On $Spin^c$-Invariants of Four-Manifolds

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A thesis submitted in

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Abstract

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The $spin^c$-invariants for a compact smooth simply-connected oriented four-manifold, as defined by Pidstrigach and Tyurin, are studied in this thesis. Unlike the Donaldson polynomial invariants, they are defined by cutting down the moduli space $\mathcal{M}'$ of '1-instantons', which is the subspace of the moduli space $\mathcal{M}$ of anti-self-dual connections parametrizing coupled ($spin^c$) Dirac operators with non-trivial kernel.

Our main goal is to study the relationship between these $spin^c$-invariants and the Donaldson polynomial invariants. The 'jumping subset' $\mathcal{M}'$ defined a cohomology class P of $\mathcal{M}$ which is given by the generalised Porteous formula. When the index l of the coupled Dirac operator is 1, the two smooth invariants are the same by definition. When $l = 0$ (or when $\mathcal{M}$ is compact), the $spin^c$-invariants are expressable as a Donaldson polynomial evaluating the 'Porteous class' P. Our main results concern the first two non-trivial cases $l = -1$ and -2, when the generalised Porteous formula can not be applied directly. Using cut-and-paste arguments to the moduli space $\mathcal{M}$, we show that for the former case the $spin^c$-invariants and the contracted Donaldson invariants differ by a correction term. It is the number of points in the immediate lower stratum of the Uhlenbeck compactification times a universal 'linking invariant' on S^4, which is obtained by computing an example (the $K3$ surface). The case when $l = -2$ and $dim\mathcal{M} = 8$ is a parametrized version of the $l = -1$ situation and the correction term, which involves the same 'linking invariant', is obtained from a suitable obstruction theory.

Contents

Chapter 1

Introduction

Using gauge theory on $SU(2)$ and $SO(3)$ bundles, Donaldson [Do] introduced polynomial invariants associated to a smooth, compact, simply-connected oriented 4-manifold X without boundary. To define such invariants, let E be an $SU(2)$ bundle over X with $c_2(E) = k > 0$. Let $\mathcal{A}_E$ be the affine space of connections on E. The gauge group $\mathcal{G}_E = Aut(E)$ acts on $\mathcal{A}_E$ and the quotient space $\mathcal{A}_E/\mathcal{G}_E$ is denoted by $\mathcal{B}_X$. Let $\mathcal{B}_X^*$ be the dense open subset of $\mathcal{B}_X$ consisting of equivalence classes of irreducible connections on E. Given a Riemannian metric g on X, write

$$\mathcal{M}_k^g = \{A \in \mathcal{A}_E \mid *_g F_A = -F_A\}/\mathcal{G}_E$$

for the moduli space of anti-self-dual (ASD) connections (or instantons) on E w.r.t. g, where F_A is the curvature tensor associated to connection A. The bundle E supports ASD connections only if $k \geq 0$. For $k = 0$, $\mathcal{M}_k^g$ is the single point $[\theta]$ carried by the trivial connection θ on X. For $k \geq 1$ and $b^+ > 0$, a moduli space $\mathcal{M}_k^g$ w.r.t. a generic metric lives inside $\mathcal{B}_X^*$ and is a smooth oriented manifold of (real) dimension $8k - 3(1 + b^+(X))$. Here the integer $b^+(X)$ is the number of positive eigenvalues in the diagonalisation of the intersection form Q_X over $\mathbb{R}$. Therefore, $\mathcal{M}_k^g$ is of even dimension $2d$ if $b^+(X)$ is odd. Roughly speaking, the polynomial invariants are defined by the pairings of the fundamental homology class of the moduli space

$\mathcal{M}_k^g$ with the cohomology classes of $\mathcal{B}_X^*$. They are the first useful invariants for distinguishing different smooth 4-manifolds of the same homotopy type.

In this thesis, the $spin^c$-invariants as constructed by Pidstrigach and Tyurin in [PT, T] are studied. Let X be as before, it is $spin^c$ as $w_1(X) = w_3(X) = 0$. A $spin^c$ structure is determined, since $H^2(X, \mathbb{Z})$ has no 2-torsion, by a complex line bundle λ with $c_1(\lambda) \equiv w_2(X)\ mod2$. Let $W^\pm$ be the corresponding positive and negative spinor bundles respectively with $detW^\pm = \lambda$. A hermitian connection α on λ induces compatible connections on $W^\pm$ which gives the ($spin^c$) Dirac operator

$$D^{\lambda,\alpha} : \Gamma(W^+) \longrightarrow \Gamma(W^-).$$

To define the $spin^c$-invariants, consider the moduli space of *1-instantons*

$$\mathcal{M}_k^{g,\alpha,1} = \{A \in \mathcal{A}_E \mid *_g F_A = -F_A \ \ \& \ \ dim_{\mathbb{C}} ker D_A^{\lambda,\alpha} \geq 1\}/\mathcal{G}_E,$$

where $indD_A^{\lambda,\alpha} \leq 0$, which is the subspace of $\mathcal{M}_k^g$ parametrizing coupled Dirac operators with kernel exhibits certain 'jumping' property. For $b^+(X) > 0$ is odd and $k > 0$, relative to a generic metric g and a generic connection α on λ, $\mathcal{M}_k^{g,\alpha,1} \subset \mathcal{B}_X^*$ is a smooth manifold, outside $\mathcal{M}_k^{g,\alpha,2}$, of dimension $2d' = dim\mathcal{M}_k^g - 2(1 - indD_A^{\lambda,\alpha})$. Here $\mathcal{M}_k^{g,\alpha,2} = \{A \in \mathcal{A}_E \mid *_g F_A = -F_A \ \ \& \ \ dim_{\mathbb{C}} ker D_A^{\lambda,\alpha} \geq 2\}/\mathcal{G}_E$ has (real) codimension ≥ 6 in $\mathcal{M}_k^{g,\alpha,1}$ and $indD_A^{\lambda,\alpha} = -k + \frac{1}{4}c_1(\lambda)^2 - \frac{1}{4}sgn(X)$ is the (complex) index of the coupled Dirac operator. The smooth points of $\mathcal{M}_k^{g,\alpha,1}$ has orientation induced from that of $\mathcal{M}_k^g$ and the normal bundle $(kerD_A^{\lambda,\alpha})^* \otimes cokerD_A^{\lambda,\alpha}$. In particular, if $d' = 0$, i.e.

$$\frac{1}{4}c_1(\lambda)^2 = -3k + \frac{3}{2}(1 + b^+(X)) + 1 + \frac{1}{4}Sgn(X),$$

then $\mathcal{M}_k^{g,\alpha,1}$ consists of isolated (smooth) points only.

Chapter 2 describes the set-up of the (numerical) $spin^c$-invariants γ' as in [PT], which are essentially the algebraic number of points in the zero-dimensional oriented space $\mathcal{M}_k^{g,\alpha,1}$. They are used in [PT] to distinguish between del-Pezzo surfaces and fake del-Pezzo surfaces by their smooth structures. However, there are 'stable range' conditions to ensure that this number is finite and one has to make sure that the definition is independent of the choice of the generic metric on X and connection on λ. As usual, there are two cases (2.3):

1) $b^+(X) \geq 2$: if $k > \frac{5}{6} + \frac{b^+(X)}{2}$, then $\mathcal{M}_k^{g,\alpha,1}$ is compact and hence consists of finite number of points only. And a cobordism argument implies that γ' is independent of the generic parameters and so is a differential invariant of X.

2) $b^+(X) = 1$: if $k \geq 2$, then $\mathcal{M}_k^{g,\alpha,1}$is compact and hence finite. However, for a generic path there is a finite number of $t \in (0,1)$ with g_t not generic, γ' in this case is a chamber invariant only, where the chambers are defined by the walls $W_e = \{x \in H^2(X, I\!\!R) \mid x \cdot e = 0\}$ with

$$\begin{cases} d \leq e^2 < 0, \quad and \\ either\ Sgn(X) - (c_1(\lambda) + 2e)^2 < 0 \\ or\ Sgn(X) - (c_1(\lambda) - 2e)^2 < 0. \end{cases}$$

In chapter 3 we describe briefly the construction of the Donaldson polynomial invariants so as to generalise the $spin^c$-invariants to the case $dim\,\mathcal{M}_k^{g,\alpha,1} > 0$ in a natural way — that is, as in [T], constructing the polynomial invariants using $\mathcal{M}_k^{g,\alpha,1}$ instead of $\mathcal{M}_k^g$. For simplicity, only the case $b^+ \geq 3$ (and odd) is considered. Let Σ be a smooth embedded surface representing the homology class $[\Sigma] \in H_2(X, Z\!\!\!Z)$. For a generic $2d$-dimensional moduli space $\mathcal{M}_k^g$ of ASD connections, a codimension 2 submanifold $\mathcal{M}_k^g \cap V_\Sigma$ can be defined which is the zero set of a transverse section s_Σ on a certain line bundle over $\mathcal{M}_k^g$. It is in fact the Poincare dual of $\mu([\Sigma])$ in $\mathcal{M}_k^g$, where $\mu : H_i(X) \longrightarrow H^{4-i}(\mathcal{B}_X^*)$ is given by the slant product pairing of the characteristic class of the universal bundle over $\mathcal{B}_X^* \times X$. Working with d surfaces $\Sigma_1, \ldots, \Sigma_d$, the 'transverse intersection' $\mathcal{I} = \mathcal{M}_k^g \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$ is then zero dimensional and consists of isolated points only. The moduli space may not be compact, but there is a natural compactification

$$\overline{\mathcal{M}}_k^g \subset \coprod_{l=0}^{k} \mathcal{M}_{k-l}^g \times sym^l(X)$$

by the weak convergence and the removable singularities theorem of ASD connections due to Uhlenbeck. Since the intermediate strata in the Uhlenbeck compactification have codimension a multiples of 4, a 'dimension counting' argument together with the 'stable range' condition $k > \frac{1}{4}(3b^+(X)+5)$ implies that the point set $\mathcal{I}$ is compact. The Donaldson polynomial invariant

$$\gamma_d : Sym^d(H_2(X,\mathbb{Z})) \longrightarrow \mathbb{Z}$$

is then defined by $\gamma_d(\Sigma_1,\ldots,\Sigma_d) = \sharp(\mathcal{M}_k^g \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d})$ counted with signs. By showing that the definition is independent of the choice of the metric and the sections s_{Σ_i}, this gives a differential invariant of X. If $\mathcal{M}_k^g$ is compact and defines a fundamental homology class $[\mathcal{M}_k^g]$, $\gamma_d(\Sigma_1,\ldots,\Sigma_d)$ is actually the pairing $< \mu([\Sigma_1]) \cup \cdots \cup \mu([\Sigma_d]), [\mathcal{M}_k^g] >$. In our discussions, the full Donaldson polynomial is needed in which the 4-dimensional class $\nu = \mu([point])$ will also be evaluated.

For the moduli space of 1-instantons $\mathcal{M}_k^{g,\alpha,1}$, there is a similar compactification

$$\overline{\mathcal{M}}_k^{g,\alpha,1} \subset \coprod_{l=0}^{k} \mathcal{M}_{k-l}^{g,\alpha,1} \times sym^l(X)$$

with the dimension of the intermediate strata decreasing in (at least) steps of 2, but this is sufficient for the same dimension counting argument to apply. Assuming the 'stable range' condition

$$k > \frac{1}{2}(3b^+(X) - \frac{1}{2}c_1(\lambda)^2 + \frac{1}{2}Sgn(X) + 7),$$

the $spin^c$-polynomial $\gamma'_d(\Sigma_1,\ldots,\Sigma_{d'})$ is then defined to be the algebraic number of points in the compact, zero dimensional space $\mathcal{M}_k^{g,\alpha,1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$. All the 'stable range' conditions above are to ensure that the trivial connection is avoided under weak limits. In the analogous $SO(3)$ construction (by lifting the gauge group to $U(2)$ so that coupled Dirac operators are defined), if the second Stiefel-Whitney class w_2 of the bundle is non-zero, these conditions can be dispensed with.

Our main goal is to investigate the possible relationship between the *spin*^{c}-invariants and the Donaldson invariants. Suppose $\mathcal{M}_k^g \subset \mathcal{B}_X^*$ is compact and hence parametrizes a (compact) family of coupled Dirac operators. By the generalised Porteous formula, the cohomology class P defined by the 'jumping subset' $\mathcal{M}_k^{g,\alpha,1}$ is given formally as a Chern class of an index bundle over $\mathcal{M}_k^g$, hence can be calculated by the Atiyah-Singer index theorem for family of operators. In this case, the *spin*^{c}-invariant is simply the Donaldson invariant evaluating P on the fundamental homology class $[\mathcal{M}_k^g]$. In particular, when the index of the coupled Dirac operator is 0 with $\mathcal{M}_k^{g,\alpha,1}$ of codimension 2, $\mathcal{M}_k^g \cap V_{\Sigma_1} \cdots \cap V_{\Sigma_{d-1}}$ is compact. Therefore, the *spin*^{c}-polynomial is actually a 'contracted' Donaldson polynomial

$$\gamma_d'(\Sigma_1, \ldots, \Sigma_{d-1}) = \gamma_d(\Sigma_1, \ldots, \Sigma_{d-1}, \sum a_i \alpha_i)$$

where $\sum a_i \mu([\alpha_i]) = P_0^1$ is the two dimensional cohomology class given by the generalised Porteous formula. But when $\mathcal{M}_k^g$ is non-compact (e.g. dimension ≥ 8) the relationship is not immediately apparent. This is because the Donaldson invariant is defined using the Uhlenbeck compactification, which involves ideal connections and so does not parametrize a family of Dirac operators. Correction terms due to lower strata in the compactification are expected.

In chapter 4, the first non-trivial case when $\mathrm{ind} D_A^\lambda = -1$ and the moduli space $\mathcal{M}_8$ of ASD connections of real dimension 8 is considered (to simplify notations, from now on subscripts are used to denote the dimensions of the moduli spaces). In this case the *spin*^{c}-polynomial is

$$\gamma_4'(\Sigma_1, \Sigma_2) = \sharp\{[A] \in \mathcal{N}_4 = \mathcal{M}_8 \cap V_{\Sigma_1} \cap V_{\Sigma_2} \mid dim_{\mathbb{C}} ker D_A^\lambda \geq 1\}.$$

However, $\mathcal{N}_4$ is non-compact, with points at infinity $([A_0], x) \in \mathcal{M}_0 \times (\Sigma_1 \cap \Sigma_2)$. Unlike the compact situation, this *spin*^{c}-polynomial is not expressable simply as a (contracted) Donaldson polynomial evaluating the 4-dimensional class P^1_{-1} given by the generalised Porteous formula. Our result is that, in this case there is a correction term coming from the immediate lower stratum in the Uhlenbeck compactification.

Theorem 1.1 (5.5.2) *Let X be a compact, smooth, simply connected, oriented Riemannian 4-manifold and $E \longrightarrow X$ an $SU(2)$ (or $SO(3)$) bundle. When the moduli space of instantons is of real dimension 8 and the index of the coupled Dirac operator is -1, there is a universal formula*

$$\gamma_4(P^1_{-1} \cup \mu(\Sigma_1) \cup \mu(\Sigma_2)) = \gamma'_4(\Sigma_1, \Sigma_2) + (\frac{1}{12}) \cdot Q(\Sigma_1, \Sigma_2) \cdot \gamma_0$$

relating the spin^c-polynomial γ'_4 and the (8-dimensional) Donaldson polynomial γ_4 evaluating the Porteous class. The correction term is essentially the number of points at infinity in the cut-down moduli space $\mathcal{M}_8 \cap V_{\Sigma_1} \cap V_{\Sigma_2}$ (up to a constant $\frac{1}{12}$).

Using the Atiyah-Singer Index theorem for family of operators, P^1_{-1} is computed in section (5.4) to be

Proposition 1.2 (5.4.4)

$$P^1_{-1} = \frac{1}{8}\mu(\alpha)^2 + \frac{1}{12}I + (\frac{1}{2} - \frac{\xi}{3})\nu,$$

where α is the poincare dual of $c_1(\lambda)$ (or of $c_1(\lambda)+c_1(F)$ in the $SO(3)$ case, where F is the $U(2)$ bundle defining the coupled Dirac operator), $\xi = c_2(E) = k$ (respectively $c_2(F) - \frac{1}{4}c_1(F)^2$), and I is the 4-dimensional class defined by the intersection form.

Similar to the computation of some blown-up formulae for Donaldson polynomials in [O], the idea is to 'compactify' the cut-down moduli space $\mathcal{N}_4$ by suitably capping off the 'ends' (so that the generalised Porteous formula can be applied). By the Taubes' gluing construction, the 'ends' of $\mathcal{N}_4$ (roughly the complement of a large compact subset), are modelled on a cone over $SO(3)$. The link $\hat{L}_{x,A_0,\zeta}$ associated to each point at infinity $([A_0], x) \in \mathcal{M}_0 \times (\Sigma_1 \cap \Sigma_2)$ is then a copy of $SO(3)$ which is obtained by gluing a family $\tilde{\eta}$ of framed connections on S^4 to A_0 at x by a gluing map of small scale ζ. Here $\tilde{\eta}$ is the $SO(3)$ family $\{[I, \sigma_\infty]\} \subset \tilde{\mathcal{B}}_{S^4}$ of classes of framed instantons, framed at $\infty \in S^4$, which is the fibre of the base-point fibration $\tilde{\mathcal{B}}_{S^4} \longrightarrow \mathcal{B}_{S^4}$ over the standard charge-1 instanton I on S^4.

Suppose $\tilde{\eta}$ is the boundary of a compact 4-family $\tilde{\tau}$ of (not-neccesarily ASD) framed connections over S^4, so that $\mathcal{T}_{x,A_0,\zeta} = [A_0 \# \tilde{\tau}]$ consists of connections that are gauge equivalent to A_0 outside a small ball about x, with $\partial \mathcal{T}_{x,A_0,\zeta} = -\hat{L}_{x,A_0,\zeta}$ (the minus sign denotes the reversed induced orientation). Capping off $\mathcal{N}_4$ by the $\mathcal{T}_{x,A_0,\zeta}$'s gives a compact 4-family $\mathcal{W}$. By evaluating the 4-dimensional 'Porteous class' P^1_{-1}, with representatives chosen to be supported away from $x \in \Sigma_1 \cap \Sigma_2$, the correction term needed will then be $\sum_{x \in \Sigma_1 \cap \Sigma_2, A_0 \in \mathcal{M}_0} \sharp \mathcal{T}'_{x,A_0,\zeta}$, where

$$\mathcal{T}'_{x,A_0,\zeta} = \{[A] \in \mathcal{T}_{x,A_0,\zeta} \mid dim_{\mathbb{C}} ker D^\lambda_A \geq 1\}.$$

This transverse jumping number $\sharp \mathcal{T}'_{x,A_0,\zeta}$ is shown in (4.3) to be a universal constant independent of X and A_0, when the scale of gluing ζ is sufficiently small. In fact, it is the transverse intersection number $\sharp \tilde{\tau}' = \sharp (j \cap \tilde{\tau})$, where

$$j = \{[a] \in \tilde{\mathcal{B}}_{S^4} \mid dim_{\mathbb{C}} ker D_a \geq 1\}$$

is the jumping locus on $\tilde{\mathcal{B}}_{S^4}$ and D is the Dirac operator for the (unique) spin structure on S^4. Since D coupled to an instanton has no non-trivial kernel, $\sharp \tilde{\tau}'$ can be regarded as the 'linking invariant' of the $SO(3)$ family $\tilde{\eta}$ in the obvious sense.

To obtain this 'linking invariant' $\sharp \tilde{\tau}'$ and hence the correction term in theorem (1.1), the relevant $spin^c$-polynomial for a $K3$ surface is computed in chapter 5, using the algebro-geometric interpretation of a 1-instanton as a stable bundle with certain section (see chapter 2, it is precisely this feature that makes the $spin^c$-invariants more computable than the Donaldson invariants). The simplicity of this computation also depends on the choice of the model for a $K3$ surface — the non-algebraic Kummer surface, with the simple structure of its Picard group fits nicely to the situation. The result is that the compact 4-dimensional moduli space of 1-instantons $\mathcal{M}'_8 \cong I\!P^2$ and the *spin*^{c}-polynomial is

Theorem 1.3 (5.3.1)

$$\gamma'_4(\Sigma_1, \Sigma_2) = \frac{1}{4}Q(\alpha, \Sigma_1)Q(\alpha, \Sigma_2),$$

where α is defined as before. On the other hand, evaluating the 'Porteous class'

$$P^1_{-1} = \frac{1}{8}\mu(\alpha) + \frac{1}{12}I - \frac{1}{3}\nu$$

using the formulae of Donaldson polynomial for $K3$ surface, the result is

Proposition 1.4 (5.5.1)

$$\gamma_4(P^1_{-1} \cup \mu(\Sigma_1) \cup \mu(\Sigma_2)) = \frac{1}{4}Q(\alpha, \Sigma_1)Q(\alpha, \Sigma_2) + \frac{1}{12}Q(\Sigma_1, \Sigma_2).$$

Since $\gamma_0 = 1$, the universal 'linking invariant' $\sharp\mathcal{T}'$ is $\frac{1}{12}$ and hence theorem (1.1) above. Note that $\sharp\mathcal{T}'$ is not an integer since only a 'rational cap' to the link $\hat{L}$ can be found. This theorem is verified in Appendix (A.2) for simply-connected elliptic surfaces S_n without multiple fibre with $n = P_g(S_n)+1$. Using Friedman's description in [F] of the relevant stable bundles, the *spin*^{c}-polynomials are

$$\gamma'_4(\Sigma_1, \Sigma_2) = \frac{1}{4}Q(\alpha + (n-2)F, \Sigma_1) \cdot Q(\alpha + (n-2)F, \Sigma_2),$$

where F is a general fibre and α is defined as before. When $n = 2$ it reduces to the case of a $K3$ surface.

Finally, the same cut-and-paste argument on moduli space applies to the case $indD^\lambda_A = -1$ (and the moduli space of instantons of any dimension) since the structure of the 'ends' are the same, so the correction term is again given by the number of points at infinity times the above universal 'linking invariant'.

Theorem 1.5 (6.1.1) *Let X be as before, with the $SU(2)$ (or $SO(3)$) moduli space of ASD connections of dimension $2d$. Suppose λ is a spin^c structure such that the index of the coupled Dirac operators have index -1, then there is a universal formula*

$$\begin{aligned}\gamma_d(P^1_{-1} \cup \mu(h)^{d-2}) &= \gamma'_d(h^{d-2}) + (\tfrac{1}{12}) \begin{pmatrix} d-2 \\ 2 \end{pmatrix} Q(h,h) \cdot \gamma_{d-4}(h^{d-4}) \\ &= \gamma'_d(h^{d-2}) + \tfrac{(d-2)(d-3)}{24} Q(h,h) \cdot \gamma_{d-4}(h^{d-4}),\end{aligned}$$

When $d < 4$, there is no such correction term. When $d = 4$ it reduces to theorem (5.5.2).

The next case when the index of the coupled Dirac operator $indD^\lambda_A = -2$ is considered for the simplest situation when the moduli space $\mathcal{M}_8$ of instantons is of dimension 8. It can be regarded as a parametrized version of the index $= -1$ case. The result is

Theorem 1.6 (6.3.3) *Let E, X and α be as in theorem (1.1), with $dim\mathcal{M}_8 = 8$ and $indD^\lambda_A = -2$, there is a universal formula*

$$\gamma_4(P^1_{-2} \cup \mu(\Sigma)) = \gamma'_4(\Sigma) + \frac{1}{24} \cdot \gamma_0 \cdot < \alpha, \Sigma >$$

relating the spin^c-polynomial γ'_4 and the (contracted) Donaldson polynomial evaluating the Porteous class

$$P^1_{-2} = -\{\frac{1}{48}\mu(\alpha)^3 + \frac{1}{24}I\mu(\alpha) + (\frac{1}{3} - \frac{\xi}{6})\nu\mu(\alpha)\}.$$

Here $\gamma'_4(\Sigma) = \sharp(J \cap \mathcal{N}_6)$ is the algebraic number of jumping instantons in the 6-dimensional transverse intersection $\mathcal{N}_6 = \mathcal{M}_8 \cap V_\Sigma$.

Again, the idea is to cap (rationally) the ends of $\mathcal{N}_6$ so that the Porteous formula can be applied. In this case, the link $\mathcal{H}_{A_0}$ associated to each $[A_0] \in \mathcal{M}_0$ is a $SO(3)$-fibre bundle over Σ. There is a 6-dimensional cap $\mathcal{R}_{A_0} \subset \mathcal{C}_{\Sigma,A_0}$, the space of connections concentrated at points in Σ with background connection A_0, given by gluing a family $\varpi = \{\tilde{\tau}_x \mid x \in \Sigma\}$ of framed charge-1 connections on S^4 to A_0 on X at points in Σ. However, unlike the index $= -1$ case, not every jumping connections in ϖ can be deformed to give jumping connections in $\mathcal{R}'_{A_0}$. There is a (complex) 1-dimensional obstruction, due to $coker D^{\lambda}_{A_0} \cong \mathbb{C}$, which defines an obstruction line bundle $\mathcal{L}$ over $\mathcal{C}_{\Sigma,A_0}$. It will be shown that $\mathcal{L}^{\otimes 2}$ is the pull-back of a line bundle L over Σ, with degree $< \alpha, \Sigma >$. Hence, the correction term corresponds to $A_0 \in \mathcal{M}_0$ is

$$\sharp \mathcal{R}'_{A_0} = (\sharp \tilde{\tau}') \cdot \frac{1}{2} < \alpha, \Sigma >= \frac{1}{24} < \alpha, \Sigma >,$$

since for generic x, $\tilde{\tau}_x \subset \tilde{\mathcal{B}}_{S^4}$ is a 4-dimensional family with $\partial \tilde{\tau}_x = -\tilde{\eta}$, the $SO(3)$-family of framed standard instanton on S^4, and so $\sharp \tilde{\tau}'_x = \frac{1}{12}$ as in the index $= -1$ case. Similarly, theorem (1.6) is verified for the case of elliptic surfaces without multiple fibre in Appendix (A.2), using Friedman's descriptions of the relevant stable bundles.

Chapter 2

The $Spin^{\mathcal{C}}$-Invariants

This chapter is devoted to give an expository account of the $spin^c$-invariants as constructed by Pidstrigach and Tyurin in [PT, T]. They are defined by counting the algebraic number of points in the moduli space of 1-instantons, i.e. the (zero dimensional) space $\mathcal{M}_k^{g,\alpha,1}$ of classes of ASD connections such that the kernel of the coupled ($spin^c$) Dirac operator exhibits certain 'jumping' property. The generalization to the case when $dim\,\mathcal{M}_k^{g,\alpha,1} > 0$ will be discussed in chapter 3.

2.1 1-Instantons

Let (X, g) be a smooth, compact, simply-connected, oriented Riemannian 4-manifold. Hence it is $spin^c$ and a $spin^c$ structure is fixed by choosing a complex line bundle λ satisfying $c_1(\lambda) \equiv w_2(X)\,mod 2$. This gives a pair of rank 2 Hermitian vector bundles W^+ and W^- (the positive and negative spinor bundles respectively) such that

$$T^*_{\mathbb{C}}X = (W^+)^* \otimes W^- \quad and \quad \bigwedge^2 W^{\pm} = \lambda.$$

A Hermitian connection ∇_0 on λ induces $U(2)$ connection on $W^{\pm}$ (compatible with the Levi-Civita connection), which gives the Dirac operator

$$D^{\lambda,\nabla_0} : \Gamma(W^+) \longrightarrow \Gamma(W^+ \otimes T^*X) \longrightarrow \Gamma(W^-),$$

where the second map is the Clifford multiplication.

Let E be an $SU(2)$ bundle over X with $c_2(E) = k$. With notations as in the introduction, by the Leibnitz rule each connection $A \in \mathcal{A}_E$ gives the coupled Dirac operator $D_A^{\lambda,\nabla_0} : \Gamma(W^+ \otimes E) \longrightarrow \Gamma(W^- \otimes E)$.

The subspaces $\{A \in \mathcal{A}_E : dim_{\mathbb{C}} ker D_A^{\lambda,\nabla_0} \geq i\}$ defines an equivariant filtration on $\mathcal{A}_E$ and therefore a filtration on $\mathcal{B}$:

$$\mathcal{B} \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \cdots \quad , \textit{ where } \mathcal{B}_i = \{[A] \in \mathcal{B} : \ dim_{\mathbb{C}} ker D_A^{\lambda,\nabla_0} \geq i\},$$

which restricts to a filtration on $\mathcal{M}_k^g$ by $\mathcal{M}_k^{g,\nabla_0,i} \stackrel{\text{def}}{=} \mathcal{M}_k^g \cap \mathcal{B}_i$.

We are interested in the subspace of the moduli space such that $\ker D_A^{\lambda,\nabla_0}$ first exhibits the 'jumping' phenomenon.

Definition 2.1.1 $\mathcal{M}_k^{g,\nabla_0,1} = \mathcal{M}_k^g \cap \mathcal{B}_1 = \{[A] \in \mathcal{M}_k^g : \ dim_{\mathbb{C}} ker D_A^{\lambda,\nabla_0} \geq 1\}$ *is the moduli space of 1-instantons.*

(The $SU(2)$ case is considered here for simplicity only. In the calculation for a $K3$ surface in chapter 5, $spin^c$-polynomials defined by *1-instantons* on a $SO(3)$ (or $U(2)$) bundle is needed which is defined as follows: Let E be a $U(2)$ bundle with adE the associated $SO(3)$ bundle, given by $\rho : U(2) \longrightarrow PU(2) = SO(3)$. Write $\mathcal{A}_E$, $\mathcal{A}_{adE}$ and $\mathcal{A}_{\bigwedge^2 E}$ for the affine spaces of connections on the bundle E, adE and $\bigwedge^2 E$ respectively, with an exact sequence of gauge groups

$$1 \longrightarrow \mathcal{G}_{\bigwedge^2 E} \longrightarrow \mathcal{G}_E \longrightarrow \mathcal{G}_{adE} \longrightarrow 1.$$

The orthogonal decomposition of Lie algebra:

$$u(2) = so(3) \oplus u(1) \quad by \quad Y = \rho_*(Y) + \frac{1}{2}Tr(Y) \cdot I$$

gives an isomorphism $\mathcal{A}_E = \mathcal{A}_{adE} \times \mathcal{A}_{\bigwedge^2 E}$, which restricted to an affine embedding $i_\beta : \mathcal{A}_{adE} \times \{\beta\} \hookrightarrow \mathcal{A}_E$, where $\beta \in \mathcal{A}_{\bigwedge^2 E}$ has harmonic curvature F_β. Let $\mathcal{A}_{F_\beta} \subset \mathcal{A}_E$ consists of connections with trace F_β. There is an isomorphism $i_\beta : (\mathcal{A}_{adE}/\mathcal{G}_{adE}) \longrightarrow (\mathcal{A}_{F_\beta}/\mathcal{G}_E)$, and the moduli space of 1-instantons is defined to be

$$\{[A] \in \mathcal{M}_{adE} \mid dim_{\mathbb{C}} ker D^{\lambda}_{i_\beta(A)} \geq 1\}$$

where $\mathcal{M}_{adE} \subset \mathcal{A}_{adE}/\mathcal{G}_{adE}$ is the moduli space of ASD connections on the $SO(3)$ bundle adE.)

