Systems of Forms in Many Variables

Simon L. Rydin Myerson

Oriel College

University of Oxford

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Abstract

We consider systems of polynomial equations and inequalities to be solved in integers. By applying the circle method, when the number of variables is large and the system is geometrically well-behaved we give an asymptotic estimate for the number of solutions of bounded size.

In the case of R homogeneous equations having the same degree d, a classic theorem of Birch provides such an estimate provided the number of variables is $R(R+1)(d-1)2^{d-1} + R$ or greater and the system is nonsingular. In many cases this conclusion has been improved, but except in the case of diagonal equations the number of variables needed has always grown quadratically in R.

We give a result requiring only $d2^dR+R$ variables, obtaining linear growth in R. When $d = 2$ or 3 we require only that the system be nonsingular; when $d > 4$ we require that the coefficients of the equations belong to a certain explicit Zariski open set. These conditions are satisfied for typical systems of equations, and can in principle be checked algorithmically for any particular system.

We also give an asymptotic estimate for the number of solutions to R polynomial inequalities of degree d with real coefficients, in the same number of variables and satisfying the same geometric conditions as in our work on equations. Previously one needed the number of variables to grow super-exponentially in the degree d in order to show that a nontrivial solution exists.

Contents

Chapter 1

Introduction

A system of Diophantine equations is a system of polynomial equations, with integer coefficients, to be solved in integers. An example is the pair of quadratic equations in six variables given by

$$x^2 + yv = z^2, \qquad xu + y^2 = w^2. \tag{1.1}$$

These equations are homogeneous, that is, every term has the same degree. The name refers to the third-century mathematician Diophantus of Alexandria. Among many other results he gave a construction [Tho14, p541] which provides infinitely many nontrivial solutions to the equations (1.1), where we call those solutions with $x = y = z = w = 0$ trivial. Determining if such a system has a (nontrivial) solution, and giving some description of the set of solutions, are problems which recur across mathematics.

When a system of Diophantine equations involves many variables one can apply the circle method. This technique originated in the years 1916–1923, with work of Hardy, Ramanujan and Littlewood, and reached its modern form in the work of Vinogradov, see Kempner [Kem23] and Cassels and Vaughan [CV85, §1]. For an example of the kind of conclusion obtained by this method, consider the Diophantine equation

$$a_1 x_1^d + \cdots + a_n x_n^d = 0, \tag{1.2}$$

where the degree d is fixed and at least 2, and the a_i are fixed nonzero integers. If d is even and all the a_i have the same sign, then the only solution is given by $x_1 = \cdots = x_n = 0$. Suppose this is not case. Provided that n sufficiently large, one can apply the circle method to show that (1.2) has nontrivial solutions, and moreover that the number of solutions with $|x_i| \leq P$ is of size around P^{n-d}.

To phrase this more formally we introduce some notation, which will be employed frequently throughout this thesis. We put $\boldsymbol{x}$ for the vector of variables $(x_1, \dots, x_n)^T$, and we let $\|\boldsymbol{x}\|_\infty$ be the maximum norm $\max_{i=1,\dots,n}|x_i|$. We write $O(A)$ to stand for a quantity whose absolute value (or, for vector quantities, whose maximum norm) is bounded by CA, where C is some implicit positive constant. Any parameters on which C depends are indicated by subscripts. We put $A \ll B$ to mean that $A = O(B)$, and $A \asymp B$ if both $A \ll B$ and $B \ll A$ hold. We say that $A = o(B)$ as $t \to \infty$ to indicate that $A/B \to 0$ as $t \to \infty$.

We can now state the following more precise result about (1.2).

Theorem 1.1. *Let $\boldsymbol{a} \in \mathbb{Z}^n$ and let $d \in \mathbb{N}$ with $d \geq 2$. Suppose either that d is odd, or that some two a_i have different signs. Suppose that $n \geq d^2 + 1$. Then there exist a positive real δ depending only on d, and a positive real constant ν depending only on $\boldsymbol{a}$ and d, such that for all $P \geq 1$ we have*

$$\#\{\boldsymbol{x} \in \mathbb{Z}^n : \|\boldsymbol{x}\|_\infty \leq P,\, a_1 x_1^d + \cdots + a_n x_n^d = 0\} = \nu P^{n-d} + O_{\boldsymbol{a},d}(P^{n-d-\delta}).$$

If $d = 2$ this result is classical, see Theorem 8.1 of Davenport [Dav05] and the subsequent comments. Otherwise the result follows by the argument used to prove Theorem 4.2 in Wooley [Woo12], together with the main conjecture in Vinogradov's mean value theorem as proved by Wooley [Woo16] in the case $d = 3$ and by Bourgain, Demeter and Guth [BDG16] when $d \geq 4$. In particular, Wooley [Woo12] refers in the comments following his Theorem 4.2 to an improved version conditional on the Vinogradov main conjecture, which includes Theorem 1.1.

We cannot weaken the condition $n \geq d^2 + 1$ in Theorem 1.1, since if $d = p - 1$ for a prime number p, then we have the Diophantine equation in d^2 variables given by

$$x_1^d + \cdots + x_d^d + p(x_{d+1}^d + \cdots + x_{2d}^d) + \cdots + p^d(x_{(d-1)d+1}^d + \cdots + x_{d^2}^d) = 0$$

which has no nontrivial solutions by an application of Fermat's Little Theorem.

The equation (1.2) is *diagonal*, that is there is no term which involves more than one of the variables x_i. Such equations are particularly well adapted to the circle method. For non-diagonal systems, one may need many more variables to obtain results similar to Theorem 1.1.

The primary goal of this thesis is to somewhat improve this situation for systems of several non-diagonal equations of the same degree. The study of such systems by the circle method began with the work of Birch [Bir62]. After briefly indicating some technical issues which such a result must necessarily address, we state Birch's theorem in §1.2.1, summarise the subsequent developments in §1.2.2 and describe our conclusions in §1.2.3.

We then turn to the problem of extending what is known for Diophantine equations to the case of inequalities with real coefficients. We outline the present state of knowledge in §1.4.1 and present our contribution in §1.4.2. We then end this introduction with an overview of the following chapters and the structure of the thesis.

1.1 Notation

The following notation and terminology will be used throughout this thesis.

1.1.1 Basic notation

As above $\boldsymbol{x}$ is a vector of n variables. We let $f_1(\boldsymbol{x}), \dots, f_R(\boldsymbol{x})$ be polynomials with real coefficients of the same degree $d \geq 2$, and we write $f_1^{[d]}(\boldsymbol{x}), \dots, f_R^{[d]}(\boldsymbol{x})$ for the degree d parts. We use vector notation for R-tuples of polynomials, so that for example $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \dots, f_R(\boldsymbol{x}))^T$ and $\boldsymbol{\alpha} \cdot \boldsymbol{f} = \sum_{i=1}^R \alpha_i f_i$.

We write $\boldsymbol{F}(\boldsymbol{x})$ for a system of R homogeneous forms of degree d in n variables with integral coefficients.

If $\boldsymbol{y}$ is a vector of k variables we write $\boldsymbol{\nabla_y}$ for $(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_k})^T$. For each $k \in \mathbb{N}$ we write λ for the Lebesgue measure on $\mathbb{R}^k$ and we let $\|\boldsymbol{t}\|_\infty = \max_i |t_i|$ be the supremum norm on $\mathbb{R}^k$. If L is a real $k \times \ell$ matrix, we set $\|L\|_\infty = \max_{i,j} |L_{ij}|$. We also define

$$\|\boldsymbol{t}\|_{\mathbb{R}/\mathbb{Z}} = \min_{\boldsymbol{u} \in \mathbb{Z}^n} \|\boldsymbol{t} - \boldsymbol{u}\|_\infty \tag{1.3}$$

and

$$\|L\|_{\mathbb{R}/\mathbb{Z}} = \min_{M} \|L - M\|_\infty \tag{1.4}$$

where the minimum is over $k \times \ell$ matrices M with integral entries. If $H(\boldsymbol{x})$ is a homogeneous form of degree d then we let $\|H\|_\infty = \frac{1}{d!} \max_{\boldsymbol{j} \in \{1,\dots,n\}^d} \left| \frac{\partial^d H(\boldsymbol{x})}{\partial x_{j_1} \cdots \partial x_{j_d}} \right|$, so that $\|H\|_\infty$ is the absolute value of the largest coefficient of H.

1.1.2 p-adic numbers

The field of *p-adic numbers* $\mathbb{Q}_p$ can be realised as the set of infinite formal sums $x = \sum_{i=k}^\infty a_i p^i$ where $k \in \mathbb{Z}$, $a_i \in \{0, \dots, p-1\}$ and $a_k \neq 0$. The p-adic integers $\mathbb{Z}_p$ are the subset defined by $k \geq 0$. Putting $|x|_p = p^{-k}$ gives an absolute value on $\mathbb{Q}_p$, which makes $\mathbb{Q}_p$ into a metric space with the p-adic topology. A system of polynomial equations $\boldsymbol{p}(\boldsymbol{x}) = \mathbf{0}$ with coefficients in $\mathbb{Z}$ has a (nonzero) solution over $\mathbb{Z}_p$ if and only if it has a (nonzero) solution modulo p^k for every $k \in \mathbb{N}$.

1.1.3 Algebraic varieties

By a projective variety we mean a reduced, not necessarily irreducible, closed subscheme of either projective space $\mathbb{P}_{\mathbb{F}}^{n-1}$, or a product of projective spaces $\mathbb{P}_{\mathbb{F}}^{n_1-1} \times \cdots \times \mathbb{P}_{\mathbb{F}}^{n_k-1}$, over a field $\mathbb{F}$. See §§4 and 5.1 in Chapter 1 of Shafarevich [Sha13a] for a classical perspective on these two cases, and for the scheme structure see Chapter 3 of the second volume [Sha13b] and in particular Example 5.19 in that chapter. Concretely these varieties have the following form. If $\boldsymbol{H}(\boldsymbol{x})$ is an R-tuple of homogeneous forms in n variables with coefficients in a field $\mathbb{F}$, then $V(\boldsymbol{H})$ will be the projective variety in $\mathbb{P}_{\mathbb{F}}^{n-1}$ defined by $\boldsymbol{H}(\boldsymbol{x}) = \mathbf{0}$. If $\boldsymbol{M}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)})$ is a system of multihomogeneous forms, where $\boldsymbol{x}^{(i)}$ is a vector of n_i variables, we may similarly define a variety $V(\boldsymbol{M})$ in $\mathbb{P}_{\mathbb{F}}^{n_1-1} \times \cdots \times \mathbb{P}_{\mathbb{F}}^{n_k-1}$ by $\boldsymbol{M} = \mathbf{0}$.

If V is a variety over $\mathbb{F}$, we denote the set of $\mathbb{F}$-points of V by $V(\mathbb{F})$. Then $V(\boldsymbol{H})(\mathbb{F})$ may be identified with the set of points $\boldsymbol{x} \in \mathbb{F}^n \setminus \{\mathbf{0}\}$ with $\boldsymbol{H}(\boldsymbol{x}) = \mathbf{0}$ modulo equivalence, where two points are equivalent if they lie on a line through the origin. When $\mathbb{F} = \mathbb{Q}$, a necessary condition for V to have a $\mathbb{Q}$-point is that it has an $\mathbb{R}$-point and a $\mathbb{Q}_p$-point for each prime p. We say V satisfies the *Hasse principle* if this necessary condition is also sufficient. We say that *weak approximation* holds for V if $V(\mathbb{Q})$ is a dense subset of $V(\mathbb{R}) \times \prod_{p \in S} V(\mathbb{Q}_p)$ in the real and p-adic topologies for any finite set of primes S.

Taking the subvarieties of a given variety V as closed sets defines a topology on V, the *Zariski topology*.

The dimension of a variety is as in §6.1 from Chapter 1 of Shafarevich [Sha13a], with the addition that if $V = \emptyset$ then we define $\dim V = -1$.

If $\boldsymbol{p}(\boldsymbol{x})$ is a system of R polynomials, then we say a solution to $\boldsymbol{p}(\boldsymbol{x}) = \mathbf{0}$ is *nonsingular* if the Jacobian matrix $(\partial p_i(\boldsymbol{x}) / \partial x_j)_{ij}$ has rank R.

If $\boldsymbol{H}(\boldsymbol{x})$ is a system of R homogeneous forms in n variables, with coefficients in a field $\mathbb{F}$, then the conditions that $\boldsymbol{H}(\boldsymbol{x}) = \mathbf{0}$ and that $(\partial p_i(\boldsymbol{x}) / \partial x_j)_{ij}$ has rank less than R define a variety in $\mathbb{P}_{\mathbb{F}}^{n-1}$ which we call $\mathrm{Sing}(\boldsymbol{H})$.

If $\mathrm{Sing}(\boldsymbol{H})$ is empty we set $\dim \mathrm{Sing}(\boldsymbol{H}) = -1$, and we say $\boldsymbol{H}$ is *nonsingular*. Equivalently, $\boldsymbol{H}$ is nonsingular if $\boldsymbol{H}(\boldsymbol{x}) = \mathbf{0}$ has no singular solutions over any field except for the trivial solution $\boldsymbol{x} = \mathbf{0}$. This occurs precisely when $V(\boldsymbol{H})$ is smooth with ideal generated by the H_i and is a complete intersection, that is, has dimension $n - R - 1$. See Shafarevich [Sha13a, pp89 and 93].

1.2 Diophantine equations in many variables

In this section we consider analogues of Theorem 1.1 for systems of one or more homogeneous forms which are not necessarily diagonal.

1.2.1 A result of Birch

We begin with a theorem which has served as a model for subsequent work applying the circle method to systems of homogeneous Diophantine equations.

Theorem 1.2 (Birch [Bir62]). *As in §1.1, let $d \geq 2$ and let $\boldsymbol{F}(\boldsymbol{x})$ be a system of R homogeneous forms $F_i(\boldsymbol{x})$, all of degree d with integer coefficients in n variables $x_1, \ldots, x_n$. Let $\mathscr{B}$ be a box in $\mathbb{R}^n$, contained in the box $[-1,1]^n$ and having sides of length at most 1 which are parallel to the coordinate axes. For each $P \geq 1$, write*

$$N_{\boldsymbol{F},\mathscr{B}}(P) = \#\{\boldsymbol{x} \in \mathbb{Z}^n : \boldsymbol{x}/P \in \mathscr{B},\ \boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}\}. \tag{1.5}$$

Let $V^\dagger_{\boldsymbol{F}}$ be the projective variety cut out in $\mathbb{P}^{n-1}_{\overline{\mathbb{Q}}}$ by the condition that the $R \times n$ Jacobian matrix $(\partial F_i(\boldsymbol{x})/\partial x_j)_{ij}$ has rank less than R. If

$$n - 1 - \dim V^\dagger_{\boldsymbol{F}} > (d-1)2^{d-1}R(R+1), \tag{1.6}$$

then for some $\mathfrak{I}_{\boldsymbol{F},\mathscr{B}}$, $\mathfrak{S}_{\boldsymbol{F}} \geq 0$ we have

$$N_{\boldsymbol{F},\mathscr{B}}(P) = \mathfrak{I}_{\boldsymbol{F},\mathscr{B}}\mathfrak{S}_{\boldsymbol{F}}P^{n-dR} + O(P^{n-dR-\delta}) \tag{1.7}$$

for all $P \geq 1$. Here the implicit constant and the constant $\mathfrak{S}_{\boldsymbol{F}}$ depend only on the forms F_i, the constant $\mathfrak{I}_{\boldsymbol{F},\mathscr{B}}$ depends only on $\boldsymbol{F}$ and $\mathscr{B}$ and δ is a positive constant depending only on d and R. If $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}$ has a nonsingular solution over $\mathbb{R}$ which lies in the interior of $\mathscr{B}$, then $\mathfrak{I}_{\boldsymbol{F},\mathscr{B}}$ is positive. If there is a nonsingular solution over $\mathbb{Q}_p$ for each prime p, then $\mathfrak{S}_{\boldsymbol{F}}$ is positive.

If $\boldsymbol{F}$ is nonsingular and (1.6) holds, then the theorem implies that the projective variety $V(\boldsymbol{F})$ satisfies the Hasse principle as defined in §1.1. In this case it also follows from Birch's methods that $V(\boldsymbol{F})$ satisfies weak approximation, as we will see when we prove our main results in §§4.1 and 5.1.

The quantities $\mathfrak{I}_{\boldsymbol{F},\mathscr{B}}$ and $\mathfrak{S}_{\boldsymbol{F}}$ are the *singular integral* and *singular series*. They measure the density of solutions over $\mathbb{R}$ and over $\mathbb{Q}_p$ for each p, and satisfy the formulae

$$\mathfrak{I}_{\boldsymbol{F},\mathscr{B}} = \lim_{P\to\infty} \tfrac{1}{P^{n-dR}}\lambda\big\{\boldsymbol{t} \in \mathbb{R}^n : \tfrac{1}{P}\boldsymbol{t} \in \mathscr{B}, \|\boldsymbol{F}(\boldsymbol{t})\|_\infty \leq \tfrac{1}{2}\big\}, \tag{1.8}$$

where as in §1.1.1 we let λ denote the Lebesgue measure, and

$$\mathfrak{S}_{\boldsymbol{F}} = \prod_p \lim_{k\to\infty} \tfrac{1}{p^{k(n-R)}} \#\big\{\boldsymbol{b} \in \{1,2,\dots,p^k\}^n : \boldsymbol{F}(\boldsymbol{b}) \equiv \boldsymbol{0} \mod p^k\big\}, \tag{1.9}$$

where the product is over primes p and converges absolutely. The asymptotic formula (1.7), with these values of the singular series and integral, can be interpreted as a case of the Manin-Peyre conjecture [FMT89, BM90, Pey95]. This gives predictions for the size of considerably more general counting functions than $N_{\boldsymbol{F},\mathscr{B}}(P)$, as described in §2.1 below.

We note that some hypothesis similar to (1.6) is required to prevent the system $\boldsymbol{F}$ from being "too singular". For instance, (1.6) rules out the quartic form

$$q(\boldsymbol{x}) = 2(x_1^2 + \cdots + x_{n-2}^2)^2 + 17x_{n-1}^4 - x_n^4,$$

since one can check that $\dim V_h^\dagger = n - 4$. Lind [Lin40] and Reichart [Rei42] showed that the hypersurface $x^4 - 17y^4 = 2z^2$ violates the Hasse principle, with nontrivial solutions over $\mathbb{R}$ and every $\mathbb{Q}_p$, but not over $\mathbb{Q}$. For $n \geq 6$, it follows by Lagrange's four-squares theorem that $q(\boldsymbol{x}) = 0$ has nonsingular solutions over $\mathbb{R}$ and every $\mathbb{Q}_p$ but not over $\mathbb{Q}$, and so cannot satisfy (1.7).

The requirement that $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}$ have *nonsingular* solutions over $\mathbb{R}$ and each $\mathbb{Q}_p$ can also be necessary. For example the quadratic form $x_1^2 + \cdots + x_{n-1}^2$ has only the singular real zeroes $(0,\dots,0,x_n)$, and consequently has too few integral zeroes to satisfy (1.7). Nonsingular p-adic solutions are guaranteed to exist whenever $\boldsymbol{F}$ is nonsingular and the number of variables is sufficiently large. Wooley [Woo98] has shown that $(d^2R)^{2^{d-1}} + 1$ variables suffices.

When $\boldsymbol{F}$ is nonsingular we can simplify the condition (1.6) as follows.

Lemma 1.3 (Browning and Heath-Brown [BHB17])**.** *Suppose that $\boldsymbol{F}$ is nonsingular. Then we have*

$$\dim V_{\boldsymbol{F}}^\dagger \leq R - 1,$$

and so we may replace the condition (1.6) *with*

$$n \geq (d-1)2^{d-1}R(R+1) + R. \tag{1.10}$$

For the proof, set $D = 2$, $r_1 = 0$, $r_2 = R$, and $F_{i,2} = F_i$ in Browning and Heath-Brown [BHB17, (1.3)]. We obtain an "equivalent optimal system" by replacing $\boldsymbol{F}$ with $A\boldsymbol{F}$ for some $A \in \mathrm{GL}_n(\mathbb{Q})$, as described after their formula (1.7). Then their (1.4) and (1.8) show that $B_2 \leq R - 1$, where $B_2 = 1 + \dim(V_{\boldsymbol{F}}^\dagger)$. This proves the lemma.

One might ask what the minimum number of variables should be for the formula (1.7) to hold. When $\boldsymbol{F}$ is nonsingular and the number of variables satisfies $n \geq 2dR+1$ one conjectures that this formula should hold based on a heuristic application of the circle method, see the comments around formula (1.5) in Browning [Bro15]. Without further conditions on $\boldsymbol{F}$ this is best possible, as can be seen when $R = 1$ and $n = 2d$ by considering the equation $x_1^d + \cdots + x_d^d = x_{d+1}^d + \cdots + x_{2d}^d$. Here the "diagonal" solutions given by $x_i = x_{i+d}$ are too numerous for the formula (1.7) to hold.

Our goal will be to weaken the hypothesis (1.10) on the number of variables, when the number of forms R is greater than one. Previous improvements of this type have required $R = 1$ or 2.

1.2.2 Subsequent work

We consider previous improvements on (1.6). We begin with conclusions concerning systems of forms of a particular degree. We then pass to results for general degrees d, and applications of Birch's methods in different contexts to Theorem 1.2. Finally we discuss the case of systems of forms with different degrees.

When $d = 2$ and $R = 1$ we have a single quadratic form F, and work of Heath-Brown [HB96] introduces new techniques based on a smooth decomposition of the indicator function of $\{0\}$ due to Duke, Friedlander and Iwaniec [DFI93]. He proves an asymptotic formula analogous to (1.7) whenever F is nonsingular and $n \geq 3$. The main term in this formula may exhibit additional logarithmic factors compared with (1.7). Building on this work, Browning and Munshi [BM13, BM15] prove (1.7) for a certain class of pairs of quadratic forms in 9 variables, and for certain pairs of diagonal quadratic forms in 8 variables. Munshi [Mun15] obtains the same conclusion when $\boldsymbol{F}$ is a nonsingular pair of quadratic forms and $n = 11$; by contrast (1.10) would require $n \geq 14$.

When $d = 2$ and $R = 2$, Heath-Brown [Hea15] proves the Hasse principle and weak approximation for $V(\boldsymbol{F})$ when $n \geq 8$ and $\boldsymbol{F}$ is nonsingular. In this case the Hasse principle holds for $V(\boldsymbol{F})$ when $n \geq 9$ regardless of whether $\boldsymbol{F}$ is nonsingular by work of Colliot-Thélène, Sansuc and Swinnerton-Dyer [CTSSD87a, CTSSD87b]. If $d = 2$ and $R = 3$ then Heath-Brown [HB17] proves the Hasse principle and weak approximation for $V(\boldsymbol{F})$ when $n \geq 19$ and $\boldsymbol{F}$ is nonsingular.

We next consider the case when $d = 3$ and $R = 1$. When F is a single nonsingular cubic form and $n = 9$, Hooley [Hoo94] proves that $V(F)$ satisfies the Hasse principle and weak approximation. Provided the box $\mathscr{B}$ does not contain a point at which the determinant of the Hessian matrix of F vanishes, he also proves that an asymptotic similar to (1.7) holds with the weaker error term $O_F(P^{n-dR}/(\log P)^{\delta})$. If $n = 9$ and the only singularities of $V(F)$ are isolated double points, then he shows that the Hasse principle holds [Hoo13]. When F is a nonsingular cubic form in 8 variables he proves the Hasse principle and weak approximation as well as the asymptotic (1.7) when $\mathscr{B}$ does not contain a zero of the Hessian determinant, assuming a Riemann hypothesis for a certain modified Hasse-Weil L-function [Hoo15]. Additionally, if F is any rational cubic form in 10 or more variables Heath-Brown [HB83] shows that $V(F)$ always has a $\mathbb{Q}$-point. In this setting Theorem 1.2 requires $n \geq 17$.

If $d = 3$ and $R = 2$, Dietmann and Wooley [DW03] show that $V(\boldsymbol{F})$ has a $\mathbb{Q}$-point provided $n \geq 827$. In general, if $d = 3$ then Schmidt [Sch82] shows $V(\boldsymbol{F})$ has a $\mathbb{Q}$-point if $n \geq (10R)^5$. In recent work Dietmann [Die08] improves this to $n \geq 400\,000R^4$ variables.

When F is a single nonsingular quartic form, Hanselmann [Han12] gives the condition $n \geq 40$ in place of the $n \geq 49$ required to apply Theorem 1.2. Work in progress of Marmon and Vishe yields a further improvement.

We turn now to results which apply to general values of d. When $d \geq 5$ and $R = 1$, a sharper condition than (1.6) is available by work of Browning and Prendiville [BP15]. For $5 \leq d \leq 9$ and F nonsingular this is essentially a reduction of 25% in the number of variables required.

Dietmann [Die15] and Schindler [Sch15] give a refinement applicable for any d and any $R \geq 2$. They show that the condition (1.6) may be replaced with $n - \sigma_{\mathbb{Z}}(\boldsymbol{F}) > (d-1)2^{d-1}R(R+1)$, where

$$\sigma_{\mathbb{Z}}(\boldsymbol{F}) = 1 + \max_{\boldsymbol{a} \in \mathbb{Z}^R \setminus \{\boldsymbol{0}\}} \dim \operatorname{Sing}(\boldsymbol{a} \cdot \boldsymbol{F}). \tag{1.11}$$

This improves on Birch's result whenever $\sigma_{\mathbb{Z}}(\boldsymbol{F}) < 1 + \dim V_{\boldsymbol{F}}^{\dagger}$. If $R = 1$ this is impossible as the two sides are equal, but Dietmann and Schindler give examples to show that it can occur if $R \geq 2$. This does not however allow one to improve the condition (1.10) for nonsingular $\boldsymbol{F}$.

Next we consider adaptations of Birch's methods to different contexts. Skinner [Ski97] allows the variables, and the coefficients of the forms F_i, to take values in the ring of integers of a number field, see Chapter 7 for more details on this and related results.

An extension to systems of bihomogeneous forms of the same degree is given by Schindler [Sch14]. Mignot [Mig15, Mig16] further develops these methods for certain trilinear forms and for hypersurfaces in toric varieties.

Liu [Liu11] proves existence of solutions in prime numbers to a quadratic equation in 10 or more variables, and asymptotic formulae for systems of equations of the same degree with prime values of the variables are considered by Cook and Magyar [CM14] and by Xiao and Yamagishi [XY15]. Magyar and Titchetrakun [MT17] extend these results to values of the variables with a bounded number of prime factors.

Brandes [Bra14, Bra17] proves theorems which count linear spaces of solutions to a system of equations of the same degree. We also mention versions of Birch's result for function fields due to Lee [Lee11] and to Browning and Vishe [BV17].

The results above all concern systems of forms of the same degree. Schmidt [Sch85] succeeds in treating the case of differing degrees, replacing (1.6) with a more arithmetic condition involving "h-invariants". When the F_i have possibly different degrees which are all odd, he applies this to give a lower bound

$$N_{\boldsymbol{F},\mathscr{B}}(P) \gg_{\boldsymbol{F}} P^{n-C}$$

for the counting function $N_{\boldsymbol{F},\mathscr{B}}(P)$ from (1.5), without any condition on the singularities of the system. Here C is an extremely large constant depending on the number of forms F_i and their degrees.

Browning and Heath-Brown [BHB17] give a version of Theorem 1.2 for nonsingular systems of forms with differing degrees, with a number of variables which is polynomial in the number of forms and exponential in their degrees. Frei and Madritsch [FM17] extend this work over number fields, and a parallel conclusion with prime values of the variables is given by Yamagishi [Yam17].

1.2.3 Our results

Our main result is the following. The proof is completed in §5.1.

Theorem 1.4. *Let $\boldsymbol{F}$ be as in §1.1. If $d = 2$ or 3, suppose that $\boldsymbol{F}$ is nonsingular. If $d \geq 4$ then suppose that $\boldsymbol{F} \in U_{d,n,R}(\mathbb{Q})$ for some explicit, nonempty Zariski open set $U_{d,n,R}$ which will be defined in Proposition 5.3 below. Then we may replace the condition* (1.6) *in Theorem 1.2 with*

$$n > d2^d R + R. \tag{1.12}$$

In particular, if we assume that $\boldsymbol{F}$ is nonsingular, that (1.12) *holds and that $\boldsymbol{F} \in U_{d,n,R}(\mathbb{Q})$ if $d \geq 4$, then $V(\boldsymbol{F})$ satisfies the Hasse principle, weak approximation and the Manin-Peyre conjecture over $\mathbb{Q}$.*

The Manin-Peyre conjecture, which was mentioned in the comments after Theorem 1.2, will be described in more detail in §2.1 below. The Zariski topology was defined in §1.1, and in particular $\boldsymbol{F} \in U_{d,n,R}(\mathbb{Q})$ means that $h_1(\boldsymbol{F}) \neq 0, \ldots, h_N(\boldsymbol{F}) \neq 0$ for some fixed list of polynomials $h_i(\boldsymbol{F})$ in the coefficients of the forms F_i. Hence the set $U_{d,n,R}(\mathbb{Q})$ contains 100% of systems $\boldsymbol{F}$. Although we do not give the polynomials h_i explicitly, the construction in Proposition 5.3 is entirely concrete and for a given $\boldsymbol{F}$ with integral coefficients the question of whether $\boldsymbol{F} \in U_{d,n,R}(\mathbb{Q})$ can in principle be decided by a finite computation.

When $d = 2$ Theorem 1.4 sharpens Birch's result for systems of 4 or more forms. For example when $R = 4$, (1.12) requires $n \geq 36$, as opposed to $n \geq 44$ for (1.10). When $d = 3$ we have an improvement for 3 or more forms and in particular, the case $R = 3$ of Theorem 1.4 applies when $n \geq 75$ while Birch's theorem requires $n \geq 99$. For higher degrees $d \geq 4$ the condition (1.12) is weaker than (1.10) as soon as $R \geq 2$.

Recall from the end of §1.2.1 that if $\boldsymbol{F}$ is nonsingular, the condition $n \geq 2dR + 1$ would conjecturally be sufficient in place of (1.6). By handling systems of forms in $O_d(R)$ variables we come, for given d, within a constant factor of this limit.

When $d = 2$ or 3 we will deduce Theorem 1.4 from the following more refined version, proved in §4.1.

Theorem 1.5. *Let $\boldsymbol{F}$ be as in §1.1. If $d = 2$ or 3, then we may replace the condition* (1.6) *in Theorem 1.2 with the condition*

$$n - \sigma_{\mathbb{R}}(\boldsymbol{F}) > d2^d R \tag{1.13}$$

where $\sigma_{\mathbb{R}}(\boldsymbol{F})$ is the element of $\{0, \ldots, n\}$ defined by

$$\sigma_{\mathbb{R}}(\boldsymbol{F}) = 1 + \max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\boldsymbol{0}\}} \dim \operatorname{Sing}(\boldsymbol{\beta} \cdot \boldsymbol{F}), \tag{1.14}$$

and $\operatorname{Sing}(\boldsymbol{\beta} \cdot \boldsymbol{F})$ is the variety cut out in $\mathbb{P}^{n-1}_{\mathbb{R}}$ by $\sum \beta_i \boldsymbol{\nabla}_{\boldsymbol{x}} F_i(\boldsymbol{x}) = \boldsymbol{0}$. In particular, if (1.13) *holds then the nonsingular locus $V(\boldsymbol{F}) \setminus \operatorname{Sing}(\boldsymbol{F})$ satisfies the Hasse principle and weak approximation over $\mathbb{Q}$.*

If $d = 2$ and $R \geq 4$, or $d = 3$ and $R \geq 3$, then (1.13) is strictly weaker than Birch's condition (1.6). This follows from the fact that we have $\operatorname{Sing}(\boldsymbol{\beta} \cdot \boldsymbol{F}) \subset V^{\dagger}_{\boldsymbol{F}}$ whenever $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\boldsymbol{0}\}$, and so

$$\sigma_{\mathbb{R}}(\boldsymbol{F}) \leq 1 + \dim V^{\dagger}_{\boldsymbol{F}}.$$

In general we have the bound

$$\sigma_{\mathbb{R}}(\boldsymbol{F}) \leq R + \dim \operatorname{Sing}(\boldsymbol{F}). \tag{1.15}$$

For if we let $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\mathbf{0}\}$ then one of the β_i, say β_R, must be nonzero and so

$$\mathrm{Sing}(\boldsymbol{F}) = \mathrm{Sing}(F_1, \dots, F_1 R - 1, \boldsymbol{\beta} \cdot \boldsymbol{F}).$$

Hence

$$V(F_1, \dots, F_{R-1}) \cap \mathrm{Sing}(\boldsymbol{\beta} \cdot \boldsymbol{F}) \subset \mathrm{Sing}(\boldsymbol{F})$$

which implies that $\dim \mathrm{Sing}(\boldsymbol{\beta} \cdot \boldsymbol{F}) - (R-1) \leq \dim \mathrm{Sing}(\boldsymbol{F})$, from which (1.15) follows by (1.14).

We remark that previous work of the author [RM15] applies whenever $\boldsymbol{F}$ is nonsingular, and allows one to replace the condition (1.10) with

$$n \geq 2^{d-1} \frac{4^d - 4}{3} R + R,$$

which is still an improvement on (1.10) when R is around $4^d/d$ in size. In forthcoming work we will update this result to conclude that for $d \geq 3$ one may use the condition

$$n - \sigma_{\mathbb{R}}(\boldsymbol{F}) \geq (4^{d-1} - 2^d) dR + R,$$

in place of (1.6), where $\sigma_{\mathbb{R}}(\boldsymbol{F})$ is as in Theorem 1.5. This is superior to (1.6) as soon as R is around 2^d in size.

We next describe the method of proof of our results above.

1.3 The Hardy-Littlewood circle method

We sketch the circle method in the form used by Birch [Bir62] to prove Theorem 1.2. This approach is based in turn on the work of Davenport [Dav59]. We employ the same approach in Chapter 3, where the following notation will be used throughout.

We let $\mathscr{B}$ be a box in $\mathbb{R}^n$ contained in the box $[-1,1]^n$, and having sides of length at most 1 which are parallel to the coordinate axes. In Proposition 3.1 below we will call such boxes *admissible*. Recall the system of R real, degree d polynomials $\boldsymbol{f}(\boldsymbol{x})$ from §1.1. For each $\boldsymbol{\alpha} \in \mathbb{R}^R$ and $P \geq 1$, we define the exponential sum

$$S(\boldsymbol{\alpha}; P) = \sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^n \\ \boldsymbol{x}/P \in \mathscr{B}}} e(\boldsymbol{\alpha} \cdot \boldsymbol{f}(\boldsymbol{x})) \tag{1.16}$$

where $e(t) = e^{2\pi i t}$. This depends implicitly on $\mathscr{B}$ and the system $\boldsymbol{f}$.

We are concerned with the case when $\boldsymbol{f}$ has integral coefficients, and our goal is to prove the asymptotic formula

$$\#\{\boldsymbol{x} \in \mathbb{Z}^n : \boldsymbol{x}/P \in \mathscr{B},\, \boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}\} = \mathfrak{S}_{\boldsymbol{f}} \mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} P^{n-dR} + O_{\boldsymbol{f}}(P^{n-dR-\delta}) \tag{1.17}$$

for all $P \geq 1$, some $\delta > 0$ and some constants $\mathfrak{S}_{\boldsymbol{f}}$ and $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ satisfying (1.8) and (1.9). Observe that when $\boldsymbol{f}(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]^R$ we have

$$\#\{\boldsymbol{x} \in \mathbb{Z}^n : \boldsymbol{x}/P \in \mathscr{B},\ \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}\} = \int_{[0,1]^R} S(\boldsymbol{\alpha}; P)\, d\boldsymbol{\alpha} \tag{1.18}$$

by (1.16) and the identity

$$\int_{[0,1]^R} e(\boldsymbol{\alpha} \cdot \boldsymbol{u})\, d\boldsymbol{\alpha} = \begin{cases} 1 & \text{if } \boldsymbol{u} = \boldsymbol{0} \\ 0 & \text{if } \boldsymbol{u} \in \mathbb{Z}^R \setminus \{\boldsymbol{0}\}. \end{cases}$$

The principle behind the circle method is that the main term in (1.18) should come from those values of $\boldsymbol{\alpha}$ which lie near to rational vectors of small height. To make this precise, for each $\Delta \in (0,1)$ and $P \geq 1$ define the *major arcs*

$$\mathfrak{M}_{P,d,\Delta} = \bigcup_{\substack{q \in \mathbb{N} \\ q \leq P^{\Delta}}} \bigcup_{\substack{0 \leq a_1,\ldots,a_R \leq q \\ (a_1,\ldots,a_R,q)=1}} \left\{ \boldsymbol{\alpha} \in [0,1)^R : \left\| \boldsymbol{\alpha} - \tfrac{\boldsymbol{a}}{q} \right\|_\infty < P^{\Delta - d} \right\}, \tag{1.19}$$

and the *minor arcs*

$$\mathfrak{m}_{P,d,\Delta} = [0,1]^R \setminus \mathfrak{M}_{P,d,\Delta}. \tag{1.20}$$

When $\boldsymbol{\alpha} \in \mathfrak{M}_{P,d,\Delta}$ we can approximate $S(\boldsymbol{\alpha}; P)$ by a more well-behaved function, as described in Lemma 5.1 of Birch, or Lemma 3.7 below. Provided that we choose Δ sufficiently small, one hopes to deduce that

$$\int_{\mathfrak{M}_{P,d,\Delta}} S(\boldsymbol{\alpha}; P)\, d\boldsymbol{\alpha} = \mathfrak{S}_{\boldsymbol{f}} \mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} P^{n-dR} + O_{\boldsymbol{f}}(P^{n-dR-\delta}) \tag{1.21}$$

for all $P \geq 1$, some $\delta > 0$ depending on d, n, R and Δ and some constants $\mathfrak{S}_{\boldsymbol{f}}$ and $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ satisfying for all $P \geq 1$, some $\delta > 0$ and some constants $\mathfrak{S}_{\boldsymbol{f}}$ and $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ satisfying (1.8) and (1.9). To prove the asymptotic (1.17) it then remains to show that

$$\int_{\mathfrak{m}_{P,d,\Delta}} S(\boldsymbol{\alpha}; P)\, d\boldsymbol{\alpha} = O_{\boldsymbol{f}}(P^{n-dR-\delta}) \tag{1.22}$$

for some $\delta > 0$ depending on d, n, R and Δ. In Birch's work, and in ours, the constraint on the number of variables n comes ultimately from the proof of (1.22).

1.3.1 Birch's approach

Birch proceeds by a "sliding scale" argument which we sketch here. Let $P \geq 1$ and $\Delta \in (0,1)$. For each $\Delta' \in [\Delta, \infty)$ define the modified major arcs

$$\mathfrak{N}_{P,d,\Delta'} = \bigcup_{\substack{q \in \mathbb{N} \\ q \leq P^{\Delta'}}} \bigcup_{\substack{0 \leq a_1,\ldots,a_R \leq q \\ (a_1,\ldots,a_R,q)=1}} \left\{ \boldsymbol{\alpha} \in [0,1)^R : \left\| \boldsymbol{\alpha} - \tfrac{\boldsymbol{a}}{q} \right\|_\infty \leq \tfrac{1}{2q} P^{\Delta' - d} \right\}.$$

Birch's Lemma 4.3 shows that for each $\Delta' \in [\Delta, \infty]$ and $\epsilon > 0$ we have

$$\sup_{\boldsymbol{\alpha} \in [0,1)^R \setminus \mathfrak{N}_{P,d,\Delta'}} |S(\boldsymbol{\alpha}; P)| \ll_{d,n,R,\epsilon} P^{n - \left(\frac{n-1-\dim V_{\boldsymbol{F}}^{\dagger}}{2^{d-1}R(d-1)} \right)\Delta' + \epsilon}, \tag{1.23}$$

where $V_{\boldsymbol{F}}^{\dagger}$ is as in Theorem 1.2. By estimating the measure of $\mathfrak{N}_{P,d,\Delta'}$, it follows that for any $\Delta_1, \Delta_2 \in (0, \infty)$ with $\Delta_2 \geq \Delta_1$, we have

$$\int_{\mathfrak{N}_{P,d,\Delta_2} \setminus \mathfrak{N}_{P,d,\Delta_1}} |S(\boldsymbol{\alpha}; P)| \, d\boldsymbol{\alpha} \ll_{d,n,R,\epsilon} P^{n - dR + (R+1)\Delta_2 - \left(\frac{n-1-\dim V_{\boldsymbol{F}}^{\dagger}}{2^{d-1}R(d-1)} \right)\Delta_1 + \epsilon}.$$

Observe that we have $\mathfrak{m}_{P,d,\Delta} \subset [0,1)^R \setminus \mathfrak{N}_{P,d,\Delta}$ and so if $\Delta_T \geq \Delta_{T-1} \geq \cdots \geq \Delta_0 = \Delta$ holds then

$$\begin{aligned} \int_{\mathfrak{m}_{P,d,\Delta}} |S(\boldsymbol{\alpha}; P)| \, d\boldsymbol{\alpha} &\leq \int_{[0,1)^R \setminus \mathfrak{N}_{P,d,\Delta}} |S(\boldsymbol{\alpha}; P)| \, d\boldsymbol{\alpha} \\ &\ll_{d,n,R,\epsilon} P^{n-dR+\epsilon} \sum_{i=0}^{T-1} P^{(R+1)\Delta_{i+1} - \left(\frac{n-1-\dim V_{\boldsymbol{F}}^{\dagger}}{2^{d-1}R(d-1)} \right)\Delta_i} \\ &\quad + P^{n - \left(\frac{n-1-\dim V_{\boldsymbol{F}}^{\dagger}}{2^{d-1}R(d-1)} \right)\Delta_T + \epsilon}. \end{aligned} \tag{1.24}$$

If we set

$$\Delta_i = \Delta + \frac{i}{\log P},$$

then the sum in (1.24) becomes a geometric progression. If $n - 1 - \dim V_{\boldsymbol{F}}^{\dagger} > R(R+1)(d-1)2^{d-1}$, in other words if (1.6) is satisfied, then we can sum the progression and let $T \to \infty$ to deduce (1.22). A nearly equivalent proof is given in Lemma 4.4 of Birch.

1.3.2 Our approach

In §3.1.1 we will follow the strategy of §1.3.1, but rather than the sets $\mathfrak{N}_{P,d,\Delta+i/\log P}$ with $i = 0, \ldots, T$ we will use sets of the form

$$D(2^i) = \{ \boldsymbol{\alpha} \in [0,1)^R : |S(\boldsymbol{\alpha}; P)| \geq 2^i P^{n+\epsilon} \}$$

where $i \in \mathbb{Z}$. In place of (1.23) we have a trivial upper bound

$$\sup_{\boldsymbol{\alpha} \in [0,1)^R \setminus D(2^i)} |S(\boldsymbol{\alpha}; P)| \leq 2^i P^{n+\epsilon},$$

and the difficulty is to show that the set $D(2^i)$ has small measure. To do this we use a *repulsion principle* of the form

$$\min\left\{ \left| \frac{S(\boldsymbol{\alpha}; P)}{P^{n+\epsilon}} \right|, \left| \frac{S(\boldsymbol{\alpha} + \boldsymbol{\beta}; P)}{P^{n+\epsilon}} \right| \right\} \ll_{\boldsymbol{f},\epsilon} \max\left\{ P^{-d} \|\boldsymbol{\beta}\|_\infty^{-1}, \|\boldsymbol{\beta}\|_\infty^{\frac{1}{d-1}} \right\}^{\mathscr{C}}, \tag{1.25}$$

valid for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$ and $P \geq 1$, some $\mathscr{C} > 0$ and some small $\epsilon > 0$. Here we interpret the right-hand side to be $+\infty$ if $\boldsymbol{\beta} = \mathbf{0}$. If both $S(\boldsymbol{\alpha}; P)$ and $S(\boldsymbol{\alpha} + \boldsymbol{\beta}; P)$ are large then (1.25) implies that one of the terms $P^{-d}\|\boldsymbol{\beta}\|_\infty^{-1}$ or $\|\boldsymbol{\beta}\|_\infty^{\frac{1}{d-1}}$ must be large. In particular, the points $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}+\boldsymbol{\beta}$ must be either very close together or somewhat far apart, hence the term "repulsion principle".

In Chapter 3 this bound allows us to cover $D(2^i)$ by a collection of small balls separated by large gaps, and thereby to show that

$$\lambda(D(2^i)) \ll_{\boldsymbol{f},\epsilon} P^{-dR} 2^{-\frac{dR}{\mathscr{C}} i} \tag{1.26}$$

where λ is the Lebesgue measure, as in §1.1.1. This will imply (1.22) provided that $\mathscr{C} > dR$. Essentially the same idea is used in Bentkus and Götze [BG99, Theorem 5.1] and also in Müller [Mül08, Lemma 4].

The proof of (1.25) given in §3.2 essentially involves only properties of the single polynomial $\boldsymbol{\beta} \cdot \boldsymbol{f}$ rather than the system of R polynomials $\boldsymbol{f}$, which enables us to obtain a result which scales well as R grows. Since $\boldsymbol{\beta} \cdot \boldsymbol{f}$ may have real rather than rational coefficients, the argument can also be applied to systems of real polynomials, which we will discuss in §2.2

1.3.3 Passing to an auxiliary counting problem

To prove the minor arc bound (1.23) Birch uses the method known as Weyl differencing, which leads to a counting problem involving a certain system of multilinear forms. We use the same techniques to treat the repulsion principle (1.25), and in this section we describe the counting problem that results.

Suppose that $f(\boldsymbol{x})$ is a degree d polynomial in n variables, and that $d \geq 2$. For each $i = 1, \ldots, n$ and $j = 1, \ldots, d-1$ we let $x_i^{(j)}$ be a variable. We let $\boldsymbol{x}^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)})^T$, so that $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}$ are n-dimensional vectors of variables. For each $i = 1, \ldots, n$ we define a multilinear form in these vectors of variables by

$$m_i^{(f)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) = \sum_{j_1, \ldots, j_{d-1}=1}^{n} x_{j_1}^{(1)} \cdots x_{j_{d-1}}^{(d-1)} \frac{\partial^d f(\boldsymbol{x})}{\partial x_{j_1} \cdots \partial x_{j_{d-1}} \partial x_i}. \tag{1.27}$$

This is well defined since the dth derivatives of $f(\boldsymbol{x})$ are independent of $\boldsymbol{x}$. We write $\boldsymbol{m}^{(f)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})$ for the vector

$$(m_1^{(f)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}), \ldots, m_n^{(f)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}))^T, \tag{1.28}$$

and we then define the following counting function.

Definition 1.6. For each $B \geq 1$ we put $N_f^{\mathrm{aux}}(B)$ for the number of $(d-1)$-tuples of vectors $\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d-1)} \in \mathbb{Z}^n$ with

$$\begin{gathered} \|\boldsymbol{x}^{(1)}\|_\infty, \dots, \|\boldsymbol{x}^{(d-1)}\|_\infty \leq B, \\ \|\boldsymbol{m}^{(f)}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d-1)})\|_\infty < \|f^{[d]}\|_\infty B^{d-2}, \end{gathered} \tag{1.29}$$

In Proposition 3.3 we will prove the repulsion estimate (1.25) provided that

$$N_{\boldsymbol{\beta}\cdot\boldsymbol{f}}^{\mathrm{aux}}(B) \ll_{\boldsymbol{f}} B^{(d-1)n-2^d\mathscr{C}} \tag{1.30}$$

for all $B \geq 1$ and $\boldsymbol{\beta} \in \mathbb{R}^R$. From the definition we have the trivial bounds

$$B^{(d-2)n} \ll_{d,n} N_{\boldsymbol{\beta}\cdot\boldsymbol{F}}^{\mathrm{aux}}(B) \ll_{d,n} B^{(d-1)n}.$$

In particular, we saw in the comments following (1.26) that we will need $\mathscr{C} > dR$, and so we must have $n > 2^d dR$ in order for this approach to succeed. Chapters 4 and 5 will develop various strategies to prove bounds of the form (1.30).

1.4 Diophantine inequalities

We now consider results comparable to Birch's Theorem 1.2 above, but for systems of polynomials with real coefficients. Throughout this section $\boldsymbol{f}(\boldsymbol{x})$ will be a system of R real polynomials in n variables with leading part $\boldsymbol{f}^{[d]}(\boldsymbol{x})$, as in §1.1. In general one would not expect the system of equations $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ to have integral solutions. We will instead consider the system of Diophantine inequalities given by

$$\|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq 1, \tag{1.31}$$

to be solved in integers $x_1, \dots, x_n$. Under suitable conditions we aim to show that

$$\begin{aligned} &\#\{\boldsymbol{x} \in \mathbb{Z}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq 1\} \\ &\qquad = (1+o(1))\lambda\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \|\boldsymbol{f}^{[d]}(\boldsymbol{x})\|_\infty \leq 1\} \end{aligned} \tag{1.32}$$

for all $P \geq 1$. This would show that the number of integral points in the region defined by (1.31) is well approximated by the measure of that region.

If it happened that one of the f_i had integral coefficients, then (1.31) would imply that $f_i(\boldsymbol{x}) = 0$ or ± 1. This will impose p-adic conditions on the variables, which should be taken into account by including a singular series $\mathfrak{S}_{\boldsymbol{f}}$ in (1.32), as in §1.2.1. We can rule out this situation as follows.

Definition 1.7. We say that $\boldsymbol{f}^{[d]}$ is *irrational* if there is no vector $\boldsymbol{\alpha} \in \mathbb{R}^R \setminus \{\boldsymbol{0}\}$ such that the linear combination $\boldsymbol{\alpha} \cdot \boldsymbol{f}^{[d]}(\boldsymbol{x}) = \sum \alpha_i f_i(\boldsymbol{x})$ has integer coefficients.

We begin by outlining some previous work on inequalities of the form (1.31) which can be compared to Theorem 1.2. We then describe some new results.

1.4.1 Previous work

We begin by summarising the principle applications of the circle method to diagonal Diophantine inequalities. We then turn to non-diagonal systems of inequalities. In this case, other than two results due to Schmidt and to Chow, the results we describe relate to cubic and quadratic inequalities.

The work of Davenport, Heilbronn, Freeman, Bentkus and Götze is particularly relevant to our present investigation and will be discussed again in §2.2.

Davenport and Heilbronn [DH46] use the circle method to give an estimate for the number of solutions to $|f(\boldsymbol{x})| \leq 1$ with $|x_i| \leq P$, when f is an irrational diagonal quadratic form in five variables. Their techniques are only applicable for a certain sequence of values of P related to Diophantine approximation properties of the coefficients of f. This is however sufficient to show that there are infinitely many integral solutions. A large body of subsequent work uses their approach to study irrational systems of diagonal inequalities. We refer the reader to Brüdern and Cook [BC92] for a summary.

Freeman [Fre01, Fre02, Fre03], inspired by the work of Bentkus and Götze [BG97, BG99] discussed below, develops a version of the Davenport-Heilbronn method which can prove asymptotic formulae for the number of solutions to systems of diagonal inequalities. His methods are discussed in §2.2 below. Parsell [Par99, Par01, Par02] extends this work to systems of diagonal equations of differing degrees. These techniques are further refined by Wooley [Woo03], showing that the number of variables required is essentially the same as for a system of diagonal equations with the same degrees.

We now turn to results for forms which are not assumed to be diagonal. A lower bound is given by Schmidt [Sch80b] for the number of solutions to the inequality $\|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq 1$ with $\|\boldsymbol{x}\|_\infty \leq P$, where $\boldsymbol{f}(\boldsymbol{x})$ is a system of R homogeneous forms of possibly different odd degrees $d_1, \dots, d_R$ in a sufficiently large number of variables.

Chow [Cho17] proves an analogue of Theorem 1.2 for the inequalities $\|\boldsymbol{F}(\boldsymbol{x} + \boldsymbol{\mu})\|_\infty \leq 1$, where $\boldsymbol{F}$ is a system of R forms of degree d with rational coefficients and $\boldsymbol{\mu}$ is a vector of irrational real numbers. The condition (1.6) remains unchanged, and an asymptotic for the number of solutions with $|x_i| \leq P$ is obtained.

More is known for cubic inequalities. In the case of a system $\boldsymbol{f}$ of R cubic forms, Freeman [Fre00, Fre04] shows that there are infinitely many solutions provided only that $n \geq (10R)^{(10R)^5}$. Chow [Cho14] shows that $|f(\boldsymbol{x})| \leq 1$ has nonzero integer solutions for all real cubic forms f if the number of variables n is at least 358 823 708. He proves the same result for forms $f(\boldsymbol{x}) + g(\boldsymbol{y})$ which split into two parts with

$n \geq 120\,897\,257$, for forms $f_1(\boldsymbol{x}^{(1)}) + \cdots + f_6(\boldsymbol{x}^{(6)})$ which split into six parts with $n \geq 77\,027$, and various intermediate cases.

