On the geometric and analytic properties of some random fractals

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Lincoln College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy



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Abstract

The heat content of a domain D of $\mathbb{R}^d$ is defined as

$$E(s) = \int_D u(s,x)dx,$$

where u is the solution to the heat equation with zero initial condition and unit Dirichlet boundary condition. This thesis studies the behaviour of $E(s)$ for small s with a particular emphasis on the case where D is a planar domain whose boundary is a random Koch curve.

When ∂D is spatially homogeneous, we show that we can recover the upper and lower Minkowski dimensions of ∂D from $E(s)$. Furthermore, in some cases where the Minkowski dimension does exist, finer fluctuations can be recovered and the heat content is controlled by $s^\alpha \exp\{f(\log(1/s)\}$ for small s, for some positive α and some regularly varying function f.

When ∂D is statistically self-similar, the heat content asymptotics are studied using a law of large numbers for the general branching process, and we show that the Minkowski dimension and content of ∂D exist and can be recovered from $E(s)$. More precisely the heat content has an almost sure expansion

$$E(s) = c_1 s^\alpha N_\infty + o(s^\alpha), \text{ a.s.}$$

for small s, for some positive c_1 and α and a positive random variable N_∞ with unit expectation.

To study the fluctuations around these asymptotics, we prove a central limit theorem for the general branching process. The proof follows a standard Taylor expansion argument and relies on the independence built into the general branching process. The limiting distribution established here is reminiscent of those arising in central limit theorems for martingales.

When ∂D is a statistically self-similar Cantor subset of $\mathbb{R}$, we discuss examples where we have and fail to have a central limit theorem for the heat content. We conclude with an open question about the fluctuations of the heat content when ∂D is a statistically self-similar Koch curve.

Preface

This document constitutes my doctoral thesis. It is submitted in partial fulfilment for the requirements of the degree of Doctor of Philosophy in Mathematics at the Mathematical Institute of the University of Oxford.

The content of this text reflects a long lasting interest in the connection between the geometry of a domain and its boundary, and how conducive the domain is to heat flowing through it. Here, we will focus essentially on the case where the domain is planar and its boundary is random, and we will study the random fluctuations of the geometry and heat flow.

The main tools that we will use are the Minkowski dimension, general branching processes, and probabilistic heat kernel estimates.

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CHAPTER 1

Introduction

1.1. Spectral problems and heat content

Let D be a bounded domain of $\mathbb{R}^d$ and Δ be the Dirichlet Laplacian. Then the spectrum Λ of $-\frac{1}{2}\Delta$ is discrete and of the form

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots,$$

where the eigenvalues are repeated according to their multiplicity. Interest in the geometric information about D encoded by Λ started a little over 100 years ago and was crystallised by Kac in his paper [**Kac66**] entitled 'Can one hear the shape of a drum?' Or more precisely, does Λ determine D up to isometry? The answer to that question is no, as shown in [**GWW92, Mil64**]; see also [**BCDS94**] for a concise presentation of a family of counterexamples.

However, some geometric information about D can be recovered. Weyl's theorem states that the *eigenvalue counting function* N defined by

$$N(\lambda) = \#\{\text{eigenvalues of } -\Delta/2 \leq \lambda\}$$

satisfies the asymptotic growth

$$N(\lambda) = c_1(d)\mathrm{vol}_d(D)\lambda^{d/2} + o(\lambda^{d/2}),$$

as $\lambda \to \infty$, for some constant $c_1(d)$ depending only on d, where vol_d denotes the d-dimensional Lebesgue measure. Furthermore, under some regularity conditions, if ∂D is smooth, then the expansion has a second order term

$$N(\lambda) = c_1(d)\mathrm{vol}_d(D)\lambda^{d/2} + c_2(d)\mathrm{vol}_{d-1}(\partial D)\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}),$$

as $\lambda \to \infty$, for some other constant $c_2(d)$ depending only on d. The reader is referred to [**Lai81, See78, See80, Vas90**] and references therein for more information. This means that, under some regularity conditions, we can recover the size of the domain and that of the boundary from the spectral asymptotics; in particular, using the isoperimetric inequality, we can determine whether D is an open ball.

Interest in the second term of the expansion of N naturally intensified when Berry studied the spectral asymptotics of domains with a fractal boundary in [**Ber79, Ber80**]. He conjectured that the Hausdorff dimension of ∂D should drive the second order term. This conjecture was disproved by Brossard and Carmona in [**BC86**] who studied the Laplace transform of N and showed that the Minkowski dimension was the relevant measure of the roughness.

As the spectral decomposition of the heat kernel with absorption on the boundary p_D indicates,[1] the heat content

$$E_D(s) = \int_D \left(1 - \int_D p_D(s, x, y)dy\right) dx$$

[1]Recall that

$$p_D(s, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n s}\phi_n(x)\phi_n(y),$$

where λ_n are the eigenvalues of $-\frac{1}{2}\Delta$ and ϕ_n the corresponding eigenfunctions; see [**Bas95**].

also provides a natural proxy for Λ. Recall that E_D may be expressed more intuitively as

$$E_D(s) = \int_D u_D(s,x)dx,$$

where u_D is the solution to the heat equation with unit Dirichlet boundary condition and zero initial condition, i.e. is the solution to

$$\begin{aligned} \partial_s u_D(s,x) &= \frac{1}{2}\Delta u_D(s,x), & (s,x) &\in (0,\infty)\times D, \\ u_D(0,x) &= u_D(0+,x) = 0, & x &\in D, \\ u_D(-,x) &= 1, & x &\in \partial D. \end{aligned} \tag{1.1}$$

We will omit the dependence on D from the notation when there is no risk of confusion.

The heat content has the advantage that it is amenable to probabilistic techniques. Furthermore, it is a convenient object to recover information about the boundary because the leading term of its short time asymptotics depends on ∂D rather than D. Indeed, in [**vdB94**], van den Berg proved that, under some regularity conditions, if the Minkowski dimension γ of ∂D exists, then

$$E(s) \asymp s^{(d-\gamma)/2}$$

for small s, where $f(x) \asymp g(x)$ means that $c^{-1}f(x) \leq g(x) \leq cf(x)$ for some positive constant c. This is the heat content analogue of the result of Brossard and Carmona for the Laplace transform of N in [**BC86**].

The short time asymptotics of the heat content have been studied for a variety of domains. Planar domains with polygonal boundary are discussed in [**vdBS88, vdBS90**]. Some domains with fractal boundary, such as the triadic Koch snowflake, are discussed in [**FLV95, LV96, vdBdH99**]. Here, we perform an analysis in the same vein for some domains with a random boundary; we study how the randomness of the geometry is encoded by the

randomness of the heat content. Using Kac's analogy, we ask how much of the distribution of the random boundary of a domain one can hear.

1.2. Outline of the content and the main results

In Chapter 2, we recall some definitions and results about fractal dimensions and contents to introduce the relevant notation. We discuss a result of [**CPS14**] on the geometry of Brownian motion with a drift which is in the spirit of results on the geometry and heat content of some random domains discussed in later chapters.

In Chapter 3, we derive various estimates on the heat content using a probabilistic representation of the solution to the heat equation along the lines of [**BC86, vdB94**]. The key result is that both the upper and lower Minkowski dimensions of the boundary of a domain can be recovered from the heat content. We also discuss a new example illustrating that the Hausdorff dimension does not dictate the short time behaviour of the heat content as well as two examples where a lot is known about the heat content asymptotics.

In Chapter 4, we introduce a class of Koch snowflakes that we call *spatially homogeneous*. We use it to construct an example of a snowflake with a boundary whose upper and lower Minkowski dimensions disagree, and to discuss what finer geometric fluctuations are encoded by the heat content.

In Chapter 5, we define the general branching process and explain how it can be used to model statistically self-similar sets, which generalise the familiar self-similar sets. The technology of branching processes will enable us to study Koch snowflakes with statistically self-similar boundary in Chapters 7 and 8.

In Chapter 6, we discuss laws of large numbers and a central limit theorem for the general branching process. These results extend those in [**Ner81**] and [**JN84**]. Particularly for the laws of large numbers, the statements given here aim to minimise the effort needed to apply them to various examples arising from spectral problems for random fractals.

In Chapter 7, we study the heat content of a family of subsets of $\mathbb{R}$ whose boundary is statistically self-similar. These sets are simple enough that we can discuss the fluctuations of the heat content around its short time asymptotics and describe some of the difficulties encountered when applying the central limit theorem for the general branching process.

Finally, in Chapter 8, we introduce a class of Koch snowflakes with statistically self-similar boundary. We study the asymptotics of its heat content as in Chapter 7 and conclude with an open question about the fluctuations of the heat content.

The chapters should be self-contained. However, the study of sets with a statistically self-similar boundary heavily relies on the theory of the general branching process. There are also two appendices at the end with results on renewal theory and heat content estimates which are included for completeness.

1.3. Notation

Most of the notation is standard and will therefore be introduced as we go along. However, for convenience, we will use the shorthand notation c_i with $i \in \mathbb{N}$ to mean some positive constant whose value is fixed for the length of a proof or a section.

CHAPTER 2

Fractal dimension and content

In this chapter, we briefly define the Hausdorff and Minkowski dimensions and contents of a subset of $\mathbb{R}^d$. We do not dwell on the generalisation of these notions to bounded metric spaces because we will not need them. For brevity, proofs are omitted and the reader is referred to [**Fal86, MP10**] from which this chapter is largely inspired.

2.1. Hausdorff dimension and measure

Let E be a bounded subset of $\mathbb{R}^d$. For $\delta \in (0, \infty]$ and $\alpha \in [0, \infty)$, put

$$\mathcal{H}^\alpha_\delta(E) = \inf\left\{\sum_{n=1}^{\infty}(\operatorname{diam} E_n)^\alpha : (E_n, n \in \mathbb{N}) \text{ covers } E \text{ and } \operatorname{diam} E_n \leq \delta \text{ for all } n\right\},$$

where $\operatorname{diam} A$ is the diameter of the set A using the Euclidian distance dist, i.e.

$$\operatorname{diam} A = \sup\{\operatorname{dist}(x, y) : x \text{ and } y \in A\}.$$

The quantity $\mathcal{H}^\alpha_\infty(E)$ is called the *α-Hausdorff content* of E. The *α-Hausdorff measure* of E is then defined as

$$\mathcal{H}^\alpha(E) = \sup_{\delta \in (0,\infty]} \mathcal{H}^\alpha_\delta(E) = \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E).$$

The Hausdorff measure is countably subadditive and therefore an outer measure.

Notice that, for $\beta \geq \alpha$, if $\mathcal{H}^\alpha(E) = 0$, then $\mathcal{H}^\beta(E) = 0$. We may therefore define the *Hausdorff dimension* of E as

$$\dim E = \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(E) = 0\} = \sup\{\alpha \geq 0 : \mathcal{H}^\alpha(E) > 0\}.$$

It is possible to check that $\mathcal{H}^\alpha(E) = 0$ if and only if $\mathcal{H}^\alpha_\infty(E) = 0$ thereby showing that we can use the Hausdorff content instead of the Hausdorff measure of E to calculate its dimension.

Thanks to its measure theoretic definition, the Hausdorff dimension has some desirable properties such as *countable stability*, i.e.

$$\dim \bigcup_{n\in\mathbb{N}} E_n = \sup\{\dim E_n : n \in \mathbb{N}\}.$$

However, it is not straightforward to calculate in general. There are many results on how to determine the Hausdorff dimension of a set; we will only use one, the mass distribution principle, which we state now.

THEOREM 2.1.1. *Let E be a Borel subset of $\mathbb{R}^d$ and $\alpha \in [0, \infty)$. Assume that there exists a measure ν supported on E such that $\nu(E) \in (0, \infty)$ and that, for some positive constants c and δ,*

$$\nu(V) \le c(\operatorname{diam} V)^\alpha$$

whenever V is a closed subset of E with $\operatorname{diam} V \le \delta$. Then,

$$\mathcal{H}^\alpha(E) \ge c^{-1}\nu(E) > 0.$$

In particular, $\dim E \ge \alpha$.

2.2. Minkowski dimension and content

Elementary definitions. Let E be a bounded subset of $\mathbb{R}^d$. We call

$$N(\epsilon, E) = \inf\{k \in \mathbb{N} : E \text{ can be covered with } k \text{ open balls of radius } \epsilon\},$$

the *covering number* of E with balls of radius ϵ. The *lower Minkowski dimension* of E is defined as

$$\underline{\dim}_M E = \liminf_{\epsilon\to 0} \frac{\log N(\epsilon, E)}{\log(1/\epsilon)}$$

and the *upper Minkowski dimension* of E is defined as

$$\overline{\dim}_M E = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon, E)}{\log(1/\epsilon)}.$$

When these two quantities are equal, we say that the Minkowski dimension exists and write

$$\dim_M E = \underline{\dim}_M E = \overline{\dim}_M E.$$

Let us briefly discuss two equivalent definitions of the Minkowski dimensions. First, we call

$$P(\epsilon, E) = \sup\{k \in \mathbb{N} : \text{there are } k \text{ disjoint balls of radius } \epsilon \text{ with centres in } E\}$$

the *packing number* of E with balls of radius ϵ. It is possible to check that

$$N(4\epsilon, E) \leq P(\epsilon, E) \leq N(\epsilon, E),$$

thereby showing that $P(-, E)$ and $N(-, E)$ have the same log asymptotics and that the packing number may be used instead of the covering number to produce an equivalent definition of the Minkowski dimension.

The second equivalent definition that we discuss is more helpful for the study of the heat content. We call

$$E_\epsilon = \{x \in \mathbb{R}^d : \operatorname{dist}(x, E) \leq \epsilon\},$$

the *ϵ-tubular neighbourhood* of E. It can be shown that

(2.1)

$$\underline{\dim}_M E = d - \limsup_{\epsilon \to 0} \frac{\log \operatorname{vol}_d(E_\epsilon)}{\log \epsilon} \quad \text{and} \quad \overline{\dim}_M E = d - \liminf_{\epsilon \to 0} \frac{\log \operatorname{vol}_d(E_\epsilon)}{\log \epsilon}.$$

When the Minkowski dimension γ of E exists, we call

$$\tilde{\mathcal{M}}_*(E) = \liminf_{\epsilon \to 0} \frac{\operatorname{vol}_d(E_\epsilon)}{\epsilon^{d-\gamma}}, \quad \text{respectively} \quad \tilde{\mathcal{M}}^*(E) = \limsup_{\epsilon \to 0} \frac{\operatorname{vol}_d(E_\epsilon)}{\epsilon^{d-\gamma}}, \tag{2.2}$$

the *lower*, respectively *upper*, *Minkowski content* of E. When these two quantities are equal we say that the Minkowski content exists and write

$$\tilde{\mathcal{M}}(E) = \tilde{\mathcal{M}}_*(E) = \tilde{\mathcal{M}}^*(E).$$

Note that the Minkowski dimension is not countably stable; it is elementary to check that the countable set

$$\{0\} \cup \{1/n : n \in \mathbb{N}\}$$

has Minkowski dimension $1/2$.

We always have the following relation for the Hausdorff and Minkowski dimensions

$$\dim E \leq \underline{\dim}_M E \leq \overline{\dim}_M E,$$

where the inequalities may all be strict.

Inner Minkowski dimension and content of the boundary of a bounded domain. It is clear from the definition of the Minkowski dimension described in (2.1) and the definition of the Minkowski content that these notions capture how the volume of the tubular neighbourhoods grows. In our applications, the sets of interest are boundaries of bounded open sets. To understand the behaviour of the heat content, the interesting quantity is the contribution of the points inside the open set to the volume of the tubular neighbourhoods; this motivates the following definitions inspired by [**BC86, vdB94**].

For a bounded open subset U of $\mathbb{R}^d$ we call $(\partial U)_\epsilon \cap U$ the *inner ϵ-tubular neighbourhood* of U. Since the volume of these sets will be used often, we will use the notation

$$\mu_U(\epsilon) = \mathrm{vol}_d((\partial U)_\epsilon \cap U).$$

We will omit the dependence on U from the notation when there is no risk of confusion.

As a natural adaptation of the definitions above, we call

$$d - \limsup_{\epsilon \to 0} \frac{\log \mu(\epsilon)}{\log \epsilon}, \quad \text{respectively} \quad d - \liminf_{\epsilon \to 0} \frac{\log \mu(\epsilon)}{\log \epsilon}$$

the *inner lower*, respectively *upper*, *Minkowski dimension* of ∂U. In all the examples discussed here, the inner Minkowski dimensions will coincide with the usual definition and so we will use the same notation as above with the same convention when the two are equal. However, when the inner Minkowski dimension γ of ∂U exists, the *inner lower*, respectively *upper*, *Minkowski content* of ∂U defined as

$$\mathcal{M}_*(\partial U) = \liminf_{\epsilon \to 0} \frac{\mu(\epsilon)}{\epsilon^{d-\gamma}}, \quad \text{respectively} \quad \mathcal{M}^*(\partial U) = \limsup_{\epsilon \to 0} \frac{\mu(\epsilon)}{\epsilon^{d-\gamma}},$$

typically differ from the definitions in (2.2). When these two quantities are equal we say that the inner Minkowski content of ∂U exists and write

$$\mathcal{M}(\partial U) = \mathcal{M}_*(\partial U) = \mathcal{M}^*(\partial U).$$

2.3. Tubular neighbourhood of Brownian motion with a drift

As a first example, we look at the volume of the tubular neighbourhoods of Brownian motion with a drift. The result is taken from [**CPS14**].

THEOREM 2.3.1. *Let B be Brownian motion in $\mathbb{R}^d$, let A be a bounded subset of $[0, \infty)$ and let $f : [0, \infty) \to \mathbb{R}^d$ be a continuous function. Write*

$$V(\epsilon, A) = \mathrm{vol}_d(\{x \in \mathbb{R}^d : \mathrm{dist}(x, (B + f)(A)) \leq \epsilon\}).$$

Then, almost surely,

$$\liminf_{\epsilon \to 0} \frac{\log V(\epsilon, A)}{\log \epsilon} = \liminf_{\epsilon \to 0} \frac{\log \mathbf{E} V(\epsilon, A)}{\log \epsilon}$$

and

$$\limsup_{\epsilon\to 0}\frac{\log V(\epsilon,A)}{\log\epsilon}=\limsup_{\epsilon\to 0}\frac{\log \mathbf{E}V(\epsilon,A)}{\log\epsilon}.$$

This theorem means that the log asymptotics of the volume of the tubular neighbourhoods of Brownian motion with a drift satisfy some kind of strong law of large numbers. In particular, the upper and lower Minkowski dimensions are constant almost surely. This makes intuitive sense when one considers Lévy's construction of Brownian motion; the same intuition motivates our construction of random fractals in the rest of this text.

In the proof, we will use the following lemma.

LEMMA 2.3.2. *Under the assumptions of Theorem 2.3.1, we have*

$$\mathbf{E}V(\epsilon,A)^2\le 2[\mathbf{E}V(\epsilon,A)]^2.$$

PROOF. For $x\in\mathbb{R}^d$, define the stopping time

$$\tau_x=\inf\{t\in A: B(t)+f(t)\in\bar{\mathcal{B}}(x,\epsilon)\},$$

where $\bar{\mathcal{B}}(x,\epsilon)$ denotes the closed ball of radius ϵ centred at x and we use the convention that the infimum over the empty set is ∞. Since $\tau_x<\infty$ if and only if $B+f$ hits $\bar{\mathcal{B}}(x,\epsilon)$ on A, we have

$$\begin{aligned}\mathbf{E}V(\epsilon,A)^2&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{P}(\tau_x<\infty,\tau_y<\infty)dydx\\&=2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{P}(\tau_x\le\tau_y<\infty)dydx\\&=2\int_{\mathbb{R}^d}\mathbf{P}(\tau_x<\infty)\int_{\mathbb{R}^d}\mathbf{P}(\tau_x\le\tau_y<\infty|\tau_x<\infty)dydx\\&=2\int_{\mathbb{R}^d}\mathbf{P}(\tau_x<\infty)\mathbf{E}[V(\epsilon,A\cap[\tau_x,\infty))|\tau_x<\infty]dx.\end{aligned}\tag{2.3}$$

By the Markov property of Brownian motion, on $\{\tau_x<\infty\}$, putting $B(t+\tau_x)-B(\tau_x)$ defines another Brownian motion independent $\mathcal{F}_{\tau_x}$. Using this, it is easy

to check that if W is another Brownian motion independent of B. Then, by translation invariance of the Lebesgue measure,

$$\begin{aligned}
&\mathbf{E}[V(\epsilon, A \cap [\tau_x, \infty))|\tau_x < \infty] \\
&\quad = \mathbf{E}[\mathrm{vol}_d(\{x \in \mathbb{R}^d : \mathrm{dist}(x, (W+f)(A \cap [\tau_x, \infty))) \leq \epsilon\})|\tau_x < \infty] \\
&\quad \leq \mathbf{E}V(\epsilon, A).
\end{aligned}$$

Inserting this into (2.3) produces the desired inequality. □

We will also require the Paley-Zygmund inequality which we recall for convenience.

THEOREM 2.3.3 (Paley-Zygmund). *Let Z be a non negative random variable with finite second moment and $\theta \in (0,1)$. Then,*

$$\mathbf{P}(Z \geq \theta \mathbf{E}Z) \geq (1-\theta)^2 \frac{(\mathbf{E}Z^2)}{\mathbf{E}(Z^2)}.$$

We may now proceed with the proof of the theorem.

PROOF OF THEOREM 2.3.1. To get the required result, it is sufficient to check that, almost surely,

$$\frac{1}{2k}\mathbf{E}V(2^{-k}, A) \leq V(2^{-k}, A) \leq k^2 \mathbf{E}V(2^{-k}, A), \quad \text{eventually.} \tag{2.4}$$

The second inequality follows from Markov's inequality and the Borel-Cantelli lemma. So we now establish the first inequality.

It is easy to check that, for every ϵ, the function

$$t \mapsto \mathbf{E}V(\epsilon, A \cap [0,t])$$

is continuous. For $k \in \mathbb{N}$ and $j \leq k$, we may therefore define

$$t_j = \inf\left\{t \geq 0 : \mathbf{E}V(2^{-k}, A \cap [0,t]) = \frac{j}{k}\mathbf{E}V(2^{-k}, A)\right\}.$$

By countable subadditivity, for $j \in \{1, \dots, k\}$,

$$\begin{aligned}\frac{1}{k}\mathbf{E}V(2^{-k}, A) &= \mathbf{E}V(2^{-k}, A \cap [0, t_j]) - \mathbf{E}V(2^{-k}, A \cap [0, t_{j-1}]) \\ &\leq \mathbf{E}V(2^{-k}, A \cap [t_{j-1}, t_j]).\end{aligned}$$

It follows that

$$\begin{aligned}&\mathbf{P}\left(V(2^{-k}, A) \leq \frac{1}{2k}\mathbf{E}V(2^{-k}, A)\right) \\ &\leq \mathbf{P}\left(V(2^{-k}, A \cap [t_{j-1}, t_j]) \leq \frac{1}{2}\mathbf{E}V(2^{-k}, A \cap [t_{j-1}, t_j]), \text{ for all } j\right) \\ &= \prod_{j=1}^{k}\left(V(2^{-k}, A \cap [t_{j-1}, t_j]) \leq \frac{1}{2}\mathbf{E}V(2^{-k}, A \cap [t_{j-1}, t_j])\right) \\ &\leq \left(\frac{7}{8}\right)^k,\end{aligned}$$

where we used the independence of the increments of Brownian motion to get the equality and the Paley-Zygmund inequality with Lemma 2.3.2 to get the last inequality.

The desired inequality in (2.4) now follows from the Borel-Cantelli lemma again and the proof is complete. □

CHAPTER 3

Heat content estimates

In this chapter, we derive bounds on the heat content using probabilistic techniques and discuss an application to a domain with a boundary whose Hausdorff and Minkowski dimensions disagree. The results presented here are taken from [**Cha14**].

3.1. The estimates

It will be convenient to be able to solve the heat equation under the conditions in (1.1) using the probabilistic representation

$$u(s,x) = \mathbf{P}_x(T_{D^c} \leq s), \tag{3.1}$$

where T_{D^c} is the hitting time of D^c of Brownian motion and $\mathbf{P}_x$ is the law of Brownian motion started at x. Therefore, we will always make the following assumption.

Assumption 3.1.1. *All the points of ∂D are regular for D^c.*

Recall that this assumption is always satisfied for simply connected planar domains; see Proposition II.1.14 of [**Bas95**]. This will cover all the examples discussed here.

We now prove a first upper bound for the heat content following the argument of [**vdB94**].

THEOREM 3.1.2. *Let D be a bounded domain in $\mathbb{R}^d$ and let $\omega : [0,\infty) \to [0,\infty)$ be an increasing function with $\omega(0) = 0$. Then, for every $s \geq 0$,*

$$E(s) \leq \mu(\omega(s)) + 2^{(d+2)/2}\mathrm{vol}_d(D)e^{-\omega(s)^2/4s}.$$

PROOF. Let $s \geq 0$ and put

$$A_s = \{x \in D : \mathrm{dist}(x, \partial D) \leq \omega(s)\}.$$

By (3.1), we have

$$\begin{aligned} E(s) &= \int_{A_s} \mathbf{P}_x(T_{D^c} \leq s)dx + \int_{D\setminus A_s} \mathbf{P}_x(T_{D^c} \leq s)dx \\ &\leq \mu(\omega(s)) + \int_{D\setminus A_s} \mathbf{P}_x(T_{D^c} \leq s)dx. \end{aligned} \tag{3.2}$$

But now, by Lévy's maximal inequality, e.g. Theorem 3.6.5 of [**Sim05**],

$$\begin{aligned} \mathbf{P}_x(T_{D^c} \leq s) &\leq \mathbf{P}_x(T_{\mathcal{B}(x,\mathrm{dist}(x,\partial D))^c} \leq s) \\ &= \mathbf{P}_0\left(\sup_{0\leq u\leq s} |B(u)| \geq \mathrm{dist}(x,\partial D)\right) \\ &\leq 2\mathbf{P}_0(|B(s)| \geq \mathrm{dist}(x,\partial D)) \\ &= 2(2\pi s)^{-d/2}\int_{|y|\geq \mathrm{dist}(x,\partial D)} e^{-|y|^2/2s}dy, \end{aligned}$$

where $\mathcal{B}(x,\delta)$ denotes the open ball of radius δ centred at x. Changing variables and standard Gaussian estimates then yield

$$\mathbf{P}_x(T_{D^c} \leq s) \leq 2^{(d+2)/2}e^{-\mathrm{dist}(x,\partial D)^2/4s}. \tag{3.3}$$

Since $\mathrm{dist}(x,\partial D) > \omega(s)$ for $x \in D \setminus A_s$, using this estimate in (3.2) completes the proof. □

The other estimates that we will use are those of Theorems 1.2 to 1.4 of [**vdB94**] which we recall here for convenience.

THEOREM 3.1.3 (van den Berg). *Let D be a bounded domain in $\mathbb{R}^d$. Then, for every $s \geq 0$,*

$$E(s) \leq 2^{d/2} s^{-1} \int_0^\infty \epsilon e^{-\epsilon^2/4s} \mu(\epsilon) d\epsilon.$$

The lower bound for the heat content is proved under a capacitary condition that we state now; see [**BC86, vdB94**] for more background. We write $\mathrm{Cap}(A)$ for the Newtonian capacity of the set A. That is, for a subset A of $\mathbb{R}^2$,

$$\mathrm{Cap}(A) = \exp\left[- \inf \left\{ \int_A \int_A g(x,y) \mu(dx) \mu(dy) : \mu(A) = 1 \right\} \right],$$

where

$$g(x,y) = -\log|y-x|;$$

and, for a subset A of $\mathbb{R}^d$ with $d \geq 3$,

$$\mathrm{Cap}(A) = \left[\inf \left\{ \int_A \int_A h(x,y) \mu(dx) \mu(dy) : \mu(A) = 1 \right\} \right]^{-1},$$

where

$$h(x,y) = |y-x|^{2-d}.$$

So, in both cases, we are taking the infimum over the set of all probability measures on A.

ASSUMPTION 3.1.4 (Capacitary density). *For the bounded domain D of $\mathbb{R}^d$ with $d \geq 2$, there exists a positive constant c such that, for all $x \in D$ and $r \in (0, \operatorname{diam} D)$,*

$$\mathrm{Cap}(\mathcal{B}(x,r) \cap \partial D) \geq c \, \mathrm{Cap}(\mathcal{B}(x,r)).$$

This assumption is usual in these problems and we say that the *capacitary density of ∂D is bounded below* when it is satisfied. This is always the case when D is a simply connected planar domain; see [**vdB94**].