In particular, when X is a compact Kähler surface, the above construction has a nice algebro-geometric interpretation.

Let K be the canonical line bundle, then $c_1(K) \equiv w_2(X) \, mod 2$ and a line bundle L is defined by $L^{\otimes 2} = K \otimes \lambda$ such that the Dirac operator

$$D^{\lambda, \nabla_0} : \Gamma(W^+) \longrightarrow \Gamma(W^-)$$

can be identified, up to a factor of $\sqrt{2}$, with the convoluted Dolbeault complex

$$\Omega^{0,0}(L) \oplus \Omega^{0,2}(L) \to \Omega^{0,1}(L).$$

If $A \in \mathcal{A}_E$ is a connection for which $(\overline{\partial}_A)^2 = 0$ (where $\overline{\partial}_A$ is the (0,1) component of the covariant derivative d_A), i.e. the (0,2) component of the curvature F_A is trivial, then it defines a holomorphic structure $\overline{\partial}_A : \Omega^{0,0}(E) \longrightarrow \Omega^{0,1}(E)$ on E. The coupled Dirac operator

$$D_A^{\lambda, \nabla_0} : \Gamma(W^+ \otimes E) \longrightarrow \Gamma(W^- \otimes E)$$

is then the convoluted Dolbeault complex of $E \otimes L$

$$\Omega^{0,0}(E \otimes L) \oplus \Omega^{0,2}(E \otimes L) \longrightarrow \Omega^{0,1}(E \otimes L).$$

Thus

$$KerD_A^{\lambda,\nabla_0} = H^0(E \otimes L) \oplus H^2(E \otimes L),$$

$$CokerD_A^{\lambda,\nabla_0} = H^1(E \otimes L)$$

and index $= \chi(E \otimes L)$. For a generic metric g such that $\mathcal{M}_k^g \subseteq \mathcal{B}^*$ consists of irreducible connections only, by the identification $\mathcal{M}_k^g \cong \mathcal{M}_S$ between the moduli space of irreducible ASD connections and that of stable holomophic structures (see [DK, Chapter 6]) we have:

$$\mathcal{M}_k^{g,\nabla_0,1} = \mathcal{M}_{1,0} \cup \mathcal{M}_{0,1},$$

where

$$\mathcal{M}_{1,0} = \{[E] \in \mathcal{M}_S : h^0(E \otimes L) \geq 1\} \ \ and$$

$$\mathcal{M}_{0,1} = \{[E] \in \mathcal{M}_S : h^2(E \otimes L) \geq 1\} = \{[E] \in \mathcal{M}_S : h^0(E^* \otimes L^* \otimes K) \geq 1\},$$

by the Serre duality ([LM, Kr]).

2.2 Transversality

Fixing the 'discrete invariants' $c_1(\lambda) \in H^2(X, \mathbb{Z})$ and $c_2(E) = k$, the moduli space of 1-instantons depends on the 'continuous invariants', i.e. the choice of a Riemannian metric g on X and the choice of a connection on the associated line bundle λ. Given a marked connection ∇_0, say the one with harmonic curvature, the space of connections on λ is isomorphic to the space $\Omega^1(X)$ of 1-forms on X. Thus the space of parameters is $\{(g, \alpha) \in \mathcal{C} \times \Omega^1(X)\}$, where $\mathcal{C}$ is the space of conformal structures on X. If the union $\bigcup_{g,\alpha} \mathcal{M}_k^{g,\alpha,1}$ of the moduli spaces of 1-instantons (from now on, the fixed background connection ∇_0 is ignored) is a Banach manifold and its projection onto the parameter space is a Fredholm map, applying Sard-Smale theorem, there exists an everywhere dense subset (generic parameters) of the parameter space such that the moduli space of 1-instantons is a manifold of dimension equals to the index of the projection.

Consider the map

$$f : \mathcal{A}_E \times \Gamma(W^+ \otimes E) \longrightarrow \Omega^2_+(\Im_E) \times \Gamma(W^- \otimes E)$$

$$f(A, \sigma) = (F^+_A, D^{\lambda,\alpha}_A(\sigma))$$

here F^+_A is the g-self-dual part of the curvature tensor and $\Im_E$ consists of skew-adjoint, trace-free endomorphisms of the rank-two vector bundle E. $\mathbb{P}(f^{-1}(0))/\mathcal{G}_E$ consists of pairs $([A], \sigma)$, with $[A] \in \mathcal{M}^{g,\alpha,1}_k$ and $\sigma \in \mathbb{P}(kerD^{\lambda,\alpha}_A)$, is fibred over $\mathcal{M}^{g,\alpha,1}_k$ with fibre $\mathbb{P}(kerD^{\lambda,\alpha}_A) \subset \mathbb{P}(\Gamma(W^+ \otimes E))$ over an 1-instanton $[A]$.[1] Consider the above smooth map of Banach manifolds, with the metric g and the 1-form α as additional parameters:

$$f : \mathcal{A}_E \times \mathcal{C} \times \Omega^1(X) \times \Gamma(W^+ \otimes E) \longrightarrow \Omega^2_+(\Im_E) \times \Gamma(W^- \otimes E)$$

Suppose the index[2] of the coupled Dirac operator is *non-positive*, the results are

Proposition 2.2.1 *The linearization δf of f is surjective at all points of the space $f^{-1}(0)$ corresponding to irreducible connections.*

[1]This larger 'space of pairs' is used for general position and bordism arguments, and could be thought of as a resolution of those singularities of $\mathcal{M}^{g,\alpha,1}_k$ which arise from a big kernel of the Dirac operator (c.f. [ACGH, chapter 3]). In fact, in [BP, P] the $spin^c$-invariants are defined using this 'moduli space of pairs'.

[2]From the Atiyah-Singer index theorem, the index of the coupled ($spin^c$) Dirac operator is

$$indD^{\lambda,\alpha}_A = -k + \frac{1}{4}c_1(\lambda)^2 - \frac{1}{4}Sgn(X) = -k + 2indD^{\lambda,\alpha}$$

(see [LM]). In general, for the $U(2)$ case,

$$indD^{\lambda,\alpha}_A = \frac{(\Lambda+\lambda)^2}{4} - \frac{1}{4}Sgn(X) + (\frac{1}{4}\Lambda^2 - c_2(E)) = \frac{1}{2}\Lambda \cdot (\Lambda + \lambda) - c_2(E) + 2indD^{\lambda},$$

where $\Lambda = c_1(E)$. Of course the index is independent of the choice of the connections A and α.

The proof can be found in [PT, Prop I.3.5]. With the transversality result for ordinary ASD connections [FU, Theorem (3.4)], it suffices to prove the surjectivity of the map

$$\Omega^1(X) \times \Gamma(W^+ \otimes E) \longrightarrow \Gamma(W^- \otimes E) \quad ; \quad (\delta\alpha, \delta\sigma) \longmapsto \delta\alpha * \sigma + D_A^{\lambda,\alpha}(\delta\sigma)$$

where $*$ is given by the Clifford multiplication. The main point in the proof is to use the uniqueness of continuation of the solutions of elliptic equations, which forces any elements perpendicular to the image to be zero.

Hence, restricting f to the irreducible connections

$$f : \mathcal{A}_E^* \times \mathcal{C} \times \Omega^1(X) \times \Gamma(W^+ \otimes E) \longrightarrow \Omega^2_+(\Im_E) \times \Gamma(W^- \otimes E),$$

we have $f^{-1}(0)$ and so, by constructing a transverse slice $N_{(A,\sigma)} = ker(d_A^* \oplus m_\sigma^*)$ of the gauge action at (A, σ) (where

$$d_A \oplus m_\sigma : \Omega^0(\Im_E) \longrightarrow \Omega^1(\Im_E) \times \Gamma(W^+ \otimes E)$$

is the linearization of the gauge action, with m_σ a multiplication of a vector spinor σ by the endomorphism), $T = I\!P(f^{-1}(0))/\mathcal{G}_E$ is a Banach manifold, cut out transversely.

Proposition 2.2.2 *The projection* $\pi : T \longrightarrow \mathcal{C} \times \Omega^1(X)$ *is a smooth Fredholm map with index* $(dim\mathcal{M}_k^g - 2(1 - indD_A^{\lambda,\alpha}))$.

The index is 2 less than the sum $(ind(d_A^* \oplus d_A^+) + 2indD_A^\lambda)$ due to projectivization, with the factor 2 appears in $indD_A^\lambda$ because the complex index of the Dirac operator is considered. Note that the codimension $2(1 - indD_A^\lambda)$ equals to that given by the generalised Porteous formula as in chapter 4.

Corollary 2.2.3 *1) The inverse image* $\pi^{-1}(g, \alpha)$ *of a generic parameter* $(g, \alpha) \in \mathcal{C} \times \Omega^1(X)$ *is a smooth manifold of (real) dimension* $dim\mathcal{M}_k^g - 2(1 - indD_A^{\lambda,\alpha})$*. Hence,* $\mathcal{M}_k^{g,\alpha,1}$ *is a smooth manifold outside* $\mathcal{M}_k^{g,\alpha,2}$ *of the same dimension.*

2) The inverse image of π of a generic path in $\mathcal{C} \times \Omega^1(X)$ is a smooth manifold of dimension $dim\mathcal{M}_k^g + 2indD_A^{\lambda,\alpha} - 1$.

2.3 (Numerical) $Spin^c$-Invariants

The case $b^+(X) = 1$ will be considered first. For $b^+(X) \geq 2$ the situation is much easier as the reducible connections can be avoided.

Suppose $A \in \mathcal{M}_k^g$ is a reducible ASD connection, which corresponds to a decomposition $E = L \oplus L^{-1}$ for a complex line bundle L over X, with $c_1(L)$ represented by an anti-self-dual form and $c_2(E) = k = -c_1(L)^2$.

When $b^+(X) = 1$, define the period map $\mathcal{P} : \mathcal{C} \longrightarrow I\!P(K_+)$, where $K_+ \subseteq H^2(X, I\!R)$ is the cone of positive vectors, by sending the metric g to the harmonic self-dual form (which is unique up to scalar multiple since $b^+(X) = 1$). The existence of a reducible g-ASD connection implies that $\mathcal{P}(g) \in W_{C_1(L)}$ for some complex line bundle L over X, where

$$W_e \stackrel{\text{def}}{=} \{x \in H^2(X, I\!R) : e \cdot x = 0\}.$$

Definition 2.3.1 *Let $d < 0$ be a negative integer. Define the set Ξ_d of chambers of type d to be the set arising on dividing up the cone K_+ by the walls W_e for all $e \in H^2(X, Z\!\!Z)$ such that $d \leq e^2 < 0$.*

It is known, by transversality argument as in [DK], that

1) For a generic metric g, the moduli space of instantons $\mathcal{M}_{k-j}^g$ and hence $\mathcal{M}_{k-j}^{g,\alpha,1}$ does not contain any reducible connections, for $0 \leq j < k$, and

2) For a generic path g_t in the space of metric, a reducible connection arises at $t = 0$ iff $\mathcal{P}(g_o) \in \bigcup_{e^2=-k} W_e$.

In fact, some of the walls can be got rid of if 1-instantons are considered.

Proposition 2.3.2 *Let $\xi_t = (g_t, \alpha_t)$ be a path in the parameter space whose image $\mathcal{P}(g_t)$ under the period map crosses a single wall $W_{c_1(L)}$ at $t = 0$. Suppose both the Dirac operators $D^{\lambda+2L}$ and $D^{\lambda-2L}$ have* non-positive *indices, then for an arbitrarily small variation $\tilde{\xi}_t$ of ξ_t, that crosses $W_{c_1(L)}$ at $t = 0$ and only differs from ξ_t in an small neighbourhood of $W_{c_1(L)}$, no reducible 1-instanton arises over $\tilde{\xi}_0$.*

To see this, consider the map

$$f_1 : \Omega^1(X) \times \Gamma(W^+) \longrightarrow \Gamma(W^-)$$

$$(\alpha, \sigma) \longmapsto D^{\lambda,\alpha}(\sigma).$$

δf_1 is surjective and in particular, $m_1 = f_1^{-1}(0)$ is a Banach manifold. Also, the projection π of m_1 into $\Omega^1(X)$ is a Fredholm map of index $2(indD^{\lambda,\alpha})$ and so the inverse image $\pi^{-1}(\alpha)$ of a generic point $\alpha \in \Omega^1(X)$ (respectively $\pi^{-1}(\alpha_t)$ of a generic path $\alpha_t \in \Omega^1(X)$) is a smooth manifold of dimension two times the (complex) index (respectively of dimension $2indD^{\lambda,\alpha} + 1$). The assumptions in the proposition imply that the operators $D^{\lambda\pm 2L}$ have zero kernel, for generic connections on $\lambda \pm 2L$. For a reducible instanton A as in the beginning of this section, ξ_t is modified such that $\tilde{\xi}_t = (\tilde{g}_t, \tilde{\alpha}_t)$ has generic $\tilde{\alpha}_0$. Then the coupled Dirac operator splits into two Dirac operators, $D_A^{\lambda,\tilde{\alpha}_0} = D^{\lambda+2L,\tilde{\alpha}_0} \oplus D^{\lambda-2L,\tilde{\alpha}_0}$ which has zero kernel, that is, the connection A is not a 1-instanton w.r.t. $\tilde{\xi}_0 = (\tilde{g}_0, \tilde{\alpha}_0)$.

Hence, by the proposition and the index formula $indD^{\lambda} = -\frac{1}{8}(Sgn(X) - c_1(\lambda)^2)$ for ($spin^c$) Dirac operator, only needs to consider the set $\Xi_{d,\lambda}$ of chambers (of type d) obtained by dividing up the cone K_+ by the (effective) walls W_e for $e \in H^2(X, \mathbb{Z})$ satisfying

$$\begin{cases} d \le e^2 < 0, \quad and \\ either\ Sgn(X) - (c_1(\lambda) + 2e)^2 < 0 \\ or\ Sgn(X) - (c_1(\lambda) - 2e)^2 < 0. \end{cases}$$

We now want to establish conditions, for $b^+ = 1$, on k that are sufficient for the following two conditions to hold:

Condition 2.3.3 *1)* $\mathcal{M}_k^{\xi,1}$ *is a compact manifold for a generic parameter* $\xi = (g, \alpha) \in \mathcal{C} \times \Omega^1(X)$.

2) the manifold $\bigcup_{t\in[-1,1]} \mathcal{M}_k^{\xi_t,1}$ *is compact for a generic path* $\xi_t : [-1,1] \longrightarrow \mathcal{C}\times\Omega^1(X)$ *not intersecting any walls in* $\mathcal{K}_+ \setminus \Xi_{-k,\lambda}$.

First note that, similar to the Uhlenbeck compactification of the moduli space of ASD connections

$$\overline{\mathcal{M}_k^g} \subseteq \coprod_{l=0}^{k} \mathcal{M}_{k-l}^g \times s^l(X),$$

there is

Compactification 2.3.4

$$\overline{\mathcal{M}_k^{g,\alpha,1}} \subseteq \coprod_{l=0}^{k} \mathcal{M}_{k-l}^{g,\alpha,1} \times s^l(X).$$

(This will be explained in section (3.2).) Note also that[3] the real dimension $dim\mathcal{M}_{k-1}^{g,\alpha,1} = dim\mathcal{M}_k^{g,\alpha,1} - 6$ for generic g and α and $k \neq 1$ (otherwise $\mathcal{M}_{k-1}^{g,\alpha,1}$ contains the trivial connection).

[3] $$\begin{aligned} dim\,\mathcal{M}_{k-l}^{g,\alpha,1} &= dim\,\mathcal{M}_{k-l}^{g} - 2(1 - index\ of\ D_A^\lambda) \\ &= (2d - 8l) - 2(1 + k - \tfrac{1}{4}c_1(\lambda)^2 + \tfrac{1}{4}Sgn(X)) + 2l \\ &= 2d - 2(1 + k - \tfrac{1}{4}c_1(\lambda)^2 + \tfrac{1}{4}Sgn(X)) - 6l \\ &= dim\,\mathcal{M}_k^{g,\alpha,1} - 6l. \end{aligned}$$

For the following definition of $spin^c$-invariants, $2d' = dim\mathcal{M}_k^{g,\alpha,1} \leq 4$ (in fact, only the case $d' = 0$ will be considered), hence the virtual dimension of all the spaces on the right-hand side of (2.3.4) with $l > 0$ are negative and even, which is sufficient for (2.3.3) to hold, except possibly for the lowest stratum $\mathcal{M}_0^{g,\alpha,1} \times s^k(X)$. For the case $b^+ = 1$,

$$d' = dim_{\mathbb{C}}\mathcal{M}_k^g - 1 + indD_A^\lambda = 4k - 3 - 1 + indD_A^\lambda$$

i.e. $indD_A^\lambda = d' - 4k + 4$ and hence $indD^\lambda = \frac{1}{2}(indD_A^\lambda + k) = 2 + \frac{d'}{2} - \frac{3}{2}k$.

By the argument as in (2.3.2), the sufficient condition, when $b^+ = 1$ and $d' \leq 2$, for the lowest stratum to be empty for a generic path in the parameter space is

Condition 2.3.5 $2 + \frac{d'}{2} - \frac{3}{2}k < 0$.

This holds, for example, if $k > 2$ or $k = 2$ and $d' < 2$. This is then also the sufficient condition for (2.3.3) to hold. In this case, together with (2.2.3) and (2.3.2), we have

Proposition 2.3.6 *Suppose there exists a path $\xi_t : [-1, 1] \longrightarrow \mathcal{C} \times \Omega^1(X)$ such that $\mathcal{P}(\xi_t)$ is in the same chamber in $\Xi_{-k,\lambda}$ for every $t \in [-1, 1]$. Assume also that the manifolds $\mathcal{M}_k^{\xi_{-1},1}$ and $\mathcal{M}_k^{\xi_1,1}$ are compact. Then there exists a small variation $\tilde{\xi}_t$ of ξ_t such that $\bigcup_{t\in[-1,1]} \mathcal{M}_k^{\tilde{\xi}_t,1}$ is a smooth compact cobordism between the two manifolds.*

For $b^+ \geq 2$, by transversality argument one can show that for a generic path in the parameter space, the corresponding family of moduli spaces of 1-instantons does not contain any reducible connections (so the arguments about chambers are not needed). Recall

$$d' = dim_{\mathbb{C}}\mathcal{M}_k^{g,\alpha,1} = dim_{\mathbb{C}}\mathcal{M}_k^g - 1 + indD_A^\lambda = 4k - \frac{3}{2}(1 + b^+) - 1 + indD_A^\lambda ,$$

$$i.e.\ \ indD^{\lambda}_{A} = d' - 4k + \frac{3}{2}b^{+} + \frac{5}{2}\ and\ so$$
$$indD^{\lambda} = \frac{1}{2}(indD^{\lambda}_{A} + k) = \frac{5}{4} + \frac{3}{4}b^{+} + \frac{d'}{2} - \frac{3}{2}k.$$

Similarly, assuming $d' \leq 2$, the sufficient condition for the lowest stratum in the compactification (2.3.4) to be empty is $indD^{\lambda} < 0$, that is $k > \frac{3}{2} + \frac{b^{+}}{2}$ (in particular, for $d' = 0$, $\mathrm{ind} D^{\lambda} < 0$ implies $k > \frac{5}{6} + \frac{b^{+}}{2}$), and there is result similar to (2.3.6).

If λ can be chosen such that

$$dim\mathcal{M}^{g}_{k} = 2(1 - indD^{\lambda}_{A}),$$

that is,

$$\frac{1}{4}C_1(\lambda)^2 = -3k + \frac{3}{2}(1 + b^{+}(X)) + 1 + \frac{1}{4}Sgn(X),$$

then $\dim\mathcal{M}^{g,\alpha,1}_{k} = 0$ and consists of smooth points only. Suppose $\mathcal{M}^{g,\alpha,1}_{k}$ is compact (for example, the condition (2.3.5) is satisfied for the case $b^{+} = 1$) and is oriented[4], then the (numerical) $spin^{c}$-invariant as defined in [PT] is the algebraic number of 1-instantons. For $b^{+} \geq 2$, this defines an 'absolute' invariant of the smooth structure of X. For $b^{+} = 1$, this is a chamber invariant, that is, there is a map $\gamma'_{X,\lambda} : \Xi_{-k,\lambda} \longrightarrow \mathbb{Z}$

[4]Consider the linearization of the Dirac equation, for a smooth point $[A] \in \mathcal{M}^{g,\alpha,1}_{k}$ there is a short exact sequence ([PT, I.2 & I.5])

$$0 \to T\mathcal{M}^{g,\alpha,1}_{k}{}_{[A]} \xrightarrow{i} H^{1}_{A} \xrightarrow{j} (Ker(D^{\lambda,\alpha}_{A}))^{*} \otimes Coker(D^{\lambda,\alpha}_{A}) \to 0,$$

where i is the inclusion into the first order deformation space of the space of (ordinary) instantons which defines the orientation of $\mathcal{M}^{g}_{k}$. For $a \in \Omega^{1}_{X}(\Im_{E})$ and $s \in KerD^{\lambda,\alpha}_{A}$, $j(a)(s) = \pi(a * s)$ where $*$ is a combination of matrix multiplication and Clifford multiplication, and π is the projection into $CokerD^{\lambda,\alpha}_{A}$. The final term has a natural complex orientation, since the Dirac operator is complex . The orientation of $\mathcal{M}^{g,\alpha,1}_{k}$ at $[A]$ is defined such that the orientation of $\mathcal{M}^{g}_{k}$ and that of $(Ker(D^{\lambda,\alpha}_{A}))^{*} \otimes Coker(D^{\lambda,\alpha}_{A})$ agree under

$$\bigwedge^{max} H^{1}_{A} = \bigwedge^{max} T\mathcal{M}^{g,\alpha,1}_{k}{}_{[A]} \otimes \bigwedge^{max} \{(Ker(D^{\lambda,\alpha}_{A}))^{*} \otimes Coker(D^{\lambda,\alpha}_{A})\}.$$

sending a chamber $\zeta \in \Xi_{-k,\lambda}$ into the algebraic number of 1-instantons for some generic parameter $\xi = (g, \alpha)$ for which the period $\mathcal{P}(g) \in \zeta$. By proposition (2.3.6), this number does not change under deformation of the metric that leaves its period in one chamber. If $f : X \longrightarrow X'$ is an orientation-preserving diffeomorphism of 4-manifolds, then the invariant γ' satisfies $\gamma'_{X,f^*\lambda}(f^*\zeta) = \gamma'_{X',\lambda}(\zeta)$.

Chapter 3

The $Spin^c$-Polynomial Invariants

In this chapter the $spin^c$-polynomial invariants will be defined for a smooth, compact, simply-connected, oriented Riemannian 4-manifold X without boundary, when the dimension of the moduli space $\mathcal{M}_k^{g,\alpha,1}$ of 1-instantons is (even and) greater than zero.

3.1 Donaldson Polynomial Invariants

This section gives a brief outline of the construction of the $SU(2)$ Donaldson polynomial invariants following [Do, DK]. Although this is now well-known, it is included since the dimension counting arguments will be used in the next section in defining the $spin^c$-polynomial invariants. Also, the representation of the cohomology classes in $\mathcal{B}_X^*$ that are 'compactly supported' in $\mathcal{M}_k^g$ will be referred to in later chapters.

Let E be a $SU(2)$ bundle over X with $c_2(E) = k$ and $\mathcal{M}_k^g$ the moduli space of g-ASD connections. For simplicity assume $b^+(X) > 1$ so that reducible connections can be avoided for a generic path of metrics (if $b^+(X) = 1$, chamber structure similar to the previous section has to be considered). Also, the rational cohomology ring of the space $\mathcal{B}_X^*$ of irreducible $SU(2)$ connections (modulo gauge equivalence) is a polynomial algebra with generators in dimensions 2 and 4. They are obtained from the homology of X via a map $\mu : H_i(X) \longrightarrow H^{4-i}(\mathcal{B}_X^*)$ defined by the slant product pairing of the characteristic class of the universal bundle over $\mathcal{B}_X^* \times X$ with the homology classes in $H_2(X)$ and $H_0(X)$. In particular, all rational cohomology of $\mathcal{B}_X^*$ lies in even dimensions. If the moduli space $\mathcal{M}_k^g$ is even dimensional, say $2d$ (i.e. $b^+(X)$ is odd), then invariants can be obtained by, roughly speaking, pairing the fundamental class $[\mathcal{M}_k^g]$ with the appropriate cohomology of $\mathcal{B}_X^*$, which then leads to a polynomial function on the homology of X. That is, let $[\Sigma_1], \ldots, [\Sigma_d]$ be classes in $H_2(X, \mathbb{Z})$, then the cup-product $\mu(\Sigma_1) \cup \cdots \cup \mu(\Sigma_d) \in H^{2d}(\mathcal{B}_X^*, \mathbb{Q})$ is 'evaluated' on $[\mathcal{M}_k^g]$, defining a number $\gamma = < \mu(\Sigma_1) \cup \cdots \cup \mu(\Sigma_d), [\mathcal{M}_k^g] >$. The difficulty here is on the possibly non-compactness of $\mathcal{M}_k^g$ which prevents us from defining the fundamental class $[\mathcal{M}_k^g]$.

X is simply-connected and so the orientation of $\mathcal{M}_k^g$ depends on the homological orientation, i.e. a choice of orientation Ω for a maximal positive subspace $H^+ \subseteq H^2(X, \mathbb{R})$ for the intersection form [DK, (5.4)]. When k is sufficiently large, $\gamma_{k,\Omega} = \gamma_{k,\Omega}(\Sigma_1, \ldots, \Sigma_d)$ can be defined and satisfies:

Properties 3.1.1 *1) $\gamma_{k,\Omega}$ depends on Σ_i only through its homology class $[\Sigma_i]$;*

2) $\gamma_{k,\Omega}(\Sigma_1, \ldots, \Sigma_d)$ is multilinear and symmetric in $[\Sigma_1], \ldots, [\Sigma_d]$;

3) $\gamma_{k,\Omega} = -\gamma_{k,-\Omega}$;

4) $\gamma_{k,\Omega}$ is independent of the choice of metric and is an invariant of the oriented diffeomorphism type of X, i.e. if $f : X \longrightarrow Y$ is an orientation-preserving diffeomorphism, then $\gamma_{k,f^*(\Omega)}(f(\Sigma_1), \ldots, f(\Sigma_d)) = \gamma_{k,\Omega}(\Sigma_1, \ldots, \Sigma_d)$.

Condition 3.1.2 *Choose k in the 'stable range' $k \geq \frac{1}{4}(3b^+ + 5)$.*[1]

[1]By using the blow-up formulae relating the polynomial invariant of X and $X \sharp \overline{\mathbb{P}^2}$, this 'stable range' restriction can be removed. In fact, the stable range condition is included to ensure that the

(One way of interpreting this constraint is to consider the Uhlenbeck compactification

$$\overline{\mathcal{M}^g_k} \ \ in \ \ \coprod_{l=0}^{k} \mathcal{M}^g_{k-l} \times s^l(X).$$

Since $\dim(\mathcal{M}^g_{k-l} \times s^l(X)) = 8(k-l) - 3(1+b^+) + 4l = 8k - 4l - 3(1+b^+) = 2d - 4l$, the dimension of each 'stratum' decreases in a factor of 4, for $l = 1, \dots, (k-1)$; except the lowest stratum $\mathcal{M}^g_0 \times s^k(X)$, which might have dimension $4k$ instead of the formal dimension $4k - 3(1+b^+)$. The stable range constraint says that the formal dimension, $2d = 8k - 3(1+b^+)$, of $\mathcal{M}^g_k$ exceeds that of $\mathcal{M}^g_0 \times s^l(X)$ by at least 2; this means $\overline{\mathcal{M}^g_k}$ is generically a manifold except at a set of codimension 2 or more, which is the usual condition for a singular complex to possess a fundamental class.)

Choose a generic metric g on X such that the ASD moduli spaces have the usual generic properties [DK, Corollary(4.3.19)]:

Condition 3.1.3 *For $0 \le j < k$, the moduli space $\mathcal{M}^g_{k-j}$ is a smooth manifold of the correct dimension, $2d - 8j$, cut out transversely by the ASD equations; and consists of irreducible solutions only.*

Choose embedded surfaces Σ_i in general position representing the homology classes in X, let $\nu(\Sigma_i)$ be tubular neighbourhoods with the property that triple intersections are empty: $\nu(\Sigma_i) \cap \nu(\Sigma_j) \cap \nu(\Sigma_l) = \emptyset$ (i, j, l are distinct). For each i, there is a line bundle $\mathcal{L}_{\Sigma_i}$ over $\mathcal{B}^*_{\nu(\Sigma_i)}$, the space of irreducible connections (modulo gauge equivalence) on $E|_{\nu(\Sigma_i)} \to \nu(\Sigma_i)$, together with a section s_i. Let $V_{\Sigma_i} = s_i^{-1}(0)$ be the zero set. It can be arranged that the closure of V_{Σ_i} in $\mathcal{B}_{\nu(\Sigma_i)}$ does not contain the trivial connection ([DK, (5.2)]). Also, by transversality argument, it can be arranged that:

Condition 3.1.4 *For any $\{i_1, \dots, i_r\} \subseteq \{1, \dots, d\}$ and any j with $0 \le j < k$, the*

trivial connection is avoided under weak limits. In later chapters, connections on $SO(3)$ bundles are considered and if w_2 of the bundle is non-zero, this condition can be dispensed with.