Chow [Cho16] also considers the system $F(\boldsymbol{x}) = \mathbf{0}, \|\boldsymbol{L}(\boldsymbol{x}) - \boldsymbol{\tau}\|_\infty \leq 1$, where F is a cubic form with integral coefficients, $\boldsymbol{L}$ is an irrational system of r linear forms and $\boldsymbol{\tau}$ is a vector of real numbers. An asymptotic formula is obtained for the number of solutions if F is nonsingular and $n \geq 16 + 8r$.

Last of all we take the case of quadratic inequalities. Margulis [Mar88, Mar89a, Mar89b] uses ideas from ergodic theory to show that if $f(\boldsymbol{x})$ is an indefinite, irrational, nonsingular real quadratic form in $n \geq 3$ variables and $t \in \mathbb{R}$, then $|f(\boldsymbol{x}) - t| < 1$ has nonzero integer solutions. This is false when $n = 3$, see Freeman [Fre99, p17]. When $n \geq 5$, or when $n = 4$ and f satisfies a certain explicit Diophantine condition, Eskin, Margulis and Mozes [EMM95, EMM98, EMM05] give an asymptotic for the number of solutions in integers between $-P$ and P. The Diophantine condition in the case $n = 4$ excludes those forms which are extremely well approximated by rational forms with signature 0 and square discriminant, and they construct forms of this type for which their conclusions do not hold. For any $n \geq 3$, Dani and Margulis [DM92, DM93] prove a lower bound for the number of solutions which exactly matches the asymptotic. Margulis and Mohammadi [MM11] extend all of the work of Margulis and collaborators described above to inhomogeneous quadratic polynomials. In the case $n = 3$ Lindenstrauss and Margulis [LM14] give a lower bound for the number of primitive integer solutions.

For forms in 9 or more variables, Bentkus and Götze [BG97, BG99] give a new proof of Eskin, Margulis and Mozes' asymptotic estimate using the circle method. They also give an asymptotic for the number of integer solutions to $B \leq f(\boldsymbol{x}) \leq B + 1$, where f is a positive definite irrational quadratic form and $B \to \infty$. These results are extended to $n \geq 5$ by Götze [Göt04] and by Götze and Margulis [GM10]. Müller [Mül05, Mül08, Mül11] estimates the number of solutions in integers between $-P$ and P to a system of inequalities $\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{t}\|_\infty \leq 1$, where $\boldsymbol{t} \in \mathbb{R}^R$ and $\boldsymbol{f}$ is a system of R quadratic forms in at least $8R + \sigma_{\mathbb{R}}(\boldsymbol{f}) + 1$ variables, with $\sigma_{\mathbb{R}}(\boldsymbol{f})$ as in Theorem 1.5.

1.4.2 Our results

Our main result on Diophantine inequalities is the following, proved in §6.1. The conditions on $\boldsymbol{f}^{[d]}$ may be compared with the hypotheses on $\boldsymbol{F}$ in Theorem 1.4 above. In particular, the condition that $\boldsymbol{f}^{[d]} \in U_{d,n,R}(\mathbb{R})$ reduces to a list of polynomial inequations $h_1(\boldsymbol{f}^{[d]}) \neq 0, \ldots, h_N(\boldsymbol{f}^{[d]}) \neq 0$ in the coefficients of the $f_i^{[d]}$.

Theorem 1.8. *If $d = 2$ or 3, suppose that $\boldsymbol{f}^{[d]}$ is nonsingular. If $d \geq 4$ then suppose that the system $\boldsymbol{f}^{[d]}$ belongs to $U_{d,n,R}(\mathbb{R})$, where the Zariski open set $U_{d,n,R}$ is as defined in Proposition 5.3 below. If $\boldsymbol{f}^{[d]}$ is irrational and we have*

$$n \geq d2^d R + R,$$

then there is $\mathfrak{I}_{\boldsymbol{f}^{[d]}} \geq 0$, depending only on $\boldsymbol{f}^{[d]}$, such that

$$\#\{\boldsymbol{x} \in \mathbb{Z}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq \tfrac{1}{2}\} = \mathfrak{I}_{\boldsymbol{f}^{[d]}} P^{n-dR} + o(P^{n-dR}) \tag{1.33}$$

as $P \to \infty$, and

$$\lambda\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \|\boldsymbol{f}^{[d]}(\boldsymbol{x})\|_\infty \leq \tfrac{1}{2}\} = \mathfrak{I}_{\boldsymbol{f}^{[d]}} P^{n-dR} + O_{\boldsymbol{f}}(P^{n-dR-\delta}) \tag{1.34}$$

for all $P \geq 1$. Here δ is some positive constant depending on d and R only, and the implicit constant in (1.34) depends at most on $\boldsymbol{f}$. If $\boldsymbol{f}^{[d]}(\boldsymbol{x}) = \mathbf{0}$ has a nonsingular real solution, then we have $\mathfrak{I}_{\boldsymbol{f}^{[d]}} > 0$.

If the system $\boldsymbol{f}$ has coefficients which are real algebraic numbers, then we may replace the $o(P^{n-dR})$ error term in (1.33) with a term $O_{\boldsymbol{f}}(P^{n-dR-\delta'})$, for some $\delta' > 0$ depending at most on d, n and R.

In the case that $d = 2$, Theorem 1.8 is essentially a restatement of results due to Bentkus and Götze [BG99, Theorem 2.2] and Müller [Mül08, Theorem 1]. The factors of $\frac{1}{2}$ in $\|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq \frac{1}{2}$ have been inserted to ensure that the constant $\mathfrak{I}_{\boldsymbol{f}^{[d]}}$ agrees with the usual singular integral found in applications of the circle method.

We can reduce the number of variables required, and remove the irrationality condition on $\boldsymbol{f}^{[d]}$, by considering a weaker condition than $\|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq \frac{1}{2}$. The theorem below is also proved in §6.1

Theorem 1.9. *If $d = 2$ or 3, suppose that $\boldsymbol{f}^{[d]}$ is nonsingular. If $d \geq 4$ then suppose that the system $\boldsymbol{f}^{[d]}$ belongs to the Zariski open set $U_{d,n,R}$ defined in Proposition 5.3 below.*

Let $\rho \in (0, d-1]$, and for $P \geq 1$ set

$$N_{\boldsymbol{f}}(P, P^\rho) = \#\{\boldsymbol{x} \in \mathbb{Z}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq \tfrac{1}{2}P^\rho\},$$
$$M_{\boldsymbol{f}^{[d]}}(P, P^\rho) = \lambda\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \|\boldsymbol{f}^{[d]}(\boldsymbol{x})\|_\infty \leq \tfrac{1}{2}P^\rho\}.$$

If we have

$$n > (d-\rho)2^d R + R - 1,$$

then there is $\mathfrak{I}_{\boldsymbol{f}^{[d]}} \geq 0$*, depending only on* $\boldsymbol{f}^{[d]}$*, such that*

$$N_{\boldsymbol{f}}(P, P^\rho) = \mathfrak{I}_{\boldsymbol{f}^{[d]}} P^{n-(d-\rho)R} + O_{\boldsymbol{f},\rho}(P^{n-(d-\rho)R-\delta}), \tag{1.35}$$

$$M_{\boldsymbol{f}^{[d]}}(P, P^\rho) = \mathfrak{I}_{\boldsymbol{f}^{[d]}} P^{n-(d-\rho)R} + O_{\boldsymbol{f}}(P^{n-(d-\rho)R-\delta}) \tag{1.36}$$

for all $P \geq 1$ *some positive* δ *depending on* d*,* n*,* R *and* ρ *only. If* $\boldsymbol{f}^{[d]}(\boldsymbol{x}) = \mathbf{0}$ *has a nonsingular real solution, then we have* $\mathfrak{I}_{\boldsymbol{f}^{[d]}} > 0$*.*

In the theorem above we assume that $\rho \leq d - 1$. Larger values of ρ would be possible but would not lead to any further improvement in the number of variables required, while other techniques would become available. For example if $\rho > d - 1$ and $\boldsymbol{f}^{[d]}$ is nonsingular with a nontrivial real zero, then it can be deduced from a result of Davenport [Dav51, Dav64] that

$$N_{\boldsymbol{f}}(P, P^\rho) = (1 + O_{d,n,R}(P^{-1}))M_{\boldsymbol{f}}(P, P^\rho).$$

If $\boldsymbol{f}^{[d]}$ is irrational with algebraic coefficients, one could also allow $\rho < 0$ in Theorem 1.9 as long as $|\rho|$ is sufficiently small. For simplicity, we omit the necessary modifications.

The results above are proved using the Davenport-Heilbronn circle method, an outline of which is given in §2.2 below.

1.5 Plan of this thesis

For reference, we give an overview of the argument used to prove the results above.

Some technical background is given in Chapter 2. Our §2.1 introduces the Manin-Peyre conjecture and relates it to the results of §1.2.1–1.4.2. We then, in §2.2, describe the Davenport-Heilbronn method used to prove Theorems 1.8 and 1.9. Finally §2.3 provides us with some lemmas from the geometry of numbers, Diophantine approximation and the elementary theory of real matrices.

In Chapter 3 we reduce the proof of Theorems 1.5 and 1.4 to the task of bounding the function $N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B)$ from §1.3.3 above. This is accomplished in two stages. Propositions 3.1 and 3.2 implement the circle method assuming a repulsion principle of the type described in §1.3.2, and Proposition 3.3 shows that such a principle follows from a bound for $N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B)$. Propositions 3.1 and 3.2 are proved in §3.1 following the outline from §1.3. After dealing with the major arcs in §3.1.2 we assemble some lemmas in §3.1.1 with which to treat the minor arcs and the proof is completed in §3.1.3. A variant of a well-known Weyl differencing argument is given in §3.2.1, which is used in §3.2.2 to prove Proposition 3.3.

We prove Theorem 1.5 at the start of Chapter 4 assuming Proposition 4.1, which is a bound for the counting function $N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B)$ from §1.3.3 and is valid when $d = 2$ or 3. In §4.2, we prove Proposition 4.1 in the simpler case when $d = 2$. When $d = 3$ we base our approach on an argument of Davenport which revolves around a Hessian matrix $H_c(\boldsymbol{x})$. This argument proceeds in four stages, described in §4.3. In §§4.4.1–4.4.4 we translate each stage into our new setting, with the final step completing the proof of the proposition.

In Chapter 5 we handle systems of forms of degree 4 or more. We use two results involving a certain quantity $\sigma^*(\boldsymbol{f}^{[d]})$ defined by equation (5.2). Proposition 5.2 bounds the function $N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B)$ from §1.3.3 in terms of $\sigma^*(\boldsymbol{f}^{[d]})$, while Proposition 5.3 shows that $\sigma^*(\boldsymbol{f}^{[d]}) < R$ holds for typical $\boldsymbol{f}$. We prove Theorem 1.4 in §5.1, assuming these results. After a preliminary lemma in §5.2.1, we prove Proposition 5.2 in §5.2.2. We then prove Proposition 5.3 in §5.3 by relating the quantity $\sigma^*(\boldsymbol{f}^{[d]})$ to the dimension of an explicit projective variety W, and parametrizing the complex points of W.

Forms with real coefficients are addressed in Chapter 6 with the Davenport-Heilbronn circle method. Here Proposition 6.1 gives an ϵ-free version of the repulsion estimate from Proposition 3.3. Proposition 6.2 then runs through the circle method assuming a bound on the function $N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B)$ from §1.3.3 and a complementary-arc bound (6.4). When $\rho = 0$ the latter bound (6.4) will be proved using a uniform upper bound given by Proposition 6.3. In §6.1 we deduce Theorems 1.8 and 1.9 from these results. In §6.2.1 we give a Weyl differencing argument analogous to §3.2.1, and Proposition 6.1 is deduced in §6.2.2. We then tackle Proposition 6.2, treating the central arc in §6.3.1 and the complementary arcs in §6.3.2 with the final steps occurring in §6.3.3. In §6.4 we invoke Lemma 2.10 from §2.3.2 to prove Proposition 6.3.

The remaining Chapter 7 is concerned with forms over number fields. After some basic definitions in §7.1, we introduce the construction known as Weil restriction in §7.2. In §7.3 we use this to prove an analogue of Theorem 1.4 over number fields, which we compare to the known results in this case.

Chapter 2

Background

In this chapter, we will set out a number of technical results for later use.

2.1 The Manin-Peyre conjecture

We reinterpret the asymptotic formula (1.7) from Chapter 1 as a result about the rational points on a variety.

Conjecture 2.1. *Assume that $\boldsymbol{H}(\boldsymbol{x})$ is a system of R forms of degrees $d_1, \dots, d_R$ in n variables with integer coefficients, and let $\boldsymbol{H}$ be nonsingular in the sense of §1.1. Write $D = \sum D_i$ and suppose that $n \geq \max\{D+2,\, R+4\}$. Set $V = V(\boldsymbol{H})$. Any point $X \in V(\mathbb{Q})$ is represented by a unique vector $\boldsymbol{x} \in \mathbb{Z}^n$ with coprime coordinates x_i, and we define the height of X by $h_V(X) = \|\boldsymbol{x}\|_\infty^{n-D}$. We say that V satisfies the* Manin-Peyre conjecture *over $\mathbb{Q}$ if there is a Zariski open set $U \subset V$ such that*

$$\#\{X \in U(\mathbb{Q}) : h_V(X) \leq B\} = \alpha_c \omega_\infty(V(\mathbb{R})) \Big(\prod_p \omega_p(V(\mathbb{Q}_p)) \Big) B \big(1 + o(1)\big) \tag{2.1}$$

as $B \to \infty$. Here ζ is the Riemann zeta function, the product is over primes p and converges absolutely and we define

$$\alpha_c = \frac{1}{n-D}, \tag{2.2}$$

$$\omega_\infty(V(\mathbb{R})) = \frac{n-D}{2} \lim_{B\to\infty} \tfrac{1}{B} \lambda\big\{\boldsymbol{t} \in \mathbb{R}^n :\ \|\boldsymbol{t}\|_\infty^{n-D} \leq B, \|\boldsymbol{H}(\boldsymbol{t})\|_\infty \leq 1\big\}, \tag{2.3}$$

$$\omega_p(V(\mathbb{Q}_p)) = (1 - p^{-(n-D)}) \lim_{k\to\infty} \tfrac{1}{p^{k(n-R)}} \#\big\{\boldsymbol{b} \in \{1, 2, \dots, p^k\}^n : \boldsymbol{H}(\boldsymbol{b}) \equiv \boldsymbol{0} \mod p^k\big\}, \tag{2.4}$$

where as in §1.1.1 we write λ for the Lebesgue measure.

As we will see below, the formula (2.1) is a special case of the conjectures of Manin and collaborators [FMT89, BM90]. Peyre [Pey95, p9 and Definition 2.1] proposes formulae for the constants α_c, $\omega_\infty(V(\mathbb{R}))$ and $\omega_p(V(\mathbb{Q}_p))$ in the setting of varieties over a general number field, and his Lemmas 2.4.4–2.4.6 imply that our definitions are special cases of his. The relation between the formula (2.1) and the asymptotic (1.7) from Theorem 1.2 is as follows.

Lemma 2.2. *Let $\boldsymbol{F}(\boldsymbol{x})$ be a nonsingular system of R forms with integral coefficients in n variables, all of the same degree d. Assume $n \geq \max\{dR+2, R+4\}$. Let $N_{\boldsymbol{F},\mathscr{B}}(P)$ be as in (1.5), let $\mathscr{B} = [-\frac{1}{2},\frac{1}{2}]^n$ and let $\delta \in (0,1)$. Suppose that for all $P \geq 1$ we have*

$$N_{\boldsymbol{F},\mathscr{B}}(P) = \mathfrak{I}_{\boldsymbol{F},\mathscr{B}}\mathfrak{S}_{\boldsymbol{F}} P^{n-dR} + O_{\boldsymbol{F}}(P^{n-dR-\delta}) \tag{2.5}$$

where the constants $\mathfrak{I}_{\boldsymbol{F},\mathscr{B}}$ and $\mathfrak{S}_{\boldsymbol{F}}$ satisfy formulae (1.8) and (1.9). Then the Manin-Peyre conjecture holds for $V(\boldsymbol{F})$ with choice $U = V(\boldsymbol{F})$ for the open set U.

Proof. Let $U = V(\boldsymbol{F})$ and observe that

$$\begin{aligned}&\#\{X \in U(\mathbb{Q}) : h_V(X) \leq B\}\\ &\qquad = \tfrac{1}{2}\#\big\{\boldsymbol{x} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} : \boldsymbol{x} \in 2B^{1/(n-dR)}\mathscr{B},\ \gcd(x_1,\dots,x_n)=1,\ \boldsymbol{F}(\boldsymbol{x})=\boldsymbol{0}\big\},\end{aligned}$$

where the leading factor of $\frac{1}{2}$ accounts for the fact that $\pm\boldsymbol{x}$ represent the same projective point. Let μ be the Möbius function, that is, μ is the function defined on natural numbers m by $\mu(m) = (-1)^r$ if m is squarefree with r distinct prime factors and $\mu(m) = 0$ otherwise. We have

$$\sum_{k|m}\mu(k) = \begin{cases}1 & \text{if } m = 1\\ 0 & \text{otherwise,}\end{cases}$$

and so on writing $\widetilde{B} = B^{1/(n-dR)}$, we have

$$\begin{aligned}\#\{X \in U(\mathbb{Q}) : h(X) \leq B\} &= \frac{1}{2}\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\setminus\{\boldsymbol{0}\}\\ \boldsymbol{x}\in 2\widetilde{B}\mathscr{B},\,\boldsymbol{F}(\boldsymbol{x})=\boldsymbol{0}}}\ \sum_{k|(x_1,\dots,x_n)}\mu(k)\\ &= \frac{1}{2}\sum_{k\leq 2B}\mu(k)\sum_{\substack{\frac{1}{k}\boldsymbol{x}\in\mathbb{Z}^n\setminus\{\boldsymbol{0}\}\\ \boldsymbol{x}\in 2\widetilde{B}\mathscr{B},\,\boldsymbol{F}(\boldsymbol{x})=\boldsymbol{0}}} 1.\end{aligned}$$

With $N_{\boldsymbol{F},\mathscr{B}}(P)$ as in (1.5), we have shown that

$$\#\{X \in U(\mathbb{Q}) : h(X) \leq B\} = \tfrac{1}{2}\sum_{k\leq 2\widetilde{B}}\mu(k)\{N_{\boldsymbol{F},\mathscr{B}}(2\widetilde{B}/k) - 1\}.$$

It thus follows from (2.5) that

$$\#\{X \in U(\mathbb{Q}) : h(X) \leq B\}$$
$$= \tfrac{1}{2}\mathfrak{I}_{\boldsymbol{F},\mathscr{B}}\mathfrak{S}_{\boldsymbol{F}} \sum_{k \leq 2\widetilde{B}} \frac{\mu(k)(2\widetilde{B})^{n-dR}}{k^{n-dR}} + O_{\boldsymbol{F}}\Big(\sum_{k \leq 2\widetilde{B}} \frac{(2\widetilde{B})^{n-dR-\delta}}{k^{n-dR-\delta}}\Big).$$

By assumption we have $n - dR \geq 2$, and so

$$\begin{aligned}\#\{X \in U(\mathbb{Q}) : h(X) \leq B\} &= \mathfrak{I}_{\boldsymbol{F},\mathscr{B}}\mathfrak{S}_{\boldsymbol{F}}2^{n-dR-1}\Big(\sum_{k=1}^{\infty} \frac{\mu(k)}{k^{n-dR}}\Big)\widetilde{B}^{n-dR} + o(\widetilde{B}^{n-dR}) \\ &= \frac{1}{\zeta(n-dR)}\mathfrak{I}_{\boldsymbol{F},\mathscr{B}}\mathfrak{S}_{\boldsymbol{F}}2^{n-dR-1}B + o(B). \qquad (2.6)\end{aligned}$$

Additionally

$$\begin{aligned}\frac{1}{\zeta(n-dR)}\mathfrak{S}_{\boldsymbol{F}} &= \Big(\prod_p \frac{1}{1-p^{-(n-dR)}}\Big)\mathfrak{S}_{\boldsymbol{F}} \\ &= \prod_p \omega_p(V(\mathbb{Q}_p)) \qquad (2.7)\end{aligned}$$

by (1.9) and (2.4), where the product converges absolutely. Finally by (1.8) and (2.3) we have $(n-D)2^{n-dR-1}\mathfrak{I}_{\boldsymbol{F},\mathscr{B}} = \omega_\infty(V(\mathbb{R}))$. Together with (2.6) and (2.7) this implies (2.1). □

2.1.1 The general setting

The formula (2.1) belongs to a wider family of predictions for the number of rational points of bounded height on varieties. In the most general setting one can consider a number field K, a projective variety V over K whose canonical class is not effective and a height function $h_{\mathcal{L}}$ associated to an ample, adelically metrised line bundle $\mathcal{L}$ on V. In this case, Manin and Batyrev [BM90, Conjecture C′] propose that

$$\#\{X \in U(K) : h_{V,K}(X) \leq B\} = C(\log B)^{t-1}B^{\alpha}(1+o(1)) \qquad (2.8)$$

as $B \to \infty$ where $\alpha \in \mathbb{R}$ and $t \in \mathbb{N}$ are explicit constants, $U \subset V$ is a suitable Zariski open subset and C is some real constant. Manin and Batyrev [BM90, Conjecture B'] and Franke, Manin and Tschinkel [FMT89, (0.2)] consider in particular the case when V is Fano, that is it has ample anticanonical class. In this case $\alpha = 1$ and $t = \operatorname{rank}\operatorname{Pic} V$, where $\operatorname{rank}\operatorname{Pic} V$ is the rank of the Picard group of V as a free Abelian group.

Peyre [Pey95, Conjecture 2.2.1] gives an explicit formula for C when V is a smooth Fano variety and $h_{\mathcal{L}}$ is the usual anticanonical height. Batyrev and Tschinkel [BT98] give such a formula whenever V is "$\mathcal{L}$-primitive". This is more general, including for example the case when $h_{\mathcal{L}}$ is the anticanonical height and V is Fano with at worst canonical singularities.

2.1.2 Counterexamples

As we have stated it, Conjecture 2.1 will fail if V violates the Hasse principle, since the right-hand side of (2.1) will then be positive and the left-hand side will be zero. More generally one expects the conjecture to fail if the points $V(\mathbb{Q})$ are not Zariski dense. One should then ask for the more general conjecture (2.8) to hold for any suitably large number field K.

This is not always the case, however. Tschinkel and Batyrev [TB96] construct a smooth Fano variety V such that (2.8) fails for every number field K containing $\sqrt{-3}$, every open subset U and every height $h_{\mathcal{L}}$ associated to a metrisation of the anticanonical bundle. Loughran [Lou15] constructs such a V for any given number field K. It may also be the case that Peyre's prediction for the constant C fails, even if the asymptotic (2.8) is correct, see Le Rudulier [LR14, Théorème 4.2].

Browning and Loughran [BL17] discuss a number of counterexamples to a uniform version of (2.8) in which the open set U is not allowed to depend on the choice of height $h_{\mathcal{L}}$. In particular, in their Example 3.6 they construct a counterexample which takes the form $V(\boldsymbol{H})$, where $\boldsymbol{H}$ is a nonsingular system of two quadratic forms in 6 variables such that $V(\boldsymbol{H})(\mathbb{Q})$ is Zariski dense. This suggests that Conjecture 2.1 may fail to hold even when $V(\mathbb{Q})$ is Zariski dense.

One may obtain a revised conjecture compatible with all of these counterexamples by following a suggestion of Peyre [Pey03, §8]. If V is a Fano variety over a number field K, and $V(K)$ is Zariski dense, then this conjecture states that

$$\#\{X \in V(K) \setminus Z : h_{V,K}(X) \leq B\} = C(\log B)^{\operatorname{rank}\operatorname{Pic}(V)-1}B(1+o(1))$$

as $B \to \infty$, where the constant C agrees with the conjectures of Peyre [Pey95] and Batyrev and Tschinkel [BT98], and Z is a *thin set* in the sense of Serre [Ser08, §3.1]. That is, Z is a finite union of sets each of which is contained either in a proper subvariety of V, or in the image of the K-points of an irreducible variety W under a generically finite dominant morphism of degree at least 2. This possibility is discussed at greater length by Browning and Loughran [BL17].

2.2 The Davenport-Heilbronn circle method

We now consider a variant of the circle method devised by Davenport and Heilbronn [DH46], which we use in Chapter 6 to prove Theorems 1.8 and 1.9. The notation defined here, in particular (2.11), (2.12), (2.16) and Definition 2.3, will be used throughout that chapter.

Let the system $\boldsymbol{f}$ be as in §1.1. As in Theorem 1.8, one hopes to prove that

$$\#\{\boldsymbol{x} \in \mathbb{Z}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq \tfrac{1}{2}\} = (1+o(1))\mathfrak{I}_{\boldsymbol{f}^{[d]}} P^{n-dR} \tag{2.9}$$

as $P \to \infty$, where $\mathfrak{I}_{\boldsymbol{f}^{[d]}}$ is a real constant.

Let the box $\mathscr{B}$ and the sum $S(\boldsymbol{\alpha}; P)$ be as in §1.3 and let φ be a continuous, compactly supported function on $\mathbb{R}^R$ which takes values in $[0,1]$. We have

$$\int_{\mathbb{R}^R} \hat{\varphi}(\boldsymbol{\alpha}) S(\boldsymbol{\alpha}; P)\, d\boldsymbol{\alpha} = \sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^n \\ \boldsymbol{x}/P \in \mathscr{B}}} \varphi(\boldsymbol{f}(\boldsymbol{x})) \tag{2.10}$$

where we write $\hat{\varphi}(\boldsymbol{\alpha}) = \int \varphi(\boldsymbol{t}) e(-\boldsymbol{\alpha} \cdot \boldsymbol{t})\, d\boldsymbol{t}$ for the Fourier transform. If $\boldsymbol{f}$ is nonsingular, n is large and $\boldsymbol{f}^{[d]}$ is irrational in the sense of Definition 1.7 then we expect the main term in (2.10) to come from small values of $\boldsymbol{\alpha}$ in the following sense. For each $P \geq 1$ and $\Delta \in (0,1)$ define

$$\mathfrak{C}_{P,d,\Delta} = [-P^{\Delta-d}, P^{\Delta-d}]^R, \tag{2.11}$$

$$\mathfrak{c}_{P,d,\Delta} = \mathbb{R}^R \setminus \mathfrak{C}_{P,d,\Delta}. \tag{2.12}$$

Historically $\mathfrak{C}_{P,d,\Delta}$ been called the major arc. Drawing on Brüdern et al [BKW12] we avoid ambiguity by referring to $\mathfrak{C}_{P,d,\Delta}$ as the *central arc* and $\mathfrak{c}_{P,d,\Delta}$ as the *complementary arcs*. Then one expects that for an appropriate choice of Δ we have

$$\int_{\mathfrak{C}_{P,d,\Delta}} S(\boldsymbol{\alpha}; P)\varphi(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = \hat{\varphi}(\boldsymbol{0})\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} P^{n-dR} + O_{\boldsymbol{f},\varphi}(P^{n-dR-\delta}) \tag{2.13}$$

for all $P \geq 1$, some $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} \geq 0$ and some $\delta > 0$. Following the circle method as described in §1.3, one might hope that

$$\int_{\mathfrak{c}_{P,d,\Delta}} S(\boldsymbol{\alpha}; P)\varphi(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \ll_{\boldsymbol{f},\varphi} P^{n-dR-\delta} \tag{2.14}$$

for all $P \geq 1$ and some $\delta > 0$. Writing $\chi_{[-\frac{1}{2},\frac{1}{2}]^R}$ for the indicator function of $[-\frac{1}{2},\frac{1}{2}]^R$, we could then try to deduce (2.9) by setting $\mathscr{B} = [-\frac{1}{2},\frac{1}{2}]^n$ and letting φ tend to $\chi_{[-\frac{1}{2},\frac{1}{2}]^R}$ pointwise.

However (2.14) may be false, for example if the coefficients of $\boldsymbol{f}$ are extremely well approximable by rational numbers. In such cases one might still hope for some bound of the form

$$\int_{\mathfrak{c}_{P,d,\Delta}} S(\boldsymbol{\alpha};P)\varphi(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} = o(P^{n-dR}). \tag{2.15}$$

Except on the central arc or major arcs, upper bounds for $S(\boldsymbol{\alpha};P)$ typically include an extra multiplicative factor of size at least $(\log P)^{O(1)}$. But the $o(1)$ term in (2.15) might tend to 0 very slowly, more slowly than $(\log P)^{-\epsilon}$ for any $\epsilon > 0$. As a result there are irrational systems $\boldsymbol{f}$ for which the bound (2.15) seems at present to be beyond reach. Largely for these reasons, the original method of Davenport and Heilbronn [DH46] is limited to a special sequence of values of P at which (2.14) holds, and cannot prove (2.9).

2.2.1 Freeman's innovations

Freeman [Fre01, Fre02, Fre03], building on Bentkus and Götze [BG97, BG99], introduced a variant Davenport-Heilbronn method which is capable of proving the asymptotic formula (2.9). Given $P \geq 1$, $\boldsymbol{\alpha} \in \mathbb{R}^R$ and an infinitely differentiable, compactly supported function ω on $\mathbb{R}^n$ taking values in $[0,1)$, we define an exponential sum

$$S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P) = \sum_{\boldsymbol{x}\in\mathbb{Z}^n} e(\boldsymbol{\alpha}\cdot\boldsymbol{f}(\boldsymbol{x}))\omega(\boldsymbol{x}/P). \tag{2.16}$$

As in (2.10) let φ be a continuous, compactly supported function on $\mathbb{R}^R$ which takes values in $[0,1]$. By Fourier inversion,

$$\sum_{\boldsymbol{x}\in\mathbb{Z}^n} \varphi\big(\boldsymbol{f}(\boldsymbol{x})\big)\omega(\boldsymbol{x}/P) = \int_{\mathbb{R}^R} S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)\hat{\varphi}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}.$$

Freeman's course of action is to show that for some $\Delta \in (0,1)$ we have

$$\int_{\mathfrak{C}_{P,d,\Delta}} S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)\varphi(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} = \hat{\varphi}(\boldsymbol{0})\mathfrak{I}_{\boldsymbol{f},\omega}P^{n-dR} + O(P^{n-dR-\delta}), \tag{2.17}$$

$$\int_{\mathfrak{c}_{P,d,\Delta}} S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)\varphi(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} = o(P^{n-dR}) \tag{2.18}$$

as $P \to \infty$. One can use Weyl differencing to obtain upper bounds for a smoothly weighted sum $S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)$ without introducing extra logarithmic factors, in contrast to the situation for $S(\boldsymbol{\alpha};P)$ described after (2.15). This makes (2.18) more easily accessible than (2.15).

One aims to deduce (2.9) from (2.16)–(2.18). This requires the error terms in (2.17) and (2.18) to depend in a controlled way on the weights φ and ω. It is convenient to introduce smoothness classes of the following kind.

Definition 2.3. Write $\boldsymbol{\kappa}$ for a sequence of positive reals $(\kappa_0, \kappa_1, \kappa_2, \dots)$. We write $\vartheta \in \mathcal{S}(\boldsymbol{\kappa}, k)$ if ϑ is an infinitely differentiable function on $\mathbb{R}^k$, supported on $[-2,2]^k$, such that

$$\left| \frac{\partial^j \vartheta(\boldsymbol{t})}{\partial t_{i_1} \cdots \partial t_{i_j}} \right| \leq \kappa_j$$

for all $\boldsymbol{t} \in \mathbb{R}^k$ and $j, i_1, \dots, i_j \in \mathbb{N}$. We write $\mathrm{Poly}_{d,n}(\boldsymbol{\kappa})$ for a polynomial in finitely many of the κ_i, with coefficients depending at most on d and n, and which may vary from line to line. We adopt the convention that the implicit constants in $O(\,\cdot\,)$ and $\ll$ notation may depend on the sequence $\boldsymbol{\kappa}$ only by having a multiplicative factor of the form $\mathrm{Poly}_{d,n}(\boldsymbol{\kappa})$, so that for example $O_{\boldsymbol{f},\boldsymbol{\kappa}}(\,\cdot\,)$ means $\mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) O_{\boldsymbol{f}}(\,\cdot\,)$.

The strategy is establish the bounds (2.17) and (2.18) for all $\varphi \in \mathcal{S}(\boldsymbol{\kappa}, R)$ and $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$ with an error term which depends polynomially on $\boldsymbol{\kappa}$. Then one chooses φ to be a function which converges pointwise as $P \to \infty$ to the indicator function of $[-\frac{1}{2}, \frac{1}{2}]^R$, and similarly one chooses ω to converge pointwise to the indicator function of $[-1,1]^n$. In order for $\varphi \in \mathcal{S}(\boldsymbol{\kappa}, R)$ and $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$ to hold one needs that $\kappa_j \to \infty$ as $P \to \infty$ for each fixed j. However the error terms in (2.17) and (2.18) will still be negligible so long as κ_j does to tend to infinity too quickly. This allows one to recover the unweighted asymptotic (2.9).

The principal challenge is the bound (2.18) for the integral over $\mathfrak{c}_{P,d,\Delta}$. When $\boldsymbol{f}$ is diagonal Freeman [Fre03] arranges for $S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)$ to factor as a product of exponential sums in one variable, and using major and minor arc estimates for these one-variable sums he is able to prove (2.18). We use a different strategy.

2.2.2 Our approach

We prove the estimate (2.17) in §6.3.1. We then tackle (2.18) in §6.3.2 via the sharp mean value (6.46). This takes the form

$$\int_E |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)| \, d\boldsymbol{\alpha} \ll P^{n-dR-\delta} + \left(\sup_{\boldsymbol{\alpha} \in E} \frac{|S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)|}{P^n} \right)^{\delta} P^{n-dR}, \tag{2.19}$$

and is valid for some $\delta > 0$, any $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$, any $P \geq 1$ and any measurable set E is contained in a hypercube of side length 1 in $\mathbb{R}^R$. In the case when $\boldsymbol{f}^{[d]}$ is irrational this is combined with the uniform upper bound (6.4) which is of the shape

$$\sup_{\substack{\boldsymbol{\alpha} \in \mathbb{R}^R \\ P^{\Delta - d} < \|\boldsymbol{\alpha}\|_\infty < \xi(P)^{-1} P^{-\rho}}} |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)| \ll \mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) \xi(P) P^n \tag{2.20}$$

where ξ is some decreasing function, taking positive real values, such that $\xi(P) \to 0$ as $P \to \infty$. Although the details vary, applications of Freeman's method can typically be rephrased in this way if desired; compare for example formulae (1.5) and (1.6) of Wooley [Woo03] and the subsequent comments.

In §6.3.2 we will prove the sharp mean value (2.19) by an ϵ-free repulsion estimate from Proposition 6.1, and deduce (2.18) using (2.20) and the rapid decay of $\hat{\varphi}$. In the case $d = 2$ this essentially reproduces work of Bentkus and Götze [BG99, Theorem 5.1] and Müller [Mül08, Lemma 4]. Then in §6.3.3 we deduce (2.9) following the strategy outlined in §2.2.1.

In fact we address a slightly more general problem. Recall that in Theorem 1.9 we considered the inequality $\|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq \frac{1}{2}P^\rho$ for some $\rho \in (0, d-1]$, rather than $\|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq \frac{1}{2}$ as in (2.9). In order to deal with this we actually carry out the argument sketched n §2.2.1 with $\varphi(\boldsymbol{f}(\boldsymbol{x}))$ replaced by $\varphi(P^{-\rho}\boldsymbol{f}(\boldsymbol{x}))$ and $\hat{\varphi}(\boldsymbol{\alpha})$ replaced by $P^{\rho R}\hat{\varphi}(P^\rho\boldsymbol{\alpha})$ for some $\rho \in [0, d-1]$. We then specialise to the case $\rho = 0$ to prove Theorem 1.8, and to the case $\rho > 0$ to prove Theorem 1.9.

2.3 Auxiliary results

In this section we collect together results which will be needed later.

2.3.1 Eigenvalues and minors

We give some results on real matrices for use in §4.4.2. We need the following relatively straightforward fact; we include a proof for the reader's convenience.

Lemma 2.4. *For each $k, \ell \in \mathbb{N}$, let*

$$T_{k,\ell} = \{\boldsymbol{a} \in \mathbb{N}^k : 1 \leq a_1 < \cdots < a_k \leq \ell\}.$$

This set has $\binom{\ell}{k}$ members. For each $k, \ell, m \in \mathbb{N}$ such that $k \leq \min\{\ell, m\}$, and each $\ell \times m$ real matrix L, define an $\binom{\ell}{k} \times \binom{m}{k}$ real matrix $L^{[k]}$ by

$$L^{[k]} = (L^{[k]}_{\boldsymbol{ab}})_{\boldsymbol{a}\in T_{k,\ell},\boldsymbol{b}\in T_{k,m}}, \qquad L^{[k]}_{\boldsymbol{ab}} = \det((L_{a_i b_j})_{1\leq i,j\leq k}),$$

so that the $L^{[k]}_{\boldsymbol{ab}}$ are the $k \times k$ minors of L. For all $\ell \times m$ matrices L, all $m \times n$ matrices M and all $k \leq \min\{\ell, m, n\}$ we have $(LM)^{[k]} = L^{[k]}M^{[k]}$. That is, we have

$$(LM)^{[k]}_{\boldsymbol{ab}} = \sum_{\boldsymbol{w}\in T_{k,m}} L^{[k]}_{\boldsymbol{aw}} M^{[k]}_{\boldsymbol{wb}}. \tag{2.21}$$

Proof. Let $\boldsymbol{e}^{(1)}, \ldots, \boldsymbol{e}^{(m)}$ be the standard basis of $\mathbb{R}^m$. Fix $L, \boldsymbol{a}, \boldsymbol{b}$; then each side of (2.21) is an alternating multilinear form in those k columns of M whose indices appear in the vector $\boldsymbol{b}$. This is some k-tuple of m-vectors.

Given the value of an alternating multilinear form at the k-tuple $\boldsymbol{e}^{(z_1)}, \ldots, \boldsymbol{e}^{(z_k)}$ for each $\boldsymbol{z} \in T_{k,m}$, one can extend by linearity and the alternating property to find its value at any k-tuple of m-vectors. In other words, it suffices to check (2.21) when, for some $\boldsymbol{z} \in T_{k,m}$, the $k \times k$ submatrix $(M_{z_i b_j})_{1 \le i,j \le k}$ is the identity and all other entries of M are zero. In this case both sides of (2.21) are equal to $M^{[k]}_{\boldsymbol{zb}}$. □

We next relate the size of the minors of a matrix M to the eigenvalues of $M^T M$.

Lemma 2.5. *Let M be a real $m \times n$ matrix. Recall that $M^T M$ is positive semidefinite and symmetric. Let the eigenvalues of $M^T M$ be $\Lambda_1^2, \cdots, \Lambda_n^2$ in decreasing order, where the Λ_i are nonnegative and in decreasing order. That is, the Λ_i are the singular values of M, listed in decreasing order.*

In particular, if M is a symmetric matrix, then the Λ_i are exactly the absolute values of the eigenvalues of M, by diagonalisation.

Given a natural number k with $k \le \min\{m, n\}$, let $\boldsymbol{\Delta}^{(k)}$ be the vector of $k \times k$ minors of M, arranged in some order. Then the maximum norm $\|\boldsymbol{\Delta}^{(k)}\|_\infty$ satisfies

$$\|\boldsymbol{\Delta}^{(k)}\|_\infty \asymp_{m,n} \Lambda_1 \cdots \Lambda_k. \tag{2.22}$$

Proof. Let the sets $T_{k,\ell}$ and the matrices $L^{[k]}$ be as in Lemma 2.4. Let O be an $n \times n$ orthogonal matrix such that $O^T M^T M O$ is diagonal. Let $\widetilde{\boldsymbol{\Delta}}^{(k)}$ be the vector of $k \times k$ minors of MO. We claim that the norms $\|\widetilde{\boldsymbol{\Delta}}^{(k)}\|_\infty$ and $\|\boldsymbol{\Delta}^{(k)}\|_\infty$ are of comparable size.

Lemma 2.4 shows that $(MO)^{[k]} = M^{[k]} O^{[k]}$, and since $(O^T)^{[k]} O^{[k]} = I^{[k]}$ and $(O^T)^{[k]}_{\alpha\beta} = O^{[k]}_{\beta\alpha}$ we see that $O^{[k]}$ is orthogonal. Hence the maximum norm of the entries satisfies

$$\|\widetilde{\boldsymbol{\Delta}}^{(k)}\|_\infty = \|(MO)^{[k]}\|_\infty \asymp_{m,n} \|M^{[k]}\|_\infty = \|\boldsymbol{\Delta}^{(k)}\|_\infty.$$

So in proving (2.22) we may assume that $M^T M$ is diagonal with diagonal entries Λ_i^2. We have

$$\begin{aligned} \sum_{\boldsymbol{a} \in T_{k,n}} \Lambda_{a_1}^2 \cdots \Lambda_{a_k}^2 &= \sum_{\boldsymbol{a} \in T_{k,n}} (M^T M)^{[k]}_{\boldsymbol{aa}} \\ &= \sum_{\substack{\boldsymbol{a} \in T_{k,n} \\ \boldsymbol{w} \in T_{k,m}}} (M^{[k]}_{\boldsymbol{wa}})^2, \end{aligned} \tag{2.23}$$

by (2.21). The left-hand side of (2.23) is $\asymp_n \Lambda_1^2 \cdots \Lambda_k^2$, and the right-hand side is $\asymp_{m,n} \|\mathbf{\Delta}^{(k)}\|_\infty^2$, so this proves (2.22). $\square$

The next lemma shows, roughly speaking, that if there is no linear space of a given dimension on which M is uniformly large, then there must be a space of greater than the complementary dimension on which M is uniformly small.

Lemma 2.6. *Let M be a real $m \times n$ matrix. Let $k \in \mathbb{N}$ such that $k \leq \min\{m, n\}$ holds. For any $C \geq 1$, either there is an $(n-k+1)$-dimensional linear subspace X of $\mathbb{R}^n$ such that*

$$\|M\boldsymbol{X}\|_\infty \leq C^{-1}\|\boldsymbol{X}\|_\infty \qquad \text{for all } \boldsymbol{X} \in X, \tag{2.24}$$

or there is a k-dimensional linear subspace V of $\mathbb{R}^n$, spanned by standard basis vectors of $\mathbb{R}^n$, such that

$$\|M\boldsymbol{v}\|_\infty \gg_{m,n} C^{-1}\|\boldsymbol{v}\|_\infty \qquad \text{for all } \boldsymbol{v} \in V. \tag{2.25}$$

Proof. Let $\Lambda_1, \ldots, \Lambda_n$ be as in Lemma 2.5. Let X be the span of the Λ_i^2-eigenvectors of $M^T M$, where i runs from k up to n. Suppose that X does not satisfy (2.24). Since $M^T M$ is orthogonally diagonalizeable , we have $\boldsymbol{X}^T M^T M \boldsymbol{X} \ll_n \|\boldsymbol{X}\|_\infty^2 \Lambda_k^2$ for all $\boldsymbol{X} \in X$. It follows that

$$\|M\boldsymbol{X}\|_\infty \ll_{m,n} \|\boldsymbol{X}\|_\infty \Lambda_k$$

for all $\boldsymbol{X} \in X$. We assumed that (2.24) is false, and so we must have

$$\Lambda_k \gg_{m,n} C^{-1}. \tag{2.26}$$

We claim that there is a k-dimensional linear space $V \subset \mathbb{R}^n$ such that

$$\|M\boldsymbol{v}\|_\infty \gg_{m,n} \|\boldsymbol{v}\|_\infty \Lambda_k, \qquad \text{for all } \boldsymbol{v} \in V, \tag{2.27}$$

and that we may take V to be a span of k standard basis vectors $\boldsymbol{e}^{(i)}$ in $\mathbb{R}^n$. Then (2.25) follows from (2.26) and (2.27), which proves the lemma.

It remains to find a span of k standard basis vectors which satisfies (2.27). As in Lemma 2.5, let $\mathbf{\Delta}^{(k)}$ be the vector of $k \times k$ minors of M.

By permuting the rows and columns of M, we may assume that

$$\|\mathbf{\Delta}^{(k)}\|_\infty = |\det(M_{ij})_{1 \leq i,j \leq k}|. \tag{2.28}$$

Let V be the span of the first k basis vectors, and let $\boldsymbol{v} \in V$. If we apply the same permutation to the v_i and to the first k rows of M, then both sides of our claim (2.27) and our assumption (2.28) remain the same. By applying such a permutation we may assume $\|\boldsymbol{v}\|_\infty = |v_1|$.

Since $v_i = 0$ for $i > k$, one has the identity

$$\left(\begin{array}{ccc} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{k1} & \cdots & M_{kn} \\ \hline 0_{(n-k)\times k} & \multicolumn{2}{|c}{I_{n-k}} \end{array}\right) \left(\begin{array}{c|c} \boldsymbol{v} & \begin{array}{c} 0_{1\times(n-1)} \\ \hline I_{n-1} \end{array} \end{array}\right) = \left(\begin{array}{cccc} (M\boldsymbol{v})_1 & M_{12} & \cdots & M_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ (M\boldsymbol{v})_k & M_{k2} & \cdots & M_{kn} \\ \hline \multicolumn{2}{c|}{0_{(n-k)\times k}} & \multicolumn{2}{c}{I_{n-k}} \end{array}\right)$$

where we have divided each matrix into three blocks, and $0_{p\times q}$ stands for a $p \times q$ block of zeroes. By (2.28) we have $\pm v_1 \cdot \|\boldsymbol{\Delta}^{(k)}\|_\infty$ for the determinant of the left-hand side. Expanding the determinant of the right-hand side in the first column, we find that it is equal to

$$\sum_{\ell=1}^{k} (-1)^{\ell+1} (M\boldsymbol{v})_\ell \det\big((M_{ij})_{\substack{i=1,\dots,k;\, i\neq \ell \\ j=2,\dots,k}} \big) \ll k \|M\boldsymbol{v}\|_\infty \|\boldsymbol{\Delta}^{(k-1)}\|_\infty.$$

So we have $\|\boldsymbol{\Delta}^{(k)}\|_\infty v_1 \ll \|M\boldsymbol{v}\|_\infty \|\boldsymbol{\Delta}^{(k-1)}\|_\infty$. By Lemma 2.5, this implies that $\Lambda_k v_1 \ll_{m,n} \|M\boldsymbol{v}\|_\infty$. As $\|\boldsymbol{v}\|_\infty = |v_1|$, we have shown that (2.27) holds. □

2.3.2 Geometry of numbers

Recall the notation $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$ defined in §1.1. The goal of this section is Lemma 2.11, which shows that if L is a real matrix and $\|L\boldsymbol{u}\|_{\mathbb{R}/\mathbb{Z}}$ is small for a large proportion of integral vectors $\boldsymbol{u}$ of bounded height, then $\|L\|_{\mathbb{R}/\mathbb{Z}}$ is small. This will be used in §6.4 to treat irrational systems of forms.

For the proof we use the geometry of numbers. We begin with some terminology. Suppose that Λ is a lattice in $\mathbb{R}^k$, that is, a set of the form

$$\Big\{ \sum_{i=1}^{k} a_i \boldsymbol{t}^{(i)} : \boldsymbol{a} \in \mathbb{Z}^k \Big\}$$

where the $\boldsymbol{t}^{(i)}$ are some k linearly independent vectors in $\mathbb{R}^k$. We call $\boldsymbol{t}^{(1)}, \dots, \boldsymbol{t}^{(k)}$ a basis of Λ. The determinant of Λ is the quantity

$$\det \Lambda = \det(\boldsymbol{t}^{(1)} \mid \dots \mid \boldsymbol{t}^{(k)}),$$

where $(\boldsymbol{t}^{(1)} \mid \dots \mid \boldsymbol{t}^{(k)})$ is the matrix with columns $\boldsymbol{t}^{(i)}$. The determinant of Λ is independent of the choice of basis $\boldsymbol{t}^{(i)}$.

If F is a non-negative real function on $\mathbb{R}^k$, then for $i = 1, \ldots, k$ we define the ith successive minimum of F with respect to Λ to be the smallest positive real number M_i such that there are i linearly independent vectors $\boldsymbol{t}$ in Λ with $F(\boldsymbol{t}) \leq M_i$. Observe that these successive minima are well defined, and that $M_1 \leq \cdots \leq M_k$. We can relate these to $\det \Lambda$ as follows.

Theorem 2.7 (Minkowski's second theorem)**.** *Suppose that F is a norm on $\mathbb{R}^k$. Let $M_1, \ldots, M_k$ be the successive minima of F with respect to a lattice Λ. Suppose that the set $\{\boldsymbol{t} \in \mathbb{R}^n : F(\boldsymbol{t}) \leq 1\}$ is convex. Then we have*

$$\frac{2^k}{k!} \det \Lambda \leq \Big(\prod_{i=1}^{k} M_i\Big) \lambda\{\boldsymbol{t} \in \mathbb{R}^k : F(\boldsymbol{t}) \leq 1\} \leq 2^k \det \Lambda,$$

where λ is the Lebesgue measure.

For a proof, see Theorem V in §VIII.4.3 of Cassels [Cas97]. It is often useful to pick a basis of the following kind.

Lemma 2.8. *If F is a norm on $\mathbb{R}^k$, and $M_1, \ldots, M_k$ are the successive minima of F with respect to Λ, then there is a basis $\boldsymbol{t}^{(1)}, \ldots, \boldsymbol{t}^{(k)}$ of Λ such that $2F(\boldsymbol{t}^{(i)}) \leq iM_i$.*

This is the Corollary to Theorem VII in §VIII.5.2 of Cassels [Cas97]. We quote one more result, relating the M_i to the counting function of the lattice Λ.

Lemma 2.9. *Let Λ be a lattice in $\mathbb{R}^k$ and let $M_1, \ldots, M_k$ be the successive minima of the norm $\|\cdot\|_\infty$ with respect to Λ. For each $r > 0$ there exists $\ell \in \{0, \ldots, k\}$ such that*

$$\#\{\boldsymbol{t} \in \Lambda : \|\boldsymbol{t}\|_\infty \leq r\} \asymp_k \frac{r^\ell}{M_1 \ldots M_\ell}, \tag{2.29}$$

where the implicit constants depend only on k.

Proof. This is Lemma 12.4 in Davenport [Dav05]. The statement of that result has a Euclidean norm $\sqrt{\boldsymbol{t}^T\boldsymbol{t}}$ in place of the maximum norm $\|\boldsymbol{t}\|_\infty$. These norms are equivalent, that is we have $\sqrt{\boldsymbol{t}^T\boldsymbol{t}} \asymp_k \|\boldsymbol{t}\|_\infty$, and so up to adjustments in the implicit constant in (2.29) this is equivalent to the result as we state it. Additionally, to simplify the proof Davenport states his result on the assumption that $\det \Lambda = 1$. We are free to scale the lattice Λ by some real constant to ensure that this condition is satisfied, and the lemma follows. □

In order to prove Lemma 2.11, treating $\|L\boldsymbol{v}\|_{\mathbb{R}/\mathbb{Z}}$ for a real matrix L and integral vectors $\boldsymbol{v}$, we specialise to a particular lattice and norm.

Lemma 2.10. *Let L be a real $m \times n$ matrix and let $C, P \geq 1$. Let F be the norm on $\mathbb{R}^m \times \mathbb{R}^n$ given by*

$$F(\boldsymbol{u}, \boldsymbol{v}) = \max\{P\|\boldsymbol{u} - L\boldsymbol{v}\|_\infty,\, P^{-1}\|\boldsymbol{v}\|_\infty\}.$$

Let $M_1, \ldots, M_{m+n}$ be the successive minima of F with respect to the lattice $\mathbb{Z}^m \times \mathbb{Z}^n$.

