open embedding, as it is a composition of open embeddings and equivalences. This works for all  $p \in U$ , and  $\{\mathcal{Y}_p : p \in U\}$  is an open cover of  $\mathcal{X}$  with  $\mathcal{Y}_p \simeq [\underline{U}_p/G_p]$  for  $\underline{U}_p$  affine and  $G_p$  finite. Hence  $\mathcal{X}$  is Deligne–Mumford. It is separated as  $\Delta_{\mathcal{X}}$  is universally closed, by assumption.  $\square$

Suppose  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  is a separated morphism of  $C^\infty$ -schemes with finite fibres. Then  $\underline{f}$  universally closed implies  $\underline{f}$  proper. Conversely, if  $\underline{X}, \underline{Y}$  are compactly generated topological spaces then  $\underline{f}$  proper implies  $\underline{f}$  universally closed. If  $\underline{X}, \underline{Y}$  are locally fair then  $\underline{X}, \underline{Y}$  are compactly generated, as they are locally homeomorphic to closed subsets of  $\mathbb{R}^n$ . Thus, in Theorem 7.15, if  $\underline{U}, \underline{V}$  are locally fair then we can replace  $\Delta_{\mathcal{X}}$  universally closed by  $\Delta_{\mathcal{X}}$  proper, yielding:

**THEOREM 7.16.** *Let  $\mathcal{X}$  be a  $C^\infty$ -stack, and suppose  $\mathcal{X}$  has an étale atlas  $\Pi : \underline{U} \rightarrow \mathcal{X}$  with  $\underline{U}$  locally fair, and the diagonal  $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is proper and separated. Then  $\mathcal{X}$  is a separated, locally fair Deligne–Mumford  $C^\infty$ -stack.*

The same holds with locally finitely presented in place of locally fair. If  $\mathcal{X} \simeq [\underline{V} \rightrightarrows \underline{U}]$  with  $\underline{U}$  a Hausdorff  $C^\infty$ -scheme then  $\underline{V}$  is Hausdorff if and only if  $\Delta_{\mathcal{X}}$  is separated. We can always choose  $\underline{U}$  Hausdorff, by replacing  $\underline{U}$  by the disjoint union of an open cover of  $\underline{U}$  by affine open subsets. Thus we can replace the condition that  $\Delta_{\mathcal{X}}$  is separated by  $\underline{U}, \underline{V}$  Hausdorff. Combining this and the results above proves:

**THEOREM 7.17. (a)** *A  $C^\infty$ -stack  $\mathcal{X}$  is separated and Deligne–Mumford if and only if it is equivalent to the stack associated to a groupoid  $[\underline{V} \rightrightarrows \underline{U}]$  where  $\underline{U}, \underline{V}$  are Hausdorff  $C^\infty$ -schemes,  $\underline{s} : \underline{V} \rightarrow \underline{U}$  is étale, and  $\underline{s} \times \underline{t} : \underline{V} \rightarrow \underline{U} \times \underline{U}$  is universally closed.*

**(b)** *A  $C^\infty$ -stack  $\mathcal{X}$  is separated, Deligne–Mumford and locally fair (or locally finitely presented) if and only if it is equivalent to some  $[\underline{V} \rightrightarrows \underline{U}]$  with  $\underline{U}, \underline{V}$  Hausdorff, locally fair (or locally finitely presented)  $C^\infty$ -schemes,  $\underline{s} : \underline{V} \rightarrow \underline{U}$  étale, and  $\underline{s} \times \underline{t} : \underline{V} \rightarrow \underline{U} \times \underline{U}$  proper.*

#### 7.4. Quotient $C^\infty$ -stacks, 1- and 2-morphisms as local models for objects, 1- and 2-morphisms in $\text{DMC}^\infty\text{Sta}$

In our next theorem, we prove that Deligne–Mumford  $C^\infty$ -stacks and their 1- and 2-morphisms are (Zariski) locally modelled on quotient  $C^\infty$ -stacks  $[\underline{X}/G]$ , quotient 1-morphisms  $[\underline{f}, \rho] : [\underline{X}/G] \rightarrow [\underline{Y}/H]$ , and quotient 2-morphisms  $[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$  from §7.1.

**THEOREM 7.18. (a)** *Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack and  $[x] \in \mathcal{X}_{\text{top}}$ , and write  $G = \text{Iso}_{\mathcal{X}}([x])$ . Then there exists a quotient  $C^\infty$ -stack  $[\underline{U}/G]$  for  $\underline{U}$  an affine  $C^\infty$ -scheme, and a 1-morphism  $i : [\underline{U}/G] \rightarrow \mathcal{X}$  which is an equivalence with an open  $C^\infty$ -substack  $\mathcal{U}$  in  $\mathcal{X}$ , such that  $i_{\text{top}} : [u] \mapsto [x] \in \mathcal{U}_{\text{top}} \subseteq \mathcal{X}_{\text{top}}$  for some fixed point  $u$  of  $G$  in  $\underline{U}$ .*

**(b)** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of Deligne–Mumford  $C^\infty$ -stacks, and  $[x] \in \mathcal{X}_{\text{top}}$  with  $f_{\text{top}} : [x] \mapsto [y] \in \mathcal{Y}_{\text{top}}$ , and write  $G = \text{Iso}_{\mathcal{X}}([x])$  and  $H = \text{Iso}_{\mathcal{Y}}([y])$ . Part (a) gives 1-morphisms  $i : [\underline{U}/G] \rightarrow \mathcal{X}$ ,  $j : [\underline{V}/H] \rightarrow \mathcal{Y}$  which are equivalences with open  $\mathcal{U} \subseteq \mathcal{X}$ ,  $\mathcal{V} \subseteq \mathcal{Y}$ , such that  $i_{\text{top}} : [u] \mapsto [x] \in \mathcal{U}_{\text{top}} \subseteq \mathcal{X}_{\text{top}}$ ,  $j_{\text{top}} : [v] \mapsto [y] \in \mathcal{V}_{\text{top}} \subseteq \mathcal{Y}_{\text{top}}$  for  $u, v$  fixed points of  $G, H$  in  $\underline{U}, \underline{V}$ .*

*Then there exists a  $G$ -invariant open neighbourhood  $\underline{U}'$  of  $u$  in  $\underline{U}$  and a quotient 1-morphism  $[\underline{f}, \rho] : [\underline{U}'/G] \rightarrow [\underline{V}/H]$  such that  $\underline{f}(u) = v$ , and  $\rho : G \rightarrow H$  is  $f_* : \text{Iso}_{\mathcal{X}}([x]) \rightarrow \text{Iso}_{\mathcal{Y}}([y])$ , fitting into a 2-commutative diagram:*

$$\begin{array}{ccc} [\underline{U}'/G] & \xrightarrow{\quad [\underline{f}, \rho] \quad} & [\underline{V}/H] \\ \downarrow i|_{[\underline{U}'/G]} & \zeta \Uparrow & \downarrow j \\ \mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y}. \end{array} \quad (7.3)$$

(c) Let  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  be 1-morphisms of Deligne–Mumford  $C^\infty$ -stacks and  $\eta : f \Rightarrow g$  a 2-morphism, let  $[x] \in \mathcal{X}_{\text{top}}$  with  $f_{\text{top}} : [x] \mapsto [y] \in \mathcal{Y}_{\text{top}}$ , and write  $G = \text{Iso}_{\mathcal{X}}([x])$  and  $H = \text{Iso}_{\mathcal{Y}}([y])$ . Part (a) gives  $i : [\underline{U}/G] \rightarrow \mathcal{X}$ ,  $j : [\underline{V}/H] \rightarrow \mathcal{Y}$  which are equivalences with open  $\mathcal{U} \subseteq \mathcal{X}$ ,  $\mathcal{V} \subseteq \mathcal{Y}$  and map  $i_{\text{top}} : [u] \mapsto [x]$ ,  $j_{\text{top}} : [v] \mapsto [y]$  for  $u, v$  fixed points of  $G, H$ .

By making  $\underline{U}'$  smaller, we can take the same  $\underline{U}'$  in (b) for both  $f, g$ . Thus part (b) gives a  $G$ -invariant open  $\underline{U}' \subseteq \underline{U}$ , quotient morphisms  $[f, \rho] : [\underline{U}'/G] \rightarrow [\underline{V}/H]$  and  $[g, \sigma] : [\underline{U}'/G] \rightarrow [\underline{V}/H]$  with  $f(u) = g(u) = v$  and  $\rho = f_* : \text{Iso}_{\mathcal{X}}([x]) \rightarrow \text{Iso}_{\mathcal{Y}}([y])$ ,  $\sigma = g_* : \text{Iso}_{\mathcal{X}}([x]) \rightarrow \text{Iso}_{\mathcal{Y}}([y])$ , and 2-morphisms  $\zeta : f \circ i|_{[\underline{U}'/G]} \Rightarrow j \circ [f, \rho]$ ,  $\theta : g \circ i|_{[\underline{U}'/G]} \Rightarrow j \circ [g, \sigma]$ .

Then there exists a  $G$ -invariant open neighbourhood  $\underline{U}''$  of  $u$  in  $\underline{U}'$  and  $\delta \in H$  such that  $\sigma(\gamma) = \delta \rho(\gamma) \circ \delta^{-1}$  for all  $\gamma \in G$  and  $g|_{\underline{U}''} = \underline{s}(\delta) \circ f|_{\underline{U}''}$ , so that  $[\delta] : [f|_{\underline{U}''}, \rho] \Rightarrow [g|_{\underline{U}''}, \sigma]$  is a quotient 2-morphism, and the following diagram of 2-morphisms in  $\mathbf{C}^\infty\mathbf{Sta}$  commutes:

$$\begin{array}{ccc} f \circ i|_{[\underline{U}''/G]} & \xRightarrow{\eta * \text{id}_{i|_{[\underline{U}''/G]}}} & g \circ i|_{[\underline{U}''/G]} \\ \Downarrow \zeta|_{[\underline{U}''/G]} & & \theta|_{[\underline{U}''/G]} \Downarrow \\ j \circ [f|_{\underline{U}''}, \rho] & \xRightarrow{\text{id}_j * [\delta]} & j \circ [g|_{\underline{U}''}, \sigma]. \end{array} \quad (7.4)$$

PROOF. In this proof we will use the theory of 2-categories from §A.1, including vertical and horizontal composition of 2-morphisms ‘\*’, ‘ $\odot$ ’, and the definition of fibre products and 2-Cartesian squares in Definition A.3.

For (a), as  $\mathcal{X}$  is Deligne–Mumford it is covered by open  $C^\infty$ -substacks  $\mathcal{V}$  equivalent to  $[\underline{V}/H]$  for  $\underline{V}$  affine and  $H$  finite, so we can choose such  $\mathcal{V}$  with  $[x] \in \mathcal{V}_{\text{top}}$ . Then  $\mathcal{V}$  has an étale atlas  $\Pi : \underline{V} \rightarrow \mathcal{V}$  and  $\Delta_{\mathcal{V}}$  is universally closed and separated by Proposition 7.13, so we can apply the proof of Theorem 7.15 to  $\mathcal{V}$  for a point  $p \in V$  with  $\Pi_*(p) = [x]$ . This constructs an open  $C^\infty$ -substack  $\mathcal{U}$  in  $\mathcal{V}$  equivalent to  $[\underline{U}/G]$ , where  $\underline{U}$  is affine and  $G = \text{Iso}_{\mathcal{X}}([x])$ , as we want.

For (b), write  $\pi_{[\underline{U}/G]} : \underline{U} \rightarrow [\underline{U}/G]$  and  $\pi_{[\underline{V}/H]} : \underline{V} \rightarrow [\underline{V}/H]$  for the projection 1-morphisms in  $\mathbf{C}^\infty\mathbf{Sta}$ . They are proper and representable. Let  $\bar{r} : G \rightarrow \text{Aut}(\underline{U})$  and  $\bar{s} : H \rightarrow \text{Aut}(\underline{V})$  be the  $G$ - and  $H$ -actions on  $\underline{U}, \underline{V}$ . Then  $\bar{r}(\gamma) : \underline{U} \rightarrow \underline{U}$  for  $\gamma \in G$  and  $\bar{s}(\delta) : \underline{V} \rightarrow \underline{V}$  for  $\delta \in H$  are the corresponding  $C^\infty$ -stack 1-morphisms, and there are natural 2-morphisms  $\lambda_\gamma : \pi_{[\underline{U}/G]} \circ \bar{r}(\gamma) \Rightarrow \pi_{[\underline{U}/G]}$  and  $\mu_\delta : \pi_{[\underline{V}/H]} \circ \bar{s}(\delta) \Rightarrow \pi_{[\underline{V}/H]}$ .

Consider the  $C^\infty$ -stack fibre product  $\underline{U} \times_{f \circ i \circ \pi_{[\underline{U}/G]}, \mathcal{Y}, j \circ \pi_{[\underline{V}/H]}} \underline{V}$ . As  $\pi_{[\underline{V}/H]}$  is representable and  $j$  is an equivalence with an open  $C^\infty$ -substack,  $j \circ \pi_{[\underline{V}/H]}$  is representable, and  $\underline{U}$  is a  $C^\infty$ -stack, so this fibre product is a  $C^\infty$ -scheme. So changing the fibre product up to equivalence, we can take  $\underline{U} \times_{\mathcal{Y}} \underline{V} = \bar{W}$  for some  $C^\infty$ -scheme  $\bar{W}$  unique up to isomorphism. The fibre product projections are 1-morphisms  $\bar{W} \rightarrow \underline{U}$  and  $\bar{W} \rightarrow \underline{V}$ , so they are 2-isomorphic to  $\bar{a}, \bar{b}$  for unique morphisms  $\bar{a} : \bar{W} \rightarrow \underline{U}$ ,  $\bar{b} : \bar{W} \rightarrow \underline{V}$ . Hence we have a 2-Cartesian square in  $\mathbf{C}^\infty\mathbf{Sta}$ , for some 2-morphism  $\omega$ :

$$\begin{array}{ccc} \bar{W} & \xrightarrow{\bar{b}} & \underline{V} \\ \downarrow \bar{a} & \omega \Uparrow & \downarrow j \circ \pi_{[\underline{V}/H]} \\ \underline{U} & \xrightarrow{f \circ i \circ \pi_{[\underline{U}/G]}} & \mathcal{Y}. \end{array} \quad (7.5)$$

We will show that the data  $\bar{r}(\gamma), \lambda_\gamma$  for  $\gamma \in G$  induces an action of  $G$  on  $\bar{W}$ . Let  $\gamma \in G$ , and apply the universal property of the 2-Cartesian square (7.5)

in Definition A.3 to the 1-morphisms  $\bar{r}(\gamma) \circ \bar{a} : \bar{W} \rightarrow \bar{U}$ ,  $\bar{b} : \bar{W} \rightarrow \bar{V}$  and 2-morphism  $\omega \odot (\text{id}_{f \circ i} * \lambda_\gamma * \text{id}_{\bar{a}}) : (f \circ i \circ \pi_{[U/G]}) \circ (\bar{r}(\gamma) \circ \bar{a}) \Rightarrow (j \circ \pi_{[V/H]}) \circ \bar{b}$ . This gives a 1-morphism  $c_\gamma : \bar{W} \rightarrow \bar{W}$ , unique up to 2-isomorphism, and 2-morphisms  $\zeta_\gamma : \bar{a} \circ c_\gamma \Rightarrow \bar{r}(\gamma) \circ \bar{a}$ ,  $\theta_\gamma : \bar{b} \circ c_\gamma \Rightarrow \bar{b}$  such that (A.6) commutes.

Now  $c_\gamma$  is 2-isomorphic to  $\bar{c}_\gamma$  for some unique  $\bar{c}_\gamma : \bar{W} \rightarrow \bar{W}$ , so we may replace  $c_\gamma$  by  $\bar{c}_\gamma$ . Then  $\zeta_\gamma : \bar{a} \circ \bar{c}_\gamma \Rightarrow \bar{r}(\gamma) \circ \bar{a}$ , so we must have  $\bar{a} \circ \bar{c}_\gamma = \bar{r}(\gamma) \circ \bar{a}$  and  $\zeta_\gamma = \text{id}_{\bar{r}(\gamma) \circ \bar{a}}$ . Similarly  $\bar{b} \circ \bar{c}_\gamma = \bar{b}$  and  $\theta_\gamma = \text{id}_{\bar{b}}$ . Therefore (A.6) reduces to  $\omega * \text{id}_{\bar{c}_\gamma} = \omega \odot (\text{id}_{f \circ i} * \lambda_\gamma * \text{id}_{\bar{a}})$ . Using  $\bar{r}(\gamma)\bar{r}(\gamma') = \bar{r}(\gamma\gamma')$  and a natural compatibility between  $\lambda_\gamma, \lambda_{\gamma'}, \lambda_{\gamma\gamma'}$  we find that  $\bar{c}_\gamma \circ \bar{c}_{\gamma'} = \bar{c}_{\gamma\gamma'}$  for  $\gamma, \gamma' \in G$ , and as  $\bar{r}(1) = \text{id}_{\bar{U}}$  and  $\lambda_1 = \text{id}_{\pi_{[U/G]}}$  we have  $\bar{c}_1 = \text{id}_{\bar{W}}$ . Hence  $\gamma \mapsto \bar{c}_\gamma$  is an action of  $G$  on  $\bar{W}$ , and  $\bar{a} \circ \bar{c}_\gamma = \bar{r}(\gamma) \circ \bar{a}$  means that  $\bar{a} : \bar{W} \rightarrow \bar{U}$  is  $G$ -equivariant.

In the same way, we obtain unique isomorphisms  $\underline{d}_\delta : \underline{W} \rightarrow \underline{W}$  for  $\delta \in H$  with  $\underline{a} \circ \underline{d}_\delta = \underline{a}$ ,  $\underline{b} \circ \underline{d}_\delta = \underline{s}(\delta) \circ \underline{b}$  and  $\omega * \text{id}_{\underline{d}_\delta} = (\text{id}_j * \mu_\delta * \text{id}_{\underline{b}}) \odot \omega$ , and  $\delta \mapsto \underline{d}_\delta$  is an action of  $H$  on  $\underline{W}$ , and  $\underline{b} : \underline{W} \rightarrow \underline{V}$  is  $H$ -equivariant. Using associativity of  $\odot$  in  $(\text{id}_j * \mu_\delta * \text{id}_{\underline{b}}) \odot \omega \odot (\text{id}_{f \circ i} * \lambda_\gamma * \text{id}_{\bar{a}})$ , we see that  $\bar{c}_\gamma$  and  $\underline{d}_\delta$  commute. Hence  $(\gamma, \delta) \mapsto \bar{c}_\gamma \circ \underline{d}_\delta$  is an action of  $G \times H$  on  $\underline{W}$ .

Since  $\pi_{[V/H]} : \bar{V} \rightarrow [V/H]$  is a principal  $H$ -bundle, and  $j : [V/H] \rightarrow \mathcal{Y}$  is an equivalence with  $\mathcal{V} \subseteq \mathcal{Y}$ , and (7.5) is 2-Cartesian, it follows that  $\bar{a} : \bar{W} \rightarrow \bar{U}$  is a principal  $H$ -bundle over the open  $C^\infty$ -subscheme  $\tilde{U}$  of  $\bar{U}$  mapped to  $\mathcal{V}$  by  $f \circ i \circ \pi_{[U/G]}$ , where the  $H$ -action for the principal  $H$ -bundle is  $\delta \mapsto \underline{d}_\delta$ . As  $u \in \tilde{U}$ , this implies that we can choose a  $G$ -invariant open neighbourhood  $U'$  of  $u$  in  $\tilde{U} \subseteq \bar{U}$  with an isomorphism  $W' = \bar{a}^{-1}(U') \cong U' \times H$ , that identifies  $\underline{d}_\delta|_{W'} : W' \rightarrow W'$  with the product of  $\text{id}_{U'}$  on  $U'$  and  $\epsilon \mapsto \delta\epsilon$  on  $H$ .

Then  $\gamma \mapsto \bar{c}_\gamma|_{W'}$  is an action of  $G$  on  $W' \cong U' \times H$ , and the projection  $U' \times H \rightarrow U'$  is  $G$ -equivariant. Since  $u \in U'$  is a fixed point of  $G$ , this implies that  $\bar{c}_\gamma$  fixes the finite subset  $\{(u, \delta) : \delta \in H\}$  in  $U' \times H$ . Define  $\rho : G \rightarrow H$  by  $\bar{c}_\gamma(u, 1) = (u, \rho(\gamma)^{-1})$  for  $\gamma \in G$ . Since  $\underline{d}_\delta$  acts by  $(u, \epsilon) \mapsto (u, \delta\epsilon)$  and  $\bar{c}_\gamma, \underline{d}_\delta$  commute, it follows that  $\bar{c}_\gamma(u, \delta) = (u, \delta\rho(\gamma)^{-1})$  for  $\gamma \in G, \delta \in H$ . Hence

$$(u, \rho(\gamma\gamma')^{-1}) = \bar{c}_{\gamma\gamma'}(u, 1) = \bar{c}_\gamma \circ \bar{c}_{\gamma'}(u, 1) = \bar{c}_\gamma(u, \rho(\gamma')^{-1}) = (u, \rho(\gamma')^{-1}\rho(\gamma)^{-1}),$$

so  $\rho(\gamma\gamma')^{-1} = \rho(\gamma')^{-1}\rho(\gamma)^{-1}$ , and  $\rho(\gamma\gamma') = \rho(\gamma)\rho(\gamma')$  for  $\gamma, \gamma' \in G$ . Thus  $\rho : G \rightarrow H$  is a group morphism.

Using  $W' \cong U' \times H$ ,  $\bar{a} \circ \bar{c}_\gamma = \bar{r}(\gamma) \circ \bar{a}$ , and  $\bar{c}_\gamma(u, \delta) = (u, \delta\rho(\gamma)^{-1})$ , we see that close to  $\{u\} \times H$ ,  $\bar{c}_\gamma|_{W'} : U' \times H \rightarrow U' \times H$  acts as  $\bar{r}(\gamma)$  on  $U'$  and  $\delta \mapsto \delta\rho(\gamma)^{-1}$  on  $H$ . Making  $U'$  smaller if necessary, we can suppose this happens on all of  $U'$ . Write  $\underline{k} : U' \hookrightarrow \underline{W}$  for the inclusion of  $U'$  as an open  $C^\infty$ -subscheme in  $\underline{W}$  via the identifications  $U' \cong U' \times \{1\} \subseteq U' \times H \cong W' \subseteq \underline{W}$ , and define  $\underline{f} = \underline{b} \circ \underline{k} : U' \rightarrow \underline{V}$ .

Let  $\gamma \in G$ . Since  $\bar{c}_\gamma|_{W'}$  acts as  $\bar{r}(\gamma)$  on  $U'$  and  $\delta \mapsto \delta\rho(\gamma)^{-1}$  on  $H$ , and  $\underline{d}_{\rho(\gamma)}$  acts as  $\delta \mapsto \rho(\gamma)\delta$  on  $H$ , we see that  $\underline{d}_{\rho(\gamma)} \circ \bar{c}_\gamma$  acts as  $\bar{r}(\gamma) \times \text{id}_1$  on  $U' \times \{1\}$ . Hence  $\underline{k} \circ \bar{r}(\gamma)|_{U'} = \underline{d}_{\rho(\gamma)} \circ \bar{c}_\gamma \circ \underline{k}$ . Composing with  $\underline{b}$  gives

$$\begin{aligned} \underline{f} \circ \bar{r}(\gamma)|_{U'} &= \underline{b} \circ \underline{k} \circ \bar{r}(\gamma)|_{U'} = \underline{b} \circ \underline{d}_{\rho(\gamma)} \circ \bar{c}_\gamma \circ \underline{k} \\ &= \underline{s}(\rho(\gamma)) \circ \underline{b} \circ \bar{c}_\gamma \circ \underline{k} = \underline{s}(\rho(\gamma)) \circ \underline{b} \circ \underline{k} = \underline{s}(\rho(\gamma)) \circ \underline{f}, \end{aligned}$$

using  $\underline{b} \circ \underline{d}_\delta = \underline{s}(\delta) \circ \underline{b}$  and  $\underline{b} \circ \bar{c}_\gamma = \underline{b}$ . We have now constructed a  $C^\infty$ -scheme morphism  $\underline{f} : U' \rightarrow \underline{V}$  and a group morphism  $\rho : G \rightarrow H$  with  $\underline{f} \circ \bar{r}(\gamma)|_{U'} = \underline{s}(\rho(\gamma)) \circ \underline{f}$  for all  $\gamma \in G$ . Thus Definition 7.2 defines  $[\underline{f}, \rho] : [U'/G] \rightarrow [V/H]$ .

Consider the diagram of 2-morphisms:

$$\begin{array}{ccc} f \circ i|_{[U'/G]} \circ \pi_{[U'/G]} \xRightarrow{\nu} j \circ [f, \rho] \circ \pi_{[U'/G]} & \xlongequal{\quad} & j \circ \pi_{[V/H]} \circ \bar{f} \\ \parallel & \xrightarrow{\omega * \text{id}_{\bar{k}}} & \parallel \\ f \circ i \circ \pi_{[U/G]} \circ \bar{a} \circ \bar{k} & \xrightarrow{\quad} & j \circ \pi_{[V/H]} \circ \bar{b} \circ \bar{k}. \end{array} \quad (7.6)$$

Here  $\omega$  is as in (7.5), and we have used  $\bar{f} = \bar{b} \circ \bar{k}$ , so that  $\bar{f} = \bar{b} \circ \bar{k}$ , and  $\pi_{[U'/G]} = \pi_{[U/G]} \circ \bar{a} \circ \bar{k}$  since  $\bar{a} \circ \bar{k}$  is the inclusion  $\bar{U}' \hookrightarrow \bar{U}$ , and  $[f, \rho] \circ \pi_{[U'/G]} = \pi_{[V/H]} \circ \bar{f}$ . Thus there is a unique 2-morphism  $\nu = \omega * \text{id}_{\bar{k}}$  making (7.6) commute.

Using  $\omega * \text{id}_{\bar{k}} = \omega \odot (\text{id}_{f \circ i} * \lambda_\gamma * \text{id}_{\bar{a}})$  for  $\gamma \in G$  we can show that  $\nu$  is  $G$ -invariant in a suitable sense, and so pushes down from  $\bar{U}'$  to  $[U'/G]$ . That is, there exists a unique 2-morphism  $\zeta : f \circ i|_{[U'/G]} \Rightarrow j \circ [f, \rho]$  with  $\nu = \zeta * \text{id}_{\pi_{[U'/G]}}$ . So (7.3) 2-commutes, completing part (b).

For (c), let  $\bar{W}, \bar{a}, \bar{b}, \omega, \bar{c}_\gamma, \bar{d}_\delta, \bar{W}', \bar{k}, \bar{f}, \bar{\rho}$  be the data constructed in (b) above for  $f : \mathcal{X} \rightarrow \mathcal{Y}$ , and let  $\hat{W}, \hat{a}, \hat{b}, \hat{\omega}, \hat{c}_\gamma, \hat{d}_\delta, \hat{W}', \hat{k}, \hat{g}, \hat{\sigma}$  be the corresponding data constructed in (b) for  $g : \mathcal{X} \rightarrow \mathcal{Y}$ . Then combining  $\eta : f \Rightarrow g$  with the analogue of (7.5) for  $g$ , we have a 2-morphism

$$(\eta * \text{id}_{i \circ \pi_{[U/G]} \circ \bar{a}}) \odot \hat{\omega} : (f \circ i \circ \pi_{[U/G]}) \circ \bar{a} \Longrightarrow (j \circ \pi_{[V/H]}) \circ \bar{b}.$$

Arguing as in the construction of  $\bar{c}_\gamma$  above, by the 2-Cartesian property of (7.5), there exists a 1-morphism  $e : \bar{W} \rightarrow \bar{W}$ , unique up to 2-isomorphism, and 2-morphisms  $\hat{\zeta} : \bar{a} \circ e \Rightarrow \bar{a}$ ,  $\hat{\theta} : \bar{b} \circ e \Rightarrow \bar{b}$  satisfying (A.6). Then  $e \cong \bar{e}$  for a unique  $\bar{e} : \bar{W} \rightarrow \bar{W}$ . Replacing  $e$  by  $\bar{e}$ , we have  $\bar{a} \circ \bar{e} = \bar{a}$ ,  $\bar{b} \circ \bar{e} = \bar{b}$ ,  $\hat{\zeta} = \text{id}_{\bar{a}}$  and  $\hat{\theta} = \text{id}_{\bar{b}}$ , and (A.6) reduces to  $\omega * \text{id}_{\bar{e}} = (\eta * \text{id}_{i \circ \pi_{[U/G]} \circ \bar{a}}) \odot \hat{\omega}$ .

By repeating this for  $\eta^{-1} : g \Rightarrow f$ , we can easily show that  $\bar{e} : \bar{W} \rightarrow \bar{W}$  is an isomorphism, and identifies  $\bar{a}, \bar{b}, \omega, \bar{c}_\gamma, \bar{d}_\delta, \bar{W}'$  with  $\hat{W}, \hat{a}, \hat{b}, \hat{\omega}, \hat{c}_\gamma, \hat{d}_\delta, \hat{W}'$ , respectively. However, the isomorphisms  $\bar{W}' \cong \bar{U}' \times H$  and  $\hat{W}' \cong \bar{U}' \times H$  involved arbitrary choices of local trivializations of the principal  $H$ -bundles  $\bar{a} : \bar{W} \rightarrow \bar{U}$  and  $\hat{a} : \hat{W} \rightarrow \bar{U}$ , so  $\bar{e}$  need not identify these isomorphisms.

Abuse notation by identifying  $\bar{W}' = \bar{U}' \times H$  and  $\hat{W}' = \bar{U}' \times H$ . Since  $\bar{a} \circ \bar{e}(u, 1) = \hat{a}(u, 1) = u$  we see that  $\bar{e}(u, 1) = (u, \delta)$  for some unique  $\delta \in H$ . As  $\bar{e}$  identifies  $\bar{d}_\epsilon$  and  $\hat{d}_\epsilon$  for  $\epsilon \in H$  we have

$$\bar{e}(u, \epsilon) = \bar{e} \circ \hat{d}_\epsilon(u, 1) = \bar{d}_\epsilon \circ \bar{e}(u, 1) = \bar{d}_\epsilon(u, \delta) = (u, \epsilon\delta). \quad (7.7)$$

Similarly, as  $\bar{e}$  identifies  $\bar{c}_\epsilon$  and  $\hat{c}_\gamma$  for  $\gamma \in G$ , and  $\bar{c}_\gamma, \hat{c}_\gamma$  act on  $\{u\} \times H$  by right multiplication by  $\rho(\gamma)^{-1}, \sigma(\gamma)^{-1}$  in  $H$ , we have

$$\bar{e}(u, \sigma(\gamma)^{-1}) = \bar{e} \circ \hat{c}_\gamma(u, 1) = \bar{c}_\gamma \circ \bar{e}(u, 1) = \bar{c}_\gamma(u, \delta) = (u, \delta\rho(\gamma)^{-1}). \quad (7.8)$$

Comparing (7.8) and (7.7) with  $\epsilon = \sigma(\gamma)^{-1}$ , we see that  $\sigma(\gamma)^{-1}\delta = \delta\rho(\gamma)^{-1}$ , so  $\sigma(\gamma) = \delta\rho(\gamma)\delta^{-1}$  for all  $\gamma \in \Gamma$ .

Since  $\bar{a} \circ \bar{e} = \hat{a}$ , regarding  $\bar{e}|_{\bar{W}'}$  as a morphism  $\bar{U}' \times H \rightarrow \bar{U}' \times H$ , we have  $\pi_{\bar{U}'} \circ \bar{e}|_{\bar{W}'} = \pi_{\bar{U}'}$ . So by (7.7),  $\bar{e}|_{\bar{W}'}$  is near  $\{u\} \times H$  the product of  $\text{id}_{\bar{U}'}$  on  $\bar{U}'$  and  $\epsilon \mapsto \epsilon\delta$  on  $H$ . Choose a  $G$ -invariant open neighbourhood  $\bar{U}''$  of  $u$  in  $\bar{U}'$  such that  $\bar{e}|_{\bar{U}'' \times H}$  is the product of  $\text{id}_{\bar{U}''}$  and  $\epsilon \mapsto \epsilon\delta$ . Then

$$g|_{\bar{U}''} = \hat{b} \circ \hat{k}|_{\bar{U}''} = \bar{b} \circ \bar{e} \circ \hat{k}|_{\bar{U}''} = \bar{b} \circ \bar{d}_\delta \circ \bar{k}|_{\bar{U}''} = \bar{s}(\delta) \circ \bar{b} \circ \bar{k}|_{\bar{U}''} = \bar{s}(\delta) \circ \bar{f}|_{\bar{U}''}.$$



Hence  $\sigma(\gamma) = \delta \rho(\gamma) \delta^{-1}$  for all  $\gamma \in G$  and  $\underline{g}|_{\underline{U}''} = \underline{s}(\delta) \circ \underline{f}|_{\underline{U}''}$ . Thus by Definition 7.3 we have a quotient 2-morphism  $[\delta] : [\underline{f}|_{\underline{U}'}, \rho] \Rightarrow [\underline{g}|_{\underline{U}'}, \sigma]$ . An argument similar to the last part of (b) then shows that (7.4) commutes.  $\square$

Using the method of Theorem 7.18(c), we can also prove:

**PROPOSITION 7.19.** *Let  $[\underline{f}, \rho], [\underline{g}, \sigma] : [\underline{X}/G] \rightarrow [\underline{Y}/H]$  be quotient 1-morphisms of quotient  $C^\infty$ -stacks in the sense of §7.1, and suppose  $[\underline{X}/G]$  is connected, that is,  $X/G$  is connected as a topological space. Then every 2-morphism  $\eta : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$  in  $\mathbf{C}^\infty\mathbf{Sta}$  is a quotient 2-morphism  $[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$  from Definition 7.3, for some unique  $\delta \in H$ .*

**PROOF.** Let  $\eta : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$  be a 2-morphism. The proof of Theorem 7.18(c) shows that for each  $[x] \in [\underline{X}/G]_{\text{top}} \cong X/G$ , there exists a unique  $\delta_{[x]} \in H$  and an open neighbourhood  $[\underline{U}_{[x]}/G]$  of  $[x]$  in  $[\underline{X}/G]$ , where  $\underline{U}_{[x]} \subseteq \underline{X}$  is  $G$ -invariant and open, such that  $\eta|_{[\underline{U}_{[x]}/G]} = [\delta_{[x]}]|_{[\underline{U}_{[x]}/G]} : [\underline{f}, \rho]|_{[\underline{U}_{[x]}/G]} \Rightarrow [\underline{g}, \sigma]|_{[\underline{U}_{[x]}/G]}$ . The map  $X/G \rightarrow H$  taking  $[x] \mapsto \delta_{[x]}$  is locally constant, as it is constant on each such open  $[\underline{U}_{[x]}/G]$ , so it is globally constant as  $X/G$  is connected, and  $\delta_{[x]} = \delta \in H$  for all  $[x] \in X/G$ . Thus,  $[\underline{X}/G]$  may be covered by open  $[\underline{U}_{[x]}/G] \subseteq [\underline{X}/G]$  with  $\eta|_{[\underline{U}_{[x]}/G]} = [\delta]|_{[\underline{U}_{[x]}/G]}$ . As 2-morphisms in  $\mathbf{C}^\infty\mathbf{Sta}$  form a sheaf, this proves that  $\eta = [\delta]$ .  $\square$

If  $\mathcal{X} = \bar{\underline{X}}$  for some  $C^\infty$ -scheme  $\underline{X}$  then  $\text{Iso}_{\mathcal{X}}([x]) \cong \{1\}$  for all  $[x] \in \mathcal{X}_{\text{top}}$ . Conversely, a Deligne–Mumford  $C^\infty$ -stack with trivial isotropy groups is a  $C^\infty$ -scheme. Note that in conventional algebraic geometry, a Deligne–Mumford stack with trivial stabilizers is an *algebraic space*, but need not be a scheme.

**THEOREM 7.20.** *Suppose  $\mathcal{X}$  is a Deligne–Mumford  $C^\infty$ -stack with  $\text{Iso}_{\mathcal{X}}([x]) \cong \{1\}$  for all  $[x] \in \mathcal{X}_{\text{top}}$ . Then  $\mathcal{X}$  is equivalent to  $\bar{\underline{X}}$  for some  $C^\infty$ -scheme  $\underline{X}$ .*

**PROOF.** As  $\text{Iso}_{\mathcal{X}}([x]) \cong \{1\}$  for all  $[x] \in \mathcal{X}_{\text{top}}$ , by Theorem 7.18(a) there is an open cover  $\{\mathcal{X}_a : a \in A\}$  of  $\mathcal{X}$  with  $\mathcal{X}_a \simeq [\underline{X}_a/\{1\}] \simeq \bar{\underline{X}}_a$  for affine  $C^\infty$ -schemes  $\underline{X}_a$ ,  $a \in A$ . Write  $i_a : \bar{\underline{X}}_a \rightarrow \mathcal{X}$  for the corresponding open embedding. As  $\Delta_{\mathcal{X}}$  is representable, for  $a, b \in A$  the fibre product  $\bar{\underline{X}}_a \times_{i_a, \mathcal{X}, i_b} \bar{\underline{X}}_b$  is represented by a  $C^\infty$ -scheme  $\underline{X}_{ab} = \underline{X}_{ba}$  with open embeddings  $i_{ab} : \underline{X}_{ab} \rightarrow \underline{X}_a$ ,  $i_{ba} : \underline{X}_{ba} \rightarrow \underline{X}_b$  identifying  $\underline{X}_{ab}$  with open  $C^\infty$ -subschemes of  $\underline{X}_a, \underline{X}_b$ .

The idea now is that the  $C^\infty$ -stack  $\mathcal{X}$  is made by gluing the  $C^\infty$ -schemes  $\underline{X}_a$  for  $a \in A$  together on the overlaps  $\underline{X}_{ab}$ , that is, we identify  $\underline{X}_a \supset i_{ab}(\underline{X}_{ab}) \cong \underline{X}_{ab} = \underline{X}_{ba} \cong i_{ba}(\underline{X}_{ba}) \subset \underline{X}_b$ . This is similar to the notion of descent for objects in §A.3, and it is easy to check that the natural 1-isomorphisms

$$\bar{\underline{X}}_{ab} \times_{\mathcal{X}} \bar{\underline{X}}_c \cong \bar{\underline{X}}_{bc} \times_{\mathcal{X}} \bar{\underline{X}}_a \cong \bar{\underline{X}}_{ca} \times_{\mathcal{X}} \bar{\underline{X}}_b \cong \bar{\underline{X}}_a \times_{\mathcal{X}} \bar{\underline{X}}_b \times_{\mathcal{X}} \bar{\underline{X}}_c$$

imply the obvious compatibility conditions of the gluing morphisms  $i_{ab}$  on triple overlaps, and that  $\underline{X}_{aa} \cong \underline{X}_a$ . So by a minor modification of the proof in Proposition 6.2 that  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$  has descent for objects, we construct a  $C^\infty$ -scheme  $\underline{X}$  with open embeddings  $j_a : \underline{X}_a \hookrightarrow \underline{X}$  such that  $\{\underline{X}_a : a \in A\}$  is an open cover of  $\underline{X}$ , and  $\underline{X}_a \times_{j_a, \underline{X}, j_b} \underline{X}_b$  is identified with  $\underline{X}_{ab}$  for  $a, b \in A$ . Then by descent for morphisms in  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$ , there exists a 1-morphism  $i : \bar{\underline{X}} \rightarrow \mathcal{X}$  with  $i_a$  2-isomorphic to  $i \circ \bar{j}_a$  for all  $a \in A$ . This  $i$  is an equivalence, so  $\mathcal{X} \simeq \bar{\underline{X}}$ , as we have to prove.  $\square$

In fact in Theorem 7.20 we can take  $\underline{X} = \underline{\mathcal{X}}_{\text{top}}$ , for  $\underline{\mathcal{X}}_{\text{top}}$  as in Definition 6.18. Recall from Definition A.14 that a 1-morphism of  $C^\infty$ -stacks  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is

*representable* if whenever  $\underline{U}$  is a  $C^\infty$ -scheme and  $g : \underline{U} \rightarrow \mathcal{Y}$  a 1-morphism then the fibre product  $\mathcal{W} = \mathcal{X} \times_{f, \mathcal{Y}, g} \underline{U}$  in  $\mathbf{C}^\infty\mathbf{Sta}$  is equivalent to a  $C^\infty$ -scheme  $\underline{V}$ .

**COROLLARY 7.21.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of Deligne–Mumford  $C^\infty$ -stacks. Then  $f$  is representable if and only if  $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \rightarrow \mathrm{Iso}_{\mathcal{Y}}([y])$  in Definition 6.21 is injective for all  $[x] \in \mathcal{X}_{\mathrm{top}}$  with  $f_{\mathrm{top}}([x]) = [y] \in \mathcal{Y}_{\mathrm{top}}$ .*

**PROOF.** Suppose  $f$  is representable, and let  $[x] \in \mathcal{X}_{\mathrm{top}}$  with  $f_{\mathrm{top}}([x]) = [y] \in \mathcal{Y}_{\mathrm{top}}$ . Then  $y : \bar{x} \rightarrow \mathcal{Y}$ , and  $\mathcal{X} \times_{f, \mathcal{Y}, y} \bar{x} \simeq [* / H]$ , where  $H = \mathrm{Ker}(f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \rightarrow \mathrm{Iso}_{\mathcal{Y}}([y]))$ . As  $f$  is representable,  $[* / H]$  is equivalent to a  $C^\infty$ -scheme, so  $H = \{1\}$ , and  $f_*$  is injective. This proves the ‘only if’ part.