'intersections'

$$\mathcal{M}^g_{k-j} \cap V_{\Sigma_{i_1}} \cap \cdots \cap V_{\Sigma_{i_r}} = \{[A] \in \mathcal{M}^g_{k-j} : [A|_{\nu(\Sigma_{i_l})}] \in V_{\Sigma_{i_l}};\ for\ l = 1, 2, \ldots, r\}$$

are transverse and is then a smooth submanifold dual to $\mu([\Sigma_{i_1}]) \cup \cdots \cup \mu([\Sigma_{i_r}])$ *in* $\mathcal{M}^g_{k-j}$. *In particular,* $\mathcal{M}^g_k \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$ *consists of isolated points.*

The construction of the line bundle $\mathcal{L}_{\Sigma_i}$'s are as follows [DK, (5.2.1)]: on the space $\tilde{\mathcal{B}}_\Sigma$ of framed connections over embedded Riemann surface Σ in X, there is the (dual) determinant line bundle $\tilde{\mathcal{L}}_\Sigma = (det\, ind(\not\partial_\Sigma, \tilde{\mathbb{E}}))^*$ of the family of coupled Dirac operators on Σ , where $\tilde{\mathbb{E}}$ is the universal bundle over $\tilde{\mathcal{B}}_\Sigma \times \Sigma$ parametrized by $\tilde{\mathcal{B}}_\Sigma$; and $\not\partial_\Sigma : \Omega^{0,0} \otimes K^{\frac{1}{2}} \longrightarrow \Omega^{0,1} \otimes K^{\frac{1}{2}}$ is the Dirac operator with $K^{\frac{1}{2}}$ a choice of square root of the canonical line bundle (and hence a choice of the spin structure on the Riemann surface Σ). By the Atiyah-Singer index theorem for family of operators, it can be shown that $c_1(\tilde{\mathcal{L}}_\Sigma) = c_2(\tilde{\mathbb{E}})/[\Sigma] \in H^2(\tilde{\mathcal{B}}_\Sigma, \mathbb{Q})$. Since the centre $\{\pm\ 1\} \subseteq SU(2)$ acts trivially on $\tilde{\mathcal{L}}_\Sigma$, this determinant line bundle descends from $\tilde{\mathcal{B}}^*_\Sigma$ to $\mathcal{B}^*_\Sigma$ and so there is a line bundle $\overline{\mathcal{L}}_\Sigma \to \mathcal{B}^*_\Sigma$ with $c_1(\overline{\mathcal{L}}_\Sigma) = \mu([\Sigma])$ in $H^2(\mathcal{B}^*_\Sigma, \mathbb{Q})$. Write $\mathcal{L}_\Sigma \to \mathcal{B}^*_{\nu(\Sigma)}$ for the pull-back of the line bundle $\overline{\mathcal{L}}_\Sigma \to \mathcal{B}^*_\Sigma$ by restriction. Tubular neighbourhood is considered so that the restriction map

$$\tau_{\nu(\Sigma)} : \mathcal{M}^g_k \longrightarrow \mathcal{B}^*_{\nu(\Sigma)}, \qquad [A] \longmapsto [A|_{\nu(\Sigma)}]$$

is well-defined, since the restriction of an irreducible ASD connection to an non-empty open set is also irreducible.

Using Neumann boundary conditions to construct slices for the action, the space $\mathcal{B}^*_{\nu(\Sigma)}$ of classes of irreducible connections over $\nu(\Sigma)$ is a manifold modelled on Hilbert spaces. Since $\mathcal{M}^g_k$ is finite dimensional, by transversality argument, there is a C^∞ section s of the line bundle $\mathcal{L}_\Sigma \to \mathcal{B}^*_{\nu(\Sigma)}$ with zero set $s^{-1}(0) = V_\Sigma$ such that the pull-back $\tau^*_{\nu(\Sigma)}(s)$ is transverse to the zero section of $\tau^*_{\nu(\Sigma)}(\mathcal{L}_\Sigma)$ on $\mathcal{M}^g_k$. Hence,

$$\mathcal{M}^g_k \cap V_\Sigma = \{[A] \in \mathcal{M}^g_k \ :\ [A|_{\nu(\Sigma)}] \in V_\Sigma\}$$

is a codimension 2 submanifold which is dual to the class $\mu([\Sigma])$ in $\mathcal{M}^g_k$.

Although the different $SU(2)$ bundles are distinguished by their second Chern classes on X, their restriction to $\nu(\Sigma)$ are isomorphic, and the argument can be extended so that the intersections $\mathcal{M}^g_k \cap V_\Sigma$ are transverse for all k. Finally, extending the result to the case of more than one surface and hence we have condition (3.1.4).

The following crucial proposition is proved by the so called 'dimension counting' argument, which will be applied again in later chapters.

Proposition 3.1.5 *The intersection* $\mathcal{M}^g_k \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$ *is compact, and hence finite.*

Proof: ([DK, (9.2.9)]) Let $\{[A_n]\}$ be a sequence in $\mathcal{M}^g_k \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$. By the compactness of $\overline{\mathcal{M}}^g_k$ in $\coprod_{l=0}^k \mathcal{M}^g_{k-l} \times s^l(X)$, w.r.t. the topology defined by the notion of weak convergence on connections, there is a subsequence $\{[A_m]\}$ which converges weakly to an ideal ASD connection $([A_\infty]; \{x_1, \dots x_l\}) \in \overline{\mathcal{M}}^g_k$. It needs to be shown that $l = 0$, so that $\{[A_m]\}$ actually converges in $\mathcal{M}^g_k$. The subsets V_{Σ_i} have the following important feature:

Alternative 3.1.6 *([DK, (9.2.10)]) For each i, either*

i) $[A_\infty]$ *is non-trivial and* $[A_\infty|_{\nu(\Sigma_i)}] \in V_{\Sigma_i}$, *or*

ii) $\nu(\Sigma_i)$ *contains one of the points* x_j.

The reason is that suppose ii) does not hold, so $\nu(\Sigma_i)$ contains none of the points x_j. Then $[A_m|_{\nu(\Sigma_i)}] \to [A_\infty|_{\nu(\Sigma_i)}]$ in $\mathcal{B}_{\nu(\Sigma_i)}$. Since it has been arranged that the closure of V_{Σ_i} in $\mathcal{B}_{\nu(\Sigma_i)}$ does not contain the trivial connection, $[A_\infty]$ is non-trivial and hence irreducible by generic choice of g as in (3.1.3). So alternative i) holds because V_{Σ_i} is closed in $\mathcal{B}^*_{\nu(\Sigma_i)}$.

The proof that $l = 0$:

Suppose $0 < l < k$, since each x_j lies in at most two of the $\nu(\Sigma_i)$ as we assume that triple intersection of tubular neighbourhoods are empty, alternative i) holds for at least $d - 2l$ surfaces, say $\Sigma_1, \dots, \Sigma_{d-2l}$. Then $[A_\infty] \in \mathcal{M}^g_{k-l} \cap \bigcap_{i=1}^{d-2l} V_{\Sigma_i}$. But this is impossible by the transversality condition (3.1.4), for the dimension of this intersection is negative:

$$dim\, \mathcal{M}^g_{k-l} - 2(d - 2l) = (2d - 8l) - 2(d - 2l) = -4l.$$

Suppose $l = k$, then A_∞ is the trivial connection and alternative ii) must hold for all $i \in \{1, \dots, d\}$, but this is impossible again as ii) must fail for at least $d - 2l$ (= $d - 2k > 0$ for k in the 'stable range') surfaces . Hence $l = 0$ and the proposition is proved.

Since both $\mathcal{M}^g_k$ and the normal bundles to the V_{Σ_i}'s are oriented, each point of the intersection $\mathcal{M}^g_k \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$ carries a sign ± 1.

Definition 3.1.7 $\gamma_{k,\Omega}(\Sigma_1, \dots \Sigma_d) = \sharp(\mathcal{M}^g_k \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d})$ *counted with signs.*

In particular, if $\mathcal{M}^g_k$ is compact, this integer is the pairing $< \mu(\Sigma_1) \cup \cdots \cup \mu(\Sigma_d), [\mathcal{M}^g_k] >$. In [DK, Theorem (9.2.12)] it is shown that this number $\gamma_{k,\Omega}$ satisfies the properties (3.1.1).

3.2 $Spin^c$-Polynomial Invariants

This section is devoted to the construction of the *spin*^{c}-polynomial invariants for 4-manifolds, as in [T], when the dimension of the moduli space $\mathcal{M}^{g,\alpha,1}_k$ of 1-instantons is (even and) greater than zero. It is parallel to the Donaldson polynomial invariants as described in the previous section, except that $\mathcal{M}^{g,\alpha,1}_k$ is used instead of the whole moduli space $\mathcal{M}^g_k$ of ASD connections. Hence, it is some kind of a 'mixed' invariants having features in both the Donaldson polynomials and the (numerical) $spin^c$-invariants as described in chapter 2.

As before, let X be a compact, simply-connected oriented 4-manifold with a Riemannian metric g and an associated line bundle λ (with a connection α). Let $D^{\lambda,\alpha} : \Gamma(W^+) \longrightarrow \Gamma(W^-)$ be the ($spin^c$) Dirac operator. For each $0 \leq j < k$, an $SU(2)$ bundle E_{k-j} over X is fixed with $c_2(E_{k-j}) = (k-j)$ and let $\mathcal{M}^g_{k-j}$ be the moduli space of g-ASD connections on E_{k-j}.

Condition 3.2.1 *As in (2.2.3), choose generic Riemannian metric g and connection α on λ such that*

i) for $0 \leq j < k$, the moduli space of 1-instantons

$$\mathcal{M}^{g,\alpha,1}_{k-j} = \{[A] \in \mathcal{M}^g_{k-j} \,|\, dim_{\mathbb{C}} Ker\, D^{\lambda,\alpha}_A \geq 1\}$$

is a manifold of dimension $(2d-8j) - 2(1-l_j)$, cut out transversely and smooth outside $\mathcal{M}^{g,\alpha,2}_{k-j}$, where $l_j = -(k-j) + \frac{1}{4}c_1(\lambda)^2 - \frac{1}{4}Sgn(X)$ is the (non-positive) index of the coupled Dirac operator

$$D^{\lambda,\alpha}_A : \Gamma(E_{k-j} \otimes W^+) \longrightarrow \Gamma(E_{k-j} \otimes W^-).$$

ii) for $0 \leq j < k$, $\mathcal{M}^g_{k-j}$ and hence $\mathcal{M}^{g,\alpha,1}_{k-j}$ consists of irreducible connections only.

Condition 3.2.2 *Choose sufficiently large[2] k lying in the 'stable range'*

$$2d' = dim_{\mathbb{R}} \mathcal{M}^{g,\alpha,1}_k = 2d - 2(1-l_0) \geq 4k+2$$

$$i.e.\ k \geq \frac{1}{2}(3b^+ - \frac{1}{2}c_1(\lambda)^2 + \frac{1}{2}Sgn(X) + 7)$$

[2]Again, using the blow-up formulae for $spin^c$- invariants as in [P], this stable range condition can be dispensed with.

which is again the requirement that the formal dimension of the moduli space of 1-instantons exceeds the dimension of the lowest stratum in the compactification (of $\mathcal{M}_k^{g,\alpha,1}$) by at least 2. Similar to (3.1.4), using transversality argument, one has

Condition 3.2.3 *For any $\{i_1,\dots,i_r\} \subseteq \{1,\dots,d' = dim_{\mathbb{C}}\mathcal{M}_k^{g,\alpha,1}\}$ and any j with $0 \le j < k$, the 'intersections'*

$$\mathcal{M}_{k-j}^{g,\alpha,1} \cap V_{\Sigma_{i_1}} \cap \cdots \cap V_{\Sigma_{i_r}} = \{[A] \in \mathcal{M}_{k-j}^{g,\alpha,1} \ : \ [A|_{\nu(\Sigma_{i_l})}] \in V_{\Sigma_{i_l}}\ ; \ for\ l = 1,2,\dots,r\}$$

are transverse[3] *and is then a submanifold dual to $\mu([\Sigma_{i_1}]) \cup \cdots \cup \mu([\Sigma_{i_r}])$ in $\mathcal{M}_{k-j}^{g,\alpha,1}$. In particular, $\mathcal{M}_k^{g,\alpha,1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$ consists of isolated points.*

Proposition 3.2.4 *The intersection $\mathcal{M}_k^{g,\alpha,1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$ is compact, hence finite.*

To see this let $\{[A_n]\}$ be a sequence in $\mathcal{M}_k^{g,\alpha,1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$, where $2d' = 2d - 2(1-l_0) = dim_{\mathbb{R}}\, \mathcal{M}_k^{g,\alpha,1}$. As before, by the compactness of $\overline{\mathcal{M}_k^g}$ in $\coprod_{l=0}^k \mathcal{M}_{k-l}^g \times s^l(X)$, there is a subsequence $\{[A_m]\}$ which converges weakly to an ideal ASD connection $([A_\infty]; \{x_1, \dots x_l\}) \in \overline{\mathcal{M}_k^g}$. We want to show that $[A_\infty] \in \mathcal{M}_{k-l}^{g,\alpha,1}$ and $l = 0$, so that $\{[A_m]\}$ actually converges in $\mathcal{M}_k^{g,\alpha,1}$. The point here is that there is an compactification of the moduli space of 1-instantons similar to the Uhlenbeck compactification of the whole moduli space $\mathcal{M}_k^g$ of ASD connections.

Let $\mu(A)$ be the first eigenvalue of the spinor Laplacian $D_A^{\lambda *} D_A^\lambda$, an important feature for the Uhlenbeck compactification is that[4]

Lemma 3.2.5 *$\mu(A_m)$ tends to $\mu(A_\infty)$ as $\{[A_m]\}$ converges weakly to $([A_\infty]; \{x_1, \dots, x_l\})$ in the compactified moduli space $\overline{\mathcal{M}_k^g}$.*

---

[3]The intersection here avoids the singularities. Strictly speaking, the 'moduli space of pairs' should be used in the transversality argument.

[4]This is proved in [DK, lemma 7.1.24], using the Weitzenböck theorem, for spin manifolds and for the simplest case that all x_i are equal.

In particular, if $A_m \in \mathcal{M}_k^{g,\alpha,1}$ (i.e. $dim_{\mathbb{C}} Ker D_{A_m}^{\lambda} > 0$), then $\mu(A_m) = 0$ for all m and by the lemma, $\mu(A_\infty) = 0$ which means $dim_{\mathbb{C}} Ker D_{A_\infty} > 0$ and so $[A_\infty] \in \mathcal{M}_{k-l}^{g,\alpha,1}$. This implies that one can define, similar to the Uhlenbeck compactification $\overline{\mathcal{M}}_k^g$, the compactification $\overline{\mathcal{M}}_k^{g,\alpha,1}$ as the closure of $\mathcal{M}_k^{g,\alpha,1}$ in the space $\coprod_{l=0}^k \mathcal{M}_{k-l}^{g,\alpha,1} \times s^l(X)$ of ideal 1-instantons.

To complete the proof of Proposition (3.2.4), the 'dimension counting argument' is used again to show that $l = 0$. The point here is that instead of decreasing in factors of 4, the dimension of the strata of the compactified moduli space of 1-instantons decrease in steps of at least 2. But this is sufficient for dimension counting argument to hold. Notice that alternative (3.1.6) still holds here. Suppose $0 < l < k$, since each x_j lies in at most 2 of the $\nu(\Sigma_i)$ as we assume that triple intersections of tubular neighbourhoods are empty, alternative (3.1.6) i) holds for at least $d' - 2l$ surfaces (remember $2d' = dim \mathcal{M}_k^{g,\alpha,1}$), say $\Sigma_1, \ldots, \Sigma_{d'-2l}$. Then $[A_\infty] \in \mathcal{M}_{k-l}^{g,\alpha,1} \cap \bigcap_{i=1}^{d'-2l} V_{\Sigma_i}$. But this is impossible by the transversality condition (3.2.1), for the dimension of this intersection is negative: $dim\, \mathcal{M}_{k-l}^{g,\alpha,1} - 2(d' - 2l) \leq (2d' - 6l) - 2(d' - 2l) = -2l$. Suppose $l = k$, then A_∞ is the trivial connection and alternative (3.1.6) ii) must hold for all $i \in \{1, \ldots, d'\}$, but this is impossible again as (3.1.6) ii) must fail for at least $d' - 2l$ ($= d' - 2k > 0$ for k in the 'stable range') surfaces. Hence $l = 0$ and the proposition is proved.

Each point of the intersection $\mathcal{M}_k^{g,\alpha,1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$ are smooth and carries a sign ± 1 from the orientation of $\mathcal{M}_k^{g,\alpha,1}$ as in (2.3). Similar to the Donaldson polynomial invariants, it can be shown that this number is independent of the choice of metrics and the section s_i defining V_{Σ_i}, hence

Definition 3.2.6 $\gamma'_{k,\Omega}(\Sigma_1, \ldots \Sigma_{d'}) = \sharp(\mathcal{M}_k^{g,\alpha,1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}})$ *counted with signs, is an invariant of the oriented diffeomorphism type of* X *which generalizes the (numerical) spin*^{c}*-invariant in chapter 2.*

Chapter 4

Donaldson Polynomials and $Spin^c$-Polynomials

4.1 Generalised Porteous Formula

The possible relationship between the Donaldson polynomial invariants and the *spin*^{c}-invariants is to be investigated. As before, let X be a smooth, simply-connected, compact oriented Riemannian 4-manifold. Let λ be the line bundle associated to a *spin*^{c} structure and $D^\lambda : \Gamma(W^+) \longrightarrow \Gamma(W^-)$ the Dirac operator. For simplicity, assume that $b^+(X)$ is odd and ≥ 3. Then for generic Riemannian metrics g the (even dimensional) moduli space of instantons consists of irreducible connections only, that is $\mathcal{M}_k^g \subseteq \mathcal{B}_X^*$.

Let E be an $SU(2)$ bundle over X with $c_2(E) = k > 0$. Suppose there is a $U(2)$ lift $\underline{\mathbb{E}}$ of the universal $SO(3)$ bundle $\mathbb{E}$ over $\mathcal{B}_X^* \times X$, which carries a family of connections and defines the μ map (see Remark at the end of this section). Let $T \subset \mathcal{B}_X^*$ be a *compact* family of connections, carried by $\underline{\mathbb{E}}|_{T\times X}$, which parametrizes a family $\{D_{A_t}^\lambda\}_{t\in T}$ of coupled Dirac operators of *non-positive* index l.

As T is compact, by a standard stabilization argument as in [DK, (5.1.3)], the formal difference $[kerD^{\lambda}_{A_t}] - [cokerD^{\lambda}_{A_t}]$ gives a well-defined element $ind(D^{\lambda}, \mathbb{E}|_{T\times X})$ of the K-theory of T. The Chern character of this index bundle for the family of coupled ($spin^c$) Dirac operators is given by the Atiyah-Singer index theorem for family ([LM]):

$$ch(ind(D^{\lambda}, \mathbb{E})) = \{ch(\mathbb{E}) \cdot \hat{A}(X) \cdot exp(\frac{1}{2}c_1(\lambda))\}/[X]$$

where $\hat{A}(X) = 1 - \frac{1}{24}p_1(X)$ for a 4-manifold.

By the generalized Porteous formula [Ko, AJ, ACGH], if T intersects the jumping locus $\{[A] \in \mathcal{B}^*_X \mid dim_{\mathbb{C}}kerD^{\lambda}_A \geq r\}$ transversely[1], then the set

$$\{t \in T : dim_{\mathbb{C}}kerD^{\lambda}_{A_t} \geq r \geq 0\} = \{t \in T : dim_{\mathbb{C}}ker(D^{\lambda}_{A_t})^* \geq s = r - l \geq 0\}$$

has the correct codimension and represents the cohomology class

$$P^r_l = (-1)^{r\cdot s}\begin{vmatrix} c_s & c_{s+1} & \cdots & c_{s+r-1} \\ c_{s-1} & c_s & & \vdots \\ \vdots & & \ddots & \vdots \\ c_{s-r+1} & \cdots & \cdots & c_s \end{vmatrix}$$

of T. Here the c_i's are the Chern classes of the index bundle $ind((D^{\lambda})^*, \mathbb{E}|_{T\times X})$.

Recall again that

$$H^*(\mathcal{B}^*_X, \mathbb{Q}) = \mathbb{Q}[\nu, \mu(\alpha_1), \ldots, \mu(\alpha_{b_2})];$$

that is, the rational cohomology ring of $\mathcal{B}^*_X$ is a polynomial algebra on the 2-dimensional generators $\mu(\alpha_i)$ ($i = 1, 2, \ldots, b_2$) and the 4-dimensional generator $\nu = \mu(\text{point})$. Using the Atiyah-Singer index theorem for family of operators, P^r_l as given by the above

[1]That is, the map $T \longrightarrow \mathcal{F}_l$ given by the family of coupled $spin^c$ Dirac operators is transversal to the submanifolds $\{h \in \mathcal{F}_l | dim_{\mathbb{C}}kerh \geq r + i\}$ for all $i \geq 0$, where $\mathcal{F}_l$ is the space of all Fredholm operators of index $l \leq 0$ from $\Gamma(W^+ \otimes E)$ to $\Gamma(W^- \otimes E)$.

generalised Porteous formula can be computed (formally) as a cohomology class in $\mathcal{B}_X^*$, and so can be written as a rational polynomial in the classes $\mu(\alpha_i)$ and ν (as will be done in section (5.4)). Suppose

$$dim\mathcal{M}_k^g = 8k - 3(1+b^+) = 2r(r-l),$$

P_l^r can be evaluated using the recipe for the Donaldson polynomial invariants as in section (3.1) to get a number $\gamma(P_l^r)$ (here the full Donaldson polynomial invariants is considered for which the 4-dimensional class ν is also evaluated, see [DK, (9.2.3)]). In particular, suppose $T = \mathcal{M}_k^g$ is *compact* (e.g. $\dim\mathcal{M}_k^g \leq 6$) and defines a fundamental homology class $[\mathcal{M}_k^g]$ in $\mathcal{B}_X^*$, when $r = 1$ and so $P_l^1 = (-1)^{1-l}c_{1-l}(ind((D^\lambda)^*, \underline{\mathbb{E}}))$, we have

$$\begin{aligned}\gamma(P_l^1) &= < P_l^1, [\mathcal{M}_k^g] > = < Polynomial(\mu(\alpha_i), \nu), [\mathcal{M}_k^g] > \\ &= \sharp\{[A] \in \mathcal{M}_k^g \,|\, dim_{\mathbb{C}} ker D_A^\lambda \geq 1\} = \sharp\mathcal{M}'.\end{aligned}$$

This is the (numerical) $spin^c$-invariant as defined in section (2.1), with the orientation of the (zero dimensional) moduli space $\mathcal{M}'$ of 1-instantons given as in (2.3), i.e. by the homology orientation of $\mathcal{M}$ and the orientation of the normal bundle $(kerD_A^\lambda)^\vee \otimes cokerD_A^\lambda$ at the smooth point $[A] \in \mathcal{M}'$. Similarly, for the general case that $\mathcal{M}_k^g \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_j}$ compact, an invariant is defined by the pairing

$$< P_l^r, [\mathcal{M}_k^g \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_j}] >$$

where $j = d - r(r-l)$. When $r = 1$, it gives the $spin^c$-polynomial invariant constructed in section (3.2).

Hence, when $\mathcal{M}_k^g$ is compact (or in general $\mathcal{M}_k^g \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_j}$ is compact), the $spin^c$-invariants is expressable, via the generalised Porteous formula, in terms of the Donaldson invariants. As an illustration, the (numerical) $spin^c$-invariant for elliptic surfaces without multiple fibre, when $\dim\mathcal{M}_k^g = 4$, is shown in Appendix (A.2) to be zero.

It will be interesting to understand the (possible) relationship between the two smooth invariants when $\mathcal{M}_k^g$ (or $\mathcal{M}_k^g \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_j}$) is non-compact. In this case, Porteous formula cannot be applied directly — the problem is that the Donaldson invariants is defined using the Uhlenbeck compactification $\overline{\mathcal{M}_k^g}$ which, involving ideal instantons, does not parametrize a family of operators. One might expect correction terms arising from the lower strata of $\overline{\mathcal{M}_k^g} \subseteq \coprod_{l=0}^k \mathcal{M}_{k-l}^g \times s^l(X)$. In the followings, It will be shown that this is indeed true for the first two non-trivial cases — when the jumping condition cut down the real dimension of $\mathcal{M}_k^g$ by 4 and by 6, that is the index of the coupled Dirac operators are -1 and -2 respectively.

Remark: As in [DK, Prop.(8.3.5)] , such $U(2)$ lift exists only if k is odd for the $SU(2)$ case. However, using the following projectivisation trick provided by Victor Pidstrigach, our rational calculations using the generalised Porteous formula should work formally by assuming such a lift.

Let $\Gamma^{\pm} = (\Gamma(W^{\pm} \otimes E) \times \mathcal{A}_{\omega}^{*} \times U(2))/\mathcal{G}_{U(2)} \longrightarrow \tilde{\mathcal{B}}_{SO(3)}^{*}$ with $U(2)$-equivariant homomorphism $\Gamma^{+} \xrightarrow{D} \Gamma^{-}$ over $\tilde{\mathcal{B}}_{SO(3)}^{*}$ (where the $U(2)$ action on $\tilde{\mathcal{B}}_{SO(3)}^{*}$ is given by $U(2) \longrightarrow SO(3) = PU(2)$).

Consider the projective bundle $\mathbb{P}(\Gamma^{+}) \xrightarrow{\pi_1} \tilde{\mathcal{B}}_{SO(3)}^{*}$. Lifting Γ^{-} by π_1 to get the bundle $Q = \pi_1^{*}(\Gamma^{-}) \otimes \Theta_{\mathbb{P}(\Gamma^{+})}(1)$ over $\mathbb{P}(\Gamma^{+})$. The coupled Dirac operator then corresponds to a section s of Q as in [PT, Chapters 1 and 3]. Let

$$s^{-1}(0) = \{(a, \mathbb{C}\cdot\sigma) \mid \mathbb{C}\cdot\sigma \subseteq Ker D_a\}$$

be the zero set of s. Then the fundamental class $[s^{-1}(0)] = c_{top}(Q)$.

Note that S^1 doesn't act on $\mathbb{P}(\Gamma^{+})$ (due to projectivisation) nor on Q (since the S^1 acts with weight 1 on $\pi_1^{*}(\Gamma^{-})$ and weight -1 on $\Theta_{\mathbb{P}(\Gamma^{+})}(1)$), so the bundle descends to

$$\pi_1^{*}(\Gamma^{-}) \otimes \Theta_{\mathbb{P}(\Gamma^{+})}(1) \longrightarrow \mathbb{P}(\Gamma^{+}) \longrightarrow \mathcal{B}_{SO(3)}^{*} .$$

Restricted to a compact set T of $\mathcal{B}^*_{SO(3)}$, and let $dim\Gamma^{\pm} = m_{\pm}$, then

$$
\begin{aligned}
& c_{m_-}(\pi_1^*(\Gamma^-) \otimes \Theta_{\mathbb{P}(\Gamma^+)}(1)) \\
& = \textstyle\sum_{i=0}^{m_-} c_i(\pi_1^*(\Gamma^-)) \cdot c_1(\Theta_{\mathbb{P}(\Gamma^+)}(1))^{m_- - i} && \text{(by formula in [Fu, remark (3.2.3)])} \\
& = \textstyle\sum_{i=0}^{m_-} c_i(\Gamma^-) \cdot s_{m_- - m_+ - i + 1}(\Gamma^+) && \text{(definition of Segre Class, [Fu, paragraph (3.1)])} \\
& = [c(\Gamma^-) \cdot s(\Gamma^+)]_{m_- - m_+ + 1} \\
& = [c(\Gamma^-) \cdot c^{-1}(\Gamma^+)]_{m_- - m_+ + 1} && \text{(definition of Segre class as inverse of Chern class)} \\
& = [c(\Gamma^- - \Gamma^+)]_{m_- - m_+ + 1}
\end{aligned}
$$

which is the Porteous class considered ([Fu, chapter 14]).

Although $\Gamma^{\pm}$ are infinite dimensional spaces, the above equality holds as in [PT, chapter 3], which is obtained by splitting the family of Dirac operators into direct sum of 2 families — one is a map between two finite dimensional bundles whereas the other one is an isomorphism. Also note that the first term $c_{m_-}(\pi_1^*(\Gamma^-) \otimes \Theta_{\mathbb{P}(\Gamma^+)}(1))$ is well-defined as the dual of zero set of a section and the last term $[c(\Gamma^- - \Gamma^+)]_{m_- - m_+ + 1}$ is simply the Chern class of the (dual) index bundle.

4.2 The Case dim$\mathcal{M} = 8$ and $indD_A^\lambda = -1$

Let X be as before. $E \to X$ an $SU(2)$ bundle (or $SO(3)$ bundle[2]) over X with $c_2(E) = k$ chosen such that the moduli space $\mathcal{M}_8$ of instantons, with respect to metric g, is of real dimension 8. Similarly, $\mathcal{M}_0$ denotes the 0-dimensional moduli space of g-instantons on a bundle over X with $c_2 = k - 1$ (From now on, for the ease of notations, the subscripts of $\mathcal{M}$ are used to denote the dimension of the moduli spaces). Generic g can be chosen such that $\mathcal{M}_0$ and $\mathcal{M}_8$ are regular and contain no reducible (Chern-Weil formula ensures that the trivial connection doesn't occur), so the transversality condition (3.1.4) holds for both moduli spaces.

[2]In the calculation for a $K3$ surface (with $b^+ = 3$) in chapter 5, in order to have a 8-dimensional moduli space, $SO(3)$ bundle with $p_1 = -10$ is considered. The arguments in this chapter apply to the $SO(3)$ case to give the universal formula (4.2.1).

Choose $spin^c$ structure on X, with associated line bundle λ, such that the index of the coupled Dirac operator $indD_A^\lambda = -1$. Suppose the $spin^c$-invariant is well-defined (for example the stable range condition (3.2.2) is satisfied, or $w_2(E) \neq 0$ for the $SO(3)$ case). To compute the $spin^c$-polynomial, as in section (3.2), $\mathcal{M}_8$ is first cut down transversely to the *compact* 4-dimensional space of 1-instantons

$$\mathcal{M}'_8 = \{[A] \in \mathcal{M}_8 | dim_{\mathbb{C}} ker D_A^\lambda \geq 1\}$$

which is then evaluated on the 4-dimensional cohomology class $\mu(\Sigma_1) \cup \mu(\Sigma_2)$ using the recipe of the Donaldson invariant as in chapter 3:

$$\gamma'_4(\Sigma_1, \Sigma_2) = \sharp(\mathcal{M}'_8 \cap V_{\Sigma_1} \cap V_{\Sigma_2}).$$

Alternatively, one can consider the non-compact 4-dimensional 'transverse intersection'

$$\mathcal{N}_4 = \mathcal{M}_8 \cap V_{\Sigma_1} \cap V_{\Sigma_2},$$

then $\gamma'_4(\Sigma_1, \Sigma_2) = \sharp \mathcal{N}'_4$, where $\mathcal{N}'_4 = \{[A] \in \mathcal{N}_4 | dim_{\mathbb{C}} ker D_A^\lambda \geq 1\}$.

Similar to [DK, Chapter (8.3)], let N_{x,A_0} be the Taubes neighbourhood in $\mathcal{M}_8$ of the ideal instanton $([A_0], x)$, where $A_0 \in \mathcal{M}_0$ and $x \in X$, with $\zeta : N_{x,A_0} \longrightarrow \mathbb{R}^+$ the scale map. Removing 'ends' of $\mathcal{N}_4$, that is, define

$$\mathcal{N}^0 = \mathcal{N}_4 \setminus \bigcup_{x \in \Sigma_1 \cap \Sigma_2, A_0 \in \mathcal{M}_0} \{v \in N_{x,A_0} \cap \mathcal{N}_4 \mid \zeta(v) < \zeta_0\}$$

for ζ_0 a small positive number. By Sard's theorem, ζ_0 can be chosen such that $\mathcal{N}^0$ is a compact manifold with boundary

$$\partial \mathcal{N}^0 = \bigcup_{x \in \Sigma_1 \cap \Sigma_2, A_0 \in \mathcal{M}_0} L_{x,A_0,\zeta_0}$$

where $L_{x,A_0,\zeta_0} = \{v \in N_{x,A_0} \cap \mathcal{N}_4 \mid \zeta(v) = \zeta_0\}$ a link of $([A_0], x)$ in $\mathcal{N}_4$.