Suppose that

$$\#\{\boldsymbol{v} \in \mathbb{Z}^n : \|\boldsymbol{v}\|_\infty \leq P,\, \|L\boldsymbol{v}\|_{\mathbb{R}/\mathbb{Z}} < P^{-1}\} \gg_{m,n} P^{n-1} \tag{2.30}$$

for some sufficiently large implicit constant depending only on m and n. Then

$$\#\{\boldsymbol{v} \in \mathbb{Z}^n : \|\boldsymbol{v}\|_\infty \leq P,\, \|L\boldsymbol{v}\|_{\mathbb{R}/\mathbb{Z}} < P^{-1}\} \asymp_{m,n} \prod_{i=1}^{n} M_i^{-1}. \tag{2.31}$$

Proof. We will begin by proving that

$$\#\{\boldsymbol{v} \in \mathbb{Z}^n : \|\boldsymbol{v}\|_\infty \leq P,\, \|L\boldsymbol{v}\|_{\mathbb{R}/\mathbb{Z}} < P^{-1}\} \asymp_{m,n} \prod_{i=1}^{n'} M_i^{-1} \tag{2.32}$$

for some $0 \leq n' \leq n + m$, where we interpret the empty product $\prod_{i=1}^{0} = 0$ as zero. We consider the lattice Λ in $\mathbb{R}^m \times \mathbb{R}^n$ defined by

$$\Lambda = \{(P(\boldsymbol{u} - L\boldsymbol{v}), P^{-1}\boldsymbol{v}) : (\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{Z}^{m+n}\}.$$

We defined the M_i to be the successive minima of F with respect to $\mathbb{Z}^m \times \mathbb{Z}^n$. But these are identical to the successive minima of $\|\cdot\|_\infty$ with respect to Λ. So Lemma 2.9 shows that

$$\#\{\boldsymbol{t} \in \Lambda : \|\boldsymbol{t}\|_\infty \leq 1\} \asymp_{m,n} \prod_{i=1}^{n'} M_i^{-1} \qquad \text{for some } 0 \leq n' \leq n + m,$$

which is exactly (2.32). In order to show that $n' = n$ in (2.32), we claim that

$$P^{-1} \leq M_i \ll_{m,n} P \qquad \text{for each } i = 1, \ldots, m + n, \tag{2.33}$$

and that

$$\prod_{i=1}^{m+n} M_i^{-1} \asymp_{m,n} P^{m-n}. \tag{2.34}$$

We have $F(\boldsymbol{u}, \boldsymbol{v}) \geq 1/P$ for any $(\boldsymbol{u}, \boldsymbol{v}) \neq (\boldsymbol{0}, \boldsymbol{0})$, which implies the lower bound in (2.33). For the upper bound, let $\boldsymbol{d}^{(1)}, \ldots, \boldsymbol{d}^{(m)}$ be the standard basis of $\mathbb{R}^m$, let

$\boldsymbol{e}^{(1)}, \dots, \boldsymbol{e}^{(n)}$ be the standard basis of $\mathbb{R}^n$ and for each i write $[L\boldsymbol{e}^{(i)}] \in \mathbb{Z}^m$ for an integer vector such that $\|L\boldsymbol{e}^{(i)} - [L\boldsymbol{e}^{(i)}]\|_\infty = \|L\boldsymbol{e}^{(i)}\|_{\mathbb{R}/\mathbb{Z}}$ holds. We then have a basis of $\mathbb{Z}^m \times \mathbb{Z}^n$ given by

$$(\boldsymbol{d}^{(1)}, \boldsymbol{0}), \dots, (\boldsymbol{d}^{(m)}, \boldsymbol{0}), ([L\boldsymbol{e}^{(1)}], \boldsymbol{e}^{(1)}), \dots, ([L\boldsymbol{e}^{(n)}], \boldsymbol{e}^{(n)}).$$

The first i vectors from this basis form a linearly independent set, and by the definition of the successive minima above Theorem 2.7 this proves the upper bound in (2.33).

For (2.34) we apply Theorem 2.7. This gives us $\prod_{i=1}^{m+n} M_i^{-1} \asymp_{m,n} \int_{F(\boldsymbol{t}) \leq 1} d\boldsymbol{t}$, and (2.34) follows since

$$\int_{F(\boldsymbol{t}) \leq 1} d\boldsymbol{t} = \int_{\|\boldsymbol{v}\|_\infty \leq P} \int_{\|\boldsymbol{u} - L\boldsymbol{v}\|_\infty \leq P^{-1}} d\boldsymbol{u}\, d\boldsymbol{v} = 2^{m+n} P^{n-m}. \tag{2.35}$$

Now suppose that $n' > n$ in (2.32). Then (2.33) and (2.34) give

$$\begin{aligned} \#\{\boldsymbol{v} \in \mathbb{Z}^n : \|\boldsymbol{v}\|_\infty \leq P,\ \|L\boldsymbol{v}\|_{\mathbb{R}/\mathbb{Z}} < P^{-1}\} &\asymp_{m,n} P^{n-m} \prod_{i=n'+1}^{m+n} M_i \\ &\ll_{m,n} P^{n-m} P^{m-1} \\ &= P^{n-1} \end{aligned}$$

which contradicts our assumption (2.30). If instead $n' < n$ in (2.32), then (2.33) gives

$$\#\{\boldsymbol{v} \in \mathbb{Z}^n : \|\boldsymbol{v}\|_\infty \leq P,\ \|L\boldsymbol{v}\|_{\mathbb{R}/\mathbb{Z}} < P^{-1}\} \asymp_{m,n} \prod_{i=1}^{n'} M_i^{-1} \leq P^{n-1},$$

again contradicting our assumption. This proves that $n' = n$, and so (2.31) holds. □

We can now state the main lemma of this section.

Lemma 2.11. *Let $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$ be as in §1.1. Let $C \geq 1$ and let L be an $m \times n$ integer matrix. Suppose that $P \gg_{m,n} C$ and that*

$$\#\{\boldsymbol{v} \in \mathbb{Z}^n : \|\boldsymbol{v}\|_\infty \leq P,\ \|L\boldsymbol{v}\|_{\mathbb{R}/\mathbb{Z}} < P^{-1}\} \geq C^{-1} P^n.$$

Then

$$\|qL\|_{\mathbb{R}/\mathbb{Z}} \ll_{m,n} CP^{-2} \tag{2.36}$$

for some $q \in \mathbb{N}$ satisfying $q = O_{m,n}(C)$.

Proof. Let the norm F and the successive minima M_i be as in Lemma 2.10. The assumptions of Lemma 2.11 ensure that

$$\#\{\boldsymbol{v} \in \mathbb{Z}^n : \|\boldsymbol{v}\|_\infty \leq P,\ \|L\boldsymbol{v}\|_{\mathbb{R}/\mathbb{Z}} < P^{-1}\} \gg_{m,n} P^{n-1}$$

for some sufficiently large implicit constant. Hence we may apply Lemma 2.10 to show that

$$\prod_{i=1}^{n} M_i^{-1} \gg_{m,n} C^{-1} P^n. \tag{2.37}$$

We apply Lemma 2.8 to find a basis $(\boldsymbol{u}^{(i)}, \boldsymbol{v}^{(i)}) \in \mathbb{Z}^m \times \mathbb{Z}^n$ $(1 \leq i \leq m+n)$ of $\mathbb{R}^m \times \mathbb{R}^n$ such that

$$\|\boldsymbol{u}^{(i)} - L\boldsymbol{v}^{(i)}\|_\infty \leq \frac{i}{2} M_i P^{-1}, \qquad \|\boldsymbol{v}^{(i)}\|_\infty \leq \frac{i}{2} M_i P. \tag{2.38}$$

We then define two integer matrices, of size $m \times n$ and $n \times n$ respectively, by

$$\begin{aligned} U &= (\boldsymbol{u}^{(1)} \mid \cdots \mid \boldsymbol{u}^{(n)}), \\ V &= (\boldsymbol{v}^{(1)} \mid \cdots \mid \boldsymbol{v}^{(n)}), \end{aligned} \tag{2.39}$$

and set

$$q = |\det V|.$$

Note that $q \in \mathbb{N} \cup \{0\}$. If we write V^{adj} for the adjoint, then we have

$$\begin{aligned} \|qL\|_{\mathbb{R}/\mathbb{Z}} &\leq \|UV^{\mathrm{adj}} - qL\|_\infty \\ &= \|(U - LV)V^{\mathrm{adj}}\|_\infty \\ &\ll_{m,n} P^{-2} \prod_{i=1}^{n} M_i P, \end{aligned}$$

by the definition (2.38). Together with (2.37) this implies that (2.36) holds. We claim that $q \neq 0$, and that $q \ll_{m,n} 1$.

Suppose for a contradiction that $q = 0$, so that $\det V = 0$. Let W be a nonsingular square submatrix of V of maximal size. Considering the columns of the adjoint W^{adj}, one can construct a vector $\boldsymbol{w} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$ with $V\boldsymbol{w} = \boldsymbol{0}$, such that the bound $\|\boldsymbol{w}\|_\infty \ll_{m,n} \prod_{i=1}^{n} \|\boldsymbol{v}^{(i)}\|_\infty$ holds. Then we have

$$\begin{aligned} \|U\boldsymbol{w}\|_\infty &= \|U\boldsymbol{w} - LV\boldsymbol{w}\|_\infty \\ &\ll_{m,n} \|LV - U\|_\infty \cdot \|\boldsymbol{w}\|_\infty \\ &\ll_{m,n} (M_n P^{-1}) \prod_{i=1}^{n} M_i P, \end{aligned}$$

by (2.38). Now, by the same argument as in (2.33), we have $M_1 \geq P^{-1}$ for each i. So we must have

$$\|U\boldsymbol{w}\|_\infty \ll_{m,n} P^{-2} \prod_{i=1}^{n} (M_i P)^2.$$

By (2.37), it follows that $\|U\boldsymbol{w}\|_\infty \ll_{m,n} P^{-2}C^{-2}$. But by assumption we have $P \gg_{m,n} C$, and so since $U\boldsymbol{w}$ is an integral vector it must vanish. But then both $U\boldsymbol{w}$ and $V\boldsymbol{w}$ vanish while $\boldsymbol{w}$ is nonzero, which is impossible as the $(\boldsymbol{u}^{(i)}, \boldsymbol{v}^{(i)})$ are linearly independent. So we have $q \neq 0$.

It remains to show that $q \ll_{m,n} C$ holds. It suffices to observe that by (2.39) we have

$$q \ll_{m,n} \prod_{i=1}^{n} \|\boldsymbol{v}^{(i)}\|_\infty$$

and by (2.38) this is

$$\ll_n \prod_{i=1}^{n} M_i P.$$

A final application of (2.37) proves the claim. □

2.3.3 Schmidt's Subspace Theorem

In §6.4 we will treat systems of forms with algebraic coefficients using a corollary to the following celebrated theorem of Schmidt.

Theorem 2.12 (The Subspace Theorem)**.** *Let $L_1(\boldsymbol{x}), \dots, L_n(\boldsymbol{x})$ be linearly independent linear forms with real or complex algebraic coefficients and let $\delta > 0$. Then there are finitely many proper linear subspaces $T_1, \dots, T_w$ of $\mathbb{Q}^n$ such that every nonzero integral vector $\boldsymbol{x} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$ satisfying*

$$|L_1(\boldsymbol{x}) \cdots L_n(\boldsymbol{x})| < \|\boldsymbol{x}\|_\infty^{-\delta}$$

is contained in one of the spaces T_i.

For a proof, see the original paper by Schmidt [Sch72] or the textbook by the same author [Sch80a, Theorem 1F, p153]. The result we use is the following.

Corollary 2.13. *Let M be a nonsingular $m \times n$ matrix with real algebraic entries, with $m \leq n$, such that $M\boldsymbol{x} = \boldsymbol{0}$ has no integral solutions. Then for all nonzero integral vectors $\boldsymbol{x} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$ we have*

$$\|M\boldsymbol{x}\|_\infty \gg_{M,\epsilon} \|\boldsymbol{x}\|_\infty^{m-n-\epsilon}. \tag{2.40}$$

Proof. We proceed by induction on n. If $n = 1$ then we must have $m = 1$, and the result is trivial, merely stating that $|Mx| \gg_{M,\epsilon} |x|^{-\epsilon}$ for all nonzero real algebraic numbers M and integers x. Suppose that $n > 1$ and that $m \leq n$.

Without loss of generality, by permuting the rows and columns of M if necessary, we may assume that the leading minor $(M_{ij})_{1\leq i,j\leq m}$ is nonsingular. Define linear forms $L_1(\boldsymbol{x}), \dots, L_n(\boldsymbol{x})$ by

$$L_i(\boldsymbol{x}) = \sum_{j=1}^{n} M_{ij}x_j \quad \text{if } 1 \leq i \leq m, \qquad L_i(\boldsymbol{x}) = x_i \quad \text{if } m+1 \leq i \leq n. \tag{2.41}$$

Our assumption that $(M_{ij})_{1\leq i,j\leq m}$ is nonsingular ensures that the L_i are linearly independent.

Let $\boldsymbol{x} \in \mathbb{Z}^n$. By the Subspace Theorem (Theorem 2.12), we either have $\boldsymbol{x} = \boldsymbol{0}$, or

$$|L_1(\boldsymbol{x}) \cdots L_n(\boldsymbol{x})| \geq \|\boldsymbol{x}\|_\infty^{-\epsilon}, \tag{2.42}$$

or else $\boldsymbol{x}$ is contained in one of a finite collection $\{T_1, \dots, T_w\}$ of proper linear subspaces of $\mathbb{Q}^n$, which depends only on M and ϵ.

If $\boldsymbol{x} = \boldsymbol{0}$ then (2.40) is trivial. If (2.42) holds, then (2.41) implies that

$$\|M\boldsymbol{x}\|_\infty^m \|\boldsymbol{x}\|_\infty^{m-n} \geq \|\boldsymbol{x}\|_\infty^{-\epsilon}$$

and hence (2.40) holds. It remains to consider the case when $\boldsymbol{x} \in T_i$ for some particular i.

Let $\dim T_i = n'$, so that $n' < n$ holds. All integral points in T_i are $\mathbb{Z}$-linear combinations of $\boldsymbol{v}^{(1)}, \dots, \boldsymbol{v}^{(n')}$, for some rational vectors $\boldsymbol{v}^{(k)} \in T_i$ depending only on M, i and ϵ. We define a matrix $V = (\boldsymbol{v}^{(1)}| \dots |\boldsymbol{v}^{(n')})$, so that the integral vector $\boldsymbol{x} \in T_i \cap \mathbb{Z}^n$ satisfies

$$\boldsymbol{x} = V\boldsymbol{y} \tag{2.43}$$

for some $\boldsymbol{y} \in \mathbb{Z}^{n'} \setminus \{\boldsymbol{0}\}$.

Consider the $m \times n'$ matrix MV. Write m' for the rank of this matrix. Let M' be an $m' \times n'$ matrix constructed by taking the first m' linearly independent rows of MV and discarding the remaining rows. Then the matrix M' is a nonsingular $m' \times n'$ matrix depending only on M, i and ϵ, and the inequalities $m' \leq n' < n$ hold. By induction we have

$$\|MV\boldsymbol{y}\|_\infty \gg_{M',\epsilon} \|\boldsymbol{y}\|_\infty^{m'-n'-\epsilon}. \tag{2.44}$$

Since $\|\boldsymbol{y}\|_\infty \asymp_{M,i,\epsilon} \|V\boldsymbol{y}\|_\infty$, and i is drawn from a finite set depending only on M and ϵ, we see from (2.43) and (2.44) that (2.40) holds. □

Chapter 3

Repulsion and the circle method

In this chapter we will estimate the number of solutions to systems of Diophantine equations, conditional on an upper bound for the counting function $N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B)$ from Definition 1.6.

Recall the system $\boldsymbol{f}$ from §1.1 and the exponential sum $S(\boldsymbol{\alpha};P)$ from (1.16). Following the plan sketched in §§1.3.2, we will be able to apply the circle method to the equations $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ provided that the repulsion principle

$$\min\left\{\left|\frac{S(\boldsymbol{\alpha};P)}{P^{n+\epsilon}}\right|, \left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P)}{P^{n+\epsilon}}\right|\right\} \leq C\max\left\{P^{-d}\|\boldsymbol{\beta}\|_\infty^{-1},\ \|\boldsymbol{\beta}\|_\infty^{\frac{1}{d-1}}\right\}^{\mathscr{C}} \tag{3.1}$$

holds for some $\mathscr{C} > dR$, $C \geq 1$ and $\epsilon > 0$ and all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$ and $P \geq 1$. Recall from (1.17) that the left-hand side of (3.1) depends implicitly on $\mathscr{B}$, and recall from the comments after (1.25) that if $\boldsymbol{\beta} = \boldsymbol{0}$ then we interpret the right-hand side of (3.1) to be $+\infty$.

Proposition 3.1. *Let $\boldsymbol{f}$ be as in §1.1 above and let $P \geq 1$. Assume $\boldsymbol{f} \in \mathbb{Z}[\boldsymbol{x}]^R$ has integral coefficients, and that the leading forms $f^{[d]}(\boldsymbol{x})$ are linearly independent. Call a box in $\mathbb{R}^n$ an* admissible box *if it is contained in the box $[-1,1]^R$, and has sides of length at most 1 which are parallel to the coordinate axes. Write*

$$N_{\boldsymbol{f},\mathscr{B}}(P) = \#\{\boldsymbol{x} \in \mathbb{Z}^n : \boldsymbol{x}/P \in \mathscr{B},\ \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}\}.$$

Suppose we are given $\mathscr{C} > dR$, $C \geq 1$ and $\epsilon > 0$ such that the bound (3.1) holds for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$, all $P \geq 1$ and all admissible boxes $\mathscr{B}$. If ϵ is sufficiently small in terms of $\mathscr{C}$, d and R, then we have

$$N_{\boldsymbol{f},\mathscr{B}}(P) = \mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}\mathfrak{S}_{\boldsymbol{f}}P^{n-dR} + O_{C,\boldsymbol{f}}(P^{n-dR-\delta})$$

for all $P \geq 1$, all admissible boxes $\mathscr{B}$, some $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} \geq 0$ depending only on $\boldsymbol{f}$ and $\mathscr{B}$, some $\mathfrak{S}_{\boldsymbol{f}} \geq 0$ depending only on $\boldsymbol{f}$ and some $\delta > 0$ depending only on $\mathscr{C}$, d and R.

Proposition 3.1 is proved in §3.1.3. The constants $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ and $\mathfrak{S}_{\boldsymbol{f}}$ have the following explicit description (compare (1.8) and (1.9) above).

Proposition 3.2. *In Proposition 3.1 we may take $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ and $\mathfrak{S}_{\boldsymbol{f}}$ to be the usual singular integral and series given by*

$$\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} = \lim_{P\to\infty} \tfrac{1}{P^{n-dR}}\lambda\big\{\boldsymbol{t}\in\mathbb{R}^n : \tfrac{1}{P}\boldsymbol{t}\in\mathscr{B}, \|\boldsymbol{f}^{[d]}(\boldsymbol{t})\|_\infty \leq \tfrac{1}{2}\big\}, \tag{3.2}$$

and

$$\mathfrak{S}_{\boldsymbol{f}} = \prod_p \lim_{k\to\infty} \tfrac{1}{p^{k(n-R)}}\#\big\{\boldsymbol{b}\in\{1,2,\ldots,p^k\}^n : \boldsymbol{f}(\boldsymbol{b})\equiv\boldsymbol{0} \mod p^k\big\}, \tag{3.3}$$

where the product is over primes p and converges absolutely. If the equation $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ has a solution lying in the box $\mathscr{B}$ for which the Jacobian matrix $(\partial f_i(\boldsymbol{x})/\partial x_j)_{ij}$ is nonsingular, then $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ is positive. If, for each prime p, the equation $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ has a solution in p-adic integers for which the Jacobian matrix $(\partial f_i(\boldsymbol{x})/\partial x_j)_{ij}$ is nonsingular, then $\mathfrak{S}_{\boldsymbol{f}}$ is positive.

Proposition 3.2 is proved in §3.1.4. Our last result of this chapter, proved in §3.2.2, gives conditions for the hypothesis (3.1) to be satisfied.

Proposition 3.3. *Let $\boldsymbol{f}$ be as in §1.1, and let $N_f^{\mathrm{aux}}(B)$ be as in Definition 1.6. Suppose that the forms $f_i^{[d]}$ are linearly independent, and that we are given $C_0 \geq 1$ and $\mathscr{C} > 0$ such that for all $\boldsymbol{\beta}\in\mathbb{R}^R$ and $B\geq 1$ we have*

$$N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B) \leq C_0 B^{(d-1)n-2^d\mathscr{C}}. \tag{3.4}$$

Let $\epsilon > 0$. Then there exists $C\geq 1$, depending only on C_0, $\boldsymbol{f}^{[d]}$ and ϵ, such that the bound (3.1) *holds for all $P\geq 1$ and all $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{R}^R$.*

3.1 The circle method

The proof of Proposition 3.1 divides into minor and major arcs, as described in §1.3.

3.1.1 The minor arcs

Our aim in this section is to assemble two lemmas which can be used to prove the minor arc bound (1.22). Lemma 3.5 shows that it suffices to have the repulsion principle (3.1), together with a bound for the supremum of $|S(\boldsymbol{\alpha};P)|$ on the minor arcs, which will be provided by Lemma 3.6.

We begin with a technical lemma concerning repulsion-type estimates of a very general kind.

Lemma 3.4. *Let $r_1 : (0,\infty) \to (0,\infty)$ be a strictly decreasing continuous function, and let $r_2 : (0,\infty) \to (0,\infty)$ be a strictly increasing continuous function. Write r_1^{-1} and r_2^{-1} for the inverses of these maps. Let $\nu > 0$ and let E_0 be a hypercube in $\mathbb{R}^R$ whose sides are of length ν and parallel to the coordinate axes. Let E be a measurable subset of E_0 and let $\varphi : E \to [0,\infty)$ be a measurable function.*

Suppose that for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$ such that $\boldsymbol{\alpha} \in E$ and $\boldsymbol{\alpha} + \boldsymbol{\beta} \in E$, we have

$$\min\{\varphi(\boldsymbol{\alpha}),\, \varphi(\boldsymbol{\alpha}+\boldsymbol{\beta})\} \leq \max\bigl\{r_1^{-1}(\|\boldsymbol{\beta}\|_\infty),\, r_2^{-1}(\|\boldsymbol{\beta}\|_\infty)\bigr\}. \tag{3.5}$$

Then, for any integers k and ℓ with $k < \ell$, we have

$$\int_E \varphi(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \ll_R \nu^R 2^k + \sum_{i=k}^{\ell-1} 2^i \left(\frac{\nu r_1(2^i)}{\min\{r_2(2^i), \nu\}} \right)^R + \left(\frac{\nu r_1(2^\ell)}{\min\{r_2(2^\ell), \nu\}} \right)^R \sup_{\boldsymbol{\alpha} \in E} \varphi(\boldsymbol{\alpha}), \tag{3.6}$$

where the implicit constant depends only on R.

Note that if we choose

$$\varphi(\boldsymbol{\alpha}) = |S(\boldsymbol{\alpha}; P)|/CP^{n+\epsilon}, \qquad r_1(t) = P^{-d} t^{-1/\mathscr{C}}, \qquad r_2(t) = t^{(d-1)/\mathscr{C}},$$

then the hypothesis (3.5) is precisely the bound (3.1).

Proof. The strategy of proof is as follows. We deduce from (3.5) that if both $\varphi(\boldsymbol{\alpha}) \geq t$ and $\varphi(\boldsymbol{\alpha}+\boldsymbol{\beta}) \geq t$ hold, then either $\|\boldsymbol{\beta}\|_\infty \leq r_1(t)$ or $\|\boldsymbol{\beta}\|_\infty \geq r_2(t)$ must hold. From this we will show that the set of points $\boldsymbol{\alpha}$ satisfying the bound $\varphi(\boldsymbol{\alpha}) \geq t$ can be covered by a collection of hypercubes of side length $2r_1(t)$, each of which is separated from the others by a gap of size $\frac{1}{2}r_2(t)$. The lemma will follow upon bounding the total Lebesgue measure of this collection of hypercubes.

For each $t > 0$ we set

$$D(t) = \{\boldsymbol{\alpha} \in E : \varphi(\boldsymbol{\alpha}) \geq t\}. \tag{3.7}$$

Observe that if $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}+\boldsymbol{\beta}$ both belong to $D(t)$, then (3.5) implies that

$$\max\bigl\{r_1^{-1}(\|\boldsymbol{\beta}\|_\infty),\, r_2^{-1}(\|\boldsymbol{\beta}\|_\infty)\bigr\} \geq t,$$

from which it follows that either $\|\boldsymbol{\beta}\|_\infty \leq r_1(t)$ or $\|\boldsymbol{\beta}\|_\infty \geq r_2(t)$ must hold.

Let $\mathfrak{a}$ be any hypercube in $\mathbb{R}^R$ whose sides are of length $\frac{1}{2}r_2(t)$ and parallel to the coordinate axes. We claim that $\mathfrak{a} \cap D(t)$ is contained in a hypercube $\mathfrak{B}$ whose sides are of length $2r_1(t)$.

To see this let $\boldsymbol{\alpha}$ be any fixed vector lying in $\mathfrak{a} \cap D(t)$, and set

$$\mathfrak{A} = \{\boldsymbol{\alpha} + \boldsymbol{\beta} : \boldsymbol{\beta} \in \mathbb{R}^R, \|\boldsymbol{\beta}\|_\infty \leq r_1(t)\}.$$

If $\boldsymbol{\alpha} + \boldsymbol{\beta}$ belongs to $\mathfrak{a} \cap D(t)$, then by definition of $\mathfrak{b}$ the bound $\|\boldsymbol{\beta}\|_\infty \leq \frac{1}{2} r_2(t)$ must hold. In particular $\|\boldsymbol{\beta}\|_\infty < r_2(t)$, so by the comments after (3.7) the bound $\|\boldsymbol{\beta}\|_\infty \leq r_1(t)$ must hold. This shows that $\boldsymbol{\alpha} + \boldsymbol{\beta} \in \mathfrak{A}$, and hence that $\mathfrak{a} \cap D(t) \subset \mathfrak{A}$, as claimed. In particular the Lebesgue measure of $\mathfrak{a} \cap D(t)$ is at most $(2r_1(t))^R$.

The set $D(t)$ is contained in E_0, a hypercube of side length ν. Thus, in order to cover the set $D(t)$ the number of boxes $\mathfrak{b}$ of side length $\frac{1}{2} r_2(t)$ one needs is at most

$$\ll_R \frac{\nu^R}{\min\{r_2(t), \nu\}^R}.$$

Summing over all the boxes $\mathfrak{b}$, it follows that

$$\lambda(D(t)) \ll_R \left(\frac{\nu r_1(t)}{\min\{r_2(t), \nu\}} \right)^R, \tag{3.8}$$

where λ is the Lebesgue measure. Consequently we have

$$\begin{aligned} \int_E \varphi(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} &= \int\limits_{E \setminus D(2^k)} \varphi(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} + \sum_{i=k}^{\ell-1} \int\limits_{E \cap (D(2^i) \setminus D(2^{i+1}))} \varphi(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \\ &\quad + \int\limits_{E \cap D(2^\ell)} \varphi(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \\ &\leq \nu^R 2^k + \sum_{i=k}^{\ell-1} 2^{i+1} \lambda(D(2^i)) + \lambda(D(2^\ell)) \sup_{\boldsymbol{\alpha} \in E} \varphi(\boldsymbol{\alpha}). \end{aligned}$$

With (3.8) this yields (3.6). □

We now apply Lemma 3.4 to deduce mean values from bounds of the form (3.1). The following result is stated in greater generality than is strictly required here, so we may apply it to forms with real coefficients in Chapter 6.

Lemma 3.5. *Let T be a complex-valued measurable function on $\mathbb{R}^R$. Let E_0 be a hypercube in $\mathbb{R}^R$ whose sides are of length ν and parallel to the coordinate axes, and let E be a measurable subset of E_0. Suppose that for some $P \geq 1$ and some $\mathscr{C} > 0$, the inequality*

$$\min\left\{ \left| \frac{T(\boldsymbol{\alpha})}{P^n} \right|, \left| \frac{T(\boldsymbol{\alpha} + \boldsymbol{\beta})}{P^n} \right| \right\} \leq \max\left\{ P^{-d} \|\boldsymbol{\beta}\|_\infty^{-1}, \|\boldsymbol{\beta}\|_\infty^{\frac{1}{d-1}} \right\}^{\mathscr{C}} \tag{3.9}$$

holds for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$*. Suppose further that*

$$\sup_{\boldsymbol{\alpha} \in E} |T(\boldsymbol{\alpha})| \leq P^{n-\delta} \tag{3.10}$$

for some $\delta \geq 0$*. Then*

$$\int_E T(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \ll \begin{cases} \nu^R P^{n-\mathscr{C}} & + P^{n-\mathscr{C}-(d-1)R} & \text{if } \mathscr{C} < R \\ \nu^R P^{n-\mathscr{C}} & + P^{n-dR} \log P & \text{if } \mathscr{C} = R \\ \nu^R P^{n-\mathscr{C}} & + P^{n-dR-\delta(1-\frac{R}{\mathscr{C}})} & \text{if } R < \mathscr{C} < dR \\ \nu^R P^{n-\mathscr{C}} \log P & + P^{n-dR-\delta(1-\frac{R}{\mathscr{C}})} & \text{if } \mathscr{C} = dR \\ \nu^R P^{n-dR-\delta(1-\frac{dR}{\mathscr{C}})} & + P^{n-dR-\delta(1-\frac{R}{\mathscr{C}})} & \text{if } \mathscr{C} > dR, \end{cases} \tag{3.11}$$

with an implicit constant depending only on $\mathscr{C}, d$ *and* R*.*

In §3.1.3 we will take $T(\boldsymbol{\alpha}) = C^{-1}P^{-\epsilon}S(\boldsymbol{\alpha}; P)$ where C is as in (3.1). We will take E to be the minor arcs $\mathfrak{m}_{P,d,\Delta}$ from (1.20). We will need to prove that $\int_{\mathfrak{m}_{P,d,\Delta}} S(\boldsymbol{\alpha}; P)\, d\boldsymbol{\alpha} \ll P^{n-dR-\delta}$, for which only the case $\mathscr{C} > dR$ of the bound (3.11) will be satisfactory.

Proof. We apply Lemma 3.4 with

$$\varphi(\boldsymbol{\alpha}) = \frac{|T(\boldsymbol{\alpha})|}{P^n}, \quad r_1(t) = P^{-d}t^{-1/\mathscr{C}}, \quad r_2(t) = t^{(d-1)/\mathscr{C}}, \tag{3.12}$$

noting that the bound (3.5) then follows from (3.9).

It remains to choose the parameters k and ℓ from (3.6). We will choose these so that the right-hand side of (3.6) is dominated by the sum $\sum_{i=k}^{\ell-1}$, rather than either of the other two terms. More precisely, take

$$k = \lfloor \log_2 P^{-\mathscr{C}} \rfloor, \qquad \ell = \lceil \log_2 P^{-\delta} \rceil, \tag{3.13}$$

observing that

$$\tfrac{1}{2}P^{-\mathscr{C}} < 2^k \leq P^{-\mathscr{C}}, \quad P^{-\delta} \leq 2^\ell < 2P^{-\delta}. \tag{3.14}$$

We may assume that $\mathscr{C} > \delta$, for otherwise the bound $\int_E T(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \leq \nu^R P^{n-\delta}$, which follows from (3.10), is stronger than any of the bounds listed in (3.11). We then have $k < \ell$ and so this choice of k, ℓ is admissible in Lemma 3.4. Hence (3.6) holds, and substituting in our choices (3.12) for the parameters yields

$$\int_E \frac{|T(\boldsymbol{\alpha})|}{P^n}\, d\boldsymbol{\alpha} \ll_R \nu^R 2^k + \sum_{i=k}^{\ell-1} 2^i \left(\frac{\nu P^{-d} 2^{-i/\mathscr{C}}}{\min\{2^{(d-1)i/\mathscr{C}}, \nu\}} \right)^R + \left(\frac{\nu P^{-d} 2^{-\ell/\mathscr{C}}}{\min\{2^{(d-1)\ell/\mathscr{C}}, \nu\}} \right)^R \sup_{\boldsymbol{\alpha} \in E} \frac{|T(\boldsymbol{\alpha})|}{P^n}. \tag{3.15}$$

By (3.10) and (3.14) we have $\sup_{\boldsymbol{\alpha}\in E}\frac{|T(\boldsymbol{\alpha})|}{P^n} \leq 2^\ell$, and so we may extend the sum in (3.15) from $\sum_{i=k}^{\ell-1}$ to $\sum_{i=k}^{\ell}$ to obtain

$$\int_E \frac{|T(\boldsymbol{\alpha})|}{P^n}\, d\boldsymbol{\alpha} \ll_R \nu^R 2^k + \sum_{i=k}^{\ell} 2^i \left(\frac{\nu P^{-d} 2^{-i/\mathscr{C}}}{\min\{2^{(d-1)i/\mathscr{C}}, \nu\}} \right)^R.$$

Since

$$\frac{P^{-d} 2^{-i/\mathscr{C}}}{\min\{2^{(d-1)i/\mathscr{C}}, \nu\}} \leq P^{-d} 2^{-di/\mathscr{C}} + \nu^{-1} P^{-d} 2^{-i/\mathscr{C}},$$

we deduce that

$$\int_E \frac{|T(\boldsymbol{\alpha})|}{P^n}\, d\boldsymbol{\alpha} \ll_R \nu^R 2^k + \sum_{i=k}^{\ell} \nu^R P^{-dR} 2^{i(1-dR/\mathscr{C})} + \sum_{i=k}^{\ell} P^{-dR} 2^{i(1-R/\mathscr{C})}. \tag{3.16}$$

Note that

$$\sum_{i=k}^{\ell} 2^{i(1-dR/\mathscr{C})} \ll_{\mathscr{C},d,R} \begin{cases} 2^{k(1-dR/\mathscr{C})} & \text{if } \mathscr{C} < dR \\ \ell - k & \text{if } \mathscr{C} = dR \\ 2^{\ell(1-dR/\mathscr{C})} & \text{if } \mathscr{C} > dR. \end{cases}$$

Recall from (3.14) that we have $2^k \geq \frac{1}{2} P^{-\mathscr{C}}$ and $2^\ell \leq 2P^{-\delta}$, and observe that by (3.13) the bound $\ell - k \leq 2 + \mathscr{C} \log_2 P$ holds. It follows that

$$\sum_{i=k}^{\ell} 2^{i(1-dR/\mathscr{C})} \ll_{\mathscr{C},d,R} \begin{cases} P^{\mathscr{C}-dR} & \text{if } \mathscr{C} < dR \\ \log P & \text{if } \mathscr{C} = dR \\ P^{-\delta(1-dR/\mathscr{C})} & \text{if } \mathscr{C} > dR, \end{cases}$$

and reasoning similarly for $\sum_{i=k}^{\ell} 2^{i(1-R/\mathscr{C})}$, we deduce from (3.16) that

$$\int_E \frac{|T(\boldsymbol{\alpha})|}{P^n}\, d\boldsymbol{\alpha}$$
$$\ll \begin{cases} \nu^R 2^k + \nu^R P^{-\mathscr{C}} & + P^{n-\mathscr{C}-(d-1)R} & \text{if } \mathscr{C} < R \\ \nu^R 2^k + \nu^R P^{-\mathscr{C}} & + P^{-dR} \log P & \text{if } \mathscr{C} = R \\ \nu^R 2^k + \nu^R P^{-\mathscr{C}} & + P^{-dR-\delta(1-R/\mathscr{C})} & \text{if } R < \mathscr{C} < dR \\ \nu^R 2^k + \nu^R P^{-\mathscr{C}} \log P & + P^{-dR-\delta(1-R/\mathscr{C})} & \text{if } \mathscr{C} = dR \\ \nu^R 2^k + \nu^R P^{-dR-\delta(1-dR/\mathscr{C})} & + P^{-dR-\delta(1-R/\mathscr{C})} & \text{if } \mathscr{C} > dR \end{cases}$$

with an implicit constant depending only on $\mathscr{C}, d$, and R. One final application of the bound $2^k \leq P^{-\mathscr{C}}$ from (3.14) completes the proof of (3.11). □

Recall from (1.11) the quantity $\sigma_{\mathbb{Z}}(\boldsymbol{F}) \in \{0, \ldots, n\}$. We have

$$\sigma_{\mathbb{Z}}(\boldsymbol{f}^{[d]}) = 1 + \max_{\boldsymbol{a}\in\mathbb{Z}^R\setminus\{\boldsymbol{0}\}} \dim \operatorname{Sing}(\boldsymbol{a}\cdot\boldsymbol{f}^{[d]}). \tag{3.17}$$

In terms of this quantity, we have the following upper bound for $S(\boldsymbol{\alpha}; P)$ on the minor arcs $\mathfrak{m}_{P,d,\Delta}$ from (1.20).

Lemma 3.6 (Dietmann [Die15], Schindler [Sch15])**.** *Suppose that the polynomials* f_i *have integer coefficients. Let* $S(\boldsymbol{\alpha}; P)$ *be as in §1.1, let* $\Delta \in (0,1)$*, let* $\mathfrak{m}_{P,d,\Delta}$ *be as in* (1.20)*, let* $\sigma_{\mathbb{Z}}(\boldsymbol{f}^{[d]})$ *be as in* (3.17) *and let* $\epsilon > 0$*. Then we have*

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{m}_{P,d,\Delta}} |S(\boldsymbol{\alpha}; P)| \ll_{d,n,R,\epsilon} P^{n-\Delta\delta_0+\epsilon}$$

where the implicit constant depends only on d, n, R *and* ϵ*, and we write*

$$\delta_0 = \frac{n - \sigma_{\mathbb{Z}}(\boldsymbol{f}^{[d]})}{(d-1)2^{d-1}R}. \tag{3.18}$$

Furthermore we have $\delta_0 \geq \frac{1}{(d-1)2^{d-1}R}$ *whenever the forms* $f_i^{[d]}$ *are linearly independent.*

Proof. This follows from Lemma 4 in Dietmann [Die15], or from Lemma 2.2 in Schindler [Sch15]. Indeed the comments at the start of the proof of Schindler's Theorem 1.1 show that those two results are equivalent. In either author's work, by choosing the parameter θ to be

$$\theta = \frac{\Delta - \epsilon}{(d-1)R}$$

we find that for each $P \geq 1$, either

$$S(\boldsymbol{\alpha}; P) \ll_{n,R,\epsilon} P^{n-\Delta\delta_0+O_{d,R}(\epsilon)}, \tag{3.19}$$

or else there exist $q \ll_{d,n,R} P^{\Delta-\epsilon}$ and $(a_1, \ldots, a_R, q) = 1$ such that

$$\|\boldsymbol{\alpha} - \tfrac{\boldsymbol{a}}{q}\|_\infty \ll_{d,n,R} P^{\Delta-d-\epsilon}.$$

Taking the parameter P to be sufficiently large in terms of d, n, R and ϵ, this shows that the inequality $\|\boldsymbol{\alpha} - \frac{\boldsymbol{a}}{q}\|_\infty < P^{\Delta-d}$ holds; since this contradicts the assumption $\boldsymbol{\alpha} \in \mathfrak{m}_{P,d,\Delta}$ we conclude that (3.19) holds.

Provided the forms $f_i^{[d]}$ are linearly independent, the variety $V(\boldsymbol{a} \cdot \boldsymbol{f}^{[d]})$ is a proper subvariety of $\mathbb{P}^{n-1}_{\bar{\mathbb{Q}}}$ for each $\boldsymbol{a} \in \mathbb{Z}^R \setminus \{\boldsymbol{0}\}$, and so $\sigma_{\mathbb{Z}}(\boldsymbol{f}^{[d]}) \leq n-1$ holds, by (3.17). This implies that $\delta_0 \geq \frac{1}{(d-1)2^{d-1}R}$, as claimed. □

3.1.2 The major arcs

Let $\Delta \in (0,1)$, and recall the major arcs $\mathfrak{M}_{P,d,\Delta}$ from (1.19). In this section we estimate $\int_{\mathfrak{M}_{P,d,\Delta}} S(\boldsymbol{\alpha}; P)\, d\boldsymbol{\alpha}$, with the goal of proving the asymptotic (1.21). We define some additional notation. For each $q \in \mathbb{N}$ and $\boldsymbol{a} \in \mathbb{Z}^R$, set

$$S_q(\boldsymbol{a}) = q^{-n} \sum_{\boldsymbol{y} \in \{1,\ldots,q\}^n} e\big(\tfrac{\boldsymbol{a}}{q} \cdot \boldsymbol{f}(\boldsymbol{y})\big). \tag{3.20}$$

This could be thought of as a "local analogue" of the sum $S(\boldsymbol{\alpha};P)$ over the ring $\mathbb{Z}/q\mathbb{Z}$, with a normalisation factor q^{-n}. Additionally, for each $\boldsymbol{\gamma}\in\mathbb{R}^R$, set

$$S_\infty(\boldsymbol{\gamma}) = \int_{\mathscr{B}} e\big(\boldsymbol{\gamma}\cdot \boldsymbol{f}^{[d]}(\boldsymbol{t})\big)\,d\boldsymbol{t}, \tag{3.21}$$

which is a local analogue of $S(\boldsymbol{\alpha};P)$ over $\mathbb{R}$. Further define local analogues of the integral $\int_{\mathfrak{M}_{P,d,\Delta}} S(\boldsymbol{\alpha};P)\,d\boldsymbol{\alpha}$ by writing

$$\mathfrak{S}(P) = \sum_{q\le P^\Delta} \sum_{\substack{\boldsymbol{a}\in\{1,\dots,q\}^R\\(a_1,\dots,a_R,q)=1}} S_q(\boldsymbol{a}) \tag{3.22}$$

and

$$\mathfrak{I}(P) = \int\limits_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^R\\ \|\boldsymbol{\alpha}\|_\infty\le P^{\Delta-d}}} P^n S_\infty(P^d\boldsymbol{\alpha})\,d\boldsymbol{\alpha}. \tag{3.23}$$

We start by approximating $\int_{\mathfrak{M}_{P,d,\Delta}} S(\boldsymbol{\alpha};P)\,d\boldsymbol{\alpha}$ in terms of these quantities.

Lemma 3.7. *Suppose that the polynomials f_i have integer coefficients. With notation as above, for all $\boldsymbol{\beta}\in\mathbb{R}^R$, all $\boldsymbol{a}\in\mathbb{Z}^R$ and all $q\in\mathbb{N}$ such that $q\le P$ we have*

$$S\big(\tfrac{\boldsymbol{a}}{q}+\boldsymbol{\beta};P\big) = P^n S_q(\boldsymbol{a}) S_\infty(P^d\boldsymbol{\beta}) + O_{\boldsymbol{f}}(qP^{n-1}\{1+P^d\|\boldsymbol{\beta}\|_\infty\}), \tag{3.24}$$

and it follows that

$$\int_{\mathfrak{M}_{P,d,\Delta}} S(\boldsymbol{\alpha};P)\,d\boldsymbol{\alpha} = \mathfrak{S}(P)\mathfrak{I}(P) + O_{\boldsymbol{f}}\big(P^{n-dR+(2R+3)\Delta-1}\big). \tag{3.25}$$

Proof. To show (3.24) we follow the proof of Lemma 5.1 in Birch [Bir62]. First observe that $\boldsymbol{\beta}\cdot\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{\beta}\cdot\boldsymbol{f}^{[d]}(\boldsymbol{x}) + O_{\boldsymbol{f}}(\|\boldsymbol{x}\|_\infty^{d-1}\|\boldsymbol{\beta}\|_\infty)$, and so

$$\begin{aligned} S(\tfrac{\boldsymbol{a}}{q}+\boldsymbol{\beta};P) &= \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ \boldsymbol{x}/P\in\mathscr{B}}} e(\boldsymbol{\beta}\cdot\boldsymbol{f}(\boldsymbol{x}))e\big(\tfrac{\boldsymbol{a}}{q}\cdot\boldsymbol{f}(\boldsymbol{x})\big)\\ &= \sum_{1\le y_1,\dots,y_n\le q} e\big(\tfrac{\boldsymbol{a}}{q}\cdot\boldsymbol{f}(\boldsymbol{y})\big) \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ \boldsymbol{x}/P\in\mathscr{B}\\ \boldsymbol{x}\equiv\boldsymbol{y}\bmod q}} e(\boldsymbol{\beta}\cdot\boldsymbol{f}^{[d]}(\boldsymbol{x}))\\ &\quad + O_{\boldsymbol{f}}(P^{n+d-1}\|\boldsymbol{\beta}\|_\infty). \end{aligned} \tag{3.26}$$

If ψ is any once continuously differentiable, complex-valued function on $\mathbb{R}^n$, then we have

$$\psi(\boldsymbol{x}) = q^{-n}\int_{\substack{\boldsymbol{u}\in\mathbb{R}^n\\ \|\boldsymbol{u}\|_\infty\le q/2}} \psi(\boldsymbol{x}+\boldsymbol{u})\,d\boldsymbol{u} + O_n\Big(q\max_{\substack{\boldsymbol{u}\in\mathbb{R}^n\\ \|\boldsymbol{u}\|_\infty\le q/2}} \|\boldsymbol{\nabla_u}\psi(\boldsymbol{x}+\boldsymbol{u})\|_\infty\Big). \tag{3.27}$$

Setting $\psi(\boldsymbol{x}) = e(\boldsymbol{\beta}\cdot \boldsymbol{f}^{[d]}(\boldsymbol{x}))$, we deduce that

$$\begin{aligned}\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ \boldsymbol{x}/P\in\mathscr{B}\\ \boldsymbol{x}\equiv\boldsymbol{y} \bmod q}} e(\boldsymbol{\beta}\cdot \boldsymbol{f}^{[d]}(\boldsymbol{x})) &= \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ \boldsymbol{x}/P\in\mathscr{B}\\ \boldsymbol{x}\equiv\boldsymbol{y} \bmod q}} q^{-n}\int_{\substack{\boldsymbol{u}\in\mathbb{R}^n\\ \|\boldsymbol{u}\|_\infty\le q/2}} e(\boldsymbol{\beta}\cdot \boldsymbol{f}^{[d]}(\boldsymbol{x}+\boldsymbol{u}))\,d\boldsymbol{u}\\ &\quad + O_{\boldsymbol{f}}(q^{1-n}P^{n+d-1}\|\boldsymbol{\beta}\|_\infty)\\ &= q^{-n}\int_{\substack{\boldsymbol{v}\in\mathbb{R}^n\\ \boldsymbol{v}/P\in\mathscr{B}}} e(\boldsymbol{\beta}\cdot \boldsymbol{f}^{[d]}(\boldsymbol{v}))\,d\boldsymbol{v}\\ &\quad + O_{\boldsymbol{f}}(q^{1-n}P^{n+d-1}\|\boldsymbol{\beta}\|_\infty + q^{1-n}P^{n-1}),\end{aligned}$$

where we have included a term $q^{1-n}P^{n-1}$ to allow for errors in approximating the boundary of the box $\mathscr{B}$. Substituting into (3.26) shows that

$$S(\tfrac{\boldsymbol{a}}{q}+\boldsymbol{\beta};P) = S_q(\boldsymbol{a})\int_{\substack{\boldsymbol{v}\in\mathbb{R}^n\\ \boldsymbol{v}/P\in\mathscr{B}}} e(\boldsymbol{\beta}\cdot \boldsymbol{f}^{[d]}(\boldsymbol{v}))\,d\boldsymbol{v} + O_{\boldsymbol{f}}(qP^{n-1}(1+P^d\|\boldsymbol{\beta}\|_\infty)).$$

To complete the proof of (3.24) it suffices to set $\boldsymbol{v} = P\boldsymbol{t}$ and use the definition of $S_\infty(\boldsymbol{\gamma})$ from the statement of the lemma.

Now (3.25) follows from (3.24) by the definition (1.19) of $\mathfrak{M}_{P,d,\Delta}$. □

Next we treat the quantity $\mathfrak{S}(P)$ from (3.22).

Lemma 3.8. *Let the polynomials f_i have integer coefficients and let $S_q(\boldsymbol{a})$ be as in (3.20). Suppose we are given $\epsilon \ge 0$ and $C\ge 1$, such that for all $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{R}^R$ and all $P\ge 1$ the bound (3.1) holds with $\mathscr{B}=[0,1]^n$. Then:*

(i) There is $\epsilon'>0$ such that $\epsilon' = O_{\mathscr{C}}(\epsilon)$ and

$$\min\{|S_q(\boldsymbol{a})|,\,|S_{q'}(\boldsymbol{a}')|\} \ll_{C,\boldsymbol{f}} (q'+q)^\epsilon \left\|\tfrac{\boldsymbol{a}}{q}-\tfrac{\boldsymbol{a}'}{q'}\right\|_\infty^{\frac{\mathscr{C}-\epsilon'}{d-1}} \tag{3.28}$$

for all $\boldsymbol{a}\in\{0,\dots,q-1\}^R$ and $\boldsymbol{a}'\in\{0,\dots,q-1\}^R$ such that $\frac{\boldsymbol{a}'}{q'}\ne\frac{\boldsymbol{a}}{q}$.

(ii) If $\mathscr{C}>\epsilon'$, then for all $t>0$ and $q_0\in\mathbb{N}$ we have

$$\#\{\tfrac{\boldsymbol{a}}{q}\in[0,1)^R : q\le q_0, |S_q(\boldsymbol{a})|\ge t\} \ll_{C,\boldsymbol{f}} (q_0^\epsilon t)^{-\frac{(d-1)R}{\mathscr{C}-\epsilon'}},$$

where it is understood that the fractions $\frac{\boldsymbol{a}}{q}$ are in lowest terms.

(iii) If the forms $f_i^{[d]}$ are linearly independent, then there is $\delta>0$ depending only on d and R, such that for all $q\in\mathbb{N}$ and $\boldsymbol{a}\in\mathbb{Z}^R$ satisfying $(a_1,\dots,a_R,q)=1$, we have

$$|S_q(\boldsymbol{a})| \ll_{\boldsymbol{f}} q^{-\delta}.$$

(iv) *Let $\Delta \in (0,1)$ and let $\mathfrak{S}(P)$ be as in* (3.22). *Suppose that ϵ is sufficiently small in terms of $\mathscr{C}$, d and R. Provided the inequality $\mathscr{C} > (d-1)R$ holds and the forms $f_i^{[d]}$ are linearly independent, we have*

$$\mathfrak{S}(P) - \mathfrak{S}_{\boldsymbol{f}} \ll_{C,\mathscr{C},\boldsymbol{f}} P^{-\Delta\delta_1} \tag{3.29}$$

for some $\mathfrak{S}_{\boldsymbol{f}} \in \mathbb{C}$ and some $\delta_1 > 0$ depending at most on $\mathscr{C}$, d and R. Moreover, we then have

$$\mathfrak{S}_{\boldsymbol{f}} = \prod_p \lim_{k\to\infty} \tfrac{1}{p^{k(n-R)}} \#\big\{\boldsymbol{b} \in \{1,2,\ldots,p^k\}^n : \boldsymbol{f}(\boldsymbol{b}) \equiv \boldsymbol{0} \mod p^k\big\}, \tag{3.30}$$

where the product is over primes p and converges absolutely.

Proof. Part (i). Provided P is sufficiently large, Lemma 3.7 will allow us to approximate the sum $S_q(\boldsymbol{a})$ by a multiple of $S\big(\boldsymbol{a}/q;P\big)$. This will enable us to transform the bound (3.1) into the bound (3.28).