THEOREM 3.1.5 (van den Berg). *Let D be a bounded domain of $\mathbb{R}^d$ and assume that either $d = 1$ or $d \geq 2$ and the capacitary density of ∂D is bounded below. Then, for every $s \geq 0$,*

$$E(s) \geq c_1 \mu(c_2 s^{1/2}).$$

These results illustrate the role of the Minkowski dimension in establishing a lower bound on $E(s)$ for small s. Furthermore, the change of variables $\eta = \epsilon^2/4$ yields

$$E(s) \leq 2^{(d+2)/2} s^{-1} \int_0^\infty e^{-\eta/s} \mu(2\eta^{1/2}) d\eta. \tag{3.4}$$

If $\mu(\epsilon)$ does not oscillate too much for small ϵ, then an Abelian theorem can be used to control the behaviour of $E(s)$ for small s as we show now.

THEOREM 3.1.6. *Let D be a bounded domain of $\mathbb{R}^d$ and assume that either $d = 1$ or $d \geq 2$ and the capacitary density of ∂D is bounded below. Assume further that there exist $\gamma \in (0, \infty)$ and a slowly varying function L such that*

$$\mu(\epsilon) = \epsilon^{d-\gamma} L(1/\epsilon)$$

for small ϵ. Then,

$$E(s) \asymp s^{(d-\gamma)/2} \tilde{L}(1/s)$$

for small s, where $\tilde{L}$ is the slowly varying function defined by

$$\tilde{L}(x) = L(x^{1/2}/2).$$

PROOF. Define

$$\tilde{\mu}(\eta) = \mu(2\eta^{1/2}) = 2^{d-\gamma} \eta^{(d-\gamma)/2} \tilde{L}(1/\eta).$$

Using (3.4), an integration by parts and applying Theorem XIII.5.3 in [**Fel68**], we have

$$E(s) \leq c_3 s^{-1} \int_0^\infty e^{-\eta/s} \tilde{\mu}(\eta) d\eta = c_3 \int_0^\infty e^{-\eta/s} \tilde{\mu}(d\eta) \sim c_4 s^{(d-\gamma)/2} \tilde{L}(1/s),$$

as $s \to 0$, where the notation $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$. A similar lower bound follows immediately using Theorem 3.1.5. □

When $\mu(\epsilon)$ oscillates too wildly, for example when the Minkowski dimension does not exist, the Abelian theorem is not applicable. But we can then rely on Theorem 3.1.2 to study the short time asymptotics of the heat content and get the following announced result.

THEOREM 3.1.7. *Let D be a bounded open subset of $\mathbb{R}^d$ and assume that either $d = 1$ or $d \geq 2$ and the capacitary density of ∂D is bounded below. Then,*

$$\liminf_{s \to 0} \left(\frac{d}{2} - \frac{\log E(s)}{\log s} \right) = \frac{1}{2} \underline{\dim}_M \partial D$$

and

$$\limsup_{s \to 0} \left(\frac{d}{2} - \frac{\log E(s)}{\log s} \right) = \frac{1}{2} \overline{\dim}_M \partial D.$$

PROOF. Put $\alpha = \underline{\dim}_M \partial D$ and $\beta = \overline{\dim}_M \partial D$ and let $\delta \in (0, \infty)$ be small.

By definition of the Minkowski dimension, we have, on the one hand, that there exists a sequence $(\epsilon_n, n \in \mathbb{N})$ converging to 0 along which

$$\mu(\epsilon_n) \leq \epsilon_n^{d-(\alpha+\delta)}$$

and, on the other hand, that

$$\mu(\epsilon) \geq \epsilon^{d-(\alpha-\delta)}$$

for ϵ small enough.

Setting $\omega(s) = \sqrt{2ds\log(1/s)}$, which is increasing around 0 and putting $\omega(s_n) = \epsilon_n$ together with Theorems 3.1.2 and 3.1.5 shows that

$$E(s_n) \leq s_n^{(d-(\alpha+\delta))/2} \log(1/s_n)^{(d-(\alpha+\delta))/2} + c_5 s_n^{d/2}$$

and

$$E(s) \geq c_6 s^{(d-(\alpha-\delta))/2}$$

for s small enough. The result for the liminf follows.

The proof for the limsup is similar, and somewhat simpler, relying on Theorems 3.1.3 and 3.1.5. □

3.2. Self-affine boundaries

We conclude this section with a brief mention of a domain whose boundary's Hausdorff and Minkowski dimensions disagree and where the results discussed here can be applied. This is an alternative example to that of [**BC86**] showing that the Minkowski dimension is the relevant measure of roughness for heat conduction problems; in the example presented here, however, the domain is connected.

Following [**PS13**], we define the boundary of the domain using the self-affine carpets of Bedford [**Bed84**] and McMullen [**McM84**] whose construction we recall briefly now. Let $1 < m < n$ be two integers. Divide the unit square $[0,1]^2$ into mn rectangles of height m^{-1} and width n^{-1}. Keep some rectangles according to a pattern P and discard the others. This produces a compact set K_1. For each rectangle of the pattern, repeat the procedure. This produces a compact subset K_2 of K_1. Continue indefinitely to get a compact set

$$K(P) = \bigcap_{j=1}^{\infty} K_j.$$

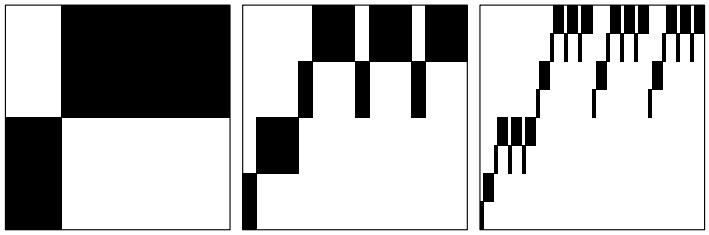


FIGURE 1. First 3 steps of the self-affine carpet for pattern A.

A natural way to represent a pattern is to use an $m \times n$ matrix with entries in $\{0, 1\}$ where each 1 indicates a rectangle that we choose to keep. The carpet corresponding to pattern P is then

$$K(P) = \left\{ \sum_{k=1}^{\infty} (a_k n^{-k}, b_k m^{-k}) : (a_k, b_k) \in D \right\},$$

where $D = \{(i, j) : P(j, i) = 1\}$. Here, the rows and columns of P are numbered starting from 0, and the rows are numbered from bottom to top. As an illustration, the first 3 iterations corresponding to the pattern

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

are shown in Figure 1.

The Hausdorff dimension of the self-affine carpet built from pattern P is

$$\log_m \left(\sum_{j=0}^{m-1} r(j)^{\log_n m} \right),$$

where $r(j)$ is the number of chosen rectangles of the pattern in row j, while its Minkowski dimension (which exists) is given by

$$1 + \log_n \left(\frac{1}{m} \sum_{j=0}^{m-1} r(j) \right);$$

see [**Bed84, McM84, Per94**].

Of course, the set $K(P)$ is not a continuous curve in general. But, for appropriate patterns, we can alter the construction of the carpet using the

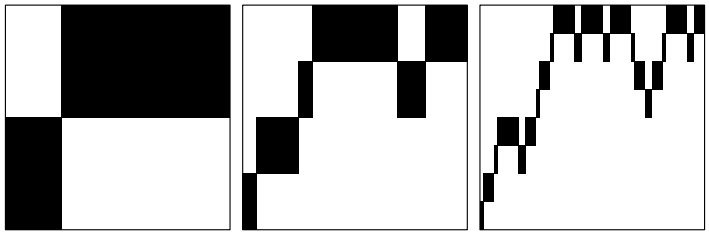


FIGURE 2. First 3 steps of the continuous self-affine curve for pattern A.

reflected pattern

$$P^r(i,j) = P(i, n-1-j)$$

to create a set that is the graph of a continuous function. We describe how to do this with pattern A, which was just defined. A similar procedure can readily be used for similar patterns to create graphs of continuous functions.

As in the construction of $K(A)$, start with pattern A. Reproduce reflected pattern A^r in the chosen rectangle of column 2 and A in the others. Repeat this procedure for chosen rectangles inside a pattern A; and, inside a pattern A^r, reproduce pattern A for the chosen rectangle of column 1 and A^r for the others. Continue indefinitely to get a compact set $K_c(A)$; this set is the graph of a continuous function $f_A : [0,1] \to [0,1]$. The first 3 iterations are shown in Figure 2. Examining the calculations in [**Per94**] shows that this alteration of the construction does not change the Hausdorff or the Minkowski dimension; see also [**PS13**].

Now define the continuous function $g : [0,2] \to [1,3]$ by

$$g(t) = 1 + f(t)\mathbf{1}_{[0,1]}(t) + (2 - f(2-t))\mathbf{1}_{(1,2]}(t),$$

and let D be the simply connected domain inside the Jordan curve defined by

$$\{(t, g(t)) : t \in [0,2]\} \cup (\{0\} \times [0,1]) \cup ([0,2] \times \{0\}) \cup (\{2\} \times [0,3]).$$

Furthermore, put

$$\nu(\epsilon) = \mathrm{vol}_2(\{x \in \mathbb{R}^2 : \mathrm{dist}(x, \{(t, g(t)) : t \in [0,2]\}) \leq \epsilon\}).$$

Because of the symmetry of g, we have

$$\nu(\epsilon) = 2\mathrm{vol}_2(\{x \in D : \mathrm{dist}(x, \{(t, g(t)) : t \in [0,2]\}) \leq \epsilon\}) + O(\epsilon^2),$$

and therefore

$$\nu(\epsilon) = 2\,\mu(\epsilon) + O(\epsilon^2).$$

It follows that the inner Minkowski dimension of ∂D is equal to the Minkowski dimension of $K_c(A)$, which is different from its Hausdorff dimension. Also, since D is simply connected, Theorem 3.1.7 can be used to show that

$$\lim_{s\to 0}\left(1 - \frac{\log E(s)}{\log s}\right) = \frac{1}{2} + \frac{1}{2}\log_n\left(\frac{1}{m}\sum_{j=0}^{m-1} r(j)\right).$$

3.3. Heat content of the unit interval

For the discussion in Chapter 7, we will need to understand the heat content of the unit interval $I = (0,1)$. We calculate it now using that the heat kernel on I with absorption on the boundary is known explicitly.

We have

$$p_I(s,x,y) = \frac{1}{2\sqrt{2\pi s}}\sum_{n=-\infty}^{\infty}\left(e^{-\frac{(x-y+2n)^2}{2s}} - e^{-\frac{(x+y+2n)^2}{2s}}\right),$$

see [**BS02**]. Therefore,

$$E(s) = \int_0^1 u(s,x)dx = 1 - \int_0^1\int_0^1 p_I(s,x,y)dxdy.$$

Recall that the reflected heat kernel, see [**BS02**],

$$p_I^r(s,x,y) = \frac{1}{2\sqrt{2\pi s}}\sum_{n=-\infty}^{\infty}\left(e^{-\frac{(x-y+2n)^2}{2s}} + e^{-\frac{(x+y+2n)^2}{2s}}\right),$$

satisfies

$$\int_0^1\int_0^1 p_I^r(s,x,y)dxdy = 1.$$

It follows that

$$E(s) = \frac{1}{\sqrt{2\pi s}} \sum_{n=-\infty}^{\infty} \int_0^1 \int_0^1 e^{-\frac{(x+y+2n)^2}{2s}} dxdy.$$

We will now rewrite this expression more explicitly as an infinite series and use this to deduce the short time asymptotics of the heat content. Notice that

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi s}} \int_0^1 \int_0^1 e^{-\frac{(x+y+2n)^2}{2s}} dxdy \\
&\quad = \int_0^1 dx \int_{x+2n}^{x+2n+1} dy \frac{e^{-y^2/2s}}{\sqrt{2\pi s}} \\
&\quad = \int_{2n}^{2n+1} dy \int_0^{y-2n} dx \frac{e^{-y^2/2s}}{\sqrt{2\pi s}} + \int_{2n+1}^{2n+2} dy \int_{y-(2n+1)}^1 dx \frac{e^{-y^2/2s}}{\sqrt{2\pi s}} \\
&\quad = \int_{2n}^{2n+1} (y-2n) \frac{e^{-y^2/2s}}{\sqrt{2\pi s}} dy + \int_{2n+1}^{2n+2} (2n+2-y) \frac{e^{-y^2/2s}}{\sqrt{2\pi s}} dy,
\end{aligned} \tag{3.5}$$

by drawing a picture of the domains of integration. Since

$$\begin{aligned}
\int_b^{b+1} (y-b) \frac{e^{-y^2/2s}}{\sqrt{2\pi s}} dy =& \sqrt{\frac{s}{2\pi}} (e^{-b^2/2s} - e^{-(b+1)^2/2s}) \\
&+ b \left(\Phi\left(\frac{b}{\sqrt{s}}\right) - \Phi\left(\frac{b+1}{\sqrt{s}}\right) \right)
\end{aligned}$$

and

$$\begin{aligned}
\int_{b-1}^{b} (b-y) \frac{e^{-y^2/2s}}{\sqrt{2\pi s}} dy =& \sqrt{\frac{s}{2\pi}} (e^{-b^2/2s} - e^{-(b-1)^2/2s}) \\
&+ b \left(\Phi\left(\frac{b}{\sqrt{s}}\right) - \Phi\left(\frac{b-1}{\sqrt{s}}\right) \right),
\end{aligned}$$

we get that the expression in (3.5) is

$$\sqrt{\frac{s}{2\pi}} (e^{-(2n)^2/2s} - 2e^{-(2n+1)^2/2s} + e^{-(2n+2)^2/2s}) \tag{3.6}$$

$$+ 2n \left(\Phi\left(\frac{2n}{\sqrt{s}}\right) - \Phi\left(\frac{2n+1}{\sqrt{s}}\right) \right) + (2n+2) \left(\Phi\left(\frac{2n+2}{\sqrt{s}}\right) - \Phi\left(\frac{2n+1}{\sqrt{s}}\right) \right). \tag{3.7}$$

Summing (3.6) over $n \in \mathbb{Z}$ yields

$$\sqrt{\frac{2s}{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-n^2/2s} = \sqrt{\frac{2s}{\pi}} \left(1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2/2s} \right).$$

And summing (3.7) over $n \in \mathbb{Z}$ yields

$$\sum_{n=-\infty}^{\infty} 2n\left(\Phi\left(-\frac{2n+1}{\sqrt{s}}\right)+\Phi\left(-\frac{2n-1}{\sqrt{s}}\right)-2\Phi\left(-\frac{2n}{\sqrt{s}}\right)\right)$$
$$= 4\sum_{n=1}^{\infty}(-1)^{n+1}n\Phi\left(-\frac{n}{\sqrt{s}}\right),$$

where we used that $\Phi(-x) = 1 - \Phi(x)$.

We have just shown that

$$E(s) = \sqrt{\frac{2s}{\pi}}\left(1+2\sum_{n=1}^{\infty}(-1)^n e^{-n^2/2s}\right) - 4\sum_{n=1}^{\infty}(-1)^n n\Phi\left(-\frac{n}{\sqrt{s}}\right). \tag{3.8}$$

To get the short time asymptotics of the heat content, we use the formula

$$\Phi(-x) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}-\frac{1}{x^3}+\frac{3}{x^5}-\cdots\right)e^{-x^2/2},$$

which is obtained by iterated integration by parts upon noticing that

$$F_n(x) = \int_x^{\infty} y^{-n}e^{-y^2/2}dy$$

satisfies

$$F_n(x) = x^{-(n+1)}e^{x^2/2} - (n+1)F_{n+2}(x).$$

Therefore,

$$\Phi\left(-\frac{n}{\sqrt{s}}\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{s^{1/2}}{n}-\frac{s^{3/2}}{n^2}+\frac{3s^{5/2}}{n^5}-\cdots\right)e^{-n^2/2s},$$

which, inserted into (3.8), yields

$$\begin{aligned} E(s) &= \sqrt{\frac{2s}{\pi}} + 2\sqrt{\frac{2s}{\pi}}\sum_{n=1}^{\infty}(-1)^n e^{-n^2/2s}\left(\frac{s}{n^2}-\frac{3s^2}{n^4}+O\left(\frac{s^3}{n^6}\right)\right) \\ &= \sqrt{\frac{2}{\pi}}s^{1/2} - 2\sqrt{\frac{2}{\pi}}s^{3/2}e^{-1/2s} + o(s^{3/2}e^{-1/2s}), \end{aligned} \tag{3.9}$$

as $s \to 0$.

3.4. Heat content of the triadic Koch snowflake

Another domain of interest to our discussion in Chapters 4 and 8 is the triadic Koch snowflake. Its heat content asymptotics are studied in [**FLV95**], where the authors use renewal theory to prove the following result.

THEOREM 3.4.1. *Let D be the triadic Koch snowflake. Then,*

$$E_D(s) = p(\log s)s^{(2-\gamma)/2} + q(\log s)s + O(e^{-r/s}),$$

as $s \to 0$, where p and q are continuous $\log 9$-periodic functions, $\gamma = \log_3 4$ is the Minkowski dimension of ∂D and r is a positive constant.

To the best of my knowledge, it is not known whether or not the functions p and q are constant.

CHAPTER 4

Spatially homogeneous snowflakes

In this chapter, we define a class of Koch snowflakes with spatial homogeneity and discuss their geometric and analytic properties. This chapter is taken from [**Cha14**].

4.1. Construction

We define spatially homogeneous random snowflakes by generalising the Koch curve. In the construction of the usual, triadic, Koch curve, the segment $[0, 1]$ is replaced by a curve $K(1)$, say, made of 4 segments of length $1/3$ arranged to produce a spike in the middle. This procedure is then iterated and produces a limiting self-similar curve.

Here we proceed similarly, but using different building blocks with different numbers of spikes. More precisely, for $a \in A$, a bounded subset of $\mathbb{N}$, put

$$m(a) = 3a + 1 \quad \text{and} \quad \ell(a) = 2a + 1,$$

and let $K(a)$ be the curve made of $m(a)$ segments of length $\ell(a)^{-1}$ arranged to produce a spikes as depicted in Figure 1. Now, let $\xi = (\xi_n, n \in \mathbb{N})$ be a sequence of elements of A. We will write $K(\xi)$ for the Koch curve where we used $K(\xi_n)$ as a building block at iteration n. For example, the curve formed from the first iterations for $\xi = (1, 3, 2, 1, \dots)$ is shown in Figure 2.



FIGURE 1. The curves $K(2)$ and $K(3)$.

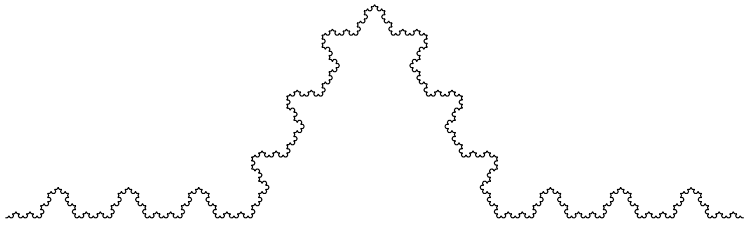


FIGURE 2. First iterations of $K(1, 3, 2, 1, \dots)$.

It will be convenient to use the notation

$$\ell_n = \ell(\xi_n), \quad m_n = m(\xi_n), \quad \epsilon_n^{-1} = L_n = \prod_{i=1}^{n} \ell_i \quad \text{and} \quad M_n = \prod_{i=1}^{n} m_i.$$

With these definitions, the nth iteration in the construction of $K(\xi)$ consists of M_n segments of length $\epsilon_n = L_n^{-1}$.

The domain $D = D(\xi)$ in which we are interested is the one enclosed by the Jordan curve made of three copies of $K(\xi)$ arranged as in the construction of the usual Koch snowflake. In particular, D is simply connected.

4.2. Fractal dimension and content

Since A is bounded, we have

$$\epsilon_n \asymp \epsilon_{n+1} \quad \text{and} \quad M_n \asymp N(\epsilon_n, K(\xi)) \asymp N(\epsilon_{n+1}, K(\xi)), \tag{4.1}$$

where $N(\epsilon, K)$ is the covering number of K by balls of radius ϵ. This observation and the mass distribution principle enable us to calculate the dimension of $K(\xi)$.

THEOREM 4.2.1. *The Hausdorff and Minkowski dimensions of the set $K(\xi)$ are given by*

$$\dim K(\xi) = \underline{\dim}_M K(\xi) = \liminf_{n\to\infty} \frac{\log M_n}{\log L_n}$$

and

$$\overline{\dim}_M K(\xi) = \limsup_{n\to\infty} \frac{\log M_n}{\log L_n}.$$

PROOF. The calculation of the upper and lower Minkowski dimension follows directly from (4.1).

To simplify the notation for the calculation of the Hausdorff dimension, put $\alpha = \underline{\dim}_M K(\xi)$. It is standard that $\dim K(\xi) \le \alpha$. To prove the other inequality, let $\delta \in (0, \infty)$. By (4.1), there exists m large such that, for every $n \ge m$, we have $M_n^{-1} \le \epsilon_n^{\alpha-\delta}$. Furthermore, the set $K(\xi)$ is made of M_m (disjoint up to $M_m - 1$ points) copies of the set

$$\tilde{K} = L_m^{-1} K(\xi_{m+1}, \xi_{m+2}, \dots).$$

Now consider the Borel measure ν assigning mass M_n^{-1} to sets of size L_n^{-1} in the obvious way for $n \ge m$. Let U be a subset of $\tilde{K}$ and let n be such that $\epsilon_{n+1} \le \operatorname{diam} U < \epsilon_n$. Then, we have

$$\nu(U) \le 3M_n^{-1} \le 3\epsilon_n^{\alpha-\delta} \le c\epsilon_{n+1}^{\alpha-\delta} \le c(\operatorname{diam} U)^{\alpha-\delta},$$

for some positive constant c, thanks to (4.1).

By the mass distribution principle, Theorem 2.1.1, it follows that $\tilde{K}$, and hence $K(\xi)$, has Hausdorff dimension at least $\alpha - \delta$. Since δ is arbitrary, this completes the proof. □

To continue the geometric description of the spatially homogeneous snowflakes, we now look at the volume of the inner tubular neighbourhoods of D. The following result immediately implies that the inner Minkowski dimensions agree with the usual definitions and calculations of Theorem 4.2.1.

LEMMA 4.2.2. *The volume of the inner tubular neighbourhoods of D satisfies*

$$\mu(\epsilon_n) \asymp M_n L_n^{-2}.$$

PROOF. On the one hand, the volume of the inner ϵ_n-tubular neighbourhood in one third of the snowflake is bounded below by the volume of the level n spike of a third of the boundary, i.e. a copy of $K(\xi)$. This means that

$$\frac{1}{3}\mu(\epsilon_n) \geq \frac{\sqrt{3}}{4} M_{n-1}\xi_n\epsilon_n^2 \asymp M_n\epsilon_n^2,$$

where we used that A is bounded.

On the other hand, this same quantity is bounded above by the volume of the set

$$\{x \in \mathbb{R}^d : \operatorname{dist}(x, K(\xi_1, \ldots, \xi_n)) \leq \epsilon_n\},$$

where $K(\xi_1, \ldots, \xi_n)$ is the curve obtained at the nth iteration in the construction of the $K(\xi)$. This means that

$$\frac{1}{3}\mu(\epsilon_n) \leq 6M_n\epsilon_n^2.$$

The result follows. □

Notice that,

$$\log L_n = n\sum_{a\in A} p_a(n)\log \ell(a) \quad \text{and} \quad \log M_n = n\sum_{a\in A} p_a(n)\log m(a),$$

where $(p_a(n), a \in A)$ is the empirical distribution for the proportions of the elements of A, i.e.

$$p_a(n) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\xi_i = a}.$$

Therefore, the key to understanding the log asymptotics of the volume inner tubular neighbourhoods, and hence the fractal dimension of ∂D, is the convergence of the empirical distribution to some limiting probability measure $(p_a, a \in A)$. This observation can easily be used to produce examples such as the following one of snowflakes whose boundary does not have a Minkowski dimension.

EXAMPLE 4.2.3. *Consider the sequence $(\xi_n, n \in \mathbb{N})$ taking values in $\{1, 2\}$ defined by*

$$\xi_n = \begin{cases} 1, & n \in S, \\ 2, & n \in \mathbb{N} \setminus S, \end{cases} \qquad \text{where} \quad S = \bigcup_{n=1}^{\infty} \{2^{2n} + 1, \ldots, 2^{2n+1}\}.$$

Then, for $a \in \{1, 2\}$, it is easy to check that

$$\liminf_{n\to\infty} p_a(n) = \frac{1}{3} \quad \text{and} \quad \limsup_{n\to\infty} p_a(n) = \frac{2}{3}.$$

Therefore,

$$\liminf_{n\to\infty} \frac{\log M_n}{\log L_n} = \liminf_{n\to\infty} \frac{n^{-1}\sum_{i\le n} \log m_i}{n^{-1}\sum_{i\le n} \log \ell_i} = \frac{\frac{1}{3}\log 4 + \frac{2}{3}\log 7}{\frac{1}{3}\log 3 + \frac{2}{3}\log 5} \simeq 1.2225,$$

while

$$\limsup_{n\to\infty} \frac{\log M_n}{\log L_n} = \limsup_{n\to\infty} \frac{n^{-1}\sum_{i\le n} \log m_i}{n^{-1}\sum_{i\le n} \log \ell_i} = \frac{\frac{2}{3}\log 4 + \frac{1}{3}\log 7}{\frac{2}{3}\log 3 + \frac{1}{3}\log 5} \simeq 1.2395.$$

So the Minkowski dimension of $K(\xi)$ does not exist.

On the other hand, when the sequence $(\xi_n, n \in \mathbb{N})$ is stationary and ergodic, we have, for every $a \in A$,

$$p_a(n) \to p_a,$$

as $n \to \infty$, for some probability measure $(p_a, a \in A)$. In this case, we have the following elementary result.

THEOREM 4.2.4. *Suppose that $(\xi_n, n \in \mathbb{N})$ is stationary and ergodic. Then, the Minkowski dimension of ∂D exists, is equal to its Hausdorff dimension, and is given by*

$$\frac{\sum_{a\in A} p_a \log m(a)}{\sum_{a\in A} p_a \log \ell(a)}.$$

In a fashion inspired by [**BH97**], let us now focus on the stationary and ergodic case and study how the speed of convergence in the ergodic theorem affects the geometry of the snowflake. Assume that $p_a(n) \to p_a$ as $n \to \infty$ for some probability measure $(p_a, a \in A)$ and that

$$\sup_{a\in A} |p_a(n) - p_a| \leq n^{-1} g(n), \tag{4.2}$$

for some function g. If $p_a \in (0, 1)$ for every $a \in A$, then

$$\limsup_{n\to\infty} |np_a(n) - np_a| > 0.$$

In particular, the best rate of convergence that one can get in general is $g(n) = O(1)$. Therefore, we will always assume that g is non-decreasing.

We now show how the rate of convergence can be used to control the volume of the inner tubular neighbourhoods.

THEOREM 4.2.5. *Suppose that $(\xi_n, n \in \mathbb{N})$ is stationary and ergodic and satisfies* (4.2) *for some function g. Then,*

$$c_1 \epsilon^{2-\gamma} e^{-c_2 g(c_3 \log(1/\epsilon))} \leq \mu(\epsilon) \leq c_4 \epsilon^{2-\gamma} e^{c_2 g(c_3 \log(1/\epsilon))}$$

for small ϵ, where $\gamma = \dim_M \partial D$.

PROOF. By Lemma 4.2.2, we know that

$$\mu(\epsilon_n) \asymp \epsilon_n^{2-\gamma} M_n L_n^{-\gamma}.$$

But, using Theorem 4.2.4, note that

$$\begin{aligned}\log(M_n L_n^{-\gamma}) &= n \sum_{a \in A} [\log m(a) - \gamma \log \ell(a)] p_a(n) \\ &= n \sum_{a \in A} [\log m(a) - \gamma \log \ell(a)] [p_a(n) - p_a] \\ &= O(g(n)).\end{aligned}$$

Putting these observations together shows that

$$c_1 \epsilon_n^{2-\gamma} e^{-c_2 g(n)} \leq \mu(\epsilon_n) \leq c_4 \epsilon_n^{2-\gamma} e^{c_2 g(n)}.$$

The result follows after using that g is non-increasing, and that $\log(1/\epsilon_n) \asymp n$ and $\epsilon_n \asymp \epsilon_{n+1}$, because A is bounded. □

One of the consequences of this result is that we can only have

$$0 < \mathcal{M}_*(\partial D) \leq \mathcal{M}^*(\partial D) < \infty,$$

if the rate of convergence is the best possible, i.e. $g(n) = O(1)$. When that is not the case, but the sharpest function g is known, one can ask about the fluctuations of the volume of the inner tubular neighbourhoods between the bounds given above. A central example where this can be done is when the sequence $(\xi_n, n \in \mathbb{N})$ is i.i.d., in which case the rate of convergence

$$g(n) = \sqrt{n \log \log n}$$

is given by the law of the iterated logarithm; we discuss this case in the following theorem.