Taubes gluing map gives $N_{x,A_0} \cap \mathcal{N}_4 \cong \mathbb{R}^+ \times SO(3)$, a cone over $SO(3)$, and hence[3] $L_{x,A_0,\zeta_0} \cong SO(3)$, with orientation induced from that of $\mathcal{N}_4$. This $SO(3)$

[3]In this case, $H^1_{A_0} = 0$, $H^2_{A_0} = 0$ (regular) and A_0 is irreducible. Hence, $N_{x,A_0} \cap \mathcal{N}_4 \cong \Psi_0^{-1}(0)$ where $\Psi_0 : \mathbb{R}^+ \times SO(3) \longrightarrow 0$ restriction of $\Psi : U_x \times \mathbb{R}^+ \times SO(3) \longrightarrow 0$ to $x \in \Sigma_1 \cap \Sigma_2$.

family is obtained by first gluing the standard instanton of charge 1 on S^4 to A_0 on X, with a gluing map of small scale ζ_0, to get a $SO(3)$-family $\hat{L}_{x,A_0,\zeta_0}$ of classes of almost ASD connections (this gluing construction will be described in next section); then a small perturbation, whose size depending on ζ_0, by elements in $\Omega^1_{X\#S^4}(\Im_E)$ gives the required ASD connections. $\hat{L}_{x,A_0,\zeta_0}$ consists of classes of connections which are gauge equivalent to A_0 outside a small ball about x of size depending on ζ_0. The $SO(3)$ corresponds to the different choice of gluing parameters [DK, (7.2),(8.2)].

Let $\mathcal{C}_{x,A_0} = \{[A] \in \mathcal{B}^*_X \mid [A|_{X\backslash B_x}] = [A_0|_{X\backslash B_x}]\}$ where $[A_0] \in \mathcal{M}_0$ and B_x is a small ball about $x \in \Sigma_1 \cap \Sigma_2$. Hence, the $SO(3)$ family $\hat{L}_{x,A_0,\zeta_0} \subset \mathcal{C}_{x,A_0}$ if the scale ζ_0 is sufficiently small. Suppose $H_3(\mathcal{C}_{x,A_0}, \mathbb{Z}) = 0$ (see Remark at the end of this section), there is compact 4-dimensional oriented family $\mathcal{T}_{x,A_0,\zeta_0} \subset \mathcal{C}_{x,A_0}$ such that $\partial\mathcal{T}_{x,A_0,\zeta_0} = -\hat{L}_{x,A_0,\zeta_0}$ (here the minus sign denotes the reverse of the orientation induced from that of L_{x,A_0,ζ_0}). The idea is to cap $\mathcal{N}^0$ by these $\mathcal{T}_{x,A_0,\zeta_0}$'s to form a closed compact 4-family of connections in $\mathcal{B}^*_X$. Then the generalised Porteous formula can be applied to this 'compactification' of the moduli space (but not the Uhlenbeck compactification which involves singular connections) to yield a formula relating the *spin*^{c}-invariant and the Donaldson invariant evaluating the cohomology class defined by the generalised Porteous formula.

The well-definedness of the two smooth invariants implies that connections in the 'ends' are immaterial and L_{x,A_0,ζ_0} can be perturbed to be $\hat{L}_{x,A_0,\zeta_0}$, when the scale ζ_0 is sufficiently small, without affecting the *spin*^{c}-invariant and the evaluation of the Donaldson invariant on the Porteous class. It is easy to see as follows.

Choose representation of the Porteous class P^1_{-1} 'supported away from' all the $x \in \Sigma_1 \cap \Sigma_2$, say $\mu(\sigma_1) \cup \mu(\sigma_2)$ with embedded surfaces σ_1 and σ_2 supported outside small balls B_x about such x. (For the 4-dimensional class $\nu = \mu(pt)$, choose the point *pt* outside B_x.) Choose ζ_1 small such that when the scale $\zeta < \zeta_1$,

$$\hat{L}_{x,A_0,\zeta} \subset \mathcal{C}_{x,A_0} = \{[A] \in \mathcal{B}^*_X \mid [A|_{X\backslash B_x}] = [A_0|_{X\backslash B_x}]\}.$$

Since $D^{\lambda}_{A_0}$ has trivial kernel generically, the first eigenvalues of the coupled Dirac operators are bounded uniformly from below by a positive number for all $[A] \in \mathcal{N}_4$ sufficiently close to the ideal instanton $([A_0], x)$ ([DK, (7.1.24)], this gives the well-definedness of the $spin^c$-invariants in section (3.2)), there is a ζ_2 such that $kerD^{\lambda}_{A+a} = 0$ for all $[A] \in L_{x,A_0,\zeta}$ where $\zeta < \zeta_2$, and $a \in \Omega^1_X(\Im_E)$ with L^2_l-norm less than certain small ϵ_1.

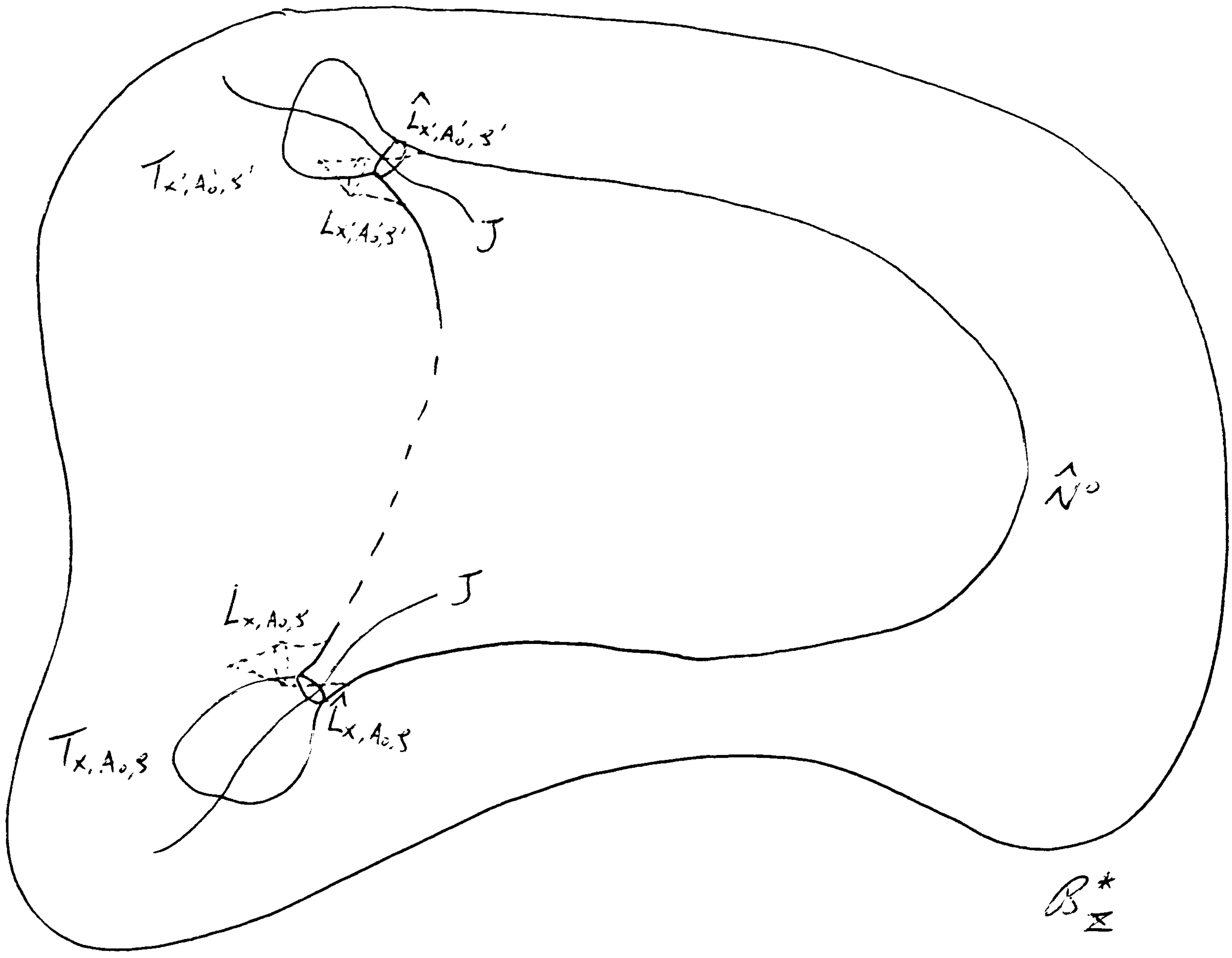
By the well-definedness of the Donaldson invariant as in section (3.1), there is a ζ_3 such that $[A] \notin \mathcal{N}_4 \cap V_{\sigma_1} \cap V_{\sigma_2}$ for all $[A] \in L_{x,A_0,\zeta}$ with $\zeta < \zeta_3$. Also, $\mathcal{M}_0 \cap V_{\sigma_1} \cap V_{\sigma_2} = \emptyset$ by transversality (3.1.4). Since V_{σ_i} are closed (in C^∞) in $\mathcal{B}^*_{\nu(\sigma_i)}$, there is a small number ϵ_2, such that for all a with L^2_l-norm $< \epsilon_2$, $[(A_0 + a)|_{\nu(\sigma_i)}] \notin V_{\sigma_i}$ for $i = 1, 2$ (notations as in (3.1.4)).

Let $\epsilon' = min\{\epsilon_1, \epsilon_2\}$, there is a ζ_4 such that whenever $\zeta < \zeta_4$, connections in $\hat{L}_{x,A_0,\zeta}$ differs from the corresponding ASD connections in $L_{x,A_0,\zeta}$ under the Taubes' gluing construction by an element in $\Omega^1_X(\Im_E)$ (identifying X with $X\#S^4$) with L^2_l norm less than ϵ'. Finally, let $\zeta'(x, A_0) = min\{\zeta_i : i = 1, \ldots, 4\}$. Then for all $\zeta < \zeta'(x, A_0)$, $L_{x,A_0,\zeta}$ can be perturbed to $\hat{L}_{x,A_0,\zeta}$ without affecting the $spin^c$-invariant and the Donaldson invariant evaluating on the Porteous class. Hence, for the scale $\zeta_0 < \zeta' = min\{\zeta'(x, A_0) | x \in (\Sigma_1 \cap \Sigma_2), A_0 \in \mathcal{M}_0\}$ sufficiently small, $\hat{\mathcal{N}}^0 \subset \mathcal{B}^*_X$ can be obtained by perturbing elements near the boundary of $\mathcal{N}^0$, such that

$$\partial \hat{\mathcal{N}}^0 = \bigcup_{x \in \Sigma_1 \cap \Sigma_2, A_0 \in \mathcal{M}_0} \hat{L}_{x,A_0,\zeta_0} \quad , \quad \gamma'_4(\Sigma_1, \Sigma_2) = \sharp(\mathcal{N}^0)' = \sharp(\hat{\mathcal{N}}^0)'$$

the transverse intersection with the jumping locus, and both $\mathcal{N}^0$ and $\hat{\mathcal{N}}^0$ give the ($SU(2)$ or $SO(3)$) Donaldson invariant $\gamma_4(P^1_{-1} \cup \mu(\Sigma_1) \cup \mu(\Sigma_2))$ when intersecting with the compactly supported representation of the 4-dimensional class P^1_{-1}.

Capping off $\hat{\mathcal{N}}^0$ by the $\mathcal{T}_{x,A_0,\zeta_0}$'s to give $\mathcal{W} = \hat{\mathcal{N}}^0 \cup_{\hat{L}_{x,A_0,\zeta_0}} \mathcal{T}_{x,A_0,\zeta_0}$, where $x \in \Sigma_1 \cap \Sigma_2$ and $A_0 \in \mathcal{M}_0$, which is closed, compact and defines a 4-dimensional cycle.



$\hat{L}_{x',A_0',\xi'}$
$T_{x',A_0',\xi'}$
$L_{x',A_0',\xi'}$
J
$\hat{N}^0$
J
$L_{x,A_0,\xi}$
$T_{x,A_0,\xi}$
$\hat{L}_{x,A_0,\xi}$
$\mathcal{B}^*_{\mathbb{Z}}$

Suppose $\mathcal{T}_{x,A_0,\zeta_0}$ can be chosen to have transverse intersection with the jumping locus $J = \{[A] \in \mathcal{B}^*_X | dim_{\mathbb{C}} ker D^\lambda_A \geq 1\}$ so that

$$\mathcal{T}'_{x,A_0,\zeta_0} = \{[A] \in \mathcal{T}_{x,A_0,\zeta_0} | dim_{\mathbb{C}} ker D^\lambda_A \geq 1\}$$

is zero-dimensional. Using notations as in (4.1), Porteous formula applies to give

$$\begin{aligned} < P^1_{-1}, [\mathcal{W}] >= \sharp \mathcal{W}' &= \sharp\{[A] \in \mathcal{W} |\ dim_{\mathbb{C}} Ker D^\lambda_A \geq 1\} \\ &= \sharp(\hat{\mathcal{N}}^0)' + \textstyle\sum_{x \in \Sigma_1 \cap \Sigma_2, A_0 \in \mathcal{M}_0} (\sharp \mathcal{T}'_{x,A_0,\zeta_0}) \end{aligned}$$

with $\sharp\mathcal{T}'_{x,A_0,\zeta_0}$ defined in the obvious way. However, since P^1_{-1} is represented by classes supported away from $x \in \Sigma_1 \cap \Sigma_2$ and connections in $\mathcal{T}_{x,A_0,\zeta_0}$ are gauge equivalent to A_0 outside small balls centred at x, $\mathcal{T}_{x,A_0,\zeta_0}$ has no contribution to $< P^1_{-1}, [\mathcal{W}] >$. So for ζ_0 sufficiently small,

$$\gamma_4(P^1_{-1} \cup \mu(\Sigma_1) \cup \mu(\Sigma_2)) = \gamma'_4(\Sigma_1, \Sigma_2) + \sum_{x \in \Sigma_1 \cap \Sigma_2, A_0 \in \mathcal{M}_0} \sharp \mathcal{T}'_{x,A_0,\zeta_0}.$$

Hence, $\sharp\mathcal{T}'_{x,A_0,\zeta_0}$ (which depends, *a priori*, on $x \in \Sigma_1 \cap \Sigma_2$, $[A_0] \in \mathcal{M}_0$ and the scale ζ_0) are (finite) numbers measuring the discrepancies between the two smooth invariants. It will be shown in the next section that when the scale ζ is sufficiently small, $\sharp\mathcal{T}'_{x,A_0,\zeta}$ is in fact a universal constant independent of X, ζ and A_0. Hence, taking orientation into accounts and with the scale of the gluing map sufficiently small, for $\operatorname{ind} D^\lambda_A = -1$ there is a universal formula

Formula 4.2.1

$$\gamma_4(P^1_{-1} \cup \mu(\Sigma_1) \cup \mu(\Sigma_2)) = \gamma'_4(\Sigma_1, \Sigma_2) + (constant\ \sharp\mathcal{T}') \cdot Q(\Sigma_1, \Sigma_2) \cdot \gamma_0$$

expressing the $spin^c$-invariant γ'_4 as the (8-dimensional) Donaldson invariant γ_4 evaluating on the Porteous class P^1_{-1}, with a correction term given, up to a constant, by the number of points at infinity $Q(\Sigma_1, \Sigma_2) \cdot \gamma_0$ of $\mathcal{N}_4$. Therefore, it involves the immediate lower stratum in the Uhlenbeck compactification $\overline{\mathcal{N}}_4$ as expected. Finally, the 'linking invariant' $\sharp\mathcal{T}'$ of $\hat{L}$ (in the sense that it is the algebraic number of intersection points in $J \cap \mathcal{T}$ where $\partial\mathcal{T} = -\hat{L}$ and $J \cap \hat{L} = \emptyset$, here J is the jumping locus) will be obtained by computing an example, the $K3$ surface, in chapter 5.

Remark: From the calculation in chapter 5, this 'linking invariant' $\sharp\mathcal{T}' = \frac{1}{12}$ is non-integral, which means that the assumption $H_3(\mathcal{C}_{x,A_0}, \mathbb{Z}) = 0$, where $x \in \Sigma_1 \cap \Sigma_2$ and $A_0 \in \mathcal{M}_0$, at the beginning of the section is incorrect. In fact, $\mathcal{C}_{x,A_0}$ is of the same homotopy type as $\tilde{\mathcal{B}}_{S^4}$ and its third homology group is shown to be $\mathbb{Z}_{12} \oplus \mathbb{Z}_2$ in Appendix B, with the $SO(3)$ family $\hat{L}_{x,A_0}$ of order 12. Hence 'twelve copies' of $\hat{L}_{x,A_0}$ can be bounded by a 4-family in $\mathcal{C}_{x,A_0}$, again denoted by $\mathcal{T}_{x,A_0}$. Capping $12\hat{\mathcal{N}}^0$ by the $\gamma_0 \cdot Q(\Sigma_1, \Sigma_2)$ number of $\mathcal{T}$, gets

$$\mathcal{W} = 12\hat{\mathcal{N}}^0 \cup_{12\hat{L}_{x,A_0}} \mathcal{T}_{x,A_0}$$

which is compact. Applying the generalised Porteous Formula to this situation, the 'correct' relationship between the two smooth invariants should read

$$< P^1_{-1}, \mathcal{W} >= 12\gamma_4(P^1_{-1} \cup \mu(\Sigma_1) \cup \mu(\Sigma_2)) = 12\gamma_4'(\Sigma_1, \Sigma_2) + (\sharp\mathcal{T}') \cdot Q(\Sigma_1, \Sigma_2) \cdot \gamma_0.$$

So the 'correct' linking invariant should be 1.

4.3 The 'Linking Invariant' is a Universal Constant

The analysis in this section follows closely that of [P], in which some blow-up formulae (i.e. glueing X with $\overline{\mathbb{CP}^2}$) for $spin^c$-polynomials are obtained. Here the gluing of X with S^4 is considered.

Recall the connected sum construction as in [DK, (7.2),(8.2)]. Let A_1, A_2 be connections on bundles E_1, E_2 (of the same structure group) over compact oriented

Riemannian four-manifolds X_1 and X_2, respectively. Identify geodesic neighbourhoods of the points x_i on X_i with neighbourhoods of zeroes in the tangent spaces $(TX_i)_{x_i}$. Let $\sigma : (TX_1)_{x_1} \longrightarrow (TX_2)_{x_2}$ be an orientation-reversing isometry. For any real number $1 > \zeta > 0$ define the gluing map of scale ζ

$$f_{\zeta,\sigma} : (TX_1)_{x_1} \setminus \{0\} \longrightarrow (TX_2)_{x_2} \setminus \{0\}$$

by $f_{\zeta,\sigma}(\eta) = \frac{\zeta^4}{|\eta|^2}\sigma(\eta)$, which is the inversion in the sphere of radius ζ^2 centred at x_1. The map induces a diffeomorphism from Ω_1 to Ω_2, where $\Omega_i \subset X_i$ are annulus centred at x_i with radii ζ and ζ^3. Removing the ζ^3-ball about x_i to get X_i', and identifying the annuli Ω_i by $f_{\zeta,\sigma}$ to get the connected sum $X_1 \# X_2 = X_1' \cup_{f_{\zeta,\sigma}} X_2'$. If the metrics in the neighbourhoods of x_i are standard, then $f_{\zeta,\sigma}$ is a conformal map.

Using the conformal equivalence $e : \mathbb{R} \times S^3 \longrightarrow \mathbb{R}^4 \setminus \{0\}$ given by $e(t,w) = e^t w$, which 'pulls back' the annulus Ω with radii ζ and ζ^3 to the tube

$$e^{-1}(\Omega) = (3log\zeta, log\zeta) \times S^3,$$

with length $t = -2log\zeta \longrightarrow \infty$ as $\zeta \longrightarrow 0$. The connected sum $X_1 \# X_2$ can be thought of as being formed by deleting x_i from X_i and regarding punctured neighbourhoods as half-cylinders and identifying the cylinders by a reflection.

Use cut-off functions to damp A_i to connection A_i' which are flat over the annuli Ω_i and equal to A_i outside a small r-ball, with $r > \zeta$, which can be done in such a way that $\|A_i - A_i'\|_{L^4(X_i)} \leq const \cdot e^{-t}$ where t is the length of the tube in the above model and the constant depends on A_i only. Choose an isomorphism (gluing parameter) of the fibres $\rho : (E_1)_{x_1} \longrightarrow (E_2)_{x_2}$. Use the flattened connections A_i' to spread this isomorphism out to give a bundle isomorphism between the E_i over Ω_i, covering $f_{\zeta,\sigma}$, which defines a bundle E_ρ over the connected sum manifold. The bundle isomorphism respects A_i' and defines a connection A_ρ on E_ρ, such that A_ρ and $A_{\rho'}$ are gauge equivalent iff ρ and ρ' are in the same orbit of the action of $\Gamma = \Gamma_1 \times \Gamma_2$ on the space of all gluing parameters, where Γ_i is the isotropy group of A_i.

In particular, as in [DK, (8.2.1)], when $X_2 = S^4$ there is a concrete description of the gluing map (which will be useful in (6.3) in describing the obstruction line bundle). Let $x \in X$ and suppose, for simplicity, that the metric is flat in a neighbourhood of x. Fix a local coordinate system identifying this neighbourhood with a neighbourhood of $0 \in \mathbb{R}^4$. For $\zeta > 0$ let

$$d_\zeta : \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \qquad d_\zeta(y) = \zeta^{-4}(y)$$

be the dilation map and let $m : \mathbb{R}^4 \longrightarrow S^4 = \mathbb{R}^4 \cup \infty$ be the standard stereographic map. Then $m \circ d_\zeta$, regarded as a map from a small annulus about $x \in X$ to a similar annulus about $\infty \in S^4$, is the same as the gluing map $f_{\zeta,\sigma}$, with σ the natural orientation-reversing isometry between the tangent spaces to S^4 at 0 and ∞.

Suppose V is a bundle supporting charge-1 connections over S^4, equipped with the round metric. Let $\tilde{\eta} = \{(I, \alpha_\infty)\}/\mathcal{G}_V$ be the $SO(3)$-family in the orbit space $\tilde{\mathcal{B}}_{S^4}$ of framed (irreducible) connections, where I is the standard instanton of charge 1 on S^4 and α_∞, which gives a trivialisation of the fibre V_∞, run through the set of all framings over $\infty \in S^4$. Hence, $\tilde{\eta}$ is the fibre over I of the base-point fibration $\tilde{\mathcal{B}}_{S^4} \longrightarrow \mathcal{B}_{S^4}$. Then the $SO(3)$-family $\hat{L}_{x,A_0,\zeta_0} = \{[A_0 \# \tilde{\eta}]\}$ with the gluing map of scale ζ_0 and the gluing parameters determined by the framings α_∞ when a trivialisation t_x of the fibre Y_x is fixed, here Y is the bundle supporting the connection $A_0 \in \mathcal{M}_0$. That is, the gluing parameters in $[A_0 \# (I, \alpha_\infty)]$ are given by the composition $V_\infty \xrightarrow{\alpha_\infty} G \xrightarrow{t_x^{-1}} Y_x$ where $G = SU(2)$ (or $SO(3)$) is the gauge group.

Recall that $\mathcal{T}_{x,A_0,\zeta_0} \subseteq \mathcal{C}_{x,A_0}$, the space of connections concentrated at x with background connection A_0, is a compact 4-dimensional family of classes of connections whose boundary is the $SO(3)$ family $\hat{L}_{x,A_0,\zeta_0}$. Therefore, $\mathcal{T}_{x,A_0,\zeta_0} = \{[A_0 \# \tilde{\tau}]\}$ (again with the same gluing map of small scale ζ_0 and the gluing parameters determined by the framings) for some compact 4-family $\tilde{\tau}$ in $\tilde{\mathcal{B}}_{S^4}$ with $\partial\tilde{\tau} = -\tilde{\eta}$. Without loss of generality, for each $x \in \Sigma_1 \cap \Sigma_2$ and $[A_0] \in \mathcal{M}_0$, the same $\tilde{\tau}$ can be used[4]

[4]In fact, using the generalised Porteous formula and the fact that $H^i(\tilde{\mathcal{B}}_{S^4}, \mathbb{Q}) = 0$, $\forall i > 0$, if two

to define the 4-family $\mathcal{T}_{x,A_0,\zeta_0}$. Again, let λ be the *spin*^{c} structure on X with $\mathrm{ind}D_A^\lambda = -1$, where $[A] \in \mathcal{M}_8$. Therefore, $\mathrm{ind}D_{A_0}^\lambda = 0$ for all $A_0 \in \mathcal{M}_0$. Note that S^4 has a (unique) spin structure with the Dirac operator D coupled to a charge-1 connection on S^4 has index -1 (i.e. additivity of indices over the connected sum). Since D coupled to the instanton I in $\tilde{\eta} = \{[I, \alpha_\infty]\}$ has trivial kernel, the jumping locus $\{[a] \in \tilde{\mathcal{B}}_{S^4} \mid dim_{\mathbb{C}} ker D_a \geq 1\}$ 'intersects $\tilde{\eta}$ transversely'. By Sard-Smale theorem, $\tilde{\tau}$ can be chosen such that $\tilde{\tau}$ and $\mathcal{T}_{x,A_0,\zeta_0}$ have transverse intersection with the corresponding jumping locus (on S^4 and $X \# S^4$ respectively), for generic small ζ_0. Therefore, $\tilde{\tau}' = \{[a] \in \tilde{\tau} | dim_{\mathbb{C}} ker D_a \geq 1\}$ is zero dimensional and, by compactness, consists of finite number of smooth points, i.e. $[a'] \in \tilde{\tau}'$ iff $dim_{\mathbb{C}} ker D_{a'} = 1$ and the map

$$T_{[a']}(\tilde{\tau}) \xrightarrow{j} (ker D_{a'})^* \otimes (coker D_{a'}), \qquad j(\delta a)(s) = \pi_{S^4}\{(\delta a) * s\}$$

is an isomorphism, where $\delta a \in T_{[a']}(\tilde{\tau})$ and $s \in ker D_{a'}$. $*$ is a combination of matrix multiplication and Clifford multiplication on bundle-valued spinors to give an element in $\Gamma(W_{S^4}^- \otimes V)$ and π_{S^4} the projection map onto $coker D_{a'}$. Since the coupled Dirac operator and the jumping condition do not depend on the framings, for simplicity $\tilde{\tau}$ is chosen such that there is a neigbourhood $\hat{U} \subset \tilde{\tau}$ of $[a'] \in \tilde{\tau}'$ consisting of framed connections with the same framing as $[a']$.

Claim 4.3.1 *The linking invariant $\sharp \mathcal{T}'_{x,A_0,\zeta} = \sharp \tilde{\tau}'$ for generic ζ sufficiently small and hence is a universal constant independent of X, ζ and A_0.*

The rest of the section is dedicated to the proof of this claim. The strategy is to show that when t is sufficiently large (or equivalently the scale ζ of the gluing map is small), for any $[a] \in \tilde{\tau}'$ there is a unique small perturbation $[a + \delta a] \in \tilde{\tau}$ such that $[A_0 \#(a + \delta a)] \in \mathcal{T}'_{x,A_0,\zeta}$; and every $[A'] \in \mathcal{T}'_{x,A_0,\zeta}$ is obtained in this way when ζ is sufficiently small. This is a special (un-obstructed) case of a more general obstruction

4-families $\tilde{\tau}_1$ and $\tilde{\tau}_2$ in $\tilde{\mathcal{B}}_{S^4}$ satisfying $\partial \tilde{\tau}_i = -\tilde{\eta} = -\{[I, \alpha_\infty]\}$, then $\sharp \tilde{\tau}'_1 = \sharp \tilde{\tau}'_2$.

theory as in [P] . In chapter (6.3), similar scheme of arguments are used when the index is -2, in that case there is a 1-dimensional obstruction to such deformation.

As in [P], see also [DK, (3.3.3)], the norm $L^{8/3}$ on the space of spinors is used. It has the property that if $g' = e^{2f}g$ is a conformal change of a metric, then

$$s \in ker(D_a^g) \Leftrightarrow s' = e^{-\frac{3}{2}f}s \in ker(D_a^{g'})$$

$$and \quad \|s\|_{L^{8/3}(g)} = \|s'\|_{L^{8/3}(g')}.$$

Bundle-valued spinors in an arbitrary conformal model g_ζ of $X\#S^4$ (with gluing map of scale ζ) are identified in this way, with g_ζ generic when restricted to X.

Suppose $\tilde{\tau}$ intersects the jumping locus transversely at $a \in \tilde{\tau}'$, and $s \in \Gamma(W_{S^4}^+ \otimes V)$ with unit $L^{8/3}$-norm such that $D_a(s) = 0$. We want to show that for ζ sufficiently small there is a unique small deformation $(a+\delta a)$ of a within $\tilde{\tau}$ which when glued to A_0 at x gives an element of $\mathcal{T}'_{x,A_0,\zeta}$. Note that $indD^\lambda_{A_0} = 0$ and so for generic metric on X and connection on λ, $kerD^\lambda_{A_0} = cokerD^\lambda_{A_0} = 0$. As in [P], the surjectivity of $D^\lambda_{A_0}$ and the transversality of $a \in \tilde{\tau}'$ (i.e. $coker(j) = 0$) will imply that there is no obstruction to such deformation.

Consider a neighbourhood $U \subset \hat{U}$ of a in $\tilde{\tau}$, consisting of elements $(a+\delta a)$ with the same framing as a and $\|\delta a\|_{L^4(S^4)} \leq \epsilon$, where ϵ is to be fixed in the proof. Let $I(\delta a) = A_0\#_\rho(a+\delta a) - A_0\#_\rho a$, with the (fixed) gluing parameter ρ given by the same framing of the elements in U. Then

$$\|I(\delta a)\|_{L^4(X\#S^4)} = \|\xi(\delta a)\|_{L^4(S^4)} \leq \|\delta a\|_{L^4(S^4)} \leq \epsilon,$$

where ξ is the cut-off function which damps the connections over S^4 to ones that are flat over the annulus in the connected sum construction.