Let $P \geq 1$ be a parameter, to be chosen later. Then (3.1) gives

$$\min\left\{\left|\frac{S\big(\frac{\boldsymbol{a}}{q};P\big)}{P^{n+\epsilon}}\right|, \left|\frac{S\big(\frac{\boldsymbol{a}'}{q'};P\big)}{P^{n+\epsilon}}\right|\right\} \leq C\max\{P^{-d}\|\tfrac{\boldsymbol{a}'}{q'}-\tfrac{\boldsymbol{a}}{q}\|_\infty^{-1}, \|\tfrac{\boldsymbol{a}'}{q'}-\tfrac{\boldsymbol{a}}{q}\|_\infty^{\frac{1}{d-1}}\}^{\mathscr{C}}. \tag{3.31}$$

Since $\mathscr{B} = [0,1]^n$ we have $S_\infty(\boldsymbol{0}) = 1$, and so (3.24) from Lemma 3.7 implies that

$$\frac{S\big(\frac{\boldsymbol{a}}{q};P\big)}{P^n} = S_q(\boldsymbol{a}) + O_{\boldsymbol{f}}(qP^{-1}), \quad \frac{S\big(\frac{\boldsymbol{a}'}{q'};P\big)}{P^n} = S_{q'}(\boldsymbol{a}') + O_{\boldsymbol{f}}(q'P^{-1}). \tag{3.32}$$

Together, (3.31) and (3.32) yield

$$\begin{aligned}&\min\big\{|S_q(\boldsymbol{a})|, |S_{q'}(\boldsymbol{a}')|\big\}\\ &\qquad\leq CP^{\epsilon-\mathscr{C}d}\|\tfrac{\boldsymbol{a}'}{q'}-\tfrac{\boldsymbol{a}}{q}\|_\infty^{-\mathscr{C}} + CP^{\epsilon}\|\tfrac{\boldsymbol{a}'}{q'}-\tfrac{\boldsymbol{a}}{q}\|_\infty^{\frac{\mathscr{C}}{d-1}} + O_{\boldsymbol{f}}((q'+q)P^{-1}).\end{aligned} \tag{3.33}$$

Observe that for P sufficiently large the term $CP^\epsilon\|\frac{\boldsymbol{a}'}{q'}-\frac{\boldsymbol{a}}{q}\|_\infty^{\mathscr{C}/(d-1)}$ dominates the right-hand side of (3.33). We claim this is the case for

$$P = (q'+q)\|\tfrac{\boldsymbol{a}'}{q'}-\tfrac{\boldsymbol{a}}{q}\|_\infty^{-\frac{1+\mathscr{C}}{d-1}}. \tag{3.34}$$

Indeed, since $\|\frac{\boldsymbol{a}'}{q'}-\frac{\boldsymbol{a}}{q}\|_\infty \leq 1$, it follows from (3.34) and (3.33) that

$$\begin{aligned}&\min\big\{|S_q(\boldsymbol{a})|, |S_{q'}(\boldsymbol{a}')|\big\}\\ &\leq CP^\epsilon(q'+q)^{-\mathscr{C}d}\|\tfrac{\boldsymbol{a}'}{q'}-\tfrac{\boldsymbol{a}}{q}\|_\infty^{\frac{\mathscr{C}+\mathscr{C}^2d}{d-1}} + CP^\epsilon\|\tfrac{\boldsymbol{a}'}{q'}-\tfrac{\boldsymbol{a}}{q}\|_\infty^{\frac{\mathscr{C}}{d-1}} + O_{\boldsymbol{f}}\Big(\|\tfrac{\boldsymbol{a}'}{q'}-\tfrac{\boldsymbol{a}}{q}\|_\infty^{\frac{1+\mathscr{C}}{d-1}}\Big)\\ &\ll_{C,\boldsymbol{f}} P^\epsilon\|\tfrac{\boldsymbol{a}'}{q'}-\tfrac{\boldsymbol{a}}{q}\|_\infty^{\frac{\mathscr{C}}{d-1}},\end{aligned}$$

which proves the result.

Part (ii). If $\epsilon' < \mathscr{C}$ is small, then by part (i), the points in the set

$$\{\tfrac{\boldsymbol{a}}{q} \in [0,1)^R : q \leq q_0, |S_q(\boldsymbol{a})| \geq t\}$$

are separated by gaps of size

$$\|\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}\|_\infty \gg_{C,\boldsymbol{f}} (q_0^{-\epsilon} t)^{\frac{d-1}{\mathscr{C}-\epsilon'}}.$$

At most $O_{C,\boldsymbol{f}}(q_0^{\epsilon} t)^{-\frac{(d-1)R}{\mathscr{C}-\epsilon'}}$ such points fit in the box $[0,1)^R$, proving the claim.

Part (iii). This is essentially identical to the proof of Lemma 5.4 in Birch [Bir62]. If $P = q^{n+1}$ and $\Delta = \frac{1}{n+1}$ then $\frac{\boldsymbol{a}}{q} \in \mathfrak{m}_{P,d,\Delta}$. Since we have assumed that the $f_i^{[d]}$ are linearly independent, Lemma 3.6 shows that

$$S_{\boldsymbol{f},\mathscr{B}}(\tfrac{\boldsymbol{a}}{q}) \ll_{d,n,R,\epsilon''} P^n q^{-\delta_0 + (n+1)\epsilon''}$$

for some $\delta_0 \geq \frac{1}{(d-1)2^{d-1}R}$ and all $\epsilon'' > 0$. Together with the case $P = q^{n+1}$ of (3.24) this yields

$$S_q(\boldsymbol{a}) \ll_{\boldsymbol{f},\epsilon''} q^{-\frac{1}{(d-1)2^{d-1}R} + (n+1)\epsilon''} + q^{-n}.$$

This proves the result.

Part (iv). In this part of the proof, whenever we write $\boldsymbol{a}/q$ it is understood that $\boldsymbol{a} \in \mathbb{Z}^R$ and $q \in \mathbb{N}$ with $(a_1, \ldots, a_R, q) = 1$. For each $Q \geq 1$, define

$$s(Q) = \sum_{\substack{\boldsymbol{a}/q \in [0,1)^R \\ Q < q \leq 2Q}} |S_q(\boldsymbol{a})|. \tag{3.35}$$

We will show below that

$$s(Q) \ll_{C,\mathscr{C},\boldsymbol{f}} Q^{-\delta_1} \tag{3.36}$$

for all $Q \geq 1$, and some $\delta_1 > 0$ depending only on $\mathscr{C}$, d and R. Since we have

$$\begin{aligned}\Bigg|\mathfrak{S}(P) - \sum_{\boldsymbol{a}/q \in [0,1)^R} S_q(\boldsymbol{a})\Bigg| &\leq \sum_{\substack{\boldsymbol{a}/q \in [0,1)^R \\ q > P^\Delta}} |S_q(\boldsymbol{a})| \\ &= \sum_{\substack{Q = 2^k P^\Delta \\ k = 0,1,\ldots}} s(Q),\end{aligned}$$

this proves (3.29) with

$$\mathfrak{S}_{\boldsymbol{f}} = \sum_{\boldsymbol{a}/q \in [0,1)^R} S_q(\boldsymbol{a}),$$

where this sum is absolutely convergent. To deduce (3.30), define

$$\mathfrak{S}_p = \sum_{k=0}^{\infty} \sum_{\substack{\boldsymbol{a} \in \{1,\ldots,p^k\}^R \\ (a_1,\ldots,a_R,p)=1}} S_{p^k}(\boldsymbol{a})$$

for each prime p. This is absolutely convergent as it is the sum of a subset of the terms of $\mathfrak{S}_{\boldsymbol{f}}$. By the Chinese Remainder Theorem, if p and q are distinct primes then we have

$$S_{p^k}(\boldsymbol{a}) S_{q^\ell}(\boldsymbol{b}) = S_{p^k q^\ell}(\boldsymbol{c}),$$

where $\boldsymbol{c}$ is the unique vector in $\{1, \ldots, p^k q^\ell\}^R$ such that $\boldsymbol{c} \equiv \boldsymbol{a}$ modulo p^k and $\boldsymbol{c} \equiv \boldsymbol{b}$ modulo p^ℓ. By absolute convergence we can then rearrange terms to conclude that $\mathfrak{S}_{\boldsymbol{f}} = \prod_p \mathfrak{S}_p$ where the product converges absolutely, and (3.30) follows since

$$\begin{aligned}
\sum_{k=0}^{K} \sum_{\substack{\boldsymbol{a} \in \{1,\ldots,p^k\}^R \\ (a_1,\ldots,a_R,p)=1}} S_{p^k}(\boldsymbol{a}) &= \sum_{\boldsymbol{a} \in \{1,\ldots,p^K\}^R} S_{p^k}(\boldsymbol{a}) \\
&= \tfrac{1}{p^{K(n-R)}} \# \big\{ \boldsymbol{b} \in \{1, 2, \ldots, p^K\}^n : \boldsymbol{f}(\boldsymbol{b}) \equiv \boldsymbol{0} \mod p^K \big\}.
\end{aligned}$$

It remains to prove (3.36). Let $\ell \in \mathbb{Z}$ and let $s(Q)$ be as in (3.35). We have

$$\begin{aligned}
s(Q) &= \sum_{\substack{\boldsymbol{a}/q \in [0,1)^R \\ |S_q(\boldsymbol{a})| \geq 2^{-\ell} \\ Q < q \leq 2Q}} |S_q(\boldsymbol{a})| + \sum_{i=\ell}^{\infty} \sum_{\substack{\boldsymbol{a}/q \in [0,1)^R \\ 2^{-i} > |S_q(\boldsymbol{a})| \geq 2^{-i-1} \\ Q < q \leq 2Q}} |S_q(\boldsymbol{a})| \\
&\leq \# \{ \tfrac{\boldsymbol{a}}{q} \in [0,1)^R : q \leq 2Q, |S_q(\boldsymbol{a})| \geq 2^{-\ell} \} \cdot \sup_{q > Q} |S_q(\boldsymbol{a})| \\
&\quad + \sum_{i=\ell}^{\infty} \# \{ \tfrac{\boldsymbol{a}}{q} \in [0,1)^R : q \leq 2Q, |S_q(\boldsymbol{a})| \geq 2^{-i-1} \} \cdot 2^{-i}. \qquad (3.37)
\end{aligned}$$

Now parts (ii) and (iii) show that

$$\# \{ \tfrac{\boldsymbol{a}}{q} \in [0,1)^R : q \leq 2Q, |S_q(\boldsymbol{a})| \geq t \} \ll_{C,\boldsymbol{f}} (Q^\epsilon t)^{-\frac{(d-1)R}{\mathscr{C} - \epsilon'}}$$

and that

$$\sup_{q > Q} |S_q(\boldsymbol{a})| \ll_{\boldsymbol{f}} Q^{-\delta}.$$

Substituting these bounds into (3.37) gives

$$s(Q) \ll_{C,\boldsymbol{f}} Q^{O_{\mathscr{C},d,R}(\epsilon) - \delta} 2^{\ell \frac{(d-1)R}{\mathscr{C} - \epsilon'}} + Q^{O_{\mathscr{C},d,R}(\epsilon)} \sum_{i=\ell}^{\infty} 2^{(i+1)\left(\frac{(d-1)R}{\mathscr{C} - \epsilon'}\right) - i}.$$

We have $\mathscr{C} > (d-1)R$ and we have assumed that ϵ' is small in terms of $\mathscr{C}$, d and R. So we may assume that $\frac{(d-1)R}{\mathscr{C}-\epsilon'} \gg_{\mathscr{C},d,R} 1$. It follows that

$$s(Q) \ll_{C,\mathscr{C},\boldsymbol{f}} Q^{O_{\mathscr{C},d,R}(\epsilon)} 2^{\ell \frac{(d-1)R}{\mathscr{C}-\epsilon'}} \big(Q^{-\delta} + 2^{-\ell}\big).$$

Picking $\ell = \lfloor \log_2 Q^{\delta} \rfloor$, shows that

$$s(Q) \ll_{C,\mathscr{C},d,R} Q^{-\delta \frac{(d-1)R-\mathscr{C}}{2\mathscr{C}} + O_{\mathscr{C},d,R}(\epsilon)}.$$

As ϵ is small in terms of $\mathscr{C}$, d and R it follows that $s(Q) \ll_{C,\mathscr{C},d,R} Q^{-\delta_1}$ for some $\delta_1 > 0$ depending only on $\mathscr{C}$, d and R. This proves (3.36). □

Next we estimate the integral $\mathfrak{I}(P)$. Together with (3.25) and (3.29) this will allow us to give an asymptotic expression for $\int_{\mathfrak{M}_{P,d,\Delta}} S(\boldsymbol{\alpha}; P)\, d\boldsymbol{\alpha}$.

Lemma 3.9. *Let $\Delta \in (0,1)$, let $S_\infty(\boldsymbol{\gamma})$ be as in (3.21) and let $\mathfrak{I}(P)$ be as in (3.23).*

(i) Suppose we are given $C \geq 1$, $\mathscr{C} > 0$ and $\epsilon \geq 0$ such that (3.1) holds for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$ and $P \geq 1$. Then for all $\boldsymbol{\gamma} \in \mathbb{R}^R$ we have

$$S_\infty(\boldsymbol{\gamma}) \ll_{C,\boldsymbol{f}} \|\boldsymbol{\gamma}\|_\infty^{-\mathscr{C}+\epsilon'}, \tag{3.38}$$

where $\epsilon' > 0$ and $\epsilon' = O_{\mathscr{C}}(\epsilon)$.

(ii) If the conclusion of part (i) holds and $\mathscr{C} - \epsilon' > R$, then there exists $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} \in \mathbb{C}$ such that for all $P \geq 1$ we have

$$\tfrac{1}{P^{n-dR}} \mathfrak{I}(P) - \mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} \ll_{\mathscr{C},C,\boldsymbol{f},\epsilon'} P^{-\Delta(\mathscr{C}-\epsilon'-R)}. \tag{3.39}$$

Furthermore we have

$$\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} = \lim_{P\to\infty} \tfrac{1}{P^{n-dR}} \lambda\big\{\boldsymbol{t} \in \mathbb{R}^n : \tfrac{1}{P}\boldsymbol{t} \in \mathscr{B}, \|\boldsymbol{f}^{[d]}(\boldsymbol{t})\|_\infty \leq \tfrac{1}{2}\big\}, \tag{3.40}$$

where λ is the Lebesgue measure.

The proof largely follows Birch [Bir62, §5-6] except when we prove (3.40), where we give an alternative treatment using a smooth weight function.

Proof. Part (i). First, for all $\boldsymbol{\beta} \in \mathbb{R}^R$ we have $|S(\boldsymbol{\beta}; P)| \leq S(\mathbf{0}; P)$, from the definition (1.16). Consequently, taking $\boldsymbol{\alpha} = \mathbf{0}$ and $\boldsymbol{\beta} = P^{-d}\boldsymbol{\gamma}$ in our hypothesis (3.1) shows that

$$|S(P^{-d}\boldsymbol{\gamma}; P)| \leq C P^{n+\epsilon} \max\{\|\boldsymbol{\gamma}\|_\infty^{-1}, P^{-\frac{d}{d-1}} \|\boldsymbol{\gamma}\|_\infty^{\frac{1}{d-1}}\}^{\mathscr{C}}.$$

Together with the case $\boldsymbol{\alpha} = P^{-d}\boldsymbol{\gamma}$ of the bound (3.1), this yields

$$S_\infty(\boldsymbol{\gamma}) \ll_{C,\boldsymbol{f}} P^\epsilon \max\{\|\boldsymbol{\gamma}\|_\infty^{-1}, P^{-\frac{d}{d-1}}\|\boldsymbol{\gamma}\|_\infty^{\frac{1}{d-1}}\}^{\mathscr{C}} + P^{-1} + P^{-1}\|\boldsymbol{\gamma}\|_\infty. \tag{3.41}$$

If we have $\|\boldsymbol{\gamma}\|_\infty \leq 1$, then we set $P = 1$ and (i) follows. Otherwise we put $P = \max\{1, \|\boldsymbol{\gamma}\|_\infty^{1+\mathscr{C}}\}$, and the result follows since (3.41) then implies

$$\begin{aligned} S_\infty(\boldsymbol{\gamma}) &\ll_{C,\boldsymbol{f}} P^\epsilon \max\{\|\boldsymbol{\gamma}\|_\infty^{-1}, \|\boldsymbol{\gamma}\|_\infty^{-1-\frac{\mathscr{C}d}{d-1}}\}^{\mathscr{C}} + \|\boldsymbol{\gamma}\|_\infty^{-1-\mathscr{C}} + \|\boldsymbol{\gamma}\|_\infty^{-\mathscr{C}} \\ &\leq 3\|\boldsymbol{\gamma}\|_\infty^{-\mathscr{C}+(1+\mathscr{C})\epsilon}. \end{aligned}$$

Part (ii). If the inequality $\mathscr{C} - \epsilon' > R$ holds, then by (3.38) we have for all $Q \geq 1$ that

$$\begin{aligned} \int\limits_{\substack{\boldsymbol{\gamma}\in\mathbb{R}^R \\ \|\boldsymbol{\gamma}\|_\infty \geq Q}} |S_\infty(\boldsymbol{\gamma})| \, d\boldsymbol{\gamma} &\ll_{C,\boldsymbol{f}} \int\limits_{\substack{\boldsymbol{\gamma}\in\mathbb{R}^R \\ \|\boldsymbol{\gamma}\|_\infty \geq Q}} \|\boldsymbol{\gamma}\|_\infty^{-\mathscr{C}+\epsilon'} \, d\boldsymbol{\gamma} \\ &\ll_{\mathscr{C},R,\epsilon} Q^{R-\mathscr{C}+\epsilon'}. \end{aligned}$$

Hence, if $\mathscr{C} - \epsilon' > R$, then

$$\begin{aligned} \Bigg(\int\limits_{\boldsymbol{\gamma}\in\mathbb{R}^R} P^{n-dR} S_\infty(\boldsymbol{\gamma}) \, d\boldsymbol{\gamma} \Bigg) - \mathfrak{I}(P) &= \int\limits_{\substack{\boldsymbol{\gamma}\in\mathbb{R}^R \\ \|\boldsymbol{\gamma}\|_\infty > P^\Delta}} P^{n-dR} S_\infty(\boldsymbol{\gamma}) \, d\boldsymbol{\gamma} \\ &\ll_{\mathscr{C},C,\boldsymbol{f},\epsilon'} P^{n-dR-\Delta(\mathscr{C}-\epsilon'-R)} \end{aligned}$$

where the integrals converge absolutely. This proves (3.39) with

$$\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} = \int\limits_{\boldsymbol{\gamma}\in\mathbb{R}^R} S_\infty(\boldsymbol{\gamma}) \, d\boldsymbol{\gamma}. \tag{3.42}$$

It remains to prove (3.40). Let $\chi : \mathbb{R}^R \to [0,1]$ be the indicator function of the box $[-\frac{1}{2}, \frac{1}{2}]^R$. We must evaluate the limit

$$\begin{aligned} &\lim_{P\to\infty} \tfrac{1}{P^{n-dR}} \lambda\big\{\boldsymbol{t} \in \mathbb{R}^n : \tfrac{1}{P}\boldsymbol{t} \in \mathscr{B}, \|\boldsymbol{f}^{[d]}(\boldsymbol{t})\|_\infty \leq \tfrac{1}{2}\big\} \\ &\qquad\qquad = \lim_{P\to\infty} \tfrac{1}{P^{n-dR}} \int_{\substack{\boldsymbol{t}\in\mathbb{R}^n \\ \boldsymbol{t}/P\in\mathscr{B}}} \chi\big(\boldsymbol{f}^{[d]}\boldsymbol{t})\big) \, d\boldsymbol{t}. \end{aligned} \tag{3.43}$$

Let φ be any infinitely differentiable, compactly supported function on $\mathbb{R}^R$, taking values in $[0,1]$. We evaluate $\frac{1}{P^{n-dR}} \int_{\boldsymbol{t}/P\in\mathscr{B}} \varphi(\boldsymbol{f}^{[d]}(\boldsymbol{t})) \, d\boldsymbol{t}$, which we think of as a smoothed version of (3.43). Fourier inversion gives

$$\begin{aligned}
\int_{\substack{\boldsymbol{t}\in\mathbb{R}^n\\ \boldsymbol{t}/P\in\mathscr{B}}} \varphi\big(\boldsymbol{f}^{[d]}(\boldsymbol{t})\big)\,d\boldsymbol{t} &= \int_{\substack{\boldsymbol{t}\in\mathbb{R}^n\\ \boldsymbol{t}/P\in\mathscr{B}}} \int_{\mathbb{R}^R} \hat{\varphi}(\boldsymbol{\alpha})e(\boldsymbol{\alpha}\cdot\boldsymbol{f}^{[d]}(\boldsymbol{t}))\,d\boldsymbol{\alpha}d\boldsymbol{t}\\
&= \int_{\mathbb{R}^R} \hat{\varphi}(\boldsymbol{\alpha}) \int_{\substack{\boldsymbol{t}\in\mathbb{R}^n\\ \boldsymbol{t}/P\in\mathscr{B}}} e(\boldsymbol{\alpha}\cdot\boldsymbol{f}^{[d]}(\boldsymbol{t}))\,d\boldsymbol{t}d\boldsymbol{\alpha}\\
&= \int_{\mathbb{R}^R} \hat{\varphi}(\boldsymbol{\alpha}) \int_{\substack{\boldsymbol{t}'\in\mathbb{R}^n\\ \boldsymbol{t}'\in\mathscr{B}}} P^n e(P^d\boldsymbol{\alpha}\cdot\boldsymbol{f}^{[d]}(\boldsymbol{t}'))\,d\boldsymbol{t}'d\boldsymbol{\alpha}\\
&= \int_{\mathbb{R}^R} \hat{\varphi}(\boldsymbol{\alpha})P^n S_\infty(P^d\boldsymbol{\alpha})d\boldsymbol{\alpha} \qquad (3.44)
\end{aligned}$$

where $\hat{\varphi}(\boldsymbol{\alpha})$ is the Fourier transform $\int_{\mathbb{R}^R}\varphi(\boldsymbol{\gamma})e(-\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})\,d\boldsymbol{\gamma}$.

Since $\mathscr{C}-\epsilon' > R$ holds by assumption, it follows from (3.38) that the function S_∞ is Lebesgue integrable. Hence (3.42) implies

$$\begin{aligned}
\hat{\varphi}(\mathbf{0})\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} &= \int_{\mathbb{R}^R} \hat{\varphi}(\mathbf{0})S_\infty(\boldsymbol{\gamma})\,d\boldsymbol{\gamma}\\
&= \lim_{P\to\infty}\int_{\mathbb{R}^R} \hat{\varphi}(P^{-d}\boldsymbol{\gamma})S_\infty(\boldsymbol{\gamma})\,d\boldsymbol{\gamma}\\
&= \lim_{P\to\infty} P^{dR}\int_{\mathbb{R}^R} \hat{\varphi}(\boldsymbol{\alpha})S_\infty(P^d\boldsymbol{\alpha})\,d\boldsymbol{\alpha}. \qquad (3.45)
\end{aligned}$$

Together, (3.44) and (3.45) show that for any infinitely differentiable, compactly supported φ taking values in $[0,1]$, we have

$$\lim_{P\to\infty}\tfrac{1}{P^{n-dR}}\int_{\substack{\boldsymbol{t}\in\mathbb{R}^n\\ \boldsymbol{t}/P\in\mathscr{B}}} \varphi(\boldsymbol{f}^{[d]}(\boldsymbol{t}))\,d\boldsymbol{t} = \hat{\varphi}(\mathbf{0})\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}. \qquad (3.46)$$

With χ as in (3.43), choose φ such that $\varphi(\boldsymbol{\gamma})\leq\chi(\boldsymbol{\gamma})$ for all $\boldsymbol{\gamma}\in\mathbb{R}^R$. Then by (3.43) and (3.46) we have

$$\liminf_{P\to\infty}\tfrac{1}{P^{n-dR}}\lambda\big\{\boldsymbol{t}\in\mathbb{R}^n : \tfrac{1}{P}\boldsymbol{t}\in\mathscr{B}, \|\boldsymbol{f}^{[d]}(\boldsymbol{t})\|_\infty\leq\tfrac{1}{2}\big\}\geq\hat{\varphi}(\mathbf{0})\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}.$$

Letting $\varphi\to\chi$ almost everywhere gives $\hat{\varphi}(\mathbf{0})\to 1$, so $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ is a lower bound for the limit inferior in (3.43). Repeating the argument with $\varphi(\boldsymbol{\gamma})\geq\chi(\boldsymbol{\gamma})$ instead of $\varphi(\boldsymbol{\gamma})\leq\chi(\boldsymbol{\gamma})$ shows that $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ is also an upper bound for the corresponding limit superior, so the limit exists and is equal to $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$. □

3.1.3 Proof of Proposition 3.1

The work of §3.1.1-§3.1.2 allows us to quickly dispose of the first proposition.

Proof of Proposition 3.1. Let $P \geq 1$ and $\Delta = \frac{1}{4R+6}$. By (1.18) we have

$$N_{\boldsymbol{f},\mathscr{B}}(P) = \int_{\mathfrak{M}_{P,d,\Delta}} S(\boldsymbol{\alpha}; P)\, d\boldsymbol{\alpha} + \int_{\mathfrak{m}_{P,d,\Delta}} S(\boldsymbol{\alpha}; P)\, d\boldsymbol{\alpha},$$

where $\mathfrak{M}_{P,d,\Delta}$ and $\mathfrak{m}_{P,d,\Delta}$ are as in (1.19) and (1.20) respectively. We apply Lemma 3.5 with

$$T(\boldsymbol{\alpha}) = C^{-1} P^{-\epsilon} S(\boldsymbol{\alpha}; P), \quad E_0 = [0,1]^R, \quad E = \mathfrak{m}_{P,d,\Delta}, \quad \delta = \Delta\delta_0,$$

where the quantity δ_0 is as in (3.18). With these choices for T, E_0, E and δ, the hypothesis (3.9) of the lemma follows from (3.1). The remaining hypothesis (3.10) follows from Lemma 3.6, possibly after increasing C if necessary. This verifies the hypotheses of Lemma 3.5, and since we have $\mathscr{C} > dR$ by assumption, the conclusion (3.11) becomes

$$\int_{\mathfrak{m}_{P,d,\Delta}} S(\boldsymbol{\alpha}; P)\, d\boldsymbol{\alpha} \ll_{C,\mathscr{C},d,R} P^{n-dR-\Delta\delta_0(1-\frac{dR}{\mathscr{C}})+\epsilon}, \tag{3.47}$$

and since the $f_i^{[d]}$ are linearly independent, we have $\delta_0 \geq \frac{1}{(d-1)2^{d-1}R}$.

We turn to the major arcs. Since $\Delta = \frac{1}{4R+6}$, it follows from Lemma 3.7 that

$$\int_{\mathfrak{M}_{P,d,\Delta}} S(\boldsymbol{\alpha}; P)\, d\boldsymbol{\alpha} = \mathfrak{S}(P)\mathfrak{I}(P) + O_{\boldsymbol{f}}\big(P^{n-dR-\frac{1}{2}}\big), \tag{3.48}$$

where $\mathfrak{S}(P)$ and $\mathfrak{I}(P)$ are as in (3.22) and (3.23) respectively. The hypotheses of Lemma 3.8(iv) and Lemma 3.9(ii) are satisfied since $\mathscr{C} > dR$ holds, the $f_i^{[d]}(\boldsymbol{x})$ are linearly independent and ϵ is small in terms of $\mathscr{C}$, d and R. Therefore (3.29) and (3.39) hold, and so

$$\mathfrak{S}(P)\mathfrak{I}(P) = \mathfrak{S}_{\boldsymbol{f}}\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}} P^{n-dR} + O_{\mathscr{C},C,\boldsymbol{f}}\big(P^{n-dR-\Delta(\mathscr{C}-R)/2}\big) + O_{\mathscr{C},C,\boldsymbol{f}}\big(P^{n-dR-\Delta\delta_1}\big), \tag{3.49}$$

where $\delta_1 > 0$ depends at most on $\mathscr{C}$, d and R. By (3.47), (3.48) and (3.49), the result holds. □

3.1.4 Proof of Proposition 3.2

To prove this proposition we use the Implicit Function Theorem and a variant of Hensel's Lemma.

Proof of Proposition 3.2. Inspecting the proof of (3.49) above, we see that $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ and $\mathfrak{S}_{\boldsymbol{f}}$ are given by formulae (3.30) and (3.40) respectively. This proves (3.2) and (3.3). It remains to show that $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ and $\mathfrak{S}_{\boldsymbol{f}}$ are positive under the given conditions.

Let $\boldsymbol{y}$ lie in the interior of $\mathscr{B}$, and suppose that $\boldsymbol{x} = \boldsymbol{y}$ is a solution to the equation $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ for which the Jacobian matrix $(\partial f_i(\boldsymbol{x})/\partial x_j)_{ij}$ is nonsingular. Apply the Implicit Function Theorem to the equations $\boldsymbol{f}^{[d]}(\boldsymbol{x}) = \boldsymbol{0}$ at the point $\boldsymbol{x} = \boldsymbol{y}$. This gives us an open subset U of the box $\mathscr{B}$, such that the set of points $\boldsymbol{x}$ in U satisfying the equation $\boldsymbol{f}^{[d]}(\boldsymbol{x}) = \boldsymbol{0}$ forms an $(n-R)$-dimensional real manifold. Considering a small neighbourhood of this manifold shows that for all $\epsilon \in (0,1]$ we have

$$\lambda\big\{\boldsymbol{s} \in U : \|\boldsymbol{f}^{[d]}(\boldsymbol{s})\|_\infty \leq \epsilon\big\} \gg_{\boldsymbol{f}} \epsilon^R.$$

Letting $\boldsymbol{t} = P\boldsymbol{s}$ and $\epsilon = \frac{1}{2}P^{-d}$, we see that

$$\lambda\big\{\boldsymbol{t} \in \mathbb{R}^n : \boldsymbol{t}/P \in U, \|\boldsymbol{f}^{[d]}(\boldsymbol{t})\|_\infty \leq \tfrac{1}{2}\big\} \gg_{\boldsymbol{f}} P^{n-dR},$$

and (3.2) then shows that $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ is positive under the conditions given in the proposition.

Let p be a prime and let $\boldsymbol{a} \in \mathbb{Z}_p^n$. Suppose that $\boldsymbol{x} = \boldsymbol{a}$ is a solution to the equation $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ for which the Jacobian matrix $(\partial f_i(\boldsymbol{x})/\partial x_j)_{ij}$ is nonsingular. Possibly after permuting the variables x_i if necessary, we can assume that the submatrix $M(\boldsymbol{x})$ consisting of the last R columns of $(\partial f_i(\boldsymbol{x})/\partial x_j)_{ij}$ is nonsingular at $\boldsymbol{x} = \boldsymbol{a}$.

The so-called valuation theoretic Implicit Function Theorem then applies to the polynomials f_i with the common zero $\boldsymbol{a}$ over the valued field $\mathbb{Q}_p$. This is essentially a version of Hensel's Lemma; see Kuhlmann [Kuh11, Theorem 25]. If we write $|\det M(\boldsymbol{a})|_p = p^{-\alpha}$, the theorem states that for all p-adic numbers $a'_1, \dots, a'_{n-R} \in \mathbb{Q}_p$ with $|a'_i - a_i|_p < p^{-2\alpha}$, there are unique p-adic numbers $a'_{n-R+1}, \dots, a'_n \in \mathbb{Q}_p$ with $|a'_i - a_i|_p < p^{-\alpha}$ such that each $f_i(\boldsymbol{a}') = 0$.

Now let $a'_1, \dots, a'_{n-R}$ be p-adic integers satisfying $a'_i \equiv a_i$ modulo $p^{2\alpha+1}$. For each $k \in \mathbb{N}$ there are $p^{(k-2\alpha-1)(n-R)}$ choices for a'_i which are distinct modulo p^k, and by the Implicit Function Theorem above each one extends to a vector of p-adic integers $\boldsymbol{a}$ satisfying $\boldsymbol{f}(\boldsymbol{a}') = 0$.

If this holds for each prime p, then $\mathfrak{S}_{\boldsymbol{f}}$ is positive. For then reducing the vectors $\boldsymbol{a}'$ modulo p^k gives $\gg_{\boldsymbol{f},p} p^{k(n-R)}$ distinct vectors $\boldsymbol{b} \in \{1, \dots, p^k\}^n$ for which $\boldsymbol{f}(\boldsymbol{b}) \equiv \boldsymbol{0}$ modulo p^k. The equality (3.3) then shows that $\mathfrak{S}_{\boldsymbol{f}} > 0$. Thus the singular series is positive under the conditions given in the proposition.

□

3.2 Large values of the sum S repel one another

In this section we verify the hypothesis (3.1), assuming a bound on the number of solutions to the "auxiliary inequality" from Definition 1.6.

3.2.1 Weyl differencing

Following Birch [Bir62], we bound $S(\boldsymbol{\alpha}; P)$ in terms of the following counting function.

Definition 3.10. For each $B \geq 1$ and $\delta > 0$, we let $U_f(B, \delta)$ be the number of $(d-1)$-tuples of integer n-vectors $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}$ such that

$$\|\boldsymbol{x}^{(1)}\|_\infty, \ldots, \|\boldsymbol{x}^{(d-1)}\|_\infty \leq B, \qquad \min_{\boldsymbol{v} \in \mathbb{Z}^n} \left\| \boldsymbol{v} - \boldsymbol{m}^{(f)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \right\|_\infty < \delta.$$

Our estimate is as follows.

Lemma 3.11. *Let $U_f(B, \delta)$ be as in Definition 3.10. For all $\epsilon > 0$, $P \geq 1$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$ and $\theta \in (0, 1]$, we have*

$$\min\left\{ \left| \frac{S(\boldsymbol{\alpha}; P)}{P^{n+\epsilon}} \right|, \left| \frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta}; P)}{P^{n+\epsilon}} \right| \right\}^{2^d} \ll_{d,n,\epsilon} \frac{U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^\theta, P^{(d-1)\theta - d})}{P^{(d-1)\theta n}} \tag{3.50}$$

where the implicit constant depends only on d, n, ϵ.

Proof. Observe that (3.50) will follow if we can prove that

$$\left| \frac{S(\boldsymbol{\alpha}; P) S(\boldsymbol{\alpha}+\boldsymbol{\beta}; P)}{P^{2(n+\epsilon)}} \right|^{2^{d-1}} \ll_{d,n,\epsilon} \frac{U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^\theta, P^{(d-1)\theta - d})}{P^{(d-1)\theta n}}.$$

First we use an idea from the proof of Theorem 5.1 in Bentkus and Götze [BG97], also found in Lemma 2.2 of Müller [Mül05], to eliminate $\boldsymbol{\alpha}$. We have

$$\begin{aligned} &S(\boldsymbol{\alpha}+\boldsymbol{\beta}; P)\overline{S(\boldsymbol{\alpha}; P)} \\ &\quad = \sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^n \\ \boldsymbol{x}/P \in \mathscr{B}}} \sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^n \\ (\boldsymbol{x}+\boldsymbol{z})/P \in \mathscr{B}}} e\big((\boldsymbol{\alpha}+\boldsymbol{\beta}) \cdot \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{\alpha} \cdot \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{z}) \big) \\ &\quad \leq \sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^n \\ \|\boldsymbol{z}\|_\infty \leq P}} \left| \sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^n \\ \boldsymbol{x}/P \in \mathscr{B}_{\boldsymbol{z}}}} e\big(\{\boldsymbol{\alpha}+\boldsymbol{\beta}\} \cdot \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{\alpha} \cdot \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{z}) \big) \right| \\ &\quad = \sum_{\substack{\boldsymbol{z} \in \mathbb{Z}^n \\ \|\boldsymbol{z}\|_\infty \leq P}} \left| \sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^n \\ \boldsymbol{x}/P \in \mathscr{B}_{\boldsymbol{z}}}} e\big(\boldsymbol{\beta} \cdot \boldsymbol{f}^{[d]}(\boldsymbol{x}) + g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z}}(\boldsymbol{x}) \big) \right| \end{aligned}$$

for some real polynomials $g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z}}(\boldsymbol{x})$ of degree at most $d-1$ in $\boldsymbol{x}$ and some boxes $\mathscr{B}_{\boldsymbol{z}} \subset \mathscr{B}$. It is a special case of Cauchy's inequality that $|\sum_{i\in\mathcal{I}} \lambda_i|^2 \leq (\#\mathcal{I}) \cdot \sum_{i\in\mathcal{I}} |\lambda_i|^2$, and it follows that

$$\begin{aligned}
&|S(\boldsymbol{\alpha}+\boldsymbol{\beta};P)\overline{S(\boldsymbol{\alpha};P)}|^{2^{d-1}} \\
&\quad \leq \Bigg(\sum_{\substack{\boldsymbol{z}\in\mathbb{Z}^n \\ \|\boldsymbol{z}\|_\infty \leq P}} \Bigg| \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ \boldsymbol{x}/P\in\mathscr{B}_{\boldsymbol{z}}}} e\big(\boldsymbol{\beta}\cdot \boldsymbol{f}^{[d]}(\boldsymbol{x}) + g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z}}(\boldsymbol{x})\big) \Bigg| \Bigg)^{2^{d-1}} \\
&\quad \ll_{d,n} P^{(2^{d-1}-1)n} \sum_{\substack{\boldsymbol{z}\in\mathbb{Z}^n \\ \|\boldsymbol{z}\|_\infty \leq P}} \Bigg| \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ \boldsymbol{x}/P\in\mathscr{B}_{\boldsymbol{z}}}} e\big(\boldsymbol{\beta}\cdot \boldsymbol{f}^{[d]}(\boldsymbol{x}) + g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z}}(\boldsymbol{x})\big) \Bigg|^{2^{d-1}}. \qquad (3.51)
\end{aligned}$$

Lemma 2.4 of Birch [Bir62] states that for all $\theta \in (0,1]$ we have[1]

$$S(\boldsymbol{\alpha};P) \ll_{d,n,\epsilon} P^{2^{d-1}n-(d-1)n\theta+\epsilon} U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(P^\theta, P^{(d-1)\theta-d}).$$

The innermost sum in (3.51) has the same form as $S(\boldsymbol{\alpha};P)$, with $\mathscr{B}_{\boldsymbol{z}}$ in place of $\mathscr{B}$ and $\boldsymbol{\beta}\cdot\boldsymbol{f}^{[d]}(\boldsymbol{x}) + g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z}}(\boldsymbol{x})$ in place of $\boldsymbol{\alpha}\cdot\boldsymbol{f}$ as the underlying polynomial. The degree of $g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z}}$ is at most $d-1$, so $\boldsymbol{\beta}\cdot\boldsymbol{f}^{[d]}(\boldsymbol{x})$ is the leading part of this polynomial. So applying Birch's result to the innermost sum in (3.51) shows

$$\begin{aligned}
&\Bigg| \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ \boldsymbol{x}/P\in\mathscr{B}_{\boldsymbol{z}}}} e\big(\boldsymbol{\beta}\cdot \boldsymbol{f}^{[d]}(\boldsymbol{x}) + g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z}}(\boldsymbol{x})\big) \Bigg|^{2^{d-1}} \\
&\qquad\qquad \ll_{d,n,\epsilon} P^{2^{d-1}n-(d-1)\theta n+\epsilon} U_{\boldsymbol{\beta}\cdot\boldsymbol{f}^{[d]}(\boldsymbol{x})+g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z}}(\boldsymbol{x})}(P^\theta, P^{(d-1)\theta-d}). \\
&\qquad\qquad = P^{2^{d-1}n-(d-1)\theta n+\epsilon} U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^\theta, P^{(d-1)\theta-d}),
\end{aligned}$$

as U_f depends only on the degree d part of f. With (3.51) this proves the result. □

3.2.2 Proof of Proposition 3.3

We prove Proposition 3.3 by comparing the two counting functions from Definitions 1.6 and 3.10.

Proof. Lemma 3.11 shows that for $\theta \in (0,1]$ we have

$$U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^\theta, P^{(d-1)\theta-d}) \gg_{d,n,\epsilon} P^{(d-1)\theta n} \min\left\{ \left|\frac{S(\boldsymbol{\alpha};P)}{P^{n+\epsilon}}\right|, \left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P)}{P^{n+\epsilon}}\right| \right\}^{2^d}. \qquad (3.52)$$

[1] Birch writes $N(P^\theta; P^{(d-1)\theta-d}; \alpha)$ and $S(\alpha)$ where we write $U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(P^\theta, P^{(d-1)\theta-d})$ and $S(\boldsymbol{\alpha};P)$, respectively.

Define θ by

$$P^{\theta} = C_1^{1/2^d\mathscr{C}} \min\left\{ \left| \frac{S(\boldsymbol{\alpha}; P)}{P^{n+\epsilon}} \right|, \left| \frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta}; P)}{P^{n+\epsilon}} \right| \right\}^{-1/\mathscr{C}}, \tag{3.53}$$

for some sufficiently large positive constant C_1 depending only on C_0, d, n and ϵ. Then we have

$$(P^{\theta})^{(d-1)n-2^d\mathscr{C}} = C_1^{-1} P^{(d-1)\theta n} \min\left\{ \left| \frac{S(\boldsymbol{\alpha}; P)}{P^{n+\epsilon}} \right|, \left| \frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta}; P)}{P^{n+\epsilon}} \right| \right\}^{2^d},$$

and hence by (3.4) we have

$$N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^{\theta}) \leq \frac{C_0}{C_1} P^{(d-1)\theta n} \min\left\{ \left| \frac{S(\boldsymbol{\alpha}; P)}{P^{n+\epsilon}} \right|, \left| \frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta}; P)}{P^{n+\epsilon}} \right| \right\}^{2^d}. \tag{3.54}$$

We consider three cases.

First case ($0 < \theta \leq 1$). In this case (3.52) holds, and on comparing this to (3.54) and choosing C_1 to be sufficiently large we can ensure that

$$N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^{\theta}) < U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^{\theta}, P^{(d-1)\theta-d}).$$

Then there must be a $(d-1)$-tuple of vectors $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)} \in \mathbb{Z}^n$ which is included in the count $U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^{\theta}, P^{(d-1)\theta-d})$ but not in $N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^{\theta})$.

Since the tuple $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})$ is counted by $U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^{\theta}, P^{(d-1)\theta-d})$, the inequality $\|\boldsymbol{x}^{(i)}\|_{\infty} \leq P^{\theta}$ holds for each $i = 1, \ldots, d-1$, and we have the bound

$$\left\| \boldsymbol{v} - \boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \right\|_{\infty} < P^{(d-1)\theta-d}, \tag{3.55}$$

for some $\boldsymbol{v} \in \mathbb{Z}^n$. Since this $(d-1)$-tuple $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})$ is not counted by $N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^{\theta})$, we must also have

$$\|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})\|_{\infty} \geq \|\boldsymbol{\beta} \cdot \boldsymbol{f}^{[d]}\|_{\infty} P^{(d-2)\theta}. \tag{3.56}$$

We use (3.55) and (3.56) to relate P^{θ} and $\|\boldsymbol{\beta}\|_{\infty}$. It follows from (3.55) that either

$$\left\| \boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \right\|_{\infty} < P^{(d-1)\theta-d} \tag{3.57}$$

or

$$\left\| \boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \right\|_{\infty} \geq \frac{1}{2}. \tag{3.58}$$

When (3.57) holds, then (3.56) implies

$$\|\boldsymbol{\beta} \cdot \boldsymbol{f}^{[d]}\|_{\infty} < \frac{P^{(d-1)\theta-d}}{P^{(d-2)\theta}} = P^{\theta-d}. \tag{3.59}$$

When on the other hand (3.58) holds, then the bound $\|\boldsymbol{x}^{(i)}\|_\infty \leq P^\theta$ implies

$$\|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})\|_\infty \ll_{d,n} \|\boldsymbol{\beta}\cdot\boldsymbol{f}^{[d]}\|_\infty P^{(d-1)\theta},$$

and it follows by (3.58) that

$$\|\boldsymbol{\beta}\cdot\boldsymbol{f}^{[d]}\|_\infty \gg_{d,n} P^{-(d-1)\theta}. \tag{3.60}$$

Either (3.59) or (3.60) holds, and since the forms $f_i^{[d]}$ are linearly independent there exist $M > \mu > 0$, depending only on $\boldsymbol{f}^{[d]}$, such that for all $\boldsymbol{\beta} \in \mathbb{R}^R$ we have

$$\mu\|\boldsymbol{\beta}\|_\infty \leq \|\boldsymbol{\beta}\cdot\boldsymbol{f}^{[d]}\|_\infty \leq M\|\boldsymbol{\beta}\|_\infty.$$

It follows that

$$P^{-\theta} \ll_{d,n,\mu,M} \max\{P^{-d}\|\boldsymbol{\beta}\|_\infty^{-1}, \|\boldsymbol{\beta}\|_\infty^{\frac{1}{d-1}}\}.$$

This estimate implies the conclusion (3.1) of the lemma upon substituting in the value of θ from (3.53) and choosing C to satisfy the bound $C \gg_{d,n,\mu,M} C_1^{1/2^d}$.

Second case ($\theta \leq 0$). We can rule this out. For if $\theta \leq 0$ then (3.53) gives

$$\min\left\{\left|\frac{S(\boldsymbol{\alpha};P)}{P^{n+\epsilon}}\right|, \left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P)}{P^{n+\epsilon}}\right|\right\} \geq C_1^{-1/2^d}. \tag{3.61}$$

To prove (3.1), we can assume without loss of generality that $P \gg_{n,\epsilon} 1$ holds. But then (3.61) is false, since $|S(\boldsymbol{\alpha};P)| \leq (P+1)^n$ by the definition (1.16).

Third case ($\theta > 1$). If $\theta > 1$, we have by (3.53) that

$$\min\left\{\left|\frac{S(\boldsymbol{\alpha};P)}{P^{n+\epsilon}}\right|, \left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P)}{P^{n+\epsilon}}\right|\right\} < C_1^{1/2^d} P^{-\mathscr{C}}. \tag{3.62}$$

Now for any $t > 0$ we have $\max\{P^{-d}t^{-1}, t^{\frac{1}{d-1}}\} \geq P^{-1}$, and hence

$$\max\{P^{-d}\|\boldsymbol{\beta}\|_\infty^{-1}, \|\boldsymbol{\beta}\|_\infty^{\frac{1}{d-1}}\}^{\mathscr{C}} \geq P^{-\mathscr{C}}.$$

So (3.1) follows from (3.62) on choosing C such that $C \geq C_1^{1/2^d}$ holds. $\square$

Chapter 4

Cubic and quadratic systems

Recall the quantity $N_f^{\mathrm{aux}}(B)$ from Definition 1.6. In this chapter we estimate $N_f^{\mathrm{aux}}(B)$ in the cases when $d = 2$ and $d = 3$. Our main result, proved in §§4.2 and 4.4.4, is as follows.

Proposition 4.1. *We call a set $\mathcal{K}$ of real degree d forms in n variables a* closed cone *if (i) for all $F \in \mathcal{K}$ and $\lambda \geq 0$ we have $\lambda F \in K$, and (ii) $\mathcal{K}$ is closed in the real linear space of cubic forms in n variables.*

Suppose that $d = 2$ or 3, let $\mathcal{K}$ be a closed cone as above with $\mathcal{K} \neq \{0\}$, and let $N_f^{\mathrm{aux}}(B)$ be as in Definition 1.6. We set

$$\sigma_{\mathcal{K}} = 1 + \max_{F \in \mathcal{K} \setminus \{0\}} \mathrm{Sing}(F), \tag{4.1}$$

where Sing(F) *is as in §1.1. In particular we have $\sigma_{\mathcal{K}} \in \{0, \dots, n-1\}$. Then*

$$N_F^{\mathrm{aux}}(B) \ll_{\mathcal{K},\epsilon} B^{(d-2)n+\sigma_{\mathcal{K}}+\epsilon} \tag{4.2}$$

for all $\epsilon > 0$, $F \in \mathcal{K}$ and $B \geq 1$.

We have phrased the result in terms of closed cones to facilitate future applications.

4.1 Deduction of Theorem 1.5

Proof of Theorem 1.5. Let $\boldsymbol{F}$ be as in §1.1. Suppose that the condition (1.13) from Theorem 1.5 holds, which is to say that

$$n - \sigma_{\mathbb{R}}(\boldsymbol{F}) \geq d2^d R + 1 \tag{4.3}$$

where $\sigma_{\mathbb{R}}(\boldsymbol{F})$ is as in (1.14). In particular the F_i must be linearly independent, or else we would have $\sigma_{\mathbb{R}}(\boldsymbol{F}) = n$, by (1.14). If we set $\mathcal{K} = \{\boldsymbol{\alpha} \cdot \boldsymbol{F} : \boldsymbol{\alpha} \in \mathbb{R}^R\}$, then clearly we have $\sigma_{\mathbb{R}}(\boldsymbol{F}) \geq \sigma_{\mathcal{K}}$. So Proposition 4.1 shows that for all $\epsilon > 0$, $F \in \mathcal{K}$ and $B \geq 1$ we have

$$N_F^{\mathrm{aux}}(B) \ll_{\boldsymbol{F},\epsilon} B^{(d-2)n+\sigma_{\mathbb{R}}(\boldsymbol{F})+\epsilon}. \tag{4.4}$$

Taking $\epsilon = \frac{1}{2}$ in (4.4) and applying (4.3), we have

$$N_F^{\mathrm{aux}}(B) \leq C_0 B^{(d-1)n-2^d\mathscr{C}}, \tag{4.5}$$

where $\mathscr{C} = dR + 2^{-d-1}$ and $C_0 \geq 1$ depends at most on $\boldsymbol{F}$.

Let $q \in \mathbb{N}$ and $\boldsymbol{a} \in \mathbb{Z}$ be parameters to be chosen later and let $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{F}(q\boldsymbol{x} - \boldsymbol{a})$. Then $N_{\boldsymbol{\beta}\cdot\boldsymbol{f}}^{\mathrm{aux}}(B) = N_{\boldsymbol{\beta}\cdot\boldsymbol{F}}^{\mathrm{aux}}(B)$ and the bound (4.5) verifies the hypothesis (3.4) of Proposition 3.3. Since $f_i^{[d]} = qF_i$ the $f_i^{[d]}$ are linearly independent. So we can apply Proposition 3.3 to the system of polynomials $\boldsymbol{f}$. This shows that for each $\epsilon > 0$ there is a constant $C \geq 1$ depending at most on $\boldsymbol{F}$, q, $\boldsymbol{a}$ and ϵ, such that

$$\min\left\{\left|\frac{S(\boldsymbol{\alpha};P)}{P^{n+\epsilon}}\right|, \left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P)}{P^{n+\epsilon}}\right|\right\} \ll_{\boldsymbol{F},\epsilon} \max\left\{P^{-d}\|\boldsymbol{\beta}\|_\infty^{-1}, \|\boldsymbol{\beta}\|_\infty^{\frac{1}{d-1}}\right\}^{\mathscr{C}}$$

for all $P \geq 1$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$. Together with the facts that the F_i are linearly independent and $\mathscr{C} > dR$, this satisfies the hypotheses of Proposition 3.1.

Taking $q = 1$ and $\boldsymbol{a} = \boldsymbol{0}$, so that $\boldsymbol{f} = \boldsymbol{F}$, proves the asymptotic formula (1.7) holds, as claimed in the theorem. Proposition 3.2 shows that the singular series and integral are positive under the conditions given in the theorem. The latter proposition also verifies that the formulae (1.8) and (1.9) from the comments after Theorem 1.2 continue to hold in the setting of Theorem 1.5.

It remains to prove that $V(\boldsymbol{F}) \setminus \mathrm{Sing}(\boldsymbol{F})$ satisfies the Hasse principle and weak approximation. Suppose that there are nonsingular real and p-adic zeroes of $\boldsymbol{F}$.

We can choose a box $\mathscr{B}$ which contains a nonsingular real zero and does not contain a singular real zero. We obtain $\mathfrak{I}_{\boldsymbol{F},\mathscr{B}}\mathfrak{S}_{\boldsymbol{F}} > 0$, and the formula (1.7) then implies that $\boldsymbol{F}$ has an integral zero which lies in a dilate of $\mathscr{B}$ and which is therefore nonsingular. This proves the Hasse principle for $V(\boldsymbol{F}) \setminus \mathrm{Sing}(\boldsymbol{F})$.

To prove weak approximation, it suffices to show the following. Choose a real $\epsilon > 0$, a natural number k, a vector $\boldsymbol{x}^{(\infty)} \in \mathbb{R}^n$ which is a nonsingular zero of $\boldsymbol{F}$ with $\|\boldsymbol{x}^{(\infty)}\|_\infty = 1$. For each p in some finite set of primes S choose a vector $\boldsymbol{x}^{(p)} \in \mathbb{Z}_p^n$ which is a nonsingular zero of $\boldsymbol{F}$. We claim that there is a vector $\boldsymbol{w} \in \mathbb{Z}^n$ which is a nonsingular zero of $\boldsymbol{F}$ and satisfies $\|\boldsymbol{x}^{(\infty)} - \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|_\infty}\|_\infty < \epsilon$ and $\boldsymbol{w} \equiv \boldsymbol{x}^{(p)} \mod p^k$ for each $p \in S$.

Let $\boldsymbol{a} \in \mathbb{Z}^n$ with $\boldsymbol{a} \equiv \boldsymbol{x}^{(p)}$ modulo p^k for each $p \in S$, let $q = \prod_{p\in S} p^k$ and set $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{F}(q\boldsymbol{x} - \boldsymbol{a})$. Let $\mathscr{B}$ be a sufficiently small box centred on $\frac{1}{2}\boldsymbol{x}^{(\infty)}$. Then $\boldsymbol{f}$ has a nonsingular real zero lying in $\mathscr{B}$ and nonsingular zeroes over the completions $\mathbb{Q}_p$ for every prime p. So by Propositions 3.1 and 3.2, for any $P \geq 1$ we have $\boldsymbol{f}(\boldsymbol{b}) = \mathbf{0}$ for some integral vector $\boldsymbol{b}$ lying in $P\mathscr{B}$. Thus $\boldsymbol{w} = q\boldsymbol{b} + \boldsymbol{a}$ is a nonsingular zero of $\boldsymbol{F}$ satisfying the required congruences, and if P is sufficiently large we will have $\|\boldsymbol{x}^{(\infty)} - \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|_\infty}\|_\infty < \epsilon$. □

4.2 The quadratic case

We will now prove the quadratic case of Proposition 4.1. The proof of the cubic case will occupy the remainder of the chapter and will be completed in §4.4.4.