Now suppose  $f_*$  is injective for all  $[x] \in \mathcal{X}_{\mathrm{top}}$ . Let  $\underline{U}$  be a  $C^\infty$ -scheme and  $g : \underline{U} \rightarrow \mathcal{Y}$  a 1-morphism, and define  $\mathcal{W} = \mathcal{X} \times_{f, \mathcal{Y}, g} \underline{U}$ , with projections  $d : \mathcal{W} \rightarrow \mathcal{X}$  and  $e : \mathcal{W} \rightarrow \underline{U}$ . Then  $\mathcal{W}$  is a Deligne–Mumford  $C^\infty$ -stack by Theorem 7.10, as  $\mathcal{X}, \mathcal{Y}, \underline{U}$  are. Let  $[w] \in \mathcal{W}_{\mathrm{top}}$ , and set  $[x] = d_{\mathrm{top}}([w])$  in  $\mathcal{X}_{\mathrm{top}}$ ,  $[u] = e_{\mathrm{top}}([w])$  in  $\underline{U}_{\mathrm{top}}$ , and  $[y] = f_{\mathrm{top}}([x]) = g_{\mathrm{top}}([u])$  in  $\mathcal{Y}_{\mathrm{top}}$ . Then by properties of fibre products of  $C^\infty$ -stacks we have a Cartesian square of groups

$$\begin{array}{ccc} \mathrm{Iso}_{\mathcal{W}}([w]) & \xrightarrow{e_*} & \mathrm{Iso}_{\underline{U}}([u]) \\ \downarrow d_* & & \downarrow g_* \\ \mathrm{Iso}_{\mathcal{X}}([x]) & \xrightarrow{f_*} & \mathrm{Iso}_{\mathcal{Y}}([y]). \end{array}$$

But  $\mathrm{Iso}_{\underline{U}}([u]) = \{1\}$  as  $\underline{U}$  is a  $C^\infty$ -scheme, and  $f_*$  is injective by assumption, so  $\mathrm{Iso}_{\mathcal{W}}([w]) = \{1\}$ , for all  $[w] \in \mathcal{W}_{\mathrm{top}}$ . Thus Theorem 7.20 shows  $\mathcal{W}$  is a  $C^\infty$ -scheme, and  $f$  is representable, proving the ‘if’ part.  $\square$

We show that  $\mathcal{X}$  being Deligne–Mumford is essential in Theorem 7.20:

**EXAMPLE 7.22.** Let the group  $\mathbb{Z}^2$  act on  $\mathbb{R}$  by  $(a, b) : x \mapsto x + a + b\sqrt{2}$  for  $a, b \in \mathbb{Z}$  and  $x \in \mathbb{R}$ . As  $\sqrt{2}$  is irrational, this is a free action. It defines a groupoid  $\mathbb{Z}^2 \times \mathbb{R} \rightrightarrows \mathbb{R}$  in  $\mathbf{Man}$  which is étale, but not proper. Applying  $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}$  gives a groupoid  $\mathbb{Z}^2 \times \mathbb{R} \rightrightarrows \mathbb{R}$  in  $\mathbf{C}^\infty\mathbf{Sch}$ , and an associated  $C^\infty$ -stack  $\mathcal{X} = [\mathbb{R} / \mathbb{Z}^2] = [\mathbb{Z}^2 \times \mathbb{R} \rightrightarrows \mathbb{R}]$ . The underlying topological space  $\mathcal{X}_{\mathrm{top}}$  is  $\mathbb{R} / \mathbb{Z}^2$ .

Since each orbit of  $\mathbb{Z}^2$  in  $\mathbb{R}$  is dense in  $\mathbb{R}$ ,  $\mathcal{X}_{\mathrm{top}}$  has the indiscrete topology, that is, the only open sets are  $\emptyset$  and  $\mathcal{X}_{\mathrm{top}}$ . Thus  $\mathcal{X}_{\mathrm{top}}$  is not homeomorphic to  $X$  for any  $C^\infty$ -scheme  $X = (X, \mathcal{O}_X)$ , as each point of  $X$  has an affine and hence Hausdorff open neighbourhood. Therefore  $\mathcal{X}$  is not equivalent to  $\bar{X}$  for any  $C^\infty$ -scheme  $\bar{X}$ . So  $\mathcal{X}$  is not Deligne–Mumford by Theorem 7.20. Hence,  $C^\infty$ -stacks with finite isotropy groups need not be Deligne–Mumford.

### 7.5. Effective Deligne–Mumford $C^\infty$ -stacks

**DEFINITION 7.23.** A Deligne–Mumford  $C^\infty$ -stack  $\mathcal{X}$  is called *effective* if whenever  $[x] \in \mathcal{X}_{\mathrm{top}}$  and  $\mathcal{X}$  near  $[x]$  is locally modelled near  $[x]$  on a quotient  $C^\infty$ -stack  $[\underline{U} / G]$  near  $[u]$ , where  $G = \mathrm{Iso}_{\mathcal{X}}([x])$  and  $u \in \underline{U}$  is fixed by  $G$ , as in Theorem 7.18(a), then  $G$  acts effectively on  $\underline{U}$  near  $u$ . That is, for each  $1 \neq \gamma \in G$ , we have  $\underline{r}(\gamma) \neq \mathrm{id}_{\underline{U}}$  near  $u$  in  $\underline{U}$ , where  $\underline{r} : G \rightarrow \mathrm{Aut}(\underline{U})$  is the  $G$ -action.

Here the  $C^\infty$ -scheme  $\underline{U}$  in Theorem 7.18(a) is determined by  $\mathcal{X}, [x]$  up to  $G$ -equivariant isomorphism locally near  $u$ . Hence to test whether  $\mathcal{X}$  is effective, it is enough to consider one choice of  $[\underline{U} / G]$  for each  $[x] \in \mathcal{X}_{\mathrm{top}}$ .

A quotient  $C^\infty$ -stack  $[\underline{X} / G]$  is effective if and only if the action  $\underline{r} : G \rightarrow \mathrm{Aut}(\underline{X})$  of  $G$  on  $\underline{X}$  is *locally effective*, that is, if for each  $1 \neq \gamma \in G$  we have  $\underline{r}(\gamma)|_{\underline{U}} \neq \mathrm{id}_{\underline{U}}$

for every nonempty open  $C^\infty$ -subscheme  $\underline{U} \subseteq \underline{X}$ . If a Deligne–Mumford  $C^\infty$ -stack  $\mathcal{X}$  is a  $C^\infty$ -scheme, it is automatically effective. Quotients  $[\ast/G]$  for  $G \neq \{1\}$  are not effective.

Here is a uniqueness property of 2-morphisms of effective Deligne–Mumford  $C^\infty$ -stacks. Embeddings and submersions of  $C^\infty$ -stacks are defined in §6.2.

**PROPOSITION 7.24.** *Let  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  be 1-morphisms of Deligne–Mumford  $C^\infty$ -stacks. Suppose any one of the following conditions hold:*

- (i)  $\mathcal{X}$  is effective and  $f$  is an embedding of  $C^\infty$ -stacks (this implies  $f_\ast : \text{Iso}_{\mathcal{X}}([x]) \rightarrow \text{Iso}_{\mathcal{Y}}(f_{\text{top}}([x]))$  is an isomorphism for each  $[x] \in \mathcal{X}_{\text{top}}$ );
- (ii)  $\mathcal{Y}$  is effective and  $f$  is a submersion; or
- (iii)  $\mathcal{Y}$  is a  $C^\infty$ -scheme.

*Then there exists at most one 2-morphism  $\eta : f \Rightarrow g$ . That is, the groupoid of such 1-morphisms is equivalent to a set.*

**PROOF.** Suppose  $\eta, \tilde{\eta} : f \Rightarrow g$  are 2-morphisms. Let  $[x] \in \mathcal{X}_{\text{top}}$  with  $f_{\text{top}}([x]) = [y] \in \mathcal{Y}_{\text{top}}$ . Apply Theorem 7.18(c) to  $\eta, \tilde{\eta}$ . This first applies (a) to  $\mathcal{X}, \mathcal{Y}$  at  $[x], [y]$ , giving  $i : [\underline{U}/G] \xrightarrow{\sim} \underline{\mathcal{U}} \subseteq \mathcal{X}$  identifying  $u \in \underline{U}$  with  $[x]$  and  $j : [\underline{V}/H] \xrightarrow{\sim} \underline{\mathcal{V}} \subseteq \mathcal{Y}$  identifying  $v \in \underline{V}$  with  $[y]$ , and then applies (b) to  $f, g$  giving  $u \in \underline{U}' \subseteq \underline{U}$  and 1-morphisms  $[f, \rho], [g, \sigma] : [\underline{U}/G] \rightarrow [\underline{V}/H]$ . Then (c) for  $\eta$  and  $\tilde{\eta}$  gives  $G$ -invariant open  $u \in \underline{U}'', \tilde{\underline{U}}'' \subseteq \underline{U}'$  and elements  $\delta, \tilde{\delta} \in H$  with 2-morphisms  $[\delta] : [f|_{\underline{U}''}, \rho] \Rightarrow [g|_{\underline{U}''}, \sigma]$ ,  $[\tilde{\delta}] : [f|_{\tilde{\underline{U}}''}, \rho] \Rightarrow [g|_{\tilde{\underline{U}}''}, \sigma]$  such that (7.4) and its analogue for  $\tilde{\eta}, \tilde{\delta}, \tilde{\underline{U}}''$  commutes. Making  $\underline{U}'', \tilde{\underline{U}}''$  smaller, we can take  $\underline{U}'' = \tilde{\underline{U}}''$ .

The 2-morphisms  $[\delta], [\tilde{\delta}] : [f|_{\underline{U}''}, \rho] \Rightarrow [g|_{\underline{U}''}, \sigma]$  imply that

$$\underline{s}(\delta) \circ \underline{f}|_{\underline{U}''} = \underline{g}|_{\underline{U}''} = \underline{s}(\tilde{\delta}) \circ \underline{f}|_{\underline{U}''}. \quad (7.9)$$

We will show that (7.9) and each of conditions (i)–(iii) force  $\delta = \tilde{\delta}$ . In case (i), as  $f$  is an embedding,  $\rho : G \rightarrow H$  is an isomorphism, and  $\underline{f} : \underline{U} \rightarrow \underline{V}$  is an embedding of  $C^\infty$ -schemes. Hence  $\delta = \rho(\gamma)$ ,  $\tilde{\delta} = \rho(\hat{\gamma})$  for  $\gamma, \hat{\gamma} \in G$ , and

$$\underline{f} \circ \underline{r}(\gamma)|_{\underline{U}''} = \underline{s}(\delta) \circ \underline{f}|_{\underline{U}''} = \underline{s}(\tilde{\delta}) \circ \underline{f}|_{\underline{U}''} = \underline{f} \circ \underline{r}(\hat{\gamma})|_{\underline{U}''}$$

by (7.9). As  $\underline{f}$  is an embedding this implies that  $\underline{r}(\gamma)|_{\underline{U}''} = \underline{r}(\hat{\gamma})|_{\underline{U}''}$ , so  $\gamma = \hat{\gamma}$  as  $G$  acts effectively on  $\underline{U}$  near  $u$  since  $\mathcal{X}$  is effective, and thus  $\delta = \tilde{\delta}$ .

In case (ii), as  $f$  is a submersion,  $\underline{f} : \underline{U} \rightarrow \underline{V}$  is surjective near  $\underline{f}(u) = v \in \underline{V}$ . Hence (7.9) implies that  $\underline{s}(\delta)|_{\underline{V}''} = \underline{s}(\tilde{\delta})|_{\underline{V}''}$  for some open neighbourhood  $\underline{V}''$  of  $v$  in  $\underline{V}$ . But  $H$  acts effectively on  $\underline{V}$  near  $v$  as  $\mathcal{Y}$  is effective, so  $\delta = \tilde{\delta}$ . In case (iii)  $H = \text{Iso}_{\mathcal{Y}}([y]) = \{1\}$  as  $\mathcal{Y}$  is a  $C^\infty$ -scheme, so  $\delta = \tilde{\delta} = 1$ . Therefore  $\delta = \tilde{\delta}$  in each case. Equation (7.4) for  $\eta, \tilde{\eta}$  now implies that  $\eta \ast \text{id}_{i|_{[\underline{U}''/G]}} = \tilde{\eta} \ast \text{id}_{i|_{[\underline{U}''/G]}}$ .

Let  $\mathcal{U}'' \subseteq \mathcal{U} \subseteq \mathcal{X}$  be the open  $C^\infty$ -substack identified with  $[\underline{U}''/G]$ . Then  $i|_{[\underline{U}''/G]} : [\underline{U}''/G] \rightarrow \mathcal{U}''$  is an equivalence, so  $\eta \ast \text{id}_{i|_{[\underline{U}''/G]}} = \tilde{\eta} \ast \text{id}_{i|_{[\underline{U}''/G]}}$  implies that  $\eta|_{\mathcal{U}''} = \tilde{\eta}|_{\mathcal{U}''}$ . Thus, each  $[x] \in \mathcal{X}_{\text{top}}$  has an open neighbourhood  $\mathcal{U}''$  in  $\mathcal{X}$  with  $\eta|_{\mathcal{U}''} = \tilde{\eta}|_{\mathcal{U}''}$ . As 2-morphisms form a sheaf on restriction to Zariski open  $C^\infty$ -substacks, this implies that  $\eta = \tilde{\eta}$ , so  $\eta : f \Rightarrow g$  is unique.  $\square$

Similar arguments show that if  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  are arbitrary 1-morphisms of Deligne–Mumford  $C^\infty$ -stacks with  $\mathcal{X}$  connected, then there are at most finitely many 2-morphisms  $\eta : f \Rightarrow g$ .

### 7.6. Orbifolds as Deligne–Mumford $C^\infty$ -stacks

Orbifolds are geometric spaces locally modelled on  $\mathbb{R}^n/G$  for  $G$  a finite group acting linearly on  $\mathbb{R}^n$ , just as manifolds are geometric spaces locally modelled on  $\mathbb{R}^n$ . Much has been written about orbifolds, and there are several definitions, as either categories or 2-categories. See Lerman [47] for a good overview.

There are three main definitions of ordinary categories of orbifolds:

- (a) Satake [62] and Thurston [65] defined an orbifold  $\mathcal{X}$  to be a Hausdorff topological space  $X$  with an atlas  $\{(V_i, \Gamma_i, \psi_i) : i \in I\}$  of orbifold charts  $(V_i, \Gamma_i, \psi_i)$ , where  $V_i$  is a manifold,  $\Gamma_i$  a finite group acting on  $V_i$ , and  $\psi_i : V_i/\Gamma_i \rightarrow X$  a homeomorphism with an open set in  $X$ . Smooth maps  $f : \mathcal{X} \rightarrow \mathcal{Y}$  between orbifolds are continuous maps  $f : X \rightarrow Y$  which lift locally to smooth maps on the charts, giving a category  $\mathbf{Orb}_{\text{ST}}$ .
- (b) Chen and Ruan [15, §4] defined orbifolds  $\mathcal{X}$  in a similar way to [62, 65], but using germs of orbifold charts  $(V_p, \Gamma_p, \psi_p)$  for  $p \in X$ . Their morphisms  $f : \mathcal{X} \rightarrow \mathcal{Y}$  are called *good maps*, giving a category  $\mathbf{Orb}_{\text{CR}}$ .
- (c) Moerdijk and Pronk [50, 51] defined a category of orbifolds  $\mathbf{Orb}_{\text{MP}}$  as *proper étale Lie groupoids* in  $\mathbf{Man}$ . Their smooth maps  $f : \mathcal{X} \rightarrow \mathcal{Y}$ , called *strong maps* [51, §5], are equivalence classes of diagrams  $\mathcal{X} \xleftarrow{\phi} \mathcal{X}' \xrightarrow{\psi} \mathcal{Y}$ , where  $\mathcal{X}'$  is a third orbifold, and  $\phi, \psi$  are morphisms of groupoids with  $\phi$  an equivalence.

A book on orbifolds in the sense of [15, 50, 51] is Adem, Leida and Ruan [2].

There are four main definitions of 2-categories of orbifolds:

- (i) Pronk [58] defines a strict 2-category  $\mathbf{LieGpd}$  of Lie groupoids in  $\mathbf{Man}$  as in (c), with the obvious 1-morphisms of groupoids, and localizes by a class of weak equivalences  $\mathcal{W}$  to get a weak 2-category  $\mathbf{Orb}_{\text{Pr}} = \mathbf{LieGpd}[\mathcal{W}^{-1}]$ .
- (ii) Lerman [47, §3.3] defines a weak 2-category  $\mathbf{Orb}_{\text{Le}}$  of Lie groupoids in  $\mathbf{Man}$  as in (c), with a non-obvious notion of 1-morphism called ‘Hilsum–Skandalis morphisms’ involving ‘bibundles’, and does not need to localize. Henriques and Metzler [33] also use Hilsum–Skandalis morphisms.
- (iii) Behrend and Xu [5, §2], Lerman [47, §4] and Metzler [49, §3.5] define a strict 2-category of orbifolds  $\mathbf{Orb}_{\text{ManSta}}$  as a class of Deligne–Mumford stacks on the site  $(\mathbf{Man}, \mathcal{J}_{\text{Man}})$  of manifolds with Grothendieck topology  $\mathcal{J}_{\text{Man}}$  coming from open covers.
- (iv) The author [39, §4.5] defines a weak 2-category of orbifolds  $\mathbf{Orb}_{\text{Kur}}$  as special examples of Kuranishi spaces.

As in Behrend and Xu [5, §2.6], Lerman [47], Pronk [58], and the author [39, Rem. 4.51(a)], approaches (i)–(iv) give equivalent weak 2-categories  $\mathbf{Orb}_{\text{Pr}}$ ,  $\mathbf{Orb}_{\text{Le}}$ ,  $\mathbf{Orb}_{\text{ManSta}}$ ,  $\mathbf{Orb}_{\text{Kur}}$ . Properties of localization imply that  $\mathbf{Orb}_{\text{MP}} \simeq \text{Ho}(\mathbf{Orb}_{\text{Pr}})$ . Thus, all of (c) and (i)–(iv) are equivalent at the level of weak 2-categories or homotopy categories.

Here is yet another definition of a strict 2-category of orbifolds  $\mathbf{Orb}_{C^\infty\text{Sta}}$ , which is similar to (iii), but defining orbifolds as a class of  $C^\infty$ -stacks, that is, as stacks on the site  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$  rather than on  $(\mathbf{Man}, \mathcal{J}_{\text{Man}})$ .

**DEFINITION 7.25.** A  $C^\infty$ -stack  $\mathcal{X}$  is called an *orbifold* if it is equivalent to the  $C^\infty$ -stack  $[\underline{V} \rightrightarrows \underline{U}]$  associated to a groupoid  $(\underline{U}, \underline{V}, s, t, u, i, m)$  in  $\mathbf{C}^\infty\mathbf{Sch}$  which is the image under  $F_{\text{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}$  of a groupoid  $(U, V, s, t, u, i, m)$  in  $\mathbf{Man}$ , where  $s : V \rightarrow U$  is an étale smooth map, and  $s \times t : V \rightarrow U \times U$  is a proper smooth map. That

is,  $\mathcal{X}$  is the  $C^\infty$ -stack associated to a *proper étale Lie groupoid* in  $\mathbf{Man}$ . Write  $\mathbf{Orb}_{C^\infty\text{Sta}}$  for the full 2-subcategory of orbifolds in  $\mathbf{C}^\infty\mathbf{Sta}$ .

As in §4.4,  $\underline{U}, \underline{V}$  are finitely presented affine  $C^\infty$ -schemes, and thus  $\mathcal{X}$  is a separated, locally finitely presented Deligne–Mumford  $C^\infty$ -stack by Theorem 7.17(b). Hence  $\mathbf{Orb}_{C^\infty\text{Sta}} \subset \mathbf{DMC}^\infty\mathbf{Sta}^{\text{lfp}}$ .

The next theorem follows from the proofs in [5, 39, 47, 58] that (i)–(iv) above are equivalent 2-categories (in particular, that orbifolds as stacks on  $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$  associated to proper étale Lie groupoids in (iii) are equivalent to (i),(ii),(iv)), and the fact that the inclusion  $\mathbf{Man} \hookrightarrow \mathbf{C}^\infty\mathbf{Sch}$  is full and faithful, with open covers  $\mathcal{J}$  in  $\mathbf{C}^\infty\mathbf{Sch}$  restricting to open covers  $\mathcal{J}_{\mathbf{Man}}$  in  $\mathbf{Man}$ .

**THEOREM 7.26.** *This 2-category of orbifolds  $\mathbf{Orb}_{C^\infty\text{Sta}}$  is equivalent to the 2-categories of orbifolds  $\mathbf{Orb}_{\text{Pr}}$ ,  $\mathbf{Orb}_{\text{Le}}$ ,  $\mathbf{Orb}_{\text{ManSta}}$ ,  $\mathbf{Orb}_{\text{Kur}}$  in [5, 39, 47, 49, 58] described in (i)–(iv) above. Also the homotopy category  $\text{Ho}(\mathbf{Orb}_{C^\infty\text{Sta}})$  is equivalent to the category of orbifolds  $\mathbf{Orb}_{\text{MP}}$  in [50, 51] described in (c) above.*

By Corollary 4.27  $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}$  takes transverse fibre products in  $\mathbf{Man}$  to fibre products in  $\mathbf{C}^\infty\mathbf{Sch}$ . As fibre products of orbifolds are locally modelled on fibre products of manifolds, and fibre products of Deligne–Mumford  $C^\infty$ -stacks are locally modelled on fibre products of  $C^\infty$ -schemes, we deduce:

**COROLLARY 7.27.** *Transverse fibre products in  $\mathbf{Orb}_{C^\infty\text{Sta}}$  agree with the corresponding fibre products in  $\mathbf{C}^\infty\mathbf{Sta}$ .*



## CHAPTER 8

### Sheaves on Deligne–Mumford $C^\infty$ -stacks

Next we discuss quasicoherent sheaves on Deligne–Mumford  $C^\infty$ -stacks  $\mathcal{X}$ , generalizing §5 for  $C^\infty$ -schemes. Some references on sheaves on orbifolds or stacks are Behrend and Xu [5, §3.1], Deligne and Mumford [17, Def. 4.10], Heinloth [32, §4], Laumon and Moret-Bailly [46, §13], and Moerdijk and Pronk [51, §2]. Our definitions are closest to [32, 51]. Almost everything in this section is an exercise in stack theory, not special to  $C^\infty$ -stacks, and would also work for sheaves (with étale descent) on other kinds of Deligne–Mumford stacks.

#### 8.1. Quasicoherent sheaves

We build our notions of sheaves on Deligne–Mumford  $C^\infty$ -stacks  $\mathcal{X}$  from those of sheaves on  $C^\infty$ -schemes  $\underline{U}$  in §5, by lifting to étale covers  $\bar{U} \rightarrow \mathcal{X}$ . Since all  $\mathcal{O}_U$ -modules on a  $C^\infty$ -scheme  $\underline{U}$  are quasicoherent by Corollary 5.22, we do not distinguish between  $\mathcal{O}_\mathcal{X}$ -modules and quasicoherent sheaves on a Deligne–Mumford  $C^\infty$ -stack  $\mathcal{X}$ , and we will just call them quasicoherent sheaves.

**DEFINITION 8.1.** Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack. Define a category  $\mathcal{C}_\mathcal{X}$  to have objects pairs  $(\underline{U}, u)$  where  $\underline{U}$  is a  $C^\infty$ -scheme and  $u : \bar{U} \rightarrow \mathcal{X}$  is an étale morphism, and morphisms  $(\underline{f}, \eta) : (\underline{U}, u) \rightarrow (\underline{V}, v)$  where  $\underline{f} : \underline{U} \rightarrow \underline{V}$  is an étale morphism of  $C^\infty$ -schemes, and  $\eta : u \Rightarrow v \circ \bar{f}$  is a 2-isomorphism. (Here  $\underline{f}$  étale is implied by  $u, v$  étale and  $u \cong v \circ \bar{f}$ .) If  $(\underline{f}, \eta) : (\underline{U}, u) \rightarrow (\underline{V}, v)$  and  $(\underline{g}, \zeta) : (\underline{V}, v) \rightarrow (\underline{W}, w)$  are morphisms in  $\mathcal{C}_\mathcal{X}$  then we define the composition  $(\underline{g}, \zeta) \circ (\underline{f}, \eta)$  to be  $(\underline{g} \circ \underline{f}, \theta) : (\underline{U}, u) \rightarrow (\underline{W}, w)$ , where  $\theta$  is the composition of 2-morphisms across the diagram:

$$\begin{array}{ccccc}
 \bar{U} & & & & \\
 \downarrow \bar{g} \circ \bar{f} & \searrow \bar{f} & & \xrightarrow{u} & \\
 \bar{V} & \xrightarrow{v} & & & \mathcal{X} \\
 \uparrow \bar{g} & \nwarrow \bar{g} & \searrow w & & \\
 \bar{W} & & & & 
 \end{array}
 \quad \begin{array}{c}
 \eta \\
 \zeta
 \end{array}$$

Define a *quasicoherent sheaf*  $\mathcal{E}$  on  $\mathcal{X}$  to assign a quasicoherent sheaf  $\mathcal{E}(\underline{U}, u)$  on  $\underline{U}$  for all objects  $(\underline{U}, u)$  in  $\mathcal{C}_\mathcal{X}$ , and an isomorphism  $\mathcal{E}_{(\underline{f}, \eta)} : \underline{f}^*(\mathcal{E}(\underline{V}, v)) \rightarrow \mathcal{E}(\underline{U}, u)$  in  $\text{qcoh}(\underline{U})$  for all morphisms  $(\underline{f}, \eta) : (\underline{U}, u) \rightarrow (\underline{V}, v)$  in  $\mathcal{C}_\mathcal{X}$ , such that for all  $(\underline{f}, \eta), (\underline{g}, \zeta), (\underline{g} \circ \underline{f}, \theta)$  as above the following diagram of isomorphisms in  $\text{qcoh}(\underline{U})$  commutes:

$$\begin{array}{ccccc}
 (\underline{g} \circ \underline{f})^*(\mathcal{E}(\underline{W}, w)) & \xrightarrow{\mathcal{E}_{(\underline{g} \circ \underline{f}, \theta)}} & & & \mathcal{E}(\underline{U}, u), \\
 \downarrow I_{\underline{f}, \underline{g}}(\mathcal{E}(\underline{W}, w)) & \searrow \underline{f}^*(\underline{g}^*(\mathcal{E}(\underline{W}, w))) & \xrightarrow{\underline{f}^*(\mathcal{E}_{(\underline{g}, \zeta)})} & \underline{f}^*(\mathcal{E}(\underline{V}, v)) & \xrightarrow{\mathcal{E}_{(\underline{f}, \eta)}} \\
 & & & & 
 \end{array}
 \quad (8.1)$$

for  $I_{f,g}(\mathcal{E})$  as in Remark 5.14.

A *morphism of quasicoherent sheaves*  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  assigns a morphism  $\phi(\underline{U}, u) : \mathcal{E}(\underline{U}, u) \rightarrow \mathcal{F}(\underline{U}, u)$  in  $\text{qcoh}(\underline{U})$  for each object  $(\underline{U}, u)$  in  $\mathcal{C}_{\mathcal{X}}$ , such that for all morphisms  $(f, \eta) : (\underline{U}, u) \rightarrow (\underline{V}, v)$  in  $\mathcal{C}_{\mathcal{X}}$  the following commutes:

$$\begin{array}{ccc} f^*(\mathcal{E}(\underline{V}, v)) & \xrightarrow{\mathcal{E}_{(f, \eta)}} & \mathcal{E}(\underline{U}, u) \\ \downarrow f^*(\phi(\underline{V}, v)) & & \downarrow \phi(\underline{U}, u) \\ f^*(\mathcal{F}(\underline{V}, v)) & \xrightarrow{\mathcal{F}_{(f, \eta)}} & \mathcal{F}(\underline{U}, u). \end{array}$$

We call  $\mathcal{E}$  a *vector bundle of rank  $n$*  if  $\mathcal{E}(\underline{U}, u)$  is a vector bundle of rank  $n$  for all  $(\underline{U}, u) \in \mathcal{C}_{\mathcal{X}}$ . Write  $\text{qcoh}(\mathcal{X})$  for the category of quasicoherent sheaves on  $\mathcal{X}$ .

**REMARK 8.2. (a)** Here is a second way to define quasicoherent sheaves, closer to [5, §3.1], [17, Def. 4.10]. Define a Grothendieck pretopology  $\mathcal{PT}_{\mathcal{X}}$  on  $\mathcal{C}_{\mathcal{X}}$  to have coverings  $\{(\underline{i}_a, \eta_a) : (\underline{U}_a, u_a) \rightarrow (\underline{U}, u)\}_{a \in A}$  where  $\underline{i}_a : \underline{U}_a \rightarrow \underline{U}$  is an open embedding for all  $a \in A$  and  $U = \bigcup_{a \in A} i_a(U_a)$ . Let  $\mathcal{J}_{\mathcal{X}}$  be the associated Grothendieck topology. Then  $(\mathcal{C}_{\mathcal{X}}, \mathcal{J}_{\mathcal{X}})$  is a site.

We can now use the standard notion of *sheaves on a site*, as in Artin [3] or Metzler [49, §2.1]. For all  $(\underline{U}, u)$  in  $\mathcal{C}_{\mathcal{X}}$ , define a  $C^\infty$ -ring  $\mathcal{O}_{\mathcal{X}}(\underline{U}, u) = \mathcal{O}_U(U)$ , where  $\underline{U} = (U, \mathcal{O}_U)$ . For all morphisms  $(f, \eta) : (\underline{V}, v) \rightarrow (\underline{U}, u)$ , define a morphism of  $C^\infty$ -rings  $\rho_{(\underline{U}, u)(\underline{V}, v)} : \mathcal{O}_{\mathcal{X}}(\underline{U}, u) \rightarrow \mathcal{O}_{\mathcal{X}}(\underline{V}, v)$  by  $\rho_{(\underline{U}, u)(\underline{V}, v)} = f_{\#}(U) : \mathcal{O}_U(U) \rightarrow \mathcal{O}_V(V)$ . Then  $\mathcal{O}_{\mathcal{X}}$  is a *sheaf of  $C^\infty$ -rings on the site  $(\mathcal{C}_{\mathcal{X}}, \mathcal{J}_{\mathcal{X}})$* .

Define a *quasicoherent sheaf*  $\mathcal{E}'$  to be a sheaf of  $\mathcal{O}_{\mathcal{X}}$ -modules on  $(\mathcal{C}_{\mathcal{X}}, \mathcal{J}_{\mathcal{X}})$ . That is,  $\mathcal{E}'$  assigns an  $\mathcal{O}_{\mathcal{X}}(\underline{U}, u)$ -module  $\mathcal{E}'(\underline{U}, u)$  for all  $(\underline{U}, u)$  in  $\mathcal{C}_{\mathcal{X}}$ , and a linear map  $\mathcal{E}'_{(f, \eta)} : \mathcal{E}'(\underline{U}, u) \rightarrow \mathcal{E}'(\underline{V}, v)$  for all  $(f, \eta) : (\underline{V}, v) \rightarrow (\underline{U}, u)$  in  $\mathcal{C}_{\mathcal{X}}$ , such that the analogue of (5.13) commutes, and the axioms for sheaves on a site hold.

If  $\mathcal{E}$  is as in Definition 8.1 then defining  $\mathcal{E}'(\underline{U}, u) = \Gamma(\mathcal{E}(\underline{U}, u))$  gives a quasicoherent sheaf in the sense of this second definition. Conversely, any quasicoherent sheaf in this second sense extends to one in the first sense uniquely up to canonical isomorphism. Thus the two definitions yield equivalent categories.

**(b)** As quasicoherent sheaves are a kind of sheaves of sets on a site, not sheaves of categories on a site as stacks are,  $\text{qcoh}(\mathcal{X})$  is a category not a 2-category.

**(c)** In Definition 8.1 we require the 1-morphisms  $u, v, w$  and morphisms  $f, g$  to be *étale*. This is important in several places below: for instance, if  $f : \underline{U} \rightarrow \underline{V}$  is étale then  $f^* : \text{qcoh}(\underline{V}) \rightarrow \text{qcoh}(\underline{U})$  is exact, not just right exact, which is needed in Proposition 8.3 to show  $\text{qcoh}(\mathcal{X})$  is abelian, and also  $\Omega_f : f^*(T^*\underline{V}) \rightarrow T^*\underline{U}$  is an isomorphism, which is needed to define the cotangent sheaf  $T^*\mathcal{X}$ . We restricted to Deligne–Mumford  $C^\infty$ -stacks  $\mathcal{X}$  in order to be able to use étale (1-)morphisms in this way. For  $C^\infty$ -stacks  $\mathcal{X}$  which do not admit an étale atlas, the approach above is inadequate and would need to be modified.

**(d)** Our notion of vector bundles  $\mathcal{E}$  over  $\mathcal{X}$  correspond to *orbifold vector bundles* when  $\mathcal{X}$  is an orbifold. That is, the isotropy groups  $\text{Iso}_{\mathcal{X}}([x])$  of  $\mathcal{X}$  are allowed to act nontrivially on the vector space fibres  $\mathcal{E}|_x$  of  $\mathcal{E}$ .

**(e)** We can also use the method of Definition 8.1 (or the approach of **(a)**) to define other kinds of sheaves on a Deligne–Mumford  $C^\infty$ -stack  $\mathcal{X}$ , such as sheaves of sets, abelian groups,  $C^\infty$ -rings,  $\dots$ , in the obvious way: we just take the  $\mathcal{E}(\underline{U}, u)$  to be a sheaf of sets,  $\dots$  on  $\underline{U}$  instead of a quasicoherent sheaf.



PROPOSITION 8.3. *Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack. Then  $\mathrm{qcoh}(\mathcal{X})$  is an abelian category.*

PROOF. We define a complex in  $\mathrm{qcoh}(\mathcal{X})$

$$0 \longrightarrow \mathcal{E} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \longrightarrow 0$$

to be *exact* if and only if

$$0 \longrightarrow \mathcal{E}(\underline{U}, u) \xrightarrow{\phi(\underline{U}, u)} \mathcal{F}(\underline{U}, u) \xrightarrow{\psi(\underline{U}, u)} \mathcal{G}(\underline{U}, u) \longrightarrow 0$$

is exact in  $\mathrm{qcoh}(\underline{U})$  for all  $(\underline{U}, u)$  in  $\mathcal{C}_{\mathcal{X}}$ . Since each  $\mathrm{qcoh}(\underline{U})$  in Definition 8.1 is abelian, and the functors  $\underline{f}^*$  in Definition 8.1 are exact (not just right exact) as  $\underline{f}$  is étale, it is easy to show this makes  $\mathrm{qcoh}(\mathcal{X})$  into an abelian category.  $\square$

EXAMPLE 8.4. Let  $\underline{X}$  be a  $C^\infty$ -scheme. Then  $\mathcal{X} = \bar{X}$  is a Deligne–Mumford  $C^\infty$ -stack. We will define an equivalence of categories  $\mathcal{I}_{\underline{X}} : \mathrm{qcoh}(\underline{X}) \rightarrow \mathrm{qcoh}(\mathcal{X})$ .

Let  $\mathcal{E}$  be an object in  $\mathrm{qcoh}(\underline{X})$ . If  $(\underline{U}, u)$  is an object in  $\mathcal{C}_{\mathcal{X}}$  then  $u : \bar{U} \rightarrow \mathcal{X} = \bar{X}$  is a 1-morphism, so as  $\mathbf{C}^\infty\mathbf{Sch}$ ,  $\bar{\mathbf{C}}^\infty\mathbf{Sch}$  are equivalent (2-)categories  $u$  is 2-isomorphic to  $\bar{u} : \bar{U} \rightarrow \bar{X}$  for some unique morphism  $\bar{u} : \bar{U} \rightarrow \bar{X}$ . Define  $\mathcal{E}'(\underline{U}, u) = \bar{u}^*(\mathcal{E})$ . If  $(\underline{f}, \eta) : (\underline{U}, u) \rightarrow (\underline{V}, v)$  is a morphism in  $\mathcal{C}_{\mathcal{X}}$  and  $\bar{u}, \bar{v}$  are associated to  $u, v$  as above, so that  $\bar{u} = \bar{v} \circ \bar{f}$ , then define

$$\mathcal{E}'_{(\underline{f}, \eta)} = I_{\bar{f}, \bar{v}}(\mathcal{E})^{-1} : \bar{f}^*(\mathcal{E}'(\underline{V}, v)) = \bar{f}^*(\bar{v}^*(\mathcal{E})) \rightarrow (\bar{v} \circ \bar{f})^*(\mathcal{E}) = \mathcal{E}'(\underline{U}, u).$$

Then (8.1) commutes for all  $(\underline{f}, \eta), (\underline{g}, \zeta)$ , so  $\mathcal{E}'$  is a quasicoherent sheaf on  $\mathcal{X}$ .

If  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  is a morphism in  $\mathrm{qcoh}(\underline{X})$  define a morphism  $\phi' : \mathcal{E}' \rightarrow \mathcal{F}'$  in  $\mathrm{qcoh}(\mathcal{X})$  by  $\phi'(\underline{U}, u) = \bar{u}^*(\phi)$  for  $\bar{u}$  associated to  $u$  as above. Then defining  $\mathcal{I}_{\underline{X}} : \mathcal{E} \mapsto \mathcal{E}'$ ,  $\mathcal{I}_{\underline{X}} : \phi \mapsto \phi'$  gives a functor  $\mathrm{qcoh}(\underline{X}) \rightarrow \mathrm{qcoh}(\mathcal{X})$ . There is a natural inverse construction: if  $\tilde{\mathcal{E}}$  is an object in  $\mathrm{qcoh}(\mathcal{X})$  then  $\tilde{\mathcal{E}}(\underline{X}, \bar{\mathrm{id}}_{\underline{X}})$  is an object in  $\mathrm{qcoh}(\underline{X})$ , and  $\tilde{\mathcal{E}}$  is canonically isomorphic to  $\mathcal{I}_{\underline{X}}(\tilde{\mathcal{E}}(\underline{X}, \bar{\mathrm{id}}_{\underline{X}}))$ . Using this we can show  $\mathcal{I}_{\underline{X}}$  is an equivalence of categories.

## 8.2. Writing sheaves in terms of a groupoid presentation

Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack. Then  $\mathcal{X}$  admits an étale atlas  $\Pi : \bar{U} \rightarrow \mathcal{X}$ , and as in §A.5 from  $\Pi$  we can construct a groupoid  $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$  in  $\mathbf{C}^\infty\mathbf{Sch}$ , with  $\underline{s}, \underline{t} : \underline{V} \rightarrow \underline{U}$  étale, such that  $\mathcal{X}$  is equivalent to the associated  $C^\infty$ -stack  $[\underline{V} \rightrightarrows \underline{U}]$ , and we have a 2-Cartesian diagram

$$\begin{array}{ccc} \bar{V} & \xrightarrow{\quad \bar{t} \quad} & \bar{U} \\ \downarrow \bar{s} & \eta \nearrow & \downarrow \Pi \\ \underline{U} & \xrightarrow{\quad \Pi \quad} & \mathcal{X}. \end{array}$$

We can now consider the objects  $(\underline{U}, \Pi)$  and  $(\underline{V}, \Pi \circ \underline{s})$  in  $\mathcal{C}_{\mathcal{X}}$ , and the two morphisms  $(\underline{s}, \mathrm{id}_{\Pi \circ \underline{s}}) : (\underline{V}, \Pi \circ \underline{s}) \rightarrow (\underline{U}, \Pi)$  and  $(\underline{t}, \eta) : (\underline{V}, \Pi \circ \underline{s}) \rightarrow (\underline{U}, \Pi)$ .

Now let  $\mathcal{E}$  be an object in  $\mathrm{qcoh}(\mathcal{X})$ . Then we have quasicoherent sheaves  $E = \mathcal{E}(\underline{U}, \Pi)$  on  $\underline{U}$  and  $E' = \mathcal{E}(\underline{V}, \Pi \circ \underline{s})$  on  $\underline{V}$ , and isomorphisms  $\mathcal{E}_{(\underline{s}, \mathrm{id}_{\Pi \circ \underline{s}})} : \underline{s}^*(E) \rightarrow E'$  and  $\mathcal{E}_{(\underline{t}, \eta)} : \underline{t}^*(E) \rightarrow E'$  in  $\mathrm{qcoh}(\underline{V})$ . Hence  $\Phi = \mathcal{E}_{(\underline{t}, \eta)}^{-1} \circ \mathcal{E}_{(\underline{s}, \mathrm{id}_{\Pi \circ \underline{s}})}$  is an isomorphism of  $\Phi : \underline{s}^*(E) \rightarrow \underline{t}^*(E)$  in  $\mathrm{qcoh}(\underline{V})$ .

$$\begin{array}{ccccc}
W & \xrightarrow{\quad} & V & & \\
\bar{\pi}_1 \swarrow & & \bar{m} \swarrow & & \searrow \bar{s} \\
\bar{V} & \xrightarrow{\quad} & \bar{U} & \xrightarrow{\quad} & \bar{U} \\
\searrow \bar{s} & & \bar{t} \swarrow & & \swarrow \Pi \\
& & \bar{V} & \xrightarrow{\quad} & \mathcal{X}, \\
& & \bar{t} \swarrow & & \swarrow \Pi \\
& & \bar{U} & \xrightarrow{\quad} & \mathcal{X},
\end{array}$$
$$\begin{array}{ccccc}
& E'' & \xleftarrow{\varepsilon_{(m, \theta_3)}} & m^*(E') & \\
\varepsilon_{(\pi_1, \theta_1)} \nearrow & & \varepsilon_{(\pi_2, \theta_2)} \nearrow & \xrightarrow{\underline{m}^*(\varepsilon_{(t, \eta)})} & \xrightarrow{\underline{m}^*(\varepsilon_{(s, \text{id}_{\Pi \circ s})})} \\
\pi_1^*(E') & \xleftarrow{\pi_1^*(\varepsilon_{(t, \eta)})} & (t \circ \pi_1)^*(E) = & \xrightarrow{\underline{m}^*(\varepsilon_{(t, \eta)})} & \xrightarrow{\underline{m}^*(\varepsilon_{(s, \text{id}_{\Pi \circ s})})} \\
& \circ I_{\pi_1, t}(E) & (t \circ \underline{m})^*(E) & \xleftarrow{\alpha} & \xrightarrow{\underline{m}^*(\varepsilon_{(s, \text{id}_{\Pi \circ s})})} \\
& & & & (s \circ \pi_2)^*(E) = \\
\pi_1^*(\varepsilon_{(s, \text{id}_{\Pi \circ s})}) \nearrow & & \xrightarrow{\gamma} \pi_2^*(E') & \xleftarrow{\pi_2^*(\varepsilon_{(s, \text{id}_{\Pi \circ s})})} & \xrightarrow{\pi_2^*(\varepsilon_{(s, \text{id}_{\Pi \circ s})})} \\
& \circ I_{\pi_1, s}(E) & \xrightarrow{\pi_2^*(\varepsilon_{(t, \eta)})} & \circ I_{\pi_2, s}(E) & \xrightarrow{\pi_2^*(\varepsilon_{(t, \eta)})} \\
& & \xrightarrow{\beta} & & \xrightarrow{\beta} \\
(\underline{s} \circ \pi_1)^*(E) = & (\underline{t} \circ \pi_2)^*(E) & \xleftarrow{\beta} & & \xrightarrow{\beta}
\end{array} \tag{8.2}$$

DEFINITION 8.5. Let  $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$  be a groupoid in  $\mathbf{C}^\infty\mathbf{Sch}$ , with  $\underline{s}, \underline{t} : \underline{V} \rightarrow \underline{U}$  étale, which we write as  $\underline{V} \rightrightarrows \underline{U}$  for short. Define a *quasicoherent sheaf* on  $\underline{V} \rightrightarrows \underline{U}$  to be a pair  $(E, \Phi)$  where  $E$  is a quasicoherent sheaf on  $\underline{U}$  and  $\Phi : \underline{s}^*(E) \rightarrow \underline{t}^*(E)$  is an isomorphism in  $\mathrm{qcoh}(\underline{V})$ , such that

in morphisms  $(\underline{s} \circ \underline{m})^*(E) \rightarrow (\underline{t} \circ \underline{m})^*(E)$  in  $\mathrm{qcoh}(\underline{W})$ . Define a *morphism*  $\phi : (E, \Phi) \rightarrow (F, \Psi)$  of such sheaves to be a morphism  $\phi : E \rightarrow F$  in  $\mathrm{qcoh}(\underline{U})$  such that  $\Psi \circ \underline{s}^*(\phi) = \underline{t}^*(\phi) \circ \Phi : \underline{s}^*(E) \rightarrow \underline{t}^*(F)$  in  $\mathrm{qcoh}(\underline{V})$ . Then quasicohherent sheaves on  $\underline{V} \rightrightarrows \underline{U}$  form an abelian category  $\mathrm{qcoh}(\underline{V} \rightrightarrows \underline{U})$ .