Let β be a cut-off function, equal to 1 on S^4 side and 0 on X side, with $\nabla\beta$ supported over the tube $S^3 \times [\theta_1 t, \theta_2 t]$ where $-\frac{1}{2} < \theta_1 < \theta_2 < \frac{1}{2}$ and $\|\nabla\beta\|_{L^4(X\#S^4)} \leq$

$\epsilon(t) \to 0$ as $t \to \infty$ ([DK, (7.2.2)]). Then βs, considered as a section in $(W^+_{X\#S^4} \otimes E)$, is close to being a harmonic spinor of $D^\lambda_{A_0\#_\rho a}$, when the gluing map is of small scale, i.e.

$$\|D^\lambda_{A_0\#_\rho a}(\beta s)\|_{L^{8/3}(X\#S^4)} \leq \epsilon'(t) \to 0, \quad as \quad t \to \infty.$$

Fix a lift $\sigma_{S^4} : cokerD_a = \Gamma(W^-_{S^4} \otimes V)/ImD_a \longrightarrow \Gamma(W^-_{S^4} \otimes V)$ of the obvious projection π_{S^4}, so that the image of σ_{S^4} consists of sections with support away from $\infty \in S^4$. Then it gives a linear map

$$\sigma : cokerD_a \longrightarrow \Gamma(W^-_{X\#S^4} \otimes E)$$

which is a lift of the projection $\pi(\eta) = \pi_{S^4}(\beta \cdot \eta)$. Define a map

$$\Phi : U \times \Gamma(W^+_{X\#S^4} \otimes E) \times cokerD_a \longrightarrow \Gamma(W^-_{X\#S^4} \otimes E)$$

$$\Phi(a + \delta a, \delta s, h) = D^\lambda_{A_0\#_\rho(a+\delta a)}(\beta s + \delta s) + \sigma(h).$$

The linearization of Φ at the point $(a, 0, 0)$ w.r.t. $(\delta s, h)$ is $D^\lambda_{A_0\#_\rho a}(\delta s) + \sigma(h)$, which has a right inverse

$$P \oplus \pi : \Gamma(W^-_{X\#S^4} \otimes E) \longrightarrow \Gamma(W^+_{X\#S^4} \otimes E) \oplus cokerD_a,$$

that is, $(D^\lambda_{A_0\#_\rho a}P \oplus \sigma\pi) = id_{\Gamma(W^-_{X\#S^4}\otimes E)}$, where P is a bounded operator from $L^{8/3}$ to $L^{8/3}_1$.

Since $\sigma\pi = pr_2$ is a projector by definition, $D^\lambda_{A_0\#_\rho a}P = pr_1$ is also a projector. One can assume that $Im(P)$ is transversal to $kerD^\lambda_{A_0\#_\rho a}$, which means $P(\eta) = P(\eta_1)$, where $\eta_i = pr_i(\eta)$. Although pr_i are not orthogonal there are estimates, say, $\|pr_i\| \leq 2$. Now we are looking for solutions of the equation $\Phi(a + \delta a, \delta s, h) = 0$ in the form $(\delta s, h) = (P(\eta), \pi(\eta)) = (P(\eta_1), \pi(\eta_2))$, i.e.

$$D^\lambda_{A_0\#_\rho(a+\delta a)}(\beta s + P(\eta_1)) + \sigma(\pi(\eta)) = 0 = D^\lambda_{A_0\#_\rho a}(\beta s) + \eta + I(\delta a) * (\beta s + P(\eta_1)),$$

which is the same as solving the system of two equations:

$$pr_1(D^\lambda_{A_0\#_\rho a}(\beta s)) + \eta_1 + pr_1\{I(\delta a) * (\beta s + P(\eta_1))\} = 0 \qquad (1)$$

$$pr_2(D^{\lambda}_{A_0\#_\rho a}(\beta s)) + \eta_2 + pr_2\{I(\delta a) * (\beta s + P(\eta_1))\} = 0. \qquad (2)$$

The first equation can be rewritten as

$$(1 + A)(\eta_1) = -pr_1\{D^{\lambda}_{A_0\#_\rho a}(\beta s)\} - pr_1\{I(\delta a) * (\beta s)\}$$

with the norm of $A(\eta_1) = pr_1(I(\delta a) * P(\eta_1))$ estimated, on $X\#S^4$, by

$$\|A(\eta_1)\|_{L^{8/3}} \leq 2\|I(\delta a)\|_{L^4}\|P(\eta_1)\|_{L^8} \leq 2 \cdot \epsilon \cdot C \cdot \|\eta_1\|_{L^{8/3}}.$$

Here, $\|P\eta\|_{L^8} \leq C \cdot \|\eta\|_{L^{8/3}}$ for some constant C, given by the Sobolev Embedding theorem. By choosing $\epsilon \leq \frac{1}{4C}$ we have $\|A(\eta_1)\|_{L^{8/3}} \leq \|\eta_1\|_{L^{8/3}}/2$. Therefore the operator $(1 + A)$ is invertible and the norm of the inverse is at most 2. This gives the existence and uniqueness of the solution

$$\eta_1 = (1 + A)^{-1}\{-pr_1(D^{\lambda}_{A_0\#_\rho a}(\beta s)) - pr_1(I(\delta a) * (\beta s))\}$$

with an estimate

$$\begin{aligned}
\|\eta_1\|_{L^{8/3}(X\#S^4)} &= \|(1 + A)^{-1}\{-pr_1(D^{\lambda}_{A_0\#_\rho a}(\beta s)) - pr_1(I(\delta a) * (\beta s))\}\|_{L^{8/3}(X\#S^4)} \\
&\leq 2\{\|pr_1(D^{\lambda}_{A_0\#_\rho a}(\beta s))\|_{L^{8/3}(X\#S^4)} + \|pr_1(I(\delta a) * (\beta s))\|_{L^{8/3}(X\#S^4)}\} \\
&\leq 4\{\|(D^{\lambda}_{A_0\#_\rho a}(\beta s))\|_{L^{8/3}(X\#S^4)} + \|(I(\delta a) * (\beta s))\|_{L^{8/3}(X\#S^4)}\} \\
&\leq 4\{\epsilon'(t) + \|\beta I(\delta a)\|_{L^4(X\#S^4)}\|s\|_{L^8(S^4)})\} \\
&\leq 4\{\epsilon'(t) + \epsilon(const_4)\}.
\end{aligned}$$

(The $\|s\|_{L^8(S^4)} \leq (const_4)\|s\|_{L^{8/3}(S^4)} = const_4$ estimate is obtained as follows: The Weitzenböck formula gives

$$(D_a)^*(D_a s) = \nabla^*_a \nabla_a s - F^+_a \cdot s + S \cdot s.$$

Although not equal to zero as $a \in \tilde{\tau}$ are not necessarily ASD, F^+_a considered as an endomorphism on positive vector spinors is bounded from above as $\tilde{\tau}$ is a compact family on S^4. Let σ be the supremum of $-F^+_a + S$, since $s \in ker D_a$,

$$\|\nabla^*_a \nabla_a s\|_{L^{8/3}(S^4)} \leq (\sigma)\|s\|_{L^{8/3}(S^4)}.$$

Since $\nabla_a^*\nabla_a$ is an elliptic operator there is *a priori* inequality

$$\|s\|_{L_2^{8/3}(S^4)} \leq (const_1)\|\nabla_a^*\nabla_a s\|_{L^{8/3}(S^4)} + (const_2)\|s\|_{L^{8/3}(S^4)}.$$

Applying Sobolev embedding theorem one gets

$$\|s\|_{L^8(S^4)} \leq (const_3)\|s\|_{L_2^{8/3}(S^4)} \leq (const_4)\|s\|_{L^{8/3}(S^4)} = (const_4).)$$

For each $(a+\delta a) \in U$, η_2 is given by equation (2) to be

$$\eta_2 = pr_2\{-D_{A_0\#_\rho a}(\beta s) - I(\delta a) * (\beta s + P(\eta_1))\}.$$

Therefore, $\pi(\eta) = \pi(\eta_2) = \pi\{-D_{A_0\#_\rho a}(\beta s) - I(\delta a) * (\beta s + P(\eta_1))\}$. Consider the linear map

$$J : U \longrightarrow coker D_a \quad , \quad \delta a \longmapsto \pi\{-I(\delta a) * (\beta s + P(\eta_1))\}$$

with $J(a) = 0$. There is estimate

$$\begin{aligned}
&\|J(\delta a)\|_{L^{8/3}(X\#S^4)} = \| - \pi\{I(\delta a) * (\beta s + P(\eta_1))\}\|_{L^{8/3}(X\#S^4)} \\
&\geq \|\pi(I(\delta a) * (\beta s))\|_{L^{8/3}(X\#S^4)} - \|I(\delta a)\|_{L^4(X\#S^4)}\|P(\eta_1)\|_{L^8(X\#S^4)} \\
&\geq \|\pi(I(\delta a) * (\beta s))\|_{L^{8/3}(X\#S^4)} - \|I(\delta a)\|_{L^4(X\#S^4)} \cdot C \cdot \|\eta_1\|_{L^{8/3}(X\#S^4)} \\
&\geq \|\pi_{S^4}((\delta a) * s)\|_{L^{8/3}(S^4)} - \|\delta a\|_{L^4(S^4)} \cdot C \cdot 4\{\epsilon'(t) + (const_4)\epsilon\} \\
&= \|j(\delta a)(s)\|_{L^{8/3}(S^4)} - \|\delta a\|_{L^4(S^4)} \cdot 4C \cdot \{\epsilon'(t) + (const_4)\epsilon\} \\
&\geq \{(const_5) - 4C \cdot (\epsilon'(t) + (const_4)\epsilon)\}\|\delta a\|_{L^4(S^4)},
\end{aligned}$$

as the injectivity of j implies

$$\|j(\delta a)(s)\|_{L^{8/3}(S^4)} \geq (const_5)\|\delta a\|_{L^4(S^4)}$$

for some positive constant $(const_5)$. Hence, when $\epsilon'(t)$ and ϵ are sufficiently small, J maps U bijectively onto an open ball $B \subset coker D_a$ with center zero . By choosing t large enough so that $\pi\{-D_{A_0\#_\rho a}(\beta s)\} \in B$, there is a unique $(a + \delta a_0) \in U$ such that the (unique) solution η to the equation $\Phi(a + \delta a_0, P(\eta), \pi(\eta)) = 0$ satisfied

$$\pi(\eta) = \pi\{-D_{A_0\#_\rho a}(\beta s) - I(\delta a) * (\beta s + P(\eta_1))\} = 0.$$

That is $D^{\lambda}_{A_0\#_\rho(a+\delta a_0)}(\beta s + P(\eta)) = 0$ and so $A_0\#_\rho(a+\delta a_0) \in \mathcal{T}'_{x,A_0,\zeta}$. This proves the first half of claim (4.3.1).

Conversely, suppose $[A] = [A_0\#a] \in \mathcal{T}'_{x,A_0,\zeta}$, where $a \in \tilde{\tau}$. Let s be a positive harmonic spinor on the connected sum with $D^{\lambda}_A(s) = 0$ and $||s||_{L^{8/3}(X\#S^4)} = 1$. Let β be a cut-off function as before. Then

$$\begin{aligned}
&||D_a(\beta s)||_{L^{8/3}(S^4)} \\
&\le ||D^{\lambda}_A(s)||_{L^{8/3}(X\#S^4)} + ||d\beta \cdot s||_{L^{8/3}(X\#S^4)} + ||(A-a)_{|supp\beta}||_{L^4(X\#S^4)}||s||_{L^8(X\#S^4)} \\
&\le ||d\beta||_{L^4(X\#S^4)}||s||_{L^8(X\#S^4)} + ||(\xi a - a)_{|supp\beta}||_{L^4(S^4)}||s||_{L^8(X\#S^4)} \\
&\le \{\epsilon(t) + const(a)\cdot e^{-t}\}||s||_{L^8(X\#S^4)} \\
&\le \{\epsilon(t) + const(a)\cdot e^{-t}\}const'\cdot ||s||_{L^{8/3}(X\#S^4)} = \epsilon_1(t)
\end{aligned}$$

where $\epsilon_1(t) \to 0$ as $t \to \infty$. (Again ξ is the cut-off function which damps the connections over S^4 to connections that are flat over the annulus in the connected sum construction, and similar to the S^4 case before, using the Weitzenböck formula and the Sobolev embedding theorem there is estimate $||s||_{L^8(X\#S^4)} \le (const')||s||_{L^{8/3}(X\#S^4)}$. In this case the Weitzenböck formula also involves a term F_∇ which is the curvature on the associated line bundle $\lambda = \bigwedge^2 W^+$.)

Similarly, consider cut-off function $(1-\beta)$ supported on the X side, one has

$$||D^{\lambda}_{A_0}(1-\beta)s||_{L^{8/3}(X)} \le \epsilon_2(t) \to 0 \quad as \quad t \to \infty.$$

However, $indD^{\lambda}_{A_0} = 0$ and so for generic metric g_t and connection on λ, $\{[A_0] \in \mathcal{M}_0|\ dim_{\mathbb{C}}kerD^{\lambda}_{A_0} \ge 1\}$ is empty (of virtual dimension -2). So, $(1-\beta)s$ can be made arbitrarily close to zero, which means the norm of βs is arbitrarily close to 1, when the scale of the gluing map $\zeta \longrightarrow 0$. Therefore, there is ζ_0 sufficiently small (with corresponding t_0 sufficiently large) such that for all $\zeta < \zeta_0$ (i.e. $t > t_0$), any $A = A_0\#a \in \mathcal{T}'_{x,A_0,\zeta}$ is given by the connected sum of A_0 with $a \in \tilde{\tau}$ which is close to an element $a' \in \tilde{\tau}'$, with $||a-a'||_{L^4(S^4)} < \epsilon$ the small number as before. Therefore, for ζ sufficiently small, $A = A_0\#a \in \mathcal{T}'_{x,A_0,\zeta}$ is the unique deformation of $A_0\#a'$ as in the first part of the proof. Hence, claim (4.3.1) that the linking invariant is a universal constant ($= \sharp\tilde{\tau}'$) follows.

Chapter 5

Calculation for a $K3$ Surface

The $spin^c$-polynomial for a $K3$ surface, when the moduli space of (ordinary) instantons is of dimension 8 and $\mathrm{ind} D_A^\lambda = -1$, is to be computed. Then the cohomology class obtained from the generalised Porteous formula is evaluated using the formula of Donaldson invariants. By comparing the two results, the linking invariant $\sharp\mathcal{T}'$ in section 4 is obtained, which is $\frac{1}{12}$.

5.1 Preliminaries

Let X be a $K3$ surface with a Kähler metric ω. Consider the (8-dimensional) moduli space of ω-ASD connections on a $SO(3)$-bundle $F \to X$ with

$$p_1(F) = -10 \quad \& \quad w_2(F)^2 \equiv 2 \; mod 4,$$

which corresponds ([DK, Proposition (6.1.13)]) to the moduli space of ω-stable $U(2)$-bundles $E \to X$ with $c_1(E) = \Lambda$ an integral lift of w_2 and $c_2(E) = \frac{10+\Lambda^2}{4}$. Hence, the $spin^c$-polynomial is well-defined since the trivial connection can be avoided under weak limits (by the Chern-Weil formula).

The $K3$ surface to be considered is the (non-algebraic) Kummer surface $X = Km(T)$, where T is a generic complex 2-torus with $Pic(T) = Pic^0(T)$, the dual torus corresponding to (flat) holomorphic line bundles with zero Chern class. Hence, $Pic(X)$ is 'generated'[1], over $\frac{1}{2}\mathbb{Z}$, by the sixteen line bundles $\{\Theta_X(\Lambda_i),\ i = 1, \ldots, 16\}$ corresponding to the sixteen effective divisors $\{\Lambda_1, \ldots, \Lambda_{16}\}$ which are (-2)-curves obtained from blowing up the points of order 2 in T, with $\Lambda_i \cdot \Lambda_j = -2\delta_{ij}$. Therefore, effective divisors are of the form $D = \sum_{i=1}^{16} d_i \Lambda_i$ with integers $d_i \geq 0\ \forall i = 1, 2, \ldots, 16$.

To compute the $spin^c$-invariant, line bundles L with $\chi(E \otimes L) = -1$ are needed to be considered. For a $K3$ surface, by the Riemann-Roch formula, this means

$$4\Lambda \cdot L + 4L^2 + \Lambda^2 + 10 = 0.$$

In the following, the case $c_1(E)^2 = \Lambda^2 = -2$ will be considered, that is

$$c_2(E) = 2 \quad , \quad \Lambda \cdot L + L^2 + 2 = 0.$$

L will be chosen such that

$$L^2 = -2 \quad and \quad \Lambda \cdot L = 0.$$

Then,

$$\lambda = 2L \quad with \quad (\Lambda + \lambda)^2 = (-2) + (-8) = -10.$$

5.2 The Moduli Space of 1-Instantons

Let ω be a Kähler metric on X (then $deg\Lambda_i > 0,\ \forall i = 1, \cdots, 16$ since Λ_i are effective divisors) such that $c_1(E) = c_1(\Theta_X(\Lambda_1))$ is the one with the smallest ω-degree among the sixteen (-2)-curves with

[1]As in [BPV, (VIII.5)], suppose Λ_t are labelled by $t \in T[2]$, the points of order 2 in T equipped with its natural structure of a 4-dimensional affine space over $\mathbb{F}_2 = \{0, 1\}$, then $Pic(X) = \{\sum a_t \Lambda_t\}$ where $a_t \in \frac{1}{2}\mathbb{Z}$ and the map $t \longmapsto 2a_t\ mod 2$ is an affine-linear function from $T[2]$ to $\mathbb{F}_2$.

Condition 5.2.1

$$0 < deg\Lambda_1 \leq deg\Lambda_2 \leq \cdots \leq deg\Lambda_{16}$$

and

$$deg\Lambda_{16} < \frac{3}{2} deg\Lambda_1.$$

(Such ω can be obtained as follows: For each order 2 element $t \in T$, the map $x \longmapsto x + t$ on T gives a well-defined map from $T/\{\pm 1\}$ to itself, which lifts to a map A_t from $X = \widetilde{T/\{\pm 1\}}$ to itself, mapping (-2)-curves to another ones. Pick any Kähler metric on X, averaging by $\{A_t\}_{t=1}^{16}$ one can obtain a Kähler metric on X such that

$$deg\Lambda_i = deg\Lambda_j \quad \forall i, j = 1, 2, \ldots, 16.$$

Then, since $\{[\omega] |\, \omega \text{ is Kähler}\} \subseteq H^{1,1}(X)$ is open, one can arrange a (generic) Kähler metric such that the degree of the 16 line bundles differ by an arbitrarily small amount and hence condition (5.2.1).)

Recall that the 1-instantons on the $K3$ surface correspond to ω-stable $U(2)$-bundles E, with $c_1(E) = \Lambda_1$ and $c_2(E) = 2$, such that

$$H^0(E \otimes L) \neq 0 \quad or \quad H^2(E \otimes L) = H^0(E^* \otimes L^*) \neq 0,$$

where L is a line bundle with $\chi(E \otimes L) = -1$, for example those L satisfy $L^2 = -2$ and $L \cdot \Lambda_1 = 0$. Possible $L \in Pic(X)$ are

$$L = \Theta(\pm\Lambda_i) \quad where \quad i = 2, 3, \ldots, 16.$$

For the case $L = \Theta(\Lambda_i)$, where $i = 2, 3, \ldots, 16$, we will see that $E \otimes L$ always has a section and so there is no cut down of dimension of the moduli space of instantons. Since E is stable with $degE > 0$ and $\chi(E) = 1$, Riemann-Roch formula gives $H^0(X, E) \neq 0$. Let $0 \neq s \in H^0(X, E)$. Suppose s vanishes on an effective divisor D

on X, then D is linearly equivalent to $d_1\Lambda_1 + \cdots + d_{16}\Lambda_{16}$ for integers $d_i \geq 0$ (but not all of them zero). Hence, $E \otimes [-D]$ has a non-trivial section, which contradicts stability as

$$degD = \sum_{i=1}^{16} d_i \cdot deg\Lambda_i > \frac{1}{2}deg\Lambda_1 = \frac{1}{2}degE.$$

Therefore, s vanishes on discrete points and Serre construction gives

Lemma 5.2.2 *Any stable $U(2)$ bundles E with $c_1(E) = \Lambda_1$ and $c_2(E) = 2$ is expressable as the following extension:*

$$0 \longrightarrow \Theta \stackrel{s}{\longrightarrow} E \longrightarrow \Theta(\Lambda_1) \otimes \mathcal{J}_Z \longrightarrow 0,$$

where $\mathcal{J}_Z$ is an ideal sheaf associated to a zero-dimensional subscheme of length two.

Tensoring $L = \Theta(\Lambda_i)$ with the above short exact sequence gets

$$0 \longrightarrow \Theta(\Lambda_i) \longrightarrow E \otimes L \longrightarrow \mathcal{J}_Z(\Lambda_1 + \Lambda_i) \longrightarrow 0,$$

Considering long exact sequence of cohomology, since $H^0(\Theta(\Lambda_i)) \neq 0$, one always have $H^0(E \otimes L) \neq 0$ for all w-stable bundles E.

For $L = \Theta(-\Lambda_2)$, $H^0(E \otimes L) = 0$ by stability of E and the choice of $c_1(E) = \Lambda_1$. (The case $i = 2$ is considered here only for simplicity, similar arguments holds for $L = \Theta(-\Lambda_i)$, $\forall i = 2, 3, \ldots, 16$). Recall that we consider Kähler metrics satisfying condition (5.2.1).

Suppose $s \in H^0(E^* \otimes L^*) \neq 0$ vanishes on an effective divisor C. So, $\exists \tilde{s} \in H^0(E^* \otimes L^* \otimes [-C])$ non-trivial section vanishes on discrete points, which induces bundle homomorphism $L \otimes [C] \longrightarrow E^*$. Hence,

$$c_2(E^* \otimes L^* \otimes [-C]) = (\Lambda_1 \cdot C) - 2\Lambda_2 \cdot C + C^2 \geq 0,$$

effectiveness limited C to be Λ_2 or $\Lambda_2 + \Lambda_i$ where $i = 2, 3, \ldots, 16$. But by the stability of E,

$$-deg\Lambda_2 + degC < \frac{1}{2}(-deg\Lambda_1)$$

both cases are impossible. Hence, s vanishes on discrete points. Since

$$c_2(E^* \otimes \Theta(\Lambda_2)) = c_2(E^*) + c_1(E^*) \cdot c_1(L^*) + c_1(L^*)^2 = c_2(E) + (-2) = 0,$$

$$c_1(E^* \otimes \Theta(\Lambda_2)) = 2\Lambda_2 - \Lambda_1,$$

we have E^* is expressable as an extension

$$0 \longrightarrow \Theta \xrightarrow{\bar{s}} E^* \otimes \Theta(\Lambda_2) \longrightarrow \Theta(2\Lambda_2 - \Lambda_1) \longrightarrow 0.$$

Conversely,

Proposition 5.2.3 *Every bundles E coming from* non-trivial *extensions above are stable.*

Proof: Denote the extension

$$0 \longrightarrow \Theta \xrightarrow{\bar{s}} E^* \otimes \Theta(\Lambda_2) = F \longrightarrow \Theta(2\Lambda_2 - \Lambda_1) \longrightarrow 0$$

by (*). Suppose $\Theta(D) \longrightarrow F$ is a potential destabilising map, that is

$$degD \geq \frac{1}{2}degF = deg\Lambda_2 - \frac{1}{2}deg\Lambda_1 > 0.$$

Tensoring $\Theta(-D)$ to (*), we get long exact sequence

$$0 \longrightarrow H^0(\Theta(-D)) \longrightarrow H^0(F \otimes \Theta(-D)) \longrightarrow H^0(\Theta(2\Lambda_2 - \Lambda_1 - D)) \longrightarrow \cdots.$$

$H^0(F \otimes \Theta(-D)) \neq 0$ implies either $H^0(\Theta(-D)) \neq 0$ (i.e. deg$D < 0$ and so not destabilising) or $H^0(\Theta(2\Lambda_2 - \Lambda_1 - D)) \neq 0$.

Suppose $H^0(\Theta(2\Lambda_2 - \Lambda_1 - D)) \neq 0$, i.e. $2\Lambda_2 - \Lambda_1 - D$ is effective. By the destabilising condition above and the choice of Λ_1 and Λ_2 in (5.2.1), only possible D are of the form $2\Lambda_2 - n\Lambda_1$, where $n \geq 1$.

However, the non-triviality of the extension (*) eliminates the case $n = 1$ (i.e. $D = 2\Lambda_2 - \Lambda_1$). For $n > 1$, the de-stabilising condition $degD \geq deg\Lambda_2 - \frac{1}{2}deg\Lambda_1 > 0$ gives $deg\Lambda_2 \geq \frac{3}{2}deg\Lambda_1$ which is impossible by our choice of Kähler metric satisfying condition (5.2.1).

That is,

Proposition 5.2.4 *The compact 4-dimensional moduli space $\mathcal{M}'_8$ of 1-instantons for $L = \Theta(-\Lambda_2)$ (i.e. the spin^c structure $\lambda = \Theta(-2\Lambda_2)$), with respect to a Kähler metric as in (5.2.1), is isomorphic to*

$$\mathcal{M}'_8 \cong \frac{H^1(\Theta(\Lambda_1 - 2\Lambda_2)) \setminus \{0\}}{\mathbb{C}^*} \cong \frac{\mathbb{C}^3 \setminus \{0\}}{\mathbb{C}^*} \cong \mathbb{P}^2$$

by the Riemann-Roch formula.

5.3 The $Spin^c$-polynomial $\gamma'_4(\sigma^2)$

To compute the *spin*^{c}-polynomial $\gamma'_4(\sigma^2)$, we need to describe $\mu(\sigma)$ in the compact 4-dimensional space $\mathcal{M}'_8$, that is

$$-\frac{1}{4}p_1(\mathbb{F})/[\sigma],$$

where $\mathbb{F}$ a universal bundle over $\mathcal{M}'_8 \times X \cong \mathbb{P}^2 \times X$ is given by the extension

$$0 \longrightarrow \pi_X^*\Theta \otimes \pi_{\mathbb{P}^2}^*(\Theta_{\mathbb{P}^2}(1)) \longrightarrow \mathbb{F} \longrightarrow \pi_X^*(\Theta(2\Lambda_2 - \Lambda_1)) \longrightarrow 0.$$

(Here $\pi_X : \mathbb{P}^2 \times X \longrightarrow X$ and $\pi_{\mathbb{P}^2} : \mathbb{P}^2 \times X \longrightarrow \mathbb{P}^2$ are the obvious projections.)

Hence, ignoring the π^*'s and write h as the hyperplane class of $\mathbb{P}^2$, we have

$$c_1(\mathbb{F}) = 2\Lambda_2 - \Lambda_1 + h,$$

$$c_2(\mathbb{F}) = (2\Lambda_2 - \Lambda_1)h = 2\Lambda_2 \otimes h - \Lambda_1 \otimes h$$

and

$$\begin{aligned} -\frac{1}{4}p_1(\mathbb{F}) &= \tfrac{-1}{4}\{c_1(\mathbb{F})^2 - 4c_2(\mathbb{F})\} \\ &= \tfrac{-1}{4}\{(-10) + (2\Lambda_1 \otimes h - 4\Lambda_2 \otimes h) + (h^2)\} \\ &\in H^4(X) \oplus (H^2(X) \otimes H^2(\mathbb{P}^2)) \oplus H^4(\mathbb{P}^2). \end{aligned}$$

By the definition of slant product [DK, section(5.1.2)],

$$\mu(\sigma) = -\frac{1}{4}p_1(\mathbb{F})/[\sigma] = \frac{1}{2}Q((2\Lambda_2 - \Lambda_1), \sigma) \cdot h$$

in $\mathcal{M}'_8 \cong \mathbb{P}^2$.

In general, an orientation of moduli space of instantons on a $SO(3)$ bundle is given, in addition to a homology orientation Ω, by an integral lift of w_2. However, for a $K3$ surface, which is a spin manifold, the orientation is independent of the lifting [DK, p.283]. Suppose Ω is chosen such that the 0-dimensional moduli space gives invariant $\gamma_0 = 1$, i.e. same as the one induced from the natural complex orientation of the moduli space $\mathcal{M}_S$ of stable bundles. As in [PT, section I.5], the orientation of $\mathcal{M}_{0,1}$ (notations as in (2.1)) differs by $(-1)^{1-\chi(E\otimes L)}$ from the natural orientation of $\mathcal{M}_S$. In our case, $1 - \chi(E \otimes L) = 1 - (-1) = 2$, i.e. the above description of $\mathcal{M}'_8$ as $\mathcal{M}_{0,1}$ has the same orientation as $\mathcal{M}_8$. So,

Theorem 5.3.1 *The spin^c-polynomial*

$$\gamma'_4(\sigma^2) = \frac{1}{4}Q((2\Lambda_2 - \Lambda_1), \sigma)^2 = \frac{1}{4}Q(\alpha, \sigma)^2,$$

where $\alpha = \Lambda_1 - 2\Lambda_2 = c_1(E) + c_1(\lambda)$.

5.4 Atiyah-Singer Index Theorem for Family of Operators

As in section (4.1), the jumping locus

$$\{[A] \in \mathcal{B}_X^* \mid dim_{\mathbb{C}} ker D_A^\lambda \geq 1\} = \{[A] \in \mathcal{B}_X^* \mid dim_{\mathbb{C}} ker(D_A^\lambda)^* \geq 1 - (-1) = 2\}$$

defines a cohomology class in $H^4(\mathcal{B}_X^*, \mathbb{Q})$ which is given by the generalised Porteous formula as $c_2(ind((D^\lambda)^*, \underline{\mathbb{E}}))$, the second Chern class of the index bundle. (Strictly speaking we should formulate in terms of compact families of connections but, as usual, to save notation we will ignore this point and compute this class formally). Here $\underline{\mathbb{E}}$ is a $U(2)$-lift of the universal $SO(3)$ bundle[2]

$$\mathbb{E} \longrightarrow \mathcal{B}_X^* \times X,$$

which defines the map $\mu : H_i(X) \longrightarrow H^{4-i}(\mathcal{B}^*)$ by

$$\mu(\beta) = -\frac{1}{4} p_1(\mathbb{E})/[\beta].$$

So $p = -\frac{1}{4}p_1(\mathbb{E}) = c_2(\underline{\mathbb{E}}) - \frac{1}{4}c_1(\underline{\mathbb{E}})^2$ and $e = c_1(\underline{\mathbb{E}})$ an integral lift of $w_2(\mathbb{E})$. By the definition of the μ map as slant product by p, we have the following:

Kunneth Decomposition 5.4.1

$$p = \nu + \Sigma + \xi \in \quad H^4(\mathcal{B}^*) \quad \oplus \quad (H^2(\mathcal{B}^*) \otimes H^2(X)) \quad \oplus \quad H^4(X)$$

$$and \qquad e = z + c_1(E) \in \quad H^2(\mathcal{B}^*) \oplus H^2(X),$$

where

$\nu = \mu(\text{point})$,

[2]Similar to [DK, (8.3.15)] for the $SU(2)$ case when the second Chern class is odd, tensoring with the determinant of the index bundle, there is always such a lift if $indD_A^\lambda = -1$.

$\Sigma = \sum_{i=1}^{b_2} \mu(\alpha_i)\alpha_i^*$ with $\{\alpha_i \mid i = 1, \ldots, b_2\}$ a basis for $H_2(X, \mathbb{Q})$ and

$\xi/[X] = c_2(E) - \frac{1}{4}c_1(E)^2 = -\frac{1}{4}p_1(F)$.