Proof of the case $d = 2$ of Proposition 4.1. Let $d = 2$ and let the matrix of each quadratic form F in the cone $\mathcal{K}$ be $M(F)$. That is, $M(F)$ is the unique real $n \times n$ symmetric matrix with

$$F(\boldsymbol{x}) = \boldsymbol{x}^T M(F)\boldsymbol{x}.$$

Then the system $\boldsymbol{m}^{(F)}$ from §1.3.3 is given by

$$\boldsymbol{m}^{(F)}(\boldsymbol{u}) = 2M(F)\boldsymbol{u},$$

so that the function $N_F^{\mathrm{aux}}(B)$ from Definition 1.6 counts vectors $\boldsymbol{u} \in \mathbb{Z}^n$ satisfying

$$\|\boldsymbol{u}\|_\infty \leq B, \qquad \|M(F)\boldsymbol{u}\|_\infty \leq \frac{1}{2}\|F\|_\infty,$$

where $\|F\|_\infty$ is as in §1.1. These integral vectors $\boldsymbol{u}$ are all contained in the box $\|\boldsymbol{u}\|_\infty \leq B$, and in the ellipsoid

$$E(F) = \{\boldsymbol{t} \in \mathbb{R}^n : \boldsymbol{t}^T M(F)^T M(F)\boldsymbol{t} < n \cdot \|F\|_\infty^2\}.$$

If $F = 0$ then $E(F) = \emptyset$ and we are done. Otherwise, this ellipsoid has principal radii $|\lambda|^{-1}\sqrt{n}\|F\|_\infty$ where λ runs over the eigenvalues of the real symmetric matrix $M(F)$, counted with multiplicity. Hence

$$N_F^{\mathrm{aux}}(B) \ll_n \prod_\lambda \min\{|\lambda|^{-1}\|F\|_\infty + 1,\, B + 1\}$$

where λ is as before. So to prove (4.2) it suffices that at least $(n - \sigma_{\mathcal{K}})$ of the λ satisfy $|\lambda| \gg_{\mathcal{K}} \|F\|_\infty$.

Suppose for a contradiction that this is false. Then there exists a sequence $F^{(i)} \in \mathcal{K}$ such that at least $\sigma_{\mathcal{K}} + 1$ of the eigenvalues of $M(F^{(i)})$ satisfy $\lambda = o(\|F^{(i)}\|_\infty)$. By passing to a subsequence, we can assume that $F^{(i)}/\|F^{(i)}\|_\infty \to F$ as $i \to \infty$, for some form F. At least $\sigma_{\mathcal{K}} + 1$ of the eigenvalues of $M(F)$ must then be zero. But then $\dim V(F) \geq \sigma_{\mathcal{K}}$ holds, which contradicts the definition (4.1). □

4.3 An argument of Davenport

We sketch the basis of the argument used for the case $d = 3$ of Proposition 4.1, and the notation used throughout the proof.

Definition 4.2. Let $c(\boldsymbol{x})$ be a cubic form in n variables with real coefficients which does not vanish uniformly, and define a symmetric matrix by

$$H_c(\boldsymbol{x}) = \frac{1}{\|c\|_\infty} \left(\frac{\partial^2 c(\boldsymbol{x})}{\partial x_i \partial x_j} \right)_{1 \le i,j \le n} \tag{4.6}$$

where $\|c\|_\infty = \frac{1}{6} \max_{i,j,k} \left| \frac{\partial^3 c(\boldsymbol{x})}{\partial x_i \partial x_j \partial x_k} \right|$, as in Definition 1.6. Thus $H_c(\boldsymbol{x})$ is the Hessian of the normalised cubic form $c(\boldsymbol{x})/\|c\|_\infty$. For each $i \in \{1, \dots, n\}$ let $\boldsymbol{\Delta}^{(c,i)}(\boldsymbol{x})$ be the vector of all $i \times i$ minors of $H_c(\boldsymbol{x})$, arranged in some order. This is a vector of degree i homogeneous forms in the variables $\boldsymbol{x}$, with real coefficients. Let $J_{\boldsymbol{\Delta}^{(c,i)}}(\boldsymbol{x})$ be the Jacobian matrix $(\partial \Delta_j^{(c,i)}(\boldsymbol{x})/\partial x_k)_{jk}$.

We aim to bound the function $N_c^{\text{aux}}(B)$ from Definition 1.6. In the case of a cubic form one sees from the definition above that $N_c^{\text{aux}}(B)$ is the number of pairs $(\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{Z}^n)^2$ with

$$\|\boldsymbol{x}\|_\infty, \|\boldsymbol{y}\|_\infty \le B, \qquad \|H_c(\boldsymbol{x})\boldsymbol{y}\|_\infty < B, \tag{4.7}$$

since the multilinear forms (1.28) are given by

$$\boldsymbol{m}^{(c)}(\boldsymbol{x}, \boldsymbol{y}) = \|c\|_\infty H_c(\boldsymbol{x})\boldsymbol{y}. \tag{4.8}$$

To prove Proposition 4.1 we adapt the argument used to prove Lemma 3 in Davenport [Dav63], and subsequently a slightly more general result in Schmidt [Sch82, §5]. These authors consider the counting function defined by

$$N_c^{\text{aux-eq}}(B) = \#\{(\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{Z}^n)^2 : \|\boldsymbol{x}\|_\infty, \|\boldsymbol{y}\|_\infty \le B,\ H_c(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{0}\}$$

for some cubic form c with coefficients in $\mathbb{Z}$. Davenport proves that either $N_c^{\text{aux-eq}}(B)$ is small, or there is a large rational linear space on which c vanishes. We briefly sketch the line of reasoning.

(1) Let $\sigma \in \{0, \dots, n-1\}$. Suppose that $N_c^{\text{aux-eq}}(B) \gg B^{n+\sigma}$ for some sufficiently large implicit constant. The contribution to this count from any one vector $\boldsymbol{x}$ is at most $O(B^{n - \operatorname{rank} H_c(\boldsymbol{x})})$. So there must be an integer b in the set $\{0, \dots, n-1\}$ such that at least $\gg B^{\sigma+b}$ integer points $\boldsymbol{x}$ satisfy both $\operatorname{rank} H_c(\boldsymbol{x}) = b$ and $\|\boldsymbol{x}\|_\infty \le B$.

(2) If σ, b are as in (1), then it follows that there is an integer point $\boldsymbol{x}^{(0)}$ such that $\operatorname{rank} H_c(\boldsymbol{x}^{(0)}) = b$ holds and the tangent space to the affine variety $\boldsymbol{\Delta}^{(c,b+1)}(\boldsymbol{x}) = \mathbf{0}$ at the point $\boldsymbol{x}^{(0)}$ has dimension $\sigma+b+1$ or more. Equivalently, $\operatorname{rank} H_c(\boldsymbol{x}^{(0)}) = b$ and $\operatorname{rank} J_{\boldsymbol{\Delta}^{(c,b+1)}}(\boldsymbol{x}^{(0)}) \leq n-\sigma-b-1$ both hold. This follows from Lemma 2 of Davenport [Dav63].

(3) If there exists a vector $\boldsymbol{x}^{(0)}$ as in (2), then it follows that there exist linear subspaces X, Y of $\mathbb{Q}^n$, with dimensions $\sigma+b+1$ and $n-b$ respectively, such that for all $\boldsymbol{X} \in X$ and $\boldsymbol{Y}, \boldsymbol{Y'} \in Y$ the equality $\boldsymbol{Y}^T H_c(\boldsymbol{X}) \boldsymbol{Y'} = 0$ holds. See Lemma 4 in Schmidt [Sch82] or the proof of Lemma 3 in Davenport [Dav63].

(4) We conclude that if $N_c^{\text{aux-eq}}(B) \gg B^{n+\sigma}$ then there are spaces X, Y as in (3). So the space Z defined by $Z = X \cap Y$ is a rational linear space, with dimension at least $\sigma+1$, such that for all $\boldsymbol{Z} \in Z$ the equality $c(\boldsymbol{Z}) = 0$ holds.

Our setting differs in three ways from that of Schmidt and Davenport. First, we consider the inequality $\|H_c(\boldsymbol{x})\boldsymbol{y}\|_\infty \leq B$ rather than the equation $H_c(\boldsymbol{x})\boldsymbol{y} = \mathbf{0}$. Second, for us the cubic form $c(\boldsymbol{x})$ may have real coefficients. And third, rather than concluding that $c(\boldsymbol{x})$ has a rational linear space of zeroes, we seek to show that the variety $V(c)$ is very singular. In §4.4.1–4.4.4 we will modify each of the four steps (1)–(4) above to accommodate these three differences.

The largest single change is as follows. Steps (1) and (2) count the number of $\boldsymbol{x}$ such that the rank of $H_c(\boldsymbol{x})$ takes a specified value. We will instead count the number of $\boldsymbol{x}$ for which the eigenvalues of $H_c(\boldsymbol{x})$ lie in specified dyadic ranges, in the following sense.

Definition 4.3. Let $\lambda_{c,1}(\boldsymbol{x}), \dots, \lambda_{c,n}(\boldsymbol{x})$ be the eigenvalues of the real symmetric matrix $H_c(\boldsymbol{x})$, listed with multiplicity and in order of decreasing absolute value.

Suppose that $B \geq 1$, that $k \in \{0, \dots, n\}$ and that $E_1, \dots, E_{k+1} \in \mathbb{R}$ such that the inequalities $E_1 \geq \dots \geq E_{k+1} \geq 1$ hold. Then we define $K_{k,B}(E_1, \dots, E_{k+1})$ to be the set of all vectors $\boldsymbol{x}$ in $\mathbb{R}^n$ satisfying the following conditions:

- the inequality $\|\boldsymbol{x}\|_\infty \leq B$ holds,
- we have $\frac{1}{2}E_i < |\lambda_{c,i}(\boldsymbol{x})| \leq E_i$ whenever $1 \leq i \leq k$ holds, and
- we have $|\lambda_{c,i}(\boldsymbol{x})| \leq E_{k+1}$ whenever $k+1 \leq i \leq n$ holds.

Our main technical result in this chapter, Lemma 4.6, provides a bound, (4.16), for the number of integer points in $K_{k,B}(E_1, \ldots, E_{k+1})$. Note that

$$|\lambda_{c,1}(\boldsymbol{x})| \leq n \|H_c(\boldsymbol{x})\|_\infty \leq n^2 \|\boldsymbol{x}\|_\infty, \tag{4.9}$$

and so we need only consider values of E_i of size $O_n(B)$.

4.4 The cubic case

Following the direction set in §4.3, we will now prove Proposition 4.1 in the case when $d = 3$.

4.4.1 Linear Diophantine inequalities with a normal matrix

In this section we show that if the counting function $N_c^{\mathrm{aux}}(B)$ from Definition 1.6 is large, then there are many integer points $\boldsymbol{x}$ for which the eigenvalues of $H_c(\boldsymbol{x})$ lie in some fixed dyadic ranges. This corresponds to step (1) from §4.3. We begin by estimating the number of solutions of a system of linear Diophantine inequalities, assuming that the underlying matrix is normal.

Lemma 4.4. *Let H be a real $n \times n$ matrix. Suppose H is normal and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues, listed with multiplicity and in order of decreasing absolute value. Let $C \geq 1$ and $B \geq 1$, and suppose that $|\lambda_1| \leq CB$ holds. Set*

$$N_H(B) = \#\{\boldsymbol{y} \in \mathbb{Z}^n : \|\boldsymbol{y}\|_\infty \leq B,\ \|H\boldsymbol{y}\|_\infty \leq B\}.$$

Then we have

$$N_H(B) \ll_{C,n} \min_{1 \leq i \leq n} \frac{B^n}{1 + |\lambda_1 \cdots \lambda_i|}.$$

Proof. The integral vectors $\boldsymbol{y}$ counted by $N_H(B)$ are contained in the box $\|\boldsymbol{y}\|_\infty \leq B$, and also in the ellipsoid

$$\{\boldsymbol{t} \in \mathbb{R}^n : \boldsymbol{t}^T H^T H \boldsymbol{t} \leq nB^2\},$$

which has principal radii $|\lambda_i|^{-1}\sqrt{n}B$. Hence

$$N_H(B) \ll_n \prod_{i=1}^n \min\{1 + |\lambda_i|^{-1}\sqrt{n}B,\ B\},$$

and as $|\lambda_i| \leq CB$ holds, this is

$$\leq \prod_{i=1}^n \min\{2C|\lambda_i|^{-1}\sqrt{n}B,\ B\}.$$

It follows that

$$N_H(B) \ll_{C,n} B^n \prod_{i=1}^{n} \min\{|\lambda_i|^{-1}, 1\}.$$

Since the inequalities $|\lambda_1| \geq \cdots \geq |\lambda_n|$ hold, we deduce that

$$\begin{aligned} N_H(B) &\ll_{C,n} B^n \min\left\{1, \frac{1}{|\lambda_1|}, \frac{1}{|\lambda_1\lambda_2|}, \ldots, \frac{1}{|\lambda_1\cdots\lambda_n|}\right\} \\ &\ll \min_{1\leq i\leq n} \frac{B^n}{1+|\lambda_1\cdots\lambda_i|}. \end{aligned}$$

□

We will now apply Lemma 4.4 to the auxiliary inequality (4.7). The result will be that the counting function $N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B)$ can be large only if one of the sets $K_{k,B}(E_1,\ldots,E_{k+1})$ from Definition 4.3 contains many integral points.

Corollary 4.5. *Let $N^{\mathrm{aux}}_c(B)$ be as in Definition 1.6, let c and $\boldsymbol{\Delta}^{(c,i)}(\boldsymbol{x})$ be as in Definition 4.2 and let $\lambda_{c,i}(\boldsymbol{x})$ and $K_{k,B}(E_1,\ldots,E_{k+1})$ be as in Definition 4.3. For any $B\geq 1$, one of the following alternatives holds. Either*

$$\frac{N^{\mathrm{aux}}_c(B)}{B^n(\log B)^n} \ll_n \#\{\mathbb{Z}^n \cap K_{0,B}(1)\}, \tag{4.10}$$

or there is $k\in\{1,\ldots,n-1\}$ and there are $e_1,\ldots,e_k\in\mathbb{N}$ such that the inequalities $\log B \gg_n e_1 \geq \cdots \geq e_k$ hold and

$$\frac{2^{e_1+\cdots+e_k} N^{\mathrm{aux}}_c(B)}{B^n(\log B)^n} \ll_n \#\big\{\mathbb{Z}^n \cap K_{k,B}(2^{e_1},\ldots,2^{e_k},1)\big\}, \tag{4.11}$$

or there are $e_1,\ldots,e_n\in\mathbb{N}$ satisfying $\log B \gg_n e_1 \geq \cdots \geq e_n$ and

$$\frac{2^{e_1+\cdots+e_n} N^{\mathrm{aux}}_c(B)}{B^n(\log B)^n} \ll_n \#\big\{\mathbb{Z}^n \cap K_{n-1,B}(2^{e_1},\ldots,2^{e_n})\big\}. \tag{4.12}$$

Proof. Note that in the case that $k=n$, there are no values of i satisfying $k+1\leq i\leq n$, so the last condition in the definition of $K_{k,B}(E_1,\ldots,E_{k+1})$ is vacuously true and can be omitted. In particular, if $k=n$ then (4.12) follows from (4.11), because

$$K_{n,B}(2^{e_1},\ldots,2^{e_n},1) \subset K_{n-1,B}(2^{e_1},\ldots,2^{e_n}).$$

So it is enough to prove that either (4.10) holds or there exist integers k and $e_1,\ldots,e_k$, satisfying the inequalities $1\leq k\leq n$ and $\log B \gg_n e_1 \geq \cdots \geq e_n$, such that (4.11) holds.

Now the set $K_{0,B}(1)$, together with the sets $K_{k,B}(2^{e_1},\ldots,2^{e_k},1)$, partition the box $\|\boldsymbol{x}\|_\infty \leq B$ into disjoint pieces. If we let $H_c(\boldsymbol{x})$ be as in Definition 4.2 and set

$$N_{H_c(\boldsymbol{x})}(B) = \#\{\boldsymbol{y}\in\mathbb{Z}^n : \|\boldsymbol{y}\|_\infty \leq B,\ \|N_{H_c(\boldsymbol{x})}(B)\boldsymbol{y}\|_\infty \leq B\},$$

then we have

$$N_c^{\mathrm{aux}}(B) = \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ \boldsymbol{x}\in K_{0,B}(1)}} N_{H_c(\boldsymbol{x})}(B) + \sum_{\substack{1\le k\le n \\ e_1\ge\cdots\ge e_k\ge 1 \\ e_1\ll_n \log B}} \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ \boldsymbol{x}\in K_{k,B}(2^{e_1},\ldots,2^{e_k},1)}} N_{H_c(\boldsymbol{x})}(B). \tag{4.13}$$

The total number of terms on the right-hand side of (4.13) is $O_n((\log B)^n)$ at most, so it follows that either

$$\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ \boldsymbol{x}\in K_{0,B}(1)}} N_{H_c(\boldsymbol{x})}(B) \gg_n \frac{N_c^{\mathrm{aux}}(B)}{(\log B)^n} \tag{4.14}$$

holds, or else there are $1 \le k \le n$ and $e_1 \ge \cdots \ge e_k \ge 1$ such that

$$\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ \boldsymbol{x}\in K_{k,B}(2^{e_1},\ldots,2^{e_k},1)}} N_{H_c(\boldsymbol{x})}(B) \gg_n \frac{N_c^{\mathrm{aux}}(B)}{(\log B)^n}. \tag{4.15}$$

If (4.14) holds then the trivial bound $N_{H_c(\boldsymbol{x})}(B) \ll_n B^n$ implies (4.10). Suppose instead that (4.15) holds.

By (4.9), for each real vector $\boldsymbol{x}$ the bound $|\lambda_{c,1}(\boldsymbol{x})| \ll_n B$ holds. So we may apply Lemma 4.4 with the choice $H = H_c(\boldsymbol{x})$ and some C depending on n only. This shows that

$$N_{H_c(\boldsymbol{x})}(B) \ll_n \frac{B^n}{2^{e_1+\cdots+e_k}}.$$

Substituting this into (4.15) we see that (4.11) holds, as claimed. □

4.4.2 Counting points in the sets $K_{k,B}(E_1,\ldots,E_{k+1})$

In this section our goal is to estimate the number of integer points in the sets $K_{k,B}(E_1,\ldots,E_{k+1})$ from Definition 4.3, in terms of the quantitites $H_c(\boldsymbol{x})$, $\boldsymbol{\Delta}^{(c,i)}(\boldsymbol{x})$ and $J_{\boldsymbol{\Delta}^{(c,i)}}(\boldsymbol{x})$ from Definition 4.2. We give the following result.

Lemma 4.6. *Suppose that $B, C \ge 1$, $\sigma \in \{0,\ldots,n-1\}$, and $k \in \{0,\ldots,n-\sigma-1\}$, and that $CB \ge E_1 \ge \cdots \ge E_{k+1} \ge 1$. Then at least one of the following holds:*

(I)$_k$ The set $K_{k,B}(E_1,\ldots,E_{k+1})$ may be covered by a collection of at most

$$O_{C,n}(B^\sigma(E_1\cdots E_{k+1})E_{k+1}^{-\sigma-k-1})$$

boxes in $\mathbb{R}^n$ of side length E_{k+1}. Such a box contains $O_n(E_{k+1}^n)$ integral points, so it follows that

$$\#\{\mathbb{Z}^n \cap K_{k,B}(E_1,\ldots,E_{k+1})\} \ll_{C,n} B^\sigma(E_1\cdots E_{k+1})E_{k+1}^{n-\sigma-k-1}. \tag{4.16}$$

(II)$_k$ There exist an integer $1 \leq b \leq k$, a point $\boldsymbol{x}^{(0)} \in K_{b,B}(E_1, \cdots, E_{b+1})$, and a $(\sigma+b+1)$-dimensional linear subspace X of $\mathbb{R}^n$ such that

$$E_{b+1} < C^{-1} E_b,$$
$$\|J_{\boldsymbol{\Delta}^{(c,b+1)}}(\boldsymbol{x}^{(0)})\boldsymbol{X}\|_\infty \leq C^{-1}\|\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}^{(0)})\|_\infty \|\boldsymbol{X}\|_\infty \quad \text{for all } \boldsymbol{X} \in X. \tag{4.17}$$

(III) There is a $(\sigma+1)$-dimensional linear subspace X of $\mathbb{R}^n$ such that

$$\|H_c(\boldsymbol{X})\|_\infty \leq C^{-1}\|\boldsymbol{X}\|_\infty \qquad \text{for all } \boldsymbol{X} \in X, \tag{4.18}$$

with $\|H_c(\boldsymbol{X})\|_\infty$ as in §4.3.

Here we have subscripted the first two items to emphasize their dependence on k; note that item (III) has no such dependence.

One could think of (II)$_k$ as saying that $H_c(\boldsymbol{x})$ is close to having rank b, while $J_{\boldsymbol{\Delta}^{(c,b+1)}}(\boldsymbol{x}^{(0)})$ is close to having rank $\leq n-\sigma-b-1$. This can be compared to the conclusion of step (2) in §4.3 above.

Proof of Lemma 4.6. The proof is by induction on k. Let c, C, B, and σ be fixed.

The case $k=0$. Let $k=0$, let $CB \geq E_1 \geq 1$ and suppose that alternative (III) does not hold. We claim that alternative (I)$_0$ holds, which is to say that $K_{0,B}(E_1)$ is covered by $O_{C,n}(B^\sigma/E_1^\sigma)$ boxes of side length E_1.

As (III) is false, applying Lemma 2.6 to the matrix of the linear map $\boldsymbol{x} \mapsto H_c(\boldsymbol{x})$ shows that there is an $(n-\sigma)$-dimensional subspace V of $\mathbb{R}^n$ with

$$\|H_c(\boldsymbol{v})\|_\infty \gg_n C^{-1}\|\boldsymbol{v}\|_\infty \qquad \text{for all } \boldsymbol{v} \in V. \tag{4.19}$$

For each $\boldsymbol{z} \in \mathbb{R}^n$, let $A_0(\boldsymbol{z})$ be the box in $\mathbb{R}^n$ defined by

$$A_0(\boldsymbol{z}) = \{\boldsymbol{z}+\boldsymbol{u}+\boldsymbol{v} : \boldsymbol{u} \in V^\perp, \boldsymbol{v} \in V, \|\boldsymbol{u}\|_\infty \leq E_1, \|\boldsymbol{v}\|_\infty \leq B\}.$$

Now $K_{0,B}(E_1)$ is contained in the box $\|\boldsymbol{x}\|_\infty \leq B$. It follows that we can cover $K_{0,B}(E_1)$ with a collection of $O_{C,n}(B^\sigma/E_1^\sigma)$ boxes of the form $A_0(\boldsymbol{z})$, each one of which is centred at a point $\boldsymbol{z}$ belonging to $K_{0,B}(E_1)$. We will show below that for each $\boldsymbol{z} \in K_{0,B}(E_1)$, the intersection $A_0(\boldsymbol{z}) \cap K_{0,B}(E_1)$ is contained in a box of side length $O_{C,n}(E_1)$. It follows that $K_{0,B}(E_1)$ is covered by $O_{C,n}(B^\sigma/E_1^\sigma)$ boxes of side length E_1, as claimed.

It remains to let $\boldsymbol{z} \in K_{0,B}(E_1)$ and let $\boldsymbol{y} \in A_0(\boldsymbol{z}) \cap K_{0,B}(E_1)$, and to deduce that $\|\boldsymbol{y}-\boldsymbol{z}\|_\infty \ll_{C,n} E_1$ must hold.

By definition of $K_{0,B}(E_1)$ we have $|\lambda_{c,1}(\boldsymbol{y})| \leq E_1$ and $|\lambda_{c,1}(\boldsymbol{z})| \leq E_1$, and the bounds $\|H_c(\boldsymbol{y})\|_\infty \ll_n E_1$ and $\|H_c(\boldsymbol{z})\|_\infty \ll_n E_1$ follow by (4.9). So we have

$$\|H_c(\boldsymbol{y}-\boldsymbol{z})\|_\infty \ll_n E_1. \tag{4.20}$$

Let $\boldsymbol{u} \in V^\perp$ and let $\boldsymbol{v} \in V$ such that $\boldsymbol{y} = \boldsymbol{z} + \boldsymbol{u} + \boldsymbol{v}$ holds. Since $\boldsymbol{y}$ lies in $A_0(\boldsymbol{z})$, we have $\|\boldsymbol{u}\|_\infty \leq E_1$, and with (4.20) this implies that

$$\|H_c(\boldsymbol{v})\|_\infty \ll_n E_1.$$

By (4.19) it follows that $\|\boldsymbol{v}\|_\infty \ll_n CE_1$, and hence that $\|\boldsymbol{y}-\boldsymbol{z}\|_\infty \ll_{C,n} E_1$, as claimed.

The inductive step. Let $k \geq 1$ and let $CB \geq E_1 \geq \cdots E_{k+1} \geq 1$. We suppose that $(\mathrm{II})_k$ and (III) are both false, and claim that $(\mathrm{I})_k$ holds.

By induction, at least one of $(\mathrm{I})_{k-1}$, $(\mathrm{II})_{k-1}$, or (III) holds. Note that of these (III) is false by assumption, and $(\mathrm{II})_{k-1}$ is false since it implies $(\mathrm{II})_k$. So in proving (I) we may assume that $(\mathrm{I})_{k-1}$ holds.

Suppose for the time being that

$$E_{k+1} < C^{-1}E_k. \tag{4.21}$$

The contrary case will be dealt with at the end of the proof. We claim that

$$K_{k,B}(E_1,\dots,E_{k+1}) = \bigcup_V K_{k,B}^{(C,V)}(E_1,\dots,E_{k+1}), \tag{4.22}$$

where V runs over all $(n-\sigma-k)$-dimensional subspaces of $\mathbb{R}^n$ which are spanned by standard basis vectors, and the set $K_{k,B}^{(C,V)}(E_1,\dots,E_{k+1})$ is given by

$$\begin{aligned} &K_{k,B}^{(C,V)}(E_1,\dots,E_{k+1}) \\ &\quad = \big\{\boldsymbol{x} \in K_{k,B}(E_1,\dots,E_{k+1}) : \|J_{\boldsymbol{\Delta}^{(c,k+1)}}(\boldsymbol{x})\boldsymbol{v}\|_\infty \geq C^{-1}\|\boldsymbol{\Delta}^{(c,k)}(\boldsymbol{x})\|_\infty \|\boldsymbol{v}\|_\infty \\ &\qquad\qquad \text{for all } \boldsymbol{v} \in V\big\}. \end{aligned} \tag{4.23}$$

We will then be able to cover each of these sets $K_{k,B}^{(C,V)}(E_1,\dots,E_{k+1})$ with a small number of boxes by using a similar argument to the case $k=0$ considered at the start of the proof.

We have assumed that $E_{k+1} < C^{-1}E_k$ and that the case $b=k$ of $(\mathrm{II})_k$ is false. So the case $b=k$ of (4.17) must be false for every $(\sigma+b+1)$-dimensional subspace X of $\mathbb{R}^n$ and every $\boldsymbol{x}^{(0)} \in K_{k,B}(E_1,\dots,E_{k+1})$. Thus for any $\boldsymbol{x}^{(0)} \in K_{k,B}(E_1,\dots,E_{k+1})$

and any $(\sigma+k+1)$-dimensional linear subspace X of $\mathbb{R}^n$, there is some $\boldsymbol{X} \in X$ such that

$$\|J_{\boldsymbol{\Delta}^{(c,k+1)}}(\boldsymbol{x}^{(0)})\boldsymbol{X}\|_\infty > C^{-1}\|\boldsymbol{\Delta}^{(c,k)}(\boldsymbol{x}^{(0)})\|_\infty\|\boldsymbol{X}\|_\infty.$$

By applying Lemma 2.6 with the choice $M = J_{\boldsymbol{\Delta}^{(c,k+1)}}(\boldsymbol{x}^{(0)})$, we deduce that for each $\boldsymbol{x}^{(0)} \in K_{k,B}(E_1,\dots,E_{k+1})$ there is an $(n-\sigma-k)$-dimensional subspace V of $\mathbb{R}^n$, spanned by standard basis vectors, such that

$$\|J_{\boldsymbol{\Delta}^{(c,k+1)}}(\boldsymbol{x}^{(0)})\boldsymbol{v}\|_\infty \geq C^{-1}\|\boldsymbol{\Delta}^{(c,k)}(\boldsymbol{x}^{(0)})\|_\infty\|\boldsymbol{v}\|_\infty \tag{4.24}$$

for all $\boldsymbol{v} \in V$. This proves (4.22), and so to prove $(\mathrm{I})_k$ it now suffices to show that for each $(n-\sigma-k)$-dimensional space V, the set (4.23) is covered by a union of $O_{C,n}(B^\sigma(E_1\cdots E_{k+1})E_{k+1}^{-\sigma-k-1})$ boxes of side length E_{k+1}.

Let $\epsilon > 0$ be a sufficiently small constant depending at most on C and n, and for each $\boldsymbol{z} \in \mathbb{R}^n$ set

$$A_k(\boldsymbol{z}) = \{\boldsymbol{z}+\boldsymbol{u}+\boldsymbol{v} : \boldsymbol{u} \in V^\perp, \boldsymbol{v} \in V, \|\boldsymbol{u}\|_\infty \leq E_{k+1}, \|\boldsymbol{v}\|_\infty \leq \epsilon E_k\}. \tag{4.25}$$

We assumed at the start of this inductive step that $(\mathrm{I})_{k-1}$ holds. Therefore the set $K_{k-1,B}(E_1,\dots,E_k)$ is covered by a collection of $O_{C,n}(B^\sigma(E_1\cdots E_k)E_k^{-\sigma-k})$ boxes of side length E_k. Since

$$K_{k,B}^{(C,V)}(E_1,\dots,E_{k+1}) \subset K_{k-1,B}(E_1,\dots,E_k),$$

the same is true for $K_{k,B}^{(C,V)}(E_1,\dots,E_{k+1})$. We subdivide each of these boxes into $O_{C,n}(E_k^{\sigma+k}/E_{k+1}^{\sigma+k})$ sub-boxes of the form $A_k(\boldsymbol{z})$. This shows that the set (4.23) may be covered by a collection of $O_{C,n}(B^\sigma(E_1\cdots E_{k+1})E_{k+1}^{-\sigma-k-1})$ boxes of the form $A_k(\boldsymbol{z})$, each of which is centred at a point $\boldsymbol{z}$ belonging to the set (4.23).

We claim that any intersection of the form $A_k(\boldsymbol{z}) \cap K_{k,B}^{(C,V)}(E_1,\dots,E_{k+1})$ is contained in a box having sides of length $O_{C,n}(E_{k+1})$. Then the set (4.23) is covered by $O_{C,n}(B^\sigma(E_1\cdots E_{k+1})E_{k+1}^{-\sigma-k-1})$ boxes of side length E_{k+1}, and by the comments after (4.24) this proves the lemma.

If we let $\boldsymbol{z} \in K_{k,B}^{(C,V)}(E_1,\dots,E_{k+1})$ and $\boldsymbol{y} \in A_k(\boldsymbol{z}) \cap K_{k,B}^{(C,V)}(E_1,\dots,E_{k+1})$, then the claim is that $\|\boldsymbol{y}-\boldsymbol{z}\|_\infty \ll_{C,n} E_{k+1}$ holds. Define $\boldsymbol{u} \in V^\perp$ and $\boldsymbol{v} \in V$ by $\boldsymbol{y} = \boldsymbol{z}+\boldsymbol{u}+\boldsymbol{v}$, and note that since $\boldsymbol{y} \in A_k(\boldsymbol{z})$, we have

$$\|\boldsymbol{u}\|_\infty \leq E_{k+1}, \qquad \|\boldsymbol{v}\|_\infty \leq \epsilon E_k. \tag{4.26}$$

Now the jth partial derivatives of the minors $\boldsymbol{\Delta}^{(c,k+1)}(\boldsymbol{x})$ are linear combinations of the minors $\boldsymbol{\Delta}^{(c,k+1-j)}(\boldsymbol{x})$ with coefficients of size at most $O_n(1)$. So we have

$$\left\|\frac{\partial^j \boldsymbol{\Delta}^{(c,k+1)}(\boldsymbol{x})}{\partial x_{i_1}\cdots\partial x_{i_j}}\right\|_\infty \ll_n \|\boldsymbol{\Delta}^{(c,k+1-j)}(\boldsymbol{x})\|_\infty, \tag{4.27}$$

and Taylor expansion shows that

$$\begin{aligned}\mathbf{\Delta}^{(c,k+1)}(\boldsymbol{z}+\boldsymbol{u}+\boldsymbol{v}) &= \mathbf{\Delta}^{(c,k+1)}(\boldsymbol{z}) + J_{\mathbf{\Delta}^{(c,k+1)}}(\boldsymbol{z})\boldsymbol{u} + J_{\mathbf{\Delta}^{(c,k+1)}}(\boldsymbol{z})\boldsymbol{v} \\ &+ O_n\Big(\|\boldsymbol{u}+\boldsymbol{v}\|_\infty^2\|\mathbf{\Delta}^{(c,k-1)}(\boldsymbol{z})\|_\infty + \cdots + \|\boldsymbol{u}+\boldsymbol{v}\|_\infty^k\|\mathbf{\Delta}^{(c,1)}(\boldsymbol{z})\|_\infty + \|\boldsymbol{u}+\boldsymbol{v}\|_\infty^{k+1}\Big).\end{aligned}$$

Now (4.27) shows that $J_{\mathbf{\Delta}^{(c,k+1)}}(\boldsymbol{z})\boldsymbol{u} = O_n(\|\boldsymbol{u}\|_\infty\|\mathbf{\Delta}^{(c,k)}(\boldsymbol{z})\|_\infty)$, and so

$$\begin{aligned}J_{\mathbf{\Delta}^{(c,k+1)}}(\boldsymbol{z})\boldsymbol{v} &= \mathbf{\Delta}^{(c,k+1)}(\boldsymbol{y}) - \mathbf{\Delta}^{(c,k+1)}(\boldsymbol{z}) \\ &+ O_n\Big(\|\boldsymbol{u}\|_\infty\|\mathbf{\Delta}^{(c,k)}(\boldsymbol{z})\|_\infty + \cdots + \|\boldsymbol{u}\|_\infty^k\|\mathbf{\Delta}^{(c,1)}(\boldsymbol{z})\|_\infty + \|\boldsymbol{u}\|_\infty^{k+1}\Big) \\ &+ O_n\Big(\|\boldsymbol{v}\|_\infty^2\|\mathbf{\Delta}^{(c,k-1)}(\boldsymbol{z})\|_\infty + \cdots + \|\boldsymbol{v}\|_\infty^k\|\mathbf{\Delta}^{(c,1)}(\boldsymbol{z})\|_\infty + \|\boldsymbol{v}\|_\infty^{k+1}\Big).\end{aligned} \tag{4.28}$$

Since $\boldsymbol{y},\boldsymbol{z}\in K_{k,B}(E_1,\dots,E_{k+1})$, Lemma 2.5 gives us the bounds

$$\|\mathbf{\Delta}^{(c,j)}(\boldsymbol{z})\|_\infty \asymp_n \prod_{i=1}^{j} E_i, \qquad \|\mathbf{\Delta}^{(c,k+1)}(\boldsymbol{y})\|_\infty \asymp_n \prod_{i=1}^{k+1} E_i, \tag{4.29}$$

and since $\boldsymbol{z}\in K_{k,B}^{(C,V)}(E_1,\dots,E_{k+1})$ it follows from (4.23) that

$$\|J_{\mathbf{\Delta}^{(c,k+1)}}(\boldsymbol{z})\boldsymbol{v}\|_\infty \gg_n C^{-1}\|\boldsymbol{v}\|_\infty \prod_{i=1}^{k} E_i. \tag{4.30}$$

Substituting (4.29) and (4.30) into (4.28) yields

$$\begin{aligned}C^{-1}\|\boldsymbol{v}\|_\infty \ll_n \prod_{i=1}^{k+1} E_i &+ \|\boldsymbol{u}\|_\infty \prod_{i=1}^{k} E_i + \cdots + \|\boldsymbol{u}\|_\infty^k E_1 + \|\boldsymbol{u}\|_\infty^{k+1} \\ &+ \|\boldsymbol{v}\|_\infty^2 \prod_{i=1}^{k-1} E_i + \cdots + \|\boldsymbol{v}\|_\infty^k E_1 + \|\boldsymbol{v}\|_\infty^{k+1}.\end{aligned}$$

Applying the bounds from (4.26) and the inequalities $E_1 \geq \cdots \geq E_{k+1}$, we deduce that

$$C^{-1}\|\boldsymbol{v}\|_\infty \ll_n \prod_{i=1}^{k+1} E_i + \epsilon\|\boldsymbol{v}\|_\infty \prod_{i=1}^{k} E_i.$$

Since ϵ is assumed to be small in terms of C and n, it follows that $\|\boldsymbol{v}\|_\infty \ll_n CE_{k+1}$ holds and hence that $\|\boldsymbol{y}-\boldsymbol{z}\|_\infty \ll_{C,n} E_{k+1}$ holds. By the comments after (4.25), this proves the lemma.

It remains to consider the case when (4.21) is false and so $E_{k+1} \geq C^{-1}E_k$ holds. At the start of the inductive step we supposed that $(\mathrm{I})_{k-1}$ holds, so $K_{k-1,B}(E_1,\dots,E_k)$ may be covered by $O_{C,n}(B^\sigma(E_1\cdots E_k)E_k^{-\sigma-k})$ boxes of side length E_k. We have

$$K_{k,B}(E_1,\dots,E_{k+1}) \subset K_{k-1,B}(E_1,\dots,E_k),$$

and so $K_{k,B}(E_1,\dots,E_{k+1})$ is also covered by these boxes. Since $E_{k+1} \geq C^{-1}E_k$, we divide each of box into $O_{C,n}(1)$ boxes of side length E_{k+1} to prove $(\mathrm{I})_k$. □

4.4.3 Small values of a trilinear form

Part (3) of Davenport's argument from §4.3 shows that if the matrices $H_c(\boldsymbol{x})$ and $J_{\boldsymbol{\Delta}^{(c,b+1)}}(\boldsymbol{x})$ have small rank, then we can find linear spaces X, Y such that for all $\boldsymbol{X} \in X$ and $\boldsymbol{Y}, \boldsymbol{Y}' \in Y$ the equation $\boldsymbol{Y}^T H_c(\boldsymbol{X})\boldsymbol{Y}' = 0$ holds. Our analogue is the following result. Here, as above, c, $H_c(\boldsymbol{x})$ and $J_{\boldsymbol{\Delta}^{(c,i)}}(\boldsymbol{x})$ are as in Definition 4.2 and $\lambda_{c,i}(\boldsymbol{x})$ is as in Definition 4.3.

Lemma 4.7. *Suppose that $b \in \{1, \dots, n-1\}$ and that $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$. Then, provided $\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}^{(0)})$ is nonzero, there exists an $(n-b)$-dimensional linear subspace Y of $\mathbb{R}^n$ such that for all $\boldsymbol{Y}, \boldsymbol{Y}' \in Y$ and all $\boldsymbol{t} \in \mathbb{R}^n$ we have*

$$\boldsymbol{Y}^T H_c(\boldsymbol{t})\boldsymbol{Y}' \ll_n \left(\frac{\|J_{\boldsymbol{\Delta}^{(c,b+1)}}(\boldsymbol{x}^{(0)})\boldsymbol{t}\|_\infty}{\|\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}^{(0)})\|_\infty} + \frac{|\lambda_{c,b+1}(\boldsymbol{x}^{(0)})| \cdot \|\boldsymbol{t}\|_\infty}{|\lambda_{c,b}(\boldsymbol{x}^{(0)})|} \right) \|\boldsymbol{Y}\|_\infty \|\boldsymbol{Y}'\|_\infty. \tag{4.31}$$

By setting $b = \operatorname{rank} H_c(\boldsymbol{x}^{(0)})$ and $X = \ker J_{\boldsymbol{\Delta}^{(c,b+1)}}(\boldsymbol{x}^{(0)})$, one may recover exactly Part (3) from §4.3. We prove Lemma 4.7 at the end of this section, after deducing

Corollary 4.8. *Let c be as in Definition 4.2, and let $N_c^{\mathrm{aux}}(B)$ be as in Definition 1.6. Let $B, C \geq 1$ and let $\sigma \in \{0, \dots, n-1\}$. Then either*

$$N_c^{\mathrm{aux}}(B) \ll_{C,n} B^{n+\sigma}(\log B)^n, \tag{4.32}$$

or else there exist positive-dimensional linear subspaces X and Y of $\mathbb{R}^n$ which satisfy $\dim X + \dim Y = n + \sigma + 1$ *and for which we have*

$$|\boldsymbol{Y}^T H_c(\boldsymbol{X})\boldsymbol{Y}'| \ll_n C^{-1}\|\boldsymbol{Y}\|_\infty \|\boldsymbol{X}\|_\infty \|\boldsymbol{Y}'\|_\infty \quad \textit{for all } \boldsymbol{X} \in X,\ \boldsymbol{Y}, \boldsymbol{Y}' \in Y. \tag{4.33}$$

Proof. The proof amounts to putting Corollary 4.5 together with Lemmas 4.6 and 4.7, and checking each of the various cases which may arise.

Lemma 4.6 shows that for any $k \in \{0, \dots, n-\sigma-1\}$ and any $E_1, \dots, E_{k+1} \in \mathbb{R}$ satisfying

$$CB \geq E_1 \geq \dots \geq E_{k+1} \geq 1,$$

one of $(\mathrm{I})_k$, $(\mathrm{II})_k$, or (III) must hold. Suppose first that in every case alternative $(\mathrm{I})_k$ holds. By (4.16), we then have

$$\#\{\mathbb{Z}^n \cap K_{k,B}(E_1, \dots, E_{k+1})\} \ll_{C,n} B^\sigma (E_1 \cdots E_{k+1}) E_{k+1}^{n-\sigma-k-1} \tag{4.34}$$

for every $k \in \{0, \dots, n-\sigma-1\}$ and every $CB \geq E_1 \geq \dots \geq E_{k+1} \geq 1$. Now Corollary 4.5 shows that either

$$\frac{N_c^{\mathrm{aux}}(B)}{B^n(\log B)^n} \ll_n \#\{\mathbb{Z}^n \cap K_{0,B}(1)\}, \tag{4.35}$$

or there are $k \in \{1, \ldots, n-1\}$ and $e_1, \ldots, e_{k+1} \in \mathbb{N}$ such that $B \gg_n 2^{e_1} \geq \cdots \geq 2^{e_{k+1}} \geq 1$ and

$$\frac{2^{e_1+\cdots+e_k} N_c^{\mathrm{aux}}(B)}{B^n (\log B)^n} \ll_n \#\{\mathbb{Z}^n \cap K_{k,B}(2^{e_1}, \ldots, 2^{e_k}, 1)\}, \tag{4.36}$$

or there are $e_1, \ldots, e_n \in \mathbb{N}$ such that $B \gg_n 2^{e_1} \geq \cdots \geq 2^{e_n} \geq 1$ and

$$\frac{2^{e_1+\cdots+e_n} N_c^{\mathrm{aux}}(B)}{B^n (\log B)^n} \ll_n \#\{\mathbb{Z}^n \cap K_{n-1,B}(2^{e_1}, \ldots, 2^{e_n})\}. \tag{4.37}$$

We may assume that C is sufficiently large in terms of n, so we may assume that $CB \geq 2^{e_i}$ in (4.36) and (4.37). Substituting the bound (4.34) into each of the estimates (4.35)–(4.37) proves the conclusion (4.32).

Suppose next that alternative (III) holds in Lemma 4.6. In this case we let $Y = \mathbb{R}^n$, and the conclusion (4.33) follows from (4.18).

It remains to treat the case when there exist $k \in \{0, \ldots, n-\sigma-1\}$ and $CB \geq E_1 \geq \cdots \geq E_{k+1} \geq 1$ such that alternative $(\mathrm{II})_k$ holds in Lemma 4.6. This means that there exist an integer $1 \leq b \leq k$, a point $\boldsymbol{x}^{(0)} \in K_{b,B}(E_1, \cdots, E_{b+1})$, and a $(\sigma+b+1)$-dimensional linear subspace X of $\mathbb{R}^n$ such that

$$E_{b+1} < C^{-1} E_b, \tag{4.38}$$

and

$$\|J_{\boldsymbol{\Delta}^{(c,b+1)}}(\boldsymbol{x}^{(0)})\boldsymbol{X}\|_\infty \leq C^{-1} \|\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}^{(0)})\|_\infty \|\boldsymbol{X}\|_\infty \qquad \text{for all } \boldsymbol{X} \in X. \tag{4.39}$$

Since $\boldsymbol{x}^{(0)} \in K_{k,B}(E_1, \ldots, E_{k+1})$, the inequalities $\frac{1}{2}E_i < \lambda_{c,i}(\boldsymbol{x}^{(0)}) \leq E_i$ hold. Therefore (4.38) implies

$$\lambda_{c,b+1}(\boldsymbol{x}^{(0)}) < 2C^{-1} \lambda_{c,b}(\boldsymbol{x}^{(0)}), \tag{4.40}$$

Note that (4.40) implies that $\lambda_{c,b}(\boldsymbol{x}^{(0)})$ is nonzero, and so $\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}^{(0)})$ is nonzero, by Lemma 2.5. Hence we may apply Lemma 4.7 to find an $(n-b)$-dimensional linear space Y such that for all $\boldsymbol{Y}, \boldsymbol{Y}' \in Y$ and all $\boldsymbol{t} \in \mathbb{R}^n$ the bound (4.31) holds. The conclusion (4.33) follows on taking $\boldsymbol{t} = \boldsymbol{X}$ in (4.31) of Lemma 4.7 and substituting in the bounds (4.39) and (4.40). □

We now proceed to prove Lemma 4.7.

Proof of Lemma 4.7. We imitate the proof of Lemma 3 in Davenport [Dav63], which begins by considering the following easy "warm-up" problem. Suppose we were to look for $n-b$ linearly independent vectors $\boldsymbol{y}$ at which $H_c(\boldsymbol{x}^{(0)})\boldsymbol{y}$ vanishes. One approach would be as follows. One can construct matrices $L^{(i)}$, $M^{(i)}$ for $i=1,\dots,n-b$, with entries in $\{0,\pm 1\}$, such that the vectors

$$\boldsymbol{y}^{(i)}(\boldsymbol{x}) = L^{(i)}\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}) \tag{4.41}$$

satisfy

$$H_c(\boldsymbol{x})\boldsymbol{y}^{(i)}(\boldsymbol{x}) = M^{(i)}\boldsymbol{\Delta}^{(c,b+1)}(\boldsymbol{x}). \tag{4.42}$$

In other words we can take the components of $\boldsymbol{y}^{(i)}(\boldsymbol{x})$ to be polynomials of the form $\pm\Delta_j^{(b)}(\boldsymbol{x})$, and we can arrange that the components of $H_c(\boldsymbol{x})\boldsymbol{y}^{(i)}(\boldsymbol{x})$ are polynomials of the form $\pm\Delta_j^{(b+1)}(\boldsymbol{x})$.

If $H_c(\boldsymbol{x}^{(0)})\boldsymbol{y} = \boldsymbol{0}$ had exactly $n-b$ linearly independent solutions $\boldsymbol{y}$, we would have $\boldsymbol{\Delta}^{(c,b+1)}(\boldsymbol{x}^{(0)}) = \boldsymbol{0}$, while $\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}^{(0)})$ would be nonzero. We would then have $n-b$ solutions $\boldsymbol{Y}^{(k)}$ defined by

$$\boldsymbol{Y}^{(k)} = \frac{\boldsymbol{y}^{(k)}(\boldsymbol{x}^{(0)})}{\|\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}^{(0)})\|_\infty} \qquad (1\le k\le n-b), \tag{4.43}$$

and if we chose our matrices $L^{(i)}, M^{(i)}$ appropriately these would be linearly independent.

We now return to the proof of the lemma. Assume for the time being that we are given $L^{(i)}$, $M^{(i)}$, $\boldsymbol{y}^{(i)}(\boldsymbol{x})$ and $\boldsymbol{Y}^{(i)}$ satisfying (4.41)–(4.43), and let $\boldsymbol{x}^{(0)}$ be as in the lemma. Let $\boldsymbol{t}\in\mathbb{R}^n$. Let $\partial_{\boldsymbol{t}}$ be the directional derivative along $\boldsymbol{t}$ defined by $\sum t_i\frac{\partial}{\partial x_i}$, and apply $\partial_{\boldsymbol{t}}$ to both sides of (4.42). This shows that

$$\big(\partial_{\boldsymbol{t}}H_c(\boldsymbol{x})\big)\boldsymbol{y}^{(i)}(\boldsymbol{x}) + H_c(\boldsymbol{x})\big(\partial_{\boldsymbol{t}}\boldsymbol{y}^{(i)}(\boldsymbol{x})\big) = M^{(i)}\big(\partial_{\boldsymbol{t}}\boldsymbol{\Delta}^{(c,b+1)}(\boldsymbol{x})\big). \tag{4.44}$$

We have

$$\partial_{\boldsymbol{t}}\boldsymbol{\Delta}^{(c,k)}(\boldsymbol{x}) = J_{\boldsymbol{\Delta}^{(c,k)}}(\boldsymbol{x})\boldsymbol{t},$$

and together with (4.41) and (4.44) this shows that

$$H_c(\boldsymbol{t})\boldsymbol{y}^{(i)}(\boldsymbol{x}) = M^{(i)}J_{\boldsymbol{\Delta}^{(c,b+1)}}(\boldsymbol{x})\boldsymbol{t} - H_c(\boldsymbol{x})L^{(i)}\partial_{\boldsymbol{t}}\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}).$$

Premultiplying by $\boldsymbol{y}^{(j)}(\boldsymbol{x})^T$ and using (4.42) gives

$$\begin{aligned}\boldsymbol{y}^{(j)}(\boldsymbol{x})^T H_c(\boldsymbol{t})\boldsymbol{y}^{(i)}(\boldsymbol{x}) &= \boldsymbol{y}^{(j)}(\boldsymbol{x})^T M^{(i)}J_{\boldsymbol{\Delta}^{(c,b+1)}}(\boldsymbol{x})\boldsymbol{t}\\ &\quad - \big(M^{(i)}\boldsymbol{\Delta}^{(c,b+1)}(\boldsymbol{x})\big)^T\big(L^{(i)}\partial_{\boldsymbol{t}}\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x})\big).\end{aligned} \tag{4.45}$$

Now Lemma 2.5 shows that

$$\frac{\|\boldsymbol{\Delta}^{(c,b+1)}(\boldsymbol{x})\|_\infty}{\|\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x})\|_\infty} \ll_n |\lambda_{c,b+1}(\boldsymbol{x})|, \qquad \frac{\|\partial_{\boldsymbol{t}}\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x})\|_\infty}{\|\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x})\|_\infty} \ll_n \frac{\|\boldsymbol{t}\|_\infty}{|\lambda_{c,b}(\boldsymbol{x})|}$$

and substituting these bounds into (4.45) gives

$$\boldsymbol{Y}^{(j)T} H_c(\boldsymbol{t}) \boldsymbol{Y}^{(i)} \ll_n \frac{\|J_{\boldsymbol{\Delta}^{(c,b+1)}}(\boldsymbol{x}^{(0)})\boldsymbol{t}\|_\infty}{\|\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}^{(0)})\|_\infty} + \frac{|\lambda_{c,b+1}(\boldsymbol{x}^{(0)})| \cdot \|\boldsymbol{t}\|_\infty}{|\lambda_{c,b}(\boldsymbol{x}^{(0)})|} \tag{4.46}$$

where the $\boldsymbol{Y}^{(k)}$ are as in (4.43).

The idea is now to let Y be the span of the $\boldsymbol{Y}^{(k)}$ and deduce (4.31) from (4.46). As we are looking for an $(n-b)$-dimensional space Y we will need $\boldsymbol{Y}^{(1)}, \ldots, \boldsymbol{Y}^{(n-b)}$ to be linearly independent. In order to prove (4.31) we require the following slightly stronger statement. We claim there are $L^{(i)}$, $M^{(i)}$, $\boldsymbol{y}^{(i)}(\boldsymbol{x})$ and $\boldsymbol{Y}^{(i)}$ satisfying (4.41)–(4.43), such that the linear combination defined by $\boldsymbol{Y} = \sum_{i=1}^{n-b} \gamma_i \boldsymbol{Y}^{(i)}$ satisfies $\|\boldsymbol{\gamma}\|_\infty \ll_n \|\boldsymbol{Y}\|_\infty$ for every vector $\boldsymbol{\gamma}$ in real $(n-b)$-space. The lemma then follows, with Y being the span of the $\boldsymbol{Y}^{(i)}$, on expressing $\boldsymbol{Y}, \boldsymbol{Y}'$ as linear combinations of the $\boldsymbol{Y}^{(i)}$ and applying (4.46).