The next theorem is proved as in Laumon and Moret-Bailly [46, Prop. 12.4.5] or Olsson [56, Prop. 4.4].

For quotient  $C^\infty$ -stacks  $[U/G]$  with  $G$  a finite group, so that  $\underline{V} = G \times \underline{U}$ , a quasicoherent sheaf  $(E, \Phi)$  on  $\underline{V} \rightrightarrows \underline{U}$  is a quasicoherent sheaf  $E$  on  $\underline{U}$  with a lift  $\Phi$

of the  $G$ -action on  $\underline{U}$  up to  $E$ . That is,  $(E, \Phi)$  is a  $G$ -equivariant quasicoherent sheaf on  $\underline{U}$ . Hence, if a Deligne–Mumford  $C^\infty$ -stack  $\mathcal{X}$  is equivalent to a quotient  $[\underline{U}/G]$  with  $G$  finite, then  $\mathrm{qcoh}(\mathcal{X})$  is equivalent to the category  $\mathrm{qcoh}^G(\underline{U})$  of  $G$ -equivariant quasicoherent sheaves on  $\underline{U}$ .

### 8.3. Pullback of sheaves as a weak 2-functor

In Definition 5.13, for a morphism of  $C^\infty$ -schemes  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  we defined a right exact functor  $\underline{f}^* : \mathrm{qcoh}(\underline{Y}) \rightarrow \mathrm{qcoh}(\underline{X})$ . As in Remarks 4.6(b) and 5.14, pullbacks cannot always be made strictly functorial in  $\underline{f}$ , that is, we do not have  $\underline{f}^*(g^*(\mathcal{E})) = (g \circ \underline{f})^*(\mathcal{E})$  for all  $\underline{f} : \underline{X} \rightarrow \underline{Y}$ ,  $g : \underline{Y} \rightarrow \underline{Z}$  and  $\mathcal{E} \in \mathrm{qcoh}(\underline{Z})$ , but instead we have canonical isomorphisms  $I_{\underline{f}, g}(\mathcal{E}) : (g \circ \underline{f})^*(\mathcal{E}) \rightarrow \underline{f}^*(g^*(\mathcal{E}))$ .

We now generalize this to pullback for sheaves on Deligne–Mumford  $C^\infty$ -stacks. The new factor to consider is that we have not only 1-morphisms  $f : \mathcal{X} \rightarrow \mathcal{Y}$ , but also 2-morphisms  $\eta : f \Rightarrow g$  for 1-morphisms  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$ , and we must interpret pullback for 2-morphisms as well as 1-morphisms.

DEFINITION 8.7. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of Deligne–Mumford  $C^\infty$ -stacks, and  $\mathcal{F} \in \mathrm{qcoh}(\mathcal{Y})$ . A *pullback* of  $\mathcal{F}$  to  $\mathcal{X}$  is  $\mathcal{E} \in \mathrm{qcoh}(\mathcal{X})$ , together with the following data: if  $\underline{U}, \underline{V}$  are  $C^\infty$ -schemes and  $u : \underline{U} \rightarrow \mathcal{X}$  and  $v : \underline{V} \rightarrow \mathcal{Y}$  are étale 1-morphisms, then there is a  $C^\infty$ -scheme  $\underline{W}$  and morphisms  $\pi_{\underline{U}} : \underline{W} \rightarrow \underline{U}$ ,  $\pi_{\underline{V}} : \underline{W} \rightarrow \underline{V}$  giving a 2-Cartesian diagram:

$$\begin{array}{ccc} \underline{W} & \xrightarrow{\pi_{\underline{V}}} & \underline{V} \\ \downarrow \pi_{\underline{U}} & \nearrow \zeta & \downarrow v \\ \underline{U} & \xrightarrow{f \circ u} & \mathcal{Y}. \end{array} \quad (8.3)$$

Then an isomorphism  $i(\mathcal{F}, f, u, v, \zeta) : \pi_{\underline{U}}^*(\mathcal{E}(\underline{U}, u)) \rightarrow \pi_{\underline{V}}^*(\mathcal{F}(\underline{V}, v))$  in  $\mathrm{qcoh}(\underline{W})$  should be given, which is functorial in  $(\underline{U}, u)$  in  $\mathcal{C}_{\mathcal{X}}$  and  $(\underline{V}, v)$  in  $\mathcal{C}_{\mathcal{Y}}$  and the 2-isomorphism  $\zeta$  in (8.3). We usually write pullbacks  $\mathcal{E}$  as  $f^*(\mathcal{F})$ .

By a similar proof to Theorem 8.6, but using descent for objects and morphisms for quasicoherent sheaves on  $C^\infty$ -schemes  $\underline{Y}$  in the étale topology rather than the open cover topology on  $\underline{Y}$ , we can prove:

PROPOSITION 8.8. *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of Deligne–Mumford  $C^\infty$ -stacks, and  $\mathcal{F}$  be a quasicoherent sheaf on  $\mathcal{Y}$ . Then a pullback  $f^*(\mathcal{F})$  exists in  $\mathrm{qcoh}(\mathcal{X})$ , and is unique up to canonical isomorphism.*

From now on we will assume that we have *chosen* a pullback  $f^*(\mathcal{F})$  for all such  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $\mathcal{F}$ . This could be done either by some explicit construction of pullbacks, as in the  $C^\infty$ -scheme case in §5.3, or by using the Axiom of Choice. As in Remark 5.14 we cannot necessarily make these choices functorial in  $f$ .

DEFINITION 8.9. Choose pullbacks  $f^*(\mathcal{F})$  for all 1-morphisms  $f : \mathcal{X} \rightarrow \mathcal{Y}$  of Deligne–Mumford  $C^\infty$ -stacks and all  $\mathcal{F} \in \mathrm{qcoh}(\mathcal{Y})$ , as above.

Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be such a 1-morphism, and  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  be a morphism in  $\mathrm{qcoh}(\mathcal{Y})$ . Then  $f^*(\mathcal{E}), f^*(\mathcal{F}) \in \mathrm{qcoh}(\mathcal{X})$ . Define the *pullback morphism*  $f^*(\phi) : f^*(\mathcal{E}) \rightarrow f^*(\mathcal{F})$  to be the morphism in  $\mathrm{qcoh}(\mathcal{X})$  characterized as follows. Let  $u : \underline{U} \rightarrow \mathcal{X}$ ,  $v : \underline{V} \rightarrow \mathcal{Y}$ ,  $\underline{W}, \pi_{\underline{U}}, \pi_{\underline{V}}$  be as in Definition 8.7, with (8.3) 2-Cartesian.

Then the following diagram of morphisms in  $\mathrm{qcoh}(\underline{W})$  commutes:

$$\begin{array}{ccc} \pi_{\underline{U}}^*(f^*(\mathcal{E})(\underline{U}, u)) & \xrightarrow{i(\mathcal{E}, f, u, v, \zeta)} & \pi_{\underline{V}}^*(\mathcal{E}(\underline{V}, v)) \\ \downarrow \pi_{\underline{U}}^*(f^*(\phi)(\underline{U}, u)) & & \downarrow \pi_{\underline{V}}^*(\phi(\underline{V}, v)) \\ \pi_{\underline{U}}^*(f^*(\mathcal{F})(\underline{U}, u)) & \xrightarrow{i(\mathcal{F}, f, u, v, \zeta)} & \pi_{\underline{V}}^*(\mathcal{F}(\underline{V}, v)). \end{array}$$

Using descent for morphisms in  $\mathrm{qcoh}(\underline{Y})$  on  $C^\infty$ -schemes  $\underline{Y}$  in the étale topology, one can show that there is a unique morphism  $f^*(\phi)$  with this property. This defines a functor  $f^* : \mathrm{qcoh}(\mathcal{Y}) \rightarrow \mathrm{qcoh}(\mathcal{X})$ .

Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $g : \mathcal{Y} \rightarrow \mathcal{Z}$  be 1-morphisms of Deligne–Mumford  $C^\infty$ -stacks, and  $\mathcal{E} \in \mathrm{qcoh}(\mathcal{Z})$ . Then  $(g \circ f)^*(\mathcal{E})$  and  $f^*(g^*(\mathcal{E}))$  both lie in  $\mathrm{qcoh}(\mathcal{X})$ . One can show that  $f^*(g^*(\mathcal{E}))$  is a possible pullback of  $\mathcal{E}$  by  $g \circ f$ . Thus as in Remark 5.14, we have a canonical isomorphism  $I_{f,g}(\mathcal{E}) : (g \circ f)^*(\mathcal{E}) \rightarrow f^*(g^*(\mathcal{E}))$ . This defines a natural isomorphism of functors  $I_{f,g} : (g \circ f)^* \Rightarrow f^* \circ g^*$ .

Let  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  be 1-morphisms of Deligne–Mumford  $C^\infty$ -stacks,  $\eta : f \Rightarrow g$  a 2-morphism, and  $\mathcal{E} \in \mathrm{qcoh}(\mathcal{Y})$ . Then we have  $f^*(\mathcal{E}), g^*(\mathcal{E}) \in \mathrm{qcoh}(\mathcal{X})$ . Let  $u : \underline{U} \rightarrow \mathcal{X}$ ,  $v : \underline{V} \rightarrow \mathcal{Y}$ ,  $\underline{W}, \pi_{\underline{U}}, \pi_{\underline{V}}$  be as in Definition 8.7. Then as in (8.3) we have 2-Cartesian diagrams

$$\begin{array}{ccc} \bar{W} & \xrightarrow{\quad} & \bar{V} \\ \downarrow \pi_{\underline{U}} & \zeta \odot (\eta * \mathrm{id}_{u \circ \pi_{\underline{U}}}) \nearrow & \downarrow \pi_{\underline{V}} \\ \underline{U} & \xrightarrow{f \circ u} & \mathcal{Y}, \end{array} \quad \begin{array}{ccc} \bar{W} & \xrightarrow{\quad} & \bar{V} \\ \downarrow \pi_{\underline{U}} & \zeta \nearrow & \downarrow \pi_{\underline{V}} \\ \underline{U} & \xrightarrow{g \circ u} & \mathcal{Y}, \end{array}$$

where in  $\zeta \odot (\eta * \mathrm{id}_{u \circ \pi_{\underline{U}}})$  ‘ $*$ ’ is horizontal composition and ‘ $\odot$ ’ vertical composition of 2-morphisms. Thus we have isomorphisms in  $\mathrm{qcoh}(\underline{W})$ :

$$\begin{array}{ccc} \pi_{\underline{U}}^*(f^*(\mathcal{E})(\underline{U}, u)) & \xrightarrow{i(\mathcal{E}, f, u, v, \zeta \odot (\eta * \mathrm{id}_{u \circ \pi_{\underline{U}}}))} & \pi_{\underline{V}}^*(\mathcal{E}(\underline{V}, v)) \\ \vdots \downarrow & & \nearrow \\ \pi_{\underline{U}}^*(g^*(\mathcal{E})(\underline{U}, u)) & \xrightarrow{i(\mathcal{E}, g, u, v, \zeta)} & \pi_{\underline{V}}^*(\mathcal{E}(\underline{V}, v)). \end{array}$$

There is a unique isomorphism ‘ $\dashrightarrow$ ’ making this diagram commute. Taken over all  $(\underline{V}, v)$ , using descent for morphisms we can show these isomorphisms are pullbacks of a unique isomorphism  $f^*(\mathcal{E})(\underline{U}, u) \rightarrow g^*(\mathcal{E})(\underline{U}, u)$ , and taken over all  $(\underline{U}, u)$  these give an isomorphism  $\eta^*(\mathcal{E}) : f^*(\mathcal{E}) \rightarrow g^*(\mathcal{E})$  in  $\mathrm{qcoh}(\mathcal{X})$ . Over all  $\mathcal{E} \in \mathrm{qcoh}(\mathcal{Y})$ , this defines a natural isomorphism  $\eta^* : f^* \Rightarrow g^*$ .

If  $\mathcal{X}$  is a Deligne–Mumford  $C^\infty$ -stack with identity 1-morphism  $\mathrm{id}_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X}$  then for each  $\mathcal{E} \in \mathrm{qcoh}(\mathcal{X})$ ,  $\mathcal{E}$  is a possible pullback  $\mathrm{id}_{\mathcal{X}}^*(\mathcal{E})$ , so we have a canonical isomorphism  $\delta_{\mathcal{X}}(\mathcal{E}) : \mathrm{id}_{\mathcal{X}}^*(\mathcal{E}) \rightarrow \mathcal{E}$ . These define a natural isomorphism  $\delta_{\mathcal{X}} : \mathrm{id}_{\mathcal{X}}^* \Rightarrow \mathrm{id}_{\mathrm{qcoh}(\mathcal{X})}$ .

The proof of the next theorem is long but straightforward. Weak 2-functors are defined in Definition A.2.

**THEOREM 8.10.** *Mapping  $\mathcal{X}$  to  $\mathrm{qcoh}(\mathcal{X})$  for objects  $\mathcal{X}$  in  $\mathbf{DMC}^\infty\mathbf{Sta}$ , and mapping 1-morphisms  $f : \mathcal{X} \rightarrow \mathcal{Y}$  to  $f^* : \mathrm{qcoh}(\mathcal{Y}) \rightarrow \mathrm{qcoh}(\mathcal{X})$ , and mapping 2-morphisms  $\eta : f \Rightarrow g$  to  $\eta^* : f^* \Rightarrow g^*$  for 1-morphisms  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$ , and the natural isomorphisms  $I_{f,g} : (g \circ f)^* \Rightarrow f^* \circ g^*$  for all 1-morphisms  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $g : \mathcal{Y} \rightarrow \mathcal{Z}$  in  $\mathbf{DMC}^\infty\mathbf{Sta}$ , and  $\delta_{\mathcal{X}}$  for all  $\mathcal{X} \in \mathbf{DMC}^\infty\mathbf{Sta}$ , together make up a weak 2-functor  $(\mathbf{DMC}^\infty\mathbf{Sta})^{\mathrm{op}} \rightarrow \mathbf{AbCat}$ , where  $\mathbf{AbCat}$  is the 2-category of abelian categories. That is, they satisfy the conditions:*

- (a) If  $f : \mathcal{W} \rightarrow \mathcal{X}$ ,  $g : \mathcal{X} \rightarrow \mathcal{Y}$ ,  $h : \mathcal{Y} \rightarrow \mathcal{Z}$  are 1-morphisms in  $\mathbf{DMC}^\infty\mathbf{Sta}$  and  $\mathcal{E} \in \mathrm{qcoh}(\mathcal{Z})$  then the following diagram commutes in  $\mathrm{qcoh}(\mathcal{X})$ :

$$\begin{array}{ccc} (h \circ g \circ f)^*(\mathcal{E}) & \xrightarrow{I_{f, h \circ g}(\mathcal{E})} & f^*((h \circ g)^*(\mathcal{E})) \\ \downarrow I_{g \circ f, h}(\mathcal{E}) & & \downarrow f^*(I_{g, h}(\mathcal{E})) \\ (g \circ f)^*(h^*(\mathcal{E})) & \xrightarrow{I_{f, g}(h^*(\mathcal{E}))} & f^*(g^*(h^*(\mathcal{E}))). \end{array}$$

- (b) If  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a 1-morphism in  $\mathbf{DMC}^\infty\mathbf{Sta}$  and  $\mathcal{E} \in \mathrm{qcoh}(\mathcal{Y})$  then the following pairs of morphisms in  $\mathrm{qcoh}(\mathcal{X})$  are inverse:

$$f^*(\mathcal{E}) = \begin{array}{c} I_{\mathrm{id}_{\mathcal{X}}, f}(\mathcal{E}) \\ \xrightarrow{\quad} \mathrm{id}_{\mathcal{X}}(f^*(\mathcal{E})) \\ \delta_{\mathcal{X}}(f^*(\mathcal{E})) \xleftarrow{\quad} \end{array} (f \circ \mathrm{id}_{\mathcal{X}})^*(\mathcal{E}), \quad f^*(\mathcal{E}) = \begin{array}{c} I_{f, \mathrm{id}_{\mathcal{Y}}}(\mathcal{E}) \\ \xrightarrow{\quad} f^*(\mathrm{id}_{\mathcal{Y}}^*(\mathcal{E})) \\ f^*(\delta_{\mathcal{Y}}(\mathcal{E})) \xleftarrow{\quad} \end{array} (\mathrm{id}_{\mathcal{Y}} \circ f)^*(\mathcal{E}).$$

Also  $(\mathrm{id}_f)^*(\mathrm{id}_{\mathcal{E}}) = \mathrm{id}_{f^*(\mathcal{E})} : f^*(\mathcal{E}) \rightarrow f^*(\mathcal{E})$ .

- (c) If  $f, g, h : \mathcal{X} \rightarrow \mathcal{Y}$  are 1-morphisms and  $\eta : f \Rightarrow g$ ,  $\zeta : g \Rightarrow h$  are 2-morphisms in  $\mathbf{DMC}^\infty\mathbf{Sta}$ , so that  $\zeta \odot \eta : f \Rightarrow h$  is the vertical composition, and  $\mathcal{E} \in \mathrm{qcoh}(\mathcal{Y})$ , then

$$\zeta^*(\mathcal{F}) \circ \eta^*(\mathcal{E}) = (\zeta \odot \eta)^*(\mathcal{E}) : f^*(\mathcal{E}) \longrightarrow h^*(\mathcal{E}) \quad \text{in } \mathrm{qcoh}(\mathcal{X}).$$

- (d) If  $f, \tilde{f} : \mathcal{X} \rightarrow \mathcal{Y}$ ,  $g, \tilde{g} : \mathcal{Y} \rightarrow \mathcal{Z}$  are 1-morphisms and  $\eta : f \Rightarrow \tilde{f}$ ,  $\zeta : g \Rightarrow \tilde{g}$  are 2-morphisms in  $\mathbf{DMC}^\infty\mathbf{Sta}$ , so that  $\zeta * \eta : g \circ f \Rightarrow \tilde{g} \circ \tilde{f}$  is the horizontal composition, and  $\mathcal{E} \in \mathrm{qcoh}(\mathcal{Z})$ , then the following commutes in  $\mathrm{qcoh}(\mathcal{X})$ :

$$\begin{array}{ccc} (g \circ f)^*(\mathcal{E}) & \xrightarrow{(\zeta * \eta)^*(\mathcal{E})} & (\tilde{g} \circ \tilde{f})^*(\mathcal{E}) \\ \downarrow I_{f, g}(\mathcal{E}) & & \downarrow I_{\tilde{f}, \tilde{g}}(\mathcal{E}) \\ f^*(g^*(\mathcal{E})) & \xrightarrow{\eta^*(g^*(\mathcal{E}))} \tilde{f}^*(g^*(\mathcal{E})) \xrightarrow{\tilde{f}^*(\zeta^*(\mathcal{E}))} & \tilde{f}^*(\tilde{g}^*(\mathcal{E})). \end{array}$$

Using Proposition 5.15 we may prove:

**PROPOSITION 8.11.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of Deligne–Mumford  $C^\infty$ -stacks. Then pullback  $f^* : \mathrm{qcoh}(\mathcal{Y}) \rightarrow \mathrm{qcoh}(\mathcal{X})$  is a right exact functor.*

#### 8.4. Cotangent sheaves of Deligne–Mumford $C^\infty$ -stacks

We now develop the analogue of the ideas of §5.6.

**DEFINITION 8.12.** Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack. Define a quasicoherent sheaf  $T^*\mathcal{X}$  on  $\mathcal{X}$  called the *cotangent sheaf* of  $\mathcal{X}$  by  $(T^*\mathcal{X})(\underline{U}, u) = T^*\underline{U}$  for all objects  $(\underline{U}, u)$  in  $\mathcal{C}_{\mathcal{X}}$  and  $(T^*\mathcal{X})_{(\underline{f}, \eta)} = \Omega_{\underline{f}} : \underline{f}^*(T^*\underline{V}) \rightarrow T^*\underline{U}$  for all morphisms  $(\underline{f}, \eta) : (\underline{U}, u) \rightarrow (\underline{V}, v)$  in  $\mathcal{C}_{\mathcal{X}}$ , where  $T^*\underline{U}$  and  $\Omega_{\underline{f}}$  are as in §5.6. Here as  $\underline{f} : \underline{U} \rightarrow \underline{V}$  is étale  $\Omega_{\underline{f}}$  is an isomorphism, so  $(T^*\mathcal{X})_{(\underline{f}, \eta)}$  is an isomorphism in  $\mathrm{qcoh}(\underline{U})$  as required. Also Theorem 5.32(a) shows that (8.1) commutes for  $\mathcal{E} = T^*\mathcal{X}$  for all such  $(\underline{f}, \eta), (g, \zeta)$ . Hence  $T^*\mathcal{X}$  is a quasicoherent sheaf.

Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of Deligne–Mumford  $C^\infty$ -stacks. Define  $\Omega_f : f^*(T^*\mathcal{Y}) \rightarrow T^*\mathcal{X}$  to be the unique morphism in  $\mathrm{qcoh}(\mathcal{X})$  characterized as follows. Let  $u : \underline{U} \rightarrow \mathcal{X}$ ,  $v : \underline{V} \rightarrow \mathcal{Y}$ ,  $\underline{W}, \pi_{\underline{U}}, \pi_{\underline{V}}$  be as in Definition 8.7, with (8.3)

Cartesian. Then the following diagram in  $\mathrm{qcoh}(\underline{W})$  commutes:

$$\begin{array}{ccc} \pi_{\underline{U}}^*(f^*(T^*\mathcal{Y})(\underline{U}, u)) & \xrightarrow{i(T^*\mathcal{Y}, f, u, v, \zeta)} & \pi_{\underline{V}}^*((T^*\mathcal{Y})(\underline{V}, v)) = \pi_{\underline{V}}^*(T^*\underline{V}) \\ \downarrow \pi_{\underline{U}}^*(\Omega_f(\underline{U}, u)) & & \Omega_{\pi_{\underline{V}}} \downarrow \\ \pi_{\underline{U}}^*((T^*\mathcal{X})(\underline{U}, u)) & \xrightarrow{(T^*\mathcal{X})_{(\pi_{\underline{U}}, \mathrm{id}_u \circ \pi_{\underline{U}})}} & (T^*\mathcal{X})(\underline{W}, u \circ \pi_{\underline{U}}) = T^*\underline{W}. \end{array}$$

This determines  $\pi_{\underline{U}}^*(\Omega_f(\underline{U}, u))$  uniquely. Over all  $(\underline{V}, v)$ , using descent for morphisms in  $\mathrm{qcoh}(\underline{U})$  on  $C^\infty$ -schemes  $\underline{U}$  in the étale topology, this determines the morphisms  $\Omega_f(\underline{U}, u)$ , and over all  $(\underline{U}, u)$  these determine  $\Omega_f$ .

If  $\mathcal{X}$  is an orbifold of dimension  $n$  then  $T^*\mathcal{X}$  is a vector bundle of rank  $n$ . Here is the analogue of Theorem 5.32:

**THEOREM 8.13. (a)** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $g : \mathcal{Y} \rightarrow \mathcal{Z}$  be 1-morphisms of Deligne–Mumford  $C^\infty$ -stacks. Then*

$$\Omega_{g \circ f} = \Omega_f \circ f^*(\Omega_g) \circ I_{f,g}(T^*\mathcal{Z}) \quad (8.4)$$

*as morphisms  $(g \circ f)^*(T^*\mathcal{Z}) \rightarrow T^*\mathcal{X}$  in  $\mathrm{qcoh}(\mathcal{X})$ .*

**(b)** *Let  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  be 1-morphisms of Deligne–Mumford  $C^\infty$ -stacks and  $\eta : f \Rightarrow g$  a 2-morphism. Then  $\Omega_f = \Omega_g \circ \eta^*(T^*\mathcal{Y}) : f^*(T^*\mathcal{Y}) \rightarrow T^*\mathcal{X}$ .*

**(c)** *Let  $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$  be Deligne–Mumford  $C^\infty$ -stacks in a 2-Cartesian square*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\ \downarrow e & \eta \nearrow & \downarrow h \\ \mathcal{X} & \xrightarrow{\quad g \quad} & \mathcal{Z} \end{array}$$

*in  $\mathbf{DMC}^\infty\mathbf{Sta}$ , so that  $\mathcal{W} = \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ . Then the following is exact in  $\mathrm{qcoh}(\mathcal{W})$ :*

$$(g \circ e)^*(T^*\mathcal{Z}) \xrightarrow{e^*(\Omega_g) \circ I_{e,g}(T^*\mathcal{Z}) \oplus -f^*(\Omega_h) \circ I_{f,h}(T^*\mathcal{Z}) \circ \eta^*(T^*\mathcal{Z})} e^*(T^*\mathcal{X}) \oplus \xrightarrow{\Omega_e \oplus \Omega_f} T^*\mathcal{W} \longrightarrow 0. \quad (8.5)$$

**PROOF.** For (a), let  $u : \bar{\underline{U}} \rightarrow \mathcal{X}$ ,  $v : \bar{\underline{V}} \rightarrow \mathcal{Y}$  and  $w : \bar{\underline{W}} \rightarrow \mathcal{Z}$  be étale. Then there is a  $C^\infty$ -scheme  $\underline{V}'$  with  $\bar{\underline{V}}' = \bar{\underline{V}} \times_{g \circ v, \mathcal{Z}, w} \bar{\underline{W}}$ , and fibre product projections  $\pi_{\underline{V}} : \underline{V}' \rightarrow \underline{V}$ ,  $\pi_{\underline{W}} : \underline{V}' \rightarrow \underline{W}$ . Define  $v' = v \circ \pi_{\underline{V}} : \underline{V}' \rightarrow \mathcal{Y}$ . Then  $v'$  is étale, as  $v$  is and  $w$  is so  $\pi_{\underline{V}}$  is. Similarly, there is a  $C^\infty$ -scheme  $\underline{U}'$  with  $\bar{\underline{U}}' = \bar{\underline{U}} \times_{f \circ u, \mathcal{X}, v'} \bar{\underline{V}}'$ , and fibre product projections  $\pi_{\underline{U}} : \underline{U}' \rightarrow \underline{U}$ ,  $\pi_{\underline{V}'} : \underline{U}' \rightarrow \underline{V}'$ . Define an étale 1-morphism  $u' = u \circ \pi_{\underline{U}} : \underline{U}' \rightarrow \mathcal{X}$ . Then we have a 2-commutative diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y} & \xrightarrow{\quad g \quad} & \mathcal{Z} \\ \downarrow u & & \downarrow v & & \downarrow w \\ \bar{\underline{U}} & \xrightarrow{\quad \eta \quad} & \bar{\underline{V}} & & \bar{\underline{W}} \\ \downarrow u' & & \downarrow v' & & \downarrow \zeta \\ \underline{U}' & \xrightarrow{\quad \pi_{\underline{U}} \quad} & \underline{V}' & \xrightarrow{\quad \pi_{\underline{W}} \quad} & \underline{W} \end{array}$$

with 2-Cartesian squares. On  $\underline{U}'$  and  $\underline{V}'$  we have commutative diagrams:

$$\begin{array}{ccc} \pi_{\underline{U}}^*(f^*(T^*\mathcal{Y})(\underline{U}, u)) & \xrightarrow[\cong]{i(T^*\mathcal{Y}, f, u, v', \eta)} & \pi_{\underline{V}'}^*((T^*\mathcal{Y})(\underline{V}', v')) = \pi_{\underline{V}'}^*(T^*\underline{V}') \\ \cong \downarrow (f^*(T^*\mathcal{Y}))_{(\pi_{\underline{U}}, \text{id}_{u'})} & & \Omega_{\pi_{\underline{V}'}} \downarrow \\ (f^*(T^*\mathcal{Y}))(\underline{U}', u') & \xrightarrow{\Omega_f(\underline{U}', u')} & (T^*\mathcal{X})(\underline{U}', u') = T^*\underline{U}', \end{array} \quad (8.6)$$

$$\begin{array}{ccc} \pi_{\underline{V}}^*(g^*(T^*\mathcal{Z})(\underline{V}, v)) & \xrightarrow[\cong]{i(T^*\mathcal{Z}, g, v, w, \zeta)} & \pi_{\underline{W}}^*(T^*\mathcal{Z}(\underline{W}, w)) = \pi_{\underline{W}}^*(T^*\underline{W}) \\ \cong \downarrow (g^*(T^*\mathcal{Z}))_{(\pi_{\underline{V}}, \text{id}_{v'})} & & \Omega_{\pi_{\underline{W}}} \downarrow \\ (g^*(T^*\mathcal{Z}))(\underline{V}', v') & \xrightarrow{\Omega_g(\underline{V}', v')} & (T^*\mathcal{Y})(\underline{V}', v') = T^*\underline{V}'. \end{array} \quad (8.7)$$

Applying  $\pi_{\underline{V}'}^*$  to (8.7) we make another commutative diagram on  $\underline{U}'$ :

$$\begin{array}{ccc} \pi_{\underline{V}'}^*(\pi_{\underline{V}}^*(g^*(T^*\mathcal{Z})(\underline{V}, v))) & \xrightarrow[\cong]{\pi_{\underline{V}'}^*(i(T^*\mathcal{Z}, g, v, w, \zeta))} & \pi_{\underline{V}'}^*(\pi_{\underline{W}}^*(T^*\underline{W})) \\ \cong \downarrow \pi_{\underline{V}'}^*((g^*(T^*\mathcal{Z}))_{(\pi_{\underline{V}}, \text{id}_{v'})}) & & \pi_{\underline{V}'}^*(\Omega_{\pi_{\underline{W}}}) \downarrow \\ \pi_{\underline{V}'}^*((g^*(T^*\mathcal{Z}))(\underline{V}', v')) & \xrightarrow{\pi_{\underline{V}'}^*(\Omega_g(\underline{V}', v'))} & \pi_{\underline{V}'}^*(T^*\underline{V}') \\ \cong \downarrow (f^*(g^*(T^*\mathcal{Z})))_{(\pi_{\underline{U}}, \text{id}_{u'})} & & (f^*(T^*\mathcal{U}))_{(\pi_{\underline{U}}, \text{id}_{u'})} \downarrow \cong \\ (f^*(g^*(T^*\mathcal{Z})))(\underline{U}', u') & \xrightarrow{(f^*(\Omega_g))(\underline{U}', u')} & (f^*(T^*\mathcal{Y}))(\underline{U}', u'). \end{array} \quad (8.8)$$

By Theorem 5.32(a) the following commutes:

$$\begin{array}{ccc} (\pi_{\underline{W}} \circ \pi_{\underline{V}'})^*(T^*\underline{W}) & \xrightarrow{\Omega_{\pi_{\underline{W}} \circ \pi_{\underline{V}'}}} & T^*\underline{U}' \\ \cong \downarrow I_{\pi_{\underline{V}'}, \pi_{\underline{W}}}(T^*\underline{W}) & & \Omega_{\pi_{\underline{V}'}} \uparrow \\ \pi_{\underline{V}'}^*(\pi_{\underline{W}}^*(T^*\underline{W})) & \xrightarrow{\pi_{\underline{V}'}^*(\Omega_{\pi_{\underline{W}}})} & \pi_{\underline{V}'}^*(T^*\underline{V}'). \end{array} \quad (8.9)$$

Using all this we obtain a commutative diagram on  $\underline{U}'$ :

$$\begin{array}{ccccc} ((g \circ f)^*(T^*\mathcal{Z}))(\underline{U}', u') & \xrightarrow{\Omega_{g \circ f}(\underline{U}', u')} & (T^*\mathcal{X})(\underline{U}', u') & & \\ \cong \downarrow (I_{f, g}(T^*\mathcal{Z}))(\underline{U}', u') & \swarrow \cong & \uparrow \cong & \nearrow \Omega_f(\underline{U}', u') & \\ & (\pi_{\underline{W}} \circ \pi_{\underline{V}'})^*(T^*\underline{W}) \rightarrow T^*\underline{U}' & & & \\ & \downarrow \cong & & & \\ & \pi_{\underline{V}'}^*(\pi_{\underline{W}}^*(T^*\underline{W})) \rightarrow \pi_{\underline{V}'}^*(T^*\underline{V}') & & & \\ \cong \swarrow & & \nwarrow \cong & & \\ (f^*(g^*(T^*\mathcal{Z}))) (\underline{U}', u') & \xrightarrow{(f^*(\Omega_g))(\underline{U}', u')} & (f^*(T^*\mathcal{Y}))(\underline{U}', u') & & \end{array} \quad (8.10)$$

Here the right hand quadrilateral of (8.10) comes from (8.6), the bottom quadrilateral from (8.8), the central square is (8.9), and the remaining two quadrilaterals are similar. Thus, the outer square of (8.10) commutes. But this is just (8.4) evaluated at  $(\underline{U}', u')$ . If  $u : \bar{\underline{U}} \rightarrow \mathcal{X}$ ,  $v : \bar{\underline{V}} \rightarrow \mathcal{Y}$  and  $w : \bar{\underline{W}} \rightarrow \mathcal{Z}$  are étale atlases then  $u' : \underline{U}' \rightarrow \mathcal{X}$  is also an étale atlas, and (8.4) evaluated on an atlas implies it in general. This proves part (a).

Part (b) is immediate from the definitions. For (c), let  $u : \bar{\underline{U}} \rightarrow \mathcal{X}$ ,  $v : \bar{\underline{V}} \rightarrow \mathcal{Y}$  and  $w : \bar{\underline{W}} \rightarrow \mathcal{Z}$  be étale. There are  $C^\infty$ -schemes  $\underline{U}', \underline{V}'$ , with  $\bar{\underline{U}}' = \bar{\underline{U}} \times_{g \circ u, \mathcal{Z}, w} \bar{\underline{W}}$ ,  $\bar{\underline{V}}' = \bar{\underline{V}} \times_{h \circ v, \mathcal{Z}, w} \bar{\underline{W}}$ , and fibre product projections  $\pi_{\underline{U}} : \underline{U}' \rightarrow \underline{U}$ ,  $\pi_{\underline{W}} : \underline{U}' \rightarrow \underline{W}$ ,  $\pi_{\underline{V}} : \underline{V}' \rightarrow \underline{V}$ ,  $\pi_{\underline{W}} : \underline{V}' \rightarrow \underline{W}$ . Then  $\pi_{\underline{U}}, \pi_{\underline{V}}$  are étale as  $w$  is. Define a  $C^\infty$ -scheme

$\underline{T} = \underline{U}' \times_{\pi_W, \bar{W}, \pi_W} \underline{V}'$ . The 1-morphisms  $u' \circ \bar{\pi}_{\underline{U}'} : \bar{\underline{T}} \rightarrow \mathcal{X}$  and  $v' \circ \bar{\pi}_{\underline{V}'} : \bar{\underline{T}} \rightarrow \mathcal{Y}$  have a natural 2-isomorphism  $g \circ (u' \circ \bar{\pi}_{\underline{U}'}) \Rightarrow h \circ (v' \circ \bar{\pi}_{\underline{V}'})$  constructed from the 2-isomorphisms in the 2-Cartesian squares defining  $\underline{U}', \underline{V}'$ . Thus as  $\mathcal{W} = \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$  there is a 1-morphism  $t : \bar{\underline{T}} \rightarrow \mathcal{W}$ , unique up to 2-isomorphism, such that  $u' \circ \bar{\pi}_{\underline{U}'} \cong e \circ t$  and  $v' \circ \bar{\pi}_{\underline{V}'} \cong f \circ t$ . Also  $t$  is étale. This gives a 2-commutative diagram

$$\begin{array}{ccccc}
 & \bar{\underline{T}} & \xrightarrow{t} & \mathcal{W} & \\
 \bar{\pi}_{\underline{U}'} \swarrow & & \searrow \bar{\pi}_{\underline{V}'} & & \searrow f \\
 \underline{\bar{U}}' & \xrightarrow{u'} & \mathcal{X} & \xleftarrow{e} & \mathcal{W} \\
 \searrow \bar{\pi}_{\underline{W}} & & \searrow \bar{\pi}_{\underline{V}'} & & \searrow f \\
 & \bar{\underline{V}}' & \xrightarrow{v'} & \mathcal{Y} & \\
 & \searrow \bar{\pi}_{\underline{W}} & & \searrow h & \\
 & \bar{\underline{W}} & \xrightarrow{w} & \mathcal{Z} & 
 \end{array}$$

in which the leftmost and rightmost squares are 2-Cartesian.

Applying Theorem 5.32(b) to the Cartesian square defining  $\underline{T}$  gives an exact sequence in  $\text{qcoh}(\underline{T})$ :

$$\begin{array}{ccc}
 (\pi_{\underline{W}} \circ \pi_{\underline{U}'})^* & \xrightarrow{\begin{array}{c} \pi_{\underline{U}'}^*(\Omega_{\pi_{\underline{W}}}) \circ I_{\pi_{\underline{U}'}, \pi_{\underline{W}}} (T^* \underline{W}) \oplus \\ -\pi_{\underline{V}'}^*(\Omega_{\pi_{\underline{W}}}) \circ I_{\pi_{\underline{V}'}, \pi_{\underline{W}}} (T^* \underline{W}) \end{array}} & \pi_{\underline{U}'}^*(T^* \underline{U}') \xrightarrow{\Omega_{\pi_{\underline{U}'}} \oplus} \\
 (T^* \underline{W}) & \xrightarrow{\quad} & \oplus \pi_{\underline{V}'}^*(T^* \underline{V}') \xrightarrow{\Omega_{\pi_{\underline{V}'}}} T^* \underline{T} \longrightarrow 0. \quad (8.11)
 \end{array}$$

By a similar argument to (a), we can use (8.11) to deduce that (8.5) evaluated at  $(\underline{T}, t)$  holds. If  $u : \underline{\bar{U}}' \rightarrow \mathcal{X}$ ,  $v : \underline{\bar{V}}' \rightarrow \mathcal{Y}$  and  $w : \underline{\bar{W}}' \rightarrow \mathcal{Z}$  are atlases then  $t : \bar{\underline{T}} \rightarrow \mathcal{W}$  is an atlas, so this implies (8.5), and proves (c).  $\square$



## CHAPTER 9

### Orbifold strata of $C^\infty$ -stacks

Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack, with topological space  $\mathcal{X}_{\text{top}}$ . Then each point  $[x] \in \mathcal{X}_{\text{top}}$  has an isotropy group  $\text{Iso}_{\mathcal{X}}([x])$ , a finite group defined up to isomorphism. For each finite group  $\Gamma$  we write  $\tilde{\mathcal{X}}_{\circ, \text{top}}^\Gamma = \{[x] \in \mathcal{X}_{\text{top}} : \text{Iso}_{\mathcal{X}}([x]) \cong \Gamma\}$ . This is a locally closed subset of  $\mathcal{X}_{\text{top}}$ , coming from a locally closed  $C^\infty$ -substack  $\hat{\mathcal{X}}_\circ^\Gamma$  of  $\mathcal{X}$  with inclusion  $\bar{O}_\circ^\Gamma(\mathcal{X}) : \hat{\mathcal{X}}_\circ^\Gamma \rightarrow \mathcal{X}$ , with

$$\mathcal{X}_{\text{top}} = \coprod_{\substack{\text{isomorphism classes} \\ \text{of finite groups } \Gamma}} \tilde{\mathcal{X}}_{\circ, \text{top}}^\Gamma. \quad (9.1)$$

One can show that for each  $\Gamma$ , the closure  $\bar{\tilde{\mathcal{X}}}_{\circ, \text{top}}^\Gamma$  of  $\tilde{\mathcal{X}}_{\circ, \text{top}}^\Gamma$  in  $\mathcal{X}_{\text{top}}$  satisfies

$$\bar{\tilde{\mathcal{X}}}_{\circ, \text{top}}^\Gamma \subseteq \coprod_{\substack{\text{isomorphism classes of finite groups } \Delta: \\ \Gamma \text{ is isomorphic to a subgroup of } \Delta}} \tilde{\mathcal{X}}_{\circ, \text{top}}^\Delta.$$

Thus (9.1) is a stratification of  $\mathcal{X}_{\text{top}}$ . The  $\tilde{\mathcal{X}}_\circ^\Gamma$  are called *orbifold strata* of  $\mathcal{X}$ .

When  $\mathcal{X}$  is an orbifold, as in §7.6, the orbifold strata are manifolds (actually, at the level of  $C^\infty$ -stacks, the alternative versions  $\hat{\mathcal{X}}_\circ^\Gamma$  below are manifolds), and are well studied. Orbifold strata of orbifolds come up in areas such as the Atiyah–Singer Index Theorem for orbifolds as in Kawasaki [42], cobordism of orbifolds as in Druschel [20], String Theory of orbifolds as in Dixon et al. [19], and (quantum) cohomology of orbifolds as in Chen and Ruan [14].

However, very little appears to have been done in considering orbifold strata from the point of view of category theory or stacks, or about orbifold strata of other kinds of Deligne–Mumford stacks. We now define and study orbifold strata of Deligne–Mumford  $C^\infty$ -stacks. Actually, almost all of §9 is an exercise in stack theory, not specific to  $C^\infty$ -stacks. But the author has been unable to find any references on it.