THEOREM 4.2.6. *Suppose that* $(\xi_n, n \in \mathbb{N})$ *is i.i.d. and that the distribution is non-degenerate, i.e.* $p_a \in (0,1)$ *for every* $a \in A$. *Then,*

$$c_1 \epsilon^{2-\gamma} e^{-c_2 \psi(1/\epsilon)} \leq \mu(\epsilon) \leq c_3 \epsilon^{2-\gamma} e^{c_2 \psi(1/\epsilon)}$$

for small ϵ, *where* $\gamma = \dim_M \partial D$ *and*

$$\psi(x) = \sqrt{\log x \log\log\log x}.$$

Furthermore,

$$\liminf_{\epsilon \to 0} \frac{\mu(\epsilon) e^{c_4 \psi(1/\epsilon)}}{\epsilon^{2-\gamma}} < \infty \quad \textit{and} \quad \limsup_{\epsilon \to 0} \frac{\mu(\epsilon) e^{-c_4 \psi(1/\epsilon)}}{\epsilon^{2-\gamma}} > 0.$$

PROOF. The first part of the result follows from Theorem 4.2.5 using the form of g for the i.i.d. case.

For the second part, notice that, by the law of the iterated logarithm,

$$\log(M_n L_n^{-\gamma}) \leq -\tilde{c} g(n), \text{ i.o.} \quad \text{and} \quad \log(M_n L_n^{-\gamma}) \geq \tilde{c} g(n), \text{ i.o.},$$

where

$$g(n) = \sqrt{n \log\log n}$$

and $\tilde{c}$ is a positive constant that dependsm on A and $(p_a, a \in A)$ in general. Proceeding as in the proof of Theorem 4.2.5 along the appropriate subsequences yields the result. □

4.3. Heat content asymptotics

We now study the heat content asymptotics of the spatially homogeneous snowflakes. We rely on the results of Chapter 3 which are readily applicable since the snowflakes are simply connected.

Let us start by finishing the discussion of Example 4.2.3, a case where the heat content has non-trivial log asymptotics reflecting that the Minkowski dimension of the boundary does not exist.

EXAMPLE 4.3.1. *For the snowflake constructed in Example 4.2.3, we get, by Theorem 3.1.7, that*

$$\liminf_{s\to 0}\left(\frac{d}{2}-\frac{\log E(s)}{\log s}\right)\simeq 0.6112,$$

while

$$\limsup_{s\to 0}\left(\frac{d}{2}-\frac{\log E(s)}{\log s}\right)\simeq 0.6198.$$

In other words, the log asymptotics of the heat content oscillate between those dictated by the upper and lower Minkowski dimensions.

As we did when we studied the volume of the inner tubular neighbourhood above, we now focus on the case where the sequence $(\xi_n, n\in\mathbb{N})$ is stationary and ergodic with a rate of convergence given by the regularly varying function g in (4.2).

Using Theorems 3.1.5, 3.1.2 or 3.1.6, and 4.2.5, it is straightforward to get upper and lower bounds for the heat content. Furthermore, when the volume of the inner tubular neighbourhoods oscillates between the upper and lower bounds given by g, then a similar reasoning along appropriate subsequences shows that the heat content oscillates in a similar fashion.

Notice, however, that dealing with the upper bound for the heat content is more delicate. This is because the function ω in Theorem 3.1.2 typically involves a logarithmic correction (see the proof of Theorem 3.1.7). As we now show, this is never a problem when g is not slowly varying; we postpone the discussion of the other situation to the next subsection.

THEOREM 4.3.2. *Suppose that $(\xi_n, n\in\mathbb{N})$ is stationary and ergodic and that g is such that*

$$\log x = o(g(x)),$$

as $x \to \infty$. Then,

$$c_1 s^{1-\gamma/2} e^{-c_2 g(c_3 \log(1/s))} \le E(s) \le c_4 s^{1-\gamma/2} e^{c_2 g(c_3 \log(1/s))} \tag{4.3}$$

for small s, where $\gamma = \dim_M \partial D$. Furthermore, if

$$\liminf_{\epsilon \to 0} \frac{\mu(\epsilon) e^{c_5 g(c_6 \log(1/\epsilon))}}{\epsilon^{2-\gamma}} < \infty \quad \textit{and} \quad \limsup_{\epsilon \to 0} \frac{\mu(\epsilon) e^{-c_5 g(c_6 \log(1/\epsilon))}}{\epsilon^{2-\gamma}} > 0, \tag{4.4}$$

then

$$\liminf_{s \to 0} \frac{E(s) e^{c_7 g(c_8 \log(1/s))}}{s^{1-\gamma/2}} < \infty \quad \textit{and} \quad \limsup_{s \to 0} \frac{E(s) e^{-c_7 g(c_8 \log(1/s))}}{s^{1-\gamma/2}} > 0. \tag{4.5}$$

Proof. The lower bound in (4.3) follows from Theorems 3.1.5 and 4.2.5. The limsup part of (4.4) implies that

$$\mu(\epsilon) \ge c_9 \epsilon^{2-\gamma} e^{c_5 g(c_6 \log(1/\epsilon))}, \text{ i.o.}$$

Together with Theorem 3.1.5, this shows that

$$E(s) \ge c_{10} s^{1-\gamma/2} e^{c_7 g(c_8 \log(1/s))}, \text{ i.o.}$$

from which the limsup part of (4.5) follows.

To get the upper bound in (4.3), we rely on Theorem 3.1.2 with the choice

$$\omega(s) = \sqrt{4s \log(1/s)} \tag{4.6}$$

and Theorem 4.2.5 to get that

$$\begin{aligned} E(s) &\le \mu(\omega(s)) + 2^{(d+2)/2} \mathrm{vol}_d(D) e^{-\omega(s)^2/4s} \\ &\le c_{11} s^{1-\gamma/2} \left(\log \frac{1}{s} \right)^{1-\gamma/2} e^{c_{12} g\left(c_{13} \log \frac{1}{s \log(1/s)}\right)} + 2^{(d+2)/2} \mathrm{vol}_d(D) s \\ &\le c_{11} s^{1-\gamma/2} \left(\log \frac{1}{s} \right)^{1-\gamma/2} e^{c_{12} g(c_3 \log(1/s))} \end{aligned} \tag{4.7}$$

for small s. Now using that $\log x = o(g(x))$, we get that

$$E(s) \le c_4 s^{1-\gamma/2} e^{c_2 g(c_3 \log(1/s))} \tag{4.8}$$

for small s, as required.

Finally, the liminf part of (4.4) implies that

$$\mu(\epsilon) \leq c_{13}\epsilon^{2-\gamma}e^{-c_{14}g(c_{15}\log(1/\epsilon))}, \text{ i.o.}$$

Reasoning as above then shows that

$$E(s) \leq c_{19}s^{1-\gamma/2}e^{-c_7 g(c_8\log(1/s))}, \text{ i.o.}$$

This establishes the liminf part of (4.5) and completes the proof. □

A particular instance of this is when $(\xi_n, n \in \mathbb{N})$ is i.i.d. and $g(x) = \sqrt{x\log\log x}$; we state it in the following corollary.

COROLLARY 4.3.3. *Suppose that* $(\xi_n, n \in \mathbb{N})$ *is i.i.d. Then,*

$$c_1 s^{1-\gamma/2}e^{-c_2\psi(1/s)} \leq E(s) \leq c_3 s^{1-\gamma/2}e^{c_2\psi(1/s)}$$

for small s*, where* $\gamma = \dim_M \partial D$ *and*

$$\psi(x) = \sqrt{\log x \log\log\log x}.$$

Furthermore,

$$\liminf_{s\to 0} \frac{E(s)e^{c_4\psi(1/s)}}{s^{1-\gamma/2}} < \infty \quad \textit{and} \quad \limsup_{s\to 0} \frac{E(s)e^{-c_4\psi(1/s)}}{s^{1-\gamma/2}} > 0.$$

4.4. Slowly varying rates of convergence in the ergodic theorem

The key element in the proof of Theorem 4.3.2 is that the logarithmic correction introduced in (4.6) is *not* felt because g is dominates. This is what enables us to go from (4.7) to (4.8), which is no longer possible if $g(x) = \log\log x$ or g is constant, for example.

These difficulties as g becomes 'closer to a constant' are expected. For example, for the triadic Koch snowflake studied extensively in [**FLV95, LP06**], it is known that the volume of the inner tubular neighbourhood behaves like

$$\mu(\epsilon) = p(\log\epsilon)\epsilon^{2-\gamma} + o(\epsilon^{2-\gamma}),$$

as $\epsilon \to 0$, for some non-constant $\log 3$-periodic function p; this corresponds to g constant. By a renewal argument, this is known to imply that

$$E(s) = q(\log s)s^{1-\gamma/2} + o(s^{1-\gamma/2}),$$

as $s \to 0$, for some $\log 9$-periodic function q. However, as pointed out in Chapter 3, it is not known whether or not q is constant. In other words, it is not known whether $E(s)$ fluctuates between the upper and lower bounds used in Theorem 4.3.2 for this constant function g.

Similar problems have also been studied by Lapidus and coauthors for the eigenvalue counting function; see [**LP93, LP96, LvF00**] and references therein. For a family of open sets called *fractal strings*, the eigenvalue counting function satisfies

$$c_1\lambda^{\gamma/2} \leq N(\lambda) - \sqrt{2/\pi}\lambda^{1/2} \leq c_2\lambda^{\gamma/2},$$

where γ is the Minkowski dimension of the boundary of the fractal string, provided

$$0 < \mathcal{M}_*(\partial U) \leq \mathcal{M}^*(\partial U) < \infty.$$

This again corresponds to a setup where g is constant; see the remark after Theorem 4.2.5. As it turns out, the question of whether $N(\lambda) - \sqrt{2/\pi}\lambda^{1/2}$ fluctuates between its upper and lower bounds is equivalent to the Riemann hypothesis; see [**LvF00**] for further information.

CHAPTER 5

Statistically self-similar sets and the general branching process

Intuitively, a set K is *statistically self-similar* if there is a random number N and some random contracting similitudes $\Phi_1, \ldots, \Phi_N$ such that

$$K = \bigcup_{i=1}^{N} \Phi_i(K_i), \tag{5.1}$$

where $K_1, \ldots, K_N$ are i.i.d. copies of K. In this chapter, we explain how such sets can be encoded by a generalisation of the Galton-Watson process called the *general branching process*.

The material in this chapter is taken from [**Cha14**].

5.1. The general branching process

Our discussion of the general branching process is inspired by [**Ham00, Jag75, Ner81**] where we refer the reader for further information.

Definition and notation. In the general branching process, the typical individual x is born at time σ_x, has offspring whose *birth times* are determined by a point process ξ_x on $(0, \infty)$, a *lifetime* modelled as a non-negative random variable L_x, and a (possibly random) càdlàg function ϕ_x on $\mathbb{R}$ called a *characteristic*.

It is natural to index the individuals of the general branching process using the address space

$$I = \bigcup_{k=0}^{\infty} \mathbb{N}^k, \quad \text{where} \quad \mathbb{N}^0 = \{\emptyset\}. \tag{5.2}$$

The ancestor $\emptyset$ is born at time $\sigma_\emptyset = 0$, and individual x has $\xi_x(0,\infty)$ offspring whose birth times $\sigma_{x,i}$ satisfy

$$\xi_x = \sum_{i=1}^{\xi_x(0,\infty)} \delta_{\sigma_{x,i}-\sigma_x},$$

where δ is the Dirac measure and x, i is the concatenation of x and i. The trace of the underlying Galton-Watson process is a random subtree of I which we denote by $\mathcal{T}$.

It is customary to assume that the triples $(\xi_x, L_x, \phi_x)_x$ are i.i.d. but we allow ϕ_x to depend on the progeny of x; we do *not*, however, make any assumptions of the joint distribution of (ξ_x, L_x, ϕ_x). When discussing a generic individual, we drop the dependence on x and write (ξ, L, ϕ).

We define

$$\xi(t) = \xi((0,t]),\ \nu(dt) = \mathbf{E}\xi(dt),\ \xi_\gamma(dt) = e^{-\gamma t}\xi(dt) \text{ and } \nu_\gamma(dt) = \mathbf{E}\xi_\gamma(dt),$$

for $\gamma \in (0,\infty)$. Furthermore, we will always assume that ν is finite on compact sets and that the general branching process has *Malthusian growth*, i.e. that there exists a *Malthusian parameter* $\gamma \in (0,\infty)$ for which

$$\nu_\gamma(\infty) = 1, \tag{5.3}$$

where we use the notation $\nu_\gamma(t) = \nu_\gamma((0,t])$; in particular $\nu_\gamma(0) = 0$. This implies, in particular, that the general branching process is super-critical, i.e. that $\nu(\infty) > 1$. We denote the first moment of the probability measure ν_γ by

$$\mu_1 = \int_0^\infty t\nu_\gamma(dt). \tag{5.4}$$

In all cases of interest to us, μ_1 we be finite. Note, however, that some convergence results still hold when $\mu_1 = \infty$, as explained in [**Ner81**].

Finally, we write

$$T_t = \#\{x \in \mathcal{T} : \sigma_x \leq t\},$$

the number of individuals born up to time t. It is standard to show that, for every t,

$$\mathbf{E}T_t = \sum_{n=0}^{\infty} \nu^{*n}(t) < \infty;$$

see, for example, [**Jag75**].

The individuals of the population are counted using the characteristic ϕ through the *characteristic counting process* Z^ϕ defined by

$$Z^\phi(t) = \sum_{x \in \mathcal{T}} \phi_x(t - \sigma_x) = \phi_\emptyset(t) + \sum_{i=1}^{\xi(\infty)} Z_i^\phi(t - \sigma_i), \tag{5.5}$$

where the Z_i^ϕ are i.i.d. copies of Z^ϕ. An important example of a characteristic is given by

$$\phi(t) = [\xi(\infty) - \xi(t)]\mathbf{1}_{t \geq 0},$$

whose associated characteristic counting process Z^ϕ is easily seen to have the property that $Z^\phi(t)$ is the number of individuals born after time t to parents born up to time t. In Chapters 7 and 8, we will define characteristics that capture the volume of tubular neighbourhoods and the heat content of a domain with a statistically self-similar boundary.

It is often natural to impose that $\phi(t) = 0$ for $t < 0$. However, we will work without this assumption unless explicitly stated otherwise.

In the rest of this section, we introduce the two key notions to study the large time asymptotics of the general branching process: a renewal equation,

and a fundamental martingale reminiscent of that central to the theory of the Galton-Watson process.

Renewal equation. Given a characteristic ϕ, define

$$z^\phi(t) = e^{-\gamma t}\mathbf{E}Z^\phi(t) \quad \text{and} \quad u^\phi(t) = e^{-\gamma t}\mathbf{E}\phi(t).$$

Using (5.5), it is easy to check that z^ϕ and u^ϕ satisfy the renewal equation

$$z^\phi(t) = u^\phi(t) + \int_0^\infty z^\phi(t-s)\nu_\gamma(ds). \tag{5.6}$$

We may therefore use renewal theory to study the behaviour of $z^\phi(t)$ for large t.

It is often easier to use renewal theory when $\phi(t)$ and therefore $u^\phi(t)$ vanish when $t < 0$. This situation can be created conveniently by putting

$$\chi_x(t) = \phi_x(t)\mathbf{1}_{t\geq 0} + \sum_{i=1}^{\xi_x(\infty)} Z^\phi_{x,i}(t-\sigma_i)\mathbf{1}_{0\leq t<\sigma_i} \tag{5.7}$$

so that

$$Z^\phi\mathbf{1}_{[0,\infty)}(t) = \chi_\emptyset(t) + \sum_{i=1}^{\xi_\emptyset(\infty)} Z^\phi_i(t-\sigma_i)\mathbf{1}_{t-\sigma_i\geq 0},$$

which means that $Z^\phi\mathbf{1}_{[0,\infty)} = Z^\chi$, the counting process of the characteristic χ. Then, we work with Z^χ instead of Z^ϕ because χ vanishes for negative times and Z^χ and Z^ϕ obviously have the same asymptotics as $t \to \infty$.

Note that even when $(\phi_x)_x$ is an i.i.d. collection, χ_x might depend on the progeny of x; this illustrates why allowing this dependence is natural and convenient.

Fundamental martingale. Define

$$\mathcal{F}_x = \sigma(\{(\xi_y, L_y) : \sigma_y \leq \sigma_x\}), \quad \mathcal{F}_t = \sigma(\mathcal{F}_x, \sigma_x \leq t)$$

and

$$\Lambda_t = \{x \in \mathcal{T} : \ x = (y, i) \text{ for some } y \in \mathcal{T}, i \in \mathbb{N}, \text{ and } \sigma_y \leq t < \sigma_x\}, \tag{5.8}$$

i.e. Λ_t is the set of individuals born after time t to parents born up to time t. Then, the process M defined by

$$M_t = \sum_{x \in \Lambda_t} e^{-\gamma \sigma_x} \tag{5.9}$$

is a non-negative càdlàg $\mathcal{F}_t$-martingale with unit expectation, see [**Jag75, Ner81**], which we shall call the *fundamental martingale*. By martingale convergence,

$$M_t \to M_\infty, \text{ a.s.},$$

as $t \to \infty$, for some random variable M_∞. Furthermore, under an $x \log x$ condition standard in the theory of branching processes, M is uniformly integrable. More precisely, in [**Don72, Don76**], Doney proved the following result.

THEOREM 5.1.1 (Doney). *The following are equivalent:*

(i) $\mathbf{E}\left[\xi_\gamma(\infty)(\log \xi_\gamma(\infty))_+\right] < \infty$*;*

(ii) $\mathbf{E}M_\infty > 0$*;*

(iii) $\mathbf{E}M_\infty = 1$*;*

(iv) $M_\infty > 0$ *almost surely on the set where there is no extinction;*

(v) M *is uniformly integrable.*

Otherwise, $M_\infty = 0$ *almost surely.*

Notice that

$$e^{\gamma t} M_t = \sum_{x \in \Lambda_t} e^{\gamma(t - \sigma_x)} = \sum_{\sigma_x \leq t} \phi_x(t - \sigma_x) = Z^\phi(t),$$

with the definition

$$\phi(t) = e^{\gamma t}[\xi_\gamma(\infty) - \xi_\gamma(t)]\mathbf{1}_{t \geq 0}.$$

Therefore, the martingale can be thought of as a rescaled counting process for an appropriate characteristic.

5.2. Applications to statistically self-similar sets

The general branching process provides a natural way to encode statistically self-similar sets discussed in [**Fal86, Gra87, MW86**] which we outline now.

To build a statistically self-similar set K in $\mathbb{R}^d$, start with the address space I defined in (5.2) and a non-empty compact set $K_\emptyset$. To each $x \in I$, we associate a random collection $(N_x, \Phi_{x,1}, \dots \Phi_{x,N_x})$, where N_x is a natural number and $\Phi_{x,i}$ are contracting similitudes whose ratios we write $R_{x,i}$. We assume that the collection is i.i.d. in x.

The random numbers $(N_x, x \in I)$ generate a random subtree $\mathcal{T}$ of I defined by $\emptyset \in \mathcal{T}$ and

$$y = y_1, \dots, y_n \in \mathcal{T} \iff y_1, \dots, y_{n-1} \in \mathcal{T} \text{ and } y_n \leq N_{y_1,\dots,y_{n-1}}.$$

For $x = x_1, \dots, x_n \in \mathcal{T}$, define

$$K_x = \Phi_{x_1} \circ \cdots \circ \Phi_{x_1,\dots,x_n}(K_\emptyset) \quad \text{and} \quad K = \bigcap_{n=1}^{\infty} \bigcup_{|x|=n} K_x,$$

where $|x|$ is the length of the word x. The set K has the intuitive property mentioned in (5.1).

Let us emphasise that the choice of $K_\emptyset$ is not unique in general. However, for technical reasons discussed in [**Fal86**], we make the following two assumptions. First, we assume that the sets $(\operatorname{int} K_x, x \in \mathcal{T})$ *form a net*, i.e.

$$x \leq y \implies \operatorname{int} K_y \subset \operatorname{int} K_x,$$

where $x \leq y$ means that the shortest path from $\emptyset$ to y contains x, and also

$$\operatorname{int} K_x \cap \operatorname{int} K_y = \emptyset \text{ if neither } x \leq y \text{ nor } y \leq x;$$

the analogue of the *open set condition* for self-similar sets. Second, we assume that the construction of K is proper, i.e. that every cut of $\mathcal{C}$ of $\mathcal{T}$ satisfies the condition: for every $x \in \mathcal{C}$, there exists a point in K_x that does not lie in any other K_y with $y \in \mathcal{C}$.

The Hausdorff dimension of statistically self-similar sets is almost surely constant and was calculated in [**Fal86, Gra87, MW86**]. It is given in the following result by a formula *à la* Moran [**Mor46**] and Hutchinson [**Hut81**] familiar from the deterministic setup.

THEOREM 5.2.1. *Let K be a statistically self-similar set as above. Write $(N, R_1, \ldots, R_N)$ for the number of similitudes and their ratios. Then, on the event that the set K is not empty,*

$$\dim K = \inf \left\{ s : \mathbf{E} \left(\sum_{i=1}^{N} R_i^s \right) \leq 1 \right\} \quad a.s.$$

To specify the corresponding general branching process, we set

$$\xi_x = \sum_{i=1}^{N_x} \delta_{-\log R_{x,i}},$$

and $L_x = \sup_i \sigma_{x,i} - \sigma_x$. With this parametrisation, the set K_x in the construction of K is born at time σ_x and has size $e^{-\sigma_x}$. Furthermore, since

$$\mathbf{E} \int_0^\infty e^{-sx} \xi(dx) = \mathbf{E} \left(\sum_{i=1}^{N} R_i^s \right),$$

the Malthusian parameter γ is equal to the Hausdorff dimension of K by definition. In applications later, we will use the general branching process to show that γ is also equal to the Minkowski dimension.

CHAPTER 6

Limit theorems for the general branching process

This chapter is devoted to the study of the asymptotic behaviour of the characteristic counting process of the general branching process. We will prove laws of large numbers and a central limit theorem. The sections on the laws of large numbers are largely inspired by [**Ner81**]. The central limit theorem discussed here extends a similar result appearing in [**JN84**] because here we allow ϕ_x to depend on the progeny of x.

6.1. Outline of the results and their proofs

Consider a general branching process $(\xi_x, L_x, \phi_x)_x$ with Malthusian parameter γ. The laws of large numbers that we prove here are results of the form

$$e^{-\gamma t} Z^{\phi}(t) \to Z_\infty, \text{ in some sense,}$$

as $t \to \infty$ for some random variable Z_∞ which is proportional to M_∞, the limit of the fundamental martingale. Such results are proved in [**Gat00b, Ner81**]; extensions to particular cases are discussed in [**CH08, Ham00**]. The proofs are divided into four steps:

(i) truncate ϕ appropriately;

(ii) establish that for the truncated version ϕ_r of ϕ, say, the expectation of $e^{-\gamma t} Z^{\phi_r}(t)$ converges to a constant c_r as $t \to \infty$;

(iii) prove that $e^{-\gamma t} Z^{\phi_r}(t) \to c_r M_\infty$ with the desired mode of convergence as $t \to \infty$ using standard probabilistic techniques; and

(iv) use suitable approximations to remove the truncation and get the result for Z^ϕ.

The second step in the description of the proofs typically relies on renewal theory. A consequence of this is that the cases when ν_γ is *lattice* (i.e. supported on a discrete additive subgroup of $\mathbb{R}$) and *non-lattice* (i.e. not lattice) need to be treated separately. Throughout this chapter, we will focus on the non-lattice case and discuss how the results may be adapted to the lattice case in Section 6.6.

The aim of Sections 6.2 and 6.3 is to recover and extend the results of [**CH08, Gat00b, Ham00, Ner81**] in a way that emphasises the separation between the steps, in the hope that the statements given here will be easier to use for other applications to statistically self-similar sets.

Note that from the point of view of the general branching process, it is desirable to have theorems with assumptions on ξ and ϕ rather than Z^ϕ. However, for many applications of interest to us, Z^ϕ is the natural object and is used to define ϕ. It is therefore useful to have a collection of results applicable to either situation and we will try to do that here.

The laws of large numbers give a precise meaning to the intuitive notion that the population of the general branching process growth like $e^{\gamma t}$. Therefore, the central limit theorem suggests that we use the scaling $e^{\gamma t/2}$ to investigate the fluctuations of $e^{-\gamma t}Z^\phi(t)$ around its limiting behaviour. This is the object of Section 6.5, where we use the independence built into the general branching process to prove a central limit theorem for the general branching process using the standard Taylor expansion technique. We show that, under some regularity

conditions,

$$e^{\gamma t/2}[e^{-\gamma t}Z^{\phi}(t) - Z_{\infty}] \to_d \text{ some distribution},$$

whose form is reminiscent of the martingale central limits theorems discussed, for example, in [**HH80**].

6.2. Weak law of large numbers

Our first aim in this section is to prove the following result.

THEOREM 6.2.1. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Malthusian parameter γ, where $\phi \geq 0$ and $\phi(t) = 0$ when $t \geq s$ for some s. Assume that ν_γ is non-lattice, that z^ϕ is bounded and that*

$$z^{\phi}(t) \to z^{\phi}(\infty),$$

a finite constant, as $t \to \infty$. Assume further that, for every t,

$$\mathbf{E}\left[\sup_{u \leq t} Z^{\phi}(u)\right] < \infty. \tag{6.1}$$

Then

$$e^{-\gamma t}Z^{\phi}(t) \to z^{\phi}(\infty)M_{\infty}, \text{ in probability},$$

as $t \to \infty$, where M_∞ is the almost sure limit of the fundamental martingale of the general branching process.

The proof relies on a few lemmas which we prove now. We will use the notation

$$\Lambda_{t,c} = \{x = (y, i) : \sigma_y \leq t, t + c < \sigma_x\}$$

and

$$Y_{t,c} = \sum_{x \in \Lambda_{t,c}} e^{-\gamma \sigma_x}.$$

Compare with (5.8) and (5.9). Finally, we will write

$$N_{t,c} = \#(\Lambda_t \setminus \Lambda_{t,c}).$$

LEMMA 6.2.2. *Assume that ν_γ is non-lattice. Then,*

$$\mathbf{E}Y_{t,c} \to \frac{1}{\mu_1}\int_c^\infty (1-\nu_\gamma(t))dt = e(c),$$

say, as $t \to \infty$, where $e(c) \to 0$, as $c \to \infty$.

This may be rewritten as

$$\lim_{c\to\infty} \limsup_{t\to\infty} \mathbf{E}Y_{t,c} = 0. \tag{6.2}$$

Intuitively, this means that if c is large, the contribution of the terms in $\Lambda_{t,c}$ to the martingale M_t are small in expectation in the limit as $t \to \infty$.

PROOF. Notice that $Y_{t,c} = e^{-\gamma t}Z^\psi(t)$ for

$$\psi(t) = e^{\gamma t}[\xi_\gamma(\infty) - \xi_\gamma(t+c)]\mathbf{1}_{t\geq 0}.$$

Put

$$z(t) = e^{-\gamma t}\mathbf{E}Z^\psi(t) \quad \text{and} \quad u(t) = e^{-\gamma t}\mathbf{E}\psi(t) = \nu_\gamma(\infty) - \nu_\gamma(t+c) = 1-\nu_\gamma(t+c).$$

The functions z and u satisfy the renewal equation (5.6). Since μ_1 is finite, u is directly Riemann integrable. Therefore, the renewal theorem in Appendix A implies that

$$z(t) \to z(\infty) = \frac{1}{\mu_1}\int_c^\infty (1-\nu_\gamma(t))dt.$$

The rest is elementary. □

LEMMA 6.2.3. *For every c such that $\nu(c) > 1$, we have, on $\{T_t \to \infty\}$,*

$$\liminf_{t\to\infty} \frac{N_{t,c}}{T_t} \geq \nu(c) - 1 > 0, \text{ a.s.}$$

The existence of a value of c satisfying the condition of the theorem is guaranteed since the general branching process is super-critical.