For the remainder of the proof we will assume for simplicity that the $b \times b$ minor of $H_c(\boldsymbol{x}^{(0)})$ with largest absolute value is the minor in the lower right-hand corner, that is, we will assume that

$$\|\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}^{(0)})\|_\infty = \left|\det\left((H_c(\boldsymbol{x}^{(0)})_{k\ell})_{\substack{k=n-b+1,\ldots,n\\ \ell=n-b+1,\ldots,n}}\right)\right|. \tag{4.47}$$

In general (4.47) holds after permuting the rows and columns of the matrix $H_c(\boldsymbol{x})$ and one can then apply the same permutations throughout the rest of our construction of $\boldsymbol{Y}^{(i)}$, every time the matrix $H_c(\boldsymbol{x})$ appears.

Define $\boldsymbol{y}^{(1)}(\boldsymbol{x}), \ldots, \boldsymbol{y}^{(n-b)}(\boldsymbol{x})$ by

$$y_j^{(i)}(\boldsymbol{x}) = \begin{cases} (-1)^{n-b} \det\left((H_c(\boldsymbol{x})_{k\ell})_{\substack{k=n-b+1,\ldots,n\\ \ell=n-b+1,\ldots,n}}\right) & \text{if } j = i, \\ (-1)^{j} \det\left((H_c(\boldsymbol{x})_{k\ell})_{\substack{k=n-b+1,\ldots,n\\ \ell=i,n-b+1,\ldots,n;\ \ell\neq j}}\right) & \text{if } j > n-b, \\ 0 & \text{otherwise} \end{cases}$$

where $(\ell = i, n-b+1, \ldots, n;\ \ell \neq j)$ means that ℓ first takes the value i and then runs over the numbers $n-b+1, \ldots, n$ with j omitted. Now this is of the form (4.41), and one can check that

$$(H_c(\boldsymbol{x})\boldsymbol{y}^{(i)}(\boldsymbol{x}))_j = \begin{cases} (-1)^{n-b} \det\left((H_c(\boldsymbol{x})_{k\ell})_{\substack{k=j,n-b+1,\ldots,n\\ \ell=i,n-b+1,\ldots,n}}\right) & \text{if } j \leq n-b \\ 0 & \text{otherwise} \end{cases}$$

which is of the form (4.42). Define a matrix Q by

$$Q = \left(\boldsymbol{Y}^{(1)} \,\middle|\, \cdots \,\middle|\, \boldsymbol{Y}^{(n-b)} \,\middle|\, \boldsymbol{e}^{(n-b+1)} \,\middle|\, \cdots \,\middle|\, \boldsymbol{e}^{(n)}\right),$$

or equivalently by

$$Q = \left(\frac{\boldsymbol{y}^{(1)}(\boldsymbol{x}^{(0)})}{\|\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}^{(0)})\|_\infty} \,\middle|\, \cdots \,\middle|\, \frac{\boldsymbol{y}^{(n-b)}(\boldsymbol{x}^{(0)})}{\|\boldsymbol{\Delta}^{(c,b)}(\boldsymbol{x}^{(0)})\|_\infty} \,\middle|\, \boldsymbol{e}^{(n-b+1)} \,\middle|\, \cdots \,\middle|\, \boldsymbol{e}^{(n)}\right),$$

so that the entries Q_{ij} have absolute value at most 1. Then one sees from (4.47) that

$$Q = \begin{pmatrix} I_{n-b} & 0_{b\times b} \\ \widetilde{Q} & I_b \end{pmatrix},$$

where $\widetilde{Q}$ is some $(n-b)\times(n-b)$ matrix. In particular $\det Q = 1$, and so the entries of Q^{-1} are bounded in terms of n. It follows that if $\boldsymbol{Y} = \sum_{i=1}^{n-b} \gamma_i \boldsymbol{Y}^{(i)}$ then $\gamma_i = (Q^{-1}\boldsymbol{Y})_i \ll_n \|\boldsymbol{Y}\|_\infty$, as claimed. □

4.4.4 Proof of Proposition 4.1

Corollary 4.8 shows that either $N_c^{\text{aux}}(B)$ is small, or there are spaces X, Y of large dimension on which $\boldsymbol{Y}^T H_c(\boldsymbol{X})\boldsymbol{Y}'$ is small. To complete the proof of Proposition 4.1 we show that the second alternative implies that c is singular. This is our analogue of Davenport's step (4), as described in §4.3.

Proof of Proposition 4.1. At the start of this chapter we proved the proposition in the case when $d = 2$ holds. Let $d = 3$, fix some $B \geq 1$ and suppose for a contradiction that the result is false. Then for every $N \in \mathbb{N}$ there is $c_N \in \mathcal{K}$ with

$$N_{c_N}^{\text{aux}}(B) \geq N B^{n+\sigma_{\mathcal{K}}} (\log B)^n.$$

We cannot have $c_N = 0$ since $N_0^{\text{aux}}(B) = 0$. So by Corollary 4.8, there are linear subspaces X_N, Y_N of $\mathbb{R}^n$ satisfying

$$\dim X_N + \dim Y_N = n + \sigma_{\mathcal{K}} + 1,$$

such that for all $\boldsymbol{X} \in X_N$ and $\boldsymbol{Y}, \boldsymbol{Y}' \in Y_N$, we have

$$|\boldsymbol{Y}^T H_{c_N}(\boldsymbol{X})\boldsymbol{Y}'| \leq N^{-1} \|\boldsymbol{Y}\|_\infty \|\boldsymbol{X}\|_\infty \|\boldsymbol{Y}'\|_\infty.$$

If we multiply c_N by a constant then the matrix $H_{c_N}(\boldsymbol{x})$ does not change. So we may assume that for each N the equality $\|c_N\|_\infty = 1$ holds. After passing to a subsequence we have $c_N \to c$ as $N \to \infty$, and it follows that there are subspaces X, Y of $\mathbb{R}^n$ such that $\dim X + \dim Y = n + \sigma_{\mathcal{K}} + 1$ and

$$\boldsymbol{Y}^T H_c(\boldsymbol{X})\boldsymbol{Y}' = 0 \qquad \text{for all } \boldsymbol{X} \in X,\ \boldsymbol{Y}, \boldsymbol{Y}' \in Y. \tag{4.48}$$

Let $b \in \{0, \ldots, n - \sigma - 1\}$ such that

$$\dim X = n - b, \qquad \dim Y = \sigma_{\mathcal{K}} + b + 1.$$

Let $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ be a basis of $\mathbb{R}^n$ such that $\boldsymbol{x}^{(b+1)}, \ldots, \boldsymbol{x}^{(n)}$ is a basis of X.

Let $[Y]$ be the projective linear space in $\mathbb{P}^{n-1}_{\mathbb{C}}$ associated to Y. Take homogeneous coordinates $\boldsymbol{y}$ on $[Y]$, so that $\boldsymbol{y}$ takes values in Y.

Let V be the projective variety cut out in $[Y]$ by the b equations

$$\boldsymbol{y}^T H_c(\boldsymbol{x}^{(i)})\boldsymbol{y} = 0 \qquad (i = 1, \ldots, b), \tag{4.49}$$

so that

$$\dim V \geq \dim[Y] - b = \sigma_{\mathcal{K}}.$$

We claim that V is contained in $\mathrm{Sing}(c)$. It follows that $\dim \mathrm{Sing}(c) \geq \sigma_{\mathcal{K}}$, which is a contradiction, by (4.1).

Now (4.48) implies that for every $\boldsymbol{y} \in Y$ we have

$$\boldsymbol{y}^T H_c(\boldsymbol{x}^{(i)})\boldsymbol{y} = 0 \qquad (i = b + 1, \ldots, n).$$

So if we let $\boldsymbol{y} \in Y$ such that (4.49) holds, then we have

$$\boldsymbol{y}^T H_c(\boldsymbol{x})\boldsymbol{y} = 0 \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n.$$

This implies that $\boldsymbol{\nabla}_{\boldsymbol{y}} c(\boldsymbol{y}) = \boldsymbol{0}$ holds, by the definition (4.6). It follows that every point of V is contained in $\mathrm{Sing}(c)$, as claimed. □

Chapter 5

Higher degree systems in general position

In Chapter 3 we saw that to apply the circle method it suffices to have a strong enough upper bound for the counting function $N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B)$ from Definition 1.6. In Chapter 4 we handled this counting function for systems of quadratic or cubic forms. In this section we treat the case of higher degrees. We then complete the proof of Theorem 1.4, describing the integral zeroes of suitably nonsingular systems of forms.

Our work in this chapter will involve a quantity $\sigma^*(\boldsymbol{H})$ defined as follows.

Definition 5.1. Suppose that $f(\boldsymbol{x})$ is a polynomial of degree d in n variables, and that $d \geq 2$. We let $\boldsymbol{m}^{(f)}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d-1)})$ be the vector of polynomials defined by (1.27) and (1.28). Then we write $J^{(f)}_{\boldsymbol{m}}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d-1)})$ for the $n \times (d-1)n$ Jacobian matrix of $\boldsymbol{m}^{(f)}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d-1)})$, that is,

$$J^{(f)}_{\boldsymbol{m}}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d-1)}) = \left(\frac{\partial \boldsymbol{m}^{(f)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})}{\partial x^{(1)}_1} \middle| \frac{\partial \boldsymbol{m}^{(f)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})}{\partial x^{(1)}_2} \middle| \cdots \middle| \frac{\partial \boldsymbol{m}^{(f)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})}{\partial x^{(d-1)}_{n-1}} \middle| \frac{\partial \boldsymbol{m}^{(f)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})}{\partial x^{(d-1)}_n} \right). \tag{5.1}$$

If $\boldsymbol{H}(\boldsymbol{x})$ is a system of R homogeneous polynomials of the same degree d in n variables, with coefficients in a field $\mathbb{F}$, then we set

$$\sigma^*(\boldsymbol{H}) = n - \min_{\boldsymbol{\beta} \in \bar{\mathbb{F}}^R \setminus \{\boldsymbol{0}\}} \min_{\substack{\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)} \in \bar{\mathbb{F}}^n \setminus \{\boldsymbol{0}\} \\ \boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{H})}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)}) = \boldsymbol{0}}} \operatorname{rank} J^{(\boldsymbol{\beta}\cdot\boldsymbol{H})}_{\boldsymbol{m}}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d-1)}) \tag{5.2}$$

where $\bar{\mathbb{F}}$ is an algebraic closure of $\mathbb{F}$.

In §5.2.2 we prove a bound for the counting function $N^{\mathrm{aux}}_{f}(B)$ from Definition 1.6 in terms of this quantity $\sigma^*(\boldsymbol{f}^{[d]})$.

Proposition 5.2. *For all $\boldsymbol{\beta} \in \mathbb{R}^R$ and all $B \geq 1$ we have*

$$N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B) \ll_{\boldsymbol{f}} B^{(d-2)n+\sigma^*(\boldsymbol{f}^{[d]})}(\log B)^{d-1}.$$

The following result, proved in §5.2, shows that $\sigma^*(\boldsymbol{H})$ is typically quite small.

Proposition 5.3. *Suppose that $n \geq R$ holds. We may consider the space of R-tuples of homogeneous degree d forms in n variables as an affine space defined over $\mathbb{Q}$. The condition that $\sigma^*(\boldsymbol{H}) \leq R-1$ defines a nonempty Zariski open set $U_{d,n,R}$ in this space.*

5.1 Deduction of Theorem 1.4

We use Propositions 5.2 and 5.3 to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $\boldsymbol{F}$ be as in the theorem. If $d \geq 4$ then Proposition 5.2 shows that for all $\epsilon > 0$, $B \geq 1$ and $\boldsymbol{\beta} \in \mathbb{R}^R$ we have

$$N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(B) \ll_{\boldsymbol{F},\epsilon} B^{(d-2)n+\sigma^*(\boldsymbol{f}^{[d]})+\epsilon} \tag{5.3}$$

while Proposition 5.3 and the assumption (1.12) show that we have

$$n - \sigma^*(\boldsymbol{F}) \geq d2^d R + 1. \tag{5.4}$$

In particular the forms F_i must be linearly independent, or else we would have $\sigma^*(\boldsymbol{F}) = n$. We can now complete the proof along identical lines to the proof of Theorem 1.5 in §4.1, with (5.3) in place of (4.4) and (5.4) in place of (4.3). In particular, when $\boldsymbol{F}$ is nonsingular this proves the Hasse principle and weak approximation for $V(\boldsymbol{F})$, and the Manin-Peyre conjecture is satisfied by Lemma 2.2.

Suppose next that $d = 2$ or 3. Theorem 1.5 and Lemma 2.2 imply the result if

$$n - \sigma_{\mathbb{R}}(\boldsymbol{F}) > d2^d R,$$

where $\sigma_{\mathbb{R}}(\boldsymbol{F})$ is as in (1.14). Since $\boldsymbol{F}$ is nonsingular, this follows from (1.15) from the introduction combined with our assumption (1.12). □

5.2 Counting solutions to the auxiliary inequality

We now prove Proposition 5.2. We begin with a technical lemma.

5.2.1 Finding spaces on which the Jacobian is large

The following result shows that, provided $\sigma^*(\boldsymbol{f}^{[d]})$ is small, then for every point where $\|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})\|_\infty$ is small, there are many ways in which we can perturb the variables $\boldsymbol{x}^{(i)}$ such that $\|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}\|_\infty$ increases rapidly.

Lemma 5.4. *Let $J_{\boldsymbol{m}}^{(f)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})$ and σ^* be as in Definition 5.1. Suppose that $\boldsymbol{\beta}\in\mathbb{R}^R\setminus\{\boldsymbol{0}\}$ and that $\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)}\in\mathbb{R}^n\setminus\{\boldsymbol{0}\}$. Then one of the following two alternatives holds: either we have*

$$\|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})\|_\infty \gg_{\boldsymbol{f}} \|\boldsymbol{\beta}\|_\infty\|\boldsymbol{x}^{(1)}\|_\infty\cdots\|\boldsymbol{x}^{(d-1)}\|_\infty,$$

or else there exist linear subspaces $U_1,\dots,U_{d-1}$ of $\mathbb{R}^n$, satisfying

$$\dim U_1+\dots+\dim U_{d-1} = n-\sigma^*(\boldsymbol{f}^{[d]}),$$

such that for all $\boldsymbol{u}^{(1)}\in U_1,\dots,\boldsymbol{u}^{(d-1)}\in U_{d-1}$, we have

$$\left\|J_{\boldsymbol{m}}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})\begin{pmatrix}\boldsymbol{u}^{(1)}\\ \hline \vdots \\ \hline \boldsymbol{u}^{(d-1)}\end{pmatrix}\right\|_\infty \gg_{\boldsymbol{f}} \|\boldsymbol{\beta}\|_\infty\|\boldsymbol{x}^{(1)}\|_\infty\cdots\|\boldsymbol{x}^{(d-1)}\|_\infty$$

Furthermore we may take the spaces U_i to be spanned by standard basis vectors of $\mathbb{R}^n$.

Proof. Suppose that $\boldsymbol{\gamma}\in\mathbb{R}^R$ and $\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(d-1)}\in\mathbb{R}^n$ such that

$$\|\boldsymbol{\gamma}\|_\infty = \|\boldsymbol{z}^{(1)}\|_\infty = \dots = \|\boldsymbol{z}^{(d-1)}\|_\infty = 1$$

holds. We will show that either

$$\|\boldsymbol{m}^{(\boldsymbol{\gamma}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(d-1)})\|_\infty \gg_{\boldsymbol{f}} 1,$$

or else

$$\|J_{\boldsymbol{m}}^{(\boldsymbol{\gamma}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(d-1)})\boldsymbol{u}\|_\infty \gg_{\boldsymbol{f}} \|\boldsymbol{u}\|_\infty \qquad \text{for all } \boldsymbol{u}\in U, \tag{5.5}$$

for some $(n-\sigma^*(\boldsymbol{f}^{[d]}))$-dimensional linear subspace U of $\mathbb{R}^{(d-1)n}$ spanned by standard basis vectors of $(d-1)n$-dimensional space. Once we have shown this, the result will follow on writing

$$\boldsymbol{\gamma}=\boldsymbol{\beta}/\|\boldsymbol{\beta}\|_\infty, \qquad \boldsymbol{z}^{(i)}=\boldsymbol{x}^{(i)}/\|\boldsymbol{x}^{(i)}\|_\infty,$$

$$U=U_1\times\dots\times U_{d-1} \qquad \boldsymbol{u}=\begin{pmatrix}\boldsymbol{u}^{(1)}/\|\boldsymbol{x}^{(1)}\|_\infty\\ \hline \vdots \\ \hline \boldsymbol{u}^{(d-1)}/\|\boldsymbol{x}^{(d-1)}\|_\infty\end{pmatrix}.$$

Let $C \geq 1$ and apply Lemma 2.6 with $k = n - \sigma^*(\boldsymbol{f}^{[d]})$ and $J_{\boldsymbol{m}}^{(\boldsymbol{\gamma}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)}, \ldots, \boldsymbol{z}^{(d-1)})$ as the matrix M. This shows that either

$$\|J_{\boldsymbol{m}}^{(\boldsymbol{\gamma}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)}, \ldots, \boldsymbol{z}^{(d-1)})\boldsymbol{u}\|_\infty \geq C^{-1}\|\boldsymbol{u}\|_\infty \qquad \text{for all } \boldsymbol{u} \in U,$$

for some $(n - \sigma^*(\boldsymbol{f}^{[d]}))$-dimensional linear subspace U of $\mathbb{R}^{(d-1)n}$ spanned by standard basis vectors, or else there is a $(1+\sigma^*(\boldsymbol{f}^{[d]}))$-dimensional linear subspace X of $\mathbb{R}^{(d-1)n}$ such that

$$\|J_{\boldsymbol{m}}^{(\boldsymbol{\gamma}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)}, \ldots, \boldsymbol{z}^{(d-1)})\boldsymbol{X}\|_\infty \leq C^{-1}\|\boldsymbol{X}\|_\infty \qquad \text{for all } \boldsymbol{X} \in X. \tag{5.6}$$

Suppose for a contradiction that (5.5) is false for every U satisfying the required conditions. Then for each $C \geq 1$ there exist vectors $\boldsymbol{\gamma}, \boldsymbol{z}^{(1)}, \ldots, \boldsymbol{z}^{(d-1)}$ with unit norm, and a space X with dimension $1 + \sigma^*(\boldsymbol{f}^{[d]})$, satisfying (5.6). Passing to a convergent subsequence, we find vectors $\boldsymbol{\gamma}, \boldsymbol{z}^{(1)}, \ldots, \boldsymbol{z}^{(d-1)}$ with unit norm and a space X with dimension $(d-2)n + 1 + \sigma^*(\boldsymbol{f}^{[d]})$, such that

$$J_{\boldsymbol{m}}^{(\boldsymbol{\gamma}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)}, \ldots, \boldsymbol{z}^{(d-1)})\boldsymbol{X} = \boldsymbol{0} \qquad \text{for all } \boldsymbol{X} \in X.$$

In other words, the matrix $J_{\boldsymbol{m}}^{(\boldsymbol{\gamma}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)}, \ldots, \boldsymbol{z}^{(d-1)})$ has rank $n - \sigma^*(\boldsymbol{f}^{[d]}) - 1$ or less. But this is impossible, by the definition (5.2) of the quantity σ^*. This proves the result. □

5.2.2 Proof of Proposition 5.2

We use Lemma 5.4 to bound the counting function $N_{\boldsymbol{\beta}\cdot\boldsymbol{f}}^{\mathrm{aux}}(B)$ by covering the set of solutions to the auxiliary inequality (1.29) with a collections of boxes of controlled size.

Proof of Proposition 5.2. If $\boldsymbol{\beta} = \boldsymbol{0}$ then $N_{\boldsymbol{\beta}\cdot\boldsymbol{f}}^{\mathrm{aux}}(B) = 0$ and the result is trivial. Let $\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\boldsymbol{0}\}$. For each $T_1, \ldots, T_{d-1} \geq 1$, define

$$\begin{aligned} &Z(T_1, \ldots, T_{d-1}) \\ &\quad = \Big\{ (\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \in (\mathbb{Z}^n)^{d-1} : T_i \leq \|\boldsymbol{x}^{(i)}\|_\infty \leq 2T_i \quad (i = 1, \ldots, d-1) \\ &\qquad\qquad \|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})\|_\infty \leq \|\boldsymbol{\beta}\|_\infty B^{d-2} \Big\}, \end{aligned}$$

so that

$$N_{\boldsymbol{\beta}\cdot\boldsymbol{f}}^{\mathrm{aux}}(B) \leq 1 + \sum_{\substack{t_1, \ldots, t_{d-1} \in \mathbb{N} \\ t_i < \log_2 B}} \# Z(2^{t_1}, \ldots, 2^{t_{d-1}}). \tag{5.7}$$

Let C_1 be a positive real number which is sufficiently large in terms of $\boldsymbol{f}$. The trivial bound $\#K(T_1, \ldots, T_{d-1}) \ll_n T_1^n \cdots T_{d-1}^n$ gives

$$\sum_{\substack{t_1,\ldots,t_{d-1}\in\mathbb{N} \\ t_1+\cdots+t_{d-1}<\log_2 C_1 B^{d-2}}} \#Z(2^{t_1}, \ldots, 2^{t_{d-1}}) \ll_{d,n} B^{(d-2)n} (\log C_1 B)^{d-1},$$

and substituting this into (5.7) gives

$$N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B) \ll_{d,n,C_1} B^{(d-2)n} (\log B)^{d-1} + \sum_{\substack{t_1,\ldots,t_{d-1}\in\mathbb{N} \\ t_i<\log_2 B \\ t_1+\cdots+t_{d-1}\geq\log_2 C_1 B^{d-2}}} \#Z(2^{t_1}, \ldots, 2^{t_{d-1}}). \tag{5.8}$$

For the remainder of the proof, we will let $T_1, \ldots, T_{d-1} \in (0, B)$ such that

$$T_1 \cdots T_{d-1} \geq C_1 B^{d-2}, \tag{5.9}$$

and we will prove that

$$\#Z(T_1, \ldots, T_{d-1}) \ll_{\boldsymbol{f}} B^{(d-2)n+R-1} \left(\frac{T_1 \cdots T_{d-1}}{B^{d-1}} \right)^{R-1}. \tag{5.10}$$

Substituting (5.10) into (5.8) will then prove the proposition.

We claim that for each $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \in Z(T_1, \ldots, T_{d-1})$, there exist linear subspaces $U_1, \ldots, U_{d-1}$ of $\mathbb{R}^n$, spanned by standard basis vectors of n-space, such that $\dim U_1 + \cdots + \dim U_{d-1} = n - \sigma^*(\boldsymbol{f}^{[d]})$ and

$$\left\| J^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}_{\boldsymbol{m}}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \begin{pmatrix} \boldsymbol{u}^{(1)} \\ \vdots \\ \boldsymbol{u}^{(d-1)} \end{pmatrix} \right\|_\infty \gg_{\boldsymbol{f}} \|\boldsymbol{\beta}\|_\infty T_1 \cdots T_{d-1} \max_{i=1,\ldots,d-1} \frac{\|\boldsymbol{u}^{(i)}\|_\infty}{T_i} \tag{5.11}$$

for all $\boldsymbol{u}^{(i)} \in U_i$. Indeed, if $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \in Z(T_1, \ldots, T_{d-1})$, then

$$\|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})\|_\infty \leq \|\boldsymbol{\beta}\|_\infty B^{d-2},$$

and by (5.9) it follows that

$$\|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})\|_\infty \leq C_1^{-1} \|\boldsymbol{\beta}\|_\infty T_1 \cdots T_{d-1}.$$

In particular,

$$\|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})\|_\infty \leq C_1^{-1} \|\boldsymbol{\beta}\|_\infty \|\boldsymbol{x}^{(1)}\|_\infty \cdots \|\boldsymbol{x}^{(d-1)}\|_\infty,$$

and since we took $C_1 \gg_{\boldsymbol{f}} 1$ sufficiently large, we can apply Lemma 5.4 to give us spaces U_i satisfying the required conditions.

Fix some particular U_i, and fix integral n-vectors $\boldsymbol{v}^{(1)},\ldots,\boldsymbol{v}^{(d-1)}$ satisfying $T_i \le \|\boldsymbol{v}^{(i)}\|_\infty \le 2T_i$ such that every $\boldsymbol{v}^{(i)}$ lies in the orthogonal complement of U_i. We then define $Z^*(T_1,\ldots,T_{d-1})$ to be the subset of $Z(T_1,\ldots,T_{d-1})$ containing those $(d-1)$-tuples $(\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(d-1)})$ which satisfy the bound (5.11) for all $\boldsymbol{u}^{(i)} \in U_i$, and for which $\boldsymbol{x}^{(i)} - \boldsymbol{v}^{(i)} \in U_i$ for each i. We claim that

$$\#Z^*(T_1,\ldots,T_{d-1}) \ll_{\boldsymbol{f}} \left(\frac{B^{d-2}}{T_1\cdots T_{d-1}}\right)^{n-\sigma^*(\boldsymbol{f}^{[d]})} T_1^{\dim U_1}\cdots T_{d-1}^{\dim U_{d-1}}. \tag{5.12}$$

Every point in the set $Z(T_1,\ldots,T_{d-1})$ lies in $Z^*(T_1,\ldots,T_{d-1})$ for some choice of the parameters U_i and $\boldsymbol{v}^{(i)}$. There are $O_{d,n}(1)$ choices for the spaces U_i, and for each one of these choices there are $O_{d,n}(T_1^{n-\dim U_1}\cdots T_{d-1}^{n-\dim U_{d-1}})$ possibilities for the vectors $\boldsymbol{v}^{(i)}$, so by summing over all the possibilities we see that (5.12) implies

$$\#Z(T_1,\ldots,T_{d-1}) \ll_{\boldsymbol{f}} \left(\frac{B^{(d-2)}}{T_1\cdots T_{d-1}}\right)^{n-\sigma^*(\boldsymbol{f}^{[d]})} T_1^{n}\cdots T_{d-1}^{n}$$

which is the desired conclusion (5.10).

Let $(\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(d-1)}),(\boldsymbol{y}^{(1)},\ldots,\boldsymbol{y}^{(d-1)}) \in Z^*(T_1,\ldots,T_{d-1})$ and for each i write $\boldsymbol{u}^{(i)} = \boldsymbol{y}^{(i)} - \boldsymbol{x}^{(i)}$, so that $\boldsymbol{u}^{(i)}$ is an integral vector lying in U_i. We suppose that

$$\|\boldsymbol{u}^{(i)}\|_\infty \le C_1^{-1}T_i \qquad \text{for all } i=1,\ldots,d-1, \tag{5.13}$$

where C_1 is the sufficiently large constant from our assumption (5.9), and we will show that

$$\|\boldsymbol{u}^{(i)}\|_\infty \ll_{\boldsymbol{f}} \frac{B^{d-2}T_i}{T_1\cdots T_{d-1}} \qquad \text{for all } i=1,\ldots,d-1. \tag{5.14}$$

From this it will follow that any box of the form

$$A(\boldsymbol{\zeta}) = \Big\{(\boldsymbol{\xi}^{(1)},\ldots,\boldsymbol{\xi}^{(d-1)}) \in (\mathbb{R}^n)^{d-1} : \text{for each } i=1,\ldots,d-1 \text{ there are } \boldsymbol{\nu}^{(i)} \in U_i \text{ such that } \boldsymbol{\xi}^{(i)} = \boldsymbol{\zeta}^{(i)} + \boldsymbol{\nu}^{(i)} \text{ and } \|\boldsymbol{\nu}^{(i)}\|_\infty \le C_1^{-1}T_i\Big\}$$

will satisfy

$$\#\big\{A(\boldsymbol{\zeta}) \cup Z^*(T_1,\ldots,T_{d-1})\big\} \ll_{\boldsymbol{f}} \left(\frac{B^{(d-2)}}{T_1\cdots T_{d-1}}\right)^{n-R+1} T_1^{\dim U_1}\cdots T_{d-1}^{\dim U_{d-1}}.$$

We need at most $O_{\boldsymbol{f}}(1)$ such boxes to cover all of $Z(T_1,\ldots,T_{d-1})$, so this implies our claim (5.12).

It remains to prove (5.14). We have

$$\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{y}^{(1)},\ldots,\boldsymbol{y}^{(d-1)}) = \boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}+\boldsymbol{u}^{(1)},\ldots,\boldsymbol{x}^{(d-1)}+\boldsymbol{u}^{(d-1)}),$$

and we will expand the right-hand side as a sum of terms of the type

$$\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{u}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(d-1)}),$$
$$\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \boldsymbol{x}^{(3)}, \ldots, \boldsymbol{x}^{(d-1)}),$$

and so on. That is, each term is equal to the system $\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}$ evaluated at a $(d-1)$-tuple of vectors, where we may take either $\boldsymbol{x}^{(i)}$ or $\boldsymbol{u}^{(i)}$ for the ith vector in the $(d-1)$-tuple. After grouping the terms together according to the number of vectors $\boldsymbol{u}^{(i)}$ occurring in the argument of $\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}$, this gives

$$\begin{aligned}
&\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{y}^{(1)}, \ldots, \boldsymbol{y}^{(d-1)}) \\
&\quad = \boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \\
&\qquad + J_{\boldsymbol{m}}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \begin{pmatrix} \boldsymbol{u}^{(1)} \\ \hline \vdots \\ \hline \boldsymbol{u}^{(d-1)} \end{pmatrix} \\
&\qquad + O_{\boldsymbol{f}}\Bigg(\|\boldsymbol{\beta}\|_\infty \sum_{1 \le i_1 < i_2 \le d-1} \|\boldsymbol{u}^{(i_1)}\|_\infty \cdot \|\boldsymbol{u}^{(i_1)}\|_\infty \prod_{k \ne i_1, i_2} T_k \Bigg) \\
&\qquad + \cdots + O_{\boldsymbol{f}}\Bigg(\|\boldsymbol{\beta}\|_\infty \sum_{1 \le i_1 < \cdots < i_{d-2} \le d-1} \|\boldsymbol{u}^{(i_1)}\|_\infty \cdots \|\boldsymbol{u}^{(i_{d-2})}\|_\infty \prod_{k \ne i_1, \ldots, i_{d-2}} T_k \Bigg).
\end{aligned} \tag{5.15}$$

By (5.13), the total error in (5.15) is

$$O_{\boldsymbol{f}}\Big(C_1^{-1} \|\boldsymbol{\beta}\|_\infty T_1 \cdots T_{d-1} \max_{i=1,\ldots,d-1} \frac{\|\boldsymbol{u}^{(i)}\|_\infty}{T_i} \Big).$$

In addition, as $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})$ and $(\boldsymbol{y}^{(1)}, \ldots, \boldsymbol{y}^{(d-1)})$ belong to $Z(T_1, \ldots, T_{d-1})$ we have

$$\|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})\|_\infty,\ \|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{y}^{(1)}, \ldots, \boldsymbol{y}^{(d-1)})\|_\infty \le \|\boldsymbol{\beta}\|_\infty B^{d-2}.$$

So by (5.15),

$$\begin{aligned}
&J_{\boldsymbol{m}}^{(\boldsymbol{\beta}\cdot\boldsymbol{f})}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \begin{pmatrix} \boldsymbol{u}^{(1)} \\ \hline \vdots \\ \hline \boldsymbol{u}^{(d-1)} \end{pmatrix} \\
&\qquad \ll_{\boldsymbol{f}} \|\boldsymbol{\beta}\|_\infty B^{d-2} + C_1^{-1} \|\boldsymbol{\beta}\|_\infty T_1 \cdots T_{d-1} \max_{i=1,\ldots,d-1} \frac{\|\boldsymbol{u}^{(i)}\|_\infty}{T_i}.
\end{aligned}$$

By (5.11) this implies that

$$\begin{aligned}
&\|\boldsymbol{\beta}\|_\infty T_1 \cdots T_{d-1} \max_{i=1,\ldots,d-1} \frac{\|\boldsymbol{u}^{(i)}\|_\infty}{T_i} \\
&\qquad \ll_{\boldsymbol{f}} \|\boldsymbol{\beta}\|_\infty B^{d-2} + C_1^{-1} \|\boldsymbol{\beta}\|_\infty T_1 \cdots T_{d-1} \max_{i=1,\ldots,d-1} \frac{\|\boldsymbol{u}^{(i)}\|_\infty}{T_i}.
\end{aligned}$$

At the start of the proof we assumed that $\boldsymbol{\beta} \neq \mathbf{0}$ and that $C_1 \gg_{\boldsymbol{f}} 1$, so this implies the conclusion (5.14). □

5.3 Proof of Proposition 5.3

In this section we will prove Proposition 5.3, bounding the quantity $\sigma^*(\boldsymbol{H})$ for typical systems $\boldsymbol{H}$. The strategy is to relate σ^* to the dimension of a certain explicit complex variety W, which we will be able to parametrise.

Proof of Proposition 5.3. If $\boldsymbol{H}$ is a system of degree d forms with coefficients in a field $\mathbb{F}$, we define a subvariety $\Sigma_{\boldsymbol{H}}$ of $\mathbb{P}_{\mathbb{F}}^{R-1} \times (\mathbb{P}_{\mathbb{F}}^{n-1})^{d-1}$ as follows. Taking $\boldsymbol{\beta}$ and the $\boldsymbol{x}^{(i)}$ as homogeneous coordinates on $\mathbb{P}_{\mathbb{F}}^{R-1}$ and $\mathbb{P}_{\mathbb{F}}^{n-1}$ respectively, $\Sigma_{\boldsymbol{H}}$ is cut out by the conditions

$$\begin{aligned} \boldsymbol{m}^{(\boldsymbol{\beta}\cdot\boldsymbol{H})}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)}) &= \mathbf{0}, \\ \operatorname{rank} J_{\boldsymbol{m}}^{(\boldsymbol{\beta}\cdot\boldsymbol{H})}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)}) &\leq n-R. \end{aligned}$$

The condition that $\Sigma_{\boldsymbol{H}}$ be nonempty cuts out a Zariski closed subset, defined over $\mathbb{Q}$, in the space of all systems $\boldsymbol{H}$. We will show that this is a proper subset. This will prove the proposition, because by (5.2) we have $\sigma^*(\boldsymbol{H}) \geq R$ precisely when the variety $\Sigma_{\boldsymbol{H}}$ has a $\overline{K}$-point.

Suppose for a contradiction that $\Sigma_{\boldsymbol{H}}$ is nonempty for every system $\boldsymbol{H}$.

Let $N(d,n)$ be the number of coefficients of a general form of degree d in n variables. The space $\mathbb{P}_{\mathbb{Q}}^{N(d,n)-1}$ parametrises degree d forms in n variables up to multiplication by a constant. Let Σ_0 be the subvariety of $\mathbb{P}_{\mathbb{Q}}^{N(d,n)-1} \times (\mathbb{P}_{\mathbb{Q}}^{n-1})^{d-1}$ defined by the two conditions

$$\begin{aligned} \boldsymbol{m}^{(H)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)}) &= \mathbf{0}, \\ \operatorname{rank} J_{\boldsymbol{m}}^{(H)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)}) &\leq n-R, \end{aligned} \tag{5.16}$$

where the form H represents a point of $\mathbb{P}_{\mathbb{C}}^{N(d,n)-1}$. Given a system $\boldsymbol{H}$ with linearly independent H_i, we can embed the variety $\Sigma_{\boldsymbol{H}}$ into Σ_0 by sending the vector $\boldsymbol{\beta}$ to the form $\boldsymbol{\beta}\cdot\boldsymbol{H}$. The image of this embedding is $\Theta \cap \Sigma_0$, where Θ is the projective linear subspace of $\mathbb{P}_{\mathbb{Q}}^{N(d,n)-1}$ spanned by the H_i. Since every variety $\Sigma_{\boldsymbol{H}}$ is nonempty by assumption, the intersection $\Theta \cap \Sigma_0$ is nonempty for every $(R-1)$-dimensional projective linear space Θ in $\mathbb{P}_{\mathbb{Q}}^{N(d,n)-1}$. So we have

$$\dim \Sigma_0 \geq N(d,n) - R. \tag{5.17}$$

Now let Σ_1 be the subvariety of $\mathbb{P}_{\mathbb{Q}}^{N(d,n)-1} \times (\mathbb{P}_{\mathbb{Q}}^{n-1})^d$ cut out by the conditions

$$\begin{aligned} \boldsymbol{m}^{(H)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) &= \boldsymbol{0}, \\ J_{\boldsymbol{m}}^{(H)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})^T \boldsymbol{x}^{(d)} &= \boldsymbol{0}, \end{aligned} \tag{5.18}$$

where H represents a point of $\mathbb{P}_{\mathbb{Q}}^{N(d,n)-1}$ and $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d)}$ are vectors of homogeneous coordinates on $\mathbb{P}_{\mathbb{Q}}^{n-1}$. Each solution of (5.16) corresponds to an R-dimensional space of vectors $\boldsymbol{x}^{(d)}$ satisfying (5.18). So each point of Σ_0 corresponds to an $(R-1)$-dimensional projective space of points on Σ_1, and by (5.17) we must have

$$\dim \Sigma_1 \geq N(d,n) - 1. \tag{5.19}$$

Consider the map

$$H \mapsto \left(\frac{J_{\boldsymbol{m}}^{(H)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})^T \boldsymbol{x}^{(d)}}{\boldsymbol{m}^{(H)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})} \right), \tag{5.20}$$

where the right-hand side is a vector with $2n$ entries obtained by concatenating two vectors with n entries each. This map is linear in the coefficients of H. Let $L(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d)})$ be the matrix of this linear map, so that $L(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d)})$ is a $(dn) \times N(d,n)$ matrix whose entries are polynomials in the $\boldsymbol{x}^{(i)}$ with rational coefficients. Given a d-tuple $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d)})$, the space of forms H satisfying (5.18) has dimension equal to

$$N(d,n) - \operatorname{rank} L(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d)}).$$

So if we let $\Lambda(k)$ be the subvariety of $(\mathbb{P}_{\mathbb{Q}}^{n-1})^d$ cut out by the condition

$$\operatorname{rank} L(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d)}) = k,$$

then each point on $\Lambda(k)$ corresponds to a $(N(d,n) - k - 1)$-dimensional projective linear space on Σ_1, and hence

$$\dim \Sigma_1 = \max_{k \in \{0, \ldots, dn\}} \dim \Lambda(k) + (N(d,n) - k - 1).$$

In particular, (5.19) implies that for some $k_0 \in \{0, \ldots, dn\}$ we have

$$\dim \Lambda(k_0) \geq k_0. \tag{5.21}$$

Let W be the variety cut out in $(\mathbb{P}_{\mathbb{Q}}^{n-1})^d \times \mathbb{P}_{\mathbb{Q}}^{dn-1}$ by the equation

$$\boldsymbol{w}^T L(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d)}) = \boldsymbol{0}, \tag{5.22}$$

where $\boldsymbol{w}$ is a vector of homogeneous coordinates on $\mathbb{P}^{dn-1}_{\mathbb{Q}}$. If we are given a d-tuple $(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d)})$ representing a point on $\Lambda(k_0)$, then the space of vectors $\boldsymbol{w}$ satisfying (5.22) has dimension

$$dn - k_0.$$

So each point on $\Lambda(k_0)$ corresponds to a $(dn-k_0-1)$-dimensional projective linear space on W, and (5.21) implies that

$$\dim W \geq dn - 1. \tag{5.23}$$

We will show that the complex points $W(\mathbb{C})$ can be parametrised by $dn-2$ complex parameters. By standard results this implies that $\dim W \leq dn-2$, see the remarks at the end of §2.3 in Chapter 2 of Shafarevich [Sha13a].

Let $(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d)},\boldsymbol{w})$ represent a $\mathbb{C}$-point of W. From the definitions (5.20) and (5.22) we see that the expression

$$\begin{aligned}\boldsymbol{w}^T\left(\frac{J^{(H)}_{\boldsymbol{m}}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})^T\boldsymbol{x}^{(d)}}{\boldsymbol{m}^{(H)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})}\right) &= (\boldsymbol{x}^{(d)})^T J^{(H)}_{\boldsymbol{m}}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})\begin{pmatrix} w_1\\ \vdots \\ w_{(d-1)n}\end{pmatrix}\\ &\quad + \boldsymbol{m}^{(H)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})^T\begin{pmatrix} w_{(d-1)n+1}\\ \vdots \\ w_{dn}\end{pmatrix}\end{aligned}$$

must vanish uniformly for all degree d forms H. In the special case when $H(\boldsymbol{x}) = (\boldsymbol{b}\cdot\boldsymbol{x})^d$, so that the form H is a dth power of a linear form, we calculate from the definition of $J^{(H)}_{\boldsymbol{m}}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})$ in (5.1) that this expression is

$$\begin{aligned}(\boldsymbol{b}\cdot\boldsymbol{w}^{(1)})(\boldsymbol{b}\cdot\boldsymbol{x}^{(2)})\cdots(\boldsymbol{b}\cdot\boldsymbol{x}^{(d)}) + (\boldsymbol{b}\cdot\boldsymbol{x}^{(1)})(\boldsymbol{b}\cdot\boldsymbol{w}^{(2)})(\boldsymbol{b}\cdot\boldsymbol{x}^{(3)})\cdots(\boldsymbol{b}\cdot\boldsymbol{x}^{(d)})&\\ + (\boldsymbol{b}\cdot\boldsymbol{x}^{(1)})\cdots(\boldsymbol{b}\cdot\boldsymbol{x}^{(d-1)})(\boldsymbol{b}\cdot\boldsymbol{w}^{(d)}),&\end{aligned}$$

where we split $\boldsymbol{w}$ into d separate n-vectors $\boldsymbol{w}^{(i)}$, given by

$$\boldsymbol{w} = \begin{pmatrix} \boldsymbol{w}^{(1)} \\ \hline \vdots \\ \hline \boldsymbol{w}^{(d)}\end{pmatrix}. \tag{5.24}$$

We may divide through by $(\boldsymbol{b}\cdot\boldsymbol{x}^{(1)})\cdots(\boldsymbol{b}\cdot\boldsymbol{x}^{(d)})$ to see that

$$\frac{\boldsymbol{b}\cdot\boldsymbol{w}^{(1)}}{\boldsymbol{b}\cdot\boldsymbol{x}^{(1)}} + \cdots + \frac{\boldsymbol{b}\cdot\boldsymbol{w}^{(d)}}{\boldsymbol{b}\cdot\boldsymbol{x}^{(d)}} = 0 \tag{5.25}$$

whenever $\boldsymbol{b}\in\mathbb{C}^n$ and all of the denominators $\boldsymbol{b}\cdot\boldsymbol{x}^{(i)} \neq 0$ are nonzero.

Below we will find $m \in \{1, \dots, d\}$, $\boldsymbol{k} \in \{1, \dots, m\}^d$, $\boldsymbol{\lambda} \in (\mathbb{C} \setminus \{0\})^d$, $\boldsymbol{\mu} \in \mathbb{C}^m$ and $\boldsymbol{y}^{(1)}, \dots, \boldsymbol{y}^{(m)} \in \mathbb{C}^n \setminus \{\boldsymbol{0}\}$ such that

$$\sum_{k_j=\ell} \lambda_j = 1 \qquad (\ell = 1, \dots, m), \tag{5.26}$$

$$\mu_1 + \cdots + \mu_m = 0, \tag{5.27}$$

$$\lambda_i \boldsymbol{x}^{(i)} = \boldsymbol{y}^{(k_i)} \qquad (i = 1, \dots, d), \tag{5.28}$$

$$\sum_{k_j=\ell} \lambda_i \boldsymbol{w}^{(j)} = \mu_\ell \boldsymbol{y}^{(\ell)} \qquad (\ell = 1, \dots, m), \tag{5.29}$$

where the $\boldsymbol{w}^{(i)}$ are as in (5.24). Given m and $\boldsymbol{k}$, we have an mn-dimensional space of parameters $(\boldsymbol{y}^{(1)}, \dots, \boldsymbol{y}^{(m)})$ and a $(d-1)$-dimensional space of parameters $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ satisfying (5.26) and (5.27). Having chosen the values of these parameters the value of each $\boldsymbol{x}^{(i)}$ is fixed uniquely by (5.28), and there is a $(d-m)n$-dimensional space of vectors $\boldsymbol{w}$ satisfying (5.29). In total then, the space of possible $(d+1)$-tuples $(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d)}, \boldsymbol{w})$ has dimension at most

$$mn + (d-1) + (d-m)n = dn + d - 1.$$

For any $u_1, \dots, u_d, v \in \mathbb{C} \setminus \{\boldsymbol{0}\}$, the $(d+1)$-tuple $(u_1\boldsymbol{x}^{(1)}, \dots, u_d\boldsymbol{x}^{(d)}, v\boldsymbol{w})$ represents the same point of $W(\mathbb{C})$ as $(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d)}, \boldsymbol{w})$. Consequently $W(\mathbb{C})$ can be parametrised with $dn - 2$ complex parameters, and by the comments after (5.23) this gives a contradiction and proves the proposition.

It remains to find, for each $(d+1)$-tuple $(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d)}, \boldsymbol{w})$ satisfying (5.22), a choice of the parameters $m, \boldsymbol{k}, \boldsymbol{\lambda}, \boldsymbol{mu}$ and $\boldsymbol{y}^{(i)}$ such that the relations (5.26)–(5.29) hold. Define an equivalence relation on the set $\{\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d)}\}$ by saying that $\boldsymbol{x}^{(i)}$ and $\boldsymbol{x}^{(j)}$ are equivalent if they are linearly dependent. Let m be the number of equivalence classes. Number them from 1 to m, and let k_i be the number of the equivalence class to which $\boldsymbol{x}^{(i)}$ belongs. All the vectors in a given equivalence class are multiples of one fixed vector, so there are nonzero scalars $\lambda_1, \dots, \lambda_d$ and nonzero vectors $\boldsymbol{y}^{(1)}, \dots, \boldsymbol{y}^{(m)}$ satisfying (5.28). By replacing each $\boldsymbol{y}^{(\ell)}$ with a scalar multiple of itself if necessary, we may assume that (5.26) holds. It remains to find $\boldsymbol{\mu} \in \mathbb{C}^m$ satisfying (5.27) and (5.29).

Substituting (5.28) into (5.25) shows that

$$\sum_{\ell=1}^{m} \frac{\boldsymbol{b} \cdot \sum_{k_i=\ell} \lambda_i \boldsymbol{w}^{(i)}}{\boldsymbol{b} \cdot \boldsymbol{y}^{(\ell)}} = 0 \tag{5.30}$$

whenever $\boldsymbol{b} \in \mathbb{C}^n$ and none of the denominators $\boldsymbol{b} \cdot \boldsymbol{y}^{(\ell)}$ vanish. Let $\ell_0 \in \{1, \ldots, m\}$, let $\boldsymbol{t} \in \mathbb{C}^n$ and suppose that $\boldsymbol{t} \cdot \boldsymbol{y}^{(\ell_0)} = 0$. Since the inequalities $m \leq d \leq n$ hold, there exist $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n$ satisfying the three conditions

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{y}^{(\ell)} &\neq 0 && \text{for all } \ell \neq \ell_0, && (5.31)\\ \boldsymbol{u} \cdot \boldsymbol{y}^{(\ell_0)} &= 0, && \text{and} \\ \boldsymbol{v} \cdot \boldsymbol{y}^{(\ell_0)} &= 1. && && (5.32) \end{aligned}$$

For some small $\epsilon > 0$ we set

$$\boldsymbol{b} = \boldsymbol{t} + \epsilon \boldsymbol{u} + \epsilon^2 \boldsymbol{v}.$$

Then the conditions (5.31) and (5.32) ensure that

$$\begin{aligned} \boldsymbol{b} \cdot \boldsymbol{y}^{(\ell)} &\gg \epsilon && \text{for all } \ell \neq \ell_0, \text{ and} \\ \boldsymbol{b} \cdot \boldsymbol{y}^{(\ell_0)} &= \epsilon^2. \end{aligned}$$

So (5.30) implies that

$$\epsilon^{-2} \boldsymbol{t} \cdot \sum_{k_i = \ell_0} \lambda_i \boldsymbol{w}^{(i)} = O(\epsilon^{-1}).$$

Letting $\epsilon \to 0$ we see that $\boldsymbol{t} \cdot \sum_{k_i = \ell_0} \lambda_i \boldsymbol{w}^{(i)} = 0$ vanishes. Recall that this holds for any $\ell_0 \in \{1, \ldots, m\}$ and any $\boldsymbol{t} \in \mathbb{C}^n$, provided only that $\boldsymbol{t} \cdot \boldsymbol{y}^{(\ell_0)}$ vanishes. So for each $\ell_0 \in \{1, \ldots, m\}$ there must be some $\mu_{\ell_0} \in \mathbb{C}$ such that

$$\sum_{k_i = \ell_0} \lambda_i \boldsymbol{w}^{(i)} = \mu_{\ell_0} \boldsymbol{y}^{(\ell_0)}.$$

This gives us an m-vector $\boldsymbol{\mu}$ satisfying (5.29). Finally, substituting (5.29) into (5.30) shows that

$$\sum_{\ell=1}^{m} \mu_\ell \frac{\boldsymbol{b} \cdot \boldsymbol{y}^{(\ell)}}{\boldsymbol{b} \cdot \boldsymbol{y}^{(\ell)}} = \mu_1 + \cdots + \mu_m = 0,$$

which proves (5.27). So (5.26)–(5.29) all hold, as required. □

Chapter 6

Diophantine inequalities

In this chapter we prove Theorems 1.8 and 1.9, concerning systems of inequalities

$$\|\boldsymbol{f}(\boldsymbol{x})\|_\infty \le \tfrac{1}{2}P^\rho, \qquad \|\boldsymbol{x}\|_\infty \le P.$$

Theorem 1.9 treats the case $\rho > 0$. The results of the previous chapters would, with some modifications, suffice to handle this problem. When ρ vanishes however, we must use the circle method in the form laid out by Freeman [Fre02] and described in §2.2. In order to do this we will need to assume a bound

$$N^{\mathrm{aux}}_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(B) \le C_0 B^{(d-1)n-2^d\mathscr{C}} \tag{6.1}$$

for all $\boldsymbol{\alpha} \in \mathbb{R}^R$ and $B \ge 1$ and some $C_0 \ge 1$ and $\mathscr{C} > 0$, where $N^{\mathrm{aux}}_f(B)$ is as in Definition 1.6.

Throughout this chapter we let $\boldsymbol{f}$ and $\boldsymbol{f}^{[d]}$ be as in §1.1, we let $S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)$ be as in (2.16) and we let $\boldsymbol{\kappa}$, $\mathcal{S}(\boldsymbol{\kappa}, k)$ and $\mathrm{Poly}_{d,n}(\boldsymbol{\kappa})$ be as in Definition 2.3. As described in that definition, we adopt the convention that all implicit constants in $\ll$ and $O(\,\cdot\,)$ notation may depend on $\boldsymbol{\kappa}$ only by a multiplicative factor $\mathrm{Poly}_{d,n}(\boldsymbol{\kappa})$.

In §6.2.2 we prove a smoothly weighted version of Proposition 3.3 from Chapter 3.

Proposition 6.1. *Let $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$, $C_0 \ge 1$ and $\mathscr{C} > 0$.*

Suppose the leading forms $f^{[d]}_i(\boldsymbol{x})$ are linearly independent, and that the bound (6.1) holds for every $\boldsymbol{\alpha} \in \mathbb{R}^R$ and $B \ge 1$. Then for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$ we have

$$\frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)}{P^n} \ll_{C_0,\boldsymbol{f},\boldsymbol{\kappa}} \max\left\{P^{-d}\|\boldsymbol{\alpha}\|_\infty^{-1},\, \|\boldsymbol{\alpha}\|_\infty^{\frac{1}{d-1}}\right\}^{\mathscr{C}}, \tag{6.2}$$

$$\min\left\{\left|\frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)}{P^n}\right|, \left|\frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}+\boldsymbol{\beta}; P)}{P^n}\right|\right\} \ll_{C_0,\boldsymbol{f},\boldsymbol{\kappa}} \max\left\{P^{-d}\|\boldsymbol{\beta}\|_\infty^{-1},\, \|\boldsymbol{\beta}\|_\infty^{\frac{1}{d-1}}\right\}^{\mathscr{C}}, \tag{6.3}$$

with implicit constants of the form $\mathrm{Poly}_{d,n}(\boldsymbol{\kappa})O_{C_0,\boldsymbol{f},\boldsymbol{\kappa}}(1)$.

We now give the conclusions of Freeman's variant of the Davenport-Heilbronn method, as described in §2.2. The following result is proved in §6.3.3.