To obtain the Chern classes of the index bundle $ind((D^\lambda)^*, \underline{\mathbb{E}})$, the Chern character is first computed using the Atiyah-Singer index theorem for a family of ($spin^c$) Dirac operators:

$$ch(ind((D_A^\lambda)^*, \underline{\mathbb{E}})) = -ch(ind((D_A^\lambda), \underline{\mathbb{E}})) = -\{ch(\underline{\mathbb{E}}) \cdot \hat{A}(X) \cdot exp(\frac{1}{2}c_1(\lambda))\}/[X].$$

where

1) $\hat{A}(X) = 1 - \frac{1}{24}p_1(X)$ for a four-manifold which means that

$$\begin{aligned} \hat{A}(X) \cdot exp(\frac{1}{2}c_1(\lambda)) &= (1 - \tfrac{1}{24}p_1(X)) \cdot (1 + \tfrac{1}{2}c_1(\lambda) + \tfrac{1}{8}c_1(\lambda)^2 + \cdots) \\ &= 1 + \tfrac{1}{2}c_1(\lambda) + (-\tfrac{1}{24}p_1(X) + \tfrac{1}{8}c_1(\lambda)^2) + \cdots . \end{aligned}$$

2) The Chern character of the $U(2)$ bundle $\underline{\mathbb{E}}$ is

$$\begin{aligned} ch(\underline{\mathbb{E}}) &= 2 + c_1 + \tfrac{1}{2}(c_1^2 - 2c_2) + \tfrac{1}{6}(c_1^3 - 3c_1c_2) \\ &\quad + \tfrac{1}{24}(c_1^4 - 4c_2c_1^2 + 2c_2^2) + \cdots . \end{aligned}$$

In terms of $p = c_2(\underline{\mathbb{E}}) - \frac{1}{4}c_1(\underline{\mathbb{E}})^2$ and $e = c_1(\underline{\mathbb{E}})$, we have

Formula 5.4.2

$$\begin{aligned} ch(\underline{\mathbb{E}}) &= 2 + e + (\tfrac{e^2}{4} - p) + (\tfrac{e^3}{24} - \tfrac{pe}{2}) \\ &\quad + (\tfrac{1}{12}p^2 - \tfrac{1}{8}pe^2 + \tfrac{1}{(24)(8)}e^4) + (\tfrac{1}{24}p^2e - \tfrac{1}{48}pe^3 + \tfrac{1}{(16)(120)}e^5) \\ &\quad + (\tfrac{-1}{360}p^3 + \tfrac{1}{96}p^2e^2 - \tfrac{1}{(48)(8)}pe^4 + \tfrac{1}{(32)(720)}e^6) + \cdots \\ &= ch^0 + ch^1 + ch^2 + ch^3 + ch^4 + ch^5 + ch^6 + \cdots , \end{aligned}$$

where $ch^j \in H^{2j}(\mathcal{B}_X^*, \mathbb{Q})$ *is the* $2j$*-dimensional part of* $ch(\underline{\mathbb{E}})$.

Notations: Let

$m_0 = 1 \ \in H^0(X)$,

$m_1 = \frac{1}{2}c_1(\lambda) \ \in H^2(X)$,

$m_2 = (-\frac{1}{24}p_1(X) + \frac{c_1(\lambda)^2}{8}) \ \in H^4(X)$ with $m_2/[X] = indD^\lambda$, and

$ch^j_k \in H^{2j-2k}(\mathcal{B}^*) \otimes H^{2k}(X) \subseteq H^{2j}(\mathcal{B}^* \times X)$ be the Kunneth factor

of ch^j of dimension $(2j-2k)$ and $(2k)$ in $\mathcal{B}^*_X$ and X respectively.

Then the Atiyah-Singer index theorem for family gives

Formula 5.4.3

$$ch^j(ind((D^\lambda)^*, \underline{\mathbb{E}})) = -(ch^j_0 \cdot m_2 + ch^{j+1}_1 \cdot m_1 + ch^{j+2}_2 \cdot m_0)/[X].$$

where $ch^j(ind((D^\lambda)^*, \underline{\mathbb{E}}))$ is the $2j$-dimensional part of the Chern character of the index bundle $ind((D^\lambda)^*, \underline{\mathbb{E}})$.[3]

To get the formulae for ch^j_k, we simply substitute the Kunneth decomposition (5.4.1) into the formula (5.4.2) for the ch^i's and grouping terms with the correct dimension, the results are:

1) $ch^1 = e = z \ + \ \Lambda \ = \ ch^1_0 \ + \ ch^1_1$

2) $ch^2 = \frac{e^2}{4} - p$
$= (\frac{z^2}{4} - \nu) \ + \ (\frac{z\Lambda}{2} - \Sigma) \ + \ (\frac{\Lambda^2}{4} - \xi)$
$= ch^2_0 \ + \ ch^2_1 \ + \ ch^2_2$

[3]Since $ch^i(ind((D^\lambda)^*, \underline{\mathbb{E}}))$ for $i = 1, 2$ is needed in order to obtain the second Chern class of the index bundle, ch^j_k for $j = 1, \cdots, 4$ and $k = 0, 1, 2$. have to be computed.

$$3)\ ch^3 = \frac{e^3}{24} - \frac{pe}{2}$$
$$= (\frac{z^3}{24} - \frac{\nu z}{2}) + (\frac{z^2\Lambda}{8} - \frac{z\Sigma}{2} - \frac{\nu\Lambda}{2})$$
$$+ (\frac{z\Lambda^2}{8} - \frac{z\xi}{2} - \frac{\Sigma\Lambda}{2}) + (\frac{\Lambda^3}{24} - \frac{\xi\Lambda}{2})$$
$$= ch_0^3 + ch_1^3 + ch_2^3 + ch_3^3$$

$$4)\ ch^4 = \frac{1}{12}p^2 - \frac{1}{8}pe^2 + \frac{1}{(24)(8)}e^4$$
$$= (\frac{1}{12}\nu^2 - \frac{1}{8}\nu z^2 + \frac{1}{(24)(8)}z^4) + (\frac{1}{6}\nu\Sigma - \frac{1}{4}\nu z\Lambda - \frac{1}{8}z^2\Sigma + \frac{1}{(24)(2)}z^3\Lambda)$$
$$+ (\frac{1}{12}\Sigma^2 + \frac{1}{6}\nu\xi - \frac{1}{8}\nu\Lambda^2 - \frac{1}{4}\Sigma z\Lambda - \frac{1}{8}z^2\xi + \frac{1}{(4)(8)}z^2\Lambda^2)$$
$$+ (\frac{1}{6}\Sigma\xi - \frac{1}{8}\Sigma\Lambda^2 - \frac{1}{4}z\Lambda\xi + \frac{1}{(24)(2)}z\Lambda^3)$$
$$+ (\frac{1}{12}\xi^2 - \frac{1}{8}\xi\Lambda^2 + \frac{1}{(24)(8)}\Lambda^4)$$
$$= ch_0^4 + ch_1^4 + ch_2^4 + ch_3^4 + ch_4^4$$

For our case, $\lambda = \Theta(-2\Lambda_2)$, $\Lambda = \Theta(\Lambda_1)$ and $\xi = c_2(E) - \frac{1}{4}c_1(E)^2$. Hence,

1) $c_1(\lambda) \cdot c_1(\lambda) = -8$, $\Lambda \cdot C_1(\lambda) = 0$ and $\Lambda^2/[X] = \Lambda_1 \cdot \Lambda_1 = -2$.

2) $\xi/[X] = \int_X \xi = 2 - \frac{1}{4}(-2) = \frac{5}{2}$.

3) $m_2/[X] = \int_X(-\frac{1}{24}p_1(X) + \frac{c_1(\lambda)^2}{8}) = -\frac{1}{8}sgn(X) + \frac{-8}{8} = \frac{16}{8} - \frac{8}{8} = 1 = indD^\lambda$.

4) We will adopt the notation:

$I = (\Sigma \cdot \Sigma)/[X]$, a 4-dimensional class defined by

the intersection form.

$$\mu = \Sigma \cdot (\Lambda + c_1(\lambda))/[X]$$
$$= \mu([\Lambda_1 - 2\Lambda_2]).$$

(Note that the dependence of the $spin^c$-polynomials on changing the $spin^c$ structure is given by

$$\gamma'_{c_1(\lambda)+2\delta,\Lambda,c_2(E)} = \gamma'_{c_1(\lambda),\Lambda+2\delta,c_2(E)+\Lambda\cdot\delta+\delta^2}$$

where $\Lambda = c_1(E)$. Therefore, the classes

$$\alpha = \Lambda + c_1(\lambda) \qquad and \qquad \xi = -\frac{1}{4}p_1(F) = c_2(E) - \frac{1}{4}c_1(E)^2$$

are invariant parameters which determined the $spin^c$-polynomial. We remark that the Porteous classes obtained below depends on these two classes only.)

Formula (5.4.3) then gives

1) $ch^1(ind) = ch^1(ind((D^\lambda)^*, \underline{E})) = -(ch_0^1 \cdot m_2 + ch_1^2 \cdot m_1 + ch_2^3 \cdot m_0)/[X]$
$= -\{z + (\frac{z\Lambda}{2} - \Sigma) \cdot \frac{1}{2}c_1(\lambda) \ + \ (\frac{z\Lambda^2}{8} - \frac{z\xi}{2} - \frac{\Sigma\Lambda}{2})\}/[X]$
$= \frac{1}{2}\mu \ + \ \frac{1}{2}z.$

2) $ch^2(ind) = ch^2(ind((D^\lambda)^*, \underline{E})) = -(ch_0^2 \cdot m_2 + ch_1^3 \cdot m_1 + ch_2^4 \cdot m_0)/[X]$
$= -\{(\frac{z^2}{4} - \nu) + (\frac{z^2\Lambda}{8} - \frac{z\Sigma}{2} - \frac{\nu\Lambda}{2}) \cdot \frac{1}{2}c_1(\lambda)$
$+ (\frac{1}{12}\Sigma^2 + \frac{1}{6}\nu\xi - \frac{1}{8}\nu\Lambda^2 - \frac{1}{4}\Sigma z\Lambda - \frac{1}{8}z^2\xi + \frac{1}{(4)(8)}z^2\Lambda^2)\}/[X]$
$= -\frac{1}{12}I \ + \ \frac{1}{3}\nu \ + \ \frac{1}{4}\mu z \ + \ \frac{1}{8}z^2.$

Hence, we have[4]

Proposition 5.4.4 *The cohomology class P_{-1}^1 in $H^4(\mathcal{B}_X^*, \mathbb{Q})$ represented by the jumping locus $\{[A] \in \mathcal{B}^* \mid dim_{\mathbb{C}}KerD_A^\lambda \geq 1\}$, as given formally by the generalised Porteous formula, is*

$$c_2(ind((D^\lambda)^*, \underline{E})) = \tfrac{1}{2}(ch^1(ind))^2 - ch^2(ind) = -\tfrac{1}{3}\nu + \tfrac{1}{12}I + \tfrac{1}{8}\mu^2.$$

[4]The expression does not involve z, and is independent of the lifting from $\mathbb{E}$ to $\underline{E}$. So, in computing $ch^i(ind(D^\lambda, \underline{E}))$ the terms involving $z \in H^2(\mathcal{B}_X^*)$ can be ignored. Therefore, $ch^1(ind) = \frac{1}{2}\mu$ and $ch^2(ind) = -\frac{1}{12}I + (-\frac{sgn(X)}{8} - \frac{\xi}{6} + \frac{(\Lambda+\lambda)^2}{8})\nu$, so that
$P_{-1}^1 = c_2(ind(D^\lambda, \underline{E})) = \frac{1}{8}\mu(\Lambda+\lambda)^2 + \frac{1}{12}I - (-\frac{sgn(X)}{8} + \frac{(\Lambda+\lambda)^2}{8} - \frac{\xi}{6})\nu$
$= \frac{1}{8}\mu(\Lambda+\lambda)^2 + \frac{1}{12}I - (\frac{1}{2}indD_A^\lambda + \frac{\xi}{2} - \frac{\xi}{6})\nu$
$= \frac{1}{8}\mu(\Lambda+\lambda)^2 + \frac{1}{12}I - (\frac{-1}{2} + \frac{\xi}{3})\nu.$

5.5 Evaluating the 'Porteous Class'

Recall the formulae for the Donaldson invariants of $K3$ surface [F] with the orientation as in (5.3.4):

$$1)\ \gamma_{2a}(\mu(h)^{2a}) = \frac{(2a)!}{2^a a!} Q(h,h)^a$$

where Q is the intersection form of the $K3$ surface X, and

$$2)\ \gamma_{d+4}(\mu(h)^a \nu^{b+2}) = 4 \cdot \gamma_d(\mu(h)^a \nu^b)$$

where $a + 2b = d$, which means that a $K3$ surface has simple type. Evaluate

$$\gamma_4(P^1_{-1} \cup \mu(\sigma)^2) = \gamma_4(\frac{1}{8}\mu^2\mu(\sigma)^2 - \frac{1}{3}\nu\mu(\sigma)^2 + \frac{1}{12}I\mu(\sigma)^2),$$

where

$$\mu = \mu(\alpha) = \mu([\Lambda_1 - 2\Lambda_2]), \quad \text{with } Q(\alpha,\alpha) = -10.$$

$$\begin{aligned} I &= (\textstyle\sum_{i=1}^{22} \mu(\alpha_i)\alpha_i^*)^2/[X] \\ &= \textstyle\sum_{i=1}^{22} (\mu(\alpha_i))^2(\alpha_i^*)^2/[X] + \sum_{i\neq j} \mu(\alpha_i)\mu(\alpha_j)(\alpha_i^*\alpha_j^*)/[X]. \end{aligned}$$

The results for each term are:

$$\begin{aligned} 1)\ \gamma_4(\tfrac{1}{8}\mu^2\mu(\sigma)^2) &= \tfrac{1}{8}\{Q(\alpha,\alpha)Q(\sigma,\sigma) + 2(Q(\alpha,\sigma))^2\} \\ &= \tfrac{1}{8}\{(-10)Q(\sigma,\sigma) + 2Q(\sigma,(\Lambda_1 - 2\Lambda_2))^2\} \\ &= -\tfrac{5}{4}Q(\sigma,\sigma) + \tfrac{1}{4}Q(\sigma,(\Lambda_1 - 2\Lambda_2))^2. \end{aligned}$$

$$\begin{aligned} 2)\ \gamma_4(\tfrac{-1}{3}\nu\mu(\sigma)^2) &= \tfrac{-1}{3}(2)\gamma_2(\mu(\sigma)^2) \\ &= (\tfrac{-1}{3})(2)Q(\sigma,\sigma) \\ &= -\tfrac{2}{3}Q(\sigma,\sigma). \end{aligned}$$

3) In terms of a basis $\{\alpha_i\}_{i=1}^{22}$ of $H_2(X)$ such that

$$\begin{cases} \alpha_i \cdot \alpha_j = 0 \qquad if \quad i \neq j \\ \alpha_i^2 = \epsilon_i = \begin{cases} 1 & \text{if i=1,2,3} \\ -1 & \text{if i=4,5,...,22} \end{cases} \end{cases}$$

we have

$$I = \sum_{i=1}^{22} \epsilon_i \, \mu(\alpha_i)^2.$$

Suppose, in terms of the above basis,

$$\sigma = \sum_i \beta_i \, \alpha_i.$$

Then,

$$Q(\sigma, \sigma) = \sum_i \beta_i^2 \, \epsilon_i$$

and

$$\mu(\sigma)^2 I = (\sum_{i=1}^{22} \beta_i^2 \mu(\alpha_i)^2 + \sum_{i \neq j} \beta_i \beta_j \mu(\alpha_i) \mu(\alpha_j)\,) \cdot (\sum_{k=1}^{22} \epsilon_k (\mu(\alpha_k))^2).$$

So, $\gamma_4(\frac{1}{12}\mu(\sigma)^2 I)$

$$\begin{aligned}
&= \tfrac{1}{12}\gamma_4\{\textstyle\sum_{i=1}^{22} \beta_i^2 \epsilon_i \mu(\alpha_i)^4 + \sum_{i\neq k} \beta_i^2 \mu(\alpha_i)^2 \epsilon_k \mu(\alpha_k)^2 \\
&\quad + \textstyle\sum_{i\neq j} \sum_k \beta_i \beta_j \mu(\alpha_i)\mu(\alpha_j)\epsilon_k(\mu(\alpha_k))^2\} \\
&= \tfrac{1}{12}\{\textstyle\sum_{i=1}^{22} \beta_i^2 \epsilon_i \gamma_4(\mu(\alpha_i)^4) + \sum_{i\neq k} \beta_i^2 \epsilon_k \gamma_4(\mu(\alpha_i)^2 \mu(\alpha_k)^2)\} \\
&= \tfrac{1}{12}\{\textstyle\sum_{i=1}^{22} \beta_i^2 \epsilon_i (3)(Q(\alpha_i, \alpha_i))^2 + \sum_{i\neq k} \beta_i^2 \epsilon_k Q(\alpha_i, \alpha_i) Q(\alpha_k, \alpha_k)\} \\
&= \tfrac{1}{12}\{\textstyle\sum_{i=1}^{22} (3)\beta_i^2 \epsilon_i + \sum_{i\neq k} \beta_i^2 \epsilon_k \epsilon_i \epsilon_k\} \\
&= \tfrac{1}{12}\{3Q(\sigma,\sigma) + \textstyle\sum_{i\neq k} \beta_i^2 \epsilon_i\} \\
&= \tfrac{1}{12}\{3Q(\sigma,\sigma) + (b_2 - 1)Q(\sigma,\sigma)\} \\
&= \tfrac{(b_2+2)}{12} Q(\sigma,\sigma) = 2Q(\sigma,\sigma).
\end{aligned}$$

Hence, we conclude that

Proposition 5.5.1 *Evaluating the 'Porteous class' $P^1_{-1} \cup \mu(\sigma)^2$ on the (8-dimensional) moduli space of instantons gives the number*

$$(-\frac{5}{4} - \frac{2}{3} + 2)Q(\sigma,\sigma) + \frac{1}{4}(Q(\sigma,\alpha))^2 = \frac{1}{12}Q(\sigma,\sigma) + \frac{1}{4}(Q(\sigma,(\Lambda_1 - 2\Lambda_2)))^2.$$

Theorem 5.5.2 *Compares (5.5.1) with (5.3.1), we have the (universal) 'linking invariant' $\sharp\mathcal{T}'$ in (4.2) is $\frac{1}{12}$. Hence, the sought-after universal formula relating the spin^c-polynomial and the (contracted) Donaldson polynomial, when $dim\mathcal{M} = 8$ and $indD^\lambda_A = -1$ is*

$$\gamma_4(P^1_{-1} \cup \mu(\Sigma_1) \cup \mu(\Sigma_2)) = \gamma_4'(\Sigma_1, \Sigma_2) + (\frac{1}{12}) \cdot Q(\Sigma_1, \Sigma_2) \cdot \gamma_0.$$

In appendix (A.1), this formula will be verified for the case of a $K3$ surface blown up at a point, and the elliptic surfaces without multiple fibre.

Chapter 6

Generalisations

In light of the results in the previous two chapters, the relationship between the $spin^c$- and the Donaldson invariants is to be studied in a more systematic way.

With notations as before, let $\mathcal{M}_{2d}$ be the $2d$-dimensional moduli space of instantons associated to a smooth, compact, simply connected oriented Riemannian 4-manifold. Suppose a $spin^c$ structure λ is chosen such that the index of the coupled Dirac operators $indD_A^\lambda = 1$, then the moduli space of 1-instantons $\mathcal{M}'_{2d} = \mathcal{M}_{2d}$ since the kernel of D_A^λ is always non-trivial. So in this case, the $spin^c$-invariant is just the Donaldson invariant evaluating on a $2d$-dimensional homology class

$$\gamma'_d(\Sigma^d) = \gamma_d(\Sigma^d).$$

For the case $indD_A^\lambda = 0$, generically $\mathcal{M}'_{2d}$ is of codimension 2. Let

$$\mathcal{N}_2^d = \mathcal{M}_{2d} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d-1}}$$

to be the transverse intersection as in (3.1.4). Use the dimension counting arguments as in (3.1.5) and (3.2.4), $\mathcal{N}_2^d$ is compact if the trivial connection can be avoided under weak limits. For example, when $w_2 \neq 0$ for the $SO(3)$ case [DK, (9.2.2)]; or when the stable range condition (3.2.2) is satisfied for the $SU(2)$ situation. In this case, the $spin^c$-invariant is defined and

$$\gamma'_d(\Sigma_1,\ldots,\Sigma_{d-1}) = \sharp\{[A]\in \mathcal{N}_2^d \mid dim_{\mathbb{C}}KerD_A^{\lambda} \geq 1\}.$$

The generalised Porteous formula applied and give

$$\gamma'_d(h^{d-1}) = \gamma_d(P_0^1 \cup \mu(h)^{d-1}),$$

where $\mathcal{M}'_{2d}$ is equipped with the correct orientation and $P_0^1 = -\frac{1}{2}\mu(\Lambda+\lambda)$ is the two dimensional cohomology class in $\mathcal{B}_X^*$ as defined in section (4.1). For example, when $d=4$ and with notations as in chapter 5, the $spin^c$-polynomial for the $K3$ surface X as given by the above formula is $\gamma'_4(h^3) = -\frac{3}{2}Q(\Lambda+\lambda,h)Q(h,h)$, which is always an integer for any $h\in H_2(X,\mathbb{Z})$ since a $K3$ surface is spin.

6.1 The Case $indD_A^{\lambda} = -1$

Suppose index of the coupled Dirac operators are -1, let $\mathcal{N}_4^d = \mathcal{M}_{2d}\cap V_{\Sigma_1}\cap\cdots\cap V_{\Sigma_{d-2}}$. As in the case index $=0$ above, suppose the trivial connection can be avoided under weak limits so that the $spin^c$-polynomial is well-defined and is given by

$$\gamma'_d(\Sigma_1,\ldots,\Sigma_{d-2}) = \sharp\{[A]\in \mathcal{N}_4^d \mid dim_{\mathbb{C}}kerD_A^{\lambda} \geq 1\}.$$

When $d<4$, $\mathcal{M}_{2d}$ is compact. When $d=4$, this is the case considered in chapter 4 and 5.

Consider $d\geq 4$. If $([A_0],x)\in \mathcal{M}_{2d-8}\times X$ is a point at infinity of $\mathcal{N}_4^d$, by dimension counting argument x must lie on at least two of the $(d-2)$ surfaces. Otherwise,

$$[A_0]\in \mathcal{M}_{2d-8}\cap V_{\Sigma_{i_1}}\cap\cdots\cap V_{\Sigma_{i_l}} \quad , \quad l\geq d-3,$$

which is impossible as

$$dim_{\mathbb{R}}(\mathcal{M}_{2d-8}\cap V_{\Sigma_{i_1}}\cap\cdots\cap V_{\Sigma_{i_l}}) = (2d-8)-2l \leq -2.$$

But as in (3.1), Σ_i's are chosen such that the tubular neighbourhood has empty triple intersection; therefore, x must lie on exactly two surfaces $\Sigma_i \cap \Sigma_j$ with

$$[A_0] \in \mathcal{M}_{2d-8} \cap V_{\Sigma_1} \cap \cdots \cap \widehat{V_{\Sigma_i}} \cap \cdots \cap \widehat{V_{\Sigma_j}} \cap \cdots \cap V_{\Sigma_{d-2}},$$

and there are altogether $\gamma_{d-4}(\Sigma_1, \ldots, \widehat{\Sigma}_i, \ldots, \widehat{\Sigma}_j, \ldots, \Sigma_{d-2})$ such background connections $[A_0]$. Similar dimension counting arguments show that no point at infinity in $\mathcal{N}_4^d$ are of the form $([A]; x_1, \ldots, x_j) \in \mathcal{M}_{2d-8j} \times Sym^j(X)$, with $j \geq 2$. So, the structure of the ends of the 4-dimensional space $\mathcal{N}_4^d$ is the same as in the $d = 4$ case considered in chapter 4 and 5, i.e. both the point of concentration and the background connection are fixed, with the compactification involves immediate lower stratum only and the ends are modelled on cones over $SO(3)$. Use the same cut-and-paste arguments to obtain a compact 4-dimensional family of connections and applying the generalised Porteous Formula, with the correct orientation, to get

Theorem 6.1.1

$$\begin{aligned}\gamma_d(P^1_{-1} \cup \mu(h)^{d-2}) &= \gamma_d'(h^{d-2}) + (\tfrac{1}{12}) \begin{pmatrix} d-2 \\ 2 \end{pmatrix} Q(h,h) \cdot \gamma_{d-4}(h^{d-4}) \\ &= \gamma_d'(h^{d-2}) + \tfrac{(d-2)(d-3)}{24} Q(h,h) \cdot \gamma_{d-4}(h^{d-4}),\end{aligned}$$

where $P^1_{-1} = \frac{1}{8}\mu(\Lambda+\lambda)^2 + \frac{1}{12}I + (\frac{1}{2} - \frac{\xi}{3})\nu$ *is the Porteous class for the case* $indD^\lambda_A = -1$ *as obtained in section (5.4).*

For $d \leq 3$, there is no correction term as the moduli space $\mathcal{M}_{2d}$ is already compact. The combinatorial factor in the formula is the number of ways to choose 2 surfaces from $(d-2)$ of them. The 'linking invariant' is the same $\frac{1}{12}$ because essentially the same 4-family $\tilde{\tau}$ on S^4, with $\partial\tilde{\tau} = -\tilde{\eta}$, is glued to $[A_0] \in \mathcal{M}_{2d-8} \cap V_{\Sigma_{i_1}} \cdots \cap V_{\Sigma_{i_{d-4}}}$ to obtain a cap of the link associated to $([A_0], x)$, and it has been shown in (4.3) that this 'linking invariant' is independent of the (fixed) background connection. In fact, it is the transverse jumping number $\sharp\{[a] \in \tilde{\tau} \mid dim_{\mathbb{C}} ker D_a \geq 1\}$ where D is the Dirac operator on S^4, when the scale of the gluing map is sufficiently small. For example, when $d = 6$, the formula gives the $spin^c$-polynomial for the $K3$ surface to be the integral valued polynomial

$$\gamma_6'(h^4) = \frac{3}{2}Q(\Lambda + \lambda, h)^2 Q(h,h).$$

To sum up, if the $spin^c$-polynomial is defined,

1) When $indD_A^\lambda = 1$, $\gamma_d'(h^d) = \gamma_d(h^d)$.

2) When $indD_A^\lambda = 0$, $\gamma_d'(h^{d-1}) = \gamma_d(P_0^1 \cup \mu(h)^{d-1})$.

3) When $indD_A^\lambda = -1$,

$$\gamma_d'(h^{d-2}) = \gamma_d(P_{-1}^1 \cup \mu(h)^{d-2}) - (\frac{1}{12}) \begin{pmatrix} d-2 \\ 2 \end{pmatrix} Q(h,h) \cdot \gamma_{d-4}(h^{d-4}).$$

6.2 The Case $indD_A^\lambda = -2$

If the moduli space of instantons $\mathcal{M}_8$ is of (real) dimension 8 and the coupled Dirac operators of index -2, then the moduli space of 1-instantons $\mathcal{M}_8'$ is generically of (real) dimension 2. Suppose the $spin^c$-polynomial is well-defined as

$$\gamma_4'(\Sigma) = \sharp(\mathcal{M}_8' \cap V_\Sigma) = \sharp(J \cap \mathcal{N}_6),$$

where $\mathcal{N}_6 = \mathcal{M}_8 \cap V_\Sigma$ the transverse intersection and J is the jumping locus in $\mathcal{B}_X^*$. Since $\mathcal{N}_6$ is of dimension 6 while the lower strata in the Uhlenbeck compactification have codimension multiples of 4, dimension counting argument implies that the closure $\overline{\mathcal{N}}_6$ meets the stratum $\mathcal{M}_0 \times X$ only (as the trivial connection is avoided under weak limit), and if $(A_0, x) \in \overline{\mathcal{N}}_6$ then $x \in \Sigma$. As in [DK, (8.2)] or [FM, chapter 3], the end of $\mathcal{N}_6$ associated to $A_0 \in \mathcal{M}_0$ is modelled on a locally trivial fibre bundle over Σ whose fibre is a cone over $SO(3)$, corresponding to gluing the standard charge-1 instanton I on S^4 to A_0 at points in Σ. Truncating the end at small gluing scale

ζ, the link (or boundary) is then a $SO(3)$ fibre bundle $\mathcal{H}_{\Sigma,A_0,\zeta} \to \Sigma$, with each fibre given as in the index$= -1$ case considered in chapter 4 and 5.

Let $\mathcal{C}_{\Sigma,A_0}$ be the space of connections on X with charge-1 concentration at a point on Σ and background connection $A_0 \in \mathcal{M}_0$. It is given by gluing $\tilde{\mathcal{B}}^0_{S^4}$, the space of charge-1 connections on S^4 framed at ∞ and centred at 0 with scale ≤ 1,[1] to A_0 at $x \in \Sigma$ by a gluing map of fixed small scale. Therefore, $H_5(\mathcal{C}_{\Sigma,A_0}, \mathbb{Q}) = 0$ as $\mathcal{C}_{\Sigma,A_0}$ can be regarded as a fibre bundle over Σ with fibre $\tilde{\mathcal{B}}^0_{S^4}$, which is of the same homotopy type as $\tilde{\mathcal{B}}_{S^4}$ and so has trivial rational (co)homology.

For ζ sufficiently small, $\mathcal{H}_{\Sigma,A_0,\zeta} \subseteq \mathcal{C}_{\Sigma,A_0}$ and so a (rational) 6-dimensional cap $\mathcal{R}_{\Sigma,A_0,\zeta} \subset \mathcal{C}_{\Sigma,A_0}$ to $\mathcal{H}_{\Sigma,A_0,\zeta}$ can be found. Therefore,

$$\partial\mathcal{R}_{\Sigma,A_0,\zeta} = -\mathcal{H}_{\Sigma,A_0,\zeta} \qquad with \qquad \mathcal{R}_{\Sigma,A_0,\zeta} = A_0\#\{\tilde{\tau}_x : x \in \Sigma\},$$

the gluing map is of scale ζ and the gluing parameters given by the framing as in (4.3). Here $\tilde{\tau}_x \subseteq \tilde{\mathcal{B}}^0_{S^4}$ is a 4-dimensional family for generic $x \in \Sigma$, with $\partial\tilde{\tau}_x = -\tilde{\eta}$ the $SO(3)$ framed family of standard instantons on S^4 as before. As in the index $= -1$ case, by Sard-Smale theorem, the 6-dimensional family $\varpi = \{\tilde{\tau}_x : x \in \Sigma\}$ can be taken to have transverse intersection ϖ' with the jumping locus

$$\{[a] \in \tilde{\mathcal{B}}_{S^4} \mid dim Ker D_a \geq 1\},$$

such that $\mathcal{R}'_{\Sigma,A_0,\zeta}$ transverse intersection with the jumping locus in $\mathcal{B}^*_X$ for generic small ζ. Therefore, $a' \in \tilde{\tau}_x$ is in the transverse two-dimensional family

$$\varpi' = \{[a] \in \varpi \mid dim_{\mathbb{C}} Ker D_a \geq 1\}$$

iff $\dim_{\mathbb{C}} Ker D_{a'} = 1$ (i.e. consisting of smooth points only) and

$$T_{a'}(\varpi_{U_x}) \xrightarrow{j} (ker D_{a'})^* \otimes (coker D_{a'}) \quad where \quad j(\delta a)(s) = \pi_{S^4}(\delta a * s),$$

[1]As in [DK, (8.2.1)], the 'centre' and the 'scale' of a charge-1 connection on S^4 can be taken as the centre and the radius of the unique minimal ball in $\mathbb{R}^4$, under stereographic projection at ∞, which contains half the total energy, i.e. $4\pi^2$. Note that the standard instanton I has centre at 0 and scale 1.

defined as before, is surjective. Here $U_x \subset \Sigma$ is a small neighbourhood of x such that $\mathcal{R}_{\Sigma,A_0,\zeta}|_{U_x}$ is given by gluing the 6-dimensional family $\varpi_{U_x} \subset \tilde{\mathcal{B}}_{S^4}$ (not neccesarily centred) to A_0 at x.

