Abstract

A group Γ is of type F_n for some $n \geq 1$ if it has a classifying complex with finite n-skeleton. These properties generalise the classical notions of finite generation and finite presentability.

We investigate the higher finiteness properties for fibre products of groups. The central conjecture is this: If $N_1 \hookrightarrow \Gamma_1 \overset{\pi_1}{\twoheadrightarrow} Q$ and $N_2 \hookrightarrow \Gamma_2 \overset{\pi_2}{\twoheadrightarrow} Q$ are short exact sequences with N_1 of type F_n, Γ_1 and Γ_2 of type F_{n+1} and Q of type F_{n+2}, then the fibre product $P = \{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \mid \pi_1(\gamma_1) = \pi_2(\gamma_2)\}$ is of type F_{n+1}. This is called the n-$(n+1)$-$(n+2)$ Conjecture.

The fibre product is the basic construction that all subgroups of direct products can be built from inductively. The conjecture therefore has implications for the finiteness properties of these subgroups. In particular, we show that the n-$(n+1)$-$(n+2)$ Conjecture for nilpotent Q implies another open conjecture, the Virtual Surjections Conjecture: A subgroup $P \leq \Gamma_1 \times \cdots \times \Gamma_n$ of a direct product of groups of type F_k is itself of type F_k if it virtually surjects to k-tuples of factors, i.e. if the image of the projection $P \to \Gamma_{i_1} \times \cdots \times \Gamma_{i_k}$ has finite index for all $i_1, \ldots, i_k$ with $1 \leq i_1 < \cdots < i_k \leq n$.

We prove the n-$(n+1)$-$(n+2)$ Conjecture in many cases, such as when the quotient Q is abelian or when the epimorphism π_2 splits. We prove a form of the conjecture where the F_n-properties are replaced by weaker homological finiteness conditions. We prove a converse which states (under slightly stronger assumptions) that the fibre product can *only* be of type F_{n+1} if the quotient Q is of type F_{n+2}. We describe an explicit map in group homology whose kernel gives an obstruction to the n-$(n+1)$-$(n+2)$ Conjecture. We prove that the general conjecture can be reduced to the special case where Γ_2 is finitely generated free and for this case we construct a concrete finite, $(n+1)$-dimensional complex with fundamental group P, which is in a certain sense close to being n-aspherical (and might actually be n-aspherical).

Our results on the n-$(n+1)$-$(n+2)$ Conjecture have consequences for the Virtual Surjections Conjecture. For example, we show that any subgroup of a direct product that virtually surjects to k-tuples, where k is greater than half the number of factors, is of type F_k. We also prove that a subgroup that virtually surjects to k-tuples satisfies a finiteness condition on k-dimensional homology.

In order to establish these result, we present different approaches to the main problem, employing a range of methods from topology and homological algebra, each of which sheds light on the conjecture from a different angle.