Proposition 6.2. *Let $\rho \in [0, d-1]$. Suppose that the $f_i^{[d]}(\boldsymbol{x})$ are linearly independent, and that the bound (6.1) holds for all $\boldsymbol{\alpha} \in \mathbb{R}^R$ and $B \geq 1$ and some $C_0 \geq 1$ and $\mathscr{C} > (d-\rho)2^d R$. Assume that for some $\Delta \in (0, \frac{1}{R+1})$ and some function ξ taking values in $(0,1]$, we have*

$$\sup_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^R \\ P^{\Delta-d}<\|\boldsymbol{\alpha}\|_\infty<\xi(P)^{-1}P^{-\rho}}} |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)| \ll_{\boldsymbol{f}} \mathrm{Poly}_{d,n}(\boldsymbol{\kappa})\xi(P)P^n \tag{6.4}$$

for all $P \geq 1$, all $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$ and all sequences of positive real numbers $\boldsymbol{\kappa}$.

Then for all such $\boldsymbol{\kappa}$ and all $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$, $\varphi \in \mathcal{S}(\boldsymbol{\kappa}, R)$ and $P \geq 1$ we have

$$\begin{aligned}\sum_{\boldsymbol{x}\in\mathbb{Z}^n} \varphi(P^{-\rho}\boldsymbol{f}(\boldsymbol{x}))\omega(\boldsymbol{x}/P) = \int_{\mathbb{R}^n} &\varphi\big(P^{-\rho}\boldsymbol{f}^{[d]}(\boldsymbol{t})\big)\omega(\boldsymbol{t}/P)\,d\boldsymbol{t} \\ &+ \mathrm{Poly}_{d,n}(\boldsymbol{\kappa})O_{C_0,\mathscr{C},\boldsymbol{f}}\big(\xi(P)^{\delta}P^{n-(d-\rho)R} + P^{n-(d-\rho)R-\delta}\big),\end{aligned} \tag{6.5}$$

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi\big(P^{-\rho}\boldsymbol{f}^{[d]}(\boldsymbol{t})\big)\omega(\boldsymbol{t}/P)\,d\boldsymbol{t} &= \hat{\varphi}(\boldsymbol{0})\mathfrak{I}_{\boldsymbol{f},\omega}P^{n-(d-\rho)R} \\ &\quad + \mathrm{Poly}_{d,n}(\boldsymbol{\kappa})O_{C_0,\mathscr{C},\boldsymbol{f}}\big(P^{n-(d-\rho)R-\delta}\big),\end{aligned} \tag{6.6}$$

where $\delta > 0$ depends only on $\mathscr{C}, \Delta, d, R$ and ρ, while $\mathfrak{I}_{\boldsymbol{f},\omega} \geq 0$ depends only on $\boldsymbol{f}^{[d]}$ and ω. Furthermore there is $\delta' > 0$ depending only on $\mathscr{C}, \Delta, d, n, R$ and ρ such that

$$\begin{aligned}\#\{\boldsymbol{x}\in\mathbb{Z}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq \tfrac{1}{2}P^\rho\} \\ = \mathfrak{I}_{\boldsymbol{f}^{[d]}}P^{n-(d-\rho)R} + O_{\mathscr{C},\boldsymbol{f},\rho}(\xi(P)^{\delta'}P^{n-(d-\rho)R} + P^{n-(d-\rho)R-\delta'}),\end{aligned} \tag{6.7}$$

$$\begin{aligned}\lambda\{\boldsymbol{x}\in\mathbb{R}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \|\boldsymbol{f}^{[d]}(\boldsymbol{x})\|_\infty \leq \tfrac{1}{2}P^\rho\} \\ = \mathfrak{I}_{\boldsymbol{f}^{[d]}}P^{n-(d-\rho)R} + O_{\mathscr{C}\,\boldsymbol{f},\rho}(P^{n-(d-\rho)R-\delta'}),\end{aligned} \tag{6.8}$$

for all $P \geq 1$, where $\mathfrak{I}_{\boldsymbol{f}^{[d]}}$ depends only on $\boldsymbol{f}^{[d]}$ and λ is the Lebesgue measure.

To prove the complementary-arc bound (6.4) we will employ Proposition 6.1, together with the following uniform upper bound in the case when $\rho = 0$. This is proved in §6.4.

Proposition 6.3. *Suppose that $\boldsymbol{f}^{[d]}$ is irrational in the sense of Definition 1.7. Then there is a function $\eta : [1,\infty) \to (0,1]$, depending only on $\boldsymbol{f}^{[d]}$, such that $\eta(P) \to 0$ as $P \to \infty$, and*

$$\sup_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^R \\ \eta(P)<\|\boldsymbol{\alpha}\|_\infty<\eta(P)^{-1}}} |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)| \leq \mathrm{Poly}_{d,n}(\boldsymbol{\kappa})\eta(P)P^n \tag{6.9}$$

for all $P \gg_{\boldsymbol{f}^{[d]}} 1$ and $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$. Moreover, if $\boldsymbol{f}^{[d]}(\boldsymbol{x})$ has real algebraic numbers for coefficients, then we can set $\eta(P) = P^{-\delta}$ for some $\delta > 0$ depending only on d and R.

In the case $d = 2$ this reproduces Lemma 3.2 of Müller [Mül05]. For higher degrees there are considerable technical complications. Our main tool is Lemma 2.11 from §2.3.2 in Chapter 2, which relates the quality of rational approximations to the matrix L and the density of integer vectors $\boldsymbol{v}$ at which $L\boldsymbol{v}$ is close to an integral vector.

6.1 Deduction of Theorems 1.8 and 1.9

We combine the propositions above with the work of Chapters 4 and 5.

Proof of Theorems 1.8 and 1.9. Throughout this proof d, n, R and $\boldsymbol{f}$ will be as in §1.1 and ρ will be a real number satisfying $\rho \in [0, d-1]$. We will write δ to stand for a positive real number which depends at most on d, n, R and ρ and which may vary from line to line. We let the smoothness classes $\mathcal{S}(\boldsymbol{\kappa}, k)$ be as in Definition 2.3, and we use the convention for implicit constants described there.

We prove the two theorems simultaneously, in the following way. If $d = 2$ or 3, we suppose that $\boldsymbol{f}^{[d]}$ is nonsingular. If $d \geq 4$ we suppose that $\boldsymbol{f}^{[d]}$ belongs to the Zariski open subset $U_{d,n,R}$ from Proposition 5.3. In either case we suppose that

$$n > (d-\rho)2^d R + R - 1. \tag{6.10}$$

Now if $\rho > 0$ we are in the setting of Theorem 1.9, while if $\rho = 0$ and $\boldsymbol{f}^{[d]}$ is irrational in the sense of Definition 1.7, then we are in the setting of Theorem 1.8. We claim that the hypotheses of Proposition 6.2 are satisfied for

$$\mathscr{C} = (d-\rho)R + \delta, \qquad \Delta = \frac{1}{4R}, \tag{6.11}$$

and some function ξ, depending only on $\boldsymbol{f}^{[d]}$ and ρ, such that

$$\xi(P) = P^{-\delta} \qquad \text{if } \rho > 0, \tag{6.12}$$

$$\xi(P) \to 0 \qquad \text{as } P \to \infty \text{ if } \rho = 0 \text{ and } \boldsymbol{f}^{[d]} \text{ is irrational, and} \tag{6.13}$$

$$\xi(P) = P^{-\delta} \qquad \text{if } \rho = 0 \text{ and } \boldsymbol{f}^{[d]} \text{ is irrational with algebraic coefficients.} \tag{6.14}$$

If $\rho = 0$ then the conclusions (1.33) and (1.34) from Theorem 1.8 follow from (6.7) and (6.8) in Proposition 6.2, while if $\rho > 0$ then the conclusions (1.35) and (1.36) from Theorem 1.9 follow in the same way. Furthermore, if $\boldsymbol{f}^{[d]}(\boldsymbol{x}) = \boldsymbol{0}$ has a nonsingular real solution then the positivity of $\mathfrak{I}_{\boldsymbol{f}^{[d]}}$ follows from (6.8) just as we deduced the positivity of $\mathfrak{I}_{\boldsymbol{f}^{[d]},\mathscr{B}}$ from (3.2) in the proof of Proposition 3.2 in §3.1.4. This will complete the proof of the theorems.

So we must show that the hypotheses of Proposition 6.2 all hold with $\mathscr{C}, \Delta$ and ξ satisfying (6.11)–(6.14). We begin by proving that the $f_i^{[d]}$ are linearly independent, and that the bound (6.1) holds with $\mathscr{C}$ as in (6.11), that is

$$N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B) \ll_{\boldsymbol{f}} B^{(d-1)n-(d-\rho)2^d R-\delta} \tag{6.15}$$

for all $B \geq 1$ and $\boldsymbol{\beta} \in \mathbb{R}^R$. If $d = 2$ or 3, then the $f_i^{[d]}$ are linearly independent since $\boldsymbol{f}^{[d]}$ is nonsingular. Moreover (6.15) will follow from Proposition 4.1 and (6.10) provided that $\sigma_{\mathcal{K}} \leq R-1$, where

$$\sigma_{\mathcal{K}} = 1 + \max_{F\in\mathcal{K}\setminus\{0\}} \dim \operatorname{Sing}(F),$$
$$\mathcal{K} = \{\boldsymbol{\beta}\cdot\boldsymbol{f}^{[d]} : \boldsymbol{\beta}\in\mathbb{R}^R\}.$$

By (1.15) from the introduction we have $\sigma_{\mathcal{K}} \leq R + \dim\operatorname{Sing}(\boldsymbol{f}^{[d]})$, and because $\boldsymbol{f}^{[d]}$ is nonsingular it follows that $\sigma_{\mathcal{K}} \leq R-1$ as required.

If instead $d \geq 4$, then Propositions 5.2 and 5.3 show that for any $B \geq 1$ and any $\epsilon > 0$ we have

$$N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B) \ll_{\boldsymbol{f},\epsilon} B^{(d-2)n+R-1+\epsilon},$$

and by (6.10) it follows that (6.15) holds. Furthermore the $f_i^{[d]}$ must be linearly independent, or else the quantity $\sigma^*(\boldsymbol{f}^{[d]})$ defined in (5.2) would be equal to n, contradicting Proposition 5.3. This proves the claim.

To satisfy the hypotheses of Proposition 6.2, it remains to show that (6.4) holds with Δ given by (6.11) and with some function ξ satisfying (6.12)–(6.14). Observe that by Proposition 6.1 we have for all $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$, all $P \geq 1$ and all $\boldsymbol{\alpha}\in\mathbb{R}^R$ that

$$\frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)}{P^n} \ll_{\boldsymbol{f},\boldsymbol{\kappa}} \max\left\{P^{-d}\|\boldsymbol{\alpha}\|_\infty^{-1},\, \|\boldsymbol{\alpha}\|_\infty^{\frac{1}{d-1}}\right\}^{\mathscr{C}}.$$

It follows that for any $A > 0$ we have

$$\sup_{P^{\frac{1}{4R}-d}\leq\|\boldsymbol{\alpha}\|_\infty\leq A} \frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)}{P^n} \ll_{\boldsymbol{f},\boldsymbol{\kappa}} P^{-\frac{\mathscr{C}}{4R}} + A^{\frac{\mathscr{C}}{d-1}}. \tag{6.16}$$

If $\rho > 0$ then we pick $A = P^{-\rho/2}$ to see that

$$\sup_{P^{\frac{1}{4R}-d}\leq\|\boldsymbol{\alpha}\|_\infty\leq P^{-\rho/2}} \frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)}{P^n} \ll_{\boldsymbol{f},\boldsymbol{\kappa}} P^{-\frac{\mathscr{C}}{4R}} + P^{-\frac{\mathscr{C}\rho}{2(d-1)}}.$$

This proves the complementary-arc bound (6.4) with Δ as in (6.11) and some ξ of the form $\xi(P) = P^{-\delta}$, as required by (6.12).

If instead $\rho = 0$ and $\boldsymbol{f}^{[d]}$ is irrational, then Proposition 6.3 shows that there is a function $\eta : [1, \infty) \to (0, 1]$ such that $\eta(P) \to 0$ as $P \to \infty$ and

$$\sup_{\eta(P) \leq \|\boldsymbol{\alpha}\|_\infty \leq \eta(P)^{-1}} \frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)}{P^n} \ll_{\boldsymbol{f},\boldsymbol{\kappa}} \eta(P) \tag{6.17}$$

for all $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$ and all $P \gg_{\boldsymbol{f}^{[d]}} 1$. Choose $A = \eta^{-1}$ in (6.16). Then (6.16) and (6.17) show that

$$\sup_{P^{\frac{1}{4R}-d} \leq \|\boldsymbol{\alpha}\|_\infty \leq \eta(P)^{-1}} \frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)}{P^n} \ll_{\boldsymbol{f},\boldsymbol{\kappa}} P^{-\frac{\mathscr{C}}{4R}} + \eta(P)^{\frac{\mathscr{C}}{(d-1)}} + \eta(P),$$

for all $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$ and all $P \geq 1$, which proves (6.4) with Δ as in (6.11), and for some ξ satisfying (6.13). Additionally, if $\boldsymbol{f}^{[d]}$ is irrational with algebraic coefficients, then we may take $\eta(P) = P^{-\delta}$ in Proposition 6.3. After adjusting the value of δ if necessary we may then take $\xi(P) = P^{-\delta}$, which justifies (6.14). By the comments after (6.14) this proves the theorems. □

6.2 Smoothly weighted sums

We consider the weighted sum $S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)$ from (2.16), where the weight ω belongs to a smoothness class $\mathcal{S}(\boldsymbol{\kappa}, n)$ as in Definition 2.3.

6.2.1 Weyl differencing

We require an analogue of Lemma 3.11 from §3.2.1.

Lemma 6.4. *Let $U_f(B, \delta)$ be as in Definition 3.10. For all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$, $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$ and $\theta \in (0, 1]$ we have*

$$\left| \frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)}{P^n} \right|^{2^{d-1}} \leq \mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) \frac{U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(P^\theta, P^{(d-1)\theta-d})}{P^{(d-1)\theta n}}, \tag{6.18}$$

$$\min\left\{ \left| \frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)}{P^{n+\epsilon}} \right|, \left| \frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}+\boldsymbol{\beta}; P)}{P^{n+\epsilon}} \right| \right\}^{2^d} \leq \mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) \frac{U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^\theta, P^{(d-1)\theta-d})}{P^{(d-1)\theta n}}. \tag{6.19}$$

Proof. We have

$$\begin{aligned}|S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)|^2 &= \left|\sum_{\boldsymbol{x}\in\mathbb{Z}^n} e\big(\boldsymbol{\alpha}\cdot\boldsymbol{f}(\boldsymbol{x})\big)\omega(\boldsymbol{x}/P)\right|^2\\ &= \sum_{\boldsymbol{x},\boldsymbol{z}\in\mathbb{Z}^n} e\big(\boldsymbol{\alpha}\cdot\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{z})-\boldsymbol{\alpha}\cdot\boldsymbol{f}(\boldsymbol{x})\big)\omega(\tfrac{\boldsymbol{x}}{P})\omega(\tfrac{\boldsymbol{x}+\boldsymbol{z}}{P})\\ &= \sum_{\boldsymbol{z}\in\mathbb{Z}^n} S_{[\boldsymbol{f}]_{\boldsymbol{z}},\omega_{\boldsymbol{z}}}(\boldsymbol{\alpha};P) \end{aligned}\tag{6.20}$$

where, following the notation of Browning and Prendiville [BP15, (3.1)], we put

$$\omega_{\boldsymbol{z}}(\boldsymbol{x}) = \omega(\boldsymbol{x}+\boldsymbol{z})\omega(\boldsymbol{x}), \qquad [\boldsymbol{f}]_{\boldsymbol{z}}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{z}) - \boldsymbol{f}(\boldsymbol{x}). \tag{6.21}$$

We define $\omega_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k)}}$ and $[\boldsymbol{f}]_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k)}}$ inductively by taking

$$\begin{aligned}\omega_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k)}}(\boldsymbol{x}) &= \omega_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k-1)}}(\boldsymbol{x}+\boldsymbol{z}^{(k)})\omega_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k-1)}}(\boldsymbol{x}), \\ [\boldsymbol{f}]_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k)}}(\boldsymbol{x}) &= [\boldsymbol{f}]_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k-1)}}(\boldsymbol{x}+\boldsymbol{z}^{(k)}) - [\boldsymbol{f}]_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k-1)}}(\boldsymbol{x}).\end{aligned}\tag{6.22}$$

We claim that for all $k \geq 1$ we have

$$\frac{|S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)|^{2^k}}{P^{2^k n}} \leq \frac{1}{P^{(k+1)n}} \sum_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k)}\in\mathbb{Z}^n} S_{[\boldsymbol{f}]_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k)}},\omega_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k)}}}(\boldsymbol{\alpha};P). \tag{6.23}$$

If $k=1$ this is just (6.20). If $k \geq 2$ then by induction we can assume that

$$\frac{|S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)|^{2^k}}{P^{2^k n}} \leq \left|\frac{1}{P^{kn}} \sum_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k-1)}\in\mathbb{Z}^n} S_{[\boldsymbol{f}]_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k-1)}},\omega_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k-1)}}}(\boldsymbol{\alpha};P),\right|^2$$

and by the special case $|\sum_{i\in\mathcal{I}}\lambda_i|^2 \leq (\#\mathcal{I})\cdot\sum_{i\in\mathcal{I}}|\lambda_i|^2$ of Cauchy's inequality, this is

$$\leq \frac{1}{P^{(k+1)n}} \sum_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k-1)}\in\mathbb{Z}^n} \left|S_{[\boldsymbol{f}]_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k-1)}},\omega_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(k-1)}}}(\boldsymbol{\alpha};P)\right|^2.$$

An application of (6.23) completes the proof of (6.23).

We will use the case $k = d-1$ of (6.23) to prove (6.18). As in formula (3.4) of Browning and Prendiville [BP15], by Taylor expansion we see that $\boldsymbol{\alpha}\cdot[\boldsymbol{f}]_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(d-1)}}(\boldsymbol{x})$ has degree at most 1 in $\boldsymbol{x}$ and that the degree 1 part satisfies

$$\begin{aligned}\boldsymbol{\alpha}\cdot[\boldsymbol{f}]^{[1]}_{\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(d-1)}}(\boldsymbol{x}) &= \sum_{i_1,\dots,i_{d-1}=1}^{n} z^{(1)}_{i_1}\cdots z^{(d-1)}_{i_k} \frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_{d-1}}}\boldsymbol{\alpha}\cdot\boldsymbol{f}(\boldsymbol{x}).\\ &= \boldsymbol{x}\cdot\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(d-1)})\end{aligned}\tag{6.24}$$

where $\boldsymbol{m}^{(f)}$ is as in §1.3.3. Now by substituting (6.24) into (6.23), we see that

$$\frac{|S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)|^{2^{d-1}}}{P^{2(d-1)n}} \leq \frac{1}{P^{dn}} \sum_{\substack{\boldsymbol{z}^{(1)},\ldots,\boldsymbol{z}^{(d-1)}\in\mathbb{Z}^n \\ \|\boldsymbol{z}^{(i)}\|_\infty\leq 4P}} \left| \sum_{\boldsymbol{x}\in\mathbb{Z}^n} e\big(\boldsymbol{x}\cdot\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)},\ldots,\boldsymbol{z}^{(d-1)})\big)\cdot\omega_{\boldsymbol{z}^{(1)},\ldots,\boldsymbol{z}^{(d-1)}}(\boldsymbol{x}/P)\right|, \tag{6.25}$$

where we are free to impose the condition $\|\boldsymbol{z}^{(i)}\|_\infty \leq 4P$ in the first sum, since $\omega \in \mathcal{S}(\boldsymbol{\kappa},n)$ is supported on $[-2,2]^n$ and so the weights $\omega_{\boldsymbol{z}^{(1)},\ldots,\boldsymbol{z}^{(d-1)}}(\boldsymbol{x}/P)=0$ from (6.21) and (6.22) vanish unless $\|\boldsymbol{z}^{(i)}\|_\infty \leq 4P$ for each i.

Observe that $\omega_{\boldsymbol{z}^{(1)},\ldots,\boldsymbol{z}^{(d-1)}} \in \mathcal{S}(\boldsymbol{\kappa}',n)$ where $\kappa_i' = \mathrm{Poly}_{d,i}(\boldsymbol{\kappa})$ is some polynomial in the κ_i depending on d,i only. We claim that for any $k\in\mathbb{N}$, $\boldsymbol{\xi}\in\mathbb{R}^n$, $\omega'\in\mathcal{S}(\boldsymbol{\kappa}',n)$ we have

$$\sum_{\boldsymbol{x}\in\mathbb{Z}^n} e(\boldsymbol{\xi}\cdot\boldsymbol{x})\omega'(\boldsymbol{x}/P) \leq \mathrm{Poly}_n(\boldsymbol{\kappa}')\cdot P^n \max\{1,\ P\|\boldsymbol{\xi}\|_{\mathbb{R}/\mathbb{Z}}\}^{-n-1} \tag{6.26}$$

where $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$ is as in §1.1 and Poly_n is a polynomial depending on n only. By Poisson summation we have

$$\sum_{\boldsymbol{x}\in\mathbb{Z}^n} e(\boldsymbol{\xi}\cdot\boldsymbol{x})\omega'(\boldsymbol{x}/P) = \sum_{\boldsymbol{y}\in\mathbb{Z}^n} P^n\hat{\omega}'(P\boldsymbol{y}-P\boldsymbol{\xi})$$

where $\hat{\omega}'$ is the Fourier transform, and since $\omega'\in\mathcal{S}(\boldsymbol{\kappa}',n)$ is $n+1$ times differentiable and the κ_i' are upper bounds for ω' and its derivatives, it follows that

$$\sum_{\boldsymbol{x}\in\mathbb{Z}^n} e(\boldsymbol{\xi}\cdot\boldsymbol{x})\omega'(\boldsymbol{x}/P) \leq \mathrm{Poly}_n(\boldsymbol{\kappa}')\sum_{\boldsymbol{y}\in\mathbb{Z}^n} P^n(P\boldsymbol{y}-P\boldsymbol{\xi})^{-n-1}.$$

By treating the points $\boldsymbol{y}$ lying close to $\boldsymbol{\xi}$ separately, we obtain

$$\begin{aligned}\sum_{\boldsymbol{y}\in\mathbb{Z}^n} P^n(P\boldsymbol{y}-P\boldsymbol{\xi})^{-n-1} &= P^{-1}\sum_{\substack{\boldsymbol{y}\in\mathbb{Z}^n\\ \|\boldsymbol{y}-\boldsymbol{\xi}\|_\infty\leq 1}}(\boldsymbol{y}-\boldsymbol{\xi})^{-n-1} + P^{-1}\sum_{\substack{\boldsymbol{y}\in\mathbb{Z}^n\\ \|\boldsymbol{y}-\boldsymbol{\xi}\|_\infty> 1}}(\boldsymbol{y}-\boldsymbol{\xi})^{-n-1}\\ &\leq P^{-1}O_n(\|\boldsymbol{\xi}\|_{\mathbb{R}/\mathbb{Z}}^{-n-1}) + P^{-1}O_n(1)\\ &\ll_n P^{-1}\|\boldsymbol{\xi}\|_{\mathbb{R}/\mathbb{Z}}^{-n-1}.\end{aligned}$$

This proves (6.26) provided that If $\|\boldsymbol{\xi}\|_{\mathbb{R}/\mathbb{Z}} > 1/P$, and in the contrary case (6.26) is trivial.

By (6.25) and (6.26) we have

$$|S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)|^{2^{d-1}} \leq \mathrm{Poly}_{d,n}(\boldsymbol{\kappa})\cdot P^{(2^{d-1}-d)n} \cdot \sum_{\substack{\boldsymbol{z}^{(i)}\in\mathbb{Z}^n\\ \|\boldsymbol{z}^{(i)}\|_\infty\leq 4P}} \max\{1,\ P\|\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)},\ldots,\boldsymbol{z}^{(d-1)})\|_{\mathbb{R}/\mathbb{Z}}\}^{-n-1}. \tag{6.27}$$

Let $U_f(B,\delta)$ as in Definition 3.10. Observe that the number of $(d-1)$-tuples of integer vectors $\boldsymbol{z}^{(i)}$ satisfying $\|\boldsymbol{z}^{(i)}\|_\infty \le 4P$ and

$$2^{i-1} \le \|\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(d-1)})\|_{\mathbb{R}/\mathbb{Z}} \le 2^i$$

is at most $U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(4P, 2^iP^{-1})$, and so we have

$$\sum_{\substack{\boldsymbol{z}^{(i)}\in\mathbb{Z}^n \\ \|\boldsymbol{z}^{(i)}\|_\infty\le 4P}} \max\{1,\ P\|\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f})}(\boldsymbol{z}^{(1)},\dots,\boldsymbol{z}^{(d-1)})\|_{\mathbb{R}/\mathbb{Z}}\}^{-n-1}$$

$$\le U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(4P,P^{-1}) + \sum_{i=1}^{\infty} 2^{-(i-1)(n+1)} U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(4P,2^iP^{-1}).$$

Together with (6.27) this yields

$$|S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)|^{2^{d-1}} \le \mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) \cdot P^{(2^{d-1}-d)n} \sum_{i=1}^{\infty} 2^{-i(n+1)} U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(4P,2^iP^{-1}). \tag{6.28}$$

By the Dirichlet box principle one sees that

$$U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(4P,2^iP^{-1}) \ll_n 2^{in} U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(P,P^{-1}),$$

and together with (6.28) this proves (6.18).

To prove (6.19), it is enough to show that

$$\left|\frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}+\boldsymbol{\beta};P)}{P^{2n}}\right|^{2^{d-1}} \le \mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) \frac{U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^\theta, P^{(d-1)\theta-d})}{P^{(d-1)\theta n}}.$$

We apply the inequality $|\sum_{i\in\mathcal{I}}\lambda_i|^2 \le (\#\mathcal{I})\cdot\sum_{i\in\mathcal{I}}|\lambda_i|^2$, which is a special case of Cauchy's inequality, a total of $d-1$ times to find

$$\begin{aligned}
&|S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}+\boldsymbol{\beta};P)\overline{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)}|^{2^{d-1}} \\
&\quad= \left|\sum_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}^n} e\big(\boldsymbol{\alpha}\cdot\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{y}) - \boldsymbol{\alpha}\cdot\boldsymbol{f}(\boldsymbol{x})\big)\omega(\tfrac{\boldsymbol{x}+\boldsymbol{y}}{P})\omega(\tfrac{\boldsymbol{x}}{P})\right|^{2^{d-1}} \\
&\quad\le P^{(2^{d-1}-1)n} \\
&\qquad\cdot \sum_{\boldsymbol{y}\in\mathbb{Z}^n}\left|\sum_{\boldsymbol{x}\in\mathbb{Z}^n} e\big((\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{y}) - \boldsymbol{\alpha}\cdot\boldsymbol{f}(\boldsymbol{x})\big)\omega(\tfrac{\boldsymbol{x}+\boldsymbol{y}}{P})\omega(\tfrac{\boldsymbol{x}}{P})\right|^{2^{d-1}} \\
&\quad= P^{(2^{d-1}-1)n} \sum_{\substack{\boldsymbol{y}\in\mathbb{Z}^n \\ \|\boldsymbol{y}\|_\infty\le 4P}} \left|\sum_{\boldsymbol{x}\in\mathbb{Z}^n} e(g_{\boldsymbol{y}}(\boldsymbol{x}))\omega(\tfrac{\boldsymbol{x}+\boldsymbol{y}}{P})\omega(\tfrac{\boldsymbol{x}}{P})\right|^{2^{d-1}}
\end{aligned} \tag{6.29}$$

where $g_{\boldsymbol{y}}$ is a polynomial depending only on $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{f}$ and $\boldsymbol{y}$, and which has the same degree d part as the polynomial $\boldsymbol{\beta}\cdot\boldsymbol{f}$. Applying (6.18) to the inner sum in (6.29) proves (6.19). □

6.2.2 Proof of Proposition 6.1

Proof. The proof of Proposition 3.3 in §3.2.2 bounds the minimum

$$\min\left\{\left|\frac{S_{\boldsymbol{f},\mathscr{B}}(\boldsymbol{\alpha})}{P^{n+\epsilon}}\right|, \left|\frac{S_{\boldsymbol{f},\mathscr{B}}(\boldsymbol{\alpha}+\boldsymbol{\beta})}{P^{n+\epsilon}}\right|\right\}^{2^d} \tag{6.30}$$

using the bound

$$\min\left\{\left|\frac{S(\boldsymbol{\alpha};P)}{P^{n+\epsilon}}\right|, \left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P)}{P^{n+\epsilon}}\right|\right\}^{2^d} \ll_{d,n,\epsilon} \frac{U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P^{\theta}, P^{(d-1)\theta-d})}{P^{(d-1)\theta n}}$$

from Lemma 3.11. This and the trivial upper bound

$$\min\left\{\left|\frac{S_{\boldsymbol{f},\mathscr{B}}(\boldsymbol{\alpha})}{P^{n+\epsilon}}\right|, \left|\frac{S_{\boldsymbol{f},\mathscr{B}}(\boldsymbol{\alpha}+\boldsymbol{\beta})}{P^{n+\epsilon}}\right|\right\}^{2^d} \ll_{d,n} 1, \tag{6.31}$$

which is employed in the second ($\theta \leq 0$) of the three cases at the end of the proof, are the only properties of the minimum (6.30) used in that proof.

The proof of the present result is precisely similar. Instead of bounding the above unweighted minimum (6.30) with Lemma 3.11, we bound the left-hand side of (6.3) or (6.2) with Lemma 6.4. Either of these quantities has a trivial upper bound of size $O_n(\kappa_0^{2^d})$ which may be used in place of (6.31). One easily verifies that the implicit constants all have the form $\mathrm{Poly}_{d,n}(\boldsymbol{\kappa})O_{C_0,\boldsymbol{f},\boldsymbol{\kappa}}(1)$, as required. □

6.3 The circle method

We now apply Freeman's variant of the Davenport-Heilbronn method, as described in §2.2. Throughout this section $S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)$ will be as in (2.16), and $\boldsymbol{\kappa}$, $\mathcal{S}(\boldsymbol{\kappa},k)$ and $\mathrm{Poly}_{d,n}(\boldsymbol{\kappa})$ will be as in Definition 2.3. We will always follow the convention for implicit constants described in Definition 2.3.

We let $\Delta \in (0,1)$ be a parameter, the central arc $\mathfrak{C}_{P,d,\Delta}$ will be as in (2.11) and the complementary arcs $\mathfrak{c}_{P,d,\Delta}$ will be as in (2.12).

6.3.1 The central arc

We first treat the $\mathfrak{C}_{P,d,\Delta}$ from (2.11). We begin by approximating $S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha};P)$ when $\boldsymbol{\alpha}$ is small, using the function defined for each $\boldsymbol{\gamma} \in \mathbb{R}^R$ by

$$S_{\omega,\infty}(\boldsymbol{\gamma}) = \int_{\mathbb{R}^n} e\big(\boldsymbol{\gamma}\cdot\boldsymbol{f}^{[d]}(\boldsymbol{t})\big)\omega(\boldsymbol{t})\,d\boldsymbol{t}. \tag{6.32}$$

Lemma 6.5. *Let $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$ and let $S_{\omega,\infty}(\boldsymbol{\gamma})$ be as in* (6.32)*. Then for each $\boldsymbol{\alpha} \in \mathbb{R}^R$ and $P \geq 1$ we have*

$$S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P) = P^n S_{\omega,\infty}(P^d \boldsymbol{\alpha}) + P^{n-1} O_{\boldsymbol{f},\boldsymbol{\kappa}}(1 + P^d \|\boldsymbol{\alpha}\|_\infty).$$

Proof. The case $q = 1$ of equation (3.27) from §3.1.2 states that if ψ is a continuously differentiable complex-valued function on $\mathbb{R}^n$, then

$$\psi(\boldsymbol{x}) = \int_{\substack{\boldsymbol{u} \in \mathbb{R}^n \\ \|\boldsymbol{u}\|_\infty \leq 1/2}} \psi(\boldsymbol{x} + \boldsymbol{u}) \, d\boldsymbol{u} + O_n\Big(\max_{\substack{\boldsymbol{u} \in \mathbb{R}^n \\ \|\boldsymbol{u}\|_\infty \leq 1/2}} \|\boldsymbol{\nabla}_{\boldsymbol{u}} \psi(\boldsymbol{x} + \boldsymbol{u})\|_\infty \Big). \tag{6.33}$$

Set $\psi(\boldsymbol{x}) = e(\boldsymbol{\alpha} \cdot \boldsymbol{f}(\boldsymbol{x}))\omega(\boldsymbol{x}/P)$, and observe that from the definition of the smoothness class $\mathcal{S}(\boldsymbol{\kappa}, n)$ we have

$$\begin{aligned} \sup_{\boldsymbol{t} \in \mathbb{R}^n} \|\boldsymbol{\nabla}_{\boldsymbol{t}} \psi(\boldsymbol{t})\|_\infty &= \max_{\substack{\boldsymbol{t} \in \mathbb{R}^n \\ \|\boldsymbol{t}\|_\infty \leq 2P}} \|\boldsymbol{\nabla}_{\boldsymbol{t}} \psi(\boldsymbol{t})\|_\infty \\ &\ll_{d,n,R} \|\boldsymbol{\alpha}\|_\infty \|\boldsymbol{f}\|_\infty P^{d-1} \kappa_0 + P^{-1} \kappa_1. \end{aligned}$$

Substituting this into (6.33) shows that

$$\begin{aligned} S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P) &= \sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^n \\ \|\boldsymbol{x}\|_\infty \leq 2P}} \psi(\boldsymbol{x}) \\ &= \int_{\mathbb{R}^n} \psi(\boldsymbol{t}) \, d\boldsymbol{t} + P^{n-1} O_{\boldsymbol{f},\boldsymbol{\kappa}}\Big(1 + P^d \|\boldsymbol{\alpha}\|_\infty\Big). \end{aligned} \tag{6.34}$$

Because $\boldsymbol{f}(\boldsymbol{t})\omega(\boldsymbol{t}/P) = \boldsymbol{f}^{[d]}(\boldsymbol{t})\omega(\boldsymbol{t}/P) + O_{\boldsymbol{f},\boldsymbol{\kappa}}(P^{d-1}\|\boldsymbol{\alpha}\|_\infty)$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \psi(\boldsymbol{t}) \, d\boldsymbol{t} &= \int_{\mathbb{R}^n} e\big(\boldsymbol{\alpha} \cdot \boldsymbol{f}^{[d]}(\boldsymbol{t})\big)\omega(\boldsymbol{t}/P) \, d\boldsymbol{t} + O_{\boldsymbol{f},\boldsymbol{\kappa}}\Big(P^{n-1+d}\|\boldsymbol{\alpha}\|_\infty\Big) \\ &= P^n \int_{\mathbb{R}^n} e\big(P^d \boldsymbol{\alpha} \cdot \boldsymbol{f}^{[d]}(\boldsymbol{t})\big)\omega(\boldsymbol{t}) \, d\boldsymbol{t} + O_{\boldsymbol{f},\boldsymbol{\kappa}}\Big(P^{n-1+d}\|\boldsymbol{\alpha}\|_\infty\Big) \end{aligned}$$

and together with (6.34) this proves the claim. □

We use Lemma 6.5 to prove a central arc estimate along the lines of (2.17) from §2.2.

Lemma 6.6. *Let $\mathfrak{C}_{P,d,\Delta}$ be as in* (2.11) *and let $S_{\omega,\infty}(\boldsymbol{\gamma})$ be as in* (6.32)*. Suppose that for some $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$, $\mathscr{C} \in (R, \infty)$ and all $P \geq 1$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$ we have*

$$\frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)}{P^n} \ll_{\boldsymbol{f}} \mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) \max\big\{P^{-d}\|\boldsymbol{\beta}\|_\infty^{-1}, \, \|\boldsymbol{\beta}\|_\infty^{\frac{1}{d-1}}\big\}^{\mathscr{C}}. \tag{6.35}$$

Then for any $\varphi \in \boldsymbol{\mathcal{S}}(\boldsymbol{\kappa}, R)$, $\rho \in (-\infty, d)$, $\Delta \in (0,1)$ and $P \geq 1$ we have

$$\int_{\mathfrak{C}_{P,d,\Delta}} S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P) P^{\rho R} \hat{\varphi}(P^{\rho}\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = \int_{\mathbb{R}^n} \varphi\big(P^{-\rho} \boldsymbol{f}^{[d]}(\boldsymbol{t})\big)\omega(\boldsymbol{t}/P)\, d\boldsymbol{t} \\ + \mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) O_{\mathscr{C},\boldsymbol{f}}(P^{n-(d-\rho)R+\Delta(R+1)-1} + P^{n-(d-\rho)R-\Delta(\mathscr{C}-R)}). \quad (6.36)$$

Furthermore the integral on the right-hand side of (6.36) *satisfies*

$$\int_{\mathbb{R}^n} \varphi\big(P^{-\rho} \boldsymbol{f}^{[d]}(\boldsymbol{t})\big)\omega(\boldsymbol{t}/P)\, d\boldsymbol{t} = \hat{\varphi}(\boldsymbol{0}) \mathfrak{I}_{\boldsymbol{f},\omega} P^{n-(d-\rho)R} \\ + \mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) O_{\mathscr{C},\boldsymbol{f}}(P^{n-(d-\rho)(R+1)} + P^{n-(d-\rho)R-d(\mathscr{C}-R)}) \quad (6.37)$$

with $\mathfrak{I}_{\boldsymbol{f},\omega} = \int_{\mathbb{R}^R} S_{\omega,\infty}(\boldsymbol{\gamma})\, d\boldsymbol{\gamma}$ absolutely convergent.

Proof. Throughout this proof we follow the convention that implicit constants will be polynomial in $\boldsymbol{\kappa}$, as described in Definition 2.3.

We begin by showing that the function $S_{\omega,\infty}(\boldsymbol{\gamma})$ from Lemma 6.5 is Lebesgue integrable. Together, (6.35) and the case $\boldsymbol{\alpha} = \boldsymbol{\beta}$ of Lemma 6.5 show that

$$P^n |S_{\omega,\infty}(P^d \boldsymbol{\beta})| \ll_{\boldsymbol{f},\boldsymbol{\kappa}} P^n \max\big\{P^{-d}\|\boldsymbol{\beta}\|_\infty^{-1}, \boldsymbol{\beta}^{\frac{1}{d-1}}\big\}^{\mathscr{C}} + P^{n-1} O_{\boldsymbol{f},\boldsymbol{\kappa}}\big(1 + P^d \|\boldsymbol{\beta}\|_\infty\big).$$

Setting $\boldsymbol{\gamma} = P^d \boldsymbol{\beta}$, we find that

$$|S_{\omega,\infty}(\boldsymbol{\gamma})| \ll_{\boldsymbol{f},\boldsymbol{\kappa}} \|\boldsymbol{\gamma}\|_\infty^{-\mathscr{C}} + O_{\boldsymbol{\gamma},\boldsymbol{f},\boldsymbol{\kappa}}(P^{-1})$$

and letting $P \to \infty$ we see that for all $\boldsymbol{\gamma} \in \mathbb{R}^R$ we have

$$|S_{\omega,\infty}(\boldsymbol{\gamma})| \ll_{\boldsymbol{f},\boldsymbol{\kappa}} \|\boldsymbol{\gamma}\|_\infty^{-\mathscr{C}}. \quad (6.38)$$

In particular, as $\mathscr{C} > R$, we see that $S_\infty : \mathbb{R}^R \to \mathbb{C}$ is Lebesgue integrable.

Next we prove (6.36). We have

$$\int_{\mathfrak{C}_{P,d,\Delta}} S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P) P^{\rho R} \hat{\varphi}(P^{\rho}\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \\ = \int_{\mathfrak{C}_{P,d,\Delta}} P^{n+\rho R} S_{\omega,\infty}(P^d \boldsymbol{\alpha}) \hat{\varphi}(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \\ + O_{\boldsymbol{f},\boldsymbol{\kappa}}(P^{n-(d-\rho)R+\Delta(R+1)-1}) \\ = \int_{\mathbb{R}^R} P^{n+\rho R} S_{\omega,\infty}(P^d \boldsymbol{\alpha}) \hat{\varphi}(P^{\rho}\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \\ + P^{n-(d-\rho)R} \cdot O_{\mathscr{C},\boldsymbol{f},\boldsymbol{\kappa}}(P^{\Delta(R+1)-1} + P^{\Delta(\mathscr{C}-R)}) \quad (6.39)$$

using Lemma 6.5 for the first equality and (6.38) for the second. Furthermore

$$\begin{aligned}
\int_{\mathbb{R}^R} & P^{n+\rho R} S_{\omega,\infty}(P^d\boldsymbol{\alpha})\hat{\varphi}(P^\rho\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \\
&= \int_{\mathbb{R}^R} P^{n+\rho R}\hat{\varphi}(P^\rho\boldsymbol{\alpha}) \int_{\mathbb{R}^n} e\big(P^d\boldsymbol{\alpha}\cdot \boldsymbol{f}^{[d]}(\boldsymbol{t})\big)\omega(\boldsymbol{t})\,d\boldsymbol{t}\,d\boldsymbol{\alpha} \\
&= \int_{\mathbb{R}^R} P^{\rho R}\hat{\varphi}(P^\rho\boldsymbol{\alpha}) \int_{\mathbb{R}^n} e(\boldsymbol{\alpha}\cdot \boldsymbol{f}^{[d]}(\boldsymbol{t}))\omega(\boldsymbol{t}/P)\,d\boldsymbol{t}\,d\boldsymbol{\alpha} \\
&= \int_{\mathbb{R}^n} \varphi(P^{-\rho}\boldsymbol{f}^{[d]}(\boldsymbol{t}))\omega(\boldsymbol{t}/P)\,d\boldsymbol{t}. \qquad (6.40)
\end{aligned}$$

by Fourier inversion. The formula (6.36) follows from (6.39) and (6.40). It remains to prove (6.37).

By Definition 2.3, we have the bound $\partial\hat{\varphi}(\boldsymbol{\alpha})/\partial\alpha_i = O_R(\kappa_0)$ for each i. It follows that $\hat{\varphi}(\boldsymbol{\alpha}) = \hat{\varphi}(\boldsymbol{0}) + O_{R,\boldsymbol{\kappa}}(\|\boldsymbol{\alpha}\|_\infty)$ and hence that

$$\begin{aligned}
\int_{\mathfrak{C}_{P,d,\Delta}} S_{\omega,\infty}(P^d\boldsymbol{\alpha})\hat{\varphi}(P^\rho\boldsymbol{\alpha})\,d\boldsymbol{\alpha} &= \int_{\|\boldsymbol{\alpha}\|_\infty\le 1} S_{\omega,\infty}(P^d\boldsymbol{\alpha})\hat{\varphi}(\boldsymbol{0})\,d\boldsymbol{\alpha} \\
&\quad + O_{R,\boldsymbol{\kappa}}\left(\int_{\mathfrak{C}_{P,d,\Delta}} |S_{\omega,\infty}(P^d\boldsymbol{\alpha})| P^\rho \|\boldsymbol{\alpha}\|_\infty\,d\boldsymbol{\alpha}\right) \\
&= \hat{\varphi}(\boldsymbol{0})\int_{\|\boldsymbol{\alpha}\|_\infty\le 1} S_{\omega,\infty}(P^d\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \\
&\quad + O_{\mathscr{C},\boldsymbol{f},\boldsymbol{\kappa}}\left(P^{\rho-d(R+1)}\right)
\end{aligned}$$

where the second equality follows from (6.38). Moreover (6.38) implies

$$\int_{\|\boldsymbol{\alpha}\|_\infty\ge 1} S_{\omega,\infty}(P^d\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \ll_{\mathscr{C},\boldsymbol{f},\boldsymbol{\kappa}} P^{-d\mathscr{C}}$$

and so

$$\begin{aligned}
\int_{\mathbb{R}^R} P^{n+\rho R} S_{\omega,\infty}(P^d\boldsymbol{\alpha})\hat{\varphi}(P^\rho\boldsymbol{\alpha})\,d\boldsymbol{\alpha} - \hat{\varphi}(\boldsymbol{0})\int_{\mathbb{R}^R} P^{n+\rho R} S_{\omega,\infty}(P^d\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \\
\ll_{C_0,\boldsymbol{f},\boldsymbol{\kappa}} \left(P^{n-(d-\rho)(R+1)} + P^{n-(d-\rho)R-d(\mathscr{C}-R)}\right). \qquad (6.41)
\end{aligned}$$

Putting $\boldsymbol{\alpha} = P^{-d}\boldsymbol{\gamma}$, we find that

$$\int_{\mathbb{R}^R} P^n S_{\omega,\infty}(P^d\boldsymbol{\alpha})\,d\boldsymbol{\alpha} = \int_{\mathbb{R}^R} S_{\omega,\infty}(\boldsymbol{\gamma})\,d\boldsymbol{\gamma} = \mathfrak{I}_{\boldsymbol{f},\omega}.$$

Together with (6.40) and (6.41), this proves (6.37). □

6.3.2 The complementary arcs

We now consider the complementary arc estimate (2.18). We follow the strategy outlined in §2.2.2.

Lemma 6.7. *Let $\mathfrak{c}_{P,d,\Delta}$ be as in (2.12). Let $\mathscr{C}$ and ρ be real numbers satisfying*

$$\mathscr{C} > \max\{d-\rho, 1\}R. \tag{6.42}$$

Suppose that for some $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$ and all $P \geq 1$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$ we have

$$\min\left\{ \left| \frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)}{P^n} \right|, \left| \frac{S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}+\boldsymbol{\beta}; P)}{P^n} \right| \right\} \ll_{\boldsymbol{f}} \mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) \max\left\{ P^{-d} \|\boldsymbol{\beta}\|_\infty^{-1}, \|\boldsymbol{\beta}\|_\infty^{\frac{1}{d-1}} \right\}^{\mathscr{C}}. \tag{6.43}$$

Let ξ be a function on $[1,\infty)$ taking values in $(0,1]$, and suppose that

$$\sup_{\substack{\boldsymbol{\alpha} \in \mathbb{R}^R \\ P^{\Delta-d} < \|\boldsymbol{\alpha}\|_\infty < \xi(P)^{-1} P^{-\rho}}} |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)| \ll_{\boldsymbol{f}} \mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) \xi(P) P^n \tag{6.44}$$

for all $P \geq 1$ and some $\Delta \in (0,1)$. Then there exists $\delta > 0$ depending only on $\mathscr{C}, d, R, \rho$ such that for any $P \geq 1$ we have

$$\int_{\mathfrak{c}_{P,d,\Delta}} S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P) P^{\rho R} \hat{\varphi}(P^\rho \boldsymbol{\alpha}) \, d\boldsymbol{\alpha} \ll_{\mathscr{C},\boldsymbol{f},\rho} \mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) \big(\xi(P)^\delta P^{n-(d-\rho)R} + P^{n-(d-\rho)R-\delta} \big). \tag{6.45}$$

Proof. Throughout this proof we write δ for a small positive constant which depends at most on $\mathscr{C}, d, R$ and ρ, and which may vary between uses. Implicit constants in $\ll$ notation will depend on $\boldsymbol{\kappa}$ only through a factor $\mathrm{Poly}_{d,n}(\boldsymbol{\kappa})$, as described in Definition 2.3.

We follow the approach described in §2.2.2. Let $\zeta_1, \ldots, \zeta_R \in \mathbb{R}$, define a box by $E_0 = \prod_{i=1}^R [\zeta_i, \zeta_i + P^{-\rho}]$ and let $E \subset E_0$ be a measurable subset. The bound (6.43) allows us to apply Lemma 3.5 with $\nu = P^{-\rho}$ and $T(\boldsymbol{\alpha}) = C^{-1} S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)$, where C is the implicit constant in (6.43). Since we have $\mathscr{C} > (d-\rho)R$ and $\mathscr{C} > R$, this shows that

$$\int_E |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)| \, d\boldsymbol{\alpha} \ll_{\mathscr{C},\boldsymbol{f},\boldsymbol{\kappa}} P^{n-dR-\delta} \log P + \Big(\sup_{\boldsymbol{\alpha} \in E} P^{-n} |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)| \Big)^\delta P^{n-dR} \tag{6.46}$$

where as above $\delta > 0$ denotes some constant depending at most on $\mathscr{C}, d, R, \rho$. Set

$$\mathfrak{b}_{P,\Delta,\boldsymbol{f}} = \{ \boldsymbol{\alpha} \in \mathbb{R}^R : P^{\Delta-d} \leq \|\boldsymbol{\alpha}\|_\infty \leq \xi(P)^{-1} P^{-\rho} \} \tag{6.47}$$
$$\mathfrak{t}_{P,\boldsymbol{f}} = \{ \boldsymbol{\alpha} \in \mathbb{R}^R : \xi(P)^{-1} P^{-\rho} \leq \|\boldsymbol{\alpha}\|_\infty \},$$

so that $\mathfrak{c}_{P,d,\Delta}$ is the union of these two pieces. Since $\varphi \in \mathcal{S}(\boldsymbol{\kappa}, R)$, we have

$$\int_{\mathfrak{t}_{P,\boldsymbol{f}}} |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)\hat{\varphi}(P^{\rho}\boldsymbol{\alpha})|\, d\boldsymbol{\alpha} \ll_{R,\boldsymbol{\kappa}} \int_{\mathfrak{t}_{P,\boldsymbol{f}}} |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)| \cdot (P^{\rho}\|\boldsymbol{\alpha}\|_{\infty})^{-R-1}\, d\boldsymbol{\alpha}. \tag{6.48}$$

By covering $\mathbb{R}^R$ with hypercubes of side length $P^{-\rho}$ and applying (6.46) on each hypercube, we deduce that

$$\begin{aligned}
&\int_{\mathfrak{t}_{P,\boldsymbol{f}}} |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)\hat{\varphi}(P^{\rho}\boldsymbol{\alpha})|\, d\boldsymbol{\alpha} \\
&\qquad \ll_{\mathscr{C},\boldsymbol{f},\boldsymbol{\kappa}} \sum_{\substack{\boldsymbol{a}\in\mathbb{Z}^R \\ \|\boldsymbol{a}\|_{\infty} \geq \xi(P)^{-1}}} (P^{n-dR-\delta}\log P + P^{n-dR})\|\boldsymbol{a}\|_{\infty}^{-R-1} \\
&\qquad \ll_{\mathscr{C},d,R,\rho} \xi(P)P^{n-dR}.
\end{aligned} \tag{6.49}$$

Now covering the set $\mathfrak{b}_{P,\Delta,\boldsymbol{f}}$ from (6.47) with hypercubes of side length $P^{-\rho}$ and applying (6.48) and (6.46), we obtain

$$\begin{aligned}
&\int_{\mathfrak{b}_{P,\Delta,\boldsymbol{f}}} |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)\hat{\varphi}(P^{\rho}\boldsymbol{\alpha})|\, d\boldsymbol{\alpha} \\
&\quad \ll_{R,\boldsymbol{\kappa}} \int_{\mathfrak{b}_{P,\Delta,\boldsymbol{f}}} |S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)| \max\{P^{\rho}\|\boldsymbol{\alpha}\|_{\infty},\, 1\}^{-R-1}\, d\boldsymbol{\alpha} \\
&\quad \ll_{\mathscr{C},\boldsymbol{f},\boldsymbol{\kappa}} \sum_{\boldsymbol{a}\in\mathbb{Z}^R} \big(P^{n-dR-\delta}\log P + \xi(P)^{\delta}P^{n-dR}\big) \max\{\|\boldsymbol{a}\|_{\infty},\, 1\}^{-R-1} \\
&\quad \ll_{\mathscr{C},d,R,\rho} P^{n-dR-\delta} + \xi(P)^{\delta}P^{n-dR}
\end{aligned} \tag{6.50}$$

The conclusion (6.45) now follows from (6.49) and (6.50). □

6.3.3 Proof of Proposition 6.2

Lemmas 6.6 and 6.7 are sufficient to prove the weighted asymptotics (6.5) and (6.6). The bulk of the following proof is therefore devoted to deducing the unweighted asymptotics (6.7) and (6.8). The requirement that our implicit constants be polynomial in $\boldsymbol{\kappa}$, as described in Definition 2.3, is essential for this last step.