PROOF. The result follows from the law of large numbers applied to the collection $(\xi_x([0,c]))_x$ and the inequality

$$N_{t,c} \geq \sum_{\sigma_x \leq t} \xi_x([0,c]) - T_t. \tag{6.3}$$ □

Our next lemma relies on the following version of the weak law of large numbers, see [**AK76, Ner81**].

THEOREM 6.2.4. *Let $(X_{t,k}, k \leq n_t, t \geq 0)$ be a family of non-negative random variables such that, for every t, the variables $(X_{t,k}, k \leq n_t)$ are independent. Assume that $n_t \to \infty$ as $t \to \infty$ and that $X_{t,k}$ are stochastically dominated by an integrable random variable X.*

Then,

$$\frac{1}{n_t} \sum_{k=1}^{n_t} (X_{t,k} - \mathbf{E}X_{t,k}) \to 0, \text{ in probability,}$$

as $t \to \infty$.

LEMMA 6.2.5. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process satisfying the assumptions of Theorem 6.2.1. Assume that c is such that $\nu(c) > 1$. Then, on $\{T_t \to \infty\}$,*

$$\sum_{x \in \Lambda_t \setminus \Lambda_{t,c}} e^{-\gamma \sigma_x} (e^{-\gamma(t+s-\sigma_x)} Z_x^{\phi}(t+s-\sigma_x) - z^{\phi}(t+s-\sigma_x)) \to 0, \text{ in probability,}$$

as $t \to \infty$.

PROOF. When $N_{t,c}$ is positive, we may write

$$\sum_{x \in \Lambda_t \setminus \Lambda_{t,c}} e^{-\gamma \sigma_x} (e^{-\gamma(t+s-\sigma_x)} Z_x^{\phi}(t+s-\sigma_x) - z^{\phi}(t+s-\sigma_x)) = N_{t,c} e^{-\gamma(t+c)} A_{t,c},$$

where

$$A_{t,c} = N_{t,c}^{-1} \sum_{x \in \Lambda_t \setminus \Lambda_{t,c}} e^{\gamma(t+c-\sigma_x)}(e^{-\gamma(t+s-\sigma_x)} Z_x^\phi(t+s-\sigma_x) - z^\phi(t+s-\sigma_x)).$$

Recall that $N_{t,c} \in \mathcal{F}_t$ and that, for every $x \in \Lambda_t \setminus \Lambda_{t,c}$, we have $\sigma_x \in \mathcal{F}_t$ while the processes Z_x^ϕ with $x \in \Lambda_t \setminus \Lambda_{t,c}$ are mutually independent and independent of $\mathcal{F}_t$. These observations and conditioning on $\mathcal{F}_t$ together with Theorem 6.2.4 and (6.1) show that, on $\{N_{t,c} \to \infty\}$, we have, for every $\epsilon \in (0, \infty)$,

$$\mathbf{P}(|A_{t,c}| > \epsilon | \mathcal{F}_t) \to 0, \text{ in probability,}$$

as $t \to \infty$. By dominated convergence, it then follows that, for every $\epsilon \in (0, \infty)$,

$$\mathbf{P}(|A_{t,c}| > \epsilon \text{ and } N_{\tau,c} \to \infty \text{ as } \tau \to \infty) \to 0, \text{ in probability,}$$

as $t \to \infty$.

Since $\sigma_x \leq t + c$ for $x \in \Lambda_t \setminus \Lambda_{t,c}$, we know that

$$e^{-\gamma(t+c)} N_{t,c} \leq M_t \to M_\infty, \text{a.s.},$$

as $t \to \infty$. Hence, on $\{N_{t,c} \to \infty\}$,

$$e^{-\gamma(t+c)} N_{t,c} A_{t,c} \to 0, \text{ in probability,}$$

as $t \to \infty$.

Finally, by Lemma 6.2.3, we know that $N_{t,c} \to \infty$ on $\{T_t \to \infty\}$, and the result follows. □

We may now proceed with the proof of the weak law of large numbers.

Proof of Theorem 6.2.1. Since $\phi(t) = 0$ when $t \geq s$, we have

$$Z^\phi(t+s) = \sum_{x \in \Lambda_t} Z_x^\phi(t+s-\sigma_x),$$

which we rewrite as

$$e^{-\gamma(t+s)}Z^\phi(t+s) = \sum_{x\in\Lambda_t\setminus\Lambda_{t,c}} e^{-\gamma\sigma_x}e^{-\gamma(t+s-\sigma_x)}Z_x^\phi(t+s-\sigma_x) + \sum_{x\in\Lambda_{t,c}} e^{-\gamma\sigma_x}e^{-\gamma(t+s-\sigma_x)}Z_x^\phi(t+s-\sigma_x).$$

Fix $\epsilon \in (0,\infty)$. We shall show that

$$\mathbf{P}(|e^{-\gamma(t+s)}Z^\phi(t+s) - z^\phi(\infty)M_\infty| \geq \epsilon) \to 0, \tag{6.4}$$

as $t \to \infty$.

Pick $\delta \in (0,\infty)$ and choose c with $\nu(c) > 1$ and $t_0 \geq 0$. Assume that c, s and t_0 are large enough to guarantee that, for $t \geq t_0$,

$$\mathbf{P}(z^\phi(\infty)|M_t - M_\infty| \geq \epsilon/4) \leq \delta/4, \tag{6.5}$$

(6.6)

$$\mathbf{P}\left(\left|\sum_{x\in\Lambda_t\setminus\Lambda_{t,c}} e^{-\gamma\sigma_x}(e^{-\gamma(t+s-\sigma_x)}Z_x^\phi(t+s-\sigma_x) - z^\phi(t+s-\sigma_x))\right| \geq \epsilon/4\right) \leq \delta/4$$

(this is possible thanks to Lemma 6.2.5),

$$\mathbf{E}Y_{t,c} \leq \frac{\delta\epsilon}{8\|z^\phi\|_\infty} \tag{6.7}$$

(this is possible thanks to Lemma 6.2.2), and, for $u \geq s - c$,

$$|z^\phi(u) - z^\phi(\infty)| \leq \frac{\delta\epsilon}{16}. \tag{6.8}$$

We now bound the probability in (6.4) by

$$\begin{aligned}&\mathbf{P}(|e^{-\gamma(t+s)}Z^\phi(t+s) - z^\phi(\infty)M_\infty| > \epsilon)\\ &\leq \mathbf{P}\left(\left|\sum_{x\in\Lambda_t} e^{-\gamma\sigma_x}e^{-\gamma(t+s-\sigma_x)}Z_x^\phi(t+s-\sigma_x) - z^\phi(\infty)\sum_{x\in\Lambda_t} e^{-\gamma\sigma_x}\right| \geq 3\epsilon/4\right)\\ &\quad + \mathbf{P}(z^\phi(\infty)|(M_t - M_\infty)| \geq \epsilon/4).\end{aligned} \tag{6.9}$$

Here we used the definition of M. The first part of the right-hand side is at most

(6.10)
$$\mathbf{P}\left(\left|\sum_{x\in\Lambda_t\setminus\Lambda_{t,c}} e^{-\gamma\sigma_x}(e^{-\gamma(t+s-\sigma_x)}Z_x^{\phi}(t+s-\sigma_x)-z^{\phi}(t+s-\sigma_x))\right|\geq\epsilon/4\right)$$

(6.11)
$$+\mathbf{P}\left(\sum_{x\in\Lambda_t\setminus\Lambda_{t,c}} e^{-\gamma\sigma_x}|z^{\phi}(t+s-\sigma_x)-z^{\phi}(\infty)|\geq\epsilon/4\right)$$

(6.12)
$$+\mathbf{P}\left(\sum_{x\in\Lambda_{t,c}} e^{-\gamma\sigma_x}|e^{-\gamma(t+s-\sigma_x)}Z_x^{\phi}(t+s-\sigma_x)-z^{\phi}(\infty)|\geq\epsilon/4\right).$$

Let $t\geq t_0$. By (6.5), the last term in (6.9) is at most $\delta/4$. By (6.6), we get that (6.10) is at most $\delta/4$. By (6.8) and Markov's inequality, (6.11) is at most $\delta/4$ since $\mathbf{E}M_t=1$. Finally, by (6.7) and Markov's inequality, (6.12) is at most $\delta/4$. Therefore, the left-hand side in (6.9) is at most δ. Since δ is arbitrary, we have established (6.4) and the proof is complete. □

To remove the assumption that $\phi(t)$ vanishes for t large, we need a way to approximate ϕ by a sequence $(\phi_n, n\in\mathbb{N})$ of characteristics that do and that satisfy $z^{\phi_n}(\infty)\to z^{\phi}(\infty)$ as $n\to\infty$. One way to ensure this is to impose that the map $t\mapsto e^{-\gamma t}\mathbf{E}\phi(t)$ be directly Riemann integrable as we show now.

THEOREM 6.2.6. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Malthusian parameter γ, where $\phi\geq 0$. Assume that ν_γ is non-lattice, that u^{ϕ} is directly Riemann integrable and that z^{ϕ} is the unique solution to the renewal equation such that*

$$z^{\phi}(-t)\to 0,$$

as $t\to\infty$. Assume further that, for every t,

$$\mathbf{E}\left[\sup_{u\leq t} Z^{\phi}(u)\right]<\infty.$$

Then,

$$z^{\phi}(t) \to z^{\phi}(\infty),$$

some finite constant, and

$$e^{-\gamma t} Z^{\phi}(t) \to z^{\phi}(\infty) M_{\infty}, \text{ in probability,}$$

as $t \to \infty$, where M_{∞} is the almost sure limit of the fundamental martingale of the general branching process.

PROOF. The existence of the limit $z^{\phi}(\infty)$ follows from the renewal theorem in Appendix A. For the rest, put

$$\phi_n(t) = \phi(t)\mathbf{1}_{t \le n} \quad \text{and} \quad u_n(t) = e^{-\gamma t}\mathbf{E}\phi_n(t).$$

Since $0 \le u_n(t) \le u^{\phi}(t)$, the functions u_n and $u^{\phi} - u_n$ are directly Riemann integrable too. Furthermore, for every n, we have $z^{\phi_n}(-\infty) = 0$. Therefore, by the renewal theorem in Appendix A,

$$\lim_{t\to\infty} z^{\phi_n}(t) = \frac{1}{\mu_1}\int_{-\infty}^{\infty} e^{-\gamma t}\mathbf{E}\phi_n(t)dt \to \frac{1}{\mu_1}\int_{-\infty}^{\infty} e^{-\gamma t}\mathbf{E}\phi(t)dt,$$

and

$$\lim_{t\to\infty} z^{\phi-\phi_n}(t) = \frac{1}{\mu_1}\int_{-\infty}^{\infty} e^{-\gamma t}\mathbf{E}[\phi(t) - \phi_n(t)]dt \to 0,$$

as $n \to \infty$, by dominated convergence.

Notice that

$$\begin{aligned} |e^{-\gamma t} Z^{\phi}(t) - z^{\phi}(\infty) M_{\infty}| \le\, & e^{-\gamma t}|Z^{\phi}(t) - Z^{\phi_n}(t)| \\ & + |e^{-\gamma t} Z^{\phi_n}(t) - z^{\phi_n}(\infty) M_{\infty}| \\ & + |z^{\phi_n}(\infty) - z^{\phi}(\infty)| M_{\infty}, \end{aligned} \tag{6.13}$$

which we now show converges to 0 in probability. First, by Markov's inequality,

$$\begin{aligned}\mathbf{P}(e^{-\gamma t}|Z^{\phi}(t) - Z^{\phi_n}(t)| \geq \epsilon) &\leq \epsilon^{-1}\mathbf{E}[e^{-\gamma t}|Z^{\phi}(t) - Z^{\phi_n}(t)|] \\ &\leq \epsilon^{-1} z^{\phi-\phi_n}(t) \\ &\to \epsilon^{-1} z^{\phi-\phi_n}(\infty),\end{aligned}$$

which can be made arbitrarily small, by choosing n large. Second,

$$\mathbf{P}(|z^{\phi_n}(\infty) - z^{\phi}(\infty)|M_\infty \geq \epsilon)$$

can be made arbitrarily small by choosing n large. And third, by Theorem 6.2.1, for every n,

$$|e^{-\gamma t} Z^{\phi_n}(t) - z^{\phi_n}(\infty) M_\infty| \to 0, \text{ in probability,}$$

as $t \to 0$.

Together, these observations show that the left hand side in (6.13) converges to 0 in probability as $t \to \infty$, as required. □

Let us conclude this section by noticing that a convenient sufficient condition for (6.1) to hold in the one-sided case is that, for every t,

$$\mathbf{E}\left[\sup_{v \leq t} \phi(v)\right] < \infty.$$

Indeed, for $u \leq t$, we then have

$$Z^{\phi}(u) = \sum_{\sigma_x \leq u} \phi_x(u - \sigma_x) \leq \sum_{\sigma_x \leq t} \sup_{v \leq t} \phi_x(v).$$

The independence of σ_x and ϕ_x and the finiteness of $\mathbf{E}T_t$ show that

$$\mathbf{E}\left[\sum_{\sigma_x \leq t} \sup_{v \leq t} \phi_x(v)\right] \leq \mathbf{E}[T_t]\mathbf{E}\left[\sup_{v \leq t} \phi(v)\right] < \infty,$$

from which (6.1) follows.

We can now state the version of the weak law of large numbers of [**Ner81**].

THEOREM 6.2.7. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process where $\phi \geq 0$ and vanishes for negative times. Assume that u^ϕ is directly Riemann integrable. Assume further that, for every t,*

$$\mathbf{E}\left[\sup_{u\leq t}\phi(u)\right] < \infty.$$

Then,

$$z^\phi(t) \to z^\phi(\infty),$$

some finite constant, as $t \to \infty$, and

$$e^{-\gamma t}Z^\phi(t) \to z^\phi(\infty)M_\infty, \text{ in probability,}$$

as $t \to \infty$, where M_∞ is the almost sure limit of the fundamental martingale of the general branching process.

Of course, using (5.7) it is possible to apply Theorem 6.2.7 to investigate the asymptotics even when the characteristic ϕ does not vanish for negative times. The condition to check then becomes that, for every t,

$$\mathbf{E}\left[\sup_{0\leq u\leq t}\left(\phi(u) + \sum_{i=1}^{\xi(\infty)} Z_i(u-\sigma_i)\mathbf{1}_{u<\sigma_i}\right)\right] < \infty. \tag{6.14}$$

This condition is a joint condition on ξ and ϕ. It reflects the care that has to be taken to avoid having many offspring x in the distant future that can still have an impact on Z^ϕ through the values that ϕ_x takes for negative times, and then violate the condition in (6.1).

The form of (6.14) also suggests that if either ξ is compactly supported, or $\phi(t) = 0$ when $t \leq s$ for some (typically negative) s, then it should be easily satisfied. This is a natural setup in some cases; see, for example, [**Ham00**].

6.3. Strong law of large numbers

Strong laws of large numbers for general branching processes require some more technical conditions which we discuss now.

CONDITION 6.3.1. *There exists a non-increasing bounded positive integrable function g on $[0,\infty)$ such that*

$$\mathbf{E}\left[\sup_{t\geq 0}\frac{\xi_\gamma(\infty)-\xi_\gamma(t)}{g(t)}\right]<\infty.$$

This condition is satisfied if there exists a non-increasing bounded positive function g such that

$$\int_0^\infty \frac{1}{g(s)}\nu_\gamma(ds)<\infty,$$

because then

$$\frac{\xi_\gamma(\infty)-\xi_\gamma(t)}{g(t)}\leq\int_t^\infty\frac{1}{g(s)}\xi_\gamma(ds)\leq\int_0^\infty\frac{1}{g(s)}\xi_\gamma(s),$$

which has finite expectation. As such, Condition 6.3.1 can be thought of as some kind of moment condition that is weaker than imposing that ν_γ have a finite second moment as is seen by taking $g(t)=1\wedge t^{-2}$. In particular, this condition is always satisfied if the expected number of offspring is finite because then, with $g(t)=1\wedge t^{-2}$, we have

$$\frac{\xi_\gamma(\infty)-\xi_\gamma(t)}{g(t)}\leq\int_t^\infty\frac{1}{g(s)}\xi_\gamma(ds)\leq\int_0^\infty\frac{1}{g(s)}\xi_\gamma(ds)\leq\sup_{u\geq 0}\{(1\vee u^2)e^{-\gamma u}\}\xi(\infty).$$

The next condition generalises Condition 6.3.1 and will be used to control the error made when truncating the characteristic. It will be particularly handy when ϕ vanishes for negative times.

CONDITION 6.3.2. *For some s, there exists a non-increasing bounded positive integrable càdlàg function h on $[s, \infty)$ such that*

$$\mathbf{E}\left[\sup_{t \geq s} \frac{e^{-\gamma t}\phi(t)}{h(t)}\right] < \infty.$$

It will be convenient, for such functions h, to extend them to the real line by setting $h(t) = 0$ when $t < s$.

To extend the law of large numbers for t on particular lattices to t in $\mathbb{R}$, we need some new notation. For $\epsilon \in (0, \infty)$, write

$$\phi_\epsilon(t) = \inf_{u \in [t-\epsilon, t+\epsilon]} \phi(u) \quad \text{and} \quad \phi^\epsilon(t) = \sup_{u \in [t-\epsilon, t+\epsilon]} \phi(u).$$

This defines càdlàg functions, and we have the following convenient property.

PROPOSITION 6.3.3. *Let $\phi \geq 0$ be a random càdlàg function on $\mathbb{R}$ satisfying Condition 6.3.2. Then so do ϕ_ϵ and ϕ^ϵ.*

PROOF. The result is obvious for ϕ_ϵ. To establish it for ϕ^ϵ, notice that

$$\sup_{t \geq s+\epsilon} \frac{e^{-\gamma t}\phi^\epsilon(t)}{h(t-\epsilon)} \leq e^{\gamma\epsilon} \sup_{u \geq s} \frac{e^{-\gamma u}\phi(u)}{h(u)},$$

which has finite expectation. Since $t \mapsto h(t-\epsilon)$ satisfies the requirements, the proof is complete. □

The last condition that will be useful concerns the regularity of the characteristic function ϕ in terms of how close ϕ_ϵ and ϕ^ϵ are to ϕ in a suitable sense.

CONDITION 6.3.4. *For some $\epsilon \in (0, \infty)$, the map $u^{\phi^\epsilon}(t) = e^{-\gamma t}\mathbf{E}\phi^\epsilon(t)$ is directly Riemann integrable.*

By Fubini's theorem and dominated convergence, this condition has the consequence that

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} e^{-\gamma t}\mathbf{E}\phi_\epsilon(t)dt = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} e^{-\gamma t}\mathbf{E}\phi^\epsilon(t)dt = \int_{-\infty}^{\infty} e^{-\gamma t}\mathbf{E}\phi(t)dt. \tag{6.15}$$

In the one-sided case when $\phi \geq 0$, Condition 6.3.2 can conveniently be used to verify Condition 6.3.4 as we show now.

PROPOSITION 6.3.5. *Let $\phi \geq 0$ be a random càdlàg function on $[0,\infty)$ satisfying Condition 6.3.2 with $s = 0$. Then, Condition 6.3.4 is satisfied.*

PROOF. Since ϕ is càdlàg, Condition 6.3.2 and dominated convergence show that $t \mapsto \mathbf{E}\phi(t)$ is also càdlàg. Furthermore,

$$e^{-\gamma t}\mathbf{E}\phi(t) \leq h(t)\mathbf{E}\left[\sup_{u\geq 0}\frac{e^{-\gamma u}\phi^\epsilon(u)}{h(u)}\right],$$

which is a bounded non-increasing integrable function. Therefore, $t \mapsto e^{-\gamma t}\mathbf{E}\phi(t)$ is directly Riemann integrable. Using Proposition 6.3.3, the same is true of ϕ_ϵ and ϕ^ϵ, which completes the proof. □

Our first aim in this section is to prove the following result.

THEOREM 6.3.6. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Malthusian parameter γ, where $\phi \geq 0$ and $\phi(t) = 0$ when $|t| \geq s$ for some s. Assume that ν_γ is non-lattice. Assume further that, for some $\epsilon \in (0,\infty)$,*

$$\mathbf{E}\left[\sup_{u\leq t} Z^{\phi^\epsilon}(u)\right] < \infty, \tag{6.16}$$

and that Conditions 6.3.1 and 6.3.4 are satisfied; in particular, z^ϕ is bounded and

$$z^\phi(t) \to z^\phi(\infty),$$

some finite constant, as $t \to \infty$.

Then,

$$e^{-\gamma t} Z^{\phi}(t) \to z^{\phi}(\infty) M_{\infty}, \ \textit{a.s.},$$

as $t \to \infty$, where M_{∞} is the almost sure limit of the fundamental martingale of the general branching process.

By monotonicity, it is clear that, under these assumptions, (6.16) holds when $\epsilon = 0$.

REMARK 6.3.7. *Unlike in Theorem 6.2.1, here, we require that $\phi(-t)$ vanish for large t. This additional requirement comes for the impossibility to control the term in* (6.12) *when proving almost sure convergence without further assumptions.*

The proof of Theorem 6.3.6 relies on a few lemmas which we prove now.

LEMMA 6.3.8. *There exists a positive constant K such that*

$$\limsup_{t\to\infty} e^{-\gamma t} T_t \leq K M_{\infty}, \ \textit{a.s.}$$

PROOF. From Lemma 6.2.3, if $\nu(c) > 1$, we have, on $\{T_t \to \infty\}$,

$$\liminf_{t\to\infty} \frac{N_{t,c}}{T_t} \geq \nu(c) - 1 > 0, \text{ a.s.}$$

Since $M_t \geq e^{-\gamma(t+c)} N_{t,c}$, we have

$$\liminf_{t\to\infty} \frac{M_t}{e^{-\gamma t} T_t} \geq (\nu(c) - 1) e^{-\gamma c}, \text{ a.s.},$$

from which the result follows. □

For the next lemma, we will use the following version of the strong law of large numbers, see [**AK76, Ner81**].

THEOREM 6.3.9. *Let $(X_{i,j}, j \leq n_i, i \in \mathbb{N})$ be a family random variables such that, for every i, the variables $(X_{i,j}, j \leq n_i)$ are independent. Assume that*

$$\liminf_{i\to\infty} \frac{n_{i+1}}{n_1 + \cdots + n_i} > 0,$$

and that $X_{i,j}$ are stochastically dominated by an integrable random variable X. Then, for every $\epsilon \in (0, \infty)$,

$$\sum_{i=1}^{\infty} \mathbf{P}\left(\left|\sum_{j=1}^{n_i} X_{ij} - \mathbf{E}X_{ij}\right| > \epsilon\right) < \infty.$$

In particular,

$$\frac{1}{n_i}\sum_{j=1}^{n_i}(X_{i,j} - \mathbf{E}X_{i,j}) \to 0, \ \ a.s.$$

as $i \to \infty$.

REMARK 6.3.10. *If the sequence $(n_i, i \in \mathbb{N})$ in Theorem 6.3.9 is random (and the condition on the limsup holds with probability one), but $X_{i,j} = Y_j$ for an i.i.d. collection of random variables $(Y_j, j \in \mathbb{N})$, then the proof given in* [**Ner81**] *still applies, and the result still holds.*

We will also need the following version of the Borel-Cantelli lemma.

THEOREM 6.3.11. *Let $(\mathcal{G}_n, n \in \mathbb{N})$ be an increasing sequence of σ-algebras and $(A_n, n \in \mathbb{N})$ be a sequence of events. Define the event*

$$E = \left\{\sum_{n=1}^{\infty} \mathbf{P}(A_n|\mathcal{G}_n) < \infty\right\}.$$

Then

$$\mathbf{P}(A_n \text{ infinitely often } |E) = 0.$$

LEMMA 6.3.12. *Assume that $(\phi_x)_x$ is i.i.d. and that $\phi(t) = 0$ when $t \leq s$ for some $s \geq 0$. Assume further that Condition 6.3.2 is satisfied. Then, there*

exists a positive constant K such that

$$\limsup_{t\to\infty} e^{-\gamma t} Z^{\phi}(t) \le K M_{\infty} \int_{s-K}^{\infty} h(t)dt.$$

PROOF. Let c be such that $\nu(c) > 1$. Since

$$T_{t+c} \ge \sum_{\sigma_x \le t} \xi([0, c]),$$

the strong law of large numbers implies that

$$\liminf_{t\to\infty} \frac{T_{t+c}}{T_t} \ge \nu(c) > 1, \text{ a.s.}$$

Therefore, by Theorem 6.3.9 and Remark 6.3.10, on $\{T_t \to \infty\}$, we have

$$\frac{1}{T_{(k+1)c} - T_{kc}} \sum_{kc < \sigma_x \le (k+1)c} U_x \to \mathbf{E}U, \text{ a.s.}, \tag{6.17}$$

as $k \to \infty$, where

$$U_x = \sup_{t\in[0,\infty)} \frac{e^{-\gamma t}\phi_x(t)}{h(t)}$$

has finite expectation, by Condition 6.3.2.

Furthermore,

$$\begin{aligned}
&e^{-\gamma(t+s)} Z^{\phi}(t+s) \\
&\quad = e^{-\gamma(t+s)} \sum_{\sigma_x \le t} \phi(t+s-\sigma_x) \\
&\quad \le \sum_{\sigma_x \le t} e^{-\gamma\sigma_x} h(t+s-\sigma_x) U_x \\
&\quad \le \sum_{\sigma_x \le t} \sum_{kc < \sigma_x \le (k+1)c} e^{-\gamma kc} h(t+s-(k+1)c) U_x \\
&\quad \le \sum_{k=0}^{\lfloor t/c \rfloor} e^{-\gamma kc} h(t+s-(k+1)c) \frac{T_{(k+1)c}}{T_{(k+1)c} - T_{kc}} \sum_{kc < \sigma_x \le (k+1)c} U_x.
\end{aligned}$$

Together with (6.17) and Lemma 6.3.8, this shows that, for any $\epsilon \in (0, \infty)$, we have, almost surely,

$$\begin{aligned}
&\limsup_{t\to\infty} e^{-\gamma(t+s)} Z^{\phi}(t+s) \\
&\qquad \leq (1+\epsilon) \limsup_{t\to\infty} \sum_{k=1}^{\lfloor t/c \rfloor} e^{-\gamma kc} h(t+s-(k+1)c) T_{(k+1)c} \\
&\qquad \leq (1+\epsilon)^2 K M_\infty \limsup_{t\to\infty} \sum_{k=1}^{\lfloor t/c \rfloor} h(t+s-(k+1)c) \\
&\qquad \leq (1+\epsilon)^2 K M_\infty \int_{s-2c}^{\infty} h(u)du,
\end{aligned}$$

for some positive constant K. The result follows. □

A particular case of this result is given in the following corollary.

COROLLARY 6.3.13. *Assume that ξ satisfies Condition 6.3.1. Then, there exists a positive constant K such that, for every $c \in (0, \infty)$,*

$$\limsup_{t\to\infty} Y_{t,c} \leq K M_\infty \int_{c-K}^{\infty} g(t)dt, \ \ a.s.$$

PROOF. Notice that $Y_{t,c} = Z^{\phi}(t+c)$ when

$$\phi(t) = e^{\gamma t}[\xi_\gamma(\infty) - \xi_\gamma(t)]\mathbf{1}_{t>c}.$$

Then apply Lemma 6.3.12 using Condition 6.3.1. □

This implies that

$$\lim_{c\to\infty} \limsup_{t\to\infty} Y_{t,c} = 0, \text{ a.s..}$$

As such, it is the analogue of (6.2) in the proof of the weak law of large numbers.

We will now prove a strong law of large numbers for the general branching process when time is restricted to certain lattices. For $c \in (0, \infty)$ and $t_0 \in [0, \infty)$, write

$$t_k = t_0 + kc, \quad k \in \mathbb{Z} \quad \text{and} \quad t_{k,n} = \frac{kc}{n}, \quad k \in \mathbb{Z},\, n \in \mathbb{N}.$$

We need the following convergence result.