We will define six variations on  $\tilde{\mathcal{X}}_\circ^\Gamma$  outlined above, Deligne–Mumford  $C^\infty$ -stacks written  $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$ , and open  $C^\infty$ -substacks  $\mathcal{X}_\circ^\Gamma \subseteq \mathcal{X}^\Gamma$ ,  $\tilde{\mathcal{X}}_\circ^\Gamma \subseteq \tilde{\mathcal{X}}^\Gamma$ ,  $\hat{\mathcal{X}}_\circ^\Gamma \subseteq \hat{\mathcal{X}}^\Gamma$ . The points and isotropy groups of  $\mathcal{X}^\Gamma, \dots, \hat{\mathcal{X}}_\circ^\Gamma$  are given by:

- (i) Points of  $\mathcal{X}^\Gamma$  are isomorphism classes  $[x, \rho]$ , where  $[x] \in \mathcal{X}_{\text{top}}$  and  $\rho : \Gamma \rightarrow \text{Iso}_{\mathcal{X}}([x])$  is an injective morphism, and  $\text{Iso}_{\mathcal{X}^\Gamma}([x, \rho])$  is the centralizer of  $\rho(\Gamma)$  in  $\text{Iso}_{\mathcal{X}}([x])$ . Points of  $\mathcal{X}_\circ^\Gamma \subseteq \mathcal{X}^\Gamma$  are  $[x, \rho]$  with  $\rho$  an isomorphism, and  $\text{Iso}_{\mathcal{X}_\circ^\Gamma}([x, \rho]) \cong C(\Gamma)$ , the centre of  $\Gamma$ .
- (ii) Points of  $\tilde{\mathcal{X}}^\Gamma$  are pairs  $[x, \Delta]$ , where  $[x] \in \mathcal{X}_{\text{top}}$  and  $\Delta \subseteq \text{Iso}_{\mathcal{X}}([x])$  is isomorphic to  $\Gamma$ , and  $\text{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x, \Delta])$  is the normalizer of  $\Delta$  in  $\text{Iso}_{\mathcal{X}}([x])$ . Points of  $\tilde{\mathcal{X}}_\circ^\Gamma \subseteq \tilde{\mathcal{X}}^\Gamma$  are  $[x, \Delta]$  with  $\Delta = \text{Iso}_{\mathcal{X}}([x])$ , and  $\text{Iso}_{\tilde{\mathcal{X}}_\circ^\Gamma}([x, \Delta]) \cong \Gamma$ .
- (iii) Points  $[x, \Delta]$  of  $\hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$  are the same as for  $\tilde{\mathcal{X}}^\Gamma, \tilde{\mathcal{X}}_\circ^\Gamma$ , but with isotropy groups  $\text{Iso}_{\hat{\mathcal{X}}^\Gamma}([x, \Delta]) \cong \text{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x, \Delta]) / \Delta$  and  $\text{Iso}_{\hat{\mathcal{X}}_\circ^\Gamma}([x, \Delta]) \cong \{1\}$ .

There are 1-morphisms  $O^\Gamma(\mathcal{X}), \dots, \hat{\Pi}_\circ^\Gamma(\mathcal{X})$  forming a 2-commutative diagram, with columns inclusions of open  $C^\infty$ -substacks:

$$\begin{array}{ccccc}
 \text{Aut}(\Gamma) \curvearrowright \mathcal{X}_\circ^\Gamma & \xrightarrow{\hat{\Pi}_\circ^\Gamma(\mathcal{X})} & \tilde{\mathcal{X}}_\circ^\Gamma & \xrightarrow{\hat{\Pi}_\circ^\Gamma(\mathcal{X})} & \hat{\mathcal{X}}_\circ^\Gamma \simeq \tilde{\hat{\mathcal{X}}}_\circ^\Gamma \\
 \downarrow \text{O}^\Gamma_\circ(\mathcal{X}) & \searrow & \downarrow \tilde{\text{O}}^\Gamma_\circ(\mathcal{X}) & & \downarrow \subset \\
 \text{Aut}(\Gamma) \curvearrowright \mathcal{X}^\Gamma & \xrightarrow{\hat{\Pi}^\Gamma(\mathcal{X})} & \tilde{\mathcal{X}}^\Gamma & \xrightarrow{\hat{\Pi}^\Gamma(\mathcal{X})} & \hat{\mathcal{X}}^\Gamma \\
 \downarrow \text{O}^\Gamma(\mathcal{X}) & \swarrow & \downarrow \tilde{\text{O}}^\Gamma(\mathcal{X}) & & \downarrow \subset
 \end{array} \quad (9.2)$$

Also  $\text{Aut}(\Gamma)$  acts on  $\mathcal{X}^\Gamma, \mathcal{X}_\circ^\Gamma$ , with  $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$ ,  $\tilde{\mathcal{X}}_\circ^\Gamma \simeq [\mathcal{X}_\circ^\Gamma / \text{Aut}(\Gamma)]$ . Note that there are in general *no natural* 1-morphisms from  $\hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$  to any of  $\mathcal{X}, \mathcal{X}^\Gamma, \mathcal{X}_\circ^\Gamma, \tilde{\mathcal{X}}^\Gamma, \tilde{\mathcal{X}}_\circ^\Gamma$ .

### 9.1. The definition of orbifold strata $\mathcal{X}^\Gamma, \dots, \hat{\mathcal{X}}_\circ^\Gamma$

We now define the orbifold strata  $\mathcal{X}^\Gamma, \dots, \hat{\mathcal{X}}_\circ^\Gamma$  and study their properties.

**DEFINITION 9.1.** Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack, and  $\Gamma$  a finite group. We will explicitly define another Deligne–Mumford  $C^\infty$ -stack  $\mathcal{X}^\Gamma$ . Since  $\mathcal{X}$  is a stack on the site  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$ ,  $\mathcal{X}$  is a category with a functor  $p_\mathcal{X} : \mathcal{X} \rightarrow \mathbf{C}^\infty\mathbf{Sch}$  satisfying many conditions. To define  $\mathcal{X}^\Gamma$  we must define another category  $\mathcal{X}^\Gamma$  and a functor  $p_{\mathcal{X}^\Gamma} : \mathcal{X}^\Gamma \rightarrow \mathbf{C}^\infty\mathbf{Sch}$ .

Define objects of the category  $\mathcal{X}^\Gamma$  to be pairs  $(A, \rho)$  satisfying:

- (a)  $A$  is an object in  $\mathcal{X}$ , with  $p_\mathcal{X}(A) = \underline{U}$  for some object  $\underline{U} \in \mathbf{C}^\infty\mathbf{Sch}$ ;
- (b)  $\rho : \Gamma \rightarrow \text{Aut}(A)$  is a group morphism, where  $\text{Aut}(A)$  is the group of isomorphisms  $a : A \rightarrow A$  in  $\mathcal{X}$ , and  $p_\mathcal{X} \circ \rho(\gamma) = \text{id}_{\underline{U}}$  for all  $\gamma \in \Gamma$ ; and
- (c) Let  $u$  be a point in  $\underline{U}$ , and  $\underline{u} : * \rightarrow \underline{U}$  the corresponding morphism in  $\mathbf{C}^\infty\mathbf{Sch}$ . Since  $p_\mathcal{X} : \mathcal{X} \rightarrow \mathbf{C}^\infty\mathbf{Sch}$  is a category fibred in groupoids, as in Definition A.5, there exists a morphism  $a_u : A_u \rightarrow A$  in  $\mathcal{X}$  with  $p_\mathcal{X}(A_u) = *$  and  $p_\mathcal{X}(a_u) = \underline{u}$ , where  $A_u$  is unique up to isomorphism in  $\mathcal{X}$ .

Having fixed  $A_u, a_u$ , Definition A.5 also implies that for each  $\gamma \in \Gamma$  there is a unique isomorphism  $\rho_u(\gamma) : A_u \rightarrow A_u$  such that  $a_u \circ \rho_u(\gamma) = \rho(\gamma) \circ a_u : A_u \rightarrow A$ , and  $p_\mathcal{X}(\rho_u(\gamma)) = \text{id}_*$ . Then  $\rho_u : \Gamma \rightarrow \text{Aut}(A_u)$  is a group morphism. We require that  $\rho_u : \Gamma \rightarrow \text{Aut}(A_u)$  should be injective for all  $u \in \underline{U}$ . This condition is independent of the choice of  $A_u, a_u$ .

Define morphisms  $c : (A, \rho) \rightarrow (B, \sigma)$  of the category  $\mathcal{X}^\Gamma$  to be morphisms  $c : A \rightarrow B$  in  $\mathcal{X}$  satisfying  $\sigma(\gamma) \circ c = c \circ \rho(\gamma) : A \rightarrow B$  in  $\mathcal{X}$  for all  $\gamma \in \Gamma$ . Given morphisms  $c : (A, \rho) \rightarrow (B, \sigma)$ ,  $d : (B, \sigma) \rightarrow (C, \tau)$  in  $\mathcal{X}^\Gamma$ , define composition  $d \circ c : (A, \rho) \rightarrow (C, \tau)$  in  $\mathcal{X}^\Gamma$  to be the composition  $d \circ c : A \rightarrow C$  in  $\mathcal{X}$ . For each object  $(A, \rho)$  in  $\mathcal{X}^\Gamma$ , define the identity morphism  $\text{id}_{(A, \rho)} : (A, \rho) \rightarrow (A, \rho)$  in  $\mathcal{X}^\Gamma$  to be  $\text{id}_A : A \rightarrow A$  in  $\mathcal{X}$ . Define a functor  $p_{\mathcal{X}^\Gamma} : \mathcal{X}^\Gamma \rightarrow \mathbf{C}^\infty\mathbf{Sch}$  by  $p_{\mathcal{X}^\Gamma} : (A, \rho) \mapsto \underline{U} = p_\mathcal{X}(A)$  on objects and  $p_{\mathcal{X}^\Gamma} : c \mapsto p_\mathcal{X}(c)$  on morphisms.

Define  $\mathcal{X}_\circ^\Gamma$  to be the full subcategory of objects  $(A, \rho)$  in  $\mathcal{X}^\Gamma$  such that  $\rho_u : \Gamma \rightarrow \text{Aut}(A_u)$  in (c) above is an isomorphism for all  $u \in \underline{U}$ . Define a functor  $p_{\mathcal{X}_\circ^\Gamma} = p_\mathcal{X}|_{\mathcal{X}_\circ^\Gamma} : \mathcal{X}_\circ^\Gamma \rightarrow \mathbf{C}^\infty\mathbf{Sch}$ . By Theorem 9.5(a) below,  $\mathcal{X}^\Gamma$  is a Deligne–Mumford  $C^\infty$ -stack, and  $\mathcal{X}_\circ^\Gamma$  is an open  $C^\infty$ -substack in  $\mathcal{X}^\Gamma$ .

DEFINITION 9.2. Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack, and  $\Gamma$  a finite group. Define a category  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$  to have objects pairs  $(A, \Delta)$  satisfying:

- (a)  $A$  is an object in  $\mathcal{X}$ , with  $p_{\mathcal{X}}(A) = \underline{U}$  for some object  $\underline{U} \in \mathbf{C}^\infty\mathbf{Sch}$ ;
- (b)  $\Delta \subseteq \text{Aut}(A)$  is a subgroup isomorphic to  $\Gamma$ , where  $\text{Aut}(A)$  is the group of isomorphisms  $a : A \rightarrow A$  in  $\mathcal{X}$ , and  $p_{\mathcal{X}}(\delta) = \text{id}_{\underline{U}}$  for all  $\delta \in \Delta$ ; and
- (c) Let  $u$  be a point in  $\underline{U}$ , and  $u : * \rightarrow \underline{U}$  the corresponding morphism in  $\mathbf{C}^\infty\mathbf{Sch}$ . Since  $p_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{C}^\infty\mathbf{Sch}$  is a category fibred in groupoids, there exists a morphism  $a_u : A_u \rightarrow A$  in  $\mathcal{X}$  with  $p_{\mathcal{X}}(A_u) = *$  and  $p_{\mathcal{X}}(a_u) = u$ , where  $A_u$  is unique up to isomorphism in  $\mathcal{X}$ . For each  $\delta \in \Delta$  there is a unique isomorphism  $\delta_u : A_u \rightarrow A_u$  such that  $a_u \circ \delta_u = \delta \circ a_u : A_u \rightarrow A$ , and  $p_{\mathcal{X}}(\delta_u) = \text{id}_*$ . Then  $\{\delta_u : \delta \in \Delta\}$  is a subgroup of  $\text{Aut}(A_u)$ , and  $\delta \mapsto \delta_u$  is a group morphism. We require that the map  $\delta \mapsto \delta_u$  should be *injective* for all  $u \in \underline{U}$ .

Define morphisms  $(A, \Delta) \rightarrow (A', \Delta')$  of  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$  to be pairs  $(c, \iota)$ , where  $c : A \rightarrow A'$  is a morphism in  $\mathcal{X}$  and  $\iota : \Delta \rightarrow \Delta'$  is a group isomorphism, satisfying  $\iota(\delta) \circ c = c \circ \delta : A \rightarrow A'$  for all  $\delta \in \Delta$ . Given morphisms  $(c, \iota) : (A, \Delta) \rightarrow (A', \Delta')$ ,  $(c', \iota') : (A', \Delta') \rightarrow (A'', \Delta'')$  in  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$ , define composition  $(c', \iota') \circ (c, \iota) = (c' \circ c, \iota' \circ \iota)$ . Define identities  $\text{id}_{(A, \Delta)} = (\text{id}_A, \text{id}_\Delta) : (A, \Delta) \rightarrow (A, \Delta)$ .

Define a functor  $p_{\mathcal{P}\tilde{\mathcal{X}}^\Gamma} : \mathcal{P}\tilde{\mathcal{X}}^\Gamma \rightarrow \mathbf{C}^\infty\mathbf{Sch}$  by  $p_{\mathcal{P}\tilde{\mathcal{X}}^\Gamma} : (A, \Delta) \mapsto \underline{U} = p_{\mathcal{X}}(A)$  on objects and  $p_{\mathcal{P}\tilde{\mathcal{X}}^\Gamma} : (c, \iota) \mapsto p_{\mathcal{X}}(c)$  on morphisms. Define  $\mathcal{P}\tilde{\mathcal{X}}_\circ^\Gamma$  to be the full subcategory of objects  $(A, \Delta)$  in  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$  with  $\{\delta_u : \delta \in \Delta\} = \text{Aut}(A_u)$  in (c) above for all  $u \in \underline{U}$ . Define a functor  $p_{\mathcal{P}\tilde{\mathcal{X}}_\circ^\Gamma} = p_{\mathcal{P}\tilde{\mathcal{X}}^\Gamma}|_{\mathcal{P}\tilde{\mathcal{X}}_\circ^\Gamma} : \mathcal{P}\tilde{\mathcal{X}}_\circ^\Gamma \rightarrow \mathbf{C}^\infty\mathbf{Sch}$ .

Although  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma, \mathcal{P}\tilde{\mathcal{X}}_\circ^\Gamma$  are in general not  $C^\infty$ -stacks, they are prestacks on the site  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$  in the sense of Definition A.6 (that is, morphisms in  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma, \mathcal{P}\tilde{\mathcal{X}}_\circ^\Gamma$  satisfy a sheaf-like condition over  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$ , but objects may not). Thus,  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma, \mathcal{P}\tilde{\mathcal{X}}_\circ^\Gamma$  have stackifications  $\hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$ , defined up to equivalence, which are stacks on the site  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$ . By Theorem 9.5(a) below,  $\hat{\mathcal{X}}^\Gamma$  is a Deligne–Mumford  $C^\infty$ -stack, and  $\hat{\mathcal{X}}_\circ^\Gamma$  is an open  $C^\infty$ -substack in  $\hat{\mathcal{X}}^\Gamma$ .

Let  $(A, \Delta), (A', \Delta')$  be objects in  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$ . Define a right action of  $\Delta$  on morphisms  $(c, \iota) : (A, \Delta) \rightarrow (A', \Delta')$  in  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$  by  $(c, \iota) \cdot \delta = (c \circ \delta, \iota^\delta)$ , where  $\iota^\delta : \Delta \rightarrow \Delta'$  maps  $\iota^\delta : \epsilon \mapsto \iota(\delta \circ \epsilon \circ \delta^{-1})$ . If  $(c', \iota') : (A', \Delta') \rightarrow (A'', \Delta'')$  is another morphism and  $\delta' \in \Delta'$ , it is easy to show that

$$((c', \iota') \cdot \delta') \circ ((c, \iota) \cdot \delta) = ((c', \iota') \circ (c, \iota)) \cdot (\iota^{-1}(\delta') \circ \delta). \quad (9.3)$$

Define a category  $\mathcal{P}\hat{\mathcal{X}}^\Gamma$  to have objects  $(A, \Delta)$  as in  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$ , and to have morphisms  $(c, \iota)\Delta : (A, \Delta) \rightarrow (A', \Delta')$  for morphisms  $(c, \iota) : (A, \Delta) \rightarrow (A', \Delta')$  in  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$ , where  $(c, \iota)\Delta = \{(c, \iota) \cdot \delta : \delta \in \Delta\}$  is the  $\Delta$ -orbit of  $(c, \iota)$ . Define composition of morphisms in  $\mathcal{P}\hat{\mathcal{X}}^\Gamma$  by  $((c', \iota')\Delta') \circ ((c, \iota)\Delta) = ((c', \iota') \circ (c, \iota))\Delta$ , where  $(c', \iota') \circ (c, \iota)$  is composition of morphisms in  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$ . Equation (9.3) shows this is well-defined. Define identity morphisms  $\text{id}_{(A, \Delta)} = (\text{id}_A, \text{id}_\Delta)\Delta : (A, \Delta) \rightarrow (A, \Delta)$  in  $\mathcal{P}\hat{\mathcal{X}}^\Gamma$ . Define a functor  $p_{\mathcal{P}\hat{\mathcal{X}}^\Gamma} : \mathcal{P}\hat{\mathcal{X}}^\Gamma \rightarrow \mathbf{C}^\infty\mathbf{Sch}$  to map  $(A, \Delta) \mapsto p_{\mathcal{X}}(A)$  on objects and  $(c, \iota)\Delta \mapsto p_{\mathcal{X}}(c)$  on morphisms.

Define  $\mathcal{P}\hat{\mathcal{X}}_\circ^\Gamma$  to be the full subcategory of  $\mathcal{P}\hat{\mathcal{X}}^\Gamma$  whose objects are objects of  $\mathcal{P}\tilde{\mathcal{X}}_\circ^\Gamma$ , and define  $p_{\mathcal{P}\hat{\mathcal{X}}_\circ^\Gamma} = p_{\mathcal{P}\hat{\mathcal{X}}^\Gamma}|_{\mathcal{P}\hat{\mathcal{X}}_\circ^\Gamma} : \mathcal{P}\hat{\mathcal{X}}_\circ^\Gamma \rightarrow \mathbf{C}^\infty\mathbf{Sch}$ . Then as for  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma, \mathcal{P}\tilde{\mathcal{X}}_\circ^\Gamma$  are prestacks on  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$ , and by Theorem 9.5(a) their stackifications  $\hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$  are Deligne–Mumford  $C^\infty$ -stacks. Furthermore, by Theorem 9.5(g) below  $\hat{\mathcal{X}}_\circ^\Gamma$  has

trivial isotropy groups, so by Theorem 7.20 there is a  $C^\infty$ -scheme  $\hat{X}_\circ^\Gamma$ , unique up to isomorphism, such that  $\hat{\mathcal{X}}_\circ^\Gamma \simeq \hat{X}_\circ^\Gamma$ .

Next, we define all the 1-morphisms in (9.2).

DEFINITION 9.3. In Definitions 9.1 and 9.2, for  $\Lambda \in \text{Aut}(\Gamma)$  define functors

$$\begin{aligned} L^\Gamma(\Lambda, \mathcal{X}) : \mathcal{X}^\Gamma &\longrightarrow \mathcal{X}^\Gamma, & O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma &\longrightarrow \mathcal{X}, & \mathcal{P}\tilde{O}^\Gamma(\mathcal{X}) : \mathcal{P}\tilde{\mathcal{X}}^\Gamma &\longrightarrow \mathcal{X}, \\ \mathcal{P}\tilde{\Pi}^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma &\longrightarrow \mathcal{P}\tilde{\mathcal{X}}^\Gamma & \text{ and } & \mathcal{P}\hat{\Pi}^\Gamma(\mathcal{X}) : \mathcal{P}\hat{\mathcal{X}}^\Gamma &\longrightarrow \mathcal{P}\hat{\mathcal{X}}^\Gamma \end{aligned}$$

on objects by

$$\begin{aligned} L^\Gamma(\Lambda, \mathcal{X}) : (A, \rho) &\mapsto (A, \rho \circ \Lambda^{-1}), & O^\Gamma(\mathcal{X}) : (A, \rho) &\mapsto A, & \mathcal{P}\tilde{O}^\Gamma(\mathcal{X}) : (A, \Delta) &\mapsto A, \\ \mathcal{P}\tilde{\Pi}^\Gamma(\mathcal{X}) : (A, \rho) &\mapsto (A, \rho(\Gamma)) & \text{ and } & \mathcal{P}\hat{\Pi}^\Gamma(\mathcal{X}) : (A, \Delta) &\mapsto (A, \Delta), \end{aligned}$$

and on morphisms by

$$\begin{aligned} L^\Gamma(\Lambda, \mathcal{X}) : c &\longmapsto c, & O^\Gamma(\mathcal{X}) : c &\longmapsto c, & \mathcal{P}\tilde{O}^\Gamma(\mathcal{X}) : (c, \iota) &\longmapsto c, \\ \mathcal{P}\tilde{\Pi}^\Gamma(\mathcal{X}) : c &\mapsto (c, \sigma \circ \rho^{-1}) & \text{ on } c : (A, \rho) &\rightarrow (B, \sigma), & \text{ and } \\ \mathcal{P}\hat{\Pi}^\Gamma(\mathcal{X}) : (c, \iota) &\mapsto (c, \iota)\Delta & \text{ on } (c, \iota) : (A, \Delta) &\rightarrow (A', \Delta'). \end{aligned}$$

It is trivial to check that these are all functors, and commute with the projections  $p_{\mathcal{X}}, p_{\mathcal{X}^\Gamma}, p_{\tilde{\mathcal{X}}^\Gamma}, p_{\hat{\mathcal{X}}^\Gamma}$  to  $\mathbf{C}^\infty\mathbf{Sch}$ . Hence  $L^\Gamma(\Lambda, \mathcal{X}), O^\Gamma(\mathcal{X})$  are 1-morphisms of  $C^\infty$ -stacks. Note that  $L^\Gamma(\Lambda, \mathcal{X}) \circ L^\Gamma(\Lambda', \mathcal{X}) = L^\Gamma(\Lambda \circ \Lambda', \mathcal{X})$  and  $L^\Gamma(\Lambda^{-1}, \mathcal{X}) = L^\Gamma(\Lambda, \mathcal{X})^{-1}$  for  $\Lambda, \Lambda' \in \text{Aut}(\Gamma)$ , so  $L^\Gamma(-, \mathcal{X})$  is an action of  $\text{Aut}(\Gamma)$  on  $\mathcal{X}^\Gamma$  by 1-isomorphisms.

Now  $\mathcal{P}\tilde{O}^\Gamma(\mathcal{X}), \mathcal{P}\tilde{\Pi}^\Gamma(\mathcal{X}), \mathcal{P}\hat{\Pi}^\Gamma(\mathcal{X})$  are 1-morphisms of prestacks, so stackifying gives 1-morphisms of  $C^\infty$ -stacks  $\tilde{O}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \rightarrow \mathcal{X}$ ,  $\tilde{\Pi}^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \rightarrow \tilde{\mathcal{X}}^\Gamma$ ,  $\hat{\Pi}^\Gamma(\mathcal{X}) : \hat{\mathcal{X}}^\Gamma \rightarrow \hat{\mathcal{X}}^\Gamma$ . Define 1-morphisms of  $C^\infty$ -stacks

$$\begin{aligned} L_\circ^\Gamma(\Lambda, \mathcal{X}) : \mathcal{X}_\circ^\Gamma &\longrightarrow \mathcal{X}_\circ^\Gamma, & O_\circ^\Gamma(\mathcal{X}) : \mathcal{X}_\circ^\Gamma &\longrightarrow \mathcal{X}, & \tilde{O}_\circ^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}_\circ^\Gamma &\longrightarrow \mathcal{X}, \\ \tilde{\Pi}_\circ^\Gamma(\mathcal{X}) : \mathcal{X}_\circ^\Gamma &\longrightarrow \tilde{\mathcal{X}}_\circ^\Gamma & \text{ and } & \hat{\Pi}_\circ^\Gamma(\mathcal{X}) : \hat{\mathcal{X}}_\circ^\Gamma &\longrightarrow \hat{\mathcal{X}}_\circ^\Gamma, \end{aligned}$$

to be the restrictions of  $L^\Gamma(\Lambda, \mathcal{X}), \dots, \hat{\Pi}^\Gamma(\mathcal{X})$  to the open  $C^\infty$ -substacks  $\mathcal{X}_\circ^\Gamma, \tilde{\mathcal{X}}_\circ^\Gamma$ . Then  $L_\circ^\Gamma(-, \mathcal{X})$  is an action of  $\text{Aut}(\Gamma)$  on  $\mathcal{X}_\circ^\Gamma$  by 1-isomorphisms.

It is easy to see that the analogue of (9.2) with prestacks  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma, \dots, \mathcal{P}\hat{\mathcal{X}}_\circ^\Gamma$  and prestack 1-morphisms  $\mathcal{P}\tilde{O}^\Gamma(\mathcal{X}), \dots, \mathcal{P}\hat{\Pi}_\circ^\Gamma(\mathcal{X})$  is strictly commutative, i.e. 2-commutative with identity 2-morphisms. Thus on stackifying, (9.2) commutes up to canonical 2-isomorphisms.

DEFINITION 9.4. Let the 1-morphisms  $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \rightarrow \mathcal{X}$ ,  $O_\circ^\Gamma(\mathcal{X}) : \mathcal{X}_\circ^\Gamma \rightarrow \mathcal{X}$  be as in Definition 9.3. We will define actions of  $\Gamma$  on  $O^\Gamma(\mathcal{X}), O_\circ^\Gamma(\mathcal{X})$  by 2-morphisms. For  $\gamma \in \Gamma$  and  $(A, \rho) \in \mathcal{X}^\Gamma$ , define an isomorphism  $E^\Gamma(\gamma, \mathcal{X})(A, \rho) : O^\Gamma(\mathcal{X})(A, \rho) \rightarrow O^\Gamma(\mathcal{X})(A, \rho)$  by  $E^\Gamma(\gamma, \mathcal{X}) = \rho(\gamma) : A \rightarrow A$ . If  $c : (A, \rho) \rightarrow (B, \sigma)$  is a morphism in  $\mathcal{X}^\Gamma$  then

$$O^\Gamma(\mathcal{X})(c) \circ E^\Gamma(\gamma, \mathcal{X})(A, \rho) = c \circ \rho(\gamma) = \sigma(\gamma) \circ \rho = E^\Gamma(\gamma, \mathcal{X})(B, \sigma) \circ O^\Gamma(\mathcal{X})(c).$$

Hence  $E^\Gamma(\gamma, \mathcal{X}) : O^\Gamma(\mathcal{X}) \Rightarrow O^\Gamma(\mathcal{X})$  is a natural isomorphism of functors. Since  $p_{\mathcal{X}}(E^\Gamma(\gamma, \mathcal{X})(A, \rho)) = p_{\mathcal{X}}(\rho(\gamma)) = \text{id}_{p_{\mathcal{X}}(A)}$  for all  $(A, \rho)$ , we have  $p_{\mathcal{X}} * E^\Gamma(\gamma, \mathcal{X}) = p_{\mathcal{X}^\Gamma}$ , so  $E^\Gamma(\gamma, \mathcal{X}) : O^\Gamma(\mathcal{X}) \Rightarrow O^\Gamma(\mathcal{X})$  is a 2-morphism of  $C^\infty$ -stacks. Clearly  $E^\Gamma(1, \mathcal{X}) = \text{id}_{O^\Gamma(\mathcal{X})}$  and  $E^\Gamma(\gamma, \mathcal{X}) \circ E^\Gamma(\delta, \mathcal{X}) = E^\Gamma(\gamma\delta, \mathcal{X})$  for all  $\gamma, \delta \in \Gamma$ , so

$E^\Gamma(-, \mathcal{X}) : \Gamma \rightarrow \text{Aut}(O^\Gamma(\mathcal{X}))$  is a group morphism. We define 2-morphisms  $E^\Gamma_\circ(\gamma, \mathcal{X}) : O^\Gamma_\circ(\mathcal{X}) \Rightarrow O^\Gamma_\circ(\mathcal{X})$  for  $\gamma \in \Gamma$  in the same way.

Here are some basic properties of these definitions.

**THEOREM 9.5. (a)**  $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$  are Deligne–Mumford  $C^\infty$ -stacks, and  $\mathcal{X}^\Gamma_\circ \subseteq \mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma_\circ \subseteq \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma_\circ \subseteq \hat{\mathcal{X}}^\Gamma$  are open  $C^\infty$ -substacks. Also  $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$  and  $\tilde{\mathcal{X}}^\Gamma_\circ \simeq [\mathcal{X}^\Gamma_\circ / \text{Aut}(\Gamma)]$ , where the  $\text{Aut}(\Gamma)$ -actions are  $L^\Gamma(-, \mathcal{X})$  and  $L^\Gamma_\circ(-, \mathcal{X})$ .

**(b)** If  $\mathcal{X}$  is separated, locally fair, locally finitely presented, or second countable, then  $\mathcal{X}^\Gamma, \mathcal{X}^\Gamma_\circ, \tilde{\mathcal{X}}^\Gamma, \tilde{\mathcal{X}}^\Gamma_\circ, \hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma_\circ$  are separated, locally fair, locally finitely presented, or second countable, respectively.

If  $\mathcal{X}$  is compact then  $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$  are compact.

**(c)** Points of  $\mathcal{X}^\Gamma_{\text{top}}$  are equivalence classes  $[x, \rho]$  of pairs  $(x, \rho)$ , where  $x : \bar{x} \rightarrow \mathcal{X}$  is a 1-morphism and  $\rho : \Gamma \rightarrow \text{Aut}(x)$  is an injective group morphism into the group  $\text{Aut}(x)$  of 2-isomorphisms  $\eta : x \Rightarrow x$ , and pairs  $(x, \rho), (x', \rho')$  are equivalent if there exists  $\zeta : x \Rightarrow x'$  with  $\zeta \odot \rho(\gamma) = \rho'(\gamma) \odot \zeta : x \Rightarrow x'$  for all  $\gamma \in \Gamma$ . They have isotropy groups

$$\text{Iso}_{\mathcal{X}^\Gamma}([x, \rho]) = \{\eta \in \text{Aut}(x) : \rho(\gamma) = \eta \rho(\gamma) \eta^{-1} \ \forall \gamma \in \Gamma\}.$$

Points of  $\mathcal{X}^\Gamma_{\circ, \text{top}}$  are  $[x, \rho]$  with  $\rho : \Gamma \rightarrow \text{Aut}(x)$  an isomorphism, and have canonical isomorphisms  $\text{Iso}_{\mathcal{X}^\Gamma_\circ}([x, \rho]) \cong C(\Gamma)$ , where  $C(\Gamma)$  is the centre of  $\Gamma$ .

**(d)** Points of  $\tilde{\mathcal{X}}^\Gamma_{\text{top}}$  are equivalence classes  $[x, \Delta]$  of pairs  $(x, \Delta)$ , where  $x : \bar{x} \rightarrow \mathcal{X}$  is a 1-morphism and  $\Delta \subseteq \text{Aut}(x)$  is a subgroup isomorphic to  $\Gamma$ , and pairs  $(x, \Delta), (x', \Delta')$  are equivalent if there exists a 2-isomorphism  $\zeta : x \Rightarrow x'$  with  $\Delta' = \zeta \odot \Delta \odot \zeta^{-1}$ . They have isotropy groups

$$\text{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x, \Delta]) \cong \{\eta \in \text{Aut}(x) : \Delta = \eta \Delta \eta^{-1}\}.$$

Points of  $\tilde{\mathcal{X}}^\Gamma_{\circ, \text{top}}$  are  $[x, \Delta]$  with  $\Delta = \text{Aut}(x)$ , and have non-canonical isomorphisms  $\text{Iso}_{\tilde{\mathcal{X}}^\Gamma_\circ}([x, \Delta]) \cong \Gamma$ .

**(e)** As topological spaces  $\hat{\mathcal{X}}^\Gamma_{\text{top}} = \tilde{\mathcal{X}}^\Gamma_{\text{top}}$  and  $\hat{\mathcal{X}}^\Gamma_{\circ, \text{top}} = \tilde{\mathcal{X}}^\Gamma_{\circ, \text{top}}$ , and  $\hat{\Pi}^\Gamma(\mathcal{X})_{\text{top}}, \hat{\Pi}^\Gamma_\circ(\mathcal{X})_{\text{top}}$  are the identity maps. For  $[x, \Delta] \in \hat{\mathcal{X}}^\Gamma_{\text{top}}$  we have

$$\text{Iso}_{\hat{\mathcal{X}}^\Gamma}([x, \Delta]) \cong \{\eta \in \text{Aut}(x) : \Delta = \eta \Delta \eta^{-1}\} / \Delta.$$

Also  $\text{Iso}_{\hat{\mathcal{X}}^\Gamma_\circ}([x, \Delta]) = \{1\}$  for all  $[x, \Delta] \in \hat{\mathcal{X}}^\Gamma_{\circ, \text{top}}$ , so  $\hat{\mathcal{X}}^\Gamma_\circ$  is a  $C^\infty$ -scheme.

**(f)**  $L^\Gamma(\Lambda, \mathcal{X}), L^\Gamma_\circ(\Lambda, \mathcal{X}), O^\Gamma(\mathcal{X}), O^\Gamma_\circ(\mathcal{X}), \tilde{O}^\Gamma(\mathcal{X}), \tilde{O}^\Gamma_\circ(\mathcal{X}), \hat{\Pi}^\Gamma(\mathcal{X}), \hat{\Pi}^\Gamma_\circ(\mathcal{X})$  are all representable, but  $\hat{\Pi}^\Gamma(\mathcal{X}), \hat{\Pi}^\Gamma_\circ(\mathcal{X})$  in general are not representable.

**(g)**  $L^\Gamma(\Lambda, \mathcal{X}), L^\Gamma_\circ(\Lambda, \mathcal{X}), O^\Gamma(\mathcal{X}), \tilde{O}^\Gamma(\mathcal{X}), \hat{\Pi}^\Gamma(\mathcal{X}), \hat{\Pi}^\Gamma_\circ(\mathcal{X}), \hat{\Pi}^\Gamma(\mathcal{X}), \hat{\Pi}^\Gamma_\circ(\mathcal{X})$  are all proper, but  $O^\Gamma_\circ(\mathcal{X}), \tilde{O}^\Gamma_\circ(\mathcal{X})$  in general are not.

**(h)**  $O^\Gamma_\circ(\mathcal{X})_{\text{top}} : \mathcal{X}^\Gamma_{\circ, \text{top}} \rightarrow \mathcal{X}_{\text{top}}$  takes  $|\text{Aut}(\Gamma)| \cdot |C(\Gamma)| / |\Gamma|$  points  $[x, \rho]$  of  $\mathcal{X}^\Gamma_{\circ, \text{top}}$  to each point  $[x] \in \mathcal{X}_{\text{top}}$  with  $\text{Iso}_{\mathcal{X}}([x]) \cong \Gamma$ . Also  $\tilde{O}^\Gamma_\circ(\mathcal{X})_{\text{top}} : \tilde{\mathcal{X}}^\Gamma_{\circ, \text{top}} \rightarrow \mathcal{X}_{\text{top}}$  is a bijection with the subset of  $[x] \in \mathcal{X}_{\text{top}}$  with  $\text{Iso}_{\mathcal{X}}([x]) \cong \Gamma$ .

**PROOF.** For (a), we first prove that  $\mathcal{X}^\Gamma$  is a Deligne–Mumford  $C^\infty$ -stack. The inertia stack of  $\mathcal{X}$  is the fibre product  $\mathcal{I}_{\mathcal{X}} = \mathcal{X} \times_{\Delta_{\mathcal{X}}, \mathcal{X} \times \mathcal{X}, \Delta_{\mathcal{X}}} \mathcal{X}$ , where  $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is the diagonal 1-morphism. There is a canonical construction of fibre products of stacks. Taking  $\mathcal{I}_{\mathcal{X}}$  to be given by this construction, by definition objects of the category  $\mathcal{I}_{\mathcal{X}}$  are triples  $(A, B, c)$  where  $A, B$  are objects in  $\mathcal{X}$  with  $p_{\mathcal{X}}(A) = p_{\mathcal{X}}(B) = \underline{U}$  in  $\mathbf{C}^\infty\mathbf{Sch}$ , and  $c : \Delta_{\mathcal{X}}(A) \rightarrow \Delta_{\mathcal{X}}(B)$  is a morphism in  $\mathcal{X} \times \mathcal{X}$

with  $p_{\mathcal{X} \times \mathcal{X}}(c) = \text{id}_{\underline{U}}$ . But  $\Delta_{\mathcal{X}}(A) = (A, A)$  and  $\Delta_{\mathcal{X}}(B) = (B, B)$ , so  $c = (c_1, c_2)$  for  $c_1, c_2 : A \rightarrow B$  morphisms in  $\mathcal{X}$  with  $p_{\mathcal{X}}(c_i) = \text{id}_{\underline{U}}$ .

Thus we may write objects of  $\mathcal{I}_{\mathcal{X}}$  as quadruples  $(A, B, c, d)$ , where  $A, B$  are objects in  $\mathcal{X}$  with  $p_{\mathcal{X}}(A) = p_{\mathcal{X}}(B) = \underline{U}$ , and  $c, d : A \rightarrow B$  are isomorphisms in  $\mathcal{X}$  with  $p_{\mathcal{X}}(c) = p_{\mathcal{X}}(d) = \text{id}_{\underline{U}}$ . Morphisms  $(A, B, c, d) \rightarrow (A', B', c', d')$  in  $\mathcal{I}_{\mathcal{X}}$  are pairs  $(a, b)$  with  $a : A \rightarrow A'$  and  $b : B \rightarrow B'$  morphisms in  $\mathcal{X}$  such that  $b \circ c = c' \circ a$  and  $b \circ d = d' \circ a$ . This forces  $p_{\mathcal{X}}(a) = p_{\mathcal{X}}(b)$ . The functor  $p_{\mathcal{I}_{\mathcal{X}}} : \mathcal{I}_{\mathcal{X}} \rightarrow \mathbf{C}^\infty\mathbf{Sch}$  acts by  $p_{\mathcal{I}_{\mathcal{X}}} : (A, B, c, d) \mapsto p_{\mathcal{X}}(A) = p_{\mathcal{X}}(B)$  on objects and  $p_{\mathcal{I}_{\mathcal{X}}} : (a, b) \mapsto p_{\mathcal{X}}(a) = p_{\mathcal{X}}(b)$  on morphisms.

Write  $i_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{I}_{\mathcal{X}}$  for the 1-morphism mapping  $A \mapsto (A, A, \text{id}_A, \text{id}_A)$  on objects and  $a \mapsto (a, a)$  on morphisms. Since  $\mathcal{X}$  is Deligne–Mumford,  $i_{\mathcal{X}}$  is an equivalence with an open and closed  $C^\infty$ -substack  $i_{\mathcal{X}}(\mathcal{X})$  in  $\mathcal{I}_{\mathcal{X}}$ . Here  $i_{\mathcal{X}}(\mathcal{X})$  is the subcategory of objects in  $\mathcal{I}_{\mathcal{X}}$  isomorphic to some  $(A, A, \text{id}_A, \text{id}_A)$ . Thus  $i_{\mathcal{X}}(\mathcal{X})$  is the full subcategory of objects  $(A, B, c, d)$  in  $\mathcal{I}_{\mathcal{X}}$  with  $c = d$ .

Since  $i_{\mathcal{X}}(\mathcal{X})$  is open and closed in  $\mathcal{I}_{\mathcal{X}}$ , its complement  $\mathcal{J}_{\mathcal{X}} = \mathcal{I}_{\mathcal{X}} \setminus i_{\mathcal{X}}(\mathcal{X})$  as a  $C^\infty$ -stack is also an open and closed  $C^\infty$ -substack in  $\mathcal{I}_{\mathcal{X}}$ . As a subcategory,  $\mathcal{J}_{\mathcal{X}}$  is not simply the complement of the subcategory  $i_{\mathcal{X}}(\mathcal{X})$ . Instead,  $\mathcal{J}_{\mathcal{X}}$  is the full subcategory of objects  $(A, B, c, d)$  in  $\mathcal{I}_{\mathcal{X}}$  satisfying the following condition (\*) analogous to Definition 9.1(c):

- (\*) Write  $\underline{U} = p_{\mathcal{X}}(A) = p_{\mathcal{X}}(B)$ , and let  $u \in \underline{U}$ , and  $\underline{u} : * \rightarrow \underline{U}$  the corresponding morphism in  $\mathbf{C}^\infty\mathbf{Sch}$ . Since  $p_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{C}^\infty\mathbf{Sch}$  is a category fibred in groupoids, there exist  $a_u : A_u \rightarrow A$ ,  $b_u : B_u \rightarrow B$  in  $\mathcal{X}$  with  $p_{\mathcal{X}}(A_u) = p_{\mathcal{X}}(B_u) = *$  and  $p_{\mathcal{X}}(a_u) = p_{\mathcal{X}}(b_u) = \underline{u}$ , and unique isomorphisms  $c_u, d_u : A_u \rightarrow B_u$  such that  $a_u \circ c_u = c \circ a_u$  and  $a_u \circ d_u = d \circ a_u$ , and  $p_{\mathcal{X}}(c_u) = p_{\mathcal{X}}(d_u) = \text{id}_*$ . We require that  $c_u \neq d_u$  for all  $u \in \underline{U}$ .