Similar to the index $= -1$ case, since connections in $\mathcal{C}_{\Sigma,A_0}$ are gauge equivalent to A_0 outside a small ball about a point (in Σ), applying the Porteous formula we get

$$\gamma_4(P^1_{-2} \cup \mu(\Sigma)) = \gamma_4'(\Sigma) + \sum_{A_0 \in \mathcal{M}_0} \sharp\mathcal{R}'_{\Sigma,A_0,\zeta}$$

for sufficiently small (and generic) ζ. Our claim is that when ζ is sufficiently small, the transverse jumping number $\sharp\mathcal{R}'_{\Sigma,A_0,\zeta}$ is independent of ζ and $A_0 \in \mathcal{M}_0$, and depends on Σ only through its homology class $[\Sigma]$. That is,

Formula 6.2.1

$$\gamma_4(P^1_{-2} \cup \mu(\Sigma)) = \gamma_4'(\Sigma) + \gamma_0 \cdot \sharp(\mathcal{R}'_\Sigma).$$

Here the relevant Porteous class defined (formally) by the jumping locus in $\mathcal{B}^*_X$ is[2]

$$P^1_{-2} = -\{\frac{1}{48}\mu^3 + \frac{1}{24}I\mu + (\frac{1}{3} - \frac{\xi}{6})\nu\mu\}$$

with notations as in (5.4). This correction term $\sharp\mathcal{R}'_\Sigma$ will be determined in the next section using results from the index$= -1$ case.

[2]Using notations as in (5.4), ignoring terms involving z,

$$\begin{aligned} ch^1(ind) &= \tfrac{1}{2}\mu(\Lambda + \lambda) \\ ch^2(ind) &= -\tfrac{1}{12}I + (-\tfrac{Sgn(X)}{8} - \tfrac{\xi}{6} + \tfrac{(\Lambda+\lambda)^2}{8})\nu \\ ch^3(ind) &= -\tfrac{1}{12}\nu\mu(\Lambda + \lambda). \end{aligned}$$

i.e.
$$\begin{aligned} P^1_{-2} = -c_3(ind) &= -\{2ch^3(ind) + \tfrac{1}{6}(ch^1(ind))^3 - ch^2(ind) \cdot ch^1(ind)\} \\ &= -\{\tfrac{1}{48}\mu^3 + \tfrac{1}{24}I\mu + (\tfrac{Sgn(X)}{16} + \tfrac{\xi}{12} - \tfrac{(\Lambda+\lambda)^2}{16} - \tfrac{1}{6})\nu\mu\} \\ &= -\{\tfrac{1}{48}\mu^3 + \tfrac{1}{24}I\mu + (\tfrac{1}{3} - \tfrac{\xi}{6})\nu\mu\}. \end{aligned}$$

6.3 The Obstruction Line Bundle

For $A_0 \in \mathcal{M}_0$, $KerD^\lambda_{A_0} = 0$ generically as $\mathrm{ind} D^\lambda_{A_0} = -1$ (and $\mathrm{ind} D_a = -1$, $\forall a \in \varpi$). As in (4.3) for the case $indD^\lambda_A = -1$, using cut-off function argument it can be shown that if $A' = A_0 \# a \in \mathcal{R}'_{\Sigma, A_0, \zeta}$, then a is close (in L^4), how close depends on ζ, to an element $a' \in \varpi'$. Therefore, for ζ sufficiently small, every jumping connections in $\mathcal{R}'_{\Sigma, A_0, \zeta}$ are given by deforming an element in the compact 2-dimensional family $\{A_0 \# \varpi'\}$. However, in this case there is a (complex) 1-dimensional obstruction to such deformation given by $cokerD^\lambda_{A_0} \cong \mathbb{C}$ (recall that there is no obstruction in the previous case $\mathrm{ind} D^\lambda_A = -1$ since the Dirac operator coupled to the background connections have trivial cokernel). The argument is just a slight modification to that in (4.3) and is given briefly as below.

Let $a' \in \tilde{\tau}_x$ be a jumping connection in the transverse 2-dimensional family ϖ'. Consider a neighbourhood U of a' in $\varpi|_{U_x}$, the 6-family of connections in $\tilde{\mathcal{B}}_{S^4}$ as in the previous section, such that U consists of elements $(a' + \delta a)$ with $A_0 \#_x (a' + \delta a) \in \mathcal{R}_{\Sigma, A_0, \zeta}|_{U_x}$ and $\|A_0 \#_x (a' + \delta a) - A_0 \#_x a'\|_{L^4(X \# S^4)} \leq \epsilon$ certain small number. The gluing is at point x with the gluing parameter given by the framings. Fix liftings

$$\sigma_{S^4} : cokerD_{a'} \longrightarrow \Gamma(W^-_{S^4} \otimes V) \quad \& \quad \sigma_X : cokerD^\lambda_{A_0} \longrightarrow \Gamma(W^-_X \otimes Y)$$

of the projections π_{S^4} and π_X, with spinors in the image of σ_{S^4} and σ_X supported away from $\infty \in S^4$ and $\Sigma \subset X$, respectively. Consider the linear map

$$\sigma_x : cokerD_{a'} \oplus cokerD^\lambda_{A_0} \longrightarrow \Gamma(W^-_{X \# S^4} \otimes E)$$

which is a lifting of the projection $\pi_x(*) = \pi_{S^4}(\beta_x *) + \pi_X((1 - \beta_x)*)$, where β_x is a cut-off function supported on the S^4-side as in (4.3), for the connected sum $X \# S^4$ at $x \in \Sigma$.

Let $s \in \Gamma(W^+_{S^4} \otimes V)$ with unit $L^{8/3}$-norm such that $D_{a'}(s) = 0$. Then $\beta_x s$, consider as an element in $\Gamma(W^+_{X \# S^4} \otimes E)$, is close to being a harmonic spinor for

$D^{\lambda}_{A_0\#_x a'}$ when the scale ζ of the gluing map is small. Again, we want to solve $\Phi_x(a'+\delta a, \delta s, h) = 0$ in the form $(\delta s, h) = (P_x(\eta), \pi_x(\eta))$, where

$$\Phi_x : U \times \Gamma(W^+_{X\#S^4} \otimes E) \times (coker D_{a'} \oplus coker D^{\lambda}_{A_0}) \longrightarrow \Gamma(W^-_{S^4\#X} \otimes E)$$

$$\Phi_x(a'+\delta a, \delta s, h) = D^{\lambda}_{A_0\#_x(a'+\delta a)}(\beta_x s + \delta s) + \sigma_x(h)$$

and

$$P_x \oplus \pi_x : \Gamma(W^-_{X\#S^4} \otimes E) \longrightarrow \Gamma(W^+_{X\#S^4} \otimes E) \oplus (coker D_{a'} \oplus coker D^{\lambda}_{A_0})$$

is a right inverse of $D^{\lambda}_{A_0\#_x a'} \oplus \sigma_x$. Using exactly the same estimate as in (4.3), there is unique solution $\eta_{\delta a} \in \Gamma(W^-_{X\#S^4} \otimes E)$ for each $\delta a \in U$. Therefore, those deformations δa with $\sigma_x \pi_x(\eta_{\delta a}) = 0$ give genuine harmonic spinors for $D^{\lambda}_{A_0\#_x(a'+\delta a)}$.

To get the proper obstruction space consider the map

$$J_x : N_{U_x} \cap U \longrightarrow coker D_{a'}\ , \qquad \delta a \longmapsto \pi_{S^4}(\beta_x \cdot \eta_{\delta a}),$$

where $N_{U_x} \subset T_{a'}(\varpi_{U_x})$ is the image of some splitting of the exact sequence

$$0 \longrightarrow ker(j) \longrightarrow T_{a'}(\varpi_{U_x}) \stackrel{j}{\longrightarrow} (Ker D_{a'})^* \otimes Coker D_{a'} \longrightarrow 0.$$

Therefore, $j|_{N_{U_x}}$ is bijective and small elements in N_{U_x} correspond to deformations of a' in ϖ in directions normal to ϖ'. With exactly the same estimate as in (4.3), when ζ and ϵ (i.e. U) is sufficiently small, J maps $N_{U_x} \cap U$ bijectively onto a small ball containing $0 \in coker D_{a'}$ and so the obstruction lies in $coker D^{\lambda}_{A_0}$.

Therefore, there is an obstruction line bundle $\mathcal{L}$ over $\mathcal{C}_{\Sigma, A_0}$, with fibres given by gluing the zero spinor in $\Gamma(W^-_{S^4} \otimes V)$ with $Coker D^{\lambda}_{A_0} \cong Ker(D^{\lambda}_{A_0})^* \subset \Gamma(W^-_X \otimes Y)$. The algebraic number of transverse zeroes of a section on the restriction $\mathcal{L}|_{\{A_0\#\varpi'\}}$ to the compact 2- dimensional family gives the required correction term $\sharp\mathcal{R}'_{\Sigma, A_0, \zeta}$ for sufficiently small ζ.

Consider the Leray-Serre spectral sequence to the fibration

$$\tilde{\mathcal{B}}^0_{S^4} \longrightarrow \mathcal{C}_{\Sigma,A_0} \xrightarrow{p} \Sigma.$$

Since $H^i(\tilde{\mathcal{B}}^0_{S^4}, \mathbb{Z}) = \mathbb{Z}_2$ for $i = 1, 2$, with the relevant d_2 are zero homomorphisms, we have

Lemma 6.3.1

$$0 \longrightarrow H^2(\Sigma, \mathbb{Z}) \longrightarrow H^2(\mathcal{C}_{\Sigma,A_0}, \mathbb{Z}) \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

is a split exact sequence and hence $H^2(\mathcal{C}_{\Sigma,A_0}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$.

Therefore, $\mathcal{L}^{\otimes 2}$ is the pull-back of a line bundle[3] L over Σ by the projection p,

$$i.e. \qquad \mathcal{L}^{\otimes 2} = p^*(L).$$

Lemma 6.3.2

$$L = det(Y \otimes W_X^+)|_\Sigma \longrightarrow \Sigma$$

and hence, with notations as in chapter 5,

$$< c_1(L), \Sigma >=< c_1(\lambda) + c_1(Y), \Sigma >=< c_1(\lambda) + c_1(E), \Sigma >=< \alpha, \Sigma > .$$

Proof: Suffices to show this is true over the link $\mathcal{H}_{\Sigma,A_0,\zeta} \subset \mathcal{C}_{\Sigma,A_0}$ (for ζ sufficiently small) which can be interpreted, using the description of the gluing map in (4.3), as the space of gluing parameters for concentrated ASD connections

$$\mathcal{H}_{\Sigma,A_0,\zeta} = Hom_{SO(3)}(\Lambda_X^+, adY)|_\Sigma \xrightarrow{p} \Sigma,$$

[3]Since $\tilde{\mathcal{B}}^0_{S^4} \sim \tilde{\mathcal{B}}_{S^4}$ has trivial rational cohomology, $H^2(\mathcal{C}_{\Sigma,A_0}, \mathbb{Q}) = H^2(\Sigma, \mathbb{Q})$, and so a certain tensor power of $\mathcal{L}$ is the pull-back of a line bundle over Σ. In fact, this is sufficient for our (rational) calculation.

where Y is the $U(2)$-bundle on X supporting A_0. This is because the standard instanton I can be obtained as the standard connection on the spinor bundle $W^-_{S^4}$, and so to make the connected sum construction an identification of the fibre adY at x with the fibre $adW^-_{S^4} \cong \Lambda^-_{S^4}$ at ∞ is needed. By the natural orientation-reversing isometry between the tangent spaces to S^4 at 0 and ∞, the latter can be identified with the fibre of $(\Lambda^+_X)^* \cong \Lambda^+_X$ at x (see [DK, (8.2.1)]).

There is a S^1-bundle over $\mathcal{H}_{\Sigma,A_0,\zeta}$

$$S^1 \longrightarrow \mathcal{P} = Hom_{U(2)}((W^+_X)^*, Y)|_\Sigma \longrightarrow \mathcal{H}_{\Sigma,A_0,\zeta}$$

which can be interpreted as the space of gluing parameters for bundle-valued spinors, using the same identificaton as above. Then the obstruction line bundle $\mathcal{L}$ restricted to $\mathcal{H}_{\Sigma,A_0,\zeta}$ is simply

$$\mathcal{P} \times_{S^1} Coker D^\lambda_{A_0} \quad \longrightarrow \quad \mathcal{H}_{\Sigma,A_0,\zeta}.$$

Since

$$\mathcal{P}^{\otimes 2} = p^*(Hom_{S^1}(\bigwedge^2((W^+_X)^*), \bigwedge^2 Y)|_\Sigma),$$

by the $2:1$ map $U(2) \longrightarrow SO(3) \times S^1$, the sought-after line bundle L over Σ is in fact $L = \{(det(W^+_X)^*)^* \otimes det(Y)\}|_\Sigma = (\lambda \otimes det(Y))|_\Sigma$. □

By the definition of the obstruction line bundle, the correction term is

$$\sharp \mathcal{R}'_{\Sigma,A_0,\zeta} = (\sharp \tilde{\tau}'_x) \cdot \frac{1}{2} < c_1(L), \Sigma >= \frac{1}{24} < \alpha, \Sigma > .$$

Here $\sharp\tilde{\tau}'_x = \frac{1}{12}$ is the linking invariant as in the index $= -1$ case since, for generic $x \in \Sigma$, $\tilde{\tau}_x \subset \tilde{\mathcal{B}}^0_{S^4}$ is a 4-dimensional family with $\partial\tilde{\tau}_x = -\tilde{\eta}$ the $SO(3)$-family of framed standard instantons. Therefore,

Theorem 6.3.3

$$\gamma_4(P^1_{-2} \cup \mu(\Sigma)) = \gamma'_4(\Sigma) + \frac{1}{24}\gamma_0 \cdot < \alpha, \Sigma > .$$

Again, this is verified in Appendix (A.2) for the case of elliptic surfaces without multiple fibre using Friedman's results.

For $\dim \mathcal{M}_{2d} = 2d \geq 8$ and $\operatorname{ind} D_A^{\lambda} = -2$, $\gamma_d'(\Sigma^{d-3}) = \#(J \cap \mathcal{N}_6)$ where $\mathcal{N}_6 = \mathcal{M}_{2d} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d-3}}$. Suppose $([A_0], x)$ is an end of $\mathcal{N}_6$, then either

1) $x \in \Sigma_k$ only and $A_0 \in \mathcal{N}_0^k = \mathcal{M}_{2d-8} \cap V_{\Sigma_1} \cap \cdots \cap \widehat{V_{\Sigma_k}} \cap \cdots \cap V_{\Sigma_{d-3}}$ or

2) $x \in \Sigma_i \cap \Sigma_j$ and $A_0 \in \mathcal{N}_2^{ij} = \mathcal{M}_{2d-8} \cap V_{\Sigma_1} \cap \cdots \cap \widehat{V_{\Sigma_i}} \cap \cdots \cap \widehat{V_{\Sigma_j}} \cap \cdots \cap V_{\Sigma_{d-3}}$.

If $\Sigma \cdot \Sigma = 0$, only 1) occurs and the same argument as for $d = 4$ gives

$$\gamma_d(P_{-2}^1 \cup \mu(\Sigma)^{d-3}) = \gamma_d'(\Sigma^{d-3}) + \frac{(d-3)}{24}\gamma_{d-4}(\Sigma^{d-4}) \cdot < \alpha, \Sigma > .$$

In general, removing ends of $\mathcal{N}_6$, the boundary of the truncated moduli space is a 5-dimensional family in $\mathcal{C}$ given by gluing in the standard charge-1 instanton I to background connections on X, here $\mathcal{C}$ is the space of connections with charge-1 concentration at a point in Σ_k ($k \in \{1, 2, \ldots, d-3\}$) and background connections in $\mathcal{N} = \bigcup_{i,j} \mathcal{N}_2^{ij}$. That is, $\mathcal{C}$ can be described as a fibre bundle

$$\tilde{\mathcal{B}}_{S^4}^0 \longrightarrow \mathcal{C} \longrightarrow \Sigma \times \mathcal{N},$$

where $\Sigma = \bigcup_k \Sigma_k \subset X$.

Again $H_5(\mathcal{C}, \mathbb{Q}) = 0$, since $\tilde{\mathcal{B}}_{S^4}^0$ has trivial rational (co-)homology, and so a 6-dimensional (rational) cap $\mathcal{R}$ to the truncated space is given by

$$\begin{aligned} \{A_0 \#_x \tilde{\tau}_{x,A_0} \mid \ & \textit{where} \quad A_0 \in \mathcal{N}_0^k \quad \textit{if} \quad x \in (\Sigma_k \setminus \textstyle\bigcup_{l \neq k} \Sigma_l) \\ & \textit{and} \quad A_0 \in \mathcal{N}_2^{ij} \quad \textit{if} \quad x \in (\Sigma_i \cap \Sigma_j) \ \}, \end{aligned}$$

here $\tilde{\tau}_{x,A_0} \subset \tilde{\mathcal{B}}_{S^4}^0$ is a 4-dimensional family such that $\partial \tilde{\tau}_{x,A_0} = -\tilde{\eta}$, for generic $x \in \Sigma$ and $A_0 \in \mathcal{N}$. Again since $\mathcal{C}$ consists of connections that are constant outside a point (in Σ), the correction term relating the *spin*^{c} and the contracted Donaldson polynomial is given by the number $\sharp\mathcal{R}'$ of jumping connections in the cap.

By the same obstruction theory as in the $d = 4$ case, since $cokerD^{\lambda}_{A_0} = \mathbb{C}$ and $kerD^{\lambda}_{A_0} = 0$, there is an obstruction line bundle $\mathcal{L}$ over $\mathcal{C}$. For a fixed $A_0 \in \mathcal{N}^k_0 \subset \mathcal{N}^{ij}_2$, $\mathcal{C}|_{\Sigma_k \times \{A_0\}}$ is then the $\mathcal{C}_{\Sigma_k, A_0}$ as in the $d = 4$ case. So, $\mathcal{L}^{\otimes 2}$ restricted to $\mathcal{C}|_{\Sigma_k \times \{A_0\}}$ is the pull-back of the line bundle $det(W^+_X \otimes Y)|_{\Sigma_k}$ over Σ_k under the projection map $\mathcal{C}|_{\Sigma_k \times \{A_0\}} \longrightarrow \Sigma_k$ as before.

Similarly, for $x \in \Sigma_i \cap \Sigma_j$ the restriction of $\mathcal{L}^{\otimes 2}$ to $\mathcal{C}|_{\{x\} \times \mathcal{N}^{ij}_2}$ is the pull-back of a line bundle L by the projection $\mathcal{C}|_{\{x\} \times \mathcal{N}^{ij}_2} \xrightarrow{p} \mathcal{N}^{ij}_2$. Our *conjecture* is that

$$c_1(L) = \mu(\alpha)|_{\mathcal{N}^{ij}_2} \quad .$$

Since $\mathcal{N}^{ij}_2$ is compact, the degree of the line bundle L is $\gamma_{d-4}(\alpha, \Sigma^{d-5})$. Therefore, by arranging that $\{A_0 \#_x \tilde{\tau}_{x, A_0} \mid x \in \Sigma_i \cap \Sigma_j \;\;\&\;\; A_0 \in \mathcal{N}^i_0, \;\; \forall i, j\}$ to contain no jumping connection,

$$\begin{aligned} \gamma_d(P^1_{-2} \cup \mu(\Sigma)^{d-3}) = \gamma'_d(\Sigma^{d-3}) &+ \tfrac{(d-3)}{24} \cdot \gamma_{d-4}(\Sigma^{d-4}) \cdot Q(\alpha, \Sigma) \\ &+ \tfrac{(d-3)(d-4)}{48} \cdot \gamma_{d-4}(\alpha, \Sigma^{d-5}) \cdot Q(\Sigma, \Sigma), \end{aligned}$$

with the obvious combinatorial factors the number of ways to choose 1 and 2 surfaces from $(d-3)$ of them. In particular, for a $K3$ surface with $d = 6$, the above formula gives the $spin^c$-polynomial to be

$$\begin{aligned} \gamma'_6(\Sigma^3) &= \gamma_6(P^1_{-2} \cup \mu(\Sigma)^3) - \tfrac{1}{4} Q(\Sigma, \Sigma) \cdot Q(\alpha, \Sigma) \\ &= -\tfrac{1}{8} Q(\alpha, \Sigma)^3. \end{aligned}$$

The reason for such conjecture is that, by definition, the fibres of the obstruction line bundle $\mathcal{L}$ is given by gluing $ker(D^{\lambda}_{A_0})^* = \mathbb{C}$ to the zero-spinor on S^4. So we expect that on $\mathcal{C}|_{\mathcal{N}^{ij}_2 \times \{x\}}$, up to twisting by a line bundle H arising from the gluing parameters for bundle-valued spinors, $\mathcal{L}^{\otimes 2} = p^*(Q^{\otimes 2})$ where $Q = ind((D^{\lambda})^*, \underline{\mathbb{F}}|_{\mathcal{N}^{ij}_2 \times X}) \to \mathcal{N}^{ij}_2$. Here $\underline{\mathbb{F}}$ is a $U(2)$-lift of the $SO(3)$ universal bundle $\mathbb{F} \longrightarrow \mathcal{M}_{2d-8} \times X$ carrying the connections on $\mathcal{M}_{2d-8}$. By the Atiyah-Singer index theorem for family as in (5.4), since $\mathrm{ind} D^{\lambda}_{A_0} = -1$ for $A_0 \in \mathcal{M}_{2d-8}$, $c_1(Q^{\otimes 2}) = \mu(\alpha)|_{\mathcal{N}^{ij}_2} + c_1(\underline{\mathbb{F}}|_{\mathcal{N}^{ij}_2 \times \{x\}})$. However, since the jumping number is independent of the lifting, the second term is expected to be cancelled out by the twisting H.

6.4 For $indD_A^\lambda \leq -3$

One would expect that similar cut-and-paste constructions on the moduli space, by suitably capping off the ends and applying the Porteous formula, will give formulae expressing the $spin^c$-polynomials in terms of the Donaldson polynomials, with the sequence of correction terms arising from lower strata in the Uhlenbeck compactification. However, unlike the previous cases where the immediate lower stratum only is involved, when index ≤ -3 the situations are more complicated.

For example, when $\operatorname{ind} D_A^\lambda = -3$ and $\dim \mathcal{M}_{2d} = 2d \quad (d \geq 4)$. Let

$$\mathcal{N}_8 = \mathcal{M}_{2d} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d-4}},$$

then the $spin^c$-polynomial, if defined, is $\gamma'_{2d}(\Sigma_1, \ldots, \Sigma_{d-4}) = \#(J \cap \mathcal{N}_8)$ where J is the jumping locus in $\mathcal{B}_X^*$. The closure of $\mathcal{N}_8$ hits two strata in the Uhlenbeck compactification in general position. Apart from the points $(A_0, x) \in \mathcal{M}_{2d-8} \times X$ with charge 1 concentration, if $d \geq 8$ there are points at infinity of the form $(A_1, x_1, x_2) \in \mathcal{M}_{2d-16} \times sym^2(X)$. Dimension counting implies that the points of concentration x_1 and x_2 must lie on at least 4 surfaces. Otherwise,

$$[A_1] \in \mathcal{I} = \mathcal{M}_{2d-16} \cap V_{\Sigma_{i_1}} \cap \cdots \cap V_{\Sigma_{i_l}}$$

where $l \geq (d-4) - 3 = d - 7$ and $\dim \mathcal{I} = (2d - 16) - 2l \leq -2$ which is impossible. But as in (3.1), since $\nu(\Sigma_i)$'s are chosen to have empty triple intersection, x_1 and x_2 must be distinct, each lies on the intersection of two surfaces, i.e.

$$x_1 \in \Sigma_p \cap \Sigma_q \quad \& \quad x_2 \in \Sigma_r \cap \Sigma_s$$

with distinct p, q, r, s and

$$[A_1] \in \mathcal{M}_{2d-16} \cap V_{\Sigma_1} \cap \cdots \cap \widehat{V_{\Sigma_p}} \cap \cdots \cap \widehat{V_{\Sigma_q}} \cap \cdots \cap \widehat{V_{\Sigma_r}} \cap \cdots \cap \widehat{V_{\Sigma_s}} \cap \cdots \cap V_{\Sigma_{d-4}}.$$

Therefore, in this situation the background connection $[A_1]$ is fixed and the link is given by a double connected sum.

When the index $= -5$, similar dimension counting argument shows that points of infinity in the third stratum, $([A_0], x_1, x_2, x_3) \in \mathcal{M}_{2d-24} \times sym^3(X)$ with x_i's distinct, will occur. Also, points of multiplicity 2 —– $([A_0], x, x) \in \mathcal{M}_{2d-16} \times sym^2(X)$ occured for the first time. Hence, more general gluing construction involving gluing instantons on S^4 of charge 2 to (family of) background connections on X, as in [FM, KM], will be needed to describe the link associated to the ends.

Appendix A

Examples

A.1 $K3$ Surface Blown Up at a Point

In this section, formula (5.5.2) (i.e. (6.1.1) for the case $d = 4$) is verified for the case of a $K3$ surface blown up at a point. As in chapter 5, $X = Km(T)$ where T is a generic complex 2-torus with $Pic(T) = Pic^0(T)$. Let $\tilde{X}$ be the blown-up Kummer surface, at a point not lying on Λ_i, $i \in \{1, \ldots, 16\}$. Then the classical invariants are $b^+ = 3$ and $b^- = 20$. Also,

$$H_2(\tilde{X}) = H_2(X) \oplus < F > \quad and \quad H^2(\tilde{X}) = H^2(X) \oplus < f >,$$

where F is the exceptional curve and $f \in H^2(\overline{\mathbb{P}}^2)$, with $f \cdot F = -1$, is the Poincare dual of F. The canonical bundle is then $K_{\tilde{X}} = \Theta_{\tilde{X}}(F)$.

With notations as in (5.1), consider the (8-dimensional) moduli space of stable $U(2)$-bundles $E \to \tilde{X}$ with

$$c_1(E) = \Lambda \;\;\&\;\; c_2(E) = \frac{10 + \Lambda^2}{4}.$$

For 1-instantons, we need to consider line bundle L with $\chi(E \otimes L) = -1$. The

Riemann-Roch formula in this case gives

$$\frac{1}{2}c_1(E\otimes L)^2 - c_2(E\otimes L) - \frac{1}{2}c_1(E\otimes L)\cdot c_1(K_{\tilde{X}}) + 4 = -1,$$

that is

$$4\Lambda\cdot L + 4L^2 + \Lambda^2 + 10 - 2(\Lambda + 2L)\cdot(F) = 0.$$

Again, the case $c_1(E) = \Lambda = \Lambda_1$ (hence $c_2(E) = 2$) and $L = \Theta(-\Lambda_2)$ will be considered. The only difference is that the line bundle $\tilde{\lambda}$ associated to the *spin*^{c} structure is given by $\tilde{\lambda}\otimes K_{\tilde{X}} = L^2$, that is

$$\tilde{\lambda} = \Theta(-2\Lambda_2 - F) \qquad \& \qquad c_1(\tilde{\lambda}) + \Lambda = \Lambda_1 - 2\Lambda_2 - F.$$

Using notations as in section (5.4), $\xi/[X] = c_2(E) - \frac{1}{4}c_1(E)^2 = \frac{5}{2}$, and so the 'Porteous class' is of the same form

$$\tilde{P}^1_{-1} = \frac{1}{8}\tilde{\mu}^2 - \frac{1}{3}\nu + \frac{1}{12}\tilde{I},$$

where $\tilde{\mu} = \Sigma\cdot(\Lambda + \tilde{\lambda})/[X] = \mu([\Lambda_1 - 2\Lambda_2 - F]) = \mu(\alpha)$,

$\nu = \mu(point)$ and

$\tilde{I} = (\Sigma\cdot\Sigma)/[X] = I - \mu(F)^2$.

Use the blow-up formulae for the Donaldson polynomial of $K3$ surface, with the orientation as in (5.5), i.e. for $h\in H_2(\tilde{X},\mathbb{Z})$,

$$\tilde{\gamma}_0 = 1, \quad \tilde{\gamma}_2(\mu(h)^2) = (\tilde{Q}(h,h) + (f\cdot h)^2),$$

$$\tilde{\gamma}_4(\mu(h)^4) = 3\tilde{Q}(h,h)^2 + 6\tilde{Q}(h,h)\cdot(f\cdot h)^2 + (f\cdot h)^4.$$

where $\tilde{Q} = Q - f^2$ is the intersection form on the blown-up $K3$ surface. Theorem (5.5.2) gives the *spin*^{c}-polynomial to be

$$\begin{aligned}\tilde{\gamma}'_4(h^2) = \ & \tilde{\gamma}_4(\tilde{P}^1_{-1}\cup\mu(h)^2) - \tfrac{1}{12}\tilde{Q}(h,h)\cdot\tilde{\gamma}_0 \\ &= \tfrac{1}{4}(\tilde{Q}(\alpha,h)^2 + 2(f\cdot h)\cdot\tilde{Q}(\alpha,h) + (f\cdot h)^2) \\ &= \tfrac{1}{4}(Q(\alpha,h))^2.\end{aligned}$$

That is, $\tilde{\gamma}_4'(F^2) = 0$ and in particular, for $h \in H_2(X, \mathbb{Z})$,

$$\tilde{\gamma}_4'(h^2) = \gamma_4'(h^2) = \frac{1}{4}Q((\Lambda_1 - 2\Lambda_2), h)^2$$

which is consistent with the blow-up formulae for *spin*^{c}-invariant in [P, Theorem 1].

A.2 Elliptic Surfaces with no Multiple Fibre

Let S be a simply connected elliptic surface such that the two multiple fibres are of odd multiplicities. By analysing the moduli space of stable rank 2 bundles with odd fibre degree, Friedman [F, part III] deduces the 4-dimensional and the 8-dimensional Donaldson polynomials for such elliptic surfaces. In particular, when S has a section σ (i.e. no multiple fibre), he gives an explicit description of the moduli spaces of stable bundles V such that $\det V$ has the same restriction to the generic fibre as σ.[1] Using the results in [F, III.4], it is easy to obtain the *spin*^{c}-polynomials for elliptic surfaces without multiple fibre, which is then used to verify formula (6.1.1), for the cases $d = 2$ and 4, and formula (6.3.3).