LEMMA 6.3.14. *Assume that $\nu(c) > 1$. Then,*

$$\mathbf{P}\left(\liminf_{k\to\infty} \frac{N_{t_{k+1},c}}{\sum_{i=1}^{k} N_{t_i,c}} > 0 \middle| T_t \to \infty \right) = 1.$$

PROOF. Since

$$\sum_{i=1}^{k} N_{t_i,c} \leq T_{t_{k+1}},$$

the result follows from Lemma 6.2.3. □

LEMMA 6.3.15. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process satisfying the assumptions of Theorem 6.3.6. Assume that $\nu(c) > 1$. Then,*

$$e^{-\gamma t_k} Z^{\phi}(t_k) \to z^{\phi}(\infty) M_{\infty}, \ \ a.s.,$$

as $k \to \infty$.

PROOF. Let n_0 be such that

$$2s \leq n_0 c.$$

For $n \geq n_0$ even, we have

(6.18)

$$|e^{-\gamma t_{k+n/2}} Z^{\phi}(t_{k+n/2}) - z^{\phi}(\infty) M_{\infty}|$$

$$\leq \left| \sum_{x \in \Lambda_{t_k} \setminus \Lambda_{t_k, nc}} e^{-\gamma \sigma_x} [e^{-\gamma(t_{k+n/2} - \sigma_x)} Z_x^{\phi}(t_{k+n/2} - \sigma_x) - z^{\phi}(t_{k+n/2} - \sigma_x)] \right| \tag{6.19}$$

$$+ \left| \sum_{x \in \Lambda_{t_k} \setminus \Lambda_{t_k, nc}} e^{-\gamma \sigma_x} z^{\phi}(t_{k+n/2} - \sigma_x) - z^{\phi}(\infty) M_{\infty} \right|. \tag{6.20}$$

Rewrite the term in (6.19) as

$$e^{-\gamma t_k} N_{t_k, nc} A_{t_k, n, c},$$

where $A_{t_k,n,c} = N_{t_k,nc}^{-1}$ is defined as

$$\left| \sum_{x \in \Lambda_{t_k} \setminus \Lambda_{t_k,nc}} e^{-\gamma(\sigma_x - t_k)} [e^{-\gamma(t_{k+n/2} - \sigma_x)} Z_x^\phi(t_{k+n/2} - \sigma_x) - z^\phi(t_{k+n/2} - \sigma_x)] \right|.$$

To control (6.19), notice first that

$$e^{-\gamma t_k} N_{t_k,nc} \leq e^{\gamma nc} M_{t_k} \to e^{2\gamma nc} M_\infty, \text{ a.s.}, \tag{6.21}$$

as $k \to \infty$. Then, reason as in the proof of Lemma 6.2.5: conditioning on $\mathcal{F}_{t_k}$ together with Theorem 6.3.9 and Remark 6.3.10, Lemma 6.3.14 and the condition in (6.16) yields that, on $\{T_t \to \infty\}$, we have, for any $\epsilon \in (0, \infty)$,

$$\sum_{k=1}^{\infty} \mathbf{P}(A_{t_k,n,c} > \epsilon | \mathcal{F}_{t_k}) < \infty.$$

By Theorem 6.3.11, this shows that, on $\{T_t \to \infty\}$,

$$A_{t_k,n,c} \to 0, \text{ a.s.},$$

as $k \to \infty$. And therefore, because of (6.21), so does the expression in (6.19).

Let us now deal with (6.20). Pick $\delta \in (0, \infty)$. Let $r \in (0, \infty)$ be large enough that

$$\limsup_{t \to \infty} Y_{t,rc} \leq \frac{\delta}{3\|z^\phi\|_\infty} M_\infty$$

(this is possible thanks to Corollary 6.3.13), and assume that $n \geq n_0$ is large enough that, for $u \geq (n/2 - r)c > 0$,

$$|z^\phi(u) - z^\phi(\infty)| \leq \delta/2.$$

Then, by definition of M, we get that, almost surely,

$$\begin{aligned}
&\limsup_{k\to\infty}\left|\sum_{x\in\Lambda_{t_k}\setminus\Lambda_{t_k,nc}} e^{-\gamma\sigma_x} z^\phi(t_{k+n/2}-\sigma_x) - z^\phi(\infty)M_\infty\right| \\
&\le \limsup_{k\to\infty}\left|\sum_{x\in\Lambda_{t_k}\setminus\Lambda_{t_k,rc}} e^{-\gamma\sigma_x}[z^\phi(t_{k+n/2}-\sigma_x) - z^\phi(\infty)]\right| \\
&\quad + \limsup_{k\to\infty}\left|\sum_{x\in\Lambda_{t_k,rc}\setminus\Lambda_{t_k,nc}} e^{-\gamma\sigma_x}[z^\phi(t_{k+n/2}-\sigma_x) - z^\phi(\infty)] - \sum_{x\in\Lambda_{t_k,nc}} e^{-\gamma\sigma_x} z^\phi(\infty)\right| \\
&\quad + \limsup_{k\to\infty}|z^\phi(\infty)M_{t_k} - z^\phi(\infty)M_\infty| \\
&\le \frac{\delta}{2}\limsup_{k\to\infty} M_{t_k} + \|z^\phi\|_\infty \limsup_{k\to\infty} Y_{t_k,rc} \\
&\le \delta M_\infty.
\end{aligned}$$

Since δ arbitrary, we conclude that (6.20) converges to 0 almost surely as $k\to\infty$. This concludes the proof. □

The convergence is easily extended for $t_{k,n}$ as we show now.

COROLLARY 6.3.16. *Let $(\xi_x, L_x, \phi_x)_x$ be a branching process satisfying the assumptions of Theorem 6.3.6. Assume that $\nu(c) > 1$. Then, almost surely,*

$$e^{-\gamma t_{k,n}} Z^\phi(t_{k,n}) \to z^\phi(\infty)M_\infty, \text{ for all } n,$$

as $k\to\infty$.

PROOF. Denote by A_q the event that the convergence in Lemma 6.3.15 occurs when $t_0 = qc$. Then, $A = \cap_{q\in\mathbb{Q}} A_q$ has probability one, and, on it, we have

$$e^{-\gamma t_{k,n}} Z^\phi(t_{k,n}) \to z^\phi(\infty)M_\infty, \text{ for all } n,$$

as $k\to\infty$. The proof is complete. □

We may now proceed with the proof of the strong law of large numbers.

PROOF OF THEOREM 6.3.6. Let c be such that $\nu(c) > 1$. By definition, for $t \in [t_{k,n}, t_{k+1,n}]$,

$$Z^{\phi_{c/n}}(t_{k,n}) \leq Z^{\phi}(t) \leq Z^{\phi^{c/n}}(t_{k+1,n}).$$

By Corollary 6.3.16 applied to $\phi_{c/n}$ and $\phi^{c/n}$, we have

$$e^{-\gamma c/n} z^{\phi_{c/n}}(\infty) M_\infty \leq \liminf_{t\to\infty} e^{-\gamma t} Z^{\phi}(t) \leq \limsup_{t\to\infty} e^{-\gamma t} Z^{\phi}(t) \leq e^{-\gamma c/n} z^{\phi^{c/n}}(\infty) M_\infty.$$

But, by Condition 6.3.4, the renewal theorem of Appendix A and dominated convergence, we have

$$z^{\phi_{c/n}}(\infty) = \frac{1}{\mu_1} \int_{-\infty}^{\infty} e^{-\gamma t} \mathbf{E}\phi_{c/n}(t) dt \uparrow \frac{1}{\mu_1} \int_{-\infty}^{\infty} e^{-\gamma t} \mathbf{E}\phi(t) dt = z^{\phi}(\infty),$$

as $n \to \infty$. A similar reasoning shows that $z^{\phi^{c/n}}(\infty) \downarrow z^{\phi}(\infty)$, as $n \to \infty$. The result follows. □

Our next step is to extend the strong law of large numbers to the case when ϕ is not compactly supported. This will, of course, require some conditions on the tails of ϕ; we will also need to strengthen Lemma 6.3.12 as follows.

LEMMA 6.3.17. *Lemma 6.3.12 still holds without the assumption that $(\phi_x)_x$ is i.i.d.*

PROOF. Define the characteristic

$$\psi(t) = \sup_{u\geq 0} \frac{e^{-\gamma u}\phi(u)}{h(u)} \mathbf{1}_{0\leq t\leq 1} = U\mathbf{1}_{0\leq t\leq 1}.$$

Note that ψ is compactly supported and the assumptions of Theorem 6.3.6 are easily checked, so we have

$$e^{-\gamma t} Z^{\psi}(t) \to z^{\psi}(\infty) M_\infty, \text{ a.s.}$$

as $t \to \infty$, where

$$z^{\psi}(\infty) = \int_0^1 e^{-\gamma t} \mathbf{E}U dt \leq \mathbf{E}U < \infty.$$

Since $\phi(t) = 0$ for $t \leq s$, we have

$$\begin{aligned} e^{-\gamma t} Z^{\phi}(t) &= e^{-\gamma t} \sum_{\sigma_x \leq t-s} \phi_x(t - \sigma_x) \\ &\leq e^{-\gamma t} \sum_{\sigma_x \leq t-c} h(t-\sigma_x) e^{\gamma(t-\sigma_x)} U_x \\ &\leq e^{-\gamma t} \sum_{k=1}^{\lfloor t-c \rfloor - 1} \sum_{k-1 < \sigma_x \leq k} h(t-\sigma_x) e^{\gamma(t-\sigma_x)} U_x \\ &\leq e^{-\gamma t} \sum_{k=1}^{\lfloor t-c \rfloor - 1} h(t-k) e^{\gamma(t-k+1)} \sum_{k-1 < \sigma_x \leq k} U_x \\ &= e^{-\gamma t} \sum_{k=1}^{\lfloor t-c \rfloor - 1} h(t-k) e^{\gamma(t-k+1)} Z^{\psi}(k). \end{aligned}$$

Therefore,

$$\limsup_{t \to \infty} e^{-\gamma t} Z^{\phi}(t) \leq e^{\gamma} \mathbf{E}(U) M_{\infty} \int_{s-2}^{\infty} h(u) du,$$

as required. □

From this, we deduce that Condition 6.3.2 may be used to remove the condition that $\phi(t)$ vanish for large t in Theorem 6.3.6. Indeed, it suffices to truncate and apply Lemma 6.3.17 to control the error caused by the truncation. This immediately produces the following result.

THEOREM 6.3.18. *Let $(\xi_x, \phi_x, L_x)_x$ be a general branching process with Malthusian parameter γ, where $\phi \geq 0$ and $\phi(t) = 0$ when $t \leq -s$ for some positive s. Assume that ν_γ is non-lattice. Assume further that, for some $\epsilon \in (0, \infty)$,*

$$\mathbf{E}\left[\sup_{u \leq t} Z^{\phi^\epsilon}(u)\right] < \infty,$$

and that Conditions 6.3.1, 6.3.2 and 6.3.4 are satisfied; in particular, z^{ϕ} is bounded and

$$z^{\phi}(t) \to z^{\phi}(\infty),$$

some finite constant as $t \to \infty$.

Then,

$$e^{-\gamma t} Z^{\phi}(t) \to z^{\phi}(\infty) M_{\infty}, \text{ a.s.},$$

as $t \to \infty$, *where* M_{∞} *is the almost sure limit of the fundamental martingale of the general branching process.*

Note the asymmetry of the statement: We need $\phi(-t)$ to vanish for large t. Indeed, unlike for the weak laws of large number, it is difficult to prove a strong law of large numbers directly when ϕ does not vanish for negative times for the reason described in Remark 6.3.7. In some applications, see for example [**Ham00**], this setup is appropriate. In other applications where $\phi(-t)$ does not vanish for large t, we typically use the method described in (5.7) to be in the one-sided case.

Using Proposition 6.3.5, we recover from Theorem 6.3.18 the strong law of large numbers of [**Ner81**].

THEOREM 6.3.19. *Let* $(\xi_x, L_x, \phi_x)_x$ *be a general branching process with Malthusian parameter* γ, *where* $\phi \geq 0$ *and* $\phi(t) = 0$ *for* $t < 0$. *Assume that* ν_{γ} *is non-lattice. Assume further that Conditions 6.3.1 and 6.3.2 are satisfied. Then,*

$$z^{\phi}(t) \to z^{\phi}(\infty),$$

some finite constant, and

$$e^{-\gamma t} Z^{\phi}(t) \to z^{\phi}(\infty) M_{\infty}, \text{ a.s.},$$

as $t \to \infty$, *where* M_{∞} *is the almost sure limit of the fundamental martingale of the general branching process.*

6.4. Convergence in L^1

As stated in Theorem 5.1.1, when a general branching process satisfies the $x \log x$ condition, its fundamental martingale is uniformly integrable. In particular, $M_t \to M_\infty$ in L^1, as $t \to \infty$. It is natural to ask when the $x \log x$ condition is sufficient to make the counting process Z^ϕ converge to its limit in L^1. A sufficient condition is for Z^ϕ to satisfy the weak law of large numbers a we show now.

THEOREM 6.4.1. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Malthusian parameter γ where $\phi \geq 0$. Assume that M is uniformly integrable. Assume further that*

$$z^\phi(t) \to z^\phi(\infty),$$

some finite constant, and

$$e^{-\gamma t} Z^\phi(t) \to z^\phi(t) M_\infty, \text{ in probability},$$

as $t \to \infty$.

Then,

$$e^{-\gamma t} Z^\phi(t) \to z^\phi(\infty) M_\infty, \text{ in } L^1,$$

as $t \to \infty$.

PROOF. Since M is uniformly integrable, $\mathbf{E}M_\infty = 1$. Therefore, we have that

$$e^{-\gamma t} Z^\phi(t) \to z^\phi(\infty) M_\infty, \text{ in probability}$$

and

$$e^{-\gamma t} \mathbf{E} Z^\phi(t) \to z^\phi(\infty) \mathbf{E} M_\infty,$$

as $t \to \infty$. This establishes that $(e^{-\gamma t} Z^{\phi}(t), t \geq 0)$ is uniformly integrable and the convergence in L^1; see, for example, [**Dur10**]. □

6.5. Central limit theorem

We start by introducing some additional notation. Consider the general branching process $(\xi_x, L_x, \bar{\zeta}_x)_x$ with Malthusian parameter γ. We assume that $\bar{\zeta}$ is such that

$$\bar{Z}(t) = Z^{\bar{\zeta}}(t)$$

has zero expectation, where the equation defines the left-hand side. In applications, $\bar{\zeta}$ is typically a suitably centred version of some characteristic ϕ.

We will use the rescaled version $\tilde{Z}$ of $\bar{Z}$ defined by

$$\tilde{Z}(t) = e^{-\gamma t/2} \bar{Z}(t) = \tilde{\bar{\zeta}}_{\emptyset}(t) + \sum_{i=1}^{\xi(\infty)} e^{-\gamma \sigma_i/2} \tilde{Z}_i(t - \sigma_i), \tag{6.22}$$

where $\tilde{\bar{\zeta}}(t) = e^{-\gamma t/2} \bar{\zeta}(t)$.

Finally, to have a proxy for the variance, we define

$$V(t) = \bar{Z}(t)^2 = \rho_{\emptyset}(t) + \sum_{i=1}^{\xi(\infty)} V_i(t - \sigma_i), \tag{6.23}$$

where

$$\rho_{\emptyset}(t) = \bar{\zeta}_{\emptyset}(t)^2 + 2\bar{\zeta}_{\emptyset}(t) \sum_{i=1}^{\xi(\infty)} \bar{Z}_i(t - \sigma_i) + 2 \sum_{i=1}^{\xi(\infty)} \sum_{j<i} \bar{Z}_i(t - \sigma_i) \bar{Z}_j(t - \sigma_j). \tag{6.24}$$

As before, the functions v and r defined by

$$v(t) = e^{-\gamma t} \mathbf{E} V(t) \quad \text{and} \quad r(t) = e^{-\gamma t} \mathbf{E} \rho(t) \tag{6.25}$$

satisfy the renewal equation

$$v(t) = r(t) + \int_0^{\infty} v(t - s) \nu_{\gamma}(ds). \tag{6.26}$$

Our central limit theorem requires the following two technical conditions.

CONDITION 6.5.1. *There exists* $\epsilon \in (0, 1/2)$ *such that*

$$e^{-\gamma t/2} \sum_{\sigma_x \leq \epsilon t} \bar{\zeta}_x(t - \sigma_x) \to 0, \text{ in probability,}$$

as $t \to \infty$.

This is a regularity condition on $\bar{\zeta}$. In applications, we typically expect $\bar{\zeta}$ to satisfy a weak law of large numbers. Therefore, the sum should grow like $e^{\gamma \epsilon t}$ and we can expect that the condition is satisfied.

CONDITION 6.5.2. *There exists* $\alpha \in (0, \infty)$ *such that*

$$\sup_{t \in \mathbb{R}} \mathbf{E}\{|\tilde{Z}(t)|^{2+\alpha}\} < \infty.$$

This is a moment condition. In applications, it is convenient to check it for integer moment, i.e. when $\alpha = 1$ or 2, because it can be done using renewal arguments.

THEOREM 6.5.3. *Let* $(\xi_x, L_x, \bar{\zeta}_x)_x$ *be a general branching process with Malthusian parameter* γ, *where* $\bar{\zeta}$ *is such that* $\mathbf{E}\bar{Z}(t) = 0$ *for every* t. *Assume that* v *is bounded and that*

$$v(t) \to v(\infty),$$

some finite constant, as $t \to \infty$. *Assume further that Conditions 6.5.1 and 6.5.2 are satisfied. Then,*

$$\tilde{Z}(t) \to \tilde{Z}_\infty, \text{ in distribution,}$$

as $t \to \infty$, *where the distribution of* $\tilde{Z}_\infty$ *is characterised by*

$$\mathbf{E}\left[e^{i\theta \tilde{Z}_\infty}\right] = \mathbf{E}\left[e^{-\frac{1}{2}\theta^2 v(\infty) M_\infty}\right].$$

In the proof, we will use that if $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ are complex numbers whose modulus is bounded by C, then

$$\left| \prod_{i=1}^{n} z_i - \prod_{i=1}^{n} w_i \right| \leq C^{n-1} \sum_{i=1}^{n} |z_i - w_i|. \tag{6.27}$$

A proof of this may be found in [**Dur10**, Lemma 3.4.3].

PROOF. For $\epsilon \in (0, 1/2)$, iterating from (6.22), the definition of $\tilde{Z}$, we get

$$\tilde{Z}(t) = \sum_{\sigma_x \leq \epsilon t} e^{-\gamma \sigma_x/2} \tilde{\zeta}_x(t - \sigma_x) + \sum_{x \in \Lambda_{\epsilon t}} e^{-\gamma \sigma_x/2} \tilde{Z}_x(t - \sigma_x). \tag{6.28}$$

The first sum appearing in (6.28) can be rewritten as

$$e^{-\gamma t/2} \sum_{\sigma_x \leq \epsilon t} \bar{\zeta}_x(t - \sigma_x),$$

which, by Condition 6.5.1, converges to 0 in probability as $t \to \infty$ if we choose ϵ appropriately. For the rest of the proof, we fix such a choice of ϵ.

We now consider the other sum appearing in (6.28), and show that it converges in distribution to $\tilde{Z}_\infty$ as $t \to \infty$. The result will then follow from Slutsky's lemma. In other words, for $\theta \in \mathbb{R}$, we want to show that

$$\mathbf{E}\left[e^{i\theta \sum_{x \in \Lambda_{\epsilon t}} e^{-\gamma \sigma_x/2} \tilde{Z}(t - \sigma_x)} - e^{-\frac{1}{2}\theta^2 v(\infty) M_\infty} \right] \to 0, \tag{6.29}$$

as $t \to \infty$. To do this, write, for an $x_0 \in (0, 1)$ that will be chosen below,

$$A_{\epsilon,t} = \left\{ \sup_{x \in \Lambda_{\epsilon t}} |\theta e^{-\gamma \sigma_x/2} \tilde{Z}_x(t - \sigma_x)| \leq x_0 \right\},$$

and split (6.29) as

$$\mathbf{E}\left[e^{i\theta \sum_{x \in \Lambda_{\epsilon t}} e^{-\gamma \sigma_x/2} \tilde{Z}_x(t - \sigma_x)} - e^{-\frac{1}{2}\theta^2 v(\infty) M_\infty}; A_{\epsilon,t}^c \right] \tag{6.30}$$

$$+ \mathbf{E}\left[e^{i\theta \sum_{x \in \Lambda_{\epsilon t}} e^{-\gamma \sigma_x/2} \tilde{Z}_x(t - \sigma_x)} - e^{-\frac{1}{2}\theta^2 v(\infty) \sum_{x \in \Lambda_{\epsilon t}} e^{-\gamma \sigma_x}}; A_{\epsilon,t} \right] \tag{6.31}$$

$$+ \mathbf{E}\left[e^{-\frac{1}{2}\theta^2 v(\infty) \sum_{x \in \Lambda_{\epsilon t}} e^{-\gamma \sigma_x}} - e^{-\frac{1}{2}\theta^2 v(\infty) M_\infty}; A_{\epsilon,t} \right]. \tag{6.32}$$

We will show that each of these terms converges to 0 as $t \to \infty$.

Fix $\theta \in \mathbb{R}$ and $\delta \in (0,1)$. Let $x_0 \in (0,1)$ be such that

$$|e^z - 1 - z| \le \delta|z| \quad \text{and} \quad \left|e^z - 1 - z - \frac{z^2}{2}\right| \le \delta|z|^2, \tag{6.33}$$

whenever $z \in \mathbb{C}$ satisfies $|z| \le x_0$. And let $\tau \in (0,\infty)$ be such that for $t \ge \tau$,

$$x_0^{-(2+\alpha)}|\theta|^{2+\alpha} \sup_{u\in\mathbb{R}} \mathbf{E}\{|\tilde{Z}(u)|^{2+\alpha}\}e^{-\gamma\epsilon\alpha t/2} \le \delta, \tag{6.34}$$

$$\theta^2\|v\|_\infty e^{-\gamma\epsilon t} \le x_0 \tag{6.35}$$

and

$$|v(t/2) - v(\infty)| \le \delta, \tag{6.36}$$

where α is given by Condition 6.5.2.

Let us start by dealing with (6.30). For $t \ge \tau$,

$$\begin{aligned}\mathbf{P}(A_{\epsilon,t}^c) &= \mathbf{P}\left(\sup_{x\in\Lambda_{\epsilon t}} |\theta e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)|^{2+\alpha} > x_0^{2+\alpha}\right)\\ &\le \mathbf{P}\left(\sum_{x\in\Lambda_{\epsilon t}} |\theta|^{2+\alpha} e^{-\gamma\sigma_x(1+\alpha/2)}|\tilde{Z}_x(t-\sigma_x)|^{2+\alpha} \ge x_0^{2+\alpha}\right),\end{aligned}$$

which, by Markov's inequality, is bounded by

$$\begin{aligned}&{x_0}^{-(2+\alpha)}|\theta|^{2+\alpha}e^{-\gamma\epsilon\alpha t/2}\mathbf{E}\left[\sum_{x\in\Lambda_{\epsilon t}} e^{-\gamma\sigma_x}|\tilde{Z}_x(t-\sigma_x)|^{2+\alpha}\right]\\ &\le x_0^{-(2+\alpha)}|\theta|^{2+\alpha}e^{-\gamma\epsilon\alpha t/2}\mathbf{E}\left[\sum_{x\in\Lambda_{\epsilon t}} e^{-\gamma\sigma_x}\mathbf{E}\{|\tilde{Z}_x(t-\sigma_x)|^{2+\alpha}|\mathcal{F}_{\epsilon t}\}\right]\\ &\le x_0^{-(2+\alpha)}|\theta|^{2+\alpha}\sup_{u\in\mathbb{R}}\mathbf{E}\{|\tilde{Z}(u)|^{2+\alpha}\}e^{-\gamma\epsilon\alpha t/2}\\ &\le \delta,\end{aligned} \tag{6.37}$$

where we have used that $\sigma_x \in \mathcal{F}_{\epsilon t}$, that $\tilde{Z}_x$ is independent of $\mathcal{F}_{\epsilon t}$ and that $\mathbf{E}M_{\epsilon t} = 1$. (This is where we need to control the moment of order $2+\alpha$ for some $\alpha \in (0,\infty)$.) Therefore, (6.30) is dominated by

$$2\mathbf{P}(A_{\epsilon,t}^c) \le 2\delta.$$

Since δ is arbitrary, it follows that

$$\mathbf{E}\left[e^{i\theta\sum_{x\in\Lambda_{\epsilon t}}e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)}-e^{-\frac{1}{2}\theta^2v(\infty)M_\infty};A^c_{\epsilon,t}\right]\to 0,$$

as $t\to\infty$, as required.

To deal with (6.32), recall that

$$\sum_{x\in\Lambda_{\epsilon t}}e^{-\gamma\sigma_x}=M_{\epsilon t}\to M_\infty,\text{ a.s.},$$

as $t\to\infty$. Therefore, dominated convergence implies that

$$\mathbf{E}\left[e^{-\frac{1}{2}\theta^2v(\infty)\sum_{x\in\Lambda_{\epsilon t}}e^{-\gamma\sigma_x}}-e^{-\frac{1}{2}\theta^2v(\infty)M_\infty};A_{\epsilon,t}\right]\to 0,$$

as $t\to\infty$, as required.