Now form the product  $\prod_{\gamma \in \Gamma} \mathcal{X}$  of  $|\Gamma|$  copies of  $\mathcal{X}$ , and write  $\Delta_{\mathcal{X}}^\Gamma : \mathcal{X} \rightarrow \prod_{\gamma \in \Gamma} \mathcal{X}$  for the diagonal 1-morphism. Consider the  $C^\infty$ -stack fibre product

$$\mathcal{Y} = \mathcal{X} \times_{\Delta_{\mathcal{X}}^\Gamma, \prod_{\gamma \in \Gamma} \mathcal{X}, \Delta_{\mathcal{X}}^\Gamma} \mathcal{X}.$$

It is a Deligne–Mumford  $C^\infty$ -stack by Theorem 7.10. As for  $\mathcal{I}_{\mathcal{X}}$ , we can take objects of  $\mathcal{Y}$  to be  $(|\Gamma|+2)$ -tuples  $(A, B, c_\gamma : \gamma \in \Gamma)$ , where  $A, B$  are objects in  $\mathcal{X}$  with  $p_{\mathcal{X}}(A) = p_{\mathcal{Y}}(B) = \underline{U}$ , and  $c_\gamma : A \rightarrow B$  for  $\gamma \in \Gamma$  are isomorphisms in  $\mathcal{X}$  with  $p_{\mathcal{X}}(c_\gamma) = \text{id}_{\underline{U}}$ . Morphisms  $(A, B, c_\gamma : \gamma \in \Gamma) \rightarrow (A', B', c'_\gamma : \gamma \in \Gamma)$  in  $\mathcal{Y}$  are pairs  $(a, b)$  with  $a : A \rightarrow A'$  and  $b : B \rightarrow B'$  morphisms in  $\mathcal{X}$  such that  $b \circ c_\gamma = c'_\gamma \circ a : A \rightarrow B'$  for all  $\gamma \in \Gamma$ . The functor  $p_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathbf{C}^\infty\mathbf{Sch}$  acts by  $p_{\mathcal{Y}} : (A, B, c_\gamma : \gamma \in \Gamma) \mapsto p_{\mathcal{X}}(A) = p_{\mathcal{X}}(B)$  on objects and  $p_{\mathcal{Y}} : (a, b) \mapsto p_{\mathcal{X}}(a) = p_{\mathcal{X}}(b)$  on morphisms.

For  $\delta, \epsilon \in \Gamma$  define  $K_{\delta, \epsilon} : \mathcal{Y} \rightarrow \mathcal{I}_{\mathcal{X}}$  to map  $(A, B, c_\gamma : \gamma \in \Gamma) \mapsto (A, B, c_{\delta\epsilon}, c_\delta \circ c_1^{-1} \circ c_\epsilon)$  on objects and  $(a, b) \mapsto (a, b)$  on morphisms. It is easy to show that  $K_{\delta, \epsilon}$  is a functor, with  $p_{\mathcal{I}_{\mathcal{X}}} \circ K_{\delta, \epsilon} = p_{\mathcal{Y}}$ . Hence  $K_{\delta, \epsilon} : \mathcal{Y} \rightarrow \mathcal{I}_{\mathcal{X}}$  is a 1-morphism of Deligne–Mumford  $C^\infty$ -stacks. Thus  $K_{\delta, \epsilon}^{-1}(i_{\mathcal{X}}(\mathcal{X}))$  is an open and closed  $C^\infty$ -substack in  $\mathcal{Y}$ , since  $i_{\mathcal{X}}(\mathcal{X})$  is open and closed in  $\mathcal{I}_{\mathcal{X}}$ .

Similarly, for  $\delta \neq \epsilon \in \Gamma$ , define  $L_{\delta, \epsilon} : \mathcal{Y} \rightarrow \mathcal{I}_{\mathcal{X}}$  to map  $(A, B, c_\gamma : \gamma \in \Gamma) \mapsto (A, B, c_\delta, c_\epsilon)$  on objects and  $(a, b) \mapsto (a, b)$  on morphisms. Then  $L_{\delta, \epsilon} : \mathcal{Y} \rightarrow \mathcal{I}_{\mathcal{X}}$  is a 1-morphism, so  $L_{\delta, \epsilon}^{-1}(\mathcal{J}_{\mathcal{X}})$  is an open and closed  $C^\infty$ -substack in  $\mathcal{Y}$ , since  $\mathcal{J}_{\mathcal{X}}$  is

open and closed in  $\mathcal{I}_\mathcal{X}$ . Define

$$\mathcal{Y}' = \bigcap_{\delta, \epsilon \in \Gamma} K_{\delta, \epsilon}^{-1}(i_\mathcal{X}(\mathcal{X})) \cap \bigcap_{\delta \neq \epsilon \in \Gamma} L_{\delta, \epsilon}^{-1}(\mathcal{J}_\mathcal{X}).$$

Then  $\mathcal{Y}'$  is an open and closed  $C^\infty$ -substack in  $\mathcal{Y}$ , as it is a finite intersection of open and closed  $C^\infty$ -substacks in  $\mathcal{Y}$ .

Define a functor  $M : \mathcal{X}^\Gamma \rightarrow \mathcal{Y}'$  to map  $M : (A, \rho) \mapsto (A, A, \rho(\gamma) : \gamma \in \Gamma)$  on objects and  $M : a \mapsto (a, a)$  on morphisms. The nontrivial claim here is that if  $(A, \rho)$  is an object in  $\mathcal{X}^\Gamma$  then  $M((A, \rho)) = (A, A, \rho(\gamma) : \gamma \in \Gamma)$  is an object in  $\mathcal{Y}'$ . The reason for this is that as  $\rho : \Gamma \rightarrow \text{Aut}(A)$  is a group morphism, for each  $\delta, \epsilon \in \Gamma$  we have  $\rho(\delta\epsilon) = \rho(\delta)\rho(\epsilon) = \rho(\delta)\rho(1)^{-1}\rho(\epsilon)$ , so  $(A, A, \rho(\gamma) : \gamma \in \Gamma)$  lies in  $K_{\delta, \epsilon}^{-1}(i_\mathcal{X}(\mathcal{X}))$ . Also, in Definition 9.1(c)  $\rho_u : \Gamma \rightarrow \text{Aut}(A_u)$  is injective, so  $\rho_u(\delta) \neq \rho_u(\epsilon)$  for  $\delta \neq \epsilon \in \Gamma$ . This is equivalent to condition  $(*)$  for  $L_{\delta, \epsilon}((A, A, \rho(\gamma) : \gamma \in \Gamma))$ , so  $(A, A, \rho(\gamma) : \gamma \in \Gamma)$  lies in  $L_{\delta, \epsilon}^{-1}(\mathcal{J}_\mathcal{X})$ .

Similarly, define a functor  $N : \mathcal{Y}' \rightarrow \mathcal{X}^\Gamma$  to map  $N : (A, B, c_\gamma : \gamma \in \Gamma) \mapsto (A, \rho)$  on objects, where we define  $\rho(\gamma) = c_1^{-1} \circ c_\gamma$  for  $\gamma \in \Gamma$ , and to map  $N : (a, b) \mapsto a$  on morphisms. The nontrivial claim is that if  $(A, B, c_\gamma : \gamma \in \Gamma)$  is an object in  $\mathcal{Y}'$  then  $N((A, B, c_\gamma : \gamma \in \Gamma)) = (A, \rho)$  is an object in  $\mathcal{X}^\Gamma$ . This holds because  $(A, B, c_\gamma : \gamma \in \Gamma) \in K_{\delta, \epsilon}^{-1}(i_\mathcal{X}(\mathcal{X}))$  forces  $\rho(\delta\epsilon) = \rho(\delta)\rho(\epsilon)$  for all  $\delta, \epsilon$ , so  $\rho : \Gamma \rightarrow \text{Aut}(A)$  is a group morphism, and  $(A, B, c_\gamma : \gamma \in \Gamma) \in L_{\delta, \epsilon}^{-1}(\mathcal{J}_\mathcal{X})$  for  $\delta \neq \epsilon$  forces  $\rho_u(\delta) \neq \rho_u(\epsilon)$  in Definition 9.1(c), so  $\rho_u$  is injective.

Now  $N \circ M = \text{id}_{\mathcal{X}^\Gamma}$ , and there is a natural transformation  $\eta : M \circ N \Rightarrow \text{id}_{\mathcal{Y}'}$  acting by  $\eta : (A, B, c_\gamma : \gamma \in \Gamma) \mapsto (\text{id}_A, c_1)$ . So  $\mathcal{X}^\Gamma, \mathcal{Y}'$  are equivalent categories. Also  $p_{\mathcal{Y}'} \circ M = p_{\mathcal{X}^\Gamma}$  and  $p_{\mathcal{X}^\Gamma} \circ N = p_{\mathcal{Y}'}$ . Therefore  $M, N$  define equivalences of  $C^\infty$ -stacks, so as  $\mathcal{Y}'$  is a Deligne–Mumford  $C^\infty$ -stack,  $\mathcal{X}^\Gamma$  is also a Deligne–Mumford  $C^\infty$ -stack equivalent to  $\mathcal{Y}'$ . This proves the first part of (a).

To see that  $\mathcal{X}_\circ^\Gamma$  is an open  $C^\infty$ -substack of  $\mathcal{X}^\Gamma$ , note that the map  $\mathcal{X}_{\text{top}} \rightarrow \mathbb{N}$  mapping  $[x] \mapsto |\text{Iso}_\mathcal{X}([x])|$  is upper semicontinuous, so the subset of points  $[x]$  in  $\mathcal{X}_{\text{top}}$  with  $|\text{Iso}_\mathcal{X}([x])| \leq |\Gamma|$  is open, and corresponds to an open  $C^\infty$ -substack  $\mathcal{X}_{\leq |\Gamma|}$  in  $\mathcal{X}$ . But then  $\mathcal{X}_\circ^\Gamma \simeq \mathcal{X}^\Gamma \times_{O^\Gamma(\mathcal{X}), \mathcal{X}, \text{inc}} \mathcal{X}_{\leq |\Gamma|}$ , so  $\mathcal{X}_\circ^\Gamma$  is the open  $C^\infty$ -substack in  $\mathcal{X}^\Gamma$  corresponding to  $\mathcal{X}_{\leq |\Gamma|}$  in  $\mathcal{X}$ , as we have to prove.

Now  $L^\Gamma(-, \mathcal{X})$  defines an action of the finite group  $\text{Aut}(\Gamma)$  on the Deligne–Mumford  $C^\infty$ -stack  $\mathcal{X}^\Gamma$  by 1-isomorphisms, so we may form the quotient  $C^\infty$ -stack  $[\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$ , which is also a Deligne–Mumford  $C^\infty$ -stack. To define  $[\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$  we first define a prestack  $\mathcal{X}^\Gamma / \text{Aut}(\Gamma)$  which is the quotient of the category  $\mathcal{X}^\Gamma$  by  $\text{Aut}(\Gamma)$ , and then  $[\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$  is its stackification. Since  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma$  was defined to be equivalent to  $\mathcal{X}^\Gamma / \text{Aut}(\Gamma)$ , its stackification  $\tilde{\mathcal{X}}^\Gamma$  is equivalent to  $[\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$ . This proves that  $\tilde{\mathcal{X}}^\Gamma$  is a Deligne–Mumford  $C^\infty$ -stack and  $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$ , as in (a). Similarly  $\tilde{\mathcal{X}}_\circ^\Gamma \subseteq \tilde{\mathcal{X}}^\Gamma$  is an open  $C^\infty$ -substack, and  $\tilde{\mathcal{X}}_\circ^\Gamma \simeq [\mathcal{X}_\circ^\Gamma / \text{Aut}(\Gamma)]$ .

To show  $\hat{\mathcal{X}}^\Gamma$  is Deligne–Mumford, we first observe that  $\mathcal{P}\hat{\mathcal{X}}^\Gamma$  is a prestack, so  $\hat{\mathcal{X}}^\Gamma$  is a stack on  $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$ , and then either note that  $\hat{\Pi}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \rightarrow \hat{\mathcal{X}}^\Gamma$  has fibre  $[\ast / \Gamma]$  and  $\tilde{\mathcal{X}}^\Gamma$  is Deligne–Mumford, or use the local models for  $\hat{\mathcal{X}}^\Gamma$  given by Theorem 9.10. Then  $\hat{\mathcal{X}}_\circ^\Gamma \subseteq \hat{\mathcal{X}}^\Gamma$  is open as for  $\mathcal{X}_\circ^\Gamma, \tilde{\mathcal{X}}_\circ^\Gamma$ . This completes (a).

For (b), if  $\mathcal{X}$  is separated, locally fair, locally finitely presented, second countable, or compact, then  $\mathcal{Y} = \mathcal{X} \times_{\prod_\gamma \mathcal{X}} \mathcal{X}$  is separated,  $\dots$ , compact, so  $\mathcal{X}^\Gamma$  are separated,  $\dots$ , compact as it is equivalent to an open and closed  $C^\infty$ -substack  $\mathcal{Y}'$

of  $\mathcal{Y}$ , and  $\mathcal{X}_\circ^\Gamma$  is separated, locally fair, locally finitely presented, or second countable (but not necessarily compact) as it is open in  $\mathcal{X}^\Gamma$ . The result for  $\tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma, \hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$  follows as  $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$ ,  $\tilde{\mathcal{X}}_\circ^\Gamma \simeq [\mathcal{X}_\circ^\Gamma / \text{Aut}(\Gamma)]$ , and  $\hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$  fibre over  $\tilde{\mathcal{X}}^\Gamma, \tilde{\mathcal{X}}_\circ^\Gamma$  with fibre  $[\bar{x}/\Gamma]$ .

For (c), there is a 1-1 correspondence between 1-morphisms  $x : \bar{x} \rightarrow \mathcal{X}$  and objects  $A_x$  in  $\mathcal{X}$  with  $p_{\mathcal{X}}(A_x) = \bar{x}$ , and if  $x, y : \bar{x} \rightarrow \mathcal{X}$  correspond to  $A_x, A_y$  in  $\mathcal{X}$  there is a 1-1 correspondence between 2-morphisms  $\eta : x \Rightarrow y$  and morphisms  $a_\eta : A_x \rightarrow A_y$  in  $\mathcal{X}$  with  $p_{\mathcal{X}}(a_\eta) = \text{id}_{\bar{x}}$ . The same correspondences hold for  $\mathcal{X}^\Gamma$ . Thus, each 1-morphism  $y : \bar{x} \rightarrow \mathcal{X}^\Gamma$  corresponds uniquely to some  $(B, \sigma)$  in  $\mathcal{X}^\Gamma$  with  $p_{\mathcal{X}}(B) = \bar{x}$ , so  $B = A_x$  for some unique 1-morphism  $x : \bar{x} \rightarrow \mathcal{X}$ , and each  $\sigma(\gamma) : A_x \rightarrow A_x$  is  $a_{\rho(\gamma)}$  for some unique 2-morphism  $\rho(\gamma) : x \Rightarrow x$ , and  $\rho : \Gamma \rightarrow \text{Aut}(x)$  is a group morphism. Definition 9.1 implies that  $\rho$  is injective.

This establishes a 1-1 correspondence between 1-morphisms  $y : \bar{x} \rightarrow \mathcal{X}^\Gamma$  and pairs  $(x, \rho)$ , where  $x : \bar{x} \rightarrow \mathcal{X}$  is a 1-morphism and  $\rho : \Gamma \rightarrow \text{Aut}(x)$  an injective group morphism. Similarly, if  $y, y' : \bar{x} \rightarrow \mathcal{X}^\Gamma$  correspond to  $(x, \rho), (x', \rho')$  then 2-morphisms  $\theta : y \Rightarrow y'$  correspond to 2-morphisms  $\zeta : x \Rightarrow x'$  with  $\zeta \odot \rho(\gamma) = \rho'(\gamma) \odot \zeta : x \Rightarrow x'$  for all  $\gamma \in \Gamma$ . Also 1-morphisms  $y : \bar{x} \rightarrow \mathcal{X}_\circ^\Gamma$  correspond to pairs  $(x, \rho)$  with  $\rho : \Gamma \rightarrow \text{Aut}(x)$  an isomorphism. Part (c) then follows. Parts (d), (e) come from the definitions of  $\mathcal{P}\tilde{\mathcal{X}}^\Gamma, \dots, \mathcal{P}\hat{\mathcal{X}}_\circ^\Gamma$  in the same way, noting that stackifying does not change 1-morphisms  $\bar{x} \rightarrow \mathcal{P}\tilde{\mathcal{X}}^\Gamma$  or their 2-morphisms.

For (f),  $L^\Gamma(\Lambda, \mathcal{X})$  is representable as it is a 1-isomorphism. Suppose  $(A, \rho)$  is an object in  $\mathcal{X}^\Gamma$  with  $p_{\mathcal{X}^\Gamma}(A, \rho) = \bar{u}$ , so that  $O^\Gamma(\mathcal{X}) : (A, \rho) \mapsto A$ , and  $a : A \rightarrow A'$  is an isomorphism in  $\mathcal{X}$  with  $p_{\mathcal{X}}(a) = \text{id}_{\bar{u}}$ . Then  $a : (A, \rho) \rightarrow (A', a \circ \rho \circ a^{-1})$  is the unique isomorphism in  $\mathcal{X}^\Gamma$  with  $O^\Gamma(\mathcal{X}) : a \mapsto a$ , so  $O^\Gamma(\mathcal{X})$  is representable. The action  $\mathcal{P}\tilde{O}^\Gamma(\mathcal{X}) : (c, \iota) \mapsto c$  of  $\mathcal{P}\tilde{O}^\Gamma(\mathcal{X})$  on 1-morphisms is injective, as  $c$  determines  $\iota$  by  $\iota(\delta) \circ c = c \circ \delta$  for  $\delta \in \Delta$ . This implies that the stackification  $\tilde{O}^\Gamma(\mathcal{X})$  is representable. Then  $\tilde{\Pi}^\Gamma(\mathcal{X})$  representable follows from  $\tilde{O}^\Gamma(\mathcal{X}) \circ \tilde{\Pi}^\Gamma(\mathcal{X}) \cong O^\Gamma(\mathcal{X})$  with  $O^\Gamma(\mathcal{X}), \tilde{O}^\Gamma(\mathcal{X})$  representable. Also  $L_\circ^\Gamma(\Lambda, \mathcal{X}), O_\circ^\Gamma(\mathcal{X}), \tilde{O}_\circ^\Gamma(\mathcal{X}), \tilde{\Pi}_\circ^\Gamma(\mathcal{X})$  are representable, as they are restrictions of  $L^\Gamma(\Lambda, \mathcal{X}), \dots, \tilde{\Pi}^\Gamma(\mathcal{X})$  to open  $C^\infty$ -substacks. The actions of  $\hat{\Pi}^\Gamma(\mathcal{X}), \hat{\Pi}_\circ^\Gamma(\mathcal{X})$  on isotropy groups have kernels isomorphic to  $\Gamma$ . So if  $\Gamma \neq \{1\}$  these actions are not injective, and  $\hat{\Pi}^\Gamma(\mathcal{X}), \hat{\Pi}_\circ^\Gamma(\mathcal{X})$  are not representable.

For (g),  $L^\Gamma(\Lambda, \mathcal{X}), L_\circ^\Gamma(\Lambda, \mathcal{X})$  are 1-isomorphisms,  $\tilde{\Pi}^\Gamma(\mathcal{X}), \tilde{\Pi}_\circ^\Gamma(\mathcal{X})$  project to quotients by  $\text{Aut}(\Gamma)$ , and  $\hat{\Pi}^\Gamma(\mathcal{X}), \hat{\Pi}_\circ^\Gamma(\mathcal{X})$  are fibrations with fibre  $[\bar{x}/\Gamma]$ , so these are all proper. We can see that  $O^\Gamma(\mathcal{X}), \tilde{O}^\Gamma(\mathcal{X})$  are proper, but  $O_\circ^\Gamma(\mathcal{X}), \tilde{O}_\circ^\Gamma(\mathcal{X})$  in general are not, using Theorem 9.10 and the fact that every Deligne–Mumford  $C^\infty$ -stack is locally of the form  $[\underline{X}/G]$ .

For (h), if  $[x] \in \mathcal{X}_{\text{top}}$  with  $\text{Iso}_{\mathcal{X}}([x]) \cong \Gamma$ , then by (c) points  $[x, \rho] \in \mathcal{X}_{\circ, \text{top}}^\Gamma$  with  $O_\circ^\Gamma(\mathcal{X})_{\text{top}} : [x, \rho] \mapsto [x]$  are given by isomorphisms  $\rho : \Gamma \rightarrow \text{Iso}_{\mathcal{X}}([x])$ . There are  $|\text{Aut}(\Gamma)|$  such  $\rho$ . If  $\rho, \rho'$  are two such isomorphisms, then (c) shows  $[x, \rho] = [x, \rho']$  if and only if  $\rho' = \rho^\alpha$  for some  $\alpha \in \Gamma$ , where  $\rho^\alpha : \gamma \mapsto \alpha\gamma\alpha^{-1}$ . For  $\alpha_1, \alpha_2 \in \Gamma$ , we see that  $\rho^{\alpha_1} = \rho^{\alpha_2}$  if and only if  $(\alpha_2^{-1}\alpha_1)\gamma = \gamma(\alpha_2^{-1}\alpha_1)$  for all  $\gamma \in \Gamma$ , that is, if  $\alpha_2^{-1}\alpha_1 \in C(\Gamma)$ . Hence, the  $\rho^\alpha$  for  $\alpha \in \Gamma$  realize  $|\Gamma|/|C(\Gamma)|$  distinct isomorphisms  $\rho' : \Gamma \rightarrow \text{Iso}_{\mathcal{X}}([x])$ . So the  $|\text{Aut}(\Gamma)|$  isomorphisms  $\rho : \Gamma \rightarrow \text{Iso}_{\mathcal{X}}([x])$  are identified in groups of  $|\Gamma|/|C(\Gamma)|$  to make  $|\text{Aut}(\Gamma)| \cdot |C(\Gamma)|/|\Gamma|$  points  $[x, \rho]$  in  $\mathcal{X}_{\circ, \text{top}}^\Gamma$ . The statement for  $\tilde{O}_\circ^\Gamma(\mathcal{X})_{\text{top}}$  is immediate as if  $[x, \Delta] \in \tilde{\mathcal{X}}_{\circ, \text{top}}^\Gamma$  then  $\Delta = \text{Aut}(x)$ , so  $[x, \Delta] \mapsto [x]$  is a 1-1 correspondence. This completes the proof of Theorem 9.5.  $\square$



EXAMPLE 9.6. Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack, and  $\mathcal{I}_{\mathcal{X}}$  the *inertia stack* of  $\mathcal{X}$ , as in the proof of Theorem 9.5. Then there is an equivalence

$$\mathcal{I}_{\mathcal{X}} = \mathcal{X} \times_{\Delta_{\mathcal{X}}, \mathcal{X} \times_{\mathcal{X}} \Delta_{\mathcal{X}}} \mathcal{X} \simeq \coprod_{k \geq 1} \mathcal{X}^{\mathbb{Z}_k}.$$

To see this, note that points of  $\mathcal{I}_{\mathcal{X}}$  are equivalence classes  $[x, \eta]$ , where  $[x] \in \mathcal{X}_{\text{top}}$  and  $\eta \in \text{Iso}_{\mathcal{X}}([x])$ . Since  $\mathcal{X}$  is Deligne–Mumford,  $\text{Iso}_{\mathcal{X}}([x])$  is a finite group, so each  $\eta \in \text{Iso}_{\mathcal{X}}([x])$  has some finite order  $k \geq 1$ , and generates an injective morphism  $\rho : \mathbb{Z}_k \rightarrow \text{Iso}_{\mathcal{X}}([x])$  mapping  $\rho : a \mapsto \eta^a$ . We may identify  $\mathcal{X}^{\mathbb{Z}_k}$  with the open and closed  $C^\infty$ -substack of  $[x, \eta]$  in  $\mathcal{I}_{\mathcal{X}}$  for which  $\eta$  has order  $k$ .

## 9.2. Lifting 1- and 2-morphisms to orbifold strata

The construction of  $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$  extends functorially to 1- and 2-morphisms.

DEFINITION 9.7. Let  $\mathcal{X}, \mathcal{Y}$  be Deligne–Mumford  $C^\infty$ -stacks,  $\Gamma$  a finite group, and  $f : \mathcal{X} \rightarrow \mathcal{Y}$  a representable 1-morphism, so that  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a functor with  $p_{\mathcal{Y}} \circ f = p_{\mathcal{X}}$ . We will define a representable 1-morphism  $f^\Gamma : \mathcal{X}^\Gamma \rightarrow \mathcal{Y}^\Gamma$ .

On objects  $(A, \rho)$  in  $\mathcal{X}^\Gamma$ , define  $f^\Gamma(A, \rho) = (f(A), f \circ \rho)$ . We must check that  $f^\Gamma(A, \rho)$  satisfies Definition 9.1(a)–(c). Parts (a),(b) hold as  $f$  is a functor with  $p_{\mathcal{Y}} \circ f = p_{\mathcal{X}}$ . For (c), if  $u \in \underline{U}$  then (c) for  $(A, \rho)$  shows that  $\rho_u : \Gamma \rightarrow \text{Aut}(A_u)$  is injective, so  $f \circ \rho_u : \Gamma \rightarrow \text{Aut}(f(A_u))$  is injective as  $f$  is representable, and this gives (c) for  $(f(A), f \circ \rho)$ . On morphisms  $c : (A, \rho) \rightarrow (B, \sigma)$  in  $\mathcal{X}^\Gamma$ , define  $f^\Gamma(c) : f^\Gamma(A, \rho) \rightarrow f^\Gamma(B, \sigma)$  by  $f^\Gamma(c) = f(c) : f(A) \rightarrow f(B)$ .

Then  $f^\Gamma : \mathcal{X}^\Gamma \rightarrow \mathcal{Y}^\Gamma$  is a functor, and  $p_{\mathcal{Y}} \circ f = p_{\mathcal{X}}$  implies that  $p_{\mathcal{Y}^\Gamma} \circ f^\Gamma = p_{\mathcal{X}^\Gamma}$ . Hence  $f^\Gamma : \mathcal{X}^\Gamma \rightarrow \mathcal{Y}^\Gamma$  is a 1-morphism of  $C^\infty$ -stacks. It is the unique such 1-morphism with  $O^\Gamma(\mathcal{Y}) \circ f^\Gamma = f \circ O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \rightarrow \mathcal{Y}$ . Also,  $f^\Gamma$  is injective on morphisms, as  $f$  is, so  $f^\Gamma$  is representable.

Now let  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  be representable, and  $\eta : f \Rightarrow g$  be a 2-morphism. Then  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  are functors, and  $\eta : f \Rightarrow g$  is a natural isomorphism. Define  $\eta^\Gamma : f^\Gamma \Rightarrow g^\Gamma$  by taking the isomorphism  $\eta^\Gamma(A, \rho) : f^\Gamma(A, \rho) \rightarrow g^\Gamma(A, \rho)$  in  $\mathcal{Y}^\Gamma$  for each object  $(A, \rho)$  in  $\mathcal{X}^\Gamma$  to be the isomorphism  $\eta^\Gamma(A, \rho) = \eta(A) : f(A) \rightarrow g(A)$  in  $\mathcal{Y}$ . Then  $\eta^\Gamma : f^\Gamma \Rightarrow g^\Gamma$  is a natural isomorphism of functors, and hence a 2-morphism in  $\mathbf{DMC}^\infty \mathbf{Sta}$ . It is the unique such 2-morphism with  $\text{id}_{O^\Gamma(\mathcal{Y})} * \eta^\Gamma = \eta * \text{id}_{O^\Gamma(\mathcal{X})}$ .

Write  $\mathbf{DMC}^\infty \mathbf{Sta}^{\text{re}}$  for the 2-subcategory of  $\mathbf{DMC}^\infty \mathbf{Sta}$  with only representable 1-morphisms. Define  $F^\Gamma : \mathbf{DMC}^\infty \mathbf{Sta}^{\text{re}} \rightarrow \mathbf{DMC}^\infty \mathbf{Sta}^{\text{re}}$  by  $F^\Gamma : \mathcal{X} \mapsto F^\Gamma(\mathcal{X}) = \mathcal{X}^\Gamma$  on objects,  $F^\Gamma : f \mapsto F^\Gamma(f) = f^\Gamma$  on representable 1-morphisms, and  $F^\Gamma : \eta \mapsto F^\Gamma(\eta) = \eta^\Gamma$  on 2-morphisms. Then  $F^\Gamma$  is a strict 2-functor, in the sense of §A.1.

DEFINITION 9.8. Let  $\mathcal{X}, \mathcal{Y}$  be Deligne–Mumford  $C^\infty$ -stacks,  $\Gamma$  a finite group, and  $f : \mathcal{X} \rightarrow \mathcal{Y}$  a representable 1-morphism. Define functors  $\mathcal{P}\hat{f}^\Gamma : \mathcal{P}\hat{\mathcal{X}}^\Gamma \rightarrow \mathcal{P}\hat{\mathcal{Y}}^\Gamma$  mapping  $(A, \Delta) \mapsto (f(A), f(\Delta))$  on objects and  $(c, \iota) \mapsto (f(c), f \circ \iota \circ f|_{\Delta}^{-1})$  on morphisms, and  $\mathcal{P}\hat{f}^\Gamma : \mathcal{P}\hat{\mathcal{X}}^\Gamma \rightarrow \mathcal{P}\hat{\mathcal{Y}}^\Gamma$  mapping  $(A, \Delta) \mapsto (f(A), f(\Delta))$  and  $(c, \iota) \mapsto (f(c), f \circ \iota \circ f|_{\Delta}^{-1})f(\Delta)$ . Then  $\mathcal{P}\hat{f}^\Gamma, \mathcal{P}\hat{f}^\Gamma$  are 1-morphisms of prestacks, so stackifying gives 1-morphisms  $\tilde{f}^\Gamma : \tilde{\mathcal{X}}^\Gamma \rightarrow \tilde{\mathcal{Y}}^\Gamma$  and  $\hat{f}^\Gamma : \hat{\mathcal{X}}^\Gamma \rightarrow \hat{\mathcal{Y}}^\Gamma$ .

If  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  are representable, and  $\eta : f \Rightarrow g$  is a 2-morphism, we define  $\mathcal{P}\hat{\eta}^\Gamma : \mathcal{P}\hat{f}^\Gamma \Rightarrow \mathcal{P}\hat{g}^\Gamma$  and  $\mathcal{P}\hat{\eta}^\Gamma : \mathcal{P}\hat{f}^\Gamma \Rightarrow \mathcal{P}\hat{g}^\Gamma$  by  $\mathcal{P}\hat{\eta}^\Gamma : (A, \Delta) \mapsto (\eta(A), \iota^\eta)$ , where  $\iota^\eta : f(\Delta) \rightarrow g(\Delta)$  maps  $\iota^\eta : f(\delta) \mapsto g(\delta) = \eta(A) \circ f(\delta) \circ \eta(A)^{-1}$  for  $\delta \in \Delta$ , and  $\mathcal{P}\hat{\eta}^\Gamma : (A, \Delta) \mapsto (\eta(A), \iota^\eta)f(\Delta)$ . Then  $\mathcal{P}\hat{\eta}^\Gamma, \mathcal{P}\hat{\eta}^\Gamma$  are 2-morphisms of prestacks, so stackifying gives 2-morphisms  $\tilde{\eta}^\Gamma : \tilde{f}^\Gamma \Rightarrow \tilde{g}^\Gamma$  and  $\hat{\eta}^\Gamma : \hat{f}^\Gamma \Rightarrow \hat{g}^\Gamma$ .

As in Definition 9.7, we would like to define 2-functors

$$\tilde{F}^\Gamma, \hat{F}^\Gamma : \mathbf{DMC}^\infty \mathbf{Sta}^{\text{re}} \longrightarrow \mathbf{DMC}^\infty \mathbf{Sta}^{\text{re}} \quad (9.4)$$

by  $\tilde{F}^\Gamma(\mathcal{X}) = \tilde{\mathcal{X}}^\Gamma$  on objects,  $\tilde{F}^\Gamma(f) = \tilde{f}^\Gamma$  on 1-morphisms,  $\tilde{F}^\Gamma(\eta) = \tilde{\eta}^\Gamma$  on 2-morphisms, and so on. But there is a difference. Stackifications of 1-morphisms of prestacks involve arbitrary choices, and are unique only up to 2-isomorphism. Therefore strict equalities of 1-morphisms of prestacks translate, on stackification, to 2-isomorphisms of their stackifications, rather than strict equalities.

For representable 1-morphisms  $f : \mathcal{X} \rightarrow \mathcal{Y}$ ,  $g : \mathcal{Y} \rightarrow \mathcal{Z}$ , in prestack 1-morphisms we have  $\mathcal{P}\tilde{g}^\Gamma \circ \mathcal{P}\tilde{f}^\Gamma = \mathcal{P}(\widetilde{g \circ f})^\Gamma$ . Thus, stackification gives a 2-isomorphism  $\tilde{F}_{g,f}^\Gamma : \tilde{g}^\Gamma \circ \tilde{f}^\Gamma \Rightarrow (\widetilde{g \circ f})^\Gamma$ , which need not be the identity. Similarly,  $\mathcal{P}(\text{id}_{\mathcal{X}})^\Gamma = \text{id}_{\mathcal{P}\tilde{\mathcal{X}}^\Gamma} : \mathcal{P}\tilde{\mathcal{X}}^\Gamma \rightarrow \mathcal{P}\tilde{\mathcal{X}}^\Gamma$ , but on stackification we get a 2-isomorphism  $\tilde{F}_{\mathcal{X}}^\Gamma : (\text{id}_{\mathcal{X}})^\Gamma \Rightarrow \text{id}_{\tilde{\mathcal{X}}^\Gamma}$ , which need not be the identity. Because of this,  $\tilde{F}^\Gamma, \hat{F}^\Gamma$  in (9.4) are weak 2-functors rather than strict 2-functors, in the sense of §A.1.

The 1-morphisms in (9.2) are compatible with  $\tilde{f}^\Gamma, \hat{f}^\Gamma$  by 2-isomorphisms

$$\tilde{O}^\Gamma(\mathcal{Y}) \circ \tilde{f}^\Gamma \cong f \circ \tilde{O}^\Gamma(\mathcal{X}), \quad \hat{\Pi}^\Gamma(\mathcal{Y}) \circ \hat{f}^\Gamma \cong \hat{f}^\Gamma \circ \hat{\Pi}^\Gamma(\mathcal{X}), \quad \hat{\Pi}^\Gamma(\mathcal{Y}) \circ \hat{f}^\Gamma \cong \hat{f}^\Gamma \circ \hat{\Pi}^\Gamma(\mathcal{X}),$$

which follow by stackifying equalities of 1-morphisms of prestacks.

**REMARK 9.9.** For  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $\Gamma$  as above, the restriction  $f|_{\mathcal{X}_\circ^\Gamma}$  need not map  $\mathcal{X}_\circ^\Gamma \rightarrow \mathcal{Y}_\circ^\Gamma$ , but only  $\mathcal{X}_\circ^\Gamma \rightarrow \mathcal{Y}^\Gamma$ , unless  $f$  induces isomorphisms on isotropy groups. Thus we do not define a 1-morphism  $f_\circ^\Gamma : \mathcal{X}_\circ^\Gamma \rightarrow \mathcal{Y}_\circ^\Gamma$ , or a 2-functor  $F_\circ^\Gamma : \mathbf{DMC}^\infty \mathbf{Sta}^{\text{re}} \rightarrow \mathbf{DMC}^\infty \mathbf{Sta}^{\text{re}}$ . The same applies for the actions of  $f$  on orbifold strata  $\tilde{\mathcal{X}}_\circ^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$ .

### 9.3. Orbifold strata of quotient $C^\infty$ -stacks $[X/G]$

The next theorem describes  $\mathcal{X}^\Gamma, \dots, \hat{\mathcal{X}}_\circ^\Gamma$  explicitly when  $\mathcal{X}$  is a quotient  $C^\infty$ -stack  $[X/G]$ , as in §7.1. We can prove it by showing the explicit constructions of Definition 7.1 and Definitions 9.1–9.2 commute up to equivalence.

**THEOREM 9.10.** *Let  $X$  be a Hausdorff  $C^\infty$ -scheme and  $G$  a finite group acting on  $X$  by isomorphisms, and write  $\mathcal{X} = [X/G]$  for the quotient  $C^\infty$ -stack, which is a Deligne–Mumford  $C^\infty$ -stack. Let  $\Gamma$  be a finite group. Then there are equivalences of  $C^\infty$ -stacks*

$$\mathcal{X}^\Gamma \simeq [(\coprod_{\text{injective group morphisms } \rho : \Gamma \rightarrow G} X^{\rho(\Gamma)})/G], \quad (9.5)$$

$$\mathcal{X}_\circ^\Gamma \simeq [(\coprod_{\text{injective group morphisms } \rho : \Gamma \rightarrow G} X_\circ^{\rho(\Gamma)})/G], \quad (9.6)$$

$$\tilde{\mathcal{X}}^\Gamma \simeq [(\coprod_{\text{subgroups } \Delta \subseteq G : \Delta \cong \Gamma} X^\Delta)/G], \quad (9.7)$$

$$\tilde{\mathcal{X}}_\circ^\Gamma \simeq [(\coprod_{\text{subgroups } \Delta \subseteq G : \Delta \cong \Gamma} X_\circ^\Delta)/G], \quad (9.8)$$

where for each subgroup  $\Delta \subseteq G$ , we write  $X^\Delta$  for the closed  $C^\infty$ -subscheme in  $X$  fixed by  $\Delta$  in  $G$ , and  $X_\circ^\Delta$  for the open  $C^\infty$ -subscheme in  $X^\Delta$  of points in  $X$  whose stabilizer group in  $G$  is exactly  $\Delta$ .

Here the action of  $G$  on  $\coprod_\rho X^{\rho(\Gamma)}$  in (9.5) is defined as follows. Let  $g \in G$  and  $\rho : \Gamma \rightarrow G$  be an injective morphism. Define another injective morphism  $\rho^g : \Gamma \rightarrow G$  by  $\rho^g : \gamma \mapsto g\rho(\gamma)g^{-1}$ . Then  $g(X^{\rho(\Gamma)}) = X^{\rho^g(\Gamma)}$ , as  $C^\infty$ -subschemes of  $X$ , and the action of  $g$  on  $\coprod_\rho X^{\rho(\Gamma)}$  maps  $X^{\rho(\Gamma)} \rightarrow X^{\rho^g(\Gamma)}$  by the restriction of  $g : X \rightarrow X$  to  $X^{\rho(\Gamma)}$ . The  $G$ -actions for (9.6)–(9.8) are similar.

We can also rewrite equations (9.5)–(9.8) as

$$\mathcal{X}^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective} \\ \text{group morphisms } \rho : \Gamma \rightarrow G}} [\underline{X}^{\rho(\Gamma)} / \{g \in G : g\rho(\gamma) = \rho(\gamma)g \ \forall \gamma \in \Gamma\}], \quad (9.9)$$

$$\mathcal{X}_\circ^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective} \\ \text{group morphisms } \rho : \Gamma \rightarrow G}} [\underline{X}_\circ^{\rho(\Gamma)} / \{g \in G : g\rho(\gamma) = \rho(\gamma)g \ \forall \gamma \in \Gamma\}], \quad (9.10)$$

$$\tilde{\mathcal{X}}^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta \subseteq G \text{ with } \Delta \cong \Gamma}} [\underline{X}^\Delta / \{g \in G : \Delta = g\Delta g^{-1}\}], \quad (9.11)$$

$$\tilde{\mathcal{X}}_\circ^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta \subseteq G \text{ with } \Delta \cong \Gamma}} [\underline{X}_\circ^\Delta / \{g \in G : \Delta = g\Delta g^{-1}\}]. \quad (9.12)$$

Here morphisms  $\rho, \rho' : \Gamma \rightarrow G$  are conjugate if  $\rho' = \rho^g$  for some  $g \in G$ , and subgroups  $\Delta, \Delta' \subseteq G$  are conjugate if  $\Delta = g\Delta'g^{-1}$  for some  $g \in G$ . In (9.9)–(9.12) we sum over one representative  $\rho$  or  $\Delta$  for each conjugacy class.

In the notation of (9.11)–(9.12), there are equivalences of  $C^\infty$ -stacks

$$\hat{\mathcal{X}}^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta \subseteq G \text{ with } \Delta \cong \Gamma}} [\underline{X}^\Delta / (\{g \in G : \Delta = g\Delta g^{-1}\} / \Delta)], \quad (9.13)$$

$$\hat{\mathcal{X}}_\circ^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta \subseteq G \text{ with } \Delta \cong \Gamma}} [\underline{X}_\circ^\Delta / (\{g \in G : \Delta = g\Delta g^{-1}\} / \Delta)]. \quad (9.14)$$

Under the equivalences (9.5)–(9.14), the 1-morphisms in (9.2) are identified up to 2-isomorphism with 1-morphisms between quotient  $C^\infty$ -stacks induced by natural  $C^\infty$ -scheme morphisms between  $\coprod_\rho \underline{X}^{\rho(\Gamma)}, \underline{X}, \dots$ . For example, the disjoint union over  $\rho$  of the inclusion  $\underline{X}^{\rho(\Gamma)} \hookrightarrow \underline{X}$  is a  $G$ -equivariant morphism  $\coprod_\rho \underline{X}^{\rho(\Gamma)} \rightarrow \underline{X}$ , inducing a 1-morphism  $[\coprod_\rho \underline{X}^{\rho(\Gamma)} / G] \rightarrow [\underline{X} / G]$ . This is identified with  $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \rightarrow \mathcal{X}$  by (9.5). Similarly,  $\tilde{\Pi}^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \rightarrow \tilde{\mathcal{X}}^\Gamma$  is identified by (9.5), (9.7) with the 1-morphism  $[\coprod_\rho \underline{X}^{\rho(\Gamma)} / G] \rightarrow [\coprod_\Delta \underline{X}^\Delta / G]$  induced by the  $C^\infty$ -scheme morphism  $\coprod_\rho \underline{X}^{\rho(\Gamma)} \rightarrow \coprod_\Delta \underline{X}^\Delta$  mapping morphisms  $\rho$  to subgroups  $\Delta = \rho(\Gamma)$ , and acting by  $\text{id}_{\underline{X}^\Delta} : \underline{X}^{\rho(\Gamma)} \rightarrow \underline{X}^\Delta$  for  $\Delta = \rho(\Gamma)$ .

#### 9.4. Sheaves on orbifold strata

Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack,  $\Gamma$  a finite group, and  $\mathcal{E} \in \text{qcoh}(\mathcal{X})$ , so that  $\mathcal{E}^\Gamma := O^\Gamma(\mathcal{X})^*(\mathcal{E}) \in \text{qcoh}(\mathcal{X}^\Gamma)$ . We will show that there is a natural representation of  $\Gamma$  on  $\mathcal{E}^\Gamma$ , and also the action of  $\text{Aut}(\Gamma)$  on  $\mathcal{X}^\Gamma$  lifts to  $\mathcal{E}^\Gamma$ , so that  $\text{Aut}(\Gamma) \ltimes \Gamma$  acts equivariantly on  $\mathcal{E}^\Gamma$ .