Let S_n denotes the simply connected elliptic surface without multiple fibre with $n = P_g(S_n) + 1$. There is a section σ s.t. $\sigma^2 = -n$ and $\sigma \cdot F = 1$ where F is a general fibre. The canonical bundle $K_{S_n} = (n-2)F$. As in [F], consider the case $n = 2k$ is even only. For an even integer d let E be an $U(2)$ bundle over S_n with

$$c_1(E) = \Lambda = \sigma - \frac{d}{2}F \quad \& \quad c_2(E) = \frac{n}{2} = k,$$

then the moduli space $\mathcal{M}$ of instantons is of (real) dimension $2d$ (The case when n odd is identical by considering $c_1 = \sigma - (\frac{d}{2}+1)F$ and $c_2 = (n-1)/2$).

[1] For some suitable choice of an ample line bundle L, using the fact that a rank 2 bundle over S with odd fibre degree is L-stable iff its restriction to a general fibre is stable, the unique L-stable bundle V_0 in the 0-dimensional moduli space is constructed. The stable bundles in the higher dimensional moduli spaces are then given by elementary modifications of V_0.

For index $indD_A^\lambda = -1$:

The $spin^c$ structure to be used is $\lambda = \Theta((d-2)-2n)F$ (with the corresponding $L = \Theta((\frac{d-n}{2}-2)F))$. The indices of the coupled Dirac operators are -1 by the Atiyah-Singer index theorem. With the notations as in (5.4),

1) $\alpha = \Lambda + \lambda = \sigma + (\frac{d}{2} - 2 - 2n)F \implies (\Lambda+\lambda)^2 = -5n + d - 4.$

2) $\xi/[X] = c_2(E) - \frac{1}{4}c_1(E)^2 = \frac{n}{2} - \frac{1}{4}(-n-d) = \frac{3n}{4} + \frac{d}{4}.$

Hence, the Porteous class is

$$\begin{aligned} P_{-1}^1 &= C_2(ind((D^\lambda)^*, \underline{E})) \\ &= \tfrac{1}{8}\mu(\Lambda+\lambda)^2 + \tfrac{1}{12}I - (-\tfrac{-1}{2} + \tfrac{\xi}{3})\nu \\ &= \tfrac{1}{8}\mu(\alpha)^2 + \tfrac{1}{12}I - (\tfrac{n}{4} - \tfrac{1}{2} + \tfrac{d}{12})\nu. \end{aligned}$$

where $I = \sum_{i=1}^{b_2} \epsilon_i \mu(\alpha_i)^2$ and $\nu = \mu(point)$.

The case $d = 2$: From the arguments in section (4.1), the (numerical) $spin^c$-invariant $\gamma_2' = \sharp\mathcal{M}_4'$ is the same as evaluating the relevant Porteous class using the recipe of the Donaldson invariants, i.e. formula (6.1.1) when $d = 2$. The relevant Porteous class in this case is

$$P_{-1}^1 = \frac{1}{8}\mu(\alpha)^2 + \frac{1}{12}I - (\frac{n}{4} - \frac{1}{3})\nu.$$

Evaluating P_{-1}^1 using the formulae of Donaldson invariant for elliptic surfaces with no multiple fibre [F]

$$\gamma_0 = 1\ , \quad \gamma_2(h^2) = (h\cdot h) + (n-2)(F\cdot h)^2,$$

the results for each terms are

1) $\frac{1}{8}\gamma_2(\mu(\alpha)^2) = \frac{1}{8}\{(\alpha\cdot\alpha) + (n-2)(F\cdot\alpha)^2\}$
$= \frac{1}{8}\{(-5n-2) + (n-2)\} = \frac{1}{8}(-4n-4).$

2) $\frac{1}{12}\gamma_2(\sum_{i=1}^{b_2}\epsilon_i\mu(\alpha_i)^2) = \frac{1}{12}\sum_{i=1}^{b_2}\epsilon_i\gamma_2(\mu(\alpha_i)^2)$
$= \frac{1}{12}\sum_{i=1}^{b_2}\epsilon_i\{(\alpha_i\cdot\alpha_i)+(n-2)(F\cdot\alpha_i)^2\}$
$= \frac{1}{12}\{b_2+(n-2)F\cdot F\} = \frac{1}{12}(b_2) = \frac{1}{12}(12n-2).$

3) $-(\frac{n}{4}-\frac{1}{3})\gamma_2(\nu) = -(\frac{n}{4}-\frac{1}{3})(2)\gamma_0 = \frac{2}{3}-\frac{n}{2}.$

since elliptic surfaces are of simple type.

So the (numerical) $spin^c$-invariant as predicted by (6.1.1) is given by

$$\gamma_2(P^1_{-1}) = (-\frac{n}{2}-\frac{1}{2})+(n-\frac{1}{6})+(\frac{2}{3}-\frac{n}{2}) = 0.$$

As in [F, Theorem (III.4.7)], the moduli space of $\mathcal{L}$-stable rank 2 bundles with $c_1 = \sigma - F$ and $c_2 = \frac{n}{2} = k$ over S_n is isomorphic to S_n, here $\mathcal{L}$ is a suitable ample line bundles in the sense of [F].[2] For n is even and $k = n/2$, if V is such a stable bundle, then either there is an exact sequence

$$0 \longrightarrow \Theta_{S_n}((k-1)F) \longrightarrow V \longrightarrow \Theta_{S_n}(\sigma - kF)\otimes m_q \longrightarrow 0$$

with m_q the maximal ideal of a point q not lying on σ, or there is a non-split exact sequence

$$0 \longrightarrow \Theta_{S_n}(kF) \longrightarrow V \longrightarrow \Theta_{S_n}(\sigma + (-k-1)F) \longrightarrow 0.$$

If V corresponds to a 1-instanton, then $H^0(V\otimes L)\neq 0$ where $L = (\lambda\otimes K_{S_n}^{-1})^{\frac{1}{2}} = \Theta_{S^n}((-1-k)F)$ (note that $H^2(V\otimes L) = 0$ since stability is w.r.t a suitable line bundle). However, tensoring $\Theta_{S_n}((-1-k)F)$ to the two exact sequences above, gets

$$0 \longrightarrow \Theta_{S_n}((-2)F) \longrightarrow V\otimes L \longrightarrow \Theta_{S_n}(\sigma + (-2k-1)F)\otimes m_q \longrightarrow 0$$

and

$$0 \longrightarrow \Theta_{S_n}(-F) \longrightarrow V\otimes L \longrightarrow \Theta_{S_n}(\sigma + (-2k-2)F) \longrightarrow 0.$$

[2]Suitable ample line bundles always exist. As in [F, Lemma (I.2.3)], they can be obtained by tensoring an ample line bundle $\mathcal{L}_0$ by $\Theta_{S_n}(lF)$ where l is large. It can be thought of as polarisations which give the section a much larger degree than the fibres.

Since $h^0(\Theta_{S_n}(-2F)) = 0 = h^0(\Theta_{S_n}(-F))$ and $h^0(\Theta_{S_n}(\sigma + (-2k-1)F)) = 0 = h^0(\Theta_{S_n}(\sigma + (-2k-2)F))$ ([F, Lemma(III.4.1)]), by considering long exact sequence of cohomology, we have $H^0(V \otimes L) = 0$ for all such stable bundles. So, the (numerical) $spin^c$-invariant $\gamma_2' = 0$ and is equal to that obtained from the Porteous formula.

The case $d = 4$**:** Let E be a $U(2)$ bundle with $c_1(E) = \sigma - 2F$ and $c_2(E) = n/2 = k$, then the moduli space of instantons is of complex dimension 4. The $spin^c$ structure to be used is $\lambda = (2-2n)F$ which gives index -1. Formula (6.1.1), when $d = 4$, gives the $spin^c$-polynomial as

$$\gamma_4'(h^2) = \gamma_4(P^1_{-1} \cup \mu(h)^2) - \frac{1}{12}\gamma_0 \cdot Q(h,h)$$

where in this case the Porteous class

$$P^1_{-1} = \frac{1}{8}\mu(\alpha)^2 + \frac{1}{12}I - (\frac{n}{4} - \frac{1}{6})\nu$$

with $\alpha = \Lambda + \lambda = \sigma - 2nF$ such that $\alpha^2 = -5n$ and $\alpha \cdot F = 1$.

Evaluating P^1_{-1} using the formulae of Donaldson invariant for elliptic surfaces with no multiple fibre (see [F])

$$\gamma_0 = 1\ , \quad \gamma_4(h^4) = 3(h\cdot h)^2 + 6(n-2)(h\cdot h)(F\cdot h)^2 + (n-2)(3n-8)(F\cdot h)^4,$$

we have the $spin^c$-polynomial predicted by theorem (6.1.1) is

$$\begin{aligned}
&\gamma_4(\tfrac{1}{8}\mu(\alpha)^2\mu(h)^2 + \tfrac{1}{12}I\mu(h)^2 - (\tfrac{1}{4}n - \tfrac{1}{6})\nu\mu(h)^2) - \tfrac{1}{12}\gamma_0 \cdot Q(h,h)\\
&= \tfrac{1}{8}\{(\alpha\cdot\alpha)(h\cdot h) + 2(\alpha\cdot h)^2 + (n-2)[(\alpha\cdot\alpha)(F\cdot h)^2 + 4(\alpha\cdot h)(F\cdot\alpha)(F\cdot h)\\
&\qquad +(h\cdot h)(F\cdot\alpha)^2] + (n-2)(3n-8)(F\cdot\alpha)^2(F\cdot h)^2\}\\
&\quad +\tfrac{1}{12}\{(12n-2)(h\cdot h) + 2(h\cdot h) + (n-2)[(12n-2)(F\cdot h)^2 + 4(F\cdot h)^2]\}\\
&\quad -(\tfrac{1}{4}n - \tfrac{1}{6})(2)\{(h\cdot h) + (n-2)(F\cdot h)^2\}\\
&\quad -\tfrac{1}{12}(1)(h\cdot h)\\
&= \{\tfrac{1}{12}(h\cdot h) + \tfrac{1}{4}(\alpha\cdot h)^2 + \tfrac{1}{2}(n-2)(\alpha\cdot h)(F\cdot h) + \tfrac{1}{4}(n-2)^2(F\cdot h)^2\} - \tfrac{1}{12}(h\cdot h)\\
&= \tfrac{1}{4}(\alpha\cdot h)^2 + \tfrac{1}{2}(n-2)(\alpha\cdot h)(F\cdot h) + \tfrac{1}{4}(n-2)^2(F\cdot h)^2\\
&= \tfrac{1}{4}Q((\alpha + (n-2)F), h)^2.
\end{aligned}$$

When $n = 2$, this reduces to the case of $K3$ surface considered in chapter 5.

As in [F, p.101], the possible types of extension for a stable bundle V, w.r.t. a suitable line bundle, with the above topological data are listed below

$$(1) \quad 0 \longrightarrow \Theta_{S_n}((k-2)F) \longrightarrow V \longrightarrow \Theta_{S_n}(\sigma - kF) \otimes \mathcal{J}_z \longrightarrow 0$$

where Z is a codimension two local complete intersection subscheme of length 2.

$$(2) \quad 0 \longrightarrow \Theta_{S_n}((k-1)F) \longrightarrow V \longrightarrow \Theta_{S_n}(\sigma + (-k-1)F) \otimes m_q \longrightarrow 0$$

where m_q the maximal ideal of a point q.

$$(3) \quad 0 \longrightarrow \Theta_{S_n}(kF) \longrightarrow V \longrightarrow \Theta_{S_n}(\sigma + (-k-2)F) \longrightarrow 0.$$

$$(4) \quad 0 \longrightarrow \Theta_{S_n}((k-1)F) \otimes m_q \longrightarrow V \longrightarrow \Theta_{S_n}(\sigma + (-k-1)F) \longrightarrow 0.$$

If V corresponds to a 1-instanton, then $H^0(V \otimes L) \neq 0$ (again $H^2(V \otimes L) = 0$ since stability is w.r.t a suitable line bundle) where $L = (\lambda \otimes K_{S_n}^{-1})^{\frac{1}{2}} = \Theta(-kF)$. Tensoring L to the exact sequences above and consider long exact sequences of cohomology, only type (3) gives $H^0(V \otimes L) \neq 0$. By [F, Corollary (III.4.5)], all the non-split extensions in type (3) are stable and the set of all non-trivial extensions can be identified with $Sym^2(\sigma) \cong \mathbb{P}^2$. Similar to (5.3), consider the universal bundle $\mathbb{F} \longrightarrow \mathbb{P}^2 \times S_n$ given by

$$0 \longrightarrow \pi_{S_n}^*(\Theta_{S_n}(kF)) \otimes \pi_{\mathbb{P}^2}^*(\Theta_{\mathbb{P}^2}(1)) \longrightarrow \mathbb{F} \longrightarrow \pi_{S_n}^*(\Theta_{S_n}(\sigma + (-k-2)F)) \longrightarrow 0,$$

with $c_1(\mathbb{F}) = \pi_{S_n}^*(\sigma - 2F) + \pi_{\mathbb{P}}^*(h)$ and $c_2(\mathbb{F}) = \pi_{S_n}^*(\sigma + (-k-2)F) \cdot (\pi_{S_n}^*(kF) + \pi_{\mathbb{P}^2}^*(h))$. Therefore,

$$-\frac{1}{4}p_1(\mathbb{F}) = -\frac{1}{4}\{c_1(\mathbb{F})^2 - 4c_2(\mathbb{F})\} = -\frac{1}{2}\{(-\sigma + (n+2)F) \otimes h + \ldots\}$$

with terms do not affect the slant product are omitted, and

$$\begin{aligned} \mu(\Sigma) &= -\tfrac{1}{4}p_1(\mathbb{F})/[\Sigma] \\ &= \{\tfrac{1}{2}(\sigma - (n+2)F) \cdot \Sigma\}h \end{aligned}$$

in $\mathcal{M}'_8 = I\!P^2$. The $spin^c$-polynomial is

$$\gamma'_4(\Sigma^2) = \frac{1}{4}Q((\sigma - (n+2)F), \Sigma)^2 = \frac{1}{4}Q((\alpha + (n-2)F), \Sigma)^2$$

where $\alpha = \sigma - 2nF = \Lambda + \lambda$, hence verifies formula (6.1.1) for the case $d = 4$ for elliptic surfaces without multiple fibre.

For index $indD_A^\lambda = -2$**:**

Recall that, for even integers d and n, $U(2)$-bundle E over S_n with

$$c_1(E) = \Lambda = \sigma - \frac{d}{2}F \qquad \& \qquad c_2(E) = \frac{n}{2} = k$$

are considered so that $\dim_{I\!R}\mathcal{M} = 2d$. The $spin^c$ structure to be used here is $\lambda = \Theta((d - 4 - 2n)F)$ (with the corresponding $L = \Theta((-3 + \frac{d}{2} - \frac{n}{2})F)$), then the index $\chi(E \otimes L) = -2$.

In particular, when $d = 4$,

$$\alpha = \Lambda + \lambda = \sigma + (-2n-2)F \implies \alpha^2 = -5n - 4 \quad \& \quad \alpha \cdot F = 1$$

$$\xi = c_2(E) - \frac{1}{4}c_1(E)^2 = \frac{3n}{4} + 1$$

and so the relevant Porteous class is

$$\begin{aligned} P^1_{-2} &= -\{\tfrac{1}{48}\mu(\alpha)^3 + \tfrac{1}{24}I\mu(\alpha) + (\tfrac{1}{3} - \tfrac{\xi}{6})\nu\mu(\alpha)\} \\ &= -\{\tfrac{1}{48}\mu^3 + \tfrac{1}{24}I\mu + (\tfrac{1}{6} - \tfrac{n}{8})\nu\mu\}. \end{aligned}$$

Evaluating the Donaldson polynomial for elliptic surfaces, gets

$$\begin{aligned} &\gamma_4(P^1_{-2} \cup \mu(\Sigma)) \\ &= -\gamma_4(\tfrac{1}{48}\mu(\alpha)^3\mu(\Sigma) + \tfrac{1}{24}I\mu(\alpha)\mu(\Sigma) + (\tfrac{1}{6} - \tfrac{n}{8})\nu\mu(\alpha)\mu(\Sigma)) \\ &= -\{(-\tfrac{n}{4} - \tfrac{3}{8})(\alpha \cdot \Sigma) + \tfrac{(n-2)(-3n-5)}{12}(F \cdot \Sigma)\} - \{\tfrac{n}{2}(\alpha \cdot \Sigma) + (n-2)(\tfrac{n}{2} + \tfrac{1}{12})(F \cdot \Sigma)\} \\ &\quad -\{(-\tfrac{n}{4} + \tfrac{1}{3})(\alpha \cdot \Sigma) + \tfrac{(n-2)(-3n+4)}{12}(F \cdot \Sigma)\} \\ &= \tfrac{1}{24}(\alpha \cdot \Sigma). \end{aligned}$$

Similar to the index $= -1$ case, by tensoring $L = \Theta((-1-k)F)$ to the four possible types of extension for stable bundles, it is easy to see that the (transverse) moduli space $\mathcal{M}'_8 = \emptyset$ and so the $spin^c$-invariant in this case

$$\gamma'_4(\Sigma) = 0, \qquad \forall\ [\Sigma] \in H_2(X).$$

Hence, verifying theorem (6.3.3).

A.3 Dolgachev Surfaces

In the hope of getting the smooth classification of Dolgachev surfaces, Bauer and Pidstrigach [BP] compute a $spin^c$-polynomial for these surfaces. Their set-up is as follows:

Let S be a simply connected Dolgachev surface, i.e. a relatively minimal elliptic surface with geometric genus $P_g = 0$. Then the canonical class is $K_S = F - F_p - F_q$ where F denotes a general fibre, F_p and F_q the two multiple fibres of coprime multiplicity p and q respectively (where p and q can be 1).

Consider $U(2)$ bundle E on S with

$$c_1(E) = K_S + 2nk \quad , \quad c_2(E) = 2$$

where $k = \frac{1}{pq}F$ is the primitive class. Then the moduli space of instantons is of dimension

$$dim\mathcal{M} = 8c_2(E) - 2c_1(E)^2 - 3(1+b^+) = 16 - 6 = 10.$$

Since the coupled Dirac operators w.r.t. the canonical $spin^c$ structure $\lambda = -K_s$ has index 0, we have the moduli space $\mathcal{M}'$ of 1-instantons is of (real) dimension 8. The $spin^c$-polynomial $q_S(n)(h^4)$, using the notation of [BP], is defined as in section (3.2).

Note that $b^+ = 1$, the $spin^c$-polynomial is a chamber invariant only (see chapter 2). As usual, the chamber containing the primitive class $k = \frac{1}{pq}F$ in its closure is used (the 'suitable chamber' in Friedman's sense). Using algebro-geometric method, their results are

$$q_S(n) = a(n)Q^2 + b(n)Qk^2 + c(n)k^4$$

where $a(n) = 3n$

$$b(n) = (2p^2q^2 - 2p^2 - 2q^2 - 1)n.$$

They failed to compute the $c(n)$ term except that $c(n) = 21n$ for $p = q = 1$.

However, $\mathcal{M} \cap V_{h_1} \cap \cdots \cap V_{h_4}$ is of dimension 2 and is compact (since p and q are relatively prime, the associated $SO(3)$ bundle has $w_2 \neq 0$ and hence the well-definedness of the $spin^c$-polynomial), the Porteous formula can be used (see (4.1)). Using notations as in section (5.4), $\Lambda = c_1(E) = K_S + 2nk$, $\lambda = -K_S$ and so

1) $(\Lambda + \lambda) = 2nk \Longrightarrow (\Lambda + \lambda)^2 = 0$

2) $\lambda \cdot \lambda = \Lambda \cdot \Lambda = 0$

3) $\xi/[X] = C_2(E) - \frac{1}{4}C_1(E)^2 = 2$

4) $m_2/[X] = \frac{-1}{8}sgn(S) = \frac{-(-8)}{8} = 1$

and the relevant Porteous class is the first Chern class of the family of coupled Dirac operators (see (5.4))

$$\begin{aligned} P_0^1 &= \{z + (\tfrac{z\Lambda}{2} - \Sigma)\tfrac{1}{2}C_1(\lambda) + (\tfrac{z\Lambda^2}{8} - \tfrac{z\xi}{2} - \tfrac{\Sigma\Lambda}{2})\}/[X] \\ &= z - \tfrac{1}{2}\Sigma \cdot \lambda - z - \tfrac{1}{2}\Sigma \cdot \Lambda \\ &= -\tfrac{1}{2}\mu(\Lambda + \lambda) \\ &= -\tfrac{1}{2}\mu(2nk) = -n \cdot \mu(k). \end{aligned}$$

Hence,

$$q_S(n)(h) = -n \cdot \gamma_5(\mu(k)\mu(h)^4)$$

where γ_5 is the 10-dimensional $SO(3)$-Donaldson invariant on S, w.r.t. a metrics in the suitable chamber. (Unfortunately, the $SO(3)$-Donaldson invariant for Dolgachev surfaces are known only when w_2 restricted to k is non-zero (see [F]), which is not this case, so the above equality cannot be used to obtain the $c(n)$ term for $q_S(n)$.)

Appendix B

$H_3(\mathcal{C}_{x,A_0}, \mathbb{Z}) = \mathbb{Z}_{12} \oplus \mathbb{Z}_2$

A sketch of a proof of the above statement, provided by J.D.S. Jones, will be given in this appendix.

As in chapter 4, $\mathcal{C}_{x,A_0} = \{[A] \in \mathcal{B}^*_X \mid [A|_{X\backslash B_x}] = [A_0|_{X\backslash B_x}]\}$ for some small ball B_x about $x \in X$ and A_0 some fixed instanton of charge 1 less than that of $[A]$. Therefore, $\mathcal{C}_{x,A_0} \sim \tilde{\mathcal{B}}_{S^4}$ are of the same homotopy type. $H_i(\tilde{\mathcal{B}}_{S^4}, \mathbb{Z}) = H^i(\tilde{\mathcal{B}}_{S^4}, \mathbb{Q})$ is shown in [DK, (5.1.14)] to be trivial for all $i > 0$ (in fact this is sufficient for our application). As in [DK, AJ], $\tilde{\mathcal{B}}_{S^4} \sim \Omega^3_1(S^3) = \Omega$ the degree 1 component of $\Omega^3(S^3)$. Two facts about Ω will be needed in order to compute its third integral homology group:

1) $\pi_1(\Omega) = \pi_2(\Omega) = \mathbb{Z}_2$ and $\pi_3(\Omega) = \mathbb{Z}_{12}$,

2) There is a map

$$f : \Omega \longrightarrow K(\mathbb{Z}_2, 1) \times K(\mathbb{Z}_2, 2) = K$$

which induces isomorphism on π_1 and π_2. Here $K_i = K(\mathbb{Z}_2, i)$ are the Eilenberg-MacLane spaces.

Consider the fibration $F \longrightarrow \Omega \xrightarrow{f} K$. Using notations as in [BT, (III.15)], there is a homology spectral sequence $(E^r_{p,q}, d^r)$ with

$$E^2_{p,q} = H_p(K) \otimes H_q(F),$$

with differential

$$d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

By considering the exact sequence of homotopy groups and using the Hurewicz theorem,

$$H_1(F, \mathbb{Z}) = H_2(F, \mathbb{Z}) = 0,$$

$$H_3(F, \mathbb{Z}) = \pi_3(F) = \mathbb{Z}_{12} = \pi_3(\Omega).$$

To compute $H_*(K)$, using the facts that

$$H_i(K_1, \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if i=1,3} \\ 0 & \text{if i=2,4} \end{cases}$$

$$H_i(K_2, \mathbb{Z}) = \begin{cases} 0 & \text{if i=1,3} \\ \mathbb{Z}_2 & \text{if i=2,4} \end{cases}$$

together with the Kunneth formula

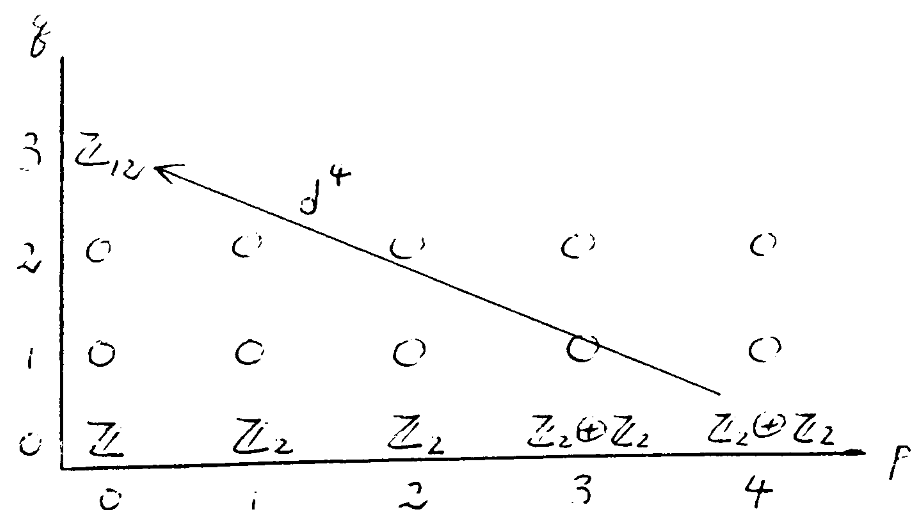
$$H_p(K) = H_p(K_1 \times K_2) = (\oplus_{i+j=p} H_i(K_1) \otimes H_j(K_2)) \oplus (\oplus_{i+j=p-1} Tor(H_i(K_1), H_j(K_2))),$$

we get

$$H_1(K, \mathbb{Z}) = H_2(K, \mathbb{Z}) = \mathbb{Z}_2,$$

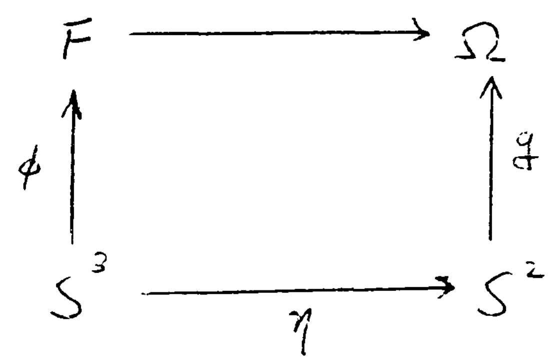
$$H_3(K, \mathbb{Z}) = H_4(K, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Hence the E^4-page is

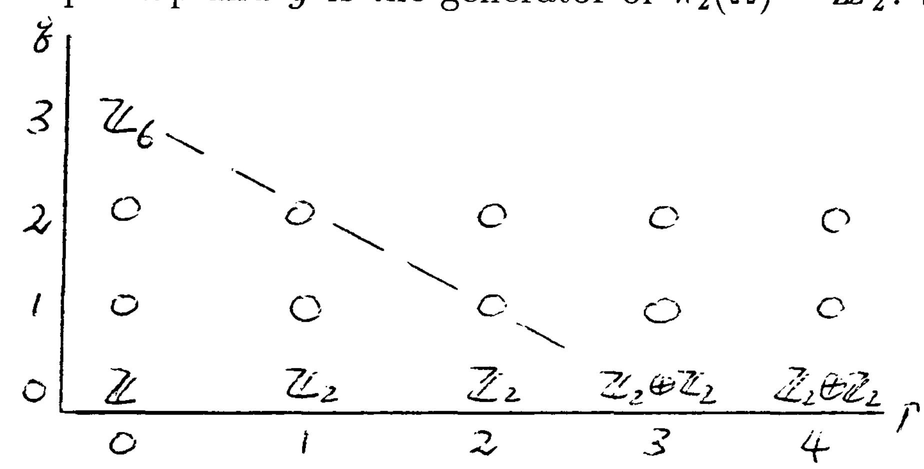


with $d^4 : E^4_{4,0} \longrightarrow E^4_{0,3}$ a non-zero differential because the kernel of the map $H_3(F) \longrightarrow$

$H_3(\Omega)$ is non-zero. This is because the generator ϕ of $H_3(F) = \pi_3(F) = \mathbb{Z}_{12}$ is actually mapped to an order 2 element in $H_3(\Omega)$, as can be seen from the commutative diagram



where η is the Hopf map and g is the generator of $\pi_2(\Omega) = \mathbb{Z}_2$. So, the E^5-page is



Hence, $H_3(\Omega, \mathbb{Z})$ is given by an extension

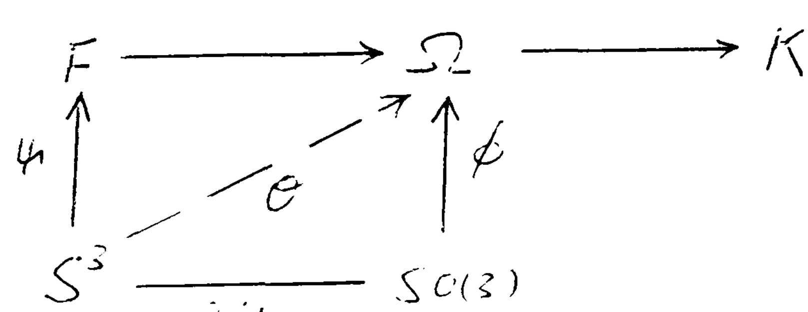
$$0 \longrightarrow \mathbb{Z}_6 \longrightarrow H_3(\Omega, \mathbb{Z}) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 0 \qquad (*).$$

Since there is an order 12 element in $H_3(\Omega, \mathbb{Z})$ as is to be shown below, this extension is non-split and so we have the result $H_3(\Omega, \mathbb{Z}) = \mathbb{Z}_{12} \oplus \mathbb{Z}_2$.

The 'framing' $SO(3) \longrightarrow \Omega$ induces a surjective map

$$\mathbb{Z} = \pi_3(SO(3)) \longrightarrow \pi_3(\Omega) = \mathbb{Z}_{12}.$$

Since $\pi_3(F) = \pi_3(\Omega) = \mathbb{Z}_{12}$, there is a commutative diagram



The order 6 element b in $\mathbb{Z}_6$ in the extension $(*)$ is mapped to an element $2h \in H_3(\Omega, \mathbb{Z})$, as the map $S^3 \longrightarrow SO(3)$ is 2:1. Hence, h is an order 12 element in $H_3(\Omega, \mathbb{Z})$. In fact, if b is the image of $\psi \in \pi_3(F) = H_3(F, \mathbb{Z})$ under the Hurewicz map, then $b = \psi_*[S^3] \in H_3(F, \mathbb{Z})$ is mapped to $\theta_*[S^3] \in H_3(\Omega, \mathbb{Z})$ by $(*)$. However, $\theta_*[S^3] = 2\phi_*[SO(3)]$. So, $h = \phi_*[SO(3)]$ is an order 12 element as required.

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