We will spend the rest of the proof dealing with (6.31), which we rewrite as

$$\mathbf{E}\left[\prod_{x\in\Lambda_{\epsilon t}}\mathbf{E}\left\{e^{i\theta e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)}\mathbf{1}_{A^x_{\epsilon,t}}\middle|\mathcal{F}_{\epsilon t}\right\}-\prod_{x\in\Lambda_{\epsilon t}}\mathbf{E}\left\{e^{-\frac{1}{2}\theta^2v(\infty)e^{-\gamma\sigma_x}}\mathbf{1}_{A^x_{\epsilon,t}}\middle|\mathcal{F}_{\epsilon t}\right\}\right]$$

using the conditional independence built into the branching structure, where

$$A^x_{\epsilon,t}=\{|\theta e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)|\le x_0\}.$$

By (6.27), the term inside the expectation satisfies

(6.38)

$$\begin{aligned}&\left|\prod_{x\in\Lambda_{\epsilon t}}\mathbf{E}\left\{e^{i\theta e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)}\mathbf{1}_{A^x_{\epsilon,t}}\middle|\mathcal{F}_{\epsilon t}\right\}-\prod_{x\in\Lambda_{\epsilon t}}\mathbf{E}\left\{e^{-\frac{1}{2}\theta^2v(\infty)e^{-\gamma\sigma_x}}\mathbf{1}_{A^x_{\epsilon,t}}\middle|\mathcal{F}_{\epsilon t}\right\}\right|\\&\quad\le\sum_{x\in\Lambda_{\epsilon t}}\left|\mathbf{E}\left\{e^{i\theta e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)}\mathbf{1}_{A^x_{\epsilon,t}}\middle|\mathcal{F}_{\epsilon t}\right\}-\mathbf{E}\left\{e^{-\frac{1}{2}\theta^2v(\infty)e^{-\gamma\sigma_x}}\mathbf{1}_{A^x_{\epsilon,t}}\middle|\mathcal{F}_{\epsilon t}\right\}\right|.\end{aligned}$$

A Taylor expansion and the second order exponential estimate yield that

$$\begin{aligned}&\left|\mathbf{E}\left\{e^{i\theta e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)}\mathbf{1}_{A^x_{\epsilon,t}}\middle|\mathcal{F}_{\epsilon t}\right\}\right.\\&\qquad\left.-\mathbf{E}\left\{\left(1+i\theta e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)-\frac{1}{2}\theta^2e^{-\gamma\sigma_x}\tilde{Z}_x(t-\sigma_x)^2\right)\mathbf{1}_{A^x_{\epsilon,t}}\middle|\mathcal{F}_{\epsilon t}\right\}\right|\\&\quad\le\delta\theta^2\|v\|_\infty e^{-\gamma\sigma_x}.\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\left|\mathbf{E}\left\{\left(1+i\theta e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)-\frac{1}{2}\theta^2 e^{-\gamma\sigma_x}\tilde{Z}_x(t-\sigma_x)^2\right)\mathbf{1}_{A^x_{\epsilon,t}}\middle|\,\mathcal{F}_{\epsilon t}\right\}\right.\\
&\qquad\left.-\,\mathbf{E}\left\{e^{-\frac{1}{2}\theta^2 v(\infty)e^{-\gamma\sigma_x}}\mathbf{1}_{A^x_{\epsilon,t}}\middle|\,\mathcal{F}_{\epsilon t}\right\}\right|\\
&\quad\le\left|\mathbf{E}\left\{\left(i\theta e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)\right)\mathbf{1}_{A^x_{\epsilon,t}}\middle|\,\mathcal{F}_{\epsilon t}\right\}\right|\\
&\qquad+\left|\mathbf{E}\left\{\left(1-\frac{1}{2}\theta^2 e^{-\gamma\sigma_x}\tilde{Z}_x(t-\sigma_x)^2-e^{-\frac{1}{2}\theta^2 v(\infty)e^{-\gamma\sigma_x}}\right)\mathbf{1}_{A^x_{\epsilon,t}}\middle|\,\mathcal{F}_{\epsilon t}\right\}\right|.
\end{aligned}$$

Now, since

$$\mathbf{E}\left[\tilde{Z}_x(t-\sigma_x)\middle|\,\mathcal{F}_{\epsilon t}\right]=0,$$

reasoning as in (6.37) shows that

$$\begin{aligned}
&\left|\mathbf{E}\left\{\left(i\theta e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)\right)\mathbf{1}_{A^x_{\epsilon,t}}\middle|\,\mathcal{F}_{\epsilon t}\right\}\right|\\
&\quad=\left|\mathbf{E}\left\{\left(i\theta e^{-\gamma\sigma_x/2}\tilde{Z}_x(t-\sigma_x)\right)\mathbf{1}_{\Omega\setminus A^x_{\epsilon,t}}\middle|\,\mathcal{F}_{\epsilon t}\right\}\right|\\
&\quad\le x_0\mathbf{E}\left\{x_0^{-1}|\theta|e^{-\gamma\sigma_x/2}|\tilde{Z}_x(t-\sigma_x)|\mathbf{1}_{\Omega\setminus A^x_{\epsilon,t}}\middle|\,\mathcal{F}_{\epsilon t}\right\}\\
&\quad\le x_0\mathbf{E}\left\{x_0^{-(2+\alpha)}|\theta|^{2+\alpha}e^{-\gamma\sigma_x(1+\alpha/2)}|\tilde{Z}_x(t-\sigma_x)|^{2+\alpha}\mathbf{1}_{\Omega\setminus A^x_{\epsilon,t}}\middle|\,\mathcal{F}_{\epsilon t}\right\}\\
&\quad\le\delta e^{-\gamma\sigma_x},
\end{aligned}\tag{6.39}$$

for $t\ge\tau$. And, since for $x\in\Lambda_{\epsilon t}$ we have

$$|v(\infty)\theta^2 e^{-\gamma\sigma_x}/2|\le\theta^2\|v\|_\infty e^{-\gamma\epsilon t}\le x_0,$$

a Taylor expansion and the first moment exponential estimate yield that

$$\begin{aligned}
&\left|\mathbf{E}\left\{\left(1-\frac{1}{2}\theta^2 e^{-\gamma\sigma_x}\tilde{Z}_x(t-\sigma_x)^2-e^{-\frac{1}{2}\theta^2 v(\infty)e^{-\gamma\sigma_x}}\right)\mathbf{1}_{A^x_{\epsilon,t}}\middle|\,\mathcal{F}_{\epsilon t}\right\}\right|\\
&\quad\le\left|\mathbf{E}\left\{-\frac{1}{2}\theta^2 e^{-\gamma\sigma_x}\tilde{Z}_x(t-\sigma_x)^2+\frac{1}{2}\theta^2 v(\infty)e^{-\gamma\sigma_x}\middle|\,\mathcal{F}_{\epsilon t}\right\}\right|+\frac{1}{2}\delta\theta^2|v(\infty)|e^{-\gamma\sigma_x}\\
&\quad\le\theta^2\{|v(t-\sigma_x)-v(\infty)|+\delta\|v\|_\infty\}e^{-\gamma\sigma_x}.
\end{aligned}$$

Notice that, for $t \geq \tau$,

$$\begin{aligned}
&\sum_{x\in\Lambda_{\epsilon t}} \theta^2\{|v(t-\sigma_x)-v(\infty)|+\delta\|v\|_\infty\}e^{-\gamma\sigma_x} \\
&\leq \theta^2 \sum_{x\in\Lambda_{\epsilon t}\setminus\Lambda_{t/2}} \delta(1+\|v\|_\infty)e^{-\gamma\sigma_x} + \theta^2 \sum_{x\in\Lambda_{\epsilon t}\cap\Lambda_{t/2}} (2+\delta)\|v\|_\infty e^{-\gamma\sigma_x} \\
&\leq \delta\theta^2(1+\|v\|_\infty)M_{\epsilon t} + \theta^2(2+\delta)\|v\|_\infty \sum_{x\in\Lambda_{\epsilon t}\cap\Lambda_{t/2}} e^{-\gamma\sigma_x}.
\end{aligned} \tag{6.40}$$

Together, Equations (6.38) to (6.40) show that (6.31) is dominated by

$$\mathbf{E}\left[\delta[1+\theta^2(1+2\|v\|_\infty)]M_{\epsilon t}\right] + \mathbf{E}\left[\theta^2(2+\delta)\|v\|_\infty \sum_{x\in\Lambda_{\epsilon t}\cap\Lambda_{t/2}} e^{-\gamma\sigma_x}\right]. \tag{6.41}$$

Fix $c \in (0,\infty)$ large. For t large enough, by Lemma 6.2.2,

$$\mathbf{E}\left[\sum_{x\in\Lambda_{\epsilon t}\cap\Lambda_{t/2}} e^{-\gamma\sigma_x}\right] \leq \mathbf{E}\left[\sum_{x\in\Lambda_{\epsilon t}\cap\Lambda_{\epsilon t+c}} e^{-\gamma\sigma_x}\right] \to \mu_1^{-1}\int_c^\infty (1-\nu_\gamma(s))ds,$$

as $t\to\infty$. The limiting expression can be made arbitrarily small by choosing c large enough. Therefore, the second term in (6.41) converges to 0. Since $\mathbf{E}M_{\epsilon t}=1$, the first term can also be made arbitrarily small by adjusting δ. The result follows. □

6.6. The lattice case

When ν_γ is supported on the lattice $\lambda\mathbb{Z}$, the $z^\phi(t)$ does not have a limit as $t\to\infty$. However, renewal theory provides conditions under which, for every y,

$$z^\phi(y+k\lambda)\to g(y),$$

as $k\to\infty$, for some λ periodic function g. This can then be used to prove laws of large numbers which state that, for every y,

$$e^{-\gamma(y+k\lambda)}Z^\phi(y+k\lambda)\to g(y)M_\infty, \text{ in probability or a.s.},$$

as $k\to\infty$.

We refrain from developing this here and refer the reader to [**Gat00a**], where Gatzouras proved analogues of the results of [**Ner81**] for the lattice case.

As for the central limit theorem, the statement given here does not depend on the non-lattice structure. Of course, when ν_γ is lattice, we cannot expect $v(t)$ to have a limit as $t \to \infty$. Instead we typically get that, for every y,

$$v(y + k\lambda) \to g(y),$$

as $k \to \infty$, where $g(y)$ is a λ-periodic function. Nevertheless, in this case, the proof presented above shows that, for every y,

$$\tilde{Z}(y + k\lambda) \to \tilde{Z}_\infty(y), \text{ in distribution,}$$

as $k \to \infty$, where the distribution of $\tilde{Z}_\infty(y)$ is characterised by

$$\mathbf{E}\left[e^{i\theta Z_\infty(y)}\right] = \mathbf{E}\left[e^{-\frac{1}{2}\theta^2 g(y) M_\infty}\right].$$

CHAPTER 7

Heat content asymptotics with Dirichlet weights

In this chapter, we define a class of bounded open subsets of the unit interval whose boundary is a statistically self-similar Cantor set. This class has the property that the construction of the boundary is determined using certain Dirichlet distributions, which makes it easier to apply the central limit theorem given in Chapter 6.

7.1. Construction of statistically self-similar sets with Dirichlet weights

Choose $\gamma \in (0,1)$, $n \geq 2$ and consider the random vector

$$(T_1, \ldots, T_n) \sim \mathrm{Dir}(\alpha),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, Dir denotes the Dirichlet distribution. That is, $(T_1, \ldots, T_n)$ takes values in the simplex

$$\{(x_1, \ldots, x_n) \in \mathbb{R}_+^n : x_1 + \cdots + x_n = 1\},$$

where it has a density given by

$$\frac{\Gamma(\alpha_0)}{\prod_{i=1}^n \Gamma(\alpha_i)} \prod_{i=1}^n x_i^{\alpha_i - 1};$$

here, Γ denotes the Gamma function and we have set

$$\alpha_0 = \alpha_1 + \cdots + \alpha_n,$$

as usual. Start with the unit interval $K_0 = [0,1]$ and replace it by n equally spaced intervals of length $T_1^{1/\gamma}, \ldots, T_n^{1/\gamma}$. Replace each of these intervals by n



FIGURE 1. First 4 iterations of the construction of K with the distribution $\mathrm{Dir}(1,1,1)$ and $\gamma = 0.6$.

intervals created independently with the same procedure. Iterating indefinitely, we obtain a decreasing sequence of compact sets $(K_j, j \geq 0)$, and $K = \cap_j K_j$ is a statistically self-similar Cantor set. Indeed, in the notation of Subsection 5.2, it suffices to set

$$(N, R_1, \ldots, R_N) = (n, T_1^{1/\gamma}, \ldots, T_n^{1/\gamma});$$

the maps $(\Phi_1, \ldots, \Phi_N)$ can easily be deduced from this. The corresponding general branching process (ξ, L) (no characteristic just yet) is obtained as detailed in Section 5.2. Figure 1 depicts the first 4 iterations in the construction of the set K with the distribution $\mathrm{Dir}(1,1,1)$.

The set in which we are interested in this chapter is $U = [0,1] \setminus K$, whose boundary is K by construction. Thanks to the Lindelöf property, U is a countable union of intervals. As such, U can be thought of as a *random string* in the sense of Lapidus and coauthors; see [**LP93, LP96, LvF00**] and references therein.

A key aspect that we will need is that the fundamental martingale M of the general branching process corresponding to this statistically self-similar set satisfies $M \equiv 1$. This follows because the Dirichlet distributions are supported on the simplex.

7.2. Fractal dimension and Minkowski content

Consider the set U defined above and notice that the Hausdorff dimension of $\partial U = K$ is γ. Our aim in this section is to show that the Minkowski dimension of K exists and is also equal to γ, and to determine the volume of the inner tubular neighbourhoods and the inner Minkowski content of U.

THEOREM 7.2.1. *The Minkowski dimension of K is almost surely equal to γ.*

PROOF. Consider the characteristic function defined by

$$\phi(t) = [\xi(\infty) - \xi(t)]\mathbf{1}_{t\geq 0}.$$

The corresponding counting process Z^ϕ counts the number of offspring born after time t to parents born up to time t. As such, $Z^\phi(t)$ is an upper bound for the covering number $N(e^{-t}, K)$ of K with balls of radius e^{-t}.

By the strong law of large numbers, we have

$$e^{-\gamma t} Z^\phi(t) \to \frac{1}{\mu_1} \int_0^\infty \mathbf{E} e^{-\gamma s} \phi(s) ds \in (0, \infty), \text{ a.s.},$$

as $t \to \infty$. This is easily used to check that $\overline{\dim}_M K \leq \gamma$ almost surely. The result follows since $\dim K = \gamma$ almost surely by Theorem 5.2.1. □

We now give a finer picture of the geometry of the set U by looking at the volume of its inner tubular neighbourhoods μ.

THEOREM 7.2.2. *Consider the set U with statistically self-similar boundary K defined above. We have*

$$\epsilon^{\gamma-1}\mu(\epsilon) \to \mathcal{M}, \text{ a.s. and in } L^1,$$

as $\epsilon \to 0$, for some positive constant $\mathcal{M}$. In particular, the inner Minkowski content of U exists, is finite, and is equal to $\mathcal{M}$.

In particular, this result implies that the inner Minkowski dimension of K coincides with the usual definition.

PROOF. By construction U is made of n i.i.d. copies U_i of itself scaled by R_i and $n-1$ intervals. Therefore,

$$\mu_U(\epsilon) = \theta(\epsilon) + \sum_{i=1}^{n} \mu_{R_i U_i}(\epsilon), \tag{7.1}$$

where $\theta(\epsilon)$ is the volume of of the inner ϵ-tubular neighbourhoods of the $n-1$ intervals and is therefore bounded by $2(n-1)\epsilon$. By scaling,

$$\mu_{R_i U_i}(\epsilon) = R_i \mu_{U_i}(R_i^{-1}\epsilon).$$

In the language of the general branching process, each $R_i U_i$ corresponds to an offspring of U born at time $\sigma_i = -\log R_i$. Therefore, putting, for $t \in \mathbb{R}$,

$$Z^\phi(t) = e^t \mu_U(e^{-t}) \quad \text{and} \quad \phi(t) = e^t \theta(e^{-t}), \tag{7.2}$$

the relations (7.1) and (7.2) combine to produce

$$Z^\phi(t) = \phi_\emptyset(t) + \sum_{i=1}^{\xi(\infty)} Z_i^\phi(t - \sigma_i),$$

where the Z_i^ϕ are i.i.d. copies of Z and ϕ is bounded. In particular, Z^ϕ is the counting process of the characteristic ϕ.

To apply the strong law of large numbers, we need to consider

$$Z^\chi(t) = Z^\phi(t)\mathbf{1}_{t\geq 0} = \chi_\emptyset(t) + \sum_{i=1}^{\xi(\infty)} Z_i^\chi(t - \sigma_i),$$

defined using (5.7).

Since $\xi(\infty) = n$ and $Z^\phi(t)$ is bounded for negative times (by 1, the measure of the unit interval), the characteristic χ is bounded as well. These observations

show that we can apply Theorem 6.3.19 to get that

$$e^{-\gamma t} Z^{\chi}(t) \to \mathcal{M}, \text{ a.s.},$$

as $t \to \infty$, where

$$\mathcal{M} = \frac{1}{\mu_1} \int_0^{\infty} e^{-\gamma s} \mathbf{E}\chi(s) ds \in (0, \infty).$$

Here we used that $M \equiv 1$ so that, in particular, $M_\infty = 1$. This also implies that the martingale is uniformly integrable, from which we get that the convergence also takes place in L^1 thanks to Theorem 6.4.1. The result follows. □

7.3. Application of the central limit theorem for the general branching process with Dirichlet weights

This section is devoted to discussing how to apply the central limit theorem for the general branching process with Dirichlet weights corresponding to the statistically self-similar set K described in Section 7.1. We write $(\xi_x, L_x, \phi_x)_x$ for this general branching process where ϕ is a characteristic that we assume regular enough to satisfy a weak law of large numbers, i.e.

$$e^{-\gamma t} Z^{\phi}(t) \to z^{\phi}(\infty), \text{ in probability},$$

as $t \to \infty$.

Preliminaries. We are interested in studying the asymptotic properties of the expression

$$e^{\gamma t/2}[e^{-\gamma t} Z^{\phi}(t) - z^{\phi}(\infty)] = e^{\gamma t/2}[e^{-\gamma t} Z^{\phi}(t) - z^{\phi}(t)] + e^{\gamma t/2}[z^{\phi}(t) - z^{\phi}(\infty)]. \tag{7.3}$$

Our aim is to apply Theorem 6.5.3 to the first term of the right-hand side. We will see that conditions making this possible ensure that the second term

in the right-hand side converges to 0. This will then produce a result on the fluctuations of $Z^\phi(t)$ of the desired form, thanks to Slutsky's lemma.

To start, notice that, in the notation of Section 6.5,

$$Z^\phi(t) - e^{\gamma t} z^\phi(t) = \bar{Z}(t) = Z^{\bar{\zeta}}(t)$$

is a centred characteristic counting process, if we define $\bar{\zeta}$ by

$$\begin{aligned} \bar{\zeta}_\emptyset(t) &= \phi_\emptyset(t) + \sum_{i=1}^n e^{\gamma(t-\sigma_i)} z^\phi(t-\sigma_i) - e^{\gamma t} z^\phi(t) \\ &= \phi_\emptyset(t) + \sum_{i=1}^n e^{\gamma(t-\sigma_i)} [z^\phi(t-\sigma_i) - z^\phi(t)], \end{aligned} \tag{7.4}$$

where we used that $e^{-\gamma\sigma_1} + \cdots + e^{-\gamma\sigma_n} = 1$. This representation can be used to get the following control on $\bar{\zeta}$.

LEMMA 7.3.1. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Dirichlet weights α. Assume that*

$$|z^\phi(t) - z^\phi(\infty)| \le c_1 e^{-\beta_1 t} \wedge c_2,$$

for some $\beta_1 \in [0, \gamma]$. Then,

$$|\bar{\zeta}(t)| \le |\phi(t)| + 2n \left(c_1 e^{(\gamma-\beta_1)t} \wedge c_2 e^{\gamma t} \right).$$

PROOF. Notice that

$$\begin{aligned} |\bar{\zeta}_\emptyset(t)| &\le |\phi_\emptyset(t)| + \sum_{i=1}^n e^{\gamma(t-\sigma_i)} \{ |z^\phi(t-\sigma_i) - z^\phi(\infty)| + |z^\phi(t) - z^\phi(\infty)| \} \\ &\le |\phi_\emptyset(t)| + \sum_{i=1}^n \left(c_1 e^{(\gamma-\beta_1)(t-\sigma_i)} \wedge c_2 e^{\gamma(t-\sigma_i)} \right) + n \left(c_1 e^{(\gamma-\beta_1)t} \wedge c_2 e^{\gamma t} \right) \\ &\le |\phi_\emptyset(t)| + \left(c_1 \sum_{i=1}^n e^{(\gamma-\beta_1)(t-\sigma_i)} \wedge c_2 \sum_{i=1}^n e^{\gamma(t-\sigma_i)} \right) + n \left(c_1 e^{(\gamma-\beta_1)t} \wedge c_2 e^{\gamma t} \right) \end{aligned}$$

The result follows upon noticing that

$$\sum_{i=1}^n e^{(\gamma-\beta_1)(t-\sigma_i)} = e^{(\gamma-\beta_1)t} \sum_{i=1}^n e^{-(\gamma-\beta_1)\sigma_i} \le n e^{(\gamma-\beta_1)t},$$

and proceeding similarly for the other sum. □

This lemma and the decomposition in (7.3) show that understanding the rate of convergence of $z^\phi(t)$ to its limit in the renewal theorem is critical to describe the fluctuations of Z^ϕ.

Convergence rate in the renewal theorem. Consider the renewal equation

$$z(t) = u(t) + \int_0^\infty z(t-s)F(ds), \tag{7.5}$$

where F is a non-lattice distribution function on $[0, \infty)$. The key to solving the renewal equation is the renewal measure

$$H = \sum_{n=0}^{\infty} F^{*n},$$

where F^{*n} denotes the n-fold convolution of F with itself, because z is typically given by

$$z(t) = \int_0^\infty u(t-y)H(dy); \tag{7.6}$$

see [**Fel68, Kar55**] among others. The renewal theorem of [**Fel68**] states that if F has a finite mean μ_1, then

$$H(t) \sim \mu_1^{-1} t,$$

where $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$. Under some regularity conditions on u, e.g. if u is directly Riemann integrable, this can be used to show that

$$z(t) \to z(\infty) = \frac{1}{\mu_1} \int_{-\infty}^{\infty} u(s)ds,$$

as $t \to \infty$; see [**Fel68**].

The error in the linear approximation of the renewal function

$$G(t) = H(t) - \mu_1^{-1} t$$

has been studied extensively. When F has a finite second moment μ_2,

$$G(t) \to \frac{\mu_2}{2\mu_1^2}, \tag{7.7}$$

as $t \to \infty$; see [**Fel68**].

The rate of convergence of $G(t)$ to its limit in this case can be studied using the Fourier transform f or F defined by

$$f(w) = \int_{-\infty}^{\infty} e^{ws} F(ds), \quad w \in \mathbb{C}.$$

We refer the reader to [**Fel68, Lea64, Sto65a, Sto65b**] and references therein for proofs; see also Appendix B of [**Kig01**] for a discussion of the lattice case. In particular, Stone proved the following theorem in [**Sto65a**].

THEOREM 7.3.2 (Stone). *Suppose that there exists $r_1 \in (0, \infty)$ such that $f(w)$ is analytic and $\neq 1$ when $\Re w \in (0, r_1)$. Then, for every $r \in (0, r_1)$,*

$$G(t) - \frac{\mu_2}{2\mu_1^2} = O(e^{-rt}),$$

as $t \to \infty$.

Since we aim to apply Lemma 7.3.1, we will be particularly interested in exponential rates of convergence of $z(t)$ to its limit. This rate of convergence is connected to that in Theorem 7.3.2 by the following lemma adapted from [**CH10**] which we include for convenience.

LEMMA 7.3.3. *Let z, u and F satisfy the renewal equation* (7.5). *Suppose that*

$$z(t) = \int_0^{\infty} u(t-y) H(dy) \to \mu_1^{-1} \int_{-\infty}^{\infty} u(y) dy,$$

as $t \to \infty$. Then,

$$z(\infty) - z(t) = \mu_1^{-1} \int_0^{\infty} u(t+y) dy - \int_0^{\infty} u(t-y) G(dy).$$

PROOF. It suffices to notice that

$$z(\infty) - z(t) = \mu_1^{-1} \int_0^\infty u(t+y)dy + \mu_1^{-1} \int_0^\infty u(t-y)dy - \int_0^\infty u(t-y)H(dy),$$

and use the definition of G. □

Explicit calculation for some Dirichlet weights. In general, it is difficult to determine the solutions to $f(w) = 1$ needed to use Theorem 7.3.2. In some cases with Dirichlet weights which we discuss now, however, we can use some properties of the Gamma function to study f and deduce convergence rates for the renewal theorem.

Let $X \sim \mathrm{Dir}(\alpha)$ be a random vector in $\mathbb{R}^n$. It is well-known that, for every i,

$$X_i \sim \mathrm{Beta}(\alpha_i, \alpha_0 - \alpha_i),$$

where Beta denotes the Beta distribution. Recall that if $Y \sim \mathrm{Beta}(\beta_1, \beta_2)$ then

$$\mathbf{E}Y^\theta = \frac{\mathrm{B}(\theta + \beta_1, \beta_2)}{\mathrm{B}(\beta_1, \beta_2)},$$

where B is the Beta function, i.e.

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Therefore, we get that

$$\psi(\theta) = \mathbf{E}\left[\sum_{i=1}^n X_i^\theta\right] = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_0+\theta)} \sum_{i=1}^n \frac{\Gamma(\alpha_i+\theta)}{\Gamma(\alpha_i)},$$

where the equation defines ψ. For the general branching process with Dirichlet weights defined above, it follows that

(7.8)

$$f(w) = \int_{-\infty}^\infty e^{ws}\nu_\gamma(ds) = \int_{-\infty}^\infty e^{(w-\gamma)s}\nu(ds) = \mathbf{E}\left[\sum_{i=1}^n e^{-\gamma\sigma_i(1-w/\gamma)}\right] = \psi(1-w/\gamma).$$

If $\alpha_0 - \alpha_i \in \mathbb{Z}$ for every i, we can use that $\Gamma(w+1) = w\Gamma(w)$ to reduce the function ψ to a rational function which may be simpler to analyse. Notice that

this assumption implies in particular that there exist some $a \in \mathbb{R}$ and $\ell_i \in \mathbb{Z}$ such that, for every i, we have $\alpha_i = a + \ell_i$. Therefore,

$$\alpha_0 = a + \ell_0 = \alpha_1 + \cdots + \alpha_n = na + \ell_1 + \cdots + \ell_n,$$

from which it follows that $(n-1)a \in \mathbb{Z}$.

Let us now see how these observations enable us to establish the desired rate of convergence for two simple cases of Dirichlet weights.

LEMMA 7.3.4. *Consider the general branching processes with Dirichlet weights α described above. Assume that*

$$\alpha_1 = \cdots = \alpha_n = \frac{k}{n-1}, \quad k \in \{1, 2\}.$$

Then the Fourier transform $f(w)$ of ν_γ is analytic and $\neq 1$ when $\Re w \in (0, \gamma]$. In particular,

$$G(t) - \frac{\mu_2}{2\mu_1^2} = O(e^{-\gamma t}),$$

as $t \to \infty$.

PROOF. For $k = 1$, a direct calculation shows that

$$\psi(\theta) = \frac{n}{1 + (n-1)\theta}, \quad \theta > -1/(n-1).$$

In particular, the only solution to $\psi(\theta) = 1$ is $\theta = 1$. Using (7.8) shows that $f(w)$ is analytic and $\neq 1$ on $\Re w \in (0, \{1 + (n-1)^{-1}\}\gamma)$.

For $k = 2$, a direct calculation shows that

$$\psi(\theta) = \frac{(\alpha+1)(\alpha+2)}{(\theta+\alpha)(\theta+\alpha+1)} = \frac{p(1)}{p(\theta)}, \quad \theta > -\alpha,$$

say, where we kept the notation $\alpha = 2/(n-1)$ for legibility. Now we know that the equation

$$p(\theta) - p(1) = \theta^2 + (2\alpha+1)\theta - 2(\alpha+1) = 0$$

has one, and therefore two (not necessarily distinct), real solutions, one of them being $\theta_1 = 1$. So the other one has to be

$$\theta_2 = \frac{-(2\alpha+1) - \sqrt{(2\alpha+1)^2 + 8(\alpha+1)}}{2} = -2(\alpha+1).$$

In particular, the only solution to $\psi(\theta) = 1$ that is different from $\theta_1 = 1$ is given by θ_2. This means that the two solutions to $f(w) = 1$ are

$$w_1 = 0 \quad \text{and} \quad w_2 = (2\alpha+3)\gamma.$$

Reasoning as above shows that $f(w)$ is analytic and $\neq 1$ on $\Re w \in (0, (1+\alpha)\gamma)$.

The rest of the statement follows from Theorem 7.3.2. □

To discuss more complicated examples and the sharpness of Theorem 7.3.2 for the general branching process with the Dirichlet weights α satisfying

$$\alpha_i = a + \ell_i \text{ for some } a \in \mathbb{R} \text{ and some } \ell_i \in \mathbb{Z}$$

described above, we need to study the Fourier transform of G.

By definition of G, we have

$$\begin{aligned} G * F(t) &= H * F(t) - \mu_1^{-1} \int_0^\infty (t-s) F(ds) \\ &= H(t) - \mathbf{1}_{t\geq 0} - \mu_1^{-1} t + 1 \\ &= G(t) + \mathbf{1}_{t<0}. \end{aligned}$$

Denoting by g the Fourier transform of G and f that of F, we thus get that

$$g(w) = \frac{1}{1-f(w)}, \quad w \in \mathbb{C}.$$

Recall from (7.8) that for the general branching process with Dirichlet weights α, the function f satisfies

$$f(w) = \psi(1 - w/\gamma) = \sum_{i=1}^{n} \frac{1}{P_i(w)},$$

where P_i is a polynomial of degree $\alpha_0 - \alpha_i$. It follows that

$$\begin{aligned} g(w) &= \frac{\prod_{i=1}^n P_i(w)}{\prod_{i=1}^n P_i(w) - \sum_{i=1}^n \prod_{j\neq i} P_j(w)} \\ &= 1 + \frac{\sum_{i=1}^n \prod_{j\neq i} P_j(w)}{\prod_{i=1}^n P_i(w) - \sum_{i=1}^n \prod_{j\neq i} P_j(w)} \\ &= 1 + \frac{R(w)}{Q(w)}, \end{aligned}$$

say. It is easy to check that

$$\deg R < (n-1)\alpha_0 = \deg Q.$$

Now, decompose g into partial fractions and write

$$g(w) = 1 + \sum_{i=1}^q \frac{Q_i(w)}{(w-\rho_i)^{m_i}},$$

where $(\rho_i, i \leq q)$ are the roots of Q with corresponding multiplicities m_i and Q_i are polynomials with $\deg Q_i < m_i$ for every i; in particular,

$$m_1 + \cdots + m_q = (n-1)\alpha_0.$$

Recall that, for $k \in \mathbb{Z}_+$,

$$\int_{-\infty}^{\infty} e^{\lambda t} t^k e^{-rt} \mathbf{1}_{t\geq 0} dt = \int_0^\infty t^k e^{(\lambda - r)t} dt = \frac{(-1)^{k+1} k!}{(\lambda - r)^{k+1}}$$

and therefore that

$$\int_0^\infty e^{\lambda t} \frac{d^k}{dt^k} \left(t^k e^{-rt}\right) dt = \frac{(-1)^{k+1} k! \lambda^k}{(\lambda - r)^{k+1}}. \tag{7.9}$$

Using this, it is easy to check that

$$G(dt) = \delta_0(t)dt + \sum_{\Re\rho_i<0} \tilde{Q}_i(t) e^{\rho_i t} \mathbf{1}_{t<0} dt + \sum_{\Re\rho_i\geq 0} \tilde{Q}_i(t) e^{-\rho_i t} \mathbf{1}_{t\geq 0} dt,$$

where the $\tilde{Q}_i$ are polynomials determined using (7.9) and satisfying $\deg \tilde{Q}_i < m_i$. Of course, since F is supported on $[0, \infty)$, so is H and therefore, by definition of G, we have

$$G(t)\mathbf{1}_{t<0} = -\mu_1^{-1} t \mathbf{1}_{t<0}.$$

Putting these considerations together shows that

$$G(dt) = \delta_0(t)dt - \mu_1^{-1}\mathbf{1}_{t<0}dt + \sum_{\Re\rho_i>0} \tilde{Q}_i(t)e^{-\rho_i t}\mathbf{1}_{t\geq 0}dt, \tag{7.10}$$

which we can integrate to study the asymptotics of G. A particular example which will guide us below is given in the following lemma.