Proof of Proposition 6.2. We first prove the bounds (6.5) and (6.6). Let $\omega \in \mathcal{S}(\boldsymbol{\kappa}, n)$, $\varphi \in \mathcal{S}(\boldsymbol{\kappa}, R)$ and $P \geq 1$ and let $\mathscr{C}$ and Δ be as in the statement of the proposition. By Fourier inversion we have

$$\begin{aligned}
\sum_{\boldsymbol{x}\in\mathbb{Z}^n} \varphi\big(P^{-\rho}\boldsymbol{f}(\boldsymbol{x})\big)\omega(\boldsymbol{x}/P) &= \int_{\mathbb{R}^R} S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)P^{\rho R}\hat{\varphi}(P^{\rho}\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \\
&= \int_{\mathfrak{C}_{P,d,\Delta}} S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)P^{\rho R}\hat{\varphi}(P^{\rho}\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \\
&\quad + \int_{\mathfrak{c}_{P,d,\Delta}} S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)P^{\rho R}\hat{\varphi}(P^{\rho}\boldsymbol{\alpha})\, d\boldsymbol{\alpha},
\end{aligned} \tag{6.51}$$

where $\mathfrak{C}_{P,d,\Delta}, \mathfrak{c}_{P,d,\Delta}$ are as in (2.11) and (2.12). We apply Lemmas 6.6 and 6.7 to estimate the last two integrals in (6.51). Proposition 6.1 shows that the hypotheses (6.35) and (6.43) of these lemmas are satisfied. Additionally, Lemma 6.6 requires that $\mathscr{C} > R$ while Lemma 6.7 has the stricter condition (6.42). These both hold, since we have assumed that $\mathscr{C} > (d-\rho)R$ and $\rho \leq d-1$. The remaining hypothesis (6.44) is exactly the complementary-arc bound (6.4), so both lemmas apply.

Now in the proposition we have $\Delta \leq \frac{1}{R+1}$ and $\mathscr{C} > R$, and this ensures that the error term in (6.36) is of size $\mathrm{Poly}_{d,n}(\boldsymbol{\kappa}) O_{\mathscr{C},\boldsymbol{f}}(P^{n-(d-\rho)R-\delta_1})$, where δ_1 is a positive constant depending at most on $\mathscr{C}, \Delta, d, R$. The first estimate (6.5) follows, by substituting (6.36) and (6.45) into (6.51) and choosing the constant δ appearing in the proposition appropriately. The second estimate (6.6) follows from (6.37), since in the proposition we have $\rho \leq d-1$ and $\mathscr{C} > R$.

It remains to prove the formulae (6.7) and (6.8) from the weighted versions (6.5) and (6.6). We first define some additional notation, then we determine the value which $\mathfrak{I}_{\boldsymbol{f}^{[d]}}$ must take in (6.8), prove (6.8) and deduce (6.7).

As in Theorem 1.9, let

$$N_{\boldsymbol{f}}(P, P^{\rho}) = \#\{\boldsymbol{x} \in \mathbb{Z}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \|\boldsymbol{f}(\boldsymbol{x})\|_\infty \leq \tfrac{1}{2}P^{\rho}\} \tag{6.52}$$

$$M_{\boldsymbol{f}^{[d]}}(P, P^{\rho}) = \lambda\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \|\boldsymbol{f}^{[d]}(\boldsymbol{x})\|_\infty \leq \tfrac{1}{2}P^{\rho}\}. \tag{6.53}$$

Pick a decreasing and infinitely differentiable function $h : \mathbb{R} \to [0,1]$ such that

$$h(t) = 1 \quad (t \leq 0), \qquad\qquad h(t) = 0 \quad (t \geq 1).$$

For each $\epsilon \in (0,1]$ define $\omega_\epsilon : \mathbb{R}^n \to [0,1]$ and $\varphi_\epsilon : \mathbb{R}^R \to [0,1]$ by

$$\omega_\epsilon(\boldsymbol{x}) = \prod_{i=1}^{n} h\Big(\frac{|x_i| - 1}{\epsilon}\Big), \qquad\qquad \varphi_\epsilon(\boldsymbol{\gamma}) = \prod_{i=1}^{R} h\Big(\frac{2|\gamma_i| - 1}{\epsilon}\Big).$$

These are increasing as functions of ϵ, and satisfy

$$\begin{aligned} \omega_\epsilon(\boldsymbol{x}) = 0 \quad \text{if} \quad \|\boldsymbol{x}\|_\infty \geq 1+\epsilon, \qquad & \omega_\epsilon(\boldsymbol{x}) = 1 \quad \text{if} \quad \|\boldsymbol{x}\|_\infty \leq 1, \\ \varphi_\epsilon(\boldsymbol{\gamma}) = 0 \quad \text{if} \quad \|\boldsymbol{\gamma}\|_\infty \geq \frac{1+\epsilon}{2}, \qquad & \varphi_\epsilon(\boldsymbol{\gamma}) = 1 \quad \text{if} \quad \|\boldsymbol{\gamma}\|_\infty \leq \frac{1}{2}. \end{aligned} \tag{6.54}$$

Further define

$$\widetilde{\omega}_\epsilon(\boldsymbol{x}) = \omega_\epsilon\big((1+\epsilon)\boldsymbol{x}\big), \qquad\qquad \widetilde{\varphi}_\epsilon(\boldsymbol{\gamma}) = \varphi_\epsilon\big((1+\epsilon)\boldsymbol{\gamma}\big), \tag{6.55}$$

noting that these are decreasing as functions of ϵ, and are supported on $[-1,1]^n$ and $[-\frac{1}{2},\frac{1}{2}]^R$ respectively. There is a sequence $\boldsymbol{\kappa}^{(\epsilon)}$ of positive real numbers with $\omega_\epsilon,\ \widetilde{\omega}_\epsilon \in \mathcal{S}(\boldsymbol{\kappa}^{(\epsilon)}, n)$ and $\varphi_\epsilon,\ \widetilde{\varphi}_\epsilon \in \mathcal{S}(\boldsymbol{\kappa}^{(\epsilon)}, R)$ and which also satisfies

$$\kappa_i^{(\epsilon)} \ll_i \epsilon^i \qquad (i = 0, 1, \dots). \tag{6.56}$$

We use these weights to identify the constant $\mathfrak{I}_{\boldsymbol{f}^{[d]}}$ in (6.8). Applying (6.8) with the weights $\omega_\epsilon, \widetilde{\omega}_\epsilon$ above and any nonzero $\varphi \in \mathcal{S}(\boldsymbol{\kappa}, R)$, we find that

$$\mathfrak{I}_{\boldsymbol{f},\omega_\epsilon} = \lim_{P\to\infty} \frac{1}{\hat{\varphi}(\mathbf{0})P^{n-dR}} \int_{\mathbb{R}^n} \varphi\big(\boldsymbol{f}(\boldsymbol{t})\big)\omega_\epsilon(\boldsymbol{t}/P)\, d\boldsymbol{t}, \tag{6.57}$$

$$\mathfrak{I}_{\boldsymbol{f},\widetilde{\omega}_\epsilon} = \lim_{P\to\infty} \frac{1}{\hat{\varphi}(\mathbf{0})P^{n-dR}} \int_{\mathbb{R}^n} \varphi\big(\boldsymbol{f}(\boldsymbol{t})\big)\omega_\epsilon((1+\epsilon)\boldsymbol{t}/P)\, d\boldsymbol{t}. \tag{6.58}$$

It follows that $\mathfrak{I}_{\boldsymbol{f},\omega_\epsilon}$ is an increasing function of ϵ, and that $\mathfrak{I}_{\boldsymbol{f},\widetilde{\omega}_\epsilon}$ is a decreasing function of ϵ. But we have $\mathfrak{I}_{\boldsymbol{f},\widetilde{\omega}_\epsilon} = (1+\epsilon)^{dR-n}\mathfrak{I}_{\boldsymbol{f},\omega_\epsilon}$, by substituting $\widetilde{P} = (1+\epsilon)P$ in (6.58) and comparing to (6.57). By letting $\epsilon \to 0$ we find that there is $\mathfrak{I}_{\boldsymbol{f}^{[d]}} \geq 0$, depending only on $\boldsymbol{f}^{[d]}$, such that

$$\mathfrak{I}_{\boldsymbol{f},\omega_\epsilon} = \mathfrak{I}_{\boldsymbol{f}^{[d]}} + O_{d,n,R}(\epsilon), \qquad \mathfrak{I}_{\boldsymbol{f},\widetilde{\omega}_\epsilon} = \mathfrak{I}_{\boldsymbol{f}^{[d]}} + O_{d,n,R}(\epsilon). \tag{6.59}$$

We now prove (6.8). We see from the definitions (6.53)–(6.55) that

$$\int_{\mathbb{R}^n} \widetilde{\varphi}_\epsilon\big(P^{-\rho}\boldsymbol{f}(\boldsymbol{t})\big)\widetilde{\omega}_\epsilon(\boldsymbol{t}/P)\, d\boldsymbol{t} \leq M_{\boldsymbol{f}}(P, P^\rho) \leq \int_{\mathbb{R}^n} \varphi\big(P^{-\rho}\boldsymbol{f}(\boldsymbol{t})\big)\omega_\epsilon(\boldsymbol{t}/P)\, d\boldsymbol{t}. \tag{6.60}$$

Applying (6.8) with $\omega = \omega_\epsilon$ and $\varphi = \varphi_\epsilon$ and inserting (6.56) gives

$$\int_{\mathbb{R}^n} \varphi\big(P^{-\rho}\boldsymbol{f}(\boldsymbol{t})\big)\omega_\epsilon(\boldsymbol{t}/P)\, d\boldsymbol{t} = \hat{\varphi}_\epsilon(\mathbf{0})\mathfrak{I}_{\boldsymbol{f},\omega_\epsilon}P^{n-(d-\rho)R} + \epsilon^{-O_{d,n}(1)}O_{\mathscr{C},\boldsymbol{f},\rho}(P^{n-(d-\rho)R-\delta}).$$

By setting $\epsilon = P^{\delta/O_{d,n}(1)}$, observing that $\hat{\varphi}_\epsilon(\mathbf{0}) = 1 + O_R(\epsilon)$ and inserting (6.59), we deduce that

$$\int_{\mathbb{R}^n} \varphi\big(P^{-\rho}\boldsymbol{f}(\boldsymbol{t})\big)\omega_\epsilon(\boldsymbol{t}/P)\, d\boldsymbol{t} = \mathfrak{I}_{\boldsymbol{f}^{[d]}}P^{n-(d-\rho)R} + O_{\mathscr{C},\boldsymbol{f},\rho}(P^{n-(d-\rho)R-\delta'})$$

for some $\delta' > 0$ depending on $\mathscr{C}, \Delta, d, n, R, \rho$ only. By repeating this argument with $\widetilde{\omega}_\epsilon, \widetilde{\varphi}_\epsilon$ in place of $\omega_\epsilon, \varphi_\epsilon$ and substituting into (6.60), we obtain

$$M_{\boldsymbol{f}}(P, P^\rho) = \mathfrak{I}_{\boldsymbol{f}^{[d]}}P^{n-(d-\rho)R} + O_{\mathscr{C},\boldsymbol{f},\rho}(P^{n-(d-\rho)R-\delta'})$$

which is exactly (6.8).

Essentially the same argument suffices to prove (6.7). Consulting the definitions (6.53), (6.54) and (6.55) we see that for all $P \geq 1$ and $\epsilon \in (0,1]$ we must have

$$\sum_{\boldsymbol{x} \in \mathbb{Z}^n} \widetilde{\varphi}_\epsilon(P^{-\rho} \boldsymbol{f}(\boldsymbol{x})) \widetilde{\omega}_\epsilon(\boldsymbol{x}/P) \leq N_{\boldsymbol{f}}(P, P^\rho) \leq \sum_{\boldsymbol{x} \in \mathbb{Z}^n} \varphi_\epsilon(P^{-\rho} \boldsymbol{f}(\boldsymbol{x})) \omega_\epsilon(\boldsymbol{x}/P). \tag{6.61}$$

Applying (6.7) and (6.8) shows that

$$\begin{aligned}\sum_{\boldsymbol{x} \in \mathbb{Z}^n} \varphi_\epsilon(P^{-\rho} \boldsymbol{f}(\boldsymbol{x})) \omega_\epsilon(\boldsymbol{x}/P) = {} & \hat{\varphi}_\epsilon(\mathbf{0}) \mathfrak{I}_{\boldsymbol{f}, \omega_\epsilon} P^{n-(d-\rho)R} \\ & + \epsilon^{-O_{d,n}(1)} O_{\mathscr{C}, \boldsymbol{f}, \rho}(\xi(P)^\delta P^{n-(d-\rho)R} + P^{n-(d-\rho)R-\delta}), \\ = {} & \mathfrak{I}_{\boldsymbol{f}^{[d]}} P^{n-(d-\rho)R} + O_{\mathscr{C}, \boldsymbol{f}, \rho}(\epsilon P^{n-(d-\rho)R}) \\ & + \epsilon^{-O_{d,n}(1)} O_{\mathscr{C}, \boldsymbol{f}, \rho}(\xi(P)^\delta P^{n-(d-\rho)R} + P^{n-(d-\rho)R-\delta}),\end{aligned} \tag{6.62}$$

where the last line follows by (6.59) and the fact that $\hat{\varphi}_\epsilon(\mathbf{0}) = 1 + O_R(\epsilon)$. Now by choosing

$$\epsilon = \max\{\xi(P)^{\delta/O_{d,n}(1)},\ P^{-\delta/O_{d,n}(1)}\},$$

we deduce from (6.62) that

$$\begin{aligned}\sum_{\boldsymbol{x} \in \mathbb{Z}^n} \varphi_\epsilon(P^{-\rho} \boldsymbol{f}(\boldsymbol{x})) \omega_\epsilon(\boldsymbol{x}/P) = {} & \mathfrak{I}_{\boldsymbol{f}^{[d]}} P^{n-(d-\rho)R} \\ & + O_{\mathscr{C}, \boldsymbol{f}, \rho}(\xi(P)^{\delta'} P^{n-(d-\rho)R} + P^{n-(d-\rho)R-\delta'}),\end{aligned}$$

for some $\delta' > 0$ depending on $\mathscr{C}, \Delta, d, n, R, \rho$ only. Repeating this argument with $\widetilde{\omega}_\epsilon$ and $\widetilde{\varphi}_\epsilon$ instead of ω_ϵ and φ_ϵ, and substituting into (6.61), we obtain

$$N_{\boldsymbol{f}}(P, P^\rho) = \mathfrak{I}_{\boldsymbol{f}^{[d]}} P^{n-(d-\rho)R} + O_{\mathscr{C}, \boldsymbol{f}, \rho}(\xi(P)^{\delta'} P^{n-(d-\rho)R} + P^{n-(d-\rho)R-\delta'}),$$

and together with (6.8) this proves (6.7). □

6.4 Irrational systems of forms

We now consider Proposition 6.3, giving an extra bound on the sum $S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)$ in the case when $\boldsymbol{f}^{[d]}$ is irrational.

6.4.1 Passage to a system of linear forms

Recall that Lemma 6.4 gave us a bound for the sum $S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)$ in terms of the counting function $U_{\boldsymbol{\alpha} \cdot \boldsymbol{f}}(B, \delta)$ from Definition 3.10 in §3.2.1. This counting function involves a system of multilinear forms of degree $d-1$.

To prove Proposition 6.3, we use repeated applications of Lemma 2.11 from §2.3.2 to relate the counting function $U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(B,\delta)$ to certain counting functions $U_{\boldsymbol{\alpha}\cdot\boldsymbol{f},k}(B,\delta)$ which we will now define.

The underlying polynomials in the counting function $U_{\boldsymbol{\alpha}\cdot\boldsymbol{f},k}(B,\delta)$ will be as follows. Given integers $d \geq 2$ and $k \in \{1,\dots,d-1\}$ and a degree d polynomial $f(\boldsymbol{x})$ with leading part $f^{[d]}(\boldsymbol{x})$, define a system of multilinear forms

$$\boldsymbol{m}^{(f;\,k)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(k)}) \in \mathbb{R}[\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(k)}]^{n^{d-k}} \tag{6.63}$$

with components

$$\sum_{j_1,\dots,j_k=1}^{n} x^{(1)}_{j_k}\cdots x^{(k)}_{j_k}\frac{\partial^d f(\boldsymbol{x})}{\partial x_{j_1}\cdots\partial x_{j_k}\partial x_{i_{k+1}}\cdots\partial x_{i_d}} \qquad (1 \leq i_{k+1},\dots,i_d \leq d) \tag{6.64}$$

arranged in some order. In particular we can take

$$\boldsymbol{m}^{(f;\,d-1)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)}) = \boldsymbol{m}^{(f)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)}) \tag{6.65}$$

to be the vector of forms defined in (1.27) and (1.28). We further define $\boldsymbol{m}^{(f;\,0)} \in \mathbb{R}^{n^d}$ to be the vector with components

$$\frac{\partial^d f(\boldsymbol{x})}{\partial x_{i_1}\cdots\partial x_{i_d}} \qquad (1 \leq i_1,\dots,i_d \leq d)$$

in some order. Thus each system of forms $\boldsymbol{m}^{(f;\,k)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(k)})$ has exactly the components of $\boldsymbol{m}^{(f;\,0)}$ as its coefficients.

Definition 6.8. Let the distance function $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$ be as in §1.1 and let the forms $\boldsymbol{m}^{(f;k)}$ be as above. For each $k \in \{1,\dots,d-1\}$, $B \geq 1$ and $\delta > 0$ define $U_{f,k}(B,\delta)$ to be the number of k-tuples of vectors $\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(k)} \in \mathbb{Z}^n$ such that

$$\|\boldsymbol{x}^{(1)}\|_\infty,\dots,\|\boldsymbol{x}^{(k)}\|_\infty \leq B, \qquad \|\boldsymbol{m}^{(f;\,k)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(k)})\|_{\mathbb{R}/\mathbb{Z}} < \delta.$$

In particular $U_{f,d-1}(B,\delta) = U_f(B,\delta)$ is the counting function from Definition 3.10.

The following lemma passes from $U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(B,\delta)$ to $U_{\boldsymbol{\alpha}\cdot\boldsymbol{f},d-1}(B,\delta)$.

Lemma 6.9. *Let $U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(B,\delta)$ be as in Definition 3.10 and let $U_{f,k}(B,\delta)$ be as in Definition 6.8. Suppose that $C,P \geq 1$ and $\boldsymbol{\alpha} \in \mathbb{R}^R$ such that*

$$C^{-1} < \|\boldsymbol{\alpha}\|_\infty < C,$$
$$U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(P,P^{-1}) > C^{-1}P^{(d-1)n}. \tag{6.66}$$

Then for each $k \in \{1, \ldots, d-1\}$, there is $C_k \geq 1$ depending only on C, d, n such that for some $q_k \in \mathbb{N}$ with $q_k = O_{d,n}(C_k)$, we either have

$$U_{q_k \boldsymbol{\alpha} \cdot \boldsymbol{f}, k}(P, P^{-1}) \geq C_k^{-1} P^{kn}, \tag{6.67}$$

or else $P \ll_{d,n} C_{k+1}$. Furthermore we may take $C_{d-1} = C$ and $C_{k-1} = O_{d,n}(C_k^2)$.

Proof. In this proof, implied constants in $\ll$, $\gg$ or $O(\,\cdot\,)$ may always depend on d, n.

By (6.65), the case $k = d-1$ of (6.67) holds with $C_{d-1} = C$ and $q_{d-1} = 1$. We suppose by induction that the lemma holds for some particular $k > 1$, and prove that it holds with k replaced by $k-1$.

We may assume that $P \gg C_{k-1}$ for some sufficiently large implicit constant, since otherwise the case $k-1$ of the lemma follows at once. After increasing C_{k-1} if necessary we may assume that $P \gg C_k$, and so we may fix a natural number q_k, with $q_k = O(C_k)$, such that (6.67) holds. Define

$$\begin{aligned}
&\mathcal{Z} = \{\boldsymbol{x} \in \mathbb{Z}^n : \|\boldsymbol{x}\|_\infty \leq P\}, \\
&\mathcal{X} = \Big\{(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k-1)}) \in \mathcal{Z}^{k-1} : \\
&\qquad\qquad \#\{\boldsymbol{x}^{(k)} \in \mathcal{Z} : \|q_k \boldsymbol{m}^{(\boldsymbol{\alpha} \cdot \boldsymbol{f}; k)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)})\|_{\mathbb{R}/\mathbb{Z}} < P^{-1}\} > \tfrac{1}{2} C_k^{-1} P^n \Big\}.
\end{aligned}$$

That is, $\mathcal{X}$ is the set of $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k-1)})$ which contribute more than $\frac{1}{2} C_k^{-1} P^n$ values of $\boldsymbol{x}^{(k)}$ to the count $U_{q_k \boldsymbol{\alpha} \cdot \boldsymbol{f}, k}(P, P^{-1})$. Then

$$\begin{aligned}
&C_k^{-1} P^{kn} \\
&\quad \leq U_{q_k \boldsymbol{\alpha} \cdot \boldsymbol{f}, k}(P, P^{-1}) \\
&\quad = \sum_{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k-1)} \in \mathcal{Z}} \#\big\{\boldsymbol{x}^{(k)} \in \mathcal{Z} : \|q_k \boldsymbol{m}^{(\boldsymbol{\alpha} \cdot \boldsymbol{f}; k)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)})\|_{\mathbb{R}/\mathbb{Z}} < P^{-1}\big\} \\
&\quad \leq \sum_{(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k-1)}) \in \mathcal{X}} P^n + \sum_{(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k-1)}) \in \mathcal{Z}^{k-1} \setminus \mathcal{X}} \tfrac{1}{2} C_k^{-1} P^n \\
&\quad \leq (\#\mathcal{X}) P^n + \tfrac{1}{2} C_k^{-1} P^{kn}
\end{aligned}$$

and so

$$\#\mathcal{X} \geq \tfrac{1}{2} C_k^{-1} P^{(k-1)n}. \tag{6.68}$$

Now $\boldsymbol{m}^{(\boldsymbol{\alpha} \cdot \boldsymbol{f}; k)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)})$ is linear in $\boldsymbol{x}^{(k)}$, so we can view it as a matrix product

$$\boldsymbol{m}^{(\boldsymbol{\alpha} \cdot \boldsymbol{f}; k)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}) = L(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k-1)}) \boldsymbol{x}^{(k)}.$$

We apply Lemma 2.11 with the choice $q_k L(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k-1)})\boldsymbol{x}^{(k)}$ for the matrix L. This shows that if $(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k-1)}) \in \mathcal{X}$, then there is $q' \in \mathbb{N}$, satisfying $q' = O(C_k)$, such that

$$\|q' q_k L(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k-1)})\|_{\mathbb{R}/\mathbb{Z}} \ll C_k P^{-2}.$$

The components of $\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f};\, k-1)}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k-1)})$ are exactly the entries of the matrix $L(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k-1)})$, by the definition of $\boldsymbol{m}^{(f,k)}$ in (6.63) and (6.64). Moreover we assumed that at the start of the proof that $P \gg C_k$ for some sufficiently large implicit constant, so we can assume that $C_k P^{-2} \ll P^{-1}$. Therefore each of the $(k-1)$-tuples in $\mathcal{X}$ satisfies

$$\left\|q' q_k \boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f};\, k-1)}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k-1)})\right\|_{\mathbb{R}/\mathbb{Z}} < P^{-1}.$$

So every member of the set $\mathcal{X}$ is counted by the counting function $U_{q' q_k, k-1}(P, P^{-1})$, for some $q' \in \mathbb{N}$ satisfying $q' = O(C_k)$. Together with (6.68), this proves (6.67) for all k by induction, with $C_{k-1} = O(C_k^2)$ and $q_{k-1} = O(C_k q_k) = O(C_{k-1})$ both bounded as required. □

6.4.2 Proof of Proposition 6.3

We are now ready to prove Proposition 6.3. We use Lemma 6.4 to bound $S_{\boldsymbol{f},\omega}(\boldsymbol{\alpha}; P)$ in terms of $U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(B, \delta)$. We then use Lemma 6.4 to pass to $U_{\boldsymbol{\alpha}\cdot\boldsymbol{f},1}(B, \delta)$, which counts solutions to a linear inequality. This we will treat using Lemma 2.11.

We treat the case of the proposition when $\boldsymbol{f}^{[d]}$ is irrational first, and then the case of algebraic coefficients later.

Proof of Proposition 6.3, irrational case. Let $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$ be as in §1.1, let $U_{\boldsymbol{\alpha}\cdot\boldsymbol{f}}(B, \delta)$ be as in Definition 3.10, let $\boldsymbol{m}^{(f;k)}$ be as in (6.63) and (6.64), and let $U_{f,k}(B, \delta)$ be as in Definition 6.8.

Let $\boldsymbol{f}^{[d]}$ be irrational in the sense of Definition 1.7. By Lemma 6.4 it suffices to find a function η, depending only on $\boldsymbol{f}^{[d]}$, such that $\eta(P) \to 0$ as $P \to \infty$, and

$$\sup_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^R \\ \eta(P) < \|\boldsymbol{\alpha}\|_\infty < \eta(P)^{-1}}} U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P, P^{-1}) \leq \eta(P) P^{(d-1)n} \tag{6.69}$$

for all $P \gg_{\boldsymbol{f}^{[d]}} 1$. Suppose there is no such function. It follows that for some $C \geq 1$ there exist arbitrarily large values of P such that some $\boldsymbol{\alpha} \in \mathbb{R}^R$ with satisfies

$$C^{-1} < \|\boldsymbol{\alpha}\|_\infty < C, \tag{6.70}$$

$$U_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(P, P^{-1}) > C^{-1} P^{(d-1)n}. \tag{6.71}$$

This is precisely the hypothesis (6.66) of Lemma 6.9. We apply that lemma with the choice $k = 1$ for the parameter k. This shows that there exists C_1 such that whenever $P \gg_{d,n} C_1$, there is $q_1 \in \mathbb{N}$ with $q_1 \ll_{d,n} C_1$ such that

$$U_{q_1\boldsymbol{\alpha}\cdot\boldsymbol{f},k}(P, P^{-1}) \geq C_1^{-1}P^{kn},$$

which we can rewrite as

$$\#\big\{\boldsymbol{x} \in \mathbb{Z}^n : \|\boldsymbol{x}\|_\infty \leq P,\ \big\|q_1\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f};\,1)}(\boldsymbol{x})\big\|_{\mathbb{R}/\mathbb{Z}} < P^{-1}\big\} \geq C_1^{-1}P^n. \tag{6.72}$$

Now $\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f};\,1)}(\boldsymbol{x}) = L\boldsymbol{x}$ where the entries of the matrix L are the components of the vector $\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f};\,0)}$, by the definitions above. We apply Lemma 2.11 to this matrix L. Together with (6.72), this shows that

$$\min_{q \ll C_1}\big\|q\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f};\,0)}\big\|_{\mathbb{R}/\mathbb{Z}} \ll_{d,n} C_1P^{-2}. \tag{6.73}$$

By assumption, this holds for pairs $(P, \boldsymbol{\alpha})$ with P arbitrarily large and $C^{-1} < \|\boldsymbol{\alpha}\|_\infty < C$. But this is impossible. If there are sequences $(P_1, P_2, \dots)$, $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \dots)$ such that $P_i \to \infty$, $C^{-1} < \|\boldsymbol{\alpha}^{(i)}\|_\infty < C$ and

$$\min_{q \ll C_1}\big\|q\boldsymbol{m}^{(\boldsymbol{\alpha}^{(i)}\cdot\boldsymbol{f};\,0)}\big\|_{\mathbb{R}/\mathbb{Z}} \ll_{d,n} C_1P^{-2},$$

then as $\|\boldsymbol{\alpha}^{(i)}\|_\infty \leq C$ we can pass to a subsequence and assume $\boldsymbol{\alpha}^{(i)} \to \boldsymbol{\alpha}$. Letting $i \to \infty$ we see that $\boldsymbol{\alpha} \cdot \boldsymbol{f}^{[d]}$ has rational coefficients. But we must have $\boldsymbol{\alpha} \neq \boldsymbol{0}$, as $\|\boldsymbol{\alpha}^{(i)}\|_\infty > C^{-1}$ for each i, so this contradicts the irrationality of $\boldsymbol{f}^{[d]}$, proving the proposition. □

Proof of Proposition 6.3, algebraic case. As in the first part of the proof we assume the notation of Definition 6.8, allow all implicit constants to depend on d and n and use the distance function $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$ defined in §1.1.

Let $\boldsymbol{f}^{[d]}$ be irrational with algebraic coefficients. As in the first part of the proof, we suppose that $C \geq 1$, $P \geq 1$ and $\boldsymbol{\alpha} \in \mathbb{R}^R$ such that (6.70) and (6.71) hold, and we will deduce that P is bounded. More precisely, we will show that $P \ll_{\boldsymbol{f}^{[d]}} C^{O_{d,R}(1)}$. By the comments above (6.71), this shows that we may take $\eta(P) = P^{-1/O(1)}$ in (6.69), and hence in the conclusion (6.9) of the proposition.

Let C_1 be as in Lemma 6.9. That lemma gives us a series of constants $C_{d-1}, \dots C_1$ satisfying $C_{d-1} = C$ and $C_{k-1} = O_{d,n}(C_k^2)$, so we have $C_1 \ll_{d,n} C^{O_{d,R}(1)}$. We claim that $P \ll_{d,n} C_1$. Suppose for a contradiction that this is false.

Since $P \gg_{d,n} C_1$, the comments below (6.71) show that the bound (6.73) will hold, that is,

$$\min_{q \ll C_1} \left\| q\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f};\,0)} \right\|_{\mathbb{R}/\mathbb{Z}} \ll_{d,n} C_1 P^{-2}. \tag{6.74}$$

Thus there exist $q \ll C_1$ and $\boldsymbol{b} \in \mathbb{Z}^{n^{d-1}}$ such that

$$\| q\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f};\,0)} - \boldsymbol{b} \|_\infty \ll_{d,n} C_1 P^{-2}, \tag{6.75}$$

where, since $\|\boldsymbol{\alpha}\|_\infty < C$ and $q \ll_{d,n} C_1$, we must have

$$\|\boldsymbol{b}\|_\infty \ll_{d,n} CC_1. \tag{6.76}$$

Let $Y \subset \mathbb{R}^{n^{d-1}}$ be the span of the R linearly independent vectors

$$\boldsymbol{m}^{(f_1;\,0)}, \ldots, \boldsymbol{m}^{(f_R;\,0)},$$

and let this space Y be cut out by the linear equations

$$L\boldsymbol{t} = \boldsymbol{0}$$

where L is an $(n^{d-1} - R) \times n^{d-1}$ real matrix depending on $\boldsymbol{f}^{[d]}$ only. Since

$$L\boldsymbol{m}^{(\boldsymbol{\alpha}\cdot\boldsymbol{f};\,0)} = \boldsymbol{0},$$

it follows from (6.75) that

$$\|L\boldsymbol{b}\|_\infty \ll_{\boldsymbol{f}^{[d]}} C_1 P^{-2}. \tag{6.77}$$

Since the $f_i^{[d]}$ have algebraic coefficients, we may assume that L has algebraic entries. As no nonzero form $\boldsymbol{\beta} \cdot \boldsymbol{f}^{[d]}$ has integral coefficients, we see that $L\boldsymbol{t} = \boldsymbol{0}$ has no integral solutions. We now apply Corollary 2.13 from §2.3.3 in Chapter 2. This shows that

$$\|L\boldsymbol{b}\|_\infty \gg_{\boldsymbol{f}^{[d]},\epsilon} \|\boldsymbol{b}\|_\infty^{-R-\epsilon}.$$

So (6.77) implies that

$$P^2 \ll_{\boldsymbol{f}^{[d]},\epsilon} C_1 \|\boldsymbol{b}\|_\infty^{R+\epsilon},$$

and on choosing $\epsilon = 1$ and applying (6.76) it follows that

$$P^2 \ll_{\boldsymbol{f}^{[d]}} C_1 (CC_1)^{R+1}.$$

Since $n \geq R$ and $C_1 \ll_{d,n} C^{O_{d,R}(1)}$ it follows that $P \ll_{\boldsymbol{f}^{[d]}} C^{O_{d,R}(1)}$, as claimed. By the comments at the start of the proof, this proves the proposition. $\square$

Chapter 7

A result over number fields

In Chapters 3–5 we considered equations to be solved in rational integers. Here we deduce a result for equations over a number field.

7.1 Preliminaries

Let K be a number field of degree m over $\mathbb{Q}$. In this section we describe the form taken by the Hasse principle and weak approximation over K.

If $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$ we let $\mathcal{N}(\mathfrak{p})$ be the norm $\#(\mathcal{O}_K/\mathfrak{p})$ and for each $x \in \mathcal{O}_K$ we let $\mathrm{ord}_{\mathfrak{p}}(x)$ be the unique non-negative integer k such that $x \in \mathfrak{p}^k \setminus \mathfrak{p}^{k+1}$, where it is understood that $\mathfrak{p}^0 = \mathcal{O}_K$. Given $y \in K$ we can write $y = \frac{a}{b}$ for some $a, b \in \mathcal{O}_K$, and we then define the $\mathfrak{p}$-adic absolute value by $|y|_{\mathfrak{p}} = \mathcal{N}(\mathfrak{p})^{\mathrm{ord}_{\mathfrak{p}}(b) - \mathrm{ord}_{\mathfrak{p}}(a)}$. We let $K_{\mathfrak{p}}$ be the completion of K as a metric space with respect to $|\cdot|_{\mathfrak{p}}$, see §5 of Chapter II in Cassels and Frölich [CF67]. When $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ this recovers the field $\mathbb{Q}_p$ with the usual p-adic metric.

Later we will use an alternative description of the $K_{\mathfrak{p}}$. By the Primitive Element Theorem there exists some $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$. Denoting by f the unique monic minimal polynomial of α over $\mathbb{Q}$, we have $K \cong \mathbb{Q}[t]/(f(t))$. Let p be a prime and let $f_1(t), \dots, f_J(t)$ be the irreducible factors of $f(t)$ over $\mathbb{Q}_p$. Up to isomorphism the fields $\mathbb{Q}_p[t]/(f_i(t))$ are exactly the fields $K_{\mathfrak{p}}$ for primes $\mathfrak{p}$ dividing the rational prime p, by the second Theorem in §10 of Chapter II from Cassels and Frölich [CF67]. So the Chinese Remainder Theorem gives

$$\mathbb{Q}_p[t]/(f(t)) \cong \bigoplus_{\mathfrak{p} | p} K_{\mathfrak{p}}, \tag{7.1}$$

where this is an isomorphism as rings, in particular as $\mathbb{Q}_p$-vector spaces and hence as topological spaces.

We can handle the Archimedean completions of K in a similar way. Let the embeddings of K into $\mathbb{R}$ be $\tau_1, \ldots, \tau_{r_1}$ and let the remaining embeddings of K into $\mathbb{C}$ be $\tau_{r_1+1}, \ldots, \tau_{r_1+r_2}$ and $\bar{\tau}_{r_1+1}, \ldots, \bar{\tau}_{r_1+r_2}$ so that $m = r_1 + 2r_2$. For each $i = 1, \ldots, r_1 + r_2$, set $|x|_{\tau_i} = |\tau_i(x)|$ and let K_{τ_i} be the completion of K with respect to $|\cdot|_{\tau_i}$. Thus $K_{\tau_i} \cong \mathbb{R}$ for $i \leq r_1$ and $K_{\tau_i} \cong \mathbb{C}$ for $i > r_1$. Over $\mathbb{R}$ the polynomial $f(t)$ has irreducible factors $t - \tau_i(\alpha)$ for $i = 1, \ldots, r_1$ and $(t - \tau_i(\alpha))(t - \bar{\tau}_i(\alpha))$ for $i = r_1 + 1, \ldots, r_1 + r_2$. As with (7.1) we deduce that

$$\mathbb{R}[t]/(f(t)) \cong \bigoplus_{i=1}^{r_1+r_2} K_{\tau_i} \tag{7.2}$$

as rings and as topological spaces.

Let M_K be the set consisting of $\tau_1, \ldots, \tau_{r_1+r_2}$ and all prime ideals of $\mathcal{O}_K$.

In earlier chapters we worked with projective varieties; here we also consider affine varieties. By an affine variety we mean a reduced, not necessarily irreducible, closed subscheme of affine space over a field. See §2 in Chapter 1 from the first volume and Chapter 3 from the second volume of Shafarevich [Sha13a, Sha13b] for a full definition.

Let U be a Zariski open subset of an affine or projective variety over K. For each $v \in M_K$ we have $U(K) \subset U(K_v)$, and so if S is a finite subset of M_K then we can embed $U(K)$ into $\prod_{v \in S} U(K_v)$. We then say that V satisfies the *Hasse principle* if the existence of a point in $U(K_v)$ for every $v \in M_k$ guarantees the existence of a point in $U(K)$. We say that U satisfies *weak approximation* if $U(K)$ is dense in $\prod_{v \in S} U(K_v)$ for any finite S, where we equip each $U(K_v)$ with the v-adic topology, that is, the topology induced by $|\cdot|_v$.

7.2 Weil restriction

Recall the result of Birch stated as Theorem 1.2 in Chapter 1. Birch [Bir62, §8] also gave a result over algebraic number fields, using the construction known as *Weil restriction* or restriction of scalars. We give an explicit description of the relevant case of this construction. For a more abstract approach see Weil [Wei82, §1.3].

As above let K be a number field of degree m, let $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$ and let $f(t)$ be the monic minimal polynomial of α. Let $\boldsymbol{G}(\boldsymbol{x})$ be a system of R forms of degree d in n variables, with coefficients in $\mathcal{O}_K$. The equation $\boldsymbol{G}(\boldsymbol{x}) = \boldsymbol{0}$ defines an affine variety in n-space over K. We let U be the Zariski open subset of this variety defined by the condition $\operatorname{rank}(\partial G_i(\boldsymbol{x})/\partial x_j)_{ij} = R$, so that the points of U correspond to the nonsingular solutions to $\boldsymbol{G}(\boldsymbol{x}) = \boldsymbol{0}$.

Let each of $\boldsymbol{y}^{(1)}, \dots, \boldsymbol{y}^{(m)}$ be a vector of n variables and put

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}^{(1)} \\ \vdots \\ \boldsymbol{y}^{(m)} \end{pmatrix}.$$

Taking $1, \dots, \alpha^{m-1}$ as a basis for K over $\mathbb{Q}$, we can write

$$\boldsymbol{G}(\boldsymbol{y}^{(1)} + \cdots + \alpha^{m-1}\boldsymbol{y}^{(m)}) = \widetilde{\boldsymbol{G}}^{(1)}(\boldsymbol{y}) + \cdots + \alpha^{m-1}\widetilde{\boldsymbol{G}}^{(m)}(\boldsymbol{y}) \tag{7.3}$$

for some systems $\widetilde{\boldsymbol{G}}^{(j)}$ of R degree d forms in mn variables with coefficients in $\mathbb{Z}$. Put

$$\widetilde{\boldsymbol{G}}(\boldsymbol{y}) = \begin{pmatrix} \widetilde{\boldsymbol{G}}^{(1)}(\boldsymbol{y}) \\ \vdots \\ \widetilde{\boldsymbol{G}}^{(m)}(\boldsymbol{y}) \end{pmatrix}.$$

The condition $\widetilde{\boldsymbol{G}}(\boldsymbol{y}) = \boldsymbol{0}$ defines an affine variety in mn-space over $\mathbb{Q}$. The open subset of this variety defined by $\operatorname{rank}(\partial\widetilde{G}_i(\boldsymbol{y})/\partial x_j)_{ij} = mR$ is the Weil restriction of U, and we denote it by $R_{K/\mathbb{Q}}U$. The points of $R_{K/\mathbb{Q}}U$ may be described as follows.

Lemma 7.1. *Let $\mathbb{F}$ be a field extending $\mathbb{Q}$. There is a bijection between the $\mathbb{F}$-points of $R_{K/\mathbb{Q}}U$ and the $\mathbb{F}[t]/(f(t))$-points of U. If $\boldsymbol{y}$ represents an $\mathbb{F}$-point of $R_{K/\mathbb{Q}}U$ then the corresponding $\mathbb{F}[t]/(f(t))$-point of U is given by $\boldsymbol{x}(t) = \boldsymbol{y}^{(1)} + \cdots + t^{m-1}\boldsymbol{y}^{(m)}$.*

Proof. By construction the equation $\widetilde{\boldsymbol{G}}(\boldsymbol{y}) = \boldsymbol{0}$ is equivalent to the polynomial congruence $\boldsymbol{G}(\boldsymbol{x}(t)) \equiv \boldsymbol{0} \mod f(t)$. It remains only to show that $\operatorname{rank}(\partial\widetilde{G}_i(\boldsymbol{y})/\partial x_j)_{ij} = mR$ if and only if $\operatorname{rank}(\partial G_i(\boldsymbol{x})/\partial x_j)_{ij}|_{\boldsymbol{x}=\boldsymbol{x}(t)} = R$, where the vector $\boldsymbol{x}(t)$ is understood to take values in $(\mathbb{F}[t]/(f(t)))^n$.

Let $g_1(t), \dots, g_K(t)$ be the irreducible factors of $f(t)$ over $\mathbb{F}$. Let $\bar{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$, and let $\alpha_{k,1}, \dots, \alpha_{k,m_k}$ be the roots of $g_k(t)$ in $\bar{\mathbb{F}}$. Define a linear change of variables over $\bar{\mathbb{F}}$ by

$$\boldsymbol{z}^{(k,\ell)} = \boldsymbol{y}^{(1)} + \alpha_{k,\ell}\boldsymbol{y}^{(2)} + \cdots + \alpha_{k,\ell}^{m-1}\boldsymbol{y}^{(m)}.$$

By the definition (7.3) this transforms the system $\widetilde{\boldsymbol{G}}(\boldsymbol{y}) = \boldsymbol{0}$ into $\boldsymbol{G}'(\boldsymbol{z}) = \boldsymbol{0}$, where

$$\boldsymbol{G}'(\boldsymbol{z}) = \begin{pmatrix} \boldsymbol{G}(\boldsymbol{z}^{(1,1)}) \\ \vdots \\ \boldsymbol{G}(\boldsymbol{z}^{(K,m_K)}). \end{pmatrix}.$$

So $\operatorname{rank}(\partial\widetilde{G}_i(\boldsymbol{y})/\partial x_j)_{ij} = mR$ if and only if $\operatorname{rank}(\partial G_i(\boldsymbol{x})/\partial x_j)_{ij}|_{\boldsymbol{x}=\boldsymbol{z}^{(k,\ell)}} = R$ for each k and ℓ. Because $\boldsymbol{y} \in \mathbb{F}^{mn}$, the vectors $\boldsymbol{z}^{(k,1)}, \dots, \boldsymbol{z}^{(k,m_k)}$ are Galois conjugates over $\mathbb{F}$ and so we need only consider $\ell = 1$. Since $\boldsymbol{z}^{(k,\ell)} = \boldsymbol{x}(\alpha_{k,\ell})$ we obtain the condition that $\operatorname{rank}(\partial G_i(\boldsymbol{x})/\partial x_j)_{ij}|_{\boldsymbol{x}=\boldsymbol{x}(\alpha_{k,1})} = R$ for each k. By the Chinese Remainder Theorem this occurs exactly when $\operatorname{rank}(\partial G_i(\boldsymbol{x})/\partial x_j)_{ij}|_{\boldsymbol{x}=\boldsymbol{x}(t)} = R$. □

Corollary 7.2. *The set $R_{K/\mathbb{Q}}U(\mathbb{Q})$ is in bijection with $U(K)$. If we identify $U(K)$ with its image in $\prod_{\mathfrak{p}|p} U(K_\mathfrak{p})$ under the diagonal embedding, then there is a homeomorphism from $R_{K/\mathbb{Q}}U(\mathbb{Q}_p)$ with the p-adic topology to $\prod_{\mathfrak{p}|p} U(K_\mathfrak{p})$ with the $\mathfrak{p}$-adic topology which sends $R_{K/\mathbb{Q}}U(\mathbb{Q})$ to $U(K)$. Similarly, there is a homeomorphism from $R_{K/\mathbb{Q}}U(\mathbb{R})$ with the real topology to $\prod_{i=1}^{r_1+r_2} U(K_{\tau_i})$ with the $|\cdot|_{\tau_i}$-topology, which maps $R_{K/\mathbb{Q}}U(\mathbb{Q})$ to the diagonally embedded copy of $U(K)$.*

Proof. The bijection from $R_{K/\mathbb{Q}}U(\mathbb{Q})$ to $U(K)$ is immediate from Lemma 7.1. Equation (7.1), this gives us a bijection from $R_{K/\mathbb{Q}}U(\mathbb{Q}_p)$ to $\prod_{\mathfrak{p}|p} U(K_\mathfrak{p})$, while by (7.2) we have a bijection from $R_{K/\mathbb{Q}}U(\mathbb{R})$ to $\prod_{i=1}^{r_1+r_2} U(K_{\tau_i})$. By the comments after (7.1) and (7.2), these maps preserve the topology. Finally, if $\boldsymbol{y} \in \mathbb{Q}^{mn}$ then we have $\boldsymbol{x}(t) \in (\mathbb{Q}[t]/(f(t)))^n \cong K^n$, so on $R_{K/\mathbb{Q}}U(\mathbb{Q})$ we recover the diagonal embedding as claimed. □

Since nonsingular solutions to $\boldsymbol{G}(\boldsymbol{x}) = \boldsymbol{0}$ correspond to points of $V(\boldsymbol{G}) \setminus \mathrm{Sing}(\boldsymbol{G})$, and likewise for $\widetilde{\boldsymbol{G}}(\boldsymbol{y}) = \boldsymbol{0}$ and $V(\widetilde{\boldsymbol{G}}) \setminus \mathrm{Sing}(\widetilde{\boldsymbol{G}})$, we deduce:

Corollary 7.3. *The Hasse principle (resp. weak approximation) holds for the projective open set $V(\boldsymbol{G}) \setminus \mathrm{Sing}(\boldsymbol{G})$ over K if and only if the Hasse principle (resp. weak approximation) holds for $V(\widetilde{\boldsymbol{G}}) \setminus \mathrm{Sing}(\widetilde{\boldsymbol{G}})$ over $\mathbb{Q}$.*

Geometrically the system $\widetilde{\boldsymbol{G}}$ may be described as follows. Let the roots of $f(t)$ in $\bar{\mathbb{Q}}$ be $\alpha_1, \dots, \alpha_m$ and define a linear change of variables over $\bar{\mathbb{Q}}$ by

$$\boldsymbol{z}^{(j)} = \boldsymbol{y}^{(1)} + \alpha_j \boldsymbol{y}^{(2)} + \cdots + \alpha_j^{m-1} \boldsymbol{y}^{(m)}.$$

As in the proof of Lemma 7.1 this transforms the system $\widetilde{\boldsymbol{G}}(\boldsymbol{y}) = \boldsymbol{0}$ into $\boldsymbol{G}'(\boldsymbol{z}) = \boldsymbol{0}$, where

$$\boldsymbol{G}'(\boldsymbol{z}) = \begin{pmatrix} \boldsymbol{G}(\boldsymbol{z}^{(1)}) \\ \vdots \\ \boldsymbol{G}(\boldsymbol{z}^{(m)}). \end{pmatrix}.$$

In particular, the quantity σ^* from Definition 5.1 satisfies

$$\sigma^*(\widetilde{\boldsymbol{G}}) = \sigma^*(\boldsymbol{G}') \leq (m-1)n + \sigma^*(\boldsymbol{G}). \tag{7.4}$$

Similarly the quantity $\sigma_\mathbb{R}$ defined in (1.14) satisfies

$$\begin{aligned} \sigma_\mathbb{R}(\widetilde{\boldsymbol{G}}) &\leq 1 + \max_{\boldsymbol{\gamma} \in \bar{\mathbb{Q}}^{mR} \setminus \{\boldsymbol{0}\}} \dim \mathrm{Sing}(\boldsymbol{\gamma} \cdot \boldsymbol{G}') \\ &\leq (m-1)n + 1 + \max_{\boldsymbol{\delta} \in \bar{\mathbb{Q}}^{R} \setminus \{\boldsymbol{0}\}} \dim \mathrm{Sing}(\boldsymbol{\delta} \cdot \boldsymbol{G}), \end{aligned}$$

and the same argument which was used to prove (1.15) in the introduction then shows that

$$\sigma_\mathbb{R}(\widetilde{\boldsymbol{G}}) \leq (m-1)n + 1 + \dim \mathrm{Sing}(\boldsymbol{G}). \tag{7.5}$$

7.3 Results

In the case of a nonsingular system of forms over a number field, we may summarise the conclusions of Birch [Bir62, §8] as follows.

Theorem 7.4 (Birch [Bir62])**.** *Let K be a number field of degree m over $\mathbb{Q}$. Let $\boldsymbol{G}$ be a system of R homogeneous forms of degree D in n variables with coefficients in $\mathcal{O}_K$. Suppose that $\boldsymbol{G}$ is nonsingular in the sense of §1.1. Then $V(\boldsymbol{G})$ satisfies the Hasse principle and weak approximation over K provided that*

$$n \geq R(mR+1)(d-1)2^{d-1} + R. \tag{7.6}$$

The right-hand side of (7.6) depends on the degree m of $K/\mathbb{Q}$, and consequently (7.6) requires more variables than the condition (1.10) from the introduction. Skinner [Ski97] is able to remedy this by introducing a variant of the circle method adapted to number fields, giving a number of variables equal to that in (1.10).

Theorem 7.5 (Skinner [Ski97])**.** *In Theorem 7.4 we may replace the condition* (7.6) *with*

$$n \geq R(R+1)(d-1)2^{d-1} + R.$$

Birch and Skinner also give an asymptotic analogous to (1.7) from the introduction. They estimate the number of solutions to $\boldsymbol{G}(\boldsymbol{x}) = \boldsymbol{0}$ as $P \to \infty$, where $\boldsymbol{x} = \omega_1 \boldsymbol{w}^{(1)} + \cdots + \omega_m \boldsymbol{w}^{(m)}$ for some integral basis $\omega_1, \ldots, \omega_m$ of K over $\mathbb{Q}$ and each $\boldsymbol{w}^{(j)}$ lies in a box of size P. This is not the most natural way to count solutions over a number field. In particular we saw in §2.1 that Theorem 1.2 implies a case of the Manin-Peyre conjecture, which is not the case for the results of Birch and Skinner above. Loughran [Lou15, §4.3] shows how to treat this conjecture with the circle method and a form of Möbius inversion, proving:

Theorem 7.6 (Loughran [Lou15])**.** *In Theorems 7.4 and 7.5 we may also conclude that $V(\boldsymbol{G})$ satisfies the Manin-Peyre conjecture over K.*

We give the following result, which improves on Theorem 7.5 when $R+1 > 2m\frac{d}{d-1}$.

Theorem 7.7. *Suppose that $d \leq 3$ and that $\boldsymbol{G}$ is nonsingular, or that $d \geq 4$ and $\boldsymbol{G} \in U_{d,n,R}(K)$ where the Zariski open set $U_{d,n,R}$ is as in Proposition 5.3. Then $V(\boldsymbol{G})$ satisfies the Hasse principle and weak approximation provided that*

$$n \geq d2^d mR + R. \tag{7.7}$$

Proof. By Corollary 7.3 it suffices to prove the Hasse principle and weak approximation for $V(\widetilde{\boldsymbol{G}})$ over $\mathbb{Q}$. If $d \leq 3$ then (7.7), (7.5) and the assumption that $\boldsymbol{G}$ is nonsingular imply that

$$mn - \sigma_{\mathbb{R}}(\widetilde{\boldsymbol{G}}) \geq d2^d mR + 1.$$

The result then follows from Theorem 1.5. If $d \geq 4$ then (7.7), (7.4) and the assumption that $\boldsymbol{G} \in U_{d,n,R}(K)$ show that

$$mn - \sigma^*(\widetilde{\boldsymbol{G}}) \geq d2^d mR + 1. \tag{7.8}$$

In particular the $\widetilde{G}_i^{(j)}$ must be linearly independent, or else $\sigma^*(\widetilde{\boldsymbol{G}}) = mn$ would hold. Recall the counting function N_F^{aux} from Definition 1.6. By Proposition 5.2 we have

$$N^{\mathrm{aux}}_{\boldsymbol{\gamma}\cdot\widetilde{\boldsymbol{G}}}(B) \ll_{\boldsymbol{G},\epsilon} B^{(d-2)n+\sigma^*(\widetilde{\boldsymbol{G}})+\epsilon} \tag{7.9}$$

for all $\boldsymbol{\gamma} \in \mathbb{R}^{mR}$, $B \geq 1$ and $\epsilon > 0$. To complete the proof of Theorem 7.7 we can follow the proof of Theorem 1.5 in §4.1, with $\widetilde{\boldsymbol{G}}$ replacing $\boldsymbol{F}$, (7.9) replacing (4.4) and (7.8) replacing (4.3). □

We expect that by adapting the arguments of Loughran one could strengthen Theorem 7.7 to show that the variety $V(\boldsymbol{G})$ satisfies the Manin-Peyre conjecture as well as the Hasse principle and weak approximation. In future work we hope to eliminate the dependence on m in the right-hand side of (7.7).

Index of notation

Mathematical symbols

Greek letters

Roman letters

Index of terms

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