**DEFINITION 9.11.** Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack, and  $\Gamma$  a finite group, so that §9.1 defines the orbifold stratum  $\mathcal{X}^\Gamma$ , a 1-morphism  $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \rightarrow \mathcal{X}$ , an action of  $\text{Aut}(\Gamma)$  on  $O^\Gamma(\mathcal{X})$  by 2-isomorphisms  $E^\Gamma(\gamma, \mathcal{X}) : O^\Gamma(\mathcal{X}) \Rightarrow O^\Gamma(\mathcal{X})$ , and an action of  $\text{Aut}(\Gamma)$  on  $\mathcal{X}^\Gamma$  by 1-isomorphisms  $L^\Gamma(\Lambda, \mathcal{X}) : \mathcal{X}^\Gamma \rightarrow \mathcal{X}^\Gamma$ .

Suppose  $\mathcal{E}$  is a quasicoherent sheaf on  $\mathcal{X}$ , and write  $\mathcal{E}^\Gamma$  for the pullback sheaf  $O^\Gamma(\mathcal{X})^*(\mathcal{E})$  in  $\text{qcoh}(\mathcal{X}^\Gamma)$ . Using the notation of Definition 8.7, for each  $\gamma \in \Gamma$  and  $\Lambda \in \text{Aut}(\Gamma)$  define morphisms  $R^\Gamma(\gamma, \mathcal{E}) : \mathcal{E}^\Gamma \rightarrow \mathcal{E}^\Gamma$  and  $S^\Gamma(\Lambda, \mathcal{E}) : L^\Gamma(\Lambda, \mathcal{X})^*(\mathcal{E}^\Gamma) \rightarrow$

$\mathcal{E}^\Gamma$  in  $\mathrm{qcoh}(\mathcal{X}^\Gamma)$  by

$$\begin{aligned} R^\Gamma(\gamma, \mathcal{E}) &= E^\Gamma(\gamma, \mathcal{X})^*(\mathcal{E}) : O^\Gamma(\mathcal{X})^*(\mathcal{E}) \longrightarrow O^\Gamma(\mathcal{X})^*(\mathcal{E}) \quad \text{and} \\ S^\Gamma(\Lambda, \mathcal{E}) &= I_{L^\Gamma(\Lambda, \mathcal{X}), O^\Gamma(\mathcal{X})}(\mathcal{E})^{-1} : L^\Gamma(\Lambda, \mathcal{X})^* \circ O^\Gamma(\mathcal{X})^*(\mathcal{E}) \longrightarrow O^\Gamma(\mathcal{X})^*(\mathcal{E}), \end{aligned}$$

where the definition of  $S^\Gamma(\Lambda, \mathcal{E})$  uses  $O^\Gamma(\mathcal{X}) \circ L^\Gamma(\Lambda, \mathcal{X}) = O^\Gamma(\mathcal{X})$ .

Since  $E^\Gamma(1, \mathcal{X}) = \mathrm{id}_{O^\Gamma(\mathcal{X})}$  and  $E^\Gamma(\gamma, \mathcal{X}) \odot E^\Gamma(\delta, \mathcal{X}) = E^\Gamma(\gamma\delta, \mathcal{X})$  for  $\gamma, \delta \in \Gamma$  as in Definition 9.4, we have

$$R^\Gamma(1, \mathcal{E}) = \mathrm{id}_{\mathcal{E}^\Gamma} \quad \text{and} \quad R^\Gamma(\gamma, \mathcal{E}) \circ R^\Gamma(\delta, \mathcal{E}) = R^\Gamma(\gamma\delta, \mathcal{E}) \quad \text{for all } \gamma, \delta \in \Gamma.$$

Hence  $R^\Gamma(-, \mathcal{E})$  is an action of  $\Gamma$  on  $\mathcal{E}^\Gamma$  by isomorphisms.

As  $L^\Gamma(\mathrm{id}_\Gamma, \mathcal{X}) = \mathrm{id}_{\mathcal{X}^\Gamma}$  and  $L^\Gamma(\Lambda, \mathcal{X}) \circ L^\Gamma(\Lambda', \mathcal{X}) = L^\Gamma(\Lambda\Lambda', \mathcal{X})$  for  $\Lambda, \Lambda' \in \mathrm{Aut}(\Gamma)$ , by properties of morphisms  $I_{*,*}(\cdot)$  we find that

$$\begin{aligned} S^\Gamma(\mathrm{id}_\Gamma, \mathcal{E}) &= \delta_{\mathcal{X}^\Gamma}(\mathcal{E}^\Gamma) : \mathrm{id}_{\mathcal{X}^\Gamma}^*(\mathcal{E}^\Gamma) \longrightarrow \mathcal{E}^\Gamma, \quad \text{and} \\ S^\Gamma(\Lambda\Lambda', \mathcal{E}) &= S^\Gamma(\Lambda', \mathcal{E}) \circ L^\Gamma(\Lambda', \mathcal{X})^*(S^\Gamma(\Lambda, \mathcal{E})) \circ I_{L^\Gamma(\Lambda', \mathcal{X}), L^\Gamma(\Lambda, \mathcal{X})}(\mathcal{E}^\Gamma). \end{aligned}$$

This means that the  $S^\Gamma(\Lambda, \mathcal{E})$  define a lift of the action of  $\mathrm{Aut}(\Gamma)$  on  $\mathcal{X}^\Gamma$  to  $\mathcal{E}^\Gamma$ , that is,  $\mathcal{E}^\Gamma$  is an  $\mathrm{Aut}(\Gamma)$ -equivariant sheaf on  $\mathcal{X}^\Gamma$ .

If  $\gamma \in \Gamma$  and  $\Lambda \in \mathrm{Aut}(\Gamma)$  then noting that  $O^\Gamma(\mathcal{X}) \circ L^\Gamma(\Lambda, \mathcal{X}) = O^\Gamma(\mathcal{X})$ , one can show from Definitions 9.3 and 9.4 that

$$E^\Gamma(\Lambda(\gamma), \mathcal{X}) * \mathrm{id}_{L^\Gamma(\Lambda, \mathcal{X})} = E^\Gamma(\gamma, \mathcal{X}) : O^\Gamma(\mathcal{X}) \Longrightarrow O^\Gamma(\mathcal{X}).$$

Pulling back  $\mathcal{E}$  by this equation and using properties of the  $I_{*,*}(\cdot)$  we find that

$$R^\Gamma(\gamma, \mathcal{E}) \circ S^\Gamma(\Lambda, \mathcal{E}) = S^\Gamma(\Lambda, \mathcal{E}) \circ L^\Gamma(\Lambda, \mathcal{X})^*(R^\Gamma(\Lambda(\gamma), \mathcal{E})). \quad (9.15)$$

This is a compatibility between the actions of  $\Gamma$  and  $\mathrm{Aut}(\Gamma)$  on  $\mathcal{E}^\Gamma$ . It says that the action of  $\mathrm{Aut}(\Gamma)$  on  $\mathcal{X}^\Gamma$  lifts to an action of  $\mathrm{Aut}(\Gamma) \ltimes \Gamma$  on  $\mathcal{E}^\Gamma$ .

Let  $\alpha : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  be a morphism in  $\mathrm{qcoh}(\mathcal{X})$ . Then  $\alpha^\Gamma := O^\Gamma(\mathcal{X})^*(\alpha) : \mathcal{E}_1^\Gamma \rightarrow \mathcal{E}_2^\Gamma$  is a morphism in  $\mathrm{qcoh}(\mathcal{X}^\Gamma)$ . Since  $E^\Gamma(\gamma, \mathcal{X})^* : O^\Gamma(\mathcal{X})^* \Rightarrow O^\Gamma(\mathcal{X})^*$  is a natural isomorphism of functors, we see that

$$\alpha^\Gamma \circ R^\Gamma(\gamma, \mathcal{E}_1) = R^\Gamma(\gamma, \mathcal{E}_2) \circ \alpha^\Gamma \quad \text{for } \gamma \in \Gamma.$$

Similarly we find that

$$\alpha^\Gamma \circ S^\Gamma(\Lambda, \mathcal{E}_1) = S^\Gamma(\Lambda, \mathcal{E}_2) \circ L^\Gamma(\Lambda, \mathcal{X})^*(\alpha^\Gamma) \quad \text{for } \Lambda \in \mathrm{Aut}(\Gamma).$$

These imply that  $R(\gamma, -)$  and  $S(\Lambda, -)$  are natural isomorphisms of functors.

Now let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a representable 1-morphism of  $C^\infty$ -stacks, so that as in §9.2 we have  $f^\Gamma : \mathcal{X}^\Gamma \rightarrow \mathcal{Y}^\Gamma$ . Let  $\mathcal{F} \in \mathrm{qcoh}(\mathcal{Y})$ . Then we may form  $f^*(\mathcal{F}) \in \mathrm{qcoh}(\mathcal{X})$  and hence  $f^*(\mathcal{F})^\Gamma = O^\Gamma(\mathcal{X})^*(f^*(\mathcal{F})) \in \mathrm{qcoh}(\mathcal{X}^\Gamma)$ , or we may form  $\mathcal{F}^\Gamma = O^\Gamma(\mathcal{Y})^*(\mathcal{F}) \in \mathrm{qcoh}(\mathcal{Y}^\Gamma)$  and hence  $(f^\Gamma)^*(\mathcal{F}^\Gamma) \in \mathrm{qcoh}(\mathcal{X}^\Gamma)$ . Since  $O^\Gamma(\mathcal{Y}) \circ f^\Gamma = f \circ O^\Gamma(\mathcal{X})$ , these are related by the canonical isomorphism

$$T^\Gamma(f, \mathcal{F}) := I_{f^\Gamma, O^\Gamma(\mathcal{Y})}(\mathcal{F}) \circ I_{O^\Gamma(\mathcal{X}), f}(\mathcal{F})^{-1} : f^*(\mathcal{F})^\Gamma \longrightarrow (f^\Gamma)^*(\mathcal{F}^\Gamma). \quad (9.16)$$

Using properties of  $I_{*,*}(\cdot)$ , it is easy to show that

$$(f^\Gamma)^*(R^\Gamma(\gamma, \mathcal{F})) \circ T^\Gamma(f, \mathcal{F}) = T^\Gamma(f, \mathcal{F}) \circ R^\Gamma(\gamma, f^*(\mathcal{F})) \quad \text{for } \gamma \in \Gamma, \quad (9.17)$$

and noting that  $f^\Gamma \circ L^\Gamma(\Lambda, \mathcal{X}) = L^\Gamma(\Lambda, \mathcal{Y}) \circ f^\Gamma$ , we also find that

$$\begin{aligned} T^\Gamma(f, \mathcal{F}) \circ S^\Gamma(\Lambda, f^*(\mathcal{F})) &= (f^\Gamma)^*(S^\Gamma(\Lambda, \mathcal{F})) \circ I_{f^\Gamma, L^\Gamma(\Lambda, \mathcal{Y})}(\mathcal{F}^\Gamma) \circ \\ &\quad I_{L^\Gamma(\Lambda, \mathcal{X}), f^\Gamma}(\mathcal{F}^\Gamma)^{-1} \circ L^\Gamma(\Lambda, \mathcal{X})^*(T^\Gamma(f, \mathcal{F})). \end{aligned}$$

This shows that the isomorphisms  $T^\Gamma(f, \mathcal{F})$  identify the  $(\text{Aut}(\Gamma) \ltimes \Gamma)$ -actions on  $f^*(\mathcal{F})^\Gamma$  and  $(f^\Gamma)^*(\mathcal{F}^\Gamma)$ .

Now let  $\mathcal{X}, \Gamma, \mathcal{X}^\Gamma, \mathcal{E}$  and  $\mathcal{E}^\Gamma$  be as above, and write  $R_0, \dots, R_k$  for the irreducible representations of  $\Gamma$  over  $\mathbb{R}$  (that is, we choose one representative  $R_i$  in each isomorphism class of irreducible representations), with  $R_0 = \mathbb{R}$  the trivial representation. Then since  $R^\Gamma(-, \mathcal{E})$  is an action of  $\Gamma$  on  $\mathcal{E}^\Gamma$  by isomorphisms, by elementary representation theory we have a canonical decomposition

$$\mathcal{E}^\Gamma \cong \bigoplus_{i=0}^k \mathcal{E}_i^\Gamma \otimes R_i \quad \text{for } \mathcal{E}_0^\Gamma, \dots, \mathcal{E}_k^\Gamma \in \text{qcoh}(\mathcal{X}^\Gamma). \quad (9.18)$$

We will be interested in splitting  $\mathcal{E}^\Gamma$  into *trivial* and *nontrivial* representations of  $\Gamma$ , denoted by subscripts ‘tr’ and ‘nt’. So we write

$$\mathcal{E}^\Gamma = \mathcal{E}_{\text{tr}}^\Gamma \oplus \mathcal{E}_{\text{nt}}^\Gamma, \quad (9.19)$$

where  $\mathcal{E}_{\text{tr}}^\Gamma, \mathcal{E}_{\text{nt}}^\Gamma$  are the subsheaves of  $\mathcal{E}^\Gamma$  corresponding to the factors  $\mathcal{E}_0^\Gamma \otimes R_0$  and  $\bigoplus_{i=1}^k \mathcal{E}_i^\Gamma \otimes R_i$  respectively. Equivalently, consider  $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} R^\Gamma(\gamma, \mathcal{E}) : \mathcal{E}^\Gamma \rightarrow \mathcal{E}^\Gamma$ . It is a projection (its square is itself), with image  $\mathcal{E}_{\text{tr}}^\Gamma$  and kernel  $\mathcal{E}_{\text{nt}}^\Gamma$ .

If  $\Gamma$  acts on  $R_i$  by  $\rho_i : \Gamma \rightarrow \text{Aut}(R_i)$ , and  $\Lambda \in \text{Aut}(\Gamma)$ , then  $\rho_i \circ \Lambda^{-1} : \Gamma \rightarrow \text{Aut}(R_i)$  is also an irreducible representation of  $\Gamma$ , and so is isomorphic to  $R_{\Lambda(i)}$  for some unique  $\Lambda(i) = 0, \dots, k$ . This defines an action of  $\text{Aut}(\Gamma)$  on  $\{0, \dots, k\}$  by permutations. One can show using (9.15) that  $S^\Gamma(\Lambda, \mathcal{E})$  acts on the splitting (9.18) by mapping  $L^\Gamma(\Lambda, \mathcal{X})^*(\mathcal{E}_i^\Gamma \otimes R_i) \rightarrow \mathcal{E}_{\Lambda^{-1}(i)}^\Gamma \otimes R_{\Lambda^{-1}(i)}$ . Since  $\Lambda(0) = 0$ , it follows that  $S^\Gamma(\Lambda, \mathcal{E})$  maps  $L^\Gamma(\Lambda, \mathcal{X})^*(\mathcal{E}_{\text{tr}}^\Gamma) \rightarrow \mathcal{E}_{\text{tr}}^\Gamma$  and  $L^\Gamma(\Lambda, \mathcal{X})^*(\mathcal{E}_{\text{nt}}^\Gamma) \rightarrow \mathcal{E}_{\text{nt}}^\Gamma$ , that is,  $S^\Gamma(\Lambda, \mathcal{E})$  preserves the splitting (9.19).

Equation (9.17) implies that  $T^\Gamma(f, \mathcal{F})$  maps  $f^*(\mathcal{F})_i^\Gamma \otimes R_i \rightarrow (f^\Gamma)^*(\mathcal{F}_i^\Gamma \otimes R_i)$  in (9.18) for  $f^*(\mathcal{F})^\Gamma, \mathcal{F}^\Gamma$ , and so maps  $f^*(\mathcal{F})_{\text{tr}}^\Gamma \rightarrow (f^\Gamma)^*(\mathcal{F}_{\text{tr}}^\Gamma)$  and  $f^*(\mathcal{F})_{\text{nt}}^\Gamma \rightarrow (f^\Gamma)^*(\mathcal{F}_{\text{nt}}^\Gamma)$  in (9.19).

The next two definitions explain to what extent this generalizes to  $\tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$ .

**DEFINITION 9.12.** Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack, and  $\Gamma$  a finite group, so that §9.1 defines the orbifold strata  $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma$  with  $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$ , and 1-morphisms  $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \rightarrow \mathcal{X}$ ,  $\tilde{O}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \rightarrow \mathcal{X}$  and  $\tilde{\Pi}^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \rightarrow \tilde{\mathcal{X}}^\Gamma$  with  $\tilde{O}^\Gamma(\mathcal{X}) \circ \tilde{\Pi}^\Gamma(\mathcal{X}) = O^\Gamma(\mathcal{X})$ .

Let us ask: how much of the structure on  $\mathcal{E}^\Gamma$  in Definition 9.11 descends to  $\tilde{\mathcal{E}}^\Gamma$ ? It turns out that  $\tilde{\mathcal{E}}^\Gamma$  does not have natural representations of  $\Gamma$  or  $\text{Aut}(\Gamma)$ , since we do not have actions of  $\Gamma$  on  $\tilde{O}^\Gamma(\mathcal{X})$  by 2-isomorphisms or of  $\text{Aut}(\Gamma)$  on  $\tilde{\mathcal{X}}^\Gamma$  by 1-isomorphisms. In effect, taking the quotient by  $\text{Aut}(\Gamma)$  in  $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$  destroys both these actions.

However, at least part of the natural decompositions (9.18)–(9.19) descends to  $\tilde{\mathcal{E}}^\Gamma$ . As in Definition 9.11, write  $R_0, \dots, R_k$  for the irreducible representations of  $\Gamma$ , so that  $\text{Aut}(\Gamma)$  acts on the indexing set  $\{0, \dots, k\}$ . Form the quotient set  $\{0, \dots, k\} / \text{Aut}(\Gamma)$ , so that points of  $\{0, \dots, k\} / \text{Aut}(\Gamma)$  are orbits  $O$  of  $\text{Aut}(\Gamma)$  in  $\{0, \dots, k\}$ . Then we may rewrite (9.18) as

$$\mathcal{E}^\Gamma \cong \bigoplus_{O \in \{0, \dots, k\} / \text{Aut}(\Gamma)} \left[ \bigoplus_{i \in O} \mathcal{E}_i^\Gamma \otimes R_i \right].$$

Since  $S^\Gamma(\Lambda, \mathcal{E})$  maps  $L^\Gamma(\Lambda, \mathcal{X})^*(\mathcal{E}_i^\Gamma \otimes R_i) \rightarrow \mathcal{E}_{\Lambda^{-1}(i)}^\Gamma \otimes R_{\Lambda^{-1}(i)}$ , we see that

$$S^\Gamma(\Lambda, \mathcal{E}) : L^\Gamma(\Lambda, \mathcal{X})^* \left( \bigoplus_{i \in O} \mathcal{E}_i^\Gamma \otimes R_i \right) \longrightarrow \bigoplus_{i \in O} \mathcal{E}_i^\Gamma \otimes R_i$$

for each  $O \in \{0, \dots, k\} / \text{Aut}(\Gamma)$ . Now the  $S^\Gamma(\Lambda, \mathcal{E})$  lift the action of  $\text{Aut}(\Gamma)$  on  $\mathcal{X}^\Gamma$  to  $\mathcal{E}^\Gamma$ , and  $\tilde{\mathcal{E}}^\Gamma$  is essentially the quotient of  $\mathcal{E}^\Gamma$  by this lifted action of  $\text{Aut}(\Gamma)$  under the equivalence  $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$ . Therefore any decomposition of  $\mathcal{E}^\Gamma$  which is invariant under  $S^\Gamma(\Lambda, \mathcal{E})$  for all  $\Lambda \in \text{Aut}(\Gamma)$  corresponds to a decomposition of  $\tilde{\mathcal{E}}^\Gamma$ . Hence there is a canonical splitting

$$\begin{aligned} \tilde{\mathcal{E}}^\Gamma &= \bigoplus_{O \in \{0, \dots, k\} / \text{Aut}(\Gamma)} \tilde{\mathcal{E}}_O^\Gamma, \quad \text{where} \\ I_{\tilde{\Pi}^\Gamma(\mathcal{X}), \tilde{O}^\Gamma(\mathcal{X})}(\mathcal{E})^{-1} [\tilde{\Pi}^\Gamma(\mathcal{X})^* (\tilde{\mathcal{E}}_O^\Gamma)] &\cong \bigoplus_{i \in O} \mathcal{E}_i^\Gamma \otimes R_i \quad \text{under (9.18)}. \end{aligned} \quad (9.20)$$

As for (9.19) we define the *trivial* and *nontrivial* parts of  $\tilde{\mathcal{E}}^\Gamma$  by  $\tilde{\mathcal{E}}_{\text{tr}}^\Gamma = \tilde{\mathcal{E}}_{\{0\}}^\Gamma$  and  $\tilde{\mathcal{E}}_{\text{nt}}^\Gamma = \bigoplus_{O \in \{1, \dots, k\} / \text{Aut}(\Gamma)} \tilde{\mathcal{E}}_O^\Gamma$ . Then

$$\begin{aligned} \tilde{\mathcal{E}}^\Gamma &= \tilde{\mathcal{E}}_{\text{tr}}^\Gamma \oplus \tilde{\mathcal{E}}_{\text{nt}}^\Gamma, \quad \text{where } I_{\tilde{\Pi}^\Gamma(\mathcal{X}), \tilde{O}^\Gamma(\mathcal{X})}(\mathcal{E})^{-1} [\tilde{\Pi}^\Gamma(\mathcal{X})^* (\tilde{\mathcal{E}}_{\text{tr}}^\Gamma)] = \mathcal{E}_{\text{tr}}^\Gamma \\ &\quad \text{and } I_{\tilde{\Pi}^\Gamma(\mathcal{X}), \tilde{O}^\Gamma(\mathcal{X})}(\mathcal{E})^{-1} [\tilde{\Pi}^\Gamma(\mathcal{X})^* (\tilde{\mathcal{E}}_{\text{nt}}^\Gamma)] = \mathcal{E}_{\text{nt}}^\Gamma. \end{aligned} \quad (9.21)$$

Each point  $[x, \Delta]$  of  $\tilde{\mathcal{X}}_{\text{top}}^\Gamma$  has isotropy group  $\text{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x, \Delta])$  with a distinguished subgroup  $\Delta$  with a noncanonical isomorphism  $\Delta \cong \Gamma$ . The fibre of  $\tilde{\mathcal{E}}^\Gamma$  at  $[x, \Delta]$  is a representation of  $\text{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x, \Delta])$ , and hence a representation of  $\Delta$ . Equation (9.21) corresponds to splitting the fibre of  $\tilde{\mathcal{E}}^\Gamma$  at  $[x, \Delta]$  into trivial and nontrivial representations of  $\Delta$ . Equation (9.20) corresponds to decomposing the fibre of  $\tilde{\mathcal{E}}^\Gamma$  at  $[x, \Delta]$  into families of irreducible representations of  $\Delta \cong \Gamma$  that are independent of the choice of isomorphism  $\Delta \cong \Gamma$ .

Now let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a representable 1-morphism of  $C^\infty$ -stacks, so that as in §9.2 we have a representable 1-morphism  $\tilde{f}^\Gamma : \tilde{\mathcal{X}}^\Gamma \rightarrow \tilde{\mathcal{Y}}^\Gamma$  with  $f \circ \tilde{O}^\Gamma(\mathcal{X}) = \tilde{O}^\Gamma(\mathcal{Y}) \circ \tilde{f}^\Gamma$ . Let  $\mathcal{F} \in \text{qcoh}(\mathcal{Y})$ , so that  $\tilde{\mathcal{F}}^\Gamma \in \text{qcoh}(\tilde{\mathcal{Y}}^\Gamma)$ ,  $f^*(\mathcal{F}) \in \text{qcoh}(\mathcal{X})$ , and  $\tilde{f}^*(\mathcal{F})^\Gamma \in \text{qcoh}(\tilde{\mathcal{X}}^\Gamma)$ . As for (9.16), we have a canonical isomorphism

$$\tilde{T}^\Gamma(f, \mathcal{F}) := I_{\tilde{f}^\Gamma, \tilde{O}^\Gamma(\mathcal{Y})}(\mathcal{F}) \circ I_{\tilde{O}^\Gamma(\mathcal{X}), f}(\mathcal{F})^{-1} : \tilde{f}^*(\mathcal{F})^\Gamma \longrightarrow (\tilde{f}^\Gamma)^*(\tilde{\mathcal{F}}^\Gamma).$$

As for  $T^\Gamma(f, \mathcal{F})$  in Definition 9.11,  $\tilde{T}^\Gamma(f, \mathcal{F})$  maps  $\tilde{f}^*(\mathcal{F})_O^\Gamma \rightarrow (\tilde{f}^\Gamma)^*(\mathcal{F}_O^\Gamma)$  in (9.20) for  $\tilde{f}^*(\mathcal{F})^\Gamma, \tilde{\mathcal{F}}^\Gamma$ , and so maps  $\tilde{f}^*(\mathcal{F})_{\text{tr}}^\Gamma \rightarrow (\tilde{f}^\Gamma)^*(\mathcal{F}_{\text{tr}}^\Gamma)$  and  $\tilde{f}^*(\mathcal{F})_{\text{nt}}^\Gamma \rightarrow (\tilde{f}^\Gamma)^*(\mathcal{F}_{\text{nt}}^\Gamma)$  in (9.21).

**DEFINITION 9.13.** Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack, and  $\Gamma$  a finite group, so that §9.1 defines the orbifold strata  $\tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$  and 1-morphisms  $\tilde{O}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \rightarrow \mathcal{X}$  and  $\hat{\Pi}^\Gamma : \tilde{\mathcal{X}}^\Gamma \rightarrow \hat{\mathcal{X}}^\Gamma$ , where  $\hat{\Pi}^\Gamma$  is non-representable, with fibre  $[\bar{x}/\Gamma]$ .

Suppose  $\mathcal{E}$  is a quasicoherent sheaf on  $\mathcal{X}$ . Since we have no 1-morphism  $\hat{\mathcal{X}}^\Gamma \rightarrow \mathcal{X}$ , we cannot pull  $\mathcal{E}$  back to  $\hat{\mathcal{X}}^\Gamma$  to define  $\hat{\mathcal{E}}^\Gamma$  in  $\text{qcoh}(\hat{\mathcal{X}}^\Gamma)$ . But we do have  $\tilde{\mathcal{E}}^\Gamma = \tilde{O}^\Gamma(\mathcal{X})^*(\mathcal{E})$  in  $\text{qcoh}(\tilde{\mathcal{X}}^\Gamma)$ , with splitting  $\tilde{\mathcal{E}}^\Gamma = \tilde{\mathcal{E}}_{\text{tr}}^\Gamma \oplus \tilde{\mathcal{E}}_{\text{nt}}^\Gamma$  as in (9.21), so we can form the pushforward  $\hat{\Pi}_*^\Gamma(\tilde{\mathcal{E}}^\Gamma)$  in  $\text{qcoh}(\hat{\mathcal{X}}^\Gamma)$ . Now pushforwards take global sections of a sheaf on the fibres of the 1-morphism. The fibres of  $\hat{\Pi}^\Gamma$  are  $[\bar{x}/\Gamma]$ . Quasicoherent sheaves on  $[\bar{x}/\Gamma]$  correspond to  $\Gamma$ -representations, and the global sections correspond to the trivial ( $\Gamma$ -invariant) part.

As the  $\Gamma$ -invariant part of  $\tilde{\mathcal{E}}^\Gamma$  is  $\tilde{\mathcal{E}}_{\text{tr}}^\Gamma$ , we see that  $\hat{\Pi}_*^\Gamma(\tilde{\mathcal{E}}_{\text{nt}}^\Gamma) = 0$ , that is,  $\mathcal{E}_{\text{nt}}^\Gamma$  and  $\tilde{\mathcal{E}}_{\text{nt}}^\Gamma$  do not descend to  $\hat{\mathcal{X}}^\Gamma$ . Define  $\hat{\mathcal{E}}_{\text{tr}}^\Gamma = \hat{\Pi}_*^\Gamma(\tilde{\mathcal{E}}_{\text{tr}}^\Gamma)$  in  $\text{qcoh}(\hat{\mathcal{X}}^\Gamma)$ . This is the natural analogue of  $\mathcal{E}_{\text{tr}}^\Gamma, \tilde{\mathcal{E}}_{\text{tr}}^\Gamma$  on  $\hat{\mathcal{X}}^\Gamma$ , and has a canonical isomorphism

$$(\hat{\Pi}^\Gamma)^*(\hat{\mathcal{E}}_{\text{tr}}^\Gamma) \cong \tilde{\mathcal{E}}_{\text{tr}}^\Gamma. \quad (9.22)$$

Now let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a representable 1-morphism of  $C^\infty$ -stacks, so that as in §9.2 we have a representable 1-morphism  $\tilde{f}^\Gamma : \tilde{\mathcal{X}}^\Gamma \rightarrow \tilde{\mathcal{Y}}^\Gamma$ . Then there is a canonical isomorphism

$$\hat{T}_{\text{tr}}^\Gamma(f, \mathcal{F}) : f^*(\widehat{\mathcal{F}})_{\text{tr}}^\Gamma \longrightarrow (\hat{f}^\Gamma)^*(\hat{\mathcal{F}}_{\text{tr}}^\Gamma),$$

the composition of the natural isomorphism  $\hat{\Pi}_*^\Gamma \circ (\tilde{f}^\Gamma)^*(\tilde{\mathcal{F}}_{\text{tr}}^\Gamma) \rightarrow (\hat{f}^\Gamma)^* \circ \hat{\Pi}_*^\Gamma(\tilde{\mathcal{F}}_{\text{tr}}^\Gamma)$  with  $\hat{\Pi}_*^\Gamma(\hat{T}_{\text{tr}}^\Gamma(f, \mathcal{F})|_{f^*(\widehat{\mathcal{F}})_{\text{tr}}^\Gamma})$ .

### 9.5. Sheaves on orbifold strata of quotients $[X/G]$

In the next theorem we take  $\mathcal{X} = [X/G]$ , and use the explicit description of  $\mathcal{X}^\Gamma$  in Theorem 9.10 to give an alternative formula for the action  $R^\Gamma(-, \mathcal{E})$  of  $\Gamma$  on  $\mathcal{E}^\Gamma$  in Definition 9.11. This then allows us to understand the splittings (9.18)–(9.22) in terms of sheaves on  $\underline{X}$ . The proof is a long but straightforward consequence of the definitions, and we leave it as an exercise.

**THEOREM 9.14.** *Let  $\underline{X}$  be a Hausdorff  $C^\infty$ -scheme,  $G$  a finite group,  $\underline{r} : G \rightarrow \text{Aut}(\underline{X})$  an action of  $G$  on  $\underline{X}$ , and  $\mathcal{X} = [\underline{X}/G]$  the quotient Deligne–Mumford  $C^\infty$ -stack. Then (9.5) gives an equivalence  $\mathcal{X}^\Gamma \simeq [\coprod_{\text{injective } \rho : \Gamma \rightarrow G} \underline{X}^{\rho(\Gamma)}/G]$ .*

Write  $\text{qcoh}^G(\underline{X})$  for the abelian category of  $G$ -equivariant quasicoherent sheaves on  $\underline{X}$ , with objects pairs  $(\mathcal{E}, \Phi)$  for  $\mathcal{E} \in \text{qcoh}(\underline{X})$  and  $\Phi(g) : \underline{r}(g)^*(\mathcal{E}) \rightarrow \mathcal{E}$  is an isomorphism in  $\text{qcoh}(\underline{X})$  for all  $g \in G$  satisfying  $\Phi(1) = \delta_{\underline{X}}(\mathcal{E})$  and

$$\Phi(gh) = \Phi(h) \circ \underline{r}(h)^*(\Phi(g)) \circ I_{\underline{r}(h), \underline{r}(g)}(\mathcal{E}) \quad \text{for all } g, h \in G,$$

and morphisms  $\alpha : (\mathcal{E}, \Phi) \rightarrow (\mathcal{F}, \Psi)$  in  $\text{qcoh}^G(\underline{X})$  are morphisms  $\alpha : \mathcal{E} \rightarrow \mathcal{F}$  in  $\text{qcoh}(\underline{X})$  with  $\alpha \circ \Phi(g) = \Psi(g) \circ \underline{r}(g)^*(\alpha)$  for all  $g \in G$ .

Then  $\text{qcoh}^G(\underline{X})$  is isomorphic to  $\text{qcoh}(G \times \underline{X} \rightrightarrows \underline{X})$  in Definition 8.5, so Theorem 8.6 gives an equivalence of categories  $F_\Pi : \text{qcoh}(\mathcal{X}) \rightarrow \text{qcoh}^G(\underline{X})$ . Using (9.5) we also get an equivalence  $F_\Pi^\Gamma : \text{qcoh}(\mathcal{X}^\Gamma) \rightarrow \text{qcoh}^G(\coprod_\rho \underline{X}^{\rho(\Gamma)})$ . These categories and functors fit into a 2-commutative diagram:

$$\begin{array}{ccc} \text{qcoh}(\mathcal{X}) & \xrightarrow{F_\Pi} & \text{qcoh}^G(\underline{X}) \\ \downarrow O^\Gamma(\mathcal{X})^* & \uparrow N^\Gamma(\mathcal{X}) & \downarrow i_{\underline{X}}^* \\ \text{qcoh}(\mathcal{X}^\Gamma) & \xrightarrow{F_\Pi^\Gamma} & \text{qcoh}^G(\coprod_\rho \underline{X}^{\rho(\Gamma)}), \end{array} \quad (9.23)$$

where  $i_{\underline{X}} : \coprod_\rho \underline{X}^{\rho(\Gamma)} \rightarrow \underline{X}$  is the union over  $\rho$  of the inclusion morphisms  $\underline{X}^{\rho(\Gamma)} \rightarrow \underline{X}$ , which is  $G$ -equivariant and so induces a pullback functor  $i_{\underline{X}}^*$  as shown, and  $N^\Gamma(\mathcal{X})$  is a natural isomorphism of functors.

Let  $(E, \Phi) \in \text{qcoh}^G(\underline{X})$ , so that  $i_{\underline{X}}^*(E, \Phi) \in \text{qcoh}^G(\coprod_\rho \underline{X}^{\rho(\Gamma)})$ . Define a morphism  $\bar{R}^\Gamma(\gamma, (E, \Phi)) : i_{\underline{X}}^*(E, \Phi) \rightarrow i_{\underline{X}}^*(E, \Phi)$  in  $\text{qcoh}^G(\coprod_\rho \underline{X}^{\rho(\Gamma)})$  for  $\gamma \in \Gamma$  such that

$$\begin{aligned} \bar{R}^\Gamma(\gamma, (E, \Phi))|_{\underline{X}^{\rho(\Gamma)}} &: i_{\underline{X}}|_{\underline{X}^{\rho(\Gamma)}}^*(E) \longrightarrow i_{\underline{X}}|_{\underline{X}^{\rho(\Gamma)}}^*(E) \text{ is given by} \\ \bar{R}^\Gamma(\gamma, (E, \Phi))|_{\underline{X}^{\rho(\Gamma)}} &= i_{\underline{X}}|_{\underline{X}^{\rho(\Gamma)}}^*(\Phi(\rho(\gamma^{-1}))) \circ I_{i_{\underline{X}}|_{\underline{X}^{\rho(\Gamma)}}, \underline{r}(\rho(\gamma^{-1}))}(\mathcal{E}) \end{aligned}$$

for each  $\rho$ , noting that  $\underline{r}(\rho(\gamma^{-1})) \circ i_{\underline{X}}|_{\underline{X}^{\rho(\Gamma)}} = i_{\underline{X}}|_{\underline{X}^{\rho(\Gamma)}}$ . Then  $\bar{R}^\Gamma(-, (E, \Phi))$  is an action of  $\Gamma$  on  $i_{\underline{X}}|_{\underline{X}^{\rho(\Gamma)}}^*(E)$  by isomorphisms. Furthermore, for each  $\mathcal{E}$  in  $\text{qcoh}(\mathcal{X})$

and  $\gamma$  in  $\Gamma$ , the following diagram in  $\mathrm{qcoh}^G(\coprod_\rho \underline{X}^{\rho(\Gamma)})$  commutes:

$$\begin{array}{ccc} F_\Pi^\Gamma(\mathcal{E}^\Gamma) & \xrightarrow{F_\Pi^\Gamma(R^\Gamma(\gamma, \mathcal{E}))} & F_\Pi^\Gamma(\mathcal{E}^\Gamma) \\ \downarrow N^\Gamma(\mathcal{X})(\mathcal{E}) & & N^\Gamma(\mathcal{X})(\mathcal{E}) \downarrow \\ i_X^* \circ F_\Pi(\mathcal{E}) & \xrightarrow{\bar{R}^\Gamma(\gamma, F_\Pi(\mathcal{E}))} & i_X^* \circ F_\Pi(\mathcal{E}). \end{array}$$

That is, the equivalences  $F_\Pi, F_\Pi^\Gamma$  in (9.23) identify the  $\Gamma$ -actions  $R^\Gamma(-, -)$  on  $O^\Gamma(\mathcal{X})^*$  and  $\bar{R}^\Gamma(-, -)$  on  $i_X^*$  by natural isomorphisms.

### 9.6. Cotangent sheaves of orbifold strata

Finally we apply these ideas to write the cotangent sheaves of  $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$  in terms of the pullbacks of  $T^*\mathcal{X}$ . The theorem illustrates the principle that when passing to orbifold strata, it is often natural to restrict to the trivial parts  $\mathcal{E}_{\mathrm{tr}}^\Gamma, \tilde{\mathcal{E}}_{\mathrm{tr}}^\Gamma, \hat{\mathcal{E}}_{\mathrm{tr}}^\Gamma$  of the pullbacks of  $\mathcal{E}$ . The nontrivial parts  $(T^*\mathcal{X})_{\mathrm{nt}}^\Gamma, (\tilde{T}^*\mathcal{X})_{\mathrm{nt}}^\Gamma$  should be interpreted as the *conormal sheaves* of  $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma$  in  $\mathcal{X}$ .

**THEOREM 9.15.** *Let  $\mathcal{X}$  be a Deligne–Mumford  $C^\infty$ -stack and  $\Gamma$  a finite group, so that §9.1 defines  $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \rightarrow \mathcal{X}$ . As in Definition 8.12 we have cotangent sheaves  $T^*\mathcal{X}, T^*(\mathcal{X}^\Gamma)$  and a morphism  $\Omega_{O^\Gamma(\mathcal{X})} : O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) \rightarrow T^*(\mathcal{X}^\Gamma)$  in  $\mathrm{qcoh}(\mathcal{X}^\Gamma)$ . But  $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) = (T^*\mathcal{X})^\Gamma$ , so by (9.19) we have a splitting  $(T^*\mathcal{X})^\Gamma = (T^*\mathcal{X})_{\mathrm{tr}}^\Gamma \oplus (T^*\mathcal{X})_{\mathrm{nt}}^\Gamma$ . Then  $\Omega_{O^\Gamma(\mathcal{X})|_{(T^*\mathcal{X})_{\mathrm{tr}}^\Gamma}} : (T^*\mathcal{X})_{\mathrm{tr}}^\Gamma \rightarrow T^*(\mathcal{X}^\Gamma)$  is an isomorphism, and  $\Omega_{O^\Gamma(\mathcal{X})|_{(T^*\mathcal{X})_{\mathrm{nt}}^\Gamma}} = 0$ .*

*Similarly, using the 1-morphism  $\tilde{O}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \rightarrow \mathcal{X}$  and the splitting (9.21) for  $(\tilde{T}^*\mathcal{X})^\Gamma$  we find that  $\Omega_{\tilde{O}^\Gamma(\mathcal{X})|_{(\tilde{T}^*\mathcal{X})_{\mathrm{tr}}^\Gamma}} : (\tilde{T}^*\mathcal{X})_{\mathrm{tr}}^\Gamma \rightarrow T^*(\tilde{\mathcal{X}}^\Gamma)$  is an isomorphism, and  $\Omega_{\tilde{O}^\Gamma(\mathcal{X})|_{(\tilde{T}^*\mathcal{X})_{\mathrm{nt}}^\Gamma}} = 0$ .*

*Also, there is a natural isomorphism  $(\widehat{T^*\mathcal{X}})_{\mathrm{tr}}^\Gamma \cong T^*(\hat{\mathcal{X}}^\Gamma)$  in  $\mathrm{qcoh}(\hat{\mathcal{X}}^\Gamma)$ .*

**PROOF.** All of the claims are local statements on  $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$ , that is, it is enough to prove them on open covers of  $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$ . As  $\mathcal{X}$  is Deligne–Mumford it is covered by open  $C^\infty$ -substacks  $\mathcal{U}$  equivalent to  $[\underline{U}/G]$  for  $\underline{U}$  an affine  $C^\infty$ -scheme and  $G$  a finite group. Then  $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$  are covered by the corresponding  $\mathcal{U}^\Gamma, \tilde{\mathcal{U}}^\Gamma, \hat{\mathcal{U}}^\Gamma$ . Thus it is sufficient to prove the theorem when  $\mathcal{X} \simeq [\underline{X}/G]$  for  $\underline{X}$  an affine  $C^\infty$ -scheme and  $G$  a finite group acting on  $\underline{X}$ . As the theorem is independent of  $\mathcal{X}$  up to equivalence, we may take  $\mathcal{X} = [\underline{X}/G]$ .

Thus we can apply Theorems 9.10 and 9.14 to translate each part of the theorem into statements about  $\underline{X}, \underline{X}^{\rho(\Gamma)}, \dots$ . For the first part, using the notation of Theorem 9.14, we find that  $F_\Pi(T^*\mathcal{X}) = (T^*\underline{X}, \Phi)$ , where  $\Phi(g) = \Omega_{T(g)}$  for  $g \in G$ . Similarly  $F_\Pi^\Gamma(T^*\mathcal{X}^\Gamma) = (T^*(\coprod_\rho \underline{X}^{\rho(\Gamma)}), \Phi^\Gamma)$ , where  $\Phi^\Gamma(g) = \coprod_\rho \Omega_{T(g)|_{\underline{X}^{\rho(\Gamma)}}}$ , and  $r(g)|_{\underline{X}^{\rho(\Gamma)}}$  maps  $\underline{X}^{\rho(\Gamma)} \rightarrow \underline{X}^{\rho(g(\Gamma))}$ .