LEMMA 7.3.5. *Assume that all the roots of Q are simple. Then,*

$$G(t) = -\mu_1^{-1}\mathbf{1}_{t<0}t + \frac{\mu_2}{2\mu_1^2}\mathbf{1}_{t\geq 0} + \sum_{\Re\rho_i>0} \frac{c_i}{\rho_i}e^{-\rho_i t}\mathbf{1}_{t\geq 0},$$

where $c_i = \mathrm{Res}(g;\rho_i)$, the residue of g at ρ_i.

Since none of the singularities can be removed and since they all have order one, all the c_i are non zero. Furthermore, since if ρ_i is a root with residue c_i then $\bar{\rho}_i$ is a root with residue $\bar{c}_i$ as $g(\bar{w}) = \overline{g(w)}$, roots with the same real part cannot cancel out. In particular, in this case, the result of Theorem 7.3.2 is sharp.

PROOF. Since all the roots of Q are simple, we have

$$g(w) = 1 + \sum_{i=1}^{q} \frac{c_i}{w-\rho_i},$$

where we must have $c_i = \mathrm{Res}(g;\rho_i)$. Integrating (7.10) then shows that

$$G(t) = \mathbf{1}_{t\geq 0} - \mu_1^{-1}\mathbf{1}_{t<0}t - \sum_{\Re\rho_i>0} \frac{c_i}{\rho_i}\mathbf{1}_{t\geq 0} + \sum_{\Re\rho_i>0} \frac{c_i}{\rho_i}e^{-\rho_i t}\mathbf{1}_{t\geq 0}.$$

Since F has a second moment, the result now follows from (7.7) after letting $t \to \infty$. □

Checking the conditions of the central limit theorem. Discussing the conditions of the central limit theorem involves growth estimates on $\bar{\zeta}$ which can be verified using Lemma 7.3.1.

We start by looking at Condition 6.5.1 in the following lemma.

LEMMA 7.3.6. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Dirichlet weights α and let $\bar{\zeta}$ be defined as in* (7.4). *Assume that*

$$|\bar{\zeta}(t)| \leq c_1 + c_2 e^{\beta t},$$

for some $\beta \in (0, \gamma/2)$. Then, Condition 6.5.1 is satisfied.

PROOF. Let $\epsilon \in (0, \gamma/2 - \beta)$. Then, for $t \geq 0$,

$$\left| e^{-\gamma t/2} \sum_{\sigma_x \leq \epsilon t} \bar{\zeta}(t - \sigma_i) \right| \leq e^{(\beta+\epsilon-\gamma/2)t} e^{-\epsilon t} \sum_{\sigma_x \leq \epsilon t} (c_1 + c_2).$$

By the weak law of large numbers for the general branching process,

$$e^{-\epsilon t} \sum_{\sigma_x \leq \epsilon t} 1 \to \frac{1}{\mu_1} \int_0^\infty e^{-\gamma t} dt = \frac{1}{\gamma \mu_1} \text{ in probability,}$$

as $t \to 0$. So the result follows since $\beta + \epsilon < \gamma/2$. □

Coupled with Lemma 7.3.1, this result illustrates that we should not expect to have a central limit theorem when $z^\phi(t) - z^\phi(\infty)$ does not decay at least as fast as $e^{-\gamma t/2}$, the threshold for the second term in the right-hand side of (7.3) to vanish.

For the discussion of the existence of the limiting variance, we assume that $(\phi_x)_x$ is i.i.d. This is sufficient for our applications and enables us to have natural sufficient conditions as above. When the characteristics are not i.i.d., a direct calculation can be used instead.

The existence of the limiting variance can be established using the following lemma.

LEMMA 7.3.7. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Dirichlet weights α. Assume that $(\phi_x)_x$ is i.i.d. and that*

$$|\bar{\zeta}(t)| \leq c_1 (e^{\beta_1 t} \mathbf{1}_{t \geq 0} + e^{\beta_2 t} \mathbf{1}_{t < 0}),$$

for some positive β_1 and β_2 satisfying $\beta_1 < \gamma/2 < \beta_2$. Then, the function v defined by the renewal equation (6.25) *has a positive finite limit $v(\infty)$ as $t \to \infty$.*

PROOF. Thanks to the independence of $(\phi_x)_x$, we have that

$$r(t) = e^{-\gamma t}\rho(t) = e^{-\gamma t}\mathbf{E}|\bar{\zeta}(t)|^2 \leq c_1^2 e^{-\tau|t|}.$$

for some positive constant τ, where r and ρ are defined in (6.25). Therefore, by the renewal theorem of [**LV96**],

$$v(t) \to v(\infty) = \frac{1}{\mu_1}\int_{-\infty}^{\infty} r(s)ds \in (0,\infty),$$

as required. □

Finally, let us discuss the moment condition for the central limit theorem. In the cases discussed here, it is convenient to check Condition 6.5.2 separately for positive and negative times, and to do it for an integer moment (e.g. $\alpha = 1$ or 2 in the statement), so that we can rely on renewal theory. The next lemmas discuss a set of sufficient assumptions for the moment condition to hold.

LEMMA 7.3.8. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Dirichlet weights α. Assume that*

$$|Z^\phi(t)| \leq c_1 e^{\gamma t/2} \text{ for } t \leq 0,$$

for some positive constant c_1 and that z^ϕ is bounded. Then,

$$\sup_{t\leq 0} \mathbf{E}|\tilde{Z}(t)|^3 < \infty.$$

PROOF. By definition,

$$|\bar{Z}(t)| \leq |Z^\phi(t)| + e^{\gamma t}|z^\phi(t)| \leq c_2 e^{\gamma t/2},$$

for $t \leq 0$. Therefore,

$$|\tilde{Z}(t)|^3 = e^{-3\gamma t/2}|\bar{Z}(t)|^3 \leq c_2^3,$$

and the result follows. □

To deal with the third moment of $\tilde{Z}$ for positive times, we will use the following observation.

LEMMA 7.3.9. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Dirichlet weights α. Then, we have*

$$\mathbf{E}\left[\sum_{x\in\mathcal{T}} e^{-y\gamma\sigma_x}\right] = \sum_{k=0}^{\infty} \psi(y)^k,$$

and the sum is finite if $y \in (1,\infty)$.

PROOF. By monotone convergence,

$$\mathbf{E}\left[\sum_{x\in\mathcal{T}} e^{-y\gamma\sigma_x}\right] = \sum_{k=0}^{\infty} \mathbf{E}\left[\sum_{|x|=k} e^{-y\gamma\sigma_x}\right].$$

Further, conditioning on the birth times, we get that

$$\begin{aligned}\mathbf{E}\left[\sum_{|x|=k} e^{-y\gamma\sigma_x}\right] &= \mathbf{E}\left[\sum_{|x|=k-1}\sum_{i=1}^{n} e^{-y\gamma\sigma_x} e^{-y\gamma(\sigma_{x,i}-\sigma_x)}\right] \\ &= \psi(y)\mathbf{E}\left[\sum_{|x|=k-1} e^{-y\gamma\sigma_x}\right].\end{aligned}$$

Iterating this and summing over k proves the equality with the infinite sum. Noting that $\psi(y) < 1$ for $y \in (1,\infty)$ completes the proof. □

Let us close the discussion of the third moment condition with the following lemma which deals with positive times. It is naturally framed in the setup where ϕ vanishes for negative times.

LEMMA 7.3.10. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Dirichlet weights α. Assume that $\phi(t) = 0$ for $t < 0$, that*

$$|\bar{\zeta}(t)| \le c_1 e^{\gamma t/2} \text{ for } t \ge 0$$

and that v is bounded. Then,

$$\sup_{t\geq 0} \mathbf{E}|\tilde{Z}(t)|^3 < \infty.$$

PROOF. Notice that

$$\bar{Z}(t)^3 = W_\emptyset(t) + \sum_{i=1}^n \bar{Z}_i(t-\sigma_i)^3,$$

where

$$W_\emptyset(t) = \bar{\zeta}_\emptyset(t)^3 + 3\bar{\zeta}_\emptyset(t)^2 \sum_{i=1}^n \bar{Z}_i(t-\sigma_i) + 3\bar{\zeta}_\emptyset(t) \sum_{i,j=1}^n \bar{Z}_i(t-\sigma_i)\bar{Z}_j(t-\sigma_j)$$
$$+ \sum_{\substack{i,j,k=1 \\ \text{not all equal}}}^3 \bar{Z}_i(t-\sigma_i)\bar{Z}_j(t-\sigma_j)\bar{Z}_k(t-\sigma_k).$$

Therefore, it is clear that $|\bar{Z}(t)|^3$ is bounded by

$$\sum_{x\in\mathcal{T}} |\bar{\zeta}_x(t-\sigma_x)|^3 \tag{7.11}$$

$$+ 3\sum_{x\in\mathcal{T}} |\bar{\zeta}_x(t-\sigma_x)|^2 \sum_{i=1}^n |\bar{Z}_{x,i}(t-\sigma_{x,i})| \tag{7.12}$$

$$+ 3\sum_{x\in\mathcal{T}} |\bar{\zeta}_x(t-\sigma_x)| \sum_{i,j=1}^n |\bar{Z}_{x,i}(t-\sigma_{x,i})|\,|\bar{Z}_{x,j}(t-\sigma_{x,j})| \tag{7.13}$$

$$+ \sum_{x\in\mathcal{T}} \sum_{\substack{i,j,k=1 \\ \text{not all equal}}}^n |\bar{Z}_{x,i}(t-\sigma_{x,i})|\,|\bar{Z}_{x,j}(t-\sigma_{x,j})|\,|\bar{Z}_{x,k}(t-\sigma_{x,k})|. \tag{7.14}$$

To prove the lemma, it is sufficient to check that $e^{-3\gamma t/2}$ times the expectation of each of these terms is bounded, which we do now.

To deal with (7.11), note that

$$e^{-3\gamma t/2}\mathbf{E}\left[\sum_{x\in\mathcal{T}} |\bar{\zeta}(t-\sigma_x)|^3\right] \leq c_1^3 \mathbf{E}\left[\sum_{x\in\mathcal{T}} e^{-\frac{3}{2}\gamma\sigma_x}\right] = c_1^3 \sum_{k=0}^\infty \psi(3/2)^k < \infty,$$

thanks to Lemma 7.3.9.

Notice that our assumption on the boundedness of v implies that

$$\mathbf{E}\bar{Z}(t)^2 \leq c_2 e^{\gamma t}.$$

Therefore, we can control the term corresponding to (7.12) using that

$$\begin{aligned}
&e^{-3\gamma t/2}\mathbf{E}\sum_{x\in\mathcal{T}}|\bar{\zeta}_x(t-\sigma_x)|^2\sum_{i=1}^{n}|\bar{Z}_{x,i}(t-\sigma_{x,i})| \\
&\quad\leq \mathbf{E}\left[\sum_{x\in\mathcal{T}}|\bar{\zeta}_x(t-\sigma_x)|^2\sum_{i=1}^{n}\mathbf{E}[\bar{Z}_{x,i}(t-\sigma_{x,i})^2|\mathcal{F}_x]^{1/2}\right] \\
&\quad\leq c_1^2c_2\mathbf{E}\left[\sum_{x\in\mathcal{T}}e^{-\gamma\sigma_i}\sum_{i=1}^{n}e^{-\gamma\sigma_{x,i}/2}\right] \\
&\quad\leq nc_1^2c_2\sum_{k=1}^{\infty}\psi(3/2)^k \\
&\quad< \infty,
\end{aligned}$$

using that $\sigma_{x,i}\geq\sigma_x$ and Lemma 7.3.9.

To deal with (7.13), we proceed similarly to get that

$$\begin{aligned}
&e^{-3\gamma t/2}\mathbf{E}\sum_{x\in\mathcal{T}}|\bar{\zeta}_x(t-\sigma_x)|\sum_{i,j=1}^{n}|\bar{Z}_{x,i}(t-\sigma_{x,i})|\,|\bar{Z}_{x,j}(t-\sigma_{x,j})| \\
&\quad\leq ne^{-3\gamma t/2}\mathbf{E}\sum_{x\in\mathcal{T}}|\bar{\zeta}_x(t-\sigma_x)|\sum_{i=1}^{n}|\bar{Z}_{x,i}(t-\sigma_{x,i})|^2 \\
&\quad\leq n^2c_1c_2\sum_{k=0}^{\infty}\psi(3/2)^k.
\end{aligned}$$

The similar argument needed to bound the term corresponding to (7.14) relies on the observation that, if i,j and k are not all equal, then, assuming without loss of generality that i is different, we have

$$\begin{aligned}
&\mathbf{E}\{|\bar{Z}_i(t-\sigma_i)|\,|\bar{Z}_j(t-\sigma_j)|\,|\bar{Z}_k(t-\sigma_k)||\mathcal{F}_\emptyset\} \\
&\quad=\mathbf{E}\{|\bar{Z}_i(t-\sigma_i)||\mathcal{F}_\emptyset\}\mathbf{E}\{|\bar{Z}_j(t-\sigma_j)|\,|\bar{Z}_k(t-\sigma_k)||\mathcal{F}_\emptyset\} \\
&\quad\leq(\mathbf{E}[\bar{Z}_i(t-\sigma_i)^2|\mathcal{F}_\emptyset])^{1/2}(\mathbf{E}[\bar{Z}_j(t-\sigma_j)^2|\mathcal{F}_\emptyset])^{1/2}(\mathbf{E}[\bar{Z}_k(t-\sigma_k)^2|\mathcal{F}_\emptyset])^{1/2}.
\end{aligned}$$

Using this and reasoning as above then shows that

$$e^{-3\gamma t/2}\mathbf{E}\sum_{x\in\mathcal{T}}\sum_{\substack{i,j,k=1\\ \text{not all equal}}}^{n}|\bar{Z}_{x,i}(t-\sigma_{x,i})|\,|\bar{Z}_{x,j}(t-\sigma_{x,j})|\,|\bar{Z}_{x,k}(t-\sigma_{x,k})|$$
$$\leq n^3 c_2\sum_{k=0}^{\infty}\psi(3/2)^k.$$

The proof is complete. □

These results are easily combined to produce the following lemma.

LEMMA 7.3.11. *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Dirichlet weights α. Assume that $(\phi_x)_x$ is i.i.d. and that*

$$|z^{\phi}(t) - z^{\phi}(\infty)| \leq c_1 e^{-\beta_1 t},$$

for some $\beta_1 > \gamma/2$, that

$$|\phi(t)| \leq c_2 + c_3 e^{\beta_3 t},$$

for some $\beta_3 < \gamma/2$ and that

$$Z^{\phi}(t) \leq c_4 e^{\gamma t/2}, \quad t \leq 0.$$

Then, all the conditions of Theorem 6.5.3 are satisfied.

PROOF. By Lemmas 7.3.1 and 7.3.6, Condition 6.5.1 is satisfied. By Lemmas 7.3.1 and 7.3.7, there is a limiting variance. By Lemma 7.3.8, the moment condition is satisfied for negative times. Furthermore, by Lemma 7.3.1, working with the characteristic vanishing for negative times defined in (5.7), Lemma 7.3.10 shows that the moment condition is satisfied for positive times. In other words, Condition 6.5.2 is satisfied. □

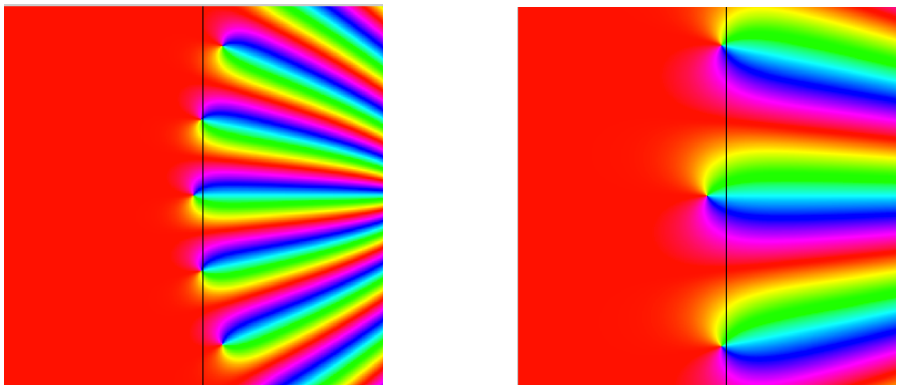


FIGURE 2. The left-hand side is the phase plot of $1 - f(\gamma w)$ for $w \in [-10, 10]+i[-10, 10]$. The right-hand side is the phase plot of $g(\gamma w)$ for $w \in [-5, 5]+i[-5, 5]$. The black line is $\{w \in \mathbb{C} : \Re w = 1/2\}$.

An example with no central limit theorem. We now discuss the case when the Dirichlet weights are $\alpha = (10, 10, 10, 10, 10)$. In that case (many similar examples can be constructed),

$$G(t) - \frac{\mu_2}{2\mu_1^2}$$

cannot be controlled by $e^{-\gamma t/2}$.

To see this, we look at the functions $f(\gamma w)$ and $g(\gamma w)$, where we rescale by γ because the discussion does not really depend on it. The corresponding phase plots are given in Figure 2.

Notice that $1 - f(\gamma w)$ has simple roots at

$$\rho_\pm \simeq 0.3832 \pm 3.965i;$$

the ones with the smallest positive real part. The residues at the corresponding poles of $g(\gamma w)$ can be evaluated numerically[1] to be

$$c_\pm \simeq -0.6297 \pm 0.1212i.$$

[1] The numerical integration was performed using the Matlab package Chebfun developed at the Mathematical Institute of the University of Oxford by the members of Nick Trefthen's numerical analysis group. Mohsin Javed's help and expertise are gratefully acknowledged.

Therefore, the reasoning in (7.9) and Lemma 7.3.5 show that, for $t \geq 0$,

$$G(t) - \frac{\mu_2}{2\mu_2^2} = d_- e^{-\rho_- \gamma t} + d_+ e^{-\rho_+ \gamma t} + o(e^{-\Re\rho_+ \gamma t}),$$

as $t \to \infty$, for some d_- and d_+ that are conjugate and non zero. In particular, in this case, Theorem 7.3.2 is sharp.

Since $\Re\rho_- = \Re\rho_+ \in (0, 1/2)$, the decomposition in (7.3) indicates that there can be no central limit theorem of the desired form. Furthermore, none of the Lemmas 7.3.1, 7.3.3, 7.3.6, 7.3.7 and 7.3.10 are applicable.

7.4. Fluctuations of the volume of the tubular neighbourhoods

In this section, we return to the setup of Section 7.2 and use the central limit theorem for the general branching process to study the fluctuations in Theorem 7.2.2. Using the results of Section 7.3, we prove the following theorem.

THEOREM 7.4.1. *Consider the set U with statistically self-similar boundary K defined in Section 7.1. Assume that the Dirichlet weights are such that*

$$G(t) - \frac{\mu_2}{2\mu_1^2} = O(e^{-\beta t}),$$

for some $\beta \in (\gamma/2, \infty)$. Then,

$$\epsilon^{-\gamma/2}[\epsilon^{\gamma-1}\mu(\epsilon) - \mathcal{M}] \to N(0, \sigma_\mu^2), \text{ in distribution}$$

as ϵ, where N stands for the normal distribution, σ_μ^2 is a positive constant and $\mathcal{M}$ is given in Theorem 7.2.2.

PROOF. We use the notation introduced in the proof of Theorem 7.2.2 and check the conditions of Theorem 6.5.3.

Notice that

$$|\phi(t)| \leq c_1(e^{-t} \vee 1).$$

Therefore, using Lemmas 7.3.3 and 7.3.1, we see that

$$|\bar{\zeta}^{\phi}(t)| \leq c_2 e^{(\gamma-\beta)t},$$

when $t \geq 0$, where we included ϕ in the notation to emphasise that we are using that characteristic. Furthermore, it is easy to check from the definition that

$$|\bar{\zeta}^{\phi}(t)| \leq c_3 e^{\gamma t},$$

when $t \leq 0$.

Since $(\phi_x)_x$ is i.i.d., Lemma 7.3.7 thus shows that there is a limiting variance $v(\infty) \in (0, \infty)$. Thanks to the renewal theorem in Appendix A, we also see that v is bounded.

To conclude, it thus suffices to check the third moment condition. For negative times, this easily done using Lemma 7.3.8. Indeed,

$$Z^{\phi}(t) = e^t \mu_U(e^{-t}) \leq c_4 e^t \leq c_4 e^{\gamma t/2},$$

when $t \leq 0$ since $\gamma \in (0, 1)$. For positive times, we need to use the characteristic χ instead and the observation that

$$\bar{Z}^{\phi}(t) = \bar{Z}^{\chi}(t) \text{ when } t \geq 0,$$

which follows from the definition in (5.7). Here again, we included the characteristic in the notation to emphasise which one is used. Therefore, the third moment condition for positive times follows from Lemma 7.3.10 since χ is bounded as observed in the proof of Theorem 7.2.2.

In other words, we have checked all the conditions of Lemma 7.3.11. We can now apply Theorem 6.5.3 with our definitions and get the required result. □

7.5. Heat content asymptotics

The study of the heat content asymptotics of the set U defined in Section 7.1 closely resembles that of the volume of the tubular neighbourhoods and we address it now. We will use that, by scaling, if D is a domain of $\mathbb{R}^d$, then for $\kappa \in (0, \infty)$, the heat content satisfies

$$E_{\kappa D}(s) = \kappa^d E_D(\kappa^{-2} s) \tag{7.15}$$

in the proof of the following theorem.

THEOREM 7.5.1. *Consider the set U with statistically self-similar boundary K defined in Section 7.1. We have*

$$s^{-(1-\gamma)/2} E(s) \to \mathcal{E}, \text{ a.s. and in } L^1,$$

as $s \to 0$, for some positive constant $\mathcal{E}$, where $\gamma = \dim_M \partial K$.

The proof largely follows that of Theorem 7.2.2 and we therefore omit some details for brevity.

PROOF. By construction of U,

$$E_U(s) = (n-1) E_{R_0 I}(s) + \sum_{i=1}^{n} E_{R_i U_i}(s),$$

where U_i are i.i.d. copies of U, $I = [0,1]$ and $(n-1)R_0 = 1 - R_1 - \cdots - R_n$ is the length of the remaining intervals.

Putting, for $t \in \mathbb{R}$,

$$Z^\phi(t) = e^t E_U(e^{-2t}) \quad \text{and} \quad \phi(t) = e^t (n-1) E_{R_0 I}(e^{-2t}),$$

the scaling (7.15) implies that

$$Z^\phi(t) = \phi(t) + \sum_{i=1}^{n} Z_i(t - \sigma_i),$$

where Z_i are i.i.d. copies of Z. Furthermore, by scaling again,

$$\phi(t) = R_0 e^t E_I(e^{-2t})$$

is bounded thanks to (3.9). The end of the proof now follows as that of Theorem 7.2.2. □

7.6. Fluctuations of the heat content

Proceeding as in Section 7.4, we can study the fluctuations of the heat content content of the set U and get the following result.

THEOREM 7.6.1. *Consider the set U with statistically self-similar boundary K defined in Section 7.1. Assume that the Dirichlet weights are such that*

$$G(t) - \frac{\mu_2}{2\mu_1^2} = O(e^{-\beta t}),$$

for some $\beta \in (\gamma/2, \infty)$. Then,

$$s^{-\gamma/4}[s^{(1-\gamma)/2}E(s) - \mathcal{E}] \to N(0, \sigma_E^2), \text{ in distribution}$$

as ϵ, where N stands for the normal distribution, σ_E^2 is a positive constant and $\mathcal{E}$ is given in Theorem 7.5.1.

PROOF. Follow the proof of Theorem 7.4.1. □

Note that these conditions are verified, for example, for the set depicted in Figure 1, as is easily checked using the results in Section 7.3.

7.7. A note on continuum random trees

Dirichlet weights simplify the applications of the central limit theorem because the fundamental martingale is equal to 1 and thus they may seem somewhat artificial. However, in [**CH08**], Croydon and Hambly showed that the

Brownian continuum random tree can be represented as a statistically self-similar sets with $\mathrm{Dir}(1/2, 1/2, 1/2)$ distribution. The study of the spectral asymptotics of the continuum random tree motivates a detailed discussion of fractals with Dirichlet weights. Further, in [**CH10**], Croydon and Hambly perform a similar analysis for stable trees. It is a natural and open question to determine if and when a stable central limit theorem for the general branching process exists and captures fluctuations of the spectral asymptotics of those fractals.

CHAPTER 8

Statistically-self similar snowflakes

In this chapter, we define a class of statistically self-similar snowflakes and study their geometry and heat content.

Let us stress that the discussion here can be extended to any other snowflakes with statistically self-similar boundary so long as the corresponding general branching process is regular enough. However, for brevity, we focus on a particular class that can easily be connected to the discussion in Chapter 4. Our aim is to emphasise that the behaviour of the heat content is qualitatively different when the boundary of the snowflake is statistically self-similar and not spatially homogeneous, because of additional spatial independence. The material in this chapter is taken from [**Cha14**].

8.1. Construction

Start with the segment $[0,1]$ and pick $a \in A$, a bounded subset of $\mathbb{N}$, randomly according to the probability distribution $(p_a, a \in A)$. Consider the building block $K(a)$ described in Chapter 4. Then replace each linear piece of $K(a)$ by a scaled i.i.d. copy of itself, i.e. again using $(p_a, a \in A)$. Iterating indefinitely, we obtain a sequence of curves converging to a statistically self-similar curve K. Indeed, in the notation of Subsection 5.2, it suffices to set

$$(N(a), R_1(a), \ldots, R_N(a)) = (m(a), \ell(a)^{-1}, \ldots, \ell(a)^{-1});$$

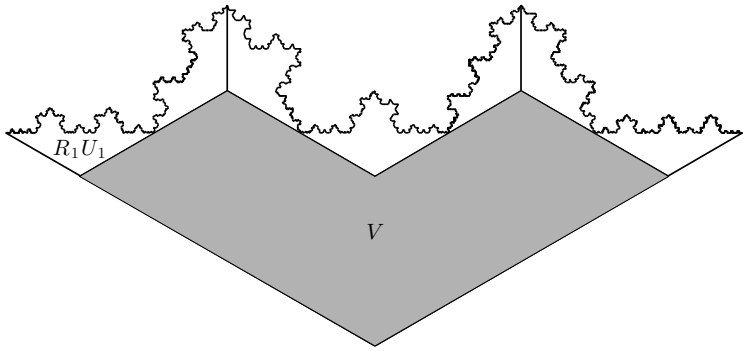


FIGURE 1. Statistically self-similar curve K for a third of the snowflake U with its natural decomposition.

the maps $(\Phi_1, \ldots, \Phi_N)$ can easily be deduced from this. The corresponding general branching process (ξ, L) (no characteristic just yet) is obtained as detailed in Section 5.2. Figure 1 contains an approximation of K when $A = \{1, 2, 3\}$ and $(p_a, a \in A)$ is uniform.

Finally, the snowflake D is defined as the simply connected interior of the Jordan curve created using three i.i.d. copies of K.

To ensure that ν_γ is non-lattice, we will make the following assumption.