Fix an injective morphism  $\rho : \Gamma \rightarrow G$ , and write  $i_X^\rho : \underline{X}^{\rho(\Gamma)} \rightarrow \underline{X}$  for the inclusion of  $\underline{X}^{\rho(\Gamma)}$  as a  $C^\infty$ -subscheme. Then  $(i_X^\rho)^*(T^*\underline{X}) = i_X^*(T^*\underline{X})|_{\underline{X}^{\rho(\Gamma)}}$  in  $\mathrm{qcoh}(\underline{X}^{\rho(\Gamma)})$ , and  $\Omega_{i_X^\rho} = \Omega_{i_X}|_{\underline{X}^{\rho(\Gamma)}}$ . Theorem 9.14 and  $\Phi(g) = \Omega_{T(g)}$  show that the  $\Gamma$ -action  $\bar{R}^\Gamma(\gamma, (T^*\underline{X}, \Phi))$  on  $(i_X^*(T^*\underline{X}), i_X^*(\Phi))$  acts on  $(i_X^\rho)^*(T^*\underline{X})$  by

$$\bar{R}^\Gamma(\gamma, (T^*\underline{X}, \Phi))|_{(i_X^\rho)^*(T^*\underline{X})} = (i_X^\rho)^*(\Omega_{T(\rho(\gamma^{-1}))}) \circ I_{i_X^\rho, r(\rho(\gamma^{-1}))}.$$



Let  $(i_X^\rho)^*(T^*\underline{X}) = (i_X^\rho)^*(T^*\underline{X})_{\text{tr}} \oplus (i_X^\rho)^*(T^*\underline{X})_{\text{nt}}$  be the decomposition into trivial and nontrivial  $\Gamma$ -representations under the action of  $\bar{R}^\Gamma(-, (T^*\underline{X}, \Phi))$ . Since Theorem 9.14 shows that the  $\Gamma$ -actions  $R^\Gamma(-, T^*\mathcal{X})$  and  $\bar{R}^\Gamma(-, (T^*\underline{X}, \Phi))$  are intertwined by  $F_\Pi^\Gamma$ , the splitting into trivial and nontrivial parts corresponds. As  $F_\Pi^\Gamma$  is an equivalence of categories by Theorem 8.6, the first part of the theorem is thus equivalent to showing that

$$\Omega_{i_X^\rho} : (i_X^\rho)^*(T^*\underline{X}) = (i_X^\rho)^*(T^*\underline{X})_{\text{tr}} \oplus (i_X^\rho)^*(T^*\underline{X})_{\text{nt}} \longrightarrow T^*\underline{X}^{\rho(\Gamma)}$$

is an isomorphism  $(i_X^\rho)^*(T^*\underline{X})_{\text{tr}} \rightarrow T^*\underline{X}^{\rho(\Gamma)}$ , and is zero on  $(i_X^\rho)^*(T^*\underline{X})_{\text{nt}}$ .

To see this, let  $x \in \underline{X}^{\rho(\Gamma)} \subseteq \underline{X}$ , and write  $\mathfrak{C}_x$  for the local  $C^\infty$ -ring  $\mathcal{O}_{X,x}$ , the stalk of  $\mathcal{O}_X$  at  $x$ . Then the action  $\rho$  of  $\Gamma$  on  $\underline{X}$  fixing  $x$  induces an action  $\phi : \Gamma \rightarrow \text{Aut}(\mathfrak{C}_x)$  of  $\Gamma$  on  $\mathfrak{C}_x$ . Since each  $\phi(\gamma) : \mathfrak{C}_x \rightarrow \mathfrak{C}_x$  acts on  $\mathfrak{C}_x$  as a  $C^\infty$ -ring isomorphism, it is an  $\mathbb{R}$ -linear map, so we may split  $\mathfrak{C}_x = \mathfrak{C}_{x,\text{tr}} \oplus \mathfrak{C}_{x,\text{nt}}$  into trivial and nontrivial  $\Gamma$ -representations. Write  $(\mathfrak{C}_{x,\text{nt}})$  for the ideal in  $\mathfrak{C}_x$  generated by  $\mathfrak{C}_{x,\text{nt}}$ , and  $\mathfrak{D}_x = \mathfrak{C}_x / (\mathfrak{C}_{x,\text{nt}})$  for the quotient  $C^\infty$ -ring, with projection  $\pi_x : \mathfrak{C}_x \rightarrow \mathfrak{D}_x$ . Then  $\mathcal{O}_{X^{\rho(\Gamma)},x} \cong \mathfrak{D}_x$  and  $i_{X,x}^\rho \cong \pi_x : \mathfrak{C}_x \rightarrow \mathfrak{D}_x$ .

We have cotangent modules  $\Omega_{\mathfrak{C}_x}, \Omega_{\mathfrak{D}_x}$  with morphisms  $\Omega_{\pi_x} : \Omega_{\mathfrak{C}_x} \rightarrow \Omega_{\mathfrak{D}_x}$  and  $(\Omega_{\pi_x})_* : \Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x \rightarrow \Omega_{\mathfrak{D}_x}$ . In stalks at  $x \in \underline{X}^{\rho(\Gamma)} \subseteq \underline{X}$  we have  $[T^*\underline{X}]_x \cong \Omega_{\mathfrak{C}_x}$ ,  $[T^*\underline{X}^{\rho(\Gamma)}]_x \cong \Omega_{\mathfrak{D}_x}$ ,  $[(i_X^\rho)^*(T^*\underline{X})]_x \cong \Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x$  and  $[\Omega_{i_X^\rho}]_x \cong (\Omega_{\pi_x})_* : \Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x \rightarrow \Omega_{\mathfrak{D}_x}$ . The  $\Gamma$ -action on  $\mathfrak{C}_x$  induces one on  $\Omega_{\mathfrak{C}_x}$ , and hence one on  $\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x$ . Thus we split into trivial and nontrivial  $\Gamma$ -representations,  $\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x = (\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{tr}} \oplus (\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{nt}}$ . This  $\Gamma$ -action is identified with that on the stalk  $[(i_X^\rho)^*(T^*\underline{X})]_x$ . Hence  $[(i_X^\rho)^*(T^*\underline{X})_{\text{tr}}]_x \cong (\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{tr}}$  and  $[(i_X^\rho)^*(T^*\underline{X})_{\text{nt}}]_x \cong (\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{nt}}$ .

We have a linear map  $d_{\mathfrak{C}_x} : \mathfrak{C}_x \rightarrow \Omega_{\mathfrak{C}_x}$ , whose image generates  $\Omega_{\mathfrak{C}_x}$  as a  $\mathfrak{C}_x$ -module. It induces a linear map  $d_{\mathfrak{C}_x} \otimes \pi_x : \mathfrak{C}_x \rightarrow \Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x$ , whose image generates  $\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x$  as a  $\mathfrak{D}_x$ -module. As  $d_{\mathfrak{C}_x} \otimes \pi_x$  is  $\Gamma$ -equivariant, it maps  $\mathfrak{C}_{x,\text{tr}}$  and  $\mathfrak{C}_{x,\text{nt}}$  to  $(\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{tr}}$  and  $(\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{nt}}$ , respectively. Hence  $(\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{tr}}$  and  $(\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{nt}}$  are generated as  $\mathfrak{D}_x$ -modules by  $(d_{\mathfrak{C}_x} \otimes \pi_x)(\mathfrak{C}_{x,\text{tr}})$  and  $(d_{\mathfrak{C}_x} \otimes \pi_x)(\mathfrak{C}_{x,\text{nt}})$ .

Since  $\mathfrak{D}_x = \mathfrak{C}_x / (\mathfrak{C}_{x,\text{nt}})$ , we see that  $(\Omega_{\pi_x})_* : \Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x \rightarrow \Omega_{\mathfrak{D}_x}$  is surjective, with kernel generated by  $(d_{\mathfrak{C}_x} \otimes \pi_x)((\mathfrak{C}_{x,\text{nt}}))$ . It is enough to use not the whole ideal  $(\mathfrak{C}_{x,\text{nt}})$ , but only the generating subspace  $\mathfrak{C}_{x,\text{nt}}$ . The  $\mathfrak{D}_x$ -submodule generated by  $(d_{\mathfrak{C}_x} \otimes \pi_x)(\mathfrak{C}_{x,\text{nt}})$  is  $(\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{nt}}$ . Thus,  $(\Omega_{\pi_x})_*$  is surjective with kernel  $(\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{nt}}$ , so  $(\Omega_{\pi_x})_*|_{(\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{tr}}} : (\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{tr}} \rightarrow \Omega_{\mathfrak{D}_x}$  is an isomorphism, and  $(\Omega_{\pi_x})_*|_{(\Omega_{\mathfrak{C}_x} \otimes_{\mathfrak{C}_x} \mathfrak{D}_x)_{\text{nt}}} = 0$ . Therefore  $[\Omega_{i_X^\rho}|_{(i_X^\rho)^*(T^*\underline{X})_{\text{tr}}}]_x : [(i_X^\rho)^*(T^*\underline{X})_{\text{tr}}]_x \rightarrow [T^*\underline{X}^{\rho(\Gamma)}]_x$  is an isomorphism, and  $[\Omega_{i_X^\rho}|_{(i_X^\rho)^*(T^*\underline{X})_{\text{nt}}}]_x = 0$ . As this holds for all  $x \in \underline{X}^{\rho(\Gamma)} \subseteq \underline{X}$ , the first part follows.

For the second part, Theorem 8.13(a) and  $\tilde{O}^\Gamma(\mathcal{X}) \circ \tilde{\Pi}^\Gamma(\mathcal{X}) = O^\Gamma(\mathcal{X})$  give a commutative diagram in  $\text{qcoh}(\mathcal{X}^\Gamma)$ :

$$\begin{array}{ccc} \tilde{\Pi}^\Gamma(\mathcal{X})^*(\tilde{O}^\Gamma(\mathcal{X})^*(T^*\mathcal{X})) = & \tilde{\Pi}^\Gamma(\mathcal{X})^*(\Omega_{\tilde{O}^\Gamma(\mathcal{X})}) & \xrightarrow{\quad} \tilde{\Pi}^\Gamma(\mathcal{X})^*(T^*(\tilde{\mathcal{X}}^\Gamma)) \\ \tilde{\Pi}^\Gamma(\mathcal{X})^*((T^*\mathcal{X})_{\text{tr}}^\Gamma) \oplus \tilde{\Pi}^\Gamma(\mathcal{X})^*((T^*\mathcal{X})_{\text{nt}}^\Gamma) & & \downarrow \Omega_{\tilde{\Pi}^\Gamma(\mathcal{X})} \\ \uparrow I_{\tilde{\Pi}^\Gamma(\mathcal{X}), \tilde{O}^\Gamma(\mathcal{X})}(\mathcal{E}) & & (9.24) \\ O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) = & \xrightarrow{\Omega_{O^\Gamma(\mathcal{X})}} & T^*(\mathcal{X}^\Gamma). \\ (T^*\mathcal{X})_{\text{tr}}^\Gamma \oplus (T^*\mathcal{X})_{\text{nt}}^\Gamma & & \end{array}$$

As  $\tilde{\Pi}^\Gamma(\mathcal{X})$  is the projection  $\mathcal{X}^\Gamma \rightarrow [\mathcal{X}^\Gamma / \text{Aut}(\Gamma)]$ , it is étale, so  $\Omega_{\tilde{\Pi}^\Gamma(\mathcal{X})}$  is an isomorphism. Also  $I_{\tilde{\Pi}^\Gamma(\mathcal{X}), \tilde{\mathcal{O}}^\Gamma(\mathcal{X})}(\mathcal{E})$  identifies ‘tr’, ‘nt’ with ‘tr’, ‘nt’ components.

Thus (9.24) and the first part show  $\tilde{\Pi}^\Gamma(\mathcal{X})^*(\Omega_{\tilde{\mathcal{O}}^\Gamma(\mathcal{X})|_{(T^*\mathcal{X})_{\text{tr}}^\Gamma}}) : \tilde{\Pi}^\Gamma(\mathcal{X})^*((\widetilde{T^*\mathcal{X}})_{\text{tr}}^\Gamma) \rightarrow \tilde{\Pi}^\Gamma(\mathcal{X})^*(T^*(\tilde{\mathcal{X}}^\Gamma))$  is an isomorphism, and  $\tilde{\Pi}^\Gamma(\mathcal{X})^*(\Omega_{\tilde{\mathcal{O}}^\Gamma(\mathcal{X})|_{(T^*\mathcal{X})_{\text{nt}}^\Gamma}}) = 0$ . As  $\tilde{\Pi}^\Gamma(\mathcal{X})$  is étale and surjective, the second part of the theorem follows. The third part is proved by a similar argument involving  $\hat{\Pi}^\Gamma$ .  $\square$

## APPENDIX A

### Background material on stacks

Finally we recall some background material on stacks needed in §6–§9. Readers unfamiliar with stacks are advised to look at an introductory text such as Olsson [57], Vistoli [68], Gomez [29], or Laumon and Moret-Bailly [46] before reading this section.

Stacks of any kind form a strict 2-category  $\mathcal{C}$ , with objects  $\mathcal{X}, \mathcal{Y}$ , 1-morphisms  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$ , and 2-morphisms  $\eta : f \Rightarrow g$ . So we begin in §A.1 with an introduction to 2-categories. Sections A.2–A.5 cover Grothendieck (pre)topologies, sites, prestacks and stacks, descent theory, properties of 1-morphisms of stacks, geometric stacks, and stacks associated to groupoids.

Our principal references were Artin [3], Behrend et al. [4], Gomez [29], Laumon and Moret-Bailly [46], Metzler [49], Noohi [55], and Olsson [57]. The topological and smooth stacks discussed by Metzler and Noohi are closer to our situation than the stacks in algebraic geometry of [4, 29, 46], so we often follow [49, 55], particularly in §A.5 which is based on Metzler [49, §3]. Heinloth [32] and Behrend and Xu [5] also discuss smooth stacks.

#### A.1. Introduction to 2-categories

A good reference on 2-categories for our purposes is Behrend et al. [4, App. B], and Borceux [8, §7] and Kelly and Street [43] are also helpful.

**DEFINITION A.1.** A *strict 2-category*  $\mathcal{C}$  consists of a proper class of *objects*  $\text{Obj}(\mathcal{C})$ , for all  $X, Y$  in  $\text{Obj}(\mathcal{C})$  a small category  $\text{Hom}(X, Y)$ , for all  $X$  in  $\text{Obj}(\mathcal{C})$  an object  $\text{id}_X$  in  $\text{Hom}(X, X)$  called the *identity 1-morphism*, and for all  $X, Y, Z$  in  $\text{Obj}(\mathcal{C})$  a functor

$$\mu_{X,Y,Z} : \text{Hom}(X, Y) \times \text{Hom}(Y, Z) \longrightarrow \text{Hom}(X, Z).$$

These must satisfy the *identity property*, that

$$\mu_{X,X,Y}(\text{id}_X, -) = \mu_{X,Y,Y}(-, \text{id}_Y) = \text{id}_{\text{Hom}(X,Y)} \quad (\text{A.1})$$

as functors  $\text{Hom}(X, Y) \rightarrow \text{Hom}(X, Y)$ , and the *associativity property*, that

$$\mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \text{id}_{\text{Hom}(Y,Z)}) = \mu_{W,X,Z} \circ (\text{id}_{\text{Hom}(W,X)} \times \mu_{X,Y,Z}) \quad (\text{A.2})$$

for all  $W, X, Y, Z$ , as functors

$$\text{Hom}(W, X) \times \text{Hom}(X, Y) \times \text{Hom}(Y, Z) \longrightarrow \text{Hom}(W, Z).$$

Objects  $f$  of  $\text{Hom}(X, Y)$  are called *1-morphisms*, written  $f : X \rightarrow Y$ . For 1-morphisms  $f, g : X \rightarrow Y$ , morphisms  $\eta \in \text{Hom}_{\text{Hom}(X,Y)}(f, g)$  are called *2-morphisms*, written  $\eta : f \Rightarrow g$ . Thus, a 2-category has objects  $X$ , and two kinds of morphisms, 1-morphisms  $f : X \rightarrow Y$  between objects, and 2-morphisms  $\eta : f \Rightarrow g$  between 1-morphisms.

A *weak 2-category*, or *bicategory*, is like a strict 2-category, except that the equations of functors (A.1), (A.2) are required to hold only up to specified natural isomorphisms, which should themselves satisfy identities. Strict 2-categories are examples of weak 2-categories in which these specified natural isomorphisms are identities. We will not give much detail on weak 2-categories, since the 2-categories of stacks we are interested in are strict.

In many examples, all 2-morphisms are 2-isomorphisms (i.e. have an inverse), so that the categories  $\text{Hom}(X, Y)$  are groupoids. Such 2-categories are called *(2,1)-categories*.

This is quite a complicated structure. There are three kinds of composition in a 2-category, satisfying various associativity relations. If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are 1-morphisms then  $\mu_{X,Y,Z}(f, g)$  is the *composition of 1-morphisms*, written  $g \circ f : X \rightarrow Z$ . If  $f, g, h : X \rightarrow Y$  are 1-morphisms and  $\eta : f \Rightarrow g$ ,  $\zeta : g \Rightarrow h$  are 2-morphisms then composition of  $\eta, \zeta$  in the category  $\text{Hom}(X, Y)$  gives the *vertical composition of 2-morphisms* of  $\eta, \zeta$ , written  $\zeta \odot \eta : f \Rightarrow h$ , as a diagram

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{f} \\ \Downarrow \eta \\ \xrightarrow{g} \\ \Downarrow \zeta \\ \xrightarrow{h} \end{array} & Y \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} X & \begin{array}{c} \xrightarrow{f} \\ \Downarrow \zeta \odot \eta \\ \xrightarrow{h} \end{array} & Y. \end{array}$$

And if  $f, \tilde{f} : X \rightarrow Y$  and  $g, \tilde{g} : Y \rightarrow Z$  are 1-morphisms and  $\eta : f \Rightarrow \tilde{f}$ ,  $\zeta : g \Rightarrow \tilde{g}$  are 2-morphisms then  $\mu_{X,Y,Z}(\eta, \zeta)$  is the *horizontal composition of 2-morphisms*, written  $\zeta * \eta : g \circ f \Rightarrow \tilde{g} \circ \tilde{f}$ , as a diagram

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{f} \\ \Downarrow \eta \\ \xrightarrow{\tilde{f}} \end{array} & Y \end{array} \quad \begin{array}{ccc} Y & \begin{array}{c} \xrightarrow{g} \\ \Downarrow \zeta \\ \xrightarrow{\tilde{g}} \end{array} & Z \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} X & \begin{array}{c} \xrightarrow{g \circ f} \\ \Downarrow \zeta * \eta \\ \xrightarrow{\tilde{g} \circ \tilde{f}} \end{array} & Z. \end{array}$$

There are also two kinds of identity: *identity 1-morphisms*  $\text{id}_X : X \rightarrow X$  and *identity 2-morphisms*  $\text{id}_f : f \Rightarrow f$ .

In a strict 2-category  $\mathcal{C}$ , composition of 1-morphisms is strictly associative,  $(g \circ f) \circ e = g \circ (f \circ e)$ , and horizontal composition of 2-morphisms is strictly associative,  $(\zeta * \eta) * \epsilon = \zeta * (\eta * \epsilon)$ . In a weak 2-category  $\mathcal{C}$ , composition of 1-morphisms is associative up to specified 2-isomorphisms.

A basic example is the *2-category of categories*  $\mathbf{Cat}$ , with objects small categories  $\mathcal{C}$ , 1-morphisms functors  $F : \mathcal{C} \rightarrow \mathcal{D}$ , and 2-morphisms natural transformations  $\eta : F \Rightarrow G$  for functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$ . Orbifolds naturally form a 2-category, as do Deligne–Mumford and Artin stacks in algebraic geometry.

In a 2-category  $\mathcal{C}$ , there are three notions of when objects  $X, Y$  in  $\mathcal{C}$  are ‘the same’: *equality*  $X = Y$ , and *isomorphism*, that is we have 1-morphisms  $f : X \rightarrow Y$ ,  $g : Y \rightarrow X$  with  $g \circ f = \text{id}_X$  and  $f \circ g = \text{id}_Y$ , and *equivalence*, that is we have 1-morphisms  $f : X \rightarrow Y$ ,  $g : Y \rightarrow X$  and 2-isomorphisms  $\eta : g \circ f \Rightarrow \text{id}_X$  and  $\zeta : f \circ g \Rightarrow \text{id}_Y$ . Usually equivalence is the most useful. For example, isomorphisms are not preserved by equivalences of 2-categories, whereas equivalences are.

Let  $\mathcal{C}$  be a 2-category. The *homotopy category*  $\text{Ho}(\mathcal{C})$  of  $\mathcal{C}$  is the category whose objects are objects of  $\mathcal{C}$ , and whose morphisms  $[f] : X \rightarrow Y$  are 2-isomorphism

classes  $[f]$  of 1-morphisms  $f : X \rightarrow Y$  in  $\mathcal{C}$ . Then equivalences in  $\mathcal{C}$  become isomorphisms in  $\text{Ho}(\mathcal{C})$ , 2-commutative diagrams in  $\mathcal{C}$  become commutative diagrams in  $\text{Ho}(\mathcal{C})$ , and so on.

*Commutative diagrams* in 2-categories should in general only commute *up to (specified) 2-isomorphisms*, rather than strictly. Then we say the diagram *2-commutes*. A simple example of a commutative diagram in a 2-category  $\mathcal{C}$  is

$$\begin{array}{ccc} & & Y \\ & \nearrow f & \searrow g \\ X & & Z \\ & \searrow h & \nearrow \end{array}$$

which means that  $X, Y, Z$  are objects of  $\mathcal{C}$ ,  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  and  $h : X \rightarrow Z$  are 1-morphisms in  $\mathcal{C}$ , and  $\eta : g \circ f \Rightarrow h$  is a 2-isomorphism.

Next we discuss 2-functors between 2-categories, following Borceux [8, §7.2, §7.5] and Behrend et al. [4, §B.4].

**DEFINITION A.2.** Let  $\mathcal{C}, \mathcal{D}$  be strict 2-categories. A *strict 2-functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  assigns an object  $F(X)$  in  $\mathcal{D}$  for each object  $X$  in  $\mathcal{C}$ , a 1-morphism  $F(f) : F(X) \rightarrow F(Y)$  in  $\mathcal{D}$  for each 1-morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ , and a 2-morphism  $F(\eta) : F(f) \Rightarrow F(g)$  in  $\mathcal{D}$  for each 2-morphism  $\eta : f \Rightarrow g$  in  $\mathcal{C}$ , such that  $F$  preserves all the structures on  $\mathcal{C}, \mathcal{D}$ , that is,

$$F(g \circ f) = F(g) \circ F(f), \quad F(\text{id}_X) = \text{id}_{F(X)}, \quad F(\zeta * \eta) = F(\zeta) * F(\eta), \quad (\text{A.3})$$

$$F(\zeta \odot \eta) = F(\zeta) \odot F(\eta), \quad F(\text{id}_f) = \text{id}_{F(f)}. \quad (\text{A.4})$$

Now let  $\mathcal{C}, \mathcal{D}$  be weak 2-categories. Then strict 2-functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  are not well-behaved. To fix this, we need to relax (A.3) to hold only up to specified 2-isomorphisms. A *weak 2-functor* (or *pseudofunctor*)  $F : \mathcal{C} \rightarrow \mathcal{D}$  assigns an object  $F(X)$  in  $\mathcal{D}$  for each object  $X$  in  $\mathcal{C}$ , a 1-morphism  $F(f) : F(X) \rightarrow F(Y)$  in  $\mathcal{D}$  for each 1-morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ , a 2-morphism  $F(\eta) : F(f) \Rightarrow F(g)$  in  $\mathcal{D}$  for each 2-morphism  $\eta : f \Rightarrow g$  in  $\mathcal{C}$ , a 2-isomorphism  $F_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$  in  $\mathcal{D}$  for all 1-morphisms  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  in  $\mathcal{C}$ , and a 2-isomorphism  $F_X : F(\text{id}_X) \Rightarrow \text{id}_{F(X)}$  in  $\mathcal{D}$  for all objects  $X$  in  $\mathcal{C}$  such that (A.4) holds, and for all  $e : W \rightarrow X$ ,  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  in  $\mathcal{C}$  the following diagram of 2-isomorphisms commutes in  $\mathcal{D}$ :

$$\begin{array}{ccccc} (F(g) \circ F(f)) \circ F(e) & \xrightarrow{F_{g,f} * \text{id}_{F(e)}} & F(g \circ f) \circ F(e) & \xrightarrow{F_{g \circ f, e}} & F((g \circ f) \circ e) \\ \downarrow \alpha_{F(g), F(f), F(e)} & & & & \downarrow F(\alpha_{g,f,e}) \\ F(g) \circ (F(f) \circ F(e)) & \xrightarrow{\text{id}_{F(g)} * F_{f,e}} & F(g) \circ F(f \circ e) & \xrightarrow{F_{g, f \circ e}} & F(g \circ (f \circ e)), \end{array}$$

and for all 1-morphisms  $f : X \rightarrow Y$  in  $\mathcal{C}$ , the following commute in  $\mathcal{D}$ :

$$\begin{array}{ccc} F(f) \circ F(\text{id}_X) & \xrightarrow{F_{f, \text{id}_X}} & F(f \circ \text{id}_X) \\ \downarrow \text{id}_{F(f)} * F_X & & \downarrow F(\beta_f) \\ F(f) \circ \text{id}_{F(X)} & \xrightarrow{\beta_{F(f)}} & F(f), \end{array} \quad \begin{array}{ccc} F(\text{id}_Y) \circ F(f) & \xrightarrow{F_{\text{id}_Y, f}} & F(\text{id}_Y \circ f) \\ \downarrow F_Y * \text{id}_{F(f)} & & \downarrow F(\gamma_f) \\ \text{id}_{F(Y)} \circ F(f) & \xrightarrow{\gamma_{F(f)}} & F(f), \end{array}$$

and if  $f, \dot{f} : X \rightarrow Y$  and  $g, \dot{g} : Y \rightarrow Z$  are 1-morphisms and  $\eta : f \Rightarrow \dot{f}$ ,  $\zeta : g \Rightarrow \dot{g}$  are 2-morphisms in  $\mathcal{C}$  then the following commutes in  $\mathcal{D}$ :

$$\begin{array}{ccc} F(g) \circ F(f) & \xrightarrow{F_{g,f}} & F(g \circ f) \\ \downarrow F(\zeta) * F(\eta) & & \downarrow F(\zeta * \eta) \\ F(\dot{g}) \circ F(\dot{f}) & \xrightarrow{F_{\dot{g}, \dot{f}}} & F(\dot{g} \circ \dot{f}). \end{array}$$

There are obvious notions of *composition*  $G \circ F$  of strict and weak 2-functors  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{E}$ , *identity 2-functors*  $\text{id}_{\mathcal{C}}$ , and so on.

If  $\mathcal{C}, \mathcal{D}$  are strict 2-categories, then a strict 2-functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  can be made into a weak 2-functor by taking all  $F_{g,f}, F_X$  to be identity 2-morphisms.

Here are some well-known facts about 2-categories and 2-functors:

- (i) Every weak 2-category  $\mathcal{C}$  is equivalent as a weak 2-category to a strict 2-category  $\mathcal{C}'$ , that is, weak 2-categories can always be strictified.
- (ii) If  $\mathcal{C}, \mathcal{D}$  are strict 2-categories, and  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a weak 2-functor, it may not be true that  $F$  is 2-naturally isomorphic to a strict 2-functor  $F' : \mathcal{C} \rightarrow \mathcal{D}$ . That is, weak 2-functors cannot necessarily be strictified.

Even if one is working with strict 2-categories, weak 2-functors are often the correct notion of functor between them.

We define fibre products in 2-categories, following [4, Def. B.13].

**DEFINITION A.3.** Let  $\mathcal{C}$  be a 2-category and  $g : X \rightarrow Z$ ,  $h : Y \rightarrow Z$  be 1-morphisms in  $\mathcal{C}$ . A *fibre product*  $X \times_Z Y$  in  $\mathcal{C}$  consists of an object  $W$ , 1-morphisms  $e : W \rightarrow X$  and  $f : W \rightarrow Y$  and a 2-isomorphism  $\eta : g \circ e \Rightarrow h \circ f$  in  $\mathcal{C}$ , so that we have a 2-commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ \downarrow e & \eta \nearrow & \downarrow h \\ X & \xrightarrow{g} & Z \end{array} \quad (\text{A.5})$$

with the following universal property: suppose  $e' : W' \rightarrow X$  and  $f' : W' \rightarrow Y$  are 1-morphisms and  $\eta' : g \circ e' \Rightarrow h \circ f'$  is a 2-isomorphism in  $\mathcal{C}$ . Then there should exist a 1-morphism  $b : W' \rightarrow W$  and 2-isomorphisms  $\zeta : e \circ b \Rightarrow e'$ ,  $\theta : f \circ b \Rightarrow f'$  such that the following diagram of 2-isomorphisms commutes:

$$\begin{array}{ccc} g \circ e \circ b & \xRightarrow{\eta * \text{id}_b} & h \circ f \circ b \\ \downarrow \text{id}_g * \zeta & & \downarrow \text{id}_h * \theta \\ g \circ e' & \xRightarrow{\eta'} & h \circ f'. \end{array} \quad (\text{A.6})$$

Furthermore, if  $\tilde{b}, \tilde{\zeta}, \tilde{\theta}$  are alternative choices of  $b, \zeta, \theta$  then there should exist a unique 2-isomorphism  $\epsilon : \tilde{b} \Rightarrow b$  with

$$\tilde{\zeta} = \zeta \odot (\text{id}_e * \epsilon) \quad \text{and} \quad \tilde{\theta} = \theta \odot (\text{id}_f * \epsilon).$$

We call such a fibre product diagram (A.5) a *2-Cartesian square*. If a fibre product  $X \times_Z Y$  in  $\mathcal{C}$  exists then it is unique up to equivalence in  $\mathcal{C}$ .

Orbifolds, and stacks in algebraic geometry, form 2-categories, and Definition A.3 is the right way to define fibre products of orbifolds or stacks, as in [4]. Given a 2-commutative diagram in a 2-category

$$\begin{array}{ccccc} U & \xrightarrow{f} & W & \xrightarrow{i} & Y \\ \downarrow e & \eta \nearrow & \downarrow h & \zeta \nearrow & \downarrow k \\ V & \xrightarrow{g} & X & \xrightarrow{j} & Z, \end{array}$$

if the two small rectangles are 2-Cartesian, then the outer rectangle is too.

### A.2. Grothendieck topologies, sites, prestacks, and stacks

Some references for this section are Olsson [57], Artin [3], Behrend et al. [4], and Laumon and Moret-Bailly [46].

DEFINITION A.4. Let  $\mathcal{C}$  be a category, and  $U \in \mathcal{C}$ . A *sieve*  $\mathcal{S}$  on  $U$  is a collection of morphisms  $\phi : V \rightarrow U$  in  $\mathcal{C}$  closed under precomposition, that is, if  $\phi : V \rightarrow U$  lies in  $\mathcal{S}$  and  $\psi : W \rightarrow V$  is a morphism in  $\mathcal{C}$  then  $\phi \circ \psi : W \rightarrow U$  lies in  $\mathcal{S}$ .

A *Grothendieck topology* on  $\mathcal{C}$  is a collection of distinguished sieves for each object  $U \in \mathcal{C}$  called *covering sieves*, satisfying some axioms we will not give. A *site*  $(\mathcal{C}, \mathcal{J})$  is a category  $\mathcal{C}$  with a Grothendieck topology  $\mathcal{J}$ .

It is often convenient to define Grothendieck topologies using Grothendieck pretopologies. A *Grothendieck pretopology*  $\mathcal{PJ}$  on  $\mathcal{C}$  is a collection of families  $\{\varphi_a : U_a \rightarrow U\}_{a \in A}$  of morphisms in  $\mathcal{C}$  called *coverings*, satisfying:

- (i) If  $\varphi : V \rightarrow U$  is an isomorphism in  $\mathcal{C}$ , then  $\{\varphi : V \rightarrow U\}$  is a covering;
- (ii) If  $\{\varphi_a : U_a \rightarrow U\}_{a \in A}$  is a covering, and  $\{\psi_{ab} : V_{ab} \rightarrow U_a\}_{b \in B_a}$  is a covering for all  $a \in A$ , then  $\{\varphi_a \circ \psi_{ab} : V_{ab} \rightarrow U\}_{a \in A, b \in B_a}$  is a covering.
- (iii) If  $\{\varphi_a : U_a \rightarrow U\}_{a \in A}$  is a covering and  $\psi : V \rightarrow U$  is a morphism in  $\mathcal{C}$  then  $\{\pi_V : U_a \times_{\varphi_a, U, \psi} V \rightarrow V\}_{a \in A}$  is a covering, where the fibre product  $U_a \times_U V$  exists in  $\mathcal{C}$  for all  $a \in A$ .

Each Grothendieck pretopology  $\mathcal{PJ}$  has an associated Grothendieck topology  $\mathcal{J}$ , in which a sieve  $\mathcal{S}$  on  $U \in \mathcal{C}$  is a covering sieve in  $\mathcal{J}$  if and only if it contains a covering  $\{\varphi_a : U_a \rightarrow U\}_{a \in A}$  in  $\mathcal{PJ}$ .

A Grothendieck pretopology  $\mathcal{PJ}$  gives a notion of *open cover* of objects in  $\mathcal{C}$ . For example, if  $\mathcal{C}$  is the category of topological spaces **Top**, we could define  $\mathcal{PJ}$  to be the collection of families  $\{\varphi_a : U_a \rightarrow U\}_{a \in A}$  in **Top** such that  $\varphi_a : U_a \rightarrow U$  is a homeomorphism with an open subset  $\varphi_a(U_a) \subseteq U$  for  $a \in A$ , with  $U = \bigcup_{a \in A} \varphi_a(U_a)$ , so that  $\{\varphi_a : U_a \rightarrow U\}_{a \in A}$  is an open cover of  $U$ .

DEFINITION A.5. Let  $\mathcal{C}$  be a category. A *category fibred in groupoids over  $\mathcal{C}$*  is a functor  $p_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{C}$ , where  $\mathcal{X}$  is a category, such that given any morphism  $g : C_1 \rightarrow C_2$  in  $\mathcal{C}$  and  $X_2 \in \mathcal{X}$  with  $p_{\mathcal{X}}(X_2) = C_2$ , there exists a morphism  $f : X_1 \rightarrow X_2$  in  $\mathcal{X}$  with  $p_{\mathcal{X}}(f) = g$ , and given commutative diagrams (on the left) in  $\mathcal{X}$ , in which  $g$  is to be determined, and (on the right) in  $\mathcal{C}$ :

$$\begin{array}{ccc}
 X_1 & \xrightarrow{\quad g \quad} & X_2 \\
 & \searrow f \quad \swarrow h & \\
 & X_3 &
 \end{array}
 \quad \xrightarrow[p_{\mathcal{X}}]{\sim} \quad
 \begin{array}{ccc}
 p_{\mathcal{X}}(X_1) & \xrightarrow{\quad g' \quad} & p_{\mathcal{X}}(X_2) \\
 & \searrow p_{\mathcal{X}}(f) \quad \swarrow p_{\mathcal{X}}(h) & \\
 & p_{\mathcal{X}}(X_3) &
 \end{array}
 \quad (A.7)$$

then there exists a unique morphism  $g$  as shown with  $p_{\mathcal{X}}(g) = g'$  and  $f = h \circ g$ . Often we refer to  $\mathcal{X}$  as the category fibred in groupoids (or prestack, or stack, etc.), leaving  $p_{\mathcal{X}}$  implicit.

If  $p_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{C}$  is a category fibred in groupoids and  $C$  is an object in  $\mathcal{C}$ , the *fibre*  $\mathcal{X}_C$  is the subcategory of  $\mathcal{X}$  with objects those  $X \in \mathcal{X}$  with  $p_{\mathcal{X}}(X) = C$ , and morphisms those  $f : X_1 \rightarrow X_2$  with  $p_{\mathcal{X}}(f) = \text{id}_C : C \rightarrow C$ . Then  $\mathcal{X}_C$  is a groupoid (i.e. a category with all morphisms isomorphisms).

DEFINITION A.6. Let  $(\mathcal{C}, \mathcal{J})$  be a site, and  $p_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{C}$  be a category fibred in groupoids over  $\mathcal{C}$ . We call  $\mathcal{X}$  a *prestack* if whenever  $\{\varphi_a : U_a \rightarrow U\}_{a \in A}$  is a covering family in  $\mathcal{J}$  and we are given commutative diagrams in  $\mathcal{X}, \mathcal{C}$  for all  $a, b \in A$ , in which

$f$  is to be determined:

$$\begin{array}{ccc}
 & X_{ab} \longrightarrow Y_{ab} & \\
 & \swarrow \quad \searrow & \\
 X_a & \longrightarrow Y_a & \\
 \searrow x_a & \swarrow x_b & \\
 & X_b & \\
 & \searrow y_a & \\
 & Y_b & \\
 & \swarrow y_b & \\
 & Y & \\
 & \xleftarrow{f} X &
 \end{array}
 \quad \xrightarrow{p_{\mathcal{X}}} \quad
 \begin{array}{ccc}
 & U_a \times_U U_b & \\
 & \swarrow \pi_{U_a} \quad \searrow \pi_{U_b} & \\
 U_a & \longrightarrow U_a & \\
 \searrow \varphi_a & \swarrow \varphi_b & \\
 & U_b & \\
 & \searrow \varphi_b & \\
 & U & \\
 & \xleftarrow{\varphi_b} U &
 \end{array}
 \quad (A.8)$$

then there exists a unique  $f : X \rightarrow Y$  in  $\mathcal{X}$  with  $p_{\mathcal{X}}(f) = \text{id}_U$  making (A.8) commute for all  $a \in A$ .

Let  $p_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{C}$  be a prestack. We call  $\mathcal{X}$  a *stack* if whenever  $\{\varphi_a : U_a \rightarrow U\}_{a \in A}$  is a covering family in  $\mathcal{J}$  and we are given commutative diagrams in  $\mathcal{X}, \mathcal{C}$  for all  $a, b, c \in A$ , with  $X_{ab} = X_{ba}$ ,  $X_{abc} = X_{bac} = X_{acb}$ , etc., in which the object  $X$  and morphisms  $x_a$  are determined:

$$\begin{array}{ccc}
 & X_{abc} \longrightarrow X_{ac} & \\
 & \swarrow \quad \searrow & \\
 X_{ab} & \longrightarrow X_a & \\
 \searrow x_{ba} & \swarrow x_{bc} & \\
 & X_b & \\
 & \searrow x_b & \\
 & X & \\
 & \xleftarrow{x_c} X_c &
 \end{array}
 \quad \xrightarrow{p_{\mathcal{X}}} \quad
 \begin{array}{ccc}
 & U_a \times_U U_b \times_U U_c & \\
 & \swarrow \quad \searrow & \\
 U_a \times_U U_b & \longrightarrow U_a & \\
 \searrow \varphi_a & \swarrow \varphi_b & \\
 & U_b & \\
 & \searrow \varphi_b & \\
 & U & \\
 & \xleftarrow{\varphi_c} U_c &
 \end{array}
 \quad (A.9)$$

then there exists  $X \in \mathcal{X}$  and morphisms  $x_a : X_a \rightarrow X$  with  $p_{\mathcal{X}}(x_a) = \varphi_a$  for all  $a \in A$ , making (A.9) commute for all  $a, b, c \in A$ .

Thus, in a prestack we have a sheaf-like condition allowing us to glue morphisms in  $\mathcal{X}$  uniquely over covers in  $\mathcal{C}$ ; in a stack we also have a sheaf-like condition allowing us to glue objects in  $\mathcal{X}$  over covers in  $\mathcal{C}$ .

**DEFINITION A.7.** Let  $(\mathcal{C}, \mathcal{J})$  be a site. A *1-morphism* between (pre)stacks  $\mathcal{X}, \mathcal{Y}$  on  $(\mathcal{C}, \mathcal{J})$  is a functor  $F : \mathcal{X} \rightarrow \mathcal{Y}$  with  $p_{\mathcal{Y}} \circ F = p_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{C}$ . If  $F, G : \mathcal{X} \rightarrow \mathcal{Y}$  are 1-morphisms, a *2-morphism*  $\eta : F \Rightarrow G$  is an isomorphism of functors with  $\text{id}_{p_{\mathcal{Y}}} * \eta = \text{id}_{p_{\mathcal{X}}} : p_{\mathcal{Y}} \circ F \Rightarrow p_{\mathcal{Y}} \circ G$ . That is, for all  $X \in \mathcal{X}$  we are given an isomorphism  $\eta(X) : F(X) \rightarrow G(X)$  in  $\mathcal{Y}$  with  $p_{\mathcal{Y}}(\eta(X)) = \text{id}_{p_{\mathcal{X}}(X)}$ , such that if  $f : X_1 \rightarrow X_2$  is a morphism in  $\mathcal{X}$  then  $\eta(X_2) \circ F(f) = G(f) \circ \eta(X_1) : F(X_1) \rightarrow G(X_2)$  in  $\mathcal{Y}$ . With these definitions, the stacks and prestacks on  $(\mathcal{C}, \mathcal{J})$  form (*strict*) 2-categories, which we write as  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$  and  $\mathbf{Presta}_{(\mathcal{C}, \mathcal{J})}$ . All 2-morphisms in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}, \mathbf{Presta}_{(\mathcal{C}, \mathcal{J})}$  are invertible, that is, are 2-isomorphisms, so  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}, \mathbf{Presta}_{(\mathcal{C}, \mathcal{J})}$  are (2,1)-categories.

A *substack*  $\mathcal{Y}$  of a stack  $\mathcal{X}$  is a strictly full subcategory  $\mathcal{Y}$  in  $\mathcal{X}$  such that  $p_{\mathcal{Y}} := p_{\mathcal{X}}|_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{C}$  is a stack. The inclusion functor  $i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{X}$  is then a 1-morphism of stacks.

**DEFINITION A.8.** Let  $(\mathcal{C}, \mathcal{J})$  be a site, and  $\mathcal{X}$  a prestack on  $(\mathcal{C}, \mathcal{J})$ , so that  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$  and  $\mathbf{Presta}_{(\mathcal{C}, \mathcal{J})}$  are 2-categories. A *stack associated to  $\mathcal{X}$* , or *stackification of  $\mathcal{X}$* , is a stack  $\hat{\mathcal{X}}$  with a 1-morphism of prestacks  $i : \mathcal{X} \rightarrow \hat{\mathcal{X}}$ , such that for every stack  $\mathcal{Y}$ , composition with  $i$  yields an equivalence of categories  $\text{Hom}(\hat{\mathcal{X}}, \mathcal{Y}) \xrightarrow{i^*} \text{Hom}(\mathcal{X}, \mathcal{Y})$ .

As in [46, Lem. 3.2], every prestack has an associated stack, just as every presheaf has an associated sheaf.



PROPOSITION A.9. *For every prestack  $\mathcal{X}$  on  $(\mathcal{C}, \mathcal{J})$  there exists an associated stack  $i : \mathcal{X} \rightarrow \hat{\mathcal{X}}$ , which is unique up to equivalence in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ .*

There is a natural construction of fibre products in the 2-category  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ :

DEFINITION A.10. Let  $(\mathcal{C}, \mathcal{J})$  be a site,  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$  be stacks on  $(\mathcal{C}, \mathcal{J})$ , and  $F : \mathcal{X} \rightarrow \mathcal{Z}, G : \mathcal{Y} \rightarrow \mathcal{Z}$  be 1-morphisms. Define a category  $\mathcal{W}$  to have objects  $(X, Y, \alpha)$ , where  $X \in \mathcal{X}, Y \in \mathcal{Y}$  and  $\alpha : F(X) \rightarrow G(Y)$  is an isomorphism in  $\mathcal{Z}$  with  $p_{\mathcal{X}}(X) = p_{\mathcal{Y}}(Y) = U$  and  $p_{\mathcal{X}}(\alpha) = \text{id}_U$  in  $\mathcal{C}$ , and for objects  $(X_1, Y_1, \alpha_1), (X_2, Y_2, \alpha_2)$  in  $\mathcal{W}$  a morphism  $(f, g) : (X_1, Y_1, \alpha_1) \rightarrow (X_2, Y_2, \alpha_2)$  in  $\mathcal{W}$  is a pair of morphisms  $f : X_1 \rightarrow X_2$  in  $\mathcal{X}$  and  $g : Y_1 \rightarrow Y_2$  in  $\mathcal{Y}$  with  $p_{\mathcal{X}}(f) = p_{\mathcal{Y}}(g) = \varphi : U \rightarrow V$  in  $\mathcal{C}$  and  $\alpha_2 \circ F(f) = G(g) \circ \alpha_1 : F(X_1) \rightarrow G(Y_2)$  in  $\mathcal{Z}$ . Then  $\mathcal{W}$  is a stack over  $(\mathcal{C}, \mathcal{J})$ .

Define 1-morphisms  $p_{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{C}$  by  $p_{\mathcal{W}} : (X, Y, \alpha) \mapsto p_{\mathcal{X}}(X)$  and  $p_{\mathcal{W}} : (f, g) \mapsto p_{\mathcal{X}}(f)$ , and  $\pi_{\mathcal{X}} : \mathcal{W} \rightarrow \mathcal{X}$  by  $\pi_{\mathcal{X}} : (X, Y, \alpha) \mapsto X$  and  $\pi_{\mathcal{X}} : (f, g) \mapsto f$ , and  $\pi_{\mathcal{Y}} : \mathcal{W} \rightarrow \mathcal{Y}$  by  $\pi_{\mathcal{Y}} : (X, Y, \alpha) \mapsto Y$  and  $\pi_{\mathcal{Y}} : (f, g) \mapsto g$ . Define a 2-morphism  $\eta : F \circ \pi_{\mathcal{X}} \Rightarrow G \circ \pi_{\mathcal{Y}}$  by  $\eta(X, Y, \alpha) = \alpha$ . Then  $\mathcal{W}, \pi_{\mathcal{X}}, \pi_{\mathcal{Y}}, \eta$  is a fibre product  $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$  in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ , in the sense of Definition A.3.

The functor  $\text{id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  is a *terminal object* in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ , and may be thought of as a point  $*$ . Products  $\mathcal{X} \times \mathcal{Y}$  in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$  are fibre products over  $*$ . If  $\mathcal{X}$  is a stack, the *diagonal 1-morphism* is the natural 1-morphism  $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ . The *inertia stack*  $I_{\mathcal{X}}$  of  $\mathcal{X}$  is the fibre product  $\mathcal{X} \times_{\Delta_{\mathcal{X}}, \mathcal{X} \times \mathcal{X}, \Delta_{\mathcal{X}}} \mathcal{X}$ , with natural *inertia 1-morphism*  $\iota_{\mathcal{X}} : I_{\mathcal{X}} \rightarrow \mathcal{X}$  from projection to the first factor of  $\mathcal{X}$ . Then we have a 2-Cartesian diagram in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ :

$$\begin{array}{ccc} I_{\mathcal{X}} & \xrightarrow{\quad} & \mathcal{X} \\ \downarrow \iota_{\mathcal{X}} & \Delta_{\mathcal{X}} \Uparrow & \Delta_{\mathcal{X}} \downarrow \\ \mathcal{X} & \xrightarrow{\quad} & \mathcal{X} \times \mathcal{X} \end{array}$$

There is also a natural 1-morphism  $j_{\mathcal{X}} : \mathcal{X} \rightarrow I_{\mathcal{X}}$  induced by the 1-morphism  $\text{id}_{\mathcal{X}}$  from  $\mathcal{X}$  to the two factors  $\mathcal{X}$  in  $I_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$  and the identity 2-morphism on  $\Delta_{\mathcal{X}} \circ \text{id}_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ .

### A.3. Descent theory on a site

The theory of descent in algebraic geometry, due to Grothendieck, says that objects and morphisms over a scheme  $U$  can be described locally on an open cover  $\{U_i : i \in I\}$  of  $U$ . It is described by Behrend et al. [4, App. A] and Olsson [57, §4], and at length by Vistoli [68]. We shall express descent as conditions on a general site  $(\mathcal{C}, \mathcal{J})$ .

DEFINITION A.11. Let  $(\mathcal{C}, \mathcal{J})$  be a site. We say that  $(\mathcal{C}, \mathcal{J})$  *has descent for objects* if whenever  $\{\varphi_a : U_a \rightarrow U\}_{a \in A}$  is a covering in  $\mathcal{J}$  and we are given morphisms  $f_a : X_a \rightarrow U_a$  in  $\mathcal{C}$  for all  $a \in A$  and isomorphisms  $g_{ab} : X_a \times_{\varphi_a \circ f_a, U, \varphi_b} U_b \rightarrow X_b \times_{\varphi_b \circ f_b, U, \varphi_a} U_a$  in  $\mathcal{C}$  for all  $a, b \in A$  with  $g_{ab} = g_{ba}^{-1}$  such that for all  $a, b, c \in A$

the following diagram commutes:

$$\begin{array}{ccc}
 (X_a \times_{\varphi_a \circ f_a, U, \varphi_b} U_b) \times_{\pi_U, U, \varphi_c} U_c & \xrightarrow{g_{ab} \times \text{id}_{U_c}} & (X_b \times_{\varphi_b \circ f_b, U, \varphi_c} U_c) \times_{\pi_U, U, \varphi_a} U_a \cong \\
 (X_a \times_{\varphi_a \circ f_a, U, \varphi_c} U_c) \times_{\pi_U, U, \varphi_b} U_b & \xleftarrow{g_{ba} \times \text{id}_{U_c}} & (X_b \times_{\varphi_b \circ f_b, U, \varphi_a} U_a) \times_{\pi_U, U, \varphi_c} U_c \\
 & \searrow g_{ca} \times \text{id}_{U_b} \quad \swarrow g_{cb} \times \text{id}_{U_a} & \\
 & (X_c \times_{\varphi_c \circ f_c, U, \varphi_a} U_a) \times_{\pi_U, U, \varphi_b} U_b \cong \\
 & (X_c \times_{\varphi_c \circ f_c, U, \varphi_b} U_b) \times_{\pi_U, U, \varphi_a} U_a, &
 \end{array}$$

then there exist a morphism  $f : X \rightarrow U$  in  $\mathcal{C}$  and isomorphisms  $g_a : X_a \rightarrow X \times_{f, U, \varphi_a} U_a$  for all  $a \in A$  such that  $f_a = \pi_{U_a} \circ g_a$  and the diagram below commutes for all  $a, b \in A$ :

$$\begin{array}{ccc}
 X_a \times_{\varphi_a \circ f_a, U, \varphi_b} U_b & \xrightarrow{g_a \times \text{id}_{U_b}} & (X \times_{f, U, \varphi_a} U_a) \times_{\varphi_a \circ \pi_{U_a}, U, \varphi_b} U_b \\
 \downarrow g_{ab} & & \cong \downarrow \\
 & & X \times_{f, U, \pi_U} (U_a \times_{\varphi_a, U, \varphi_b} U_b) \\
 & & \cong \downarrow \\
 X_b \times_{\varphi_b \circ f_b, U, \varphi_a} U_a & \xleftarrow{g_b^{-1} \times \text{id}_{U_a}} & (X \times_{f, U, \varphi_b} U_b) \times_{\varphi_b \circ \pi_{U_b}, U, \varphi_a} U_a.
 \end{array}$$

Furthermore  $X, f$  should be unique up to canonical isomorphism. Note that all the fibre products used above exist in  $\mathcal{C}$  by Definition A.4(iii).

**DEFINITION A.12.** Let  $(\mathcal{C}, \mathcal{J})$  be a site. We say that  $(\mathcal{C}, \mathcal{J})$  *has descent for morphisms* if whenever  $\{\varphi_a : U_a \rightarrow U\}_{a \in A}$  is a covering in  $\mathcal{J}$  and  $f : X \rightarrow U$ ,  $g : Y \rightarrow U$  and  $h_a : X \times_{f, U, \varphi_a} U_a \rightarrow Y \times_{g, U, \varphi_a} U_a$  for all  $a \in A$  are morphisms in  $\mathcal{C}$  with  $\pi_{U_a} \circ h_a = \pi_{U_a}$  and for all  $a, b \in A$  the following diagram commutes:

$$\begin{array}{ccc}
 (X \times_{f, U, \varphi_a} U_a) \times_{\varphi_a \circ \pi_{U_a}, U, \varphi_b} U_b & \xrightarrow{h_a \times \text{id}_{U_b}} & (Y \times_{g, U, \varphi_a} U_a) \times_{\varphi_a \circ \pi_{U_a}, U, \varphi_b} U_b \\
 \downarrow \cong & & \cong \downarrow \\
 X \times_{f, U, \pi_U} (U_a \times_{\varphi_a, U, \varphi_b} U_b) & & Y \times_{g, U, \pi_U} (U_a \times_{\varphi_a, U, \varphi_b} U_b) \\
 \downarrow \cong & & \cong \downarrow \\
 (X \times_{f, U, \varphi_b} U_b) \times_{\varphi_b \circ \pi_{U_b}, U, \varphi_a} U_a & \xrightarrow{h_b \times \text{id}_{U_a}} & (Y \times_{g, U, \varphi_b} U_b) \times_{\varphi_b \circ \pi_{U_b}, U, \varphi_a} U_a,
 \end{array}$$

then there exists a unique  $h : X \rightarrow Y$  in  $\mathcal{C}$  with  $h_a = h \times \text{id}_{U_a}$  for all  $a \in A$ .

Then [4, Prop.s A.12, A.13 & §A.6] show that descent holds for objects and morphisms for affine schemes with the fppf topology, but for arbitrary schemes with the fppf topology, descent holds for morphisms and fails for objects.

#### A.4. Properties of 1-morphisms

Objects  $V$  in  $\mathcal{C}$  yield stacks  $\bar{V}$  on  $(\mathcal{C}, \mathcal{J})$ .

**DEFINITION A.13.** Let  $(\mathcal{C}, \mathcal{J})$  be a site, and  $V$  an object of  $\mathcal{C}$ . Define a category  $\bar{V}$  to have objects  $(U, \theta)$  where  $U \in \mathcal{C}$  and  $\theta : U \rightarrow V$  is a morphism in  $\mathcal{C}$ , and to have morphisms  $\psi : (U_1, \theta_1) \rightarrow (U_2, \theta_2)$  where  $\psi : U_1 \rightarrow U_2$  is a morphism in  $\mathcal{C}$  with  $\theta_2 \circ \psi = \theta_1 : U_1 \rightarrow V$ . Define a functor  $p_{\bar{V}} : \bar{V} \rightarrow \mathcal{C}$  by  $p_{\bar{V}} : (U, \theta) \mapsto U$  and  $p_{\bar{V}} : \psi \mapsto \psi$ . Note that  $p_{\bar{V}}$  is *injective on morphisms*. It is then automatic that  $p_{\bar{V}} : \bar{V} \rightarrow \mathcal{C}$  is a category fibred in groupoids, since in (A.7) we can take  $g = g'$ . It is also automatic that  $p_{\bar{V}} : \bar{V} \rightarrow \mathcal{C}$  is a prestack, since in (A.8) we must have

$X_a = Y_a = (U_a, \theta_a)$ ,  $x_a = y_a = \varphi_a$ ,  $X = Y = (U, \theta)$ , etc., and the unique solution for  $f$  is  $f = \text{id}_U$ .

The site  $(\mathcal{C}, \mathcal{J})$  is called *subcanonical* if  $\bar{V}$  is a stack for all objects  $V \in \mathcal{C}$ . If descent for morphisms holds for  $(\mathcal{C}, \mathcal{J})$  then  $(\mathcal{C}, \mathcal{J})$  is subcanonical. Most sites used in practice are subcanonical. Suppose  $(\mathcal{C}, \mathcal{J})$  is a subcanonical site. If  $\bar{f} : V \rightarrow W$  is a morphism in  $\mathcal{C}$ , define a 1-morphism  $\bar{f} : \bar{V} \rightarrow \bar{W}$  in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$  by  $\bar{f} : (U, \theta) \mapsto (U, f \circ \theta)$  and  $\bar{f} : \psi \mapsto \psi$ . Then the (2-)functor  $V \mapsto \bar{V}$ ,  $f \mapsto \bar{f}$  embeds  $\mathcal{C}$  as a full discrete 2-subcategory of  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ .

**DEFINITION A.14.** Let  $(\mathcal{C}, \mathcal{J})$  be a subcanonical site. A stack  $\mathcal{X}$  over  $(\mathcal{C}, \mathcal{J})$  is called *representable* if it is equivalent in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$  to a stack of the form  $\bar{V}$  for some  $V \in \mathcal{C}$ . A 1-morphism  $F : \mathcal{X} \rightarrow \mathcal{Y}$  in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$  is called *representable* if for all  $V \in \mathcal{C}$  and all 1-morphisms  $G : \bar{V} \rightarrow \mathcal{Y}$ , the fibre product  $\mathcal{X} \times_{F, \mathcal{Y}, G} \bar{V}$  in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$  is a representable stack.

**REMARK A.15.** For stacks in algebraic geometry, one often takes a different definition of representable objects and 1-morphisms:  $(\mathcal{C}, \mathcal{J})$  is a category of schemes with the étale topology, but stacks are called representable if they are equivalent to an *algebraic space* rather than a scheme. This is because schemes are not general enough for some purposes, e.g. the quotient of a scheme by an étale equivalence relation may be an algebraic space but not a scheme.

In our situation, we will have no need to enlarge  $C^\infty$ -schemes to some category of ‘ $C^\infty$ -algebraic spaces’, as  $C^\infty$ -schemes are already general enough, e.g. the quotient of a locally fair  $C^\infty$ -scheme by an étale equivalence relation is a locally fair  $C^\infty$ -scheme. This is because the natural topology on  $C^\infty$ -schemes is much finer than the Zariski or étale topology on schemes, for instance, affine  $C^\infty$ -schemes are always Hausdorff.

**DEFINITION A.16.** Let  $(\mathcal{C}, \mathcal{J})$  be a subcanonical site. Let  $\mathbf{P}$  be a property of morphisms in  $\mathcal{C}$ . (For instance, if  $\mathcal{C}$  is the category **Top** of topological spaces, then  $\mathbf{P}$  could be ‘proper’, ‘open’, ‘surjective’, ‘covering map’, ...). We say that  $\mathbf{P}$  is *invariant under base change* if for all Cartesian squares in  $\mathcal{C}$

$$\begin{array}{ccc} W & \xrightarrow{\quad f \quad} & Y \\ \downarrow e & & \downarrow h \\ X & \xrightarrow{\quad g \quad} & Z, \end{array}$$

if  $g$  is  $\mathbf{P}$ , then  $f$  is  $\mathbf{P}$ . We say that  $\mathbf{P}$  is *local on the target* if whenever  $f : U \rightarrow V$  is a morphism in  $\mathcal{C}$  and  $\{\varphi_a : V_a \rightarrow V\}_{a \in A}$  is a covering in  $\mathcal{J}$  such that  $\pi_{V_a} : U \times_{f, V, \varphi_a} V_a \rightarrow V_a$  is  $\mathbf{P}$  for all  $a \in A$ , then  $f$  is  $\mathbf{P}$ .

Let  $\mathbf{P}$  be invariant under base change and local in the target, and let  $F : \mathcal{X} \rightarrow \mathcal{Y}$  be a representable 1-morphism in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ . If  $W \in \mathcal{C}$  and  $G : \bar{W} \rightarrow \mathcal{Y}$  is a 1-morphism then  $\mathcal{X} \times_{F, \mathcal{Y}, G} \bar{W}$  is equivalent to  $\bar{V}$  for some  $V \in \mathcal{C}$ , and under this equivalence the 1-morphism  $\pi_{\bar{W}} : \mathcal{X} \times_{F, \mathcal{Y}, G} \bar{W} \rightarrow \bar{W}$  is 2-isomorphic to  $\bar{f} : \bar{V} \rightarrow \bar{W}$  for some unique morphism  $f : V \rightarrow W$  in  $\mathcal{C}$ . We say that  $F$  *has property  $\mathbf{P}$*  if for all  $W \in \mathcal{C}$  and 1-morphisms  $G : \bar{W} \rightarrow \mathcal{Y}$ , the morphism  $f : V \rightarrow W$  in  $\mathcal{C}$  corresponding to  $\pi_{\bar{W}} : \mathcal{X} \times_{F, \mathcal{Y}, G} \bar{W} \rightarrow \bar{W}$  has property  $\mathbf{P}$ .

We define *surjective* 1-morphisms without requiring them representable.

**DEFINITION A.17.** Let  $(\mathcal{C}, \mathcal{J})$  be a site, and  $F : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ . We call  $F$  *surjective* if whenever  $Y \in \mathcal{Y}$  with  $p_Y(Y) = U \in \mathcal{C}$ , there exists

a covering  $\{\varphi_a : U_a \rightarrow U\}_{a \in A}$  in  $\mathcal{J}$  such that for all  $a \in A$  there exists  $X_a \in \mathcal{X}$  with  $p_{\mathcal{X}}(X_a) = U_a$  and a morphism  $g_a : F(X_a) \rightarrow Y$  in  $\mathcal{Y}$  with  $p_{\mathcal{Y}}(g_a) = \varphi_a$ .

Following [46, Prop. 3.8.1, Lem. 4.3.3 & Rem. 4.14.1], [55, §6], we may prove:

PROPOSITION A.18. *Let  $(\mathcal{C}, \mathcal{J})$  be a subcanonical site, and*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow e & \nearrow \eta & \downarrow h \\ \mathcal{X} & \xrightarrow{g} & \mathcal{Z} \end{array}$$

be a 2-Cartesian square in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ . Let  $\mathbf{P}$  be a property of morphisms in  $\mathcal{C}$  which is invariant under base change and local in the target. Then:

- (a) If  $h$  is representable, then  $e$  is representable. If also  $h$  is  $\mathbf{P}$ , then  $e$  is  $\mathbf{P}$ .
- (b) If  $g$  is surjective, then  $f$  is surjective.

Now suppose also that  $(\mathcal{C}, \mathcal{J})$  has descent for objects and morphisms, and that  $g$  (and hence  $f$ ) is surjective. Then:

- (c) If  $e$  is surjective then  $h$  is surjective, and if  $e$  is representable, then  $h$  is representable, and if also  $e$  is  $\mathbf{P}$ , then  $h$  is  $\mathbf{P}$ .

### A.5. Geometric stacks, and stacks associated to groupoids

The 2-category  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$  of all stacks over a site  $(\mathcal{C}, \mathcal{J})$  is usually too general to do geometry with. To obtain a smaller 2-category whose objects have better properties, we impose extra conditions on a stack  $\mathcal{X}$ :

DEFINITION A.19. Let  $(\mathcal{C}, \mathcal{J})$  be a site. We call a stack  $\mathcal{X}$  on  $(\mathcal{C}, \mathcal{J})$  *geometric* if the diagonal 1-morphism  $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is representable, and there exists  $U \in \mathcal{C}$  and a surjective 1-morphism  $\Pi : \bar{U} \rightarrow \mathcal{X}$ , which we call an *atlas* for  $\mathcal{X}$ . Write  $\mathbf{GSta}_{(\mathcal{C}, \mathcal{J})}$  for the full 2-subcategory of geometric stacks in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ . Here  $\Delta_{\mathcal{X}}$  representable implies  $\Pi$  is representable.

To obtain nice classes of stacks, one usually requires further properties  $\mathbf{P}$  of  $\Delta_{\mathcal{X}}$  and  $\Pi$ . For example, in algebraic geometry with  $(\mathcal{C}, \mathcal{J})$  schemes with the étale topology, we assume  $\Delta_{\mathcal{X}}$  is quasicompact and separated, and  $\Pi$  is étale for Deligne–Mumford stacks  $\mathcal{X}$ , and  $\Pi$  is smooth for Artin stacks  $\mathcal{X}$ .

The following material is based on Metzler [49, §3.1 & §3.3], Laumon and Moret-Bailly [46, §§2.4.3, 3.4.3, 3.8, 4.3], and Lerman [47, §4.4].

We can characterize geometric stacks  $\mathcal{X}$  up to equivalence solely in terms of objects and morphisms in  $\mathcal{C}$ , using the idea of *groupoid objects* in  $\mathcal{C}$ .

DEFINITION A.20. A *groupoid object*  $(U, V, s, t, u, i, m)$  in a category  $\mathcal{C}$ , or simply *groupoid* in  $\mathcal{C}$ , consists of objects  $U, V$  in  $\mathcal{C}$  and morphisms  $s, t : V \rightarrow U$ ,  $u : U \rightarrow V$ ,  $i : V \rightarrow V$  and  $m : V \times_{s, U, t} V \rightarrow V$  satisfying the identities

$$\begin{aligned} s \circ u &= t \circ u = \text{id}_U, \quad s \circ i = t, \quad t \circ i = s, \quad s \circ m = s \circ \pi_2, \quad t \circ m = t \circ \pi_1, \\ m \circ (i \times \text{id}_V) &= u \circ s, \quad m \circ (\text{id}_V \times i) = u \circ t, \\ m \circ (m \times \text{id}_V) &= m \circ (\text{id}_V \times m) : V \times_U V \times_U V \longrightarrow V, \\ m \circ (\text{id}_V \times u) &= m \circ (u \times \text{id}_V) : V = V \times_U U \longrightarrow V, \end{aligned} \tag{A.10}$$

where we suppose all the fibre products exist.

Groupoids in  $\mathcal{C}$  are so called because a groupoid in **Sets** is a groupoid in the usual sense, that is, a category with invertible morphisms, where  $U$  is the set of

objects,  $V$  the set of morphisms,  $s : V \rightarrow U$  the *source* of a morphism,  $t : V \rightarrow U$  the *target* of a morphism,  $u : U \rightarrow V$  the *unit* taking  $X \mapsto \text{id}_X$ ,  $i$  the *inverse* taking  $f \mapsto f^{-1}$ , and  $m$  the *multiplication* taking  $(f, g) \mapsto f \circ g$  when  $s(f) = t(g)$ . Then (A.10) reduces to the usual axioms for a groupoid.

From a geometric stack with an atlas, we can construct a groupoid in  $\mathcal{C}$ .

DEFINITION A.21. Let  $(\mathcal{C}, \mathcal{J})$  be a subcanonical site, and suppose  $\mathcal{X}$  is a geometric stack on  $(\mathcal{C}, \mathcal{J})$  with atlas  $\Pi : \bar{U} \rightarrow \mathcal{X}$ . Then  $\bar{U} \times_{\Pi, \mathcal{X}, \Pi} \bar{U}$  is equivalent to  $\bar{V}$  for some  $V \in \mathcal{C}$  as  $\Pi$  is representable. Hence we can take  $\bar{V}$  to be the fibre product, and we have a 2-Cartesian square

$$\begin{array}{ccc} \bar{V} & \xrightarrow{\quad \bar{t} \quad} & \bar{U} \\ \downarrow \bar{s} & \nearrow \eta & \downarrow \Pi \\ \bar{U} & \xrightarrow{\quad \Pi \quad} & \mathcal{X} \end{array} \quad (\text{A.11})$$

in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ . Here as  $(\mathcal{C}, \mathcal{J})$  is subcanonical, any 1-morphism  $\bar{V} \rightarrow \bar{U}$  in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$  is 2-isomorphic to  $\bar{f}$  for some unique morphism  $f : V \rightarrow U$  in  $\mathcal{C}$ . Thus we may write the projections in (A.11) as  $\bar{s}, \bar{t}$  for some unique  $s, t : V \rightarrow U$  in  $\mathcal{C}$ .

By the universal property of fibre products there exists a 1-morphism  $H : \bar{U} \rightarrow \bar{V}$ , unique up to 2-isomorphism, with  $\bar{s} \circ H \cong \text{id}_{\bar{U}} \cong \bar{t} \circ H$ . This  $H$  is 2-isomorphic to  $\bar{u} : \bar{U} \rightarrow \bar{V}$  for some unique morphism  $u : U \rightarrow V$  in  $\mathcal{C}$ , and then  $s \circ u = t \circ u = \text{id}_U$ . Similarly, exchanging the two factors of  $U$  in the fibre product we obtain a unique morphism  $i : V \rightarrow V$  in  $\mathcal{C}$  with  $s \circ i = t$  and  $t \circ i = s$ . In  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$  we have equivalences

$$\overline{V \times_{s, U, t} V} \simeq \bar{V} \times_{\bar{s}, \bar{U}, \bar{t}} \bar{V} \simeq (\bar{U} \times_{\mathcal{X}} \bar{U}) \times_{\bar{U}} (\bar{U} \times_{\mathcal{X}} \bar{U}) \simeq \bar{U} \times_{\mathcal{X}} \bar{U} \times_{\mathcal{X}} \bar{U}.$$

Let  $m : V \times_{s, U, t} V \rightarrow V$  be the unique morphism in  $\mathcal{C}$  such that  $\bar{m}$  is 2-isomorphic to the projection  $\overline{V \times_{s, U, t} V} \rightarrow \bar{V} = \bar{U} \times_{\mathcal{X}} \bar{U}$  corresponding to projection to the first and third factors of  $\bar{U}$  in the final fibre product. It is now not difficult to verify that  $(U, V, s, t, u, i, m)$  is a groupoid in  $\mathcal{C}$ .

Conversely, given a groupoid in  $\mathcal{C}$  we can construct a stack  $\mathcal{X}$ .

DEFINITION A.22. Let  $(\mathcal{C}, \mathcal{J})$  be a site with descent for morphisms, and  $(U, V, s, t, u, i, m)$  be a groupoid in  $\mathcal{C}$ . Define a prestack  $\mathcal{X}'$  on  $(\mathcal{C}, \mathcal{J})$  as follows: let  $\mathcal{X}'$  be the category whose objects are pairs  $(T, f)$  where  $f : T \rightarrow U$  is a morphism in  $\mathcal{C}$ , and morphisms are  $(p, q) : (T_1, f_1) \rightarrow (T_2, f_2)$  where  $p : T_1 \rightarrow T_2$  and  $q : T_1 \rightarrow V$  are morphisms in  $\mathcal{C}$  with  $f_1 = s \circ q$  and  $f_2 \circ p = t \circ q$ . Given morphisms  $(p_1, q_1) : (T_1, f_1) \rightarrow (T_2, f_2)$  and  $(p_2, q_2) : (T_2, f_2) \rightarrow (T_3, f_3)$  the composition is  $(p_2, q_2) \circ (p_1, q_1) = (p_2 \circ p_1, m \circ (q_1 \times (q_2 \circ p_2)))$ , where  $q_1 \times (q_2 \circ p_2) : T_1 \rightarrow V \times_{t, U, s} V$  is induced by the morphisms  $q_1 : T_1 \rightarrow V$  and  $q_2 \circ p_2 : T_1 \rightarrow V$ , which satisfy  $t \circ q_1 = f_2 \circ p_1 = s \circ (q_2 \circ p_2)$ .

Define a functor  $p_{\mathcal{X}'} : \mathcal{X}' \rightarrow \mathcal{C}$  by  $p_{\mathcal{X}'} : (T, f) \mapsto T$  and  $p_{\mathcal{X}'} : (p, q) \mapsto p$ . Using the groupoid axioms (A.10) we can show that  $p_{\mathcal{X}'} : \mathcal{X}' \rightarrow \mathcal{C}$  is a category fibred in groupoids. Since  $(\mathcal{C}, \mathcal{J})$  has descent for morphisms, we can also show  $\mathcal{X}'$  is a prestack. But in general it is not a stack. Let  $\mathcal{X}$  be the associated stack from Proposition A.9. We call  $\mathcal{X}$  the *stack associated to the groupoid*  $(U, V, s, t, u, i, m)$ . It fits into a natural 2-commutative diagram (A.11).

Groupoids in  $\mathcal{C}$  are often written  $V \rightrightarrows U$ , to emphasize  $s, t : V \rightarrow U$ , leaving  $u, i, m$  implicit. The associated stack is then written as  $[V \rightrightarrows U]$ .

Our next theorem is proved by Metzler [49, Prop. 70] when  $(\mathcal{C}, \mathcal{J})$  is the site of topological spaces with open covers, but examining the proof shows that all he uses about  $(\mathcal{C}, \mathcal{J})$  is that fibre products exist in  $\mathcal{C}$  and  $(\mathcal{C}, \mathcal{J})$  has descent for objects and morphisms. See also Lerman [47, Prop. 4.31]. If fibre products may not exist in  $\mathcal{C}$  then one must also require the morphisms  $s, t$  in  $(U, V, s, t, u, i, m)$  to be *representable* in  $\mathcal{C}$ , that is, for all  $f : T \rightarrow U$  in  $\mathcal{C}$  the fibre products  $T_{f,U,s}V$  and  $T_{f,U,t}V$  exist in  $\mathcal{C}$ .

**THEOREM A.23.** *Let  $(\mathcal{C}, \mathcal{J})$  be a site, and suppose that all fibre products exist in  $\mathcal{C}$ , and that descent for objects and morphisms holds in  $(\mathcal{C}, \mathcal{J})$ . Then the constructions of Definitions A.21, A.22 are inverse. That is, if  $(U, V, s, t, u, i, m)$  is a groupoid in  $\mathcal{C}$  and  $\mathcal{X}$  is the associated stack, then  $\mathcal{X}$  is a geometric stack, and the 2-commutative diagram (A.11) is 2-Cartesian, and  $\Pi$  in (A.11) is surjective and so an atlas for  $\mathcal{X}$ , and  $(U, V, s, t, u, i, m)$  is canonically isomorphic to the groupoid constructed in Definition A.21 from the atlas  $\Pi : \bar{U} \rightarrow \mathcal{X}$ . Conversely, if  $\mathcal{X}$  is a geometric stack with atlas  $\Pi : \bar{U} \rightarrow \mathcal{X}$ , and  $(U, V, s, t, u, i, m)$  is the groupoid in  $\mathcal{C}$  constructed from  $\Pi$  in Definition A.21, and  $\mathcal{X}$  is the stack associated to  $(U, V, s, t, u, i, m)$  in Definition A.22, then  $\mathcal{X}$  is equivalent to  $\mathcal{X}$  in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ . Thus every geometric stack is associated to a groupoid.*

In the situation of Theorem A.23 we have 2-Cartesian diagrams

$$\begin{array}{ccc}
 \bar{V} & \xrightarrow{\quad \bar{t} \quad} & \bar{U} \\
 \downarrow \bar{s} & \Uparrow & \downarrow \Pi \\
 \bar{U} & \xrightarrow{\quad \Pi \quad} & \mathcal{X},
 \end{array}
 \quad
 \begin{array}{ccc}
 \bar{V} & \xrightarrow{\quad \Pi \circ \bar{s} \quad} & \mathcal{X} \\
 \downarrow \bar{s} \times \bar{t} & \Uparrow & \downarrow \Delta_{\mathcal{X}} \\
 \bar{U} \times \bar{U} & \xrightarrow{\quad \Pi \times \Pi \quad} & \mathcal{X} \times \mathcal{X},
 \end{array}
 \quad (A.12)$$

$$\begin{array}{ccc}
 \bar{V} \times_{\bar{s} \times \bar{t}, \bar{U} \times \bar{U}, \Delta_{\bar{U}}} \bar{U} & \xrightarrow{\quad \Pi \circ \bar{s} \times \Pi \circ \bar{t} \quad} & I_{\mathcal{X}} \\
 \downarrow \pi_{\bar{U}} & \Uparrow & \downarrow \iota_{\mathcal{X}} \\
 \bar{U} & \xrightarrow{\quad \Pi \quad} & \mathcal{X},
 \end{array}
 \quad
 \begin{array}{ccc}
 \bar{U} & \xrightarrow{\quad \Pi \quad} & \mathcal{X} \\
 \downarrow \bar{u} \times \text{id}_{\bar{U}} & \Uparrow & \downarrow j_{\mathcal{X}} \\
 \bar{V} \times_{\bar{s} \times \bar{t}, \bar{U} \times \bar{U}, \Delta_{\bar{U}}} \bar{U} & \xrightarrow{\quad \Pi \circ \bar{s} \times \Pi \circ \bar{t} \quad} & I_{\mathcal{X}},
 \end{array}$$

with surjective rows. So from Proposition A.18 we deduce:

**COROLLARY A.24.** *In the situation of Theorem A.23, let  $\mathbf{P}$  be a property of morphisms in  $\mathcal{C}$  which is invariant under base change and local in the target. Then  $\bar{\Pi} : \bar{U} \rightarrow \mathcal{X}$  is  $\mathbf{P}$  if and only if  $s : V \rightarrow U$  is  $\mathbf{P}$ , and  $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is  $\mathbf{P}$  if and only if  $s \times t : V \rightarrow U \times U$  is  $\mathbf{P}$ , and  $\iota_{\mathcal{X}} : I_{\mathcal{X}} \rightarrow \mathcal{X}$  is  $\mathbf{P}$  if and only if  $\pi_U : V \times_{s \times t, U \times U, \Delta_U} U \rightarrow U$  is  $\mathbf{P}$ , and  $j_{\mathcal{X}} : \mathcal{X} \rightarrow I_{\mathcal{X}}$  is  $\mathbf{P}$  if and only if  $u \times \text{id}_U : U \rightarrow V \times_{s \times t, U \times U, \Delta_U} U$  is  $\mathbf{P}$ .*

We can describe atlases for fibre products of geometric stacks.

**EXAMPLE A.25.** Suppose  $(\mathcal{C}, \mathcal{J})$  is a subcanonical site, and all fibre products exist in  $\mathcal{C}$ . Let

$$\begin{array}{ccc}
 \mathcal{W} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\
 \downarrow e & \eta \Uparrow & \downarrow h \\
 \mathcal{X} & \xrightarrow{\quad g \quad} & \mathcal{Z}
 \end{array}$$

be a 2-Cartesian diagram in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ , where  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$  are geometric stacks. Let  $\Pi_{\mathcal{X}} : \bar{U}_{\mathcal{X}} \rightarrow \mathcal{X}$  and  $\Pi_{\mathcal{Y}} : \bar{U}_{\mathcal{Y}} \rightarrow \mathcal{Y}$  be atlases. As  $\Delta_{\mathcal{Z}}$  is representable the fibre product  $\bar{U}_{\mathcal{X}} \times_{g \circ \Pi_{\mathcal{X}}, \mathcal{Z}, h \circ \Pi_{\mathcal{Y}}} \bar{U}_{\mathcal{Y}}$  is represented by an object  $U_{\mathcal{W}}$  of  $\mathcal{C}$ . Then we have

a 2-commutative diagram, where we omit 2-morphisms:

$$\begin{array}{ccccc}
 & & \bar{U}_W := \bar{U}_X \times_Z \bar{U}_Y & \xrightarrow{\pi_1} & \mathcal{X} \times_Z \bar{U}_Y & \xrightarrow{\pi_2} & \mathcal{W} \\
 & \swarrow & \searrow & & \swarrow & & \searrow \\
 \bar{U}_X & \xrightarrow{\Pi_X} & \mathcal{X} & \xleftarrow{e} & \mathcal{W} & & \\
 & \searrow & \swarrow & & \swarrow & & \searrow \\
 & & Z & \xleftarrow{h} & \mathcal{Y} & & \\
 & & & & \swarrow & & \searrow \\
 & & & & \mathcal{Y} & & 
 \end{array}
 \quad (A.13)$$

Here the five squares in (A.13) are 2-Cartesian. Define  $\Pi_W = \pi_2 \circ \pi_1 : \bar{U}_W \rightarrow \mathcal{W}$ , where  $\pi_1, \pi_2$  are as in (A.13). Proposition A.18(a),(b) imply that  $\pi_1, \pi_2$  are representable and surjective, since  $\Pi_X, \Pi_Y$  are. Hence  $\Pi_W = \pi_2 \circ \pi_1$  is also representable and surjective, so  $\mathcal{W}$  is a geometric stack, and  $\Pi_W$  is an atlas for  $\mathcal{W}$ . In the same way, if  $\mathbf{P}$  is a property of morphisms in  $\mathcal{C}$  which is invariant under base change and local in the target and closed under compositions, and  $\Pi_X, \Pi_Y$  are  $\mathbf{P}$ , then  $\Pi_W$  is  $\mathbf{P}$ .

Now let  $\bar{V}_W = \bar{U}_W \times_{\mathcal{W}} \bar{U}_W$  and complete to a groupoid  $(U_W, V_W, s_W, t_W, u_W, i_W, m_W)$  in  $\mathcal{C}$  as above, with  $\mathcal{W} \simeq [V_W \rightrightarrows U_W]$ , and do the same for  $\mathcal{X}, \mathcal{Y}$ . Then by a diagram chase similar to (A.13) we can show that

$$\bar{V}_W \cong \bar{V}_X \times_Z \bar{V}_Y \quad \text{and} \quad V_W \cong (U_W \times_{U_X} V_X) \times_{U_Y} V_Y. \quad (A.14)$$

**COROLLARY A.26.** *Suppose  $(\mathcal{C}, \mathcal{J})$  is a subcanonical site, and all fibre products exist in  $\mathcal{C}$ . Then the 2-subcategory  $\mathbf{GSta}_{(\mathcal{C}, \mathcal{J})}$  of geometric stacks is closed under fibre products in  $\mathbf{Sta}_{(\mathcal{C}, \mathcal{J})}$ .*





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## Glossary of Notation

- AC<sup>∞</sup>Sch** category of affine  $C^\infty$ -schemes, 35  
**AC<sup>∞</sup>Sch<sup>fa</sup>** category of fair affine  $C^\infty$ -schemes, 35  
**AC<sup>∞</sup>Sch<sup>fp</sup>** category of finitely presented affine  $C^\infty$ -schemes, 35  
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 $\mathfrak{C} \otimes_{\infty} \mathfrak{D}$  coproduct of  $C^\infty$ -rings  $\mathfrak{C}, \mathfrak{D}$ , 10  
 $\mathfrak{C}^G$   $C^\infty$ -subring fixed by finite group  $G$  acting on  $C^\infty$ -ring  $\mathfrak{C}$ , 16  
 $\mathfrak{C}\text{-mod}$  abelian category of modules over a  $C^\infty$ -ring  $\mathfrak{C}$ , 49  
**C<sup>∞</sup>Rings** category of  $C^\infty$ -rings, 9  
**C<sup>∞</sup>Rings<sup>fa</sup>** category of fair  $C^\infty$ -rings, 15  
**C<sup>∞</sup>Rings<sup>fg</sup>** category of finitely generated  $C^\infty$ -rings, 12  
**C<sup>∞</sup>Rings<sup>fp</sup>** category of finitely presented  $C^\infty$ -rings, 12  
**C<sup>∞</sup>RS** category of  $C^\infty$ -ringed spaces, 29  
**C<sup>∞</sup>Sch** category of  $C^\infty$ -schemes, 35  
**C<sup>∞</sup>Sch<sup>lf</sup>** category of locally fair  $C^\infty$ -schemes, 35  
**C<sup>∞</sup>Sch<sup>lfp</sup>** category of locally finitely presented  $C^\infty$ -schemes, 35  
 $\bar{\mathbf{C}}^\infty\mathbf{Sch}$  2-subcategory of  $\mathcal{X}$  in **C<sup>∞</sup>Sta** equivalent to a  $C^\infty$ -scheme  $\bar{X}$ , 65  
 $\bar{\mathbf{C}}^\infty\mathbf{Sch}^{\text{lf}}$  2-subcategory of  $\mathcal{X}$  in **C<sup>∞</sup>Sta** equivalent to  $\bar{X}$  for  $X$  locally fair, 65  
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**DMC<sup>∞</sup>Sta<sup>lf</sup>** 2-category of locally fair Deligne–Mumford  $C^\infty$ -stacks, 77  
**DMC<sup>∞</sup>Sta<sup>lfp</sup>** 2-category of locally finitely presented Deligne–Mumford  $C^\infty$ -stacks, 77  
**DMC<sup>∞</sup>Sta<sup>re</sup>** 2-category of Deligne–Mumford  $C^\infty$ -stacks with representable 1-morphisms, 111  
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 $F_{\mathbf{C}^\infty\mathbf{Sch}}^{\mathbf{C}^\infty\mathbf{Sta}} : \mathbf{C}^\infty\mathbf{Sch} \rightarrow \mathbf{C}^\infty\mathbf{Sta}$  inclusion from  $C^\infty$ -schemes to  $C^\infty$ -stacks, 66  
 $f_*(\mathcal{E})$  pushforward (direct image) sheaf, 27  
 $f^{-1}(\mathcal{E})$  pullback (inverse image) sheaf, 27  
 $\underline{f}^*(\mathcal{E})$  pullback of sheaf of  $\mathcal{O}_Y$ -modules under  $\underline{f} : \underline{X} \rightarrow \underline{Y}$ , 55  
 $f^*(\mathcal{E})$  pullback of quasicoherent sheaf  $\mathcal{E}$  under  $f : \mathcal{X} \rightarrow \mathcal{Y}$ , 97  
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- $f_{\sharp} : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$  morphism of sheaves of  $C^\infty$ -rings in  $f : \underline{X} \rightarrow \underline{Y}$ , 29  
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 $\mathfrak{m}_X^\infty$  flat ideal of  $f \in C^\infty(\mathbb{R}^n)$  vanishing to all orders on  $X \subseteq \mathbb{R}^n$ , 19  
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