ASSUMPTION 8.1.1. *The set A contains two elements a_1 and a_2 such that*

$$p_{a_1} \text{ and } p_{a_2} \in (0,1) \quad \text{and} \quad \frac{\log \ell(a_1)}{\log \ell(a_2)} \notin \mathbb{Q}.$$

8.2. Fractal dimension and Minkowski content

Now that we have defined the statistically self-similar snowflakes, we will use the general branching process to study their geometry. First, note that the proof of Theorem 7.2.1 applies here without a change to show that the Minkowski dimension of ∂D exists and is equal to γ almost surely. We therefore move directly to the study of the volume of the inner tubular neighbourhoods

of D with the following result which implies that for this example, the inner Minkowski dimension of ∂D coincides with the usual definition.

THEOREM 8.2.1. *For the statistically self-similar snowflake D, we have*

$$\epsilon^{\gamma-2}\mu(\epsilon) \to \mathcal{M}N_\infty, \text{ a.s. and in } L^1,$$

as $\epsilon \to 0$, for some positive constant $\mathcal{M}$ and positive random variable N_∞ with unit expectation. In particular, the inner Minkowski content exists, is finite, and given by $\mathcal{M}N_\infty$ almost surely.

PROOF. The snowflake D is built from three i.i.d. copies of U, the open set whose boundary is made of two linear pieces and K, as depicted in Figure 1.

We will focus on how to deal with one such third U for now. To do that, put

$$\tilde{\mu}_U(\epsilon) = \mathrm{vol}_2(\{x \in U : d(x, K) \leq \epsilon\}),$$

i.e. $\tilde{\mu}$ measures the volume of the part of the inner tubular neighbourhood close to the fractal part K of the boundary of U only.

By construction of K, the set U is made of a polygonal region V (shaded in Figure 1), say, and $m(a)$ copies U_i of itself scaled by a random factor R_i. Therefore,

$$\tilde{\mu}_U(\epsilon) = \theta(\epsilon) + \sum_{i=1}^{m(a)} \tilde{\mu}_{R_iU_i}(\epsilon), \tag{8.1}$$

where $\theta(\epsilon)$ is an error term bounded by $c_1\epsilon^2$, as is clear from Figure 1 (the existence of the constant c_1 also uses that the set A and therefore $m(a)$ is bounded). Furthermore, by scaling,

$$\tilde{\mu}_{R_iU_i}(\epsilon) = R_i^2\tilde{\mu}_{U_i}(R_i^{-1}\epsilon). \tag{8.2}$$

In the language of the general branching process, each R_iU_i corresponds to an offspring of the fractal K born at time $\sigma_i = -\log R_i$. Therefore, putting, for $t \in \mathbb{R}$,

$$Z^\phi(t) = e^{2t}\tilde{\mu}_U(e^{-t}) \quad \text{and} \quad \phi(t) = e^{2t}\theta(e^{-t}),$$

the relations (8.1) and (8.2) combine to produce

$$Z^\phi(t) = \phi(t) + \sum_{i=1}^{\xi(\infty)} Z_i^\phi(t - \sigma_i),$$

where the Z_i^ϕ are i.i.d. copies of Z and ϕ is bounded. In particular, Z^ϕ is the counting process of the characteristic ϕ.

To apply the strong law of large numbers, we need to consider

$$Z^\chi(t) = Z^\phi(t)\mathbf{1}_{t\geq 0} = \chi(t) + \sum_{i=1}^{\xi(\infty)} Z_i^\chi(t - \sigma_i),$$

defined using (5.7).

Since $\xi(\infty)$ is bounded (because A is) and $Z^\phi(t)$ is bounded for negative times (by $\mathrm{vol}_2(U)$), the characteristic χ must be bounded as well. The observations imply that Conditions 6.3.1 and 6.3.2 are satisfied and that M is uniformly integrable thanks to Theorem 5.1.1. Therefore, by Theorems 6.3.19 and 6.4.1,

$$e^{-\gamma t}Z^\chi(t) \to z^\chi(\infty)M_\infty \text{ a.s. and in } L^1,$$

as $t \to \infty$, where

$$z^\chi(\infty) = \frac{\int_0^\infty u^\chi(t)dt}{\int_0^\infty t\nu_\gamma(dt)} \in (0, \infty).$$

By definition of Z^χ, this shows that

$$\epsilon^{\gamma-2}\tilde{\mu}_U(\epsilon) \to z^\chi(\infty)M_\infty. \tag{8.3}$$

As μ is the sum of three i.i.d. copies of $\tilde{\mu}_U$, we get the desired result by putting $\mathcal{M} = 3z^\chi(\infty)$ and setting N_∞ to be a third of the sum of the three i.i.d. copies of M_∞ given by the general branching process. □

8.3. Heat content asymptotics

Following the analysis performed above for the volume of the inner tubular neighbourhoods of D, we now use the general branching process to study the heat content of D. Our aim is to prove the following result.

THEOREM 8.3.1. *For the statistically self-similar Koch snowflake D*

$$s^{\gamma/2-1}E(s) \to \mathcal{E}N_\infty, \text{ a.s. and in } L^1,$$

as $s \to 0$, for some positive constant $\mathcal{E}$ and positive random variable N_∞ with unit expectation, where $\gamma = \dim_M \partial D$.

The random variable appearing in this theorem is the same as that appearing in Theorem 8.2.1. So this theorem implies that, for statistically self-similar snowflakes, one can recover almost surely both the Minkowski dimension and the inner Minkowski content from short time asymptotics of the heat content.

In the proof, we will use the elementary fact that if D is a domain in $\mathbb{R}^d$ and $r \in (0, \infty)$, then

$$E_{rD}(s) = r^d E_D(r^{-2}s). \tag{8.4}$$

PROOF. As in the proof for the volume of the inner tubular neighbourhood, we first only consider a third U of the snowflake and let w be the solution of the heat equation with the boundary condition

$$w(s,x) = \begin{cases} 1, & x \text{ in the fractal part of } \partial U, \\ 0, & x \text{ in the linear part of } \partial U, \end{cases}$$

and the initial condition

$$w(0,x) = 0, \quad x \in U.$$

Now write F_U for the heat content of U with this altered boundary condition, i.e.

$$F_U(s) = \int_U w(s,x)dx.$$

Before we can use the theory of general branching processes, we need some notation to understand the effect of adding extra cooling inside U. More precisely, write $\tilde{w}$ for the solution of the heat equation with boundary condition

$$\tilde{w}(s,x) = \begin{cases} 1, & x \text{ in the fractal part of } \partial U, \\ 0, & x \text{ in the linear part of } \partial U \text{ or in } \partial V, \end{cases}$$

where V is the polygonal region defined in the proof of Theorem 8.2.1, and the initial condition

$$\tilde{w}(0,x) = 0, \quad x \in U.$$

Let $\tilde{F}_U$ be the corresponding heat content

$$\tilde{F}_U(s) = \int_U \tilde{w}(s,x)dx.$$

With these definitions, we have

$$F_U(s) = F_U(s) - \tilde{F}_U(s) + \sum_{i=1}^{m(a)} F_{R_iU_i}(s) = \psi(s) + \sum_{i=1}^{m(a)} F_{R_iU_i}(s),$$

say. By scaling (8.4),

$$F_{R_iU_i}(s) = R_i^2 F_{U_i}(R_i^{-2}s) = e^{-2\sigma_i} F_{U_i}(e^{2\sigma_i}s).$$

Putting, for $t \in \mathbb{R}$,

$$Z^\phi(t) = e^{2t}F_U(e^{-2t}) \quad \text{and} \quad \phi(t) = e^{2t}\psi(e^{-2t}),$$

yields

$$Z^\phi(t) = \phi(t) + \sum_{i=1}^{\xi(\infty)} Z_i^\phi(t - \sigma_i),$$

where the Z_i^ϕ are i.i.d. copies of Z^ϕ.

Let us now show that ϕ is bounded. This is done using the results of [**vdBG98**] about the impact on the heat content of imposing extra cooling. Put

$$\lambda(\epsilon) = \mathrm{vol}_2(\{x \in U : d(x,S) < \epsilon/\sqrt{2} \text{ and } d(x,V) < \epsilon/\sqrt{2}\}),$$

where S is the fractal part of ∂U, i.e.

$$S = \overline{\{x \in \partial U : w(-,x) = 1\}}.$$

It is easy to check (by drawing a picture) that

$$\lambda(\epsilon) \leq c_1 \epsilon^2.$$

Therefore, by Corollary 1.3 of [**vdBG98**] and an integration by parts,

$$0 \leq \psi(s) \leq c_2 \int_0^\infty e^{-\epsilon^2/4s} \lambda(d\epsilon) = c_2 2^{-1} s^{-1} \int_0^\infty \lambda(\epsilon) \epsilon e^{-\epsilon^2/4s} d\epsilon \leq c_3 s \wedge c_4.$$

From this, it follows that ϕ is bounded.

Now reasoning as in the proof of Theorem 8.2.1 shows that

$$s^{\gamma/2-1} F_U(s) \to \mathcal{F} M_\infty, \text{ a.s. and in } L^1,$$

as $s \to 0$ for some positive constant $\mathcal{F}$.

To conclude, recall that D is the union of three i.i.d. copies of U, say $U^{(1)}, \ldots, U^{(3)}$. Furthermore, by the estimate of [**vdBG98**] again, we have

$$E_D(s) = \sum_{i=1}^3 F_{U^{(i)}}(s) + O(s),$$

as $s \to 0$. It follows that

$$s^{\gamma/2-1} E_D(s) \to \mathcal{E} N_\infty, \text{ a.s. and in } L^1,$$

as $s \to 0$, where $\mathcal{E} = 3\mathcal{F}$ and N_∞ is defined as in the proof of Theorem 8.2.1, as required. □

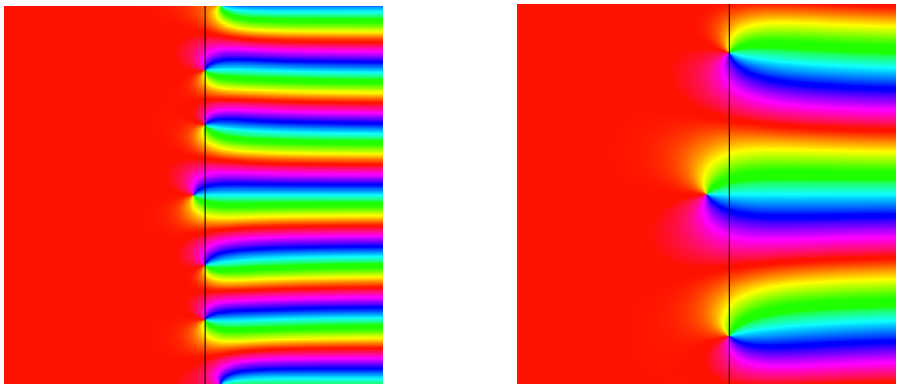


FIGURE 2. The left hand side is the phase plot of $1 - f(w)$ for $w \in [-10, 10] + i[-10, 10]$. The right-hand side is the phase plot for $g(w)$ with $w \in [-5, 5] + i[-5, 5]$. The black line is $\{w \in \mathbb{C} : \Re w = \gamma/2\}$.

8.4. Fluctuations of the heat content

More work on general branching processes is required before the central limit theorem can be applied to statistically self-similar snowflakes, where the fundamental martingale is not equal to 1. However, the discussion in Chapter 7 naturally leads to the following conjecture.

CONJECTURE 8.4.1. *There is a non-trivial class of statistically self-similar snowflakes D for which*

$$s^{-\gamma/4}(s^{\gamma/2-1}E_D(s) - \mathcal{E}N_\infty) \to_d Y_\infty,$$

where $\mathcal{E}$ and N_∞ are defined as above and Y_∞ is a random variable whose characteristic function has the form

$$\mathbf{E}\left[e^{i\theta Y_\infty}\right] = \mathbf{E}\left[e^{-\frac{1}{2}\theta^2\sigma^2 N_\infty}\right],$$

for some $\sigma \in (0, \infty)$.

Comparing this with Theorem 3.4.1 shows that if and when this conjecture holds, the random fluctuations are of an order strictly bigger than the deterministic fluctuations around the first order behaviour that occur for the triadic Koch snowflake.

Rather surprisingly, for the statistically self-similar snowflakes built from the construction detailed in Section 8.1, the problem with the speed of convergence of $z^{\phi}(t)$ to its limit typically seems to occur. For example, when $A = \{1, 2, 3\}$ and $(p_a, a \in A)$ is uniform, the Minkowski dimension γ of the boundary can be evaluated numerically and we have

$$\gamma \simeq 1.2114.$$

The corresponding phase plots shown in Figure 2 illustrate the presence of simple roots of $1 - f(w)$ at

$$0, \quad 0.5768 \pm 6.5831i \quad \text{and} \quad 0.5877 \pm 3.7127i.$$

These correspond to simple poles in the phase plot of g which are non-removable and therefore have non-zero residue. While the arguments in Section 7.3 rely on the polynomial form of the Fourier transform $f(w)$, the numerical evidence suggests there is little hope of getting a central limit theorem for this simple example.

Open question. Can we easily characterise the class of snowflakes for which Conjecture 8.4.1, for example, using only a condition on the rate of convergence in the renewal theorem?

There are many different ways to construct snowflakes with a statistically self-similar boundary. The example presented here was chosen to illustrate

the qualitative impact of the additional spatial randomness, compared the the spatially homogeneous case.

What is needed is to find a construction for which the rate of convergence in the renewal theorem for the corresponding renewal equation is fast enough. Typically, however, the fundamental martingale of the corresponding general branching process will not be equal to one. Therefore, a better understanding of the property of the fundamental martingale of the general branching process, and its speed of convergence to its limit in L^p would also be required.

APPENDIX A

Renewal theory

In this appendix, we present some results on renewal theory. The approach here is largely taken from Feller [**Fel68**]. We do not reproduce all of Feller's proofs but give some details when they are left to the reader.

A.1. One-sided renewal theory

Let F be a probability distribution with $F(0) = 0$. We assume that F is non-lattice; the lattice case is discussed in [**Fel68**]. We also assume that F has a finite mean, i.e. that

$$\mu_1 = \int_0^\infty yF(dy) = \int_0^\infty (1 - F(y))dy < \infty.$$

Let $(T_j, j \in \mathbb{N})$ be a sequence of i.i.d. random variables with distribution F and put

$$S_n = T_1 + \cdots + T_n.$$

It is intuitively clear that

$$N(x) = \sum_{n=0}^{\infty} \mathbf{1}_{S_n \leq x} < \infty, \text{ a.s.}$$

But in fact, the expectation of $N(x)$ is finite: we now show that the measure defined by $U(x) = \mathbf{E}N(x)$ is *transient*, i.e., that, for every x,

$$U(x) = \mathbf{E}N(x) = \sum_{n=0}^{\infty} \mathbf{P}(S_n \leq x) = \sum_{n=0}^{\infty} F^{*n}(x) < \infty,$$

where F^{*n} is the nth convolution of F with itself. For this, we will use that, for every x,

$$F^{*n}(x) \to 0,$$

as $n \to \infty$.

PROPOSITION A.1.1. *The measure U is transient.*

PROOF. Let $x \geq 0$. There exists r such that $F^{*r}(x) < 1$. Furthermore, $n \mapsto F^{*n}(x)$ is non-increasing, and, since F^{*r} is supported on $\mathbb{R}_+$, for every n,

$$F^{*rn}(x) \leq (F^{*r}(x))^n.$$

Therefore,

$$U(x) \leq r \sum_{n=0}^{\infty} F^{*rn}(x) \leq \frac{r}{1 - F^{*r}(x)} < \infty. \qquad \square$$

Given a function z and a measure F, the *renewal equation* is the relation

$$Z = z + F * Z,$$

where z and Z are functions that vanish on $(-\infty, 0)$. Explicitly, this means that

$$Z(x) = z(x) + \int_0^{\infty} Z(x-y)F(dy) = z(x) + \int_0^{x} Z(x-y)F(dy),$$

since $F(0) = 0$ and Z vanishes on $(-\infty, 0)$.

It is clear that if Z is the solution of the renewal equation, then so is $Z+c$ for every constant c. However, we have the following uniqueness result for solutions vanishing for $x < 0$, which are often the solutions of interest in applications.

PROPOSITION A.1.2. *There exists a unique solution Z to the renewal equation that vanishes for $x < 0$ and is bounded on compact sets.*

PROOF. Let Z and Y be two solutions to the renewal equation satisfying the assumptions above. Put $V = Z - Y$; we shall show that $V \equiv 0$. It is clear that $V = F * V$, that V vanishes for $x < 0$ and that V is bounded on compact intervals. Therefore,

$$\begin{aligned} |V(x)| &= |F * V(x)| = |F^{*n} * V(x)| \\ &\leq \int_0^x |V(x-y)| F^{*n}(dy) \\ &\leq \sup_{y \in [0,x]} \{|V(y)|\} F^{*n}(x) \\ &\to 0, \end{aligned}$$

as $n \to \infty$, and the proof is complete. □

Let us emphasise that this result relies on Z vanishing when $x < 0$ to end up with an integral over a compact set; it is false otherwise, which hints that two-sided renewal equations are more delicate.

The solution of the renewal equation can be described explicitly as follows.

THEOREM A.1.3. *Let z be a bounded function vanishing on $(-\infty, 0)$. Then the unique solution to the renewal equation that vanishes for $(-\infty, 0)$ and is bounded on compact intervals is given by $Z = U * z$. More explicitly,*

$$Z(x) = \int_0^x z(x-y) U(dy).$$

PROOF. It suffices to check that Z satisfies the renewal equation, which is easily done using Proposition A.1.1 to see that everything is well-defined; see [**Fel68**] also. □

Results about the behaviour of $Z(x)$ and $U(x)$ for x large are called *renewal theorems* and we conclude this section by giving some such results. The proofs are omitted; see [**Fel68**] for the details.

THEOREM A.1.4. *The following two statements are equivalent:*

(i) *For every $h > 0$, we have*

$$U(x) - U(x-h) \to \mu_1^{-1} h,$$

as $x \to \infty$, and;

(ii) *for every directly Riemann integrable function z that vanishes on $(-\infty, 0)$, we have*

$$Z(x) \to \frac{1}{\mu_1} \int_0^\infty z(y)dy,$$

as $x \to \infty$, where Z is the only solution that is bounded on compact intervals and vanishes on $(-\infty, 0)$.

The following result is Feller's version of the renewal theorem.[1]

THEOREM A.1.5. *For every $h > 0$, we have*

$$U(x) - U(x-h) \to \mu_1^{-1} h,$$

as $x \to \infty$.

When $\mu_1 = \infty$, there are still some result with convergence to 0. And when the distribution F has finite higher moments, or decays quickly, sharper results on the behaviour of U are known. See [**Fel68, Sto65a, Sto65b**] for more details.

The proof relies on the following lemma which is of independent interest.

LEMMA A.1.6. *Let F be a non-lattice probability distribution not concentrated at the origin, and let ζ be a bounded continuous function such that*

$$\zeta(x) = (F * \zeta)(x) = \int_{-\infty}^{\infty} \zeta(x-y)F(dy).$$

Then, ζ is constant.

[1]Often, in the literature, renewal theorems are about Z rather than U.

A.2. Two-sided renewal theory

In the applications of renewal theory discussed in this document, F is always satisfies $F(0) = 0$. However, in the two-sided case, this assumption is not made in general; instead, we require that U be *transient*, i.e. that, for every bounded interval I,

$$U(I) = \sum_{n=0}^{\infty} F^{*n}(I) < \infty.$$

We will keep working with the assumption that F has a finite mean μ_1.

In the two-sided case, given a function z and a measure F, the *renewal equation*

$$Z = z + F * Z,$$

where z and Z do not necessarily vanish on $(-\infty, 0)$, becomes

$$Z(x) = z(x) + \int_{-\infty}^{\infty} Z(x-y)F(dy).$$

When z is compactly supported (and continuous), the function

$$Z(x) = (U * z)(x) = \int_{-\infty}^{\infty} z(x-y)U(dy)$$

is well-defined, bounded (and continuous). And then we get the following uniqueness result from Lemma A.1.6

PROPOSITION A.2.1. *Let z be continuous and compactly supported. Then, up to an additive constant, $Z = U * z$ is the unique solution to the renewal equation $Z = z + F * Z$ that is bounded and continuous.*

We now detail the proof of the analogue of Theorem A.1.4.

THEOREM A.2.2. *The following two statements are equivalent:*

(i) *For every $h > 0$, we have*

$$U(x) - U(x-h) \to \mu_1^{-1} h,$$

as $x \to \infty$ and

$$U(x) - U(x-h) \to 0,$$

as $x \to -\infty$; and

(ii) *for every directly Riemann integrable function z, we have*

$$Z(x) \to \frac{1}{\mu_1} \int_{-\infty}^{\infty} z(y)dy,$$

as $x \to \infty$, provided that $Z = U * z$ and $Z(x) \to 0$ as $x \to -\infty$.

PROOF. Let us start by proving that (ii) implies (i). Consider the function $z = \mathbf{1}_{[0,h]}$ and put $Z = U * z$ so that

$$Z(x) = \int_{-\infty}^{\infty} z(x-y)U(dy) = \int_{-\infty}^{\infty} \mathbf{1}_{[x-h,t]}(y)U(dy) = U(x) - U(x-h).$$

Because of the assumptions on Z, we have established (i).

Let us now show that (i) implies (ii). Fix h and, for $n \in \mathbb{Z}$, put

$$z_n = \mathbf{1}_{[(n-1)h,nh]}.$$

Reasoning as above, it is clear that $Z_n = U * z_n$ satisfies $Z_n(x) \to 0$ as $x \to -\infty$. Therefore,

$$Z_n(x) = U(t-(n-1)h) - U(t-nh) = Z_0(t-(n-1)h) \to \mu_1^{-1} h = \int_{-\infty}^{\infty} z_n(y)dy,$$

as $x \to \infty$.

Now suppose that Z has the form

$$Z = \sum_{n \in \mathbb{Z}} a_n z_n,$$

where $(a_n, n \in \mathbb{Z})$ is a sequence of non-negative numbers such that

$$\sum_{n\in\mathbb{Z}} a_n < \infty.$$

Then,

$$Z = U * z = \sum_{n\in\mathbb{Z}} a_n Z_n.$$

But since $Z_n(x) \leq M$ for some constant M, we have

$$\sum_{|n|\leq N} a_n Z_n(x) \leq Z(x) \leq \sum_{|n|\leq N} a_n Z_n(x) + M \sum_{|n|>N} a_n.$$

Letting $x \to \infty$ and $N \to \infty$ shows that

$$Z(x) \to \frac{h}{\mu_1} \sum_{n\in\mathbb{Z}} a_n = \frac{1}{\mu_1} \int_{-\infty}^{\infty} z(y)dy.$$

For z a non-negative directly Riemman integrable function, put

$$\underline{m}_h(n) = \inf_{[(n-1)h,nh]} z(y) \quad \text{and} \quad \bar{m}_h(n) = \sup_{[(n-1)h,nh]} z(y)$$

and define

$$\underline{z}_h(x) = \sum_{n\in\mathbb{Z}} \underline{m}_h(n)\mathbf{1}_{[(n-1)h,nh]}(x) \quad \text{and} \quad \bar{z}_h(x) = \sum_{n\in\mathbb{Z}} \bar{m}_h(n)\mathbf{1}_{[(n-1)h,nh]}(x).$$

Notice that

$$\begin{aligned}\liminf_{h\to 0} \frac{h}{\mu} \sum_{n\in\mathbb{Z}} \underline{m}_h(n)h \leq \lim_{x\to\infty} \underline{Z}_h(x) \leq \liminf_{x\to\infty} Z(x) \\ \leq \limsup_{x\to\infty} Z(x) \leq \lim_{x\to\infty} \bar{Z}_h(x) \leq \limsup_{h\to 0} \frac{h}{\mu} \sum_{n\in\mathbb{Z}} \bar{m}_h(n)h,\end{aligned}$$

where $\underline{Z} = U * \underline{z}$ and $\bar{Z} = U * \bar{z}$. And the result follows because the leftmost term and rightmost term are equal by direct Riemann integrability.

For an arbitrary directly Riemann integrable function, it suffices to write it as the difference of its positive and negative parts and use this to get the result. □

We conclude this section with the two-sided renewal theorem of Feller.

THEOREM A.2.3. *For every $h > 0$, we have*

$$U(x) - U(x-h) \to \mu_1^{-1} h,$$

as $x \to \infty$, and

$$U(x) - U(x-h) \to 0,$$

as $x \to -\infty$.

As in the one-sided case, there are results $\mu_1 = \infty$, see [**Fel68**]. Other relevant versions of two-sided renewal theorems may be found in [**Kar55**] and [**LV96**].

APPENDIX B

Heat content estimates with extra cooling

In this appendix, we reproduce, without proofs, the comparison estimates for the heat equation of [**vdBG98**]. They provide a relevant version of the 'principle of not feeling the boundary'.

Let D be a domain of $\mathbb{R}^d$ and let $\phi : \partial D \to \mathbb{R}_+$ be a bounded continuous function. Let u be the solution to the heat equation with Dirichlet boundary condition ϕ and zero initial condition, i.e.

$$\begin{aligned} \partial_t u(s,x) &= \frac{1}{2}\Delta u(s,x), && (s,x) \in (0,\infty) \times D, \\ u(0,x) &= u(0+,x) = 0, && x \in D, \\ u(-,x) &= \phi(x), && x \in \partial D. \end{aligned} \tag{B.1}$$

The heat content of D is the function

$$E(s) = \int_D u(s,x)dx.$$

Now let C be a closed subset of $\bar{D}$ and consider solving the heat equation as before, but with the additional requirement that the temperature of C remain 0 at all times, i.e. we are looking for $\tilde{u}$ such that

$$\begin{aligned} \partial_t \tilde{u}(s,x) &= \frac{1}{2}\Delta \tilde{u}(s,x), && (s,x) \in (0,\infty) \times D, \\ \tilde{u}(0,x) &= \tilde{u}(0+,x) = 0, && x \in D, \\ \tilde{u}(-,x) &= \phi(x), && x \in \partial D \setminus C, \\ \tilde{u}(-,x) &= 0, && x \in C. \end{aligned} \tag{B.2}$$

We will denote the heat content with the extra cooling by

$$\tilde{E}(s) = \int_D \tilde{u}(s,x)dx.$$

The following three results estimate the differences between $u-\tilde{u}$ and $E-\tilde{E}$, i.e. the effect of the extra cooling on the heat flow inside D.

We will use the notation

$$S = \overline{\{x \in \partial D : \phi(x) = 0\}}.$$

We start with an estimate on the difference between the solutions to the heat equation.

THEOREM B.0.4. *Let p and q be such that $1/p + 1/q = 1$. Then, for all $x \in D$ and $s \in (0,\infty)$,*

$$0 \leq u(s,x) - \tilde{u}(s,x) \leq 2^{(d+2)/2}\|\phi\|_\infty \exp\left\{-\frac{\operatorname{dist}(x,S)^2}{4ps} - \frac{\operatorname{dist}(x,C)^2}{4qs}\right\}.$$

The second result is an estimate on u that is very intuitively interpreted as a principle of not feeling the boundary in short time.

THEOREM B.0.5. *Suppose that $\phi \equiv 1$. Then, for all $x \in D$ and $t \in (0,\infty)$,*

$$u(s,x) \leq 2^{(d+2)/2} \exp\left\{-\frac{\operatorname{dist}(x,\partial D)^2}{4s}\right\}.$$

To state the final result about the difference between the heat contents, we use need the following notation. For p and q such that $1/p + 1/q = 1$, put

$$\mu(\epsilon) = \operatorname{vol}_d(\{x \in D \setminus C : \operatorname{dist}(x,S) \leq \sqrt{p}\epsilon \text{ and } \operatorname{dist}(x,C) \leq \sqrt{q}\epsilon\})$$

and

$$\nu(\epsilon) = \operatorname{vol}_d(\{x \in C : \operatorname{dist}(x,S) \leq \sqrt{p}\epsilon\}).$$

THEOREM B.0.6. *Suppose that, for every* $s \in (0, \infty)$,

$$\int_D e^{-\operatorname{dist}(x,S)^2/(4ps)} dx < \infty.$$

Then,

$$0 \leq E(s) - \tilde{E}(s) \leq 2^{(d+2)/2} \|\phi\|_\infty \int_0^\infty e^{-\epsilon^2/(4s)} (\mu + \nu)(d\epsilon).$$

Furthermore, if $\operatorname{vol}_d(C) = 0$, *then*

$$0 \leq E(s) - \tilde{E}(s) \leq 2^{(d+2)/2} \|\phi\|_\infty \int_0^\infty e^{-\epsilon^2/(4s)} \mu(d\epsilon).$$

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